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Algebraic topological invariants for Morse-Smale vector fields

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Coordinatore:
Chiar.mo Prof. **Leonardo Biliotti**

Tutore:
Chiar.ma Prof.ssa **Claudia Landi**

Dottorando: **Clemens Bannwart**

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Abstract

The goal of this thesis is to introduce new algebraic methods for the topological analysis of fields. We consider different special cases like gradient fields, gradient-like Morse-Smale vector fields, or Morse-Smale vector fields with closed orbits. We describe different ways for assigning a chain complex to a vector field and discuss connections to known invariants from topological data analysis.

We start by studying the category of tame epimorphic parametrized chain complexes, factored chain complexes for short, generalizing results from persistent homology to this setting, namely the structure theorem and the isometry theorem. We present a pipeline that produces a factored chain complex from a weighted based chain complex and apply this to the Morse complex in both the smooth and discrete settings. In combination with the structure theorem, this yields a tagged barcode for gradient-like Morse-Smale vector fields on compact Riemannian manifolds. We prove local stability and combinatorial approximation results, and describe connections to the classical persistence barcode in the special case of the gradient of a scalar field.

Going beyond the gradient-like case, we recall a method by Franks [27] to replace a closed orbit by a pair of fixed points via a local perturbation. We show that there are multiple non-equivalent ways of following this procedure and and describe the consequences of this non-uniqueness to the endeavour of assigning CW complexes or chain complexes to Morse-Smale vector fields in a canonical way.

In order to assign a chain complex to any Morse-Smale vector field, we use the filtration of the underlying manifold by unstable manifolds used by Smale in [50] and consider the spectral sequence in Čech homology associated with this filtration. We show that the terms on the first page of this spectral sequence admit canonical bases corresponding to the fixed points and closed orbits of the vector field. In the 2D case, we present a method to rearrange the algebraic information of the spectral sequence so as to obtain a canonical chain complex, whose homology agrees with the singular homology of the manifold. We derive the Morse inequalities from [50] from this chain complex and present a method for endowing it with bases in each degree.

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Introduction

In topological data analyis (TDA), the main idea is to take methods, constructions, and results from the field of algebraic topology and apply them to analyse and visualize various types of data. The main pipeline thereby goes as follows. We start with the data, which stems from some application. We find a way to interpret this data in a geometric way (which is sometimes more obvious and sometimes more far fetched) and then use this interpretation to build some kind of topological structure from the data, like a filtered topological space, a simplicial complex, etc. We then apply an algebraic invariant (very often homology with field coefficients) in order to transform the topological structure into an algebraic structure, allowing us to use knowledge from linear algebra, homological algebra, representation theory, or other fields of algebra, to decompose the algebraic object into indecomposables. Depending on the type of algebraic objects, we can characterize these indecomposables by some simple geometric objects, like intervals, and then use the resulting multiset of these geometric objects as a summary of the data we started with.

Different types of data

Data can come in different forms. A common form is that of a scalar field, where one is given a domain and to each point of this domain there is an associated value. For example, the domain might be some portion of the 3-dimensional space and the value is the temperature at each point. Or, one may consider a 2-dimensional area and let the associated value describe the altitude, giving a topographic description of a landscape. Mathematically, this is equivalent to studying a function $f\colon M \to \mathbb{R}$, where M is the domain, usually modelled as some topological space. Often, M has more structure than just a topology, for example M might be a smooth manifold endowed with a Riemannian metric. In these cases it can be helpful to take into account this additional structure when studying the function f.

From here we can generalize in various directions. For example, if one looks at multiple functions at the same time, then one is basically studying a function $f\colon M \to \mathbb{R}^n$, where n is the number of functions. In the case where M is a smooth manifold, we can also think of data as appearing in the form of a vector field, which means that to every point $p \in M$ we associate a vector in the tangent space T_pM.

Generalizing all of these cases into a unified theory, we can say that all the mentioned examples are instances of tensor fields. However, since they are so different in their appearance, very different methods are used to analyze each case.

Existing approaches in TDA

When studying data that comes in the form of a continuous function $f\colon M \to \mathbb{R}$, also called a scalar field, it is usual to analyze it through its persistence barcode, which is built in three steps: first, one filters M with respect to the f-sublevel sets; secondly, one considers homology with field coefficients in order to assign a vector space to each sublevel set, thus obtaining a persistence module; finally, one exploits the fact that tame persistence modules can be decomposed as finite direct sums of simpler persistence modules, whose support is given by real intervals, to define the persistence barcode of f as the multiset of such intervals [44]. Thanks to properties like stability and invariance, persistence barcodes have received much attention not only as a mathematical construct but also in applications outside of mathematics [32]. Stability refers to the fact that it is possible to endow both the set of functions as well as the set of persistence modules with distances so that the above-described pipeline is continuous. Invariance refers to the fact that precomposing f with a homeomorphism does not change the final persistence barcode.

In the case of multiple scalar fields, i.e. maps $f\colon M \to \mathbb{R}^n$, the first two steps of the persistence pipeline offer no difficulties and are well-understood in multi-parameter persistence, problems arising however when trying to produce a barcode from a multiparameter persistence module [12].

The main obstacle in generalizing persistence theory to vector fields arises already at the first step of the pipeline because of the lack of a natural filtration defined by a vector field. To circumvent this problem, one could think of embedding the manifold into $\mathbb{R}^n$ and treating the vector field as a multi-scalar field. This approach has the drawback of forgetting the contravariant nature of vector fields. Alternatively, following [18, 55], one can study a vector field through the sublevel sets of its norm function with the so-called well groups. A drawback of this approach is that it disregards the topological information carried by indices of singular points. Another recent approach for the study of vector fields has been presented in [23], where a chain complex is assigned to a generalized Morse-Smale vector field on a manifold. A problem with this approach have been pointed out in [1], namely that the boundary operator may not always square to zero.

Some recent papers are suggesting that it is more convenient to obtain persistence barcodes by directly decomposing the parametrized chain complex associated with the sublevel set filtration rather than decomposing persistence modules, see e.g. [53, 14, 40]. This has the advantage that additional information can be taken into acount that would otherwise be lost when applying homology. An often considered special case of this is the one of filtered chain complexes, where all the internal maps of a parametrized chain complex are monomorphisms, i.e. inclusions of one chain complex into another.

Our contribution

The goal of this thesis is to lay the groundwork for a theory of persistence for vector fields on Riemannian manifolds. We are restricting our attention to Morse-Smale vector fields. Roughly speaking, these are vector fields which a have a finite number of fixed points and closed orbits, such that in a neighbourhood of each of these the vector field takes a simple form, and globally these interact in a generic way. These vector fields are well-studied and much is known about their structure. They are structurally stable and thus preserve their structural properties under small perturbations. In two dimensions they lie densely in the space of all vector fields, i.e. any vector field can be approximated arbitrarily well by Morse-Smale vector fields [45]. We further differentiate between the gradient-like case, where there are no closed orbits, and the general case.

Our setup is that of working with chain complexes associated with Morse-Smale vector fields. Where they are already available from the literature such as the Morse complexes of gradient-like vector fields (see e.g. [3, 49]), we propose constructions to turn them into parametrized chain complexes, with the difference that our parametrized chain complexes have epimorphic rather than the usual monomorphic internal maps that come from filtrations. Even if this may look very different from the usual persistence approach based on filtrations, we point out that quotients can be seen as simplifications of the initial chain complex, in line with the original idea of persistence as a topological simplification method [21]. Sometimes, when no canonical chain complex is available from the literature as in the case of Morse-Smale vector fields with closed orbits, we propose a construction with particular attention to their topological significance. More precisely, the structure of the thesis and its main results are as follows.

In Chapter 1, we review the necessary background about parametrized objects, the generalized interleaving and bottleneck distances, dynamical systems, and Morse homology in both the smooth and discrete setting.

In Chapter 2, we study the category of tame epimorphic parametrized chain complexes, which we call factored chain complexes for brevity. These are functors that take real values to chain complexes, with internal maps all epimorphisms, and that may fail to be isomorphisms at most at finitely many times. We show that objects in this category can be decomposed into simple direct summands enumerated by a multiset of tagged real intervals, thus yielding a tagged barcode (see Section 2.3). After extending the standard interleaving and bottleneck distances to this setting, we prove the isometry theorem stating that the interleaving distance between two factored chain complexes is equal to the bottleneck distance between their tagged barcodes (see Section 2.4). We give general procedures that start from a chain complex with chosen bases and weights on them, and constructs a factored chain complex (see Section 2.5). We prove stability for these constructions, in the sense that, if we start from two isomorphic based chain complexes, equipped with a generic set of weights, the corresponding factored chain complexes have tagged barcodes whose bottleneck distance is a continuous function of the weights (see Section 2.6).

In Chapter 3 we confine ourselves to gradient-like Morse-Smale vector fields on a compact Riemannian manifold, to which we can assign a barcode of tagged intervals, using our construction. In this case, the chain complex is the Morse complex and the weights are given by the distances between singular points of the field. Interestingly, a genericity condition on the weights is now needed, in the form of no two pairs of singular points having the same distance, to guarantee the uniqueness of the resulting tagged barcode (cf. Figure 3.3). This marks a key difference with the gradient case, where it is not needed. We show that the map taking such a vector field to its tagged barcode is continuous, thus proving stability (see Section 3.3). Moreover, two topologically equivalent vector fields with isometric singular sets have the same barcode, thus yielding invariance. Thanks to the generality of the approach, we can also apply it to combinatorial gradient-like vector fields [25]. Our third contribution is that the tagged barcode of a smooth vector field on a compact Riemannian manifold can be approximated arbitrarily well, in terms of bottleneck distance, by the tagged barcodes of combinatorial vector fields defined on sufficiently refined triangulations of the manifold (see Section 3.4). We exemplify how to derive a barcode of tagged intervals for a scalar field f, when f is a Morse-Smale function, starting from its usual Morse complex generated by critical points. In this case, the weights are given by the differences in function values and the resulting tagged barcode can be related to the classical persistence barcode of f (see Section 3.5).

In Chapter 4, we start examining Morse-Smale vector fields with closed orbits. First, we recall a method by Franks [27] to replace a closed orbit of index k by a pair of fixed points of indices k and $k + 1$ via a local perturbation. Applying this to every closed orbit, this yields a method of turning any Morse-Smale vector field into a gradient-like Morse-Smale vector field. We present examples that showcase the fact that there is no a priori canonical way of doing this replacement and that different choices can lead to different gradient-like vector fields, for which the associated CW-decompositions or chain complexes do not agree. In particular, this points out errors in [27] and [23]. We give an alternative proof that the differential defined in [23] squares to zero in two dimensions. The resulting homology may differ from the singular homology of the underlying manifold, however. We also present a three-dimensional example, where the differential does not square to zero. These difficulties motivate the use of a different strategy in the case with closed orbits.

In Chapter 5, we present an original approach to assign a chain complex to a Morse-Smale vector field. For this, we first recall some facts about Čech homology, spectral sequences, and linear algebra (see Sections 5.1 to 5.3). We then consider the induced filtration of the underlying manifold by unstable manifolds, to which we associate the corresponding spectral sequence. The first page of this spectral sequence consists of the relative Čech homology of adjacent steps of the filtration and we show that they can be endowed with canonical bases corresponding to the fixed points and closed orbits of the vector field (see Section 5.4). In the two-dimensional case, by rearranging the topological information of the spectral sequence, we obtain a chain complex (see Sections 5.5 and 5.6). Some important properties of this chain complex are listed in Theorem 5.28, namely that the homology of the chain complex is isomorphic to the homology of the underlying manifold, and the dimensions of the vector spaces can be obtained by summing the fixed points and closed orbits of certain indices. From this chain complex we can derive the Morse inequalities from [50] (Corollary 5.30). We also present a way to endow this chain complex with bases in each degree. Finally, as a proof of concept, we write down the chain complex explicitly with respect to these bases in two different examples of Morse-Smale vector fields on the 2-sphere.

The results presented in this thesis are partially available as preprints on arXiv. The content of Chapters 2 and 3 can be found in [2] and the content from Chapter 4 can be found in [1].

Chapter 1

Preliminaries

In this section, we explain the necessary background material and fix our notation. We provide either proofs or references for all results.

We use the convention that 0 is a natural number, i.e. $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. We fix an arbitrary field $\mathbb{F}$. For specific examples we will use the choice $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. We denote by $[0, \infty)$ the poset category of the totally ordered set of non-negative real numbers. We will often use the symbol $\mathbb{1}$ to denote the identity. The symbol $\cong$ is used to denote isomorphisms in any category.

1.1 Parametrized objects

Parametrized objects are a generalization of persistence modules and have been described in [10] under the name of generalized persistence modules. This setting is convenient for us because it allows us to introduce some concepts in a higher generality so that we can then apply them in the case of parametrized vector spaces and also parametrized chain complexes.

Let $\mathcal{C}$ be a category. A **parametrized object of $\mathcal{C}$** is a functor $X\colon [0, \infty) \to \mathcal{C}$. This means that for all $t \geq 0$ we have an object X^t of $\mathcal{C}$ and for any pair $s \leq t \in [0, \infty)$ we have a morphism $X^{s\leq t}\colon X^s \to X^t$ in $\mathcal{C}$. These morphisms are called the **internal morphisms** (or **internal maps**) of X. The functoriality translates to the conditions that $X^{t\leq t}$ is the identity on X^t and whenever $0 \leq r \leq s \leq t$, then we have $X^{r\leq t} = X^{s\leq t} \circ X^{r\leq s}$. A **parametrized morphism** $\varphi\colon X \to Y$ between two parametrized objects X, Y of $\mathcal{C}$ is a natural transformation of functors. This means that for all $t \geq 0$, $\varphi^t\colon X^t \to Y^t$ is a morphism in $\mathcal{C}$ and for $s \leq t$ we have $\varphi^t \circ X^{s\leq t} = Y^{s\leq t} \circ \varphi^s$.

By Vect we denote the category of finite dimensional vector spaces and linear maps and by Ch we denote the category of compact, non-negative chain complexes with coefficients in $\mathbb{F}$, where a chain complex $C_\bullet$ is called **non-negative**, if $C_i = 0$ for $i < 0$, and **compact**, if $\dim C_i < \infty$ for all i and $C_i = 0$ for all but finitely many i. In this paper, we assume all chain complexes to be compact, non-negative, and with field coefficients, so we will usually omit these adjectives. We will only consider parametrized objects in the categories Vect, where we talk about parametrized vector spaces and parametrized (linear) maps, and Ch, where we talk about parametrized chain complexes and parametrized (chain) maps.

Definition 1.1. Let $X\colon [0, \infty) \to \mathcal{C}$ be a parametrized object of a category $\mathcal{C}$. We say that X is

- **left-constant at** $t \in [0, \infty)$, if either $t = 0$ or there exists a real number $\varepsilon > 0$ such that for all $0 \leq \delta \leq \varepsilon$, the internal map $X^{t-\delta \leq t}$ is an isomorphism,
- **right-constant at** $t \in [0, \infty)$, if there exists a real number $\varepsilon > 0$ such that for all $0 \leq \delta \leq \varepsilon$, the internal map $X^{t \leq t+\delta}$ is an isomorphism,
- **tame**, if X is right-constant at every $t \in [0, \infty)$, and left-constant everywhere but on a finite set.
- **epimorphic**, if all the internal maps are epimorphisms,
- **monomorphic**, if all the internal maps are monomorphisms.

We denote the category of parametrized objects and parametrized morphisms of $\mathcal{C}$ by P$\mathcal{C}$. We denote by TP$\mathcal{C}$ and TEP$\mathcal{C}$ the full subcategories of tame parametrized objects and tame epimorphic parametrized objects, respectively. Note that a parametrized object X is tame if and only if there exists a sequence

$$0 = t_0 < t_1 < \cdots < t_r < t_{r+1} = \infty$$

such that, for any $i = 0, \ldots, r$ and $t_i \leq s \leq t < t_{i+1}$, the map $X^{s \leq t}$ is an isomorphism. Since we are working mostly with the category of chain complexes, we introduce also some shorter terminology for that case: We say **filtered chain complex** for a tame monomorphic parametrized chain complex and **factored chain complex** for a tame epimorphic parametrized chain complex. We note that of these two terms only the former is well-established in the literature.

1.1.A The generalized interleaving distance.

The interleaving distance was originally introduced as a distance between persistence modules [16], but was later generalized in various ways. Relevant for us is the case of parametrized objects [11, 10]. We repeat the definition here.

Given a parametrized object $X \colon [0, \infty) \to \mathcal{C}$ and $\varepsilon > 0$, we define its **ε-shift** as X_ε, where $(X_\varepsilon)^t := X^{t+\varepsilon}$ and $(X_\varepsilon)^{s \leq t} := X^{s+\varepsilon \leq t+\varepsilon}$. Given a parametrized morphism $\phi \colon X \to Y$, we define its **ε-shift** as $\phi_\varepsilon \colon X_\varepsilon \to Y_\varepsilon$, where $(\phi_\varepsilon)^t := \phi^{t+\varepsilon}$. Given two parametrized objects $X, Y \in$ P$\mathcal{C}$, an **ε-interleaving** between X and Y is a pair (ϕ, ψ) of parametrized morphisms $\phi \colon X \to Y_\varepsilon$ and $\psi \colon Y \to X_\varepsilon$, such that $(\psi_\varepsilon \circ \phi)^t = X^{t \leq t+2\varepsilon}$ and $(\phi_\varepsilon \circ \psi)^t = Y^{t \leq t+2\varepsilon}$ for all $t \geq 0$. If there exists an ε-interleaving between X and Y we say that X and Y are **ε-interleaved**. Note that if $\delta > \varepsilon$ and X and Y are ε-interleaved, than it follows that they are also δ-interleaved. We say that X and Y are **interleaved** if there exists a real number $\varepsilon \geq 0$ such that X and Y are ε-interleaved.

Given two parametrized objects $X, Y \in$ P$\mathcal{C}$, their **interleaving distance** is defined as

$$d_I(X, Y) := \inf\{\varepsilon > 0 | \ \exists \ \varepsilon\text{-interleaving between } X \text{ and } Y\},$$

where we use the convention that $\inf \emptyset = \infty$. The interleaving distance is an extended pseudometric on P$\mathcal{C}$ and induces an extended metric on isomorphism classes of TP$\mathcal{C}$ as can be seen by adapting the proof of [17, Proposition 5.3] or the arguments after [44, Definition 3.3] by replacing the index set $\mathbb{R}$ by $[0, \infty)$ and vector spaces and linear maps by objects and morphisms of $\mathcal{C}$. See also [10] for a proof in the categorical setting.

1.1.B The generalized bottleneck distance

Similar to the interleaving distance, also the bottleneck distance has been generalized [39]. We give the basic definitions in large generality here but will later consider only two special cases: Multisets of intervals for the barcodes of parametrized vector spaces, and multisets of tagged intervals for the barcodes of factored chain complexes.

A **multiset** is a pair $\mathcal{A} = (A, m)$, where A is a set and $m\colon A \to \mathbb{N}_{\geq 1}$ is a function. For any element $I \in A$, the value $m(I)$ represents the multiplicity of I in $\mathcal{A}$. Following [4], we define the **representation** of $\mathcal{A}$ as the set

$$\operatorname{Rep}(\mathcal{A}) := \{(I, k) \in A \times \mathbb{N}_{\geq 1} \mid m(I) \geq k\}.$$

Given a set S, we denote by $\operatorname{Mult}(S)$ the set of all multisets (A, m) for which $A \subset S$. We sometimes treat multisets on S as if they were sets and leave it to the reader to make the necessary adjustments. Explicitly, this can usually be done by identifying all multisets with their representations.

Assume that $c\colon S \times S \to [0, \infty]$ is an extended pseudometric on S and $W\colon S \to [0, \infty]$ is a function on S, such that c and W are **compatible**, which means that for all $s_1, s_2 \in S$ we have

$$|W(s_1) - W(s_2)| \leq c(s_1, s_2).$$

Then we can define the corresponding bottleneck distance between multisets on S. Given $\mathcal{A}, \mathcal{B} \in \operatorname{Mult}(S)$, and identifying $\mathcal{A}, \mathcal{B}$ with their representations, a **matching** between them is a subset $\mathcal{M} \subseteq \mathcal{A} \times \mathcal{B}$ such that for all $I \in \mathcal{A}$ there is at most one $J \in \mathcal{B}$ such that $(I, J) \in \mathcal{M}$ and for all $J' \in \mathcal{B}$ there is at most one $I' \in \mathcal{A}$ such that $(I', J') \in \mathcal{M}$. If $(I, J) \in \mathcal{M}$, then we say that I and J are **matched**, all other elements of $\mathcal{A}$ and $\mathcal{B}$ are called **unmatched**. The **cost** of the matching $\mathcal{M}$ is defined as

$$\operatorname{cost}(\mathcal{M}) := \max \left\{ \sup_{(I,J) \in \mathcal{M}} c(I, J), \sup_{I \in \mathcal{A} \cup \mathcal{B} \text{ unmatched}} W(I) \right\}.$$

Given $\varepsilon > 0$, we call $\mathcal{M}$ an **ε-matching** if $\operatorname{cost}(\mathcal{M}) \leq \varepsilon$. The **bottleneck distance** between $\mathcal{A}$ and $\mathcal{B}$ is defined as

$$\begin{aligned} d_B(\mathcal{A}, \mathcal{B}) &:= \inf\{\operatorname{cost}(\mathcal{M}) \mid \mathcal{M} \subseteq \mathcal{A} \times \mathcal{B} \text{ is a matching}\} \\ &= \inf\{\varepsilon > 0 \mid \exists\ \varepsilon\text{-matching between } \mathcal{A} \text{ and } \mathcal{B}\}. \end{aligned}$$

Remark 1.2. If we are given a set S, an extended pseudometric $c\colon S \times S \to [0, \infty]$, and a function $W\colon S \to [0, \infty]$, such that c and W are compatible, then the bottleneck distance d_B is an extended pseudometric on $\operatorname{Mult}(S)$.

1.1.C Parametrized vector spaces.

Now we review the structure theorem and the isometry theorem from persistent homology in the language of parametrized vector spaces. This has two purposes. One is to motivate the analogous results for factored chain complexes and the other is that we want to use the results for parametrized vector spaces when proving the corresponding results for factored chain complexes.

In TDA literature, parametrized vector spaces are usually called **persistence modules**. Often another totally ordered set is used instead of $[0, \infty)$. Popular choices are $\mathbb{R}$, $\mathbb{Z}$,

$\mathbb{N}$, or finite totally ordered sets. This choice of poset leads to small technical differences, but the overall flavour of the resulting theory is the same.

In this paper, by an **interval** we always mean a half-open interval of non-negative real numbers, i.e. a subset $I \subseteq [0, \infty)$ of the form $I = [s, t)$ for some $0 \leq s < t \leq \infty$. We denote the set of all intervals by $\mathcal{I}$. Now we define the interval functors in TPVect.

Definition 1.3. Given $0 \leq s < t \leq \infty$, we define the **interval functor in** TPVect, denoted $\mathbb{F}[s, t)$, to be the parametrized vector space whose vector spaces and internal maps are given by

$$\mathbb{F}[s,t)^r := \begin{cases} \mathbb{F}, & \text{if } s \leq r < t, \\ 0, & \text{otherwise,} \end{cases} \qquad \mathbb{F}[s,t)^{q \leq r} := \begin{cases} \mathbb{1}_{\mathbb{F}}, & \text{if } s \leq q \leq r < t, \\ 0, & \text{otherwise.} \end{cases}$$

We state the structure theorem in TPVect. A proof can be found in [44].

Theorem 1.4 (Structure theorem in TPVect)**.** *Any tame parametrized vector space V is isomorphic to a finite direct sum of interval functors in* TPVect*, i.e. there exists a unique finite multiset* Bar = Bar(V) $\in$ Mult($\mathcal{I}$) *of intervals such that*

$$V \cong \bigoplus_{[s,t) \in \mathrm{Bar}} \mathbb{F}[s,t).$$

The multiset Bar *is called the* ***persistence barcode*** *of V.*

Let us now recall the isometry theorem in TPVect, which is another classical result in TDA. For this, we first introduce the interleaving distance and the bottleneck distance for tame parametrized vector spaces. The former is defined by applying the more general definition given in Section 1.1.A above to the case of the category $\mathcal{C} = \mathrm{Vect}$. For the latter, we want to apply the definition presented in Section 1.1.B. We thus need to define a pseudometric and a weight function for multisets of intervals. Given two intervals $I = [s, t)$ and $I' = [s', t')$, we define the **cost of matching** I **to** I' as $c(I, I') := \max\{|s-s'|, |t-t'|\}$. The **weight of** I is defined as $W(I) := \frac{t-s}{2}$. One can check that c defines an extended metric on multisets of intervals and that c and W are compatible. We thus get a bottleneck distance d_B on multisets of intervals by Remark 1.2.

The following theorem is another classical result in TDA, a proof can be found in [44]. The inequality $\geq$ was first shown in [16] and the inequality $\leq$ was first shown in [37].

Theorem 1.5 (Isometry Theorem in TPVect)**.** *If V and W are tame parametrized vector spaces, then*

$$d_I(V, W) = d_B(\mathrm{Bar}(V), \mathrm{Bar}(W)).$$

1.2 Decomposition of chain complexes.

If we have not explicitly assigned a symbol to the differential in a chain complex $C_\bullet$, then we write ∂_n^C for the differential in degree n. If the degree and/or the chain complex are clear from the context, then we sometimes write only ∂^C or ∂_n or even just ∂ instead.

Given $n \geq 1$, the n**-disk of** Ch is the chain complex $D^n = (D^n_\bullet, \partial_\bullet)$ with

$$D^n_k = \begin{cases} \mathbb{F}, & \text{if } k = n, n-1, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \partial_k = \begin{cases} \mathbb{1}_{\mathbb{F}}, & \text{if } k = n, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{for all } k \in \mathbb{Z}.$$

Given $n \geq 0$, the n**-sphere of** Ch is defined as the chain complex $S^n = (S^n_\bullet, \partial_\bullet)$ with

$$S^n_k = \begin{cases} \mathbb{F}, & \text{if } k = n, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \partial_k = 0, \qquad \text{for all } k \in \mathbb{Z}.$$

It is known (see e.g. [56]) that every chain complex $X \in$ Ch can be written as a finite direct sum of disks and spheres in a unique way up to isomorphism, since we are working with field coefficients. In particular,

- # of n-spheres $= \dim(H_n(X))$,
- # of n-disks $= \dim(\operatorname{im}(\partial_n \colon X_n \to X_{n-1}))$.

Given a chain complex $C_\bullet$ and a basis $\mathcal{B} = \{b_1, \ldots, b_r\}$ for C_{k-1}, we denote by $\langle \cdot, \cdot \rangle$ the scalar product on C_{k-1} induced by the basis $\mathcal{B}$. If we are given an element $a \in C_k$ and a basis element $b_j \in \mathcal{B}$, writing $\langle \partial a, b_j \rangle \neq 0$ hence means that $\partial a = \lambda_1 b_1 + \cdots + \lambda_r b_r$ with $\lambda_j \neq 0$, i.e. b_j appears in the boundary of a. Given a vector space V and $x \in V$, we denote by $\operatorname{Span}(x)$ the linear subspace of V generated by x.

Later, we will need the following result, whose proof is well-known (see, e.g., [35]).

Lemma 1.6. *If $C_\bullet$ is a chain complex and $a \in C_n$ with $\partial a \neq 0$, then the sequence $\overline{C}_\bullet$ of linear maps*

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r] & C_n / \operatorname{Span}(a) \arrow[r] & C_{n-1} / \operatorname{Span}(\partial a) \arrow[r] & C_{n-2} \arrow[r] & \cdots \end{tikzcd}$$

naturally induced by the differentials in $C_\bullet$ is a chain complex in Ch*. Moreover, there exists an epimorphic chain map $q \colon C_\bullet \to \overline{C}_\bullet$,*

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & C_{n-2} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & C_{n+1} \arrow[r] & C_n / \operatorname{Span}(a) \arrow[r] & C_{n-1} / \operatorname{Span}(\partial a) \arrow[r] & C_{n-2} \arrow[r] & \cdots , \end{tikzcd}$$

where the vertical maps are either the identity maps or quotient maps. Moreover, assume that we are given a basis $\mathcal{B}_k$ for each C_k, such that $a \in \mathcal{B}_n$. Further, we are given $b \in \mathcal{B}_{n-1}$ such that $\lambda := \langle \partial a, b \rangle \neq 0$. Then, defining

$$\overline{\mathcal{B}}_n = \{[a'] \mid a' \in \mathcal{B}_n \backslash \{a\}\}, \qquad \overline{\mathcal{B}}_{n-1} = \{[b'] \mid b' \in \mathcal{B}_{n-1} \backslash \{b\}\}, \qquad \overline{\mathcal{B}}_k = \mathcal{B}_k \text{ for } k \neq n, n-1,$$

yields bases $\overline{\mathcal{B}}_k$ for the vector spaces $\overline{C}_k$. If, for all k, we denote by M_k the matrix that represents $\partial^C_k \colon C_k \to C_{k-1}$ with respect to the bases $\mathcal{B}_k$ and $\mathcal{B}_{k-1}$ and by $\overline{M}_k$ the matrix that represents $\partial^{\overline{C}}_k \colon \overline{C}_k \to \overline{C}_{k-1}$ with respect to the bases $\overline{\mathcal{B}}_k$ and $\overline{\mathcal{B}}_{k-1}$, then:

(i) $\overline{M}_k = M_k$ for $k \geq n+2$ and for $k \leq n-2$.

(ii) $\overline{M}_{n+1}$ is obtained from M_{n+1} by deleting the row corresponding to $a \in \mathcal{B}_n$.

(iii) To obtain $\overline{M}_n$ from M_n, multiply the row of b by $\langle \partial a, b \rangle^{-1}$, and then subtract from the row of every $b' \in \mathcal{B}_{n-1}$, the row of b multiplied by $\langle \partial a, b' \rangle$. Finally, delete the column corresponding to $a \in \mathcal{B}_n$ and the row corresponding to $b \in \mathcal{B}_{n-1}$.

(iv) $\overline{M}_{n-1}$ is obtained from M_{n-1} by deleting the column corresponding to $b \in \mathcal{B}_{n-1}$.

The proof consists of standard linear algebra arguments, but we include it for the sake of completeness.

Proof. The chain complex $D_\bullet$ defined by $D_n = \operatorname{Span}(a)$, $D_{n-1} = \operatorname{Span}(\partial a)$, and $D_k = 0$ for $k \neq n, n-1$, is a subcomplex of $C_\bullet$. The chain complex $\overline{C}_\bullet$ is equal to the quotient chain complex $C_\bullet/D_\bullet$ and the chain map q is equal to the quotient map. It thus follows from the general theory of chain complexes that $\overline{C}_\bullet$ is a chain complex and that q is an epimorphic chain map.

We now check that $\overline{\mathcal{B}}_k$ is a basis of $\overline{C}_k$ for all i. The only non-trivial cases are $k = n$ and $k = n-1$. In both of these, the dimension of $\overline{C}_k$ is one less that the dimension of C_k and also the size of $\overline{\mathcal{B}}_k$ is one less than the size of $\mathcal{B}_k$, so it suffices to check that $\overline{\mathcal{B}}_k$ generates all of $\overline{C}_k$. For this it remains to show that $[a] \in \operatorname{Span}(\overline{\mathcal{B}}_n)$ and $[b] \in \operatorname{Span}(\overline{\mathcal{B}}_{n-1})$. The former is true because $[a] = 0$ in $\overline{C}_n$. For the latter, assume that $\mathcal{B}_{n-1} = \{b_1, \ldots, b_r\}$ with $b = b_j$. Then, since $\langle \partial a, b\rangle \neq 0$, we have $\partial a = \lambda_1 b_1 + \cdots + \lambda_r b_r$ with $\lambda_j \neq 0$. Therefore $0 = [\partial a] = \lambda_1[b_1] + \cdots + \lambda_r[b_r]$ and so $[b_j] = -\lambda_j^{-1}(\lambda_1[b_1] + \cdots + \lambda_r[b_r]) \in \operatorname{Span}(\overline{\mathcal{B}}_{n-1})$.

It remains to prove the statements about the representation matrices. (i) holds because in these cases, $\overline{C}_k = C_k$, $\overline{C}_{k-1} = C_{k-1}$, $\overline{\mathcal{B}}_k = \mathcal{B}_k$, $\overline{\mathcal{B}}_{k-1} = \mathcal{B}_{k-1}$ and $\partial_k^{\overline{C}} = \partial_k^C$. (ii) follows from elementary linear algebra arguments.

As for (iii), note that replacing b by ∂a in $\mathcal{B}_{n-1}$ yields a new basis $\mathcal{B}'_{n-1}$ for C_{n-1}. The matrix M'_n that represents ∂_n^C with respect to $\mathcal{B}_n$ and $\mathcal{B}'_{n-1}$ can be obtained from M_n by dividing the row of b by $\langle \partial a, b\rangle$ and then, for the row of every $b' \in \mathcal{B}_{n-1}$, subtracting from it $\langle \partial a, b'\rangle$ times the row of b. Now, since $\mathcal{B}'_{n-1}$ contains ∂a as a basis element, to obtain the representation matrix of the induced map from $C_n/\operatorname{Span}(a)$ to $C_{n-1}/\operatorname{Span}(\partial a)$ we can just delete the column corresponding to a and the row corresponding to ∂a.

To check (iv), we can consider the basis $\mathcal{B}'_{n-1}$ of C_{n-1} again and denote by M'_{n-1} the matrix that represents ∂_{n-1}^C with respect to $\mathcal{B}'_{n-1}$ and $\mathcal{B}_{n-2}$. This matrix can be obtained from M_{n-1} by scaling the column in M_{n-1} and adding to it scalar multiples of other columns. Since $\partial(\partial a) = 0$, the column in M'_{n-1} corresponding to ∂a is zero. The matrix $\overline{M}_{n-1}$ can be obtained from M'_{n-1} by deleting that column. Since all other columns of M'_{n-1} agree with M_{n-1}, we can also obtain $\overline{M}_{n-1}$ directly from deleting the column corresponding to b from M_{n-1}, which is what we wanted to show. ☐

Using the epimorphic chain map $q\colon C_\bullet \to \overline{C}_\bullet$ from Lemma 1.6, we get the following result, the content of which is well-known as well (see e.g. [56, Section 1.4]). Again we include a proof for the sake of completeness.

Lemma 1.7. *Let $C_\bullet$ be a chain complex and let $\mathcal{B}_k$ be a basis of C_k, for all k. Let $a \in \mathcal{B}_n$ and $b \in \mathcal{B}_{n-1}$ such that $\lambda := \langle \partial a, b\rangle \neq 0$. Then, there is a short exact sequence of chain complexes*

$$0 \longrightarrow D^n \xrightarrow{\ \iota\ } C_\bullet \xrightarrow{\ q\ } \overline{C}_\bullet \longrightarrow 0,$$

where ι is defined by mapping the generator $1_n \in D^n_n$ to $a \in C_n$ and the generator $1_{n-1} \in D^n_{n-1}$ to $\partial a \in C_{n-1}$. Moreover, the sequence splits.

Proof. It is straightforward to check that ι is a monomorphic chain map whose image is equal to $\ker(q)$, thus yielding a short exact sequence. To see that the sequence splits, we define a left inverse $\psi\colon C_\bullet \to D^n$ as follows. Define $\psi_k = 0$ for all $k \neq n, n-1$. Define $\psi_{n-1}(b) = \lambda^{-1} \cdot 1_{n-1}$ and $\psi_{n-1}(b') = 0$ for all other basis elements $b' \in \mathcal{B}_{n-1}$, and extend linearly to any C_{n-1}. For any $a' \in \mathcal{B}_n$, let $\lambda' := \langle \partial a', b\rangle$ and define $\psi_n(a') = \frac{\lambda'}{\lambda} \cdot 1_n$, extend linearly to all of C_n. Direct calculations show that ψ is indeed a chain map and that $\psi \circ \iota$ is the identity on D^n. ☐

1.3 Dynamical systems

We introduce some concepts from dynamical systems that we are going to use, see e.g. [45] or [52] for more details. Given a closed smooth manifold M, we denote by $\mathfrak{X}(M)$ the set of smooth vector fields on M. By $\mathfrak{X}^1(M)$ we denote the topological space with underlying set $\mathfrak{X}(M)$ endowed with the Whitney C^1-topology.

In this section, we fix a vector field $v \in \mathfrak{X}(M)$ and consider the dynamics of the flow generated by v. We define everything with a subscpript v, but this might be dropped in cases where it is clear from the context which vector field we are considering. We write $\phi_v \colon \mathbb{R} \times M \to M$ for the corresponding flow. The flow satisfies

$$\phi_v(s, \phi_v(t,p)) = \phi_v(s+t,p), \qquad \phi_v(0,p) = p \qquad \text{and} \qquad \left.\frac{d}{dt}\right|_{t=0} \phi_v(t,p) = v(p)$$

for all $p \in M$ and is uniquely determined by these properties. We will sometimes write $\phi_v^t(p)$ instead of $\phi_v(t,p)$, especially when we want to fix a value of t and consider the resulting map $\phi_v^t \colon M \to M$. Note that in the third equation above, the opposite is done, namely a value for p is fixed, so that we get a map $\mathbb{R} \to M$, to which we can apply $\frac{d}{dt}$.

Given any point $p \in M$, the **orbit** of p (w.r.t. v) is the set

$$\mathcal{O}_v(p) := \phi_v(\mathbb{R}, p) = \{\phi_v(t,p) \mid t \in \mathbb{R}\}.$$

A **fixed point** (also called **singular point** or **rest point**) of v is a point $p \in M$ such that $\phi_v(t,p) = p$ for all $t \in \mathbb{R}$, i.e. $\mathcal{O}_v(p) = \{p\}$. This is equivalent to $v(p) = 0$. A **periodic point** of v is a point $p \in M$ which is not a fixed point such that there exists $T > 0$ with $\phi_v(T,p) = p$. The smallest such T is called the **period** of p. The orbit of a periodic point is called a **closed orbit** (or **periodic orbit**). One can check that all points on the same closed orbit have the same period. We denote by $\mathrm{Fix}(v)$ the set of all fixed points of v and by $\mathrm{Orb}(v)$ the set of all closed orbits of v. Moreover, we define $\mathrm{Sing}(v) := \mathrm{Fix}(v) \sqcup \mathrm{Orb}(v)$ and call its elements the **singular elements** of v.

Definition 1.8. The **α-limit set** and the **ω-limit set** of p (w.r.t. v) are defined as

$$\alpha_v(p) := \bigcap_{-\infty < t \leq 0} \overline{\phi_v((-\infty, t], p)} = \{q \in M \mid \exists \text{ sequence } t_i \to -\infty \text{ with } \phi_v(t_i, p) \to q\},$$

$$\omega_v(p) := \bigcap_{0 \leq t < \infty} \overline{\phi_v([t, \infty), p)} = \{q \in M \mid \exists \text{ sequence } t_i \to \infty \text{ with } \phi_v(t_i, p) \to q\}.$$

A subset $A \subseteq M$ is called **ϕ_v-invariant** if $\phi_v(t, A) = A$ for all $t \in \mathbb{R}$. The orbit of any point $p \in M$ is a ϕ_v-invariant subset of M.

Proposition 1.9 (Proposition 1.4 of [45]). *Let $p \in M$. Then $\alpha_v(p)$ and $\omega_v(p)$ are non-empty, closed, connected and ϕ_v-invariant subsets of M.*

Definition 1.10. A point $p \in M$ is called **chain-recurrent** (w.r.t. v) if for all $T > 0$ and $\varepsilon > 0$ there exist $x_0, x_1, \ldots, x_m \in M$, with $x_0 = x_m = p$, such that for all $i = 0, \ldots, m-1$ there exists $t_i > T$ with $d_M(\phi(t_i, x_i), x_{i+1}) \leq \varepsilon$.

Remark 1.11. If $p \in M$ is either a fixed or periodic point, then p is chain-recurrent.

Definition 1.12. Given a fixed point p of v, its **stable manifold** and its **unstable manifold** are defined by

$$\begin{aligned} W_v^s(p) &:= \{q \in M \mid \phi_v(t,q) \to p \text{ as } t \to \infty\} = \{q \in M \mid \omega_v(q) = p\}, \\ W_v^u(p) &:= \{q \in M \mid \phi_v(t,q) \to p \text{ as } t \to -\infty\} = \{q \in M \mid \alpha_v(q) = p\}. \end{aligned}$$

Given a closed orbit γ of v, we define its **stable manifold** and its **unstable manifold** are defined by

$$\begin{aligned} W_v^s(\gamma) &:= \{q \in M \mid \exists p \in \gamma \text{ such that } d_M(\phi_v(t,p), \phi_v(t,q)) \to 0 \text{ as } t \to \infty\}, \\ W_v^u(\gamma) &:= \{q \in M \mid \exists p \in \gamma \text{ such that } d_M(\phi_v(t,p), \phi_v(t,q)) \to 0 \text{ as } t \to -\infty\}. \end{aligned}$$

Definition 1.13. A set $\Lambda \subseteq M$ is called **hyperbolic** (w.r.t. v) if there exist two subbundles $E^s, E^u \subseteq T_\Lambda M$ such that

- $T_pM = \langle v(p) \rangle \oplus E_p^s \oplus E_p^u$ for all $p \in \Lambda$,
- there exist $0 < \lambda < 1$ and $C > 0$ such that for all $p \in \Lambda$ and $t \geq 0$ we have

  $$\left\| D\phi_v^t(p)x \right\| \leq C\lambda^t \left\| x \right\| \qquad \text{and} \qquad \left\| D\phi_v^{-t}(p)y \right\| \leq C\lambda^t \left\| y \right\|,$$

  where $x \in E_p^s$ and $y \in E_p^u$ and the norms are taken with respect to some Riemannian metric. Due to compactness of M, the definition is independent of this choice.

The two subbundles E^s and E^u are uniquely determined. The dimension of E^u is called the **index** of Λ and denoted by $\operatorname{ind}_v(\Lambda)$.

In this paper we only consider the case where Λ is either a fixed point or a closed orbit of v. For a proof of the following theorem see [52] and the references therein.

Theorem 1.14. *The stable and unstable manifolds of hyperbolic fixed points and hyperbolic closed orbits are injectively immersed submanifolds of M.*

1.3.A Morse-Smale vector fields

We now give the definition of a Morse-Smale vector field. We restrict ourselves to these vetor fields, for two reasons. The first one is that the toplogical structure they induce on a manifold is simpler than for general vector fields, which allows us to apply different methods. They are also structurally stable, meaning that they are indifferent to small perturbations (see Theorem 1.17 below). Finally, in two dimensions, Morse-Smale vector fields form a dense subset of the set of all vector fields [46].

Two injectively immersed submanifolds $K, N \subseteq M$ are said to **intersect transversally**, if for every point $p \in K \cap N$ we have $T_pM = T_pK + T_pN$. Note that if $\dim(K) + \dim(N) < \dim(M)$, then K and N intersect transversally if and only if $K \cap N = \emptyset$.

Definition 1.15. A vector field $v \in \mathfrak{X}(M)$ is called a **Morse-Smale vector field**, if it satisfies the following conditions:

(i) The set of chain-recurrent points consists of finitely many fixed points and closed orbits, all of which are hyperbolic.

(ii) The stable and unstable manifolds of fixed points and closed orbits intersect transversally.

We call a Morse-Smale vector field **gradient-like** if it has no closed orbits. We denote by $\mathfrak{X}_{MS}(M)$ the set of Morse-Smale vector fields on M, and by $\mathfrak{X}_{gMS}(M)$ the set of gradient-like Morse-Smale vector fields.

Hyperbolic fixed points and hyperbolic closed orbits are isolated. Thus, on a closed manifold, we could drop the condition that there are only finitely many fixed points thanks to compactness.

Given a Morse-Smale vector field $v \in \mathfrak{X}_{gMS}(M)$, all of its singular elements have a well-defined index, so we can decompose the set of fixed points and closed orbits accordingly. We write $\mathrm{Fix}_k(v)$ for the set of fixed points of index k and $\mathrm{Orb}_k(v)$ for the set of closed orbits of index k. If $\dim(M) = m$, then we get decompositions of the set of fixed points and the set of closed orbits,

$$\mathrm{Fix}(v) = \bigsqcup_{k=0}^{m} \mathrm{Fix}_k(v) \qquad \text{and} \qquad \mathrm{Orb}(v) = \bigsqcup_{k=0}^{m-1} \mathrm{Orb}_k(v).$$

Remark 1.16. Note that a gradient-like Morse-Smale vector field is not necessarily the gradient of a function. Some authors say that a vector field v is gradient-like with respect to a function f if $\mathrm{Fix}(v) = \mathrm{Crit}(f)$ and $Df(p)[v(p)] < 0$ for all $p \in M \setminus \mathrm{Fix}(v)$. Note that such a function f is not unique. With respect to any Riemannian metric on M this is equivalent to $\langle v(p), -\nabla f(p)\rangle > 0$, i.e. the vector fields v and $-\nabla f$ point in the same half-space. Such a function indeed exists for every gradient-like Morse-Smale vector field (see [51]), but v is not guaranteed to coincide with the gradient of any such f.

Two vector fields $v, w \in \mathfrak{X}(M)$ are **topologically equivalent** if there exists a homeomorphism $h\colon M \to M$ that maps the orbits of v to the orbits of w, preserving their orientations. The homeomorphism h is called a **topological equivalence** between v and w. A vector field $v \in \mathfrak{X}(M)$ is **structurally stable**, if for every $\varepsilon > 0$ there exists a neighbourhood $\mathcal{N}$ of v in $\mathfrak{X}^1(M)$ such that for all $w \in \mathcal{N}$ there exists a topological equivalence $\varphi\colon M \to M$ between v and w such that $d_M(p, \varphi(p)) \leq \varepsilon$ for all $p \in M$.

The following result basically follows from [48]. Since there it is written in a slightly different form, we include a proof in which we explain how to adapt that proof.

Theorem 1.17. *Morse-Smale vector fields are structurally stable.*

Proof. Note that any Morse-Smale vector field satisfies Axiom A and the strong transversality condition, so we can apply the results from [48]. By numbers in brackets we mean the corresponding statements in [48]. Definitions for undefined terms can be found there.

By (1.6) the vector field v is infinitesimally stable, so we can apply (1.5), which implies that v is structurally semistable. This means that we can find a neighbourhood $\mathcal{N}$ of v such that all $w \in \mathcal{N}$ are semiconjugate to v. The semiconjugacy is a continuous map $h\colon M \to M$, which is onto, but may not be a homeomorphism. It is given by $h = \exp(u)$, where u is some vector field on M.

Moreover, using that v is of class C^2, it follows that for any $\delta > 0$, we can choose the neighbourhood $\mathcal{N}$ small enough, such that the vector field u has norm smaller than δ. The norm used on u is based on a choice, but we can choose the supremum norm induced by the Riemannian metric.

If we choose δ smaller than the injectivity radius of M, then for $h = \exp(u)$ it holds that $d_M(x, h(x)) = |u(x)|$. Thus, it follows that h moves points less than δ. This concludes the proof. □

1.3.B The smooth Morse complex

The main theorem of Morse theory states that if we have a Morse function $f\colon M \to \mathbb{R}$, then M is homotopy equivalent to a CW complex with one k-cell for each critical point of f of index k. The proof uses the flow of the negative gradient vector field $-\nabla f$. The idea of Morse homology is to make the connection between the flowlines of $-\nabla f$ and the topology of M more explicit by constructing a chain complex $C_\bullet$ whose vector space in degree k is generated by the critical points of f (i.e. zeroes of $-\nabla f$ of index k) and whose differential can be computed by counting certain flowlines of $-\nabla f$, such that the homology of that chain complex is the homology of M.

The construction can be extended to gradient-like vector fields, as we show in the following. We give this definition only in the case of $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, in order to reduce the amount of technicalities. Everything we do is also possible with coefficients in an arbitrary field, see e.g. [3] or [49].

Definition 1.18. If $v \in \mathfrak{X}(M)$ is a gradient-like Morse-Smale vector field, then the associated **smooth Morse complex** $\mathrm{MC}_\bullet(v) \in \mathrm{Ch}$ (Morse complex, for brevity) is defined as follows:

1. For $k \geq 0$, $\mathrm{MC}_k(v)$ is the free $\mathbb{F}$-vector space generated by the fixed points of index k of v.
2. The boundary operator $\partial\colon \mathrm{MC}_k(v) \to \mathrm{MC}_{k-1}(v)$ is defined by counting the flowlines between fixed points of adjecent index (mod 2). Explicitly, for all $p \in \mathrm{Fix}_k(v)$ define

$$\partial(p) := \sum_{q \in \mathrm{Fix}_{k-1}(v)} \alpha(p,q) \cdot q,$$

where $\alpha(p,q)$ is the number of flowlines from p to q (mod 2), i.e. the number of connected components of $W^u(p) \cap W^s(q)$ (mod 2).

What we call the Morse complex here is sometimes called Morse-Smale-Witten complex (e.g. in [3]) or Thom-Smale complex (e.g. in [31]). Standard references for this include [3] and [49]. See also [8] for the historical development and [30] for a beginner-friendly introduction with some explicit examples.

Since a Morse-Smale vector field v has only finitely many fixed points, $\mathrm{MC}_k(v)$ is a finite-dimensional vector space for all k. Also $\mathrm{MC}_k(v) = 0$ for $k < 0$ and $k > \dim(M)$. In order to see that the boundary operator ∂ is well-defined, observe that, by compactness, for any two fixed points p and q with $\mathrm{ind}_v(p) = \mathrm{ind}_v(q) + 1$, there can exist only finitely many flowlines from p to q.

Lemma 1.19. *If $v, w \in \mathfrak{X}_{gMS}(M)$, then a topological equivalence $h\colon M \to M$ between v and w induces an isomorphism between $\mathrm{MC}_\bullet(v)$ and $\mathrm{MC}_\bullet(w)$.*

Proof. A topological equivalence h between v and w establishes a bijection between the fixed points of v and the fixed points of w, respecting the indices. This induces bijections $\mathrm{Fix}_k(v) \to \mathrm{Fix}_k(w)$ and thus isomorphisms $\mathrm{MC}_k(v) \to \mathrm{MC}_k(w)$ for all k. Moreover, h maps the stable and unstable manifolds of any fixed point p of v to the stable and unstable manifolds of the corresponding fixed point $h(p)$ of w. Therefore $\alpha(p,q) = \alpha(h(p), h(q))$ for any pair $(p,q) \in \mathrm{Fix}_k(v) \times \mathrm{Fix}_{k-1}(v)$ and thus the isomorphisms $\mathrm{MC}_k(v) \to \mathrm{MC}_k(w)$ commute with the boundary operators in $\mathrm{MC}_\bullet(v)$ and $\mathrm{MC}_\bullet(w)$, so we get an isomorphism of chain complexes. □

The usual result that the differential squares to zero in the Morse complex of a Morse-Smale function can be transferred to the Morse complex for gradient Morse-Smale vector fields.

Theorem 1.20. *Let $v \in \mathfrak{X}(M)$ be a gradient-like Morse-Smale vector field. Then $(\mathrm{MC}_\bullet(v), \partial)$ is a chain complex, i.e. $\partial^2 = 0$. Moreover, the homology of this chain complex is isomorphic to the singular homology of M.*

Proof. If v is the gradient of a Morse-Smale function $f \colon M \to \mathbb{R}$ with respect to some Riemannian metric on M, then we refer to [3] for a proof. In order to reduce to that case when v is gradient-like, note that by combining Lemma 2 from [43] and Theorem B from [51], there exists a function $f \colon M \to \mathbb{R}$ which is Morse-Smale with respect to some Riemannian metric and v is topologically equivalent to $-\nabla f$. Since the result holds for $-\nabla f$, it also holds for v by Lemma 1.19. □

Later we will parametrize this chain complex. In other words, given a gradient-like Morse-Smale vector field $v \in \mathfrak{X}(M)$, we will construct a factored chain complex $X(v)$, such that $X(v)^0 = \mathrm{MC}_\bullet(v)$. We will do this by applying algebraic simplifications to the Morse complex of v.

1.3.C The combinatorial Morse complex

Discrete Morse theory has been introduced by Forman in [26, 25], transferring many definitions and results from the smooth to the combinatorial setting, also the Morse complex. Let us start by recalling some basic definitions. Given a simplicial complex K, a **combinatorial vector field** on K is a function $V \colon K \to K \cup \{0\}$ satisfying the following conditions:

- for all $\sigma \in K$: either $V(\sigma) = 0$ or $\dim V(\sigma) = \dim \sigma + 1$ and σ is a face of $V(\sigma)$,
- $V(V(\sigma)) = 0$,
- $|V^{-1}(\sigma)| \leq 1$ for all $\sigma \in K$.

Cells $\sigma \in K$ with $V(\sigma) = 0$ and $V^{-1}(\sigma) = \emptyset$ are called **critical cells** for V. We write $\mathrm{Crit}(V)$ for the set of all critical cells. We denote by $\overline{\mathfrak{X}}(K)$ the set of all combinatorial vector fields on K. If V is a combinatorial vector field on K, then, following [31], a **V-path of dimension** k is a sequence $\sigma_0, \sigma_1, \ldots, \sigma_r$ of k-simplices, such that $\sigma_i \neq \sigma_{i+1}$ and σ_{i+1} is a hyperface of $V(\sigma_i)$ for all $i = 0, \ldots, r-1$. If $r = 0$, then this V-path is called **stationary** and if $\sigma_r = \sigma_0$, then it is called **closed**. See [31] for more details. A combinatorial vector field V with no closed non-stationary V-paths is called **gradient-like**. We denote by $\overline{\mathfrak{X}}_g(K)$ the set of all gradient-like combinatorial vector fields on K.

Definition 1.21. If $V \in \overline{\mathfrak{X}}_g(K)$ is a gradient-like combinatorial vector field on a simplicial complex K, then the associated **combinatorial Morse complex** $\overline{\mathrm{MC}}_\bullet(V) \in \mathrm{Ch}$ is defined as follows:

1. For $k \geq 0$, $\overline{\mathrm{MC}}_k(V)$ is the $\mathbb{F}$-vector space generated by the critical k-cells of V.
2. The boundary operator $\partial \colon \overline{\mathrm{MC}}_k(V) \to \overline{\mathrm{MC}}_{k-1}(V)$ is defined by counting (mod 2) the number of V-paths between each pair of critical cells of indices k and $k-1$.

We refer to [31] for a proof that this is indeed a chain complex (called under the different name of a Thom-Smale complex) and that the homology of this chain complex is isomorphic to the simplicial homology of K.

1.3.D Relating the smooth and combinatorial Morse complexes

In order to prove our combinatorial approximation result in Section 3.4, we rely on two known results. The first one says that, given a gradient-like Morse-Smale vector field, one can construct a combinatorial vector field on a triangulation of the manifold, so that the Morse complexes of these vector fields are isomorphic.

If M is a smooth manifold, then a **triangulation** of M is a pair (M', ϕ), where M' is a simplicial complex and $\phi\colon |M'| \to M$ is a homeomorphism from the geometric realization of M' to M. A **triangulated manifold** is a triple (M, M', ϕ), where M is a smooth manifold and (M', ϕ) is a triangulation of M. We say **triangulated Riemannian manifold** if M is endowed with a Riemannian metric. The following result is proven by Gallais in [31], with a small gap filled by Benedetti in [5].

Theorem 1.22. *Let M be a smooth closed oriented Riemannian manifold and let $v \in \mathfrak{X}_{gMS}(M)$. Then there exists a triangulation (M', ϕ) of M and a gradient-like combinatorial vector field $V \in \overline{\mathfrak{X}}_g(M')$ such that*

(i) For every k there exists a bijection between the fixed points of v of index k and the critical cells of V of dimension k. If $p \in \mathrm{Fix}(v)$ corresponds to $\sigma \in \mathrm{Crit}(V)$, then p lies in the geometric realization of the corresponding critical cell, i.e. $p \in \phi(|\sigma|)$.

(ii) For a pair (p, q) of fixed points of v of index $k+1$ and k and the corresponding critical cells (τ, σ) of V we have a bijection between flow lines from p to q and V-paths from a hyperface of τ to σ.

(iii) The bijection from (i) induces an isomorphism $\mathrm{MC}_\bullet(v) \cong \overline{\mathrm{MC}}_\bullet(V)$.

The second result states that if we are given a combinatorial vector field on some simplicial complex, then we can construct a new combinatorial vector field on the barycentric subdivision, such that the two Morse complexes are isomorphic.

Given a simplicial complex K, we denote by $\Delta(K)$ its barycentric subdivision. The k-simplices of $\Delta(K)$ correspond to chains of length $k+1$ $\sigma_0 \subsetneq \cdots \subsetneq \sigma_k$ of simplices in K. We write $\Delta^n(K)$ for the nth iteration. Assume that moreover we are given $V \in \overline{\mathfrak{X}}(K)$ and for every critical k-cell σ of V we have chosen a k-cell of $\Delta(K)$ that lies in σ. Then it is shown by Zhukova in [58] that V and these choices of cells induce a combinatorial vector field on $\Delta(K)$ in a canonical way. We denote this combinatorial vector field by $\Delta(V) \in \overline{\mathfrak{X}}(\Delta(K))$, hiding the dependence on the choices of cells from the notation. Equivalently, the choice upon which $\Delta(V)$ depends amounts to choosing an ordering of the vertices of each critical simplex of V. The following result from [58] lists some of the properties of $\Delta(V)$.

Theorem 1.23. *Let K be a simplicial complex and $V \in \overline{\mathfrak{X}}_g(K)$. Assume we have chosen an ordering of the vertices of every critical cell of V (and thus $\Delta(V)$ is defined). Then*

(i) $\Delta(V) \in \overline{\mathfrak{X}}_g(\Delta(K))$, i.e. there are no closed non-stationary $\Delta(V)$-paths.

(ii) The critical simplices of $\Delta(V)$ are exactly the chosen ones. This yields a bijection between the critical k-cells of V and the critical k-cells of $\Delta(V)$ for all k.

(iii) There is a one-to-one correspondence between gradient paths (see [58] for the definition) of V and of $\Delta(V)$, respecting the bijections from (ii).

In particular, the bijections from (ii) yield an isomorphism $\overline{\mathrm{MC}}_\bullet(V) \cong \overline{\mathrm{MC}}_\bullet(\Delta(V))$.

Chapter 2

Factored chain complexes

In this section we develop the general theory of tame epimorphic parametrized chain complexes, i.e. factored chain complexes. Many of the results are analogous to the case of tame parametrized vector spaces, i.e. persistence modules.

We start by a classification of the chain maps between disks and spheres in Ch, which are the simplest chain complexes. This will help us describe the simplest possible parametrized chain complexes and parametrized chain maps between them. Note that in the category Ch, an epimorphism is a chain map that is surjective in every degree.

Proposition 2.1. *Any non-zero chain map between disks and/or spheres in* Ch *is a scalar multiple of one of the following maps:*

(i) The identity $\mathbb{1}_{D^n} \colon D^n \to D^n$. This map is an epimorphism.

(ii) The identity $\mathbb{1}_{S^n} \colon S^n \to S^n$. This map is an epimorphism.

(iii) The inclusion $\iota^n \colon S^n \hookrightarrow D^{n+1}$. This map is not an epimorphism.

(iv) The chain map $\Phi^n \colon D^n \to D^{n+1}$, which is the identity in degree n and zero in all other degrees. This map is not an epimorphism.

(v) The chain map $\Psi^n \colon D^n \to S^n$, which is the identity in degree n and zero in all other degrees. This map is an epimorphism.

Proof. Non-zero chain maps can only exist between disks and spheres that are non-zero in at least one common degree. This leaves us with the possible options $D^n \to D^n$, $S^n \to S^n$, $S^n \to D^{n+1}$, $D^n \to D^{n+1}$, $D^n \to S^n$, $D^n \to S^{n-1}$, $S^n \to D^n$, and $D^{n+1} \to D^n$. The last three options are in fact not possible, which can be checked by hand. The other five options are all possible and correspond to the chain maps listed above. Since all the involved vector spaces are one-dimensional, these are the only possibilities up to scalar multiplication. The identity maps on D^n and S^n are isomorphisms and therefore in particular epimorphisms. The chain map Ψ^n is surjective in degree n, which is the only degree where S^n is non-zero, so it is an epimorphism. The chain maps ι^n and Ψ^n both fail to be surjective in the degree $n+1$ and are thus not epimorphisms. $\square$

This is what the chain map $\Psi^n \colon D^n \to S^n$ looks like:

$$\begin{tikzcd}
D^n \arrow[d, "\Psi^n"] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{F} \arrow[r] \arrow[d] & \mathbb{F} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
S^n & \cdots \arrow[r] & 0 \arrow[r] & \mathbb{F} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

We use the convention that linear maps $\mathbb{F} \to \mathbb{F}$ in any diagram are always assumed to be the identity map, if not stated otherwise.

2.1 Interval functors in TEPCh

In this section, we start to develop the general theory of functors in TEPCh, i.e. the category of tame epimorphic parametrized chain complexes, called factored chain complexes for short (see Section 1.1 for the relevant definitions). We describe the simplest possible factored chain complexes, which turn out to be the building blocks for all other factored chain complexes. Analogously to how the simplest parametrized vector spaces are characterized by intervals, these factored chain complexes are characterized by tagged intervals, which are intervals with an additional choice of one point. This point marks the time of collapsing from a disk to a sphere.

Definition 2.2. A **tagged interval** is a tuple consisting of a real interval $[0,t)$, where $0 < t \leq \infty$, together with a distinguished point $s \in [0,t]$. We denote this tagged interval also by $[0,s,t)$. We write $\mathcal{J}$ for the set of all tagged intervals.

Definition 2.3. Let $n \in \mathbb{N}$ and $[0,s,t) \in \mathcal{J}$. In the case $n = 0$ we additionally assume that $s = 0$. Then we define the **interval functor** $\mathcal{I}^n[0,s,t)$ **in** TEPCh by

$$(\mathcal{I}^n[0,s,t))^r = \begin{cases} D^n, \text{ if } 0 \leq r < s, \\ S^n, \text{ if } s \leq r < t, \\ 0, \text{ otherwise,} \end{cases} \quad \text{and} \quad (\mathcal{I}^n[0,s,t))^{q\leq r} = \begin{cases} \mathbb{1}_{D^n}, \text{ if } 0 \leq q \leq r < s, \\ \Psi^n, \text{ if } 0 \leq q < s \leq r < t, \\ \mathbb{1}_{S^n}, \text{ if } s \leq q \leq r < t, \\ 0, \text{ otherwise.} \end{cases}$$

Note that the interval functor $\mathcal{I}^n[0,s,t)$ is non-zero on $[0,t)$ in degree n and on $[0,s)$ in degree $n-1$, hence one may think of the tagged interval $[0,s,t)$ as representing the pair $[0,s) \subseteq [0,t)$.

The following diagram shows a diagrammatic depiction of the interval functor $\mathcal{I}^n[0,s,t)$ in TEPCh, where chain complexes are drawn as columns.

$$\begin{tikzcd}
 & 0 & & & s & & & t & \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
n & \mathbb{F} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{F} \arrow[r] \arrow[d] & \mathbb{F} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{F} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
n-1 & \mathbb{F} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{F} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
 & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
 & \vdots & & \vdots & \vdots & & \vdots & \vdots &
\end{tikzcd}$$

For a pictorial representation of $\mathcal{I}^2[0,0,\infty)$, $\mathcal{I}^1[0,1,1)$ and $\mathcal{I}^0[0,0,\infty)$, see Figure 3.1 (right). We now prove some results describing parametrized chain maps between different interval functors in TEPCh. This will later be useful for the structure theorem. We introduce a notation that will be convenient in many proofs and make statements more readable: For $X \in \mathrm{TPCh}$, $a \in X_n^s$ and $t \geq s$, we write $a^t := X_n^{s \leq t}(a) \in X_n^t$. Moreover, we use the convention that $X_\bullet^\infty$ denotes the zero chain complex, so that $X_n^{s \leq \infty}$ is the zero map for all $0 \leq s \leq \infty$ and $n \in \mathbb{N}$.

Lemma 2.4. *Let $[0, s, t) \in \mathcal{J}$, let $1_n \in \mathcal{I}^n[0, s, t)_n^0$ be a generator of $\mathbb{F}$ and let $X \in \mathrm{PCh}$ be any parametrized chain complex. Then,*

$$\mathrm{Hom}(\mathcal{I}^n[0, s, t), X) \cong \ker(X_n^{0 \leq t}) \cap \ker(X_{n-1}^{0 \leq s} \circ \partial_n^0) \subseteq X_n^0,$$

where the isomorphism is given by $\varphi \mapsto \varphi_n^0(1_n)$.

Proof. Assigning $\varphi \mapsto \varphi_n^0(1_n)$ yields a linear map $\mathrm{Hom}(\mathcal{I}^n[0, s, t), X) \to X_n^0$, so it suffices to check that this map is injective and has image $\ker(X_n^{0 \leq t}) \cap \ker(X_{n-1}^{0 \leq s} \circ \partial_n^0)$.

For injectivity, note that every non-zero element $x \in \mathcal{I}^n[0, s, t)_k^r$ is a scalar multiple of either 1_n^r or $\partial 1_n^r$ under the internal maps. Therefore, if $\varphi_n^0(1_n) = 0$, then also $\varphi = 0$.

To show that the image is contained in $\ker(X_n^{0 \leq t}) \cap \ker(X_{n-1}^{0 \leq s} \circ \partial_n^0)$, note that $\partial 1_n^s = 0$ and $1_n^t = 0$, thus if $\varphi \colon \mathcal{I}^n[0, s, t) \to X$ is any parametrized chain map with $\varphi_n^0(1_n) = a$, then

$$X_{n-1}^{0 \leq s}(\partial_n a) = X_{n-1}^{0 \leq s}(\partial_n \varphi_n^0(1_n)) = X_{n-1}^{0 \leq s}(\varphi_{n-1}^0(\partial_n 1_n)) = \varphi_{n-1}^s\left(\partial 1_n^s\right) = \varphi_{n-1}^s(0) = 0$$

and

$$X_n^{0 \leq t}(a) = X_n^{0 \leq t}(\varphi_n^0(1_n)) = \varphi_n^t\left(1_n^t\right) = \varphi_n^t(0) = 0,$$

hence $a \in \ker(X_n^{0 \leq t}) \cap \ker(X_{n-1}^{0 \leq s} \circ \partial_n^0)$.

On the other hand, if $a \in \ker(X_n^{0 \leq t}) \cap \ker(X_{n-1}^{0 \leq s} \circ \partial_n^0)$, then we can define a parametrized chain map $\varphi \colon \mathcal{I}^n[0, s, t) \to X$ explicitly by defining

$$\varphi_n^r(1_n^r) := a^r \quad \text{and} \quad \varphi_{n-1}^r(\partial 1_n^r) := \partial a^r,$$

and then extending linearly to all of $\mathcal{I}^n[0, s, t)$. This is enough since the elements 1_n^r and $\partial 1_n^r$ are generators. This is well-defined because $\partial a^s = 0$ and $a^t = 0$ and thus yields $\varphi \in \mathrm{Hom}(\mathcal{I}^n[0, s, t), X)$ with $\varphi_n^0(1_n) = a$. □

Lemma 2.5. *Let $n, m \in \mathbb{N}$ and $[0, s, t), [0, s', t') \in \mathcal{J}$. Then*

$$\mathrm{Hom}(\mathcal{I}^n[0, s, t), \mathcal{I}^m[0, s', t')) \cong \begin{cases} \mathbb{F}, & \textit{if } m = n \textit{ and } s \geq s',\ t \geq t', \\ \mathbb{F}, & \textit{if } m = n + 1 \textit{ and } t \geq s' > 0, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. For $X = \mathcal{I}^m[0, s', t')$ we have

$$\ker(X_n^{0 \leq t}) \cong \begin{cases} \mathbb{F}, & \text{if } m = n \text{ and } t \geq t', \\ \mathbb{F}, & \text{if } m = n + 1 \text{ and } t \geq s' > 0, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and}$$

$$\ker(X_{n-1}^{0 \leq s} \circ \partial_n^0) \cong \begin{cases} \mathbb{F}, & \text{if } m = n \text{ and } s \geq s', \\ \mathbb{F}, & \text{if } m = n + 1 \text{ and } s' > 0, \\ 0, & \text{otherwise.} \end{cases}$$

By using Lemma 2.4 and intersecting the two kernels, the result follows. □

Proposition 2.6. *Interval functors* $\mathcal{I}^n[0,s,t)$ *are indecomposable in* TEPCh.

Proof. Consider a decomposable factored chain complex $X \in$ TEPCh, i.e. $X = Y \oplus Z$ for $Y, Z \in$ TEPCh nonzero. If we denote by ϕ and ψ the projections on the first and second summand, then these are two nonzero elements of $\mathrm{End}(X)$ none of which is a scalar multiple of the other, so $\mathrm{End}(X) \ncong \mathbb{F}$. However, it follows from Lemma 2.5 that $\mathrm{End}(\mathcal{I}^n[0,s,t)) \cong \mathbb{F}$, so $\mathcal{I}^n[0,s,t)$ must be indecomposable. □

2.2 Parametrized vector spaces induced from parametrized chain complexes

Given a parametrized chain complex X, we present two important ways how we can construct parametrized vector spaces from X. One is by taking homology and the other considers the image of the differential in X. If X is tame and epimorphic, then the two of them together in some sense contain all the information about X, as we will see later.

Given a chain complex $C_\bullet$, we denote by $H_n(C_\bullet)$ the homology of $C_\bullet$ in degree n with coefficients in $\mathbb{F}$.

Definition 2.7. If $X\colon [0,\infty) \to \mathrm{Ch}$ is a parametrized chain complex, its n**-homology** is the parametrized vector space $H_n(X)\colon [0,\infty) \to \mathrm{Vect}$, defined by

- $H_n(X)^t := H_n(X^t)$,
- $H_n(X)^{s \le t} := H_n(X^{s \le t})$.

Given a parametrized chain map $\varphi\colon X \to Y$, we define the parametrized linear map $H_n(\varphi)\colon H_n(X) \to H_n(Y)$ by

- $H_n(\varphi)^t := H_n(\varphi^t)$.

Definition 2.8. If $X\colon [0,\infty) \to \mathrm{Ch}$ is a parametrized chain complex, its n**-boundary** is the parametrized vector space $\partial_n(X)\colon [0,\infty) \to \mathrm{Vect}$, defined by

- $\partial_n(X)^t := \mathrm{im}(\partial_n^{X^t}) \subseteq X_{n-1}^t$,
- $\partial_n(X)^{s \le t} := X_{n-1}^{s \le t}|_{\partial_n(X)^s}$.

Given a parametrized chain map $\varphi\colon X \to Y$, we define the parametrized linear map $\partial_n(\varphi)\colon \partial_n(X) \to \partial_n(Y)$ by

- $\partial_n(\varphi)^t := \varphi^t|_{\partial_n(X)^t}$.

Remark 2.9. There are functors $H_n, \partial_n\colon \mathrm{Ch} \to \mathrm{Vect}$, such that for all $X \in$ PCh we have $H_n(X) = H_n \circ X$ and $\partial_n(X) = \partial_n \circ X$. From this it follows, that $H_n(X), \partial_n(X) \in$ PVect and moreover, the assignments $X \mapsto H_n(X)$ and $X \mapsto \partial_n(X)$ are functors PCh $\to$ PVect. Explicitly, this **functoriality** means that H_n and ∂_n have the following properties.

(*i*) If $X \in$ PCh, then $H_n(X), \partial_n(X) \in$ PVect.

(*ii*) If $\varphi\colon X \to Y$ is a parametrized chain map, then $H_n(\varphi)\colon H_n(X) \to H_n(Y)$ and $\partial_n(\varphi)\colon \partial_n(X) \to \partial_n(Y)$ are parametrized linear maps.

(*iii*) Identity morphisms get mapped to identity morphisms, i.e. $H_n(\mathbb{1}_X) = \mathbb{1}_{H_n(X)}$ and $\partial_n(\mathbb{1}_X) = \mathbb{1}_{\partial_n(X)}$ for all $X \in \mathrm{PCh}$.

(*iv*) If $\psi\colon Y \to Z$ is another parametrized chain map, then $H_n(\psi \circ \varphi) = H_n(\psi) \circ H_n(\varphi)$ and $\partial_n(\psi \circ \varphi) = \partial_n(\psi) \circ \partial_n(\varphi)$.

(*v*) If $X \cong Y$ in PCh, then $H_n(X) \cong H_n(Y)$ and $\partial_n(X) \cong \partial_n(Y)$ in PVect.

2.3 The structure theorem for factored chain complexes

In this section we study decompositions of factored chain complexes and prove the structure theorem. This result is analogous to the structure theorem for persistence modules Theorem 2.17, but even more so to the decomposition result for filtered chain complexes given in [13]. In fact, it would be possible to derive our structure theorem from the latter using a duality argument. We refrain to do so, on one hand for self-containment, and on the other hand because a direct argument seems to give more insight why the statement is true.

Let X be a parametrized chain complex. A **parametrized subcomplex** of X is a parametrized chain complex Y such that Y^t is a subcomplex of X^t for all t and the internal maps in Y are given by the restrictions of the internal maps in X. We write $Y \subseteq X$. Given parametrized chain complexes X, Y, their **direct sum** $X \oplus Y$ is the parametrized chain complex defined by $(X \oplus Y)^t := X^t \oplus Y^t$ and $(X \oplus Y)^{s \le t} := X^{s \le t} \oplus Y^{s \le t}$. The direct sum between any finite number of parametrized chain complexes is defined analogously.

One can check that if X, Y are parametrized chain complexes, then $X \oplus Y$ is again a parametrized chain complex. Since the direct sum of epimorphisms is an epimorphism, it follows that the direct sum of epimorphic parametrized chain complexes is again epimorphic. Also if X and Y are both tame, then also $X \oplus Y$ is tame. Note that the set where $X \oplus Y$ fails to be left-constant is the union of the corresponding sets for X and Y. All of these arguments continue to hold when we pass to direct sums of arbitrary (but still finite) size. To summarize, we can say that the categories PCh, EPCh, and TEPCh are closed under finite direct sums.

Lemma 2.10. *Let X be a parametrized chain complex and let $Y \subseteq X$ with Y epimorphic.*

(*i*) *If $X^{s \le t}$ is an isomorphism, then also $Y^{s \le t}$ is an isomorphism.*

(*ii*) *If X is right-constant at $t \in [0, \infty)$, then Y is also right-constant at t.*

(*iii*) *If X is left-constant at $t \in [0, \infty)$, then Y is also left-constant at t.*

(*iv*) *If $X \in \mathrm{TEPCh}$, then also $Y \in \mathrm{TEPCh}$.*

Proof. The statement (*i*) follows from the fact that the restriction of an injective chain map to a subcomplex is again injective. The statements (*ii*) and (*iii*) follow from (*i*). In order to show (*iv*), we need to show that under the given assumptions, Y is tame. We know that X is tame, i.e. right-constant everywhere and left-constant everywhere but on a finite set of points. By (*ii*), also Y is right-constant everywhere and by (*iii*), Y can fail to be left-constant only at points where also X fails to be left-constant, thus also only on a finite set of points. It follows that also Y is tame and thus $Y \in \mathrm{TEPCh}$. □

Now we prove some properties of the functors $H_n, \partial_n \colon$ TEPCh $\to$ TPVect. In words, we show that both H_n and ∂_n preserve tameness and direct sums, while ∂_n additionally preserves the property of being epimorphic. Applying H_n or ∂_n to interval functors in TEPCh yields interval functors in TPVect.

Proposition 2.11. *Let X be a parametrized chain complex.*

(*i*) *If X is tame, then both $H_n(X)$ and $\partial_n(X)$ are tame.*

(*ii*) *If X is epimorphic, then $\partial_n(X)$ is epimorphic.*

(*iii*) *If $X = Y \oplus Z$, then $H_n(X) = H_n(Y) \oplus H_n(Z)$ and $\partial_n(X) = \partial_n(Y) \oplus \partial_n(Z)$.*

(*iv*) *Interval functors in* TEPCh *induce interval functors in* TPVect *in the following way, where we use the convention that $\mathbb{F}[t,t) = 0$ for any t.*

$$H_k(\mathcal{I}^n[0,s,t)) = \begin{cases} \mathbb{F}[s,t) & \text{if } k = n, \\ 0 & \text{if } k \neq n, \end{cases} \quad \text{and} \quad \partial_k(\mathcal{I}^n[0,s,t)) = \begin{cases} \mathbb{F}[0,s) & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases}$$

Proof. To see that (*i*) holds, note that if $X^{s\le t}$ is an isomorphism of chain complexes, then both $H_n(X^{s\le t})$ and the restriction of $X^{s\le t}_{n-1}$ to $\partial_n(X^s_n)$ are isomorphisms of vector spaces.

To show (*ii*), let $b \in \partial_n(X^t_n) \subseteq X^t_{n-1}$. Then $b = \partial a$ for some $a \in X^t_n$. Since $X^{s\le t}$ is an epimorphism, there exists $a' \in X^s_n$ with $X^{s\le t}_n(a') = a$. Thus we have $b = X^{s\le t}_{n-1}(\partial a')$. This shows that the internal map $\partial_n(X)^{s\le t}$ is an epimorphism for any $s \le t \in [0,\infty)$.

It follows from standard arguments in homological algebra that homology and boundary both preserve direct sums. This proves (*iii*).

In order to show (*iv*), note that for the n-disk D^n and the n-sphere S^n in Ch it holds that

$$H_k(D^n) = 0, \text{ for all } k, \qquad H_k(S^n) = \begin{cases} \mathbb{F}, \text{ if } k = n, \\ 0, \text{ if } k \neq n, \end{cases}$$

$$\partial_k(D^n) = \begin{cases} \mathbb{F}, \text{ if } k = n, \\ 0, \text{ if } k \neq n, \end{cases} \qquad \partial_k(S^n) = 0, \text{ for all } k.$$

From this and the fact that both homology and the boundary map the identity to the identity, (*iv*) follows. ☐

We introduce a new notation for the parametrized subcomplex induced by a single element. Note that by applying Lemma 2.4 in the case $s = t = \infty$, we get an isomorphism

$$\mathrm{Hom}(\mathcal{I}^n[0,\infty,\infty), X) \xrightarrow{\cong} X^0_n.$$

Denote the inverse of this isomorphism by $\Phi \colon X^0_n \to \mathrm{Hom}(\mathcal{I}^n[0,\infty,\infty), X)$.

Definition 2.12. Let $X \in$ PCh and $a \in X^0_n$. Then we define the parametrized subcomplex $\langle a \rangle$ of X as the image of the parametrized chain map $\Phi(a) \colon \mathcal{I}^n[0,\infty,\infty) \to X$.

Explicitly, this means that for $t \in [0,\infty)$, the parametrized chain complex $\langle a \rangle^t$ is given by

$$\langle a \rangle^t_n = \mathrm{Span}(X^{0\le t}_n(a)), \qquad \langle a \rangle^t_{n-1} = \mathrm{Span}(X^{0\le t}_{n-1}(\partial a)), \qquad \langle a \rangle^t_k = 0 \text{ if } k \neq n, n-1.$$

The differential in degree n is given by the restriction of the differential $\partial_n^{X^t}\colon X_n^t \to X_{n-1}^t$, all the other differentials are zero. The internal maps $\langle a\rangle^{s\leq t}$ are given by the restrictions of the internal maps of X. Note that the dependence of $\langle a\rangle$ from X is hidden from the notation. Alternatively, $\langle a\rangle$ can be characterized as the smallest parametrized subcomplex of X containing a.

Given a parametrized chain complex X and an element $a \in X_n^0$, we define its **death time** as $d(a) := \inf\{s \geq 0 \mid X_n^{0\leq s}(a) = 0\}$. We also say that a **dies at time** $d(a)$. We start by observing some properties of death times that will be useful later.

Lemma 2.13. *Consider a parametrized chain complex $X \in \mathrm{PCh}$, elements $a_1, \dots, a_k \in X_n^0$, and scalars $\lambda_1, \dots, \lambda_k \in \mathbb{F}$.*

- *(i)* $d(\lambda_1 a_1 + \cdots + \lambda_k a_k) \leq \max\{d(a_i) \mid \lambda_i \neq 0\}$.
- *(ii) If $d(a_1), \dots, d(a_k)$ are distinct, then $d(\lambda_1 a_1 + \cdots + \lambda_k a_k) = \max\{d(a_i) \mid \lambda_i \neq 0\}$.*
- *(iii) If $d(a_1), \dots, d(a_k)$ are distinct and none of them is zero, then $a_1, \dots, a_k$ are linearly independent.*

Proof. The inequality in (i) holds simply because if $s \geq d(a_i)$ for all i, then

$$X_n^{0\leq s}(\lambda_1 a_1 + \cdots + \lambda_k a_k) = \lambda_1 a_1^s + \cdots + \lambda_k a_k^s = 0.$$

For (ii), assume now that the death times are all distinct. Without loss of generality assume that $d(a_1) < \cdots < d(a_k)$, with $\lambda_k \neq 0$. Then, for any $d(a_{k-1}) < s < d(a_k)$, we have $X_n^{0\leq s}(\lambda_1 a_1 + \cdots + \lambda_k a_k) = \lambda_k a_k^s \neq 0$. This shows that $d(\lambda_1 a_1 + \cdots + \lambda_k a_k) \geq d(a_k)$.

Finally, for (iii), assume by contradiction that $a_1, \dots, a_k$ are linearly dependent, i.e. there exists i such that we can write a_i as a linear combination of the other a_j. Then, by (ii), the death time of a_i is among the death times of the other a_j, contradicting the fact that they are k distinct values. ☐

We now show that the parametrized subcomplex $\langle a\rangle$ generated by an element $a \in X_n^0$ is always isomorphic to an interval functor in TEPCh. Note that we do not have to assume that X is epimorphic for this.

Proposition 2.14. *Let X be a parametrized chain complex and let $0 \neq a \in X_n^0$. If X is right-constant at $s := d(\partial a)$ and at $t := d(a)$, then $\langle a\rangle$ is isomorphic to $\mathcal{I}^n[0,s,t)$, with $t > 0$. In particular, $\langle a\rangle$ is a factored chain complex.*

Proof. Since X is right-constant at s and t, it follows that $X_{n-1}^{0\leq s}(\partial a) = 0$ and $X_n^{0\leq t}(a) = 0$. Therefore, by Lemma 2.4, there exists a parametrized chain map $\varphi\colon \mathcal{I}^n[0,s,t) \to X$ with $\varphi_n^0(1_n) = a$. One can check that the image of φ is precisely $\langle a\rangle$ and that φ_k^r is injective for all r and k. Therefore, it follows that φ is an isomorphism from $\mathcal{I}^n[0,s,t)$ to $\langle a\rangle$. Note that $t > 0$, because otherwise $a = 0$ would follow, since X is right-constant at t. ☐

We want to formulate and prove a structure theorem for factored chain complexes. More precisely, we want to prove that every factored chain complex is isomorphic to a direct sum of interval functors in TEPCh. We begin with a statement from linear algebra that will turn out to be useful for this.

Proposition 2.15. *Let $f\colon V \twoheadrightarrow W$ be a surjective linear map and $A, B \subseteq V$ linear subspaces such that $V = A \oplus B$ and $\ker(f) \subseteq B$. Then $W = f(A) \oplus f(B)$.*

Proof. Denote by $\tilde{f}\colon V/\ker(f) \to W$ the isomorphism induced by f. Note that the decomposition $V = A \oplus B$ implies that $V/\ker(f) = \frac{A}{A\cap\ker(f)} \oplus \frac{B}{B\cap\ker(f)} = A \oplus B/\ker(f)$. Therefore,

$$W = \tilde{f}(V/\ker(f)) = \tilde{f}(A \oplus B/\ker(f)) = \tilde{f}(A) \oplus \tilde{f}(B/\ker(f)) = f(A) \oplus f(B). \qquad \square$$

Next we state a result that says that we can split off an interval functor from any factored chain complex. This is the heart of the existence part of the structure theorem.

Lemma 2.16. *Let $X \in \mathrm{TEPCh}$ and $a \in X_n^0 \setminus \operatorname{im}(\partial_{n+1})$ satisfying the condition*

$$d(a) = \min\{d(a') \mid a' \in X_n^0,\ d(\partial a') = d(\partial a)\}.$$

Then there exists a parametrized subcomplex $Y \subseteq X$ such that $X \cong \langle a\rangle \oplus Y$.

Proof. For any $r \geq 0$, we denote the differentials in the chain complex X^r by $\partial_k^r\colon X_k^r \to X_{k-1}^r$. Moreover we write $a^r := X_n^{0\leq r}(a)$ and $\partial a^r := X_{n-1}^{0\leq r}(\partial_n^0 a) = \partial_n^r(a^r)$. Since $a \notin \operatorname{im}(\partial_{n+1})$, we have $a \neq 0$ and thus, by Proposition 2.14, there exists a tagged interval $[0, s_1, t_1) \in \mathcal{J}$, such that $\langle a\rangle \cong \mathcal{I}^n[0, s_1, t_1)$. By tameness of X, if $s_1 > 0$, there exist $s_0 < s_1$ and $t_0 < t_1$, such that for all $s_0 \leq s < s_1$ and for all $t_0 \leq t < t_1$, the internal chain maps $X^{s_0\leq s}$ and $X^{t_0\leq t}$ are isomorphisms. In the case $s_1 = 0$, we do not choose s_0 and in the case $s_1 = t_1$, we may choose $s_0 = t_0$. We construct a direct summand Y of X, complementary to $\langle a\rangle$, in five steps.

1. We define Y to be all of X in all degrees where $\langle a\rangle$ is zero. Explicitly: For $k \neq n, n-1$ and any $r \geq 0$, we define $Y_k^r := X_k^r$.

2. Let $Y_n^0 \subseteq X_n^0$ be a linear subspace such that

$$\ker(X_n^{0\leq t_0}) \subseteq Y_n^0, \qquad \ker(\partial_n^{s_0} \circ X_n^{0\leq s_0}) \subseteq Y_n^0, \qquad \text{and} \qquad X_n^0 = \langle a\rangle_n^0 \oplus Y_n^0$$

   (in the case $s_1 = 0$ we simply drop the second condition). In terms of death times, the first two conditions are saying that any element that dies before a or whose boundary dies before ∂a has to be contained in Y_n^0. Of course we need to argue why such a subspace exists. Note that $\langle a\rangle_n^0 \cap (\ker(X_n^{0\leq t_0}) + \ker(\partial_n^{s_0} \circ X_n^{0\leq s_0})) = \{0\}$, since otherwise we could write $a = b_1 + b_2$ with $b_1^{t_0} = 0$ and $\partial b_2^{s_0} = 0$. In particular $d(b_1) < t_1 = d(a)$ and $d(\partial b_2) < s_1 = d(\partial a)$. This would imply that $d(\partial b_1) = d(\partial a - \partial b_2) = \max\{d(\partial a), d(\partial b_2)\} = d(\partial a)$ by Lemma 2.13, hence violating the condition that the death time of a is minimal among all elements whose boundaries have the same death time as ∂a. By standard linear algebra arguments there exists a subspace $Y_n^0 \subseteq X_n^0$ that contains $\ker(X_n^{0\leq t_0}) + \ker(\partial_n^{s_0} \circ X_n^{0\leq s_0})$ and is a direct summand of $\langle a\rangle_n^0$.

3. For any $r \geq 0$, we define $Y_n^r := X_n^{0\leq r}(Y_n^0)$. It follows from the previous step and Proposition 2.15 that $X_n^r = \langle a\rangle_n^r \oplus Y_n^r$ for all $r < t_1$. For larger values of r, it follows from $a^r = 0$ and surjectivity of $X_n^{0\leq r}$ that $Y_n^r = X_n^r$.

4. In the case $s_1 = 0$, we define $Y_{n-1}^0 := X_{n-1}^0$. Otherwise, let $Y_{n-1}^0 \subseteq X_{n-1}^0$ be a linear subspace such that

$$\partial_n^0(Y_n^0) \subseteq Y_{n-1}^0, \qquad \ker(X_{n-1}^{0\leq s_0}) \subseteq Y_{n-1}^0, \qquad \text{and} \qquad X_{n-1}^0 = \langle a\rangle_{n-1}^0 \oplus Y_{n-1}^0.$$

Again, for such a subspace to exist it is sufficient to show that $\langle a\rangle_{n-1}^0 \cap (\partial_n^0(Y_n^0) + \ker(X_{n-1}^{0\leq s_0})) = \{0\}$. If that were not the case, then $\partial a = \partial b + c$ for some $b \in Y_n^0$ and $c \in X_{n-1}^0$ with $c^{s_0} = 0$. In particular $d(c) < d(\partial a)$. Defining $a' := a - b$ would then yield an element of X_n^0 with $d(\partial a') = d(\partial a - \partial b) = d(c) < d(\partial a)$. This would mean that $(\partial a')^{s_0} = 0$, thus $a' \in Y_n^0$. But, since $b \in Y_n^0$, this would imply that $a = a' - b \in Y_n^0$, contradicting the fact that $\langle a\rangle_n^0 \cap Y_n^0 = 0$. As in step 2 we can now construct $Y_{n-1}^0 \subseteq X_{n-1}^0$ by standard linear algebra arguments.

5. For any $r \geq 0$, we define $Y_{n-1}^r := X_{n-1}^{0\leq r}(Y_{n-1}^0)$. If follows from the previous step and Proposition 2.15 that $X_{n-1}^r = \langle a\rangle_{n-1}^r \oplus Y_{n-1}^r$ for all $r < s_1$. For larger values of r, it follows from $\partial a^r = 0$ and surjectivity of $X_{n-1}^{0\leq r}$ that $Y_{n-1}^r = X_{n-1}^r$.

So far we have constructed linear subspaces $Y_k^r \subseteq X_k^r$ for all r, k with the property that $X_k^r = \langle a\rangle_k^r \oplus Y_k^r$. The internal maps of Y, both vertical and horizontal, are defined to be the restrictions of the corresponding maps in X. It remains to check that Y is indeed a tame epimorphic parametrized subcomplex of X.

Let us first check that Y^0 is a subcomplex of X^0, i.e. $\partial_k^0(Y_k^0) \subseteq Y_{k-1}^0$ for all k. We only need to check those degrees where $Y_{k-1}^0 \neq X_{k-1}^0$, so we need to check that $\partial_{n+1}^0(Y_{n+1}^0) \subseteq Y_n^0$ and $\partial_n^0(Y_n^0) \subseteq Y_{n-1}^0$. The latter is true by construction of Y_{n-1}^0 in step 4 above. If $s_1 = 0$, then the former holds because in that case $Y_{n-1}^0 = X_{n-1}^0$. Otherwise, it holds because

$$\partial_{n+1}^0(Y_{n+1}^0) = \partial_{n+1}^0(X_{n+1}^0) \subseteq \ker(\partial_n^0) \subseteq \ker(X_{n-1}^{0\leq s_0} \circ \partial_n^0) = \ker(\partial_n^{s_0} \circ X_n^{0\leq s_0}) \subseteq Y_n^0,$$

where the last inclusion holds by construction of Y_n^0 in step 2 above. To show that Y^r is a subcomplex of X^r for any $r \geq 0$, first note that the restriction of $X_k^{0\leq r}$ to Y_k^0 yields a surjective map $Y_k^0 \twoheadrightarrow Y_k^r$, by construction of Y_k^r. We can use this fact together with the knowldege that Y^0 is a subcomplex of X^0 to show that Y^r is a subcomplex of X^r. Thus we have shown that Y is an epimorphic parametrized subcomplex of X. Tameness of Y follows from Lemma 2.10 (*iv*). ☐

Now we are ready to prove the structure theorem in TEPCh.

Theorem 2.17 (Structure theorem in TEPCh)**.** *Any factored chain complex X is isomorphic to a finite direct sum of interval functors in* TEPCh*, i.e. for every $n \in \mathbb{N}$, there exists a unique finite multiset* $\mathrm{tBar}_n = \mathrm{tBar}_n(X) \in \mathrm{Mult}(\mathcal{J})$ *of tagged intervals, such that*

$$X \cong \bigoplus_{n\in\mathbb{N}} \bigoplus_{[0,s,t)\in\mathrm{tBar}_n} \mathcal{I}^n[0,s,t).$$

The collection of multisets $\mathrm{tBar}(X) = (\mathrm{tBar}_n(X))_{n\in\mathbb{N}}$ is called the **tagged barcode** of X. Sometimes we will call $\mathrm{tBar}_n(X)$ the tagged barcode in degree n (or n-th tagged barcode).

Proof of existence. The existence of such a decomposition can be shown by iterated use of Lemma 2.16. Formally, we do a proof by induction over the number $m(X) := \sum_k \dim(X_k^0)$.

If $m(X) = 0$, then $X = 0$ and the statement holds for X, since then, X is isomorphic to the empty direct sum, i.e. $\mathrm{tBar}_n(X) = \emptyset$ for all n.

Assume now that $m(X) = m > 0$ and that we have already proven the statement for all $Z \in$ TEPCh with $m(Z) < m$. Since $X \neq 0$, there must exist some $n \geq 0$ and $a \in X_n^0 \setminus \mathrm{im}(\partial_{n+1})$ whose death time is minimal among all elements whose boundary has the same death time as ∂a. By tameness of X, there can be only finitely many different death times among the elements of X_n^0, thus an element with minimal death time always exists (since X_n^0 is finite-dimensional, this also follows from Lemma 2.13 *(iii)* even without the tameness assumption). By Lemma 2.16, we have $X = \langle a\rangle \oplus Y$ for some $Y \in$ TEPCh. Clearly $m(Y) < m(X)$ and thus Y can be written as a finite direct sum of interval functors in TEPCh. Furthermore, $\langle a\rangle$ is isomorphic to an interval functor in TEPCh by Proposition 2.14, thus also X is isomorphic to a finite direct sum of interval functors in TEPCh. □

Proof of uniqueness. Assume that there are two collections of multisets tBar_k and tBar'_k, $k \in \mathbb{N}$, such that

$$X := \bigoplus_{k\in\mathbb{N}} \bigoplus_{[0,s,t)\in\mathrm{tBar}_k} \mathcal{I}^k[0,s,t) \cong \bigoplus_{k\in\mathbb{N}} \bigoplus_{[0,s,t)\in\mathrm{tBar}'_k} \mathcal{I}^k[0,s,t) =: Y.$$

Fix a number $n \in \mathbb{N}$. By functoriality of $H_n(\square)$ and $\partial_n(\square)$, this implies that $H_n(X) \cong H_n(Y)$ and $\partial_n(X) \cong \partial_n(Y)$. By Proposition 2.11 *(iii)* and *(iv)*, we have

$$H_n(X) = \bigoplus_{\substack{[s,t)\in\mathrm{tBar}_n,\\ s<t}} \mathbb{F}[s,t) \qquad \text{and} \qquad \partial_n(X) = \bigoplus_{\substack{[s,t)\in\mathrm{tBar}_n,\\ s>0}} \mathbb{F}[0,s).$$

This implies that we can obtain the barcodes of $H_n(X)$ and $\partial_n(X)$ by adding, for every tagged interval $[0,s,t) \in \mathrm{tBar}_n$, an interval of the form $[0,s)$ to $\mathrm{Bar}(\partial_n(X))$ and, if $s \neq t$, an interval of the form $[s,t)$ to $\mathrm{Bar}(H_n(X))$. Applying the definition that a multiset is a function, where the function value is interpreted as the multiplicity of a (tagged) interval, the barcodes of $H_n(X)$ and $\partial_n(X)$ are given by

$$\mathrm{Bar}(H_n(X))([s,t)) = \begin{cases} \mathrm{tBar}_n([0,s,t)), \text{ if } s<t, \\ 0, \text{ if } s=t, \end{cases}$$

$$\mathrm{Bar}(\partial_n(X))([r,s)) = \begin{cases} \sum_{s\leq t} \mathrm{tBar}_n([0,s,t)), \text{ if } r=0, \\ 0, \text{ if } r>0. \end{cases}$$

We can apply the same reasoning to Y to get the analogous expressions for $\mathrm{Bar}(H_n(Y))$ and $\mathrm{Bar}(\partial_n(Y))$, just with tBar_n replaced by tBar'_n. By the uniqueness in Theorem 1.4 we have $\mathrm{Bar}(H_n(X)) = \mathrm{Bar}(H_n(Y))$ and $\mathrm{Bar}(\partial_n(X)) = \mathrm{Bar}(\partial_n(Y))$. The first equality implies that

$$\mathrm{tBar}_n([0,s,t)) = \mathrm{tBar}'_n([0,s,t)) \text{ for all } s<t \in [0,\infty],$$

so it only remains to show that tBar_n and tBar'_n also agree on tagged intervals of the form $[0,s,s)$. For this, consider

$$\begin{array}{c} \mathrm{Bar}(\partial_n(X))([0,s)) = \sum_{s\leq t} \mathrm{tBar}_n([0,s,t)) = \mathrm{tBar}_n([0,s,s)) + \sum_{s<t} \mathrm{tBar}_n([0,s,t)) \\ \| \\ \mathrm{Bar}(\partial_n(Y))([0,s)) = \sum_{s\leq t} \mathrm{tBar}'_n([0,s,t)) = \mathrm{tBar}'_n([0,s,s)) + \sum_{s<t} \mathrm{tBar}'_n([0,s,t)). \end{array}$$

Since we have seen before that tBar_n and tBar'_n agree on tagged intervals $[0,s,t)$ with $s<t$, it follows from this chain of equations that also $\mathrm{tBar}_n([0,s,s)) = \mathrm{tBar}'_n([0,s,s))$. Thus we have shown that $\mathrm{tBar}_n = \mathrm{tBar}'_n$ and since n was chosen arbitrarily, this concludes the proof. □

The following corollary gives sufficient conditions for the existence of certain elements in the tagged barcode of a factored chain complex.

Corollary 2.18. *Let $X \in \mathrm{TEPCh}$ be such that all of its internal chain maps induce isomorphisms in homology in all degrees and let $s < t \in [0, \infty)$ be such that*

(i) for all $s < s' < t$ the internal chain map $X^{s \leq s'}$ is an isomorphism,

(ii) $\dim(X_k^t) = \dim(X_k^s) - 1$ for $k = n, n-1$ and $\dim(X_k^t) = \dim(X_k^s)$ for all other values of k.

Then the tagged interval $[0, t, t)$ appears with multiplicity 1 in the tagged barcode $\mathrm{tBar}_n(X)$ and does not appear (i.e. has multiplicity 0) in all other tagged barcodes $\mathrm{tBar}_m(X)$, with $m \neq n$. Moreover, for any $s < t' < t$, the tagged interval $[0, t', t')$ does not appear in the tagged barcode of X.

Proof. By Theorem 2.17, X is a direct sum of interval functors in TEPCh. Since all internal chain maps of X induce isomorphisms in homology, only tagged intervals of the form $[0, t', t')$, for $0 < t' \leq \infty$, can appear in the tagged barcode of X. By (ii), in the decomposition of X there must be precisely one interval functor $\mathcal{I}^n[0, t', t')$, and no interval functor $\mathcal{I}^m[0, t', t')$ for $m \neq n$, with $s < t' \leq t$. Condition (i) prevents the case $t' < t$, so the only remaining possibility is what was claimed. ☐

2.4 The isometry theorem in TEPCh

The goal of this section is to generalize the classical isometry theorem (Theorem 1.5) to the case of factored chain complexes. For this, we first describe how the interleaving distance and the bottleneck distance look like in this setup. In both cases, these definitions are special cases of the definitions given in Section 1.1.

The interleaving distance. The interleaving distance in TEPCh is precisely the one that we introduced in larger generality in Section 1.1.A, applied to the category $\mathcal{C} = \mathrm{Ch}$. Here we give some explicit formulae for computing the interleaving distance between interval functors in TEPCh.

Remark 2.19. It follows from the definition of $\mathcal{I}^n[0, s, t)$ and the ε-shift that for any $[0, s, t) \in \mathcal{J}$ and $\varepsilon \geq 0$ we have

$$\mathcal{I}^n[0, s, t)_\varepsilon = \mathcal{I}^n[0, \max\{s - \varepsilon, 0\}, \max\{t - \varepsilon, 0\}),$$

where $\mathcal{I}^n[0, 0, 0) := 0$.

Lemma 2.20. *Given $\varepsilon > 0$ and two tagged intervals $J = [0, s, t)$ and $J' = [0, s', t')$, the corresponding interval functors in TEPCh, i.e. $\mathcal{I}^n[0, s, t)$ and $\mathcal{I}^n[0, s', t')$, are ε-interleaved if and only if*

$$\varepsilon \geq \min \left\{ \max \{ |s - s'|, |t - t'| \}, \max \left\{ \frac{t}{2}, \frac{t'}{2} \right\} \right\},$$

where we use the convention that $|\infty - \infty| = 0$.

Proof. If $\frac{t}{2}, \frac{t'}{2} \leq \varepsilon$, then the internal maps of length 2ε are zero in both $\mathcal{I}^n[0, s, t)$ and $\mathcal{I}^n[0, s', t')$. Therefore, the zero maps are an ε-interleaving between $\mathcal{I}^n[0, s, t)$ and $\mathcal{I}^n[0, s', t')$.

If $|s - s'|, |t - t'| \leq \varepsilon$, then, by Remark 2.19 and Lemma 2.5, there exist non-zero parametrized chain maps $\phi\colon \mathcal{I}^n[0, s, t) \to \mathcal{I}^n[0, s', t')_\varepsilon$ and $\psi\colon \mathcal{I}^n[0, s', t') \to \mathcal{I}^n[0, s, t)_\varepsilon$. More explicitly, we let ϕ and ψ be the maps constructed in the proof of Lemma 2.5. One can then check that the compositions $\psi_\varepsilon \circ \phi$ and $\phi_\varepsilon \circ \psi$ agree with the internal maps of length 2ε in $\mathcal{I}^n[0, s, t)$ and $\mathcal{I}^n[0, s', t')$.

Let us now assume that $\varepsilon < \min\left\{\max\left\{|s - s'|, |t - t'|\right\}, \max\left\{\frac{t}{2}, \frac{t'}{2}\right\}\right\}$ and show that in this case there exists no ε-interleaving. Since $\varepsilon < \max\left\{\frac{t}{2}, \frac{t'}{2}\right\}$, we can assume without loss of generality that $\varepsilon < \frac{t}{2}$. Therefore, there exist internal maps in $\mathcal{I}^n[0, s, t)$ of length 2ε which are non-zero. Therefore, any ε-interleaving must consist of non-zero morphisms. However, since $\varepsilon < \max\left\{|s - s'|, |t - t'|\right\}$, it follows from Remark 2.19 and Lemma 2.5 that no non-zero morphism with a shift of ε can exist in one of the two directions. □

Corollary 2.21. *Two interval functors $\mathcal{I}^n[0, s, t)$ and $\mathcal{I}^n[0, s', t')$ in* TEPCh *are interleaved if and only if one of the following cases holds:*

(i) $t, t' < \infty$,

(ii) $t = t' = \infty$ *and* $s, s' < \infty$,

(iii) $t = t' = s = s' = \infty$.

More generally, if $X, Y \in$ TEPCh *are interleaved, then it follows that the tagged barcodes of X and Y in each degree have the same amount of intervals of the form $[0, s, \infty)$ with $s < \infty$ and of the form $[0, \infty, \infty)$.*

Proof. By Lemma 2.20, the two interval functors in TEPCh are interleaved if and only if $\max\{|s - s'|, |t - t'|\} < \infty$ or $\max\left\{\frac{t}{2}, \frac{t'}{2}\right\} < \infty$. Note that the former is equivalent to saying that s and s' are either both finite or both infinite and t and t' are either both finite or both infinite. The statement $\max\left\{\frac{t}{2}, \frac{t'}{2}\right\} < \infty$ is equivalent to saying that t and t' are both finite, so it is a stronger statement and we can ignore it. Thus we get the three cases described in the statement: Either t, t', s, s' are all finite, or t, t' are infinite and s, s' are finite, or t, t', s, s' are all infinite.

In order to prove the more general statement, note that by tameness there exists $T > 0$ such that for all $T \leq s \leq t$, the internal maps $X^{s \leq t}$ and $Y^{s \leq t}$ are isomorphisms. From this it follows that the chain complexes X^T and Y^T are isomorphic and must thus have the same number of disks and spheres in their decomposition. The number of n-disks in the decomposition of X^T is equal to the number of copies of $\mathcal{I}^n[0, \infty, \infty)$ in the decomposition of X and the number of n-spheres in the decomposition of X^T is equal to the number of copies of $\mathcal{I}^n[0, s, \infty)$ for any $s < \infty$ in the decomposition of X. The same holds for Y^T and Y, hence the result follows. □

The bottleneck distance. Similar to the interleaving distance, the bottleneck distance is a special case of the generalized bottleneck distance described in Section 1.1.B. We only need to give a pseudometric and a weight function on the set $\mathcal{J}$ of tagged intervals.

Definition 2.22. Given two tagged intervals $J = [0, s, t)$ and $J' = [0, s', t')$, we define the **cost of matching J to J'** as $c(J, J') := \max\{|s - s'|, |t - t'|\}$. The **weight of J** is defined as $W(J) := \frac{t}{2}$.

One can check that c is an extended metric on $\mathcal{J}$ and that c and W are compatible. Therefore, by Remark 1.2, the bottleneck distance d_B is an extended pseudometric on $\mathrm{Mult}(\mathcal{J})$.

Note that the bottleneck distance of two multisets of tagged intervals from $\mathcal{J}$ is finite if and only if they contain the same number of tagged intervals of the form $[0, \infty, \infty)$ and of the form $[0, s, \infty)$ with $s < \infty$, respectively. This is analogous to the Corollary 2.21 about interleaving distances between interval functors in TEPCh.

The isometry theorem. We are now ready to state the isometry theorem and prove one of the two inequalities.

Theorem 2.23 (Isometry Theorem in TEPCh)**.** *If X and Y are factored chain complexes, then*

$$d_I(X, Y) = \max_{n\in\mathbb{N}} d_B(\mathrm{tBar}_n(X), \mathrm{tBar}_n(Y)).$$

Proof of the inequality $\leq$. It suffices to prove that if for some $\varepsilon > 0$ there exist ε-matchings between $\mathrm{tBar}_n(X)$ and $\mathrm{tBar}_n(Y)$ for all n, then there exists an ε-interleaving between X and Y. By Theorem 2.17, we have that

$$X \cong \bigoplus_{n\in\mathbb{N}} \left(\bigoplus_{[0,s,t)\in\mathrm{tBar}_n(X)} \mathcal{I}^n[0, s, t) \right) \qquad \text{and} \qquad Y \cong \bigoplus_{n\in\mathbb{N}} \left(\bigoplus_{[0,s',t')\in\mathrm{tBar}_n(Y)} \mathcal{I}^n[0, s', t') \right).$$

Fix a number $\varepsilon > 0$ and assume that there exists an ε-matching between $\mathrm{tBar}_n(X)$ and $\mathrm{tBar}_n(Y)$ for each n. From this, we want to construct an ε-interleaving between X and Y. If $[0, s, t) \in \mathrm{tBar}_n(X)$ and $[0, s', t') \in \mathrm{tBar}_n(Y)$ are matched, then $\max\{|s - s'|, |t - t'|\} \leq \varepsilon$. Therefore $\mathcal{I}^n[0, s, t)$ and $\mathcal{I}^n[0, s', t')$ are ε-interleaved by Lemma 2.20. For intervals $[0, s, t) \in \mathrm{tBar}_n(X)$ and $[0, s', t') \in \mathrm{tBar}_n(Y)$ that are unmatched we have $t, t' \leq 2\varepsilon$, and therefore the internal maps of length 2ε in $I^n[0, s, t)$ and $\mathcal{I}^n[0, s', t')$ are zero, thus they are ε-interleaved with the zero parametrized chain complex.

We now construct an ε-interleaving between X and Y. Given a interval $[0, s, t) \in \mathrm{tBar}_n(X)$ that is matched with $[0, s', t') \in \mathrm{tBar}_n(Y)$, we define ϕ on the summand $\mathcal{I}^n[0, s, t)$ to be the interleaving constructed in the proof of Lemma 2.20. For any unmatched interval $[0, s, t) \in \mathrm{tBar}_n(X)$ we define ϕ to be zero on $\mathcal{I}^n[0, s, t)$. By extending linearly, this defines a parametrized chain map $\phi\colon X \to Y_\varepsilon$. We define $\psi\colon Y \to X_\varepsilon$ analogously. It follows that (ϕ, ψ) is an ε-interleaving between X and Y. □

For the other inequality, we mimic the proof given in [6] as suggested by previous attempts [15]. We begin by setting up the notation.

Definition 2.24. Given $X \in$ TEPCh and $\varepsilon > 0$, we define $\mathrm{tBar}^\varepsilon_n(X) := \{I \in \mathrm{tBar}_n(X) \mid W(I) > \varepsilon\}$. Given $X, Y \in$ TEPCh, we define for $I \in \mathrm{tBar}^\varepsilon_n(X)$ and for $A \subseteq \mathrm{tBar}^\varepsilon_n(X)$

$$\mu_\varepsilon(I) := \{J \in \mathrm{tBar}_n(Y) \mid c(I, J) \leq \varepsilon\} \quad \text{and} \quad \mu_\varepsilon(A) := \bigcup_{I\in A} \mu(I).$$

For $J \in \mathrm{tBar}^\varepsilon_n(Y)$ and $A \subseteq \mathrm{tBar}^\varepsilon_n(Y)$ the sets $\mu_\varepsilon(J) \subseteq \mathrm{tBar}_n(X)$ and $\mu_\varepsilon(A) \subseteq \mathrm{tBar}_n(X)$ are defined analogously.

The proof of Theorem 2.23 uses Hall's theorem. For this we recall some notions from graph theory. Given a bipartite graph H with vertex set $L \sqcup R$, for any subset $A \subseteq L$ we define $\mu(A)$ to be the set of those vertices in R that have an edge to at least one vertex in A. For $A \subseteq R$ we define $\mu(A)$ analogously. A **partial matching** $\mathcal{M}$ on H is a set of edges of H such that no two edges share a common vertex. We say that a matching $\mathcal{M}$ **covers** a set $A \subseteq L$ (or $A \subseteq R$) if every vertex of A is contained in an edge of $\mathcal{M}$. Now we can state Hall's marriage theorem.

Theorem 2.25 (Hall's marriage theorem [33]). *Let H be a finite bipartite graph on the vertex set $L \sqcup R$. Then there exists a partial matching on H that covers L if and only if, for any subset $A \subseteq L$, we have $|\mu(A)| \geq |A|$.*

We will need a slightly stronger version of this theorem, which we formulate in the following corollary.

Corollary 2.26. *Let H be a finite bipartite graph on the vertex set $L \sqcup R$ and assume we are given two subsets $L^* \subseteq L$ and $R^* \subseteq R$. Then there exists a partial matching on H that covers both L^* and R^* if and only if, for any subset $A \subseteq L^*$ or $A \subseteq R^*$, we have $|\mu(A)| \geq |A|$.*

Proof. As in the original Hall's theorem, it is not hard to see that if there exists a partial matching covering L^* and R^*, then this condition is satisfied. It thus remains to show the other implication. Denote by H_1 the subgraph of H resulting from restricting the vertex set to $L^* \sqcup R$ and denote by H_2 the subgraph of H resulting from restricting the vertex set to $L \sqcup R^*$. By Theorem 2.25 there exists a matching $\mathcal{M}_1$ on H_1 covering L^* and a matching $\mathcal{M}_2$ on H_2 covering R^*. We now view both $\mathcal{M}_1$ and $\mathcal{M}_2$ as partial matchings on H and we want to combine them into a partial matching that covers both L^* and R^*.

We start with $\mathcal{M}_1$. This covers all vertices of L^*, but it may not cover all vertices of R^*. If we are given such a vertex $x \in R^*$, it must be covered in $\mathcal{M}_2$, so we can start drawing a path through H, starting at x and alternating between edges from $\mathcal{M}_1$ and $\mathcal{M}_2$. By finiteness, this path must end at some point, and it can only end outside of L^* and R^*. The reason for this is that when we follow the path up to a vertex $y \in L^*$, then we must have arrived there by an edge from $\mathcal{M}_2$. However, this vertex y is covered in the matching $\mathcal{M}_1$, so the path cannot end there. An analogous argument shows why it cannot end in R^*. We can thus replace all the edges from $\mathcal{M}_1$ along this path with the edges from $\mathcal{M}_2$ along the path and get a matching which still covers L^* and additionally covers the vertex y. Repeating this argument eventually yields a partial matching that covers both L^* and R^*. ☐

The following result indicates how we are going to apply Hall's theorem.

Lemma 2.27. *If $X, Y \in \mathrm{TEPCh}$ are ε-interleaved, then for any subset $A \subseteq \mathrm{tBar}_n^{\varepsilon}(X)$ or $A \subseteq \mathrm{tBar}_n^{\varepsilon}(Y)$, we have $|\mu_{\varepsilon}(A)| \geq |A|$.*

Before we prove Lemma 2.27, we explain how it helps us finish the proof of the isometry theorem.

Proof of the inequality $\geq$ in Theorem 2.23. Let $\varepsilon > 0$ and assume that X and Y are ε-interleaved. We need to show that for any $n \in \mathbb{N}$, there exists an ε-matching between $\mathrm{tBar}_n(X)$ and $\mathrm{tBar}_n(Y)$. We construct a finite bipartite graph H as follows. Let the vertex set of H be given by $\mathrm{tBar}_n(X) \sqcup \mathrm{tBar}_n(Y)$. Connect $I \in \mathrm{tBar}_n(X)$ and $J \in \mathrm{tBar}_n(Y)$

with an edge if and only if $c(I, J) \leq \varepsilon$. Thus in the graph H, it holds that for any subset $A \subseteq \mathrm{tBar}_n(X)$ or $A \subseteq \mathrm{tBar}_n(Y)$ we have $\mu(A) = \mu_\varepsilon(A)$. Therefore, by Lemma 2.27, the conditions for Corollary 2.26 are satisfied and there exists a partial matching in H covering both $\mathrm{tBar}^\varepsilon_n(X)$ and $\mathrm{tBar}^\varepsilon_n(Y)$. Such a matching corresponds exactly to an ε-matching of the tagged barcodes $\mathrm{tBar}_n(X)$ and $\mathrm{tBar}_n(Y)$, thus we are done. $\square$

It thus remains only to prove Lemma 2.27. For this we define $\alpha([0, s, t)) := s + t$ and we first show the following result.

Lemma 2.28. *Let* $I_1 = [0, s_1, t_1), I_2 = [0, s_2, t_2), I_3 = [0, s_3, t_3) \in \mathcal{J}$ *with* $\alpha(I_1) \leq \alpha(I_3)$ *and let*

$$f\colon \mathcal{I}^n[0, s_1, t_1) \to \mathcal{I}^n[0, s_2, t_2)_\varepsilon \quad \text{and} \quad g\colon \mathcal{I}^n[0, s_2, t_2) \to \mathcal{I}^n[0, s_3, t_3)_\varepsilon$$

be parametrized chain maps which are both non-zero. Then $c(I_1, I_2) \leq \varepsilon$ *or* $c(I_2, I_3) \leq \varepsilon$.

Proof. By Lemma 2.5, the existence of f implies that $s_2 \leq s_1 + \varepsilon$ and $t_2 \leq t_1 + \varepsilon$. Assume that $\mathcal{I}^n[0, s_1, t_1)$ and $\mathcal{I}^n[0, s_2, t_2)$ are not ε-interleaved. Then, by Lemma 2.20, $s_2 < s_1 - \varepsilon$ or $t_2 < t_1 - \varepsilon$. In the first case we have $\alpha(I_1) = s_1 + t_1 > s_2 + \varepsilon + t_1 \geq s_2 + t_2 = \alpha(I_2)$. Similarly, in the second case, we have $\alpha(I_1) = s_1 + t_1 > s_1 + t_2 + \varepsilon \geq s_2 + t_2 = \alpha(I_2)$. Assuming that $\mathcal{I}^n[0, s_2, t_2)$ and $\mathcal{I}^n[0, s_3, t_3)$ are not ε-interleaved either, with a similar computation we get $\alpha(I_2) > \alpha(I_3)$. Hence $\alpha(I_1) > \alpha(I_2) > \alpha(I_3)$, which contradicts the hypothesis. $\square$

We introduce more notation.

Definition 2.29. If $X \in \mathrm{TEPCh}$, $I = [0, s, t) \in \mathrm{tBar}_n(X)$ and $\varepsilon > 0$, then we denote by $\eta^\varepsilon_I\colon \mathcal{I}^n[0, s, t) \to \mathcal{I}^n[0, s, t)_\varepsilon$ the canonical parametrized chain map given by the internal maps of $\mathcal{I}^n[0, s, t)$.

Let $X, Y \in \mathrm{TEPCh}$. For any $I = [0, s, t) \in \mathrm{tBar}_n(X)$, we write $\iota^X_I : \mathcal{I}^n[0, s, t) \hookrightarrow X$ for the inclusion and $\pi^X_I : X \twoheadrightarrow \mathcal{I}^n[0, s, t)$ for the projection. Analogously, for $J \in \mathrm{tBar}_n(Y)$, we write ι^Y_J and π^Y_J for the inclusion and projection in Y. Now let (ϕ, ψ) be an ε-interleaving between X and Y. For any $I \in \mathrm{tBar}_n(X)$ and $J \in \mathrm{tBar}_{n'}(Y)$, we write $\phi_{I,J} := (\pi^Y_J)_\varepsilon \circ \phi \circ \iota^X_I$ and $\psi_{J,I} := (\pi^X_I)_\varepsilon \circ \psi \circ \iota^Y_J$.

As a final preparation for the proof of Lemma 2.27, recall that by Lemma 2.4, any morphism $\varphi\colon \mathcal{I}^n[0, s, t) \to \mathcal{I}^n[0, s', t')$ is uniquely determined by the scalar $\varphi(1^0_n) \in \mathcal{I}^n[0, s', t')^0_n = \mathbb{F}$. We denote this scalar by $\lambda(\varphi)$. This scalar has the following properties:

- $\lambda(\varphi' + \varphi) = \lambda(\varphi') + \lambda(\varphi)$ for $\varphi'\colon \mathcal{I}^n[0, s, t) \to \mathcal{I}^n[0, s', t')$,
- $\lambda(\varphi' \circ \varphi) = \lambda(\varphi') \cdot \lambda(\varphi)$ for $\varphi'\colon \mathcal{I}^n[0, s', t') \to \mathcal{I}^n[0, s'', t'')$,
- $\lambda(\varphi_\varepsilon) = \lambda(\varphi)$ for $0 < \varepsilon < t'$,
- $\lambda(\eta^{2\varepsilon}_I) = 1$ whenever $I \in \mathrm{tBar}^\varepsilon_n(X)$ for some $X \in \mathrm{TEPCh}$.

Proof of Lemma 2.27. Let (ϕ, ψ) be an ε-interleaving between X and Y. We prove the statement for a subset $A \subseteq \mathrm{tBar}^\varepsilon_n(X)$, the case where $A \subseteq \mathrm{tBar}^\varepsilon_n(Y)$ is analogous. Not that for any $I, I' \in A$ with $\alpha(I) \leq \alpha(I')$ we have

$$(\pi^X_{I'})_{2\varepsilon} \circ \psi_\varepsilon \circ \phi \circ \iota^X_I = \sum_{m \in \mathbb{N}} \sum_{J \in \mathrm{tBar}_m(Y)} (\psi_{J,I'})_\varepsilon \circ \phi_{I,J} = \sum_{J \in \mathrm{tBar}_n(Y)} (\psi_{J,I'})_\varepsilon \circ \phi_{I,J} = \sum_{J \in \mu_\varepsilon(A)} (\psi_{J,I'})_\varepsilon \circ \phi_{I,J}, \tag{2.1}$$

where the second equality follows from Lemma 2.5 and the third one holds since, for $J \notin \mu_\varepsilon(A)$, at least one of the two morphisms $\psi_{J,I'}$ and $\phi_{I,J}$ must be zero by Lemma 2.28. The left-hand side of Equation (2.1) is equal to $\eta_I^{2\varepsilon}$ if $I = I'$ and equal to zero if $I \neq I'$. In particular, $\mu_\varepsilon(A) \neq \emptyset$ when $A \neq \emptyset$.

We order the elements of $A = \{I_1 = [0, s_1, t_1), \dots, I_r = [0, s_r, t_r)\}$ so that $\alpha(I_i) \leq \alpha(I_{i'})$ for $i \leq i'$, and we set $\mu_\varepsilon(A) = \{J_1 = [0, s_1', t_1'), \dots, J_q = [0, s_q', t_q')\}$. The goal is to show that $q \geq r$.

Computing λ on both sides of Equation (2.1) applied to $I = I' = I_i \in A$, and using the properties of λ, we get

$$1 = \lambda(\eta_{I_i}^{2\varepsilon}) = \sum_{J \in \mu_\varepsilon(A)} \lambda(\psi_{J,I_i}) \cdot \lambda(\phi_{I_i,J}) = \sum_{k=1}^{q} \lambda(\psi_{J_k,I_i}) \cdot \lambda(\phi_{I_i,J_k}).$$

If, on the other hand, we apply Equation (2.1) to $I = I_i$ and $I' = I_{i'}$, for $i < i'$, we get

$$0 = \sum_{J \in \mu_\varepsilon(A)} \lambda(\psi_{J,I_{i'}}) \cdot \lambda(\phi_{I_i,J}) = \sum_{k=1}^{q} \lambda(\psi_{J_k,I_{i'}}) \cdot \lambda(\phi_{I_i,J_k}).$$

We can put these equations into the form of a matrix equation, which yields

$$\begin{pmatrix} \lambda(\psi_{J_1,I_1}) & \cdots & \lambda(\psi_{J_q,I_1}) \\ \vdots & \ddots & \vdots \\ \lambda(\psi_{J_1,I_r}) & \cdots & \lambda(\psi_{J_q,I_r}) \end{pmatrix} \begin{pmatrix} \lambda(\phi_{I_1,J_1}) & \cdots & \lambda(\phi_{I_r,J_1}) \\ \vdots & \ddots & \vdots \\ \lambda(\phi_{I_1,J_q}) & \cdots & \lambda(\phi_{I_r,J_q}) \end{pmatrix} = \begin{pmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Note that for $i > i'$ we make no statement, which is why there are unknown entries on the upper right triangle of the matrix on the right-hand side. This matrix has rank r and the product of the two matrices on the left-hand side has rank at most q, therefore $q \geq r$. ☐

2.5 Parametrizing chain complexes

We are now going to describe two methods of transforming a chain complex into a factored chain complex. This requires additional input. In the first place, we need to use based chain complexes. A **based chain complex** is a chain complex $C_\bullet$ together with a basis $\mathcal{B}_k$ for every C_k. The other thing we need is to equip pairs of basis elements of adjacent degrees with weights. Formally, this is dealt with in the following definition.

Definition 2.30. A **weighted based chain complex** is a based chain complex $C_\bullet$, with bases $\mathcal{B}_k$, additionally equipped with functions $\mathrm{w}\colon \mathcal{B}_k \times \mathcal{B}_{k-1} \to [0, \infty)$ for all k. The values $\mathrm{w}(a, b)$, for $a \in \mathcal{B}_k$ and $b \in \mathcal{B}_{k-1}$, are called **weights**. A weighted based chain complex $(C_\bullet, \mathcal{B}_\bullet, \mathrm{w})$ is called **ordered**, if for every $t \in [0, \infty)$, a total order is given on the set

$$\mathrm{w}^{-1}(t) := \bigcup_k \{(a, b) \in \mathcal{B}_k \times \mathcal{B}_{k-1} \mid \mathrm{w}(a, b) = t\}.$$

A weighted based chain complex is called **generic** if all the weights are non-zero and pairwise different. We denote by wbCh (resp. wb$^+$Ch) the class of (ordered) weighted based chain complexes.

A generic weighted based chain complex is in particular ordered, since the sets $\mathrm{w}^{-1}(t)$ have cardinality zero or one for all values of t.

The main idea for constructing a factored chain complex from an ordered weighted based chain complex is the following: Repeatedly simplify the chain complex with respect to the pair of minimal weight (breaking the tie by using the ordering, if needed) until we are left with a chain complex whose differential is zero in all degrees. The difference between the two constructions we present is how we build a factored chain complex from this sequence of simplifications. In the first construction, represented by the letter X, the resulting functor $X\colon [0,\infty) \to \mathrm{Ch}$ corresponds closely to this sequence of simplifications, in the sense that the chain complexes X^t are given by the result of applying some of the simplifications, and the internal chain maps are given by compositions of the quotient maps. The parameter $t \in [0,\infty)$ can be thought of as time, in which case we can interpret the construction as applying one simplification after the other, always waiting for a time equal to the weight of a pair before simplifying that pair.

We present the first construction only for the case of strictly positive weights. Pairs of weight zero could be allowed, resulting in multiple simplifications happening at the same time. However, we leave it to the reader to fill in the details for this, as it adds some technicalities and hinders the readability of the exposition.

Construction 2.31. Given an ordered weighted based chain complex $C_\bullet = (C_\bullet, \mathcal{B}_\bullet, \mathrm{w})$ with strictly positive weights, we define the factored chain complex $X = X(C_\bullet)$ as follows.

(1) Let $t_0 := 0$ and $X^{t_0} := C_\bullet$.

(2) Assume that we have already defined a sequence of numbers $0 = t_0 < t_1 < \ldots < t_r < \infty$ and a functor $X\colon [0, t_r] \to \mathrm{Ch}$. Further assume that X^{t_r} is an ordered weighted based chain complex, i.e. we have a basis $\mathcal{B}^r_k$ for each $X^{t_r}_k$ and totally ordered weights $\mathrm{w}(a, b)$ between basis elements of adjacent degrees.

If all the differentials in the chain complex X^{t_r} are zero, go to step (3).

Otherwise, consider the set

$$A := \bigcup_{i\geq 1} \{(a', b') \in \mathcal{B}^r_i \times \mathcal{B}^r_{i-1} \mid \langle \partial a', b' \rangle \neq 0\}$$

and let $(a, b) \in A$ be the unique pair of minimum weight, or, if the minimal weight is not unique, the pair of minimal weight which is minimal with respect to the order. Define

$$t_{r+1} := t_r + \mathrm{w}(a, b).$$

Define $X^{t_{r+1}}$ to be the chain complex resulting from Lemma 1.6, when applied to the chain complex X^{t_r} and the element $a \in \mathcal{B}^r_k$. We endow the vector spaces $X^{t_{r+1}}_i$ with the bases $\mathcal{B}^{r+1}_i$ coming from Lemma 1.6, when applied to the chain complex X^{t_r}, the bases $\mathcal{B}^r_i$, and the elements $a \in \mathcal{B}^r_k$, $b \in \mathcal{B}^r_{k-1}$. Consider basis elements $a' \in \mathcal{B}^r_i$ and $b' \in \mathcal{B}^r_{i-1}$, both different from a and b. We define $\mathrm{w}([a'], [b']) := \mathrm{w}(a', b')$, thus the weights on the bases $\mathcal{B}^r_\bullet$ induce weights on the bases $\mathcal{B}^{r+1}_\bullet$. The same holds for the orderings of pairs with the same weight.

For $t_r < t < t_{r+1}$, define $X^t := X^{t_r}$ and for $t_r \leq s \leq t < t_{r+1}$, let $X^{s\leq t} := \mathbb{1}$ and define $X^{s\leq t_{r+1}}$ to be the epimorphism from Lemma 1.6. The remaining internal chain maps are defined by composition, thus we have extended X to a functor $X\colon [0, t_{r+1}] \to \mathrm{Ch}$. Update r to $r + 1$ and repeat step (2).

(3) If all the differentials in the chain complex X^{t_r} are zero, then we define $X^t := X^{t_r}$ for all $t_r < t < \infty$ and $X^{t \leq s} = \mathbb{1}$ for all $t_r \leq t \leq s < \infty$. The remaining internal maps are defined by composition, so we have extended X to a functor $X \colon [0, \infty) \to \mathrm{Ch}$.

Proposition 2.32. *Construction 2.31 assigns a factored chain complex to any ordered weighted based chain complex with strictly positive weights.*

Proof. First note that, since $C_\bullet$ is ordered, the set A contains a unique minimal pair in the first iteration. Since then we only get rid of one pair in each iteration and do not change any of the other weights, this holds also for later iterations of step (2).

We need to check that the conditions from the beginning of step (2) are satisfied when we go to another iteration of step (2). They are satisfied for the first iteration of step (2), when $r = 0$, due to the definitions in step (1). If $r > 0$, the conditions are satisfied at the beginning of step (2) and the differentials in X^{t_r} are not all zero, then it follows from Lemma 1.6 that the conditions are also satisfied for the next iteration.

We also need to show that the definition stops at some point, i.e. that we do not get an infinite loop of repeating step (2). This holds because with each iteration we reduce the total dimension of the chain complex X^{t_r} and since $X^{t_0} = C_\bullet$ has finite total dimension, we can do this only a finite number of times.

Once we get to step (3), we get a factored chain complex $X = X(C_\bullet) \in \mathrm{PCh}$. It remains to check that X is tame and epimorphic. Indeed X is epimorphic since the only internal maps that are not isomorphisms are quotient maps of the form given by Lemma 1.6, or compositions of such maps. Also, by construction, X is right-constant everywhere and fails to be left-constant exactly at the points $t_1, \dots, t_{r_{\max}}$, where $r_{\max}$ is the number of times we have repeated step (2). There are only a finite amount of these points, since we have shown before that we repeat step (2) only a finite number of times. Thus, X is tame. ☐

Proposition 2.33. *If $C_\bullet = (C_\bullet, \mathcal{B}_\bullet, \mathrm{w})$ is an ordered weighted based chain complex with strictly positive weights and $(a_1, b_1) \in \mathcal{B}_{n_1} \times \mathcal{B}_{n_1 - 1}, \dots, (a_r, b_r) \in \mathcal{B}_{n_r} \times \mathcal{B}_{n_r - 1}$ are the pairs that get simplified in the construction of $X = X(C_\bullet)$, then*

$$X(C_\bullet) \cong \bigoplus_{i=1}^{r} \mathcal{I}^{n_i}[0, t_i, t_i) \oplus \bigoplus_{k=0}^{\infty} (\mathcal{I}^k[0, 0, \infty))^{\beta_k},$$

where $\beta_k = \dim(H_k(C_\bullet))$. The numbers t_i are given by

$$t_i = \mathrm{w}(a_1, b_1) + \dots + \mathrm{w}(a_i, b_i).$$

Proof. For any $i = 1, \dots, r$, picking the i-th pair (a_i, b_i) yields a short exact sequence

$$0 \longrightarrow D^{n_i} \xrightarrow{\iota} X^{t_{i-1}} \xrightarrow{q} X^{t_i} \longrightarrow 0,$$

where ι is the unique chain map that sends $D^{n_i}_{n_i} \ni 1_{n_i} \mapsto a \in C_{n_i}$ and q is the quotient map. This sequence splits by Lemma 1.7, thus $X^{t_{i-1}} = X^{t_i} \oplus D^{n_i} = X^{t_i} \oplus \ker X^{t_{i-1} \leq t_i}$ and the result follows by induction. ☐

Now we present our second construction for assigning a factored chain complex to an ordered weighted based chain complex, represented by the letter Y. It is formulated without further conditions and has the advantage that when a pair of small weight gets simplified, this always yields a small tagged interval in the tagged barcode, even if this pair gets simplified at a later stage of the construction. In particular, pairs of weight zero, if they exist, do not contribute to the tagged barcode.

Construction 2.34. Given an ordered weighted based chain complex $C_\bullet = (C_\bullet, \mathcal{B}_\bullet, \mathrm{w})$, we define the factored chain complex $Y = Y(C_\bullet)$ as follows.

(1) Let $Z_\bullet := C_\bullet \in \mathrm{wbCh}$.

(2) Initialize Y by setting it to be equal to the zero factored chain complex discretized by $t_0 = 0$ and $t_1 = +\infty$ with only zero vector spaces and null linear maps.

(3) Find the pair $(a, b) \in \mathcal{B}_k \times \mathcal{B}_{k-1}$ in $Z_\bullet$, among all k, that has the smallest weight $\mathrm{w}(a, b)$ (in case of multiple pairs sharing the smallest weight, pick the minimal one with respect to the order) and satisfies the condition: $\langle \partial a, b \rangle \neq 0$. If we cannot find such a pair, then all boundaries are zero in $Z_\bullet$ and in that case, we go to Step (7).

(4) Update Y by summing to it the interval functor $\mathcal{I}^n[0, t, t)$ in TEPCh, where $t = \mathrm{w}(a, b)$ and $(a, b) \in \mathcal{B}_n \times \mathcal{B}_{n-1}$. If $\mathrm{w}(a, b) = 0$, leave Y as it is.

(5) Update $Z_\bullet$ by quotienting it by the n-disk of Ch generated by a and ∂a as in Lemma 1.6. Update the bases according to Lemma 1.6, applied to the elements $a \in \mathcal{B}_n$ and $b \in \mathcal{B}_{n-1}$. Given two basis elements $[a']$ and $[b']$ in the updated bases, in adjacent degrees, we define $\mathrm{w}([a'], [b']) := \mathrm{w}(a', b')$, thereby inducing weights on the updated bases. The new orderings of pairs with the same weight are defined analogously.

(6) Repeat from Step (3).

(7) Add as many interval functors $\mathcal{I}^n[0, 0, \infty)$ to Y as needed to reach the dimension of Z_n for each $n \geq 0$.

It immediately follows from the fact that $Y(C_\bullet)$ is constructed as the direct sum of factored chain complexes that Construction 2.34 defines an assignment $Y\colon \mathrm{wb}^+\mathrm{Ch} \to \mathrm{TEPCh}$.

Proposition 2.35. *If* $C_\bullet = (C_\bullet, \mathcal{B}_\bullet, \mathrm{w})$ *is an ordered weighted based chain complex and* $(a_1, b_1) \in \mathcal{B}_{n_1} \times \mathcal{B}_{n_1-1}, \ldots, (a_r, b_r) \in \mathcal{B}_{n_r} \times \mathcal{B}_{n_r-1}$ *are the pairs that get simplified in the construction of* $Y = Y(C_\bullet)$*, then*

$$Y(C_\bullet) \cong \bigoplus_{i=1}^{r} \mathcal{I}^{n_i}[0, s_i, s_i) \oplus \bigoplus_{k} (\mathcal{I}^k[0, 0, \infty))^{\beta_k},$$

where $\beta_k = \dim(H_k(M; \mathbb{F}))$ *and* $s_i = \mathrm{w}(a_i, b_i)$.

Comparing Construction 2.31 and Construction 2.34, we see that for any $C_\bullet \in \mathrm{wb}^+\mathrm{Ch}$, the same pairs get simplified in the construction of $X(C_\bullet)$ and $Y(C_\bullet)$, but at different times. Note that the sequence of simplification times is increasing by construction for X but not necessarily for Y. The barcode resulting from the Y construction always has bars of length given by the weights of the pairs, while the lenghts of the bars in X accumulate. Thus, if we simplify a pair of small weight, this will always result in a small change of the barcode of Y, which is not necessarily true for X (compare also the different upper bounds given in Lemma 2.39).

Based on Lemma 1.6, the explicit computation of the tagged barcode from Construction 2.34 consists of Gaussian elimination and deletion of rows and columns. We include the pseudocode for completeness, see Algorithm 1. The pseudocode for computing the tagged barcode from Construction 2.31 would look very similar, with the only difference being that we would initialize the variable t in the beginning by setting it to zero and then update it in every step of the while loop by adding $\mathrm{w}(b_j, b_i)$ to it in line 4.

Algorithm 1 Tagged barcode computation (in $\mathbb{Z}/2\mathbb{Z}$)

```
Input: $(C_\bullet, \mathcal{B}_\bullet, \mathrm{w}_\bullet) \in \mathrm{wb}^+\mathrm{Ch}$
Output: Tagged barcode of $Y(C_\bullet, \mathcal{B}_\bullet, \mathrm{w}_\bullet)$ and sequence of simplified pairs
tBar := $\emptyset$ and Pairs := $\emptyset$
$d$ := total boundary matrix of $C_\bullet$ w.r.t. bases $\mathcal{B}_\bullet$
while a non-zero column $j$ exists in $d$ do
    $i, j$ := pair with $d[i, j] \neq 0$ and $t := \mathrm{w}(b_j, b_i)$ minimal
    Append $(n, [0, t, t))$ to tBar and $(b_j, b_i)$ to Pairs, with $n = \deg(b_j)$
    for $k$ with $d[k, j] \neq 0$ do
        Add the $i$-th row to the $k$-th row
    end for
    Delete rows $i, j$ and columns $i, j$ from $d$
end while
for all indices $j$ of remaining columns in $d$ do
    Append $(n, [0, 0, +\infty))$ to tBar with $n = \deg(b_j)$
end for
Return tBar and Pairs
```

2.6 Stability

We want to give a stability result that bounds the interleaving distance between the factored chain complexes assigned by Construction 2.31 and Construction 2.34 to sufficiently similar weighted based chain complexes. To do this, we start by introducing reparametrizations, since in the case of Construction 2.31, the two factored chain complexes will be related by a reparametrization.

Definition 2.36. A map $\alpha\colon [0, \infty) \to [0, \infty)$ is called a **reparametrization** if it is a homeomorphism (which implies that $\alpha(0) = 0$ and that α is order-preserving) and $\varepsilon_\alpha := \sup_{t\in[0,\infty)} |\alpha(t) - t| < \infty$. If $X\colon [0, \infty) \to \mathrm{Ch}$ is a parametrized chain complex, then we call $X \circ \alpha\colon [0, \infty) \to \mathrm{Ch}$ the **reparametrization of** X **by** α. This is again a parametrized chain complex, with $(X \circ \alpha)^t = X^{\alpha(t)}$ and $(X \circ \alpha)^{s\leq t} = X^{\alpha(s)\leq\alpha(t)}$.

Lemma 2.37. *The inverse α^{-1} of a reparametrization is again a reparametrization and $\varepsilon_{\alpha^{-1}} = \varepsilon_\alpha$.*

Proof. The inverse of a homeomorphism is again a homeomorphism, so the only thing to show is that $\varepsilon_{\alpha^{-1}} = \varepsilon_\alpha$. This follows because for any $t \in [0, \infty)$, we have $|\alpha^{-1}(\alpha(t)) - \alpha(t)| = |t - \alpha(t)| = |\alpha(t) - t|$. □

Proposition 2.38. *Given α and X as in the definition above, we have that X and $X \circ \alpha$ are ε_α-interleaved.*

Proof. Let $\varepsilon := \varepsilon_\alpha$ and $Y := X \circ \alpha$. The internal maps of Y are given by $Y^{s\leq t} = X^{\alpha(s)\leq\alpha(t)}$. We construct an ε-interleaving between X and Y as follows.

Since $\varepsilon = \varepsilon_\alpha = \varepsilon_{\alpha^{-1}}$, we have $\alpha(t) \leq t + \varepsilon$ and $t \leq \alpha(t + \varepsilon)$ for all $t \in [0, \infty)$, where the second inequality results from applying α, which is order preserving, to the inequality $\alpha^{-1}(t) \leq t + \varepsilon$. Defining a morphism $\psi\colon Y \to X_\varepsilon$ means defining, for every $t \geq 0$, a map from Y^t to $X^{t+\varepsilon}$ commuting with the internal maps. Since $Y^t = X^{\alpha(t)}$ and $\alpha(t) \leq t+\varepsilon$, we can take ψ^t to be the internal map of X, i.e. $\psi^t := X^{\alpha(t)\leq t+\varepsilon}$. Similarly we get $\phi\colon X \to Y_\varepsilon$ by $\phi^t := X^{t\leq\alpha(t+\varepsilon)}$. One can check that (ϕ, ψ) is an interleaving pair for X, Y. □

We are now ready to prove our stability result for generic weighted based complexes. The following lemma gives an upper bound on the interleaving distance between the parametrizations of two weighted based chain complexes which are isomorphic and whose weights are similar enough. In the next section it will be used in the proof of Theorem 3.10 and Theorem 3.16.

Lemma 2.39. *Assume that we are given generic weighted based chain complexes* $(C_\bullet, \mathcal{B}_\bullet, \mathrm{w})$, $(C'_\bullet, \mathcal{B}'_\bullet, \mathrm{w}')$ *and bijections* $\varphi\colon \mathcal{B}_k \to \mathcal{B}'_k$, *inducing an isomorphism of chain complexes* $C_\bullet \cong C'_\bullet$. *Assume further that the bijections* φ *respect the ordering of the weights in* $\mathcal{B}_\bullet$ *and* $\mathcal{B}'_\bullet$, *i.e. for all* $(a,b) \in \mathcal{B}_k \times \mathcal{B}_{k-1}$ *and* $(c,d) \in \mathcal{B}_j \times \mathcal{B}_{j-1}$ *we have* $\mathrm{w}(a,b) < \mathrm{w}(c,d)$ *if and only if* $\mathrm{w}'(\varphi(a),\varphi(b)) < \mathrm{w}'(\varphi(c),\varphi(d))$. *If in the construction of* $X(C_\bullet)$ *and* $Y(C_\bullet)$ *the pairs*

$$(a_1, b_1), \quad \dots, \quad (a_n, b_n),$$

get simplified in this order, then in the construction of $X(C'_\bullet)$ *and* $Y(C'_\bullet)$ *the pairs*

$$(\varphi(a_1), \varphi(b_1)), \quad \dots, \quad (\varphi(a_n), \varphi(b_n)),$$

get simplified in this order. Setting

$$d_\varphi(C_\bullet, C'_\bullet) := \max\{|\,\mathrm{w}(a,b) - \mathrm{w}'(\varphi(a),\varphi(b))| \mid a \in \mathcal{B}_k,\ b \in \mathcal{B}_{k-1}\},$$

the following inequalities hold:

$$d_I(X(C_\bullet), X(C'_\bullet)) \leq n d_\varphi(C_\bullet, C'_\bullet),$$
$$d_I(Y(C_\bullet), Y(C'_\bullet)) \leq d_\varphi(C_\bullet, C'_\bullet).$$

Proof. Since the orders of the weights in $C_\bullet$ and $C'_\bullet$ correspond via φ, the same is true in particular for the pair of basis elements with the smallest weight. This also does not change after any of the simplifications, since we update the weights by simply deleting some basis elements and not changing the other weights. It follows that whenever we simplify a pair (a,b) in the construction of $X(C_\bullet)$, we simplify the corresponding pair $(\varphi(a),\varphi(b))$ in the construction of $X(C'_\bullet)$. Since we simplify the same pairs in the construction of X and Y, the same is true also for Y.

To prove the inequalities we use two different strategies. We begin with the second inequality, for which we apply Theorem 2.23, so that we can use the bottleneck distance instead of the interleaving distance. From Construction 2.34 it is straightforward to see that the k-th tagged barcode of $Y(C_\bullet)$ consists of one copy of $[0,t,t)$ for any pair $(a,b) \in \mathcal{B}_k \times \mathcal{B}_{k-1}$ that gets simplified, where $t = \mathrm{w}(a,b)$. We can match each such tagged interval with the corresponding tagged interval $[0,t',t')$ in the k-th tagged barcode of $Y(C'_\bullet)$, where $t' = \mathrm{w}'(\varphi(a),\varphi(b))$. This yields a matching whose cost is upper bounded by $d_\varphi(C_\bullet, C'_\bullet)$.

To show the first inequality we will use Proposition 2.38, so we want to construct a reparametrization $\alpha\colon [0,\infty) \to [0,\infty)$ such that $X(C'_\bullet)\circ\alpha \cong X(C_\bullet)$ and $\varepsilon_\alpha \leq n d_\varphi(C_\bullet, C'_\bullet)$.

For $i = 1, \dots, n$, let $\mathrm{w}_i := \mathrm{w}(a_i, b_i)$ and $\mathrm{w}'_i := \mathrm{w}'(\varphi(a_i), \varphi(b_i))$. Next we define

$$t_i := \mathrm{w}_1 + \cdots + \mathrm{w}_i \quad \text{and} \quad t'_i := \mathrm{w}'_1 + \cdots + \mathrm{w}'_i\,.$$

We construct the reparametrization $\alpha\colon [0,\infty) \to [0,\infty)$ as follows. We define $\alpha(0) := 0$ and $\alpha(t_i) := t'_i$ for all i. In between t_i and t_{i+1} we interpolate linearly. Above t_n we define $\alpha(t_r + s) := t'_r + s$ for all $s \geq 0$. One can check that this indeed defines a reparametrization and that the supremum of $|\alpha(t)-t|$ is attained at one of the t_i and is bounded by $nd_\varphi(C_\bullet, C'_\bullet)$, since for all $i = 1, \dots, n$ we have

$$|t_i - t'_i| \leq |\, \mathrm{w}_1 - \mathrm{w}'_1 \,| + \cdots + |\, \mathrm{w}_i - \mathrm{w}'_i \,| \leq id_\varphi(C_\bullet, C'_\bullet) \leq nd_\varphi(C_\bullet, C'_\bullet).$$

It remains to check that $X(C'_\bullet) \cong X(C_\bullet) \circ \alpha$. By assumption the bijections $\varphi\colon \mathcal{B}_k \to \mathcal{B}'_k$ induce an isomorphism $X(C_\bullet)^0 \cong X(C'_\bullet)^0$. The isomorphism type of $X(C_\bullet)^t$ changes when we do the simplifications, which is exactly at the time t_i. For $X(C'_\bullet)$ it is at the time t'_i instead. Since we do the same simplification moves in the same order, the complexes $X(C_\bullet)^t$ and $X(C'_\bullet)^{\alpha(t)}$ remain isomorphic for all $t \in [0, \infty)$. □

Chapter 3

The tagged barcode of a gradient-like Morse-Smale vector field

Let M be a closed Riemannian manifold of dimension n and denote by d_M the distance on M induced by the Riemannian structure. The goal of this section is to assign a factored chain complex to a gradient-like Morse-Smale vector field in general position (defined below) on M. For this, we apply either of our two constructions from Section 2.5 to the Morse complex $\mathrm{MC}_\bullet(v)$. This yields a sequence of algebraic simplifications, depending on the distances between the fixed points of v, resulting in a factored chain complex.

3.1 General position

In our construction of a factored chain complex from a weighted based chain complex, we need that the weights are ordered. The simplest way to satisfy this condition is to demand that they are pairwise different, i.e. the weighted based chain complex is generic. We thus formulate an analogous condition for the distances between the fixed points of the vector fields that we study, i.e. that no two of these distances are the same. We show that this condition is not too restricting, in the sense that the set of vector fields satisfying it forms an open and dense subset of all gradient-like Morse-Smale vector fields.

Definition 3.1. Given a finite set of points $S \subseteq M$, we define

$$\xi(S) := \min\{|d_M(a,b) - d_M(a',b')| \mid a,b,a',b' \in S \text{ and } a \neq b,\ a' \neq b',\ \{a,b\} \neq \{a',b'\}\}.$$

If $\xi(S) > 0$, then we say that the points in S are **in general position**. In plain words, this means that no two pairs of points of S have the same distance. A gradient-like Morse-Smale vector field is said to be **in general position** if its set of fixed points is in general position. We denote by $\mathfrak{X}_{gMS+}(M)$ the set of gradient-like Morse-Smale vector fields in general position.

Proposition 3.2. *The set $\mathfrak{X}_{gMS+}(M)$ is open and dense in $\mathfrak{X}^1_{gMS}(M)$.*

The part about openness is fairly straightforward, but for denseness we will first prove two lemmas. The first lemma states that if we are given some points on M, we can put them into general position by moving them by an arbitrarily small amount.

Lemma 3.3. *Given distinct points $p_1, \ldots, p_n \in M$ and $\delta > 0$, there exist distinct points $q_1, \ldots, q_n \in M$ such that $d_M(p_i, q_i) < \delta$ for all i and the points $q_1, \ldots, q_n$ are in general position.*

Proof. For this proof, we call each instance of four points p_i, p_j, p_r, p_s with $d_M(p_i, p_j) = d_M(p_r, p_s)$, where $p_i \neq p_j$, $p_r \neq p_s$ and $\{p_i, p_j\} \neq \{p_r, p_s\}$, a problem. We show that we can get rid of any given problem by moving just one point by an arbitrarily small amount. In order to see this, note that there are two essentially different cases. The first case is the one where all four points are distinct. In the second case we have just three distinct points, one of which appears on both sides of the equation. However, in both cases there is a point which appears only once. Without loss of generality let p_i be that point. Then, consider a length minimizing geodesic γ from p_i to p_j. If we move p_i along γ towards p_j, we reduce $d_M(p_i, p_j)$ but do not change $d_M(p_r, p_s)$. If the amount we move p_i is small enough, then we will also not introduce any new problems. Repeating this procedure finitely many times, we can get rid of all the problems one by one. For each point p_i we denote the position where it ended up by q_i, which yields points in general position. ☐

Let us recall the definitions of the Whitney C^0- and C^1-topologies on $\mathfrak{X}(M)$. We give the definitions in the same spirit as they are given in [45], however we define the C^1-topology on $\mathfrak{X}(M)$ directly, instead of using an embedding $M \hookrightarrow \mathbb{R}^s$ and then topologizing $\mathfrak{X}(M)$ as a closed subspace of $C^1(M, \mathbb{R}^s)$. The resulting topology is the same.

We start by saying that, for any matrix $A \in \mathbb{R}^{p\times m}$, we write $\|A\|$ for the Frobenius norm of A, i.e. the square root of the sum of all squared entries of A. In the special case where $p = 1$ or $m = 1$, this equals the standard Euclidean norm on R^m or $\mathbb{R}^p$, respectively. Recall that the Frobenius matrix norms are submultiplicative, i.e. for any $A \in \mathbb{R}^{k\times p}$ and $B \in \mathbb{R}^{p\times m}$ we have $\|AB\| \leq \|A\| \cdot \|B\|$.

For any $r > 0$, we denote by $B_r(0) \subseteq \mathbb{R}^m$ the ball with radius r centered around the origin. We will consider smooth functions $F\colon B_4(0) \to \mathbb{R}^p$, for arbitrary values of p. This general setup allows us to define, at the same time, the C^0-norms and C^1-norms of functions $f\colon B_4(0) \to \mathbb{R}$, vector fields $\vec{v}\colon B_4(0) \to \mathbb{R}^m$, etc. Note that the differential of a vector field on $B_4(0)$ is a function $D\vec{v}\colon B_4(0) \to \mathbb{R}^{m\times m}$, and we identify $\mathbb{R}^{m\times m} \cong \mathbb{R}^{m^2}$ in order to include also this in our definition. Hence, for any smooth function $F\colon B_4(0) \to \mathbb{R}^p$ we define

$$\|F\|_0 := \sup_{x\in B_2(0)} \|F(x)\|,$$
$$\|F\|_1 := \max\{\|F\|_0, \|DF\|_0\}.$$

Note that, even though we are considering functions defined on $B_4(0)$, in the definition of these norms we only look at the values on $B_2(0)$. Thus they are technically speaking not norms, but defining them in this way will be convenient for us when defining the C^0- and C^1-norms for vector fields on manifolds.

Let us cover M by sets $V_1, \dots, V_k$ such that each of the sets is contained in a larger set $V_i \subseteq U_i$ on which we are given a chart $\sigma_i\colon U_i \to B_4(0)$ with $\sigma_i(V_i) = B_2(0)$. Given these charts, any vector field $v \in \mathfrak{X}(M)$ induces Euclidean vector fields $\vec{v}_i\colon B_4(0) \to \mathbb{R}^m$ by

$$\begin{tikzcd} B_4(0) \arrow[r, "\sigma_i^{-1}"] \arrow[rrrr, bend right, "=:\vec{v}_i"'] & U_i \arrow[r, "v|_{U_i}"] & TU_i \arrow[r, "\cong"] & U_i \times \mathbb{R}^m \arrow[r, "\mathrm{pr}_2"] & \mathbb{R}^m, \end{tikzcd}$$

where the diffeomorphism $TU_i \cong U_i \times \mathbb{R}^m$ is induced by the chart σ_i. Explicitly, if we denote by $(x_i^1, \dots, x_i^m)$ the local coordinates on U_i induced by σ_i, then $\vec{v}_i(x) = \begin{pmatrix} v_i^1(\sigma_i^{-1}(x)) \\ \vdots \\ v_i^m(\sigma_i^{-1}(x)) \end{pmatrix}$, where the functions $v_i^j \colon U_i \to \mathbb{R}$ are defined through the equation $v(p) = \sum_j v_i^j(p) \cdot \frac{\partial}{\partial x_i^j}|_p$. We define the C^0-norm and the C^1-norm of $v \in \mathfrak{X}(M)$ by

$$\|v\|_0 := \max_{i=1,\dots,k} \|\vec{v}_i\|_0 , \qquad \|v\|_1 := \max_{i=1,\dots,k} \|\vec{v}_i\|_1 .$$

One can check that this defines a norm on $\mathfrak{X}(M)$ and that the topology induced from this norm does not depend on the choices that are involved. We now state and prove the second lemma needed for Proposition 3.2. It will be used to show that it is possible to C^1-perturb a vector field and move a fixed point to an arbitrary point in a small neighbourhood.

Lemma 3.4. *Let $\vec{v} \colon B_4(0) \to \mathbb{R}^m$ be a smooth map and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that for all $y \in B_\delta(0)$ there exists a diffeomorphism $\varphi = \varphi_y \colon B_4(0) \to B_4(0)$ that satisfies*

- $\varphi(x) = x$ *for* $\|x\| \geq 1$,
- $\varphi(y) = 0$,
- $\|\vec{v} - \vec{v} \circ \varphi\|_1 < \varepsilon$.

Proof. We start by picking a smooth function $\rho \colon \mathbb{R}^m \to \mathbb{R}$ with the following properties:

- $\rho(0) = 1$,
- $0 \leq \rho(x) \leq 1$ for $\|x\| \leq \frac{1}{2}$,
- $\rho(x) = 0$ for $\|x\| \geq \frac{1}{2}$.

We can explicitly choose δ to be

$$\delta := \min\left\{\frac{1}{2}, \frac{\varepsilon}{L_1}, \frac{1}{\|\rho\|_1}, \frac{\varepsilon}{L_2 + \|\vec{v}\|_1 \cdot \|\rho\|_1}\right\},$$

where the constants $L_1 = L_1(\vec{v})$ and $L_2 = L_2(\vec{v})$ are the Lipschitz constants of $\vec{v}$ and $D\vec{v}$, i.e.

$$L_1 := \sup\left\{\frac{\|\vec{v}(x) - \vec{v}(y)\|}{\|x-y\|} \mid x \neq y \in B_2(0)\right\} \quad \text{and}$$
$$L_2 := \sup\left\{\frac{\|D\vec{v}(x) - D\vec{v}(y)\|}{\|x-y\|} \mid x \neq y \in B_2(0)\right\}.$$

The reason for this choice will become clear later. The values of L_1 and L_2 are bounded by the C^1-norms of $\vec{v}$ and $D\vec{v}$, respectively. In particular, they are finite, as the closure of $B_2(0)$ is compact. We now define the map $\varphi = \varphi_y \colon B_4(0) \to B_4(0)$ by $\varphi(x) := x - \rho(x-y) \cdot y$. For $\|y\| < \delta$ and $x \in B_2(0)$, we have

$$\begin{aligned}\|D\varphi(x)\| = \|\mathbb{1} - D(\rho(x-y)\cdot y)\| = \left\|\mathbb{1} - \nabla\rho(x-y)\cdot y^T\right\| &\geq \|\mathbb{1}\| - \left\|\nabla\rho(x-y)\cdot y^T\right\| \\ &\geq 1 - \|\nabla\rho(x-y)\| \cdot \left\|y^T\right\| \\ &\geq 1 - \|\rho\|_1 \cdot \|y\| > 0.\end{aligned}$$

For $\|x\| \geq 1$ we have $\varphi(x) = x$ since for $\|y\| \leq \frac{1}{2}$ and $\|x\| \geq 1$ we have $\|x - y\| \geq \frac{1}{2}$ and thus $\rho(x-y) = 0$. Therefore, φ is a diffeomorphism and it has the first two of the desired properties. For the third property we need to show that

$$\sup_{x \in B_2(0)} \|\vec{v}(x) - \vec{v}(\varphi(x))\| < \varepsilon \quad \text{and} \quad \sup_{x \in B_2(0)} \|D\vec{v}(x) - D(\vec{v} \circ \varphi)(x)\| < \varepsilon.$$

Fix an arbitrary point $x \in B_2(0)$. We compute

$$\|\vec{v}(x) - \vec{v}(\varphi(x))\| \leq L_1 \|x - \varphi(x)\| = L_1 \|\rho(x-y) \cdot y\| \leq L_1 \|y\| < L_1 \delta \leq \varepsilon,$$

which proves the first inequality. The second one follows from the computation

$$\begin{aligned}
\|D\vec{v}(x) - D(\vec{v} \circ \varphi)(x)\| &= \|D\vec{v}(x) - D\vec{v}(\varphi(x)) \cdot D\varphi(x)\| \\
&\leq \|D\vec{v}(x) - D\vec{v}(\varphi(x))\| + \|D\vec{v}(\varphi(x)) - D\vec{v}(\varphi(x)) \cdot D\varphi(x)\| \\
&\leq L_2 \|x - \varphi(x)\| + \|D\vec{v}(\varphi(x))\| \cdot \|\mathbb{1} - D\varphi(x)\| \\
&\leq L_2 \|y\| + \|\vec{v}\|_1 \cdot \left\|(\nabla\rho(x-y)) \cdot y^T\right\| \\
&\leq L_2 \|y\| + \|\vec{v}\|_1 \cdot \|\rho\|_1 \cdot \|y\| \\
&= (L_2 + \|\vec{v}\|_1 \cdot \|\rho\|_1) \|y\| < (L_2 + \|\vec{v}\|_1 \cdot \|\rho\|_1)\delta \leq \varepsilon,
\end{aligned}$$

where in the third inequality we use that

$$\mathbb{1} - D\varphi(x) = D(x - \varphi(x)) = D(\rho(x-y) \cdot y) = \nabla\rho(x-y) \cdot y^T.$$

This finishes the proof. □

We are now ready to prove Proposition 3.2.

Proof of Proposition 3.2. Let $v \in \mathfrak{X}_{gMS+}(M)$ and choose $\varepsilon > 0$ small enough, such that for any $a, b, c, d \in \text{Fix}(v)$ with $a \neq b$, $c \neq d$ and $\{a, b\} \neq \{c, d\}$ we have $\varepsilon < |d_M(a,b) - d_M(c,d)|$. By Theorem 1.17 there exists a neighbourhood $\mathcal{N}$ of v in $\mathfrak{X}^1(M)$ such that for all $w \in \mathcal{N}$, the fixed points of w are within ε-distance from the fixed points of v. By our choice of ε this implies that $\mathcal{N} \subseteq \mathfrak{X}_{gMS+}(M)$, so we have shown that $\mathfrak{X}_{gMS+}(M)$ is open.

It remains to show that $\mathfrak{X}_{gMS+}(M)$ is dense in $\mathfrak{X}^1_{gMS}(M)$. For this, let $v \in \mathfrak{X}^1_{gMS}(M)$ and let $\mathcal{N} \subseteq \mathfrak{X}^1_{gMS}(M)$ be an open neighbourhood of v. Denote by $p_1, \dots, p_n \in M$ the fixed points of v. We want to construct $w \in \mathcal{N}$ with fixed points $q_1, \dots, q_n$ which are in general position.

We cover M by $V_1, \dots, V_k$ ($k \geq n$), contained in charts (U_i, σ_i), satisfying

- $\sigma_i \colon U_i \to B_4(0) \subseteq \mathbb{R}^m$ with $\sigma_i(V_i) = B_2(0)$,
- $W_i \cap U_j = \emptyset$ for $i \neq j$ with $W_i := \sigma_i^{-1}(B_1(0))$,
- $p_i \in V_i$ and $\sigma_i(p_i) = 0$ for $i = 1, \dots, n$,
- $\|\sigma_i(q)\| = d_M(p_i, q)$ for all $i = 1, \dots, n$ and for all $q \in W_i$.

The last condition can be satisfied by choosing the σ_i to agree with the exponential map at p_i for nearby points. The topology induced on $\mathfrak{X}(M)$ by the norm $\|\cdot\|_1$ yields $\mathfrak{X}^1(M)$. Now since $\mathcal{N}$ is a neighbourhood of v in $\mathfrak{X}^1(M)$, there exists $\varepsilon > 0$ such that $\{w \in \mathfrak{X}^1(M) \mid \|v - w\|_1 < \varepsilon\} \subseteq \mathcal{N}$. Thus it suffices to show that there exists $w \in \mathfrak{X}^1_{gMS+}(M)$ with $\|v - w\|_1 < \varepsilon$. We want to show the following claim.

Claim. There exists $\delta > 0$ such that, for arbitrarily chosen $q_1, \dots, q_n \in M$ with $d_M(p_i, q_i) < \delta$ for all i, there exists $w \in \mathfrak{X}^1(M)$ with $\|v - w\|_1 < \varepsilon$ and $\mathrm{Fix}(w) = \{q_1, \dots, q_n\}$.

Assuming this claim, the theorem follows since by Lemma 3.3 we can choose the points $q_i \in M$ in such a way that $w \in \mathfrak{X}^1_{gMS+}(M)$. It thus remains only to show the claim. For this we apply Lemma 3.4 to each of the local vector fields $\vec{v}_i \colon B_4(0) \to \mathbb{R}^m$ and obtain $\delta_i > 0$ such that for $y \in B_{\delta_i}(\sigma_i(p_i))$ there exists $\varphi_i \colon B_4(0) \to B_4(0)$ with the properties listed in Lemma 3.4. Define $\delta := \min_i \delta_i$. Thus for $q_i \in M$ with $d_M(p_i, q_i) < \delta$, there exists $\varphi_i \colon B_4(0) \to B_4(0)$ such that φ_i is the identity outside of $B_1(0)$, $\varphi_i(\sigma_i(q_i)) = 0$, and $\|\vec{v}_i - \vec{v}_i \circ \varphi_i\|_1 < \varepsilon$.

We now define $w \in \mathfrak{X}(M)$ as follows. We define $w_i \in \mathfrak{X}(U_i)$ by

$$\begin{tikzcd} U_i \arrow[r, "\Delta"] \arrow[rrrrr, bend right, "=:w_i"'] & U_i \times U_i \arrow[r, "\mathrm{id} \times \sigma_i"] & U_i \times B_4(0) \arrow[r, "\mathrm{id} \times \varphi_i"] & U_i \times B_4(0) \arrow[r, "\mathrm{id} \times \vec{v}_i"] & U_i \times \mathbb{R}^m \arrow[r, "\cong"] & TU_i. \end{tikzcd}$$

Note that $w_i = v$ on $U_i \setminus W_i$. Thus, for $i \neq j$, since $U_i \cap W_j = \emptyset$ and $W_i \cap U_j = \emptyset$, we have $w_i = w_j = v$ on $U_i \cap U_j$, thus we can glue together the w_i to obtain $w \in \mathfrak{X}(M)$ with $w|_{U_i} = w_i$ for all i.

We first check that $\mathrm{Fix}(w) = \{q_i\}$. Outside of the sets W_i, $i = 1, \dots, n$, we have $w = v \neq 0$. Inside W_i, v has exactly one zero, namely p_i. For any $q \in W_i$ we thus have

$$\begin{aligned} w(q) = 0 \iff w_i(q) = 0 \iff \vec{v}_i(\varphi_i(\sigma_i(q))) = 0 \iff \varphi_i(\sigma_i(q)) = 0 &\iff \sigma_i(q) = \sigma_i(q_i) \\ &\iff \quad q = q_i. \end{aligned}$$

Thus $\mathrm{Fix}(w) = \bigcup_i \mathrm{Fix}(w) \cap W_i = \bigcup_i \{q_i\}$. It remains to show that $\|v - w\|_1 < \varepsilon$. Indeed, we have

$$\|v - w\|_1 = \max_i \|\vec{v}_i - \vec{w}_i\|_1 = \max_i \|\vec{v}_i - \vec{v}_i \circ \varphi_i\|_1 < \varepsilon.$$

This proves the claim and hence the proposition. □

3.2 Assigning a factored chain complex to a vector field

Now we are in the position to assign a factored chain complex to a gradient-like Morse-Smale vector field in general position $v \in \mathfrak{X}_{gMS+}(M)$. The idea is to take the Morse complex $\mathrm{MC}_\bullet(v)$ and apply a sequence of simplifications, each determined by a pair (a, b) of fixed points of index k and $k-1$, for some $k \in \{1, \dots, n\}$. This yields a factored chain complex with constant homology whose tagged barcode consists of finite bars of the form $[0, t, t)$ and infinite bars $[0, 0, \infty)$. Each finite bar corresponds to a pair of fixed points and the value t is related to their distance on M. The infinite bars correspond to fixed points which are not paired up, and they can be viewed as generators for the homology of M.

More precisely, we endow $\mathrm{MC}_\bullet(v)$ with some additional information in order to apply Construction 2.31 or Construction 2.34. We view the Morse complex $\mathrm{MC}_\bullet(v)$ as a based chain complex, where the basis in each degree is given by the fixed points of that index. If we are moreover given a distance d_M on M (for example the shortest path distance induced from a Riemannian metric), then we view $\mathrm{MC}_\bullet(v)$ as a weighted based chain complex, with the weight of a pair of basis elements given by the distance of the corresponding fixed points on M. Note that in case M is not connected, the shortest path distance is not defined between points of different connected components. In this case, we set it to be any fixed number greater or equal to the maximum distance between any two points from the same component. One can check that this yields a metric. We will slightly abuse notation and always write $\mathrm{MC}_\bullet(v)$ for the Morse complex of v, but consider it to be a chain complex, or a based chain complex, or a weighted based chain complex, depending on the context. If v is in general position, then $\mathrm{MC}_\bullet(v)$ is generic, thus in particular ordered, so we can apply Construction 2.31 or Construction 2.34 and construct

$$X(v) := X(\mathrm{MC}_\bullet(v)) \in \mathrm{TEPCh} \qquad \text{or} \qquad Y(v) := Y(\mathrm{MC}_\bullet(v)) \in \mathrm{TEPCh},$$

respectively. Note that the basis elements that appear in Construction 2.31 or Construction 2.34 always correspond to fixed points of v, even after multiple iterations. We therefore get a sequence of pairs of fixed points $(a_1, b_1), \ldots, (a_r, b_r)$, whose indices always differ by one. We say that the pairs $(a_1, b_1), \ldots, (a_r, b_r)$ are simplified in the construction of $X(v)$ or $Y(v)$.

The resulting tagged barcode, together with the pairs of fixed points that were simplified in the construction, then gives a topological invariant of the vector field. It can be interpreted as a sequence of algebraic simplifications that one can do in order to simplify the Morse complex of the vector field.

Example 3.5. In Figure 3.1 we see a gradient-like Morse-Smale vector field $v \in \mathfrak{X}_{gMS+}(S^2)$. It is defined on the 2-sphere, which is represented as a square, where all of the boundary is contracted to a point x, which is a fixed point of index 0 for v. We denote the remaining fixed points by p, q, s. Let us assume that S^2 is equipped with a Riemannian metric such that $d(p, s) = 1$, $d(q, s) = 2$, and also all other distances are pairwise different and larger than 1, so that v is in general position. The Morse complex of v is

$$\mathbb{F}^2 = \mathrm{Span}(p, q) \xrightarrow{[1,1]} \mathbb{F} = \mathrm{Span}(s) \xrightarrow{[0]} \mathbb{F} = \mathrm{Span}(x).$$

The weights are given by the distances, so for example $\mathrm{w}(p, s) = 1$ and $\mathrm{w}(q, s) = 2$. The only pairs that we could possibly simplify are (p, s) and (q, s), so since $\mathrm{w}(p, s) < \mathrm{w}(q, s)$ we will simplify (p, s) at time 1, giving rise to the tagged interval $[0, 1, 1)$ in degree 2 the tagged barcode of $X(v)$. After this simplification, all differentials are zero, so q and x cannot be simplified, giving rise to the tagged intervals $[0, 0, \infty)$ in the degrees 2 and 0 respectively. On the right of Figure 3.1 we visualize the barcode in the following way: Finite bars $[0, t, t)$ in degree n are showed as a gray block spanning from degree $n - 1$ to degree n on the vertical axis and from 0 to t on the horizontal axis. Infinite bars $[0, 0, \infty)$ in degree n are drawn as a black horizontal line on the height n.

Remark 3.6. One could also use other weights than the distance between the fixed points. One may try to use the length of a flow line or the maximal value of the norm of the vector field along a flow line, instead. However, in these cases, one would have weights only between points that are connected by a flow line. It is then not clear a priori how one would update the weights after a simplification of the Morse complex. For this reason, we are using the distance between the points. However, thanks to the generality of the definitions and results of Section 2.5, other choices for the weights are not in principle precluded.

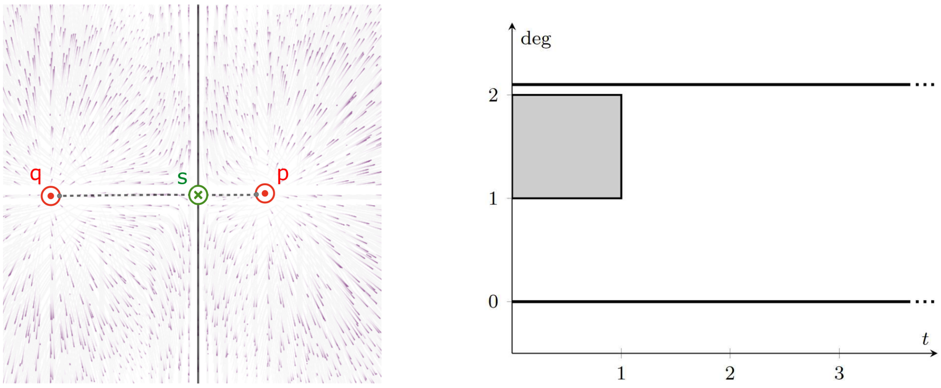


Figure 3.1: Left: Gradient-like Morse-Smale vector field v on the 2-sphere with two fixed points p, q of index 2, one fixed point s of index 1, and one fixed point x of index 0. The point x is represented in the image by the whole boundary. Right: Visualization of the resulting tagged barcode $\mathrm{tBar}(Y(v))$.

In the case where we are studying a Morse-Smale function $f\colon M \to \mathbb{R}$ via the Morse complex of its gradient, $\mathrm{MC}_\bullet(-\nabla f)$, there is another obvious choice for the weights, namely the distance of function values. In that case, the tagged barcode one obtains is closely related to the persistence barcode of f, see Section 3.5. Since we are also allowing non-conservative vector fields, however, this approach does not work more generally.

Remark 3.7 (Invariance)**.** Usually, when talking about (topological/geometric) invariants, one would like to describe a set of transformations of the input objects under which the outcome is invariant. In our case, if two vector fields $v, w \in \mathfrak{X}_{gMS+}(M)$ are topologically equivalent and the topological equivalence can be realized by a homeomorphism $h\colon M \to M$ that restricts to an isometry of the fixed points, then $X(v) \cong X(w)$ and $Y(v) \cong Y(w)$.

Remark 3.8 (Non-triviality)**.** Non-triviality refers to the fact that an invariant is able to distinguish input objects if they are "different enough" in some sense. In our case, we can say that the constructions X and Y are sensitive enough to distinguish between the vector fields v and w in either of the following situations:

(i) The two vector fields v and w are defined on manifolds M and M' whose singular homology groups are not isomorphic.

(ii) The two vector fields v and w are both defined on M, but there exists $1 \leq k \leq \dim M$ such that $|\operatorname{Fix}_k(v)| \neq |\operatorname{Fix}_k(w)|$.

(iii) The two vector fields v and w are both defined on M, and $|\operatorname{Fix}_k(v)| = |\operatorname{Fix}_k(w)|$ for all k, but the distance values between fixed points of v and those of w are all distinct.

Even beyond these cases, the constructions may distinguish between different vector fields, see Example 3.9.

Example 3.9. Consider the gradient-like Morse-Smale vector fields on S^2 displayed in Figure 3.2. This figure is presented in the same way as the images of Chapter 4, see the explanations given on page 57. Denote by v the vector field on the left and by w the

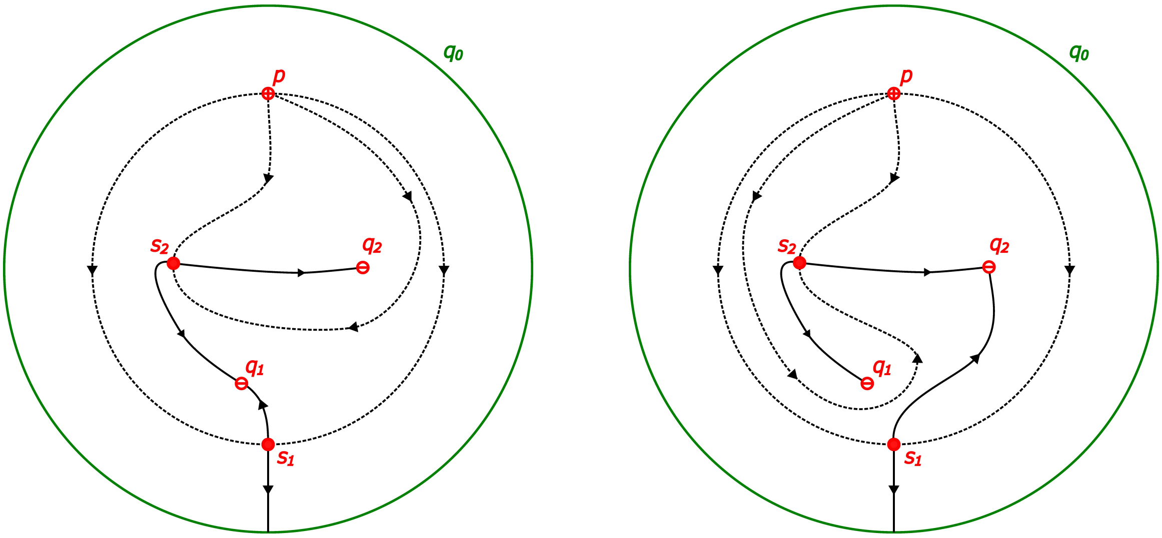


Figure 3.2: Two gradient-like Morse-Smale vector fields on the 2-sphere that have the same fixed points but are topologically different

one on the right. The fixed points of v and w coincide, they both have one source p, two saddles s_1, s_2, and three sinks q_0, q_1, q_2. The point q_0 is represented by the green boundary. However, the topology of these two vector fields are not the same, which can be seen by the fact that the flow lines are connecting the fixed points in different ways. Applying Algorithm 1 to the Morse complexes of both v and w reflects this difference. The relevant distances between the fixed points are $d(s_1, q_1) = 1.5$, $d(s_1, q_2) = 4.8$, $d(s_2, q_1) = 3.3$, and $d(s_2, q_2) = 4.5$. We assume that $d(s_1, q_0)$ and $d(s_2, q_0)$ are larger than all of these distances, even if it may appear otherwise in the images.

In the case of v, we first simplify the pair (s_1, q_1) and then, in the second step, the pair (s_2, q_2). The resulting decomposition of $Y(v)$ is thus

$$Y(v) \cong \mathcal{I}^0[0,0,\infty) \oplus \mathcal{I}^1[0,1.5,1.5) \oplus \mathcal{I}^1[0,4.5,4.5) \oplus \mathcal{I}^2[0,0,\infty).$$

For w on the other hand, we first simplify the pair (s_2, q_1) and then, in the second step, the pair (s_1, q_2). The resulting decomposition of $Y(w)$ is then

$$Y(w) \cong \mathcal{I}^0[0,0,\infty) \oplus \mathcal{I}^1[0,3.3,3.3) \oplus \mathcal{I}^1[0,4.8,4.8) \oplus \mathcal{I}^2[0,0,\infty).$$

3.3 Local stability

Recall that $\mathfrak{X}^1_{gMS+}(M)$ is the space of gradient-like Morse-Smale vector fields on M in general position, endowed with the Whitney C^1 topology. We endow TEPCh with the topology induced by the interleaving distance. The goal of this section is to prove the following statement.

Theorem 3.10. *The maps $X, Y \colon \mathfrak{X}^1_{gMS+}(M) \to \mathrm{TEPCh}$, which assign to a gradient-like Morse-Smale vector field in general position v the factored chain complexes $X(v)$ and $Y(v)$, built as in Construction 2.31 and Construction 2.34, respectively, are continuous.*

Proof. In order to show that the map $X \colon \mathfrak{X}^1_{gMS+}(M) \to \mathrm{TEPCh}$ is continuous, it suffices to show that for every $v \in \mathfrak{X}^1_{gMS+}(M)$ and every $\varepsilon > 0$ there exists a neighbourhood $\mathcal{N}$

of v in $\mathfrak{X}^1_{gMS+}(M)$ such that for all $w \in \mathcal{N}$, we have $d_I(X(v), X(w)) \leq \varepsilon$. Denote by

$$(a_1, b_1), \quad \dots, \quad (a_n, b_n),$$

the pairs of fixed points that are simplified by applying the construction to v, in this order. Let $\varepsilon' > 0$ be a positive number that satsifies

$$2n\varepsilon' \leq \varepsilon, \qquad 4\varepsilon' \leq \xi(\mathrm{Fix}(v)), \qquad 2\varepsilon' \leq d_M(p, q) \text{ for all } p \neq q \in \mathrm{Fix}(v).$$

By Theorem 1.17, there exists a neighbourhood $\mathcal{N}$ of v in $\mathfrak{X}^1_{gMS+}(M)$ such that for all $w \in \mathcal{N}$ there exists a topological equivalence φ between v and w such that $d_M(p, \varphi(p)) \leq \varepsilon'$ for all $p \in \mathrm{Fix}(v)$. By Lemma 1.19, φ induces an isomorphism between the Morse complexes of v and w. We now check that the conditions for Lemma 2.39 are satisfied, i.e. we want to show that for any $a, b, c, d \in \mathrm{Fix}(v)$, if $d_M(a, b) < d_M(c, d)$, then $d_M(\varphi(a), \varphi(b)) < d_M(\varphi(c), \varphi(d))$.

To show this, note that, if $d_M(a, b) < d_M(c, d)$, then

$$\begin{aligned} d_M(\varphi(a), \varphi(b)) - d_M(\varphi(c), \varphi(d)) &\leq d_M(\varphi(a), a) + d_M(a, b) + d_M(b, \varphi(b)) \\ &\quad + d_M(c, \varphi(c)) - d_M(c, d) + d_M(d, \varphi(d)) \\ &< d_M(a, b) - d_M(c, d) + 4\varepsilon' \\ &\leq d_M(a, b) - d_M(c, d) + \xi(\mathrm{Fix}(v)) \leq 0, \end{aligned}$$

from which it follows that $d_M(\varphi(a), \varphi(b)) < d_M(\varphi(c), \varphi(d))$. We have thus shown that the weights of pairs from $\mathrm{Fix}(v)$ are in the same order as the weights of the corresponding pairs from $\mathrm{Fix}(w)$, thus, by Lemma 2.39, in the construction of $X(w)$ the pairs

$$(\varphi(a_1), \varphi(b_1)), \quad \dots, \quad (\varphi(a_r), \varphi(b_r)),$$

get simplified in this order and $d_I(\mathrm{MC}_\bullet(v), \mathrm{MC}_\bullet(w)) \leq n d_\varphi(\mathrm{MC}_\bullet(v), \mathrm{MC}_\bullet(w))$. Note that for any fixed points $a, b \in \mathrm{Fix}(v)$, we have

$$|d(a, b) - d(\varphi(a), \varphi(b))| \leq d(a, \varphi(a)) + d(b, \varphi(b)) \leq 2\varepsilon',$$

thus $d_\varphi(\mathrm{MC}_\bullet(v), \mathrm{MC}_\bullet(w)) \leq 2\varepsilon'$. Combining this yields the desired inequality. If we replace X by Y, the same proof works and when choosing ε' suffices that $2\varepsilon' \leq \varepsilon$. ☐

Example 3.11. Consider the vector field v, defined on the 2-sphere, from Figure 3.3. Again, the whole boundary of the image represents a single point y of S^2, which is a sink for v. The other fixed points of v are two sources p, q, two saddle points r, s, and a sink x. The relevant flow lines for the definition of the Morse complex are visible in the picture. Assume the 2-sphere is equipped with a Riemannian metric such that $d(p, r) = d(p, s) = 1$, $d(s, x) = 2$, $d(r, x) = \sqrt{8}$, whereas all other pairs of fixed points have larger distances. As weights are given by distances, the pairs (p, r) and (p, s) both have the same weight, so our constructions depend on the ordering of these two pairs. If we simplify the pair (p, r) first, then we next simplify the pair (s, x), which has weight 2. In contrast, if we simplify the pair (p, s) first, then we next simplify the pair (r, x), which has weight $\sqrt{8} \approx 2.8$. This shows that if the weights are not unique, then it can happen that different orderings lead to different tagged barcodes. In Section 3.5 we will see that in the case of scalar fields, the situation is different, i.e. we can define a slightly different tagged barcode which is independent of the ordering of the pairs.

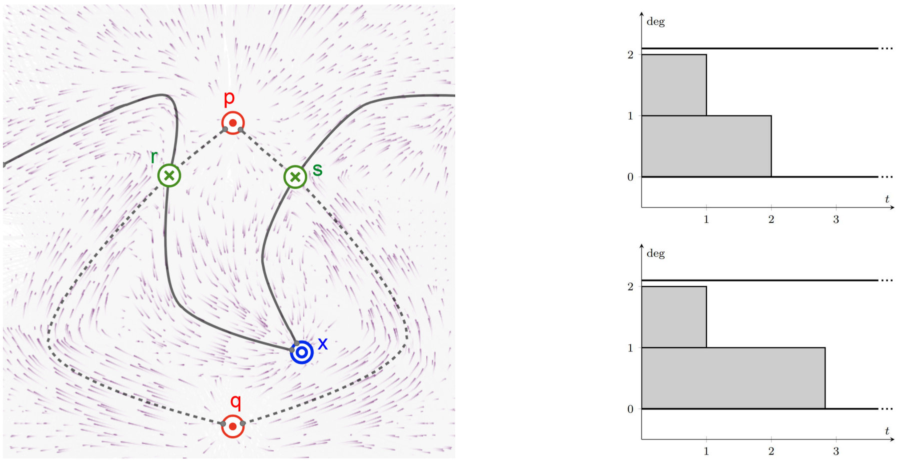


Figure 3.3: Different orderings of the pairs (p, r) and (p, s) yield different tagged barcodes for this vector field, as explained in Example 3.11.

3.4 Combinatorial approximations

Given a smooth gradient-like Morse-Smale vector field v on a smooth manifold M, one can approximate it by a sequence of triangulations of M and combinatorial vector fields on these triangulations. First, Theorem 1.22 asserts that there exists a triangulation T of M and a combinatorial vector field V on it such that: (1) there is a bijection between the set of fixed points p of v and the set of critical cells σ_p of V such that $p \in \sigma_p$ and $\mathrm{ind}_v(p) = \dim(\sigma_p)$, (2) for each pair of critical cells σ_p and σ_q such that $\dim(\sigma_p) = \dim(\sigma_q) + 1$, V-paths from hyperfaces of σ_p to σ_q are in bijection with integral curves of v up to renormalization connecting q to p, (3) this bijection induces an isomorphism between the Morse complexes $\mathrm{MC}_\bullet(v)$ and $\overline{\mathrm{MC}}_\bullet(V)$ associated with v and V, respectively. Then, Theorem 1.23 states that if we replace T by its barycentric subdivision $\Delta(T)$, the combinatorial vector field V on T induces a combinatorial vector field $\Delta(V)$ on the barycentric subdivision $\Delta(T)$. By iterating this process we get a combinatorial approximation theorem.

As we did before with the smooth Morse complex, we want to view the combinatorial Morse complex as a weighted based chain complex. To do this, we assume that some extra structure is present on the simplicial complex.

Definition 3.12. A **metric simplicial complex** is a simplicial complex equipped with a metric on its set of cells. A metric simplicial complex is called **generic**, if all the distances are pairwise different.

Remark 3.13. Given a simplicial complex K, the combinatorial Morse complex can be viewed as an assignment $\overline{\mathfrak{X}}_g(K) \to \mathrm{b\,Ch}$, where the bases on $\overline{\mathrm{MC}}_\bullet(V)$ are given by the canonical bases of critical cells.

If K is moreover a metric simplicial complex, then we view the combinatorial Morse complex as a map $\overline{\mathfrak{X}}_g(K) \to \mathrm{wbCh}$. The bases of $\overline{\mathrm{MC}}_\bullet(V)$ is given as before by the critical cells and the weights are taken to be the distances between those cells. Thus in that case we can assign a factored chain complex to the combinatorial Morse complex, according to either Construction 2.31 or Construction 2.34. We again shorten our notation by writing $X(V) := X(\overline{\mathrm{MC}}_\bullet(V))$ and $Y(V) := Y(\overline{\mathrm{MC}}_\bullet(V))$ for any $V \in \overline{\mathfrak{X}}_g(K)$.

If K is a simplicial complex and σ is a k-cell of K, then $|\sigma| \subseteq |K|$ has a well-defined **barycenter**. We denote this point by $b(\sigma) \subseteq |\sigma|$.

Using the barycenters, every triangulated Riemannian manifold induces a metric simplicial complex in the following way: If (M, M', ϕ) is a triangulated Riemannian manifold, then we have the simplicial complex M' and for the distance on the cells of M' we take the distance between the barycenters, i.e. we define $d(\sigma, \tau) := d_M(\phi(b(\sigma)), \phi(b(\tau)))$.

Since there is a canonical homeomorphism between $|K|$ and $|\Delta(K)|$, we will identify these two spaces. Therefore, if (M, M', ϕ) is a triangulated manifold, then so is $(M, \Delta(M'), \phi)$.

Definition 3.14. Let M and v be as in Theorem 1.22. A **triangulation of** v is a triple (M', ϕ, V) such that

- (M', ϕ) is a triangulation of M,
- $V \in \overline{\mathfrak{X}}_g(M')$,
- V satisfies the conditions (i), (ii), (iii) from Theorem 1.22.

Lemma 3.15. *Let M and v be as in Theorem 1.22 and let (M', ϕ, V) be a triangulation of v. Then it is possible to choose orderings of the vertices of the critical cells of V in such a way that also $(\Delta(M'), \phi, \Delta(V))$ is a triangulation of v.*

Proof. Since (M', ϕ, V) is a triangulation of v, for every fixed point p of v there exists a critical cell σ of V such that $p \in \sigma$. Since the k-cells of $\Delta(M')$ cover the k-cells of M', there exists a cell σ' in $\Delta(M')$, with $|\sigma'| \subseteq |\sigma|$, such that $p \in \sigma'$. We can choose this cell σ' (which may not be unique) for the definition of $\Delta(V)$. Doing so for every fixed point of v implies, by Theorem 1.23, that $\Delta(V)$ again satisfies all the conditions from Theorem 1.22 and so $(\Delta(M'), \phi, \Delta(V))$ is a triangulation of v. $\square$

Now we are ready to state and prove our combinatorial approximation theorem.

Theorem 3.16. *Let M be an oriented Riemannian manifold and let $v \in \mathfrak{X}_{gMS+}(M)$. Let (M', ϕ, V) be a triangulation of v. Then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have*

$$d_I\left(X(v), X(\Delta^n(V))\right) < \varepsilon.$$

The statement of the theorem is also true when we replace X with Y. The proof is analogous. Also, when writing $\Delta^n(V)$, we are hiding from the notation the dependence on some choices. The precise statement would be that there exists a sequence of N choices, so that for any choice we do afterwards, the statement from the theorem holds.

Proof. By iterated application of Lemma 3.15, $(\Delta^n(M'), \phi, \Delta^n(V))$ is a triangulation of v for all n, given the right choices in the definition of $\Delta(V)$, $\Delta^2(V)$, etc. Since the cells in $\Delta^n(V)$ become smaller as n gets larger, the barycenters of $\sigma_p^{(n)}$ converge to p as n goes to infinity, where by $\sigma_p^{(n)}$ we denote the barycenter of the critical cell in $\Delta^n(M')$ that contains p. If we choose N large enough such that $\phi(|\sigma_p^{(N)}|) \subseteq B_{\varepsilon'}(p)$, where $\varepsilon' > 0$ is a small enough positive number such that the conditions of Lemma 2.39 are satisfied for $\mathrm{MC}_\bullet(V)$ and $\overline{\mathrm{MC}}_\bullet(\Delta^N(V))$. The conditions for such ε' are analogous to the conditions in the proof of Theorem 3.10. $\square$

3.5 The case of scalar fields and connections to persistent homology

We now turn our attention to scalar fields, i.e. functions, on a Riemannian manifold. As it is often done, we restrict ourselves to the case of Morse-Smale functions. The gradient of such a function is a gradient-like Morse-Smale vector field. The goal of this section is twofold. First, we relate the classical persistence barcode of the function to the tagged barcode of its gradient field. Secondly, we show that, in contrast to what happens for gradient-like vector fields (see Example 3.11), for gradient fields we do not need further genericity conditions.

Given a Morse-Smale function, we can endow the Morse complex of its gradient with a different set of weights. Instead of distances between fixed points we can use the difference in function values. For convenience, we will assume that f takes only non-negative values. Since we consider only closed manifolds, this can always be achieved by adding a constant.

We now introduce a modification of Definition 2.30 for scalar fields. This is analogous to [22].

Definition 3.17. A **based chain complex with a filter** is a based chain complex $(C_\bullet, \mathcal{B}_\bullet)$ together with a **monotone** function $f\colon \bigcup_k \mathcal{B}_k \to [0, \infty)$, that is, for all $a \in \mathcal{B}_k$, $b \in \mathcal{B}_{k-1}$, if $\langle \partial a, b\rangle \neq 0$, then $f(b) < f(a)$.

Therefore, if we are given a Morse-Smale function f, then we can view the Morse complex $\mathrm{MC}_\bullet(-\nabla f)$ as both a based chain complex with a filter, where the filter is given by the function values of f on the fixed points, and a weighted based chain complex, where the weights are equal to $|f(a) - f(b)|$.

Recall that a filtered chain complex is a parametrized chain complex which is tame and where all the internal chain maps are monomorphisms. A based chain complex with a filter $(C_\bullet, \mathcal{B}_\bullet, f)$ induces a filtered chain complex $F(C_\bullet, \mathcal{B}_\bullet, f)\colon [0, \infty) \to \mathrm{Ch}$, $t \mapsto C_\bullet^{f\leq t}$, in the following way. The chain complex $C_\bullet^{f\leq t}$ consists of the vector spaces $C_k^{f\leq t} := \mathrm{Span}(\{b \in \mathcal{B}_k \mid f(b) \leq t\}) \subseteq C_k$. The differential $\partial\colon C_k^{f\leq t} \to C_{k-1}^{f\leq t}$ is given by the restriction of the differential in $C_\bullet$. This is well-defined because the function f is monotone.

Applying homology then yields persistence modules $H_n(F(C_\bullet, \mathcal{B}_\bullet, f))$, one for each n, as explained in Section 2.2. These are parametrized vector spaces and hence have persistence barcodes due to Theorem 1.4.

On the other hand, given a filtered based chain complex $(C_\bullet, \mathcal{B}_\bullet, f)$, this induces a weighted based chain complex $(C_\bullet, \mathcal{B}_\bullet, \mathrm{w})$, where the weights are given by $\mathrm{w}(a, b) := |f(a) - f(b)|$. Because it has different weights than in the previous sections, the factored chain complexes resulting from applying either Construction 2.31 or Construction 2.34 are hence also different. However, if we apply them in this setting, the result is related to the persistence barcode of f. More precisely, we will show that it is possible to obtain the tagged barcode of $Y(C_\bullet, \mathcal{B}_\bullet, \mathrm{w})$ by mapping finite intervals $[s, t)$ from $\mathrm{Bar}(H_n(F(C_\bullet, \mathcal{B}_\bullet, f)))$ to $[0, t - s, t - s)$ in $\mathrm{tBar}_{n+1}(Y(C_\bullet, \mathcal{B}_\bullet, \mathrm{w}))$ and infinite intervals $[s, \infty)$ to $[0, 0, \infty)$ in $\mathrm{tBar}_n(Y(C_\bullet, \mathcal{B}_\bullet, \mathrm{w}))$. The proof strategy is by induction over the number of pairs that get simplified in our construction and show that if we simplify a pair $(a, b) \in \mathcal{B}_n \times \mathcal{B}_{n-1}$, then there is an interval present in $\mathrm{Bar}(H_{n-1}(F(C_\bullet, \mathcal{B}_\bullet, f)))$ of the form $[f(b), f(a))$. Moreover, the persistence barcode resulting from that simplification is equal to the persistence barcode of the original filtered chain complex with that interval removed.

Formally, we want to define the dashed map in the following diagram that makes it commute:

$$\begin{tikzcd}
(C_\bullet, \mathcal{B}_\bullet, f) \arrow[r, mapsto, "\mathrm{w}(\cdot,\cdot):=|f(\cdot)-f(\cdot)|"] \arrow[d, mapsto, "F"] & (C_\bullet, \mathcal{B}_\bullet, \mathrm{w}) \arrow[d, mapsto, "Y"] \\
F((C_\bullet, \mathcal{B}_\bullet, f)) \arrow[d, mapsto, "\mathrm{Bar}"] & Y((C_\bullet, \mathcal{B}_\bullet, \mathrm{w})) \arrow[dd, mapsto, "\mathrm{tBar}"] \\
\mathrm{Bar}\left(\bigoplus_{n\in\mathbb{N}} H_n(F(C_\bullet, \mathcal{B}_\bullet, f))\right) \arrow[d, equal] & \\
\bigcup_{n\in\mathbb{N}} \mathrm{Bar}(H_n(F(C_\bullet, \mathcal{B}_\bullet, f)) \arrow[r, mapsto, dashed] & \bigcup_{n\in\mathbb{N}} \mathrm{tBar}_n(Y(C_\bullet, \mathcal{B}_\bullet, \mathrm{w}))
\end{tikzcd}$$

This will be done in Theorem 3.19. In order to state this result, we use **filtered interval spheres** [13]. This is the monomorphic analogue to the interval functors in TEPCh. Recall that $\iota^n \colon S^n \hookrightarrow D^{n+1}$ denotes the inclusion of chain complexes. Given $n \in \mathbb{N}$, $0 \leq s < \infty$ and $s \leq t \leq \infty$, we define $\mathbb{I}^n[s,t) \in \mathrm{TPCh}$ by

$$(\mathbb{I}^n[s,t))^r = \begin{cases} 0, \text{ if } 0 \leq r < s, \\ S^n, \text{ if } s \leq r < t, \\ D^{n+1}, \text{ if } t \leq r, \end{cases} \qquad \text{and} \quad (\mathbb{I}^n[s,t))^{q\leq r} = \begin{cases} 0, \text{ if } q < s, \\ \mathbb{1}_{S^n}, \text{ if } s \leq q \leq r < t, \\ \iota^n, \text{ if } s \leq q < t \leq r, \\ \mathbb{1}_{D^{n+1}}, \text{ if } t \leq q. \end{cases}$$

The following lemma is based on Proposition 3.2 in [13]. In the proof, we explain the necessary adjustments to apply that result.

Lemma 3.18. *Let $(C_\bullet, \mathcal{B}_\bullet, f)$ be a based chain complex with a filter. Let $(a,b) \in \mathcal{B}_n \times \mathcal{B}_{n-1}$, such that $\langle \partial a, b\rangle \neq 0$ and*

(i) $f(b) = \max\{f(b') \mid \langle \partial a, b'\rangle \neq 0\}$,

(ii) $f(a) = \min\{f(a') \mid \langle \partial a', b\rangle \neq 0\}$.

Denote by $(\overline{C}_\bullet, \overline{\mathcal{B}}_\bullet)$ the based chain complex resulting from applying Lemma 1.6 to the pair (a,b). Define the filter $\overline{f}$ on $(\overline{C}_\bullet, \overline{\mathcal{B}}_\bullet)$ by $\overline{f}([c]) := f(c)$ for $c \in \mathcal{B}_i$. Then we have an isomorphism of filtered chain complexes

$$F(C_\bullet, \mathcal{B}_\bullet, f) \cong F(\overline{C}_\bullet, \overline{\mathcal{B}}_\bullet, \overline{f}) \oplus \mathbb{I}^{n-1}[f(b), f(a)).$$

Proof. Proposition 3.2 of [13] is formulated in terms of a filtered chain complex X and a pair of generators (x_i, x_j) satisfying the so-called split conditions. The conclusion states that $X \cong \mathbb{I}^{n-1}[s,e) \oplus X'$, where n is the degree of x_j, s is the entrance time of x_j, e is the entrance time of x_i and X' is another filtered chain complex. Note that in [13], the pair of generators (x_i, x_j) is written in increasing degree, whereas we write pairs in decreasing degree. We apply the result to $X = F(C_\bullet, \mathcal{B}_\bullet, f)$ and the pair of generators (b,a), noting that the split conditions from [13] are equivalent to our conditions $\langle \partial a, b\rangle \neq 0$ and $(i-ii)$.

It remains to show that $\overline{f}$ indeed defines a monotone function on $(\overline{C}_\bullet, \overline{\mathcal{B}}_\bullet)$ and that the filtered chain complex $F(\overline{C}_\bullet, \overline{\mathcal{B}}_\bullet, \overline{f})$ coincides with X' from [13]. To check the former, consider a pair $(c, d) \in \mathcal{B}_k \times \mathcal{B}_{k-1}$ with $\langle \partial^{\overline{C}}[c], [d] \rangle \neq 0$ with respect to the basis $\overline{\mathcal{B}}_{k-1}$. If also $\langle \partial c, d \rangle \neq 0$ with respect to $\mathcal{B}_{k-1}$, then we have $\overline{f}([c]) - \overline{f}([d]) = f(c) - f(d) > 0$, so $\overline{f}$ satisfies the monotonicity condition for the pair $([c], [d])$. Note that for $k \neq n$, $\langle \partial^{\overline{C}}[c], [d] \rangle \neq 0$ implies that already $\langle \partial c, d \rangle \neq 0$ with respect to the basis $\mathcal{B}_{k-1}$ (in the case $k = n - 1$ this uses the fact that we quotient out ∂a, which satisfies $\partial(\partial a) = 0$). The only case remaining is thus $k = n$ and $d \notin \partial c$. This implies that $\langle \partial c, b \rangle \neq 0$ and $\langle \partial a, d \rangle \neq 0$ (see Lemma 1.6 (*iii*)). Thus, by (*i*) and (*ii*), we have $f(c) \geq f(a)$ and $f(d) \leq f(b)$, and thus

$$\overline{f}([c]) - \overline{f}([d]) = f(c) - f(d) \geq f(a) - f(b) > 0,$$

so $\overline{f}$ is monotone. The fact that $F(\overline{C}_\bullet, \overline{\mathcal{B}}_\bullet, \overline{f})$ is isomorphic to X' from [13] can be checked by comparing the differential δ' from [13] with the induced differential from Lemma 1.6.

□

Due to the additivity of homology, this implies that the persistence barcodes of $H_i(F(C_\bullet, \mathcal{B}_\bullet, f))$ can be obtained from those of $H_i(\overline{C}_\bullet, \overline{f})$ by adding one copy of $[f(b), f(a))$ to $\mathrm{Bar}(H_{n-1}(\overline{C}_\bullet, \overline{f}))$ in degree $n - 1$.

Theorem 3.19. *Let $(C_\bullet, \mathcal{B}_\bullet, f)$ be a based chain complex with filter and let $(C_\bullet, \mathcal{B}_\bullet, \mathrm{w})$ be the corresponding weighted based chain complex, with the weights given by $\mathrm{w}(a, b) = |f(a) - f(b)|$. In case of multiple pairs having the same weight, we choose an arbitrary order. Then there is a bijection of multisets*

$$\bigcup_{n\in\mathbb{N}} \mathrm{Bar}(H_n(F(C_\bullet, \mathcal{B}_\bullet, f)) \longrightarrow \bigcup_{n\in\mathbb{N}} \mathrm{tBar}_n(Y(C_\bullet, \mathcal{B}_\bullet, \mathrm{w})),$$

defined by

$$\begin{aligned} \mathrm{Bar}(H_n(F(C_\bullet, \mathcal{B}_\bullet, f))) \ni [s, t) &\longmapsto [0, t - s, t - s) \in \mathrm{tBar}_{n+1}(Y(C_\bullet, \mathcal{B}_\bullet, \mathrm{w})), \\ \mathrm{Bar}(H_n(F(C_\bullet, \mathcal{B}_\bullet, f)) \ni [s, \infty) &\longmapsto [0, 0, \infty) \in \mathrm{tBar}_n(Y(C_\bullet, \mathcal{B}_\bullet, \mathrm{w})). \end{aligned}$$

In words, the theorem says that we can obtain the tagged barcode of $Y(C_\bullet, \mathcal{B}_\bullet, \mathrm{w})$ from the persistence barcode of $\bigoplus_n H_n(F(C_\bullet, \mathcal{B}_\bullet, f))$ by shifting all the bars to the left until they start at zero. The finite bars change degree by one, the infinite bars remain in the same degree.

Proof. For brevity, we write $F(C_\bullet) := F(C_\bullet, \mathcal{B}_\bullet, f)$ and $Y(C_\bullet) := Y(C_\bullet, \mathcal{B}_\bullet, \mathrm{w})$ during this proof. Note that $Y(C_\bullet)$ a priori depends on the chosen ordering of the pairs of equal weight, since this influences which pairs get simplified in Construction 2.34. We do an induction over the number m of finite tagged intervals in $\mathrm{tBar}(Y(C_\bullet))$.

If $m = 0$, that means that the differential in $C_\bullet$ is zero. In that case, for any $n \in \mathbb{N}$, the tagged barcode $\mathrm{tBar}_n(Y(C_\bullet))$ consists of one copy of $[0, 0, \infty)$ for each element of the basis $\mathcal{B}_n$. In the persistence barcode $\mathrm{Bar}(H_n(F(C_\bullet))$, on the other hand, each $b \in \mathcal{B}_n$ yields a bar of the form $[f(b), \infty)$. Thus the claim of the theorem holds true in this case.

For the induction step, let $m \geq 1$ and assume that we have proven the theorem for $m - 1$. Since $m \geq 1$, the differential in $C_\bullet$ is non-zero, hence at least one pair gets simplified in the construction of $Y(C_\bullet)$. Let $(a, b) \in \mathcal{B}_n \times \mathcal{B}_{n-1}$ be the first pair that gets simplified. This means that $\mathrm{w}(a, b)$ is minimal among all pairs (a', b') with $\langle \partial a', b' \rangle \neq 0$, which implies that the conditions from Lemma 3.18 are satisfied for the pair (a, b). Denote by $(\overline{C}_\bullet, \overline{\mathcal{B}}_\bullet)$ the based chain complex resulting from applying Lemma 1.6 to the pair (a, b). Note that this is equal to the based chain complex from Lemma 3.18 as well as the one from Construction 2.34. Also the filter $\overline{f}$ from Lemma 3.18 induces the same weights $\overline{\mathrm{w}}$ as the ones from Construction 2.34. Therefore, by the induction hypothesis, we have the above-described bijection of multisets

$$\bigcup_{i\in\mathbb{N}} \mathrm{Bar}(H_i(F(\overline{C}_\bullet))) \longrightarrow \bigcup_{i\in\mathbb{N}} \mathrm{tBar}_i(Y(\overline{C}_\bullet)).$$

Now we want to extend the bijection to $F(C_\bullet)$ and $Y(C_\bullet)$. By construction of $Y(C_\bullet)$, it is clear that $\mathrm{tBar}_i(Y(C_\bullet)) = \mathrm{tBar}_i(Y(\overline{C}_\bullet))$ for $i \neq n$ and $\mathrm{tBar}_n(Y(C_\bullet)) = \mathrm{tBar}_n(Y(\overline{C}_\bullet)) \sqcup \{[0, \mathrm{w}(a,b), \mathrm{w}(a,b))\}$. By Lemma 3.18 and additivity of homolgy, we have

$$\mathrm{Bar}(H_i(F(C_\bullet))) = \begin{cases} \mathrm{Bar}(H_i(F(\overline{C}_\bullet))) & \text{if } i \neq n-1, \\ \mathrm{Bar}(H_{n-1}(\overline{C}_\bullet)) \sqcup \{[f(b), f(a))\} & \text{if } i = n-1. \end{cases}$$

Thus, to extend the bijection of multisets, it is enough to map

$$[f(b), f(a)) \mapsto [0, f(a) - f(b), f(a) - f(b)) = [0, \mathrm{w}(a,b), \mathrm{w}(a,b)),$$

yielding a bijection of multisets between the persistence barcode of $\bigoplus_{i\in\mathbb{N}} H_i(F(C_\bullet))$ and the tagged barcode of $Y(C_\bullet)$. This completes the proof. □

Corollary 3.20. *Given* $(C_\bullet, \mathcal{B}_\bullet, \mathrm{w}) \in \mathrm{wb}^+\mathrm{Ch}$*, where the weights are induced from a monotone function, then the tagged barcode of* $X(C_\bullet)$ *and* $Y(C_\bullet)$ *does not depend on the ordering of the weights.*

Proof. If the weights are induced from a monotone function f, then by Theorem 3.19 we can compute the tagged barcode of $Y(C_\bullet)$ also from the persistence barcode of $\bigoplus_n H_n(F(C_\bullet, \mathcal{B}_\bullet, f))$. This barcode comes from the decomposition of a filtered vector space, which by Theorem 1.4 is unique, and hence does not depend on the ordering of the pairs needed to compute $Y(C_\bullet)$. Thus, also the tagged barcode of $Y(C_\bullet)$ must be independent of the order. Since the tagged barcode of $X(C_\bullet)$ can be obtained from the barcode of $Y(C_\bullet)$ by adding up the lenghts of the intervals (compare Proposition 2.33 for $X(C_\bullet)$ with Proposition 2.35 for $Y(C_\bullet)$), this extends also to $X(C_\bullet)$. □

Chapter 4

Non-uniqueness when removing closed orbits

The content of this chapter is in some sense a justification for the following chapter. In Section 2.5 we gave a method for for obtaining a factored chain complex (and thus by Theorem 2.17 a tagged barcode) from a chain complex that is equipped with some extra structure. In Chapter 3 we then explained how to apply this to the Morse complex of a gradient-like Morse-Smale vector field, making use of the fact that in that case, a chain complex is readily available by the Morse complex. If such a complex existed also in the case of Morse-Smale vector fields with closed orbits, we would only have to endow it with bases and weights and could apply the results from Chapter 2.

In this chapter we describe difficulties with some approaches for defining such a complex. We trace these back to a method introduced by Franks for replacing a closed orbit by a pair of fixed points, pointing out the non-uniqueness of this procedure. We discuss how this yields obstructions to the endeavor of assigning topological or algebraic invariants to general Morse-Smale vector fields.

As a result of this, we construct our own chain complex in the following chapter, making use of the spectral sequence associated with a filtration induced by the vector field.

4.1 Why and how to remove closed orbits

Assume that M is a closed smooth manifold and X is a Morse-Smale vector field on M. A lot of information can be gained by studying the singular elements of X, i.e. the fixed points and closed orbits. An index can be assigned to each singular element, describing the local behaviour of X. Generalized Morse inequalities were proven by Smale, relating the number of singular elements of each index of X to the Betti numbers of M [50]. A special case that is better understood is when X is gradient-like, i.e. has no closed orbits. In that case it was pointed out in [27], that by following [41] we can construct a CW complex Y whose k-cells are in one-to-one correspondence with the fixed points of index k of X. This CW complex is defined uniquely up to cell equivalence and there exists a homotopy equivalence $M \to Y$ that maps the unstable manifold of every fixed point into the closure of the corresponding cell. Another structure that can be defined in terms of the singular elements of X is the Morse complex (also sometimes denoted by some combination of the names Morse, Smale, Witten, Thom, Floer, see [8] for a historical discussion). This is a chain complex with coefficients in some field $\mathbb{F}$ (often $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$

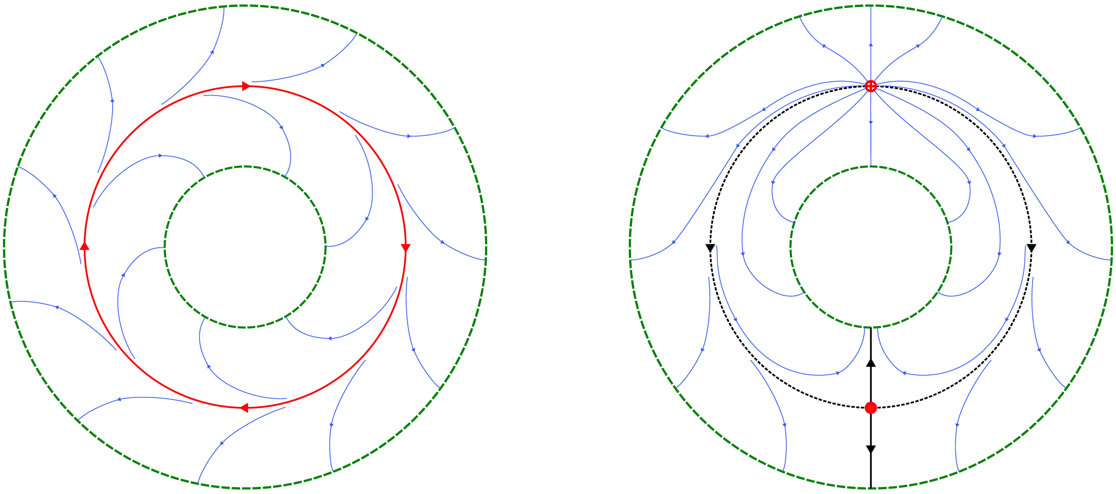


Figure 4.1: By Proposition 4.1, we can replace a closed orbit of index k (left panel) by two fixed points of index $k+1$ and k (right panel). The vector field stays unchanged outside of a neighbourhood of the orbit, which can be chosen arbitrarily small.

is chosen for easier computations), such that for every k, the vector space in degree k has a basis corresponding to the fixed points of index k of X. The differential of this chain complex is defined by counting certain flow lines and its homology agrees with the singular homology of M, see e.g. [3] for details.

The question arises whether we can define analogous structures more generally for Morse-Smale vector fields also in the presence of closed orbits. An indicator for this is the following result by Franks, allowing us to replace a closed orbit of index k by a pair of fixed points (called rest points in [27]) of index $k+1$ and k, respectively.

Proposition 4.1 ([27, Proposition 5.1])**.** *Suppose X is a Morse-Smale vector field on an orientable manifold whose flow has a closed orbit γ of index k in standard form. Then given a small neighborhood U of γ, there exists a Morse-Smale vector field X' which agrees with X outside U and in U has rest points p, q of index $k+1$ and k but no other rest points or closed orbits. Also, q is a successor to p and their connecting manifold consists of two points framed with opposite orientations if the original closed orbit γ was untwisted and otherwise framed with the same orientation. Moreover the unstable manifold for γ will be equal to $W^u(p) \cup W^u(q)$.*

Roughly speaking, this works by reversing the direction of X on one half of the orbit, thereby creating two fixed points that lie on opposite ends of the original closed orbit. The other points of what before was a closed orbit now form two flow lines from the fixed point of index $k+1$ to the fixed point of index k, see Figure 4.1. By repeating this for every closed orbit, we can replace X by a gradient-like vector field X'.

This yields the following pipeline: Start with a Morse-Smale vector field X. Replace all closed orbits by pairs of fixed points according to Proposition 4.1 to obtain a gradient-like Morse-Smale vector field X'. Use the known results to assign some structure to X' (e.g. a CW complex or a chain complex). Consider the choices that were made when perturbing X to X' and check that the resulting structure is unique (up to some equivalence relation).

We first examine the uniqueness question for CW complexes assigned to Morse-Smale vector fields. It was claimed in [27, Corollary 5.2] that this can be used to assign a CW complex to X which is unique up to cell equivalence. In Section 4.2 we review the definition of cell equivalence and some relevant results, then provide examples that illustrate that the CW complex resulting from this procedure may not be unique up to cell equivalence, showing that the uniqueness part in [27, Corollary 5.2] is incorrect.

Next we examine the uniqueness question for assigning chain complexes to Morse-Smale vector fields. In Section 4.3 we consider the vector spaces and linear maps defined in [23]. We show that in dimension two, these maps square to zero, but the resulting homology may differ from the singular homology of the underlying manifold. Then we present an example in dimension three where these maps do not square to zero, thereby showing that [23, Theorem 2.5] is incorrect.

An algebraic structure that can be assigned to more general flows is given by the connection matrix, whose existence has been proven by Franzosa [29]. For gradient-like Morse-Smale vector fields it was shown to be unique and coincide with the Morse complex by Reineck [47]. The examples presented in [29, 47] involving closed orbits show the non-uniqueness of the connection matrix for general Morse-Smale vector fields and resonate with our examples about the non-uniqueness of assigned CW decompositions and chain complexes.

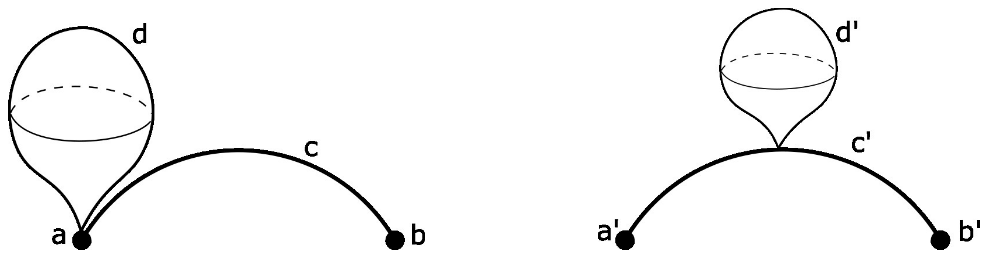
Let us make some remarks about how we display our examples. We draw fixed points and closed orbits in red. For fixed points, we use different symbols, according to the index. Fixed points of index 0 (sinks) are drawn as a minus sign. Fixed points of index 1 (saddles) are drawn as a filled dot. Fixed points of index 2 (sources) are drawn as a plus sign. In Figure 4.6b we have a three-dimensional vector field, where we draw the fixed points of index 3 as a hollow dot. Figures 3.2 and 4.3 to 4.5 display vector fields on the 2-sphere, which we visualize as a disk, the green boundary representing a single point of the sphere. This point is a fixed point of index 0 of the vector field in each case. Flow lines flowing to and from saddle points are drawn in black. Other flow lines are drawn in blue, with thinner lines. In some cases, these additional flow lines are not drawn, as their exact shape is not relevant for our purposes.

4.2 Non-uniqueness for CW complexes

We recall the definition of cell equivalence, since it is a rather specific notion of [27]. We start by introducing the face poset. If e and e' are cells of a CW complex Y, we will say that $e' \leq e$ if the closure of e contains any part of the interior of e'. If we make this relation transitive (and denote the new transitive relation by $\leq$ as well) then we obtain a partial ordering on the cells of Y. If S is a subset of the cells of Y with the property that $e \in S$ and $e' \leq e$ implies $e' \in S$, then the union of the cells in S forms a subcomplex of Y. In particular, if e is a cell of Y we define the **base** of e, denoted $Y(e)$, to be the smallest subcomplex of Y containing e. Thus $Y(e)$ is the union of all cells e' in Y such that $e' \leq e$.

Definition 4.2 ([27, Definition 2.1])**.** Two finite CW complexes Y and Y' will be called **cell equivalent** providing there is a homotopy equivalence $h\colon Y \to Y'$ with the property that there is a one-to-one correspondence between cells of Y and cells of Y' such that if $e \subseteq Y$ corresponds to $e' \subseteq Y'$ then h maps $Y(e)$ to $Y'(e')$ and is a homotopy equivalence of these subcomplexes.

There is a problem with this definition of cell equivalance, namely that it does not yield an equivalence relation. This is showcased by Figure 4.2, where we have two CW complexes, Y, Y', such that a cell equivalence $h\colon Y \to Y'$ exists, but not in the other direction. We thus present an alternative version of the definition of cell equivalence,



(a) The CW complex Y consists of the 0-cells a and b, the 1-cell c, and the 2-cell d, which is attached to the 0-cell a.

(b) The CW complex Y' consists of the 0-cells a' and b', the 1-cell c', and the 2-cell d', which is attached in the middle of the 1-cell c'.

Figure 4.2: In Y, the base of the cell d consists only of the cells d and a. In Y', the base of the cell d' is all of Y'. The map $h\colon Y \to Y'$ that maps $a \mapsto a'$, $b \mapsto b'$, $c \mapsto c'$, the bottom half of d to the left half of c', and the upper half of d to d', is a cell equivalence according to Definition 4.2. However, no cell equivalence exists in the other direction, so it does not satisfy the stronger Definition 4.3.

where we demand the existence of a homotopy inverse that is also a cell equivalence and that the base of any cell gets mapped surjectively onto the base of its corresponding cell.

Definition 4.3. Two finite CW complexes Y and Y' are called **cell equivalent** if there are maps $f\colon Y \to Y'$ and $g\colon Y' \to Y$ that are homotopy inverses to each other, such that there exists a one-to-one correspondence between the cells of Y and the cells of Y' such that if $e \subset Y$ corresponds to $e' \subset Y$, then $f(Y(e)) = Y'(e')$, $g(Y'(e')) = Y(e)$, and the restrictions of f to $Y(e)$ and of g to $Y'(e')$ are homotopy inverses to each other.

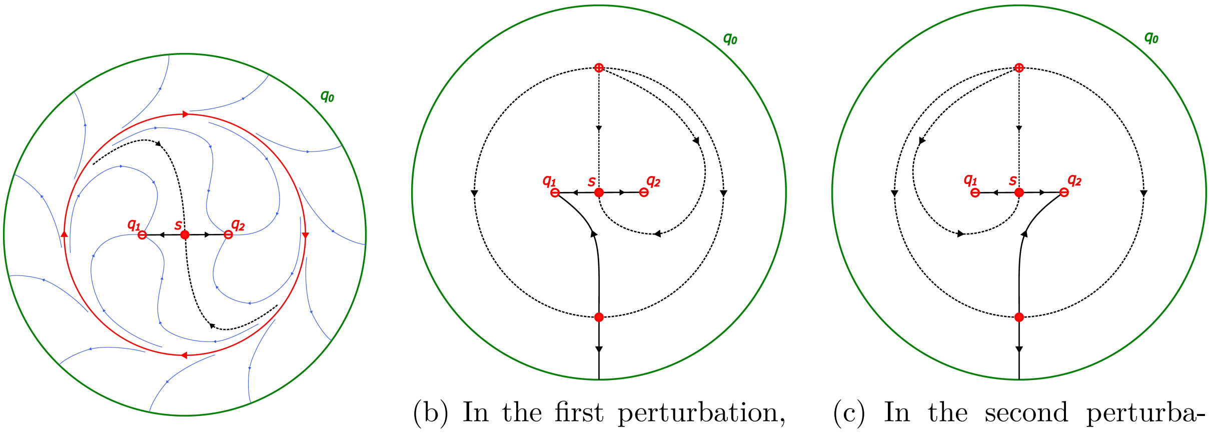
In this version, symmetry is already built into the definition, so it becomes straightforward to check that it is indeed an equivalence relation on finite CW complexes. Moreover, if Y and Y' are cell equivalent in the sense of Definition 4.3, it follows that their face posets are isomorphic.

Given a closed smooth manifold M and a singular element β of a vector field on M, we denote by $W^s(\beta)$ and $W^u(\beta)$ its stable and unstable manifolds. See [27] for the necessary definitions and background. The following statement is a reformulation of a fundamental result from Morse theory, stating that we can assign to each gradient-like Morse-Smale vector field a CW complex Y unique up to cell equivalence, with the homotopy type of the underlying manifold.

Theorem 4.4 ([27, Theorem 2.3])**.** *If X is a gradient-like Morse-Smale vector field on M, then there exists a CW complex Y, unique up to cell equivalence, and a homotopy equivalence $g\colon M \to Y$ such that for each rest point p of index k, $g(W^u(p))$ is contained in the base $Y(e)$ of a single k-cell e.*

In this way g establishes a one-to-one correspondence between rest points of X of index k and k-cells of Y. Moreover, the partial order $<$ on rest points defined by $q \leq p$ if and only if $W^s(q) \cap W^u(p) \neq \emptyset$ corresponds to the partial order $\leq$ on the cells of Y.

Combining Theorem 4.4 with Proposition 4.1, we get the following result, stating that we can do the same thing also for general Morse-Smale vector fields. The underlined part about uniqueness is not true, as it is showcased next by Examples 4.6 and 4.7. If we delete that part the statement becomes correct. The author of [27] agrees with this assessment [28].



(a) Before the perturbation, the vector field contains a closed orbit.

(b) In the first perturbation, the unstable manifold of the new saddle intersects the stable manifold of q_1.

(c) In the second perturbation, the unstable manifold of the new saddle intersects the stable manifold of q_2.

Figure 4.3: This vector field has three sinks q_0, q_1, q_2, one saddle s and one repelling orbit γ. Due to Proposition 4.1 the repeller can be replaced by a source and a saddle. It is not clear a priori if the newly created saddle connects to q_1 or q_2. Note that in Figure 4.3a, we draw some of additional flow lines in order to visualize the flow of the vector field. In Figure 4.3b and Figure 4.3c we omit them.

Corollary 4.5 ([27, Corollary 5.2])**.** *For any Morse-Smale vector field X on a compact manifold M there is a CW complex Y homotopy equivalent to M and unique up to cell equivalence associated to X. Corresponding to each rest point of X of index k there is a k-cell of Y, and to each closed orbit of X of index k there are cells of Y of dimension $k+1$ and k attached by a map of degree 0 or 2 depending on whether or not the orbit is twisted.*

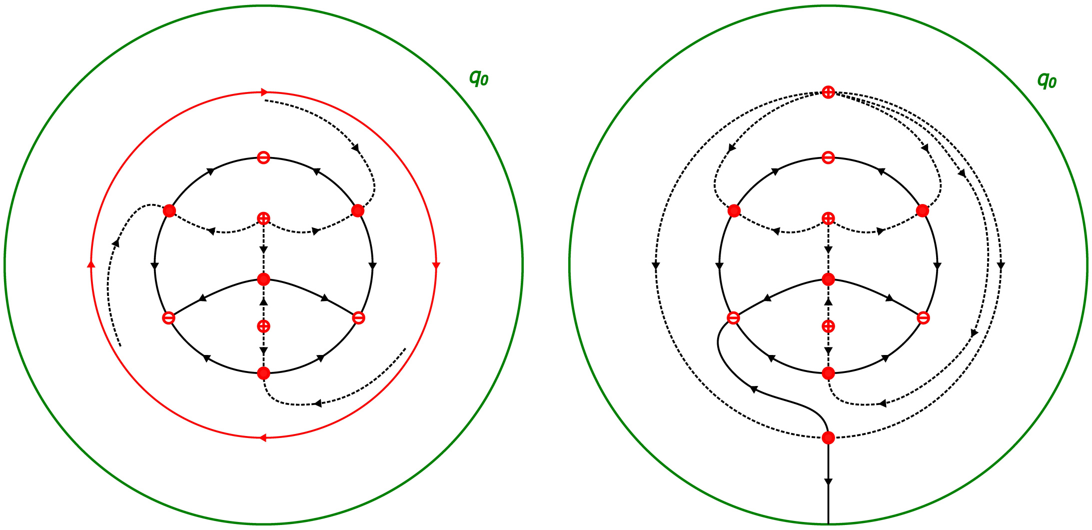
The proof of Corollary 4.5 (without the underlined part) goes as follows: Given a Morse-Smale vector field X, apply Proposition 4.1 to each closed orbit in order to obtain a gradient-like vector field X'. Every fixed point of X corresponds to a fixed point of X' of the same index and every closed orbit of index k of X corresponds to two fixed points of X', of index $k+1$ and k, respectively. We can then apply Theorem 4.4 to X' in order to get a CW complex.

The CW complex assigned to X' is unique up to cell equivalence, but when replacing X by X' there are some choices involved, which can result in different CW structures. We now present two examples that illustrate this.

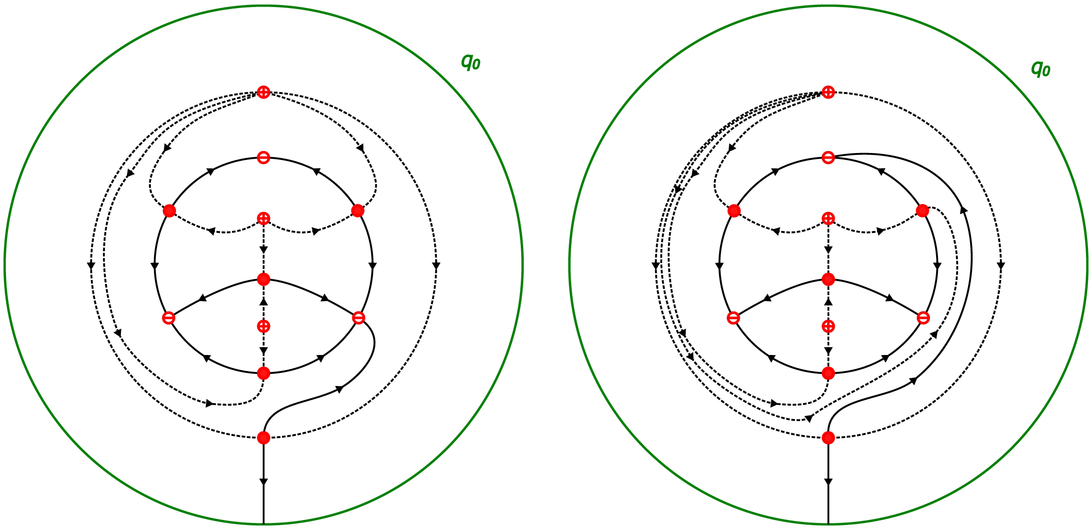
Example 4.6. In Figure 4.3, we describe the minimal example illustrating the choice when removing a closed orbit. However, in this example, both choices still lead to two cell equivalent CW complexes. The cell equivalence is given by swapping the cells corresponding to q_1 and q_2.

We now present a slightly more complicated example, where we end up with two CW complexes that are not cell equivalent.

Example 4.7. In the vector field X displayed in Figure 4.4, we again have one closed orbit, but this time we have more additional fixed points. There are three sinks on the inside of the closed orbit, so when we replace it by a source and a saddle, it is not clear a priori to which of these three sinks the flow line starting at the newly introduced



(a) The original vector field X (b) The perturbation X_1



(c) The perturbation X_2 (d) The perturbation X_3

Figure 4.4: In this example, the CW complexes assigned to the perturbations X_1 and X_2 are cell equivalent to each other, but not to the CW complex of X_3. We omit drawing any additional flow lines in order not to overload the pictures.

saddle should connect to. The first two choices X_1 and X_2 lead to cell equivalent CW decompositions, but the CW complex resulting from the third perturbation X_3 is not cell equivalent to the other two. To see why this is the case, note that for X_1 and X_2 the number of saddles that each sink is connected to is 1,2,3,4. For X_3 however it is 1,3,3,3. This implies that the the posets given by the relation $\leq$ on the cells introduced earlier are not isomorphic, hence the CW complexes cannot be cell equivalent.

4.3 Non-uniqueness for chain complexes

In [23], given a generalized Morse-Smale vector field, Eidi and Jost construct a sequence of vector spaces and linear maps. This is done directly in terms of the singular elements and their intersections of stable and unstable manifolds, without referring to any perturbations. We repeat the main definitions here, for the simpler case of Morse-Smale vector fields (i.e. we exclude homoclinic orbits). In this section we work with coefficients in the field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

Given a Morse-Smale vector field X and two singular elements β_1, β_2 of X (i.e. fixed points or closed orbits), we define $\alpha(\beta_1, \beta_2)$ to be the number of connected components (mod 2) of $W^u(\beta_1) \cap W^s(\beta_2)$.

Definition 4.8. Given a Morse-Smale vector field X on a closed smooth manifold M, the Eidi-Jost complex $(\mathrm{C}^{EJ}_\bullet, \partial^{EJ}_\bullet)$ is defined as follows:

- For any $k \geq 0$, $\mathrm{C}^{EJ}_k = \mathrm{C}^{EJ}_k(X)$ is the free $\mathbb{F}$-vector space generated by all fixed points of index k, all closed orbits of index k and all closed orbits of index $k-1$. Denote by $\mathcal{B}^{EJ}_k = \mathcal{B}^{EJ}_k(X)$ this generating set (i.e. the canonical basis for C^{EJ}_k).
- The linear map $\partial^{EJ}_k = \partial^{EJ}_k(X)\colon \mathrm{C}^{EJ}_k \to \mathrm{C}^{EJ}_{k-1}$ is defined by setting, for all $\beta \in \mathcal{B}^{EJ}_k$,
$$\partial^{EJ}_k(\beta) := \sum_{\beta' \in \mathcal{B}^{EJ}_{k-1}} \alpha(\beta, \beta')\beta'$$
and extending linearly.

Note that a closed orbit γ of index k contributes one copy to $\mathcal{B}^{EJ}_k$ and $\mathcal{B}^{EJ}_{k+1}$ each. We distinguish the two by adding a minus and a plus sign, i.e. we write $\gamma^- \in \mathcal{B}^{EJ}_k$ and $\gamma^+ \in \mathcal{B}^{EJ}_{k+1}$.

There is an issue with the proof that $\partial^{EJ}_\bullet$ squares to zero presented in [23]. Proposition 4.1 is used in that proof and it seems as though the uniqueness (which does not hold due to our examples from Section 4.2) is assumed implicitly. We present an alternative proof that works only in the case of two-dimensional manifolds.

Proposition 4.9. *If M is a two-dimensional closed smooth manifold and X is a Morse-Smale vector field on M, then $\partial^{EJ}_1 \circ \partial^{EJ}_2 = 0$.*

Proof. Let M be a 2-dimensional manifold and let X be a Morse-Smale vector field on M. Write $(C_\bullet, \partial_\bullet)$ and $\mathcal{B}_\bullet$ for the Eidi-Jost complex and the corresponding bases. We want to show that $\partial_1 \circ \partial_2 = 0$. We do an induction over the number m of closed orbits of X.

If $m = 0$, then $(C_\bullet, \partial_\bullet)$ is equal to the usual chain complex from Morse homology, for which it is known that the differential squares to zero [3].

We now assume $m > 0$ and that we have shown the statement for any Morse-Smale vector field with $m-1$ closed orbits. Let γ be a closed orbit of X of index k (where either $k = 0$ or $k = 1$). We apply Proposition 4.1 to obtain a new Morse-Smale vector field X' which agrees with X outside of a small neighbourhood of γ and that instead of the closed orbit γ has two fixed points x and y of index $k+1$ and k, respectively. These two fixed points are connected by two flow lines, going from x to y. We denote by $(C'_\bullet, \partial'_\bullet)$ and $\mathcal{B}'_\bullet$ the Eidi-Jost complex and the corresponding bases for X'. Note that $\mathcal{B}'_{k+1} = (\mathcal{B}_{k+1} \setminus \{\gamma^+\}) \sqcup \{x\}$ and $\mathcal{B}'_k = (\mathcal{B}_k \setminus \{\gamma^-\}) \sqcup \{y\}$. By the induction hypothesis we know that $\partial'_1 \circ \partial'_2 = 0$.

Let us from now on assume that $k = 1$, i.e. γ is a repelling orbit. The proof for $k = 0$ is analogous and we omit it. For simplicity we identify $\partial_2, \partial_1, \partial'_2, \partial'_1$ with the corresponding representation matrices with respect to the bases $\mathcal{B}_\bullet$ and $\mathcal{B}'_\bullet$, which we assume to be ordered in some way. We claim that the following hold.

(i) The row corresponding to γ^- in ∂_2 and the row corresponding to y in ∂_2' are both zero.

(ii) The matrices ∂_2 and ∂_2' are the same.

(iii) The matrices ∂_1 and ∂_1' differ only in the column corresponding to γ^- (in ∂_1) and y (in ∂_1').

As for (i), note that γ is a repeller, so $\alpha(\beta, \gamma) = 0$ for all other singular elements β. Thus the row corresponding to γ^- in ∂_2 is zero. For ∂_2' on the other hand, there are two flow lines from x to y and no other flow lines from any other singular element to y (because y is a saddle point). These two flow lines cancel out due to $\mathbb{Z}_2$ coefficients and thus also the row corresponding to y in ∂_2' is zero.

For (ii) and (iii), note that all remaining entries are defined by counting intersections of stable and unstable manifolds of singular elements different from γ. Assuming that we have chosen small enough the neighbourhood of γ on which X and X' differ, these intersection numbers are the same for X and X', hence the according entries in the boundary matrices are the same.

From these three claims it follows now that $\partial_1 \circ \partial_2 = \partial_1' \circ \partial_2'$, since the only difference between the two is in one column of ∂_1 (resp. ∂_1') that gets multiplied by a zero row in ∂_2 (resp. ∂_2') anyway. Hence, as we knew already that $\partial_1' \circ \partial_2' = 0$, it follows that $\partial_1 \circ \partial_2 = 0$. □

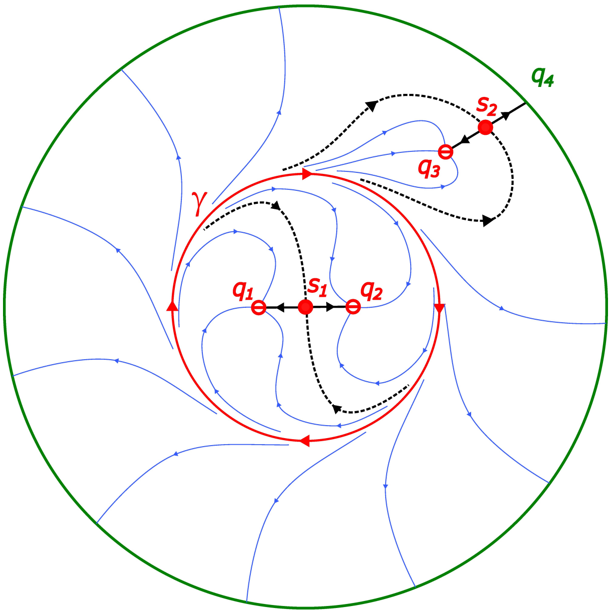
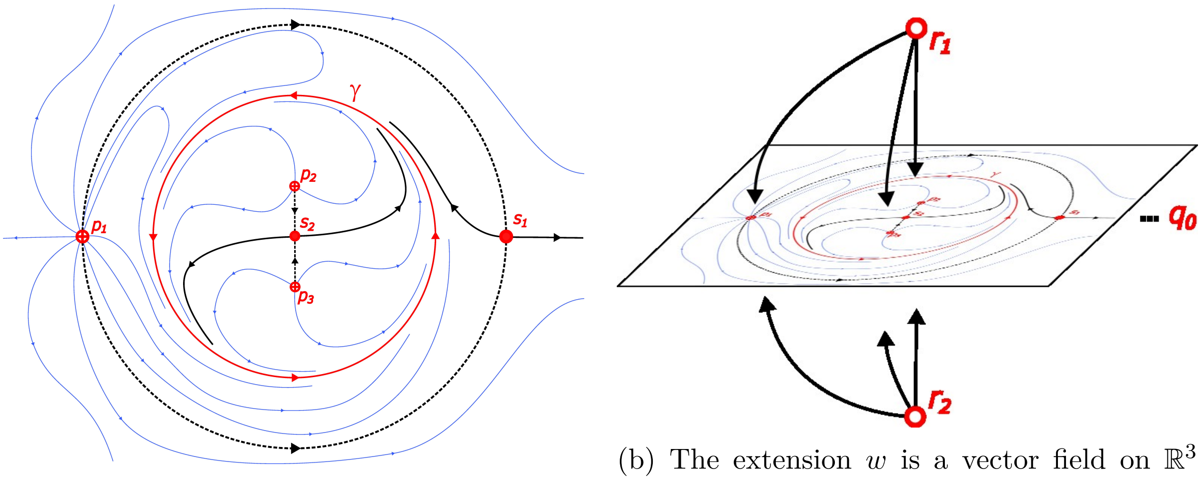


Figure 4.5: This is an example of a Morse-Smale vector field on the 2-sphere, where the homology of the Eidi-Jost complex differs from the singular homology of S^2.

Here we show by an example that the homology of the Eidi-Jost complex of a Morse-Smale vector field on a surface can be different from the singular homology of the surface, implying that the proof of [23, Theorem 2.5] is flawed (this has been acknowledged by the authors of [23] by private communication).

Example 4.10. For the vector field depicted in Figure 4.5, the associated Eidi-Jost complex is given by



(a) The Morse-Smale vector field X on $\mathbb{R}^2$

(b) The extension w is a vector field on $\mathbb{R}^3$ with a sink at infinity.

Figure 4.6: This is an example of a Morse-Smale vector field on S^3 for which the Eidi-Jost differential does not square to zero.

$$\begin{tikzcd}
\mathbb{F} = \operatorname{Span}(\gamma^+) \arrow[d, "\partial_2^{EJ} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}"] \\
\mathbb{F}^3 = \operatorname{Span}(s_1, s_2, \gamma^-) \arrow[d, "\partial_1^{EJ} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}"] \\
\mathbb{F}^4 = \operatorname{Span}(q_1, q_2, q_3, q_4).
\end{tikzcd}$$

Indeed, since ∂_1^{EJ} in this example has rank 2, it follows that we get a one-dimensional homology group in degree 1 and two-dimensional homology in degree 0. This does not agree with the homology of the 2-sphere.

Moreover, in the following example, the Eidi-Jost differential does not square to zero.

Example 4.11. We build a 3-dimensional Morse-Smale vector field V as follows. Consider first the vector field X on $\mathbb{R}^2$ from Figure 4.6a. It has sources p_1, p_2, p_3, saddles s_1, s_2, and a closed orbit γ of index 0. Outside of what is shown in the image, the vector field continues to the rest of $\mathbb{R}^2$ without any further singular elements, all of the flow lines going towards infinity. More precisely, we may assume that the point s_2 lies at the origin and that for $\|\vec{x}\|$ large enough, $X(\vec{x}) = \vec{x}$.

We now describe how to extend $X = (X_1, X_2)$ to a vector field V on $\mathbb{R}^3$. The idea is that on the plane $\mathbb{R}^2 \times \{0\} \subseteq \mathbb{R}^3$, V agrees with X and outside of this plane, in a small neighbourhood of $\mathbb{R}^2 \times \{0\}$ everything flows toward that plane, and at the points $(0, 0, 1)$ and $(0, 0, -1)$ we have two fixed points of index 3, see Figure 4.6b. More explicitly, we choose a value $0 < \varepsilon < \frac{1}{2}$ and demand that V has the following properties:

$$V(x,y,z) = \begin{cases} (X_1(x,y), X_2(x,y), -z) & \text{if } |z| < \varepsilon \\ (x,y,z-1) & \text{if } z > \frac{3}{4} \\ (x,y,z+1) & \text{if } z < -\frac{3}{4}. \end{cases}$$

On the slices $\mathbb{R}^2 \times [\varepsilon, \frac{3}{4}]$ and $\mathbb{R}^2 \times [-\frac{3}{4}, -\varepsilon]$ we interpolate somehow, the exact way we do this is not important as long as we introduce no additional singular elements. All the singular elements of X become singular elements of V. One can check that they are still hyperbolic and that their unstable manifolds stay the same, while their stable manifolds get one extra dimension, locally perpendicular to the plane $\mathbb{R}^2 \times \{0\}$. This means that they keep the same index. Additionally, V has the fixed points r_1 and r_2 of index 3. The vector field V is thus again a Morse-Smale vector field.

We can rescale V so that $\|V(\vec{x})\| \to 0$ when $\|\vec{x}\| \to \infty$, which allows us to add a fixed point for V of index 0, call it q_0, at infinity and thus view V as a vector field on S^3. In order to write down the Eidi-Jost complex, note that $\alpha(r_i, p_j) = 1$ for $i = 1, 2$ and $j = 1, 2, 3$. The other relevant values of α can be read off Figure 4.6a. The resulting sequence is

$$\begin{tikzcd}
\operatorname{Span}(r_1, r_2) \arrow[d, "\partial_3^{EJ} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}"] \\
\operatorname{Span}(p_1, p_2, p_3) \arrow[d, "\partial_2^{EJ} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}"] \\
\operatorname{Span}(s_1, s_2, \gamma^+) \arrow[d, "\partial_1^{EJ} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}"] \\
\operatorname{Span}(q_0, \gamma^-),
\end{tikzcd}$$

and we see that $\partial_2^{EJ} \circ \partial_3^{EJ} \neq 0$.

In conclusion, we have highlighted uniqueness issues arising when removing closed orbits from Morse-Smale vector field.

Chapter 5

A chain complex for 2D Morse-Smale vector fields with closed orbits

In this chapter we present a method to assign a chain complex to a Morse-Smale vector field v on a surface M. The homology of this chain complex will turn out to be isomorphic to the singular homology of M, with no arbitrary choices involved. To do this, we first define a filtration of M following [50]. Then we consider the spectral sequence associated with this filtration and construct a chain complex from it.

We fix a field $\mathbb{F}$. All homology groups and vector spaces are understood to have coefficients in $\mathbb{F}$, which we usually do not make explicit in our notation.

5.1 Čech homology

All the topological spaces are assumed to be Hausdorff and we usually do not state this explicitly. The spaces we consider are subsets of manifolds or quotients of Hausdorff spaces by compact subsets and thus again Hausdorff.

Given a pair of spaces (X, A) we write $\check{H}_k(X, A)$ for its Čech homology in degree k with coefficients in a field $\mathbb{F}$. We write $\check{H}_k(X)$ for $\check{H}_k(X, \emptyset)$. This is defined by taking the inverse limit of the simplicial homology of the nerves of all open coverings of (X, A). See [24] for details. We recall some definitions and list some properties that we are going to use, giving references for proofs.

The main reason why we use Čech homology instead of singular homology is that it satisfies a stronger version of the excision axiom, namely invariance under relative homeomorphisms. From this it follows that the Čech homology for an arbitrary compact pair (X, A) is isomorphic to the reduced one of the quotient space X/A. Singular homology has this property only for nice enough pairs (A needs to be a deformation retract of some neighbourhood in X), but the pairs appearing in the filtration assigned to a vector field are not always nice in that sense.

Definition 5.1. A map of pairs $f\colon (X, A) \to (Y, B)$ is called a **relative homeomorphism** if it maps $X \setminus A$ homeomorphically to $Y \setminus B$.

The following result is [24, Chapter X, Lemma 5.2].

Lemma 5.2. *If $f\colon (X, A) \to (Y, B)$ is a map of compact pairs and f maps $X \setminus A$ bijectively onto $Y \setminus B$, then f is a relative homeomorphism.*

As an immediate corollary we get that the quotient map that collapses a closed subspace A of a compact space X to a point is a relative homeomorphism.

Corollary 5.3. *If (X, A) is a compact pair, then the quotient map $q\colon (X, A) \to (X/A, \{*\})$ is a relative homeomorphism.*

The following result is [24, Chapter X, Theorem 5.4], stating that Čech homology is invariant under relative homeomorphisms.

Theorem 5.4. *If $f\colon (X, A) \to (Y, B)$ is a relative homeomorphism between compact pairs, then the induced map $f_*\colon \check{H}_*(X, A) \to \check{H}_*(Y, B)$ is an isomorphism.*

A **pointed space** $(X, *)$ consists of a topological space X together with a chosen point $* \in X$, called the **basepoint**. Usually we denote the basepoints of all pointed spaces with the same symbol $*$, unless it is helpful to do otherwise. When considering a quotient space X/A, if not otherwise specified, the canonical basepoint is the image of A under the quotient map $X \to X/A$.

The **wedge sum** $X_1 \vee X_2$ of two pointed spaces $(X_1, *_1)$ and $(X_2, *_2)$ is defined as the quotient space $(X_1 \sqcup X_2)/(\{*_1\} \sqcup \{*_2\})$. One can check that this operation is associative, i.e. $(X_1 \vee X_2) \vee X_3$ and $X_1 \vee (X_2 \vee X_3)$ are naturally homeomorphic. Thus we can define the n-fold wedge sum by iteration, with the according choice of the canonical basepoint.

Proposition 5.5. *Given compact pointed spaces $(X_1, *_1), \dots, (X_n, *_n)$, we have*

$$\check{H}_k(X_1 \vee \dots \vee X_n, *) \cong \check{H}_k(X_1, *_1) \oplus \dots \oplus \check{H}_k(X_n, *_n).$$

Proof. By an induction argument, it is enough to prove the statement for $n = 2$. Consider the quotient map $q\colon (X_1 \sqcup X_2, \{*_1\} \sqcup \{*_2\}) \to (X_1 \vee X_2, *)$. By definition of the wedge sum, q is a relative homeomorphism and hence by Theorem 5.4 induces an isomorphism in Čech homology. When considering Čech homology of the disjoint union $X_1 \sqcup X_2$, we can restrict to open covers where every open set is a subset of either X_1 or X_2, since the set of such covers is cofinal [24, Chapter VIII, Corollary 3.16]. The nerves of such covers are disjoint unions of the corresponding nerves of the covers for X_1 and X_2 and thus the simplicial homology decomposes as a direct sum. Since inverse limits commute with direct sums, this property stays true also when taking the inverse limit and it follows that $\check{H}_k(X_1 \sqcup X_2, \{*_1\} \sqcup \{*_2\}) \cong \check{H}_k(X_1, \{*_1\}) \oplus \check{H}_k(X_2, \{*_2\})$. □

For nice enough pairs (X, A), Čech homology agrees with singular homology. Nice enough in this case means **triangulable**, i.e. there exists a simplicial pair (K, L) and a homeomorphism $h\colon (|K|, |L|) \to (X, A)$. We now state two results [24, Chapter IX, Theorem 9.3] and [24, Chapter VII, Theorem 10.1], which combined tell us that such a triangulation induces isomorphisms from the Čech homology of (X, A) to the simplicial homology of (K, L) and from there to the singular homology of (X, A).

Theorem 5.6. *A triangulation $h\colon (|K|, |L|) \to (X, A)$ induces a covering of (X, A) which yields isomorphisms*

$$\check{H}_q(X, A; G) \xrightarrow{\cong} H^{\Delta}_q(K, L; G)$$

between the Čech homology of (X, A) and the simplicial homology of (K, L) for any coefficient group G, for all q. These isomorphisms commute with the connecting homomorphisms for the long exact sequences of the pairs (K, L) and (X, A), respectively.

Theorem 5.7. *A triangulation $h\colon (|K|,|L|) \to (X,A)$ induces isomorphisms*

$$H_q^{\Delta}(K,L;G) \xrightarrow{\cong} H_q(X,A;G)$$

from the simplicial homology of (K,L) to the singular homology of (X,A) over any coefficient group G, for all q. These isomorphisms commute with the connecting homomorphisms for the long exact sequences of the pairs (K,L) and (X,A), respectively.

By combining these two results, we get that Čech homology is isomorphic to singular homology for triangulable pairs.

Corollary 5.8. *If (X,A) is a triangulable pair, then there is a natural isomorphism $H_*(X,A) \cong \check{H}_*(X,A)$, commuting with the connecting homomorphisms from the long exact sequences.*

Denote by $X^\star$ the **one-point compactification** of a space X. See either [24] or [42] for the precise definition. When considering $X^\star$ as a pointed space, we choose the newly added point $*$, often called the **point at infinity**, as the basepoint. We repeat the statement [24, Chapter X, Lemma 6.3], which gives a criterion to check when a space is homeomorphic to a one-point compactification.

Lemma 5.9. *Let X, Y be locally compact, $A \subseteq X$ a closed subset and $f\colon X \setminus A \to Y$ a proper map (i.e. inverse images of compact sets under f are compact). Define $\overline{f}\colon X \to Y^\star$ by setting $\overline{f}(x) = f(x)$ for $x \in X \setminus A$ and $f(a) = *$ for $a \in A$. Then, $\overline{f}$ is a continuous extension of f. If f is a homeomorphism, then $\overline{f}$ is a relative homeomorphism between the pairs (X,A) and $(Y^\star, \{*\})$. If, moreover, X is compact, and A is a single point, then $\overline{f}$ is a homeomorphism.*

Using this lemma, we can prove that for compact pairs (X,A), collapsing A to a point is the same as first removing A and then adding a point at infinity.

Proposition 5.10. *If (X,A) is a pair of compact Hausdorff spaces, then $X/A \cong (X \setminus A)^\star$.*

Proof. Let $f\colon X \setminus A \to X \setminus A$ be the identity map, where the domain is considered as a subspace of X/A and the target as a subspace of X. By tracing the definitions of the subset topology and the quotient topology, one can verify that the open sets are the same in both spaces, namely open subsets of X disjoint from A (since A is closed), and thus f is a homeomorphism. Since X is compact, also X/A is compact. Therefore, it follows from Lemma 5.9 that the map $\overline{f}\colon X/A \to (X \setminus A)^\star$, which is the identity on $X \setminus A$ and sends the point A to $*$, is a homeomorphism. □

A disadvantage of using Čech homology is that the long sequence of homology groups of a pair (X,A) may not be exact. By Corollary 5.8, the sequence is exact for a triangulable pair, but we are interested also in pairs that are not triangulable. Luckily, since we are working with closed sets of compact manifolds, we are only considering compact pairs. Thanks to this and to the fact that we are working with field coefficients, we do indeed get long exact sequences of pairs, as stated in [24, Chapter IX, Theorem 7.6], which we repeat here for convenience.

Theorem 5.11. *If (X,A) is a compact pair and G is a vector space over a field $\mathbb{F}$, then the long sequence in Čech homology with coefficients in G associated to the pair (X,A) is exact.*

5.2 Spectral sequences

We state some definitions and results from the theory of spectral sequences. We adapt the definitions to our needs, since we are only working with field coefficients. More details (and more generality) can be found in [56]. Consider also that there are some errors with the indices in [56, Section 5.9], see [57] for a list of corrections.

Definition 5.12. A **spectral sequence** starting at page a consists of the following data:

- vector spaces $E^r_{p,q}$ for all $r \geq a$ and $p, q \in \mathbb{Z}$,
- linear maps $d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-r,q+r-1}$ for all p, q, satisfying $d^r_{p-r,q+r-1} \circ d^r_{p,q} = 0$,
- isomorphisms between E^{r+1} and the homology of E^r, i.e.

$$E^{r+1}_{p,q} \cong \ker(d^r_{p,q}) / \operatorname{im}(d^r_{p+r,q-r+1}).$$

The **total degree** of a term $E^r_{p,q}$ in a spectral sequence is the number $p+q$. A spectral sequence is called **bounded** if for each n and r, there are only finitely many non-zero terms of total degree n in the r-th page, and **bounded below** if for every n there exists an integer $f(n)$ such that for $p < f(p+q)$ we have $E_{p,q} = 0$. Note that this implies that for every p, q there exists r_0 such that for all $r \geq r_0$ we have $E^r_{p,q} = E^{r+1}_{p,q}$. We denote this vector space by $E^\infty_{p,q}$. The collection of $E^\infty_{p,q}$ for all p, q is called the ∞**-page** of the spectral sequence.

Definition 5.13. A bounded spectral sequence is said to **converge to** H_*, where $H_* = (H_n)_{n\in\mathbb{Z}}$ is a sequence of vector spaces, if for each n there exists a finite filtration

$$0 = H^s_n \subseteq \cdots \subseteq H^{p-1}_n \subseteq H^p_n \subseteq H^{p+1}_n \subseteq \cdots \subseteq H^t_n = H_n$$

and isomorphisms $E^\infty_{p,q} \cong H^p_n / H^{p-1}_n$, where $n = p + q$. In symbolic notation, we write $E \implies H_*$.

Definition 5.14. An **exact couple** $\mathcal{E}$,

$$\begin{tikzcd} \mathcal{E}\colon & D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & & E, \arrow[ul, "k"] & \end{tikzcd}$$

consists of bigraded vector spaces $D = \bigoplus_{p,q} D_{p,q}$ and $E = \bigoplus_{p,q} E_{p,q}$ together with bigraded linear maps i, j, k with bidegrees $(1, -1), (1-r, r-1), (-1, 0)$, for some $r \geq 0$, satisfying $\ker(i) = \operatorname{im}(k)$, $\ker(j) = \operatorname{im}(i)$, $\ker(k) = \operatorname{im}(j)$.

An exact couple is called **bounded below** if for each n there exists an integer $f(n)$ such that for $p < f(p+q)$ we have $D_{p,q} = 0$. The next result is [56, Proposition 5.9.2].

Proposition 5.15. *An exact couple $\mathcal{E}$ in which i, j, and k have bidegrees $(1, -1)$, $(-a, a)$, and $(-1, 0)$ determines a spectral sequence $\{E^r_{p,q}\}_{r\geq a+1}$ starting at page $a+1$. A morphism of exact couples induces a morphism of the corresponding spectral sequences.*

The next result is [56, Classical Convergence Theorem 5.9.7].

Theorem 5.16. *If an exact couple is bounded below, then the spectral sequence is bounded below and converges to* $(H_n)_{n\in\mathbb{Z}}$*, where* $H_n = \lim\limits_{p\to\infty} D_{p,n-p}$ *is the direct limit along the maps* $i_{p,n-p}\colon D_{p,n-p} \to D_{p+1,n-p-1}$.

We now apply this to a finite filtration of a topological space, inducing a spectral sequence in Čech homology that converges to the Čech homology of the space. Note that additional to the spaces in the filtration being compact, we are making use of the fact that we are working with field coefficients, in order to guarantee the existence of the long exact sequences in Čech homology.

Proposition 5.17. *If* $\emptyset \subseteq L_0 \subseteq L_1 \subseteq \cdots \subseteq L_N = M$ *is a filtration of a compact topological space* M*, such that each* L_p *is compact, then this determines a spectral sequence* $\{E^r_{\bullet,\bullet}\}_{r\geq 1}$ *whose first page is given by* $E^1_{p,q} = \check{H}_{p+q}(L_p, L_{p-1})$ *and that converges to* $\check{H}_*(M)$.

Proof. Let $n := p+q$. We formally extend the filtration to an infinite filtration by defining $L_p := \emptyset$ for $p < 0$ and $L_p := M$ for $p > N$. Since we are using field coefficients and the spaces L_p are compact, by Theorem 5.11, for each pair (L_p, L_{p-1}) there exists a long exact sequence in Čech homology

$$\cdots \longrightarrow \check{H}_n(L_{p-1}) \xrightarrow{\ i\ } \check{H}_n(L_p) \xrightarrow{\ j\ } \check{H}_n(L_p, L_{p-1}) \xrightarrow{\ k\ } \check{H}_{n-1}(L_{p-1}) \longrightarrow \cdots,$$

where i is the map induced in Čech homology from the inclusion $L_{p-1} \hookrightarrow L_p$, j is the map induced in Čech homology from the inclusion of pairs $(L_p, \emptyset) \hookrightarrow (L_p, L_{p-1})$, and k is the connecting homomorphism from Čech homology. We define $D_{p,q} = \check{H}_{p+q}(L_p)$ and $E_{p,q} = \check{H}_{p+q}(L_p, L_{p-1})$ and endow i, j, k with indices such that

$$i_{p,q}\colon \underbrace{\check{H}_n(L_p)}_{=D_{p,q}} \longrightarrow \underbrace{\check{H}_n(L_{p+1})}_{=D_{p+1,q-1}}, \qquad j_{p,q}\colon \underbrace{\check{H}_n(L_p)}_{=D_{p,q}} \longrightarrow \underbrace{\check{H}_n(L_p, L_{p-1})}_{=E_{p,q}},$$

$$k_{p,q}\colon \underbrace{\check{H}_n(L_p, L_{p-1})}_{=E_{p,q}} \longrightarrow \underbrace{\check{H}_{n-1}(L_{p-1})}_{=D_{p-1,q}}.$$

Note that $D_{p,q}$ and $E_{p,q}$ are defined for all $p, q \in \mathbb{Z}$, but many of the terms are zero, namely $D_{p,q} = 0$ if $p < 0$ or $q < -p$, and $E_{p,q} = 0$ if $p < 0$ or $p > N$ or $q < -p$. We can put all of this information together, by defining the vector spaces $D = \bigoplus_{p,q} D_{p,q}$ and $E = \bigoplus_{p,q} E_{p,q}$ and the linear maps $i = \bigoplus_{p,q} i_{p,q}$, $j = \bigoplus_{p,q} j_{p,q}$ and $k = \bigoplus_{p,q} k_{p,q}$. We thus get an exact couple

$$\mathcal{E}\colon \quad \begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E, \arrow[ul, "k"] & \end{tikzcd}$$

where i, j, k have the bidegrees $(1,-1), (0,0), (-1,0)$, respectively. That this couple is exact follows immediately from the exactness of the long exact sequence in homology for each pair (L_p, L_{p-1}). By Proposition 5.15 we thus get a spectral sequence. Moreover, since this exact couple is bounded, we can apply Theorem 5.16, which tells us that the spectral sequence is bounded below and converges to the direct limit $\lim_{p\to\infty} D_{p,n-p}$. By the definition of the $D_{p,q}$, this is equal to $\check{H}_n(M)$. ☐

5.3 Linear algebra – canonical choice of basis

This preliminary section deals with the question of how to endow a vector space with a basis, in a way which is as canonical as possible. More precisely, we deal with the scenario, where we are given a vector space W and a subspace $V \subseteq W$. We assume the existence of an ordered basis of W and want to produce from this a basis either for V or for W/V.

Later, we will use this in the following way. When constructing a chain complex from the spectral sequence associated to a Morse-Smale vector field v, we will show that the first page of the spectral sequence can be endowed with bases that correspond to the singular elements of v in a canonical way. The chain complex that we construct in each degree consists of direct sums of either subspaces or quotients of the terms on the first page of the spectral sequence, so we apply the methods from this section to endow them each with a basis.

5.3.A Bases for subspaces

We make use of the reduced row echelon form to endow subspaces of a given vector space with canonical bases. A more detailed description of the row echelon form and algorithms for computing it can be found in many linear algebra textbooks, see e.g. [36]. For this method to work, an ordered basis for the total vector space needs to be fixed.

Definition 5.18. A matrix is in **reduced row echelon form** if it satisfies the following conditions:

(i) The leading entry in each nonzero row is 1.

(ii) All zero rows lie at the bottom of the matrix and the positions of the leading entries of the nonzero rows are strictly increasing.

(iii) Each column containing a leading 1 has zeroes in all its other entries.

Any matrix A can be put into reduced row echelon form by transforming Gauss-Jordan elimination on its rows. These operations do not change the subspace that is spanned by the rows of A and the resulting matrix is unique. Given two matrices, their resulting reduced row echolon forms agree if and only if the subspaces generated by their rows are the same.

Hence, if we are given a subspace V of a vector space W, and we have fixed an ordered basis of W, then we can choose an arbitrary set of generators for V, consider the matrix whose rows consist of these generators with respect to the basis of W, bring the matrix to reduced row echelon form and the rows of this matrix form a canonical ordered basis for V.

5.3.B Bases for quotient spaces

The method for endowing quotient spaces with a canonical basis is even simpler than the one for subspaces. Assume we are given a vector space W, with an ordered basis $\mathcal{B} = \{b_1, ..., b_n\}$, and a linear subspace V of W. We produce a complement U of V in W (i.e. $W = U \oplus V$), together with a basis, as follows:

Define U_0 to be the zero subspace. If U_i is defined already, define U_{i+1} as follows. If b_{i+1} lies in $U_i + V$, then $U_{i+1} = U_i$. Otherwise $U_{i+1} = \mathrm{Span}(U_i \cup \{b_{i+1}\})$. In the end we let $U = U_n$ and a basis for U is given by those b_k that were used to generate U.

This can be used to endow the quotient space W/V with a basis, since any complement U of V is canonically isomorphic to W/V via the map $u \mapsto u + V$.

5.4 The filtration induced by a Morse-Smale vector field

Let M be a closed smooth manifold of dimension m. Given a Morse-Smale vector field v on M, we can decompose M as a disjoint union of the unstable manifolds of the fixed points and closed orbits of v, see Section 1.3 for the definitions. The idea of this filtration is that in each step we add all those unstable manifolds that attach to what we have already added before.

To make this more precise, we first introduce the boundary of an unstable manifold. If β is either a fixed point or a closed orbit of index k, we denote by $\overline{W^u(\beta)}$ the closure of the unstable manifold of β in M. We define $\partial W^u(\beta) := \overline{W^u(\beta)} \setminus W^u(\beta)$. Note that this is equivalent to the definition given in [50]. Now let $L_{-1} := \emptyset$ and, for $i \geq 1$, let L_i be the union of all those $W^u(\beta)$, where β is a singular element of v, such that $\partial W^u(\beta) \subseteq L_{i-1}$. Note that all singular elements with an unstable manifold of dimension k (i.e. fixed points of index k or closed orbits of index $k-1$) are contained in L_k, but it is possible that they already appear in an earlier step of the filtration.

We will show how this filtration determines a spectral sequence, with the terms on the first page given by $E^1_{p,q} = \check{H}_{p+q}(L_p, L_{p-1})$. The direct sum of the entries along diagonals of the infinity page of this spectral sequence will give us the homology of M (since we are using field coefficients). By construction $L_m = M$, where m is the dimension of M, hence the $(m+1)$-th page is the infinity page.

5.4.A Preliminary homology computations

In this section we compute the relative Čech homology groups of the unstable manifolds of fixed points and closed orbits with respect to their boundaries. This will be helpful in the next section for endowing the relative Čech homology of the pairs coming from the filtration. We start by stating [50, Lemma 3.8].

Lemma 5.19. *If β is a fixed point of index $k \geq 0$, then $W^u(\beta) \cong \mathbb{R}^k$. If β is a closed orbit of index $k \geq 0$, then $W^u(\beta) \cong \mathbb{R}^k \times S^1$.*

We first treat the easier case of the unstable manifold of a fixed point.

Corollary 5.20. *If β is a fixed point of index $k \geq 0$, then*

$$\check{H}_r(\overline{W^u(\beta)}, \partial W^u(\beta)) \cong \begin{cases} \mathbb{F}, & \textit{if r=k}, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. This follows from

$$\check{H}_r(\overline{W^u(\beta)}, \partial W^u(\beta)) \cong \check{H}_r(\overline{W^u(\beta)}/\partial W^u(\beta), *) \cong \check{H}_r(W^u(\beta)^\star, *) \cong \check{H}_r(S^k, *) \cong H_r(S^k, *).$$

The first and second isomorphisms follow from Theorem 5.4 together with Corollary 5.3 and Proposition 5.10, respectively. Note that $\overline{W^u(\beta)}$ is compact as it is a closed subset of M, which is compact. The third isomorphism holds by Lemma 5.19 and because $(\mathbb{R}^k)^\star \cong S^k$. The fourth isomorphism follows from Corollary 5.8. ☐

Now we want to do the analogous computation for the unstable manifold of a closed orbit. This is more difficult because the one-point compactification of $\mathbb{R}^k \times S^1$ is a more complicated space. We therefore use a different compactification, namely $D^k \times S^1$, where D^k denotes the closed unit disk of dimension k. The ”boundary” of this compactification is given by $S^{k-1} \times S^1$, so eventually we want to compute the relative homology of the pair $(D^k \times S^1, S^{k-1} \times S^1)$. We start by computing the homology of $S^{k-1} \times S^1$.

Lemma 5.21. *The following isomorphisms hold:*

(i) $H_r(S^0 \times S^1) \cong \begin{cases} \mathbb{F} \oplus \mathbb{F}, & \textit{if } r = 0, 1, \\ 0, & \textit{otherwise.} \end{cases}$

(ii) $H_r(S^1 \times S^1) \cong \begin{cases} \mathbb{F}, & \textit{if } r = 0, 2, \\ \mathbb{F} \oplus \mathbb{F}, & \textit{if } r = 1, \\ 0, & \textit{otherwise.} \end{cases}$

(iii) For $k \geq 3$*, we have* $H_r(S^{k-1} \times S^1) \cong \begin{cases} \mathbb{F}, & \textit{if } r = 0, 1, k-1, k, \\ 0, & \textit{otherwise.} \end{cases}$

Proof. For any $k \geq 1$, the homology of the product $S^{k-1} \times S^1$ can be computed by using the Künneth formula, yielding

$$H_r(S^{k-1} \times S^1) \cong \bigoplus_{i+j=r} H_i(S^{k-1}) \otimes H_j(S^1).$$

From this, the result follows using the well-known homology groups of the spheres. □

Now we are ready to compute the relative homology of the pair $(D^k \times S^1, S^{k-1} \times S^1)$. Note that here we are using singular homology, but we could also use Čech homology instead, since these pairs are triangulable and thus the two are isomorphic.

Lemma 5.22. *For any* $k \geq 1$*, we have*

$$H_r(D^k \times S^1, S^{k-1} \times S^1) \cong \begin{cases} \mathbb{F}, & \textit{if } r = k, k+1, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. We use the long exact sequence in homology of the pair $(D^k \times S^1, S^{k-1} \times S^1)$. We start with the case $k \geq 3$. By Lemma 5.21, the homology of $S^{k-1} \times S^1$ is non-zero only in degrees $0, 1, k-1, k$, while $D^k \times S^1$ is homotopy equivalent to S^1 and thus has non-zero homology only in degrees $0, 1$. In these non-zero degrees the homology is always isomorphic to $\mathbb{F}$. One can check that the embedding $S^{k-1} \times S^1 \hookrightarrow D^k \times S^1$ induces isomorphisms in homology in degrees 0 and 1. Thus the relevant parts of the long exact sequence are as follows:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & H_{k+1}(D^k \times S^1, S^{k-1} \times S^1) \arrow[dll] \\
\underbrace{H_k(S^{k-1} \times S^1)}_{\cong \mathbb{F}} \arrow[r] & 0 \arrow[r] & H_k(D^k \times S^1, S^{k-1} \times S^1) \arrow[dll] \\
\underbrace{H_{k-1}(S^{k-1} \times S^1)}_{\cong \mathbb{F}} \arrow[r] & 0 \arrow[r] & H_{k-1}(D^k \times S^1, S^{k-1} \times S^1) \arrow[dl] \\
& \cdots \arrow[dl] & \\
\underbrace{H_1(S^{k-1} \times S^1)}_{\cong \mathbb{F}} \arrow[r, "\cong"] & \underbrace{H_1(D^k \times S^1)}_{\cong \mathbb{F}} \arrow[r] & H_1(D^k \times S^1, S^{k-1} \times S^1) \arrow[dll] \\
\underbrace{H_0(S^{k-1} \times S^1)}_{\cong \mathbb{F}} \arrow[r, "\cong"] & \underbrace{H_0(D^k \times S^1)}_{\cong \mathbb{F}} \arrow[r] & H_0(D^k \times S^1, S^{k-1} \times S^1) \arrow[dll] \\
0 & &
\end{tikzcd}$$

It now follows from exactness of the sequence that $H_r(D^k \times S^1, S^{k-1} \times S^1) = 0$ for $r \neq k, k+1$ and also that the connecting homomorphism in degrees $k+1$ and k is an isomorphism, hence completing the proof for the case $k \geq 3$.

The cases $k = 1$ and $k = 2$ can be done analogously, only that the two non-zero parts of the long exact sequence overlap. □

The computations we just did are useful because of the following result, stating that there exists a relative homeomorphism between the pair $(D^k \times S^1, S^{k-1} \times S^1)$ and the the unstable manifold of a closed orbit with its boundary collapsed to a point.

Proposition 5.23. *If β is a closed orbit of index $k \geq 1$, then there exists a relative homeomorphism from $(D^k \times S^1, S^{k-1} \times S^1)$ to $(\overline{W^u(\beta)}/\partial W^u(\beta), *)$.*

Proof. Denote by $f\colon \mathbb{R}^k \times S^1 \to W^u(\beta)$ the diffeomorphism from Lemma 5.19 (the definition can be found in [50, Section 2.2]). Identifying the interior of D^k with $\mathbb{R}^k$ via some fixed homeomorphism, we can view f as a map from $\mathrm{Int}(D^k) \times S^1$ to $W^u(\beta)$. By Lemma 5.9, we can extend this to a relative homeomorphism $\overline{f}\colon (D^k \times S^1, S^{k-1} \times S^1) \to (W^u(\beta)^\star, *)$. Composing this with the homeomorphism $W^u(\beta)^\star \cong \overline{W^u(\beta)}/\partial W^u(\beta)$ from Proposition 5.10, the result follows. □

Now we are ready to prove the analogous result to Corollary 5.20 also for closed orbits.

Corollary 5.24. *If β is a closed orbit of index $k \geq 0$, then*

$$\check{H}_r(\overline{W^u(\beta)}, \partial W^u(\beta)) \cong \begin{cases} \mathbb{F}, & \text{if } r{=}k{+}1,\ k, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. If the index of β is $k = 0$, then $\overline{W^u(\beta)} = W^u(\beta) = \beta \cong S^1$ and $\partial W^u(\beta) = \emptyset$. In that case, the result is clear. For $k \geq 1$, we have

$$\begin{aligned}\check{H}_r(\overline{W^u(\beta)}, \partial W^u(\beta)) \cong \check{H}_r(\overline{W^u(\beta)}/\partial W^u(\beta), *) &\cong \check{H}_r(D^k \times S^1, S^{k-1} \times S^1) \\ &\cong H_r(D^k \times S^1, S^{k-1} \times S^1).\end{aligned}$$

The first isomorphism follows from Theorem 5.4 and Corollary 5.3. The second isomorphism follows from Theorem 5.4 and Proposition 5.23. The third isomorphism follows from Corollary 5.8, since these are triangulable spaces. The result then follows from Lemma 5.22. ☐

5.4.B Giving canonical bases for relative Čech homology

Recall that $\mathrm{Fix}_k(v)$ is the set of fixed points of index k and $\mathrm{Orb}_k(v)$ is the set of closed orbits of index k. Recall further that in the beginning of Section 5.4 we described a filtration $\emptyset = L_{-1} \subseteq L_0 \subseteq \cdots \subseteq L_m = M$ in terms of unstable manifolds of the singular elements of v. Inspired by the proof of [50, Lemma 5.1] and making use of the computations done in Section 5.4.A, we show that one can get canonical bases for the relative Čech homology groups of the pairs (L_p, L_{p-1}), where each fixed point contributes one basis element and each periodic orbit contributes two. This is useful for us since these will be the terms appearing on the first page of our spectral sequence.

Given $\beta \in \mathrm{Fix}(v)$, we denote by $\mathbb{F}\langle\beta\rangle$ the vector space of formal $\mathbb{F}$-linear combinations of β. If $\beta \in \mathrm{Orb}(v)$ is a closed orbit, we formally create two copies β^+ and β^-, and denote by $\mathbb{F}\langle\beta^+\rangle$ and $\mathbb{F}\langle\beta^-\rangle$ the vector spaces of their formal $\mathbb{F}$-linear combinations, respectively.

Proposition 5.25. *For every $p, r \geq 0$, we have*

$$\check{H}_r(L_p, L_{p-1}) \cong \bigoplus_{\substack{\beta\in\mathrm{Fix}_r(v)\\ \beta\in L_p\setminus L_{p-1}}} \mathbb{F}\langle\beta\rangle \oplus \bigoplus_{\substack{\beta\in\mathrm{Orb}_{r-1}(v)\\ \beta\subseteq L_p\setminus L_{p-1}}} \mathbb{F}\langle\beta^+\rangle \oplus \bigoplus_{\substack{\beta\in\mathrm{Orb}_{r}(v)\\ \beta\subseteq L_p\setminus L_{p-1}}} \mathbb{F}\langle\beta^-\rangle.$$

Proof. For $p = 0$ we have $L_p = \mathrm{Fix}_0(v) \sqcup \mathrm{Orb}_0(v)$. Thus,

$$\begin{aligned}\check{H}_r(L_p, L_{p-1}) = \check{H}_r(L_0, \emptyset) &= \check{H}_r(\mathrm{Fix}_0(v) \sqcup \mathrm{Orb}_0(v)) \\ &= \bigoplus_{\beta\in\mathrm{Fix}_0(v)} \check{H}_r(\{\beta\}) \oplus \bigoplus_{\beta\in\mathrm{Orb}_0(v)} \check{H}_r(\beta).\end{aligned}$$

Since $\beta \in \mathrm{Fix}_0(v)$ is a point and $\beta \in \mathrm{Orb}_0(v)$ is homeomorphic to a circle, the result in the case $p = 0$ follows. For $p \geq 1$, we have

$$\begin{aligned}
\check{H}_r(L_p, L_{p-1}) &\cong \check{H}_r(L_p/L_{p-1}, *) \\
&\cong \check{H}_r\left(\bigvee_{\substack{\beta\in\mathrm{Fix}(v)\\ \beta\in L_p\setminus L_{p-1}}} \overline{W^u}(\beta)/\partial W^u(\beta) \vee \bigvee_{\substack{\beta\in\mathrm{Orb}(v)\\ \beta\subseteq L_p\setminus L_{p-1}}} \overline{W^u}(\beta)/\partial W^u(\beta), * \right) \\
&\cong \bigoplus_{\substack{\beta\in\mathrm{Fix}(v)\\ \beta\in L_p\setminus L_{p-1}}} \check{H}_r\left(\overline{W^u}(\beta)/\partial W^u(\beta), *\right) \oplus \bigoplus_{\substack{\beta\in\mathrm{Orb}(v)\\ \beta\subseteq L_p\setminus L_{p-1}}} \check{H}_r\left(\overline{W^u}(\beta)/\partial W^u(\beta), *\right) \\
&\cong \bigoplus_{\substack{\beta\in\mathrm{Fix}_r(v)\\ \beta\in L_p\setminus L_{p-1}}} \mathbb{F}\langle\beta\rangle \oplus \bigoplus_{\substack{\beta\in\mathrm{Orb}_{r-1}(v)\\ \beta\subseteq L_p\setminus L_{p-1}}} \mathbb{F}\langle\beta^+\rangle \oplus \bigoplus_{\substack{\beta\in\mathrm{Orb}_r(v)\\ \beta\subseteq L_p\setminus L_{p-1}}} \mathbb{F}\langle\beta^-\rangle.
\end{aligned}$$

The first isomorphism follows from Theorem 5.4 and Corollary 5.3. The second one follows from the definition of the L_p, since $L_p \setminus L_{p-1}$ is the disjoint union of all those unstable manifolds that are not contained in L_{p-1} but whose boundary is. The third one follows from Proposition 5.5. The fourth isomorphism follows from Corollaries 5.20 and 5.24. ☐

5.5 The chain complex assigned to a spectral sequence

This section is purely algebraic. Let us consider a 3×3 spectral sequence $E = \{E^r_{\bullet,\bullet}\}_{r\geq 1}$, i.e. a spectral sequence that is zero everywhere outside of a 3×3 grid. This algebraic setup is motivated by the case where we are considering a Morse-Smale vector field on a surface, see Section 5.6. Given such a spectral sequence, we describe a method to define a chain complex.

We start by displaying the first and second pages, respectively. We do not write those differentials that are obviously zero because they start or end outside the 3×3 grid:

$$\begin{tikzcd}
E^1_{0,1} & E^1_{1,1} \arrow[l, "d^1_{1,1}"'] & E^1_{2,1} \arrow[l, "d^1_{2,1}"'] \\
E^1_{0,0} & E^1_{1,0} \arrow[l, "d^1_{1,0}"'] & E^1_{2,0} \arrow[l, "d^1_{2,0}"'] \\
E^1_{0,-1} & E^1_{1,-1} \arrow[l, "d^1_{1,-1}"'] & E^1_{2,-1} \arrow[l, "d^1_{2,-1}"']
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
E^2_{0,1} & E^2_{1,1} & E^2_{2,1} \\
E^2_{0,0} & E^2_{1,0} & E^2_{2,0} \arrow[ull, "d^2_{2,0}" description] \\
E^2_{0,-1} & E^2_{1,-1} & E^2_{2,-1} \arrow[ull, "d^2_{2,-1}" description]
\end{tikzcd}$$

Writing the vector spaces appearing on the second page in terms of the differentials from the first page, we get

$$E^2_{0,q} = E^1_{0,q}/\operatorname{im}(d^1_{1,q}), \qquad E^2_{1,q} = \ker(d^1_{1,q})/\operatorname{im}(d^1_{2,q}), \qquad E^2_{2,q} = \ker(d^1_{2,q}),$$

where $q = 1, 0, -1$.

We want to decompose some of the entries of the first page so that they contain the entries of the second page as subspaces and hence we can define the differential of the second page on the first page already. This yields

$$\begin{tikzcd}
E^1_{0,1}/\operatorname{im}(d^1_{1,1}) \oplus \operatorname{im}(d^1_{1,1}) & E^1_{1,1} \arrow[l, "\hat{d}^1_{1,1}"'] & E^1_{2,1} \arrow[l, "d^1_{2,1}"'] \\
E^1_{0,0}/\operatorname{im}(d^1_{1,0}) \oplus \operatorname{im}(d^1_{1,0}) & E^1_{1,0} \arrow[l, "\hat{d}^1_{1,0}"'] & \ker(d^1_{2,0}) \oplus E^1_{2,0}/\ker(d^1_{2,0}) \arrow[l, "\hat{d}^1_{2,0}"'] \arrow[ull, "\hat{d}^2_{2,0}"'] \\
E^1_{0,-1} & E^1_{1,-1} \arrow[l, "d^1_{1,-1}"'] & \ker(d^1_{2,-1}) \oplus E^1_{2,-1}/\ker(d^1_{2,-1}), \arrow[l, "\hat{d}^1_{2,-1}"'] \arrow[ull, "\hat{d}^2_{2,-1}"']
\end{tikzcd}$$

where the maps denoted by $\hat{d}^1_{p,q}$ and $\hat{d}^2_{p,q}$ are defined by

$$\hat{d}^1_{1,1} = \begin{bmatrix} 0 \\ d^1_{1,1} \end{bmatrix}, \qquad \hat{d}^2_{2,0} = \begin{bmatrix} d^2_{2,0} & 0 \\ 0 & 0 \end{bmatrix}, \qquad \hat{d}^1_{1,0} = \begin{bmatrix} 0 \\ d^1_{1,0} \end{bmatrix},$$

$$\hat{d}^1_{2,0} = \begin{bmatrix} 0 & \overline{d}^1_{2,0} \end{bmatrix}, \qquad \hat{d}^2_{2,-1} = \begin{bmatrix} d^2_{2,-1} & 0 \\ 0 & 0 \end{bmatrix}, \qquad \hat{d}^1_{2,-1} = \begin{bmatrix} 0 & \overline{d}^1_{2,-1} \end{bmatrix}.$$

The bar sign denotes the map induced on the quotient. Explicitly, the map

$$\overline{d}^1_{2,0} \colon E^1_{2,0}/\ker(d^1_{2,0}) \longrightarrow E^1_{1,0}$$

is defined by $\overline{d}^1_{2,0}([a]) := d^1_{2,0}(a)$ and the map

$$\overline{d}^1_{2,-1} \colon E^1_{2,-1}/\ker(d^1_{2,-1}) \longrightarrow E^1_{1,-1}$$

is defined by $\overline{d}^1_{2,-1}([a]) := d^1_{2,-1}(a)$. It thus follows that

$$d^1_{1,0} \circ \overline{d}^1_{2,0} = 0 \quad \text{and} \quad d^1_{1,-1} \circ \overline{d}^1_{2,-1} = 0. \tag{5.1}$$

By taking direct sums along the diagonals, we can obtain a chain complex which we denote by $C_\bullet = C_\bullet(E)$. This chain complex can be non-zero only in degrees between -1 and 3. Explicitly, it looks as follows:

$$\begin{tikzcd}
C_3 = E^1_{2,1} \arrow[d, "{\begin{bmatrix} d^1_{2,1} \\ 0 \\ 0 \end{bmatrix}}"] \\
C_2 = E^1_{1,1} \oplus \ker(d^1_{2,0}) \oplus E^1_{2,0}/\ker(d^1_{2,0}) \arrow[d, "{\begin{bmatrix} 0 & d^2_{2,0} & 0 \\ d^1_{1,1} & 0 & 0 \\ 0 & 0 & \overline{d}^1_{2,0} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}"] \\
C_1 = E^1_{0,1}/\operatorname{im}(d^1_{1,1}) \oplus \operatorname{im}(d^1_{1,1}) \oplus E^1_{1,0} \oplus \ker(d^1_{2,-1}) \oplus E^1_{2,-1}/\ker(d^1_{2,-1}) \arrow[d, "{\begin{bmatrix} 0 & 0 & 0 & d^2_{2,-1} & 0 \\ 0 & 0 & d^1_{1,0} & 0 & 0 \\ 0 & 0 & 0 & 0 & \overline{d}^1_{2,-1} \end{bmatrix}}"] \\
C_0 = E^1_{0,0}/\operatorname{im}(d^1_{1,0}) \oplus \operatorname{im}(d^1_{1,0}) \oplus E^1_{1,-1} \arrow[d, "{\begin{bmatrix} 0 & 0 & d^1_{1,-1} \end{bmatrix}}"] \\
E^1_{0,-1}.
\end{tikzcd}$$

Using matrix multiplication, together with $(d^1)^2 = 0$, $(d^2)^2 = 0$, and Equation (5.1), we see that the differential squares to zero and thus $C_\bullet$ is indeed a chain complex.

Proposition 5.26. *If E is a 3×3 spectral sequence that converges to H_*, then the homology of the associated chain complex is isomorphic to H_*.*

Proof. Since the spectral sequence E is 3×3, the third page is already the ∞-page, i.e. $E^\infty_{p,q} = E^3_{p,q}$ for all p, q. Some terms of the ∞-page are even the same as on the second page, more precisely we have $E^\infty_{p,q} = E^2_{p,q}$ for $(p,q) = (1,1), (2,1), (1,0), (0,-1), (1,-1)$. In conclusion, the ∞-page looks as follows,

$$\begin{array}{ccc}
E^2_{0,1}/\operatorname{im}(d^2_{2,0}) & \ker(d^1_{1,1})/\operatorname{im}(d^1_{2,1}) & \ker(d^1_{2,1}) \\
\\
E^2_{0,0}/\operatorname{im}(d^2_{2,-1}) & \ker(d^1_{1,0})/\operatorname{im}(d^1_{2,0}) & \ker(d^2_{2,0}) \\
\\
E^1_{0,-1}/\operatorname{im}(d^1_{1,-1}) & \ker(d^1_{1,-1})/\operatorname{im}(d^1_{2,-1}) & \ker(d^2_{2,-1}).
\end{array}$$

Denoting by $C_\bullet$ the chain complex constructed from E, a direct computation shows that it has the following homology:

$$\begin{aligned}
H_3(C_\bullet) &= \ker(d^1_{2,1}),\\
H_2(C_\bullet) &= \ker(d^1_{1,1})/\operatorname{im}(d^1_{2,1}) \oplus \ker(d^2_{2,0}),\\
H_1(C_\bullet) &= E^2_{0,1}/\operatorname{im}(d^2_{2,0}) \oplus \ker(d^1_{1,0})/\operatorname{im}(d^1_{2,0}) \oplus \ker(d^2_{2,-1}),\\
H_0(C_\bullet) &= E^2_{0,0}/\operatorname{im}(d^2_{2,-1}) \oplus \ker(d^1_{1,-1})/\operatorname{im}(d^1_{2,-1}),\\
H_{-1}(C_\bullet) &= E^1_{0,-1}/\operatorname{im}(d^1_{1,-1}).
\end{aligned}$$

These are exactly the direct sums along the diagonals of E^∞, i.e. $H_n(C_\bullet) \cong \bigoplus_{p+q=n} E^\infty_{p,q}$. Since $E \implies H_*$, there exist filtrations of each H_n by subspaces H^p_n such that $E^\infty_{p,q} \cong H^p_n/H^{p-1}_n$. Since we are working with field coefficients, this implies the existence of (non-canonical) isomorphisms $H_n \cong \bigoplus_{p+q=n} E^\infty_{p,q}$, and thus $H_n \cong H_n(C_\bullet)$. □

5.5.A Endowing the chain complex with bases

Note that the vector spaces C_n appearing in the chain complex $C_\bullet$ constructed from a 3×3 spectral sequence E consist of the direct sums of the terms from the first page of E, some of these terms being further decomposed as the direct sum of a subspace (either a kernel or an image of d^1) plus the corresponding quotient space. In other words, there exist (non-canonical) isomorphisms

$$C_n \cong \bigoplus_{p+q=n} E^1_{p,q}. \tag{5.2}$$

Assume now that we are given an ordered basis $\mathcal{B}_{p,q}$ of $E^1_{p,q}$, for all p, q. Then, by the isomorphisms from Equation (5.2), we can turn $C_\bullet$ into a based chain complex, but not in a canonical way. For some applications this yields no problem. See for example Section 5.6.B, where we reprove the Morse inequalities from [50] by using only the dimensions of the C_n.

For some other applications, especially with a view towards real-world applications, we may want to write down the chain complex explicitly by fixing an explicit basis and expressing the differentials as matrices with respect to these bases. Since there is no a priori canonical way to do this, we describe a pipeline to use the methods from section 5.3 to at least do it in a well-defined way. More precisely, according to Sections 5.3.A and 5.3.B, the ordered bases $\mathcal{B}_{p,q}$ induce bases of all subspaces and quotients of terms of the first page of the spectral sequence. Since the chain complex $C_\bullet$ constructed from E in each degree consists of direct sums of either terms from E^1, or subspaces or quotients thereof, this means that we get an ordered basis for C_k for each k.

5.6 The case of vector fields on surfaces

Let v be a Morse-Smale vector field on a closed smooth manifold M. We want to assign a chain complex $C_\bullet = C_\bullet(v)$ to v according to what we have described in these previous sections.

5.6.A The chain complex assigned to a 2D Morse-Smale vector field

Consider the filtration $\emptyset \subseteq L_0 \subseteq L_1 \subseteq L_2 = M$ described in Section 5.4. By Proposition 5.17, this induces a spectral sequence $E = E(v)$, with $E^1_{p,q} = \check{H}_{p+q}(L_p, L_{p-1})$ and $E \implies \check{H}_*(M)$. Since M is a manifold and thus triangulable, we have $\check{H}_*(M) \cong H_*(M)$. By Proposition 5.25, we get can endow the terms on the first page of E with bases given by the fixed points and closed orbits of v. In particular, this implies many zero terms, as we show in the following proposition.

Proposition 5.27. *If* $\dim(M) = 2$ *and* $\emptyset = L_{-1} \subseteq L_0 \subseteq L_1 \subseteq L_2 = M$ *is the filtration defined with respect to a Morse-Smale vector field* v *as described above, then the following holds:*

(i) $\check{H}_i(L_0, \emptyset) = 0$ *for* $i \geq 2$*,*

(ii) $\check{H}_i(L_1, L_0) = \check{H}_i(L_2, L_1) = 0$ *for* $i \geq 3$*,*

(iii) $\check{H}_0(L_1, L_0) = \check{H}_0(L_2, L_1) = 0$*.*

Proof. For (i), note that L_0 is a disjoint union of finitely many points and circles, in particular it is triangulable, so Čech homology agrees with singular homology by Corollary 5.8 and is potentially non-zero only in degrees zero and one.

On a two-dimensional manifold, the index of a fixed point can be at most 2 and the index of a closed orbit can be at most 1, i.e. $\mathrm{Fix}_r(v) = \emptyset$ for $r \geq 3$ and $\mathrm{Orb}_r(v) = \emptyset$ for $r \geq 2$. Thus it follows from Proposition 5.25 that $\check{H}_i(L_1, L_0) = \check{H}_i(L_2, L_1) = 0$ for $i \geq 3$. This proves (ii).

Note that $\check{H}_0(L_p, L_{p-1})$ is generated by fixed points of index zero and closed orbits of index zero that are present in L_p but not in L_{p-1}. But if β is a fixed point or closed orbit of index zero, then $\partial W^u(\beta) = \emptyset$ and thus $W^u(\beta) \subseteq L_0$. From this, (iii) follows. □

As a consequence, $E = E(v)$ is a 3×3 spectral sequence, with $E^1_{p,q} = \check{H}_{p+q}(L_p, L_{p-1})$, so we are in the setting from Section 5.5. In fact, even some terms inside the 3×3 grid are zero. Explicitly, the first page of E (where we do not write the zero terms) looks as follows:

$$\begin{tikzcd}
\check{H}_1(L_0,\emptyset) & \check{H}_2(L_1,L_0) \arrow[l] & \\
\check{H}_0(L_0,\emptyset) & \check{H}_1(L_1,L_0) \arrow[l] & \check{H}_2(L_2,L_1) \arrow[l] \\
 & & \check{H}_1(L_2,L_1).
\end{tikzcd}$$

The chain complex $C_\bullet = C_\bullet(v) := C_\bullet(E(v))$ assigned to v, as described in Section 5.5, in this case simplifies to

$$\begin{tikzcd}
C_2 = E^1_{1,1} \oplus \ker(d^1_{2,0}) \oplus E^1_{2,0}/\ker(d^1_{2,0}) \arrow[d, "\begin{bmatrix} 0 & d^2_{2,0} & 0 \\ d^1_{1,1} & 0 & 0 \\ 0 & 0 & \overline{d}^1_{2,0} \\ 0 & 0 & 0 \end{bmatrix}"] \\
C_1 = E^1_{0,1}/\operatorname{im}(d^1_{1,1}) \oplus \operatorname{im}(d^1_{1,1}) \oplus E^1_{1,0} \oplus E^1_{2,-1} \arrow[d, "\begin{bmatrix} 0 & 0 & 0 & d^2_{2,-1} \\ 0 & 0 & d^1_{1,0} & 0 \end{bmatrix}"] \\
C_0 = E^1_{0,0}/\operatorname{im}(d^1_{1,0}) \oplus \operatorname{im}(d^1_{1,0}).
\end{tikzcd}$$

In conclusion, we have constructed a chain complex from a Morse-Smale vector field, such that the homology of the chain complex agrees with the homology of the underlying manifold, and the vector spaces in the chain complex are generated by the singular elements of the vector field. We summarize this in the following theorem.

Theorem 5.28. *If v is a Morse-Smale vector field on a closed smooth 2-dimensional manifold M, then the chain complex $C_\bullet = C_\bullet(v)$ has the following properties:*

(i) $H_*(C_\bullet) \cong H_*(M)$,

(ii) $\dim(C_k) = |\operatorname{Fix}_k(v)| + |\operatorname{Orb}_{k-1}(v)| + |\operatorname{Orb}_k(v)|$ *for* $k = 0, 1, 2$.

Proof. By Proposition 5.17, the spectral sequence E constructed from the filtration $\emptyset \subseteq L_0 \subseteq L_1 \subseteq L_2 = M$ converges to $\check{H}_*(M)$, which equals $H_*(M)$, since M is a manifold and thus triangulable. By Proposition 5.26, it follows that $H_*(C_\bullet) \cong H_*(M)$.

It remains to show the formula for the dimensions. Let us introduce additional notation to further decompose the sets of fixed points and closed orbits, according to the time at which they enter the filtration of M by the sets L_p. More precisely, for $r = 0, 1, 2$, we have $\operatorname{Fix}_r(v) = \operatorname{Fix}^0_r(v) \sqcup \operatorname{Fix}^1_r(v) \sqcup \operatorname{Fix}^2_r(v)$, and for $r = 0, 1$, we have $\operatorname{Orb}_r(v) = \operatorname{Orb}^0_r(v) \sqcup \operatorname{Orb}^1_r(v) \sqcup \operatorname{Orb}^2_r(v)$, where

$$\begin{aligned}
\operatorname{Fix}^p_r(v) &:= \{\beta \in \operatorname{Fix}_r(v) \mid \beta \in L_p \setminus L_{p-1}\}, \\
\operatorname{Orb}^p_r(v) &:= \{\beta \in \operatorname{Orb}_r(v) \mid \beta \subseteq L_p \setminus L_{p-1}\}.
\end{aligned}$$

We can now compute the dimension of C_k, namely

$$\begin{aligned}
\dim(C_k) = \sum_{p+q=k} \dim(E^1_{p,q}) &= \sum_{p+q=k} \dim(\check{H}_k(L_p, L_{p-1})) \\
&= \sum_{p+q=k} \dim(|\operatorname{Fix}^p_k(v)| + |\operatorname{Orb}^p_{k-1}(v)| + |\operatorname{Orb}^p_k(v)|) \\
&= |\operatorname{Fix}_k(v)| + |\operatorname{Orb}_{k-1}(v)| + |\operatorname{Orb}_k(v)|,
\end{aligned}$$

where the first equality follows from Equation (5.2), the second by the construction of E, the third from Proposition 5.25, and the fourth from realizing that the sum over $p+q = k$ is equivalent to simply a sum over all p and then reordering the terms. ☐

We now prove that the isomorphism type of the chain complex assigned to a Morse-Smale vector field is invariant under topological equivalence.

Proposition 5.29. *If v and w are topologically equivalent Morse-Smale vector fields on a closed smooth manifold M, then the chain complexes $C_\bullet(v)$ and $C_\bullet(w)$ are isomorphic.*

Proof. Let $\emptyset \subseteq L_0 \subseteq L_1 \subseteq L_2 = M$ be the filtration associated with v and let $\emptyset \subseteq L'_0 \subseteq L'_1 \subseteq L'_2 = M$ be the filtration associated with w, as described in Section 5.4. Denote by $h\colon M \to M$ the topological equivalence between v and w. This means that h is a homeomorphism that maps fixed points of v to fixed points of w and closed orbits of v to closed orbits of w, preserving the indices. Thus h restricts to homeomorphisms $L_0 \cong L'_0$, $L_1 \cong L'_1$, $L_2 \cong L'_2$. It follows that we have an isomorphism of spectral sequences $E(v) \cong E(w)$. Since the construction of $C_\bullet(v)$ from $E(v)$ and of $C_\bullet(w)$ from $E(w)$ is purely algebraic and follows the same step, this induces an isomorphism of chain complexes $C_\bullet(v) \cong C_\bullet(w)$. □

5.6.B The Morse inequalities

We now want to use the chain complex $C_\bullet = C_\bullet(v)$ assigned to a Morse-Smale vector field v on a surface M in order to derive the strong Morse inequalities from [50]. Let us first recall the algebraic Morse inequalities, namely for any compact chain complex $C_\bullet \in \mathrm{Ch}$, and any $q \geq 0$, we have

$$\sum_{i=0}^{q}(-1)^{q+i}\dim(C_i) \geq \sum_{i=0}^{q}(-1)^{q+i}\dim(H_i(C_\bullet)). \tag{5.3}$$

These inequalities can be shown by standard linear algebra arguments, doing an induction over q. Moreover, if q is the maximal value for which $C_q \neq 0$, then the inequality from (5.3) becomes an equality. The value on both sides of the equation in that case is the Euler characteristic. Given these algebraic Morse inequalities, we can derive the Morse inequalities from [50, Theorem 1.1], using the mere existence of a chain complex as in Theorem 5.28.

Corollary 5.30. *If v is a Morse-Smale vector field on a surface M, for $q \geq 0$, write $R_q := \dim(H_q(M))$ and $M_q := |\operatorname{Fix}_q(v)| + |\operatorname{Orb}_{q-1}(v)| + |\operatorname{Orb}_q(v)|$. Then, the following inequalities hold:*

$$\sum_{i=0}^{q}(-1)^{q+i}M_i \geq \sum_{i=0}^{q}(-1)^{q+i}R_i.$$

Proof. From Theorem 5.28 and Equation (5.3) it follows that

$$\sum_{i=0}^{q}(-1)^{q+i}M_i = \sum_{i=0}^{q}(-1)^{q+i}\dim(C_i) \geq \sum_{i=0}^{q}(-1)^{q+i}\dim(H_i(C_\bullet)) = \sum_{i=0}^{q}(-1)^{q+i}R_i. \qquad \square$$

5.6.C Matrix representations for the boundary operators

As mentioned in Section 5.5.A, for some applications it may be useful to have a based chain complex, in order to write it down explicitly, representing the differentials as matrices. The terms of the first page of the spectral sequence E can be endowed with canonical bases according to Proposition 5.25. The chain complex constructed in Section 5.6 is made up of subspaces and quotients from the first page of the spectral sequence, so in principle we can apply the methods from Section 5.3. The only obstacle is that these methods need *ordered* bases to work. We thus start by presenting a method for ordering the singular elements of v of each index with respect to each other.

In this section we assume that M is additionally endowed with a Riemannian metric, so that we can talk about distances via the shortest path distance, denoted by d_M. Given two subsets $\beta, \beta' \subseteq M$, we define $D_M(\beta, \beta') := \inf_{x \in \beta, y \in \beta'} d_M(x, y)$. Note that, if β and β' are both compact and disjoint, then $D_M(A, B) > 0$. Also $D_M(\beta, \beta') = D_M(\beta', \beta)$. It almost seems as though this could be a metric, but one can easily find situations where the triangle inequality is not satisfied. However, this is no problem for our purposes.

Definition 5.31. A Morse-Smale vector field v on M is said to be **in general position**, if for any $\beta_1, \beta_2, \beta_3, \beta_4 \in \mathrm{Sing}(v)$, with $\beta_1 \neq \beta_2$, we have $D_M(\beta_1, \beta_2) = D_M(\beta_3, \beta_4)$ if and only if $\{\beta_1, \beta_2\} = \{\beta_3, \beta_4\}$.

For each $\beta \in \mathrm{Sing}(v)$, let $\ell(\beta)$ be the list of length n whose entries consist of the numbers $D(\beta, \beta')$ for all $\beta' \in \mathrm{Sing}(v) \setminus \{\beta\}$, in increasing order. Formally, these lists are elements of $\mathbb{R}^n$ and we put a total order on them by using the lexicographic order. This means that given $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$, let $i_0 = \min\{i \mid a_i \neq b_i\}$ and write $a < b$ if $a_{i_0} < b_{i_0}$.

We use these lists to induce a total order on the elements of $\mathrm{Sing}(v)$, by saying $\beta < \beta'$ if $\ell(\beta) < \ell(\beta')$ (note that for a closed orbit γ, we do not need to order γ^- and γ^+ with respect to each other, since they do not appear as basis elements of the same vector space). Note that the assumption that $\mathrm{Sing}(v)$ has at least three elements was necessary, since if we had only two singular elements, then these two would have the same list. This however yields no big restriction, since vector fields with only two singular elements are very simple and one can check that in the spectral sequence associated to such a vector field, the differential on the first page is already zero.

In other words, the above explanations prove the following result.

Proposition 5.32. *If v is in general position, then it is possible to define a total order on the singular elements of v, so that ordered bases can be uniquely assigned to the vector spaces in each degree of $C_\bullet(v)$, allowing us to express the entries in the differential matrices in matrix form.*

5.6.D Two examples

We now give two examples of Morse-Smale vector fields v on S^2 for which we compute the chain complex $C_\bullet(v)$. To simplify the expressions, we assume that $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

Example 5.33. Consider the vector field $v \in \mathfrak{X}_{MS}(S^2)$ displayed in Figure 5.1. Note that this is the same vector field as in Example 4.6. It has three sinks q_0, q_1, q_2, one saddle s and one closed orbit γ of index 1. The filtration $\emptyset =\subseteq L_0 \subseteq L_1 \subseteq L_2 = S^2$ is the following: $L_0 = \{q_0, q_1, q_2\}$, $L_1 \setminus L_0 = W^u(s)$, $L_2 \setminus L_1 = W^u(\gamma)$. In this example, the spaces L_p are all triangulable, so their Čech homology agrees with singular homology. According to Proposition 5.17, there is a spectral sequence E converging to $H_*(M)$ with the first page given by $E^1_{p,q} = H_{p+q}(L_p, L_{p-1})$. We endow these terms with canonical bases according to Proposition 5.25. The first page of the spectral sequence thus looks as follows:

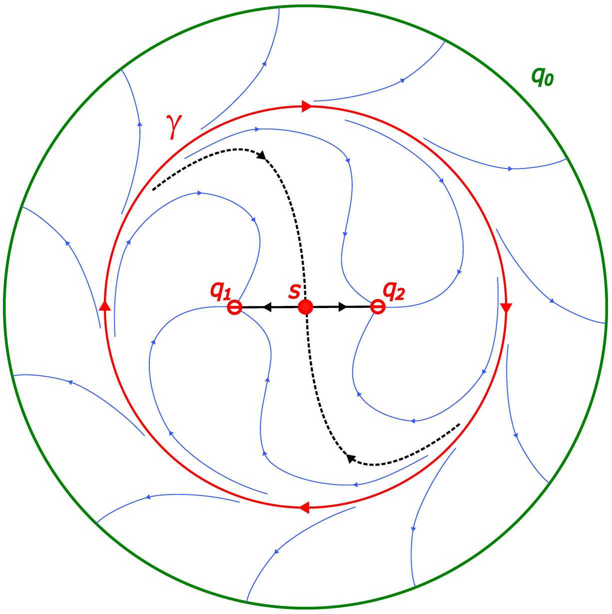


Figure 5.1: Morse-Smale vector field on S^2 with three sinks, one saddle, and one repelling orbit

$$\begin{tikzcd}
\underbrace{H_1(L_0,\emptyset)}_{=0} & \underbrace{H_2(L_1,L_0)}_{=0} \arrow[l] & \\
\underbrace{H_0(L_0,\emptyset)}_{=\mathbb{F}\langle q_0\rangle\oplus\mathbb{F}\langle q_1\rangle\oplus\mathbb{F}\langle q_2\rangle} & \underbrace{H_1(L_1,L_0)}_{=\mathbb{F}\langle s\rangle} \arrow[l, "d^1_{1,0}"'] & \underbrace{H_2(L_2,L_1)}_{=\mathbb{F}\langle\gamma^+\rangle} \arrow[l, "d^1_{2,0}"'] \\
 & & \underbrace{H_1(L_2,L_1)}_{=\mathbb{F}\langle\gamma^-\rangle},
\end{tikzcd}$$

where the differentials with respect to these bases are given by $d^1_{1,0} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $d^1_{2,0} = \begin{bmatrix} 0 \end{bmatrix}$.

The second page looks as follows:

$$\begin{tikzcd}
0 & 0 & \\
E^2_{0,0} = E^1_{0,0}/\operatorname{im}(d^1_{1,0}) & \overbrace{\ker(d^1_{1,0})}^{=0} & E^2_{2,0} = E^1_{2,0} = \mathbb{F}\langle\gamma^+\rangle \\
 & & E^2_{2,-1} = E^1_{2,-1} = \mathbb{F}\langle\gamma^-\rangle. \arrow[ull, "d^2_{2,-1}"]
\end{tikzcd}$$

Note that $E^2_{0,0} \cong \langle q_0, q_1, q_2 \mid q_1 = q_2\rangle$. Using the method described in Section 5.3.B, the basis assigned to $E^2_{0,0}$ is $\{[q_0],[q_1]\}$. The resulting matrix for $d^2_{2,-1}$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. To $\operatorname{im}(d^1_{1,0})$

we assign the basis $\{q_1 + q_2\}$ according to Section 5.3.A. In conclusion, this yields the following based chain complex:

$$\begin{tikzcd}
C_2 = \ker(d^1_{2,0}) = \langle \gamma^+ \rangle \arrow[d, "\begin{bmatrix} 0 \\ 0 \end{bmatrix}"] \\
C_1 = E^1_{1,0} \oplus E^1_{2,-1} = \langle s, \gamma^- \rangle \arrow[d, "\begin{bmatrix} 0 & d^2_{2,-1} \\ d^1_{1,0} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}"] \\
C_0 = E^1_{0,0} / \operatorname{im}(d^1_{1,0}) \oplus \operatorname{im}(d^1_{1,0}) = \langle [q_0], [q_1], q_1 + q_2 \rangle .
\end{tikzcd}$$

Example 5.34. Consider the vector field $v \in \mathfrak{X}_{MS}(S^2)$ displayed in Figure 5.2. Note that this is the same vector field as in Example 4.7. It has four sinks q_0, q_1, q_2, q_3, four saddles s_1, s_2, s_3, s_4, two sources p_1, p_2, and one closed orbit γ of index 1. The filtration $\emptyset = \subseteq L_0 \subseteq L_1 \subseteq L_2 = S^2$ is the following:

$$L_0 = \{q_0, q_1, q_2, q_3\}, \qquad \begin{aligned} L_1 \setminus L_0 &= W^u(s_1) \sqcup W^u(s_2) \sqcup W^u(s_3) \sqcup W^u(s_4), \\ L_2 \setminus L_1 &= W^u(p_1) \sqcup W^u(p_2) \sqcup W^u(\gamma). \end{aligned}$$

As in the previous example, these are all triangulable, so we may consider singular homology instead of Čech homology. Again by Propositions 5.17 and 5.25 we get a spectral sequence $E(v)$ with the first page looking as follows:

$$\begin{tikzcd}
\underbrace{H_1(L_0, \emptyset)}_{=0} & \underbrace{H_2(L_1, L_0)}_{=0} \arrow[l] & \\
\underbrace{H_0(L_0, \emptyset)}_{=\bigoplus_{i=0}^3 \mathbb{F}\langle q_i \rangle} & \underbrace{H_1(L_1, L_0)}_{=\bigoplus_{i=1}^4 \mathbb{F}\langle s_i \rangle} \arrow[l, "d^1_{1,0}"'] & \underbrace{H_2(L_2, L_1)}_{=\mathbb{F}\langle p_1 \rangle \oplus \mathbb{F}\langle p_2 \rangle \oplus \mathbb{F}\langle \gamma^+ \rangle} \arrow[l, "d^1_{2,0}"'] \\
 & & \underbrace{H_1(L_2, L_1)}_{=\mathbb{F}\langle \gamma^- \rangle} .
\end{tikzcd}$$

The differentials with respect to these bases are given by

$$d^1_{1,0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \qquad \text{and} \qquad d^1_{2,0} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} .$$

Assume that the distances between the singular elements are such that the ordering produced from the method described in Section 5.6.C order them according to how we have listed them here. We need to determine bases for the subspaces $\operatorname{im}(d^1_{1,0}) \subseteq E^1_{0,0}$, $\ker(d^1_{2,0}) \subseteq E^1_{2,0}$ and for the quotient spaces $E^1_{0,0} / \operatorname{im}(d^1_{1,0})$, $E^1_{2,0} / \ker(d^1_{2,0})$.

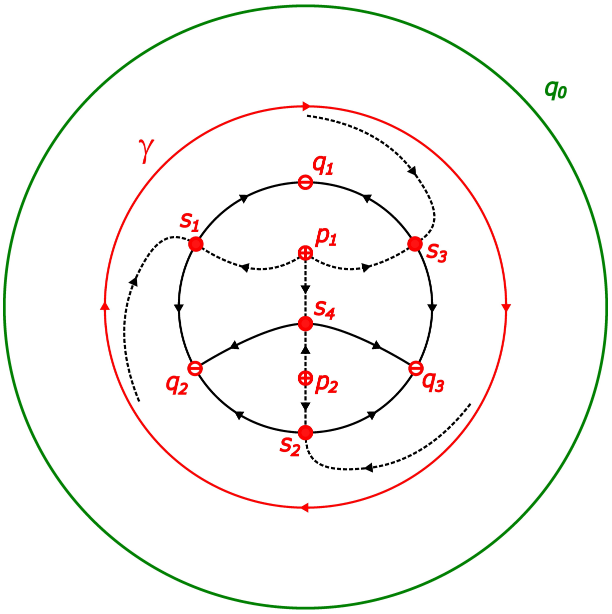


Figure 5.2: Morse-Smale vector field on S^2 with four sinks, fours saddles, two sources, and one repelling orbit

Basis for subspaces: According to Section 5.3.A, we take an arbitrary generating set, write their representation vectors as rows into a matrix, and bring the matrix to reduced echelon form. The non-zero rows of the resulting matrix will be the basis. In the case of $\operatorname{im}(d^1_{1,0}) \subseteq E^1_0 0$, we can start with the columns of $d^1_{1,0}$. This yields the following row echelon form:

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The resulting basis for $\operatorname{im}(d^1_{1,0})$ is thus $\{q_1 + q_3, q_2 + q_3\}$. For $\ker(d^1_{2,0}) \subseteq E^1_{2,0}$, a basis is given by the $\{p_1 + p_2 + \gamma^+\}$. In this case there is nothing more to do, since the basis consists only of one element.

Basis for quotient spaces: Following the procedure described in Section 5.3.B yields the basis $\{[q_0], [q_1]\}$ for $E^2_{0,0} = E^1_{0,0}/\operatorname{im}(d^1_{1,0})$ and the basis $\{[p_1], [p_2]\}$ for $E^1_{2,0}/\ker(d^1_{2,0})$. We can thus write the second page:

$$\begin{tikzcd}
0 & 0 & \\
E^2_{0,0} = \mathbb{F}\langle [q_0], [q_1] \rangle & \overbrace{\ker(d^1_{1,0})}^{=0} & E^2_{2,0} = \mathbb{F}\langle p_1 + p_2 + \gamma^+ \rangle \\
 & & E^2_{2,-1} = E^1_{2,-1} = \mathbb{F}\langle \gamma^- \rangle. \arrow[ull, "d^2_{2,-1}"']
\end{tikzcd}$$

The matrix representation of the non-zero differential with respect to these bases is $d^2_{2,-1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Hence, we get the following chain complex:

$$\begin{tikzcd}
C_2 = \ker(d^1_{2,0}) \oplus E^1_{2,0}/\ker(d^1_{2,0}) = \mathbb{F}\langle p_1 + p_2 + \gamma^+ \rangle \oplus \mathbb{F}\langle [p_1] \rangle \oplus \mathbb{F}\langle [p_2] \rangle \arrow[d, "{\begin{bmatrix} 0 & \overline{d}^1_{2,0} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}"] \\
C_1 = E^1_{1,0} \oplus E^1_{2,-1} = \mathbb{F}\langle s_1 \rangle \oplus \mathbb{F}\langle s_2 \rangle \oplus \mathbb{F}\langle s_3 \rangle \oplus \mathbb{F}\langle s_4 \rangle \oplus \mathbb{F}\langle \gamma^- \rangle \arrow[d, "{\begin{bmatrix} 0 & d^2_{2,-1} \\ d^1_{1,0} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}}"] \\
C_0 = E^1_{0,0}/\operatorname{im}(d^1_{1,0}) \oplus \operatorname{im}(d^1_{1,0}) = \mathbb{F}\langle [q_0] \rangle \oplus \mathbb{F}\langle [q_1] \rangle \oplus \mathbb{F}\langle q_1 + q_3 \rangle \oplus \mathbb{F}\langle q_2 + q_3 \rangle.
\end{tikzcd}$$

Conclusion and outlook

We have described the category of factored chain complexes, proving some analogues to classical results for persistence modules. We then used these results to assign tagged barcodes to gradient-like Morse-Smale vector fields. We showcased via examples the difficulties arising due to the presence of closed orbits and presented a new approach via spectral sequences to assign a chain complex to general Morse-Smale vector fields on surfaces.

Beyond local stability

In Section 3.3 we present a local stability result, in the sense that the assignment of the factored chain complex to a Morse-Smale vector field in general position yields a continuous map. In other words, for every such vector field v there exists a neighbourhood $\mathcal{N}$ of v such that for all $w \in \mathcal{N}$ the interleaving distance between the factored complexes assigned to v and w is bounded in terms of the distance between v and w. A global stability result would mean having upper bounds for any two vector fields, but Example 3.11 shows that this does not hold.

While we cannot ensure that the lack of global stability is unavoidable, we think this is the fundamental difference between gradient-like and gradient vector fields. For the latter, it is indeed possible to adjust the weights so that one achieves global stability, as we show in Section 3.5. In the more general case, we have not found an analogous choice of weights with this property. However, we believe that we have presented the subject with large generality, allowing for different choices of chain complexes, bases, weights, and parameterizations, so to leave open the way to further discoveries in this direction.

Beyond chain complexes

In Chapter 3, the Morse complex assigned to a gradient-like Morse-Smale vector field was used in order to apply the constructions from Section 2.5, yielding a tagged barcode in that case. This used the fact that the Morse complex is by definition equipped with canonical bases, so in order to define weights it only remained to additionally consider the information about the distances between the fixed points.

Since in Chapter 5 we assign a chain complex to a general Morse-Smale vector field, and describe a method for endowing it with bases in Section 5.6.C, it would be interesting to explore how to incorporate the distance between the singular elements also here. This could induce a weighted based chain complex, allowing us to apply the methods from Section 2.5 also in this case, thus resulting in a tagged barcode for general Morse-Smale vector fields.

Beyond surfaces

The results from Chapter 3 apply to a gradient-like Morse-Smale vector field on a manifold of any dimension. In Chapter 5, however, we restricted ourselves to dimension two. The reason for this is that in higher dimensions, the filtration assigned to a Morse-Smale vector field has more steps and the corresponding spectral sequence is not contained in a 3×3 grid. It seems plausible that our construction from Section 5.5 could be generalized (to $m \times m$ grids possibly), so that this restriction would no longer be necessary. All the other steps would then follow through and we could generalize also this approach to arbitrary dimensions.

Beyond vector fields

As in the case of vector fields, there is a classification of the singularities for generic rank 2 tensor fields on surfaces, see e.g. [9, 7, 38]. The genericity condition can be viewed as an analog of the Morse-Smale condition for vector fields. In order to talk about the topology of a tensor field, one may consider the two induced line fields that arise from considering the eigenvalues at every point [20]. One can define separatrices that start at the singularities, dividing the underlying manifold into sections. The Poincaré-Hopf theorem has been extended to line fields, relating the Euler characteristic of the manifold to the sum of topological indices of the singularities [19]. It would be interesting to investigate if one can construct a chain complex also in this case, containing topological information of the tensor field in an algebraic form. As it was the case for the Morse inequalities for vector fields (see Corollary 5.30), it might be possible to derive the Poincaré-Hopf theorem from the existence of such a chain complex. A related, but different, approach that exists already in this direction, is to apply the concept of robustness to 2D tensor fields [54, 34].

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