On representation of preferences à la Debreu

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Abstract

A representation theorem proven by G. Debreu in 1960, although somehow neglected by the literature, implies several deep and unexplored consequences both for Economics and for Decision Theory. This paper focuses on some of them. In particular, possible decompositions of state-dependent utilities à la Debreu (which may equivalently be seen as "utility-dependent probabilities") are analyzed, showing that Debreu's representation is based upon a "joint" probability/utility function. It is illustrated how Debreu's Theorem can provide a neat geometrical interpretation of Castagnoli and LiCalzi's "benchmarking" representation of preferences. (Conditional) Certainty Equivalents are defined and studied, and possible implications for attempting representation of incomplete preferences are discussed.

Keywords: Debreu's Theorem; Representation of preferences; Sure Thing principle; Representation of preferences; State-dependent utility; Benchmarking; Certainty equivalents; Incomplete preferences.

JEL classification: D81 - C02

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1 Introduction

In 1960, Gerard Debreu (1960) proved a very deep theorem about representation of preferences among random variables. His key axiom was the so-called *Sure Thing Principle*, originally introduced by Savage (1951), which asks for a preference to completely neglect possible common parts between two alternatives (see also Fishburn, 2018). With topological methods, G. Debreu proved that a preference satisfies the Sure Thing Principle if and only if it can be represented by means of a (utility) functional which turns out to be additively decomposable with respect to the "states of the world", *i.e.*, the elementary events in the state space.

Unlike the classical results about representation of preferences, such as von Neumann and Morgenstern (1944), Savage (1951), Schmeidler (1986), Gilboa and Schmeidler (1989), and many others, Debreu's representation does not explicitly identify a probability measure. This absence makes Debreu's work look "inadequate" in some sense, and it may explain why his result is scarcely used both in Economics and in Decision Theory. Indeed, only a few papers in the literature are related to Debreu's Theorem: the first significant reference appears in Mas-Colell (1990). An extension of Debreu's Theorem was given by Chew and Wakker (1996) and Wakker and Zank (1999), while the first significant applications to Decision Theory were proposed by Nau (2003, 2006). More recently, a full extension to a generic state space was obtained by Castagnoli and LiCalzi (2006), with consequences explored by Castagnoli *et al.* (2016), and an investigation about intertemporal preferences inspired by Debreu's utility was presented in Maggis and Maran (2021).

However, the consequences of Debreu's Theorem have not been fully explored. The present paper aims at investigating some of them. We analyse the relationship between Debreu's functional and the classical expected utility, we show how Debreu's Theorem can provide a neat geometric interpretation of Castagnoli and LiCalzi's "benchmarking" representation of preferences and we define and study the concept of Certainty Equivalent for Debreu's utility. Finally, we provide a first investigation on the case of incomplete preferences.

In Section 2, we present Debreu's Theorem. In Section 3, we discuss the meaning of the representation functional; in particular, we show how Debreu's representation is based upon a "joint" probability/utility function. Section 4 is dedicated to the connection between Debreu's Theorem and the "benchmarking" representation *à la* Castagnoli - LiCalzi, jointly with some geometric considerations. Section 5 is devoted to Certainty Equivalent (also "conditional" to some particular event) for Debreu's functionals. Finally, Section 6 hosts an analysis about possible relationships between Debreu's Theorem and incomplete preferences. Conclusions, and outlines for further research, are gathered in Section 7.

2 Debreu's Theorem and first considerations

As usual, a *random variable* is a function $X\colon \Omega \to \mathbb{R}$ defined on a set Ω of "states of the world", usually called the *state space*, with suitable measurability conditions. For the purpose of this paper, random variables can be assumed to represent random money results. We shall follow the original setting by Debreu (1960), by supposing the state space to be a finite set $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$. In such a case, every random variable $X\colon \Omega \to \mathbb{R}$ can be naturally identified with the column vector $[x_1 \;\; x_2 \;\; \cdots \;\; x_m]^{\mathrm{T}} \in \mathbb{R}^m$, with $x_i := X(\omega_i)$ for every $i = 1, 2, \ldots, m$. The finiteness hypothesis allows us to deal more intuitively with the objects involved in this paper, but it is important to underline that the same line of reasoning exposed by Castagnoli and LiCalzi (2006) extends all of the results to the fully general case. We suppose some set $\mathcal{X} \subseteq \mathbb{R}^m$ of random variables defined on Ω to be given.

Given a subset $A \subseteq \Omega$, we use the notation $\mathbb{I}_A$ to denote the *indicator function* of the subset A, *i.e.*, the random variable that takes value 1 on the elements of A and 0 on the elements of $A^{\mathrm{c}} := \Omega \backslash A$. In the case when $A = \{\omega_i\}$ is a singleton, we write $\mathbb{I}_{\omega_i}$ instead of $\mathbb{I}_{\{\omega_i\}}$. If $x, y \in \mathcal{X}$ are random variables and $A \subseteq \Omega$, we indicate with xAy the random variable that takes the same values of x on A and of y on A^{c}: in symbols, $xAy = x \cdot \mathbb{I}_A + y \cdot \mathbb{I}_{A^{\mathrm{c}}}$, where "$\cdot$" denotes the pointwise product of functions. For the sake of simplicity, we suppose that the set $\mathcal{X}$ be closed under such an operator, *i.e.*, that $xAy \in \mathcal{X}$ for every $x, y \in \mathcal{X}$ and every $A \subseteq \Omega$. We also suppose that $0 \in \mathcal{X}$, with 0 the null (degenerate) random variable.

Definition 1. A binary relation $\succcurlyeq$ on $\mathcal{X}$ is called a *preference* if it is:

- *complete*, *i.e.*, for every $x, y \in \mathcal{X}$, either $x \succcurlyeq y$ or $y \succcurlyeq x$;
- *reflexive*, *i.e.*, $x \succcurlyeq x$ for every $x \in \mathcal{X}$;
- *transitive*, *i.e.*, if $x, y, z \in \mathcal{X}$ are such that $x \succcurlyeq y$ and $y \succcurlyeq z$, then $x \succcurlyeq z$.

Note that reflexivity is an immediate consequence of completeness.

The relation $x \succcurlyeq y$ is read as "x is (weakly) *preferred* to y". In the case when $x, y \in \mathcal{X}$ are such that both $x \succcurlyeq y$ and $y \succcurlyeq x$, it is said that x and y are *indifferent* to the decision maker, and the notation $x \sim y$ is used; finally, $x \succ y$ indicates the case when $x \succcurlyeq y$ and $x \nsim y$.

The state $\omega_i \in \Omega$ is called *inessential* (with respect to the preference $\succcurlyeq$) if $x \cdot \mathbb{I}_{\omega_i} \sim y \cdot \mathbb{I}_{\omega_i}$ for every $x, y \in \mathcal{X}$. Intuitively, this means that the decision makers do not value the amount that the random variables take on ω_i, that is to say, that they deem the state ω_i to have null probability. The state ω_i is called *essential* if it is not inessential, *i.e.*, if there are two random variables $x, y \in \mathcal{X}$ such that $x \cdot \mathbb{I}_{\omega_i} \succ y \cdot \mathbb{I}_{\omega_i}$.

The key hypothesis to Debreu's Theorem is the so-called *Sure Thing Principle* (originally introduced by Savage, 1951, as "Axiom P2"): it states that, whenever $x, y \in \mathcal{X}$ and $A \subseteq \Omega$ are such that $xAz \succcurlyeq yAz$ for some $z \in \mathcal{X}$, then $xAw \succcurlyeq yAw$ for *every* $w \in \mathcal{X}$. The Sure Thing Principle requires the decision makers to neglect the "common parts" of two random variables when comparing them: since xAz and yAz coincide outside of A (where they both take the same values as z), only the values they take on A matter to the decision makers, and therefore their preferences do not change if the "common" part of the two random variables is modified in any possible way.

The Sure Thing Principle might resemble the classical *Independence Axiom* of von Neumann and Morgenstern (1944), but such a resemblance is smaller than it might appear. The Independence Axiom deals with distribution functions of the lotteries and with their probabilistic mixtures, whereas the Sure Thing Principle deals with the pointwise functional definition of random variables. We purposefully used the different terms "lottery" and "random variable". In the classical setting based on *lotteries* (*i.e.*, on probability laws), an "hidden axiom" is accepted, according to which two random variables which share the same law must be considered indifferent to the decision maker (because, as a matter of fact, they are indistinguishable by the model). This seems too much of a restriction in several decision problems, such as, for instance, insurance problems, where a decision maker is quite naturally interested in the actual events where the policy is triggered and not only on their probability, or such as financial models, where it is quite different to get a $1,000$\$ dividend in a scenario when all of the stocks in the market yield high returns than it is to get it on a "black Friday". Debreu's setting allows to overcome such a possibly undesired restriction.

Besides the Sure Thing Principle, two more hypotheses are involved in Debreu's Theorem. The preference $\succcurlyeq$ is required to be *continuous* with respect to the Euclidean norm on $\mathbb{R}^m$, in the sense that if a sequence $(x_k)_{k\in\mathbb{N}}$ in $\mathcal{X}$ is given such that $x_k \to x \in \mathcal{X}$ and if $y \in \mathcal{X}$ is such that $x_k \succcurlyeq y$ for every $k \in \mathbb{N}$, then $x \succcurlyeq y$ as well. Finally, the preference $\succcurlyeq$ has to be *monotonic* with respect to the pointwise partial order on $\mathbb{R}^m$: if $x \geqq y$ (meaning that $x_i \geqslant y_i$ for every $i = 1, 2, \dots, m$), then $x \succcurlyeq y$. We are now ready to state the following

Theorem 1 (Debreu). *Let $\mathcal{X}$ be a set of random variables on the finite state space $\Omega = \{\omega_1, \omega_2, \dots, \omega_m\}$ $(m \geqslant 3)$ and $\succcurlyeq$ a continuous and monotonic preference on $\mathcal{X}$ such that there exist at least three essential states of the world.*

The preference $\succcurlyeq$ agrees with the Sure Thing Principle if, and only if, it is represented by an additively decomposable real valued function $F\colon \mathcal{X} \to \mathbb{R}$, i.e., *such that*

$$F(y) = F\left(\sum_{i=1}^{m} y_i \mathbb{I}_{\omega_i}\right) = \sum_{i=1}^{m} u_i(y_i)$$

with $u_1, u_2, \dots, u_m \colon \mathbb{R} \to \mathbb{R}$ increasing[1] *real valued functions which are unique up to additive constants and a common increasing linear transformation.*

Proof. See Debreu (1960). □

Remarks 2. (i) The case when there is a single essential state of the world is irrelevant, because it corresponds to absence of uncertainty. The case when there are only two essential states is indeed quite complicated. In such a case, according to Karni and Safra (1998), Debreu's Theorem still holds, provided that a very technical requirement, called the *Hexagon Condition*, is satisfied (see also Köbberling, 2003).

(ii) Since additive constants on the u_is do not matter in the representation of $\succcurlyeq$, it is always possible to take $u_i(0) = 0$ for every $i = 1, 2, \dots, m$. In particular, if ω_i is inessential, then $u_i \equiv 0$.

From now on, we shall assume without loss of generality that $u_i(0) = 0$ for every $i = 1, 2, \dots, m$. This way, Debreu's representations of a preference become unique up to a common increasing linear transformation only.

Example 1. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$: it can for instance represent the set of the possible results of the roll of a die. A random variable X on Ω can be naturally identified with the column vector $x = [x_1 \;\; x_2 \;\; x_3 \;\; x_4 \;\; x_5 \;\; x_6]^{\mathrm{T}}$ where x_i is the value associated to the outcome $\omega_i = i$, $i = 1, 2, \dots, 6$. Assume that there exists a preference $\succcurlyeq$ with Debreu's representation $F(y) = \sum_{i=1}^{6} u_i(y_i)$ where the six utility functions are $u_i(y_i) = i y_i$, $i = 1, 2, \dots, 6$. Then

$$F(y) = \sum_{i=1}^{6} i y_i$$

[1] Here and in what follows, by "increasing function" we mean "strictly increasing function".

It is obvious that if $y \geqq x$, then $\sum_{i=1}^{6} iy_i \geqslant \sum_{i=1}^{6} ix_i$. Hence $y \succcurlyeq x$, that is, the preference is monotonic. It can be easily shown that it is continuous as well. Moreover, if there exist $A \subset \Omega$ and a random variable z such that $yAz \succcurlyeq xAz$ for some z, then

$$\sum_{i\in A} iy_i + \sum_{i\in A^c} iz_i \geqslant \sum_{i\in A} ix_i + \sum_{i\in A^c} iz_i$$

which implies that $yAw \succcurlyeq xAw$ for all $w \in \mathcal{X}$, namely the preference $\succcurlyeq$ agrees with the Sure Thing Principle.

3 Debreu's Theorem and expected utility

For every $i = 1, 2, \ldots, m$, the value $u_i(y_i)$ represents the *utility* of getting the amount y_i in the state ω_i. We stress out that such a utility depends not only on the amount y_i at stake, but also on the state ω_i in which such an amount is obtained. In order to emphasise such a feature, Debreu's decomposition can be written as $F(y) = \sum_{i=1}^{m} u(y_i; \omega_i)$, with the function $u \colon \mathbb{R} \times \Omega \to \mathbb{R}$ called a *state-dependent utility*.

The main difference between Debreu's representation and the classical expected utility one lies in the fact that there is no natural way to decompose the state-dependent utility $u(y_i; \omega_i)$ into the product $u(y_i)p(\omega_i)$ of a utility $u(y_i)$ and a probability $p(\omega_i)$. But this does not mean that *no* decomposition is possible, but rather than *almost every* decomposition can be, even if not all of them are meaningful in order to compare two random variables.

Take indeed any probability p on Ω such that $p(\omega_i) > 0$ for every essential state ω_i. It is of course possible to write

$$F(y) = \sum_{i=1}^{m} u(y_i; \omega_i) = \sum_{i=1}^{m} \frac{u(y_i; \omega_i)}{p(\omega_i)} \cdot p(\omega_i)$$

(where the fraction is conventionally set equal to zero on inessential states), and this amounts to decompose the state-dependent utility $u(y_i; \omega_i)$ into the product of the probability $p(\omega_i)$ and the ratio $\frac{u(y_i;\omega_i)}{p(\omega_i)} = u_p(y_i; \omega_i)$, which can be interpreted as a utility that depends on the given probability p. With this decomposition, $y \succcurlyeq x$ if and only if $\mathrm{E}[u_p(y)] \geqslant \mathrm{E}[u_p(x)]$.

In a similar way, take any increasing (utility) function $v \colon \mathbb{R} \to \mathbb{R}$ such that $v(0) = 0$ and $v(k) \neq 0$ for every $k \neq 0$. We can write

$$F(y) = \sum_{i=1}^{m} u(y_i; \omega_i) = \sum_{i=1}^{m} v(y_i) \cdot \frac{u(y_i; \omega_i)}{v(y_i)}$$

(where, again, the fraction is conventionally set equal to zero if $y_i = 0$. Note that the fraction is always nonnegative, given the monotonicity of v and of the u_is.). Since utility functions can be arbitrarily multiplied by a positive constant, it is possible to scale v in such a way that $\sum_{i=1}^{m} \frac{u(y_i;\omega_i)}{v(y_i)} = 1$, and then the ratio $\frac{u(y_i;\omega_i)}{v(y_i)}$ can be interpreted as a probability which depends on the given utility function v, hence on the random variable. Note that, if v is not linear, the normalization condition can give different probabilities and, consequently, different utilities for different random variables.

Such a lack of a unique "strong" decomposition is probably the main reason for the lack of popularity suffered by Debreu's Theorem. But we observe that such a strong decomposition is not "natural" at all, and that indeed in most applications it results either out of sheer luck or even out of brute force. Indeed, in several models, it is perfectly sensible for the utility of an outcome to depend on the scenario which yields it (see also Montesano, 2021).

Example 2. Consider again the preference introduced in Example 1. The six separate utility functions $u_1, u_2, \ldots, u_6$ in the Debreu's representation can be rewritten through the single state-dependent utility $u(y_i;\omega_i) = \omega_i y_i$. Assuming the die to be fair, we take $p(i) = 1/6$ for every $i = \omega_i \in \Omega$, and

$$F(y) = \sum_{i=1}^{6} i y_i = \sum_{i=1}^{6} \frac{\omega_i y_i}{1/6} \cdot \frac{1}{6} = \sum_{i=1}^{6} 6u(y_i;\omega_i) \cdot \frac{1}{6}$$

thus seeing the Debreu's decomposition of $F(y)$ as the expected value of the *new* utility function $6u(y_i;i)$ with respect to the uniform probability on Ω. Given the random variable x, we have that

$$y \succcurlyeq x \quad \Longleftrightarrow \quad F(y) \geqslant F(x) \quad \Longleftrightarrow \quad E\left[6u(y)\right] \geqslant E\left[6u(x)\right] \ .$$

The equivalence between the Debreu's representation of the preference and the expected utility representation still holds if we take a different probability. Take, for instance, $\widetilde{p}$ such that $\widetilde{p}(i) = \frac{i}{21}$. Then $F(y) \geqslant F(x)$ if and only if $\mathrm{E}[\widetilde{u}(y)] \geqslant \mathrm{E}[\widetilde{u}(x)]$ where the expectation is computed with respect to the probability $\widetilde{p}$ and $\widetilde{u}(y_i;\omega_i) = 21u_i(y_i)/i$.

On the other hand, consider now the (linear) utility function $v(x) = ax$, $a > 0$: from the condition

$$\sum_{i=1}^{m} \frac{u(y_i;i)}{ay_i} = 1$$

we obtain $a = 21$, $p(i) = i/21$ for every i and again $F(y) = \mathrm{E}\left[v(y)\right]$. The same holds for every random variable of $\mathcal{X}$.

However, if we consider a non linear v, not only we obtain different values for a, but, if we consider different random variables, also for the probabilities and for the consequent utilities. Take, for instance, $v(x) = a(1 - e^{-0.01x})$, $x^1 = [1\ 1\ 1\ 1\ 1\ 0]^{\mathrm{T}}$, and $x^2 = [1\ 1\ 1\ 1\ 0\ 1]^{\mathrm{T}}$. From the normalization condition, we find two different values for a:

$$a(x^1) = \frac{15}{1 - e^{-0.01}} \qquad a(x^2) = \frac{16}{1 - e^{-0.01}}$$

which give different probabilities and utilities functions depending on x^1 and x^2:

$$p_{x^1}(i) = \frac{i}{15} \text{ for } i = i = 1, 2, \ldots, 5 \qquad p_{x^1}(6) = 0$$

$$p_{x^2}(i) = \frac{i}{16} \text{ for } i = i = 1, 2, 3, 4, 6 \qquad p_{x^2}(5) = 0$$

and:

$$v_{_{x^1}}(x) = \frac{15}{1 - e^{-0.01}}\left(1 - e^{-0.01x}\right) \qquad v_{_{x^2}}(x) = \frac{16}{1 - e^{-0.01}}\left(1 - e^{-0.01x}\right)$$

Of course, the proposed interpretation of the Debreu's functional, as an expected value with respect to a probability depending on a given utility function, is to be intended as a formal possibility of a traditional reading, but it cannot always be exploited for a comparison between random variables. The essence of the 'utility' of Debreu is, in some way, oblique to the classical one (or with more general *ambitions*): preferences among random variables have to be read directly on F.

4 Debreu's Theorem and benchmarking

Debreu's result admits an interesting geometric interpretation.

Let $y = [y_1 \;\; y_2 \;\; \cdots \;\; y_m]^{\mathrm{T}}$ be a random variable on $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$; suppose that $y \geqq 0$. The (*truncated*) *hypograph* of y is defined as

$$\text{hypo}(y) := \bigcup_{i=1}^{m} \{\omega_i\} \times [0, y_i] \subseteq \Omega \times \mathbb{R} :$$

in other words, it is the "histogram" formed by m rectangles, each of them y_i high and associated to the "basis point" ω_i. In the case when y is not positive, the definition can be extended by taking into consideration the interval $[y_i, 0]$ whenever $y_i < 0$, although the name "hypograph" is no longer fully appropriate for the set that obtains. A better way, indeed, would be to decompose the random variable y as the sum $y^+ - y^-$ of its positive and negative parts, and to deal separately with the two parts.

Starting from a paper by Chateauneuf (1999), Castagnoli and Favero (2010) pointed out that the expected utility functional is nothing but a measure ν on $\Omega \times \mathbb{R}_+$ (or on $\Omega \times \mathbb{R}$, in the general, nonpositive, case) coming from the product of the probability p on Ω and the Lebesgue-Stiltjes measure μ_u on $\mathbb{R}$ defined by the utility function u:

$$\mathrm{E}\left[u(y)\right] = \sum_{i=1}^{m} u(y_i) \cdot p(\omega_i) = \sum_{i=1}^{m} \left[p\left(\{\omega_i\}\right) \cdot \int_0^{y_i} \mathrm{d}u(y)\right] = [p \otimes \mu_u](\text{hypo}y)$$

Such a measure $\nu = p \otimes \mu_u$ is additive with respect to the states of the world, but not with respect to the values taken by the random variables.

It is immediate to realise that Debreu's Theorem is nothing but the most general case of such a representation: the preference "$\succcurlyeq$" can be represented by a measure ν on $\Omega \times \mathbb{R}$, which is still additive with respect to the elements of Ω

but, unlike what happens in the expected utility case, need not be a product measure (see also Castagnoli *et al.*, 2016).

Secondly, let $y \in \mathcal{X}$ be a positive random variable (for a generic $y \in \mathcal{X}$, the positive and negative parts are separately considered). On the half-line of nonnegative real numbers $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geqslant 0\}$ consider the identity function $q(s) = s$ (any injective function q might be used as well): it is possible to see q as a real-valued, positive random variable defined on the state space $\mathbb{R}_+$ (that is, on a state space which does not coincide with Ω).

The hypograph of y can be rewritten as

$$\text{hypo}(y) = ((\omega, s) \in \Omega \times \mathbb{R}_+ : y(\omega) \geqslant q(s)) \subseteq \Omega \times \mathbb{R}$$

indeed, the pair (ω_i, s) belongs to the above set if and only if $y(\omega_i) = y_i$ is greater than or equal to $q(s) = s$, *i.e.*, if and only if $s \in [0, y_i]$. The representing function $F = \nu\,(\text{hypo}(\cdot))$ thus becomes

$$F(y) = \nu\big(\{(\omega, s) : y(\omega) \geqslant q(s)\}\big) = \nu\{y \geqq q\}$$

In other words, it represents the measure of the set of the states of the world such that the random variable y is greater than the random variable q or, briefly, the measure of the event such that y "outperforms" q. Note that, if ν is normalised (this is possible under very reasonable assumptions, such as, for instance, the fact that all of the random variables in $\mathcal{X}$ take their values in a bounded interval $[a, b] \subseteq \mathbb{R}$), such a measure becomes the probability that y outperforms q.

The interpretation is quite fruitful (see, for instance, Bordley and LiCalzi, 2000, Della Vigna and LiCalzi, 2001, and Castagnoli and LiCalzi, 2006): the decision makers have a "reference" random variable q of their choice (defined on a state space which is generally not Ω) and "rank" the random variables in $\mathcal{X}$ depending on the probability to yield a "better result" than q. Roughly speaking, the choice of the reference function q essentially corresponds to the choice of the "scale" on the value axis of the hypograph. For instance, taking $q(s) = \mathrm{e}^s - 1$ corresponds to "drawing" the "histograms" that make $\text{hypo}(y)$ in a quasi-logarithmic scale. Indeed, $y(\omega) \geqslant q(s)$ amounts to saying $\mathrm{e}^s - 1 \leqslant y(\omega)$, that is $s \leqslant \ln\big(y(\omega) + 1\big)$: in other words, instead of the hypograph defined at the beginning of this section, the set

$$\text{hypo}'(y) := \bigcup_{i=1}^{m} \{\omega_i\} \times \big[0, \ln(y_i + 1)\big] \subseteq \Omega \times \mathbb{R}$$

is considered.

Thus ν turns out to be a product measure if and only if y and q are stochastically independent of each other. In such a case, indeed, all of the pairs of events $\{\omega \in \Omega : y(\omega) = t\}$ and $\{s \in \mathbb{R}_+ : t \geqslant s\}$ are independent and, therefore,

$$\nu\left((\omega, s) : y(\omega) \geqslant q(s)\right) = \sum_{t \in \mathbb{R}_+} \nu_1\left(\{\omega : y(\omega) = t\}\right) \nu_2\left(\{s : t \geqslant s\}\right)$$

(the sum is actually a finite one, because Ω is finite and therefore the set $\{y = t\}$ is not empty only for finitely many $t \in \mathbb{R}_+$). Note that, if ν is normalised, this is also the case for the marginals ν_1 and ν_2, which turn then out to be probabilities on Ω and $\mathbb{R}_+$ respectively; in such a case, moreover, $\nu_2\{s : t \geqslant s\}$ is nothing but the cumulative distribution function $Q(t)$ of q evaluated at t. This way,

$$\nu\left((\omega, s) : y \geqslant q\right) = \sum_{t \in \mathbb{R}_+} p\{y = t\}\ Q(t) = \mathrm{E}_p\left[Q(y)\right]$$

and we come back to the expected utility representation, with the interesting interpretation (still due to Castagnoli and LiCalzi, 2006) that the utility function Q is the cumulative distribution function of the decision maker's benchmark: for instance, the linear utility function $v(x) = \frac{x-a}{b-a}$ on the interval $[a, b]$ corresponds to a uniform random benchmark, and the utility function $v(x) = 1 - \mathrm{e}^{0.01x}$ on $\mathbb{R}_+$ corresponds to an exponential random benchmark.

Such an independence can be read straightaway from Debreu's Theorem: indeed, Debreu's representation coincides with the classical expected utility if and only if $u(y_i, \omega_i)$ can be decomposed into the product $u(y_i)p(\omega_i)$, which is the same as asking that the dependence of u on the state only amounts to multiplication by the probability of the state itself. Decision makers do not care at all on the particular state ω_i onto which the outcome y_i is attached, and their evaluation of the pair (ω_i, y_i) simply consists in the probability of the state ω_i multiplied by the utility $u(y_i)$ evaluated according to a single, given utility function u independent of the state. Shortly, we recover the same conclusion that decision makers act according to the classical expected (state-independent) utility criterion if and only if their preferences are independent of the states of the world.

5 (Conditional) Certainty Equivalents

The *certainty equivalent* of a random variable $y \in \mathcal{X}$ is classically defined to be a constant $C(y) \in \mathbb{R}$ such that

$$C(y) \sim y\ .$$

The decision maker is indifferent between getting the random variable y or the certain amount $C(y)$. The certainty equivalent can be equivalently defined in terms of the representation functional as the constant $C(y) \in \mathbb{R}$ such that

$$F\big([C(y)\ \ C(y)\ \ \cdots\ \ C(y)]^{\mathrm{T}}\big) = F(y)\ .$$

Of course, since the certainty equivalent depends on the preference and not on its representation à la Debreu, it will be the same for any $u_1, \ldots, u_m$ which satisfy Theorem 1 in Debreu (1960).

Given a (deterministic) amount $k \in \mathbb{R}$ on a subset $A \subseteq \Omega$, we introduce the notation

$$u(k; A) := F(k\mathbb{I}_A) = \sum_{\omega_i \in A} u(k; \omega_i)$$

to indicate the utility of the certain amount k "restricted" to the event A. In particular,

$$u(k;\Omega) = \sum_{i=1}^{m} u(k;\omega_i) = F\left([k \;\; k \;\; \cdots \;\; k]^{\mathrm{T}}\right)$$

is the utility of the degenerate random variable (certain amount) k.

This way, for every $A \subseteq \Omega$, a real function $u_A \colon \mathbb{R} \to \mathbb{R}$ is defined by the position $u_A(k) := u(k;A)$. We remark that the monotonicity of the preference $\succcurlyeq$ ensures that all of the u_As are increasing functions; moreover, it is straightforward to prove that, if $\succcurlyeq$ is continuous, then the u_As are continuous as well.

The equality which defines the certainty equivalent can be written as

$$u_\Omega\left(C(y)\right) = F(y)$$

and the existence of the certainty equivalent is ensured if u_Ω is continuous (see, for instance, Kreps (1988)). A direct definition of the certainty equivalent may be given by means of the *generalised inverse* of u_Ω defined as $u_\Omega^{\leftarrow}(k) = \inf_x\{x \in \mathbb{R} : u_\Omega(x) \geqslant k\}$ (see, for instance, Embrechts and Hofert, 2010):

$$C(y) := u_\Omega^{\leftarrow}\left(F(y)\right) = u_\Omega^{\leftarrow}\left(\nu(\mathrm{hypo}(y)\right)\;.$$

The geometric interpretation of the certainty equivalent is immediate: $C(y)$ is a "height" such that the "rectangular histogram" whose bars are all $C(y)$ high has exactly the same measure ν as the "histogram" individuated by the original random variable y.

We introduce the following:

Definition 2. The *certainty equivalent of $y \in \mathcal{X}$ conditional to the event $A \subseteq \Omega$* is the constant $C_A(y) \in \mathbb{R}$ such that

$$\sum_{\omega_i \in A} u(C_A(y);\omega_i) = u_A\left(C_A(y)\right) = F(y\mathbb{I}_A) = \sum_{\omega_i \in A} u(y_i;\omega_i).$$

The decision maker is indifferent, in the case that the event A prevails, between receiving the amount $C_A(y)$ in every $\omega_i \in A$ or the original random variable y (taking value y_i in ω_i). The conditional certainty equivalent can again be defined directly by

$$C_A(y) := u_A^{\leftarrow}\left(F(y \cdot \mathbb{I}_A)\right) = u_A^{\leftarrow}\left(\nu(\mathrm{hypo}(y \cdot \mathbb{I}_A)\right)$$

Geometrically, $C_A(y)$ is the height such that the "rectangle" with "basis" A has exactly the same measure ν as the "histogram" with the heights y_i associated to each $\omega_i \in A$.

The conditional certainty equivalent is monotonic, that is $C_A(x) \leqslant C_A(y)$ if $xI_A \leqq yI_A$ and is obviously non negative for non-negative random variables. Three other interesting properties for conditional certainty equivalents can be derived. In order to prove them, we first introduce the following notation: let $\mathcal{A} = \{A_1, A_2, \dots, A_n\}$ ($n < m$) be a partition of Ω (*i.e.*, $A_i \cap A_j = \varnothing$ whenever $i \neq j$ and $\bigcup_{j=1}^{n} A_j = \Omega$), and define *the certainty equivalent of y conditional to the partition $\mathcal{A}$* as the random variable $y|\mathcal{A}$

$$y|\mathcal{A} := \sum_{j=1}^{n} C_{A_j}(y)\mathbb{I}_{A_j} .$$

Roughly speaking, $y|\mathcal{A}$ is obtained by "replacing" the values taken by y with the conditional certainty equivalent "block by block" on each element of the partition $\mathcal{A}$.

Proposition 3. *The conditional certainty equivalent satisfies the following properties:*

(i) Associative property*: $F(y|\mathcal{A}) = F(y)$, namely $y|\mathcal{A} \sim y$.*

(ii) Compatibility property over different events*: if $A, B \subseteq \Omega$ are disjoint events* (i.e., *such that $A \cap B = \varnothing$) and if $C_A(y) = C_B(y)$, then $C_{A\cup B}(y) = C_A(y) = C_B(y)$.*

(iii) Additivity property: *If $x, y \in \mathcal{X}$ are such that $x_i \cdot y_i = 0$ for all i, then $u_A(C_A(x + y)) = u_A(C_A(x)) + u_A(C_A(y))$ for all $A \subseteq \Omega$. Moreover, if the utility functions in the representation of F are linear, $C_A(x + y) = C_A(x) + C_A(y)$ for all $A \subseteq \Omega$.*

Proof. (i) Since for every $j = 1, 2, \dots, n$ the definition of conditional certainty equivalent ensures that $\sum_{\omega_i \in A_j} u\left(C_{A_j}(y); \omega_i\right) = \sum_{\omega_i \in A_j} u(y_i; \omega_i)$, it is immediate to conclude that

$$\begin{aligned} F(y|\mathcal{A}) &= \sum_{i=1}^{m} u(y|\mathcal{A}; \omega_i) = \sum_{j=1}^{n} \sum_{\omega_i \in A_j} u(C_{A_j}(y); \omega_i) = \\ &= \sum_{j=1}^{n} \sum_{\omega_i \in A_j} u(y_i; \omega_i) = \sum_{i=1}^{m} u(y_i; \omega_i) = F(y) \ . \end{aligned}$$

(ii) Denote $C_A(y) = C_B(y) = c$. The following chain of equalities hold:

$$\begin{aligned} u_{A\cup B}(C_{A\cup B}(y)) &= \sum_{\omega_i \in A\cup B} u(y_i; \omega_i) = \sum_{\omega_i \in A} u(y_i; \omega_i) + \sum_{\omega_i \in B} u(y_i; \omega_i) = \\ &= \sum_{\omega_i \in A} u(C_A(y); \omega_i) + \sum_{\omega_i \in B} u(C_B(y); \omega_i) = \\ &= \sum_{\omega_i \in A\cup B} u(c; \omega_i) \ . \end{aligned}$$

(iii) Since there is no $\omega_i \in \Omega$ such that both x_i and y_i are simultaneously nonzero (and $u_i(0) = 0$ for all i), we can write

$$\begin{aligned} u_A(C_A(x+y)) &= \sum_{\omega_i \in A} u(x_i + y_i; \omega_i) = \\ &= \sum_{\omega_i \in A : x_i \neq 0} u(x_i; \omega_i) + \sum_{\omega_i \in A : y_i \neq 0} u(y_i; \omega_i) = \\ &= \sum_{\omega_i \in A} u(x_i; \omega_i) + \sum_{\omega_i \in A} u(y_i; \omega_i) = \\ &= \sum_{\omega_i \in A} u(C_A(x); \omega_i) + \sum_{\omega_i \in A} u(C_A(y); \omega_i) = \\ &= u_A(C_A(x)) + u_A(C_A(y)) \; . \end{aligned}$$

It is then immediate that, if the u_is are linear, $C_A(x+y) = C_A(x) + C_A(y)$ as well. □

Remarks 4. (i) The associative property states that the decision maker does not change the evaluation of y if all of the values that y takes on a given event $A \subseteq \Omega$ are replaced with the certainty equivalent $C_A(y)$ conditional to that event. In such a case, we can derive a "conditional decomposition" of F as

$$F(y|\mathcal{A}) = \sum_{j=1}^{n} u_{A_j}\left(C_{A_j}(y)\right) = \sum_{j=1}^{n} u(C_{A_j}(y); A_j)$$

that is, a decomposition by means of " event-dependent" utility functions rather than of state-dependent ones. Moreover, since $y \sim y|\mathcal{A}$, we can rewrite the above decomposition as

$$u_\Omega(C(y)) = \sum_{j=1}^{n} u_{A_j}\left(C_{A_j}(y)\right) .$$

(ii) The compatibility property formalises the fact that if a decision maker is indifferent between receiving an amount k or the original random variable y in case that the event A prevails, and the same amount is considered equivalent to y in case that some other event B, alternative to A, prevails, then k is the certainty equivalent of y if either A or B occurs. If the two events are not disjoint, it is straightforward to show that the certainty equivalent on $A \cup B$ is the same as on the single events if it is the same if both A and B occur, namely $C_{A\cup B}(y) = C_A(y) = C_B(y)$ if and only if $C_{A\cap B}(y) = C_A(y) = C_B(y)$ as well.

(iii) The additivity property states that if a random variable can be split into the sum of two (or more) *orthogonal* random variables, then F is additively decomposable with respect to the certainty equivalents of the addenda of the sum of the random variables. Furthermore, the certainty equivalent is additively decomposable as well if F (or equivalently the Debreu's functions u_i) is linear.

Proposition 3 highlights some important properties of the conditional certainty equivalent, that allow the decision maker to exploit the knowledge that some event will prevail to evaluate a random amount according to her/his preferences. This reminds the notion of conditional expectation, where the knowledge that some event has occurred is exploited to calculate the expectation of a random variable. The two concepts are indeed strictly connected and share many similarities, starting from the definition of conditional certainty equivalent, that resembles the definition of the *conditional expected value* of a random variable with respect to a given algebra, where the functional F replaces the expectation[2]. Both the conditional certainty equivalent and the conditional expected value have the standard properties of monotonicity and non-negativity. In finite sets, algebrae and partitions are in one-to-one correspondence (see also Section 6), and the conditional expected value of a random variable simply turns out to be its "piecewise constant" approximation obtained by replacing its values with the conditional expected values on the minimal events of the given partition, namely $E[y|\mathcal{A}] = \sum_{j=1}^{n} c_{A_j}(y)$ where $c_{A_j}(y) = E[y|A_j]$. The conditional certainty equivalent behaves exactly in the same way in finite state spaces, as the associative property shows. Moreover, if $\mathcal{A}'$ is a less fine partition less than $\mathcal{A}$ (that is the algebra generated by $\mathcal{A}'$ is a subset of the algebra generated by $\mathcal{A}$), the conditional certainty equivalent of $y|\mathcal{A}$ with respect to the partition $\mathcal{A}'$ coincides with $y|\mathcal{A}'$, mimicking the law of iterated expectation. The compatibility property resembles another property of the conditional expectation: if a random variable has the same conditional expectation on two (or more) disjoint events ($E[y|A] = E[y|B]$ where A and B are disjoint), then it has the same conditional expectation on the union of these events, namely $E[y|A \cup B] = E[y|A] = E[y|B]$. Lastly, though the certainty equivalent, unlike the conditional expectation, is not a linear operator, the additivity property is reminiscent of the property of conditional expectation according to which, given a function u such that $u(0) = 0$, and two random variables x, y such that $x(\omega) \cdot y(\omega) = 0$ for all $\omega \in \Omega$, $\mathrm{E}\left[u(x+y) \mid A\right] = \mathrm{E}\left[u(x) \mid A\right] + \mathrm{E}\left[u(y) \mid A\right]$ for all $A \subseteq \Omega$.

We illustrate the concept of conditional certainty equivalent and the properties stated in Proposition 3 by means of some examples.

Example 3. Consider again the same setting of the previous examples. Let $y = [100 \;\; 200 \;\; 300 \;\; 400 \;\; 500 \;\; 600]^{\mathrm{T}}$; then $F(y) = 9,100$.

1. The certainty equivalent $C(y)$ of y is such that

$$9{,}100 = F(y) = u_\Omega(C(y)) = \sum_{i=1}^{m} u(C(y); i) = 21C(y) \;,$$

   which gives $C(y) = 433.\overline{3}$.

2. Consider the subset $A = \{1, 2\} \subset \Omega$. For the deterministic amount $k = 150$, it is

$$u(150; A) = u(150; 1) + u(150; 2) = u_1(150) + u_2(150) = 450 \;;$$

[2]For sake of the reader, we recall that the *expected value* $y \in \mathcal{X}$ *conditional to the event* $A \subseteq \Omega$, $E[y|A]$, is the constant $c_A(y) \in \mathbb{R}$ such that $E[y\mathbb{I}_A] = E[c_A(y)\mathbb{I}_A]$.

conversely, the certainty equivalent $C_A(y)$ conditional to A is such that

$$450 = u_A(C_A(y)) = \sum_{i \in A} u(C_A(y); i) = 3C_A(y) \ ,$$

which gives $C_A(y) = 150$.

3. As for the associative property, consider the partition $\mathcal{A} = \{A_1, A_2, A_3\}$ of Ω with $A_1 = \{1\}$, $A_2 = \{2, 3\}$, $A_3 = \{4, 5, 6\}$. It is straightforward that

$$C_{A_1}(y) = 100 \ , \quad C_{A_2}(y) = 260 \ , \quad C_{A_3}(y) = 513.\overline{3}$$

and therefore the conditional random variable $y|\mathcal{A}$ turns out to be

$$y|\mathcal{A} = [100 \ \ 260 \ \ 260 \ \ 513.\overline{3} \ \ 513.\overline{3} \ \ 513.\overline{3}]^{\mathrm{T}} \ .$$

Note that:

$$F(y|\mathcal{A}) = 100 + (2+3) \cdot 260 + (4+5+6) \cdot 513.\overline{3} = 9{,}100 = F(y) \ ,$$

which precisely shows that $y|\mathcal{A} \sim y$.

Moreover, let $\mathcal{A}' = \{A_1 \cup A_3, A_2\}$. Then:

$$C_{A_1 \cup A_3}(y) = C_{A_1 \cup A_3}(y|\mathcal{A}) = 487.5 \ , \quad C_{A_2}(y) = C_{A_2}(y|\mathcal{A}) = 260 \ ,$$

namely $(y|\mathcal{A})|\mathcal{A}' = y|\mathcal{A}'$.

4. Let us now look at the compatibility property. Consider the random variable $z = [2 \ \ 2 \ \ 5 \ \ 17 \ \ -10 \ \ 12]^{\mathrm{T}}$ and the subsets $A = \{1, 2\}$, $B = \{5, 6\}$, $F = \{1, 5, 6\}$, and $G = \{1, 4, 5\}$ of Ω. It is

$$\begin{aligned}
A \cap B = \varnothing \ &: \quad C_A(z) = C_B(z) = 2 = C_{A \cup B}(z) \ ; \\
A \cap F \neq \varnothing \ &: \quad C_A(z) = C_F(z) = 2, \quad C_{A \cup F}(z) = 2 = C_{A \cap F}(z) \ ; \\
F \cap G \neq \varnothing \ &: \quad C_F(z) = C_G(z) = 2, \quad C_{F \cup G} = 5.75 \neq -8 = C_{F \cap G} \ .
\end{aligned}$$

5. Finally, for the additivity property, consider the random variables $r = [20 \ \ 0 \ \ 4 \ \ 0 \ \ 10 \ \ 0]^{\mathrm{T}}$ and $s = [0 \ \ 5 \ \ 0 \ \ 2 \ \ 0 \ \ 4]^{\mathrm{T}}$ with disjoint support: it is

$$\begin{gathered}
C_B(r) = 4.\overline{54}, \quad C_B(s) = 2.\overline{18} \ ; \\
u_B(C_B(r+s)) = 74 = 50 + 24 = u_B(C_B(r)) + u_B(C_B(s)) \ ; \\
C_B(r+s) = 6.\overline{72} = C_B(r) + C_B(s) \ .
\end{gathered}$$

Lastly, consider the non linear $u_i(y_i) = \sqrt{y_i}$, $i = 1, 2, \ldots, 6$. It is:

$$\begin{gathered}
C_B(r) = 2.5, \quad C_B(s) = 1 \ ; \\
u_B(C_B(r+s)) = \sqrt{10} + 2 = u_B(C_B(r)) + u_B(C_B(s)) \ ; \\
C_B(r+s) = \sqrt{10} + 3.5 \neq 3.5 = C_B(r) + C_B(s) \ .
\end{gathered}$$

6 Debreu's Theorem and incomplete preferences

An *incomplete preference* $\succcurlyeq$ is a reflexive and transitive binary relation on $\mathcal{X}$ (see, among others, Karni and Zhou, 2021).

Proposition 5. *If the (incomplete) preference $\succcurlyeq$ is monotonic, then there exists at least a maximal algebra $\mathcal{A}$ of Ω such that the restriction of $\succcurlyeq$ to the $\mathcal{A}$-measurable random variables is complete.*

Proof. Start by noticing that, if $\succcurlyeq$ is complete when restricted to a algebra, it maintains completeness when restricted to coarser algebrae: therefore, it is sensible and natural to look for a maximal one. Note, furthermore, that $\succcurlyeq$ is of course complete if restricted to the trivial algebra $\{\varnothing, \Omega\}$: in such a case, indeed, the only measurable random variables are the degenerate (*i.e.*, constant) ones, and completeness follows from the monotonicity of $\succcurlyeq$. Therefore, at least one maximal algebra with the required property is bound to exist. (A straightforward application of Zorn's Lemma ensures that the thesis holds even for non-finite state spaces.) □

Remark 6. In this section, we use for denoting algebrae the same symbol $\mathcal{A}$ used to indicate partitions in Section 5. This is due to the fact that, being Ω a finite set, algebrae on Ω and partitions of Ω are in one-to-one correspondence. Namely, the minimal nonempty elements of an algebra $\mathcal{A}$ on Ω turn out to form a partition of Ω; conversely, given a partition $\{A_1, A_2, \ldots, A_n\}$, the 2^n possible (disjoint) unions of the n "parts" turn out to make an algebra on Ω (whose minimal elements are exactly $A_1, A_2, \ldots, A_n$).

Furthermore, the conditional decomposition $y|\mathcal{A}$ defined above with respect to a partition $\mathcal{A}$ is, as a random variable, measurable with respect to the algebra generated by $\mathcal{A}$. It is therefore perfectly sensible to use the same notation $y|\mathcal{A}$ also for the case when $\mathcal{A}$ is an algebra, by simply considering the sets $A_1, A_2, \ldots, A_n$ involved in the definition of $y|\mathcal{A}$ to be the nonempty minimal sets of the algebra $\mathcal{A}$.

Let us fix a maximal $\mathcal{A}$ such that the monotonic preference $\succcurlyeq$ is complete when restricted to $\mathcal{A}$-measurable random variables, and recall that a random variable $x \in \mathcal{X}$ is $\mathcal{A}$-measurable if and only if (it takes constant values over the minimal events of $\mathcal{A}$, *i.e.*, if and only if) it can be written as $x = \sum_{j=1}^{n} x_j \mathbb{I}_{A_j}$, with $\{A_1, A_2, \ldots, A_n\}$ the unique partition generating $\mathcal{A}$. If $\succcurlyeq$ is continuous and agrees with the Sure Thing Principle, Debreu's Theorem ensures that it can be represented by a function F which can be written, according to the "conditional" representation seen in Section 5, as

$$F(x) = \sum_{j=1}^{n} u(x_j; A_j).$$

Consider any $y \in \mathcal{X}$, and define the two subsets of $\mathcal{X}$

$$Y^* := \{z \in \mathcal{X} : z \text{ is } \mathcal{A}\text{-measurable}, \; z \geqq y\}$$

$$Y_* := \{z \in \mathcal{X} : z \text{ is } \mathcal{A}\text{-measurable}, \ z \leqq y\}$$

We could say that the elements of Y^* and Y_* *super*- and *sub*-replicate y respectively. Since Ω is finite, both sets Y^* and Y_* are not empty (because y only takes finitely many values and such sets contain at least the constant random variables equal, respectively, to the maximum and to the minimum value taken by y). Therefore, we can define

$$F^*(y) := \inf_z \{F(z) : z \in Y^*\} \quad , \qquad F_*(y) := \sup_z \{F(z) : z \in Y_*\} \ .$$

Moreover for every $y^* \in Y^*$ and every $y_* \in Y_*$ it is $y^* \geqq y_*$, so that it is always $F(y^*) \geqslant F(y_*)$ and, therefore, $F^*(y) \geqslant F_*(y)$. Roughly speaking, then, $F^*(y)$ and $F_*(y)$ are an upper and a lower measure for y, deduced from the "complete" restriction of $\succcurlyeq$. These measures associate to every $y \in \mathbb{R}^m$ the interval $[F_*(y), F^*(y)]$, which of course collapses into a singleton if y is $\mathcal{A}$-measurable.

Since Ω is finite, $F^*(y)$ and $F_*(y)$ are in fact attained (hence they are respectively a minimum and a maximum). We can indeed define $y^* = \sum_{j=1}^n y_j^* I_{A_j}$ and $y_* = \sum_{j=1}^n y_{*j} I_{A_j}$ where

$$y_j^* = \max_{\omega_i \in A_j} y_i \qquad\qquad y_{*j} = \min_{\omega_i \in A_j} y_i.$$

It is clear that $y^* \in Y^*$ and $z \geqq y^*$ for all $z \in Y^*$. Similarly, $y_* \in Y_*$ and $z \leqq y^*$ for all $z \in Y_*$. As a result, $F^*(y) = F(y^*)$ and $F_*(y) = F(y_*)$.

Such measures provide a criterion to "extend" the preference $\succcurlyeq$ by setting $x \succcurlyeq' y$ whenever $F_*(x) \geqslant F^*(y)$. Suppose indeed that $F_*(x) > F^*(y)$: then, there exist $x_* \in X_*$ and $y^* \in Y^*$ such that $x \geqq x_* \succcurlyeq y^* \geqq y$, which makes natural to derive that x should be preferred to y: $x \succcurlyeq' y$ implies $x \succcurlyeq y$.

Note that the definition of F^* and F_* can be applied to infinite state spaces as well, although in such a case the inf and sup no longer need to be a min and a max. Setting $x \succcurlyeq' y$ whenever $F_*(x) \geqslant F^*(y)$ remains a good definition, as it is immediate to realise by applying the properties of the inf and the sup.

Remark 7. This criterion can be formulated also in terms of certainty equivalent. Let indeed:

$$C^*(y) = \inf_z \{C(z) : z \in Y^*\} = C(y^*), \quad C_*(y) = \sup_z \{C(z) : z \in Y_*\} = C(y_*).$$

In some sense, decision makers cannot decide between getting y or any of the certain amounts in the interval $[C_*(y), C^*(y)]$. However, they prefer x to y if $C_*(x) \geqslant C^*(y)$.

We used inverted commas when stating that $\succcurlyeq'$ extends $\succcurlyeq$ because, although it is clear that the definition above may actually make possible for the decision maker to choose among random variables that were not comparable in the original preference $\succcurlyeq$, the new preference $\succcurlyeq'$ obtained this way might not be compatible with the original one. For such a compatibility to hold, it should be $F_*(x) \geqslant F^*(y)$ whenever $x \succcurlyeq y$, but unfortunately this is not the case, as the following example shows.

Example 4. On $\mathbb{R}^3$, define the preference

$$x \succcurlyeq y \qquad \Longleftrightarrow \qquad \begin{cases} x_1 + x_2 \geqslant y_1 + y_2 \\ x_1 + x_3 \geqslant y_1 + y_3 \end{cases} .$$

Such a preference is plainly monotonic and reflexive; its definition makes quite patent that it is continuous as well. Since $[1\ 0\ 0]^{\mathrm{T}}$, $[0\ 1\ 0]^{\mathrm{T}}$ and $[0\ 0\ 1]^{\mathrm{T}}$ are all $\succ [0\ 0\ 0]^{\mathrm{T}}$, all of the states in Ω are essential.

It is straightforward to realise that $\succcurlyeq$ agrees with the Sure Thing Principle. Let us start with an example: if $A = \{\omega_1, \omega_3\}$, then $xAz = [x_1\ \ z_2\ \ x_3]^{\mathrm{T}}$ and $yAz = [y_1\ \ z_2\ \ y_3]^{\mathrm{T}}$: therefore, $xAz \succcurlyeq yAz$ if and only if

$$\begin{cases} x_1 + z_2 \geqslant y_1 + z_2 \\ x_1 + x_3 \geqslant y_1 + y_3 \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} x_1 \geqslant y_1 \\ x_1 + x_3 \geqslant y_1 + y_3 \end{cases}$$

and it is plain to see that, in such a case, $xAw \succcurlyeq yAw$ for every $w \in \mathbb{R}^3$. It should be now evident that, whatever $A \subseteq \Omega$ is, a comparable situation arises: the components relative to the states $\omega_i \notin A$ simply "cancel out" of the inequalities that define $\succcurlyeq$ and, therefore, do not matter in determining the preference among the resulting random variables.

The preference $\succcurlyeq$ is not complete: for instance, $x = [1\ 0\ 1]^{\mathrm{T}}$ and $y = [1\ 1\ 0]^{\mathrm{T}}$ are not comparable, because $x_1 + x_2 = 1 \leqslant 2 = y_1 + y_2$ and $x_1 + x_3 = 2 \geqslant 1 = y_1 + y_3$. However, $\succcurlyeq$ becomes complete if restricted to random variables measurable with respect to $\mathcal{A}_1 = \big\{\varnothing, \{\omega_1\}, \{\omega_2, \omega_3\}, \Omega\big\}$, because when $x_2 = x_3$ and $y_2 = y_3$ the two inequalities above become a single one.

Consider now

$$\begin{aligned} x &= [1\ \ {-4}\ \ 4]^{\mathrm{T}} \ : \qquad x_1 + x_2 = -3 \ , \quad x_1 + x_3 = 5 \ ; \\ y &= [4\ \ {-3}\ \ 2]^{\mathrm{T}} \ : \qquad y_1 + y_2 = 1 \ , \quad\ \ y_1 + y_3 = 6 \ . \end{aligned}$$

It is $y \succ x$. The $\mathcal{A}_1$-measurable "upper and lower approximations"of x and y, respectively, are:

$$\begin{aligned} x_* &= [1\ \ {-4}\ \ {-4}]^{\mathrm{T}} \ , \quad x^* = [1\ \ 4\ \ 4]^{\mathrm{T}} \ ; \\ y_* &= [4\ \ {-3}\ \ {-3}]^{\mathrm{T}} \ , \quad y^* = [4\ \ 2\ \ 2]^{\mathrm{T}} \end{aligned}$$

and $x^* \succ y_*$ (because $1 + 4 = 5 > 1 = 4 - 3$), so it cannot be $F_*(y) > F^*(x)$.

Indeed, it is not difficult to see that the complete preference on A_1 is represented by the function $F(x) = x_1 + x_2$ where $x = [x_1\quad x_2\quad x_2]^{\mathrm{T}}$, and the certainty equivalent for such random variable is $C(x) = \frac{x_1 + x_2}{2}$, hence $F_*(y) = 1$, $F^*(x) = 5$, $C_*(y) = 1/2$, and $C^*(x) = 5/2$.

Note that $\mathcal{A}_1$ is indeed the unique nontrivial algebra on Ω which makes the preference complete upon restriction. The only two other nontrivial algebrae of Ω are indeed $\mathcal{A}_2 = \big\{\varnothing, \{\omega_1, \omega_3\}, \{\omega_2\}, \Omega\big\}$ and $\mathcal{A}_3 = \big\{\varnothing, \{\omega_1, \omega_2\}, \{\omega_3\}, \Omega\big\}$. The two random variables $x'' = [2\ \ 0\ \ 2]^{\mathrm{T}}$ and $y'' = [1\ \ 5\ \ 1]^{\mathrm{T}}$ are $\mathcal{A}_2$-measurable and they are not comparable with respect to $\succcurlyeq$, because $x''_1 + x''_2 = 2 < 6 = y''_1 + y''_2$, but $x''_1 + x''_3 = 4 > 2 = y''_1 + y''_3$. Analogously, $x''' = [2 \ 2 \ 0]^{\mathrm{T}}$ and $y''' = [1 \ 1 \ 5]^{\mathrm{T}}$ are $\mathcal{A}_3$-measurable and $x'''_1 + x'''_2 = 4 > 2 = y'''_1 + y'''_2$, but $x'''_1 + x'''_3 = 2 < 6 = y'''_1 + y'''_3$.

Remark 8. In the classical identification of an algebra of Ω with a certain state of information of the decision makers, completeness of $\succcurlyeq$ with respect to $\mathcal{A}$ amounts to saying that decision makers can choose among random variables which are "compatible" with respect to their state of information, but they cannot make a choice between random variables that require more information to be fully "understood".

In such a situation, decision makers can nevertheless give a minimal and a maximal evaluation of the undecidable random variables with a kind of "worst case/best case" approach, and of course they can safely assume that y is better than x whenever the worst case of y is better than the best case of x.

Finally, note that the maximal algebra $\mathcal{A}$ need not be unique and that, in principle, the extensions of $\succcurlyeq$ built upon the restrictions to two different maximal algebrae cannot be guaranteed to be compatible with each other. In such a case, a more restrictive "extension" of the preference might be defined, by saying that y is preferred to x if the worst cases of y according to the two (or more) maximal algebrae turn *all* to be better than the best cases of x according to the two (or more) maximal algebrae.

7 Conclusions

The present paper examined some consequences of a theorem by G. Debreu, based upon the *Sure Thing Principle*.

The result proved by Debreu is that a complete, monotonic and continuous preference satisfies the Sure Thing Principle if and only if it can be represented by a real-valued functional, which turns out to be additively decomposable with respect to the "states of the world". Thus the functional turns out to be a *state-dependent utility*. We highlight that Debreu's representation induces a (joint) measure on the product space $\Omega \times \mathbb{R}$, thus simultaneously evaluating events *and* values taken by the random variables. If the probability measure on Ω is assigned, it is always possible to pretend the joint measure to be the product of such a probability by a state-dependent utility function. On the other hand, it is also possible to assign the other "marginal measure", that is, the utility function on $\mathbb{R}$, thus looking at Debreu's representation as an expected utility with respect to an "utility-dependent probability". Moreover, Debreu's representation can be read as the probability of outperforming a given benchmark that, this time, need not be stochastically independent of the random variables under consideration. More precisely, the benchmark turns out to be independent from the random variables if and only if the measure induced by the Debreu's representation is a product measure.,

Furthermore, the statewise decomposability of Debreu's representation can be exploited to introduce, in quite a natural way, the concept of a *conditional* *certainty equivalent* with respect to an event $A \subseteq \Omega$, defined to be the constant amount that, if A prevails, the decision maker deems equally desirable as the random variable. Suitable properties hold.

Section 6 discusses possible connections between Debreu's Theorem and incomplete preferences. Given any incomplete preference $\succcurlyeq$, there has to be at least a maximal algebra $\mathcal{A}$ on Ω such that $\succcurlyeq$ is complete over $\mathcal{A}$-measurable random variables and, therefore, such a restriction can be represented by means of a Debreu's functional. Given a random variable y in the original space, it is then reasonable to take into consideration its upper and lower $\mathcal{A}$-measurable approximations and to define an "upper" and "lower" measure of y as the the values taken on such approximations by the Debreu's representation of the complete restriction of $\succcurlyeq$. Such a definition, of course, induces a new preference, call it $\succcurlyeq'$, by setting $x \succcurlyeq' y$ whenever the lower measure of x is greater than the upper measure of y; unfortunately, while $x \succcurlyeq' y$ implies that $x \succcurlyeq y$, the opposite implication does not hold. The problem of finding an extended representation for incomplete preferences remains therefore open and stimulating.

Further research could take again into consideration all of the possible (σ-)algebrae which make $\succcurlyeq$ complete by restriction, and then to "paste together" the resulting representations, to be seen as "projections" *à la* Rumbos (2001).

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