UNIVERSITÀ DEGLI STUDI DI PARMA

Dottorato di Ricerca in Tecnologie dell'Informazione

XXIX Ciclo

Synthesis and Analysis of Virtual Holonomic Constraints for Lagrangian Mechanical Systems

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Dicembre 2016

To those who have
always believed in me,

Contents

Introduction **1**

I Virtual Holonomic Constraints on Generalized Cylinders **9**

1 State of the Art **11**

1.1 Notation 12

1.1.1 Lagrangian Mechanical Systems with Generalized Cylindrical Configuration Space 14

1.2 Literature on Virtual Holonomic Constraints 17

1.3 Virtual Holonomic Constraints as Controlled Invariant 22

1.3.1 VHCs for Mechanical Systems with Underactuation Degree Equal to One 30

2 Synthesis of One Dimensional Virtual Holonomic Constraints **45**

2.1 Synthesis of Regular VHCs 49

2.1.1 Proofs 52

2.1.2 Example 62

2.2 Synthesis of Stable VHCs 66

2.2.1 Proofs 70

2.2.2 Example 75

II Coordinate Free Approach to Virtual Holonomic Constraints **79**

3 Coordinate Free Formulation of Virtual Holonomic Constraints **81**
3.1 Mechanical Systems and Riemannian Geometry 82
3.1.1 Notation 82
3.1.2 Lagrangian Mechanical Systems 90
3.2 Description of the Reduced Dynamics of VHCs 94
3.2.1 Induced Connection 97
3.2.2 Coordinate Representation 100
3.3 Examples 103

4 Inverse Lagrangian Problem **119**
4.1 General Conditions 121
4.1.1 Conditions for Metrizability 123
4.2 Case with Underactuation Degree Equal to One 124
4.2.1 Examples 127
4.3 Case with Underactuation Degree Equal to Two 128
4.3.1 Examples 130

Conclusion **135**

Bibliography **137**

Acknowledgements **145**

List of Figures

1 Control acceleration required for maintaining the mass on the unit circle. 6

1.1 Example 1.1. The Lagrangian mechanical system (1.9) subjected to the relations h_1 and h_2. 24

1.2 Example 1.2. Asymptotical stabilization of h_2 through the input-output linearizing feedback (1.14). 27

1.3 Pendulum on a cart. 28

1.4 Example 1.3. Asymptotical stabilization of h through the input-output linearizing feedback (1.14). The solution is depicted in blue, the virtual constraint h^{-1} is depicted in red. 29

1.5 Double pendulum. 37

1.6 Example 1.4. The regular VHC γ obtained by Procedure 1.1 setting $\gamma_2(\lambda) = \lambda$ and $d = 0$. 39

1.7 Example 1.4. The regular VHC γ obtained by Procedure 1.1 setting $\gamma_2(\lambda) = \sin\lambda$ and $d = 2$. 41

1.8 Example 1.4. The regular VHC γ obtained by Procedure 1.1 setting $\gamma_2(\lambda) = \lambda - \cos\lambda$ and $d = 1$. 42

1.9 Example 1.4. Some examples of regular VHCs generated by Procedure 1.1. 43

2.1 PVTOL aircraft model. 63

2.2 Example 2.1. Synthesis of a regular VHC starting from curve γ^1. 64

2.3 Example 2.1. Synthesis of a regular VHC starting from curve γ^2. 65
2.4 Example 2.2. Synthesis of a regular VHC starting from curve γ^3 and possessing a stable limit cycle. 77
2.5 Example 2.2. Synthesis of a regular VHC starting from curve γ^4 and possessing unstable reduced dynamics. 77
2.6 Examples 2.1 and 2.2. Illustrations of the VHCs obtained in these examples though the synthesis method of Theorem 2.1. . 78

3.1 Representation of the constraint connection. 94
3.2 Decomposition of a vector $v \in T_q\mathcal{Q}$. 98
3.3 Representation of the induced connection. 100
3.4 Example 3.1 case a). Representation of a vector field which generates the input distribution. 105
3.5 Example 3.1 case a). Phase portrait on the constraint manifold, with initial condition $\dot{z} > 0$. 106
3.6 Example 3.1 case b). Representation of the input distribution (3.21) with $d = 2$ on $\mathcal{C}$ and illustration of the reduced dynamics on $\mathcal{C}$. 107
3.7 Example 3.1 case c) and case d). 109
3.8 Example 3.2. Representation of the input distribution on $\mathcal{C}$. . . 110
3.9 Example 3.2. Some solutions of (3.27). 111
3.10 Double pendulum on cart. 112
3.11 Example 3.3 case a) and $\rho = 0$: solution of the reduced dynamics with initial conditions $z = (0, 10^{-6})^T$, $\dot{z} = (0, 0)^T$. 115
3.12 Example 3.3 case a) and $\rho = 0.2$: the component functions of the reduced dynamics. 115
3.13 Example 3.3 case a) and $\rho = 0.2$: solution of the reduced dynamics with initial conditions $z = (0, 10^{-6})^T$, $\dot{z} = (0, 0)^T$. . . . 116
3.14 Example 3.3 case b): plot of function ρ, solution of (3.34). . . . 117
3.15 Example 3.3 case b): the component functions of the reduced dynamics. 117

3.16 Example 3.3 case b): solution of the reduced dynamics with initial conditions $z = (0, 10^{-6})^T$, $\dot{z} = (0, 0)^T$. 118

4.1 Example 4.3. Functions obtained by (4.10) for the systems (3.23) and (3.24). 129

4.2 Example 4.5. Representation of the induced metric $\hat{g}$ and the induced potential $\hat{P}$ for the double pendulum on a cart with $k = 0$. 133

Introduction

This thesis relies on Analytical Mechanics which is a formulation of Classical Mechanics, developed by Lagrange at the end of the eighteenth century [1]. This formalism describes motion in a system by means of the configuration space. The configuration space is a differentiable smooth manifold, which can be locally represented by *generalized coordinates*. The *degrees of freedom* of a mechanical system, (DOF), are the dimension of the configuration space. The set of the generalized coordinates defines the *configuration* of the system, q, and its derivative with respect to the time is denoted by $\dot{q}$. The *state* of the system is the pair $(q, \dot{q})$.

A Lagrangian mechanical system is described by the *configuration space*, and by a scalar function, the *Lagrangian function*, which depends on the state. We consider Lagrangian functions with the following special structure

$$\mathscr{L}(q, \dot{q}) = K(q, \dot{q}) - P(q)$$

where K is the kinetic energy and P is the potential energy. The motion of a Lagrangian mechanical system is given by the solutions $q(t)$ of the *Euler-Lagrange equations* (for details see [2])

$$\frac{d}{dt}\frac{\partial \mathscr{L}(q, \dot{q})}{\partial \dot{q}} - \frac{\partial \mathscr{L}(q, \dot{q})}{\partial q} = 0.$$

As a consequence, the motion of a mechanical system is described by a second order differential equation.

In this thesis, we consider control affine mechanical systems. Let m denote the number of linear independent controls acting on the system. The motion of a Lagrangian mechanical control system is given by the solutions of the *forced Euler-Lagrange equations*

$$\frac{d}{dt}\frac{\partial \mathscr{L}(q,\dot{q})}{\partial \dot{q}} - \frac{\partial \mathscr{L}(q,\dot{q})}{\partial q} = \sum_{i=1}^{m} B_i(q)\,\tau_i,$$

where $B_i, i \in \{1,\dots,m\}$, are vector fields which specify the directions of the control forces and $\tau_i, i \in \{1,\dots,m\}$, are real functions which specify the control force magnitudes. If m is equal to the degrees of freedom, then the controlled system is called *fully actuated*. On the other hand, if m is less than the degrees of freedom, the controlled system is called *underactuated*. The difference between the degrees of freedom of the system and the number of actuators is called the *underactuation degree*.

One of the most interesting challenges in the study of *Lagrangian mechanical control systems* is *motion control*. Traditionally, the resolution of this problem is divided in two parts, the generation of a reference signal, the *motion planning*, and the design of the feedback law making the mechanical system track the signal. This paradigm has been successfully employed in the control of fully actuated mechanical systems. Instead, for underactuated mechanical systems, the lack of direct control of each degree of freedom implies hard restrictions in motion planning. Indeed, the control design for fully actuated systems is an old area of research whereas the control design for underactuated systems has a more recent origin.

This thesis is focused on a particular motion control paradigm which differs from traditional approaches. It has been introduced by Grizzle *et al.* in [3] to solve a specific application problem: the locomotion of a bipedal robot. In particular they have shown that an effective way to encode a desired gait is to enforce desired relations between the joint angles by control, reducing the degrees of freedom of the controlled system. This control approach is called *virtual holonomic constraints* (VHCs). More specifically, a virtual holonomic constraint is a relationship among the coordinates of the system (as a physical holonomic constraint) which is made invariant via feedback control. The difference between a physical holonomic constraint and a VHC is that, in the first, the relationship is maintained by the physical structure of the constraint, instead, in the latter, the relationship is maintained by virtue of actuators.

Few years later, Shiriaev, Canudas-de-Wit and collaborators initiated a study on VHCs. In [4], they studied VHCs on general mechanical systems with n degrees of freedom, $n-1$ actuators and a cylindrical configuration space. The aim of that work is to enforce a specific stable oscillation. They represented the VHC as a closed curve in the configuration space. Then, they showed that the dynamics of the constrained system (namely the dynamics on the closed curve) are governed by an unforced second-order system. Then, they developed a methodology for stabilizing a closed curve based on a time-varying linearization.

Consolini, Maggiore and collaborators investigated on VHCs applied to n dimensional Lagrangian mechanical systems with $n-1$ actuators and a cylindrical configuration space. Their works cover some unaddressed questions on this control paradigm. First, they gave a formal and explicit definition of VHCs. They investigated conditions for the feasibility of VHCs and developed a systematic procedure for generating feasible VHCs [5]. The authors focused on the study of the dynamics on the constrained system, called reduced dynamics, and noted that, under some conditions, the constrained system is not a Lagrangian mechanical system [6]. This mean that the reduced dynamics may not be obtained through the Euler-Lagrange equations of a suitable Lagrangian function. The problem of finding the Lagrangian function is known as the *inverse Lagrangian problem*. The intuitive explanation of this behavior is that, in the case of VHCs, the relation between the coordinates is enforced by the actuators, which may do work on the system. Instead, in case of physical holonomic constraints, the forces do not do work on the system, since they are always orthogonal to the velocity vectors (as stated in the d'Alembert Principle). Thus, the constrained system is a Lagrangian mechanical system. An in-depth study of this topic is given by Mohammadi *et al.* in [7, 8]. The authors provided necessary and sufficient conditions under which the reduced dynamics admit a Lagrangian function.

The following toy example allows to understand the meaning of VHCs. Consider a unit point mass in $\mathbb{R}^2$ with configuration q and subject to a control force τ

$$\ddot{q}(t) = R(\alpha)q(t)\tau(t) \; , \tag{1}$$

where $R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ is the rotation matrix associated to angle α, which is an assigned constant. This system has two configuration variables and one control, corresponding to an underactuation degree of one. Assume that $q(0) = (1,0)^T$, $\dot{q}(0) = (0,1)^T$. We want to design a feedback control law for τ such that the mass remains on the unit circle, that represents the VHC we want to enforce. Namely, we require the existence of a real-valued function λ, representing the mass position along the circle, such that

$$q(t) = \begin{pmatrix} \cos\lambda(t) \\ \sin\lambda(t) \end{pmatrix} .$$

It is easy to show that, if $\cos\alpha \neq 0$, the control law $\tau(t) = -(\cos\alpha)^{-1}\dot{\lambda}(t)^2$ is such that the solution q of (1) remains on the unit circle. The intuition of this control law is depicted in Figures 1a, 1b, 1c: the normal component of the control acceleration corresponds to the centripetal acceleration $\ddot{q}_n$ required to maintain the mass on the unit circle. Since, if $\sin\alpha \neq 0$, the control force is not normal to the circle, a tangential acceleration $\ddot{q}_t$ component is also present.

By substituting this control law in (1), we obtain the reduced dynamics

$$\ddot{\lambda} = -\dot{\lambda}^2 \, \tan\alpha.$$

Their explicit solution is given by

$$\dot{\lambda}(t) = \frac{1}{1 + t\tan\alpha} \, .$$

At this point, we can distinguish three cases.

1. If $\alpha = 0$ (Figure 1a), the control force is orthogonal to the circle and corresponds to the centripetal force that is required to keep the mass on the circle. The control force does not do work on the system, the mass velocity is constant and the reduced dynamics admit a Lagrangian function. This corresponds to the case in which the mass is maintained on the unit circle by a physical holonomic constraint.

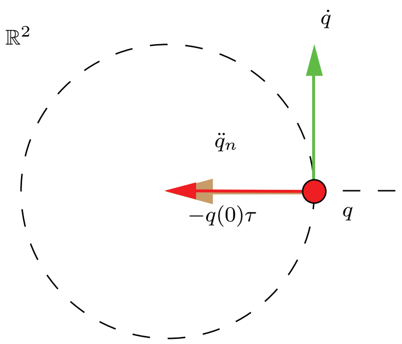
2. If $\tan \alpha > 0$ (Figure 1b), $\lim_{t \to +\infty} \dot{\lambda}(t) = 0$, that is the mass asymptotically stops.

3. If $\tan \alpha < 0$ (Figure 1c), the mass speed goes to infinity at the finite escape time $-(\tan \alpha)^{-1}$.

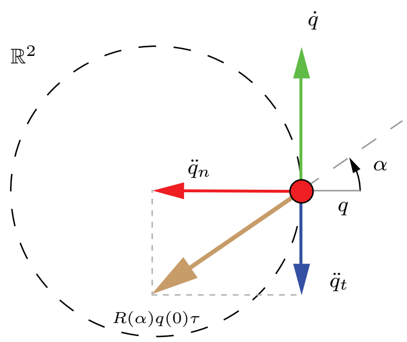
It is evident that the reduced dynamics in case 2) and 3) are not Lagrangian.

As shown above, a feasible VHC may exhibit unstable dynamics and, clearly, the implementation of a such VHC in a real application is unpractical. The first objective of this thesis is to provide a procedure for synthesizing feasible VHCs whose reduced dynamics possess an asymptotically stable limit cycle. The proposed synthesis procedure is an extension of [9]. Given a closed curve, it allows to generate, under some hypotheses, a feasible VHC. Moreover, we present conditions under which it is possible to find a curve passing through a given configuration such that the corresponding VHC possesses an asymptotically stable limit cycle.

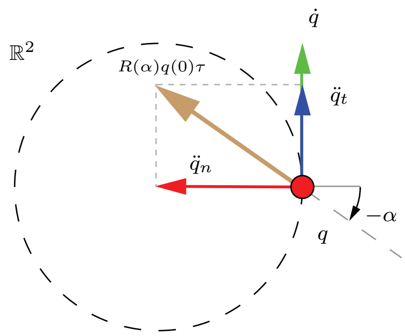
The results presented in [6, 5] apply to systems whose configuration space is a generalized cylinder. This is a limitation since it is not satisfied by several mechanical systems of practical relevance (for instance, manipulators with spherical joints, UAV and cranes). The second objective of this thesis is to provide a coordinate free formulation that allows extending the results presented in [6, 5] with two respects: firstly, the configuration space is a generic manifold, secondly, the underactuation degree may be greater than one. Even though the treatment of Lagrangian mechanics with a coordinate free formalism is widely used, at least to our knowledge, this has not been done before in virtual constraints theory.



(a) Case $\alpha = 0$.



(b) Case $\alpha > 0$.



(c) Case $\alpha < 0$.

Figure 1: Control acceleration required for maintaining the mass on the unit circle.

Finally, we address the inverse Lagrangian problem in this coordinate free formalism. Using renown results from differential geometry, we provide necessary and sufficient conditions for a constrained system to be a Lagrangian mechanical system.

This thesis is divided in two parts. The first concerns the study of VHCs for mechanical systems with a cylindrical configuration space and an underactuation degree of one. Chapter 1 gives an introduction on mechanical systems and introduces the notion and the formalism used in this first part. It describes the state of the art on VHCs. The generation of feasible and stable VHCs is presented in Chapter 2. This chapter shows a generalization of the synthesis method developed in [9] and proposes conditions for choosing a feasible VHC whose dynamics possess a stable limit cycle.

The second part of this thesis is the core of this work. Chapter 3 introduces the notion of a Lagrangian system with a Riemannian manifold. It defines the concept of *induced connection* associated to the application of a VHC on a mechanical system. Several examples are shown in order to help the reader to understand the key passages. Chapter 4 deals with the inverse Lagrangian problem.

Part I

Virtual Holonomic Constraints on Generalized Cylinders

Chapter 1

State of the Art

To live with honor you have to pine, trouble, contend, mistake,
start again from the beginning and throw everything away
and begin again and fight and eternally lose.
Calm is a poltroonery of the soul.

– Lev Tolstoj

In this chapter, we describe the state of the art on virtual holonomic constraints. In Section 1.1, we introduce the notation and summarize the main concepts. The notation presented there refers to Part I of this thesis. In Section 1.2, we discuss some results on VHCs literature, focusing on the motivation and presenting some recent applications. In Section 1.3 we focus on some results presented by Consolini and Maggiore in [6, 5, 10]. Even though the material presented in this section is not an original contribution of this thesis, it is necessary for understanding the contributions presented in next chapters.

1.1 Notation

We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ the sets of natural numbers, integer numbers and real numbers, respectively. We denote by $\mathbb{S}$ the unit circle, that is

$$\mathbb{S} = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\right\}.$$

Let $T \in \mathbb{R}$ be a constant value, we denote by $[\mathbb{R}]_T$, the quotient space of $\mathbb{R}$ with respect to the equivalence relation $\sim$ on $\mathbb{R}$ such that $x \sim y$ if and only if there exists $k \in \mathbb{Z}$ such that $x - y = T\,k$. Note that, $[\mathbb{R}]_T$ is diffeomorphic to $\mathbb{S}$. For vectors $f, g \in \mathbb{R}^n$, $\langle f, g\rangle$ denotes the Euclidean inner product and $\|f\| = \sqrt{\langle f, f\rangle}$ the Euclidean norm of f. The distance between $p \in \mathbb{R}^n$ and a subset $S \subset \mathbb{R}^n$ is denoted by $||p||_S$.

Let $\mathcal{M}$ be an m-dimensional smooth manifold, we denote by $T\mathcal{M}$ its tangent bundle. Given $p \in \mathcal{M}$, we denote by $T_p\mathcal{M}$ the tangent space to $\mathcal{M}$ at p. Let $\mathcal{N}$ be an n-dimensional smooth manifold and let $F : \mathcal{M} \to \mathcal{N}$ be a smooth map, we denote by $dF : T\mathcal{M} \to T\mathcal{N}$ the *global differential* of F. We denote by $dF(p) : T_p\mathcal{M} \to T_{F(p)}\mathcal{N}$ the *differential* of F at $p \in \mathcal{M}$, that is the restriction of dF to the tangent space $T_p\mathcal{M}$.

In this part of the thesis, we consider only manifolds defined as follows. The n-dimensional smooth manifold $\mathcal{Q}$ is a **generalized cylinder** if there exist $p, l \in \mathbb{N}$ such that $p + l = n$ and $\mathcal{Q} = (\mathbb{S})^p \times \mathbb{R}^l$. In other words, a generalized cylinder is a manifold whose coordinates are linear displacements or angles. The tangent bundle of a generalized cylinder is given by $T\mathcal{Q} = \mathcal{Q} \times \mathbb{R}^n$. Map $\bar{p} : \mathbb{R} \to \mathbb{S}$, defined as $\bar{p}(x) = (\sin(x), \cos(x))$, is a covering map from $\mathbb{R}$ to $\mathbb{S}$. A covering map $p : \mathbb{R}^n \to \mathcal{Q}$ is defined by $p((x_1, \ldots, x_n)) = (\bar{p}(x_1), \bar{p}(x_2), \ldots, \bar{p}(x_p), x_{p+1}, \ldots, x_n)$.

If $\mathcal{Q}$ is a generalized cylinder, a **vector field** v on $\mathcal{Q}$ is a function $v : \mathcal{Q} \to \mathbb{R}^n$. We use the notation v_q in place of $v(q)$ to represent the vector obtained by evaluating v at q. We denote by v_i, with $i \in \{1, \ldots, n\}$, the i-th element of the vector field v, which is a function $v_i : \mathcal{Q} \to \mathbb{R}$. If $v_q \neq 0$,$\forall q \in \mathcal{Q}$, $\hat{v}$ is the corresponding normalized vector field (i.e., $\hat{v}_q = v_q\|v_q\|^{-1}$, $\forall q \in \mathcal{Q}$). If v, w are vector fields on $\mathcal{Q}$, their **Lie Bracket** is the vector field $[v, w] = (c_1, \ldots, c_n)^T$ where

$$c_i = \sum_{k=1}^{n} \left(v_k \frac{\partial w_i}{\partial q_k} - w_k \frac{\partial v_i}{\partial q_k} \right) .$$

Given a vector field v on $\mathcal{Q}$ and a vector $f \in \mathbb{R}^n$, the **directional derivative** of v along f is given by the vector field $\mathcal{L}v$ on $\mathcal{Q}$ such that

$$(\mathcal{L}_f v)_q = \frac{d}{dz} v_{q+zf}, \, \forall q \in Q \, .$$

A **distribution** on $\mathcal{Q}$ is a map U that assigns to each $q \in \mathcal{Q}$ a subspace $U(q)$. Given vector fields $u^1, u^2, \ldots, u^m$ on $\mathcal{Q}$, the distribution $U(q) = \mathrm{span}\{u^1_q, u^2_q, \ldots, u^m_q\}$ is **involutive** at $q_0 \in \mathcal{Q}$ if, for any $i, j = 1, \ldots, m$, $[u^i, u^j]_{q_0} \subset U(q_0)$.[1]

Let $M : \mathcal{Q} \to \mathbb{R}^{m \times p}$ be a **matrix-valued function** on $\mathcal{Q}$, we denote by $M_{i,j}$, $i \in 1, \ldots, n$ and $j \in 1, \ldots, m$, the ij-th element of the matrix-valued function M, which is a scalar function $M_{i,j} : \mathcal{Q} \to \mathbb{R}$. For simplicity of notation, we refer to the matrix-valued function with the term matrix. We define the directional derivative of the matrix M as

$$(\mathcal{L}_f M)(q) = \frac{d}{dz} M(q + zf), \, \forall q \in Q \, .$$

Let $I \subset \mathbb{R}$ be an interval, then a **curve** γ in a manifold $\mathcal{Q}$ is a smooth map $\gamma : I \to \mathcal{Q}$. A **vector field along the curve** γ is a smooth map $V : I \to \mathbb{R}^n$, such that $V(t) \in T_{\gamma(t)}\mathcal{Q}$ for all $t \in I$. We denote by $\dot{\gamma}$ the **velocity of the curve** γ, which is the vector field along γ defined as

$$\dot{\gamma}_i(t) = \frac{d\gamma_i(t)}{dt}, \, \forall i \in \{1, \ldots, n\}.$$

The curve γ is called *regular* if $\dot{\gamma}$ does not vanish on the interval I.

Consider $x \in \mathcal{Q}$, and let v be a vector field on $\mathcal{Q}$. The differential equation

$$\dot{x} = v_x$$

[1]Note that $[u^i, u^j]_{q_0}$ is the vector obtained by evaluating the vector field $[u^i, u^j]$ at q_0.

is the *ordinary differential equation* associated with v and x is the *state* of the system. Let $x^0 \in \mathcal{Q}$ be the initial state and let $I \subset \mathbb{R}$ be an interval containing 0. The curve $\gamma : I \rightarrow \mathcal{Q}$ is an *integral curve of v through x^0* if $\gamma(0) = x^0$ and $\dot{\gamma}(t) = v_{x(t)}$. The set $O(x^0) = \gamma(I)$, namely the image of the integral curve, is called the **orbit of $\boldsymbol{v}$ through $\boldsymbol{x_0}$**.

1.1.1 Lagrangian Mechanical Systems with Generalized Cylindrical Configuration Space

A Lagrangian system is given by an $n-$*dimensional* smooth manifold $\mathcal{Q}$, called **configuration space**, and a function defined on its tangent bundle $T\mathcal{Q}$, called **Lagrangian function** $\mathscr{L} : T\mathcal{Q} \rightarrow \mathbb{R}$. In this part of the thesis we consider only Lagrangian systems for which $\mathcal{Q}$ is a **generalized cylinder**. Thus, the generalized coordinates, $q_i, i \in \{1, \dots, n\}$, are either linear displacements in $\mathbb{R}$ or angular displacement in $[\mathbb{R}]_T$. The set of the generalized coordinates $q = (q_1, \dots, q_n) \in \mathcal{Q}$, is the *configuration* of the system. We denote by $\dot{q} \in T_q\mathcal{Q}$ the derivative with respect to time of the configuration.

The Lagrangian system is **mechanical** if the Lagrangian function has the following special form

$$\mathscr{L}(q, \dot{q}) = \frac{1}{2}\dot{q}^T D(q)\dot{q} - P(q), \tag{1.1}$$

where $D : \mathcal{Q} \rightarrow \mathbb{R}^{n\times n}$ is the *inertial matrix*, which is symmetric and positive definite, and $P : \mathcal{Q} \rightarrow \mathbb{R}$ is the *potential function*. The *motion* of the system is given by the curves which satisfy the *Euler-Lagrange equations*

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}} - \frac{\partial \mathscr{L}}{\partial q} = 0. \tag{1.2}$$

These curves are called **solutions** of (1.2).

We call **Lagrangian mechanical control system** a Lagrangian mechanical system endowed with controls. Let m be the number of controls and let $\tau \in \mathbb{R}^m$ be the control inputs vector. The motion of a Lagrangian mechanical control system is given by the solutions of the *forced Euler-Lagrange equations*

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}} - \frac{\partial \mathscr{L}}{\partial q} = B(q)\tau, \tag{1.3}$$

where $B : \mathcal{Q} \to \mathbb{R}^{n\times m}$ is a matrix assumed to have **full rank** for all $q \in \mathcal{Q}$, that is $\mathrm{rank}(B) = m$.

This system is called *fully actuated* if $n = m$ and *underactuated* if $n > m$.

Definition 1.1 (Underactuation degree). The difference between the degrees of freedom n and the number of linear independent actuators m is called underactuation degree.

With the above mentioned assumptions, the Lagrangian mechanical control system takes on the form of a system of n second order differential equations[2]

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla P_q = B(q)\tau \tag{1.4}$$

where $C(q,\dot{q}) : T\mathcal{Q} \to \mathbb{R}^{n\times n}$ is the *Coriolis force matrix* and $\nabla P_q : \mathcal{Q} \to \mathbb{R}^n$ is the gradient of the potential (a vector field). Each component of the Coriolis force matrix can be obtained by equation

$$C_{c,b}(q,\dot{q}) = \sum_{a=1}^{n} \Gamma_{c,a,b}(q)\dot{q}_a,\ \forall (q,\dot{q}) \in T\mathcal{Q}.$$

where $\Gamma_{a,b,c} : \mathcal{Q} \to \mathbb{R}$, $a,b,c \in \{1,\dots n\}$, are the *Christoffel symbols of the first kind* defined by

$$\Gamma_{c,a,b}(q) = \frac{1}{2}\left(\frac{\partial D_{c,a}(q)}{\partial q_b} + \frac{\partial D_{c,b}(q)}{\partial q_a} - \frac{\partial D_{a,b}(q)}{\partial q_c}\right),\ \forall q \in \mathcal{Q}.$$

Since the inverse of the inertial matrix is well defined on $\mathcal{Q}$, we can rewrite (1.4) as

$$\ddot{q} = -D^{-1}(q)\left(C(q,\dot{q})\dot{q} + \nabla P_q\right) + D^{-1}B(q)\tau \tag{1.5}$$

[2]Note that, $\ddot{q}$ in (1.5) defines an ODE on $T\mathcal{Q} = \mathcal{Q} \times \mathbb{R}^n$. Let $x = (q,\dot{q})^T$. Then, given an initial condition $x^0 = (q^0,\dot{q}^0)$ the integral curve of (1.5) through x^0 is given by the curve q and its velocity $\dot{q}$. Since the curve q identifies directly its velocity, the integral curve of (1.5) can be identified univocally by the curve q, which is the solution of (1.3) with initial condition $(q^0,\dot{q}^0)$.

where $D^{-1}B(q)$ represents the term $D^{-1}(q)B(q)$ and is called the *input distribution matrix*. By the previous assumptions and the Sylvester's rank inequality, the **rank** of $D^{-1}B$ is **full**. Let $u^1, \ldots, u^m$ be the vector fields such that $D^{-1}B(q) = (u_q^1, \ldots, u_q^m)$, the m-dimensional subspace

$$U(q) = \text{span}\{u_q^1, u_q^2, \ldots, u_q^m\}$$

is called **input distribution** and represents the set of acceleration vectors that can be obtained with all possible choices of the control input vector τ.

Let $I \subset \mathbb{R}$ be an interval and let v be a vector field defined along a curve $\gamma : I \to \mathcal{Q}$. We denote by $\nabla_{\dot{\gamma}} v$ the *covariant derivative of* v *along* γ (see [11] and [12]). It is a vector field defined along γ and its components are defined by the following equation

$$\big(\nabla_{\dot{\gamma}(t)} v_{\gamma(t)}\big)_k = \big((\mathcal{L}_{\dot{\gamma}(t)} v)_{\gamma(t)}\big)_k + \sum_{i,j=1}^{n,n} \Gamma_{i,j}^k(\gamma(t))\, \dot{\gamma}_i(t)\, \dot{v}_i(\gamma(t)),\ \forall t \in I$$

where $\Gamma_{i,j}^k : \mathcal{Q} \to \mathbb{R}$, with $k, i, j \in \{1, \ldots, n\}$, are the *Christoffel symbols of the* **second kind** and are defined as

$$\Gamma_{i,j}^k = \sum_{l=1}^{n} D_{k,l}^{-1}(q)\, \Gamma_{l,i,j}(q),\ \forall q \in \mathcal{Q}.$$

Consider a curve $q : I \to \mathcal{Q}$ and its velocity $\dot{q} : I \to \mathbb{R}^n$. Then the covariant derivative of $\dot{q}$ along q is given by

$$(\nabla_{\dot{q}}\dot{q})_k = \ddot{q}_k + \sum_{i,j=1}^{n,n} \Gamma_{i,j}^k\, \dot{q}_i\, \dot{q}_j,\ k \in \{1, \ldots, n\}.$$

Equation (1.5) can be compactly rewritten as

$$\nabla_{\dot{q}}\dot{q} = -D^{-1}\nabla P_q + D^{-1}B(q)\tau, \tag{1.6}$$

where $D^{-1}\nabla P_q$ denotes the vector field given by $D^{-1}(q)\nabla P_q, \forall q \in \mathcal{Q}$, that is the acceleration due to the potential force. The covariant derivative includes the accelerations due to the Coriolis forces, defined by the following equation

$$(D^{-1}(q)C(q,\dot{q}))_{i,j} = \sum_{k=1}^{n} \Gamma_{j,k}^i(q)\, \dot{q}_k,\ \forall (q,\dot{q}) \in T\mathcal{Q}.$$

A **holonomic constraint** of order k for (1.6) is a relation $H(q) = 0$, where $H : \mathcal{Q} \to \mathbb{R}^k$ is a smooth map and $\mathrm{rank}(dH(q)) = k$, $\forall q \in H^{-1}(0)$. Since dH has full rank, map H is a submersion, and by the *submersion level set theorem* ((5.13) in [13]), $H^{-1}(0)$ is a smooth manifold of dimension $n-k$. We call **constrained system** a Lagrangian mechanical (control) system subjected to a holonomic constraint H. In particular, the configuration space of the constrained system is $H^{-1}(0)$ and its motion is given by an $(n-k)$-dimensional second order differential equation.

The constrained system is a Lagrangian mechanical (control) system if there exists a Lagrangian function with the form (1.1), defined on the tangent bundle of $H^{-1}(0)$, such that the motion of the constrained system satisfies (1.2) or (1.3). The problem of finding this Lagrangian function is called the **inverse Lagrangian problem**.

1.2 Literature on Virtual Holonomic Constraints

Enforcing constraints by actuators is a concept that can be tracked back to the work of Paul Appell [14] and, more explicitly, to the work of Henri Béghin [15] in the early part of the 20th century. This concept is known as *servoconstraints* and has been intensively investigated afterward, for instance in [16], and more recently in [17, 18]. For almost one century, the studies on servoconstraints concerned the field of physics. In the last decade, this concept has emerged in control engineering as a valuable tool to solve motion control problems.

Motion control has always been one of the main challenges in robotic applications. Traditionally, *trajectory tracking* has been the dominant paradigm for designing motion control algorithms. In general, it consists of a two-step design: the generation of a timed reference trajectory signal using specific planning algorithms and the implementation of a suitable control method, which makes the system follow the reference trajectory signal (for instance, with a PID controller or a sliding mode controller). The popularity of this paradigm is justified in industrial manufacturing settings, where the environment is highly structured. Recently, emerging robotic applications, such as bio-inspired robots and mobile robots, have shown the limitations of this approach. Namely, the high number of DOF and the underactuated nature of these systems make the generation of reference trajectory signals an extremely difficult task. Moreover, unstructured environments may inject notable disturbances in the controller and desynchronize the motion prescribed by the reference trajectory signal. As a consequence, if the controller attempts to regain the synchronism when the tracking error is high, the overcompensation may cause damage to the system itself and to the environment around it.

An early trace of the virtual constraint idea in applications can be found in the work of Nakanishi and collaborators [19]. The group enforced through feedback a constraint on the angles of an acrobot (a double pendulum with an actuator at the elbow), in order to imitate the pendulum-like motion of an ape's brachiation.

Meanwhile, the virtual constraint paradigm was pioneered by Grizzle and collaborators for solving a specific applicative challenge, the motion control of a planar bipedal robot [3, 20, 21]. The planar bipedal model considered in their previous works is a planar open kinematical chain, which is composed by a single joint, the *hip*, which connects two identical open chains called *legs* and a third chain called *torso*. Two torques are applied between the legs and the torso. In [3] the legs are simple links, while in [22] the legs have knees, namely they are composed by two joined links, and for each knee a torque is applied. It is assumed that the motion takes place in the sagittal plane on a level surface and the robot can touch the surface only with the end of the legs.

Grizzle *et al.* modeled the walking cycle as successive phases of single support, with a stance leg touching the ground and a swing leg, and double support, with both legs touching the ground. Our interest concerns the single support phase, in which the dynamics are described by a second order n-dimensional differential equation. In this phase, the system can be described by $n-1$-angular coordinates representing the joints of the robots and an angular coordinate representing the position of the stance leg with respect to the ground. Hence, the configuration space Q is the generalized cylinder, $\mathbb{S} \times \cdots \times \mathbb{S} \in Q$.

Even though every joint of the robot is actuated, the overall system is underactuated, indeed actuators do not act on the coordinate representing the angle between the stance leg and the ground. In order to simplify the motion design, instead of generating a trajectory, Grizzle et al. defined a set of outputs, equal in number to the inputs, and designed a feedback controller that asymptotically drives the outputs to zero. The novelty was in the definition of these outputs, which were defined as a relationship between the coordinates of the system, in this way, the overall procedure corresponds to enforcing a holonomic constraint. Moreover, the subset of $\mathcal{Q}$ enforced by actuators is required to be an *attractive invariant set*.

The stability analysis of the walking cycle is the principal issue considered in [3] and the *method of Poincaré section* is the natural tool to study its stability. Unfortunately, the hypersurface transversal to the orbit has dimension n, and the computation of the Poincaré map is impractical. The intuition was to analyze only the internal dynamics of the system compatible with an identically null output, called *the zero dynamics*. The state space on which the zero dynamics evolve is called *zero dynamics manifold* and, in this specific case, is diffeomorphic to $\mathbb{S} \times \mathbb{R}$. Therefore, the computation of the Poincaré map is easier and the result allows understanding the stability property of the closed orbit on the zero dynamics manifold. The analysis and study of the zero dynamics inspired the development of the virtual holonomic constraints theory. An extensive survey on the work of Grizzle *et al.* can be found in [23], in particular in Chapter 5, where the authors started to call this approach *virtual constraint*. Further works of the group extended the results to 3D bipedal robot [24, 25, 26] and, recently, the group started the investigation of virtual non holonomic constraints [27].

Shiriaev, Canudas-de-Wit and collaborators developed a procedure for stabilizing repetitive behaviors on underactuated mechanical systems based on the virtual constraint idea. Indeed, an early appearance of *virtual constraints* term can be found in [28], although that paper refers also to [4] (which was published a year later). A hint of this terminology can be found also in [29]. In [28], the term *virtual constraint* is defined as a set of relations among the coordinates (or links) of the system imposed through feedback control.

In [4], the authors considered mechanical systems with underactuation degree of one. The aim was to determine a family of feedback control laws and conditions which ensure exponential stability of the virtual constraint and of a target periodic solution orbit γ which lies on the zero dynamics manifold. The existence of γ has been assumed. The group gave an explicit expression of the zero dynamics.[3] Let θ be a parametrization variable of the orbit γ, then the zero dynamics are the solutions of the following second order differential equation

$$\alpha(\theta(t))\ddot{\theta}(t) + \beta(\theta(t))\dot{\theta}^2 + \gamma(\theta(t)) = 0 \tag{1.7}$$

where $\alpha(t), \beta(t), \theta(t)$ are smooth scalar functions. Note that, function α is assumed to be not vanishing in a neighborhood of γ. They also found a first integral of the zero dynamics (detailed information can be found in [30]).

The main contribution of [4] extends the results of [29] on which Canudas-de-Wit and collaborators proposed a technique for finding virtual constraints and a control law that stabilizes a desired closed orbit. In [4] the group developed a constructive procedure based on a time-varying linearized feedback such that the given virtual constraints and the desired T-periodic orbit θ_γ are exponentially stable. In [31], the transverse linearization technique is generalized to the case of mechanical systems with underactuation degree greater than one. Shiriaev *et al.* applied their results to various experimental set-ups, for instance a Furuta pendulum [32], a pendubot [33], a passive walker robot [34] and a simplified helicopter [35].

Consolini, Maggiore and collaborators started investigating on the virtual constraint paradigm motivated by Grizzle's approach. At the time, despite the growing interest on this paradigm in control applications, several question remained unaddressed. The group began formalizing the definition of *virtual*

[3]In their works they refer to the zero dynamics with the term *virtual limit system*.

holonomic constraint as a relationship among the generalized variables of the system which is made *invariant* by feedback. Of notable importance for the development of their framework is the definition of a *regular* VHC. A regular VHC is a VHC which, under mild hypotheses, can be made invariant with an input-output linearizing feedback. The zero dynamics[4] of a regular VHC is well defined and, in case of one dimensional VHCs, has the form of (1.7).

For Lagrangian mechanical systems with underactuation degree of one, the group has extensively studied the properties of the associated zero dynamics, focusing on the inverse Lagrangian problem. In particular, they showed that, if the zero dynamics manifold[5] is diffeomorphic to $\mathbb{S} \times \mathbb{R}$, there may not exist a Lagrangian function defined as (1.1) on $\mathbb{S} \times \mathbb{R}$, such that the zero dynamics satisfy the Euler-Lagrange equations (1.2). This means that the reduced system, which is defined by the zero dynamics and the zero dynamics manifold, is not a Lagrangian mechanical system. Sufficient and necessary conditions for a reduced system to be Lagrangian can be found in [7].

Focusing on the pendubot, Consolini, Maggiore and collaborators investigated on the stabilization of a specific orbit on the zero dynamics manifold, preserving its invariance. In case of Lagrangian constrained systems, the first integral found in [30] can be viewed as an energy function. Given suitable initial conditions, the orbit lies on a level set of the zero dynamics manifold such that the value of the that function remains constant. The aim is to stabilize a target orbit, that corresponds to a certain value of the energy function, taking into account the lack of control input on the zero dynamics. The group presented two different approaches which dynamically change the geometry of the VHC. The first approach, presented in [36], is based on the transversal linearization, while the second one, presented in [37], is based on passivity.

Recently, the VHCs approach has been employed for the motion control of

[4]In the works of Consolini and Maggiore the zero dynamics are denoted with the terms *constraint dynamics* or *reduced dynamics*. In this thesis we will use the latter term.

[5]In the works of Consolini and Maggiore the zero dynamics manifold is called *constraint manifold*.

snake holonomic robots [38, 39, 40, 41, 42]. Even more recently, Čelikovský has been investigating on VHCs for mechanical systems with degree of underactuation greater than one [43, 44].

1.3 Virtual Holonomic Constraints as Controlled Invariant

This section reports some results developed by Consolini and Maggiore.

Definition 1.2 (Virtual holonomic constraint). A virtual holonomic constraint (VHC) of order k for system (1.6) is a relation $h(q) = 0$, where $h(q) : \mathcal{Q} \to \mathbb{R}^k$ is a smooth map, $\mathrm{rank}(dh(q)) = k$, $\forall q \in h^{-1}(0)$, and the set

$$\Gamma = \{(q, \dot{q}) : h(q) = 0, dh(q)\dot{q} = 0\} \tag{1.8}$$

is **controlled invariant**. That is, there exists a smooth feedback $\tau(q, \dot{q})$ such that Γ is positively invariant for the closed-loop system.
The set Γ is called the **constraint manifold** associated with the VHC $h(q) = 0$. A VHC is called **stabilizable** if there exists a smooth feedback $\tau(q, \dot{q})$ that asymptotically stabilizes Γ. In this case, the feedback $\tau(q, \dot{q})$ is said to **enforce the VHC**.

Since $dh(q)$ has full rank for all $q \in \mathcal{Q}$, the map h is a *submersion*. As a consequence, by the *submersion level set theorem* (5.13 in [13]), $h^{-1}(0)$ is a smooth manifold of dimension $n-k$. The relation $h(q) = 0$ univocally identifies $h^{-1}(0)$, for simplicity we will call also the latter a VHC.

The controlled invariance of Γ means that, if the control system is initialized on Γ, that is $(q(0), \dot{q}(0)) \in \Gamma$, there exists a smooth feedback which enforces $(q(t), \dot{q}(t)) \in \Gamma$, $\forall\, t \in \mathbb{R}^+$. The controlled invariance of Γ is a necessary condition for the asymptotical stability of Γ.

The manifold Γ is **asymptotically stable** for the closed-loop system if, for all $\epsilon > 0$, there exists $\delta > 0$ such that, for all $(q(0), \dot{q}(0)) \in ||(q, \dot{q})||_\Gamma < \delta$, the solution of the closed-loop system remains in $||(q(t), \dot{q}(t))||_\Gamma < \epsilon$, $t > 0$ and asymptotically converges to Γ ($\lim_{t\to+\infty} ||(q(t), \dot{q}(t))||_\Gamma = 0$).

Example 1.1. Consider a unitary mass particle lying on $\mathbb{R}^2$. Its configuration is $q = (q_1, q_2)^T \in \mathcal{Q} = \mathbb{R}^2$. The Lagrangian function is $\mathscr{L}(q, \dot{q}) = \frac{1}{2}\dot{q}^T D(q)\dot{q} - P(q)$, where $D(q)$ is the identity matrix and the potential is $P(q) = q_2^2$. Moreover, $\tau \in \mathbb{R}$ is the control input and $B(q) = (q_1^2 + q_2^2, 0)^T$, so that the system has underactuation degree equal to one. Since D is the identity matrix, the input distribution U is spanned by B. Following (1.5), the dynamics of the system are given by

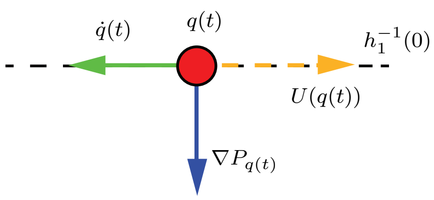
$$\begin{aligned} \ddot{q}_1 = & \left(q_1^2 + q_2^2\right)\tau \\ \ddot{q}_2 = & \quad -2q_2. \end{aligned} \tag{1.9}$$

Consider the relation $h_1(q) = q_2 - 1$, that is, we would like to constraint the motion on the horizontal line $q_2 = 1$. Let Γ_1 be the set defined as (1.8) and let $(q(0), \dot{q}(0)) \in \Gamma_1$ be the configuration at $t = 0$. The set Γ_1 is a controlled invariant if there exists a smooth feedback such that $\frac{d}{dt}(dh(q)\dot{q}) = 0$, that is $\ddot{q}_2 = 0$. Thus, relation h_1 is not a VHC because there does not exist any control input τ that makes h_1 a controlled invariant.

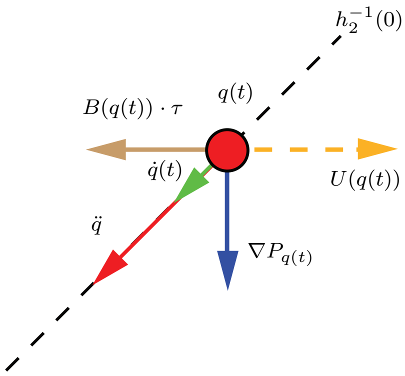
Then, consider the relation $h_2(q) = q_1 - q_2 - 1$, that is we would like to constraint the motion on the diagonal line passing through $(0, -1)^T$. Let Γ_2 be the set defined as (1.8), it is a controlled invariant if there exists a smooth feedback such that $\ddot{q}_1 = \ddot{q}_2$. The relation h_2 is a VHC since Γ_2 can be made invariant by the feedback $\tau = \frac{-2q_2}{q_1^2+q_2^2}$.

Intuitively, in the first case, the control input can not compensate for the acceleration due to gravity. The reason is that the input distribution is tangential to the manifold $h_1^{-1}(0)$, as shown in Figure 1.1a. In the second case, the control is transversal to $h_2^{-1}(0)$ and can modify the acceleration of the particle such that its velocities remain on the tangent space of $h_2^{-1}(0)$, as shown in Figure 1.1b.

Definition 1.3 (Regular VHC). A relation $h(q) = 0$ is a regular VHC if the



(a) No choice of τ can maintain the the particle on Γ_1. Indeed, the control input can not compensate for the acceleration due to potential.



(b) The control law τ is able to enforce the system on Γ_2.

Figure 1.1: Example 1.1. The Lagrangian mechanical system (1.9) subjected to the relations h_1 and h_2.

output function $e = h(q)$ yields vector relative degree $\{2, \dots, 2\}$ everywhere on the set Γ (1.8).

Let $h(q)$ be a regular VHC, then system (1.6) endowed with output $e = h(q)$ is **input-output feedback linearizable**, and the corresponding zero dynamics manifold is Γ. Then, Γ is controlled invariant and $h(q)$ is a VHC.

The following proposition gives a sufficient and necessary condition for a smooth map $h : \mathcal{Q} \to \mathbb{R}^k$ to be a regular VHC.

Proposition 1.1. *Consider (1.6), let n be its dimension and let m be the number of its actuators. Let $h : \mathcal{Q} \to \mathbb{R}^k$ be a smooth map with* $\text{rank}(dh(q)) = k, \forall q \in h^{-1}(0)$ *and $k \leq m$. The relation $h(q) = 0$ is a regular VHC of order k for (1.6) if and only if*

$$\text{rank}\left(dh(q)\, D^{-1}B(q)\right) = k,\ \forall q \in h^{-1}(0). \tag{1.10}$$

Proof. Consider a Lagrangian control mechanical system (1.6) endowed with output $e = h(q)$. We can rewrite this system as a $2n$-dimensional nonlinear affine control system with configuration variable $x = (q, \dot{q})^T$, namely

$$\begin{aligned} \dot{x} = & \begin{pmatrix} \dot{q} \\ f(q,\dot{q}) \end{pmatrix} + \begin{pmatrix} 0 \\ D^{-1}B(q) \end{pmatrix} \tau \\ e = & \qquad\qquad h(q) \end{aligned}$$

where function $f : T\mathcal{Q} \to \mathbb{R}^n$ represents the unforced part of equation (1.5). This nonlinear system has **relative degree 2** for all components of the output if the following two conditions are satisfied (see 12.2 of [45])

$$\frac{\partial h(q)}{\partial x} \begin{pmatrix} 0 \\ D^{-1}B(q) \end{pmatrix} = 0 \tag{1.11}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial h(q)}{\partial x}\begin{pmatrix} \dot{q} \\ f(q,\dot{q}) \end{pmatrix}\right)\begin{pmatrix} 0 \\ D^{-1}B(q) \end{pmatrix} \neq 0 \tag{1.12}$$

For a Lagrangian mechanical system condition (1.11) is always satisfied, indeed

$$\begin{pmatrix} \frac{\partial h(q)}{\partial q} & \frac{\partial h(q)}{\partial \dot{q}} \end{pmatrix} \begin{pmatrix} 0 \\ D^{-1}B(q) \end{pmatrix} = \begin{pmatrix} dh(q) & 0 \end{pmatrix} \begin{pmatrix} 0 \\ D^{-1}B(q) \end{pmatrix} = 0, \, \forall q \in \mathcal{Q}.$$

Condition (1.12) can be rewritten as

$$\begin{aligned} \frac{\partial}{\partial x}\left(\begin{pmatrix} dh & 0 \end{pmatrix}\begin{pmatrix} \dot{q} \\ f(q,\dot{q}) \end{pmatrix}\right)\begin{pmatrix} 0 \\ D^{-1}B(q) \end{pmatrix} &= \frac{\partial}{\partial x}(dh(q)\dot{q})\begin{pmatrix} 0 \\ D^{-1}B(q) \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial(dh(q)\dot{q})}{\partial q} & \frac{\partial(dh(q)\dot{q})}{\partial \dot{q}} \end{pmatrix}\begin{pmatrix} 0 \\ D^{-1}B(q) \end{pmatrix} \\ &= dh(q)\, D^{-1}B(q), \end{aligned}$$

then condition (1.12) can be satisfied if and only if rank $\big(dh(q)\, D^{-1}B(q)\big)$ is maximum for all $q \in h^{-1}(0)$. □

Corollary 1.1. *Let $h : \mathcal{Q} \to \mathbb{R}^k$ be a smooth map with full rank. For all $q \in h^{-1}(0)$ the tangent space of $h^{-1}(0)$ is $\ker(dh(q))$ which is a subspace of $\mathbb{R}^n$. Condition (1.10) is equivalent to the following condition*

$$\dim\left(\ker(dh(q)) \cap \operatorname{Im}\left(D^{-1}B(q)\right)\right) = m - k, \forall q \in h^{-1}(0). \tag{1.13}$$

A regular VHC has an important property: under mild hypotheses it is a **stabilizable VHC**. If there exist increasing functions $\alpha, \beta : [0, r) \to [0, +\infty)$ with $r > 0$, such that the map $H : (q, \dot{q}) \to (h(q), dh(q)\dot{q})$ is bounded as $\alpha||(q, \dot{q})||_\Gamma \leq H(q, \dot{q}) \leq \beta||(q, \dot{q})||_\Gamma$, then an **input-output linearizing feedback asymptotically stabilizes** Γ. Consider a Lagrangian control mechanical system represented as a $2n$-dimensional nonlinear system as shown in proof of proposition 1.1. The following input-output linearizing feedback[6] (see Cap.12 of [45] for more details)

$$\tau = \left(dh(q)D^{-1}B(q)\right)^{-1}\left(-\begin{pmatrix}\frac{\partial}{\partial q}\left(dh(q)\dot{q}\right) & dh(q)\end{pmatrix}\begin{pmatrix}\dot{q} \\ f(q, \dot{q})\end{pmatrix} + v\right)$$

renders the input-output map equal to $\ddot{e} = v$, where v is the auxiliary control signal.

In order to asymptotically stabilize manifold Γ, a suitable choice of v is given by a PD regulator, that is $v = -k_1 e - k_2 \dot{e}$, so that $e(t) \to 0$ exponentially (it has been done in [6]). Then, given a regular VHC $h(q) = 0$, satisfying the mild assumptions above, a suitable feedback which render $h(q) = 0$ **stable** is

$$\begin{aligned}\tau = \left(dh(q)D^{-1}B(q)\right)^{-1}\Big\{-dh(q)\left(-D^{-1}C(q, \dot{q}) - D^{-1}\nabla P_q\right) + \\ -\frac{\partial\left(dh(q)\dot{q}\right)}{\partial q}\dot{q} - k_1 h(q) - k_2 dh(q)\dot{q}\Big\}\end{aligned} \tag{1.14}$$

where $k_1, k_2 \in \mathbb{R}$ are positive constants.

Example 1.2. Consider the system presented in Example 1.1 and the smooth map $h_2(q) = q_1 - q_2 - 1$. The relation $h_2(q) = 0$ is a regular VHC, indeed

$$\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}q_1^2 + q_2^2 \\ 0\end{pmatrix} = q_1^2 + q_2^2 \neq 0, \forall q \in h_2^{-1}(0).$$

Equation (1.14) gives a feedback which exponentially stabilize the relation

[6] It is also known as *computed torque method*.

$h_2(q) = 0$, that is

$$\tau = \frac{1}{q_1^2 + q_2^2} \left(- \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -2q_2 \end{pmatrix} - k_1(q_1 - q_2 - 1) - k_2 \begin{pmatrix} 1 & -1 \end{pmatrix} \dot{q} \right)$$
$$= \frac{-2q_2 - k_1(q_1 - q_2 - 1) - k_2(\dot{q}_1 - \dot{q}_2)}{q_1^2 + q_2^2}.$$

Figure 1.2 shows the support of the solution curve (in blue) of the closed-loop system with feedback constants $k_1 = 10, k_2 = 3$, passing through $q(0) = (2, -1)^T$, $\dot{q}(0) = (1, -1)^T$. The red line represents the desired VHC.

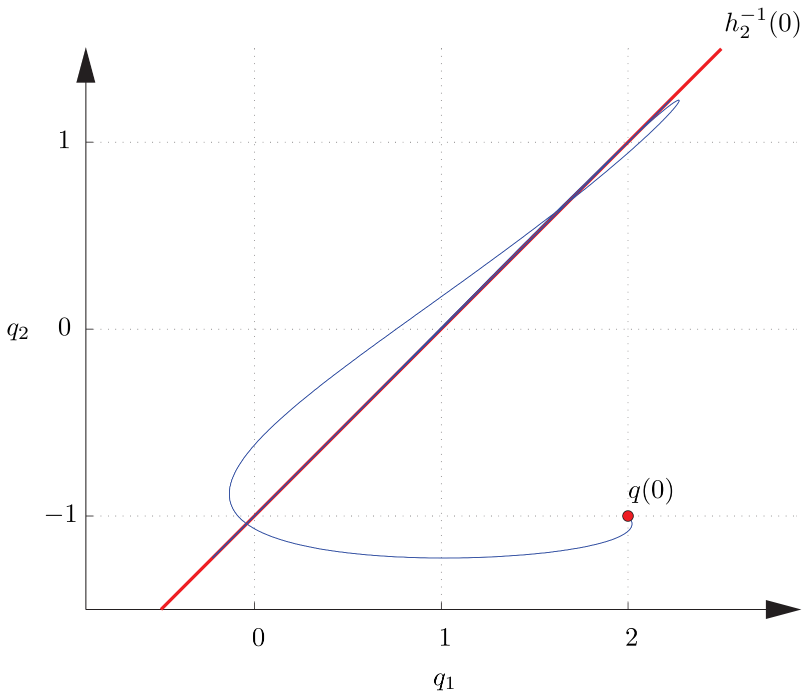


Figure 1.2: Example 1.2. Asymptotical stabilization of h_2 through the input-output linearizing feedback (1.14).

Example 1.3. The pendulum on a cart is a set of two particles on $\mathbb{R}^2$, where the first particle can move only on the horizontal axis, and the second maintains a fixed distance with respect to the first. Namely, the two particles are connected by a link.[7] We assume that both particles have unitary mass. The gravitational force acts on both particles.

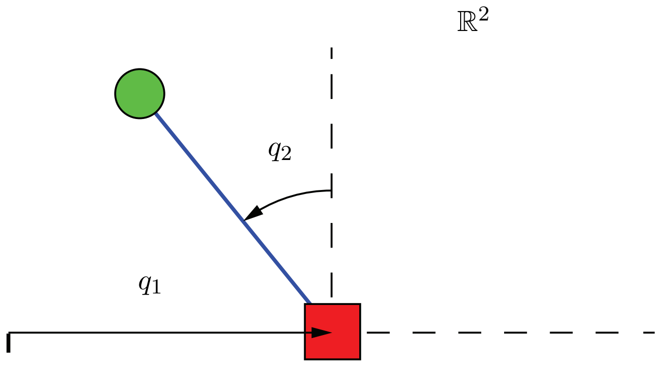


Figure 1.3: Pendulum on a cart.

The configuration space of this Lagrangian mechanical system is the cylinder $\mathcal{Q} = \mathbb{R} \times \mathbb{S}$. We choose the coordinates as shown in Figure 1.3, where $q_1 \in \mathbb{R}$ is the displacement along the x-axis and $q_2 \in [\mathbb{R}]_{2\pi}$ is the orientation of the link. The Lagrangian function is given by (1.1) and, for this choice of coordinates, the inertial matrix is

$$D = \begin{pmatrix} 2 & -\cos q_2 \\ -\cos q_2 & 1 \end{pmatrix}$$

and the potential function[8] is $P = \cos q_2$. A control input τ acts on the first particle, thus $B(q) = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$. The forced Euler-Lagrange equations are given by

$$\begin{aligned} 2\ddot{q}_1 - \ddot{q}_2 \cos q_2 + \dot{q}_2^2 \sin q_2 &= \tau \\ \ddot{q}_2 - \ddot{q}_1 \cos q_2 - \sin q_2 &= 0. \end{aligned} \tag{1.15}$$

Consider the relation $h(q) = q_1 - L \sin q_2 = 0$, with $L \in \mathbb{R}$. If $L \geq 1$, this relation is not a regular VHC since condition (1.10) is not satisfied on all

[7]The link has unitary length and negligible mass.
[8]Without lost of generality, we consider unitary gravitational acceleration.

$q \in h^{-1}(0)$, indeed

$$dh(q)\,D^{-1}B(q) = L - \frac{2L-1}{\sin^2(q_2)+1}$$

vanishes when $q_2 = \pm\arcsin\sqrt{1-\frac{1}{L}} + k\pi$, where $k = \{0,1\}$. For example, consider h with $L=-2$ and the control input τ given by (1.14) with $k_1 = 15$ and $k_2 = 5$. Figure 1.4 shows the support of the solution curve of the closed loop system (1.15) with initial condition $q(0) = (2, \frac{\pi}{4})^T$, $\dot{q}(0) = (0,0)^T$.

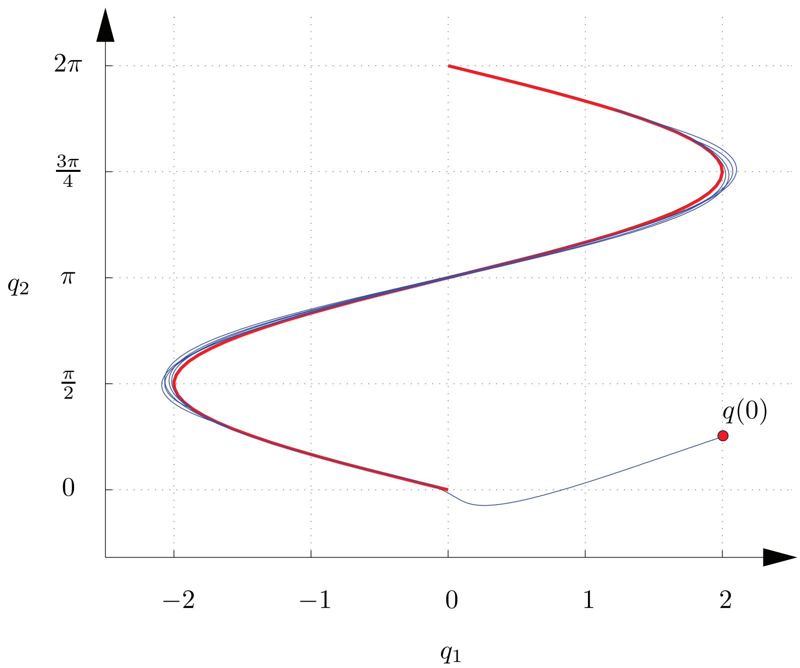


Figure 1.4: Example 1.3. Asymptotical stabilization of h through the input-output linearizing feedback (1.14). The solution is depicted in blue, the virtual constraint h^{-1} is depicted in red.

1.3.1 VHCs for Mechanical Systems with Underactuation Degree Equal to One

In this subsection, we consider Lagrangian controlled mechanical systems with underactuation degree equal to **one**. This class of underactuated systems includes popular nonlinear benchmarks such as the pendubot, the acrobot, the pendulum on a cart, and robotic applications such as biped robots and bicycles. The analysis provided in this subsection is not only relevant for presenting the state of the art on VHCs, but it is also necessary for understanding the next Chapter 2. We consider only VHCs of order $n-1$.

As before, we consider Lagrangian mechanical control systems whose configuration spaces are generalized cylindrical manifolds. For VHC of order $n-1$, the set $h^{-1}(0)$ is a one dimensional manifold without self intersection. Namely, $h^{-1}(0)$ is a regular curve embedded in $\mathcal{Q}$ and it is diffeomorphic to either the real line $\mathbb{R}$ (open curve) or the unit circle $\mathbb{S}$ (closed curve). In some circumstances, it might be convenient to represent the manifold $h^{-1}(0)$ as a parametric curve.

Definition 1.4 (Parametric VHC). Let $h(q) = 0$ be a VHC of order $n-1$ for a Lagrangian mechanical control system. Let $\gamma : \Theta \to \mathcal{Q}$ be a regular parametrization of $h^{-1}(0)$, where $\Theta = \mathbb{R}$ if $h^{-1}(0)$ is diffeomorphic to $\mathbb{R}$, or $\Theta = [\mathbb{R}]_T$ is $h^{-1}(0)$ is diffeomorphic to $\mathbb{S}$. Then, a **parametric VHC** is a regular embedded curve

$$q = \gamma(\lambda),\ \lambda \in \Theta$$

such that $\mathrm{Im}(\gamma) = h^{-1}(0)$ and $\lambda \in \Theta$ is the parametrizing variable of the VHC.

Remark 1.1. The VHCs studied in [6, 5] are parametric VHCs where the parametrizing variable is one of the configuration variables. Let $q_n \in \Theta$ be a configuration variable of $\mathcal{Q}$, a parametric VHC is the curve

$$\gamma(q_n) = (\phi_1(q_n), \ldots, \phi_{n-1}(q_n), q_n)^T,$$

where ϕ_i are smooth functions.

We assume the existence of a left annihilator of the input distribution represented by matrix $D^{-1}B$. We denote by $\eta : \mathcal{Q} \to \mathbb{R}^n$ the vector field which is a basis of such left annihilator, namely, for all $q \in \mathcal{Q}$, $\eta_q^T D^{-1}B(q) = 0$. The following proposition presents a sufficient and necessary condition for a parametric VHC to be a regular VHC.

Proposition 1.2. *Let $h : \mathcal{Q} \to \mathbb{R}^{n-1}$ be a smooth map with full rank and let $\gamma : \Theta \to \mathcal{Q}$ be a parametric VHC of $h^{-1}(0)$. Then, h is a* **regular VHC** *of order $n-1$ if and only if*

$$\left\langle \eta_{\gamma(\lambda)}, \gamma'(\lambda) \right\rangle \neq 0, \forall \lambda \in \Theta. \tag{1.16}$$

Proof. It is a direct consequence of Corollary 1.1. Indeed

$$\mathrm{Im}(\gamma'(\lambda)) = \ker(dh(\gamma(\lambda))$$

for all $\lambda \in \Theta$, then (1.13) is satisfied if and only if

$$\mathrm{Im}(\gamma'(\lambda)) \cap \mathrm{Im}(D^{-1}B(\gamma(\lambda))) = \{\emptyset\},\ \lambda \in \Theta$$

which corresponds to condition (1.16). □

Reduced Dynamics

Consider the smooth map $h : \mathcal{Q} \to \mathbb{R}^{n-1}$ with full rank and such that $h(q) = 0$ is a *regular VHC*. The dynamics of the system subject to the VHC are called the **reduced dynamics**. Since the feedback law $\tau(q, \dot{q})$ that stabilizes Γ is unique, the reduced dynamics are represented by an autonomous system.

The following proposition gives an explicit formulation for the reduced dynamics for regular VHC of order $n-1$ on underactuated mechanical system with underactuation degree one.

Proposition 1.3. *Consider a n-dimensional Lagrangian control mechanical system with underactuation degree of one. Let $\gamma : \Theta \to \mathcal{Q}$ be a parametric*

regular VHC, *with parametrizing variable* $\lambda \in \Theta$, *with either* $\Theta = \mathbb{R}$ *or* $\Theta = [\mathbb{R}]_T$. *The reduced dynamics are given by*

$$\ddot{\lambda} = \psi_1(\lambda) + \psi_2(\lambda)\dot{\lambda}^2 \tag{1.17}$$

where

$$\psi_1(\lambda) = -\frac{\left\langle \eta_{\gamma(\lambda)}, D^{-1}\nabla P_{\gamma(\lambda)} \right\rangle}{\left\langle \eta_{\gamma(\lambda)}, \gamma'(\lambda) \right\rangle} \tag{1.18}$$

$$\psi_2(\lambda) = -\frac{\left\langle \eta_{\gamma(\lambda)}, \nabla_{\gamma'(\lambda)}\gamma'(\lambda) \right\rangle}{\left\langle \eta_{\gamma(\lambda)}, \gamma'(\lambda) \right\rangle} \tag{1.19}$$

Being a regular VHC, functions ψ_1, ψ_2 *are well defined by condition (1.16).*

Proof. Following [6], the reduced dynamics is obtained multiplying (1.6) by η and substituting q with $\gamma(\lambda)$, that is

$$\left\langle \eta_{\gamma(\lambda)}, \nabla_{\left(\gamma'(\lambda)\dot{\lambda}\right)} \left(\gamma'(\lambda)\dot{\lambda}\right) \right\rangle = -\left\langle \eta_{\gamma(\lambda)}, D^{-1}\nabla P_{\gamma(\lambda)} \right\rangle.$$

From the properties of the covariant derivative (see definition 2.1 of [12]) the following equality holds

$$\nabla_{\left(\gamma'(\lambda)\dot{\lambda}\right)} \left(\gamma'(\lambda)\dot{\lambda}\right) = \ddot{\lambda}\gamma'(\lambda) + \dot{\lambda}\nabla_{\left(\dot{\lambda}\gamma'(\lambda)\right)}\gamma'(\lambda) = \gamma'(\lambda)\ddot{\lambda} + \dot{\lambda}^2\nabla_{\gamma'(\lambda)}\gamma'(\lambda).$$

Then, (1.17) is obtained by the following straightforward manipulations

$$\begin{aligned}
\left\langle \eta_{\gamma(\lambda)}, \gamma'(\lambda)\ddot{\lambda} + \dot{\lambda}^2\nabla_{\gamma'(\lambda)}\gamma'(\lambda) \right\rangle &= -\left\langle \eta_{\gamma(\lambda)}, D^{-1}\nabla P_{\gamma(\lambda)} \right\rangle \\
\left\langle \eta_{\gamma(\lambda)}, \gamma'(\lambda)\ddot{\lambda} \right\rangle + \left\langle \eta_{\gamma(\lambda)}, \dot{\lambda}^2\nabla_{\gamma'(\lambda)}\gamma'(\lambda) \right\rangle &= -\left\langle \eta_{\gamma(\lambda)}, D^{-1}\nabla P_{\gamma(\lambda)} \right\rangle \\
\ddot{\lambda}\left\langle \eta_{\gamma(\lambda)}, \gamma'(\lambda) \right\rangle = -\dot{\lambda}^2\left\langle \eta_{\gamma(\lambda)}, \nabla_{\gamma'(\lambda)}\gamma'(\lambda) \right\rangle &- \left\langle \eta_{\gamma(\lambda)}, D^{-1}\nabla P_{\gamma(\lambda)} \right\rangle.
\end{aligned}$$

□

Shiriaev *et al.* in [30] provide a first integral function. Setting

$$M(\lambda) = \mathrm{e}^{-2\int_0^\lambda \psi_2(\mu)d\mu} \tag{1.20}$$

$$V(\lambda) = -\int_0^\lambda \psi_1(\mu)M(\mu)d\mu, \tag{1.21}$$

function $E(\lambda, \dot{\lambda}) = \frac{1}{2}M(\lambda)\dot{\lambda}^2 + V(\lambda)$ is an integral of motion for (1.17). This fact may seem to imply that the reduced dynamics satisfy Euler-Lagrange function (1.2) for the Lagrangian function $\mathscr{L}(\lambda, \dot{\lambda}) = \frac{1}{2}M(\lambda)\dot{\lambda}^2 - V(\lambda)$. Actually, this is true only if $\Theta = \mathbb{R}$, in fact, if $\Theta = [\mathbb{R}]_T$ the energy function E may be multivalued. Indeed function M and V are not necessarily T-periodic functions. In this case, there not exist a Lagrangian function of the form (1.1) defined on $T[\mathbb{R}]_T$ such that the solutions of the corresponding Euler-Lagrangian equations are the same solutions of the reduced dynamics, thus the reduced system is not a Lagrangian mechanical system.

For VHCs of order $n-1$, this problem has been deeply investigated by Mohammadi *et al.* in [7]. The following theorems report the main results of that work.

Theorem 1.1 (Theorem 3.3 in [7])**.** *If $\Theta = \mathbb{R}$, the motion of the Lagrangian mechanical system with Lagrangian function $\mathscr{L}(\lambda, \dot{\lambda}) = \frac{1}{2}M(\lambda)\dot{\lambda}^2 - V(\lambda)$ and configuration space $\mathbb{R}$ satisfies the reduced dynamics (1.17).*

Theorem 1.2 (Theorem 3.5 in [7])**.** *If $\Theta = [\mathbb{R}]_T$, the motion of the Lagrangian mechanical system with Lagrangian function $\mathscr{L}(\lambda, \dot{\lambda}) = \frac{1}{2}M(\lambda)\dot{\lambda}^2 - V(\lambda)$ and configuration space $[\mathbb{R}]_T$ satisfies the reduced dynamics (1.17) if and only if M and V are T-periodic, that is $M(\lambda) = M(\lambda + kT)$, $V(\lambda) = V(\lambda + kT)$ for all $k \in \mathbb{N}$ and for all $\lambda \in [\mathbb{R}]_T$.*

Note that, if $\Theta = [\mathbb{R}]_T$ and Theorem 1.2 holds, the reduced dynamics (1.17) are completely characterized by V. A closed orbit σ of (1.17) is said to be a *rotation of* λ, if σ is homeomorphic to a circle $\{(\lambda, \dot{\lambda}) \in [\mathbb{R}]_T \times \mathbb{R} : \dot{\lambda} = c\}$, for a given constant c. The closed orbit σ of (1.17) is said to be an *oscillation of* λ, if σ is homeomorphic to a circle $\{(\lambda, \dot{\lambda}) \in [\mathbb{R}]_T \times \mathbb{R} : \lambda^2 + \dot{\lambda}^2 = c\}$, for a given constant c.

Generation of Regular VHCs

Here, we present a constructive procedure for finding *regular* VHCs for Lagrangian mechanical systems with degree of underactuation of one. This procedure is a slightly modified version of the procedure reported in [5], in which the VHCs are expressed in the parametric form described in Remark 1.1. Here, we consider VHCs diffeomorphic to $\mathbb{S}$ expressed in the form reported in Definition 1.4.

By Proposition 1.2, a VHC is *regular* if and only if the following equation holds

$$\sum_{i=1}^{n} \eta_i(\gamma(\lambda))\, \gamma_i'(\lambda) = \delta(\lambda),\ \lambda \in [\mathbb{R}]_T \tag{1.22}$$

where η_i is a component of the vector field η and $\delta : [\mathbb{R}]_T \to \mathbb{R} \setminus \{0\}$ is a continuous function bounded away from zero. Let $\{i_1, \dots, i_n\}$ be a permutation of the set $\{1, \dots, n\}$, and consider i_1 such that η_{i_1} is bounded away from zero.[9] Besides, consider an arbitrarily selection of $n-1$ continuous functions $\gamma_{i_2}, \dots, \gamma_{i_n}$, such that functions $\gamma_i : [\mathbb{R}]_T \to \mathbb{R}$ are T-periodic.[10] Equation (1.22) can be rewritten as a first order differential equation of γ_{i_1} with independent variable λ,

$$\gamma_{i_1}'(\lambda) = \frac{1}{\eta_{i_1}(\gamma(\lambda))} \left(-\sum_{j=2}^{n} \eta_{i_j}(\gamma(\lambda))\, \gamma_{i_j}'(\lambda) + \delta(\gamma_{i_1}(\lambda), \lambda) \right) \tag{1.23}$$

where we allow δ to depend also on γ_{i_1} for greater flexibility. Given an initial condition $\gamma_{i_1}(0)$, it is possible to find the missing function γ_{i_1} solving the differential equation (1.23) over the interval $[0, T]$. The solution is valid if and only if it is T-*periodic*, namely the solution has to be a continuous function on $[\mathbb{R}]_T$. In detail, if $\gamma_{i_1} : [\mathbb{R}]_T \to \mathbb{R}$ the solution is valid if $\gamma_{i_1}(0) = \gamma_{i_1}(T)$, otherwise if $\gamma_{i_1} : [\mathbb{R}]_T \to [\mathbb{R}]_{T_i}$ the solution is valid if $\gamma_{i_1}(0) = d\, T_i\, \gamma_{i_1}(T)$, with $d \in \mathbb{Z}$. The number d is called *degree* of the function $\gamma_{i_1} : [\mathbb{R}]_T \to [\mathbb{R}]_{T_i}$ and represents the number of revolution of the angle $\gamma_{i_1}(\lambda)$ per each revolution of λ.

[9] In general, this condition is not essential. Example 3.4 in [5] shows a way to overcome this limitation.

[10] In case of $\gamma_i : [\mathbb{R}]_T \to [\mathbb{R}]_{T_i}$ the continuity property implies that $\gamma_i(0) = k\, T_i\, \gamma_i(T)$, with $k \in \mathbb{N}$.

The following lemma, thoroughly inspired by Lemma 3.1 in [5], gives necessary and sufficient conditions for the existence of a function δ such that the solution of (1.23) is T-periodic.

Lemma 1.1. *Consider equation (1.23), suppose that η_{i_1} is bounded away from zero, and that for each q_{i_j}, $j \in \{2, \dots, n\}$ the function $q_{i_1} \to \eta|_{q=\gamma(\lambda)}$ is bounded. Fix an initial condition $\gamma_{i_1}(0) = \gamma_{i_1}^0$ and, if $\gamma_{i_1} : [\mathbb{R}]_T \to [\mathbb{R}]_{T_i}$, a desired degree $d \in \mathbb{Z}$ (otherwise $d = 0$). Then, there exists a continuous function $\delta : \Theta \times [\mathbb{R}]_T \to \mathbb{R} \setminus \{0\}$ such that the solution γ_{i_1} of (1.23) is T-periodic with degree d if, and only if, the solution of (1.23) with $\delta = 0$, denoted by $\bar{\gamma}_{i_1}$, satisfies*

$$\bar{\gamma}_{i_1}(T) - \bar{\gamma}_{i_1}(0) \neq d\, T_{i_1}. \tag{1.24}$$

Moreover, for any $\mu : \Theta \times [\mathbb{R}]_T \to \mathbb{R} \setminus \{0\}$, there exists a unique $\epsilon \in \mathbb{R} \setminus \{0\}$ such that the solution of (1.23) with $\delta = \epsilon\, \mu$ is T-periodic.

Remark 1.2. The assumption that $q_{i_1} \to \eta$ is bounded for all q_{i_j}, $j \in \{2, \dots, n\}$ is mild, indeed typically B is constant and D, the inertial matrix, is bounded. For instance, all robot manipulators with revolute joints have bounded inertial matrix.

The following procedure can be used to generate regular VHCs for a cyclic variable.

Procedure 1.1 (Generation of regular VHCs). 1) Choose a component of η such that it is bounded away from 0. Given $\{i_1, \dots, i_n\}$ a permutation of the set $\{1, \dots, n\}$, we denote the index of this component with i_1.

2) Choose $n - 1$ arbitrary continuous functions $\gamma_{i_j}, j \in \{2, \dots, n\}$, such that functions $\gamma_i : [\mathbb{R}]_T \to \mathbb{R}$ are T-periodic.

3) Write the first order differential equation (1.23).

4) Fix an initial condition $\gamma_{i_1}^0$ and, if $\gamma_i : [\mathbb{R}]_T \to [\mathbb{R}]_{T_i}$ a desired degree d (otherwise fix $d = 0$).

5) Check whether the solution of (1.23) with $\delta = 0$, denoted by $\bar{\gamma}_{i_1}$, satisfies (1.24). If it does not, change the function γ_{i_j}.

6) Set $\delta(\gamma_{i_1}, \lambda) = \epsilon\, \mu(\gamma_{i_1}, \lambda)$, where $\mu(\gamma_{i_1}, \lambda)$ is a continuous and T-periodic function bounded away form zero. Find the unique $\epsilon \in \mathbb{R} \setminus \{0\}$ such that the solution of (1.23) with initial condition $\gamma_{i_1}^0$ satisfies

$$\gamma_{i_1}(T) - \gamma_{i_1}(0) = d\,T_{i_1}, \text{ if } \gamma_{i_1} : [\mathbb{R}]_T \to [\mathbb{R}]_{T_{i_1}}$$
$$\gamma_{i_1}(T) - \gamma_{i_1}(0) = 0, \text{ if } \gamma_{i_1} : [\mathbb{R}]_T \to \mathbb{R}.$$

The check in step 5) can be done using numerical integration, and the value ϵ in 6) can be found with a one dimensional search based on the numerical integration of (1.23).

Assumption 1.1. *For some $\bar{q} \in \mathcal{Q}$, it holds that $D(q)$, $P(q)$ and $B(q)$ in (1.6) are even with respect to $\bar{q}$, that is, for all $q \in \mathcal{Q}$, $D(\bar{q}+q) = D(\bar{q}-q)$, $P(\bar{q}+q) = P(\bar{q}-q)$, $B(\bar{q}+q) = B(\bar{q}-q)$.*

Remark 1.3. Several robotic system satisfy Assumption 1.1, such as double pendulum, Furuta pendulum and planar biped robot.

Lemma 1.2 (Lemma 3.3 in [5])**.** *Suppose that Assumption 1.1 holds and that the functions in* 2) *are chosen to be also* odd*. Consider Procedure 1.1 and fix the initial condition $\gamma_{i_1}^0 = 0$. Then, the corresponding T-periodic solution of (1.23) is odd and γ is an odd parametric VHC.*

The following proposition gives a sufficient condition for a reduced system to be Lagrangian, and together with Lemma 1.2 can be used in a procedure for finding regular VHCs whose constrained system is Lagrangian.

Proposition 1.4 (Proposition 4.4 in [6])**.** *Suppose that Assumption 1.1 holds and that γ is an odd parametric VHC. Then function $\mathscr{L}(\lambda, \dot{\lambda}) = \frac{1}{2}M(\lambda)\dot{\lambda}^2 - V(\lambda)$ is the Lagrangian function of the constrained system on $\mathbb{S} \times \mathbb{R}$, where $M(\lambda)$ and $V(\lambda)$ are given by (1.20) and (1.21), respectively.*

Example 1.4. The double pendulum is a set of two particles on $\mathbb{R}^2$, where the first particle maintains a fixed distance with respect to a fixed point, and the latter maintains a fixed distance with respect to the first one. Namely, a link connects the first particle to the origin and a second one the second particle to the first.[11] We assume that both particles have unitary mass and are subject to gravity.[12]

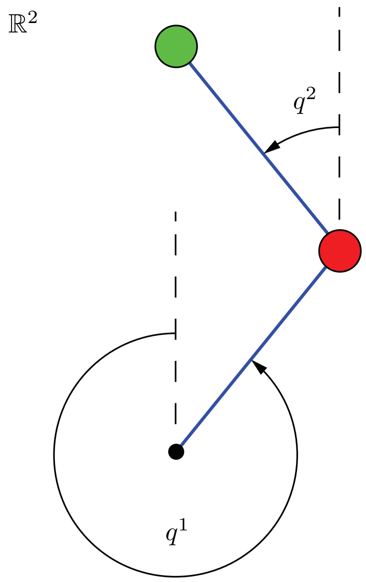


Figure 1.5: Double pendulum.

The configuration space of the system is a torus $\mathcal{Q} = \mathbb{S} \times \mathbb{S}$. We choose the coordinates as depicted in Figure 1.5, where $q_1 \in [\mathbb{R}]_{2\pi}$ denotes the orientation of the first link and $q_2 \in [\mathbb{R}]_{2\pi}$ denotes the orientation of the second one. The Lagrangian function has the form (1.1), where the inertial matrix is

$$D(q) = \begin{pmatrix} 2 & \cos(q_1 - q_2) \\ \cos(q_1 - q_2) & 1 \end{pmatrix}$$

[11] The masses of these two links are considered negligible.
[12] The gravity acceleration is considered unitary.

and the potential function is $P(q) = 2\cos q_1 + \cos q_2$. A control input τ acts on the first link, thus $B(q) = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$. This Lagrangian mechanical control system is known also as *pendubot*. The motion of the system satisfies the following forced Euler-Lagrange equations

$$\begin{aligned} 2\ddot{q}_1 + \cos(q_1 - q_2)\ddot{q}_2 + \sin(q_1 - q_2)\dot{q}_2^2 - 2\sin q_1 &= \tau \\ \cos(q_1 - q_2)\ddot{q}_1 + \ddot{q}_2 - \sin(q_1 - q_2)\dot{q}_1^2 - \sin q_1 &= 0. \end{aligned}$$

Now, we want to find some regular VHCs γ for this system, parametrized with respect to variable $\lambda \in [\mathbb{R}]_{2\pi}$. Consider Procedure 1.1 with the annihilator vector field

$$\eta_q = \begin{pmatrix} \cos(q_1 - q_2) \\ 1 \end{pmatrix}.$$

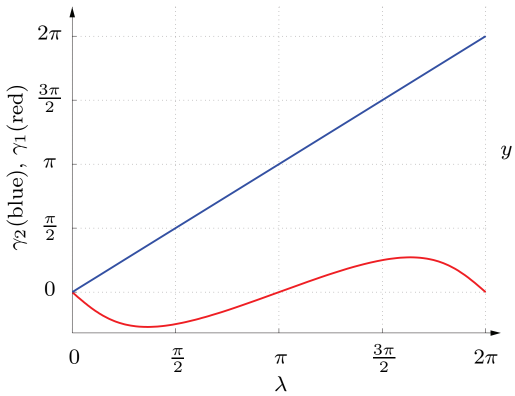
Following the procedure, set $\gamma_2 = \gamma_{i_1}$, $\gamma_1 = \gamma_{i_2}$. We choose the function[13] $\gamma_2(\lambda) = \lambda$ then, the first order differential equation (1.23) becomes

$$\gamma_1'(\lambda) = -\cos(\lambda - \gamma_1(\lambda)) + \delta. \tag{1.25}$$

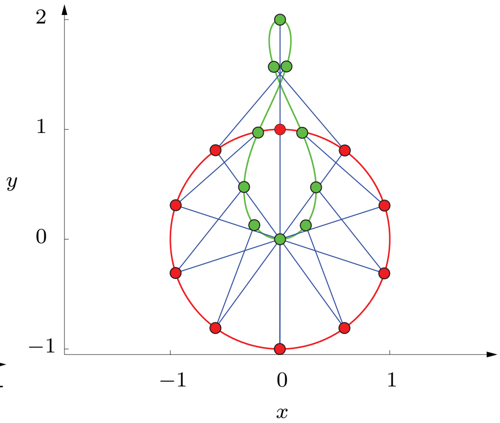
Moreover, we choose the initial condition $\gamma_2(0) = 0$ and the degree of γ_1 is chosen to be $d = 0$. The solution of (1.25) with $\delta = 0$ is not a 2π-periodic function, therefore (1.24) is satisfied. Finally, setting $\mu = 1$, we found that the solution of (1.25) is 2π-periodic if $\epsilon = 0.4142$. Thus, the VHC given by $\gamma = (\gamma_2, \gamma_1)^T$ is *regular*. Figure 1.6a shows the component functions of the obtained regular VHC. Figure 1.6b depicts the configurations of the double pendulum satisfying the VHC.

By Proposition 1.4, the constrained system is a Lagrangian mechanical system with configuration space $\mathbb{S}$. The Lagrangian function has the form (1.1) with inertial matrix M given by (1.20) and potential V given by (1.21). Figure 1.6c shows the functions M and V, which are 2π-periodic as expected by the statement of Theorem 1.2. Figure 1.6d depicts the orbit of the reduced dynamics such that $\lambda(0) = 0.1$ and $\dot{\lambda} = 0$.

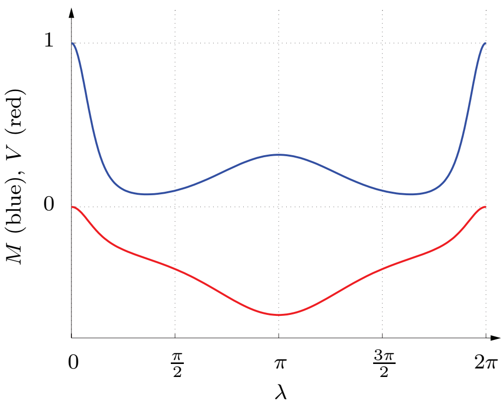
[13] Note that the function is continuous and odd.



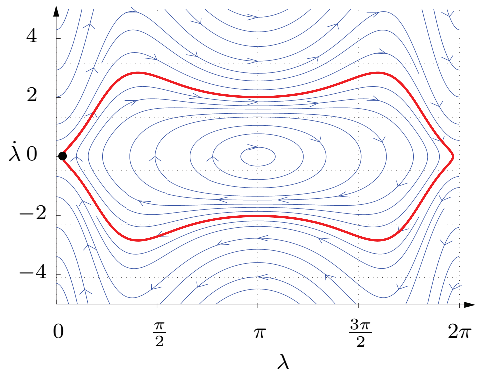
(a) Component functions of γ.



(b) The two particles subjected to the VHC.



(c) Functions M and V for the constrained system.



(d) Orbit of the reduced dynamics with initial condition $\lambda(0) = 0.1$, $\dot{\lambda}(0) = 0$.

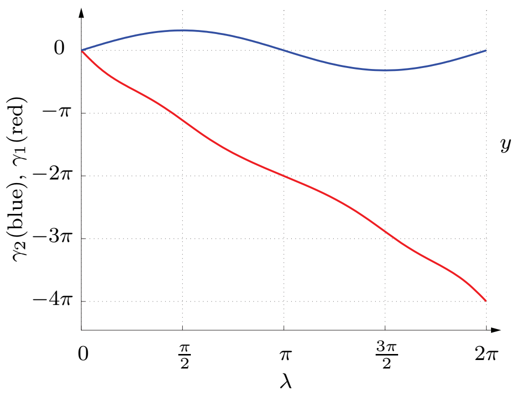
Figure 1.6: Example 1.4. The regular VHC γ obtained by Procedure 1.1 setting $\gamma_2(\lambda) = \lambda$ and $d = 0$.

In the following examples, we try to generate regular VHCs following Procedure 1.1. As before, we choose the second component of η, the initial condition $\gamma_1(0) = 0$ and the function $\mu = 0$.

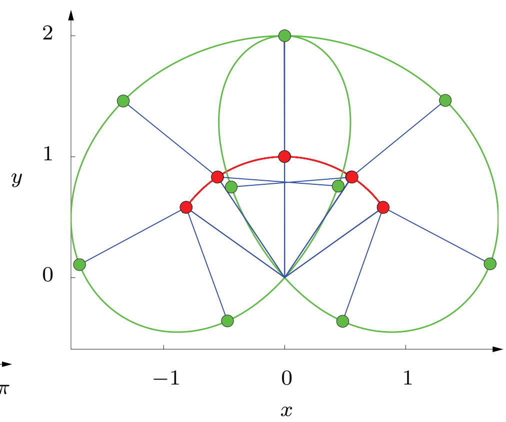
Case $\boldsymbol{\gamma_2 = \sin\lambda}$.
Setting $\gamma_2 = \sin\lambda$ and $d = 0$ condition (1.24) is not satisfied. Instead, setting $d = 2$, condition (1.24) holds. Then, the regular VHC is given by $\gamma = (\gamma_2, \gamma_1)^T$ where γ_2 is given and γ_1 is the solution of (1.23) with $\epsilon = 2.3391$. Figure 1.7a shows the component functions of the obtained regular VHC. Figure 1.7b depicts the configurations of the double pendulum on the VHC. Since the hypotheses of Proposition 1.4 are satisfied, the constrained system is a Lagrangian mechanical system on $\mathbb{S}$. Figure 1.7c shows functions M and V. Figure 1.7d shows the orbit on the constraint manifold for the initial condition $\lambda(0) = 0.1$, $\dot{\lambda} = 0$.

Case $\boldsymbol{\gamma_2 = \lambda - \cos\lambda}$.
Set $\gamma_2 = \lambda - \cos\lambda$ and $d = 1$. Condition (1.24) is satisfied, and for $\epsilon = -1.3420$ the solution of (1.23) is 2π-periodic, so that $\gamma = (\gamma_2, \gamma_1)^T$ is a regular VHC. Since γ_1 is not an odd function, the hypotheses of Proposition 1.4 do not hold. Moreover, functions M and V given by (1.20) and (1.21), are **not** 2π-periodic (as shown in Figure 1.8c). Then, by Theorem 1.2, the constrained system is not a Lagrangian mechanical system on $\mathbb{S}$. As shown in Figure 1.8d, the reduced dynamics are unstable.

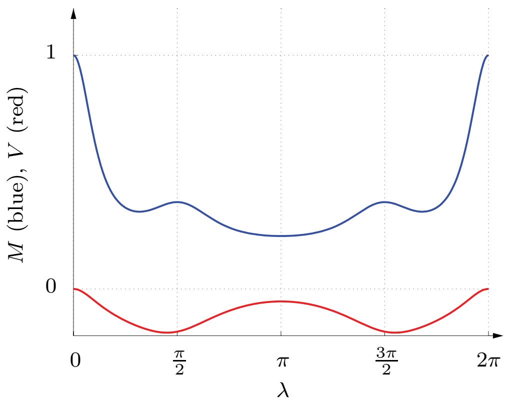
Other examples.
Figure 1.9 depicts the configuration of the double pendulum subjected to other regular VHCs generated by Procedure 1.1.



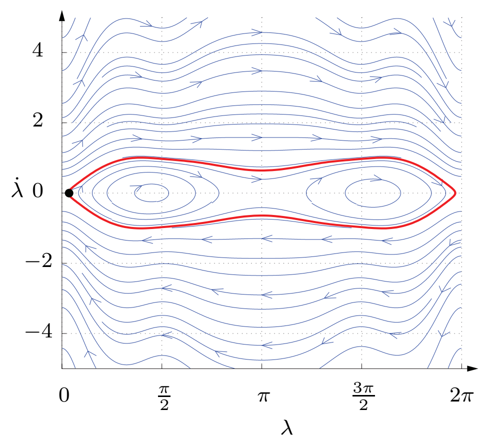
(a) Component functions of γ.



(b) The two particles subjected to the VHC.

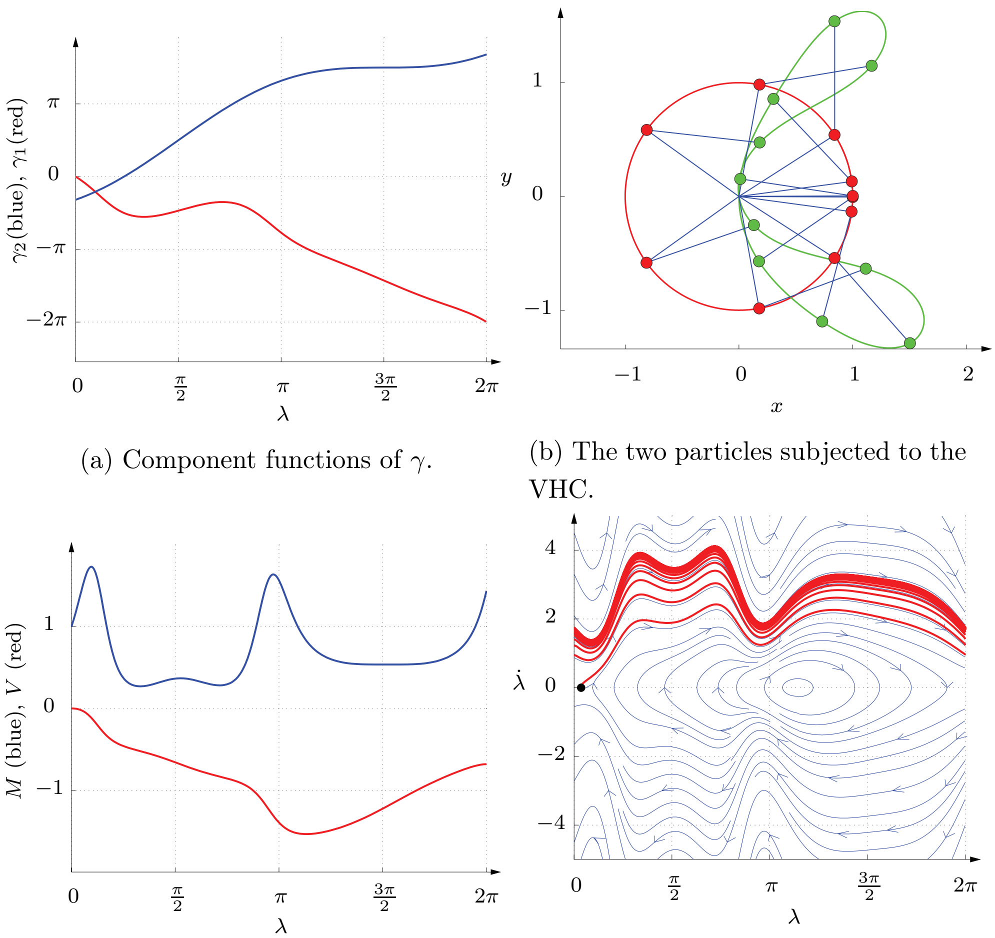


(c) Functions M and V for the constrained system.



(d) Orbit of the reduced dynamics with initial condition $\lambda(0) = 0.1$, $\dot{\lambda}(0) = 0$.

Figure 1.7: Example 1.4. The regular VHC γ obtained by Procedure 1.1 setting $\gamma_2(\lambda) = \sin \lambda$ and $d = 2$.



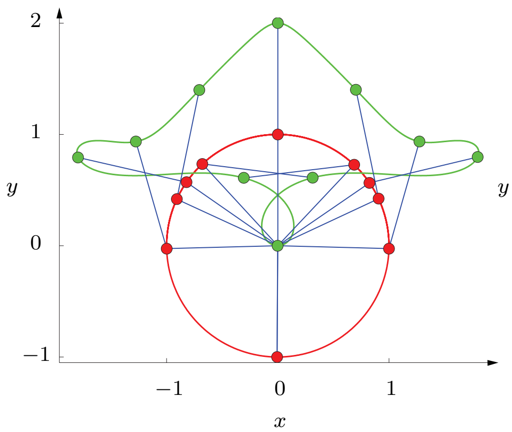
(a) Component functions of γ.

(b) The two particles subjected to the VHC.

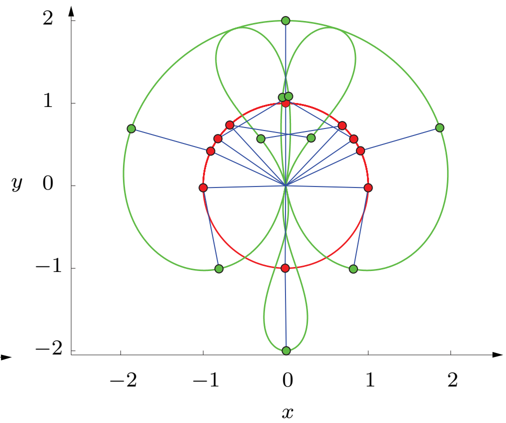
(c) Functions M and V for the constrained system.

(d) Orbit of the reduced dynamics with initial condition $\lambda(0) = 0.1$, $\dot{\lambda}(0) = 0$.

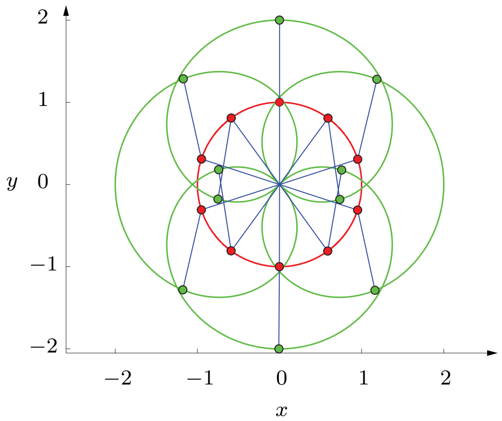
Figure 1.8: Example 1.4. The regular VHC γ obtained by Procedure 1.1 setting $\gamma_2(\lambda) = \lambda - \cos\lambda$ and $d = 1$.



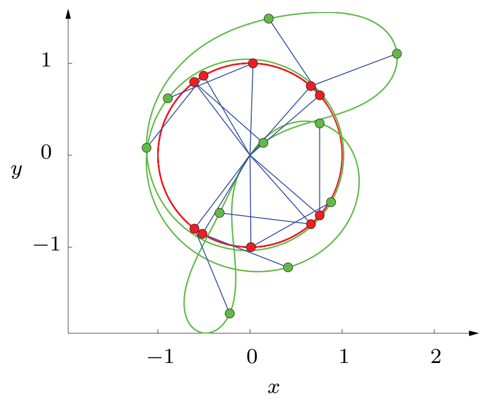
(a) The two particles subjected to the VHC generated by setting $\gamma_2(\lambda) = \lambda - 3\sin\lambda$ and $d = 0$.



(b) The two particles subjected to the VHC generated by setting $\gamma_2(\lambda) = \lambda - 3\sin\lambda$ and $d = 2$.



(c) The two particles subjected to the VHC generated by setting $\gamma_2(\lambda) = \lambda$ and $d = -5$.



(d) The two particles subjected to the VHC generated by setting $\gamma_2(\lambda) = \lambda + 4\cos\lambda$ and $d = -3$.

Figure 1.9: Example 1.4. Some examples of regular VHCs generated by Procedure 1.1.

Chapter 2

Synthesis of One Dimensional Virtual Holonomic Constraints

If people do not believe that mathematics is simple,
it is only because they do not realize how complicated life is.

– John von Neumann

In this chapter, we consider Lagrangian mechanical control systems with underactuation degree equal to **one** and a configuration space given by an n-dimensional **generalized cylindrical manifold**, with $n \geq 3$. We present a synthesis method that, under some hypotheses, allows to generate **regular VHCs**, with order $n-1$, **whose reduced dynamics possess an asymptotically stable limit cycle**. Remember that a regular VHC is a VHC that, under mild hypotheses, is stabilizable with an input-output linearized feedback, and possesses well defined reduced dynamics. Some of the results in this chapter have been presented in [46].

In Subsection 1.3.1, we reported a synthesis approach developed by Consolini and Maggiore in [5]. Under some hypotheses, if $n-1$ components of a parametric VHC are freely assigned, Procedure 1.1 allows to compute the remaining component as the solution of a scalar differential equation, so that the resulting VHC is regular.

The synthesis approach presented in this chapter is different with respect to Procedure 1.1. It transforms a given periodic curve γ, starting at an assigned configuration q_0, into a regular VHC, denoted by $G[\gamma]$, similar to γ. This procedure is a generalization of the procedure developed in [9] and it is described in Section 2.1.

The main contributions of this chapter are presented in Section 2.2. In particular, Theorem 2.2 and Proposition 2.4 present sufficient conditions on γ such that the reduced dynamics of $G[\gamma]$ have an asymptotically stable limit cycle or are unstable. Finally, Theorem 2.3 presents a constructive procedure for finding a VHC, starting at an assigned configuration q_0, that possesses an asymptotically stable limit cycle.

The synthesis method presented here is valid only for system with $n \geq 3$ and number of actuators $n-1$. Moreover, the method is valid only for parametric VHCs diffeomorphic to $\mathbb{S}$. We refer to the notation of Section 1.1, and, in the following, we add some useful definitions.

A **closed regular curve** is a smooth function $\gamma : [\mathbb{R}]_{2\pi} \to \mathcal{Q}$ such that γ', its velocity, never vanishes. The curve is **simple** if it does not have self-intersections. The **length function** of γ is $L_\gamma : [0, 2\pi] \to \mathbb{R}$ defined as $L_\gamma(\lambda) = \int_0^\lambda \|\gamma'(\mu)\| d\mu$ and the length of γ is given by $L_\gamma = L_\gamma(2\pi)$. Curve γ is **planar** if there exists a two dimensional subspace $V \subset \mathbb{R}^n$ such that $\gamma'(\lambda) \in V$, $\forall \lambda \in [0, 2\pi]$. For a regular, closed, planar and simple curve γ, the **isoperimetric quotient** is the ratio

$$Q_\gamma = \frac{4\pi A_\gamma}{L_\gamma^2},$$

where A_γ is the area enclosed by curve γ. In other words, Q_γ is the ratio of the area enclosed by γ and the area of the circle with the same perimeter.

The following assumptions define the class of systems that we consider in this part of the thesis.

Assumption 2.1. *Each component q_i, $i = 1, \dots, n$, is either a linear displacement in $\mathbb{R}$ or an angular displacement in $[\mathbb{R}]_{2\pi}$.*

Assumption 2.2. *There exists a left annihilator of $D^{-1}B$, that is, there exists a smooth vector field η which does not vanish in $\mathcal{Q}$ and is such that $\eta_q^T D^{-1}(q)B(q) = 0$, $\forall q \in \mathcal{Q}$.*

Remark 2.1. Given $\gamma : [\mathbb{R}]_{2\pi} \to \mathbb{R}^n$, define $\bar{\gamma}(\lambda) = p(\gamma(\lambda))$, where p is the covering map from $\mathbb{R}^n \to \mathcal{Q}$. We say that γ is a regular VHC if $\bar{\gamma}$ is a regular VHC. In this way it is possible to define VHCs as closed curves on $\mathbb{R}^n$ instead of $\mathcal{Q}$. Note that this can be done since $\mathcal{Q}$ is a generalized cylinder.

A **limit cycle** is a nontrivial closed isolated orbit. Consider the differential equation given by the reduced dynamics such that $(\lambda, \dot{\lambda}) \in \Theta \times \mathbb{R}$. The limit cycle $\hat{\lambda}$ is **asymptotically stable** if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $(\lambda(0), \dot{\lambda}(0)) \in ||(\lambda, \dot{\lambda})||_{\hat{\lambda}} < \delta$ the integral curve of the system remains in $||(\lambda(t), \dot{\lambda}(t))||_{\hat{\lambda}} < \epsilon, t > 0$ and asymptotically converges to $\hat{\gamma}$, if $\lim_{t \to +\infty} ||(\lambda(t), \dot{\lambda}(t))||_{\hat{\lambda}} = 0$.

Since the VHC is regular, the reduced dynamics are given by Proposition 1.3. As discussed in [7], the stability properties of the reduced dynamics (1.17) depend on ψ_1 and ψ_2. These dynamics can be Lagrangian, such as the case considered in [47], have an asymptotically stable limit cycle, such as for the bicycle model discussed in [48], or be unstable. The last two cases are characterized by the following proposition, which completes Proposition 4.1 of [48].

Proposition 2.1. *Let be given the differential equation*

$$\ddot{\lambda} = \psi_1(\lambda) + \psi_2(\lambda)\dot{\lambda}^2 \tag{2.1}$$

where ψ_1 and ψ_2 are continuous 2π-periodic functions, and $\psi_1(\lambda) > 0$, $\forall \lambda \in [0, 2\pi]$, then:

1) *If $\int_0^{2\pi} \psi_2(\lambda) d\lambda < 0$, (2.1) has two limit cycles λ^+, λ^-, with $\dot{\lambda}^+(t) > 0$ and $\lambda^-(t) = \lambda^+(-t)$, $\forall t \in \mathbb{R}$. Moreover λ^+ is asymptotically stable, while λ^- is unstable.*

2) If $\int_0^{2\pi}\psi_2(\lambda)d\lambda > 0$, (2.1) has no limit cycles and satisfies $\lim_{t\to\infty}\dot{\lambda}(t) = +\infty$.

Proof. **Part 1).** Consider the differential equation

$$\begin{cases} z'(\lambda) = 2\psi_1(\lambda) + 2\psi_2(\lambda)z \\ z(0) = z_0 \end{cases} \tag{2.2}$$

and set $m(\lambda) = e^{\int_0^\lambda 2\psi_2(\lambda)d\lambda}$. The solution of (2.2) satisfies $z(2\pi) - z_0 = f(z_0)$, where

$$f(z_0) = m(2\pi)z_0 + \int_0^{2\pi} m(2\pi)m(\tau)^{-1}\psi_1(\tau)d\tau - z_0\,.$$

Since $f(0) = \int_0^{2\pi} m(2\pi)m(\tau)^{-1}\psi_1(\tau)d\tau > 0$ and $f'(z) = m(2\pi) - 1 < 0$, by continuity of f, there exists $\bar{z}_0 > 0$ such that $f(\bar{z}_0) = 0$. Let $\bar{z}$ be the solution of (2.2) with the initial condition $z_0 = \bar{z}_0$ (note that $z(t) > 0$, $\forall t \in \mathbb{R}$) and let λ^+ be the solution of $\dot{\lambda}^+(t) = \sqrt{\bar{z}(\lambda^+(t))}$, with the initial condition $\lambda(0) = 0$. Then λ is a 2π - periodic solution of (2.1) with $\dot{\lambda}^+(t) > 0$, $\forall t \in \mathbb{R}$. Set $\lambda^-(t) = \lambda^+(-t)$, then a direct substitution shows that also λ^- is a solution of (2.1). The asymptotic stability of λ^+ is a consequence of Proposition 4.1 of [48]. Finally, since $\lambda^-(t) = \lambda^+(-t)$, λ^- is an unstable limit cycle.

Part 2). Consider the differential equation

$$\begin{cases} z'(\lambda) = 2\psi_1(\lambda) + 2\psi_2(\lambda)z \\ z(0) = \dot{\lambda}(0)^2\,, \end{cases} \tag{2.3}$$

the assumptions on ψ_1 and ψ_2 (i.e. $\psi_1(\lambda) > 0$, $\forall t \in \mathbb{R}$ and $\int_0^{2\pi}\psi_2(\lambda)d\lambda > 0$) imply that $\lim_{\lambda\to+\infty} z(\lambda) = +\infty$ and $\lim_{\lambda\to-\infty} z(\lambda) = -\infty$. Note that the solution of (2.3) satisfies

$$\dot{\lambda}^2(t) - z(\lambda(t)) = 0\,. \tag{2.4}$$

By continuity of ψ_1 and ψ_2, there exists $\rho > 0$ such that the set $\Omega = \{(\lambda, \dot{\lambda}) : |\dot{\lambda}| \leq \rho\}$ is such that $\psi_1(\lambda) + \psi_2(\lambda)\dot{\lambda}^2 > 0$, $\forall(\lambda, \dot{\lambda}) \in \Omega$. There exists a time $\hat{t}$

for which $\dot{\lambda}(\hat{t}) \geq -\rho$. In fact, if this is not the case, then $\lim_{t\to+\infty} \lambda(t) = -\infty$ and $\lim_{t\to\infty} z(\lambda(t)) = \lim_{\lambda\to-\infty} z(\lambda) = -\infty$ which contradicts the fact that $z(\lambda) > 0$ as a consequence of (2.4). Hence, by the definition of set Ω, there exists a time $\bar{t}$ such that the solution of (2.1) satisfies $\dot{\lambda}(t) > \rho$, $\forall t \geq \bar{t}$. This implies that $\lim_{t\to\infty} \lambda(t) = +\infty$, and, by (2.4), that $\lim_{t\to\infty} \dot{\lambda}(t) = +\infty$. $\square$

In case 1) of the statement of Proposition 2.1, we will say that the reduced dynamics **have an asymptotically stable limit cycle**, in case 2) we will say that the reduced dynamics are **unstable**.

Remark 2.2. The case in which $\psi_1(\lambda) < 0$, $\forall \lambda \in \mathbb{R}$, can be reduced to the case discussed in Proposition 2.1. Namely, setting $\nu = -\lambda$, equation (2.1) becomes $\ddot{\zeta} = -\psi_1(-\nu) - \psi_2(-\nu)\dot{\nu}^2$. Setting $\bar{\psi}_1(\nu) = -\psi_1(-\nu)$ and $\bar{\psi}_2(\nu) = -\psi_2(-\nu)$, the system has the form (2.1), with $\bar{\psi}_1(\nu) > 0$, $\forall \nu \in \mathbb{R}$.

2.1 Synthesis of Regular VHCs

In this section, we address the problem of determining a closed VHC that starts at an assigned configuration, namely:

Problem 2.1. *Let $q_0 \in \mathcal{Q}$ be an assigned configuration, find a closed regular curve $\gamma : [\mathbb{R}]_{2\pi} \to \mathcal{Q}$, that defines a regular VHC and such that $\gamma(0) = q_0$.*

As anticipated in Section 1.1.1, we denote by U the **input distribution** of the Lagrangian mechanical control system. For an assigned configuration $q_0 \in \mathcal{Q}$, we consider a curve γ that satisfies the following assumptions.

Assumption 2.3. *Curve $\gamma : [\mathbb{R}]_{2\pi} \to \mathbb{R}^n$ is smooth, regular, planar and simple, moreover $p(\gamma(0)) = q_0$ and $\gamma'(\lambda) \in U(q_0)$, $\forall \lambda \in [0, 2\pi]$. Note that the function p is the covering map $p : \mathbb{R}^n \to \mathcal{Q}$ (see Section 1.1).*

In particular, the last assumption requires γ to belong to the affine subspace passing at q_0 and parallel to $U(q_0)$. We will define a map G that determines a new closed curve, denoted by $G[\gamma]$, that starts at q_0 and defines a regular VHC.

In the following, we will assume that the following regularity properties are satisfied.

Assumption 2.4. *There exist real constants D_0, D_1, D_2, D_3, D_4 such that, $\forall q \in \mathcal{Q}$, $r, s \in \mathbb{R}^n$,*

$$
\begin{aligned}
\|\eta_q\| &\leq D_0\,, \\
\|(\mathcal{L}_s\hat{\eta})_q\| &\leq D_1\|s\|\,, && (2.5)\\
\|(\mathcal{L}_r(\mathcal{L}_s\hat{\eta}))_q\| &\leq D_2\|r\|\|s\|\,, \\
\|(\mathcal{L}_s\nabla P)_q\| &\leq D_3\|s\|\,, && (2.6)\\
\|(\mathcal{L}_s D)(q)\| &\leq D_4\|s\|\,. && (2.7)
\end{aligned}
$$

For $q \in \mathcal{Q}$, denote by $\Pi_q : \mathbb{R}^n \to \mathbb{R}^n$ the orthogonal projection onto the input subspace at q, namely, $\forall v \in \mathbb{R}^n$,

$$\Pi_q(v) = v - \hat{\eta}_q \langle \hat{\eta}_q, v\rangle \,, \tag{2.8}$$

and let the complementary projection $\Pi^\perp$ be defined as $\Pi_q^\perp = I - \Pi_q$, $\forall q \in \mathbb{R}^n$. For the sake of simplicity, if $q \in \mathbb{R}^n$, we write $\Pi_{p(q)}$ as Π_q.

Let $\delta \in \mathbb{R}^n$ be an assigned parameter and define a curve γ_δ as the solution of the following differential equation

$$\begin{cases} \gamma'_{\delta(\lambda)} = \Pi_{\gamma_\delta(\lambda)}\left(\gamma'(\lambda)\right) + \delta\|\gamma'(\lambda)\| \\ \gamma_\delta(0) = \gamma(0)\,. \end{cases} \tag{2.9}$$

The form of equation (2.9) comes from the observation that, since

$$\langle \eta_q, \Pi_q(v)\rangle \;=\; 0, \forall q \in Q, \forall v \in \mathbb{R}^n,$$

the transversality condition (1.16) for curve γ_δ evaluates as

$$\left\langle \eta_{\gamma_\delta(\lambda)}, \gamma'_\delta(\lambda)\right\rangle = \left\langle \eta_{\gamma_\delta(\lambda)}, \delta\right\rangle \|\gamma'(\lambda)\| \neq 0\,; \tag{2.10}$$

this implies that, if

$$\gamma_\delta(0) = \gamma_\delta(2\pi) \text{ and } \left\langle \eta_{\gamma_\delta(\lambda)}, \delta\right\rangle \neq 0,\ \forall\lambda \in [0, 2\pi]\,, \tag{2.11}$$

then γ_δ is a closed curve that defines a regular VHC.

The following theorem shows that, if the input distribution is not involutive at q_0, then, for every curve γ of sufficiently small length, whose isoperimetric constant is bounded from below by a positive constant, there exists a value of δ such that conditions (2.11) hold and, consequently, curve γ_δ defined in (2.9) is a regular VHC.

Theorem 2.1. *Let v, w be vector fields on $\mathcal{Q}$ such that $v, w \subset U$ and $[v, w]_{q_0} \notin U(q_0)$. Then, for every real positive constant C, there exist a real constant L such that, for any closed planar regular curve γ satisfying Assumptions 2.3 and such that $\gamma'(\lambda) \in span\{v_{q_0}, w_{q_0}\}$, $\mathrm{L}_\gamma \leq L$ and $Q_\gamma \geq C$, there exists a vector δ^* for which the solution of (2.9) is 2π-periodic and γ_{δ^*} defines a regular VHC. Moreover δ^* is the limit as $i \to \infty$ of the iteration*

$$\begin{cases} \delta(i+1) = \delta(i) - \frac{\gamma_{\delta(i)}(2\pi) - q_0}{L_\gamma} \\ \delta(0) = 0\,. \end{cases} \tag{2.12}$$

Hence, if the hypotheses of Theorem 2.1 are satisfied, iteration (2.12) and equation (2.9) allow finding a modified trajectory γ_{δ^*} that solves Problem 2.1. As a consequence of Theorem 2.1, for all curves γ satisfying the hypotheses of Theorem 2.1, operator

$$G[\gamma] = \gamma_{\delta^*}$$

is well defined. In this way, operator G transforms an assigned curve γ into a new curve $G[\gamma]$ that defines a regular VHC. For this reason, we refer to operator G as the **virtual constraint generator**; note the structure similar to equation (6) in [5]. We postpone the proof of Theorem 2.1 to Section 2.1.1.

The following proposition, that will also be proved in Section 2.1.1, is a direct consequence of Theorem 2.1. It shows that, if the input distribution U is **not involutive** at q_0, then it is always possible to choose a curve γ such that $G[\gamma]$ is well defined and is a regular VHC.

Proposition 2.2. *Assume that the input distribution U is not involutive at $q_0 \in \mathcal{Q}$, then there exists a curve γ that satisfies Assumption 2.3 and such that iteration (2.12) is convergent (i.e., $G[\gamma]$ is well defined and is a regular VHC).*

2.1.1 Proofs

***Note* 2.1.** The proof of Theorem 2.1 can be split in two part. The first part proves that, given a curve γ and under the assumptions of the claim, there exists a vector δ^* such that the curve obtained solving (2.9), γ_{δ^*}, is a closed curve. The second part proves that such curve is a regular VHC.

The first part is given by part *i*) of Lemma 2.3, which proves that there exists a length L such that, if the length of γ is less than L, function (2.12) is a contraction.

The second part is given by Proposition 2.3. The proof of that proposition consider condition (2.11), which can be written as

$$\Pi^{\perp}_{\gamma_{\delta^*}(\lambda)}(\delta^*) \neq 0, \forall \lambda \in [\mathbb{R}]_{2\pi}. \tag{2.13}$$

The left member of 2.13 can be written as the sum of three terms (see 2.33). By *ii*) and *iii*) of Lemma 2.3, two of these terms are upper bounded, whereas, by Lemma 2.4 and the condition (2.32), the remaining term is lower bounded. These bounds depend on the length of γ. If the length of γ is less than a constant L and condition (2.32) is satisfied, the last member is greater than the sum of the other two. In this case, the sum of these three members does not change sign on γ and (2.11) is satisfied. The hypotheses of Theorem 2.1 on the involutivity of q_0 and on the isoperimetrical quotient imply condition (2.32) and, as a consequence, the regularity of the curve γ_{δ^*}.

Note that, conditions *ii*) and *iii*) of Lemma 2.3 are consequences of Lemma 2.2 when δ is chosen equal to δ^*. Lemma 2.1 and Remark 2.3 are instrumental to the next results of this section.

For all $q \in \mathcal{Q}$, let $X_q : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function. Analogously as before, the **directional derivative** of X at q_0 with respect to $v \in \mathbb{R}^n$ is the linear function $(\mathcal{L}_v(X))_{q_0} : \mathbb{R}^n \to \mathbb{R}^m$, such that

$$(\mathcal{L}_v(X))_{q_0}(w) = \frac{d}{dz}(X_{q_0+zv}(w)), \; \forall w \in \mathbb{R}^n .$$

Remark 2.3. A straightforward computation shows that Assumption 2.4 implies that, $\forall q, r \in \mathcal{Q}, v, x, w \in \mathbb{R}^n$,

$$\begin{aligned} &\|(\mathcal{L}_v \Pi)_q(x)\| \leq 2D_1 \|v\| \|x\| \,, \\ &\|(\mathcal{L}_w \mathcal{L}_v \Pi)_q(x)\| \leq 2(D_2 + D_1^2) \|v\| \|w\| \|x\| \,, \end{aligned} \tag{2.14}$$

$$\|\Pi_q(v) - \Pi_r(v)\| \leq 2D_1 \|q - r\| \|v\| \,. \tag{2.15}$$

The following lemma characterizes the distance between the solutions of similar differential equations.

Lemma 2.1. *Let C, M be real positive constants and $f, g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $s : \mathbb{R} \to \mathbb{R}$ be smooth functions such that*

$$\|f(\lambda, x) - g(\lambda, y)\| \leq C\|x - y\| + s(\lambda), \; \forall x, y \in \mathbb{R}^n, \forall \lambda \in [0, M] \,. \tag{2.16}$$

Let $x, y : \mathbb{R} \to \mathbb{R}^n$ be solutions of the differential equations

$$\dot{x}(\lambda) = f(\lambda, x), \; \dot{y}(\lambda) = g(\lambda, y) \,,$$

with initial conditions $x(0) = y(0)$. Then, it follows that

$$\|x(\lambda) - y(\lambda)\| \leq e^{MC} \left(\int_0^\lambda |s(\nu)| d\nu \right), \; \forall \lambda \in [0, M] \,.$$

Proof. Set $e = x - y$, then

$$\begin{cases} \dot{e}(\lambda) = f(\lambda, x) - g(\lambda, y) \\ e(0) = 0 \,, \end{cases}$$

by hypothesis (2.16)

$$\|\dot{e}(\lambda)\| \leq C\|e(\lambda)\| + s(\lambda) \,,$$

by integration

$$\|e(\lambda)\| \leq \int_0^\lambda e^{C(\lambda - \nu)} |s(\nu)| d\nu \,.$$

Then, $\forall \lambda \leq M$

$$\|e(\lambda)\| \leq e^{MC} \int_0^\lambda |s(\nu)| d\nu \,.$$

$\square$

Consider the following function

$$z_\delta(\lambda) = \gamma(\lambda) + \int_0^\lambda (\mathcal{L}_{\gamma(\mu)-q_0}\Pi)_{q_0}\left(\gamma'(\mu)\right) d\mu + L_\gamma(\lambda)\delta\,. \tag{2.17}$$

The next Lemma shows that function (2.17) is an approximation of the solution of (2.9) and gives an upper bound of the difference.

Lemma 2.2. *There exists a positive real constant C_1 such that, for any $\delta \in \mathbb{R}^n$ and any closed curve $\gamma : [0, 2\pi] \to \mathbb{R}^n$ such that*

$$\Pi_{q_0}\gamma'(\lambda) = \gamma'(\lambda)\,, \tag{2.18}$$

the following inequality holds

$$\|\gamma_\delta(\lambda) - z_\delta(\lambda)\| \le C_1(L_\gamma(\lambda)^3 + L_\gamma(\lambda)^2\|\delta\|),\ \forall\lambda \in [0, 2\pi]\,, \tag{2.19}$$

where γ_δ is the solution of (2.9) and z_δ defined as (2.17). Moreover

$$\|z_0(2\pi) - z_0(0)\| \le D_1 L_\gamma^2\,. \tag{2.20}$$

Proof. Inequality (2.20) follows from (2.14) since

$$\begin{aligned}\|z_0(2\pi) - z_0(0)\| = \left\|\int_0^{2\pi} (\mathcal{L}_{\gamma(\mu)-q_0}\Pi)_{q_0}\left(\gamma'(\mu)\right) d\mu\right\| \\ \le 2D_1\int_0^{2\pi} L_\gamma(\mu)\ \left\|\gamma'(\mu)\right\| d\mu \le D_1 L_\gamma^2\,.\end{aligned}$$

By (2.18), differentiating (2.17):

$$z'_\delta(\lambda) = \Pi_{q_0}\left(\gamma'(\lambda)\right) + (\mathcal{L}_{\gamma(\lambda)-q_0}\Pi)_{q_0}\left(\gamma'(\lambda)\right) + \|\gamma'(\lambda)\|\delta,$$

then

$$\begin{aligned}(\gamma_\delta - z_\delta)'(\lambda) = \Pi_{\gamma_\delta(\lambda)}\left(\gamma'(\lambda)\right) - \Pi_{q_0}\left(\gamma'(\lambda)\right) - (\mathcal{L}_{\gamma(\lambda)-q_0}\Pi)_{q_0}\left(\gamma'(\lambda)\right) = \\ \left(\Pi_{\gamma_\delta(\lambda)} - \Pi_{z_\delta(\lambda)}\right)\left(\gamma'(\lambda)\right) + \left(\Pi_{z_\delta(\lambda)} - \Pi_{\gamma(\lambda)}\right)\left(\gamma'(\lambda)\right) + \\ \left(\Pi_{\gamma(\lambda)} - \Pi_{q_0}\right)\left(\gamma'(\lambda)\right) - (\mathcal{L}_{\gamma(\lambda)-q_0}\Pi)_{q_0}\left(\gamma'(\lambda)\right)\,.\end{aligned}$$

Moreover, by (2.15)

$$\left\| \left(\Pi_{\gamma_\delta(\lambda)} - \Pi_{z_\delta(\lambda)} \right) (\gamma'(\lambda)) \right\| \leq 2D_1 \left\| \gamma_\delta(\lambda) - z_\delta(\lambda) \right\| \max_{\lambda \in [0, 2\pi]} \left\| \gamma'(\lambda) \right\|,$$

$$\begin{aligned}
&\left\| \left(\Pi_{z_\delta(\lambda)} - \Pi_{\gamma(\lambda)} \right) \left(\gamma'(\lambda) \right) \right\| \\
&\qquad \leq 2D_1 \left\| \int_0^\lambda (\mathcal{L}_{\gamma(\mu) - q_0} \Pi)_{q_0} \left(\gamma'(\mu) \right) d\mu + L_\gamma(\lambda) \delta \right\| \left\| \gamma'(\lambda) \right\| \\
&\qquad \leq \left(2D_1^2 \int_0^\lambda L_\gamma(\mu) \left\| \gamma'(\mu) \right\| d\mu + 2D_1 L_\gamma(\lambda) \left\| \delta \right\| \right) \left\| \gamma'(\lambda) \right\| \\
&\qquad\qquad\qquad \leq D_1 \left(D_1 L_\gamma(\lambda)^2 + 2L_\gamma(\lambda) \left\| \delta \right\| \right) \left\| \gamma'(\lambda) \right\|.
\end{aligned}$$

Note that, by Taylor's theorem with remainder and (2.14),

$$\begin{aligned}
&\| \Pi_{\gamma(\lambda)} \left(\gamma'(\lambda) \right) - \Pi_{q_0} \left(\gamma'(\lambda) \right) - (\mathcal{L}_{\gamma(\lambda) - q_0} \Pi)_{q_0} \left(\gamma'(\lambda) \right) \| \leq \\
&\qquad\qquad (D_2 + D_1^2) L_\gamma(\lambda)^2 \| \gamma'(\lambda) \|, \ \forall \lambda \in [0, 2\pi].
\end{aligned}$$

From the previous inequalities, it follows that

$$\begin{aligned}
&\| (\gamma_\delta - z_\delta)'(\lambda) \| \leq 2D_1 \left(\max_{\lambda \in [0, 2\pi]} \| \gamma'(\lambda) \| \right) \| (\gamma_\delta - z_\delta)(\lambda) \| + \\
&\qquad\qquad \left((2D_1^2 + D_2) L_\gamma(\lambda) + 2D_1 \| \delta \| \right) L_\gamma(\lambda) \| \gamma'(\lambda) \|,
\end{aligned}$$

then, (2.19) is a consequence of Lemma 2.1, with C_1 defined appropriately in terms of D_1 and D_2. □

The following proposition shows that, if curve γ is sufficiently short, then (2.12) is a contraction, moreover, if δ^* is its limit as $i \to +\infty$, $G[\gamma] = \gamma_{\delta^*}$ is 2π-periodic. Further, it provides a closed-form approximation of δ^* and a maximum bound between the closed curves γ_{δ^*} and γ. These are mainly derived by previous Lemma 2.2. It is convenient to rewrite (2.12) as $\delta(k+1) = T(\delta(K))$, where $T : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$T(\delta) = \delta - \frac{\gamma_\delta(2\pi) - \gamma_\delta(0)}{\mathrm{L}_\gamma}, \tag{2.21}$$

and define

$$\bar{\delta} = \frac{z_0(0) - z_0(2\pi)}{\mathrm{L}_\gamma}.$$

Lemma 2.3. *There exist positive real constants L, C_2, C_2, C_4, C_5 such that for every closed curve γ, with $\mathrm{L}_\gamma < L$, the following properties hold:*

i) T is a contraction and converges to a fixed point δ^*, such that $G[\gamma] = \gamma_{\delta^*}$ is 2π-periodic;

ii) the following bounds hold

$$\|\delta^*\| \leq C_2 L_\gamma, \; \left\|\bar{\delta} - \delta^*\right\| \leq C_3 \mathrm{L}_\gamma^2\,; \tag{2.22}$$

iii) the following bounds hold

$$\|G[\gamma](\lambda) - \gamma(\lambda)\| \leq C_4 L_\gamma(\lambda)^2, \; \|G[\gamma](\lambda) - q_0\| \leq C_5 L_\gamma(\lambda), \; \forall \lambda \in [0, 2\pi]\,. \tag{2.23}$$

Proof. Let $\delta_1, \delta_2 \in \mathbb{R}^n$ and define $e_{\delta_1,\delta_2} : \mathbb{R} \to \mathbb{R}^n$ as

$$e_{\delta_1,\delta_2}(\lambda) = (\delta_1 - \delta_2)\,\frac{L_\gamma(\lambda)}{\mathrm{L}_\gamma} - \frac{\gamma_{\delta_1}(\lambda) - \gamma_{\delta_2}(\lambda)}{\mathrm{L}_\gamma}. \tag{2.24}$$

Note that $T(\delta_1) - T(\delta_2) = e_{\delta_1,\delta_2}(2\pi)$. The derivative of e is given by

$$\begin{aligned}
e'_{\delta_1,\delta_2}(\lambda) &= (\delta_1 - \delta_2)\frac{\|\gamma'(\lambda)\|}{\mathrm{L}_\gamma} - \frac{\gamma'_{\delta_1}(\lambda) - \gamma'_{\delta_2}(\lambda)}{\mathrm{L}_\gamma} \\
&= (\delta_1 - \delta_2)\frac{\|\gamma'(\lambda)\|}{\mathrm{L}_\gamma} - \frac{1}{\mathrm{L}_\gamma} \\
&\qquad \left[\left(\Pi_{\gamma_{\delta_1}(\lambda)} - \Pi_{\gamma_{\delta_2}(\lambda)}\right)\left(\gamma'(\lambda)\right) + (\delta_1 - \delta_2)\|\gamma'(\lambda)\|\right] \\
&= -\frac{\left(\Pi_{\gamma_{\delta_1}(\lambda)} - \Pi_{\gamma_{\delta_2}(\lambda)}\right)(\gamma'(\lambda))}{\mathrm{L}_\gamma}
\end{aligned}$$

then, by (2.15),

$$\left\|e'_{\delta_1,\delta_2}(\lambda)\right\| \leq \frac{2D_1}{\mathrm{L}_\gamma}\,\|\gamma_{\delta_1}(\lambda) - \gamma_{\delta_2}(\lambda)\|\,\|\gamma'(\lambda)\|\,. \tag{2.25}$$

From (2.24), it follows that

$$\gamma_{\delta_1}(\lambda) - \gamma_{\delta_2}(\lambda) = -e_{\delta_1,\delta_2}(\lambda) L_\gamma + (\delta_1 - \delta_2)\, L_\gamma(\lambda)\,,$$

therefore (2.25) implies

$$\left\|e'_{\delta_1,\delta_2}(\lambda)\right\| \leq (\|e_{\delta_1,\delta_2}(\lambda)\| + \|\delta_1 - \delta_2\|)\, 2D_1 \|\gamma'(\lambda)\|.$$

By integration,

$$\|e_{\delta_1,\delta_2}(\lambda)\| \leq (e^{2D_1 L_\gamma(\lambda)} - 1)\, \|\delta_1 - \delta_2\|\ ,$$

setting $\lambda = 2\pi$, it follows that

$$\|T(\delta_1) - T(\delta_2)\| \leq (e^{2D_1 L_\gamma} - 1)\, \|\delta_1 - \delta_2\|\,, \tag{2.26}$$

hence, if $e^{2D_1 L_\gamma} - 1 < 1$, that is if $\mathrm{L}_\gamma < \frac{\log 2}{2D_1}$, T is a contraction and admits a unique fixed-point, δ^*. Moreover, $\delta^* = T(\delta^*)$ implies $\gamma_{\delta^*}(2\pi) = \gamma_{\delta^*}(0)$, which proves part i) of the thesis.

By (2.26),

$$\begin{aligned} \|T(\delta^*) - T(0)\| &\leq (e^{2D_1 L_\gamma} - 1)\, \|\delta^* - 0\| \\ \|\delta^* - T(0)\| &\leq (e^{2D_1 L_\gamma} - 1)\, \|\delta^*\| \end{aligned} \tag{2.27}$$

therefore, by triangle inequality

$$\begin{aligned} \|\delta^*\| - \|T(0)\| &\leq (e^{2D_1 L_\gamma} - 1)\, \|\delta^*\| \\ \|\delta^*\| &\leq \frac{\|T(0)\|}{2 - e^{2D_1 L_\gamma}}\,, \end{aligned}$$

moreover, by (2.27),

$$\|\delta^* - T(0)\| \leq e^{2D_1 L} 2D_1 L_\gamma\, \|\delta^*\|\ . \tag{2.28}$$

By (2.20) of Lemma 2.2, it follows that $\|\bar{\delta}\| \leq D_1 L_\gamma$, moreover, by (2.19) evaluated with $\delta = 0$,

$$\begin{aligned} \|T(0) - \bar{\delta}\| &= \left\| \frac{\gamma_0(2\pi) - \gamma_0(0)}{\mathrm{L}_\gamma} - \frac{z_0(2\pi) - z_0(0)}{\mathrm{L}_\gamma} \right\| \\ &= \left\| \frac{\gamma_0(2\pi) - z_0(2\pi)}{\mathrm{L}_\gamma} \right\| \leq C_1 \mathrm{L}_\gamma^2\,, \end{aligned} \tag{2.29}$$

thus $\|T(0)\| \leq C_1 L_\gamma^2 + D_1 L_\gamma$ which implies the first of (2.22). The second of (2.22) follows from bound,

$$\begin{aligned}\left\|\bar{\delta} - \delta^*\right\| &\leq \left\|\bar{\delta} - T(0) + T(0) - \delta^*\right\| \\ &\leq \left\|\bar{\delta} - T(0)\right\| + \|T(0) - \delta^*\|\end{aligned}$$

inequalities (2.29), (2.28) and the first of (2.22).

The first of (2.23) is a consequence of (2.19), (2.22), the definition of z_{δ^*} in (2.17), and the bound

$$\|G[\gamma](\lambda) - \gamma(\lambda)\| \leq \|\gamma_{\delta^*}(\lambda) - z_{\delta^*}(\lambda)\| + \|z_{\delta^*}(\lambda) - \gamma(\lambda)\|\,,$$

and the second of (2.23) holds since the definition of γ_{δ^*} in (2.9) and (2.22) imply

$$\|\gamma_{\delta^*}(\lambda) - q_0\| \leq L_\gamma + C_2 L_\gamma^2\,.$$

$\square$

At this point we need some additional definitions. Let X, Y be vector fields on $\mathcal{Q}$ such that $X, Y \in U$ and $q \in \mathcal{Q}$, define the 2-forms

$$\begin{aligned}h_q^+(X,Y) &= \Pi_q^\perp(\mathcal{L}_X Y + \mathcal{L}_Y X)_q\,,\\ h_q^-(X,Y) &= \Pi_q^\perp(\mathcal{L}_X Y - \mathcal{L}_Y X)_q) = \Pi_q^\perp[X,Y]_q\,.\end{aligned}$$

Form h^+ is called the **second fundamental form** of the distribution U, while form h^- is called the **integrability tensor** (for more details, see chapter 2.1 of [49]). Further, set $h(X,Y)_q = \Pi_q^\perp(\mathcal{L}_X Y)_q$, note that

$$h(X,Y)_q = \frac{1}{2}(h_q^+(X,Y) + h_q^-(X,Y)),$$

hence $h(X,Y)$ is also a tensor (for each $q \in \mathcal{Q}$ it is a linear function with respect to vectors X_q, Y_q).

Note that, given a vector field v on $\mathcal{Q}$ such that $\Pi(v) = v$, for any $w \in \mathbb{R}^n$,

$$\begin{aligned}\Pi^\perp(\mathcal{L}_w(v)) =& \Pi^\perp(\mathcal{L}_w(\Pi(v))\\ =& \Pi^\perp((\mathcal{L}_w(\Pi)(v)) + \Pi(\mathcal{L}_w v))\\ =& \Pi^\perp((\mathcal{L}_w(\Pi))(v))\,.\end{aligned} \tag{2.30}$$

Given a simple closed curve $\gamma = (\gamma_1, \gamma_2)^T : [0, 2\pi] \to \mathbb{R}^2$,

$$P = \int_0^\lambda \gamma_1(\lambda)\gamma_2'(\lambda) - \gamma_2(\lambda)\gamma_1'(\lambda) d\lambda$$

represents the area enclosed by γ, positive if γ is positively oriented. Similarly, for a simple closed curve $\gamma : [0, 2\pi] \to \mathbb{R}^n$, $P_{i,j} = \int_0^\lambda \gamma_i(\lambda)\gamma_j'(\lambda) - \gamma_j(\lambda)\gamma_i'(\lambda) d\lambda$ is the (signed) area of the projection of the curve on the plane of coordinates (i, j).

The following proposition computes $\Pi_{q_0}^\perp(z_\delta(2\pi) - z_\delta(0))$, which is the component of vector $z_\delta(2\pi) - z_\delta(0)$ orthogonal with respect to $U(q_0)$. This result will be of key importance to prove that $G[\gamma]$ is a regular VHC.

Lemma 2.4. *Let $q_0 \in Q$, let vectors v_i, $i = 1, \dots, n-1$ be an orthonormal base of the image of Π_{q_0} and let γ be a curve that satisfies Assumptions 2.3, then*

$$\Pi_{q_0}^\perp(z_\delta(2\pi) - z_\delta(0)) = \sum_{\substack{i,j=1,\dots,n-1\\ i>j}} h^-(v_i, v_j)_{q_0} P_{i,j} + \mathrm{L}_\gamma \Pi_{q_0}^\perp(\delta)\,,$$

where $P_{i,j}$ denotes the (signed) area of the projection of curve γ onto the (v_i, v_j) plane.

Proof. Since $\gamma(2\pi) = \gamma(0)$, it follows that

$$\Pi_{q_0}^\perp(z_\delta(2\pi) - z_\delta(0)) = \Pi_{q_0}^\perp\left(\int_0^{2\pi} (\mathcal{L}_{\gamma(\lambda)-q_0}\Pi)_{q_0}\gamma'(\lambda)d\lambda\right) + L_\gamma \Pi_{q_0}^\perp(\delta)\,.$$

Write $\gamma(\lambda) - q_0 = \sum_{i=1}^{n-1} v_i\gamma_i(\lambda)$ and $\gamma'(\lambda) = \sum_{j=1}^{n-1} v_j\gamma_j'(\lambda)$, then, by (2.30),

$$\begin{aligned}\Pi_{q_0}^\perp\left(\int_0^{2\pi} (\mathcal{L}_{\gamma(\lambda)-q_0}\Pi)_{q_0}\gamma'(\lambda)d\lambda\right) &= \sum_{i,j=1,\dots,n-1} \Pi_{q_0}^\perp((\mathcal{L}_{v_i}v_j)_{q_0}) \int_0^{2\pi} \gamma_i(\lambda)\gamma_j'(\lambda)d\lambda \\ &= \sum_{i,j=1,\dots,n-1} h(v_i, v_j)_{q_0} s_{i,j}\,, \end{aligned} \tag{2.31}$$

where $s_{i,j} = \int_0^\lambda \gamma_i(\lambda)\gamma_j'(\lambda)d\lambda$ and we have used the definition of h. Decompose $h = h^+ + h^-$ and consider the following expressions, where $i, j = 1, \dots, n-1$

and $i \geq j$,

$$\begin{aligned}\big(h^+(v_i, v_j)s_{i,j} + h^+(v_j, v_i)s_{j,i}\big) &= h^+(v_i, v_j)\int_0^{2\pi} \big(\gamma_i(\lambda)\gamma_j'(\lambda) + \gamma_j(\lambda)\gamma_i'(\lambda)\big)\, d\lambda \\ &= h^+(v_i, v_j)(\gamma_i(2\pi)\gamma_j(2\pi) - \gamma_i(0)\gamma_j(0)) \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\big(h^-(v_i, v_j)s_{i,j} + h^-(v_j, v_i)s_{j,i}\big) &= h^-(v_i, v_j)\int_0^{2\pi} \big(\gamma_i(\lambda)\gamma_j'(\lambda) - \gamma_j(\lambda)\gamma_i'(\lambda)\big)\, d\lambda \\ &= h^-(v_i, v_j)P_{i,j}\,.\end{aligned}$$

where $P_{i,j} = s_{i,j} - s_{j,i}$ is the signed area of the projection of γ onto the (v_i, v_j) plane. Hence, the symmetric part h^+ of h cancels out from (2.31) and

$$\Pi_{q_0}^{\perp}(z_\delta(2\pi) - z_\delta(0)) = \sum_{\substack{i,j=1,\dots,n-1 \\ i>j}} h^-(v_i, v_j)_{q_0} P_{i,j}\,.$$

$\square$

The following Proposition presents a sufficient condition for $G[\gamma]$ to be a regular VHC. This result is based on Lemma 2.3 and Lemma 2.4.

Proposition 2.3. *For any $q_0 \in \mathcal{Q}$ and any real positive constant K, there exists a positive real constant L such that for any closed regular curve γ that satisfies Assumptions 2.3, such that $L_\gamma \leq L$ and*

$$\left| \sum_{i,j=1,\dots,n,\, i>j} h^-(v_i, v_j)_{q_0} P_{i,j} \right| > K L_\gamma^2, \tag{2.32}$$

iteration (2.21) converges to a solution δ^* and $G[\gamma] = \gamma_{\delta^*}$ is a regular VHC.

Proof. By (2.9)

$$\begin{aligned}\Pi^{\perp}_{\gamma_{\delta^*}(\lambda)}\left(\gamma'_{\delta^*}(\lambda)\right) &= \Pi^{\perp}_{\gamma_{\delta^*}(\lambda)}\left(\Pi_{\gamma_{\delta^*}(\lambda)}\left(\gamma'(\lambda)\right) + \|\gamma'(\lambda)\|\delta^*\right)\\ &= \|\gamma'(\lambda)\|\Pi^{\perp}_{\gamma_{\delta^*}(\lambda)}\left(\delta^*\right)\\ &= \|\gamma'(\lambda)\|\Big(\left(\Pi^{\perp}_{\gamma_{\delta^*}(\lambda)} - \Pi^{\perp}_{q_0}\right)(\delta^*) + \Pi^{\perp}_{q_0}\left(\delta^* - \bar{\delta}\right) + \Pi^{\perp}_{q_0}\left(\bar{\delta}\right)\Big).\end{aligned} \tag{2.33}$$

Note that, by ii) of Lemma 2.3, (2.15) and (2.23), $\forall\lambda \in [0, 2\pi]$,

$$\left\|\left(\Pi^{\perp}_{\gamma_{\delta^*}(\lambda)} - \Pi^{\perp}_{q_0}\right)(\delta^*)\right\| \leq 2D_1 \left\|\gamma_{\delta^*}(\lambda) - q_0\right\| \|\delta^*\| \leq 2D_1(C_5 L_\gamma)C_2 L_\gamma\,,$$

by (2.22) of Lemma 2.3, $\left\|\Pi^{\perp}_{q_0}\left(\delta^* - \bar{\delta}\right)\right\| \leq C_3 \mathrm{L}^2_\gamma$. By Lemma 2.4 and hypothesis (2.32), $\Pi^{\perp}_{q_0}\left(\bar{\delta}\right) = \Pi^{\perp}_{q_0}\frac{z_0(0)-z_0(2\pi)}{L_\gamma} \geq K L_\gamma$. Hence, $\forall\lambda \in [0, 2\pi]$,

$$\|\Pi^{\perp}_{\gamma_{\delta^*}(\lambda)}\left(\gamma'_{\delta^*}(\lambda)\right)\| \geq L_\gamma(K - 2D_1(C_5C_2 + C_3)L_\gamma)\,,$$

and, choosing $L < \frac{K}{2D_1(C_5C_2+C_3)}$, for any curve γ such that $\mathrm{L}_\gamma \leq L$,

$$\|\Pi^{\perp}_{\gamma_{\delta^*}(\lambda)}\left(\gamma'_{\delta^*}(\lambda)\right)\| \geq 0, \forall\lambda \in [0, 2\pi]\,,$$

hence γ_{δ^*} is a regular VHC.

□

Finally, the hypothesis of Theorem 2.1 imply the existence of a K in (2.32).

Proof of Theorem 2.1.

By hypothesis $(\Pi^{\perp}[v,w])_{q_0} = 2h^-(v_{q_0}, w_{q_0})_{q_0} \neq 0$, hence, since $|A| \geq \frac{CL^2_\gamma}{4\pi}$, where A is the signed area enclosed by γ,

$$\left|\sum_{i,j=1,\dots,n,\, i>j} h^-(v_i, v_j)_{q_0} P_{i,j}\right| = \left|h^-(v_{q_0}, w_{q_0})_{q_0} A\right| \geq |h^-(v_{q_0}, w_{q_0})_{q_0}|\frac{CL^2_\gamma}{4\pi}\,,$$

and the thesis is a consequence of Proposition 2.3. ■

Proof of Proposition 2.2.

Since the input distribution U is not involutive at q_0, there exist vector fields

$X, Y \in U$ such that $(\Pi^{\perp}[X,Y])_{q_0} = 2h^{-}(X,Y) \neq 0$. Let γ be a closed planar simple curve such that $\gamma(0) = q_0$ and $\gamma'(\lambda) \in \text{span}\{X_{q_0}, Y_{q_0}\}$ with signed area P_{12}. Let $v_1, \ldots, v_{n-1}$ be a base of $U(q_0)$ with $v_1 = X_{q_0}$, $v_2 = Y_{q_0}$, consider the family of planar closed curves $\gamma^{\epsilon} = q_0 + \epsilon(\gamma - q_0)$, with $\epsilon \in (0, +\infty)$ and let $P_{12,\epsilon}$ denote its area. Note that $L_{\epsilon\gamma} = \epsilon L_{\gamma}$ and $P_{12,\epsilon} = \epsilon^2 P_{12}$, then, $\forall \epsilon \in (0, +\infty)$

$$\begin{aligned}
\frac{\sum_{i,j=1,\ldots,n,\, i>j} h^{-}(v_i, v_j) P_{i,j,\epsilon}}{L^2_{\gamma^{\epsilon}}} &= \frac{\sum_{i,j=1,\ldots,n,\, i>j} h^{-}(v_i, v_j) P_{i,j}}{L^2_{\gamma}} \\
&= h^{-}(X_{q_0}, Y_{q_0}) \frac{P_{12}}{L^2_{\gamma}} \neq 0\,,
\end{aligned}$$

since $P_{i,j} = 0$ in the above sum if $(i,j) \neq (1,2)$, due to the fact that γ belongs to the affine plane parallel to $\text{span}\{v_1, v_2\}$, passing at q_0. Then, by Proposition 2.3, it is possible to choose a sufficiently small ϵ such that the fixed point iteration (2.21) converges to a solution δ^* and γ_{δ^*} is a regular VHC. ■

2.1.2 Example

Example 2.1. As an application of Theorem 2.1, consider the simplified PVTOL aircraft model (see [50])

$$\begin{cases} \ddot{q}_1 = -\sin q_3\ \tau_1 + \nu \cos q_3\ \tau_2 \\ \ddot{q}_2 = \cos q_3\ \tau_1 + \nu \sin q_3\ \tau_2 - 1 \\ \ddot{q}_3 = \tau_2 \end{cases} \tag{2.34}$$

where $q_1, q_2 \in \mathbb{R}$ are the horizontal and vertical coordinates of the center of mass, while $q_3 \in \mathbb{S}$ is the rotation angle. Figure 2.1 shows the chosen generalized coordinates. The controls are given by τ_1 and τ_2. The parameter ν is the coupling between the rolling moment and the lateral acceleration of the aircraft. The constant -1 is the normalized gravitational acceleration. System (2.34) is

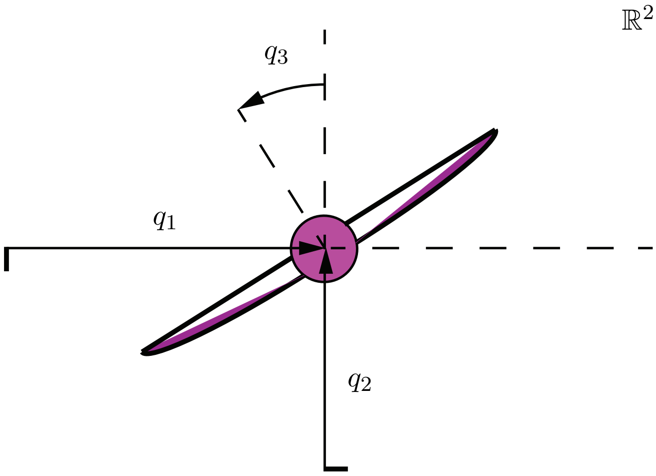


Figure 2.1: PVTOL aircraft model.

in the form (1.5), with $D(q) = I$, $C(q, \dot{q}) = 0$, $P = q_2$, and

$$B(q) = \begin{pmatrix} -\sin q_3 & \nu \cos q_3 \\ \cos q_3 & \nu \sin q_3 \\ 0 & 1 \end{pmatrix}.$$

Note that

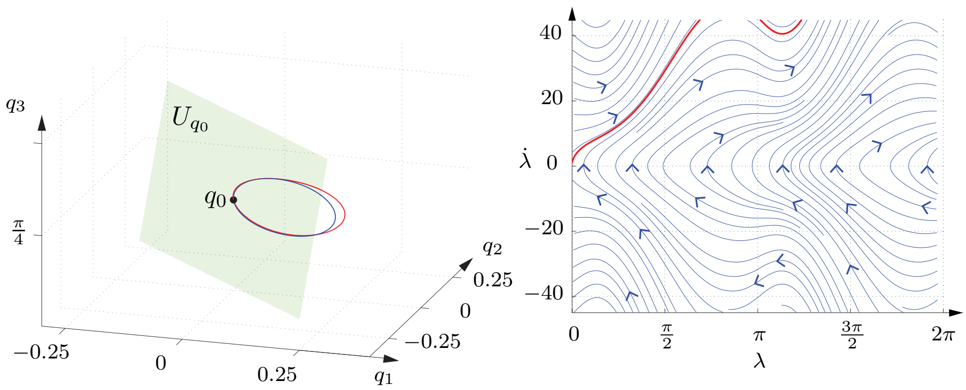
$$\eta_q = \begin{pmatrix} \cos q_3 \\ \sin q_3 \\ -\nu \end{pmatrix}.$$

In the following, we consider $\nu = 0.1$. Let u^1, u^2 be the vector fields on $\mathcal{Q}$ such that $B(q) = (u^1_q, u^2_q)$, $\forall q \in \mathcal{Q}$, then $[u^1, u^2] = (-\cos q_3, -\sin q_3, 0)^T$. Set $q_0 = (0, 0, \frac{\pi}{4})^T$, the configuration in which the center of mass is at the origin and the rotation angle is $\frac{\pi}{4}$. Note that the input distribution is not involutive at q_0.

As a first case, define curve $\gamma^1 : [\mathbb{R}]_{2\pi} \to \mathbb{R}^3$ as

$$\gamma^1(\lambda) = B(q_0) \begin{pmatrix} 1 & \frac{1}{3} \\ -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 0.2\,(\cos\lambda - 1) \\ 0.1 \sin\lambda \end{pmatrix} + q_0\,. \tag{2.35}$$

Note that γ^1 is an ellipse passing through q_0 and belonging to the affine plane parallel to $U(q_0) = \text{Im } B(q_0)$, hence γ^1 satisfies Assumption 2.3. Iteration (2.12) converges to $\delta^* = (-0.0412, -0.0538, 0.0068)^T$. The resulting curve $G[\gamma^1]$ defines a regular VHC since condition (1.16) is satisfied. Figure 2.2a represents γ^1 (blue), $G[\gamma^1]$ (red), and the affine plane $\{q_0 + U(q_0)\}$ (green surface). In this case $\psi_1(\lambda) > 0, \lambda \in [\mathbb{R}]_{2\pi}$ and $\int_0^{2\pi} \psi_2(\lambda) d\lambda = 2.1173$, therefore, in accordance with Proposition 2.1, the reduced dynamics are unstable. Figure 2.2b shows the phase portrait of the reduced dynamics and the positive semi orbit of (1.17) with the initial condition $\lambda(0) = 0, \dot{\lambda}(0) = 1$.



(a) Curves γ^1 (blue), and $G[\gamma^1]$ (red).

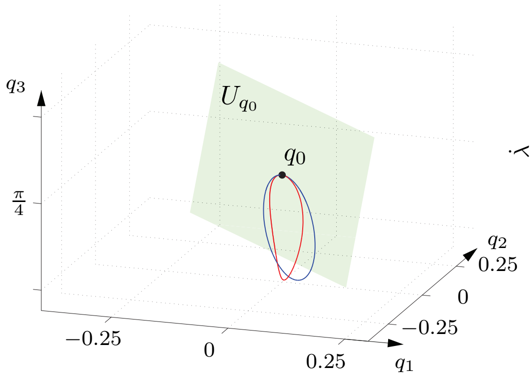
(b) Phase portrait of the reduced dynamics on $G[\gamma^1]$.

Figure 2.2: Example 2.1. Synthesis of a regular VHC starting from curve γ^1.

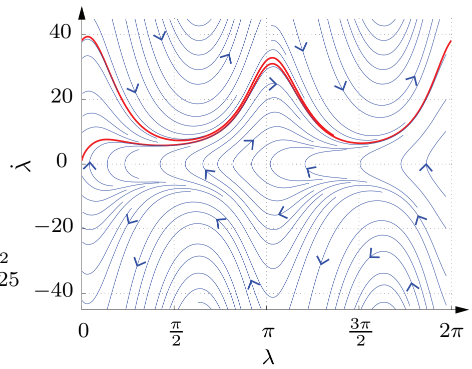
As a second case, consider the curve $\gamma^2 : [\mathbb{R}]_{2\pi} \to \mathbb{R}^3$ defined as

$$\gamma^2(\lambda) = B(q_0) \begin{pmatrix} \frac{1}{3} & -1 \\ 1 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0.2\,(\cos\lambda - 1) \\ 0.1 \sin\lambda \end{pmatrix} + q_0,$$

which, again, is an ellipse passing through q_0 and belonging to the affine plane parallel to $U(q_0)$. The fixed-point iteration (2.12) converges to $\delta^* = (-0.0618, -0.0251, 0.0066)^T$, and, also in this case, curve $G[\gamma^2]$ defines a regular VHC since condition (1.16) is satisfied. Figure 2.3a depicts curves γ^2 and $G[\gamma^2]$. Since $\psi_1(\lambda) > 0, \lambda \in [\mathbb{R}]_{2\pi}$ and $\int_0^{2\pi} \psi_2(\lambda) d\lambda = -2.3103$, in accordance with Proposition 2.1, the reduced dynamics on $G[\gamma^2]$ have an exponentially stable limit cycle. Figure 2.3b shows the phase portrait of the reduced dynamics and the positive semi orbit of (1.17) with the initial condition $\lambda(0) = 0, \dot{\lambda}(0) = 1$. The representation of the PVTOL configuration on the VHCs is given by Figure 2.6a. In particular the green curve represents $G[\gamma^1]$ and the yellow curve represents $G[\gamma^2]$.



(a) Curves γ^2 (blue), and $G[\gamma^2]$ (red).



(b) Phase portrait of the reduced dynamics on $G[\gamma^2]$.

Figure 2.3: Example 2.1. Synthesis of a regular VHC starting from curve γ^2.

Note that, even if ellipses γ^1 and γ^2 are similar, the VHCs $G[\gamma^1]$ and $G[\gamma^2]$ **have different stability properties**: $G[\gamma^2]$ has an asymptotically stable limit cycle, while $G[\gamma^1]$ is unstable. Hence, $G[\gamma^2]$ induces a stable periodic motion on the controlled system that can be maintained with bounded control forces, while $G[\gamma^1]$ has reduced dynamics with unbounded speed and requires unbounded control forces to be enforced. The next section shows that, in some cases, the stability properties of the VHC $G[\gamma]$ can be inferred **a priori** from the curve γ without the actual computation of $G[\gamma]$. Moreover, it shows that, in some cases, it is possible to choose curve γ in such a way that the resulting VHC $G[\gamma]$ has reduced dynamics with an exponentially stable limit cycle.

2.2 Synthesis of Stable VHCs

The discussion of Example 2.1 shows that the synthesis method presented in Section 2.1 has one important limitation: the reduced dynamics of $G[\gamma]$ can be evaluated only **a posteriori** with respect to the choice of curve γ. Since VHCs with unstable dynamics are not practically desirable, it is natural to formulate the following problem:

Problem 2.2. *Let $q_0 \in \mathcal{Q}$ be an assigned configuration, find a closed regular planar curve $\gamma : [\mathbb{R}]_{2\pi} \to \mathcal{Q}$, with $\gamma(0) = q_0$, such that $G[\gamma]$ exists and defines a regular VHC whose reduced dynamics possess an* **exponentially stable** *limit cycle.*

We present a possible solution to this problem based on Proposition 2.1, which gives sufficient conditions for the presence of a stable limit cycle in the reduced dynamics that depend on the signs of $\psi_1(\lambda), \lambda \in [0, 2\pi]$ and $\int_0^{2\pi} \psi_2(\lambda) d\lambda$. Roughly speaking, we will show that, if γ is of sufficiently small length, curves γ and $G[\gamma]$ are very similar, so that the signs of $\psi_1(\lambda)$ and $\int_0^{2\pi} \psi_2(\lambda) d\lambda$ can be inferred directly from γ, without the actual computation of $G[\gamma]$.

As a first step, in the following lemma, we rewrite term $\int_0^{2\pi} \psi_2(\lambda) d\lambda$ in a more convenient form.

Lemma 2.5. *Let ψ_2 be given by (1.19) and set*

$$\psi_3(\lambda) = \frac{\big\langle (\nabla_{\gamma'(\lambda)} \eta)_{\gamma(\lambda)}, \gamma'(\lambda) \big\rangle}{\big\langle \eta_{\gamma(\lambda)}, \gamma'(\lambda) \big\rangle}, \tag{2.36}$$

then $\int_0^{2\pi} \psi_2(\lambda) d\lambda = \int_0^{2\pi} \psi_3(\lambda) d\lambda$.

Proof. Proof of Lemma 2.5. The numerator of ψ_2 contains one of the two terms of the derivative of the denominator:

$$\nabla_{\gamma'(\lambda)} \langle \eta_{\gamma(\lambda)}, \gamma'(\lambda) \rangle = \big\langle (\nabla_{\gamma'(\lambda)} \eta)_{\gamma(\lambda)}, \gamma'(\lambda) \big\rangle + \big\langle \eta_{\gamma(\lambda)}, \nabla_{\gamma'(\lambda)} \gamma'(\lambda) \big\rangle$$

by adding and removing the missing term from the numerator, we rewrite ψ_2 as

$$\begin{aligned} \psi_2(\lambda) &= -\nabla_{\gamma'(\lambda)} \log(\big\langle \eta_{\gamma(\lambda)}, \gamma'(\lambda) \big\rangle) + \frac{(\big\langle (\nabla_{\gamma'(\lambda)} \eta)_{\gamma(\lambda)}, \gamma'(\lambda) \big\rangle}{\big\langle \eta_{\gamma(\lambda)}, \gamma'(\lambda) \big\rangle} \\ &= -\nabla_{\gamma'(\lambda)} \log(\big\langle \eta_{\gamma(\lambda)}, \gamma'(\lambda) \big\rangle) + \psi_3(\lambda) \,. \end{aligned}$$

The thesis follows from the fact that the integral of $-\nabla_{\gamma'(\lambda)} \log(\langle \eta_{\gamma(\lambda)}, \gamma'(\lambda) \rangle)$ over $[0, 2\pi]$ is 0, since it is the integral on a closed curve of the covariant derivative of a scalar function. $\square$

Then, it is convenient to adopt the following notations.

Definition 2.1. Define function $w(q) : \mathcal{Q} \to \mathbb{R}$ as

$$w(q) = \langle \eta_q, D^{-1}(q) \nabla P_q \rangle .$$

Definition 2.2. Define the 2-form $\hat{M} : T_q\mathcal{Q} \times T_q\mathcal{Q} \to \mathbb{R}$ as

$$\hat{M}_q(x, y) = \langle (\nabla_x \eta)_q, y \rangle$$

and define $M_q : U(q) \times U(q) \to \mathbb{R}$ as the restriction of $\hat{M}$ to the image of the input distribution (i.e., $M_q(x, y) = \hat{M}_q(x, y), \forall x, y \in U(q)$).

In this way, equations (1.18) and (2.36) are rewritten as

$$\psi_1(\lambda) = -\frac{w(\gamma(\lambda))}{\langle \eta_{\gamma(\lambda)}, \gamma'(\lambda) \rangle}\,, \psi_3(\lambda) = \frac{\hat{M}_{\gamma(\lambda)}(\gamma'(\lambda), \gamma'(\lambda))}{\langle \eta_{\gamma(\lambda)}, \gamma'(\lambda) \rangle}\,.$$

Assume that, for an assigned curve γ, the fixed point iteration (2.12) converges and that the resulting curve $G[\gamma]$ defines a regular VHC. Let ψ_1^*, ψ_3^* be the functions ψ_1 and ψ_3 computed on the VHC $G[\gamma]$, namely

$$\psi_1^*(\lambda) = -\frac{w(G[\gamma](\lambda))}{\langle \eta_{G[\gamma](\lambda)}, G[\gamma]'(\lambda) \rangle}\,, \psi_3^*(\lambda) = \frac{\hat{M}_{G[\gamma](\lambda)}(G[\gamma]'(\lambda), G[\gamma]'(\lambda))}{\langle \eta_{G[\gamma](\lambda)}, G[\gamma]'(\lambda) \rangle}\,.$$

Our main results are based on the following observations, that will be proved in Section 2.2.1.

1) If γ satisfies the hypotheses of Theorem 2.1 and is of sufficiently small length, there exists a constant C, independent on γ, such that $\|G[\gamma](\lambda) - \gamma(\lambda)\| \leq CL_\gamma(\lambda)^2$, $\forall \lambda \in [0, 2\pi]$, that is $G[\gamma]$ is very similar to γ (see Lemma 2.3).

2) By (2.10), the denominators of ψ_1^* and ψ_3^* have a constant sign, for $\lambda \in [0, 2\pi]$.

3) Since w is smooth, if $w(q_0) \neq 0$, by continuity, if γ is sufficiently small, the sign of the numerator of ψ_1^*, is equal to the sign of $w(q_0)$.

4) Since the two-form M_q is smooth with respect to q, if γ is sufficiently small, the numerator of ψ_3^* can be approximated by $M_{q_0}(\gamma'(\lambda), \gamma'(\lambda))$.

In some cases, for a closed curve γ of sufficiently small length, these observations allow inferring **a priori** the stability properties of the reduced dynamics for the VHC $G[\gamma]$. This fact is shown in the following two theorems, whose proofs are reported in Section 2.2.1.

Theorem 2.2. *Assume that the hypotheses of theorem 2.1 hold and that one of the following conditions is satisfied*

a) $M_{q_0} > 0, w(q_0) > 0$ *or* $M_{q_0} < 0, w(q_0) < 0$,

b) $M_{q_0} > 0, w(q_0) < 0$ *or* $M_{q_0} < 0, w(q_0) > 0$.

Then, if a) holds, the reduced dynamics of $G[\gamma]$ *have an exponentially stable limit cycle, if b) holds, the reduced dynamics of* $G[\gamma]$ *are unstable.*

Proposition 2.4. *Let* $q_0 \in \mathcal{Q}$ *be an assigned configuration and let* V *be a 2-dimensional subspace of* $U(q_0)$ *such that there exist two vector fields* f, g *on* $\mathcal{Q}$ *such that* $V = span\{f_{q_0}, g_{q_0}\}$ *and* $[f, g]_{q_0} \notin U$*. Let* γ *be a curve that satisfies Assumption 2.3 and set*

$$I = \int_0^{2\pi} M_{q_0}(\hat{\gamma}'(\lambda), \hat{\gamma}'(\lambda)) d\lambda \,. \tag{2.37}$$

Then, there exists $\bar{\epsilon} > 0$ *such that,* $\forall \epsilon \in (0, \bar{\epsilon}]$*, iteration (2.12) is convergent for curve*

$$\gamma^\epsilon = q_0 + \epsilon(\gamma - q_0) \tag{2.38}$$

and $G[\gamma^\epsilon]$ *defines a regular VHC. Moreover,* *a) If $I > 0, w(q_0) > 0$ or $I < 0, w(q_0) < 0$, the reduced dynamics of $G[\gamma^\epsilon]$ have a stable limit cycle.*

b) If $I > 0, w(q_0) < 0$ or $I < 0, w(q_0) > 0$, the reduced dynamics $G[\gamma^\epsilon]$ are unstable.

In particular, Theorem 2.2 shows that, if $w(q_0) \neq 0$ and if M_{q_0} is positive or negative defined, the stability properties of the reduced dynamics associated to $G[\gamma]$ are the same for **all** curves γ of sufficiently small length. Proposition 2.4 shows that, if $w(q_0) \neq 0$ and $I \neq 0$, then, there exists a value $\bar{\epsilon} > 0$, sufficiently small such that, for any $\epsilon \in (0, \bar{\epsilon})$, curve γ^ϵ defined in (2.38) is such that iteration (2.12) converges and the resulting curve $G[\gamma^\epsilon]$ defines a regular VHC, whose reduced dynamics have stability properties that depend only on the signs of $w(q_0)$ and I. Note that γ^ϵ represents a rescaling of γ: in other words, in the hypotheses of Theorem 2.2, it is always possible to rescale γ so that $G[\gamma^\epsilon]$ is well defined.

The following theorem completes the previous results. In particular, it shows that, if $w(q_0) \neq 0$ and M_{q_0} is indefinite, then it is possible to find a curve γ such that the resulting VHC $G[\gamma]$ has reduced dynamics characterized by the presence of an exponentially stable limit cycle.

Theorem 2.3. *If M_{q_0} is indefinite and $w(q_0) \neq 0$, then, for any positive real constant L, it is possible to find three VHCs γ^1, γ^2, γ^3 such that, for $i = 1, 2, 3$, $q_0 = \gamma_i(0)$, $L_{\gamma_i} \leq L$, and*

a) the reduced dynamics of γ^1 have an exponentially stable limit cycle,

b) the reduced dynamics of γ^2 are unstable,

c) the reduced dynamics of γ^3 are SEL.[1]

The following remark illustrates how to choose γ in such a way that the reduced dynamics of $G[\gamma]$ have a stable limit cycle. It is a direct consequence of Theorems 2.2 and 2.3.

[1] For the definition of SEL (singular Euler-Lagrange) systems, we refer the reader to definition 3.1 in [7].

Remark 2.4. Assume that f, g are vector fields on $\mathcal{Q}$ such that $[f, g]_{q_0} \notin U(q_0)$, set $V = \text{span}\{f_{q_0}, g_{q_0}\}$ and $N = M_{q_0}|_V$. Assume that one of the following two cases holds:

1) $N > 0$, $w(q_0) > 0$ or $N < 0$, $w(q_0) < 0$.

2) N is indefinite.

In case 1), let γ be any curve satisfying Assumption 2.3 and such that $\gamma'(\lambda) \in V$, $\forall \lambda \in [0, 2\pi]$. Then, there exists $\bar{\epsilon}$, sufficiently small, such that, $\forall \epsilon \in (0, \bar{\epsilon}]$, iteration (2.12) converges and curve $G[\gamma^\epsilon]$ defines a regular VHC, whose reduced dynamics have an exponentially stable limit cycle by Theorems 2.1 and 2.2.

In case 2) Denote with μ^+ the positive eigenvalue of N, with μ^- the negative eigenvalue, and with v^+, v^- the corresponding eigenvectors. Consider a curve defined as

$$\gamma_{\rho,\theta} = q_0 + \rho \left(v^+ \cos\theta\, (\cos\lambda - 1) + v^- \sin\theta\, \sin\lambda \right) \tag{2.39}$$

where $\rho \in \mathbb{R}^+$ and $\theta \in [0, \frac{\pi}{2}]$ are two parameters. Consider function

$$F(\theta) = \int_0^{2\pi} N(\hat{\gamma}'_{\rho,\theta}(\lambda), \hat{\gamma}'_{\rho,\theta}(\lambda)) d\lambda \,,$$

with $\hat{\gamma}' = \frac{\hat{\gamma}'}{\|\hat{\gamma}'\|}$. If $w(q_0) > 0$ (respectively, $w(q_0) < 0$), choose θ such that $\theta \in (0, \frac{\pi}{2})$ and $F(\theta) > 0$, (respectively, $F(\theta) < 0$). The existence of such values of θ is discussed in the proof of Theorem 2.3, in Section 2.2.1.

Then, there exist a sufficiently small $\bar{\rho}$, such that $\forall \rho \in (0, \bar{\rho}]$, iteration (2.12) is convergent and curve $G[\gamma_{\rho,\theta}]$ defines a regular VHC, whose reduced dynamics have a stable limit cycle (see the proof of Theorem 2.3 in Section 2.2.1).

2.2.1 Proofs

Note 2.2. Let γ be a given curve satisfying Assumption 2.3 and let $\hat{\gamma}$ be the corresponding regular VHC obtained by (2.12) (that is, $\bar{\gamma} = G[\gamma]$). By Proposition 2.1 the stability properties of the reduced dynamics of $\bar{\gamma}$ depend on signs of ψ_1^* and ψ_2^*. Moreover, Lemma 2.5 states that there exists a function ψ_3^* such that $\int_0^{2\pi} \psi_2^*(\lambda)d\lambda = \int_0^{2\pi} \psi_3^*(\lambda)d\lambda$. Our aim is to estimate the signs of these functions, in order to understand the stability properties of $\hat{\gamma}$, from the knowledge of the curve γ. Set $q_0 = \gamma(0)$.

Consider the following approximations: $\frac{w(q_0)}{\langle \eta_{q_0}, \hat{\delta} \rangle} \approx \psi_1^*(\lambda)$ and $\frac{M_{q_0}(\gamma'(\lambda), \gamma'(\lambda))}{\langle \eta_{q_0}, \hat{\delta} \rangle} \approx \psi_3^*(\lambda)$, $\forall \lambda \in [\mathbb{R}]_{2\pi}$. By Lemma 2.7, the difference between them is upper bounded by the length of the curve γ. Then, there exists a constant L such that, if the length of γ is less than L, the signs of the approximations are equal to the signs of the functions ψ_1^* and $\psi_3^*(\lambda)$. If M_{q_0} is positive or negative definite, the sign of (2.37) is positive (or negative) for any choice curves γ. On the other hand, if M_{q_0} is indefinite, the sign of (2.37) depends on the choice of the curve γ. An iterative procedure for the construction of regular VHC with stable reduced dynamics is shown in Remark 2.4.

Lemma 2.6. *Let be given functions $N, N', D, D', d, n : \mathbb{R} \to \mathbb{R}$ such that, $\forall \lambda \in \mathbb{R}$,*

$$|N(\lambda) - N'(\lambda)| \leq n(\lambda),\ |D(\lambda) - D'(\lambda)| \leq d(\lambda),\ |D(\lambda) - d(\lambda)| \neq 0\,, \quad (2.40)$$

then

$$\left| \frac{N(\lambda)}{D(\lambda)} - \frac{N'(\lambda)}{D'(\lambda)} \right| \leq \frac{|N(\lambda)|d(\lambda) + |D(\lambda)|n(\lambda)}{|D(\lambda)|(|D(\lambda)| - d(\lambda))}\,.$$

Proof. Note that

$$\frac{N(\lambda)}{D(\lambda)} - \frac{N'(\lambda)}{D'(\lambda)} = \frac{N(\lambda)D'(\lambda) - N'(\lambda)D(\lambda)}{D(\lambda)D'(\lambda)}\,,$$

then Assumptions (2.40) imply

$$\begin{aligned} |D'(\lambda)N(\lambda) - N'(\lambda)D(\lambda)| &= |D'(\lambda)N(\lambda) - N'(\lambda)D(\lambda) \\ &\qquad + N(\lambda)D(\lambda) - N(\lambda)D(\lambda)| \\ &= |D(\lambda)(N(\lambda) - N'(\lambda)) + N(\lambda)(D'(\lambda) - D(\lambda))| \\ &\leq |D(\lambda)|n(\lambda) + |N(\lambda)|d(\lambda) \\ |D(\lambda)D'(\lambda)| &\geq |D(\lambda)||D(\lambda) - d(\lambda)|\,. \end{aligned}$$

□

Lemma 2.7. *There exists positive real constants C_6, C_7, C_8 such that, for any curve γ for which the fixed point iteration (2.21) converges to a value $\delta = \delta^*$ and $\bar{\gamma} = G[\gamma]$ is a feasible VHC, the following bounds hold*

$$|w(\bar{\gamma}(\lambda)) - w(q_0)| \leq C_6 L_\gamma(\lambda)\,, \tag{2.41}$$

$$\left| M_{\bar{\gamma}(\lambda)}\left(\frac{\bar{\gamma}'(\lambda)}{\|\gamma'(\lambda)\|}, \frac{\bar{\gamma}'(\lambda)}{\|\gamma'(\lambda)\|}\right) - M_{q_0}(\hat{\gamma}'(\lambda), \hat{\gamma}'(\lambda)) \right| \leq C_7 L_\gamma\,. \tag{2.42}$$

$$\left|\left\langle \eta_{\bar{\gamma}(\lambda)}, \delta^* \right\rangle - \left\langle \eta_{q_0}, \bar{\delta} \right\rangle\right| \leq C_8 L_\gamma^2\,, \tag{2.43}$$

Proof. By assumptions (2.5), (2.6), (2.7), there exists a positive real constant E_1 such that function $w(q) = \left\langle \eta_q, D^{-1}(q)\nabla P_q \right\rangle$ is such that, for any $q_0, q \in \mathcal{Q}$, $\|w(q) - w(q_0)\| \leq E_1 \|q - q_0\|$, hence (2.41) follows from the second of (2.23).

Further, by assumptions (2.5), (2.7) and the definition of the Christoffel's symbols, there exist positive real constants E_2, E_3 such that function $H(q, v) = \langle (\nabla_v \eta)_q, v \rangle$ is such that, for any $q, q_0 \in \mathcal{Q}$, $v, w \in \mathbb{R}^n$,

$$\begin{aligned} \|H(q,v) - H(q_0,w)\| &\leq \|H(q,v) - H(q,w)\| + \|H(q,w) - H(q_0,w)\| \\ &\leq E_2 \|v - w\| + E_3 \|q - q_0\| \|w\|^2. \end{aligned} \tag{2.44}$$

Being $M_{q_0}(\dot{q}, \dot{q}) = H(q_0, \dot{q})$, (2.44) implies that

$$\begin{aligned} &\left\| M_{\bar{\gamma}(\lambda)}\left(\frac{\bar{\gamma}'(\lambda)}{\|\gamma'(\lambda)\|}, \frac{\bar{\gamma}'(\lambda)}{\|\gamma'(\lambda)\|}\right) - M_{q_0}\left(\hat{\gamma}'(\lambda), \hat{\gamma}'(\lambda)\right) \right\| \\ &\qquad \leq E_2 \frac{\|\Pi_{\bar{\gamma}(\lambda)}(\gamma'(\lambda)) + \delta^* \|\gamma'(\lambda)\| - \gamma'(\lambda)\|}{\|\gamma'(\lambda)\|} + E_3 \|\bar{\gamma}(\lambda) - q_0\| \\ &\qquad\quad \leq E_2 \left\| \delta^* - \Pi^{\perp}_{\bar{\gamma}(\lambda)}(\hat{\gamma}'(\lambda)) \right\| + E_3 \|\bar{\gamma}(\lambda) - q_0\| \\ &\qquad\quad \leq E_2 \|\delta^*\| + E_2 \left\| \left(\Pi^{\perp}_{\bar{\gamma}(\lambda)} - \Pi^{\perp}_{q_0} \right) \right\| \|\hat{\gamma}'(\lambda)\| + E_3 \|\bar{\gamma}(\lambda) - q_0\|\,, \end{aligned}$$

property (2.42) then follows from (2.15), (2.22) and the second of (2.23).

Finally (2.43) is a consequence of bound

$$\begin{aligned}&\left|\left\langle \eta_{\bar\gamma(\lambda)}, \delta^*\right\rangle - \left\langle \eta_{q_0}, \bar\delta\right\rangle\right| \\ &\qquad\qquad \leq \left|\left\langle \eta_{\bar\gamma(\lambda)}, \delta^*\right\rangle - \langle \eta_{q_0}, \delta^*\rangle\right| + \left|\langle \eta_{q_0}, \delta^*\rangle - \left\langle \eta_{q_0}, \bar\delta\right\rangle\right| \\ &\qquad\qquad\qquad\qquad \leq D_1\|\bar\gamma(\lambda) - q_0\|\|\delta^*\| + D_0\|\delta^* - \bar\delta\|\end{aligned}$$

and (2.23), (2.22). □

Proof of Theorem 2.2.
Set $\alpha = \left\langle \eta_{q_0}, \bar\delta\right\rangle$ and $\psi_1^0 = L_\gamma \frac{\omega(q_0)}{\alpha}$. Assume $M_{q_0} > 0$, $\omega(q_0) > 0$, $\alpha > 0$, the other cases are analogous. Note that, by Lemma 2.4, $|\alpha| = |h^-(v_{q_0}, w_{q_0})_{q_0} \frac{Q_\gamma}{4\pi} L_\gamma|$, then, setting $d_1 = |h^-(v_{q_0}, w_{q_0})_{q_0} \frac{Q_\gamma}{4\pi}|$, $d_2 = |h^-(v_{q_0}, w_{q_0})_{q_0} \frac{1}{4\pi}|$, it follows that

$$d_1 L_\gamma \leq \alpha \leq d_2 L_\gamma\,.$$

Note that $\psi_1^0 \geq \frac{\omega(q_0)}{d_2} > 0$, moreover

$$\psi_1^0 - L_\gamma\|\gamma'(\lambda)\|\psi_1(\lambda) = L_\gamma\left(\frac{\omega(q_0)}{\alpha} - \frac{\omega(\gamma_{\delta^*})}{\left\langle \eta_{\gamma_{\delta^*}}, \delta^*\right\rangle}\right)\,.$$

By (2.22), Lemma 2.6, Lemma 2.7, it follows that

$$\left\|\frac{\omega(q_0)}{\alpha} - \frac{\omega(\gamma_{\delta^*})}{\left\langle \eta_{\gamma_{\delta^*}}, \delta^*\right\rangle}\right\| \leq \frac{\alpha C_6 L_\gamma + |\omega(q_0)| C_8 L_\gamma^2}{\alpha(\alpha - C_8 L_\gamma^2)} \leq \frac{d_2 C_6 + |\omega(q_0)| C_8}{d_1(d_1 - C_8 L_\gamma)}\,.$$

If needed, reduce L such that $d_1 - C_8 L > \frac{d_1}{2}$, then, setting $d_3 = 2\frac{d_2 C_6 + |\omega(q_0)| C_8}{d_1^2}$ the previous inequality implies that

$$|\psi_1^0 - L_\gamma\|\gamma'(\lambda)\|\psi_1(\lambda)| \leq d_3 L_\gamma\,. \tag{2.45}$$

Set $\psi_3^0(\lambda) = L_\gamma \frac{M_{q_0}(\gamma'(\lambda), \gamma'(\lambda))}{\alpha\|\gamma'(\lambda)\|^2}$. Assume that $M_{q_0} > 0$ (the case $M_{q_0} < 0$ is analogous) and let μ_1, μ_2 be its minimum and maximum eigenvalues. Note that $\psi_3^0 \geq \frac{\mu_1}{d_2} > 0$, moreover

$$\psi_3^0 - \psi_3 \frac{L_\gamma}{\|\gamma'(\lambda)\|} = L_\gamma\left(\frac{M_{q_0}(\hat\gamma'(\lambda), \hat\gamma'(\lambda))}{\alpha} - \frac{M_{\gamma_{\delta^*}}\left(\frac{\gamma'_{\delta^*}(\lambda)}{\|\gamma'(\lambda)\|}, \frac{\gamma'_{\delta^*}(\lambda)}{\|\gamma'(\lambda)\|}\right)}{\left\langle \eta_{\gamma_{\delta^*}}, \delta^*\right\rangle}\right)\,.$$

By (2.22), Lemma 2.6 and Lemma 2.7, it follows that

$$\left\|\psi_3^0 - \psi_3 \frac{L_\gamma}{\|\gamma'(\lambda)\|}\right\| \leq L_\gamma \frac{\alpha C_7 L_\gamma + \mu_2 C_8 L_\gamma^2}{\alpha(\alpha - C_8 L_\gamma^2)} \leq \frac{d_2 C_7 + \mu_2 C_8 L_\gamma}{d_1(d_1 - C_8 L_\gamma)},$$

hence, setting $d_4 = 2\frac{d_2 C_7}{d_1^2}$, $d_5 = 2\frac{\mu_2 C_8}{d_1^2}$

$$\left|\psi_3^0 - \psi_3 \frac{L_\gamma}{\|\gamma'(\lambda)\|}\right| \leq d_4 L_\gamma + d_5 L_\gamma^2 . \tag{2.46}$$

Thus, by (2.45), (2.46), there exists L sufficiently small such that, if $L_\gamma < L$, $\psi_3(\lambda) > 0$ and $\psi_1(\lambda) > 0$, $\forall \lambda \in [0, 2\pi]$, which imply the thesis by Proposition 2.1. ■

Proof of Proposition 2.4.
Set $\alpha = \langle \eta_{q_0}, \bar{\delta} \rangle$ and define $\psi_1^0 = -L_\gamma \frac{\omega(q_0)}{\alpha}$ and $\psi_3^0 = L_\gamma \frac{M_{q_0}(\gamma'(\lambda), \gamma'(\lambda))}{\alpha \|\gamma'(\lambda)\|^2}$. By Theorem 2.1, there exists a sufficiently small $\bar{\epsilon}$, such that, $\forall \epsilon \in (0, \bar{\epsilon}]$, the fixed point iteration (2.21) converges to a value $\delta^*(\epsilon)$ and curve $G[\gamma^\epsilon]$ defines a regular VHC.

Let $\psi_1(\lambda, \epsilon)$ and $\psi_3(\lambda, \epsilon)$ be the functions defined in (1.18) and (2.36) for γ^ϵ. As a consequence of (2.45) and (2.46), there exists a sufficiently small $\hat{\epsilon}$ such that, $\forall \epsilon \in (0, \hat{\epsilon})$, the sign of ψ_1^0 is equal to the sign of $\psi_1(\lambda, \epsilon)$, $\forall \lambda \in [0, 2\pi]$ and the sign of $\int_0^{2\pi} \psi_3^0(\lambda) d\lambda = \frac{L_\gamma}{\alpha} I$ is equal to the sign of $\int_0^{2\pi} \psi_3(\lambda, \epsilon) d\lambda$. Then, the thesis is a consequence of Proposition 2.1. ■

Proof of Theorem 2.3.
Let v_1, v_2 be the eigenvectors of M_{q_0}, with associated eigenvalues $\mu_1 > 0$, $\mu_2 < 0$. Let $\gamma_{\rho,\theta} : [0, 2\pi] \to \mathbb{R}^n$ be the closed curve

$$\gamma_{\rho,\theta}(\lambda) = q_0 + \rho(\cos\theta\,(\cos(\lambda) - 1) v_1 + \sin\theta\, \sin(\lambda)\, v_2),$$

where ρ is a positive real constant and $\theta \in [0, \frac{\pi}{2}]$. Define the real valued function

$$F(\theta) = \int_0^{2\pi} \frac{M_{q_0}(\gamma'_{\rho,\theta}(\lambda), \gamma'_{\rho,\theta}(\lambda))}{\|\gamma'_{\rho,\theta}(\lambda)\|^2} d\lambda, .$$

Note that F is constant with respect to ρ, $F(0) = \mu_1 2\pi$, $F(\frac{\pi}{2}) = \mu_2 2\pi$, hence, by continuity, there exist $c > 0$ and $\theta_1, \theta_2 \in (0, \frac{\pi}{2})$ such that $F(\theta_1) > c$, $F(\theta_2) < -c$. Note that $Q_{\gamma_{\rho,\theta}}$ is independent on ρ and that $\min_{\theta\in[\theta_1,\theta_2]} Q_{\gamma_{\rho,\theta}} > 0$. Hence, there exists $\rho_0 > 0$ such that the family of curves $\{\gamma_{\rho,\theta}, \rho \in (0, \rho_0], \theta \in [\theta_1, \theta_2]\}$ satisfies the hypothesis of Theorem 2.1. Let $\psi_1(\lambda, \rho, \theta)$, $\psi_3(\lambda, \rho, \theta)$ be the corresponding functions defined as in (1.18), (2.36) and set $\psi_1^0(\theta) = \frac{\omega(q_0)}{\langle \eta_{q_0}, \bar{\delta}(\theta)\rangle}$, $\psi_3^0(\theta) = \frac{F(\theta)}{\langle \eta_{q_0}, \bar{\delta}(\theta)\rangle}$.

By (2.46), (2.45), there exists a positive constant $\bar{\rho}$ such that, $\forall \rho \in (0, \bar{\rho})$, integrals $\int_0^{2\pi} \psi_3(\lambda, \theta_1, \rho) d\lambda$ and $\int_0^{2\pi} \psi_3(\lambda, \theta_2, \rho) d\lambda$ have opposite signs, while integrals $\int_0^{2\pi} \psi_1(\lambda, \theta_1, \rho) d\lambda$ and $\int_0^{2\pi} \psi_1(\lambda, \theta_2, \rho) d\lambda$ have the same sign. Moreover, by continuity, there exists a function of $\theta_3(\rho)$ such that $\int_0^{2\pi} \psi_3(\lambda, \theta_3(\rho), \rho) d\lambda = 0$, $\forall \rho \in (0, \bar{\rho})$. Hence, there exists a sufficiently small ρ such that $G[\gamma_{\rho,\theta_1}]$ defines a regular VHC whose reduced dynamics have an asymptotically stable limit cycle, $G[\gamma_{\rho,\theta_2}]$ defines a regular VHC whose reduced dynamics are unstable, and $G[\gamma_{\rho,\theta_3(\rho)}]$ defines a regular VHC whose reduced dynamics are SEL. ■

2.2.2 Example

Example 2.2. We continue the discussion of Example 2.1 related to the PVTOL system with the initial configuration $q_0 = (0, 0, \frac{\pi}{4})^T$. Note that $w(q_0) = \sin(q_3) = 0.7071$ and

$$\hat{M}_{q_0}(v, w) = v^T \begin{pmatrix} 0 & 0 & -0.351 \\ 0 & 0 & 0.351 \\ -0.351 & 0.351 & 0 \end{pmatrix} w\,.$$

Since the input distribution has dimension 2, $V = U(q_0)$, moreover $M = \hat{M}_{q_0}|_V = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$ is indefinite, with eigenvalues $\mu^+ = \frac{1}{2}$, $\mu^- = -\frac{1}{2}$ and eigenvectors $v^+ = (0.707,\ 0.707)^T$, $v^- = (-0.707,\ 0.707)^T$.

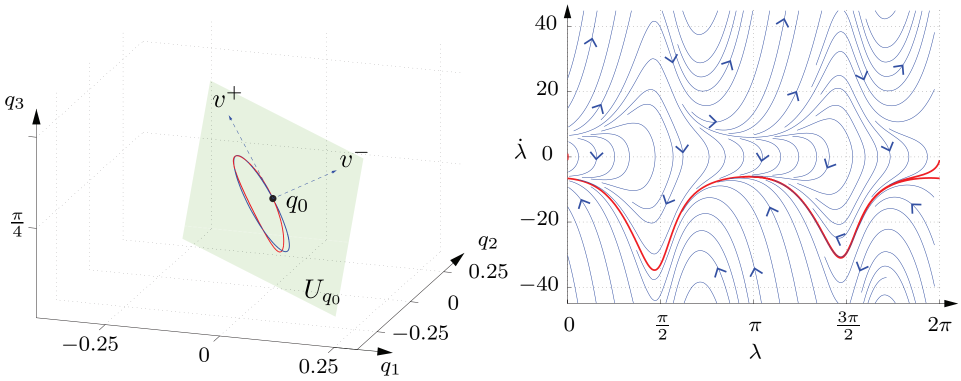
Consider the curves γ^1, γ^2 defined in Example 2.1. For γ^2, the index defined in (2.37) is given by $I = 0.6244$. Since I and $w(q_0)$ are positive, in accordance with Proposition 2.4, there exists $\bar{\epsilon} > 0$, such that, $\forall \epsilon \in (0, \bar{\epsilon})$, the fixed point iteration (2.12) is convergent and $G[\gamma^{2,\epsilon}]$ defines a regular VHC whose reduced dynamics have a stable limit cycle.

For γ^1, the index defined in (2.37) is $I = -0.6260$. In accordance with Proposition 2.4, there exists $\bar{\epsilon} > 0$, such that, $\forall \epsilon \in (0, \bar{\epsilon})$, the fixed point iteration (2.12) is convergent and $G[\gamma^{1,\epsilon}]$ defines a regular VHC whose reduced dynamics are unstable.

For instance, choosing $\epsilon = 1$, $G[\gamma^1]$ and $G[\gamma^2]$ are well defined and the stability properties of the associated reduced dynamics are in accordance with the results of Theorem 2.2.

Now we want to generate a VHC such that the reduced dynamics have a stable limit cycle. In accordance with Remark 2.4, since $w(q_0) > 0$ define $\gamma^3 = \gamma_{\rho,\theta}$ as in (2.39) with $\theta = 0.3$, $\rho = 0.2$. The fixed-point iteration (2.12) converges to $\delta^* = (0.0318,\, 0.0269,\, -0.0042)^T$, and $G[\gamma^3]$ is a regular VHC. The reduced dynamics have a stable limit cycle, indeed $\psi_1(\lambda) < 0, \lambda \in [0, 2\pi]$ and $\int_0^2 \psi_2(\lambda)d\lambda = 7.1648$. The curves obtained are depicted in Figure 2.4a, (in blue the curve γ^3 and in red the curve $G[\gamma^3]$, while the faded green plane represents the affine subset $\{q_0 + U(q_0)\}$). Figure 2.4b shows the phase portrait of the reduced dynamics and the positive semi orbit of (1.17) through $\lambda(0) = 0, \dot{\lambda}(0) = 1$. Curve $G[\gamma^3]$ is depicted in Figure 2.6a with the color black.

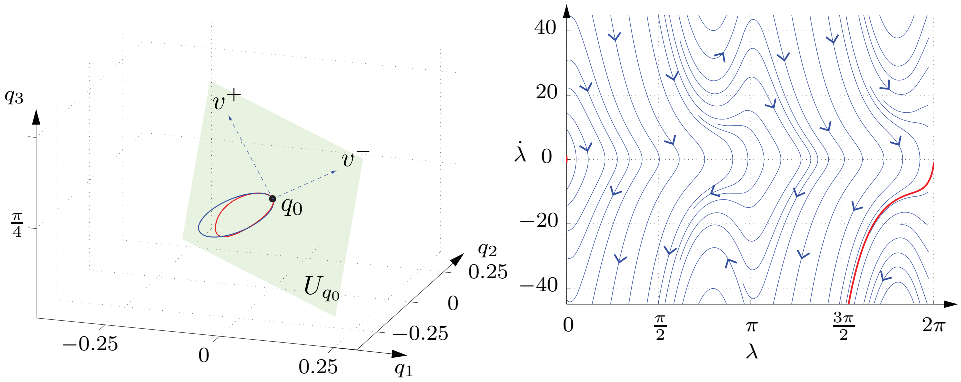
As a second example, set $\gamma^4 = \gamma_{\rho,\theta}$ as in (2.39) with $\theta = \frac{\pi}{2} - 0.3$ and $\rho = 0.2$. The fixed-point iteration (2.12) converges to $\delta^* = (0.0363,\, 0.0204,\, -0.0042)^T$, and $G[\gamma^4]$ defines a regular VHC (Fig. 2.5a). Figure 2.5b shows the phase portrait of the reduced dynamics and the positive semi orbit of (1.17) through $\lambda(0) = 0, \dot{\lambda}(0) = 1$. As expected, $G[\gamma^4]$ has unstable reduced dynamics, indeed $\psi_1(\lambda) < 0, \lambda \in [\mathbb{R}]_{2\pi}$ and $\int_0^2 \psi_2(\lambda)d\lambda = -7.1636$. In Figure 2.6a curve $G[\gamma^4]$ is depicted in blue.



(a) Curves γ^3 (blue), and $G[\gamma^3]$ (red).

(b) Phase portrait of the reduced dynamics on $G[\gamma^3]$.

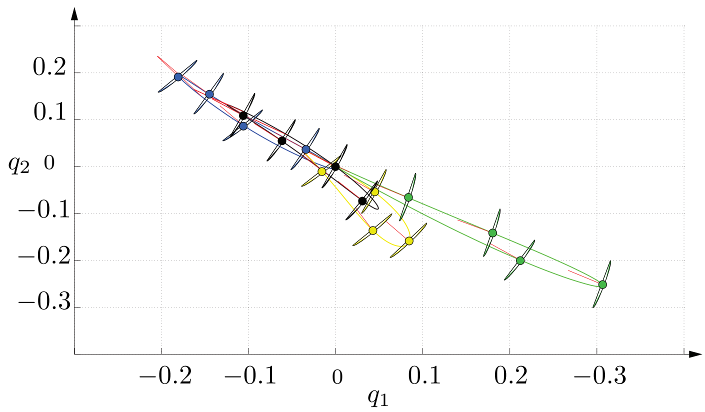
Figure 2.4: Example 2.2. Synthesis of a regular VHC starting from curve γ^3 and possessing a stable limit cycle.



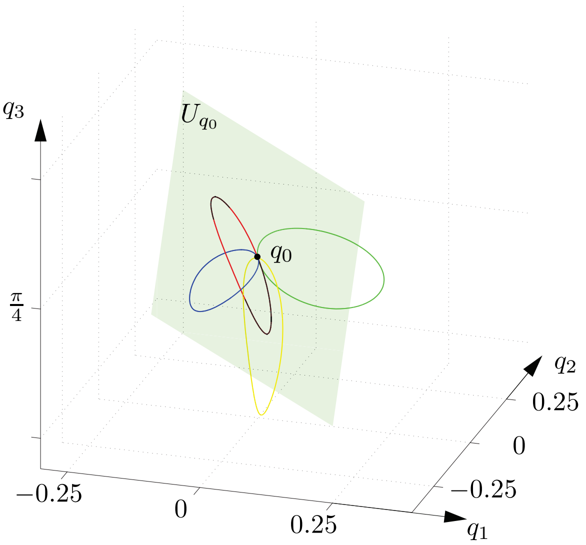
(a) Curves γ^4 (blue), and $G[\gamma^4]$ (red).

(b) Phase portrait of the reduced dynamics on $G[\gamma^4]$.

Figure 2.5: Example 2.2. Synthesis of a regular VHC starting from curve γ^4 and possessing unstable reduced dynamics.



(a) PVTOL configurations for $G[\gamma^1]$ (green), $G[\gamma^2]$ (yellow) $G[\gamma^3]$ (black), and $G[\gamma^4]$ (blue).



(b) VHCs $G[\gamma^1]$ (green), $G[\gamma^2]$ (yellow), $G[\gamma^3]$ (black) and $G[\gamma^4]$ (blue).

Figure 2.6: Examples 2.1 and 2.2. Illustrations of the VHCs obtained in these examples though the synthesis method of Theorem 2.1.

Part II

Coordinate Free Approach to Virtual Holonomic Constraints

Chapter 3

Coordinate Free Formulation of Virtual Holonomic Constraints

Mathematics is the art of giving
the same name to different things.

– Henri Poincaré

In this chapter, we present a coordinate free formulation for VHCs. This formalism is based on a differential and Riemannian geometry setting. A Lagrangian mechanical system can be naturally identified with a Riemannian manifold. With this approach, the trajectories of the Lagrangian mechanical system correspond to the geodesic curves of the associated Levi-Civita connection. Even though this identification is natural, to our knowledge, it has not been used before in VHCs theory.

Following this formalism, we extend some results presented in [6, 5] (and summarized in Chapter 1 of this thesis) to Lagrangian mechanical systems on a generic manifold. We present the contributions on virtual constraint theory in Section 3.2. The first result, Proposition 3.1, shows that a VHC is a controlled invariant set if its tangent bundle and the input distribution are transversal. This condition is a straightforward generalization of the one presented in Propositions 1.2. Theorem 3.1 shows that the projection of the Levi-Civita connection on the tangent bundle is still a connection, called the *induced connection*. Then, Proposition 3.2 shows that the reduced dynamics correspond to the geodesics of the induced connection. In Section 3.3 we present some examples of VHCs described with this formalism. Some of these results have been presented in [51].

3.1 Mechanical Systems and Riemannian Geometry

In this section, we present a coordinate free description of Lagrangian mechanical systems. In the first subsection, we summarize the basic concepts of differential and Riemannian geometry that we will use in this part of the thesis. The notation has been adapted from [13] and [11].

3.1.1 Notation

In this part of the thesis we adopt the **Einstein summation convention**. Following this convention, if in a given expression the same index appears twice, as an upper index of a term and the as a lower index of another one, these terms are understood to be summed over all possible values of that index. For instance, $x^i y_i$ means $\sum_i x^i y_i$ and $z^i_{k,j,l} x^k y^j w^l u_i$ means $\sum_k x^k \sum_j y^j \sum_i \sum_l z^i_{k,j,l} w^l_i$.

Differential Geometry Notation

In this thesis, we deal with manifolds which are smooth, Hausdorff and second countable. Let $\mathcal{M}$ be an n-dimensional manifold, a **coordinate chart on $\boldsymbol{\mathcal{M}}$** is a pair $(\mathcal{U}, \phi)$, where $\mathcal{U}$ is an *open* subset of $\mathcal{M}$, called **coordinate neighborhood**, and $\phi : \mathcal{U} \to \bar{\mathcal{U}}$ is a diffeomorphism, called **local coordinate map**, with $\bar{\mathcal{U}}$ an *open* subset of $\mathbb{R}^n$. The component functions $(x^1, \dots, x^n)$ of ϕ, denoted by $\phi(q) = (x^1(q), \dots, x^n(q))$, $q \in \mathcal{M}$, are called **local coordinates on $\boldsymbol{\mathcal{U}}$**. Given $p \in \mathcal{U}$, if $\phi(0) = 0$, we say that the coordinate chart is centered at p. Two coordinate charts, $(\mathcal{U}, \phi)$ and $(\mathcal{V}, \psi)$, are said to be *smoothly compatible* if either $\mathcal{U} \cap \mathcal{V} = \emptyset$ or $\psi \circ \phi^{-1}$ is a diffeomorphism. A **smooth atlas** $\mathcal{A}$ for $\mathcal{M}$ is a collection of charts whose domains cover $\mathcal{M}$ and any two charts in $\mathcal{A}$ are smoothly compatible with each other.

Let $\mathcal{M}$ be an n-dimensional manifold, $f : \mathcal{M} \to \mathbb{R}^k$ is a **smooth function** if, for each $p \in \mathcal{M}$, there exists a coordinate char $(\mathcal{U}, \phi)$ such that $p \in \mathcal{U}$ and $f \circ \phi^{-1}$ is smooth to the open set $\bar{\mathcal{U}} = \phi(\mathcal{U}) \subset \mathbb{R}^n$. We call the smooth function $\bar{f} : \bar{\mathcal{U}} \to \mathbb{R}^k$, defined as $\bar{f}(x) = f \circ \phi^{-1}(x)$, the *coordinate representation of f*. We denote the family of all smooth scalar function on $\mathcal{M}$, $f : \mathcal{M} \to \mathbb{R}$, by $C^\infty(\mathcal{M})$.

Let $\mathcal{M}, \mathcal{N}$ be smooth manifolds, $F : \mathcal{M} \to \mathcal{N}$ is a **smooth map** if, for each $p \in \mathcal{M}$ there exists a coordinate char on $\mathcal{M}$, $(\mathcal{U}, \phi)$ containing $p \in \mathcal{U}$, and a coordinate char on $\mathcal{N}$, $(\mathcal{V}, \psi)$ containing $F(p) \in (\mathcal{V}, \psi)$, such that $F(\mathcal{U}) \subseteq \mathcal{V}$ and the composite map $\psi \circ F \circ \phi^{-1}$ is smooth from $\phi(\mathcal{U})$ to $\psi(\mathcal{V})$.[1] We call the smooth map $\bar{F} : \phi(\mathcal{U}) \to \psi(\mathcal{V})$, defined by $\bar{F}(x) = \psi \circ F \circ \phi^{-1}(x)$, the *coordinate representation of F*.

Let $\mathcal{M}$ be a smooth manifold and let $p \in \mathcal{M}$. A **derivation at p** is a linear map $v : C^\infty(\mathcal{M}) \to \mathbb{R}$ satisfying $v(fg) = f(p)vg + g(p)vf, \forall f, g \in C^\infty(\mathcal{M})$. The set of all derivation of $C^\infty(\mathcal{M})$ at p is a *n-dimensional vector space* called **tangent space to $\mathcal{M}$ at p**, and it is denoted by $T_p\mathcal{M}$. The elements of $T_p\mathcal{M}$ are called **tangent vector at p**. Given a coordinate char $(\mathcal{U}, \phi)$, the local coordinates $(x^1, \ldots, x^n)$ give a basis for $T_p\mathcal{M}$, consisting of the partial derivative operators $\frac{\partial}{\partial x^i}$, called **coordinate basis of $T_p\mathcal{M}$**. When there is no confusion about which coordinates are meant, we abbreviate the coordinate basis by the notation ∂_i. Then a tangent vector $v \in T_p\mathcal{M}$ can be described by $v = v^i \partial_i$, where v^i are its components with respect to the coordinate basis.

Let $\mathcal{M}$ be an n-dimensional smooth manifold. The disjoint union of the tangent spaces to $\mathcal{M}$ at all points of $\mathcal{M}$ is called **tangent bundle of $\mathcal{M}$** and it is denoted by $T\mathcal{M}$. The tangent bundle is a $2n$-dimensional smooth manifold, and its elements are denoted by the pair (p, v). Moreover, the tangent bundle

[1] Note that, a smooth function is a special case of smooth map.

is endowed with a natural projection map $\pi : T\mathcal{M} \to \mathcal{M}$.

Let $\mathcal{M}, \mathcal{N}$ be smooth manifolds and let $F : \mathcal{M} \to \mathcal{N}$ be a **smooth map**. For each $p \in \mathcal{M}$ the map $dF_p : T_p\mathcal{M} \to T_{F(p)}N$ is called **differential of $\boldsymbol{F}$ at $\boldsymbol{p}$**. Let $(\mathcal{U}, \phi)$ and $(\mathcal{V}, \psi)$ be the coordinate charts on $\mathcal{M}$ and $\mathcal{N}$, respectively. Denote by x the local coordinates on $\mathcal{U}$ and by y the local coordinates on $\mathcal{V}$. Then, for each $p \in \mathcal{U}$, the action of dF_p on the coordinate basis $\frac{\partial}{\partial x^i}$ is given by

$$dF_p \left(\left. \frac{\partial}{\partial x^i} \right|_p \right) = \frac{\partial \bar{F}^j}{\partial x^i}(\phi(p)) \left. \frac{\partial}{\partial y^j} \right|_{F(p)}.$$

Namely, dF_p is represented in coordinate basis by the Jacobian matrix of the coordinate representation of F. By putting together the differentials of F at all the points of $\mathcal{M}$, we obtain the **global differential of $\boldsymbol{F}$**, which is a smooth map $dF : T\mathcal{M} \to T\mathcal{N}$.

Let $\mathcal{M}$ be an m-dimensional smooth manifold and let $\mathcal{N}$ be an n-dimensional smooth manifold. Consider a smooth map $F : \mathcal{M} \to \mathcal{N}$ and $p \in \mathcal{M}$, then the *rank of* $\boldsymbol{F}$ *at* $\boldsymbol{p}$ is the rank of dF_p (that is, the rank of the Jacobian matrix). The map F has *full rank in* p if the rank is equal to the minimum of $\{\dim \mathcal{M}, \dim \mathcal{N}\}$. The map F has **full rank** if it has full rank for all $p \in \mathcal{M}$. The map $F : \mathcal{M} \to \mathcal{N}$ is a **smooth submersion** if its rank is equal to $\dim \mathcal{N}$ (that is its differential is surjective for each point). The map $F : \mathcal{M} \to \mathcal{N}$ is a **smooth immersion** if its rank is equal to $\dim \mathcal{M}$ (that is its differential is injective for each point). If $\dim(\mathcal{M}) = \dim(\mathcal{N})$ and the smooth map F has a smooth inverse, F is a **diffeomorphism** from $\mathcal{M}$ to $\mathcal{N}$. The n-dimensional manifolds $\mathcal{M}, \mathcal{N}$ are *diffeomorphic* if there exists a diffeomorphism between them.

In this thesis we consider a particular subfamily of smooth immersions, the embeddings. A **smooth embedding** is both a smooth immersion and a topological embedding. Let $\mathcal{M}$ and $\mathcal{N}$ be smooth manifolds and let $F : \mathcal{M} \to \mathcal{N}$ be a smooth embedding. The image of this map, $\mathcal{S} = F(\mathcal{M})$, is an **embedded submanifold** of $\mathcal{N}$ with the property that F is a *diffeomorphism* onto its image. The embedding $\iota : \mathcal{S} \to \mathcal{M}$ is called *inclusion map* and it is defined as $\iota = F(F^{-1})$. We define **codimension of $\mathcal{S}$ in $\mathcal{M}$** the difference

$\dim\mathcal{S} - \dim\mathcal{M}$. For each $p \in \mathcal{M}$ and $q = F(p) \in \mathcal{S}$, the image of the injective linear map $dF_p : T_p\mathcal{M} \to T_{F(p)}\mathcal{N}$, is a linear subspace of $T_{F(p)}\mathcal{M}$ and it is denoted by $T_{F(p)}\mathcal{S}$, namely $T_{F(p)}\mathcal{S} = dF_p(T_p\mathcal{M})$. Let $\mathcal{M}$ be an n-dimensional manifold and $\mathcal{S} \subseteq \mathcal{M}$ an embedded submanifold with codimension m. By Theorem 5.8 of [13], for all $p \in \mathcal{S}$, there exists a coordinate chart $(\mathcal{U}, \phi)$ such that $\phi(\mathcal{S}) = \{x \in \phi(\mathcal{U}) : x^{n-m} = \cdots = x^n = 0\}$. In other words, in a neighborhood of p, $\mathcal{S}$ is locally parametrized as the subset of $\mathcal{M}$ on which the last m components of the local coordinates are null.

A **smooth vector field** is a smooth map $X : \mathcal{M} \to T\mathcal{M}$, with the property $X_p \in T_p\mathcal{M}$, $\forall p \in \mathcal{M}$. The *support of* X is the closure of the set $\{p \in \mathcal{M} : X_p \neq 0\}$. Let $(\mathcal{U}, \phi)$ be a coordinate chart on $\mathcal{M}$ and let $(x^1, \ldots, x^n)$ be the local coordinates. For each $p \in \mathcal{U}$ we can write X in terms of the coordinate basis $X_p = X^i(p)\partial_i$. The n functions $X^i : \mathcal{U} \to \mathbb{R}$ are the *component functions* of X on the chart. We denote by $\mathfrak{X}(\mathcal{M})$ the set of all vector fields on $\mathcal{M}$.

Let $\mathcal{M}$ and $\mathcal{N}$ be smooth n-dimensional manifolds and let $F : \mathcal{M} \to \mathcal{N}$ be a diffeomorphism. Then, the **pushforward of $\boldsymbol{X \in \mathfrak{X}(\mathcal{M})}$ by $\boldsymbol{F}$** is a vector field $F_*X \in \mathfrak{X}(\mathcal{N})$ defined as

$$(F_*X)_q = dF_{F^{-1}(q)}\left(X_{F^{-1}(q)}\right), \forall q \in \mathcal{N}.$$

Let $\dim\mathcal{N} > \dim\mathcal{M}$ and let $F : \mathcal{M} \to \mathcal{N}$ be an embedding with $\mathcal{S} = F(\mathcal{M})$. Since F is a diffeomorphism from $\mathcal{M}$ to $\mathcal{S}$, for any $X \in \mathfrak{X}(\mathcal{M})$, F_*X is a vector field which lies, for each $q \in S$, in the subspace T_qS.

Let V be a vector space, a **covector on $\boldsymbol{V}$** is a linear map $\omega : V \to \mathbb{R}$. The space of all convectors is itself a vector space, V^*, which is called the *dual space of* V. Let $\mathcal{M}$ be an n-dimensional smooth manifold and let $p \in \mathcal{M}$, the **cotangent space to $\boldsymbol{\mathcal{M}}$ at $\boldsymbol{p}$** is the dual space of $T_p\mathcal{M}$, denoted by $T_p^*\mathcal{M}$. Let $(\mathcal{U}, \phi)$ be a coordinate chart on $\mathcal{M}$ and let $(x^1, \ldots, x^n)$ the local coordinates on $\mathcal{U}$. For each $p \in \mathcal{U}$, the coordinate basis of $T_p^*\mathcal{M}$, called **coordinate dual basis**, is denoted by dx^i such that $\forall i, j \in \{1, n\}$ $\delta^i_j = dx^i\partial_j$, with δ the Kronecker delta.

Let $\mathcal{M}$ be a smooth manifold, the disjoint union of the cotangent spaces to $\mathcal{M}$ at all points of $\mathcal{M}$ is called the **cotangent bundle of $\mathcal{M}$** and it is denoted by $T^*\mathcal{M}$. A **smooth covector field** is a smooth map $\omega : \mathcal{M} \to T^*\mathcal{M}$, with the property $\omega_p \in T_p^*\mathcal{M}$, $\forall p \in \mathcal{M}$. Let $(\mathcal{U}, \phi)$ be a coordinate chart on $\mathcal{M}$ and let $(x^1, \dots, x^n)$ be the local coordinates. For each $p \in \mathcal{U}$ we can write ω in terms of the dual basis $\omega_p = \omega_i(p)dx^i$. The n functions $\omega_i : \mathcal{U} \to \mathbb{R}$ are the *component functions* of ω on the chart. We denote by $\mathfrak{U}(\mathcal{M})$ the set of all covector fields on $\mathcal{M}$. Then, given $X \in \mathfrak{X}(\mathcal{M})$ and $\omega \in \mathfrak{U}(\mathcal{M})$, $\omega(X) : \mathcal{M} \to \mathbb{R}$ is a smooth function on $\mathcal{M}$, defined as $\omega_p(X_p) = \omega_i(p)X^i(p)$, $\forall p \in \mathcal{M}$.

Let $\mathcal{M}$ and $\mathcal{N}$ be smooth manifolds and let $F : \mathcal{M} \to \mathcal{N}$ be a smooth map.[2] Given $\omega \in \mathfrak{U}(\mathcal{N})$, the covector field $F^*\omega \in \mathfrak{U}(\mathcal{M})$ is the **pullback of $\omega \in \mathfrak{U}(\mathcal{M})$ by F**, defined by

$$\left(F^*\omega\right)_p = dF_p^*\left(\omega_{F(p)}\right).$$

Let $\mathcal{M}$ be a smooth manifold and let $f : \mathcal{M} \to \mathbb{R}$ be a smooth function. The **differential of f** is the covector field denoted by df. Considering the dual basis, the differential of f can be written by $df_q = \frac{\partial f}{\partial x^i}(q)\, dx^i, \forall p \in \mathcal{M}$. A covector filed $\omega \in \mathfrak{U}(\mathcal{M})$ is called **exact** on $\mathcal{M}$ if there exists a smooth function $f \in C^\infty(\mathcal{M})$ such that $\omega = df$. In this case, the function is called a **potential for ω**.

Let V be a vector space and let V^* be its dual space. A **(k, l)-tensor F** is a multilinear smooth map which associates to k vectors and l covectors a real number, $F : V^k \times (V^*)^l \to \mathbb{R}$. Given a n-dimensional manifold $\mathcal{M}$, a **smooth (k, l)-tensor field T on $\mathcal{Q}$** is a smooth map which assigns a tensor to each element of $\mathcal{Q}$. We use the notation T_p to represent the tensor obtained evaluating T at $p \in \mathcal{M}$, that is a linear map $T_p : (T_p\mathcal{M})^k \times (T_p^*\mathcal{M})^l \to \mathbb{R}$. Let $(\mathcal{U}, \phi)$ be a coordinate chart, and let $(x^1, \dots, x^n)$ be the local coordinates on $\mathcal{U}$. In terms of the coordinate basis (∂_i) and the coordinate dual basis (dx^i), T has the coordinate expression

$$T_p = T_{i_1,\dots,i_k}^{j_1,\dots,j_l}(p)\, \partial_{j_1} \times \dots \partial_{j_l} \times \dots \times dx^{i_1} \times \dots dx^{i_k},\, \forall p \in \mathcal{U}$$

[2]For pullbacks, it is not necessary that F is a diffeomorphism.

where the $n(k+l)$ functions $F^{j_1,\ldots,j_l}_{i_1,\ldots,i_k} : \mathcal{M} \to \mathbb{R}$ are the *component functions* of T on the chart. Note that a scalar function is a $(0,0)$-tensor field, a vector field a $(0,1)$-tensor field, a covector field a $(1,0)$-tensor field.

A ***k*-dimensional smooth distribution (codistribution) $\boldsymbol{D}$** on a n-dimensional (m > k) smooth manifold $\mathcal{M}$ is a smooth k-subbundle of $T\mathcal{M}$ ($T^*\mathcal{M}$).[3] Namely, for each $p \in \mathcal{M}$, D_p is a subspace of dimension k of $T_p\mathcal{M}$ ($T^*_p\mathcal{M}$). A k-dimensional smooth distribution (codistribution) D can be described by the span of k vector (covector) fields, called **generators of $\boldsymbol{D}$**, ${}^iX \in \mathfrak{X}(\mathcal{M})$ (${}^i\omega \in \mathfrak{U}(\mathcal{M})$), $i \in \{0,\ldots,k\}$. These vector (covector) field are the coordinate basis of D_p on $T_p\mathcal{M}$ ($T^*_p\mathcal{M}$) for all $p \in \mathcal{M}$.

An **annihilator of $\boldsymbol{D}$**, $\mathrm{ann}(D)$, is a $(n-k)$-dimensional smooth codistribution (distribution), such that its generators and the generators of D commute. Namely, given a distribution generated by ${}^iX \in \mathfrak{X}(\mathcal{M})$, the covector fields ${}^i\omega \in \mathfrak{U}(\mathcal{M})$ is a generator of $\mathrm{ann}(D)$ if ${}^i\omega_p({}^iX_p) = 0, \forall p \in \mathcal{M}$.

We denote by the operator $[,]$ the Lie bracket of vector fields. A smooth distribution D is **involutive** if, for all $p \in \mathcal{M}$, and for each pair of generators of D, X, Y, $[X,Y] \in D$. A **maximal local integral manifold** through $p_0 \in \mathcal{M}$ for D is an immersed smooth submanifold $\mathcal{S}$ containing p_0, such that $T_p\mathcal{S} = D_p$ for all $p \in \mathcal{S}$. The smooth distribution D is **integrable** if there exists a maximal local integral manifold for each $p \in \mathcal{M}$. By Frobenius's Theorem (cf. Theorem 3.90 in [52] or see Ch.19 of [13]), a smooth distribution is integrable if and only if it is involutive.

Riemannian Geometry Notation

A **Riemannian manifold** is a pair $(\mathcal{Q}, g)$, where $\mathcal{Q}$ is a smooth manifold and g is a smooth (2,0)-tensor field on $\mathcal{Q}$, symmetric (i.e., $g(X,Y) = g(Y,X)$, for any $X, Y \in \mathfrak{X}(\mathcal{Q})$) and positive definite, called **Riemannian metric**. In this way, for each $q \in \mathcal{Q}$, $g_q : T_qQ \times T_q\mathcal{Q} \to \mathbb{R}$ is an inner product on $T_q\mathcal{Q}$.

[3]Note that, these distributions represent a subclass of the distributions described in 3.88 of [52], possessing the regularity property and being continuously differentiable.

The **flat** map associates to $X \in \mathfrak{X}(\mathcal{Q})$ the unique covector field $X^\flat \in \mathfrak{U}(\mathcal{Q})$ such that $X^\flat(Y) = g(X, Y)$, $\forall Y \in \mathfrak{X}(\mathcal{Q})$. Its inverse, the **sharp** map, associates to $\omega \in \mathfrak{U}(\mathcal{Q})$ the unique vector field $\omega^\# \in \mathfrak{X}(\mathcal{Q})$ such that $\omega(X) = g(\omega^\#, X)$, $\forall X \in \mathfrak{X}(\mathcal{Q})$. In terms of coordinate basis and dual basis, given $X \in \mathfrak{X}(\mathcal{Q})$, the component functions of the unique covector field obtained by its flat map are given by $(X^\flat)_i = g_{i,j}X^j$. Given $\omega \in \mathfrak{U}(\mathcal{Q})$, the component functions of the unique covector filed obtained by its sharp map are given by $(\omega^\sharp)^i = g^{i,j}X_j$, where the component functions $g^{i,j}$ satisfy $g^{i,j}g_{j,k} = \delta^i_k$.

Now, we want to differentiate vector fields in a coordinate free way. Since the values of a vector field *live* on different tangent spaces, we need a way to compare or *connect* these values. An **affine connection**[4] ∇ on $\mathcal{Q}$ is a map $\nabla : \mathfrak{X}(\mathcal{Q}) \times \mathfrak{X}(\mathcal{Q}) \to \mathfrak{X}(\mathcal{Q})$ which assigns to the pair $X, Y \in \mathfrak{X}(\mathcal{Q})$, a vector field $\nabla_X Y \in \mathfrak{X}(\mathcal{Q})$, such that the following properties are satisfied

$$\begin{aligned} &\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z, \\ &\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z, \\ &\nabla_X(fY) = f\nabla_X Y + X(f)Y, \end{aligned} \tag{3.1}$$

where $Z \in \mathfrak{X}(\mathcal{Q})$ and f, g are real-valued smooth functions. The **torsion** of $\nabla_X Y$ is given by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Let $(\mathcal{U}, \phi)$ be a coordinate chart on $\mathcal{Q}$ and let ∂_i be the coordinate basis on $\mathcal{U}$, for any choice of i, j, the affine connection is

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{i,j}\partial_k, \tag{3.2}$$

where the n^3 functions $\Gamma^k_{i,j} : \mathcal{U} \to \mathbb{R}$ are called **Christoffel symbols of ∇**. Given $X, Y \in \mathfrak{X}(\mathcal{Q})$, the affine connection $\nabla_X Y$ in coordinate basis is given by

$$(\nabla_X Y)_{(p)} = \left(X(Y^k(p)) + \Gamma^k_{i,j}(p)X^i(p)Y^j(p)\right)\partial_k$$

for all $p \in \mathcal{U}$.

[4] In [11] it is denoted by *linear* connection.

The vector field $\nabla_X Y$ is also called **covariant derivative of Y in direction of X**. Actually, any affine connection induces the covariant derivative of a given tensor in the direction of a given vector field. Let T be a (k,l)-tensor field, the **total covariant derivative of T** is a $(k+1,l)$-tensor field, denoted by ∇T. In coordinate basis, for all $p \in \mathcal{U}$, it is denoted by

$$\left(T^{j_1,\dots,j_l}_{i_1,\dots,i_k;m}\right)_{(p)} = \frac{T^{j_1,\dots,j_l}_{i_1,\dots,i_k}(p)}{\partial x^i} + \sum_{s=1}^{l} T^{j_1\dots e\dots j_l}_{i_1\dots i_k}(p)\,\Gamma^{j_s}_{m,e}(p) - \sum_{s=1}^{k} T^{j_1\dots j_l}_{i_1\dots e\dots i_k}(p)\,\Gamma^{e}_{m,i_s}(p).$$

A **curve** is a smooth map $\gamma : I \to \mathcal{M}$, with $I \subset \mathbb{R}$ is an interval. For any $t \in I$, the **velocity of the curve** is a vector field with support on $\gamma(I)$ defined as $\dot\gamma(t) = \frac{\gamma(t)}{dt}$.[5] The *acceleration* of a curve in a coordinate free way is given by the **covariant derivatives along the curve**, which is defined by $\nabla_{\dot\gamma(t)}\dot\gamma(t)$. A **geodesic** is a curve satisfying $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0, \forall t \in I$. Let $(\mathcal{U}, \phi)$ be a coordinate chart on $\mathcal{Q}$ and let ∂_i be the coordinate basis on $\mathcal{U}$, the coordinate representation of the curve is $\bar\gamma : \bar I \subseteq I \to \mathbb{R}^n$, defined by $\bar\gamma(t) = \phi(\gamma(t))$. The interval $\bar I$ satisfies $\gamma(\bar I) \in \mathcal{U}$. The velocity vector is written in terms of coordinate basis as $\dot\gamma(t) = \dot\gamma^i(t)\partial_i = \frac{\phi(\gamma(t))}{dt}\partial_i$. The covariant derivative is written in terms of coordinate basis as

$$(\nabla_{\dot\gamma(t)}\dot\gamma(t))_{\gamma(t)} = \left(\frac{\dot\gamma^k(t)}{dt} + \Gamma^k_{i,j}(\gamma(t))\,\dot\gamma^i(t)\,\dot\gamma^j(t)\right)\partial_k,\ \forall t \in \bar I. \tag{3.3}$$

The Fundamental Theorem of Riemannian Geometry (see Theorem 5.4 of [11]), states that, in a Riemannian manifold $(\mathcal{Q}, g)$, there exists a unique affine connection, the **Levi-Civita connection**, which is *compatible* with the Riemannian metric, that is $\nabla g = 0$, and *symmetric*, that is its torsion is equal to 0. We denote the Levi-Civita connection by $\overset{g}{\nabla}$. Given a coordinate chart $(\mathcal{U}, \phi)$, the Christoffel symbols of the Levi-Civita connection are given by

$$\Gamma^k_{i,j}(p) = \frac{1}{2} g^{k,l}(p)\left(\frac{\partial g_{j,l}(p)}{\partial x^i} + \frac{\partial g_{i,l}(p)}{\partial x^j} - \frac{\partial g_{i,j}(p)}{\partial x^l}\right),\ \forall p \in \mathcal{U}.$$

[5] This is the same concept of vector field along curves cited in 1.1.

Remark on Notation

Smooth manifolds and embedded submanifold are denoted by a calligraphic capital letter, for instance $\mathcal{M}$. The evaluation of a smooth map/function, F, in a given point of a manifold, p, is represented by $F(p)$. The evaluation of a smooth tensor field, T, in a given point of a manifold, p, is represented by T_p.

The coordinate representation of a smooth map/function, F, is denoted by the bar, $\bar{F}$. The representation in coordinate basis and/or dual basis of a smooth tensor, T, is given by $T_i^j \partial_i \times dx^j$. The evaluation of the component functions at a given point of the manifold, p, is written as $T_i^j(p)$. Moreover, since these component functions are univocally identified by the choice of the coordinates, we avoid to add the bar.

3.1.2 Lagrangian Mechanical Systems

In this subsection, following the book [52], we present a coordinate free definition of some concepts of mechanics.

Definition 3.1 (Lagrangian mechanical system). A Lagrangian mechanical system is a triple $(\mathcal{Q}, g, P)$, where

- $\mathcal{Q}$ is a n-dimensional configuration manifold, called configuration space,
- g is a smooth symmetric and positive definite (2,0)-tensor field on $\mathcal{Q}$,
- $P : \mathcal{Q} \to \mathbb{R}$ is a scalar function.

A Lagrangian system is naturally associated to the Riemannian manifold $(\mathcal{Q}, g)$, where, for all $q \in \mathcal{Q}$, the kinetic energy is given by $\frac{1}{2} g_q(\dot{q}, \dot{q})$ and the potential energy is given by $P(q)$. The Lagrangian function is given by

$$\mathscr{L}(q, \dot{q}) = \frac{1}{2} g_q(\dot{q}, \dot{q}) - P(q).$$

The motion of a Lagrangian system is given by the curves which *minimize* the following function, called *action* associated with $\mathscr{L}$,

$$A_{\mathscr{L}}(q(t), \dot{q}(t)) = \int_I \mathscr{L}(q(t), \dot{q}(t)) dt, \tag{3.4}$$

where I is an open interval of $\mathbb{R}$ and $q : I \to \mathcal{Q}$ is a curve on $\mathcal{Q}$. The curve $q(t)$ minimizes (3.4) if, for any coordinate chart $(\mathcal{U}, \phi)$ such that $q(t) \in U$, its coordinate representation $x(t) = \phi(q(t))$ is a *solution* of the *Euler-Lagrange equations* (see Theorem 4.38 in [52])

$$\frac{d}{dt}\frac{\partial \bar{\mathscr{L}}(x,\dot{x})}{\partial \dot{x}} - \frac{\partial \bar{\mathscr{L}}(x,\dot{x})}{\partial x} = 0,$$

where $\bar{\mathscr{L}}(x,\dot{x}) = \mathscr{L}(q,\dot{q})$.

Definition 3.2 (Lagrangian mechanical control system). A Lagrangian mechanical control system is a quadruple $(\mathcal{Q}, g, P, B)$, where

- $(\mathcal{Q}, g, P)$ is a Lagrangian mechanical system;
- B is a m-dimensional codistribution generated by covector fields ${}^1B, \ldots, {}^m B$, it is assumed that B is smooth and that B_q has dimension m, $\forall q \in \mathcal{Q}$.

The solutions of a Lagrangian control system satisfy (see Proposition 4.57 in [52])

$$\frac{d}{dt}\frac{\partial \bar{\mathscr{L}}(x,\dot{x})}{\partial \dot{x}} - \frac{\partial \bar{\mathscr{L}}(x,\dot{x})}{\partial x} = \sum_{i=1}^{m} {}^iB_x \tau_i, \tag{3.5}$$

where ${}^iB_x$ is the representation in coordinate basis of ${}^iB_q$. Note that equation (1.3) has the same form of (3.5). Setting $\mathrm{grad}P = (dP)^{\#}$ and ${}^iF = ({}^iB)^{\#}$, a coordinate free representation of (3.5) is given by the following equation (see Proposition 4.82 of [52])

$$\overset{g}{\nabla}_{\dot{q}}\, \dot{q} = -\mathrm{grad}P(q) + \sum_{i=1}^{m} {}^iF_q \tau_i(t), \;\; \forall q \in \mathcal{Q}, \tag{3.6}$$

where $\overset{g}{\nabla}$ is the *Levi-Civita connection* of the Riemannian manifold $(\mathcal{Q}, g)$. The distribution F, defined as $F_q =$ span $\{{}^1F_q, {}^2F_q, \ldots, {}^mF_q\}$, is called *input distribution.*

Physical Holonomic Constraints

Following Ch.4.5 of [52], let $\mathcal{Q}$ be a smooth manifold. A physical constraint is described by a k-dimensional smooth distribution D on $\mathcal{Q}$, called **constraint distribution**. A curve $\gamma : I \to \mathcal{Q}$, with $I \subset \mathbb{R}$ an interval, *satisfies* the constraint distribution D if $\dot\gamma(t) \in D_{\gamma(t)}, \forall t \in I$. Given a constraint distribution D, a constraint force for D is a force taking values in the codistribution $\mathrm{ann}(D)$. Given $\gamma : I \to \mathcal{Q}$, the **constraint force along $\boldsymbol{\gamma}$** is a covector field $\alpha : I \to T^*\mathcal{Q}$ along γ such that $\alpha_{\gamma(t)} \in \mathrm{ann}(D)$, $\forall t \in I$.

The Lagrangian mechanical control system $(\mathcal{Q}, g, P, B)$ is called **physically constrained by $\boldsymbol{D}$**, if for each smooth curve $\gamma : I \to \mathcal{Q}$ there exists a constraint force α along γ such that

- γ satisfies D;
- γ satisfies the *Lagrange-d'Alembert Principle* for the force $B\tau + \alpha$ and the Lagrangian function (see Proposition 4.59 of [52]).

We denote that system with $(\mathcal{Q}, g, P, B, D)$.

Let $D^\perp$ be the g-orthogonal complement to D in $T\mathcal{Q}$. For each $q \in \mathcal{Q}$, we denote by $\Pi_q : T_q\mathcal{Q} \to D_q$ and by $\Pi_q^\perp : T_q\mathcal{Q} \to D_q^\perp$ the unique smooth submersions such that, $\forall v \in T_q\mathcal{Q}$, $v = P_q(v) + P_q^\perp(v)$. Then $\Pi_q, \Pi_q^\perp$ are the g-orthogonal projections onto D_q and $D_q^\perp$, respectively. The solutions of $(\mathcal{Q}, g, P, B, D)$ are given by the curves satisfying the following equations

$$\overset{g}{\nabla}_{\dot q}\, \dot q = -\mathrm{grad}P(q) + \sum_{i=1}^{m} {}^iF_q \tau_i(t) + (\alpha)^\sharp, \tag{3.7}$$

$$\Pi_q^\perp(\dot q) = 0, \tag{3.8}$$

where $(\alpha)^\sharp$ is the acceleration due to the constraint force along a solution. The equation 3.8 is satisfied if $(\alpha)^\sharp$ holds the following form[6]

$$(\alpha)^\sharp = -\Pi_q^\perp\left(-\text{grad}P(q) + \sum_{i=1}^m {}^iF_q\tau_i(t)\right) - (\overset{g}{\nabla}\,\Pi^\perp)_q(\dot{q},\dot{q}).$$

Note that $(\alpha)^\sharp$ annihilates the g-orthogonal projections onto $D_q^\perp$ of the acceleration forces and of the covariant derivative. Then the solutions of (3.7) are equal to the solutions of the g-orthogonal projection of (3.7) on D, that is

$$\tilde{\nabla}_{\dot{q}}\dot{q} = \Pi_q\left(-\text{grad}P(q)\right) + \Pi_q\left(\sum_{i=1}^m {}^iF_q\tau_i(t)\right) \tag{3.9}$$

where $\tilde{\nabla}_{\dot{q}}\dot{q} = \Pi_q(\overset{g}{\nabla}_{\dot{q}}\,\dot{q})$, $\forall \dot{q}\in D_q$, is called the **constrained connection**.

If the constraint distribution D is integrable, the constraint is called **holonomic**. Let $\mathcal{S}$ be a maximal local integral manifold for D, which is assumed to be an embedded submanifold, $\mathcal{S}\subseteq\mathcal{Q}$. Let $g_0 = \iota^*g$ be the pullback of the metric tensor on $\mathcal{S}$, then the solutions of physical constrained by the holonomic distribution D are given by equation (3.9), where the constraint connection is the *Levi-Civita connection* compatible with g_0, that is $\tilde{\nabla}_{\dot{q}}\dot{q} = \overset{g_0}{\nabla}_{\dot{q}}\,\dot{q}, \forall(q,\dot{q})\in T\mathcal{S}$ (see Proposition 4.97 in [52]). Moreover, $(\mathcal{S}, g_0)$ is a Riemannian manifold and $(\mathcal{S}, g_0, P, \iota^{*i}B)$ is a Lagrangian mechanical control system.

Remark 3.1. For $q\in\mathcal{S}$, it is possible to decompose the tangent space $T_q\mathcal{Q}$ into two *orthogonal* subbundles, $T_q\mathcal{S}$ and $N_q\mathcal{S}$. Along $\mathcal{S}$, the covariant derivative can be decomposed as follows (The Gauss Formula, Theorem 8.2 of [11])

$$\overset{g}{\nabla}_{\dot{q}}\,\dot{q} = (\overset{g}{\nabla}_{\dot{q}}\,\dot{q})^T + \mathrm{I\!I}(\dot{q},\dot{q}),$$

where $(\overset{g}{\nabla}_{\dot{q}}\,\dot{q})^T = \tilde{\nabla}_{\dot{q}}\dot{q}\in T_q\mathcal{S}$ and $\mathrm{I\!I} : T_q\mathcal{S}\times T_q\mathcal{S}\to N_q\mathcal{S}$ is the **second fundamental form of** $\mathcal{S}$. In order to satisfy (3.8), the constraint force α annihilates $\mathrm{I\!I}$. Note that, the dynamics of a physically constrained system are given by the orthogonal projection on $T\mathcal{S}$ of the dynamics (3.6).

[6]Note that, $\Pi^\perp$ is a $(1,1)$-tensor field, then $\overset{g}{\nabla}\,\Pi^\perp$ is a $(2,1)$-tensor field and $\overset{g}{\nabla}\,\Pi^\perp(\dot{q},\dot{q})$ is a $(0,1)$-tensor field, that is a vector field.

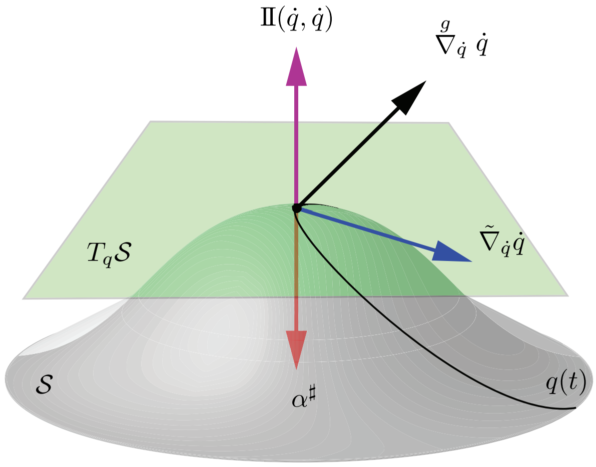


Figure 3.1: Representation of the constraint connection.

3.2 Description of the Reduced Dynamics of VHCs

Consider a Lagrangian mechanical control system $(\mathcal{Q}, g, P, B)$, where $\mathcal{Q}$ denote a smooth manifold of dimension n. Let E be a smooth manifold of dimension $p = n - m$ and let $f : E \to \mathcal{Q}$ be a smooth injective map of rank p. Under these hypotheses f is a smooth embedding (see Proposition 4.22 of [13]), the we denotes by $\mathcal{C}$ denotes an embedded submanifold of $\mathcal{Q}$, that is $\mathcal{C} = f(E)$. Since f is a smooth embedding, $T\mathcal{C}$ is a subbundle of $T\mathcal{Q}$ on $\mathcal{C}$.

Definition 3.3 (Controlled invariant manifold). Let $\mathcal{C}$ be a closed and embedded submanifold of $\mathcal{Q}$, manifold $T\mathcal{C}$ is controlled invariant for system (3.6) if, for any $(q_0, \dot q_0) \in T\mathcal{C}$, there exists a control function $\hat\tau : T\mathcal{C} \to \mathbb{R}^m$ such that the solution of (3.6) satisfies

$$\begin{cases} q(t) \in \mathcal{C}, \\ (q(0), \dot q(0)) = (q_0, \dot q_0)\,, \end{cases}$$

for all t belonging to the interval of existence of the solution. For simplicity, we also say that $\mathcal{C}$ is controlled invariant.

Definition 3.4 (Virtual Holonomic Constraint). A virtual holonomic constraint is a closed and embedded submanifold $\mathcal{C}$ such that $T\mathcal{C}$ is controlled invariant for system (3.6).

Remark 3.2. Definition 3.4 generalizes definition 1.2. In fact, in the special case in which $\mathcal{Q}$ is a generalized cylinder, definition 3.4, with $\mathcal{C} = h^{-1}(0)$, is equivalent to Definition 1.2.

Two subbundles of the tangent space of $\mathcal{Q}$ are transversal if their sum is the full tangent space of $\mathcal{Q}$, that is:

Definition 3.5 (Transversality). Let H be a subbundle of the tangent bundle of a manifold $\mathcal{Q}$ and let $\mathcal{C}$ be an embedded submanifold of $\mathcal{Q}$. We say that H and $\mathcal{C}$ are *transversal* if

$$H_q \oplus T_q\mathcal{C} = T_qQ, \ \ \forall q \in \mathcal{C}.$$

A sufficient condition for $\mathcal{C}$ to be a VHC is the transversality of the input subbundle F and the tangent subbundle $T\mathcal{C}$.

The following proposition, which generalizes Proposition 3.2 of [6], states a sufficient condition for an embedded manifold to be a controlled invariant.

Proposition 3.1. *Let $\mathcal{C}$ be a closed embedded submanifold of Q, if the input subbundle F and $\mathcal{C}$ are transversal, then $\mathcal{C}$ is controlled invariant for system (3.6). Moreover, the feedback law $\hat{\tau}$ that renders $\mathcal{C}$ invariant is* unique.

Proof. Let $\Pi : \mathbb{R}^n \to \mathbb{R}^m$ be the projection on the last m coordinates. Since $\mathcal{C}$ is a p-dimensional embedded submanifold of $\mathcal{Q}$, by Theorem 5.8 of [13], there exists a chart $(\mathcal{U}, \phi)$, such that $q \in \mathcal{U} \subset \mathcal{Q}$ and $\Pi(\phi(\mathcal{C} \cap \mathcal{U})) = 0$.
For each $r \in \mathcal{U}$, define map $\sigma_r : F_r \to \mathbb{R}^m$ as $\sigma_r(v) = \Pi((d\phi)_r(v))$. Since F_r and $\mathcal{C}$ are transversal, $F_r + T_r\mathcal{C} = T_r\mathcal{Q}$. Since $(d\phi)_r$ is bijective, $\Pi((d\phi)_r(F_r + T_r\mathcal{C})) = \Pi(\mathbb{R}^n) = \mathbb{R}^m$. Being $\Pi((d\phi)_r(T_r\mathcal{C})) = 0$ (from the definition of ϕ), it follows that $\Pi((d\phi)_r(F_r)) = \mathbb{R}^m$. Hence, σ_r is injective and, being a map between vector spaces of the same size, it is a bijection. Set $z(t) = \Pi(\phi(q(t))$, then

$\dot{z}(t) = \Pi\left((d\phi)_{q(t)}(\dot{q}(t))\right)$, and

$$\begin{aligned}
\ddot{z}(t) &= \Pi\left((\overset{g}{\nabla}_{\dot{q}(t)}\ d\phi)_{q(t)}(\dot{q}(t)) + (d\phi)_{q(t)}(\overset{g}{\nabla}_{\dot{q}}\ \dot{q}(t)) \right) \\
&= \Pi\left((\overset{g}{\nabla}_{\dot{q}(t)}\ d\phi)_{q(t)}(\dot{q}(t)) + (d\phi)_{q(t)}(-\mathrm{grad}P_{q(t)} + F_{q(t)}\tau(t)) \right) \\
&= \Pi\left((\overset{g}{\nabla}_{\dot{q}(t)}\ d\phi)_{q(t)}(\dot{q}(t)) + (d\phi)_{q(t)}(-\mathrm{grad}P_{q(t)}) \right) + \sigma_{q(t)}\left(F_{q(t)}\tau(t)\right) ,
\end{aligned}$$

which we rewrite as $\ddot{z}(t) = f(q(t), \dot{q}(t), \tau(t))$. Since, for all $q \in \mathcal{Q}$, σ_q is a bijection, there exists (one and only one) control $\hat{\tau}(q, \dot{q})$ for which $f(q, \dot{q}, \hat{\tau}(q, \dot{q})) = 0$, $\forall q \in \mathcal{C}$.

By continuity, there exists a positive real constant $\bar{t}$ such that the solution of (3.6) with $\tau(t) = \hat{\tau}(q(t), \dot{q}(t))$, satisfies $q(t) \in \mathcal{U}$, $\forall t \in [0, \bar{t}]$. Hence, $z(t)$ is well defined for $t \in [0, \bar{t}]$. Being $(q(0), \dot{q}(0)) \in T\mathcal{C}$, z satisfies the differential equation $\ddot{z}(t) = 0, z(0) = \dot{z}(0) = 0$, so that $z(t) = 0$, $\forall t \in [0, \bar{t}]$, which implies that $\Pi(\phi(q(t))) = 0$, $\forall t \in [0, \bar{t}]$.

Let $\hat{t}$ be the maximum interval of existence of the solution of (3.6) with control $\tau(q, \dot{q})$ defined above. Set $\bar{t} = \sup_t\{q(s) \in \mathcal{C}, \forall s \in [0, t]\}$. To complete the proof we need to show that $\bar{t} = \hat{t}$. By contradiction, assume that $\bar{t} < \hat{t}$. There exists a sequence t_i such that $\lim_{i\to\infty} t_i = \bar{t}$. Since $T\mathcal{C}$ is closed, the sequence $(q(t_i), \dot{q}(t_i))$ is convergent to an element of $T\mathcal{C}$, set $c = \lim_{i\to\infty}(q(t_i), \dot{q}(t_i))$. Then, since $c \in T\mathcal{C}$, by the first part of this proof there exists an interval $I = [\bar{t}, \bar{t} + \epsilon]$ such that $q(t) \in \mathcal{C}$, $\forall t \in I$, contradicting the definition of $\bar{t}$. □

The previous proposition justifies the following definition.

Definition 3.6 (Regular Virtual Holonomic Constraint). A VHC $\mathcal{C}$ is regular if the input distribution F and $\mathcal{C}$ are transversal.

Remark 3.3. Definition 3.6 generalizes definition 1.3. In fact, if $\mathcal{Q}$ is a generalized cylinder, the transversality of F and $\mathcal{C}$ is equivalent to the fact that the relative degree of the output function $e = h(q)$ is well-defined and equal to 2, $\forall q \in \mathcal{C}$.

Since the feedback law $\hat{\tau}$ for which $\mathcal{C}$ is invariant is unique, the solution of (3.6), with input $\tau(t) = \hat{\tau}(q(t), \dot{q}(t))$ and initial conditions $(q(0), \dot{q}(0)) \in T_q\mathcal{C}$, belongs to $T\mathcal{C}$, $\forall t \in [0, \bar{t}]$, the associated dynamics are called the **reduced dynamics** and are defined by

$$\overset{g}{\nabla}_{\dot{q}}\, \dot{q} = -\mathrm{grad} P_q + \sum_{i=1}^{m} (F_i)_q \hat{\tau}_i(q, \dot{q}). \tag{3.10}$$

3.2.1 Induced Connection

In this section, we rewrite dynamics (3.10) in a different form. As previously shown on physical holonomic constraints, the solutions for constrained Lagrangian mechanical systems may be represented as the geodesic curves of an affine connection. In this section, using basic tools from the theory of affine connections, we show that this fact also holds for constraints enforced by control forces, that, differently from constraint forces, are generically *not orthogonal* to the tangent space of $\mathcal{C}$ but *only transversal*. Namely, we will show that a VHC $\mathcal{C}$ naturally defines a connection on $\mathcal{C}$ (*the induced connection*), that allows representing the trajectories of the reduced dynamics (3.10) as geodesic curves.

Lemma 3.1. *If $\mathcal{C}$ is a regular VHC, for any $q \in \mathcal{C}$ and any $v \in T_q\mathcal{Q}$ there exist unique smooth submersions $\sigma_q : T_q\mathcal{Q} \to T_q\mathcal{C}$, $\nu_q : T_q\mathcal{Q} \to F_q$ such that $v = \sigma_q(v) + \nu_q(v)$.*

Proof. The existence and the uniqueness of the two submersions is a direct consequence of the fact that F_q and $\mathcal{C}$ are transversal. They are smooth since F and $\mathcal{C}$ are smooth. □

Figure 3.2 represents the decomposition of vector v in the two components $\sigma_q(v)$ and $\nu_q(v)$.

Theorem 3.1 (Induced Connection for VHC)**.** *If $\mathcal{C}$ is a regular VHC, for any $X, Y \in \mathfrak{X}(\mathcal{C})$, set*

$$\hat{\nabla}_X Y = \sigma(\nabla_X Y)\,,$$

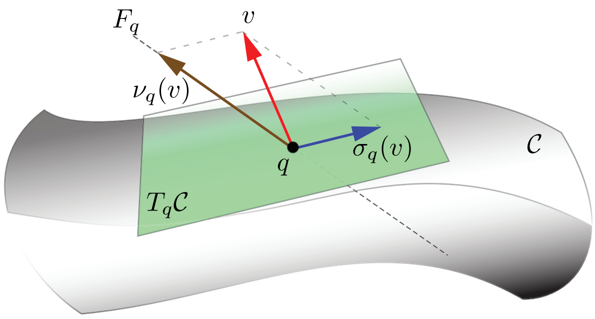


Figure 3.2: Decomposition of a vector $v \in T_q\mathcal{Q}$.

then

i) $\hat{\nabla}$ *is a connection on* $\mathcal{C}$*,*

ii) if ∇ *is torsionless then* $\hat{\nabla}$ *is torsionless.*

Proof. Let $X, Y, Z \in \mathfrak{X}(\mathcal{C})$ and let $f, g : \mathcal{C} \to \mathbb{R}$ be continuous functions. To prove i) we check conditions (3.1). Since ∇ is a connection and σ is a linear map,

$$\hat{\nabla}_{fX+gY}Z = \sigma(\nabla_{fX+gY}Z) = f\sigma(\nabla_X Z) + g\sigma(\nabla_Y Z) = f\hat{\nabla}_X Z + g\hat{\nabla}_Y Z\,,$$

$$\begin{aligned}\hat{\nabla}_X(Y+Z) = \sigma(\nabla_X(Y+Z)) = \sigma(\nabla_X Y + \nabla_X Z) = \\ \sigma(\nabla_X Y) + \sigma(\nabla_X Z) = \hat{\nabla}_X Y + \hat{\nabla}_X Z\,,\end{aligned}$$

$$\begin{aligned}\hat{\nabla}_X(fY) = \sigma(\nabla_X(fY)) = \sigma(f\nabla_X(Y) + X(f)Y) = \\ f(\sigma\nabla_X(fY)) + X(f)\sigma(Y)) = f\hat{\nabla}_X(Y) + X(f)Y\end{aligned}$$

where $\sigma(Y) = Y$ since Y is a vector field on $\mathcal{C}$.

To prove ii), denote by $T, \hat{T}$ the torsion tensors of connections ∇ and $\hat{\nabla}$, then

$$\begin{aligned}\hat{T}(X,Y) = \hat{\nabla}_X Y - \hat{\nabla}_Y X - [X,Y] = \\ \sigma(\nabla_X Y - \hat{\nabla}_Y X) - [X,Y]) = \sigma(T(X,Y)) = 0,\end{aligned}$$

where we used the fact that $\sigma([X,Y]) = [X,Y]$, since $[X,Y] \in T\mathcal{C}$ (see Corollary 8.31 in [13]) and the hypothesis that ∇ is torsionless. $\square$

Connection $\hat{\nabla}$ is called the *induced connection* on $\mathcal{C}$. Note that Proposition 3.1 is an adaptation to our setting on VHC of a basic result in the theory of affine connections, see for instance Section 3.1, Chapter 9, of [53]. The induced connection $\hat{\nabla}$ allows writing in a compact form the reduced dynamics (3.10).

Proposition 3.2. *The reduced dynamics (3.10) can be rewritten as*

$$\hat{\nabla}_{\dot{q}}\dot{q} = -\sigma_q(gradP_q)\,. \tag{3.11}$$

Proof. From Lemma 3.1, $\forall w \in F_q, w = \nu_q(w)$. Since for any vector field v, $(\sigma+\nu)v = v$

$$\begin{aligned}\overset{g}{\nabla}_{\dot{q}}\,\dot{q} = (\sigma+\nu)\overset{g}{\nabla}_{\dot{q}}\,\dot{q} &= \hat{\nabla}_{\dot{q}}\dot{q} + \nu(-(\mathrm{grad}P_q) + F_q\hat{\tau}(q)) \\ &= \hat{\nabla}_{\dot{q}}\dot{q} - (1-\sigma_q)\mathrm{grad}P_q + F_q\hat{\tau}(q))\,,\end{aligned}$$

hence

$$\hat{\nabla}_{\dot{q}}\dot{q} + \sigma(\mathrm{grad}P_q) = \overset{g}{\nabla}_{\dot{q}}\,\dot{q} + \mathrm{grad}P_q - F_q\hat{\tau}(q)\,.$$

$\square$

Remark 3.4. If $P = 0$, the constrained dynamics (3.11) reduce to

$$\hat{\nabla}_{\dot{q}}\dot{q} = 0\,, \tag{3.12}$$

thus, the reduced dynamics correspond to the geodesic curves of the induced connection $\hat{\nabla}$.

Remark 3.5. The induced connection can be rewritten as

$$\hat{\nabla}_{\dot{q}}\dot{q} = \sigma(\overset{g}{\nabla}_{\dot{q}}\,\dot{q})^T + \sigma(\mathrm{II}(\dot{q},\dot{q})) = \tilde{\nabla}_{\dot{q}}\dot{q} + \sigma(\mathrm{II}(\dot{q},\dot{q})).$$

The induced connection can be viewed as the sum of the constrained connection (see Subsection 3.1.2) and the projection σ on $T\mathcal{C}$ of the second fundamental form II.

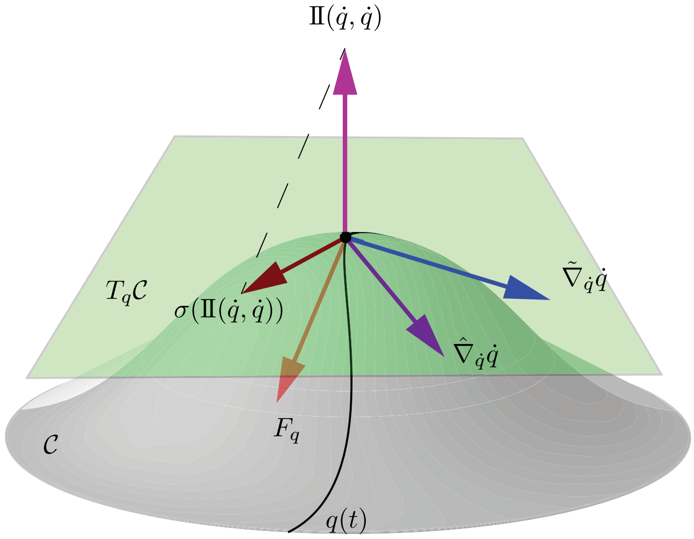


Figure 3.3: Representation of the induced connection.

3.2.2 Coordinate Representation

Let $\mathcal{E}$ be a p-dimensional smooth manifold and $\mathcal{Q}$ be a n-dimensional smooth manifold. Let $\mathcal{C}$ be a virtual holonomic constraint, defined as the image set of the smooth embedding $f : \mathcal{E} \to \mathcal{C} \subset \mathcal{Q}$. For each $e \in \mathcal{E}$, there exist smooth charts $(\mathcal{V}, \psi)$ for $\mathcal{E}$ centered at e, and $(\mathcal{U}, \phi)$, centered at $q = f(e)$, such that $f(\mathcal{V}) \subset \mathcal{U}$. The associated local coordinates are denoted by $z = (z^1, \ldots, z^p)$, $z = \psi(e)$ and $x = (x^1, \ldots, x^n)$, $x = \phi(q)$.

Denote by $\bar{f} : \psi(\mathcal{V}) \to \phi(\mathcal{U})$ the injective map from the local coordinates of $\mathcal{E}$ to the local coordinates of $\mathcal{Q}$. Then, $\bar{f} = \phi \circ f \circ \psi^{-1}$, as is apparent from the following commutative diagram

$$\begin{array}{ccc} \mathcal{V} \subset \mathcal{E} & \xrightarrow{f} & \mathcal{U} \subset \mathcal{Q} \\ \Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle \phi} \\ \psi(\mathcal{V}) \subset \mathbb{R}^p & \xrightarrow{\bar{f}} & \phi(\mathcal{U}) \subset \mathbb{R}^n. \end{array}$$

A chart for $\mathcal{C}$, centered ad q, is given by $(\mathcal{C} \cap \mathcal{U}, \vartheta)$, where $\vartheta : \mathcal{C} \cap \mathcal{U} \to \mathbb{R}^p$, $\vartheta = \psi \circ f^{-1}$, then $\vartheta(f(p)) = \psi(p), \forall p \in \mathcal{V}$.

The differential of f is denoted by df. Remember that, for each $e \in \mathcal{V}$, df_e represents in coordinate basis the Jacobian matrix of $\bar{f}$. Since f is an embedding, the image of $df : T\mathcal{E} \to T\mathcal{C}$ is a p-dimensional smooth distribution on $\mathcal{C}$. In coordinate basis the component functions of df are given by $df^i_j : \psi(\mathcal{V}) \to \mathbb{R}$, then the component functions of $\dot{x}$ are given by

$$\dot{x}^i = df^i_j(z)\,\dot{z}^j,\ \forall z \in \psi(\mathcal{V}),\ \forall i \in \{1,\dots,n\}. \tag{3.13}$$

In this subsection, we omit the domain of the component functions when it is clear from the context.

We denote by $\mathcal{H}f$ the differential of df. Then, for each $e \in \mathcal{V}$, $\mathcal{H}f_e$ represents in coordinate basis the Hessian matrix of $\bar{f}$. Moreover, the component functions of $\mathcal{H}f$ are given by $(\mathcal{H}f)^k_{i,j} : \psi(\mathcal{V}) \to \mathbb{R}$, then the component functions of $\ddot{x}$ are given by

$$\ddot{x}^k = df^k_i\,\ddot{z}^i + (\mathcal{H}f)^k_{i,j}\,\dot{z}^i\,\dot{z}^j,\ \forall k \in \{1,\dots,n\}. \tag{3.14}$$

For each $q \in \mathcal{C}$, the coordinate representation $\bar{\sigma}_{\phi(q)} : \mathbb{R}^n \to \mathbb{R}^p$ of projection σ_q satisfies the following commutative diagram

$$\begin{tikzcd} T_q\mathcal{Q} \arrow[r, "d\phi_q"] \arrow[d, "\sigma_q"] & \mathbb{R}^n \arrow[d, "\hat{\sigma}_{\phi(q)}"] \\ T_q\mathcal{C} \arrow[r, "d\vartheta_q"] & \mathbb{R}^p. \end{tikzcd}$$

Thus, $\bar{\sigma}_{\phi(q)} = d\vartheta_q \circ \sigma_q \circ d\phi^{-1}_{\phi(q)}$. Alternatively, let $M_x \in \mathbb{R}^{n\times p}, N_x \in \mathbb{R}^{p\times n}$ be such that $\mathrm{Im}\ M_x = (d\phi)_{\phi^{-1}(x)}(T_{\phi^{-1}(x)}\mathcal{C})$ and $\mathrm{Ker}\ N_x = (d\phi)_{\phi^{-1}(x)}(F_{\phi^{-1}(x)})$, then

$$\bar{\sigma}_{\phi(q)} = (N_x M_x)^{-1}\,N_x\,.$$

Let $F^\perp$ be the annihilator of the input distribution F. Consider ${}^iF^\perp, i \in \{1,\dots,n\}$ the generator of the codistribution $F^\perp$. Then we denote by $(F^\perp)^i_j : \phi(\mathcal{U}) \to \mathbb{R}$ the component functions of $F^\perp$ in coordinate basis, namely $(F^\perp)^i_j = {}^iF^\perp_j$. Since $F^\perp_q$ and df_q are bases of $\ker F_q$ and $T_q\mathcal{C}$ for each $q \in \mathcal{C}$, the coordinate representation of the component functions of $\bar{\sigma}$, $\bar{\sigma}^k_i : \phi(\mathcal{U}) \to \mathbb{R}$, can be expressed by the following equation

$$\bar{\sigma}_i^k = \left((F^\perp)_j^i \, df_l^j\right)^{-1} (F^\perp)_l^k, \; i \in \{1, \dots, n\}, \; k \in \{1, \dots, p\}. \tag{3.15}$$

Let $q \in \mathbb{S}$, then for any vector $v \in T_q\mathcal{Q}$ there exists an unique vector $w \in T_{F^{-1}(q)}\mathcal{E}$ such that, in coordinate basis,

$$w^i = \bar{\sigma}_j^i v^j, \; \forall i \in \{1, \dots, p\}.$$

The coordinate representation $\hat{\bar{\bar{\nabla}}} : \mathfrak{X}(\mathbb{R}^p) \times \mathfrak{X}(\mathbb{R}^p) \to \mathfrak{X}(\mathbb{R}^p)$ of the induced connection $\hat{\nabla} = \sigma(\nabla) : \mathfrak{X}(\mathcal{C}) \times \mathfrak{X}(\mathcal{C}) \to \mathfrak{X}(\mathcal{C})$ satisfies the following commutative diagram

$$\begin{tikzcd}
\mathfrak{X}(\mathbb{R}^n) \times \mathfrak{X}(\mathbb{R}^n) \arrow[d, "\hat{\bar{\bar{\nabla}}}"] & \mathfrak{X}(\mathcal{C}) \times \mathfrak{X}(\mathcal{C}) \arrow[l, "\vartheta_* \times \vartheta_*"'] \arrow[d, "\bar{\sigma}=\sigma(\nabla)"] \\
\mathfrak{X}(\mathbb{R}^p) & \mathfrak{X}(\mathcal{C}) \arrow[l, "\vartheta_*"']
\end{tikzcd}$$

Thus, for $V, W \in \mathfrak{X}(\mathbb{R}^p)$

$$\hat{\bar{\bar{\nabla}}}_V W = \vartheta_* \sigma(\nabla_{\vartheta_*^{-1} V}(\vartheta_*^{-1} W)) \,.$$

The Christoffel symbols $\hat{\Gamma}_{i,j}^k$ of the induced connection can be computed from (3.2), that is $\hat{\bar{\bar{\nabla}}}_{(\partial z)_i}(\partial z)_j = \hat{\Gamma}_{i,j}^k (\partial z)_k$, where $\{(\partial z)_1, \dots, (\partial z)_p\}$ is the coordinate basis of $\mathbb{R}^p$. Thus,

$$\hat{\Gamma}_{i,j}^k = \left(\hat{\bar{\bar{\nabla}}}_{(\partial z)_i}(\partial z)_j\right)^k, \; \forall k, i, j \in \{1, \dots, p\}. \tag{3.16}$$

A more explicit formulation of the Christoffel symbols of the induced connection is given by the following lemma.

Lemma 3.2. *Let $(\mathcal{U}, \phi)$ and $(\mathcal{V}, \psi)$ be coordinate chart for $\mathcal{Q}$ and $\mathcal{E}$ respectively, and let $x = \phi(q), q \in \mathcal{U}$, $z = \psi(e), \mathrm{e} \in \mathcal{V}$ be their local coordinates. Let $f : \mathcal{E} \to \mathcal{Q}$ be a smooth embedding such that $\mathcal{C}$ is an embedded submanifold $f(\mathcal{E}) = \mathcal{C}$. Let $\sigma : T_q\mathcal{Q} \to T_q\mathcal{C}$, $\forall q \in \mathcal{C}$ be a local map with coordinate representation given by (3.15). Then, the covariant derivative is written in coordinate basis by*

$$\hat{\nabla}_{\dot{z}} \dot{z} = \left(\ddot{z}^l + \hat{\Gamma}_{i,j}^l \dot{z}^i \dot{z}^j\right) (\partial z)_l$$

where the Christoffel symbols $\hat{\Gamma}^k_{i,j} : \psi(\mathcal{V}) \to \mathbb{R}$ *are given by*

$$\hat{\Gamma}^l_{i,j} = \bar{\sigma}^l_k \left((\mathcal{H}f)^k_{i,j} + \Gamma^k_{m,o}\, df^m_i\, df^o_j \right), \forall i, j, k \in \{1, \dots, p\}. \tag{3.17}$$

Proof. By Theorem 3.1 the induced connection in coordinate basis is given by $(\hat{\nabla}_{\dot{z}} \dot{z}) = \bar{\sigma}(\overset{g}{\nabla}_{\dot{x}}\, \dot{x})$ for all $z \in T_{\psi^{-1}(z)}\mathcal{E}$ and $\dot{x} = f_* \dot{z}$, such that $\dot{x} \in T_{f(\psi^{-1}(z))}\mathcal{C}$. Then, by (3.3), we can rewrite, in coordinate basis, the component functions of the covariant derivative as

$$(\hat{\nabla}_{\dot{z}} \dot{z})^l = \bar{\sigma}^l_k \left(\overset{g}{\nabla}_{\dot{x}}\, \dot{x} \right)^k,$$
$$(\hat{\nabla}_{\dot{z}} \dot{z})^l = \bar{\sigma}^l_k \left(\ddot{x}^k + \Gamma^k_{m,o}\, \dot{x}^m\, \dot{x}^o \right),$$

where $l \in \{1, \dots, p\}$. Considering (3.13), (3.14) and the equality $\delta^l_i = \bar{\sigma}^l_k df^k_i$, the thesis follows from the following equivalences

$$(\hat{\nabla}_{\dot{z}} \dot{z})^l = \bar{\sigma}^l_k \left(df^k_i\, \ddot{z}^i + (\mathcal{H}f)^k_{i,j}\, \dot{z}^i\, \dot{z}^j + \Gamma^k_{m,o}\, df^m_i\, \dot{z}^i\, df^o_j\, \dot{z}^j \right),$$
$$\ddot{z}^l + \hat{\Gamma}^l_{ij}\, \dot{z}^i\, \dot{z}^j = \ddot{z}^l + \bar{\sigma}^l_k \left((\mathcal{H}f)^k_{i,j} + \Gamma^k_{m,o}\, df^m_i\, df^o_j \right) \dot{z}^i\, \dot{z}^j,$$

where $l \in \{1, \dots, p\}$. □

3.3 Examples

In this section we present some examples of VHCs described with the formalism previously presented. In Example 3.1, we analyse a simple mechanical system, a particle lying in $\mathbb{R}^2$, enforced to lie on a circle by a single actuator. This example is the same one considered in the Introduction but presented in the formalism of this chapter. In Example 3.2, a particle lying in a tridimensional space is enforced to lie on a sphere by a single actuator. This example shows the application of VHCs on a system with underactuation degree greater than one. Moreover, the configuration space of the system is not a generalized cylinder. Example 3.3 presents a double pendulum on a cart, with an underactuation degree equal to 2.

Example 3.1. Particle in $\mathbb{R}^2$ constrained on $\mathbb{S}$. Consider again the system presented in the Introduction: a unit point mass in $\mathcal{Q} = \mathbb{R}^2$ with configuration $q \in \mathbb{R}^2$. The metric tensor is given by the Euclidean metric $g_q(\dot{q}, \dot{q}) = \|\dot{q}\|^2$. The dynamics of this system are obtained from (3.6) and can be written in coordinates as

$$\ddot{q}(t) = F_{q(t)}\tau(t) - \mathrm{grad}P \tag{3.18}$$

where $\tau(t) \in \mathbb{R}$ is the control input. We consider a VHC given by the embedding $f : [\mathbb{R}]_{2\pi} \to \mathcal{C} \subset \mathbb{R}^2$ defined as

$$f(z) = \begin{pmatrix} \cos z \\ \sin z \end{pmatrix}$$

whose image is a unit circle, centered at the origin and is the VHC we want to enforce. In the following, we compute the reduced dynamics on f for different choices of the input distribution and the potential function.

Case a)

The input distribution is the image of the vector field

$$F_q = R(\alpha)q, \tag{3.19}$$

where $R(\alpha)$ is the rotation matrix associated to angle α, which is an assigned constant. No potential forces act on the particle, that is $P = 0$. Figure 3.4 represents vector field F for $\alpha = 0.5$.

By Proposition 3.1, the embedded submanifold is a regular VHC if $T_q\mathcal{C}$ and F_q (defined in (3.19)) are linearly independent for each $q \in \mathcal{C}$. Since the image of $df_{f^{-1}(q)} = \left(-q^2, q^1\right)^T$ is a basis of $T_q\mathcal{C}$, this is equivalent to condition

$$\det[df_{f^{-1}(q)}, F_q] = -\cos\alpha \neq 0\,,$$

so that $\mathcal{C}$ is feasible if $\cos\alpha \neq 0$.

We compute the reduced dynamics according to Subsection 3.2.2. The Hessian of the VHC f is $\mathcal{H}f = (-\cos z, -\sin z)^T$ and the annihilator of the input distribution evaluated on the constraint is $F^{\perp}_{f(z)} = (\sin z, -\cos z)\, R(-\alpha)$.

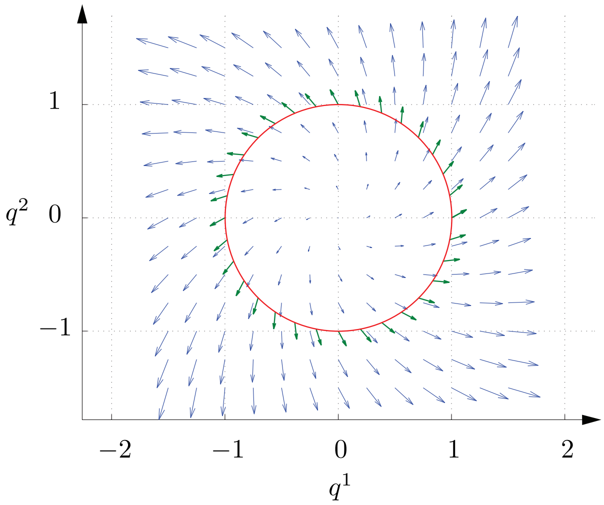


Figure 3.4: Example 3.1 case a). Representation of a vector field which generates the input distribution.

By (3.15),

$$\begin{aligned}\bar{\sigma}_z &= \left((\sin z, -\cos z)\ R(-\alpha) \begin{pmatrix} -\sin z \\ \cos z \end{pmatrix} \right)^{-1} (\sin z, -\cos z)\, R(-\alpha) \\ &= -\frac{(\sin z, -\cos z)\ R(-\alpha)}{\cos\alpha}\end{aligned}$$

and the induced Christoffel symbol is given by (3.17), that is

$$\begin{aligned}\hat{\Gamma}^z_{z,z}(z) &= -\frac{(\sin z, -\cos z)\ R(-\alpha)}{\cos z} \begin{pmatrix} -\cos\alpha \\ -\sin\alpha \end{pmatrix} \\ &= \frac{-\sin\alpha}{-\cos\alpha} = \tan\alpha.\end{aligned}$$

Then, the dynamics on the constraint submanifold are given by

$$\ddot{z} = -\tan\alpha\, \dot{z}^2. \tag{3.20}$$

Figure 3.5 shows the phase portrait of the reduced dynamics for $\alpha = 0.5$. Note

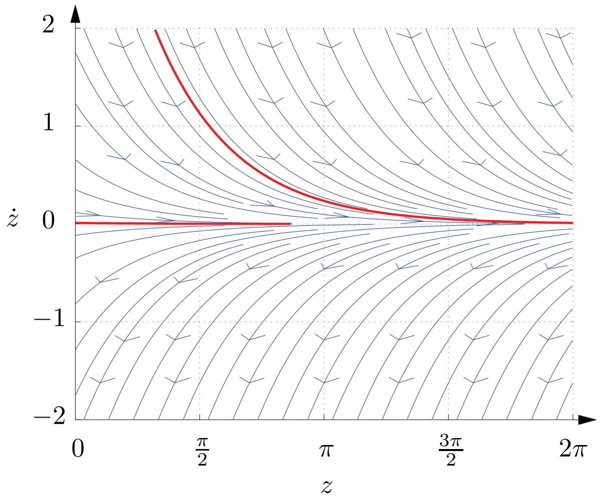


Figure 3.5: Example 3.1 case a). Phase portrait on the constraint manifold, with initial condition $\dot{z} > 0$.

that $\{(z, \dot{z}) : \dot{z} = 0\}$ is a set of unstable equilibria and that, if the initial condition satisfy $\dot{z}(0) > 0$, then $\lim_{t\to\infty} \dot{z}(t) = 0$ (see the red curve in Figure 3.5). This implies that system (3.20) is not Lagrangian for $\alpha = 0.5$.

Case b)

The input distribution is the image of the vector field

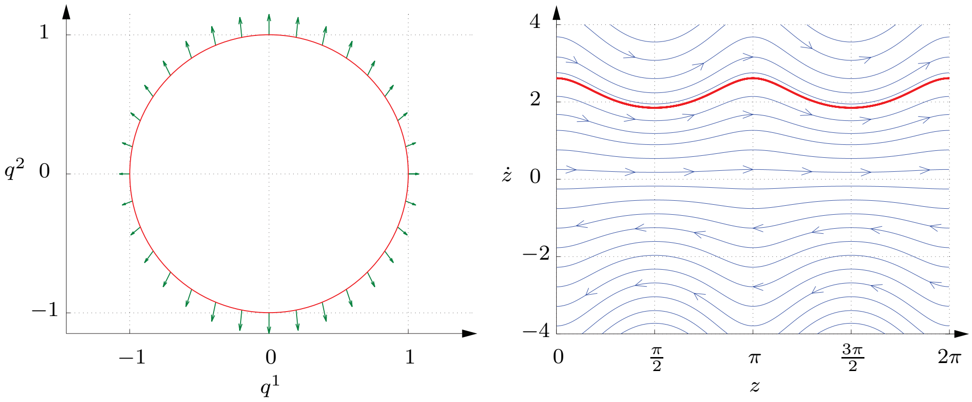
$$F_q = \begin{pmatrix} q^1 \\ d\, q^2 \end{pmatrix} \tag{3.21}$$

where $d \in \mathbb{R}$ is a constant. Figure 3.6a shows the vector field F for $d = 2$. No potential acts on the system.

As shown above, the transversality of $\mathcal{C}$ and the input distribution can be checked by the following inequality

$$\det[df_z,\, F_{f^{-1}(q)}] = \sin^2 z \neq \frac{1}{1-d}, \forall z \in [\mathbb{R}]_{2\pi}.$$

Then, the VHC is regular if and only if $d > 0$.



(a) Representation of a vector field which generates the input distribution.

(b) Phase portrait on the constraint manifold, with initial condition $z = 1$, $\dot{z} = 2$.

Figure 3.6: Example 3.1 case b). Representation of the input distribution (3.21) with $d = 2$ on $\mathcal{C}$ and illustration of the reduced dynamics on $\mathcal{C}$.

The reduced dynamics can be computed following (3.15) and (3.17), and are given by

$$\ddot{z} = -\frac{(d-1)\sin(2z)}{2\left[(d-1)\sin^2 z + 1\right]}\dot{z}^2. \tag{3.22}$$

For $d = 2$, Figure 3.6b shows the orbit of the reduced dynamics with initial condition $z = 2$, $\dot{z} = 1$.

Case c)

The input distribution is given by (3.21) with $d = 2$. The potential is given by $P = q^1 + q^2$, so that

$$\text{grad}P_z = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Figure 3.7a shows a representation of the input distribution and the vector field grad$P$ on $\mathcal{C}$. The reduced dynamics are given by

$$\ddot{z} = -\frac{\sin(2z)}{2\sin^2 z + 2}\dot{z}^2 + \frac{\cos z - 2\sin z}{\cos^2 z - 2}. \tag{3.23}$$

Figure 3.7b shows the orbits of the reduced dynamics with initial condition $z = 1$, $\dot{z} = 2$ and $z = \pi$, $\dot{z} = 0$.

Case d)

The input distribution is given by (3.21) with $d = 2$. The potential function is given by

$$P = q^1 q^2,$$

so that

$$\mathrm{grad} P_z = \begin{pmatrix} q^2 \\ q^1 \end{pmatrix}.$$

Figure 3.7c shows a representation of the input distribution and the vector field grad$P$ on $\mathcal{C}$. The reduced dynamics are given by

$$\ddot{z} = -\frac{\sin(2z)}{2\sin^2 z + 2}\dot{z}^2 - \frac{4}{\sin^2 z + 1} + 3. \tag{3.24}$$

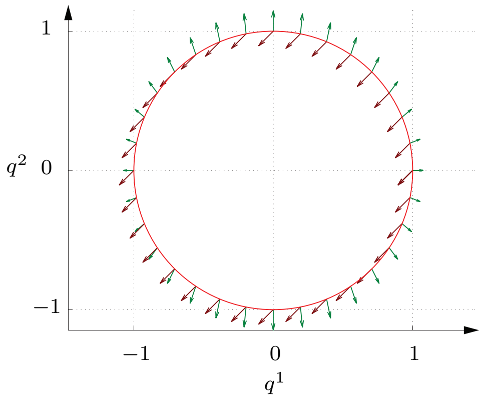
Figure 3.7d shows the orbits of the reduced dynamics with initial condition $z = 1$, $\dot{z} = 2$ and $z = \pi$, $\dot{z} = 0$.

Example 3.2. Particle in $\mathbb{R}^3$ constrained on $\mathbb{S}^2$. Consider a unit point mass in $\mathcal{Q} = \mathbb{R}^3$ with configuration $q \in \mathbb{R}^3$. The metric tensor is Euclidean ($g_q(\dot{q}, \dot{q}) = \|\dot{q}\|^2$) and the related Christoffel's symbols are null ($\Gamma^k_{i,j} = 0$). No potential forces act on the particle, that is $P = 0$. Consider as input distribution the image of the vector field $F_q = (q^1, q^2, 2\,q^3)^T$ then the dynamics of this system are obtained from (3.6) and can be written in coordinates as

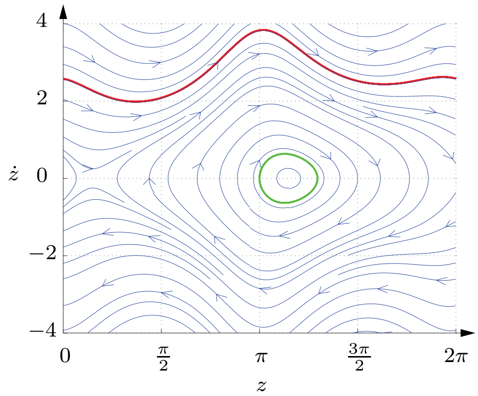
$$\ddot{q}(t) = F_{q(t)}\tau(t), \tag{3.25}$$

where $\tau(t) \in \mathbb{R}$ is the control input. System (3.25) has underactuation degree equal to two, since it has only one control input. Let $\mathcal{C} = \mathbb{S}^2$ be the VHC to be enforced, that is a unit sphere centered at the origin. Let $P = \{(0,0,1)^T, (0,0,-1)^T\}$ denote the poles of $\mathbb{S}^2$. Then, if $q \notin P$, $T_q\mathcal{C} = \mathrm{Im}\ M_q$, where

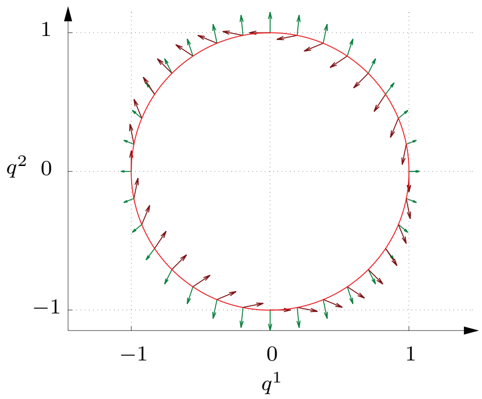
$$M_q = \begin{pmatrix} -q^2 & 0 \\ q^1 & -q^3 \\ 0 & q^2, \end{pmatrix}$$



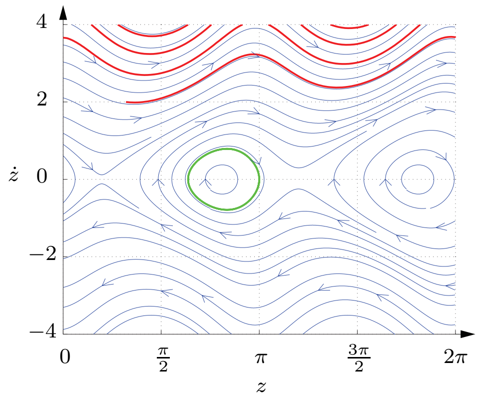
(a) Case c) The green arrows represent the vector field which generates the input distribution on $\mathcal{C}$. The red arrows represent the acceleration due to the potential, grad$P$, on $\mathcal{C}$.



(b) Case c) Phase portrait on the constraint manifold. The red curve represents the orbit with initial condition $z = 1$, $\dot{z} = 2$. The green curve represents the orbit with initial condition $z = \pi$, $\dot{z} = 0$.



(c) Case d) The green arrows represent the vector field which generates the input distribution on $\mathcal{C}$. The red arrows represent the acceleration due to the potential, grad$P$, on $\mathcal{C}$.



(d) Case d) Phase portrait on the constraint manifold. The red curve represents the orbit with initial condition $z = 1$, $\dot{z} = 2$. The green curve represents the orbit with initial condition $z = \pi$, $\dot{z} = 0$.

Figure 3.7: Example 3.1 case c) and case d).

if $q \in P$,

$$T_q\mathcal{C} = \mathrm{Im}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Since $\det[M_q, F_q] \neq 0$, $\forall q \in \mathbb{S}^2$, F and $\mathcal{C}$ are transversal and $\mathcal{C}$ is a VHC by Proposition 3.1. Figure 3.8 shows the input distribution on the VHC $\mathcal{C}$.

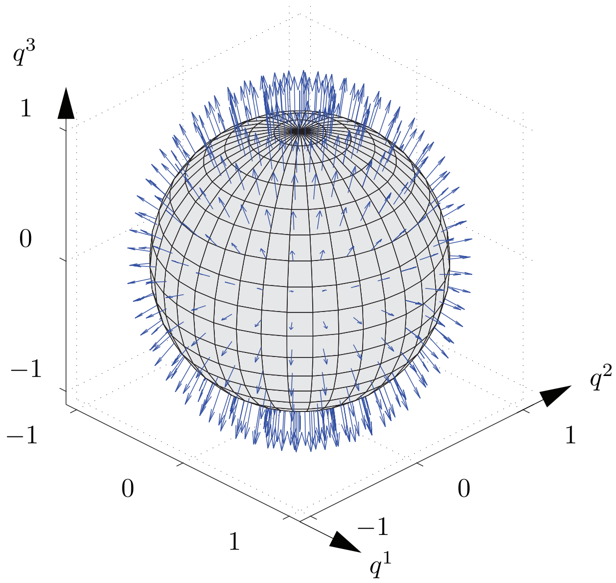


Figure 3.8: Example 3.2. Representation of the input distribution on $\mathcal{C}$.

The sphere $\mathbb{S}^2$ has a local chart $\psi : \mathcal{C} \setminus P \to (0, \pi) \times [0, 2\,\pi)$, defined as

$$\psi(q) = \begin{pmatrix} \arccos q^3 \\ \mathrm{atan2}\ (q^2, q^1) \end{pmatrix}. \tag{3.26}$$

Moreover, $F_q = \ker N_q$, with

$$N_q = \begin{pmatrix} q^2 & -q^1 & 0 \\ 2q^3 & 0 & -q^1 \end{pmatrix}.$$

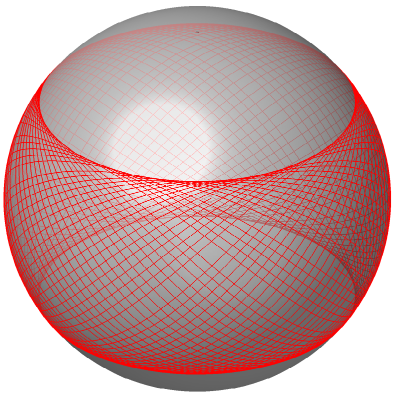
In local coordinates $z = \psi(q)$, the induced Christoffel symbols are given by (3.16):

$$\hat{\Gamma}^{z^1}_{z^1,z^1} = -\frac{\sin(2z^1)}{2(\cos^2(z^1)+1)},$$
$$\hat{\Gamma}^{z^1}_{z^2,z^2} = \frac{\sin(2z^1)}{\sin^2(z^1)+1},$$
$$\hat{\Gamma}^{z^2}_{z^1,z^2} = \hat{\Gamma}^{z^2}_{z^2,z^1} = \cot z^1.$$

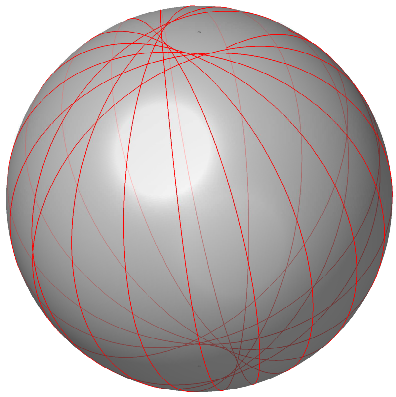
Thus, the reduced dynamics (3.11) are given by

$$\begin{cases} \ddot{z}^1 &= \frac{\sin(2z^1)}{2(\cos^2(z^1)+1)}(\dot{z}^1)^2 - \frac{\sin(2z^1)}{\sin^2(z^1)+1}(\dot{z}^2)^2 \\ \ddot{z}^2 &= -2\cot z^1\ \dot{z}^1\ \dot{z}^2. \end{cases} \tag{3.27}$$

Figure 3.9 shows some geodesics for the system (3.27).



(a) Solution of (3.27) with initial condition $z(0) = (\frac{\pi}{2}, 0)$, $\dot{z}(0) = (1, 1)$.



(b) Solution of (3.27) with initial condition $z(0) = (\frac{\pi}{16}, 0)$, $\dot{z}(0) = (0, 1)$.

Figure 3.9: Example 3.2. Some solutions of (3.27).

Example 3.3. Double pendulum on a cart. Consider the double pendulum on a cart depicted in Figure 3.10. The configuration manifold of this system is $\mathcal{Q} = \mathbb{R} \times \mathbb{S} \times \mathbb{S}$, where q^1 is the horizontal displacement of the cart, while q^2

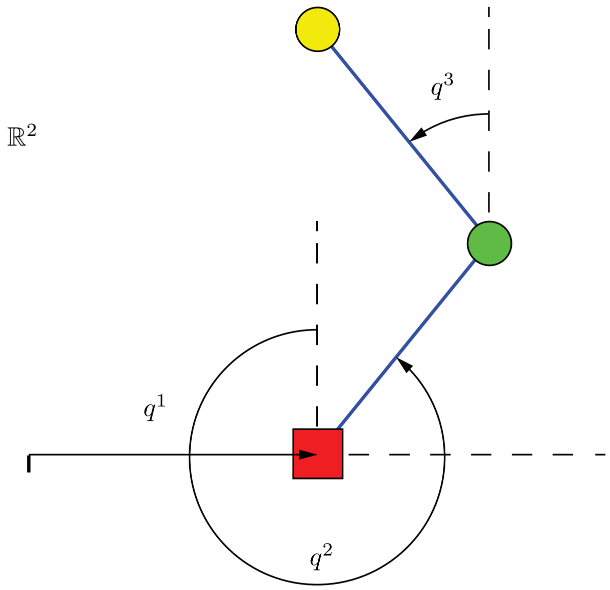


Figure 3.10: Double pendulum on cart.

and q^3 are the joint angles. All masses and rods lengths are assumed to be unitary. The kinetic energy is given by $g_q(\dot{q},\dot{q})$, where

$$g_q = \begin{pmatrix} 3 & -2\,\cos q^2 & -\cos q^3 \\ -2\,\cos q^2 & 2 & \cos(q^2-q^3) \\ -\cos q^3 & \cos(q^2-q^3) & 1 \end{pmatrix}$$

and the nonzero Christoffel symbols are

$$\begin{aligned}
\Gamma^{q^1}_{q^2,q^2} &= -\frac{4\,\sin q^2}{2\,\cos(2\,q^2)+\cos(2\,q^2-2\,q^3)-5}, & \Gamma^{q^1}_{q^3,q^3} &= -\frac{\sin(2\,q^2-q^3)+\sin q^3}{2\,\cos(2\,q^2)+\cos(2\,q^2-2\,q^3)-5}, \\
\Gamma^{q^2}_{q^2,q^2} &= -\frac{2\,\sin(2\,x_2)+\sin(2\,q^2-2q^3)}{2\,\cos(2\,q^2)+\cos(2\,q^2-2\,q^3)-5}, & \Gamma^{q^2}_{q^3,q^3} &= -\frac{3\,\sin(q^2-q^3)+\sin(q^2+q^3)}{2\,\cos(2\,q^2)+\cos(2\,q^2-2\,q^3)-5}, \\
\Gamma^{q^3}_{q^2,q^2} &= \frac{4\,\sin(q^2-q^3)}{2\,\cos(2\,q^2)+\cos(2\,q^2-2\,q^3)-5}, & \Gamma^{q^3}_{q^3,q^3} &= \frac{\sin(2\,q^2-2\,q^3)}{2\,\cos(2\,q^2)+\cos(2\,q^2-2\,q^3)-5}\,.
\end{aligned}$$

The gradient of the potential force is

$$\nabla P = \begin{pmatrix} 0 & -2\,g\sin q^2 & -g\sin q^3 \end{pmatrix}^T,$$

with $g = 9.8\frac{m}{s^2}$. Set $\mathcal{E} = \mathbb{R} \times \mathbb{S}$, $z \in \mathcal{E}$, and consider as VHC the image of the embedding $f : \mathcal{E} \to \mathcal{C} \subset \mathcal{Q}$

$$f(z) = \begin{pmatrix} z^1 \\ \rho(z^2) \\ z^2 \end{pmatrix} \tag{3.28}$$

where $\rho : \mathbb{S} \to \mathbb{S}$ is a smooth function that relates the angles of the two joints. We want to choose function ρ so that $\mathcal{C}$ is a VHC. Note that, for all $q \in \mathcal{C}$, $T_q\mathcal{C} = \text{Im } M_q$, where

$$M_q = \begin{pmatrix} 1 & 0 \\ 0 & \rho'(q^3) \\ 0 & 1 \end{pmatrix} .$$

We consider the presence of a single actuator (that corresponds to an underactuation degree on 2) with two possible configurations:

a) $B = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$, that is the control is a force on the cart,

b) $B = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$, that is the control is a torque on the second joint.

Case a), control force on cart

Consider the input codistribution $B = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$, so that $F_q = g_q^{-1}B = \ker N_q$ with

$$N_q = \begin{pmatrix} -2\cos q^2 & 2 & \cos(q^2 - q^3) \\ -\cos q^3 & \cos(q^2 - q^3) & 1 \end{pmatrix} .$$

$T_q\mathcal{C}$ and F_q are transversal if and only if $\det(N_q M_q \neq 0)$, that is

$$\begin{aligned} &\cos z^2 \cos(\rho(z^2) - z^2) - 2\cos\rho(z^2) \\ &\qquad + 2\rho'(z^2)\left(\cos z^2 - \cos\rho(z^2)\cos(\rho(z^2) - z^2)\right) \neq 0, \ \forall z \in \mathcal{E}. \end{aligned} \tag{3.29}$$

We need to choose function ρ in order to satisfy this condition. Note that, if $\rho = 0$ the condition is satisfied, hence, by continuity, it is still satisfied if the image of ρ is contained in a sufficiently small neighborhood of 0. For instance we can set, for $k \in \mathbb{R}$,

$$\rho(z^2) = k \sin z^2 \,, \tag{3.30}$$

then condition (3.29) is satisfied if

$$|\cos k| > \frac{1}{2} + 2|k| \,. \tag{3.31}$$

The choice $k = 0$ corresponds to a configuration in which the first rod is vertical at all times. The reduced dynamics (3.11) can be written in coordinates as

$$\begin{aligned} \ddot{z}^1 &= -\hat{\Gamma}^{z^1}_{z^2,z^2}(z)\, \dot{z}^2 \dot{z}^2 + P^{z^1}(z) \\ \ddot{z}^2 &= -\hat{\Gamma}^{z^2}_{z^2,z^2}(z)\, \dot{z}^2 \dot{z}^2 + P^{z^2}(z) \end{aligned} \tag{3.32}$$

where the induced Christoffel symbols are given by

$$\hat{\Gamma}^{z^1}_{z^2,z^2}(z) = \frac{\sin z^2}{\sin^2(z^2)+1}, \quad \hat{\Gamma}^{z^2}_{z^2,z^2}(z) = \frac{\sin(2z^2)}{\sin^2(z^2)+1},$$

and

$$P^{z^1}(z) = -\frac{g}{2}\frac{\sin(2z^2)}{\sin^2(z^2)+1}, \quad P^{z^2}(z) = -2g\frac{\sin z^2}{\sin^2(z^2)+1}.$$

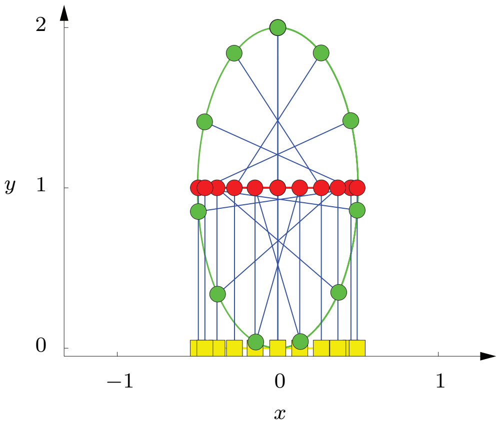
Figure 3.11a depicts the configuration of the double pendulum on cart on the VHC obtained by setting $k = 0$ in (3.30).

For $k > 0$ satisfying (3.31), the reduced dynamics are such that the only nonzero Christoffel symbols are $\hat{\Gamma}^{z^1}_{z^2,z^2}$ and $\hat{\Gamma}^{z^2}_{z^2,z^2}$ as in (3.32). For example, with $k = 0.2$, the induced Christoffel symbols are depicted in Figure 3.12a and the induced potential accelerations are depicted in Figure 3.12b. Figure 3.13a shows the configuration of the double pendulum on cart on the VHC obtained with $k = 0.2$.

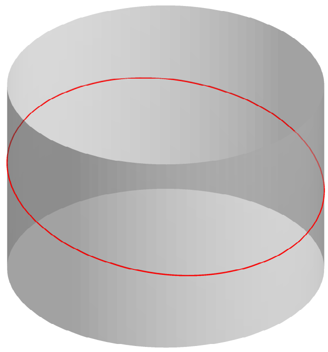
Case b) , control force on second joint

In this case $B = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$. Consider again the VHC defined in (3.28), in this case the transversality condition is given by

$$6\rho'(z^2) - 4\rho'(z^2)\cos^2(\rho(z^2)) + 3\cos(\rho(z^2) - z^2) - 2\cos\rho(z^2)\cos z^2 \neq 0, \; \forall z \in \mathcal{E}. \tag{3.33}$$

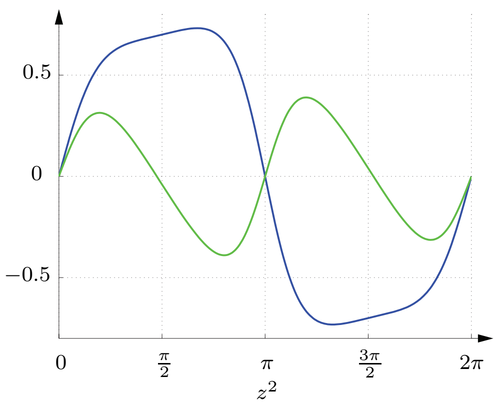


(a) Double pendulum on a cart subject to (3.28).

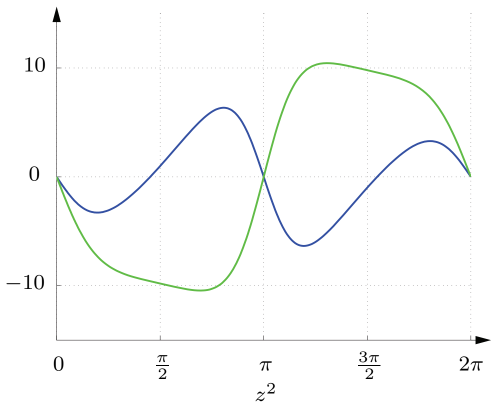


(b) The solution on the constraint manifold.

Figure 3.11: Example 3.3 case a) and $\rho = 0$: solution of the reduced dynamics with initial conditions $z = (0, 10^{-6})^T$, $\dot{z} = (0, 0)^T$.

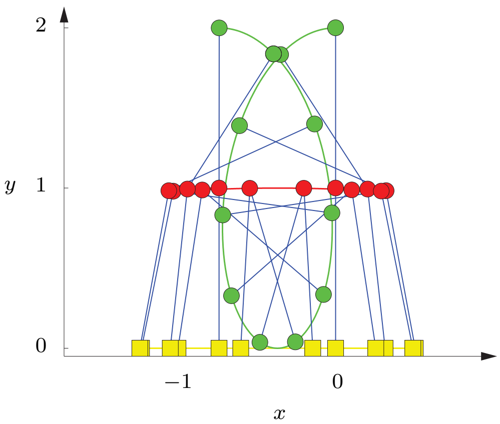


(a) Functions $\hat{\Gamma}^{z^1}_{z^2,z^2}$ (blue) and $\hat{\Gamma}^{z^2}_{z^2,z^2}$ (green).

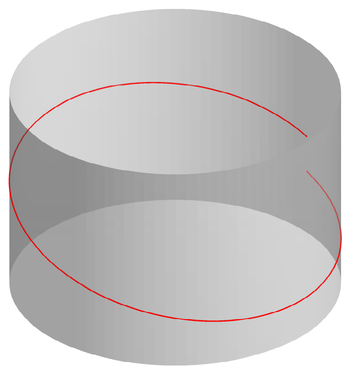


(b) Functions P^{z^1} (blue) and P^{z^2} (green).

Figure 3.12: Example 3.3 case a) and $\rho = 0.2$: the component functions of the reduced dynamics.



(a) Double pendulum on a cart subject to (3.28).



(b) The solution on the constraint manifold.

Figure 3.13: Example 3.3 case a) and $\rho = 0.2$: solution of the reduced dynamics with initial conditions $z = (0, 10^{-6})^T$, $\dot{z} = (0, 0)^T$.

Note that the condition above depends only on coordinate q^2. One possible way to find ρ is to use the same idea of the *virtual constraint generator* (see [37]). Namely, we set the right hand side of (3.33) to a real constant $\delta \neq 0$, obtaining a differential equation of the form

$$\xi_1(\rho, z^2)\rho'(z^2) + \xi_2(\rho, z^2) = \delta \,. \tag{3.34}$$

Since $\xi_1(\rho, z^2) = 6 - 4\cos^2(z^2) \neq 0, \forall z^2 \in \mathbb{S}$, by Proposition 2.3 of [37], there exists a value of δ such that the solution of (3.34) satisfies $\rho(0) = \rho(2\pi)$. In particular, by numerical computation, $\delta = -0.5683$ and the corresponding function ρ is depicted in Figure 3.16a.

The reduced dynamics have the same structure as (3.32). Figures 3.15a and 3.15b show the numerically computed nonzero Christoffel symbols of the induced connection on $\mathcal{C}$ and the induced potential accelerations respectively. Figure 3.16a shows the configuration of the system on the submanifold $\mathcal{C}$.

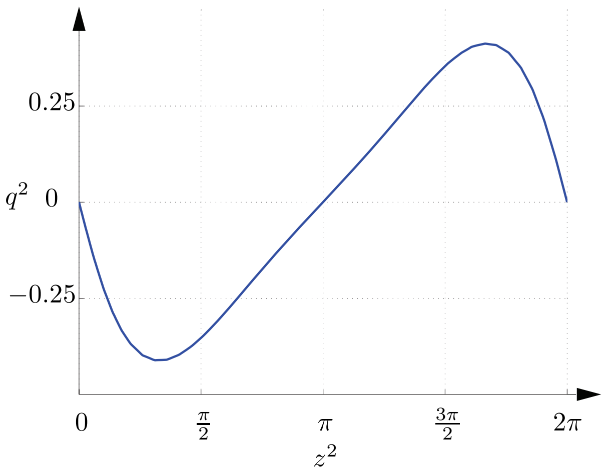
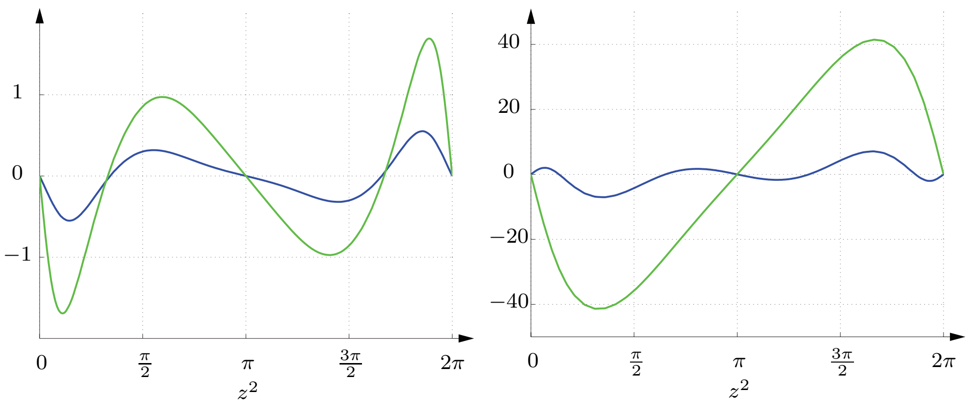


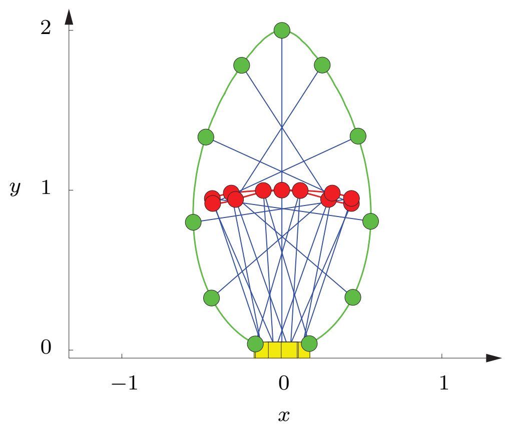
Figure 3.14: Example 3.3 case b): plot of function ρ, solution of (3.34).



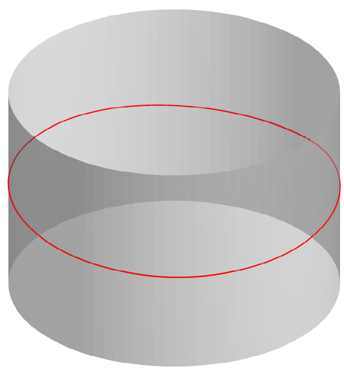
(a) Functions $\hat{\Gamma}^{z^1}_{z^2,z^2}$ (blue) and $\hat{\Gamma}^{z^2}_{z^2,z^2}$ (green).

(b) Functions P^{z^1} (blue) and P^{z^2} (green).

Figure 3.15: Example 3.3 case b): the component functions of the reduced dynamics.



(a) Double pendulum on a cart subject to (3.28).



(b) The solution on the constraint manifold.

Figure 3.16: Example 3.3 case b): solution of the reduced dynamics with initial conditions $z = (0, 10^{-6})^T$, $\dot{z} = (0, 0)^T$.

Chapter 4

Inverse Lagrangian Problem

Ognuno sta solo sul cuor della terra
trafitto da un raggio di Sole:
ed è subito sera.

– Salvatore Quasimodo

In the previous chapters we presented some examples in which the reduced dynamics exhibit behaviors that are not possible for Lagrangian mechanical systems. In fact, as shown in Chapter 2 it is possible to find constraints in which the reduced dynamics possess stable limit cycles. On the other hand, a system constrained by physical holonomic constraints is always a Lagrangian mechanical system. Then, the following question arises naturally:

- under which conditions the reduced dynamics of a VHC are a Lagrangian mechanical system?

As reported in Chapter 1, for mechanical systems whose configuration space is a generalized cylinder, with an underactuation degree equal to one, an answer has already been given in [7]. Here, using the formalism presented in Chapter 3, we address this same question in a more general setting.

The answer to this question is related to the inverse problem of Lagrangian mechanics. This problem consists in determining whether a system of differential equations can be obtained by the Euler-Lagrangian equations of some Lagrangian function. The sufficient and necessary conditions for the existence of such a Lagrangian function are called *Helmholtz conditions*. These conditions were found by Helmholtz towards the end of the nineteenth century and are formalized as an algebraic-differential system. These conditions have been extensively studied in mathematical literature in the subsequent years [54]. In relation to the publication of the Santilli's book [55] and the growing interest in symmetries and first integrals for second order differential equations, this problem has received a renewed interest in the last decades and some researchers recast the conditions into a coordinate free formalism [56].

In the following, we complete the notation introducing the concept of *Lagrangian second order differential equation* and formalize the above question. Let $\pi : T\mathcal{Q} \to \mathcal{Q}$, $\pi(q, \dot{q}) = q$ be the natural projection map, a *second order differential equation on* $\mathcal{Q}$ is defined by a smooth vector field on the tangent bundle, $X \in \mathfrak{X}(T\mathcal{Q})$, such that $(d\pi)(X_{(q,\dot{q})}) = \dot{q}$. Given $(q_0, \dot{q}_0) \in T\mathcal{Q}$, a solution of the second order differential equation on $\mathcal{Q}$ is a differentiable function $\gamma : I \to T\mathcal{Q}$, where I is an open interval of $\mathbb{R}$, such that $\gamma(0) = (q_0, \dot{q}_0)$ and $\frac{d\gamma}{dt} = X_{(q_0,\dot{q}_0)}$. Let (U, ϕ) be a coordinate chart centered at $q \in \mathcal{Q}$ with local coordinates $x = (x^1, \ldots, x^n)$, then the solutions of a second order ordinary differential equation satisfy

$$\ddot{x} = f(x, \dot{x})$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is the local representation of X.

Definition 4.1 (Lagrangian second order differential equation). A second order differential equation X on $\mathcal{Q}$ is called *Lagrangian* if there exist a symmetric, definite positive $(2, 0)$-tensor $g_q : T_q\mathcal{Q} \times T_q\mathcal{Q} \to \mathbb{R}, q \in \mathcal{Q}$ and a potential function $P : \mathcal{Q} \to \mathbb{R}$, such that every solution $q(t)$ of X satisfies

$$\overset{g}{\nabla}_{\dot{q}}\, \dot{q} = -\text{grad } P_q,$$

where $\overset{g}{\nabla}$ is the Levi-Civita connection of the Riemannian manifold (Q, g).

A second order differential equation on $\mathcal{Q}$ is called *Lagrangian* if its solutions are the solutions of a Lagrangian mechanical system. Then, we can formally state the problem concerning this chapter.

Problem 4.1. *Give necessary and sufficient conditions such that the reduced dynamics are Lagrangian.*

An affine connection is called **metrizable** if it is the Levi-Civita connection of a Riemannian metric. In the following, Proposition 4.1 shows that the metrizability of the induced connection, together with a condition on potential function, is equivalent to the fact that the reduced dynamics are Lagrangian. The conditions for the metrizability of the induced connection are presented in Subsection 4.1.1, where we adapt well-known results from the literature on the metrizability problem. Sections 4.2 and 4.3 presents explicit conditions for the metrizability of reduced dynamics of dimension 1 and 2. In particular, in dimension 1, it shows a different derivation of some of the conditions presented in [7].

4.1 General Conditions

The following proposition gives a first answer to Problem 4.1.

Proposition 4.1. *The following statements are equivalent:*

a) the induced connection $\hat{\nabla}$ is metrizable with associated metric $\hat{g}$ and the following equation has a solution $\hat{P}$ on $\mathcal{C}$

$$\sigma(gradP) = grad\hat{P}\,. \tag{4.1}$$

b) the reduced dynamics (3.11) are Lagrangian.

Proof. a) $\rightarrow$ b)

Let $\hat{g}$ be the metric associated to the induced connection and define the Lagrangian, $\mathscr{L} = \frac{1}{2}\hat{g}(\dot{q}, \dot{q}) - \hat{P}$. The solution of the EL equations are given by

$\hat{\nabla}_{\dot{q}}\dot{q} = -\mathrm{grad}\hat{P}_q$, by (4.1) this equation can be rewritten as $\hat{\nabla}_{\dot{q}}\dot{q} = -\sigma_q(\mathrm{grad}P_q)$, which corresponds to the reduced dynamics (3.11).

b) → a)

By hypothesis, there exists a Riemannian metric $\hat{g}$ and a potential function $\hat{P}$ such that the dynamics are obtained from the EL equation for a Lagrangian $\mathscr{L} = \frac{1}{2}\hat{g}(\dot{q}, \dot{q}) - \hat{P}$ and are given by

$$\bar{\nabla}_{\dot{q}}\dot{q} = -\mathrm{grad}\hat{P}_q\,, \tag{4.2}$$

where $\bar{\nabla}$ is the Levi-Civita connection of $\hat{g}$. If $\dot{q} = 0$, (4.2) and (3.11) imply that $\ddot{q} = \sigma_q(g_q^{-1}(dP)_q) = \mathrm{grad}\hat{P}_q$ which corresponds to (4.1). Subtracting (3.11) from (4.2) it follows that $(\hat{\nabla}_{\dot{q}} - \bar{\nabla}_{\dot{q}})\dot{q} = 0$ which implies that the connections $\hat{\nabla}$ and $\bar{\nabla}$ determine the same geodesic curves. Since $\hat{\nabla}$ and $\bar{\nabla}$ are torsion-less, they are the same connection (see the first proposition of Section 3 of [57]), hence $\hat{\nabla} = \bar{\nabla}$ and $\hat{\nabla}$ is metrizable. □

If $P = 0$, we obtain the following corollary.

Corollary 4.1. *If $P = 0$, the following statements are equivalent:*

a) the reduced dynamics (3.12) are Lagrangian,

b) $\hat{\nabla}$ is a metrizable connection.

Note that, in the general case, the induced connection $\hat{\nabla}$ is not metrizable. Indeed, in general, control forces *do work* on the system and the constrained system may exhibit features (e.g., presence of limit cycles) that are not possible for a Lagrangian system. One important case in which $\hat{\nabla}$ is metrizable is when the input codistribution is the orthogonal bundle of $T\mathcal{C}$. This corresponds to the case of physical holonomic constraints, in which the control forces are orthogonal to $T\mathcal{C}$ and do not do work on the system. In this case, the affine connection $\hat{\nabla}$ is the Levi-Civita connection of the restriction of the metric tensor g on $T\mathcal{C}$ (see for instance Section 4.5.7 of [52]).

4.1.1 Conditions for Metrizability

The problem of deciding if a connection is metrizable corresponds to the inverse of the Fundamental Theorem of Riemannian geometry. This relevant problem has received a considerable degree of attention in literature, but, in general, it is not of easy solution. In this section we make use of some of the related main results.

We briefly recall the definition of the holonomy group of a connection ∇. Roughly speaking, it is the group of transformations that can be obtained by parallel transport of a vector on a closed curve (see for instance [58]). In the following, we say that a piecewise smooth closed curve $\gamma : [0, 2\pi] \to \mathcal{Q}$ is a loop based on q if $\gamma(0) = \gamma(2\pi) = q$. For any $v \in T_q\mathcal{Q}$, consider the differential equation

$$\begin{cases} \nabla_{\dot\gamma(\lambda)} x(\lambda) = 0 \\ x(0) = v \ , \end{cases} \tag{4.3}$$

and set $P_\gamma(v) = x(2\pi)$. Thus, $P_\gamma(v)$ is the result of the parallel transport of vector v on γ. Moreover, $P_\gamma(v)$ is a linear and invertible function of v. The holonomy group of $\mathcal{C}$ based at q is given by

$$\mathrm{Hol}_q(\nabla) = \{P_\gamma | \gamma \text{ is a loop based on } q\} .$$

while the restricted holonomy group is made only by contractible loops (i.e., loops that are homotopic to a point), that is

$$\mathrm{Hol}_q^0(\nabla) = \{P_\gamma | \gamma \text{ is a contractible loop based on } q\} .$$

Note that Hol_q^0 is a normal subgroup of Hol_q (see Theorem 4.2 of [59]), so that the quotient group $\mathrm{Hol}_q/\mathrm{Hol}_q^0$ is well defined. Let $\Pi_1(\mathcal{Q})$ be the first homotopy group of $\mathcal{Q}$ with reference point q, there exists a surjective homomorphism $f : \Pi_1(\mathcal{Q}) \to \mathrm{Hol}_q/\mathrm{Hol}_q^0$, such that $f(\gamma)$ is the equivalence class of P_γ.

A fundamental result on metrizability is due to Schmidt; it relates metrizability to the properties of the holonomy group of the connection:

Proposition 4.2 (see [60]). *The following statements are equivalent:*

a) ∇ *is a metrizable connection,*

b) the holonomy group of ∇ *keeps a non degenerate and symmetric* $(2,0)$*-tensor field* g *invariant, that is,*

$$g(T(v), T(v)) = g(v, v), \forall q \in \mathcal{Q}, T \in Hol_q(\nabla) \ .$$

Note that the verification of condition *b)* of Proposition 4.2 can be a very complex task since it involves the computation of the holonomy group of the induced connection.

In general, the subgroup Hol_q^0 can be computed from the curvature tensor with the Ambrose-Singer Theorem (see Theorem 8.1 of [59]). In particular, if the curvature tensor of the connection is null (i.e., the connection is flat) Hol_q^0 is trivial. In this case, to satisfy ii) of Proposition 4.2, it is sufficient to verify that g is invariant with respect to the loops representing a set of generators of the fundamental group of $\mathcal{Q}$. Thus, the following proposition holds, which is a direct consequence of Theorem 2 of [61].

Proposition 4.3. *Let* G *be a set of generators for the group* $\Pi_1(\mathcal{Q})$*. A flat connection* ∇ *on a path-connected manifold* $\mathcal{Q}$ *is metrizable if and only if there exists a symmetric positive definite* $(2,0)$*-tensor field* g *such that, for any* $\zeta \in G$*, there exists a loop* $\gamma \in \zeta$ *for which*

$$g(P_\gamma(v), P_\gamma(v)) = g(v, v) \, . \tag{4.4}$$

The following discussion shows that simpler conditions hold if $\mathcal{C}$ has dimension 1 or 2.

4.2 Case with Underactuation Degree Equal to One

If $\mathcal{C}$ is a smooth connected one dimensional manifold without boundaries, then it is diffeomorphic either to the circle $\mathbb{S}$ or to the set of real numbers $\mathbb{R}$ (see the Appendix of [62]). Any affine connection of a one dimensional manifold is flat, then its restricted holonomy group is trivial.

We represent the manifold $\mathcal{C}$ in coordinate by $z \in \mathbb{R}$ if $\mathcal{C}$ is diffeomorphic to $\mathbb{R}$ or by $z \in [\mathbb{R}]_{2\pi}$ if $\mathcal{C}$ is diffeomorphic to $\mathbb{S}$. The coordinate representation of the reduced dynamics (3.11) can be written in the following form

$$\ddot{z} = -\bar{\sigma}_z(\mathrm{grad} P_z) - \hat{\Gamma}^z_{z,z}(z)\,\dot{z}^2 = \psi_1(z) + \psi_2(z)\dot{z}^2 \,, \tag{4.5}$$

where $z \in \mathbb{R}$. Note that (4.5) is equal to (1.17).

In this case, a solution of the inverse Lagrangian problem has been provided in Theorems 3.3, 3.5 of [8], which, in the notation of this chapter, lead to the following result

Theorem 4.1. *a) If $\mathcal{C}$ is diffeomorphic to $\mathbb{R}$, then system (4.5) is Lagrangian.*

b) If $\mathcal{C}$ is diffeomorphic to $\mathbb{S}$, then the following statements are equivalent:

i) System (4.5) is Lagrangian.

ii) The following identities hold

$$\int_0^{2\pi} \psi_2(x)dx = 0, \tag{4.6}$$

$$\int_0^{2\pi} \psi_1(x)e^{-2\int_0^x \psi_2(\mu)d\mu}dx = 0\,. \tag{4.7}$$

In the following, we presents a different derivation of Theorem 4.1 based on Propositions 4.1 and 4.3. First, consider the following result on metrizability.

Proposition 4.4. *Let $\mathcal{C}$ be a one dimensional smooth manifold with an affine connection ∇:*

a) if $\mathcal{C}$ is diffeomorphic to $\mathbb{R}$, connection ∇ is metrizable.

b) if $\mathcal{C}$ is diffeomorphic to $\mathbb{S}$, connection ∇ is metrizable if and only if

$$\int_0^{2\pi} \Gamma^z_{zz}(z)dz = 0\,. \tag{4.8}$$

Proof. a) Since $\mathbb{R}$ is flat and simply connected, $\mathrm{Hol}(\nabla) = \mathrm{Hol}^0(\nabla)$ is trivial, hence ∇ is always metrizable (see Proposition 2.2.6 of [63]).

b) The fundamental group of $\mathbb{S}$ is isomorphic to $(\mathbb{Z}, +)$, one generator being the loop $\gamma : [0, 2\pi] \to \mathbb{S}$, defined as $\gamma(\lambda) = [\cos\lambda, \sin\lambda]^T$, that winds once around the circle in a counterclockwise direction. Being $\mathbb{S}$ flat, $\mathrm{Hol}^0(\nabla)$ is trivial. Map $P_\gamma(v)$ is given by (4.3), that is $P_\gamma(v) = x(2\pi)$ where

$$\begin{cases} \dot{x} = -\Gamma^z_{zz} x \\ x(0) = v \end{cases}$$

that is

$$P_\gamma(v) = \exp\left(-\int_0^{2\pi} \Gamma^z_{zz}(z) dz \right) v \,.$$

By Proposition 4.3, ∇ is metrizable if and only if there exists a symmetric and positive definite $(2,0)$-tensor g such that $g(v,v) = g(P_\gamma(v), P_\gamma(v))$, $\forall v \in \mathbb{R}$. Since $T_q\mathbb{S}$ has dimension 1, it is satisfied if $P_\gamma(v) = v$, that is if condition (4.8) holds. $\square$

If an affine connection $\hat{\nabla}$ is metrizable there exists a Riemannian metric $\hat{g}$ such that $\hat{\nabla}$ is the Levi-Civita connection of $\hat{g}$. Then, the metric can be found from the compatibility condition of the Levi-Civita connection, that is $\hat{\nabla}\hat{g} = 0$. The metric $\hat{g}$ satisfies the following differential equation

$$\begin{cases} \frac{\partial \hat{g}(z)}{\partial z} = -\hat{\Gamma}^z_{z,z}(z)\, \hat{g}(z) \\ \hat{g}(0) = c \end{cases}$$

where c is a positive real constant. In case of $\mathcal{C}$ diffeomorphic to $\mathbb{S}$, setting $c = 1$, the coordinate representation of the induced metric $\hat{g}$ is given by

$$\hat{g}(z) = e^{2\int_0^z \hat{\Gamma}^z_{z,z}(\mu) d\mu}. \tag{4.9}$$

Equation (4.9) can be rewritten as $\hat{g}_{1,1} = e^{-2\int_0^{2\pi} \psi_2(\mu) d\mu}$, which is equal to (1.20). The following proof of Theorem 4.1 present a different approach with respect to [7]. Here, the proof is a consequence of Proposition 4.1.

Proof of Theorem 4.1.

a): Equation (4.1) can be rewritten as $d\hat{P} = \sigma(\text{grad}P)^\flat$. Being $\hat{\nabla}$ metrizable by *a)* of Proposition 4.4, the 1-form $\sigma(\text{grad}P)^\flat$ is well defined and, being a 1-form on $\mathbb{R}$, it is exact (see Corollary 16.27 in [13]), then equation (4.1) admits a solution $\hat{P}$.

b): $\Rightarrow$, equation (4.6) follows from Propositions 4.1 and 4.4. Equation (4.7) follows from the fact that the 1-form

$$\sigma(\text{grad}P)^\flat = g_z\psi_1(z) = e^{\int_0^z \psi_2(\mu)d\mu}\,\psi_1(z)$$

is exact by hypothesis.

$\Leftarrow$, the first of (4.6) implies that $\hat{\nabla}$ is metrizable by Proposition 4.4. Since $\Pi_1(\mathbb{S})$ is generated by the loop γ defined before, the second of (4.7) implies that the 1-form

$$\sigma(\text{grad}P)^\flat = g_z\psi_1(z) = e^{\int_0^z \psi_2(\mu)d\mu}\psi_1(z)$$

is exact. Hence, equation (4.1) admits a solution and proposition 4.1 implies the thesis. ∎

If the induced connection is metrizable and the 1-form $\sigma(\text{grad}P)^\flat$ is exact, the coordinate representation of the induced potential is given by

$$\hat{P}(z) = \int_0^z \hat{g}(x)\,\hat{\sigma}(\text{grad}P)^\flat_x dx. \tag{4.10}$$

and can be rewritten as $\hat{P}(z) = -\int_0^z \hat{g}(x)\,\psi_1(x)dx$, which corresponds to (1.21).

4.2.1 Examples

Example 4.1 (Cont. Example 3.1 case a)). Consider again system (3.18) and the embedded submanifold $\mathbb{S}$. Since $\hat{\Gamma}^z_{z,z} = \tan\alpha$, condition (4.8) implies that the induced connection is metrizable if and only if $\tan\alpha = 0$. This corresponds to the case in which the input codistribution is orthogonal to the tangent space of $\mathcal{C}$ (that is $\alpha = 0, \pi$).

Example 4.2 (Cont. Example 3.1 case b)). In this case the Christoffel symbol is given by $\hat{\Gamma}^z_{z,z}(z) = \frac{(d-1)\sin(2z)}{2\left[(d-1)\sin^2 z+1\right]}$. Since $P = 0$, by Corollary 4.1, the system is Lagrangian if and only if the induced connection is metrizable. For $d > 0$, condition (4.8) is always satisfied, thus system (3.22) is Lagrangian.

The induced metric is given by

$$\hat{g}(z) = e^{-2\mathrm{arctanh}\left(\frac{(d-1)\,\sin^2 z}{(d-1)\,\sin^2 z+2}\right)},$$

in particular, for $d = 2$, $\hat{g}(z) = e^{-2\mathrm{arctanh}\left(\frac{\sin^2 z}{\sin^2 z+2}\right)}$.

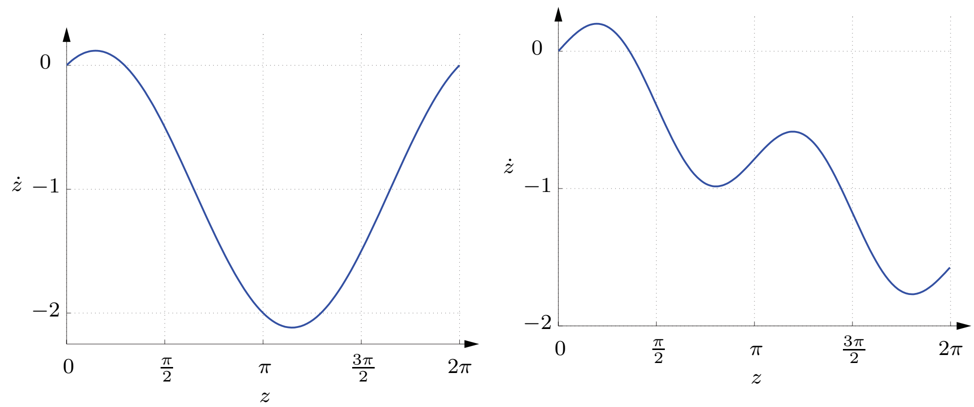
Example 4.3 (Cont. Example 3.1 cases c), d)). As shown in Example 4.2, in these two cases the induced connection is metrizable, therefore condition (4.6) is satisfied. Moreover, since $P \neq 0$, the reduced dynamics are Lagrangian if and only if condition (4.7) holds.

We verified this condition by numerical integration as follows. For case c), condition (4.7) is satisfied, then $(\mathrm{grad}P)^\flat$ of (3.23) is an exact 1-form and the reduced dynamics are Lagrangian. The induced potential, $\hat{P}$, can be obtained by (4.10). Figure 4.1a shows the induced potential function $\hat{P}$ obtained by numerical integration. Instead, for case d), condition (4.7) is not satisfied. The result of the numerical integration is shown in Figure 4.1b, and it is not a continuous function. Then, $(\mathrm{grad}P)^\flat$ is not an exact 1-form and the reduced dynamics (3.24) are not Lagrangian. Indeed, the red orbit depicted in Figure 3.7d shows that the reduced dynamics are unstable.

4.3 Case with Underactuation Degree Equal to Two

The inverse Fundamental Theorem of Riemannian geometry has a relatively simple formulation in the case of manifold with dimension equal to two (see [64]). The result makes use of the Ricci tensor $\mathcal{R}$ related to the induced connection. We briefly recall its definition. The curvature of a connection ∇ is the $(1, 3)$ tensor field R such that (see [59], Section 3, Theorem 4.1)

$$R(X, Y)Z = \nabla_X \nabla_y Z - \nabla_Y \nabla_x Z - \nabla_{[X,Y]} Z\ .$$



(a) The potential function $\hat{P}$ of system (3.23).

(b) This function is not a continuos function, then can not be a potential for (3.24).

Figure 4.1: Example 4.3. Functions obtained by (4.10) for the systems (3.23) and (3.24).

The Ricci tensor is then defined as the trace of the linear operator $Z \to R(Z,Y)X$, that is

$$\mathcal{R}(X,Y) = \mathrm{Tr}[Z \to R(Z,Y)X] \, .$$

A tensor T is called recurrent if there exists a one-form ω such that the covariant derivative of T satisfies

$$\nabla T = \omega \otimes T \, , \tag{4.11}$$

where $\otimes$ is the tensor product (see [13]).

The following result can be obtained by applying Theorem 3.3 of [64] to Corollary 4.1.

Proposition 4.5. *Consider a two dimensional configuration space and* $P = 0$. *If the Ricci curvature of the induced connection is symmetric, non-degenerate and recurrent and if the one-form* ω *for which* $\nabla\mathcal{R} = \omega \otimes \mathcal{R}$ *is exact with* $\omega = dh$, *where* h *is a scalar function, then* *a)* ∇ *is metrizable and the metric is related to the Ricci tensor by* $g = e^{-h}\mathcal{R}$*,*

b) the constrained dynamics are Lagrangian.

Proposition 4.5 allows checking if the induced connection is metrizable. The involved computation is not trivial since it requires the covariant derivation of the Ricci tensor. Proposition 4.5 also gives the explicit form of the induced metric.

4.3.1 Examples

Example 4.4 (Cont. Example 3.2). Since $\mathbb{S}^2$ is a 2-dimensional manifold, its metrizability can be determined with Proposition 4.5. Using chart ψ defined in (3.26), the coordinate representation of the Ricci curvature tensor for $\mathbb{S}^2$, with respect to the induced connection, is given by

$$\mathcal{R} = \begin{pmatrix} \frac{1}{\cos^2(z^1)+1} & 0 \\ 0 & \frac{2\sin^2(z^1)}{\sin^2(z^1)-2} \end{pmatrix},$$

which is symmetric and non-degenerate. The total covariant derivative of the tensor $\mathcal{R}$ with respect to the coordinates are given by

$$\hat{\nabla}_{z^1}\mathcal{R} = \begin{pmatrix} \frac{2\sin(2z^1)}{(\cos^2(z^1)+1)^2} & 0 \\ 0 & \frac{-8\cos z^1 \sin^3(z^1)}{(\sin^2(z^1)-2)^3} \end{pmatrix}, \quad \hat{\nabla}_{z^2}\mathcal{R} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then, $\mathcal{R}$ is recurrent because (4.11) holds with the one-form

$$\omega = \begin{pmatrix} \frac{2\sin^2(2z^1)}{(\cos^2(z^1)+1)^2} & 0 \end{pmatrix}$$

moreover, this one-form is exact since $\omega = dh$ with $h = -4\operatorname{arctanh}\left(\frac{\sin^2(z^1)}{\sin^2(z^1)-4}\right)$. Note that, since any closed path on $\mathbb{S}^2$ is contractible to a point, ω is exact if and only if it is closed. By Proposition 4.5, the induced connection $\hat{\nabla}$ is metrizable and the metric is given by

$$\hat{g} = \begin{pmatrix} \frac{\cos^2(z^1)+1}{4} & 0 \\ 0 & \frac{\sin^2(z^1)}{2} \end{pmatrix}.$$

Example 4.5 (Cont. Example 3.3 case a)). We use Proposition 4.1 to determine if the reduced dynamics are Lagrangian. First, we check if the induced connection $\hat{\nabla}$ is metrizable. Note that $\mathcal{C}$ is isomorphic to $\mathbb{R} \times \mathbb{S}$, the fundamental group of $\mathbb{R}$ is trivial, while the fundamental group of $\mathbb{S}$ is isomorphic to $(\mathbb{Z}, +)$. Thus, one generator of the fundamental group of $\mathcal{C}$ is the loop $\gamma : [0, 2\pi] \to \mathbb{R} \times \mathbb{S}$, defined as $\gamma(\lambda) = [0, \cos\lambda, \sin\lambda]^T$. Curve γ represents a trajectory in which the position of the cart remain fixed, while the angle q^3 performs once a full rotation in a counterclockwise direction. Since the only nonzero Christoffel symbols are $\hat{\Gamma}^{z^1}_{z^2,z^2}$ and $\hat{\Gamma}^{z^2}_{z^2,z^2}$ and these are functions of z^2 only, the curvature tensor is null, so that the induced connection is flat. Then, by Proposition 4.3, $\hat{\nabla}$ is metrizable if and only if there exists a symmetric, positive definite $(2,0)$-tensor $\hat{g}$ such that $\hat{g}_0(v,v) = \hat{g}_0(P_\gamma(v), P_\gamma(v))$, $\forall v \in T_q\mathcal{C} = \mathbb{R}^2$.

By (4.3), $P_\gamma(v) = x(2\pi)$, with

$$\begin{cases} \begin{pmatrix} \dot{x}^1(\lambda) \\ \dot{x}^2(\lambda) \end{pmatrix} + \begin{pmatrix} \hat{\Gamma}^{z^1}_{z^2,z^2} \\ \hat{\Gamma}^{z^2}_{z^2,z^2} \end{pmatrix} x^2 = 0 \\ x(0) = v\,, \end{cases}$$

Set $I(\lambda) = \int_0^\lambda \hat{\Gamma}^{z^2}_{z^2,z^2}(\mu)d\mu$, $M(\lambda) = \int_0^\lambda \hat{\Gamma}^{z^1}_{z^2,z^2} I(\mu)d\mu$, then

$$x^2(\lambda) = I(\lambda)v^2,\ x^1(\lambda) = M(\lambda)v^2 + v_1,\ P_\gamma(v) = \begin{pmatrix} 1 & M(2\pi) \\ 0 & I(2\pi) \end{pmatrix}.$$

Let $\hat{g} = \begin{pmatrix} 1 & a \\ a & b \end{pmatrix}$ be the coordinate representation of a generic $(2,0)$ symmetric tensor, then condition (4.4) is given by $P_\gamma^T \hat{g} P_\gamma = \hat{g}$, that is

$$a(1-I) = M,\ b(1-I^2) = M(M + 2aI)\,.$$

If $I \neq 1$ the only solution is given by

$$a = \frac{M}{1-I},\ b = a^2\,,$$

however, the resulting $\hat{g}$ is not positive definite. Hence, for $I \neq 1$ the induced connection is not metrizable but admits a degenerate metric tensor. If $I = 1$, then it must be $M = 0$, then condition (4.4) is satisfied for all choices of a, b. Note that, in the VHC defined in (3.28), $\hat{\Gamma}^{z^1}_{z^2,z^2}, \hat{\Gamma}^{z^2}_{z^2,z^2}$ are odd functions, which imply that $I = 1$ and $M = 0$, so that the induced connection is metrizable. The associated metric can then be computed by parallel transport of the $(2,0)$-tensor $\hat{g}$ using relation $\nabla\hat{g} = 0$.

By Proposition 4.1, if the induced connection is metrizable, the reduced dynamics are Lagrangian if and only if condition (4.1) holds. This is equivalent to check if the 1-form $\omega = \sigma(\mathrm{grad}P)^\flat$ is exact, or that ω is closed and $\int_\gamma \omega = 0$, where γ is the path defined above.

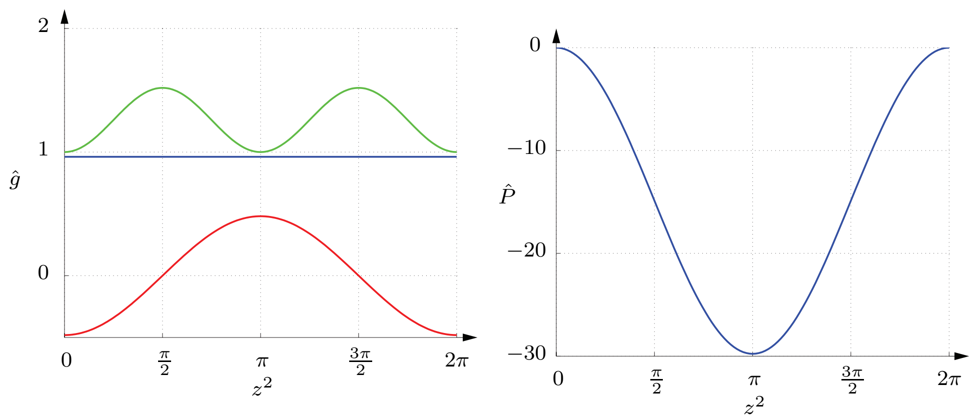
The 1-form ω is closed if its mixed partial derivatives are equal (Proposition 11.45 of [13]), then, since ω depends only on z^2, this condition is satisfied if and only if

$$\frac{\partial\left(\hat{g}_{z^1,z^1}\,\sigma(\mathrm{grad}P)^{z^1} + \hat{g}_{z^1,z^2}\,\sigma(\mathrm{grad}P)^{z^2}\right)}{\partial z^2} = 0. \tag{4.12}$$

If the equation above is satisfied, ω is exact if and only if

$$\int_\gamma \omega = \int_0^{2\pi} \hat{g}_{z^2,z^1}\,\sigma(\mathrm{grad}P)^{z^1} + \hat{g}_{z^2,z^2}\,\sigma(\mathrm{grad}P)^{z^2}dz^2 = 0\,. \tag{4.13}$$

The explicit computation of the above conditions (4.12), (4.13) is rather complex, it was done with a symbolic manipulation program. It turns out that they are satisfied in case a) (control on the cart) for $\rho = 0$. In this case, we numerically found $a(0) = -0.5, b(0) = 1.0398$. Figure 4.2a depicts the induced metric components: in blue the function a, in red the functions b and in green the function c. Figure 4.2b depicts the potential function obtained solving (4.13). For $\rho \neq 0$, conditions (4.12), (4.13) are not satisfied, hence the induced connection is metrizable but the reduced dynamics are not Lagrangian.



(a) Induced metric components. In blue a, in red b and in green c.

(b) Induced potential.

Figure 4.2: Example 4.5. Representation of the induced metric $\hat{g}$ and the induced potential $\hat{P}$ for the double pendulum on a cart with $k = 0$.

Conclusion

In this thesis we have considered two aspects of virtual holonomic constraint theory: the synthesis of regular VHCs and the coordinate free description of VHCs.

Regarding the first topic, our main contribution is a synthesis procedure that generates regular VHCs whose reduced dynamics possess a stable limit cycle. This fact has relevance in applications.

The second topic of this thesis concerns the description of VHCs in a coordinate free formalism. This formalism is based on a Riemannian geometry setting, which allows to represent a wide class of Lagrangian mechanical systems. We have extended some results of [6, 5] for such systems. In particular, we have extended the regularity condition and we have given a coordinate free representation of the reduced dynamics (introducing the induced connection). Finally, we have addressed the inverse Lagrangian problem. In particular, we have shown that it is strongly related to the metrizability of the induced connection.

A possible future development concerns the extension of Noether's Theorem to symmetric Lagrangian mechanical systems subject to symmetric virtual holonomic constraints. Another important development is the investigation of possible industrial and control applications of this theory. For instance, formation control problem for unmanned aerial vehicles could be addressed within this framework.

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Acknowledgements

First and foremost, I sincerely thank professor Luca Consolini for supporting and motivating me during these years. His enthusiasm, intuition and talent have inspired me and have been the driving force through this work and my graduate career. He has taught me several things, from the geometry and control theory to patience and humility. I would like to thank professor Aurelio Piazzi for his constant help and for the trust that he has shown in my capabilities since my BSc thesis. I express my sincere gratitude to professor Manfredi Maggiore for his hospitality at the University of Toronto and for having taught me mathematical rigor in control and the passion for fountain pens. I would like to sincerely thank the external reviewers, professor Fabio Morbidi and professor Manfredi Maggiore himself, for their insightful observations on this thesis.

I am thankful to all my friends and colleagues with whom I share many experiences of the PhD student life. I wish to thank my former roommate Federico *Weka* and my officemates Fabjan and Giorgio for the invaluable time spent together. I thank the odd citizens of the building one: Giuse of the *GiuseLab*, the *Golden Boy* Andrea, the *King Laurin* Mattia, *Professor Money* Francesco, *Cabba* Federico, the *mountaineer* Michele, *Dano* Daniele, *mr.P* Piero, Marina, Fabio. I express my sincere gratitude to the people who made my stay in Toronto a beautiful experience. In particular I thank Alireza Mohammadi and my roommates of there Cristóbal, Kazunari and Ulises.

I warmly thank my girlfriend Mascia, for giving me the energy needed to take on happily the life. She has always been there, supporting and cheering me through the good times and bad. I am grateful to her family and my family for the support during these years. This thesis would never have taken form without the loving support of my parents, my brother Riccardo and my grandmother Vanda. They have always been there for me and I appreciate them for having always believed in me and in my capabilities. Finally, I want to remember my grandparents and my friend Lando who left an empty place in my heart.

Grazie.