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Complex Dynamics inside Fatou Sets

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Complex Dynamics inside Fatou Sets

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Abstract

This thesis investigates the behavior of orbits inside Fatou sets in one dimension and higher dimensions.

Suppose $f(z)$ is a polynomial of degree $N \geq 2$ on $\mathbb{C}$, p is an attracting fixed point of $f(z)$, Ω_1 is the immediate basin of attraction of p , $\{f^{-1}(p)\} \cap \Omega_1 \neq \{p\}$, $\mathcal{A}(p)$ is the basin of attraction of p , $\Omega_i (i = 1, 2, \dots)$ are the connected components of $\mathcal{A}(p)$. Then there is a constant C so that for every point z_0 inside any Ω_i , there exists a point $q \in \cup_k \{f^{-k}(p)\}$ inside Ω_i such that $d_{\Omega_i}(z_0, q) \leq C$, where d_{Ω_i} is the Kobayashi distance on Ω_i . This result implies that the orbit of z_0 is tracked by the orbit of some point $q \in \cup_k \{f^{-k}(p)\}$.

In addition, we prove a variety of different results related to the same questions about attracting basins for rational functions on $\hat{\mathbb{C}}$ and entire functions on $\mathbb{C}$, parabolic basins of holomorphic polynomials on $\mathbb{C}$, and finally, focus on attracting basins for some holomorphic polynomial maps on $\mathbb{C}^2$.

Estratto

Questa tesi indaga il comportamento delle orbite all'interno degli insiemi di Fatou in una dimensione e in dimensioni superiori.

Supponiamo che $f(z)$ sia un polinomio di grado $N \geq 2$ su $\mathbb{C}$, p è un punto fisso attrattivo di $f(z)$, Ω_1 è l'immediata basin di attrazione di p , $\{f^{-1}(p)\} \cap \Omega_1 \neq \{p\}$, $\mathcal{A}(p)$ è il bacino di attrazione di p , $\Omega_i (i = 1, 2, \dots)$ sono le componenti connesse di $\mathcal{A}(p)$. Allora esiste una costante C tale che per ogni punto z_0 all'interno di qualsiasi Ω_i , esiste un punto $q \in \cup_k \{f^{-k}(p)\}$ all'interno di Ω_i tale che $d_{\Omega_i}(z_0, q) \leq C$, dove d_{Ω_i} è la distanza di Kobayashi su Ω_i . Questo risultato implica che viene tracciata l'orbita di z_0 dall'orbita di un punto $q \in \cup_k \{f^{-k}(p)\}$.

Inoltre, dimostriamo numerosi risultati diversi relativi all'attrazione di bacini per funzioni razionali su $\hat{\mathbb{C}}$ e funzioni intere su $\mathbb{C}$, bacini parabolici di polinomi olomorfi su $\mathbb{C}$ e, infine, ci concentriamo sull'attrazione di bacini per alcune mappe polinomiali olomorfe su $\mathbb{C}^2$.

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Preface

This thesis is submitted in partial fulfillment of the requirements for *Philosophiae Doctor* degree at the University of Parma. The research presented here was conducted at the University of Parma under the supervision of Professor John Erik Fornaess.

The thesis consists of four papers, listed in two categories: one dimension Paper I, Paper II, Paper III, and higher dimension Paper IV, and provides background information and motivation for the work and a summary of each paper in the introductory chapter.

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List of Papers

Paper I

Mi Hu: Interior Dynamics of Fatou Sets. *Annali di Matematica Pura ed Applicata* (1923 -), **202**, 2903–2914 (2023),
doi:10.1007/s10231-023-01344-9.

Paper II

John Erik Fornæss, Mi Hu: Orbits inside Fatou sets. Preprint (2023), available online at doi:10.48550/arXiv.2309.08334.

Paper III

Mi Hu: Dynamics inside Parabolic Basins. Preprint (2022), available online at doi:10.48550/arXiv.2208.03756.

Paper IV

Mi Hu: Dynamics inside Fatou sets in higher dimensions. *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze*, 2023.
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Chapter 1

Introduction

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1.1 Introduction

This thesis consists of four papers, listed in two categories:

1. One Dimension: Paper I, II, III.
2. Higher Dimension: Paper IV.

First, I will provide background information and motivation for the work. I will also give a summary of each paper, mainly presenting the Theorems, the ideas of the proofs, and the main difficulties in the proofs.

1.2 Background

A general goal in discrete dynamical systems is to qualitatively and quantitatively describe the possible dynamical behavior under the iteration of maps. The first two papers on complex dynamics were published by the German mathematician Ernst Schröder [36, 37] in 1870 and 1871. In 1917, Fatou [17, 18] and Julia [29] proved some results related to the iteration of rational functions of one complex variable in the Comptes Rendus of the French Academy of Sciences. More than sixty years later, Sullivan [39] proved the non wandering domains theorem, which complete the Fatou classification theorem. There also have been many contributions from famous researchers [1, 5, 6, 10, 12–15, 22, 25, 28, 33, 40, 41] on topics and applications in complex dynamics. About more background contents, we refer to the books [4, 5, 12, 34, 38]. Here, we only introduce some definitions which are used in our papers.

1.2.1 Complex dynamics in one dimension

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a nonconstant holomorphic map, and $f^n : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be its n -fold iterate.

Definition 1.2.1. [34] For $z \in \hat{\mathbb{C}}$, the set

$$\{z_n\} = \{z_1 = f(z_0), z_2 = f^2(z_0), \dots\}$$

is called the **orbit of the point** $z = z_0$.

If $z_N = z_0$ for some integer N , we say that z_0 is a **periodic point** of f with period N . If $N = 1$, we say that z_0 is a **fixed point** of f .

Definition 1.2.2. [12] Let z_0 be a fixed point of f . The number $\lambda = f'(z_0)$ is called the **multiplier** of f at z_0 . We classify fixed points according to λ as follows:

If $0 < |\lambda| < 1$, we call z_0 an **attracting fixed point**;

If $|\lambda| = 0$, we call z_0 a **superattracting fixed point**;

If $|\lambda| = 1$ and $\lambda^n = 1$ for some integer n , we call z_0 a **rationally neutral fixed point**;

If $|\lambda| = 1$ and $\lambda^n \neq 1$ for any integer n , we call z_0 an **irrationally neutral fixed point**;

If $|\lambda| = 1$, $\lambda^n = 1$ for some integer n , and no iterate of f is the identity map, we call z_0 a **parabolic fixed point**;

If $|\lambda| > 1$, we call z_0 a **repelling fixed point**.

In addition, for the periodic orbit, we can use the chain rule to classify the attracting, neutral or repelling behaviour in the following:

Suppose $\{z_1, z_2, \dots, z_n\}$ is a periodic orbit of order n , then it is attracting, neutral, or repelling depends on whether $|(f^n)'(z_1)| = \prod_{i=1}^n |f'(z_i)|$ is less than 1, equal to 1, or greater than 1, respectively.

Definition 1.2.3. [34] A collection of maps $\{\mathcal{F}\} : U^{Open} \subset \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ is called a **normal family** if every infinite sequence of maps from $\mathcal{F}$ contains a subsequence which converges locally uniformly to a map from $U \rightarrow \hat{\mathbb{C}}$.

In complex dynamics, two crucial disjoint invariant sets are associated with f , the **Julia set** and the **Fatou set** [34], which partition the sphere $\hat{\mathbb{C}}$.

Definition 1.2.4. [34] The **Fatou set** of f is defined as the largest open set where the family of iterates is normal. The complement of the Fatou set is called the **Julia set**.

Definition 1.2.5. [34] Let p be an attracting periodic point of period N , we define the **basin of attraction** to be the open set $\mathcal{A}(p) \subset \hat{\mathbb{C}}$ consisting of all points $z \in \hat{\mathbb{C}}$ such that the iterates $f^N(z), f^{2N}(z), \dots$ converge locally uniformly to p .

The **immediate basin of attraction** is defined to be the connected component of $\mathcal{A}(p)$ which contains p and is denoted by $\mathcal{A}_0$.

Let us also recall some definitions and results about holomorphic dynamics of a polynomial $f = z + a_2 z^2 + a_3 z^3 + \dots$ in a neighborhood of the parabolic fixed point 0.

Definition 1.2.6. [34] Let $f(z) = z + az^{m+1} + (\text{higher terms})$, $m \geq 1, a \neq 0$. A complex number $\mathbf{v}$ will be called an **attraction vector** at the origin if $m\mathbf{a}\mathbf{v}^m = -1$, and a **repulsion vector** at the origin if $m\mathbf{a}\mathbf{v}^m = 1$. Note here that $\mathbf{v}$ should be thought of as a tangent vector at the origin. We say that some orbit $z_0 \mapsto z_1 \mapsto z_2 \mapsto \dots$ for the map f converges to zero **nontrivially** if $z_n \rightarrow 0$ as $n \rightarrow \infty$, but no z_n is actually equal to zero. There are m equally spaced attraction vectors at the origin.

Lemma 1.2.7. [34] If an orbit $f : z_0 \mapsto z_1 \mapsto \dots$ converges to zero nontrivially, then z_k is asymptotic to $\mathbf{v}_j / \sqrt[m]{k}$ as $k \rightarrow +\infty$ for one of the m attraction vectors $\mathbf{v}_j$.

Proof. See the proof in chapter 10 of Milnor's book [34]. □

Definition 1.2.8. [34] If an orbit $z_0 \mapsto z_1 \mapsto \cdots$ under f converges to zero with $z_k \sim \mathbf{v}_j / \sqrt[k]{k}$, we say that this orbit $\{z_k\}$ **tends to zero from the direction $\mathbf{v}_j$** .

Definition 1.2.9. [34] Given an attraction vector $\mathbf{v}_j$ in the tangent space of $\mathbb{C}$ at 0, the associated **parabolic basin of attraction** $\mathcal{A}_j = \mathcal{A}(0, \mathbf{v}_j)$ is defined to be the set consisting of all $z_0 \in \mathbb{C}$ for which the orbit $z_0 \mapsto z_1 \mapsto \cdots$ converges to 0 from the direction $\mathbf{v}_j$.

Definition 1.2.10. [34] Suppose f is defined and univalent on some neighborhood N of 0. An open set $\mathcal{P}_j \subset N$ is called an **attracting petal** for f for the vector $\mathbf{v}_j$ at 0 if

1. f maps $\mathcal{P}_j$ into itself, and
2. an orbit $z_0 \mapsto z_1 \mapsto \cdots$ under f is eventually absorbed by $\mathcal{P}_j$ if and only if it converges to 0 from the direction $\mathbf{v}_j$.

Definition 1.2.11. [34] The connected components of the Fatou set of f are called **Fatou components**. A Fatou component $\Omega \subset \hat{\mathbb{C}}$ of f is said to be **invariant** if $f(\Omega) = \Omega$.

At the beginning of the 20th century, Fatou [19–21] classified all possible invariant Fatou components of rational functions on the Riemann sphere. He proved that only the following three cases could occur.

Theorem 1.2.12 (The Fatou classification). [12, 34] Suppose Ω is an invariant Fatou component. Then one of the following possibilities occur:

1. **(Attracting Basin)** Ω contains a fixed point p , and the orbit of every point in Ω converges to p .
2. **(Parabolic Basin)** $\partial\Omega$ contains a fixed point p , and the orbit of every point in Ω converges to p .
3. **(Rotation Domain)** Ω is conformally equivalent to the unit disk or an annulus, and the map is conjugate to an irrational rotation.

The classification of Fatou components was completed in the 1980s when Sullivan proved the non-existence of non wandering domains for rational functions on $\hat{\mathbb{C}}$.

Theorem 1.2.13 (Non wandering domains). [39] Every Fatou component of a rational map is **preperiodic**, i.e., there are $n, m \in \mathbb{N}$ such that $f^{n+m}(\Omega) = f^m(\Omega)$.

For more details and results, we refer the reader to [39].

Definition 1.2.14. [30, 34] Let $\Omega_1, \Omega_2 \subset \mathbb{C}$. Then Ω_1 and Ω_2 are **conformally isomorphic** if and only if there is a homeomorphism from Ω_1 onto Ω_2 which is holomorphic.

Theorem 1.2.15 (The Riemann mapping theorem). [30] Let Ω be a simply connected non-empty open subset of $\mathbb{C}$ and $\Omega \neq \mathbb{C}$. Then Ω is conformally equivalent to the open unit disc $\Delta(0, 1)$.

Definition 1.2.16. [12, 26, 34] Let $e^{i\theta}$ be a complex number on the unit circle $\partial\Delta$, and $a_1, a_2, \dots, a_m \in \Delta$ be complex numbers. Then

$$g(z) = e^{i\theta} \prod_{j=1}^m \frac{z - a_j}{1 - \bar{a}_j z}$$

is called a **Blaschke product** of degree m .

1.2.2 Complex dynamics in higher dimensions

Compared with one dimension, there are very interesting results about dynamics inside attracting basins in higher dimensions. Let $F(z, w) = (P(z, w), Q(z, w))$, where P, Q are two polynomials of degree $m_1, m_2 \geq 2$ on $\mathbb{C}$, and $F^n : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be its n -fold iterate.

Definition 1.2.17. [23, 24] For $(z, w) \in \mathbb{C}^2$, the set $\{(z_n, w_n)\} = \{(z_1, w_1) = F(z_0, w_0), (z_2, w_2) = F^2(z_0, w_0), \dots\}$ is called the **orbit of the point** $(z, w) = (z_0, w_0)$.

If $(z_N, w_N) = (z_0, w_0)$ for some integer N , we say that (z_0, w_0) is a **periodic point** of F of period N .

If $N = 1$, then (z_0, w_0) is a **fixed point** of F .

In complex dynamics, two crucial disjoint invariant sets are associated with F , the **Julia set** and the **Fatou set**, which partition $\mathbb{C}^2$.

Definition 1.2.18. [23, 24] The **Fatou set** of F is defined as the largest open set where the family of iterates is locally normal. In other words, for any point $(z, w) \in \mathbb{C}^2$, there exists some neighborhood U of (z, w) so that the sequence of iterates of the map restricted to U forms a normal family, so the iterates are well behaved.

The complement of the Fatou set is called the **Julia set**.

Definition 1.2.19. [40] Let F be a holomorphic map with a fixed point $(z_0, w_0) \in \mathbb{C}^2$. We say that

1. (z_0, w_0) is an **attracting fixed point** of F if both the eigenvalues of the Jacobian matrix $F'((z_0, w_0))$ are less than 1 in absolute value.
2. (z_0, w_0) is a **repelling fixed point** of F if both the eigenvalues of the Jacobian matrix $F'((z_0, w_0))$ are bigger than 1 in absolute value.
3. (z_0, w_0) is a **semi-parabolic fixed point** of F if one of the eigenvalues of the Jacobian matrix $F'((z_0, w_0))$ is equal to 1 and the other one is less than 1 in absolute value.
4. (z_0, w_0) is a **semi-repelling fixed point** of F if one of the eigenvalues of the Jacobian matrix $F'((z_0, w_0))$ is equal to 1 and the other one is bigger than 1 in absolute value.

5. (z_0, w_0) is a **saddle fixed point** of F if one of the eigenvalues of the Jacobian matrix $F'((z_0, w_0))$ is less than 1 and the other one is bigger than 1 in absolute value.
6. (z_0, w_0) is a **parabolic fixed point** of F if F is tangent to the identity.

Definition 1.2.20. [6, 23, 40] Let (z_0, w_0) be a fixed point of $F(z, w)$. The **basin of** (z_0, w_0) is defined to be the largest open set $\mathcal{A}(z_0, w_0) \subset \mathbb{C}^2$ on which the iterates converge uniformly on compact subsets to (z_0, w_0) .

If (z_0, w_0) is an attracting fixed point, then the **basin of** (z_0, w_0) is called the **basin of attraction**. Similarly, for the **semi-parabolic basin** and the **parabolic basin**.

1.2.2.1 Polynomial skew products

Definition 1.2.21. [28, 32, 35] A polynomial skew product on $\mathbb{C}^2$ is a map of the form $F(z, w) = (P(z), Q(z, w))$, where $P(z)$ and $Q(z, w)$ are polynomials.

1.2.3 The Euclidean distance

Let $d_E(z_1, z_2) := |z_1 - z_2|$ denote the Euclidean distance between any two points $z_1, z_2 \in \mathbb{C}$. In this thesis, we will mostly use a more refined distance, see the next subsection.

1.2.4 The Kobayashi metric

In this thesis, we basically use the Kobayashi metric. The reason is because the Kobayashi metric is an important tool in complex dynamics, see examples in [1, 2, 8–10, 42], it has many good properties. It is distance decreasing and invariant under holomorphic conjugation while Euclidean distance is not. In particular, we can measure the Kobayashi distance between any two points inside any simply connected open set by sending them to the unit disk using the Riemann mapping theorem, and moreover, there is an explicit formula to calculate the Kobayashi distance on the unit disk.

Furthermore, there have been many studies about probability measures that can describe the dynamics on the Julia set. For example, if z is any non-exceptional point, the inverse orbits $\{f^{-n}(z)\}$ equidistribute toward the Green measure μ which lives on the Julia set. This was proved already by Brodin [11] in 1965, and many improvements and generalizations have been made [13, 16]. However, this equidistribution toward μ is in the weak sense, and hence it is with respect to the Euclidean metric. Therefore, it is a reasonable question to ask how dense $\{f^{-n}(z)\}$ is in $\mathcal{F}$ near the boundary of $\mathcal{F}$ if we use finer metrics, for instance, the Kobayashi metric.

Definition 1.2.22. Let $\hat{\Omega} \subset \mathbb{C}$ be a domain. We choose a point $z \in \hat{\Omega}$ and a vector ξ which is tangent to the plane at the point z . Let Δ denote the unit disk in the complex plane. We define the **Kobayashi metric** [31]

$$F_{\hat{\Omega}}(z, \xi) := \inf\{\lambda > 0 : \exists f : \Delta \xrightarrow{\text{hol}} \hat{\Omega}, f(0) = z, \lambda f'(0) = \xi\}.$$

Let $\gamma : [0, 1] \rightarrow \hat{\Omega}$ be a piecewise smooth curve. The **Kobayashi length** of γ is defined to be

$$L_{\hat{\Omega}}(\gamma) = \int_{\gamma} F_{\hat{\Omega}}(z, \xi) |dz| = \int_0^1 F_{\hat{\Omega}}(\gamma(t), \gamma'(t)) |\gamma'(t)| dt.$$

For any two points z_1 and z_2 in $\hat{\Omega}$, the **Kobayashi distance** between z_1 and z_2 is defined to be

$$d_{\hat{\Omega}}(z_1, z_2) = \inf\{L_{\hat{\Omega}}(\gamma) : \gamma \text{ is a piecewise smooth curve connecting } z_1 \text{ and } z_2\}.$$

Note that $d_{\hat{\Omega}}(z_1, z_2)$ is defined where z_1, z_2 are in the same connected component of $\hat{\Omega}$.

We know that if $\hat{\Omega} = \Delta$, then the Kobayashi metric is the same as the **Poincaré metric** (see page 9 in [22] and sections 0 and 3 in [31]). And

$$F_{\Delta}(z, \xi) = \frac{|\xi|}{1 - |z|^2}.$$

Note that it is common to call $F_{\hat{Q}}$ a "Poincaré metric" for any $\hat{Q} \subset \mathbb{C}$, but for generalization to higher dimensions, and also for the basin of ∞ , we call it a "Kobayashi metric" even in dimension one to be more consistent.

Proposition 1.2.23 (The distance decreasing property of the Kobayashi Metric [31]). Suppose Ω_1, Ω_2 are domains in $\mathbb{C}$, $z, w \in \Omega_1, \xi \in \mathbb{C}$, and $f : \Omega_1 \rightarrow \Omega_2$ is holomorphic. Then

$$F_{\Omega_2}(f(z), f'(z)\xi) \leq F_{\Omega_1}(z, \xi), \quad d_{\Omega_2}(f(z), f(w)) \leq d_{\Omega_1}(z, w).$$

Corollary 1.2.24. Suppose $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{C}$. Then for any $z, w \in \Omega_1$ and $\xi \in \mathbb{C}$, we have

$$F_{\Omega_2}(z, \xi) \leq F_{\Omega_1}(z, \xi), \quad d_{\Omega_2}(z, w) \leq d_{\Omega_1}(z, w).$$

Theorem 1.2.25 (The completeness of the Kobayashi Metric [30]). If $\Omega \subseteq \mathbb{C}$ is a bounded domain with C^2 boundary, then Ω is complete in the Kobayashi metric.

In this thesis, the domains usually have fractal boundaries, which are not C^2 smooth. Hence, the boundary behavior of the Kobayashi metric is much more complicated and it is the major difficulty in Paper IV.

1.3 Motivation

In section 1.2, Theorem 1.2.13 recalled that Sullivan completed the classification of Fatou components. It tells us every Fatou component of a rational map of degree bigger than one is preperiodic in $\hat{\mathbb{C}}$. Therefore, all Fatou components are preperiodic to attracting basins, parabolic basins and rotation domains. The orbits in the rotation domains are easy to describe since the functions are conjugate to

irrational rotations. The orbits in attracting basins and parabolic basins are much more complicated. There are many results about the orbits near the boundary of an immediate basin $\mathcal{A}_0$ of an attracting fixed point p and also close to the attracting fixed point p using hyperbolicity. As far as we know, there is no systematic study of the rest of the orbits. This includes not only the behavior in the immediate basin, but also all the Fatou components that are preimages of the immediate attracting basin. There are similar problems for parabolic basins. However, in this case, we have the additional difficulty in the lack of hyperbolicity. These are the topics in Papers I, II and III. This problem is present in many other dynamical systems as well. One application arises from Newton's method in [5, 27]. It is of practical interest to know how many times Newton's method must be iterated to get the desired approximation of the root. In this thesis, we will also investigate this problem in complex dimension two in Paper IV.

In addition, if z_0 is an arbitrary starting point inside $\mathcal{A}(p)$, then $z_n = f^n(z_0)$ may land on p for some n . It is also possible that z_n getting closer and closer to p , but never land on p , i.e., the Euclidean distance $d_E(f^n(z_0), p) = |f^n(z_0) - p| \neq 0$. To understand the forward iteration of the orbits behavior of any arbitrary z_0 , it is natural to think if it is equivalent to consider the backward orbits of the fixed point p . If $d_E(f^n(z_0), p) = 0$, then the answer is yes. Otherwise, some preimages in $\{f^{-k}(p)\}$ might be close to z_0 after k iterations. There also might never be such preimages close to z_0 .

Therefore, there are two crucial questions:

1. Whether there always exists some preimage $q \in \{f^{-n}(p)\}$ nearby any point z_0 in Kobayashi distance, which goes strictly to the fixed point p (see the following Figure 1.1)?

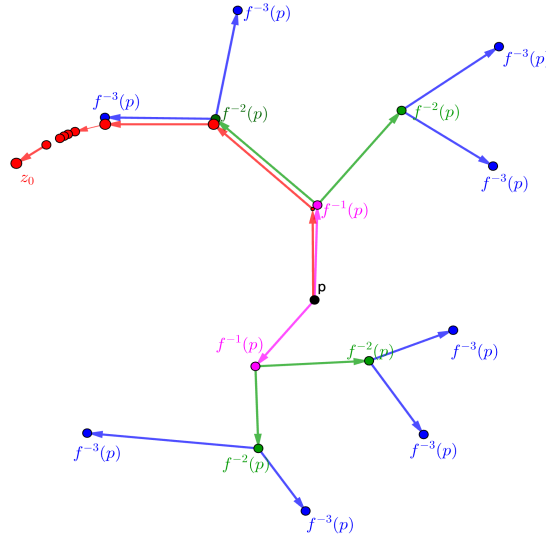


Figure 1.1: The orbit of preimages of p .

2. Suppose there exists some preimage $q \in \{f^{-k}(p)\}$ close to z_0 . Whether the behavior of the forward orbit

$$\{f^{-k}(p), f(f^{-k}(p)), \dots, f^k(f^{-k}(p))\}$$

can track the orbit of

$$\{z_0, f(z_0), \dots, f^k(z_0)\}?$$

Note that here, "track" means the Kobayashi distance $d_{\mathcal{A}(p)}(f^i(z_0), f^i(f^{-k}(p)))$ is uniformly bounded for any $0 \leq i \leq k$, $f^0(z_0) := z_0$, $f^0(p) := p$, $f^0(f^{-k}(p)) := f^{-k}(p)$. The reason why we want to use the Kobayashi metric is because the Kobayashi metric is an important tool in complex dynamics. It has many good properties including being distance decreasing! See subsection 1.2.4 for more details.

If the answer to the first question is yes, then instead of considering the forward orbit of $\{z_0\}$, we can consider the forward orbit of q . Note that it is also possible that there are many preimages in some region in $\mathcal{A}(p)$, but there are a few in the other region (see the following Figure 1.2). For instance, for the polynomial

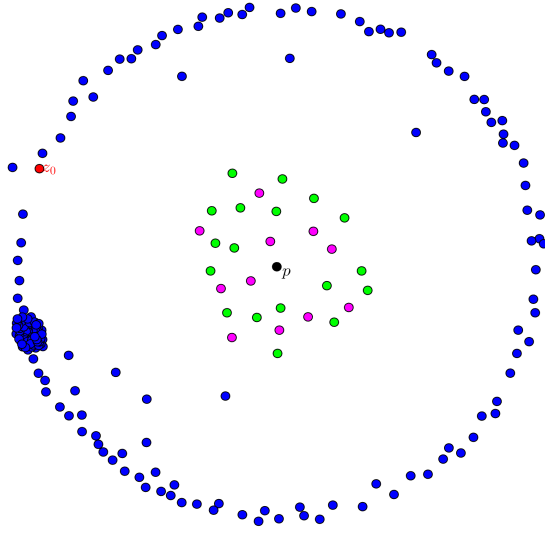


Figure 1.2: An example of the preimages of the fixed point p .

$f = z + \frac{1}{2}z^2 - 2z^3 + \varepsilon^2$, there are many preimages of $p = \frac{1}{4} + 8\varepsilon^2 + o(\varepsilon^3)$ near the eggbeater (see the introduction in the paper [6] and the following Figure 1.3). Hence, if z_0 is not close to the eggbeater, then it is not so obvious to find a preimage

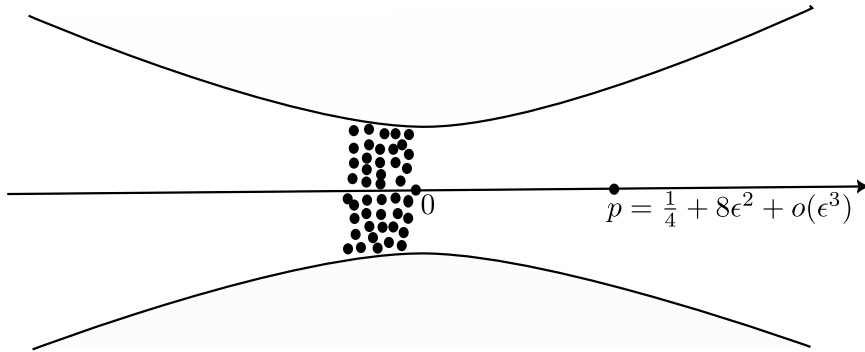


Figure 1.3: Orbits of z_0 near the eggbeater.

$q \in \{f^{-k}(p)\}$ close to z_0 .

1.4 Summary of results

Paper I. We give positive answers to the questions in section 1.3 for the attracting basins of attracting fixed points of holomorphic polynomials in $\mathbb{C}$. We investigate the precise behavior of orbits inside the immediate attracting basins of an attracting fixed point for holomorphic polynomials in $\mathbb{C}$. See Theorem A below. Furthermore, we also study the orbits inside all the Fatou components, which are preimages of the immediate attracting basin, see Theorem B.

Theorem A. Suppose $f(z)$ is a polynomial of degree $N \geq 2$ on $\mathbb{C}$, Ω is the immediate attracting basin of $f(z)$, and p is an attracting fixed point inside Ω , $\{f^{-1}(p)\} \cap \Omega \neq \{p\}$. Then there is a constant C such that for every point $z_0 \in \Omega$, there exists a point $q \in \cup_k f^{-k}(p)$, $k \geq 0$ so that $d_\Omega(z_0, q) \leq C$, d_Ω is the Kobayashi distance on Ω .

To prove Theorem A, we can apply the Riemann mapping theorem to conjugate the immediate basin Ω to the unit disk and send the attracting fixed point p to 0. Furthermore, this conjugation preserves the Kobayashi metric on Ω and is equivalent to the Kobayashi metric on the unit disk (see Sect. 3 in [31]). Hence, we should study the proper holomorphic self-maps of the unit disk. Moreover, the proper maps on the unit disk can be written as $g(z) = e^{i\theta} \prod_{j=1}^m \frac{z-a_j}{1-\bar{a}_j z}$ for $e^{i\theta}$ on the unit circle $\partial\Delta$ (see Lemma 15.5 on page 163 in [34]), and $m \geq 2$ since there is at least one critical point inside Ω (see Theorem 2.2 on page 59 in [12], and constants $|a_j| < 1$ with at least one of the $a_j = 0$ since f has an attracting fixed point p which is sent to 0. Note that the degree of g depends on how many critical points are inside Ω , so $2 \leq m \leq N$. Therefore, instead of considering the orbits of polynomials on the attracting basin with an attracting fixed point at p , we only need to consider the orbits of finite Blaschke products on the unit disk with an attracting fixed point at 0.

First, we discuss the simplest case when all $a_j = 0$. This is the case when $\{g^{-1}(0)\} \cap \Omega = \{0\}$. Then, we proved a suitably modified version of Theorem A in Paper I. We choose a point p' , which is very close to 0. Then since the formula of $g(z)$ is specific, we can exactly calculate q . Hence, it is easy to show that any z_0 , $q \in g^{-k}(p') \in \Delta$, the Kobayashi distance between z_0 and q is uniformly bounded by calculating the following formula

$$d_\Delta(z_0, q) = d_\Delta(0, \frac{q - z_0}{1 - \bar{z}_0 q}) = \log \frac{1 + |\frac{q - z_0}{1 - \bar{z}_0 q}|}{1 - |\frac{q - z_0}{1 - \bar{z}_0 q}|}. \quad (1.4.1)$$

Hence, we proved that there is a constant C and a point p' which is very close to p , so that for every point z_0 inside any Ω , there exists a point $q \in \cup_k f^{-k}(p')$ inside Ω such that $d_\Omega(z_0, q) \leq C$, where d_Ω is the Kobayashi distance on Ω . This implies that Theorem A is still valid for the attracting basin of ∞ if this basin is simply connected, see the following Corollary and its proof in Paper I.

Corollary. I.2.5 *Let f be a polynomial of degree $m \geq 2$, considered as a map on $\hat{\mathbb{C}}$. Suppose that $\Omega(\infty)$ is the basin of ∞ and is simply connected. If $p \in \Omega \setminus \{\infty\}$,*

then there exists a constant $C_0 > 0$ such that for every point $z_0 \in \Omega$, there exists $q \in \cup_k f^{-k}(p)$, $k \geq 0$ satisfying $d_\Omega(z_0, q) \leq C_0$, where d_Ω is the Kobayashi distance on Ω .

However, if some $a_j \neq 0$, then the formula of $g(z)$ is more complicated. Hence, we can not measure whether the Kobayashi distance $d_\Delta(z_0, q)$ is bounded or not by calculating directly using the formula (1.4.1). Especially, either z_0 or q goes to the boundary of the unit disk, then $d_\Delta(z_0, q)$ might go to infinity. Hence, the main difficulty is to measure the Kobayashi distance between q and z_0 , especially, when z_0 is very close to the boundary of the unit disk for the case when some $a_j \neq 0$. To overcome this difficulty, we used only basic knowledge of complex analysis [3] to explore how orbits go in annuli lying in the unit disc and with the unit circle as their outer boundaries. We appreciate one referee who gave a very nice comment that an alternative approach would be to use the Blaschke products near the unit circle, which can be conformally conjugated to a power map in an annulus lying in the unit disc and with the unit circle as a boundary. More details are on page 5 in [7] and Theorem VI.5.1 in [12]. However, this alternative approach only works for proving Theorem A. It cannot help to prove our more general situation in the following Theorem C when Ω is the whole basin of attraction.

Theorem B. Suppose $f(z)$ is a polynomial of degree $N \geq 2$ on $\mathbb{C}$, p is an attracting fixed point of $f(z)$, Ω_1 is the immediate basin of attraction of p , $\{f^{-1}(p)\} \cap \Omega_1 \neq \{p\}$, $\mathcal{A}(p)$ is the basin of attraction of p , $\Omega_i (i = 1, 2, \dots)$ are the connected components of $\mathcal{A}(p)$. Then there is a constant $\tilde{C}$ so that for every point z_0 inside any Ω_i , there exists a point $q \in \cup_k f^{-k}(p)$ inside Ω_i such that $d_{\Omega_i}(z_0, q) \leq \tilde{C}$, where d_{Ω_i} is the Kobayashi distance on Ω_i .

This Theorem B essentially shows that any arbitrary forward orbit that goes to the attracting fixed point can be tracked by an orbit of one preimage of the fixed point p . Hence, to consider the orbit of any arbitrary $z_0 \in \Omega_i$, it would be enough to consider the forward orbit of some preimage of the fixed point p , which is equivalently studying the backward orbit of the fixed point p as well.

Paper II. We give positive answers to the questions in section 1.3 for the attracting basins of attracting fixed points of holomorphic rational functions on $\hat{\mathbb{C}}$ and entire functions in $\mathbb{C}$. We have the following Theorem A and Theorem II.3.4.

Theorem A. Let $R(z)$ be a rational function on $\mathbb{P}^1$, $\mathcal{A}(p)$ be the basin of attraction of an attracting fixed point p of R , and $\Omega_i (i = 1, 2, \dots)$ be the connected components of $\mathcal{A}(p)$. Assume Ω_1 contains p . Let $p_0 \in \Omega_1$ close to p . Then there exists a constant C so that for any $z_0 \in \Omega_i$, there is a point $q \in \cup_k R^{-k}(p_0)$, $k \geq 0$ so that the Kobayashi distance $d_{\Omega_i}(z_0, q) \leq C$.

Remark: In the next section II.2.2, we will clarify what we mean with p_0 close to p in this Theorem A and Theorem II.3.4 in the following section II.3. There we will discuss also the dependence of C on the choice of p_0 .

This theorem generalized the results in Paper I which are for polynomials in $\mathbb{C}$. The key difference with rational functions is that the basins might not be simply

connected. This applies as well to the basin of infinity for a polynomial. It was observed in Paper I that the theorem holds if the basin of attraction of ∞ of a polynomial is simply connected in $\hat{\mathbb{C}}$.

To prove Theorem A in Paper II, firstly, we investigate basins of infinity of polynomials in $\hat{\mathbb{C}}$ which are not simply connected. And we proved the following Theorem II.3.4.

Theorem. II.3.4 *Let f be a polynomial in $\mathbb{P}^1$ and $\mathcal{A}(\infty)$ be the basin of attraction of infinity. Let $p \in \mathcal{A}(\infty)$ close to ∞ , $p \neq \infty$. Then there exists a constant C so that for any $z_0 \in \mathcal{A}(\infty)$, there is a point $q \in \cup_k f^{-k}(p)$, $k \geq 0$ so that the Kobayashi distance $d_{\mathcal{A}(\infty)}(z_0, q) \leq C$.*

Compared with the proof of Theorem A in Paper I, we can not use the Riemann mapping theorem to conjugate the basin of infinity to a unit disc when the basin of infinity is not simply connected. To solve this problem, let U_0 be a small neighborhood of ∞ containing $p \in \mathcal{A}(\infty)$. We let, in fact, U_0 be a disc in Böttcher coordinates centered at ∞ . Let $U_n = f^{-n}(U_0)$. There exists an n_0 so that U_{n_0} contains all critical points inside $\mathcal{A}(\infty)$. Set $W_{n+1} = U_{n+1} \setminus \bar{U}_n$.

Note that we need to be careful about the topology. Firstly, the sets W_{n+1} might have many connected components. We need to make sure that the orbit of z_0 lands in the same connected component as q , where $q \in \cup_k \{f^{-k}(p)\}$, $k \geq 0$. Secondly, when $f^n(z_0)$ and q is in some connected component $W_{m,i}$ of W_m , we also need to avoid when $f^n(z_0)$ and q land on the boundary of $W_{m,i}$. Thirdly, we need to consider when z_0 inside $\mathcal{A}(\infty)$ or z_0 is far away from ∞ and close to the boundary of $\mathcal{A}(\infty)$. The former case is easy to prove since we can choose a compact subset of $\mathcal{A}(\infty)$ which includes z_0 and ∞ . However, in the latter case, it is complicated. We need to carefully deal with the critical points in $\mathcal{A}(\infty)$.

However, for rational functions on $\hat{\mathbb{C}}$, there is another problem: there are preimages of the immediate basin of attraction that are not simply connected. The essential idea to solve this problem is the same as the proof of Theorem II.3.4.

For an entire function f in $\mathbb{C}$, we show that Theorem A in Paper II fails in general. The idea is that we construct an entire function $f(z)$ satisfying the following properties so that Theorem A in Paper II fails:

Proposition. II.5.1 There exists an entire function with the following properties:

1. $f(0) = 0, 0 < |f'(0)| < 1$;
2. There exists an increasing sequence of strictly positive numbers $A_n \rightarrow \infty$, so that $f(\{A_{2n} < |z| < A_{2n+1}\}) \subset \subset \{A_{2n+2} < |z| < A_{2n+3}\}$;
- (3) There exists a sequence $z'_n, A_{2n+1} < |z'_n| < A_{2n+2}$ so that $f(z'_n) = 0$ and f vanishes to order n at z'_n ;
- (4) On a small disc $|z - z'_n| < \delta_n$, $f = (z - z'_n)^n k_n(z)$ where k_n is approximately $\frac{2A_2}{\delta_n^n}$.

The proof of this Proposition is that we construct an entire function in the following way:

$$\begin{aligned} f_1 &= \left[\frac{1}{2}z + \left(\frac{2z}{A_2} \right)^N \right] (z - z'_1) h_1 k_1, \\ &\dots \\ f_n &= \left[f_{n-1} + (z - z'_1) \cdots (z - z'_{n-1})^{n-1} \left(\frac{2z}{A_{2n}} \right)^N \right] (z - z'_n)^n h_n k_n, \end{aligned}$$

where h_n are polynomials approximating $\frac{1}{(z-z'_n)^n}$ very well on $\{|z| < A_{2n+1}\}$.

Let $f := \lim_{n \rightarrow \infty} f_n$ satisfies all properties above and f is entire on $\mathbb{C}$. It has infinitely many preimages of 0, meanwhile, all these z'_n are also critical points of f .

Therefore, we construct infinitely many simply connected Fatou components $\Omega_n \subset \{A_{2n+1} < |z| < A_{2n+2}\}$ and including z'_n . Let Ω_0 be the immediate basin of attraction of 0. Then the map $f : \Omega_n \rightarrow \Omega_0$ is conjugated to a map F on the unit disc with the property that $F(z) = z^n g$. By property (4) Proposition II.5.1, g will have no zeroes and must be uniformly close to 1 in absolute value. If we choose a point $\frac{1}{2} \in \Delta$, then $F^{-1}(\frac{1}{2}) \rightarrow \partial\Delta$. Then $d_\Delta(0, F^{-1}(\frac{1}{2})) \rightarrow \infty$ as $n \rightarrow \infty$. Let z^* be the preimage of $F^{-1}(\frac{1}{2}) \in \Omega_n$. Then we know $d_{\Omega_n}(z'_n, z^*) = d_\Delta(0, F^{-1}(\frac{1}{2})) \rightarrow \infty$ as $n \rightarrow \infty$. Hence, the Kobayashi distance will go to infinity.

On the other hand, we have the following positive results:

Theorem. II.5.3 *Suppose that $f(z)$ is an entire function with 0 as an attracting fixed point with a nonzero derivative at the origin. Suppose that f has finitely many critical points. Suppose also that all Fatou components are bounded open sets. Then Theorem A in Paper II holds.*

This situation is essentially the same as for a polynomial.

Paper III. We give negative answers to the questions in section 1.3 for the parabolic basins of a parabolic fixed point of a holomorphic polynomial in $\mathbb{C}$.

Theorem A. Let $f(z) = z + z^2$. We choose an arbitrary constant $C > 0$ and the point $q = -\frac{1}{2} \in \mathcal{A}$. Then there exists a point $z_0 \in \mathcal{A}$ so that for any $\tilde{q} \in Q := \cup_{l,k=0}^{\infty} \{f^{-l}(f^k(-\frac{1}{2}))\}$ the Kobayashi distance $d_{\mathcal{A}}(z_0, \tilde{q}) \geq C$.

Here $\mathcal{A} := \mathcal{A}(0, -1)$ in Theorem A in Paper III is the parabolic basin of $f = z + z^2$ with the attraction vector $\mathbf{v} = -1$, see Definitions 1.2.6 and 1.2.9. Note that for $f = z + z^2$, there is only one attracting vector, that is $\mathbf{v}_j = \mathbf{v} = -1$.

Note that in this Theorem A in Paper III, we cannot directly choose to iterate the inverse of the parabolic fixed point for a parabolic basin. This slightly differs from attracting basins in Theorem A in Paper I. However, we still want to choose a point that can be very close to the parabolic fixed point. Hence, we choose $q \in \mathcal{A}$ and iterate it k times, then $f^k(q)$ is getting very close to the parabolic fixed point 0. Then we iterate the inverse of $f^k(q)$. To simplify, here we choose $q = -\frac{1}{2}$.

To prove Theorem A in Paper III, firstly, we use a conformal mapping to obtain that the Kobayashi metric on $\mathbb{C} \setminus \mathbb{R}^+$ is $F_{\mathbb{C} \setminus \mathbb{R}^+} = \frac{1}{2r \sin \frac{\theta}{2}} |dz|$ for $z = re^{i\theta} \in \mathbb{C} \setminus \mathbb{R}^+, i.e., \theta \in (0, 2\pi)$. Secondly, we define a *Left Pac-Man* $D_R := \{z = re^{i\theta}, 0 < r \leq R, \theta_0 < \theta < 2\pi - \theta_0\}$ and a *Right Pac-Man* $\tilde{D}_R := \{z = re^{i\theta}, 0 < r \leq R, -\pi + \theta_0 < \theta < \pi - \theta_0\}$. Thirdly, we find a point $z_0 \in D_\varepsilon \setminus \mathbb{R}^-$ such that for any $\tilde{q} \in Q := \cup_{l,k=0}^\infty \{f^{-l}(f^k(-\frac{1}{2}))\}$, we have $d_{\mathcal{A}}(z_0, \tilde{q}) \geq C$ by calculating three Kobayashi distances on $\mathbb{C} \setminus \mathbb{R}^+$.

Furthermore, using the distance decreasing property (see Proposition III.2.7) of the Kobayashi metric, Theorem A in Paper III implies the following Corollary, which says the set of all z_0 is dense in the boundary of the parabolic basin of f :

Corollary. III.3.4 *Let $X \subset \mathcal{A}$ be the set of all $z_0 \in \mathcal{A}$ so that $d_{\mathcal{A}}(z_0, \tilde{q}) \geq C$ for any $\tilde{q} \in \mathcal{A}$. If $z \in X$, then any point $w \in f^{-1}(z)$ is in X . Hence X is dense in the boundary of $\mathcal{A}$.*

We also generalize Theorem A in Paper III to the case of several petals inside the parabolic basin in the following Theorem B:

Theorem B. Let $f(z) = z + az^{m+1}, m \geq 1, a \neq 0$, and Ω_j be the immediate basin of $\mathcal{A}_j$. We choose an arbitrary constant $C > 0$ and a point $q \in \mathbf{v}_j \cap \mathcal{P}_j$. Then there exists a point $z_0 \in \Omega_j$ so that for any $\tilde{q} \in Q := (\cup_{l=0}^\infty f^{-l}(f^k(q))) \cap \Omega_j$ (l, k are non-negative integers), the Kobayashi distance $d_{\Omega_j}(z_0, \tilde{q}) \geq C$, where d_{Ω_j} is the Kobayashi metric.

Here $\mathbf{v}_j$ is an attraction vector in the tangent space of $\mathbb{C}$ at 0, $\mathcal{P}_j$ is an attracting petal for f for the vector $\mathbf{v}_j$ at 0, $\mathcal{A}_j = \mathcal{A}(0, \mathbf{v}_j)$ is the parabolic basin of attraction associated to $\mathbf{v}_j$, see Definitions 1.2.6, 1.2.9, and 1.2.10.

The idea of proof of Theorem B in Paper III is the same as for Theorem A in Paper III. The difference is that $f(z) = z + az^{m+1}, m \geq 1, a \neq 0$ has m different petals. Let $S := \{z = re^{i\theta}, r > 0, 0 < \theta < \frac{2\pi}{m}\}$ be the sector with angle $2\pi/m$, including the attracting petal $\mathcal{P}_j$, then the angle between $\mathbf{v}_j$ and $\mathbb{R}^+$ is $\frac{\pi}{m}$. Then, instead of considering orbits behavior in $\mathbb{C} \setminus \mathbb{R}^+$ of $f = z + z^2$, here we consider it in S .

In the end, we consider the behavior of orbits inside parabolic basins of general polynomials.

Theorem C. Let $f(z) = z + az^{m+1} + (\text{higher order terms}), m \geq 1, a \neq 0$, and Ω_j be the immediate basin of $\mathcal{A}_j$. We choose an arbitrary constant $C > 0$ and a point $q \in \mathbf{v}_j \cap \mathcal{P}_j$. Then there exists a point $z_0 \in \Omega_j$ so that for any $\tilde{q} \in Q := (\cup_{l=0}^\infty f^{-l}(f^k(q))) \cap \Omega_j$ (l, k are non-negative integers), the Kobayashi distance $d_{\Omega_j}(z_0, \tilde{q}) \geq C$, where d_{Ω_j} is the Kobayashi metric.

These theorems are very different from results in Paper I. They are more interesting and surprising results.

Paper IV. We give some positive and negative answers to the questions in section 1.3 for holomorphic polynomials in $\mathbb{C}^2$. We first consider the maps $F(z, w) = (P(z), Q(w))$ where $P(z), Q(w)$ are two polynomials of degree $m_1, m_2 \geq 2$ on $\mathbb{C}$. Then we also consider polynomial skew products in $\mathbb{C}^2$.

Theorem. IV.3.2. Suppose $F(z, w) = (P(z), Q(w))$, where $P(z), Q(w)$ are two polynomials of degree $m_1, m_2 \geq 2$ on $\mathbb{C}$, $P(0) = Q(0) = 0$, and $0 < |P'(0)|, |Q'(0)| < 1$. Let Ω be the immediate attracting basin of $F(z, w)$. Then there is a constant C such that for every point $(z_0, w_0) \in \Omega$, there exists a point $(\tilde{z}, \tilde{w}) \in \cup_k F^{-k}(0, 0), k \geq 0$ so that $d_\Omega((z_0, w_0), (\tilde{z}, \tilde{w})) \leq C$, d_Ω is the Kobayashi distance on Ω .

However, Theorem IV.3.2 is not valid for any of the following cases:

Let $F(z, w) = (P(z), Q(w))$, where $P(z), Q(w)$ are two polynomials of degree $m_1, m_2 \geq 2$ on $\mathbb{C}$ and $P(0) = Q(0) = 0$.

- (1) $P(z) = z^{m_1}, Q(w) = w^{m_2}$;
- (2) $P(z) = z^m, 0 < |Q'(0)| < 1$, i.e., $P'(0) = 0$;
- (3) $P(z) = z^m, Q'(0) = 1$, i.e., $P'(0) = 0$;
- (4) $0 < |P'(0)| < 1, Q'(0) = 1$;
- (5) $P'(0) = Q'(0) = 1$.

The proof of these results can be done by calculating the Kobayashi distance $d_\Omega((z_0, w_0), (\tilde{z}, \tilde{w}))$ by analyzing $d_{\Omega_P}(z_0, \tilde{z}), d_{\Omega_Q}(w_0, \tilde{w})$, separately and using the results in Paper I and Paper III. Here, Ω_P and Ω_Q are denoted by the immediate basin of attraction of $P(z)$ and $Q(w)$, respectively.

We also considered polynomial skew products. Suppose F is a polynomial skew product, $F(z, w) = (P(z), Q(z, w))$, where $P(z), Q(z, w)$ are two polynomials of degree $m_1, m_2 \geq 2$ on $\mathbb{C}$ and $P(0) = 0, Q(0, 0) = 0$. We proved that, in the following cases Theorem IV.3.2 fails as well:

- (1) $P(z) = z^2, Q(z, w) = w^2 + az, a \in \mathbb{C}$;
- (2) $P(z) = az + z^2, Q(z, w) = w^2 + cw + bz, 0 < |a|, |b|, |c| \ll 1$ and $|a| \gg |c|, |a| \gg |b|, |c| \gg |ab|$.

However, for $F(z, w) = (P(z), Q(z, w))$, we can not calculate the Kobayashi distance $d_\Omega((z_0, w_0), (\tilde{z}, \tilde{w}))$ by analyzing $d_{\Omega_P}(z_0, \tilde{z}), d_{\Omega_Q}(w_0, \tilde{w})$ separately as same as for $F(z, w) = (P(z), Q(w))$ since Ω_Q also depends on the z -coordinate. It is very difficult to calculate or even analyze the boundary behavior of the Kobayashi distance because of the complicated fractal nature of the boundary of the basin of the attracting fixed point $(0, 0)$.

Suppose first that $P(z) = z^2, Q(z, w) = w^2 + az, a \in \mathbb{C}$. We choose an arbitrary constant $C > 0$. It is easy to see that the z -coordinate of any point in $F^{-k}(\varepsilon, 0)$ approaches $\partial\Delta$ as $k \rightarrow \infty$. Hence there is an l so that if $k > l$, then $d_\Omega((0, w_0), F^{-k}(\varepsilon, 0)) \geq C$ for any w_0 . Let $(z_j, w_j), j = 1, \dots, N$ be the points in $\{F^{-k}(\varepsilon, 0)\}_{k \leq l}$. Next, we want to show that there is a w_0 so that $(0, w_0) \in \Omega$ and $d_\Omega((0, w_0), (z_j, w_j)) \geq C$ for any $j = 1, \dots, N$. The idea to prove this is to construct a holomorphic function $h(z, w) = w - f_\infty(z)$ so that $h : \Omega \rightarrow \Delta(0, R) \setminus \{0\}, R > 1$ and $h(z_j, w_j) \mapsto c_j \in \Delta \setminus \{0\}$ and $h(0, w_n) \mapsto 0$.

Hence, the points $h(0, w_n)$ goes to 0, i. e., $h(0, w_n)$ are very close to the boundary point 0 of the punctured disk as $n \rightarrow \infty$. Then let $w_0 = w_n$ for some larger n .

Suppose second that $P(z) = az + z^2, Q(z, w) = w^2 + cw + bz, 0 < |a|, |b|, |c| < 1$ and $|a| \gg |c|, |a| \gg |b|, |c| \gg |ab|$. The essential idea is similar to the case above. We choose an arbitrary constant $C > 0$. Let z_N be a preimage of $-a \in P^{-1}(0)$, i.e., $z_N \in \{P^{-N}(-a)\}$. We find a point $(z_0, w_0) = (z_N, 0)$ so that for any $(\tilde{z}, \tilde{w}) \in \cup_k^\infty \{F^{-k}(0, 0)\}, k \geq 0$, the Kobayashi distance $d_\Omega((z_0, w_0), (\tilde{z}, \tilde{w})) \geq C$ for all $i \in \mathbb{N}$.

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Papers

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Paper I

Interior Dynamics of Fatou Sets

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Abstract

In this paper, we investigate the precise behavior of orbits inside attracting basins. Let f be a holomorphic polynomial of degree $m \geq 2$ in $\mathbb{C}$, $\mathcal{A}(p)$ be the basin of attraction of an attracting fixed point p of f , and $\Omega_i (i = 1, 2, \dots)$ be the connected components of $\mathcal{A}(p)$. Assume Ω_1 contains p and $\{f^{-1}(p)\} \cap \Omega_1 \neq \{p\}$. Then there is a constant C so that for every point z_0 inside any Ω_i , there exists a point $q \in \cup_k f^{-k}(p)$ inside Ω_i such that $d_{\Omega_i}(z_0, q) \leq C$, where d_{Ω_i} is the Kobayashi distance on Ω_i . In paper [10], we proved that this result is not valid for parabolic basins.

Keywords: Basin of attraction, Blaschke products, Kobayashi metric

I.1 Introduction

A general goal in discrete dynamical systems is to qualitatively and quantitatively describe the possible dynamical behaviour under the iteration of maps satisfying certain conditions.

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a nonconstant holomorphic map, and $f^n : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be its n -fold iterate. In complex dynamics, two crucial disjoint invariant sets are associated with f , the *Julia set* and the *Fatou set* [12], which partition the sphere $\hat{\mathbb{C}}$. The Fatou set of f is defined as the largest open set where the family of iterates is normal. In other words, for any point $z \in \hat{\mathbb{C}}$, there exists some neighborhood U of z so that the sequence of iterates of the map restricted to U forms a normal family, so the iterates are well-behaved. The complement of the Fatou set is called the Julia set. The connected components of the Fatou set of f are called *Fatou components*. A Fatou component $\Omega \subset \hat{\mathbb{C}}$ of f is *invariant* if $f(\Omega) = \Omega$. For $z \in \hat{\mathbb{C}}$, the set $\{z_n\} = \{z_1 = f(z_0), z_2 = f^2(z_0), \dots\}$ is called the orbit of the point $z = z_0$. If $z_N = z_0$ for some integer N , we say that z_0 is a periodic point of f of period N . If $N = 1$, then z_0 is a fixed point of f .

At the beginning of the 20th century, Fatou [4–6] classified all possible invariant Fatou components of rational functions on the Riemann sphere. He proved that only the following three cases could occur:

1. (attracting case) Ω contains a fixed point p , and the orbit of every point in Ω converges to p .
2. (parabolic case) $\partial\Omega$ contains a fixed point p , and the orbit of every point in Ω converges to p .

3. (rotation domain) Ω is conformally equivalent to the unit disk or an annulus, and the map is conjugate to an irrational rotation.

The classification of Fatou components was completed in the 1980s when Sullivan proved that every Fatou component of a rational map is preperiodic, i.e., there are $n, m \in \mathbb{N}$ such that $f^{n+m}(\Omega) = f^m(\Omega)$. For more details and results, we refer the reader to [13].

However, there has been no detailed study until now of the more precise behavior of orbits inside the Fatou set. For example, let $\mathcal{A}(p) := \{z \in \mathbb{C}; f^n(z) \rightarrow p\}$ be the *basin of attraction* of an attracting fixed point p . One can ask when z_0 is close to $\partial\mathcal{A}(p)$, what orbits $\{z_n\}$ of z_0 going from z_0 to near the attracting fixed point p look like, or how many iterations are needed to reach this. One application arises from Newton's method in [1]. It is of practical interest to know how many times Newton's method must be iterated to get the desired approximation of the root.

These kinds of questions are the main topics of this paper. In the second section, we study the dynamics of holomorphic polynomials on the attracting basins and prove our main Theorem A and Theorem B:

Theorem A. Suppose $f(z)$ is a polynomial of degree $N \geq 2$ on $\mathbb{C}$, Ω is the immediate attracting basin of $f(z)$, and p is an attracting fixed point inside Ω , $\{f^{-1}(p)\} \cap \Omega \neq \{p\}$. Then there is a constant C such that for every point $z_0 \in \Omega$, there exists a point $q \in \cup_k f^{-k}(p)$, $k \geq 0$ so that $d_\Omega(z_0, q) \leq C$, d_Ω is the Kobayashi distance on Ω .

To prove Theorem A, we use conjugation to consider the orbits of Blaschke products on the unit disk instead of considering the orbits of polynomials on the attracting basin. This implies that Theorem A is still valid for the attracting basin of ∞ if this basin is simply connected, see Corollary I.2.5 in the following section. Furthermore, we also generalized Theorem A to the following Theorem B when Ω is the whole basin of attraction.

Theorem B. Suppose $f(z)$ is a polynomial of degree $N \geq 2$ on $\mathbb{C}$, p is an attracting fixed point of $f(z)$, Ω_1 is the immediate basin of attraction of p , $\{f^{-1}(p)\} \cap \Omega_1 \neq \{p\}$, $\mathcal{A}(p)$ is the basin of attraction of p , $\Omega_i (i = 1, 2, \dots)$ are the connected components of $\mathcal{A}(p)$. Then there is a constant $\tilde{C}$ so that for every point z_0 inside any Ω_i , there exists a point $q \in \cup_k f^{-k}(p)$ inside Ω_i such that $d_{\Omega_i}(z_0, q) \leq \tilde{C}$, where d_{Ω_i} is the Kobayashi distance on Ω_i .

Note that Theorem A and Theorem B are no longer valid for parabolic basins, see details in [10].

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I.2 Dynamics of holomorphic polynomials inside attracting basins

Let $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a holomorphic map and Ω be an invariant Fatou component. It follows from the classification above in section I.1 that we have a complete understanding of the long-term behaviour of all orbits in Ω , but how precisely the iterates of a point move inside Ω is presently still unknown. This section will more precisely describe how iterates of a point move inside an invariant Fatou component and the whole basin of attraction. Then we give our main results in Theorem A and Theorem B.

I.2.1 The Kobayashi metric

Definition I.2.1. Let $\hat{\Omega} \subset \mathbb{C}$ be a domain. We choose a point $z \in \hat{\Omega}$ and a vector ξ which is tangent to the plane at the point z . Let Δ denote the unit disk in the complex plane. We define the Kobayashi metric [11]

$$F_{\hat{\Omega}}(z, \xi) := \inf\{\lambda > 0 : \exists f : \Delta \xrightarrow{hol} \hat{\Omega}, f(0) = z, \lambda f'(0) = \xi\}.$$

Let $\gamma : [0, 1] \rightarrow \hat{\Omega}$ be a piecewise smooth curve. The Kobayashi length of γ is defined to be

$$L_{\hat{\Omega}}(\gamma) = \int_{\gamma} F_{\hat{\Omega}}(z, \xi) |dz| = \int_0^1 F_{\hat{\Omega}}(\gamma(t), \gamma'(t)) |\gamma'(t)| dt.$$

For any two points z_1 and z_2 in $\hat{\Omega}$, the Kobayashi distance between z_1 and z_2 is defined to be

$$d_{\hat{\Omega}}(z_1, z_2) = \inf\{L_{\hat{\Omega}}(\gamma) : \gamma \text{ is a piecewise smooth curve connecting } z_1 \text{ and } z_2\}.$$

Note that $d_{\hat{\Omega}}(z_1, z_2)$ is defined where z_1, z_2 are in the same connected component of $\hat{\Omega}$.

Let $d_E(z_1, z_2)$ denote the Euclidean metric distance for any two points $z_1, z_2 \in \Delta$.

We know that if $\hat{\Omega} = \Delta$, then the Kobayashi metric is the same as the Poincaré metric (see page 9 in [7] and sections 0 and 3 in [11]). And

$$F_{\Delta}(z, \xi) = \frac{|\xi|}{1 - |z|^2}.$$

Note that it is common to call $F_{\hat{Q}}$ a "Poincaré metric" for any $\hat{Q} \subset \mathbb{C}$, but for generalization to higher dimensions, and also for the basin of ∞ , we call it a "Kobayashi metric" even in dimension one to be more consistent.

Proposition I.2.2 (The distance decreasing property of the Kobayashi Metric [11]). *Suppose Ω_1, Ω_2 are domains in $\mathbb{C}$, $z, w \in \Omega_1, \xi \in \mathbb{C}$, and $f : \Omega_1 \rightarrow \Omega_2$ is holomorphic. Then*

$$F_{\Omega_2}(f(z), f'(z)\xi) \leq F_{\Omega_1}(z, \xi), \quad d_{\Omega_2}(f(z), f(w)) \leq d_{\Omega_1}(z, w).$$

Corollary I.2.3. *Suppose $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{C}$. Then for any $z, w \in \Omega_1$ and $\xi \in \mathbb{C}$, we have*

$$F_{\Omega_2}(z, \xi) \leq F_{\Omega_1}(z, \xi), \quad d_{\Omega_2}(z, w) \leq d_{\Omega_1}(z, w).$$

I.2.2 Main results about orbits inside an attracting basin

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a polynomial of degree N and Ω be an immediate basin of attraction that contains an attracting fixed point p of f . Then Ω is a connected component of $\mathcal{A}(p)$. In addition, an immediate basin of attraction of a holomorphic polynomial is simply connected. We can apply the Riemann mapping theorem to conjugate the immediate basin of attraction to the unit disk and send the attracting fixed point p to 0. Furthermore, this conjugation preserves the metric on Ω and is equivalent to the metric on the unit disk (see section 3 in [11]). Hence we should study the proper holomorphic self-maps of the unit disk. Moreover, the proper maps on the unit disk can be written as $g(z) = e^{i\theta} \prod_{j=1}^m \frac{z-a_j}{1-\bar{a}_j z}$ for $e^{i\theta}$ on the unit circle $\partial\Delta$ (see Lemma 15.5 on page 163 in [12]), and $m \geq 2$ since there is at least one critical point inside Ω (see Theorem 2.2 on page 59 in [3]), and constants $|a_j| < 1$ with at least one of the $a_j = 0$ since f has an attracting fixed point p which is sent to 0. Note that the degree of g depends on how many critical points are inside Ω , so $2 \leq m \leq N$. Therefore, instead of considering the orbits of polynomials on the attracting basin with an attracting fixed point at p , we only need to consider the orbits of Blaschke products on the unit disk with an attracting fixed point at 0.

First, we discuss the simplest case when all $a_j = 0$. Then for $g = e^{i\theta} z^m$, we have the following theorem.

Theorem I.2.4. *Suppose $g(z) = e^{i\theta} z^m, m \geq 2$, we pick a point $\hat{p} \in \Delta \setminus \{0\}$. Then there exists a constant $C_0 > 0$ such that for every point $z_0 \in \Delta$, there exists $q \in \cup_k g^{-k}(\hat{p}), k \geq 0$ satisfying $d_\Delta(z_0, q) \leq C_0$, where d_Δ is the Kobayashi distance on the unit disk Δ .*

Proof. It is easy to describe the dynamics of $g(z) = e^{i\theta} z^m, m \geq 2$ since all its iterates can be written very explicitly as $g^n(z) = e^{i\frac{1-m^n}{1-m}\theta} z^{m^n}$. Let the point $\hat{p} = \rho e^{i\theta_0}, 0 < \rho < 1$, then the inverse images of $\hat{p}$ are

$$T_k := s e^{i\varphi}, s = \rho^{1/m^k}, \varphi = \frac{2j\pi + \theta_0 - \frac{1-m^k}{1-m}\theta}{m^k}, j = 0, \dots, m^k - 1.$$

We need to show that all points are at most a uniformly bounded distance from this sequence in the Kobayashi metric of the disk.

The k th preimages under this map of an annulus of the form $\{\rho \leq |z| \leq \rho^{1/m}\}$ can be represented as m^k essentially disjoint ‘boxes’ each consisting of a sector of

I.2. Dynamics of holomorphic polynomials inside attracting basins

the annulus $\{\rho^{1/m^k} \leq |z| \leq \rho^{1/m^{k+1}}\}$ with angle $2\pi/m^k$. Each of these boxes must contain a preimage of the given point $\hat{p}$, for a suitable choice of ρ . The length of any path in such a box w.r.t. the Kobayashi metric (on the unit disk, it is equal to the Poincaré metric) is bounded above by its Euclidean length multiplied by a simple upper bound for the density of the Kobayashi metric in the box. This gives an upper bound for the box diameter w.r.t. the Kobayashi metric of

$$\left((1 - \rho^{1/m^k}) + \frac{2\pi}{m^k} \right) \frac{1}{1 - \rho^{1/m^{k+1}}}. \quad (\text{I.2.1})$$

Then the estimates

$$1 - \frac{1}{2}x \geq e^{-x} \geq 1 - x, \text{ for } 0 \leq x \leq 1, \text{ with } x = \frac{\ln(1/\rho)}{m^k}, \frac{\ln(1/\rho)}{m^{k+1}},$$

for k sufficiently large, gives

$$\frac{\ln(1/\rho)}{2m^{k+1}} \leq 1 - \rho^{1/m^{k+1}} \leq 1 - \rho^{1/m^k} \leq \frac{\ln(1/\rho)}{m^k},$$

which, using (I.2.1), shows that these boxes have a uniformly bounded diameter. $\square$

Corollary I.2.5. *Let f be a polynomial of degree $m \geq 2$, considered as a map on $\hat{\mathbb{C}}$. Suppose that $\Omega(\infty)$ is the basin of ∞ and is simply connected. If $p \in \Omega \setminus \{\infty\}$, then there exists a constant $C_0 > 0$ such that for every point $z_0 \in \Omega$, there exists $q \in \cup_k g^{-k}(p)$, $k \geq 0$ satisfying $d_\Omega(z_0, q) \leq C_0$, where d_Ω is the Kobayashi distance on Ω .*

Proof. Using Theorem 9.5 in [12], we know that $\Omega(\infty)$ is conformally isomorphic to the exterior of the closed unit disk $\bar{\Delta}$ under an isomorphism

$$\Phi : \Omega(\infty) \rightarrow \hat{\mathbb{C}} \setminus \bar{\Delta},$$

which conjugates f on $\Omega(\infty)$ to the m th power map $w \mapsto w^m$.

Then by Theorem I.2.4, the proof is done. $\square$

Next, we generalize Theorem I.2.4 to general Blaschke products $g(z) = e^{i\theta} \prod_{j=1}^m \frac{z - a_j}{1 - \bar{a}_j z}$, with some $a_j \neq 0$. However, there is at least one of the $a_j = 0$ since 0 is a fixed point of g . In this case, $\hat{p} \in \Delta \setminus \{0\}$ in Theorem I.2.4 can now be replaced by 0.

Theorem I.2.6. *Suppose*

$$g(z) = e^{i\theta} z^{m_1} \prod_{l=m_1+1}^m \frac{z - a_l}{1 - \bar{a}_l z}, \quad m \geq 2, \quad 1 \leq m_1 < m, \quad a_l \neq 0, \quad a_l \in \Delta, \quad e^{i\theta} \in \partial\Delta.$$

There exists a constant C_0 such that for every point $z_0 \in \Delta$, there exists $q \in \cup_k g^{-k}(0)$ satisfying $d_\Delta(z_0, q) \leq C_0$.

Proof. First of all, if $|z_0| \leq \rho < 1$ for some constant ρ to be chosen later, then we choose $q = 0$. And by the formula (on page 21 in [12]) for the Kobayashi metric on the disk, we have

$$d_\Delta(z_0, q) = d_\Delta\left(0, \frac{z_0 - q}{1 - z_0 \bar{q}}\right) = \ln \frac{1 + |z_0|}{1 - |z_0|} \leq \ln \frac{1 + \rho}{1 - \rho} := C'_0. \quad (\text{I.2.2})$$

Therefore, there is a constant C'_0 such that for every point $z_0 \in \Delta$ and $|z_0| \leq \rho$, there exists a $q = 0$ satisfying $d_\Delta(z_0, q) \leq C'_0$.

Secondly, if $|z_0| > \rho$, then we will show that there still exists a point $q \in \cup_k g^{-k}(0)$ such that $d_\Delta(z_0, q)$ is uniformly bounded even when z_0 is very close to $\partial\Delta$.

Let us first remind the reader why the preimages of 0 are dense in the Euclidean distance on the boundary of the unit disk intersecting the unit disk, i.e., the set $\cup_{n \rightarrow \infty} \{g^{-n}(0)\}$ clusters at every point in $\partial\Delta \cap \Delta$. First of all, we know that the Blaschke product g carries Δ onto Δ and $\hat{\mathbb{C}} \setminus \bar{\Delta}$ onto $\hat{\mathbb{C}} \setminus \bar{\Delta}$, and the Julia set of g is the unit circle. See Theorem 1.8 on page 57 and the corresponding example on page 58 in [3]. Second, by Theorem 4.10 on page 47 in [12], we know that for any $\tilde{z}_0 \in \partial\Delta$, there exists a sequence $\{g^{-n}(0)\}$ that converges to $\tilde{z}_0$. Therefore, the preimages of 0 are dense in $\partial\Delta \cap \Delta$.

Note that this does not finish the proof because the Kobayashi distance from $\{g^{-n}(0)\}$ to points near the boundary might still be arbitrarily large. We still need to show that this is not the case.

Before continuing with the proof of Theorem I.2.6, we have the following two lemmas for the orbits near the boundary of the unit disk.

Lemma I.2.7. $|g'(z)| > 1$ for all $z \in \partial\Delta$.

Proof. Lemma 3.4.3 on page 46 in [9] shows that $g'(z) \neq 0$ for all $z \in \partial\Delta$. However, $g(z) = e^{i\theta} z^{m_1} \prod_{l=m_1+1}^m \frac{z - a_l}{1 - \bar{a}_l z}$. Then $a_l = 0$ when $l = 1, 2, \dots, m_1$. Hence

$$|g'(z)| = m_1 + \sum_{l=m_1+1}^m \frac{1 - |a_l|^2}{|z - a_l|^2} > m_1 \geq 1,$$

for all $z \in \partial\Delta$. We refer the readers to the proof of lemma 3.4.3 in [9] for more details. □

Therefore, we can choose $r_0 \in (0, 1)$ so that $|g'(z)| > 1 + \varepsilon$ for any z inside the annulus $\mathbb{A} := \{z : r_0 < |z| < 1\}$ and some fixed $\varepsilon > 0$.

Lemma I.2.8. $|g^{-1}(z)| > |z|$ for all $z \in \mathbb{A}$.

Proof. See the Schwarz lemma and Lemma I.2.7 above. □

We continue with the proof of Theorem I.2.6. We consider the inverse orbit of 0 near the boundary of the unit disk precisely, especially we investigate the inverse orbit inside $\mathbb{A}$.

Let $D_0 = g^{-1}(\Delta \setminus \mathbb{A})$, and choose $r_1 > r_0$ such that $D_0 \subset \mathbb{B}_0 = \{z : |z| \leq r_1 < 1\}$. Let $D_1 = g^{-1}(\mathbb{B}_0)$, then by Lemma I.2.8, we can choose $r_2 > r_1$ such that $D_1 \subset \mathbb{B}$ where $\mathbb{B} = \{z : |z| < r_2 < 1\}$. Let $\mathbb{B}_1 := \{z : r_1 < |z| < r_2\}$, then for any $z_0 \in \Delta \setminus \mathbb{B}$, there exists a point $z'_0 := g^{M_1}(z_0) \in \mathbb{B}_1$, $z'_1 := g^{M_1-1}(z_0) \in \Delta \setminus \mathbb{B}$ for some integer M_1 (See figure 1.4). Since the preimages of 0 are dense in the

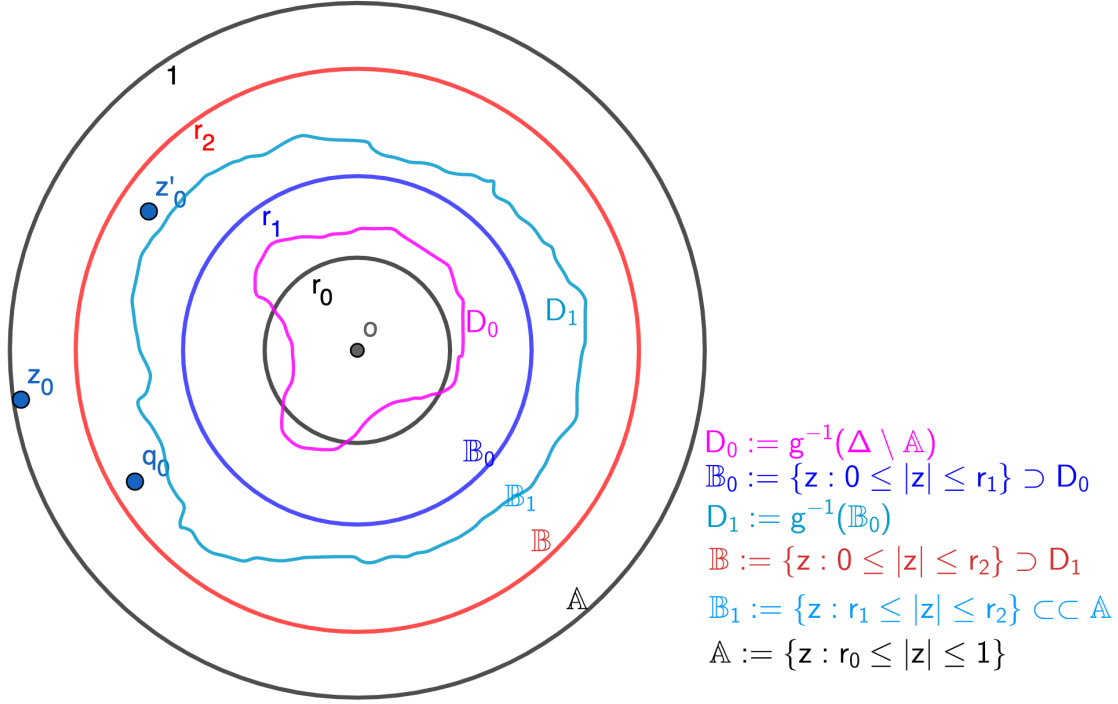


Figure 1.4: Domains near the boundary of the unit disk.

boundary of the unit disk, we can calculate all preimages of 0 and choose some integer M_2 such that $q_0 := g^{-M_2}(0) \in \mathbb{B}_1$, $q_1 := g^{-1}(q_0) \in \Delta \setminus \mathbb{B}$.

Now, we need to choose a sector $\mathbb{A}^+$ such that $z'_0, q_0 \in \mathbb{A}^+$, but z'_0, q_0 are neither on $\partial\mathbb{A}^+$ nor very close to $\partial\mathbb{A}^+$: first, choose a sector S with angle $\frac{\pi}{3}$ and one boundary line of S passes through 0 and z'_0 such that $q_0 \notin S$. Secondly, choosing a sector $\tilde{S}$ inside S with angle $\frac{\pi}{5}$ such that $\tilde{S}$ and S have the same angle bisector region of $\mathbb{A}$ not including z'_0 and q_0 . Then we let $\mathbb{A}^+ := \mathbb{A} \setminus (\tilde{S} \cap \mathbb{A})$ (see the whole green area, including the red area, in Figure 1.5).

Since we know precisely the orbit $\{z_n\}$, we can choose the inverse map to be the branch of $g^{-1}(z)$ on $\mathbb{A}^+$ such that $g^{-1}(z'_0) = g^{-1}(g^{M_1}(z_0)) = g^{M_1-1}(z_0) = z'_1$ sending q_0 to q_1 . In addition, there are no critical points inside $\mathbb{A}$ since $|g'(z)| > 1 + \varepsilon$ and $0 < |(g^{-1}(z))'| < \frac{1}{1+\varepsilon}$ for all $z \in \mathbb{A}$. Then the branch of the inverse map $g^{-1}(z)$ which we choose is well defined in $\mathbb{A}^+$. Inductively, we can, in the same way, define $g^{-n}(z)$ in $\mathbb{A}^+$. We obtain that the inverse map $g^{-n}(z) : \mathbb{A}^+ \rightarrow \mathbb{A}$ is analytic. And the annulus $\Delta \setminus \mathbb{B}$ contains $z'_1, z'_2, z'_3, \dots, z'_n$ and $q_1, q_2, q_3, \dots, q_n$.

Next, we estimate the Kobayashi distance. Choosing a sector $\tilde{S} \subset S$ with an

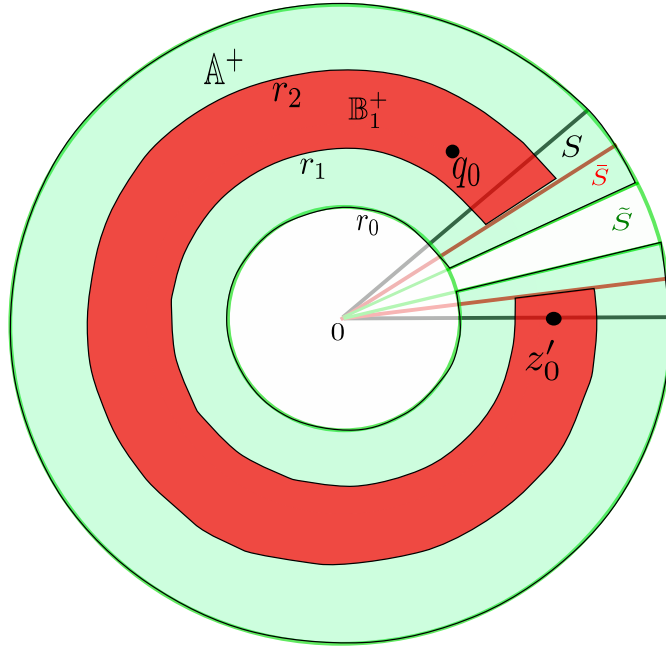


Figure 1.5: The sectors $\mathbb{A}^+$ and $\mathbb{B}_1^+$.

angle $\frac{\pi}{4}$, then let $\mathbb{B}_1^+ := \mathbb{B}_1 \setminus (\bar{S} \cap \mathbb{B}_1) \subset \mathbb{A}^+$ (see the red area in Figure 1.5), then $\mathbb{B}_1^+ \subset \subset \mathbb{A}$ including z'_0, q_0 . Hence there is a constant C_0 such that

$$d_{\mathbb{B}_1^+}(z'_0, q_0) \leq C_0. \quad (\text{I.2.3})$$

In addition, by Corollary I.2.3, we have

$$d_{\mathbb{A}^+}(z'_0, q_0) \leq d_{\mathbb{B}_1^+}(z'_0, q_0). \quad (\text{I.2.4})$$

Inductively, $g^{-M_1}(z) : \mathbb{A}^+ \rightarrow \mathbb{A}$ sends the points $z'_0, q_0 \in \mathbb{A}^+$ to the points $z_0, q_{M_1} \in \mathbb{A}$, respectively. The Kobayashi distance is decreasing by Proposition I.2.2, then we obtain

$$d_{\mathbb{A}}(z_0, q_{M_1}) \leq d_{\mathbb{A}^+}(z'_0, q_0). \quad (\text{I.2.5})$$

Since $\mathbb{A} \subset \Delta$, using Corollary I.2.3 once again, we know

$$d_{\Delta}(z_0, q_{M_1}) \leq d_{\mathbb{A}}(z_0, q_{M_1}). \quad (\text{I.2.6})$$

Therefore, from equations (I.2.3) to (I.2.6), there is a constant C_0 and a point $q := q_{M_1} \in \cup_k g^{-k}(0)$ such that

$$d_{\Delta}(z_0, q) \leq d_{\mathbb{A}}(z_0, q) = d_{\mathbb{A}}(z_0, q_{M_1}) \leq d_{\mathbb{A}^+}(z'_0, q_0) \leq d_{\mathbb{B}_1^+}(z'_0, q_0) \leq C_0, \quad (\text{I.2.7})$$

for some integer k . Choosing $\rho = r_2$, we finally prove the theorem.

Remark: An alternative approach would be to use that g near the unit circle is conformally conjugate to a power map in an annulus lying in the unit disc and with the unit circle as a boundary component. More precisely, on page 5 in [2], Bergweiler and Morosawa showed that there exists a compact set $E \subset \Delta$, $\rho_0 \in (0, 1)$ and a homeomorphism $\varphi : \{\rho_0 < |z| < 1\} \rightarrow \Delta \setminus E$ such

that $g(\varphi(z)) = \varphi(z^m)$ for $\{\rho_0^{1/m} < |z| < 1\}$. Moreover, φ is also analytic on $\{\rho_0 < |z| < 1\}$ by Theorem VI.5.1 in [3]. Then our Theorem I.2.6 holds by Theorem I.2.4 for $|z_0| > \rho$ using the localization theorem for the Kobayashi metric as in paper [8].

Our proof above actually used only basic knowledge of complex analysis to explore how orbits go in annuli lying in the unit disc and with the unit circle as their outer boundaries. And we also showed how big ρ needs to be chosen exactly. $\square$

Now, we can obtain our main theorems as follows:

Theorem A. Suppose $f(z)$ is a polynomial of degree $N \geq 2$ on $\mathbb{C}$, Ω is the immediate attracting basin of $f(z)$, and p is an attracting fixed point inside Ω , $\{f^{-1}(p)\} \cap \Omega \neq \{p\}$. Then there is a constant C such that for every point $z_0 \in \Omega$, there exists a point $q \in \cup_k f^{-k}(p)$, $k \geq 0$ so that $d_\Omega(z_0, q) \leq C$, d_Ω is the Kobayashi distance on Ω .

This theorem would imply that the orbit of z_0 behaves like the orbit of some preimage of the fixed point p . More precisely, the orbit of z_0 is shadowed by the orbit of the point $f^{-k}(p)$ for some positive integer k . Hence it would be enough to consider only these preimage points $f^{-k}(p)$. But then, instead of considering the forward orbit of the preimage of the fixed point p , we can equivalently study the backward orbit of the fixed point p . Note that this is very useful for practical purposes because these inverse orbits can be color plotted.

Proof. Since Ω is an immediate attracting basin of $f(z)$, p is an attracting fixed point inside Ω , we know that $f(\Omega) = \Omega$ and Ω is simply connected. By the Riemann mapping theorem, there exists a conformal mapping φ such that we can conjugate f on Ω to $g = e^{i\theta} \prod_{j=1}^m \frac{z-a_j}{1-\bar{a}_j z}$ on the unit disk, i.e., $\varphi \circ f = g \circ \varphi$. And g is a proper self-map of the unit disk Δ , with an attracting fixed point at the origin. We refer the reader to Lemma 15.5 on page 163 in [12] for more details. In addition, for any two points $a, b \in \Omega$, we have $d_\Omega(a, b) = d_\Delta(\varphi(a), \varphi(b))$ by the Riemann mapping theorem (see the Remark on page 8 in [11]). Then this theorem is true because of Theorem I.2.6. $\square$

In the remainder of this section, we will prove that Theorem A still holds when Ω is the whole basin of attraction.

Theorem B. Suppose $f(z)$ is a polynomial of degree $N \geq 2$ on $\mathbb{C}$, p is an attracting fixed point of $f(z)$, Ω_1 is the immediate basin of attraction of p , $\{f^{-1}(p)\} \cap \Omega_1 \neq \{p\}$, $\mathcal{A}(p)$ is the basin of attraction of p , $\Omega_i (i = 1, 2, \dots)$ are the connected components of $\mathcal{A}(p)$. Then there is a constant $\tilde{C}$ so that for every point z_0 inside any Ω_i , there exists a point $q \in \cup_k f^{-k}(p)$ inside Ω_i such that $d_{\Omega_i}(z_0, q) \leq \tilde{C}$, where d_{Ω_i} is the Kobayashi distance on Ω_i .

Proof. Since f is a polynomial with an attracting fixed point $p \in \Omega_1$, there are two possible cases.

(1) $\mathcal{A} = \Omega_1$. Then this theorem is essentially as same as Theorem A.

(2) $\mathcal{A}$ has at least two connected components, then $\mathcal{A}$ has infinitely many connected components: Suppose Ω_2 is another connected component of f which is distinct from Ω_1 and $f(\Omega_2) = \Omega_1$. We know $f(\Omega_1) = \Omega_1$, and then there must be a third component Ω_3 which can be mapped to Ω_2 , and so on. It implies that $\mathcal{A}$ has infinitely many connected components.

For the second situation, let us first consider the orbit between two connected components, Ω_1 and Ω_2 , where $f(\Omega_2) = \Omega_1$. If the start point z_0 is inside Ω_1 , then the proof is done by Theorem A. If the start point z_0 is inside Ω_2 , then there is a point $\hat{z}_0 := f(z_0) \in \Omega_1$. By the above Theorem A, we know that there is a constant C such that for every point $\hat{z}_0 \in \Omega_1$, there exists a point $\hat{q} \in \cup_k f^{-k}(p)$, $k \geq 0$ in Ω_1 so that $d_{\Omega_1}(\hat{z}_0, \hat{q}) \leq C$, d_{Ω_1} is the Kobayashi distance on Ω_1 . Next, we need to show that there is a constant C' such that for every point $z_0 \in \Omega_2$, there exists a point $q := f^{-1}(\hat{q}) \in \cup_k f^{-k}(p)$, $k \geq 0$ in Ω_2 so that $d_{\Omega_2}(z_0, q) \leq C'$. Note that f has only finitely many critical points. If Ω_2 contains no critical points, then $f : \Omega_2 \rightarrow \Omega_1$ is an isometry, hence, we can choose $C' = C$. We only need to show the existence of C' for the cases when Ω_2 contains critical points of f .

Since Ω_1 and Ω_2 are simply connected, by the Riemann mapping theorem, there are two biholomorphic maps, $\psi_1 : \Omega_1 \rightarrow \Delta$ and $\psi_2 : \Omega_2 \rightarrow \Delta$. Then f is conjugate with $g = \psi_1 \circ f \circ \psi_2^{-1}$, which is a proper self-map of the unit disk Δ . Hence g is a Blaschke product. Then these three points $z_0 \in \Omega_2$, $\hat{z}_0 = f(z_0) \in \Omega_1$, $\hat{q} \in \Omega_1$ are sent to Δ , we denote them by $Z_0, \hat{Z}_0, \hat{Q} \in \Delta$, respectively, and $d_{\Delta}(\hat{Z}_0, \hat{Q}) = d_{\Omega_1}(\hat{z}_0, \hat{q}) < C$. Therefore, it is equivalent to prove that there exists a point $Q = g^{-1}(\hat{Q}) \in \Delta$ and a constant C' (independent of z_0) such that $d_{\Delta}(Z_0, Q) < C'$.

We know that g has finitely many critical points. We can choose a disk $\Delta(0, r_0) = \{z : |z| \leq r_0 < 1\}$ including all critical points of g . We denote $D_0 = g(\Delta(0, r_0))$ and choose a disk $\Delta(0, R_0) = \{z : |z| \leq R_0 < 1\}$ such that $D_0 \subseteq \Delta(0, R_0)$, then let $D_1 := \Delta \setminus \Delta(0, R_0)$, see the following Figure 1.6. Then

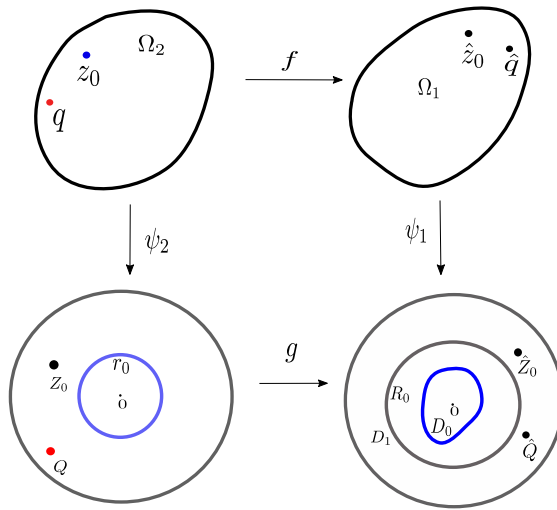


Figure 1.6: The conjugation.

there are four cases for distributing $\hat{Z}_0$ and $\hat{Q}$.

Case 1: if $\hat{Z}_0, \hat{Q} \in \Delta(0, R_0)$, we choose a disk $\Delta(0, r_1) = \{z : |z| \leq r_1 < 1\}$ such that $g^{-1}(\Delta(0, R_0)) \subseteq \Delta(0, r_1)$, then $Z_0 \in \Delta(0, r_1)$ and there exists a point $Q = g^{-1}(\hat{Q}) \in \Delta(0, r_1)$. Hence $d_\Delta(Z_0, Q) \leq C'$ for some uniform constant C' .

Case 2: if $\hat{Z}_0 \in \Delta(0, R_0), \hat{Q} \in D_1$, then there exists a disk $\Delta(0, R_1) = \{z : |z| \leq R_1 < 1\}$ including $\hat{Q}$ and $\hat{Z}_0$ since $d_\Delta(\hat{Z}_0, \hat{Q}) \leq C$. Hence there is a disk $\Delta(0, r_2) = \{z : |z| \leq r_2 < 1\}$ so that $g^{-1}(\Delta(0, R_1)) \subseteq \Delta(0, r_2)$. Then letting Q be any point of $g^{-1}(\hat{Q})$ inside $\Delta(0, r_2)$. Hence $d_\Delta(Z_0, Q) \leq C''$ for some uniform constant C'' .

Case 3: if $\hat{Z}_0 \in D_1, \hat{Q} \in \Delta(0, R_0)$, this situation is the same as case 2.

Case 4: if $\hat{Z}_0, \hat{Q} \in D_1$. We know that $g^{-1}(z)$ is locally holomorphic from a subset of D_1 to Δ . Then we can choose the branch of the inverse map of g such that $Z_0 = g^{-1}(\hat{Z}_0) \in \Delta \setminus \Delta(0, r_0)$, then there exists a point $Q := g^{-1}(\hat{Q}) \in \Delta \setminus \Delta(0, r_0)$. Next, we need to show that the Kobayashi distance $d_\Delta(Z_0, Q)$ is still uniformly bounded. The way to prove this is similar to the proof of Theorem I.2.6.

If $r_0 \leq |Z_0| \leq \rho < 1, r_0 \leq |Q| \leq \rho < 1$, then using equation (I.2.2), we conclude that there is a constant C' such that $d_\Delta(Z_0, Q) \leq C'$. But if either $|Z_0| > \rho$ or $|Q| > \rho$, we can prove that $d_\Delta(Z_0, Q)$ is still uniformly bounded as follows.

We choose a sector D_1^+ inside D_1 such that $\hat{Z}_0, \hat{Q} \in D_1^+$, but $\hat{Z}_0, \hat{Q}$ are neither on ∂D_1^+ nor very close to ∂D_1^+ . The way to choose D_1^+ is the same as choosing $\mathbb{A}^+$ in the proof of Theorem I.2.6. Then we have

$$d_\Delta(Z_0, Q) \leq d_{D_1^+}(\hat{Z}_0, \hat{Q})$$

since $g^{-1}(z) : D_1^+ \rightarrow \Delta$ is holomorphic.

Now, we need to show that $d_{D_1^+}(\hat{Z}_0, \hat{Q})$ is bounded by some constant. In Wold's paper [14] (Theorem 3.4), he proved that $F_{D_1^+}(z, \xi_1) - F_\Delta(z, \xi_2) = O(\delta(z))$, where δ denotes the boundary distance. Then

$$\begin{aligned} d_{D_1^+}(\hat{Z}_0, \hat{Q}) &\leq d_\Delta(\hat{Z}_0, \hat{Q}) + d_E(\gamma(t), \partial\Delta) \leq C + \int_0^1 O\left(\left|\gamma(t) - \frac{\gamma(t)}{|\gamma(t)|}\right|\right) \gamma'(t) dt \\ &\leq C + \int_0^1 |\gamma'(t)| dt \leq C + |\hat{Z}_0 - \hat{Q}| < C + 2, \end{aligned}$$

where $\gamma(t)$ is a suitable curve from $\hat{Z}_0$ to $\hat{Q}$. Hence there exists a constant $C' := C + 2$ such that $d_\Delta(Z_0, Q) \leq d_{D_1^+}(\hat{Z}_0, \hat{Q}) < C'$.

Therefore, this theorem is true for all these four cases, i.e., there is a constant $\tilde{C}$ so that there exists a point $q = f^{-1}(\hat{q}) \in \Omega_2$ such that $d_{\Omega_2}(z_0, q) \leq \tilde{C}$.

Let us continuously consider the orbit between more connected components for the case (2), i.e., $\mathcal{A}$ has infinitely many connected components. Suppose the starting point z_0 is inside some connected component $\Omega_{i_1}, i_1 = 3, 4, 5, \dots$. Then there is a positive integer N_0 such that $\hat{z}_0 := f^{N_0}(z_0) \in \Omega_1$. Note that if $N_0 = 1$, it is the same as the orbit between two connected components, so the rest is to consider when $N_0 \geq 2$. By Theorem A, we have that there is a constant C such that for every point $\hat{z}_0 \in \Omega_1$, there exists a point $\hat{q} \in \cup_k f^{-k}(p), k \geq 0$ in Ω_1 so that $d_{\Omega_1}(\hat{z}_0, \hat{q}) \leq C$. Then we only need to show that there is a point $q \in \Omega_{i_1}$ such that $d_{\Omega_{i_1}}(z_0, q)$ is uniformly bounded. When finding q , which is some point of iterating

the inverse of $\hat{q} \in \Omega_1$, we need to be careful in dealing with the critical points when it appears in the inverse orbit.

We know that f has finitely many critical points, so there are only finitely many Ω_i containing critical points. Let Ω_{i_2} be a connected component satisfies $f(\Omega_{i_2}) = \Omega_1$ and $f^{N_0-1}(\Omega_{i_1}) = \Omega_{i_2}$. If there are some critical points in Ω_{i_2} , then we do the same procedure as above to find that there is a point $f^{-1}(\hat{q})$ such that $d_{\Omega_{i_2}}(f^{-1}(\hat{z}_0), f^{-1}(\hat{q}))$ is bounded by some constant C' . If there are no critical points inside Ω_{i_2} , then the Kobayashi metric is an isometry, thus we can choose $C' = C$, see the following Figure 1.7. Inductively, after N_0 times of iterating the inverse

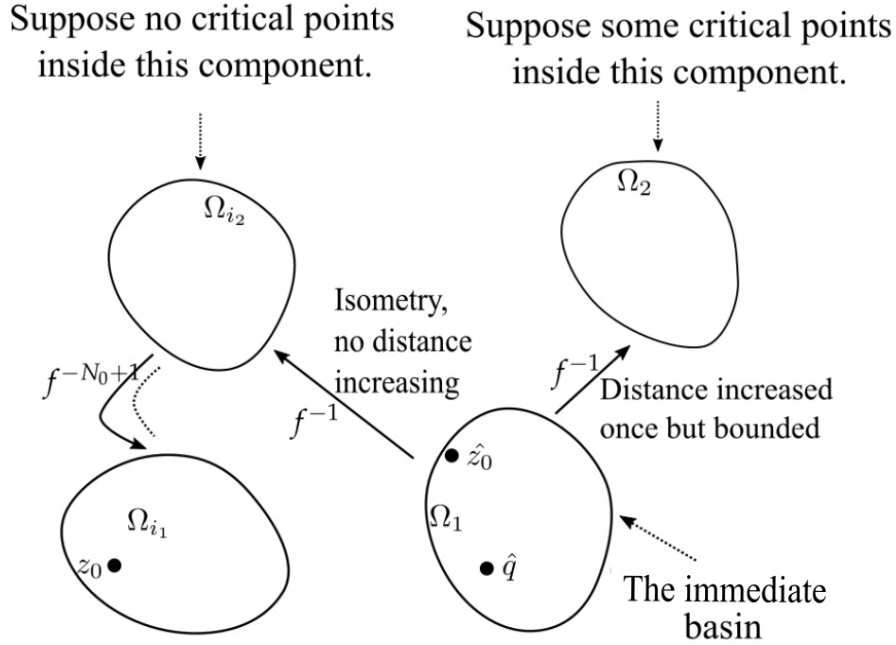


Figure 1.7: The Kobayashi distance increased finitely many times between infinitely many components.

orbit of $\hat{q}$, we definitely can find a constant so that for every point $z_0 \in \Omega_{i_1}$, there exists a point $q := f^{-N_0}(\hat{q}) \in \Omega_{i_1}$ such that $d_{\Omega_{i_1}}(z_0, q)$ is uniformly bounded. Note that the constant we want to find only increases when there are critical points in the component Ω_i by f^{-1} from a previous component, but there are only finitely many Ω_i containing critical points, it will only increase finitely many times. Hence, the uniform constant we need can be very large but it always exists.

Therefore, no matter how many connected components $\mathcal{A}(p)$ has, there is a constant $\tilde{C}$ so that for every point z_0 inside any Ω_i , there exists a point $q \in \cup_k f^{-k}(p)$ inside Ω_i such that $d_{\Omega_i}(z_0, q) \leq \tilde{C}$, where d_{Ω_i} is the Kobayashi distance on Ω_i .

□

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Paper II

Orbits inside Fatou sets

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Abstract

In this paper, we investigate the precise behavior of orbits inside attracting basins of rational functions on $\mathbb{P}^1$ and entire functions f in $\mathbb{C}$.

Let $R(z)$ be a rational function on $\mathbb{P}^1$, $\mathcal{A}(p)$ be the basin of attraction of an attracting fixed point p of R , and Ω_i ($i = 1, 2, \dots$) be the connected components of $\mathcal{A}(p)$. Assume Ω_1 contains p . Let $p_0 \in \Omega_1$ close to p . Then there exists a constant C so that for any $z_0 \in \Omega_i$, there is a point $q \in \cup_k R^{-k}(p_0)$, $k \geq 0$ so that the Kobayashi distance $d_{\Omega_i}(z_0, q) \leq C$.

For entire functions f , we generally can not have similar results as for rational functions. However, if f has finitely many critical points, then similar results hold.

II.1 Introduction

Let $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, $R : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a nonconstant holomorphic map, and $R^n : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be its n -fold iterate. In complex dynamics, two crucial disjoint invariant sets are associated with R , the *Julia set* and the *Fatou set* [1, 3, 15], which partition the sphere $\mathbb{P}^1$. The Fatou set of R is defined as the largest open set where the family of iterates is locally normal. In other words, for any point $z \in \mathbb{P}^1$, there exists some neighborhood U of z so that the sequence of iterates of the map restricted to U forms a normal family, so the iterates are well-behaved. The complement of the Fatou set is called the Julia set. The connected components of the Fatou set of R are called *Fatou components*. A Fatou component $\Omega \subset \mathbb{P}^1$ of R is *invariant* if $R(\Omega) = \Omega$.

The Fatou classification of invariant Fatou components [1, 3, 15] was studied by Fatou [6–8] and Julia in the early 20th century. In the 1930's, Cremer [4] completed the classification of rotation domains using the Uniformization Theorem. The existence of rotation domains (that is, Siegel disks and Herman rings) was proved even later than that, by Siegel [16] and Herman [12]. The classification of Fatou components was completed in the 1980s when Sullivan [17] proved that every Fatou component of a rational map is preperiodic, i.e., if R is a rational map of degree $d \geq 2$ on the Riemann sphere and Ω is an invariant component of the Fatou set, there are $n, m \in \mathbb{N}$ such that $R^{n+m}(\Omega) = R^m(\Omega)$. For a comprehensive understanding of additional details and results, see the paper [17] for further information. Therefore, all Fatou components are preperiodic to attracting basins, parabolic basins and rotation domains. The orbits in the rotation domains are easy

to describe since the functions are conjugate to irrational rotations. The orbits in attracting basins and parabolic basins are much more complicated. There are many results about the orbits near the boundary of an immediate basin $\mathcal{A}_0$ of an attracting fixed point p and also close to the attracting fixed point p using hyperbolicity. As far as we know, there is no systematic study of the rest of the orbits. This includes not only the behavior in the immediate basin, but also all the Fatou components that are preimages of the immediate attracting basin.

In addition, there have been few detailed studies until now of the more precise behavior of orbits inside the Fatou set. For example, let $\mathcal{A}(p) := \{z \in \mathbb{C}; f^n(z) \rightarrow p\}$ be the *basin of attraction* of an attracting fixed point p . An intriguing question arises when z_0 is inside of $\mathcal{A}(p)$ but very close to $\partial\mathcal{A}(p)$: what is the behavior of the orbits z_n starting from z_0 and approaching to the attracting fixed point p ? Specifically, if an orbit z_n starting from z_0 eventually reaches the point p , we can gain insight into the behavior of the orbit of z_0 by examining the backward orbit of the fixed point p . However, there are cases where the orbit $\{z_n\}$ of z_0 approaches p indefinitely without ever reaching it. In such situations, when considering the behavior of the orbit of z_0 , it becomes uncertain whether we can rely on the backward orbit of the fixed point p instead. Consequently, determining how to handle these situations becomes an interesting question. Furthermore, one might wonder how many iterations are required to reach this uncertain state. One application arises from Newton's method in [1]. It is of practical interest to know how many times Newton's method must be iterated to get the desired approximation of the root.

In the paper [9], Hu investigated the precise behavior of orbits inside attracting basins of polynomials in $\mathbb{C}$. Let f be a holomorphic polynomial of degree $m \geq 2$ in $\mathbb{C}$, $\mathcal{A}(p)$ be the basin of attraction of an attracting fixed point p of f , and Ω_i ($i = 1, 2, \dots$) be the connected components of $\mathcal{A}(p)$. Assume Ω_1 contains p and $\{f^{-1}(p)\} \cap \Omega_1 \neq \{p\}$. Then there is a constant C so that for every point z_0 inside any Ω_i , there exists a point $q \in \cup_k f^{-k}(p)$ inside Ω_i such that $d_{\Omega_i}(z_0, q) \leq C$, where d_{Ω_i} is the Kobayashi distance on Ω_i . It shows that in an attracting basin of a complex polynomial in $\mathbb{C}$, the backward orbit of the attracting fixed point either is the point itself or accumulates at the boundary of all the components of the basin in such a way that all points of the basin lie within a uniformly bounded distance of the backward orbit, measured with respect to the Kobayashi metric. However, in paper [10], Hu proved that this result is not valid for parabolic basins of polynomials in $\mathbb{C}$. Hu [11] also studied similar problems in higher dimensions and obtained a variety of interesting results.

It is natural to inquire about the precise behavior of orbits within the attracting basins of rational functions on $\mathbb{P}^1$. Exploring these questions forms the primary focus of this paper. In the second section, we recall some basic definitions of the Kobayashi metric and local coordinates near attracting fixed points. Then, we discuss when a basin of infinity is not simply connected for polynomials in the third section. In the fourth section, we investigate the dynamics of rational functions within the attracting basins and establish our main result, namely Theorem A.

Theorem A. Let $R(z)$ be a rational function on $\mathbb{P}^1$, $\mathcal{A}(p)$ be the basin of attraction of an attracting fixed point p of R , and Ω_i ($i = 1, 2, \dots$) be the connected

components of $\mathcal{A}(p)$. Assume Ω_1 contains p . Let $p_0 \in \Omega_1$ close to p . Then there exists a constant C so that for any $z_0 \in \Omega_i$, there is a point $q \in \cup_k R^{-k}(p_0)$, $k \geq 0$ so that the Kobayashi distance $d_{\Omega_i}(z_0, q) \leq C$.

Remark: In the next section II.2.2, we will clarify what we mean with p_0 close to p in Theorem A and Theorem II.3.4 in the following section II.3. There we will discuss also the dependence of C on the choice of p_0 .

Theorem A generalize the results in [9] which are attracting basins of fixed point p of polynomials in $\mathbb{C}$, the key property is that the attracting basins of polynomials in $\mathbb{C}$ are simply connected. This also works for rational functions in the case that the attracting basins are simply connected. In this paper, we considered the attracting basins which can have Fatou components that are not simply connected. This is the key difference between polynomials in $\mathbb{C}$ and rational functions in $\mathbb{P}^1$. This applies as well to the basin of infinity for a polynomial. It was observed in [9] that the theorem holds if the basin of attraction of ∞ of a polynomial is simply connected in $\mathbb{P}^1$.

In the fifth section, we also discuss the corresponding result for entire functions in $\mathbb{C}$. There, we show that Theorem A fails in general. Nevertheless, the theorem still holds after some mild conditions. (Finitely many critical points.)

II.2 Preliminary

II.2.1 The Kobayashi metric

First of all, let us recall some definitions of the Kobayashi metric.

Definition II.2.1. Let $\Omega \subset \mathbb{P}^1$ be a domain. We choose a point $z \in \Omega$ and a vector ξ which is tangent to the plane at the point z . Let Δ denote the unit disk in the complex plane. We define the Kobayashi metric [13]

$$F_{\Omega}(z, \xi) := \inf\{\lambda > 0 : \exists f : \Delta \xrightarrow{hol} \Omega, f(0) = z, \lambda f'(0) = \xi\}.$$

And $d_{\Omega}(z_1, z_2)$ is denoted as the Kobayashi distance between z_1 and z_2 , for any two points z_1 and z_2 in Ω .

Here we do not repeat the definition of Kobayashi distance, we refer readers for more details in the paper [9]. Note that $d_{\Omega}(z_1, z_2)$ is defined where z_1, z_2 are in the same connected component of Ω . And if $\Omega = \Delta$, then the Kobayashi metric is the same as the Poincaré metric (see page 9 in [5] and sections 0 and 3 in [13]).

We still want to repeat the following properties of Kobayshi metrics here since it is very important and we will use them many times in our prove later.

Proposition II.2.2 (The distance decreasing property of the Kobayashi Metric [13]). Suppose Ω_1, Ω_2 are domains in $\mathbb{P}^1$, $z, w \in \Omega_1, \xi \in \mathbb{C}$, and $f : \Omega_1 \rightarrow \Omega_2$ is holomorphic. Then

$$F_{\Omega_2}(f(z), f'(z)\xi) \leq F_{\Omega_1}(z, \xi), \quad d_{\Omega_2}(f(z), f(w)) \leq d_{\Omega_1}(z, w).$$

Corollary II.2.3. Suppose $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{C}$. Then for any $z, w \in \Omega_1$ and $\xi \in \mathbb{C}$, we have

$$F_{\Omega_2}(z, \xi) \leq F_{\Omega_1}(z, \xi), \quad d_{\Omega_2}(z, w) \leq d_{\Omega_1}(z, w).$$

II.2.2 Local coordinates near attracting fixed points

Let f be a holomorphic function in a neighborhood of an attracting fixed point p . There are classically two cases:

1. The attracting case where $0 < |f'(p)| < 1$. Then there exists a neighborhood U_p and a biholomorphic map $\varphi : U_p \rightarrow \Delta$ so that $\varphi(p) = 0$ and we have the commutative diagram

$$\varphi \circ f(z) = f'(p)\varphi(z).$$

This is the Linearization theorem of Kœnigs [14].

2. The superattracting case where $f'(p) = 0$. Then $f(z) = a_n(z - p)^n + \dots$ for some constant $a_n \neq 0$ and integer $n \geq 2$. Then there exists a neighborhood U_p and a biholomorphic map $\varphi : U_p \rightarrow \Delta$ so that $\varphi(p) = 0$ and we have the commutative diagram

$$\varphi \circ f(z) = (\varphi(z))^n.$$

This is the theorem by Böttcher [2].

In these two cases, let $K_r = \varphi^{-1}\{0 < |w| \leq r\}$, $0 < r < 1$. In the following theorems, we will let p_0 be any point in K_r . And then the constant C will be bounded independently of p_0 , but it depends on r .

II.3 Dynamics of polynomials inside attracting basins in $\mathbb{P}^1$

In the paper [9], for a polynomial f in $\mathbb{C}$ or if the basin of attraction $\mathcal{A}(\infty)$ of f is simply connected, Hu proved the following theorem and corollary:

Theorem II.3.1. *Suppose $f(z)$ is a polynomial of degree $N \geq 2$ on $\mathbb{C}$, p is an attracting fixed point of $f(z)$, Ω_1 is the immediate basin of attraction of p , $\{f^{-1}(p)\} \cap \Omega_1 \neq \{p\}$, $\mathcal{A}(p)$ is the basin of attraction of p , Ω_i ($i = 1, 2, \dots$) are the connected components of $\mathcal{A}(p)$. Then there is a constant $\tilde{C}$ so that for every point z_0 inside any Ω_i , there exists a point $q \in \cup_k f^{-k}(p)$ inside Ω_i such that $d_{\Omega_i}(z_0, q) \leq \tilde{C}$, where d_{Ω_i} is the Kobayashi distance on Ω_i .*

Corollary II.3.2. *Let f be a polynomial of degree $m \geq 2$, considered as a map on $\hat{\mathbb{C}}$. Suppose that $\Omega(\infty)$ is the basin of ∞ and is simply connected. If $p \in \Omega \setminus \{\infty\}$, then there exists a constant $C_0 > 0$ such that for every point $z_0 \in \Omega$, there exists $q \in \cup_k f^{-k}(p)$, $k \geq 0$ satisfying $d_{\Omega}(z_0, q) \leq C_0$, where d_{Ω} is the Kobayashi distance on Ω .*

It is very natural to ask, when the basin of infinity is not simply connected, whether we can still obtain analogous results. To answer it, the following lemma comes into play first:

Lemma II.3.3. *Let U_0 be a small disc in Böttcher coordinates centered at ∞ containing p and $U_n = f^{-n}(U_0)$. Set $W_{n+1} := U_{n+1} \setminus \bar{U}_n$. Then, the sets W_n have finitely many connected components $W_{n,i}$. There is at least one preimage of p in each connected component of each W_n .*

Proof. Since U_0 is a small disc in the Böttcher coordinates, we know that f is equivalent to z^N (N is actually the degree of f) near infinity. We then get that $U_1 = f^{-1}(U_0)$ is also such a disc with larger radius and f is a branched cover from U_1 to U_0 of order N . Note that for unbranched covers, the covering map is an isometry in the Kobayashi metric. However, this does not hold for branched covers. Inductively $U_n = f^{-n}(U_0)$ is a branched cover of order N of U_{n-1} . Then, all the U_n will be connected.

Now, we need to know whether W_n is connected. This is true when n is small, say $n \leq n'$. But then $W_{n'+1}$ will have at most N different components, each of which is a cover of $W_{n'}$. Similarly, W_n will have finitely many different components for all larger $n > n' + 1$.

The map $f : W_{n+1} \rightarrow W_n$ is a branched or unbranched cover of order N . Each W_n is a finite union of connected components $W_{n,i}$. For each $W_{n,i}$ there are finitely many connected components of W_{n+1} which are branched or unbranched covers of $W_{n,i}$.

Suppose that for each component $W_{n,i}$ of W_n , there is a preimage of the form $f^{-m}(p) \in W_{n,i}$. Pick any connected component $W_{n+1,j}$ of W_{n+1} . Then this is a covering of some $W_{n,i}$. Hence there exists a preimage $f^{-m-1}(p)$ in this component $W_{n+1,j}$. By induction, there is at least one preimage of p in each connected component of each W_n . □

This lemma implies that when $f^n(z_0)$ is in some connected component $W_{m,i}$ of W_m , there always exists a preimage $q \in \cup_k f^{-k}(p)$, $k \geq 0$ in the same component $W_{m,i}$.

Next, we show that we can still have similar results with Theorem II.3.1, even the basin of infinity is not simply connected.

Theorem II.3.4. *Let f be a polynomial in $\mathbb{P}^1$ and $\mathcal{A}(\infty)$ be the basin of attraction of infinity. Let $p \in \mathcal{A}(\infty)$ close to ∞ , $p \neq \infty$. Then there exists a constant C so that for any $z_0 \in \mathcal{A}(\infty)$, there is a point $q \in \cup_k f^{-k}(p)$, $k \geq 0$ so that the Kobayashi distance $d_{\mathcal{A}(\infty)}(z_0, q) \leq C$.*

Proof. We use the same notation as in Lemma II.3.3, by continuity of the Kobayashi distance, it is sufficient to prove the Theorem for z_0 in a dense subset of Ω_1 . Hence we can assume that the orbit of z_0 never intersects any ∂U_n .

By Lemma II.3.3, we know that the orbit of z_0 lands in the same connected component as one of the $f^{-k}(p)$. We know there exists an n_0 so that U_{n_0} contains all critical points inside $\mathcal{A}(\infty)$. Since we need to calculate the Kobayashi distance between q and $f^n(z_0)$ on the same component $W_{m,i}$, we need to avoid any preimages q landing on the boundary of any components of $W_{m,i}$. Thus, we add the extra assumption that there is no orbit of a critical point containing a point on the boundary of U_0 , and there is no preimage of p on the boundary of U_0 . Then the boundary of each U_n consists of smooth curves.

We can then also find a curve γ_n in the same component which connects q to $f^n(z_0)$. By induction we can lift γ_n to γ_k connecting $f^k(z_0)$ to some preimage $f^{-\ell}(p)$. This whole curve will then be inside the same component of $W_{n-k,i}$.

Let $V_n = \mathcal{A}(\infty) \setminus \overline{U}_n$. Then the map $f : V_{N_0+1} \rightarrow V_{N_0}, N_0 > n_0$ is an unbranched cover. Thus, for any two points z, w in V_{N_0} , we can find preimages $z' = f^{-1}(z), w' = f^{-1}(w)$ in V_{N_0+1} so that

$$d_{V_{N_0+1}}(z', w') = d_{V_{N_0}}(z, w).$$

By Corollary II.2.3, we know

$$d_{V_{N_0}}(z', w') \leq d_{V_{N_0+1}}(z', w').$$

Hence,

$$d_{V_{N_0}}(z', w') \leq d_{V_{N_0}}(z, w). \quad (\text{II.3.1})$$

Next, we need to consider z_0 is very close to the boundary of $\mathcal{A}(\infty)$ or not.

If z_0 is far away from ∞ and close to $\partial\mathcal{A}(\infty)$, we know that $f^{N_1}(z_0) \in W_1$ for some possibly large integer N_1 . We can then choose some $q' := f^{-M_1}(p) \in W_1$, M_1 is a positive integer.

Then we let $\gamma'_1 \in W_1$ be a curve connecting $f^{N_1}(z_0)$ and q' . We lift γ'_1 to γ'_2 , where γ'_2 connects $f^{N_1-1}(z_0)$ and some preimage $f^{-1}(q')$. This whole curve γ'_2 will be inside the same component of W_2, i . Then by induction, γ'_{N_1} connects z_0 and some preimage $f^{-N_1}(q')$ will be inside some component of $W_{N_1, j}$ as well.

Suppose $\gamma'_{\tilde{N}} \in W_{\tilde{N}}$ connecting $f^{N_1+1-\tilde{N}}(z_0)$ and some preimage $f^{-\tilde{N}+1}(q')$ is the first curve of $\gamma'_1, \dots, \gamma'_{N_1}$ landed in V_{n_0+1} for some integer $1 < \tilde{N} < N_1$. Since $W_{\tilde{N}} \subset \subset V_{n_0+1} \subset \mathcal{A}(\infty)$, there is a constant C_0 so that $d_{W_{\tilde{N}}}(f^{N_1+1-\tilde{N}}(z_0), f^{-\tilde{N}+1}(q')) \leq C_0$.

Then by Corollary II.2.3, we have

$$d_{\mathcal{A}(\infty)}(z_0, f^{-N_1}(q')) \leq d_{V_{n_0+1}}(z_0, f^{-N_1}(q')), \quad (\text{II.3.2})$$

and

$$d_{V_{n_0+1}}(f^{N_1+1-\tilde{N}}(z_0), f^{-\tilde{N}+1}(q')) \leq d_{W_{\tilde{N}}}(f^{N_1+1-\tilde{N}}(z_0), f^{-\tilde{N}+1}(q')). \quad (\text{II.3.3})$$

By the inequalities (II.3.1), we have

$$d_{V_{n_0+1}}(z_0, f^{-N_1}(q')) \leq d_{V_{n_0+1}}(f(z_0), f^{-N_1+1}(q')) \leq \dots \leq d_{V_{n_0+1}}(f^{N_1+1-\tilde{N}}(z_0), f^{-\tilde{N}+1}(q')). \quad (\text{II.3.4})$$

Hence, we have

$$d_{\mathcal{A}(\infty)}(z_0, f^{-N_1}(q')) \leq C_0.$$

If z_0 inside any $U_n \subset \mathcal{A}(\infty), n \leq n_0$. We can choose some $q = p$. In this case, since $U_n \subset \subset \mathcal{A}(\infty)$, there is a constant C'_0 so that $d_{\mathcal{A}(\infty)}(z_0, q) \leq C'_0$.

Therefore, let $C = \max\{C_0, C'_0\}$, there always exist a point $q \in \cup_k f^{-k}(p), k \geq 0$ such that

$$d_{\mathcal{A}(\infty)}(z_0, q) \leq C$$

for any $z_0 \in \mathcal{A}(\infty)$.

□

II.4 Dynamics of rational functions inside attracting basins in $\mathbb{P}^1$

In the third section, we considered the basin of infinity for polynomials, this is a simplest case of basins of attraction for rational functions which can be not simply connected. In this section, we will consider the general cases for rational functions.

Theorem A. Let $R(z)$ be a rational function on $\mathbb{P}^1$, $\mathcal{A}(p)$ be the basin of attraction of an attracting fixed point p of R , and Ω_i ($i = 1, 2, \dots$) be the connected components of $\mathcal{A}(p)$. Assume Ω_1 contains p . Let $p_0 \in \Omega_1$ close to p . Then there exists a constant C so that for any $z_0 \in \Omega_i$, there is a point $q \in \cup_k R^{-k}(p_0)$, $k \geq 0$ so that the Kobayashi distance $d_{\Omega_i}(z_0, q) \leq C$.

Proof. If z_0 is inside Ω_1 , then the proof is just a small variation of Theorem II.3.4 in section 3. The tiny difference is that ∞ can be an attracting fixed point of R but maybe not a superattracting fixed point.

If $z_0 \in \Omega_2$ and $R(\Omega_2) = \Omega_1$. We know $R(z_0) \in \Omega_1$. There exists a constant C so that for every $R(z_0) \in \Omega_1$, there exists a point $q \in \cup_k R^{-k}(p_0) \in \Omega_1$ such that $d_{\Omega_1}(R(z_0), q) \leq C$. Then we need to consider if we can find a preimage of q inside Ω_2 such that the Kobayashi distance between $R^{-1}(q)$ and z_0 can still be uniformly bounded.

Let $S_1 \subset \Omega_1$ be a nonempty compact subset which contains the images of the critical points of R in Ω_2 . For $i \geq 2$, we inductively let $S_i \subset \Omega_1$ be a compact subset with $S_{i-1} \subset S_i$ and which contains all points a in Ω_1 for which $d_{\Omega_1}(a, b) \leq C$ for all $b \in S_{i-1}$. We also let $T_i \subset \Omega_2$ be the inverse image of S_i (see the following Figure 2.8). Then $R : \Omega_2 \setminus T_i \rightarrow \Omega_1 \setminus S_i$ is an unbranched covering for each i .

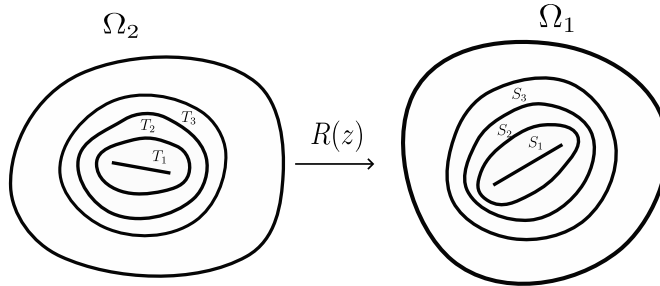


Figure 2.8: Increasing compact subsets inside Ω_1 and Ω_2 .

There is a constant L so that the Kobayashi metric

$$F_{\Omega_1 \setminus S_1}(a, \xi) \leq L F_{\Omega_1}(a, \xi)$$

for all (a, ξ) with $a \in \Omega_1 \setminus S_2$ and a tangent vector ξ at a .

Next we show that there exists a constant C' so that for every $z_0 \in \Omega_2$ there exists a $q_0 \in \cup_k R^{-k}(p_0) \in \Omega_2$ for which $d_{\Omega_2}(z_0, q_0) < C'$.

1. We suppose that $z_0 \in T_3$. Then $R(z_0) \in S_3$. Hence there exists a point $q \in \cup_k R^{-k}(p_0) \in \Omega_1$ for which $d_{\Omega_1}(R(z_0), q) < C$. This implies that $q \in S_4$.

Hence all preimages of q in Ω_2 are in T_4 . We can let q_0 be any of these. Since both z_0 and q_0 are in a compact set, the Kobayashi distance is bounded by some fixed constant C_1 .

2. Assume that $z_0 \in \Omega_2 \setminus T_3$. Then $R(z_0) \in \Omega_1 \setminus S_3$. There then exists a point $q \in \cup_k R^{-k}(p_0) \in \Omega_1$ for which $d_{\Omega_1}(R(z_0), q) < C$. This implies that $q \in \Omega_1 \setminus S_2$. Hence,

$$d_{\Omega_1 \setminus S_1}(R(z_0), q) < Ld_{\Omega_1}(R(z_0), q) \leq LC.$$

Since R is an unbranched covering over $\Omega_1 \setminus S_1$, there is a preimage q_0 of q so that $d_{\Omega_2 \setminus T_1}(z_0, q_0) < LC$. Since $\Omega_2 \setminus T_1 \subset \Omega_2$, it follows that $d_{\Omega_2}(z_0, q_0) < LC$. Finally we let $C' = \max\{C_1, LC\}$.

If z_0 is inside some other component $\Omega_i (i \geq 3)$ so that $R^j(\Omega_i) = \Omega_1$, we repeat the procedure above, we can prove that this theorem still holds.

Therefore, there is a constant C for every point z_0 inside any Ω_i , there exists a point $q \in \cup_k R^{-k}(p)$ inside Ω_i such that $d_{\Omega_i}(z_0, q) \leq C$, where d_{Ω_i} is the Kobayashi distance on Ω_i .

□

II.5 Dynamics of entire functions inside attracting basins in $\mathbb{C}$

In this section, we also discuss the corresponding result for entire functions in $\mathbb{C}$. There, we show that the theorem fails in general. Nevertheless, the theorem still holds after some mild conditions. (Finitely many critical points.)

We construct an entire function $f(z)$ with an attracting fixed point so that Theorem A fails.

Proposition II.5.1. *There exists an entire function $f(z)$ with the following properties:*

- (1) $f(0) = 0, 0 < |f'(0)| < 1$;
- (2) *There exists an increasing sequence of strictly positive numbers $A_n \rightarrow \infty$, so that $f(\{A_{2n} < |z| < A_{2n+1}\}) \subset \subset \{A_{2n+2} < |z| < A_{2n+3}\}$;*
- (3) *There exists a sequence $z'_n, A_{2n+1} < |z'_n| < A_{2n+2}$ so that $f(z'_n) = 0$ and f vanishes to order n at z'_n ;*
- (4) *On a small disc $|z - z'_n| < \delta_n, f = (z - z'_n)^n k_n(z)$ where k_n is approximately $\frac{2A_2}{\delta_n^n}$.*

Proof. Let $f_0(z) = \frac{1}{2}z$, then $f_0(z)$ satisfies (1) in Proposition II.5.1. We choose $A_2 = 1, A_3 = 2, A_4 = 4$. Let $\tilde{f}_1(z) = f_0(z) + \left(\frac{2z}{A_2}\right)^N$, N is some positive integer can be chosen later largely. Hence,

$$\tilde{f}_1(\{A_2 < |z| < A_3\}) \subset \subset \{A_4 < |z| < A_5\}$$

due to the term $\left(\frac{2z}{A_2}\right)^N$. Thus, $\tilde{f}_1(z)$ satisfies (1) and (2) in Proposition II.5.1. We choose $z'_1 = 3.5 \in \{A_3 < |z| < A_4\}$. Next, we need to construct a function vanishes at z'_1 but we do not want change the function values of $\tilde{f}_1$ too much since we need the sequence of f_n we constructed can be convergent. Hence, let $\hat{f}_1 = \tilde{f}_1 \cdot (z - z'_1)h_n$ where h_n approximately $\frac{1}{z - z'_1}$ very well on $\{|z| < A_3\}$. Thus, $\hat{f}_1$ satisfies (1), (2) and (3) in Proposition II.5.1. Let $f_1 = \hat{f}_1 k_1$ to satisfy (4) in Proposition II.5.1. Therefore, by induction, we construct a sequence

$$\begin{aligned} f_1 &= \left[\frac{1}{2}z + \left(\frac{2z}{A_2} \right)^N \right] (z - z'_1)h_1 k_1, \\ &\dots \\ f_n &= \left[f_{n-1} + (z - z'_1) \cdots (z - z'_{n-1})^{n-1} \left(\frac{2z}{A_{2n}} \right)^N \right] (z - z'_n)h_n k_n \end{aligned}$$

where h_n are polynomials approximating $\frac{1}{(z - z'_n)^n}$ very well on $\{|z| < A_{2n+1}\}$.

Let $f := \lim_{n \rightarrow \infty} f_n$ satisfies all properties above. Then f is entire on $\mathbb{C}$ if the limit exists. We know when $|z| < \frac{A_{2n}}{4} < A_{2n+1}$, hence h_n is arbitrary close to $\frac{1}{(z - z'_n)^n}$, i.e., $|(z - z'_n)^n h_n - 1| < \varepsilon_n$ for all $|z| < \frac{A_{2n}}{4} < A_{2n+1}$. Then

$$\begin{aligned} &|f_n - f_{n-1}| \\ &\leq |f_{n-1}(z - z'_n)^n h_n k_n - f_{n-1}| + \left| (z - z'_1) \cdots (z - z'_{n-1})^{n-1} \left(\frac{2z}{A_{2n}} \right)^N (z - z'_n)^n h_n k_n \right| \\ &\leq |f_{n-1}[(z - z'_n)^n h_n k_n - 1]| + \frac{1}{2^{n+1}} < \frac{1}{2^n} \end{aligned}$$

if h_n is chosen close enough to $\frac{1}{(z - z'_n)^n}$, k_n is close enough to 1 and N is large enough. Hence, there exists an entire function satisfies Proposition II.5.1. $\square$

Theorem II.5.2. *Let f be the entire function satisfying four properties as above. Let $\mathcal{A}(0)$ be the basin of attraction of 0 of f , and $\Omega_n (n = 1, 2, 3, \dots)$ be the connected components containing z'_n of $\mathcal{A}(0)$, and Ω_0 contain 0. We choose an arbitrary constant $C > 0$. Then there exists a point z_0 inside some Ω_n 's so that for any $q \in \cup_{k=0}^{\infty} \{f^{-k}(0)\}$ (k is non negative integer) and q inside the same component Ω_n with z_0 , the Kobayashi distance $d_{\Omega_n}(z_0, q) \geq C$, where d_{Ω_n} is the Kobayashi distance on Ω_n .*

Proof. The function f has infinitely many preimages of 0 by the property (3) of Proposition II.5.1, meanwhile, all these z'_n are also critical points of f . Since Ω_0 is the immediate basin of attraction, Ω_0 is simply connected. Since Ω_n are the Fatou components containing z'_n . Then $\Omega_n \subset \{A_{2n+1} < |z| < A_{2n+2}\}$. By the maximum principle, it must be simply connected.

$$\begin{array}{ccc} \Omega_n, \{f^{-1}(\hat{z}_0), z'_n\} & \xrightarrow{f} & \Omega_0, \{\hat{z}_0, 0\} \\ \varphi \downarrow & & \downarrow \psi \\ \Delta, \{F^{-1}(\frac{1}{2}), 0\} & \xrightarrow{F=z^n g} & \Delta, \{\frac{1}{2}, 0\} \end{array}$$

The map $f : \Omega_n \rightarrow \Omega_0$ is conjugate to a map F on the unit disc with the property that $F(z) = z^n g$. By property (4) Proposition II.5.1, here g will have no zeroes and must be uniformly close to 1 in absolute value. If we choose a point $\frac{1}{2} \in \Delta$, then $F^{-1}(\frac{1}{2}) \rightarrow \partial\Delta$. Then $d_\Delta(0, F^{-1}(\frac{1}{2})) \rightarrow \infty$ as $n \rightarrow \infty$. Let z^* be the preimage of $F^{-1}(\frac{1}{2}) \in \Omega_n$. Then we know $d_{\Omega_n}(z'_n, z^*) = d_\Delta(0, F^{-1}(\frac{1}{2})) \rightarrow \infty$ as $n \rightarrow \infty$. Hence, the Kobayashi distance will go to infinity. $\square$

On the other hand, we have the following positive results:

Theorem II.5.3. *Suppose that $f(z)$ is an entire function with 0 as an attracting fixed point with nonzero derivative at the origin. Suppose that f has finitely many critical points. Suppose also that all Fatou components are bounded open sets. Then Theorem A holds.*

Proof. Since all Fatou components are bounded, they are also simply connected. On the immediate basin of attraction, the map behaves similarly with a polynomial on the immediate basin of attraction. If we consider the map $f : \Omega_1 \rightarrow \Omega_2$ on preimages of the immediate basin, f will be biholomorphic if there is no critical point in Ω_1 . Furthermore, only for finitely many choices of Ω_1 will there be a critical point inside. So the situation is the same as for a polynomial. $\square$

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Paper III

Dynamics inside Parabolic Basins

Mi Hu

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Abstract

In this paper, we investigate the behavior of orbits inside parabolic basins. Let $f(z) = z + az^{m+1} + (\text{higher terms})$, $m \geq 1$, $a \neq 0$, and Ω_j be the immediate basin of $\mathcal{A}_j$. We choose an arbitrary constant $C > 0$ and a point $q \in \mathbf{v}_j \cap \mathcal{P}_j$. Then there exists a point $z_0 \in \Omega_j$ so that for any $\tilde{q} \in Q := (\cup_{l=0}^{\infty} f^{-l}(f^k(q))) \cap \Omega_j$ (l, k are non-negative integers), the Kobayashi distance $d_{\Omega_j}(z_0, \tilde{q}) \geq C$, where d_{Ω_j} is the Kobayashi distance on Ω_j . In a previous paper [15], we showed that this result is not valid for attracting basins.

Keywords: Parabolic basins, Kobayashi metric, Complex Dynamics

III.1 Introduction

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a nonconstant holomorphic map, and $f^n : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be its n -fold iterate. In complex dynamics [3, 7, 9, 10, 17], there are two crucial disjoint invariant sets, the *Julia set*, and the *Fatou set* of f , which partition the sphere $\hat{\mathbb{C}}$. The Fatou set of f is defined as the largest open set $\mathcal{F}$ where the family of iterates is locally normal. In other words, for any point $z \in \mathcal{F}$, there exists some neighborhood U of z so that the sequence of iterates of the map restricted to U forms a normal family, so the orbits of iteration are well-behaved. The complement of the Fatou set is called the Julia set.

For $z \in \hat{\mathbb{C}}$, we call the set $\{z_n\} = \{z_1 = f(z_0), z_2 = f^2(z_0), \dots\}$ the orbit of the point $z = z_0$. If $z_N = z_0$ for some integer N , we say that z_0 is a periodic point of f . If $N = 1$, then z_0 is a fixed point of f .

There have been many studies about probability measures that can describe the dynamics on the Julia set. For example, if z is any non-exceptional point, the inverse orbits $\{f^{-n}(z)\}$ equidistribute toward the Green measure μ which lives on the Julia set. This was proved already by Brolin [6] in 1965, and many improvements and generalizations have been made [8, 11]. However, this equidistribution toward μ is in the weak sense, and hence it is with respect to the Euclidean metric. Therefore, it is a reasonable question to ask how dense $\{f^{-n}(z)\}$ is in $\mathcal{F}$ near the boundary of $\mathcal{F}$ if we use finer metrics, for instance, the Kobayashi metric. The Kobayashi metric is an important tool in complex dynamics, see examples in [1, 2, 4, 5].

There are some classical results about the behavior of a rational function on the Fatou set $\mathcal{F}$ as well. The connected components of the Fatou set of f are called

Fatou components. A Fatou component $\Omega \subset \hat{\mathbb{C}}$ of f is invariant if $f(\Omega) = \Omega$. At the beginning of the 20th century, Fatou [12–14] classified all possible invariant Fatou components of rational functions on the Riemann sphere. He proved that only three cases can occur: attracting, parabolic, and rotation. And in the '80s, Sullivan [19] completed the classification of Fatou components. He proved that every Fatou component of a rational map is eventually periodic, i.e., there are $n, m \in \mathbb{N}$ such that $f^{n+m}(\Omega) = f^m(\Omega)$. Furthermore, there are many investigations about the boundaries of the connected Fatou components. In a recent paper by Roesch and Yin [18], they proved that if f is a polynomial, any bounded Fatou component, which is not a Siegel disk, is a Jordan domain.

In the paper [15], we investigated the behavior of orbits inside attracting basins, especially near the boundary, and obtained the following theorem:

Theorem III.1.1. *Suppose $f(z)$ is a polynomial of degree $N \geq 2$ on $\mathbb{C}$, p is an attracting fixed point of $f(z)$, Ω_1 is the immediate basin of attraction of p , $\{f^{-1}(p)\} \cap \Omega_1 \neq \{p\}$, and $\mathcal{A}(p)$ is the basin of attraction of p , $\Omega_i (i = 1, 2, \dots)$ are the connected components of $\mathcal{A}(p)$. Then there is a constant $\tilde{C}$ so that for every point z_0 inside any Ω_i , there exists a point $q \in \cup_k \{f^{-k}(p)\}$ inside Ω_i such that $d_{\Omega_i}(z_0, q) \leq \tilde{C}$, where d_{Ω_i} is the Kobayashi distance on Ω_i .*

For the case when $\{f^{-1}(p)\} \cap \Omega_1 = \{p\}$, we proved a suitably modified version of Theorem III.1.1 in our paper [15]. In conclusion, we proved that there is a constant C and a point p' ($p' = p$ or p' is very close to p) so that for every point z_0 inside any Ω_i , there exists a point $q \in \cup_k \{f^{-k}(p')\}$ inside Ω_i such that $d_{\Omega_i}(z_0, q) \leq C$, where d_{Ω_i} is the Kobayashi distance on Ω_i . This Theorem III.1.1 essentially shows that any arbitrary orbit can be tracked by an orbit of one preimage of the fixed point p .

It is a natural question to ask if the same statement holds for all parabolic basins. Our main results as following in this paper show that theorem III.1.1 in paper [15] is no longer valid for parabolic basins:

Theorem A. Let $f(z) = z + z^2$. We choose an arbitrary constant $C > 0$ and the point $q = -\frac{1}{2} \in \mathcal{A}$. Then there exists a point $z_0 \in \mathcal{A}$ so that for any $\tilde{q} \in Q := \cup_{l,k=0}^{\infty} \{f^{-l}(f^k(-\frac{1}{2}))\}$ (l, k are non negative integer), the Kobayashi distance $d_{\mathcal{A}}(z_0, \tilde{q}) \geq C$, where $d_{\mathcal{A}}$ is the Kobayashi distance on $\mathcal{A}$.

Here $\mathcal{A} = \mathcal{A}(0, -1)$ in Theorem A is the parabolic basin of $f = z + z^2$ with the attraction vector $\mathbf{v} = -1$, see Definitions III.2.1 and III.2.4. Note that for $f = z + z^2$, there is only one attracting vector, that is $\mathbf{v}_j = \mathbf{v} = -1$.

In this Theorem A, we cannot directly choose to iterate the inverse of the parabolic fixed point for a parabolic basin. This slightly differs from attracting basins in Theorem III.1.1. However, we still want to choose a point that can be very close to the parabolic fixed point. Hence, we choose $q \in \mathcal{A}$ and iterate it k times, then $f^k(q)$ is getting very close to the parabolic fixed point. Then we iterate the inverse of $f^k(q)$. To simplify, here we choose $q = -\frac{1}{2}$.

By Theorem A and Proposition III.2.7, we have the following Corollary which says the set of all z_0 clusters at every point in the boundary of the parabolic basin of $f = z + z^2$:

Corollary. III.3.4 *Let $X \subset \mathcal{A}$ be the set of all $z_0 \in \mathcal{A}$ so that $d_{\mathcal{A}}(z_0, \tilde{q}) \geq C$ for any $\tilde{q} \in \mathcal{A}$. If $z \in X$, then any point $w \in f^{-1}(z)$ is in X . Hence X is dense in the boundary of $\mathcal{A}$.*

We also generalize Theorem A to the case of several petals inside the parabolic basin in Theorem B:

Theorem B. Let $f(z) = z + az^{m+1}$, $m \geq 1$, $a \neq 0$, and Ω_j be the immediate basin of $\mathcal{A}_j$. We choose an arbitrary constant $C > 0$ and a point $q \in \mathbf{v}_j \cap \mathcal{P}_j$. Then there exists a point $z_0 \in \Omega_j$ so that for any $\tilde{q} \in Q := (\cup_{l=0}^{\infty} f^{-l}(f^k(q))) \cap \Omega_j$ (l, k are non-negative integers), the Kobayashi distance $d_{\Omega_j}(z_0, \tilde{q}) \geq C$, where d_{Ω_j} is the Kobayashi distance on Ω_j .

Here $\mathbf{v}_j$ is an attraction vector in the tangent space of $\mathbb{C}$ at 0, $\mathcal{P}_j$ is an *attracting petal* for f for the vector $\mathbf{v}_j$ at 0, $\mathcal{A}_j = \mathcal{A}(0, \mathbf{v}_j)$ is the parabolic basin of attraction associated to $\mathbf{v}_j$, see Definitions III.2.1, III.2.4 and III.2.5.

In the end, we consider the behavior of orbits inside parabolic basins of general polynomials.

Theorem C. Let $f(z) = z + az^{m+1} + (\text{higher terms})$, $m \geq 1$, $a \neq 0$, and Ω_j be the immediate basin of $\mathcal{A}_j$. We choose an arbitrary constant $C > 0$ and a point $q \in \mathbf{v}_j \cap \mathcal{P}_j$. Then there exists a point $z_0 \in \Omega_j$ so that for any $\tilde{q} \in Q := (\cup_{l=0}^{\infty} f^{-l}(f^k(q))) \cap \Omega_j$ (l, k are non-negative integers), the Kobayashi distance $d_{\Omega_j}(z_0, \tilde{q}) \geq C$, where d_{Ω_j} is the Kobayashi distance on Ω_j .

This paper is organized as follows: in section 2, we recall some definitions and results [17] about holomorphic dynamics of polynomials in a neighborhood of the parabolic fixed point and the Kobayashi metric; in section 3, we prove our main results, Theorem A, Theorem B, and Theorem C.

Acknowledgement

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III.2 Preliminary

III.2.1 Holomorphic dynamics of polynomials in a neighborhood of a parabolic fixed point.

Let us first recall some definitions and results [17] about holomorphic dynamics of a polynomial $f = z + a_2 z^2 + a_3 z^3 + \dots$ in a neighborhood of the parabolic fixed point 0.

Definition III.2.1. Let $f(z) = z + az^{m+1} + (\text{higher terms})$, $m \geq 1$, $a \neq 0$. A complex number $\mathbf{v}$ will be called an attraction vector at the origin if $ma\mathbf{v}^m = -1$, and a repulsion vector at the origin if $ma\mathbf{v}^m = 1$. Note here that $\mathbf{v}$ should be thought of as a tangent vector at the origin. We say that some orbit $z_0 \mapsto z_1 \mapsto z_2 \mapsto \cdots$ for the map f converges to zero nontrivially if $z_n \rightarrow 0$ as $n \rightarrow \infty$, but no z_n is actually equal to zero. There are m equally spaced attraction vectors at the origin.

Lemma III.2.2. If an orbit $f : z_0 \mapsto z_1 \mapsto \cdots$ converges to zero nontrivially, then z_k is asymptotic to $\mathbf{v}_j / \sqrt[m]{k}$ as $k \rightarrow +\infty$ for one of the m attraction vectors $\mathbf{v}_j$.

Proof. See the proof in chapter 10 of Milnor's book [17]. □

Definition III.2.3. If an orbit $z_0 \mapsto z_1 \mapsto \cdots$ under f converges to zero with $z_k \sim \mathbf{v}_j / \sqrt[m]{k}$, we will say that this orbit $\{z_k\}$ tends to zero from the direction $\mathbf{v}_j$.

Definition III.2.4. Given an attraction vector $\mathbf{v}_j$ in the tangent space of $\mathbb{C}$ at 0, the associated parabolic basin of attraction $\mathcal{A}_j = \mathcal{A}(0, \mathbf{v}_j)$ is defined to be the set consisting of all $z_0 \in \mathbb{C}$ for which the orbit $z_0 \mapsto z_1 \mapsto \cdots$ converges to 0 from the direction $\mathbf{v}_j$.

Definition III.2.5. Suppose f is defined and univalent on some neighborhood $N \in \mathbb{C}$. An open set $\mathcal{P}_j \subset N$ is called an attracting petal for f for the vector $\mathbf{v}_j$ at 0 if

- (1) f maps $\mathcal{P}_j$ into itself, and
- (2) an orbit $z_0 \mapsto z_1 \mapsto \cdots$ under f is eventually absorbed by $\mathcal{P}_j$ if and only if it converges to 0 from the direction $\mathbf{v}_j$.

III.2.2 The Kobayashi metric

Definition III.2.6. Let $\hat{\Omega} \subset \mathbb{C}$ be a domain. We choose a point $z \in \hat{\Omega}$ and a vector ξ tangent to the plane at the point z . Let Δ denote the unit disk in the complex plane. We define the Kobayashi metric

$$F_{\hat{\Omega}}(z, \xi) := \inf\{\lambda > 0 : \exists f : \Delta \xrightarrow{\text{hol}} \hat{\Omega}, f(0) = z, \lambda f'(0) = \xi\}.$$

Let $\gamma : [0, 1] \rightarrow \hat{\Omega}$ be a piecewise smooth curve. The Kobayashi length of γ is defined to be

$$L_{\hat{\Omega}}(\gamma) = \int_{\gamma} F_{\hat{\Omega}}(z, \xi) |dz| = \int_0^1 F_{\hat{\Omega}}(\gamma(t), \gamma'(t)) |\gamma'(t)| dt.$$

For any two points z_1 and z_2 in $\hat{\Omega}$, the Kobayashi distance between z_1 and z_2 is defined to be

$$d_{\hat{\Omega}}(z_1, z_2) = \inf\{L_{\hat{\Omega}}(\gamma) : \gamma \text{ is a piecewise smooth curve connecting } z_1 \text{ and } z_2\}.$$

Note that $d_{\hat{\Omega}}(z_1, z_2)$ is defined when z_1 and z_2 are in the same connected component of $\hat{\Omega}$.

Proposition III.2.7 (The distance decreasing property of the Kobayashi Metric [16]). *Suppose Ω_1 and Ω_2 are domains in $\mathbb{C}$, $z, w \in \Omega_1, \xi \in \mathbb{C}$, and $f : \Omega_1 \rightarrow \Omega_2$ is holomorphic. Then*

$$F_{\Omega_2}(f(z), f'(z)\xi) \leq F_{\Omega_1}(z, \xi), \quad d_{\Omega_2}(f(z), f(w)) \leq d_{\Omega_1}(z, w).$$

Corollary III.2.8. *Suppose $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{C}$. Then for any $z, w \in \Omega_1$ and $\xi \in \mathbb{C}$, we have*

$$F_{\Omega_2}(z, \xi) \leq F_{\Omega_1}(z, \xi), \quad d_{\Omega_2}(z, w) \leq d_{\Omega_1}(z, w).$$

III.3 Proof of the main theorems

In this section, we will prove our main theorems: Theorem A, Theorem B, and Theorem C.

III.3.1 Dynamics inside the parabolic basin of $f(z) = z + z^2$

Let us recall the statement of our main Theorem A:

Theorem A. Let $f(z) = z + z^2$. We choose an arbitrary constant $C > 0$ and the point $q = -\frac{1}{2} \in \mathcal{A}$. Then there exists a point $z_0 \in \mathcal{A}$ so that for any $\tilde{q} \in Q := \cup_{l,k=0}^{\infty} \{f^{-l}(f^k(-\frac{1}{2}))\}$ (l, k are non negative integer), the Kobayashi distance $d_{\mathcal{A}}(z_0, \tilde{q}) \geq C$, where $d_{\mathcal{A}}$ is the Kobayashi distance on $\mathcal{A}$.

As pointed out above, this is opposite to Theorem III.1.1 in the paper [15] for attracting basins. Note that in this Theorem A, we cannot directly choose to iterate the inverse of the parabolic fixed point for a parabolic basin. This slightly differs from attracting basins in Theorem III.1.1. However, we still want to choose a point that can be very close to the parabolic fixed point. Hence, we choose $q \in \mathcal{A}$ and iterate it k times, then $f^k(q)$ is getting very close to the parabolic fixed point. Then we iterate the inverse of $f^k(q)$. To simplify, here we choose $q = -\frac{1}{2}$.

Proof. Let $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^- = (-\infty, 0]$ be the positive and negative real axis, respectively. Then the parabolic basin $\mathcal{A} \subsetneq (\mathbb{C} \setminus \mathbb{R}^+)$. Let $\mathbb{H}$ be the upper half-plane, $\varphi(z) : \mathbb{C} \setminus \mathbb{R}^+ \rightarrow \mathbb{H}$ with $\varphi(z) = \sqrt{z}$ (see Figure 3.8). We know the Kobayashi metric on the upper half plane is $F_{\mathbb{H}} = \frac{|d\omega|}{\text{Im } \omega}$ for $\omega \in \mathbb{H}$. Hence the Kobayashi metric on $\mathbb{C} \setminus \mathbb{R}^+$ is $F_{\mathbb{C} \setminus \mathbb{R}^+} = \frac{|\varphi'|}{\text{Im } \varphi} |dz| = \frac{1}{2|\sqrt{z}|\text{Im } \sqrt{z}} |dz| = \frac{1}{2r \sin \frac{\theta}{2}} |dz|$ for $z = re^{i\theta} \in \mathbb{C} \setminus \mathbb{R}^+, i.e., \theta \in (0, 2\pi)$.

We know that $f^k(q)$ will always be on the negative real axis for any integer $k \geq 0$. We choose $\theta_0 > 0$ and two rays $l_1 := \{z = re^{i\theta_0}, r > 0\}, l_2 := \{z = re^{-i\theta_0}, r > 0\}$. Then we denote by $T := \{z = re^{i\theta}, r > 0, \theta_0 \leq \theta \leq 2\pi - \theta_0\}$, a sector inside $\mathbb{C} \setminus \mathbb{R}^+$.

Before continuing with the proof of Theorem A, we have the following well-known Lemma III.3.2. For the reader's convenience and to introduce notation, we include the proof and define a Left/Right Pac-Man for an easy explanation of the proof.

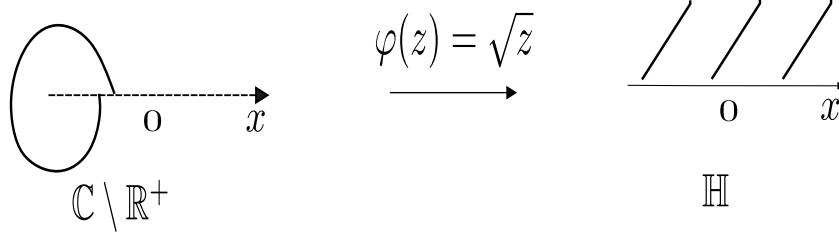


Figure 3.9: The map φ

Definition III.3.1. We call a domain $D_R := \{z = re^{i\theta}, 0 < r \leq R, \theta_0 < \theta < 2\pi - \theta_0\}$ a Left Pac-Man and $\tilde{D}_R := \{z = re^{i\theta}, 0 < r \leq R, -\pi + \theta_0 < \theta < \pi - \theta_0\}$ a Right Pac-Man.

Lemma III.3.2. There exists a Left Pac-Man D_{R_0} such that $D_{R_0} \subsetneq T \cap \mathcal{A}$.

Proof. We want to know how orbits go precisely near the parabolic fixed point at 0. Let $\omega = \varphi(z) = -1/z$ send 0 to ∞ , then the conjugated map has the expansion

$$F(\omega) = \varphi \circ f \circ \varphi^{-1}(\omega) = \omega + 1 + o(1) \quad \text{as } |\omega| \rightarrow \infty.$$

And we have $l_1(l_2)$ is mapped to two new rays $l_1^\omega := \{z = re^{i(\pi-\theta_0)}, r > 0\}$ ($l_2^\omega := \{z = re^{i(-\pi+\theta_0)}\}$); T is mapped to $T' = \{re^{i\theta}, r > 0, -\pi + \theta_0 \leq \theta \leq \pi - \theta_0\}$; the Left Pac-Man D_R is mapped to $T' \setminus \tilde{D}_{\frac{1}{R}}$ for any radius R (see Figure 3.10).

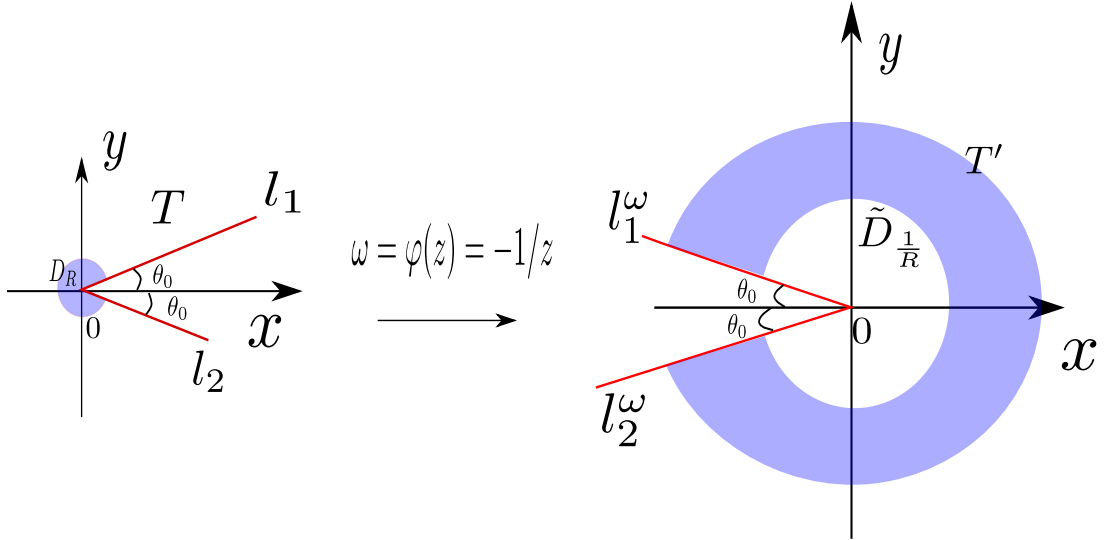


Figure 3.10: The image of a Left Pac-Man

We choose a Right Pac-Man $\tilde{D}_{\frac{1}{r_0}}$ such that for any $\omega \in T' \setminus \tilde{D}_{\frac{1}{r_0}}$, we have $|o(1)| < \frac{\theta_0}{3}$. Note here, actually, $r_0 = r_0(\theta_0)$ should be sufficiently small. Then we draw the upper tangent line L_0 of $\tilde{D}_{\frac{1}{r_0}}$ such that the angle between L_0 and the

real axis is $\frac{\theta_0}{2}$. Then L_0 will intersect l_1^ω and the real axis, we denote these two intersect points A_0 and B_0 , respectively. Let $\frac{1}{r} = \max\{|OA_0|, |OB_0|\} > \frac{1}{r_0}$, here O is the origin zero, then we choose the Right Pac-Man $\tilde{D}_{\frac{1}{r}}$ (see Figure 3.11).

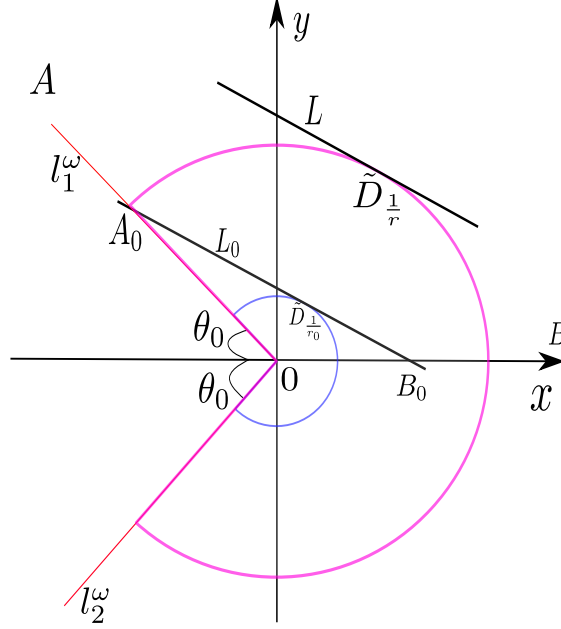


Figure 3.11: The choice of the Right Pac-Man

If we take any $\omega_0 \in T' \setminus \tilde{D}_{\frac{1}{r}}$, $F^n(\omega_0) = \omega_0 + n + o(1)$ for all positive integers n such that $|F^n(\omega_0)| \geq |\omega_0 + n - n\frac{\theta_0}{3}|$, then we know that $F^n(\omega_0)$ will never go inside $\tilde{D}_{\frac{1}{r_0}}$.

Therefore, let $R_0 = r, \tilde{R}_0 = r_0$, then we have for any $z_0 \in D_{R_0}$, $f^n(z_0) \rightarrow 0$, hence $D_{R_0} \subsetneq T \cap \mathcal{A}$, and we also know that $f^n(D_{R_0}) \subseteq D_{\tilde{R}_0}$. $\square$

Lemma III.3.3. *We can choose a Left Pac-Man $D_{R'_0} \subseteq D_{R_0}$ such that $f^n(D_{R'_0}) \subseteq D_{R_0}$ for all $n = 1, 2, \dots$*

Proof. On the procedure for proving Lemma III.3.2, we can draw another upper tangent line L of $\tilde{D}_{R_0}$ such that the angle between L and the real axis is $\frac{\theta_0}{2}$. Then L will also intersect l_1^ω and the real axis, we denote these two intersect points A and B respectively. Let $\frac{1}{r'} = \max\{|OA|, |OB|\}$, then we choose the Right Pac-Man $\tilde{D}_{\frac{1}{r'}}$. If we take any $\omega \in T' \setminus \tilde{D}_{\frac{1}{r'}}$, we know that $F^n(\omega)$ will never go inside of $\tilde{D}_{\frac{1}{r}}$. Hence, let $R'_0 = \frac{1}{r'}$, we have $f^n(D_{R'_0}) \subseteq D_{R_0}$. $\square$

We continue with the proof of Theorem A. The idea of the proof is to find a point $z_0 \in D_\varepsilon \setminus \mathbb{R}^-$ such that for any $\tilde{q} \in Q$, we have $d_{\mathcal{A}}(z_0, \tilde{q}) \geq C$.

Now, we will consider the following cases of $\tilde{q}$ inside three subsets of $\mathcal{A}$ (see Figure 3.12):

Case 1: $\tilde{q} \in ((T \cap \mathcal{A}) \setminus D_{R'_0}) \setminus \mathbb{R}^-$. Let $d_{\mathcal{A}}^1(z_0, \tilde{z})$ be the Kobayashi distance between z_0 and any point $\tilde{z} \in \partial D_{R'_0} \setminus R^-$ (see the blue curve in Figure 3.12). Then we prove that $d_{\mathcal{A}}^1(z_0, \tilde{z}) \geq C$.

Case 2: $\tilde{q} \in \mathbb{R}^-$. Let $d_{\mathcal{A}}^2(z_0, z')$ be the Kobayashi distance between z_0 and any point $z' \in \mathbb{R}^- \cap \mathcal{A}$ (see the pink curve in Figure 3.12). Then we prove that $d_{\mathcal{A}}^2(z_0, z') \geq C$.

Case 3: $\tilde{q} \in \mathcal{A} \cap \{S' := \{z = re^{i\theta}, r > 0, 0 < \theta \leq \theta_0\}\}$. Let $d_{\mathcal{A}}^3(z_0, \hat{z})$ be the Kobayashi distance between z_0 and any point $\hat{z} \in l_1$ (see the green curve in Figure 3.12). Then we prove that $d_{\mathcal{A}}^3(z_0, \hat{z}) \geq C$.

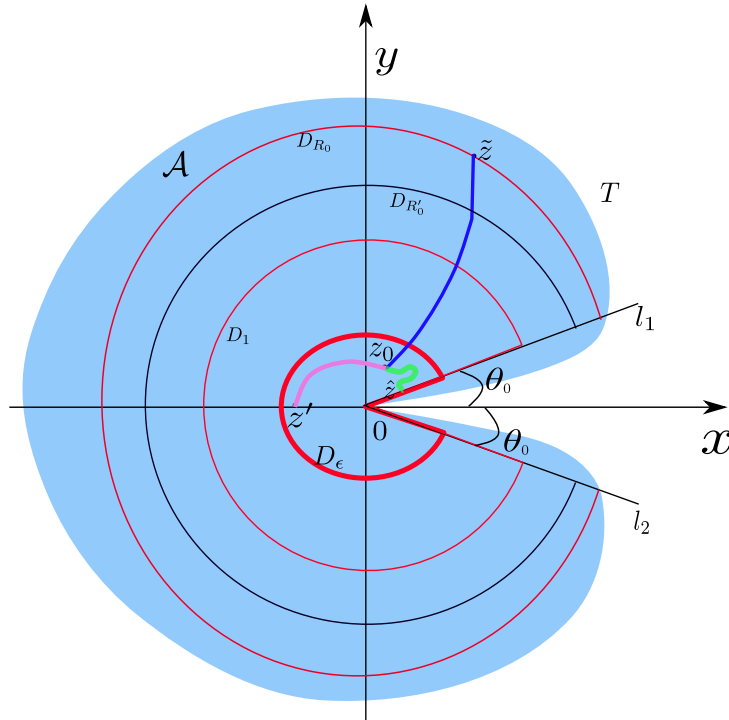


Figure 3.12: The three Kobayashi distances in $\mathcal{A}$

Remark:

1) In the investigation of these three cases, it will become clear how small θ_0 needs to be.

2) $\partial D_{R'_0}$ means the boundary of the Left Pac-Man $D_{R'_0}$. It is the circular curve of $D_{R'_0}$, not including the mouth of $D_{R'_0}$ which belongs to l_1 and l_2 ;

3) $\tilde{q}$ can never be inside $D_{R'_0} \setminus \mathbb{R}^-$: If $\tilde{q} \in D_{R'_0} \setminus \mathbb{R}^-$, more precisely, suppose $f^{-l}(f^k(q)) \in D_{R'_0} \setminus \mathbb{R}^-$ for some integers $l, k \geq 0$, we iterate l times of $f^{-l}(f^k(q))$, we have $f^k(q) \in D_{R_0} \setminus \mathbb{R}^-$ since we know $f^n(D_{R'_0}) \subseteq D_{R_0}$ for any positive integer n by Lemma III.3.2 and $f^k(q) \notin \mathbb{R}^-$ since $R'_0 < R_0 = r < r_0$, and f is biholomorphic near 0. Hence $\text{Im}(f^k(q)) \neq 0$. However, this contradicts $\text{Im}(f^k(q)) = \text{Im}(f^k(-\frac{1}{2})) = 0$. Thus $\tilde{q} \notin D_{R'_0} \setminus \mathbb{R}^-$;

4) if $\tilde{q} \in \mathcal{A} \setminus D_{R'_0}$, $\tilde{q}$ is far away from $\partial D_{R'_0}$, then we know that $d_{\mathcal{A}}(z_0, \tilde{q}) \geq d_{\mathcal{A}}^1(z_0, \tilde{z})$. This is because $d_{\mathcal{A}}^1(z_0, \tilde{z})$ is the minimum distance between z_0 and any $\tilde{q} \in ((T \cap \mathcal{A}) \setminus D_{R'_0}) \setminus \mathbb{R}^-$.

Hence, we need to prove that we can choose z_0 so that all these three Kobayashi distances $d_{\mathcal{A}}^1(z_0, \tilde{z}), d_{\mathcal{A}}^2(z_0, z'), d_{\mathcal{A}}^3(z_0, \tilde{z}) \geq C$ for the given constant C . Next, we will estimate these three Kobayashi distances.

First, we estimate $d_{\mathcal{A}}^1(z_0, \tilde{z})$. Suppose $z_0 \in D_\varepsilon$,

$$\begin{aligned} d_{\mathcal{A}}^1(z_0, \tilde{z}) &\geq d_{\mathbb{C} \setminus \mathbb{R}^+}(z_0, \tilde{z}) \\ &= \inf \int_{\gamma(t)} F_{\mathbb{C} \setminus \mathbb{R}^+}(\gamma(t)) |\gamma'(t)| dt \\ &= \inf \int_{\gamma(t)} \frac{1}{2|\gamma(t)| \sin \frac{\arg(\gamma(t))}{2}} |\gamma'(t)| dt \\ &\geq \inf \int_{\gamma(t)} \frac{1}{2|\gamma(t)|} |\gamma'(t)| dt \\ &= \frac{1}{2} \inf \int_\varepsilon^{R_0} \frac{|dr|}{r} \\ &\geq \frac{1}{2} (\ln R_0' - \ln \varepsilon), \end{aligned}$$

where $\gamma(t)$ is a smooth path joining z_0 to $\tilde{z}$. The last inequality holds since there might have some derivatives of the path $\gamma(t)$ are negative in some pieces. In addition, we can see that $d_{\mathcal{A}}^1(z_0, \tilde{z}) \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

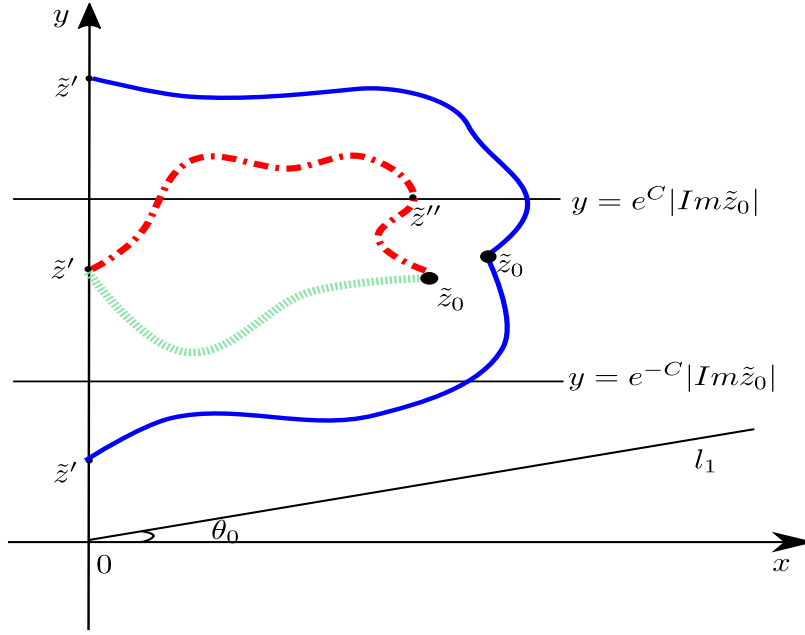
Second, we estimate $d_{\mathcal{A}}^2(z_0, z')$. Let $\varepsilon = \varepsilon_0$, i.e., fix ε , and let $D_1 \subset T$ be a scaling of D_{ε_0} by $S(z) = \frac{z}{|z_0|}$, sending z_0, z' to $\tilde{z}_0 := \frac{z_0}{|z_0|}, \frac{z'}{|z_0|}$, respectively. By homogeneity, we know the Kobayashi distance $d_{\mathbb{C} \setminus \mathbb{R}^+}(z_0, z') = d_{\mathbb{C} \setminus \mathbb{R}^+}(\tilde{z}_0, z'/|z_0|)$. Since we hope to prove $d_{\mathcal{A}}^2(z_0, z') \geq C$, we need z_0 to be far from $\mathbb{R}^-$, and so does $\tilde{z}_0$. Let $S_T := \{z = e^{i\theta}, \theta_0 < \theta < \frac{\pi}{2} - \theta_0\}$, assume $\tilde{z}_0 \in S_T$ and $\operatorname{Re} \tilde{z}_0 > \frac{1}{2}$, then any curve from $\tilde{z}_0$ to $\frac{z'}{|z_0|}$ must pass through a point $\tilde{z}'$ on the positive imaginary axis, i.e., $\operatorname{Re} \tilde{z}' = 0$. For simplicity, we assume this curve and $\tilde{z}'$ lie in the upper half plane. Hence $d_{\mathcal{A}}^2(z_0, z') \geq d_{\mathbb{C} \setminus \mathbb{R}^+}(z_0, z') = d_{\mathbb{C} \setminus \mathbb{R}^+}(\tilde{z}_0, z'/|z_0|) \geq d_{\mathbb{C} \setminus \mathbb{R}^+}(\tilde{z}_0, \tilde{z}')$. We have

$$\begin{aligned} d_{\mathcal{A}}^2(z_0, z') &\geq d_{\mathbb{C} \setminus \mathbb{R}^+}(\tilde{z}_0, \tilde{z}') \\ &= \inf \int_{\tilde{z}_0}^{\tilde{z}'} F_{\mathbb{C} \setminus \mathbb{R}^+}(z) \\ &= \inf \int_{\tilde{z}_0}^{\tilde{z}'} \frac{|dz|}{|z| 2 \sin(\theta/2)} \\ &\geq \sqrt{2} \inf \int_{\tilde{z}_0}^{\tilde{z}'} \frac{|dz|}{|z| \sin \theta} \\ &\geq \sqrt{2} \inf \int_{\tilde{z}_0}^{\tilde{z}'} \frac{|dz|}{\operatorname{Im} z} \\ &> \sqrt{2} |\ln(\operatorname{Im} \tilde{z}') - \ln(\operatorname{Im} \tilde{z}_0)|. \end{aligned}$$

Then there are three situations for the Kobayashi distance between $\tilde{z}_0$ and $\tilde{z}'$:

1) If $\operatorname{Im} \tilde{z}' \geq e^C |\operatorname{Im} \tilde{z}_0|$ or $\operatorname{Im} \tilde{z}' \leq \frac{|\operatorname{Im} \tilde{z}_0|}{e^C}$ for the constant $C > 0$ (see the blue curves on Figure 3.13), then $|\ln(\operatorname{Im} \tilde{z}') - \ln(\operatorname{Im} \tilde{z}_0)| \geq C$, hence $d_{\mathcal{A}}^2(z_0, z') \geq C$ is true.

2) If $\tilde{z}' \in \mathcal{L} := \{z = x + iy, \frac{|\operatorname{Im} \tilde{z}_0|}{e^C} < y < e^C |\operatorname{Im} \tilde{z}_0|\}$ (see the green curve on Figure 3.13). We prove that $d_{\mathcal{A}}(\tilde{z}_0, \tilde{z}') \geq \sqrt{2} \inf \int_{\tilde{z}_0}^{\tilde{z}'} \frac{|dz|}{\operatorname{Im} z} \geq C$ for $z \in \mathcal{L}$.


 Figure 3.13: Three situations for the Kobayashi distance between $\tilde{z}_0$ and $\tilde{z}'$

Let $z = x + iy \in \mathcal{L}$, then $\text{Im } z \leq e^C |\text{Im } \tilde{z}_0|$. Hence we have

$$\int_{\tilde{z}_0}^{\tilde{z}'} \frac{|dz|}{\text{Im } z} \geq \int_{\tilde{z}_0}^{\tilde{z}'} \frac{|dx|}{\text{Im } z} \geq \int_{\tilde{z}_0}^{\tilde{z}'} \frac{|dx|}{e^C |\text{Im } \tilde{z}_0|} = \frac{\text{Re } \tilde{z}_0}{e^C} \frac{1}{|\text{Im } \tilde{z}_0|} > \frac{1}{2e^C |\text{Im } \tilde{z}_0|}.$$

Note that the last inequality holds because we choose $\text{Re } \tilde{z}_0 > 1/2$.

As long as $\theta_0 < \frac{1}{2Ce^C}$, we can choose $\tilde{z}_0$ so that $\text{Im } \tilde{z}_0 < \frac{1}{2Ce^C}$, hence $\int_{\tilde{z}_0}^{\tilde{z}'} \frac{|dz|}{\text{Im } z} > C$.

3) If $\tilde{z}' \in \mathcal{L}$, but the curve γ between $\tilde{z}_0$ and $\tilde{z}'$ gets outside of $\mathcal{L}$ starting at some point $\tilde{z}'' \in \gamma \cap \mathcal{L}$ for a while (see the red curve in Figure 3.13), then enter back to $\mathcal{L}$ again, then we still have $d_{\mathcal{A}}^2(z_0, z') \geq C$ is true because $d_{\mathbb{C} \setminus \mathbb{R}^+}(z_0, z') \geq d_{\mathbb{C} \setminus \mathbb{R}^+}(z_0, \tilde{z}'') \geq C$. The last inequality holds since 1) is valid.

After these calculations, we fix z_0 so that $d_{\mathcal{A}}^1(z_0, \tilde{z})$ and $d_{\mathcal{A}}^2(z_0, z')$ are both bigger or equal to C . To obtain $d_{\mathcal{A}}^3(z_0, \hat{z}) \geq C$, we will need an even smaller θ'_0 , see the following calculation.

We know $d_{\mathcal{A}}^3(z_0, \hat{z}) \geq d_{\mathbb{C} \setminus \mathbb{R}^+}(z_0, \hat{z}) = d_{\mathbb{C} \setminus \mathbb{R}^+}(\tilde{z}_0, \hat{z}/|z_0|)$. Hence to have $d_{\mathcal{A}}^3(z_0, \hat{z}) \geq C$, we need to estimate $d_{\mathbb{C} \setminus \mathbb{R}^+}(\tilde{z}_0, \hat{z}/|z_0|)$.

When we estimate $d_{\mathcal{A}}^2(z_0, z')$, we send z_0 to $\tilde{z}_0$ and choose $\text{Re } \tilde{z}_0 > 1/2$ and $\tilde{z}$ close to l_1 . Hence $d_{\mathbb{C} \setminus \mathbb{R}^+}(\tilde{z}_0, \hat{z}/|z_0|)$ might be very small. To handle this situation, we first choose a disk $\Delta_{\mathbb{C} \setminus \mathbb{R}^+}^K(\tilde{z}_0, C)$ centered at $\tilde{z}_0$ with radius C in the Kobayashi distance. Since this disk $\Delta_{\mathbb{C} \setminus \mathbb{R}^+}^K(\tilde{z}_0, C)$ is a compact subset of $\mathbb{C} \setminus \mathbb{R}^+$, there exists a sector $S'' := \{z = re^{i\theta}, r > 0, 0 < \theta < \theta'_0\}$ such that $S'' \cap \Delta_{\mathbb{C} \setminus \mathbb{R}^+}^K(\tilde{z}_0, C) = \emptyset$, here we can assume that $\theta'_0 < \theta_0$. Therefore, $d_{\mathbb{C} \setminus \mathbb{R}^+}(\tilde{z}_0, \hat{z}/|z_0|) \geq C$ for any $\hat{z}/|z_0| \in \mathcal{A} \cap S''$. Furthermore, for this new θ'_0 , it does not change the conclusion of the estimation of $d_{\mathcal{A}}^1(z_0, \tilde{z})$ and $d_{\mathcal{A}}^2(z_0, z')$.

Therefore, there is a point z_0 such that all these three distances $d_{\mathcal{A}}^1(z_0, \tilde{z}), d_{\mathcal{A}}^2(z_0, z'), d_{\mathcal{A}}^3(z_0, \hat{z}) \geq C$ for the given constant C .

□

Corollary III.3.4. *Let $X \subset \mathcal{A}$ be the set of all $z_0 \in \mathcal{A}$ so that $d_{\mathcal{A}}(z_0, \tilde{q}) \geq C$ for any $\tilde{q} \in \mathcal{A}$. If $z \in X$, then any point $w \in f^{-1}(z)$ is in X . Hence X is dense in the boundary of $\mathcal{A}$.*

Proof. By Theorem A, we know that $X \neq \emptyset$. Suppose $w \notin X$, then there exists $\tilde{q} \in Q$ so that $d_{\mathcal{A}}(w, \tilde{q}) < C$. Since f is distance decreasing, see Proposition III.2.7, we have $d_{\mathcal{A}}(f(w), f(\tilde{q})) = d_{\mathcal{A}}(z, \tilde{q}) < C$. This contradicts $z \in X$.

Furthermore, we know that $\{f^{-n}(z)\}$ clusters at every point in Julia set. In particular, this is true if $z \in X$. More precisely, $\{f^{-n}(z)\}$ equidistributes toward the Green measure. Therefore, the closure of X contains the boundary of $\mathcal{A}$. $\square$

III.3.2 Dynamics inside the parabolic basin of $f(z) = z + az^{m+1}, m \geq 1, a \neq 0$

In this subsection, we generalize Theorem A to the case of several petals inside the parabolic basin. Let us recall the statement of our main Theorem B:

Theorem B. Let $f(z) = z + az^{m+1}, m \geq 1, a \neq 0$, and Ω_j be the immediate basin of $\mathcal{A}_j$. We choose an arbitrary constant $C > 0$ and a point $q \in \mathbf{v}_j \cap \mathcal{P}_j$. Then there exists a point $z_0 \in \Omega_j$ so that for any $\tilde{q} \in Q := (\cup_{l=0}^{\infty} f^{-l}(f^k(q))) \cap \Omega_j$ (l, k are non-negative integers), the Kobayashi distance $d_{\Omega_j}(z_0, \tilde{q}) \geq C$, where d_{Ω_j} is the Kobayashi distance on Ω_j .

Proof. We conjugate f using rotation:

$$\begin{array}{ccc} z & \xrightarrow{f=z+az^{m+1}} & f(z) \\ e^{i\theta}z \downarrow & & \downarrow e^{i\theta}z \\ w & \xrightarrow{g} & g(w) \end{array}$$

$w \mapsto e^{i\theta}(e^{-i\theta}w + a(e^{-i\theta}w)^{m+1}) = w + ae^{-im\theta}w^{m+1}$. Suppose $a = re^{i\psi}$, then we can choose $\theta = \frac{\psi}{m}$ such that $ae^{-im\theta}$ is a real positive number. Then we can assume $S := \{z = re^{i\theta}, r > 0, 0 < \theta < \frac{2\pi}{m}\}$ be the sector with angle $2\pi/m$, including the attracting petal $\mathcal{P}_j$, then the angle between $\mathbf{v}_j$ and $\mathbb{R}^+$ is $\frac{\pi}{m}$. We denote the boundary rays of S by $l_{\mathbb{R}^+}^1 := \{z = r > 0\}$ and $l_{\mathbb{R}^+}^2 := \{z = re^{i\frac{2\pi}{m}}, r > 0\}$. We choose $\theta_0 > 0$ and two rays $l_1 := \{z = re^{i\theta_0}, r > 0\}, l_2 := \{z = re^{i(\frac{2\pi}{m} - \theta_0)}, r > 0\}$. Then we denote by $T := \{z = re^{i\theta}, r > 0, \theta_0 < \theta < \frac{2\pi}{m} - \theta_0\}$ a sector inside S .

Let $\varphi(z) : S \rightarrow \mathbb{H}$ with $\varphi(z) = z^{m/2}$. Then the Kobayashi metric F_S is

$$F_S = \frac{|\varphi'|}{\operatorname{Im} \varphi} |dz| = \frac{m}{2r \sin(\frac{\theta m}{2})} |dz|.$$

As in the proof of Theorem A, we can choose two analogous "Pac-Man" $D_{R'_0}^m := \{z = re^{i\theta}, 0 < r < R'_0, \theta_0 < \theta < \frac{2\pi}{m} - \theta_0\}, D_{R_0}^m := \{z = re^{i\theta}, 0 < r < R_0, \theta_0 < \theta < \frac{2\pi}{m} - \theta_0\}$ central at 0 with radius $R'_0, R_0 > 0 (R'_0 < R_0)$, respectively,

such that $D_{R_0}^m \subsetneq T \cap \Omega_j$ and $f^n(D_{R_0}^m) \subset D_{R_0}^m$. Then similarly, we need to estimate the three Kobayashi distances from z_0 to any point $\tilde{z} \in \partial D_{R_0}^m$ (see the blue curve in Figure 3.14), $z' \in \mathbf{v}_j$ (see the pink curve in Figure 3.14), and $\hat{z} \in \Omega_j \cap \{S' := \{z = re^{i\theta}, r > 0, 0 < \theta < \theta_0\}\}$ (see the green curve in Figure 3.14), and show that all of them are not less than C .

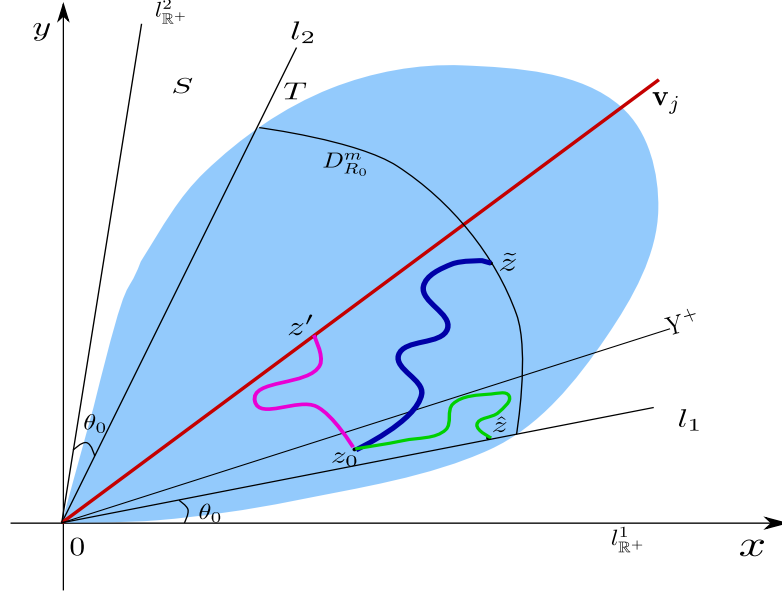


Figure 3.14: The three Kobayashi distances in Ω_j

First, suppose $z_0 \in D_\varepsilon^m := \{z = re^{i\theta}, 0 < r < \varepsilon, \theta_0 < \theta < \frac{2\pi}{m} - \theta_0\}, \varepsilon \ll R_0'$. Let us estimate the Kobayashi distance from z_0 to any point $\tilde{z}$ on the boundary of $D_{R_0}^m$, and we denote this distance by $d_{\Omega_j}^1(z_0, \tilde{z})$.

$$\begin{aligned} d_{\Omega_j}^1(z_0, \tilde{z}) &\geq d_S(z_0, \tilde{z}) = \inf_{\gamma(t)} \int F_S(\gamma(t)) |\gamma'(t)| dt \\ &= \inf_{\gamma(t)} \int \frac{m}{2|\gamma(t)| \sin \frac{m \arg(\gamma(t))}{2}} |\gamma'(t)| dt \geq \inf_{\gamma(t)} \int \frac{m}{2|\gamma(t)|} |\gamma'(t)| dt \\ &= \frac{m}{2} \inf_{\varepsilon} \int_{\varepsilon}^{R_0} \frac{|dr|}{r} \geq \frac{m}{2} (\ln R_0 - \ln \varepsilon), \end{aligned}$$

where $\gamma(t)$ is a smooth path joining z_0 to $\tilde{z}$. In addition, we can see that $d_{\Omega_j}^1(z_0, \tilde{z}) \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

Second, we calculate the Kobayashi distance from z_0 to any point $z' \in \mathbf{v}_j$ denoted by $d_{\Omega_j}^2(z_0, z')$. Let $\varepsilon = \varepsilon_0$, i.e., fix ε , and $D_1^m \subset T$ be a scaling of $D_{\varepsilon_0}^m$ by $S(z) = \frac{z}{|z_0|}$, sending z_0, z' to $\tilde{z}_0 := \frac{z_0}{|z_0|}, \frac{z'}{|z_0|}$, respectively. By homogeneity, we know the Kobayashi distance $d_S(z_0, z') = d_S(\tilde{z}_0, z'/|z_0|)$. Since we hope to prove $d_{\Omega_j}^2(z_0, z') > C$, we need z_0 to be far from $\mathbf{v}_j$, and so does $\tilde{z}_0$. Let $S_T := \{z = e^{i\theta}, \theta_0 < \theta < \frac{\pi}{2m} - \theta_0\}$, assume $\tilde{z}_0 \in S_T$ and $\text{Re } \tilde{z}_0$ is sufficiently big, then any curve from $\tilde{z}_0$ to $\frac{z'}{|z_0|}$ must pass through a point $\tilde{z}'$ on the ray $Y^+ := \{re^{i\frac{\pi}{2m}}, r > 0\}$. Hence $d_{\Omega_j}^2(z_0, z') \geq d_S(z_0, z') = d_S(\tilde{z}_0, z'/|z_0|) \geq d_S(\tilde{z}_0, \tilde{z}')$.

Then we have

$$\begin{aligned}
 d_{\Omega_j}^2(z_0, z') &\geq d_S(\tilde{z}_0, \tilde{z}') = \inf_{\tilde{z}_0}^{\tilde{z}'} F_S(z) \\
 &= \inf_{\tilde{z}_0}^{\tilde{z}'} \frac{m|dz|}{2r \sin(m\theta/2)} \\
 &= \inf_{\tilde{z}_0}^{\tilde{z}'} \frac{\frac{m\theta}{2}}{\sin(\frac{m\theta}{2})} \cdot \frac{m|dz|}{2r(m\theta/2)} \\
 &= c_1 \inf_{\tilde{z}_0}^{\tilde{z}'} \frac{|dz|}{r\theta} \\
 &= c_1 \inf_{\tilde{z}_0}^{\tilde{z}'} \frac{\sin \theta}{\theta} \cdot \frac{|dz|}{r \sin \theta} \\
 &= c_1 c_2 \inf_{\tilde{z}_0}^{\tilde{z}'} \frac{|dz|}{r \sin \theta} \\
 &= c_1 c_2 \inf_{\tilde{z}_0}^{\tilde{z}'} \frac{|dz|}{\operatorname{Im} z} \\
 &\geq c_1 c_2 |\ln(\operatorname{Im} \tilde{z}') - \ln(\operatorname{Im} \tilde{z}_0)|.
 \end{aligned}$$

where $c_1 := \inf \frac{\frac{m\theta}{2}}{\sin \frac{m\theta}{2}}, c_2 := \inf \frac{\sin \theta}{\theta}$.

Then there are three situations for the Kobayashi distance between $\tilde{z}_0$ and $\tilde{z}'$:

1) If $\operatorname{Im} \tilde{z}' \geq e^C |\operatorname{Im} \tilde{z}_0|$ or $\operatorname{Im} \tilde{z}' \leq \frac{|\operatorname{Im} \tilde{z}_0|}{e^C}$ for some constant $C > 1$, then $|\ln(\operatorname{Im} \tilde{z}') - \ln(\operatorname{Im} \tilde{z}_0)| \geq C$, hence $d_{\Omega_j}^2(z_0, z') \geq C$ is true.

2) If $\tilde{z}' \in \mathcal{L} := \{z = x + iy, \frac{|\operatorname{Im} \tilde{z}_0|}{e^C} < y < e^C |\operatorname{Im} \tilde{z}_0|\}$. We need to prove that $d_{\Omega_j}(\tilde{z}_0, \tilde{z}') \geq c_1 c_2 \inf_{\tilde{z}_0}^{\tilde{z}'} \frac{|dz|}{\operatorname{Im} z} \geq C$ for $z \in \mathcal{L}$.

Let $z = x + iy \in \mathcal{L}$, then $\operatorname{Im} z \leq e^C |\operatorname{Im} \tilde{z}_0|$. Hence we have

$$\int_{\tilde{z}_0}^{\tilde{z}'} \frac{|dz|}{\operatorname{Im} z} \geq \int_{\tilde{z}_0}^{\tilde{z}'} \frac{|dx|}{\operatorname{Im} z} \geq \int_{\tilde{z}_0}^{\tilde{z}'} \frac{|dx|}{e^C |\operatorname{Im} \tilde{z}_0|} = \frac{|\operatorname{Re} \tilde{z}' - \operatorname{Re} \tilde{z}_0|}{e^C} \frac{1}{|\operatorname{Im} \tilde{z}_0|} > \frac{1}{2e^C |\operatorname{Im} \tilde{z}_0|},$$

Note that the last inequality holds since we choose $\operatorname{Re} \tilde{z}_0$ sufficiently big so that $\tilde{z}_0$ is close to l_1 , and $|\operatorname{Re} \tilde{z}' - \operatorname{Re} \tilde{z}_0| > 1/2$ because $\tilde{z}'$ will have to be close to 0 since it lies on $Y^+ \cap \mathcal{L}$ and its imaginary part is close to 0, which makes $d_{\Omega_j}^2(z_0, z')$ as big as we want.

In other words, as long as $\theta_0 < \frac{1}{2Ce^C}$, we have $\operatorname{Im} \tilde{z}_0 < \frac{1}{2Ce^C}$. In addition, $|\operatorname{Re} \tilde{z}' - \operatorname{Re} \tilde{z}_0| > 1/2$, we obtain $\int_{\tilde{z}_0}^{\tilde{z}'} \frac{|dz|}{\operatorname{Im} z} > C$.

3) If $\tilde{z}' \in \mathcal{L}$, but the curve γ between $\tilde{z}_0$ and $\tilde{z}'$ get outside of $\mathcal{L}$ starting at some point $\tilde{z}'' \in \gamma \cap \mathcal{L}$ for a while, then enter back to $\mathcal{L}$ again, then we still have $d_{\Omega_j}^2(z_0, z') \geq C$ is true because $d_S(z_0, z') \geq d_S(\tilde{z}_0, \tilde{z}'') \geq C$. The last inequality holds since 1) is valid. We have a conclusion as same as $d_{\mathcal{A}}^2(z_0, z') \geq C$ in the proof of Theorem A.

At last, we estimate the Kobayashi distance from z_0 to any point $\hat{z} \in \Omega_j \cap \{S' := \{z = re^{i\theta}, r > 0, 0 < \theta < \theta_0\}\}$ denoted by $d_{\Omega_j}^3(z_0, \hat{z})$. We know $d_{\Omega_j}^3(z_0, \hat{z}) \geq d_S(z_0, \hat{z}) = d_S(\tilde{z}_0, \hat{z}/|z_0|)$.

We use the method for computing $d_{\Omega_j}^3(z_0, \hat{z})$ as same as $d_{\mathcal{A}}^3(z_0, \hat{z}) \geq C$ in the proof of Theorem A. We first choose a disk $\Delta_S^K(\tilde{z}_0, C)$ centered at $\tilde{z}_0$ with radius C in the Kobayashi distance. Since this disk $\Delta_S^K(\tilde{z}_0, C)$ is a compact subset of S ,

there exists a sector $S'' := \{z = re^{i\theta}, r > 0, \pi - \frac{\pi}{m} < \theta < \pi - \frac{\pi}{m} + \theta'_0\}$ such that $S'' \cap \Delta_S^K(\tilde{z}_0, C) = \emptyset$, here we can assume that $\theta'_0 < \theta_0$. Therefore, $d_S(\tilde{z}_0, \hat{z}/|z_0|) \geq C$ for any $\hat{z}/|z_0| \in \Omega_j \cap S''$. Furthermore, for this new θ'_0 , it does not change the conclusion of the estimation of $d_{\Omega_j}^1(z_0, \tilde{z})$ and $d_{\Omega_j}^2(z_0, z')$. $\square$

III.3.3 Dynamics inside the parabolic basin of $f(z) = z + az^{m+1} + (\text{higher order terms})$, $m \geq 1, a \neq 0$

Finally, in this subsection, we consider the behavior of orbits inside parabolic basins of general polynomials. Let us recall the statement of our main Theorem C:

Theorem C. Let $f(z) = z + az^{m+1} + (\text{higher order terms})$, $m \geq 1, a \neq 0$, and Ω_j be the immediate basin of $\mathcal{A}_j$. We choose an arbitrary constant $C > 0$ and a point $q \in \mathbf{v}_j \cap \mathcal{P}_j$. Then there exists a point $z_0 \in \Omega_j$ so that for any $\tilde{q} \in Q := (\cup_{l=0}^{\infty} f^{-l}(f^k(q))) \cap \Omega_j$ (l, k are non-negative integers), the Kobayashi distance $d_{\Omega_j}(z_0, \tilde{q}) \geq C$, where d_{Ω_j} is the Kobayashi distance on Ω_j .

The essential idea to prove Theorem C is the same as the proof of Theorem A and Theorem B. However, we cannot draw the parabolic basin of $f(z) = z + az^{m+1} + (\text{higher order terms})$ directly as Figure 3.12 and Figure 3.14 since there are higher order terms of f , the parabolic basin can be more complicated.

To simplify the discussion, we first consider the case $m = 1$:

$$f(z) = z + z^2 + (\text{higher order terms}).$$

When there are no higher order terms, the crucial estimate of the Kobayashi metric comes from the fact that the parabolic basin is contained in $\mathbb{C} \setminus \mathbb{R}^+$. Hence we could compare it with the Kobayashi metric on $\mathbb{C} \setminus \mathbb{R}^+$.

In the case of higher order terms, the parabolic basin can be more complicated. However, we can, instead of $\mathbb{C} \setminus \mathbb{R}^+$, use the double sheeted domain

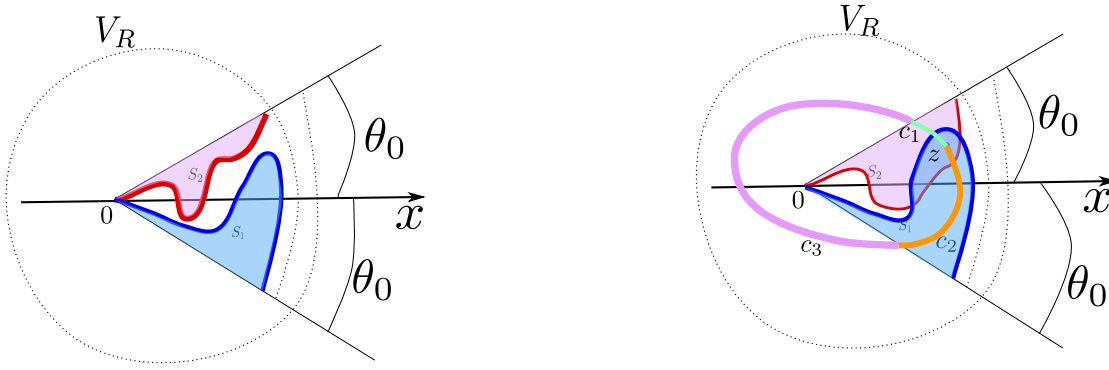
$$V_R := \{z = re^{i\theta}, 0 < r < R, -\theta_0 < \theta < 2\pi + \theta_0\}.$$

Next, we investigate the properties of V_R to explain why we choose the double sheeted domain V_R as above.

Proposition III.3.5. Let $\bar{D}_R := \{z = re^{i\theta}, 0 < r < R, -\theta_0 < \theta < \theta_0\}$, $\mathcal{A}$ be the whole basin of f , S_1 be the connected component of $\mathcal{A} \cap \bar{D}_R$ which contains $\{z = re^{i\theta_0}, 0 < r < R\}$, and S_2 be the connected component of $\mathcal{A} \cap \bar{D}_R$ which contains $\{z = re^{-i\theta_0}, 0 < r < R\}$. Then any two pieces S_1, S_2 (see the left of Figure 3.15) are disjoint in $\bar{D}_R$.

Proof. We know that, inside V_R and near the origin, $\mathcal{A}$ contains the Left Pac-Man $D_R := \{z = re^{i\theta}, 0 < r < R, \theta_0 < \theta < 2\pi - \theta_0\}$.

If S_1 intersects S_2 , then there is a point $z \in S_1 \cap S_2$. We can draw three curves, c_1 from z to $l_1 := \{z = re^{i\theta_0}\}$, c_2 from z to $l_2 := \{z = re^{-i\theta_0}\}$, and $c_3 \in D_R$ which

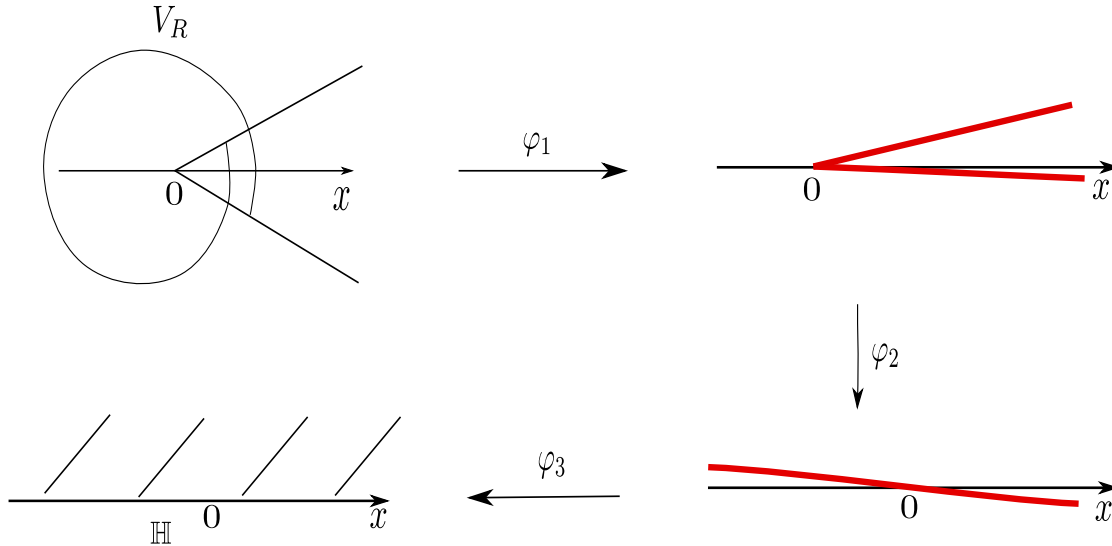

 Figure 3.15: Two pieces S_1 and S_2 in $\bar{D}_R$

connect c_1 and c_2 . Hence $\mathcal{A}$ contains a closed curve $\gamma_c := c_1 + c_2 + c_3$ with the winding number 1 around the origin (see the right of Figure 3.15).

We know that $f^n(z) \rightarrow 0$ when $z \in \gamma_c$, since $\gamma_c \in \mathcal{A}$. In addition, by the maximum principle, we have $f^n(z) \rightarrow 0$ when z is inside the domain bounded by γ_c . Hence $\mathcal{A}$ contains a neighborhood of 0, then 0 is an attracting fixed point. However, this contradicts that 0 is a parabolic fixed point of f . $\square$

By Proposition III.3.5, we can use the Kobayashi metric on V_R instead of $\mathbb{C} \setminus \mathbb{R}^+$.

First, we know that V_R can be mapped to a sector $S := \{z = re^{i\theta}, r > 0, \theta_1 < \theta < \theta_2, \theta_1 \ll \theta_2 < \pi/2\}$ by $\varphi_1(z) = z^{c/2}$ when c is sufficiently small. Second, we can change c such that $\theta_2 = \pi - \theta_1$ by some map $\varphi_2(z)$. At last, by some rotation map φ_3 , we can map S to the upper half plane $\mathbb{H}$ (see the Figure 3.16).


 Figure 3.16: The maps φ_1, φ_2 and φ_3

Therefore, the map $\varphi(z) := \varphi_3 \circ \varphi_2 \circ \varphi_1$ from V_R to the upper half plane $\mathbb{H}$ becomes $\varphi(z) = e^{i\psi} z^{\frac{c}{2}}$, instead of $\varphi(z) = z^{1/2}$, where c is very close to 1. Then, with the above setting, the rest of the estimation goes through as in Theorem B.

If $m > 1$, it is difficult to draw the specific parabolic basins of f or the attracting petals. Let

$$V_R =: \{z = re^{i\theta}, r > 0, -\theta_0 < \theta < \frac{2\pi}{m} + \theta_0\}.$$

We use Figure 3.17 to illustrate how we can choose V_R (see the domain with pink curves as its argument). We want to map V_R to the upper half-plane. First, Let $z \rightarrow e^{i\theta_0}z$, then V_R is mapped to

$$V'_R := \{z = re^{i\theta}, r > 0, 0 < \theta < \frac{2\pi}{m} + 2\theta_0\}.$$

We define $\varphi(z) = (e^{i\theta_0}z)^{\frac{\pi}{\frac{2\pi}{m}+2\theta_0}}$. Then the Kobayashi metric on V_R is

$$F_{V_R} = \frac{|\varphi'(z)|}{\text{Im } \varphi} |dz|.$$

And similarly, we have the same properties of S_1, S_2 as in Proposition III.3.5 (see the red curve and blue curve on Figure 3.17). Then the rest of the estimation

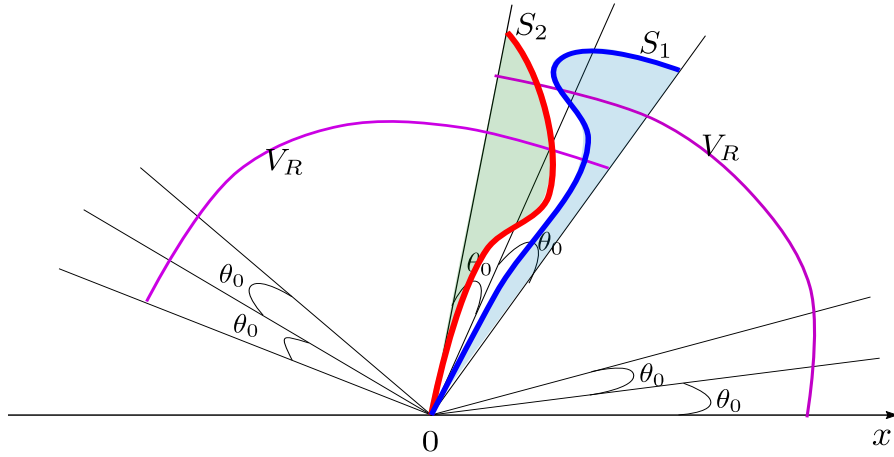


Figure 3.17: Two pieces S_1 and S_2

goes through as in Theorem B. Thus, we are done.

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Paper IV

Dynamics inside Fatou sets in higher dimensions

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Abstract

In this paper, we investigate the behavior of orbits inside attracting basins in higher dimensions. Suppose $F(z, w) = (P(z), Q(w))$, where $P(z), Q(w)$ are two polynomials of degree $m_1, m_2 \geq 2$ on $\mathbb{C}$, $P(0) = Q(0) = 0$, and $0 < |P'(0)|, |Q'(0)| < 1$. Let Ω be the immediate attracting basin of $F(z, w)$. Then there is a constant C such that for every point $(z_0, w_0) \in \Omega$, there exists a point $(\tilde{z}, \tilde{w}) \in \cup_k F^{-k}(0, 0)$, $k \geq 0$ so that $d_\Omega((z_0, w_0), (\tilde{z}, \tilde{w})) \leq C$, d_Ω is the Kobayashi distance on Ω . However, for many other cases, this result is invalid.

Keywords: Basin of attractions, Kobayashi metric, Fatou sets

Mathematics Subject Classification (2020): 32H50 (primary); 32F45; 32A19 (secondary).

IV.1 Introduction

In discrete dynamical systems, we are interested in qualitatively and quantitatively describing the possible dynamical behavior under the iteration of maps satisfying certain conditions. In our paper [5], Hu studied the dynamics of holomorphic polynomials on attracting basins and obtained Theorem IV.1.1:

Theorem IV.1.1. [Hu, 2022] *Suppose $f(z)$ is a polynomial of degree $N \geq 2$ on $\mathbb{C}$, p is an attracting fixed point of $f(z)$, Ω_1 is the immediate basin of attraction of p , $\{f^{-1}(p)\} \cap \Omega_1 \neq \{p\}$, $\mathcal{A}(p)$ is the basin of attraction of p , $\Omega_i (i = 1, 2, \dots)$ are the connected components of $\mathcal{A}(p)$. Then there is a constant $\tilde{C}$ so that for every point z_0 inside any Ω_i , there exists a point $q \in \cup_k f^{-k}(p)$ inside Ω_i such that $d_{\Omega_i}(z_0, q) \leq \tilde{C}$, where d_{Ω_i} is the Kobayashi distance on Ω_i .*

Theorem IV.1.1 shows that in an attracting basin of a complex polynomial, the backward orbit of the attracting fixed point either is the point itself or accumulates at the boundary of all the components of the basin in such a way that all points of the basin lie within a uniformly bounded distance of the backward orbit, measured with respect to the Kobayashi metric. This is an interesting and innovative problem and result. There are no publications by other researchers to the best of our knowledge.

However, Hu [6] proved that Theorem IV.1.1 is no longer valid for parabolic basins of polynomials in one dimension. This is a more interesting and surprising result.

Compared with one dimension, there are very interesting results about dynamics inside attracting basins in higher dimensions. Let $F(z, w) = (P(z), Q(w))$, where $P(z), Q(w)$ are two polynomials of degree $m_1, m_2 \geq 2$ on $\mathbb{C}$, and $F^n : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be its n -fold iterate. In complex dynamics, two crucial disjoint invariant sets are associated with F , the *Julia set* and the *Fatou set* [11, 12], which partition $\mathbb{C}^2$. The Fatou set of F is defined as the largest open set where the family of iterates is locally normal. In other words, for any point $(z, w) \in \mathbb{C}^2$, there exists some neighborhood U of (z, w) so that the sequence of iterates of the map restricted to U forms a normal family, so the iterates are well behaved. The complement of the Fatou set is called the Julia set.

The classification of Fatou components in one dimension was completed in the '80s when Sullivan [15] proved that every Fatou component of a rational map is preperiodic, i.e., there are $n, m \in \mathbb{N}$ such that $f^{n+m}(\Omega) = f^m(\Omega)$. For more details and results, we refer the reader to [2, 3, 11]. In higher dimensions, the dynamics is quite different and more fruitful than in $\mathbb{C}$. We refer the reader to [10, 12–14] for more details and results.

The connected components of the Fatou set of F are called *Fatou components*. A Fatou component $\Omega \subset \mathbb{C}^2$ of F is *invariant* if $F(\Omega) = \Omega$. For $(z, w) \in \mathbb{C}^2$, the set $\{(z_n, w_n)\} = \{(z_1, w_1) = F(z_0, w_0), (z_2, w_2) = F^2(z_0, w_0), \dots\}$ is called the orbit of the point $(z, w) = (z_0, w_0)$. If $(z_N, w_N) = (z_0, w_0)$ for some integer N , we say that (z_0, w_0) is a periodic point of F of period N . If $N = 1$, then (z_0, w_0) is a fixed point of F . There have been few detailed studies until now of the more precise behavior of orbits inside the Fatou set. For example, let $\mathcal{A}(z', w') := \{(z, w) \in \mathbb{C}^2; F^n(z, w) \rightarrow (z', w')\}$ be the *basin of attraction* of an attracting fixed point (z', w') . One can ask when (z_0, w_0) is close to $\partial\mathcal{A}(z', w')$, what orbits $\{(z_n, w_n)\}$ of (z_0, w_0) going from (z_0, w_0) to near the attracting fixed point (z', w') look like, or how many iterations are needed.

In this paper, we will investigate how the orbits behave inside the attracting basin of $F(z, w)$ in $\mathbb{C}^2$. We obtain the following, Theorem IV.3.2 in section 3:

Theorem. IV.3.2. *Suppose $F(z, w) = (P(z), Q(w))$, where $P(z), Q(w)$ are two polynomials of degree $m_1, m_2 \geq 2$ on $\mathbb{C}$, $P(0) = Q(0) = 0$, and $0 < |P'(0)|, |Q'(0)| < 1$. Let Ω be the immediate attracting basin of $F(z, w)$. Then there is a constant C such that for every point $(z_0, w_0) \in \Omega$, there exists a point $(\tilde{z}, \tilde{w}) \in \cup_k F^{-k}(0, 0), k \geq 0$ so that $d_\Omega((z_0, w_0), (\tilde{z}, \tilde{w})) \leq C$, d_Ω is the Kobayashi distance on Ω .*

However, Theorem IV.3.2 is not valid for any of the following cases:

Let $F(z, w) = (P(z), Q(w))$, where $P(z), Q(w)$ are two polynomials of degree $m_1, m_2 \geq 2$ on $\mathbb{C}$ and $P(0) = Q(0) = 0$.

- (1) $P(z) = z^{m_1}, Q(w) = w^{m_2}$;
- (2) $P(z) = z^m, 0 < |Q'(0)| < 1$, i.e., $P'(0) = 0$;
- (3) $P(z) = z^m, Q'(0) = 1$, i.e., $P'(0) = 0$;
- (4) $0 < |P'(0)| < 1, Q'(0) = 1$;

$$(5) \quad P'(0) = Q'(0) = 1.$$

Polynomial skew products, for example, [7, 8], have been useful test cases for complex dynamics in two dimensions. This allows one to use one variable results in one of the variables: Suppose F is a polynomial skew product, $F(z, w) = (P(z), Q(z, w))$, where $P(z), Q(z, w)$ are two polynomials of degree $m_1, m_2 \geq 2$ on $\mathbb{C}$ and $P(0) = 0, Q(0, 0) = 0$. We will prove that, in the following cases in section 4, Theorem IV.3.2 fails as well:

- (1) $P(z) = z^2, Q(z, w) = w^2 + az, a \in \mathbb{C};$
- (2) $P(z) = az + z^2, Q(z, w) = w^2 + cw + bz, 0 < |a|, |b|, |c| \ll 1$ and $|a| \gg |c|, |a| \gg |b|, |c| \gg |ab|.$

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IV.2 Preliminaries

First of all, let us recall the definition of the Kobayashi metric, for example, [?, 1, 9, 16].

Definition. Let $\hat{\Omega} \subset \mathbb{C}^n$ be a domain. We choose a point $z \in \hat{\Omega}$ and a vector ξ tangent to $\mathbb{C}^n$ at the point z . Let Δ denote the unit disk in the complex plane $\mathbb{C}$. We define the *Kobayashi metric*

$$F_{\hat{\Omega}}(z, \xi) := \inf\{\lambda > 0 : \exists f : \Delta \xrightarrow{hol} \hat{\Omega}, f(0) = z, \lambda f'(0) = \xi\}.$$

Let $\gamma : [0, 1] \rightarrow \hat{\Omega}$ be a piecewise smooth curve. The *Kobayashi length* of γ is defined to be

$$L_{\hat{\Omega}}(\gamma) = \int_{\gamma} F_{\hat{\Omega}}(z, \xi) |dz| = \int_0^1 F_{\hat{\Omega}}(\gamma(t), \gamma'(t)) |\gamma'(t)| dt.$$

For any two points z_1 and z_2 in $\hat{\Omega}$, the *Kobayashi distance* between z_1 and z_2 is defined to be

$$d_{\hat{\Omega}}(z_1, z_2) = \inf\{L_{\hat{\Omega}}(\gamma) : \gamma \text{ is a piecewise smooth curve connecting } z_1 \text{ and } z_2\}.$$

Note that $d_{\hat{\Omega}}(z_1, z_2)$ is defined when z_1 and z_2 are in the same connected component of $\hat{\Omega}$.

Let $d_E(z_1, z_2)$ denote the Euclidean metric distance between any two points $z_1, z_2 \in \Delta$.

IV.3 Dynamics of holomorphic polynomial maps inside Fatou sets of $F(z, w) = (P(z), Q(w))$

In this section, we study how precisely the orbit goes inside attracting basins of $F(z, w) = (P(z), Q(w))$, where $P(z), Q(w)$ are two polynomials of degree $m_1, m_2 \geq 2$ on $\mathbb{C}$.

IV.3.1 Dynamics of $F(z, w) = (z^m, w^m), m \geq 2$, i.e., $P'(0) = Q'(0) = 0$

Theorem IV.3.1. *Let $F(z, w) = (z^m, w^m), m \geq 2$. We choose an arbitrary constant $C > 0$ and the point $(\varepsilon, \varepsilon), 0 < \varepsilon \ll 1$. Then there exists a point $(z_0, w_0) \in \mathcal{A}$ so that for any $(\tilde{z}, \tilde{w}) \in \cup_k^\infty \{F^{-k}(\varepsilon, \varepsilon)\}, k \geq 0$, the Kobayashi distance $d_{\mathcal{A}}((z_0, w_0), (\tilde{z}, \tilde{w})) \geq C$.*

Proof. We know $(\tilde{z}_k, \tilde{w}_k) = (\varepsilon^{1/m^k}, \varepsilon^{1/m^k})$ for some positive integer k . Then $|\tilde{z}| = |\tilde{w}|$. We take $(z_0, w_0) = (1 - \delta, \delta)$, δ is sufficiently small depending on C . Then

$$\begin{aligned} d_{\Delta \times \Delta}((z_0, w_0), (\tilde{z}_k, \tilde{w}_k)) &= d_{\Delta \times \Delta}((z_0, w_0), (\varepsilon^{1/m^k}, \varepsilon^{1/m^k})) \\ &= \max \left\{ d_{\Delta}(z_0, \tilde{z}_k), d_{\Delta}(w_0, \tilde{w}_k) \right\} \\ &= \max \left\{ d_{\Delta}\left(0, \frac{z_0 - \tilde{z}_k}{1 - z_0 \tilde{z}_k}\right), d_{\Delta}\left(0, \frac{w_0 - \tilde{w}_k}{1 - w_0 \tilde{w}_k}\right) \right\} \\ &= \max \left\{ \ln \frac{1 + \left| \frac{z_0 - \tilde{z}_k}{1 - z_0 \tilde{z}_k} \right|}{1 - \left| \frac{z_0 - \tilde{z}_k}{1 - z_0 \tilde{z}_k} \right|}, \ln \frac{1 + \left| \frac{w_0 - \tilde{w}_k}{1 - w_0 \tilde{w}_k} \right|}{1 - \left| \frac{w_0 - \tilde{w}_k}{1 - w_0 \tilde{w}_k} \right|} \right\} \\ &= \max \left\{ \ln \frac{1 + \left| \frac{1 - \delta - \varepsilon^{1/m^k}}{1 - (1 - \delta)\varepsilon^{1/m^k}} \right|}{1 - \left| \frac{1 - \delta - \varepsilon^{1/m^k}}{1 - (1 - \delta)\varepsilon^{1/m^k}} \right|}, \ln \frac{1 + \left| \frac{\delta - \varepsilon^{1/m^k}}{1 - \delta\varepsilon^{1/m^k}} \right|}{1 - \left| \frac{\delta - \varepsilon^{1/m^k}}{1 - \delta\varepsilon^{1/m^k}} \right|} \right\}. \end{aligned}$$

Hence there are two cases to consider.

Case 1: Notice that $d_{\Delta \times \Delta}((z_0, w_0), (\tilde{z}_k, \tilde{w}_k)) \geq \ln \frac{1 + \left| \frac{\delta - \varepsilon^{1/m^k}}{1 - \delta\varepsilon^{1/m^k}} \right|}{1 - \left| \frac{\delta - \varepsilon^{1/m^k}}{1 - \delta\varepsilon^{1/m^k}} \right|} \geq \ln \frac{1}{1 - \left| \frac{\delta - \varepsilon^{1/m^k}}{1 - \delta\varepsilon^{1/m^k}} \right|} \rightarrow \infty$, when $k \rightarrow \infty$. Hence $d_{\Delta \times \Delta}((z_0, w_0), (\tilde{z}_k, \tilde{w}_k)) \geq C$ if $k > k_0$ for some integer k_0 and δ is smaller than $1/2$.

Case 2: Also, $d_{\Delta \times \Delta}((z_0, w_0), (\tilde{z}_k, \tilde{w}_k)) \geq \ln \frac{1 + \left| \frac{1 - \delta - \varepsilon^{1/m^k}}{1 - (1 - \delta)\varepsilon^{1/m^k}} \right|}{1 - \left| \frac{1 - \delta - \varepsilon^{1/m^k}}{1 - (1 - \delta)\varepsilon^{1/m^k}} \right|} \rightarrow \infty$ when $\delta \rightarrow 0$ for each fixed k . Hence $d_{\Delta \times \Delta}((z_0, w_0), (\tilde{z}_k, \tilde{w}_k)) \geq C$ if $k \leq k_0$ and δ small enough. $\square$

$$F(z, w) = (P(z), Q(w))$$

IV.3.2 Dynamics of $F(z, w) = (P(z), Q(w))$, $P(0) = Q(0) = 0$, $0 < |P'(0)|, |Q'(0)| < 1$

Theorem IV.3.2. Suppose $F(z, w) = (P(z), Q(w))$, where $P(z), Q(w)$ are two polynomials of degree $m_1, m_2 \geq 2$ on $\mathbb{C}$, $P(0) = Q(0) = 0$, and $0 < |P'(0)|, |Q'(0)| < 1$. Let Ω be the immediate attracting basin of $F(z, w)$. Then there is a constant C such that for every point $(z_0, w_0) \in \Omega$, there exists a point $(\tilde{z}, \tilde{w}) \in \cup_k F^{-k}(0, 0)$, $k \geq 0$ so that $d_\Omega((z_0, w_0), (\tilde{z}, \tilde{w})) \leq C$, d_Ω is the Kobayashi distance on Ω .

Proof. Let the immediate basin of attraction of $P(z), Q(w)$ be denoted by Ω_P, Ω_Q , respectively. Then $\Omega = \Omega_P \times \Omega_Q$.

By the conclusion of Theorem A in [5]. i. e., Theorem IV.1.1 in the introduction section above, we know that there is a constant $C_P(C_Q)$ such that for every point $z_0(w_0) \in \Omega_P(\Omega_Q)$, there exists a point $\tilde{z}(\tilde{w}) \in \cup_k P^{-k}(0)(\cup_k Q^{-k'}(0))$, $k, k' \geq 0$ so that $d_{\Omega_P}(z_0, \tilde{z}) \leq C_P, (d_{\Omega_Q}(w_0, \tilde{w}) \leq C_Q)$, where $d_{\Omega_P}(d_{\Omega_Q})$ is the Kobayashi distance on $\Omega_P(\Omega_Q)$. Suppose $K = \max\{k, k'\}$ and $C = \max\{C_P, C_Q\}$, then $\tilde{z} \in \cup_k P^{-K}(0), \tilde{w} \in \cup_k Q^{-K}(0)$. Hence $(\tilde{z}, \tilde{w}) \in \cup_K F^{-K}(0, 0) \subset \Omega$. Therefore, $d_\Omega((z_0, w_0), (\tilde{z}, \tilde{w})) \leq C$, d_Ω is the Kobayashi distance on Ω . □

IV.3.3 Dynamics of $F(z, w) = (P(z), Q(w))$, $P(0) = Q(0) = 0$

- (1) $P(z) = z^m, 0 < |Q'(0)| < 1$, i.e., $P'(0) = 0$;
- (2) $P(z) = z^m, Q'(0) = 1$, i.e., $P'(0) = 0$;
- (3) $0 < |P'(0)| < 1, Q'(0) = 1$;
- (4) $P'(0) = Q'(0) = 1$.

Let Ω be the basin of $(0, 0)$, namely, for case (1), Ω is the attracting basin, for case (2) and (3), Ω is the semi-parabolic basin, for case (4), Ω is the parabolic basin.

Theorem IV.3.3. Suppose $F(z, w) = (P(z), Q(w))$ is as in any of the above four cases. Then there exists a point $(z_0, w_0) \in \Omega$ so that for any $(\tilde{z}, \tilde{w})$ inside the preimage set under $\{F^{-k}\}$ of the fixed point $(0, 0)$ or a point very close to $(0, 0)$, the Kobayashi distance $d_\Omega((z_0, w_0), (\tilde{z}, \tilde{w})) \geq C$.

Proof. If $P'(0) = 0, 0 < |Q'(0)| < 1$, by Theorem IV.3.1 and IV.3.2, we know $\Omega = \Delta \times \Omega_Q$. We choose $z_0 = 1 - \delta$, δ small enough. Then we know that $\tilde{z} = \varepsilon^{1/m^k}$ and

$$\begin{aligned} d_\Omega((z_0, w_0), (\tilde{z}, \tilde{w})) &= \max(d_\Delta(z_0, \tilde{z}), d_{\Omega_Q}(w_0, \tilde{w})) \\ &\geq d_\Delta(z_0, \tilde{z}) \\ &= \ln \frac{1 + \left| \frac{1 - \delta - \varepsilon^{1/m^k}}{1 - (1 - \delta)\varepsilon^{1/m^k}} \right|}{1 - \left| \frac{1 - \delta - \varepsilon^{1/m^k}}{1 - (1 - \delta)\varepsilon^{1/m^k}} \right|} \\ &\rightarrow \infty \end{aligned}$$

as $\delta \rightarrow 0$. Then the proof is done.

If $P(z) = z^m$, $Q'(0) = 1$ or $0 < |P'(0)| < 1$, $Q'(0) = 1$, by the result in the paper [6], we know this theorem is valid since $d_{\Omega_Q}(w_0, \tilde{w})$ is unbounded. Hence the same reason for $P'(0) = Q'(0) = 1$.

□

IV.4 Dynamics of Polynomial skew products inside attracting basins of $F(z, w) = (P(z), Q(z, w))$

Polynomial skew products have been useful test cases for complex dynamics in two dimensions. This allows one to use one variable results in one of the variables. However, for $F(z, w) = (P(z), Q(z, w))$, we can not calculate the Kobayashi distance $d_{\Omega}((z_0, w_0), (\tilde{z}, \tilde{w}))$ by analyzing $d_{\Omega_P}(z_0, \tilde{z}), d_{\Omega_Q}(w_0, \tilde{w})$ separately as for $F(z, w) = (P(z), Q(w))$ since Ω_Q also depends on the z -coordinate. In this section, we consider the dynamics of $F(z, w)$ near $(0, 0)$ and study two simple cases.

IV.4.1 Dynamics of $F(z, w) = (z^2, w^2 + az)$

Theorem IV.4.1. *Suppose $F(z, w) = (z^2, w^2 + az)$, $a \neq 0$. Let Ω be the immediate attracting basin of $(0, 0)$ and $0 < \varepsilon \ll 1$. We choose an arbitrary constant $C > 0$. Then there exists a point $(z_0, w_0) \in \Omega$ so that for any $(\tilde{z}, \tilde{w}) \in \cup_k^\infty \{F^{-k}(\varepsilon, 0)\}$, $k \geq 0$, the Kobayashi distance $d_{\Omega}((z_0, w_0), (\tilde{z}, \tilde{w})) \geq C$.*

Proof. We choose $z_0 = 0$. Note that the projection map $\pi : \Omega \rightarrow \Delta$, $\pi(z, w) = z$ is distance decreasing in the Kobayashi metric. In addition, we know that the z -coordinate of any point in $F^{-k}(\varepsilon, 0)$ approaches to $\partial\Delta$ as $k \rightarrow \infty$. Hence there is an l so that if $k > l$, then $d_{\Omega}((0, w_0), F^{-k}(\varepsilon, 0)) \geq C$ for any w_0 . Let $(z_j, w_j), j = 1, \dots, N$ be the points in $\{F^{-k}(\varepsilon, 0)\}_{k \leq l}$. Next, we want to show that there is a w_0 so that $(0, w_0) \in \Omega$ and $d_{\Omega}((0, w_0), (z_j, w_j)) \geq C$ for any $j = 1, \dots, N$.

We first deal with a small a and then with general a . When a is small, we have the following lemma.

Lemma IV.4.2. *Let $D := \{(z, w); |z| < 1, |w| < 3/4\}$ be a bidisc. If $|a| < 3/16$, then $D \subset \Omega$.*

Proof. If $|w| < 3/4$, then $|w^2| < 9/16$. Hence we have $|w^2 + az| < 9/16 + |a| < 3/4$.

□

Now we continue to prove Theorem IV.4.1. Let $f(z) \equiv w_0 = 2/3$ for $|z| < 1$. Then $\gamma_0 = (z, f(z))$ is a graph over $|z| < 1$ inside Ω . Then $F^{-1}(\gamma_0) = F^{-1}(z, f(z)) = (\sqrt{z}, \sqrt{f(z) - a\sqrt{z}})$. Let us choose $\gamma_1 = (\sqrt{z}, \sqrt{f(z) - a\sqrt{z}}) := (z, \sqrt{f(z^2) - az}) = (z, f_1(z))$. By induction, we can have $\gamma_2 = (z, \sqrt{f_1(z^2) - az}) = (z, f_2(z)); \dots; \gamma_n = (z, f_n(z))$, here we always choose $f_n(z)$ so that $f_n(0) > 0$. Note that inductively $f_n(z) \geq 2/3$, hence $f_n(z^2) - az$ never has any zeros, which

IV.4. Dynamics of Polynomial skew products inside attracting basins of

$$F(z, w) = (P(z), Q(z, w))$$

means any branches of f_n cannot meet at some points of $z \in D$, i.e., none of any two graphs of γ_n intersect each other in D . Then $\lim_{n \rightarrow \infty} \gamma_n \subset \partial\Omega$ since γ_0 does not go through $(0, 0)$ and $\gamma_0 \subset \Omega$ so that the backward orbits of γ_0 converge to $\partial\Omega$. By Montel's theorem, there is a convergent subsequence $f_n(z) \rightarrow f_\infty(z)$, then $(z, f_\infty(z)) \subseteq \partial\Omega$ and $f_\infty(0) = 1$. This implies that for any z , $\gamma_\infty(z) \in \partial\Omega_z$ for every Ω_z , where Ω_z is the slice of Ω at z .

Let $h(z, w) = w - f_\infty(z)$, then $h(z, w)$ is holomorphic on Ω . And we know that $h(0, w) = w - 1$, then $\lim_{w \rightarrow 1} h(0, w) = 0$. Hence $h(z, w)$ is vanishing at $(0, 1)$ and $h(\Omega) \subset \Delta(0, R) \setminus \{0\}$ for some constant radius $R > 1$ since Ω is a bounded set.

Let us recall the Kobayashi metric on the punctured disk, see example 2.8 in [11]. The universal covering surface of $\Delta(0, R) \setminus \{0\}$ can be identified with the left half-plane $\{w = u + iv; u < 0\}$ under the exponential map $w \mapsto z = Re^w \in \Delta(0, R) \setminus \{0\}$ with $dw = \frac{dz}{z}$. Hence the Kobayashi metric $|\frac{dw}{u}|$ on the left half-plane corresponds to the metric $\left| \frac{dz}{r \ln \frac{r}{R}} \right|$ on the punctured disk $\Delta(0, R) \setminus \{0\}$, where

$$r = |z| \text{ and } u = \ln \frac{r}{R}.$$

Then let $x := h(0, w_0) := \lim_{w \rightarrow 1} h(0, w)$ and $y := h(z_j, w_j)$ for $j = 1, \dots, N$. Then

$$\begin{aligned} d_\Omega((0, w_0), (z_j, w_j)) &\geq d_{\Delta(0, R) \setminus \{0\}}(x, y) \\ &= \left| \int_x^y \frac{1}{|z| \ln \frac{|z|}{R}} dz \right| \\ &= |\ln(|\ln y/R|) - \ln(|\ln x/R|)| \\ &\rightarrow \infty, \end{aligned}$$

as $x \rightarrow 0$.

For general a , although we cannot choose a cylinder as in Lemma IV.4.2, we can first choose $D_0 := \{(z, w); |z| < \eta \ll 1, |w| < 3/4\}$. Let $\hat{f}(z) = \hat{w}_0 = 2/3$ for $|z| < \eta$, then $\hat{\gamma}_0 = (z, \hat{f}(z))$ is a graph over $|z| < \eta$. We have $F^{-1}(\hat{\gamma}_0) = F^{-1}(z, \hat{w}_0) = (\sqrt{z}, \sqrt{\hat{w}_0^2 - a\sqrt{z}})$. Then we choose $\hat{f}_1 := \sqrt{\hat{f}(z^2) - az}$ restricted to $|z| < \eta$. Inductively, $\hat{f}_{n+1}(z) = \sqrt{\hat{f}_n(z^2) - az}$ is restricted to $|z| < \eta$ as well. Then one gets $\hat{f}_n(z) \rightarrow \hat{f}_\infty(z)$, then $(z, \hat{f}_\infty(z)) \subseteq \partial\Omega$ and $\hat{f}_\infty(0) = 1$.

Second, let $D_1 := \{(z, w); |z| < \sqrt{\eta}, w \in \mathbb{C}\}$. We choose $g_0(z) = w'_0 = \hat{f}_\infty(z)$ where $|z| < \eta$. And $\gamma'_0 = (z, g_0(z))$ is a graph over $|z| < \eta$. Then we have $F^{-1}(z, w'_0) := (z, g_1(z))$ for $|z| < \eta$. However, there are two cases:

(1) If $g_0 - az \neq 0$ for $|z| < \sqrt{\eta}$. Then there are two solutions of g_1 , we denote them by $g_{1,1} := \sqrt{w_0'^2 - a\sqrt{z}}$; $g_{1,2} := -\sqrt{w_0'^2 - a\sqrt{z}}$, $|z| < \sqrt{\eta}$. We let g_1 be one of them and $D_2 := \{(z, w); |z| < \sqrt{\eta}, w \in \mathbb{C}\}$.

(2) If $g_0 - az = 0$ has one or more zeros on $|z| < \sqrt{\eta}$. We let g_1 denote the multivalued function. We repeat this for $g_2(z) = \sqrt{g_1(z^2) - az}$, etc. We continue this process until we obtain a multivalued function $g_n(z)$ well defined on $|z| < \eta^{1/2^n}$, here n will be determined below. Hence, $g_n(z)$ will have at most 2^n sheets. Meanwhile, we let $D_{n+2} := \{(z, w); |z| < \eta^{1/2^n}, w \in \mathbb{C}\}$.

Then one gets $g_n(z) \rightarrow g_\infty(z)$, then $(z, \lim_{n \rightarrow \infty} g_n(z)) \subseteq \partial\Omega$ and $\lim_{n \rightarrow \infty} g_n(0) = 1$. However, we cannot simply choose $\hat{h}(z, w) = w - g_\infty(z)$.

The reason is there are 2^n sheets of $g_n(z)$ for every integer n , and it is possible that some of them meet at some $z \in D_{n+1}$, eg. $g_{n,1}(z') = g_{n,100}(z') = 0$ for some $z' \in \{z; |z| < \eta^{1/2^n}\}$, i.e., $g_{n-1}(z^2) - az$ has zeroes in D_n . Hence we let $\hat{h}(z, w) := \prod_{i=1, \dots, 2^n} (w - g_{n,i}(z))$. Then h is holomorphic on D_{n+1} and $\lim_{w \rightarrow 1} \hat{h}(0, w) = 0$. Thus, $\hat{h}$ vanishes at $(0, 1)$ and $\hat{h}(\Omega) \subset \Delta(0, R) \setminus \{0\}$ for some constant radius $R > 1$ since Ω is a bounded set. In addition, we can choose n big enough so that all finitely many (z_j, w_j) are inside D_{n+1} .

Then let $x' := \hat{h}(0, w_0) := \lim_{w \rightarrow 1} \hat{h}(0, w)$ and $y' := \hat{h}(z_j, w_j)$ for $j = 1, \dots, N$. Then

$$\begin{aligned} d_{D_{n+1}}((0, w_0), (z_j, w_j)) &\geq d_{\Delta(0, R) \setminus \{0\}}(x', y') \\ &= \left| \int_{x'}^{y'} \frac{1}{|z| \ln \frac{|z|}{R}} dz \right| \\ &= |\ln(|\ln y'/R|) - \ln(|\ln x'/R|)| \\ &\rightarrow \infty, \end{aligned}$$

as $x' \rightarrow 0$.

In the end, we still need to show that for any two points $(0, w_0), (z_j, w_j)$ inside D_{n+1} , the Kobayashi distance $d_{D_m}((0, w_0), (z_j, w_j))$ is approximately equal to $d_\Omega((0, w_0), (z_j, w_j))$ as long as $D_{n+1} \subset\subset D_m$, and D_m is very close to Ω . We prove the following localization result for the Kobayashi metric (see [4]).

Lemma IV.4.3. *For $0 < s < 1$, let $\Omega_s = \{(z, w) \in \Omega; |z| < s\}$. Fix $0 < r < 1$, let $0 < c < 1$. Then there exists an R ($r < R < 1$) so that for every $p \in \Omega_r$ and ξ , we have*

$$cK_\Omega(p, \xi) \geq K_{\Omega_R}(p, \xi) \geq K_\Omega(p, \xi).$$

Proof. By the Definition in section 2, we know

$$F_{\Omega_r}(p, \xi) := \inf\{\lambda > 0 : \exists f : \Delta \xrightarrow{hol} \Omega_r, f(0) = p, \lambda f'(0) = \xi\}.$$

We choose $r < R < 1$,

$$F_{\Omega_R}(p, \xi) := \inf\{\mu > 0 : \exists g : \Delta \xrightarrow{hol} \Omega_r, g(0) = p, \mu g'(0) = \xi\}.$$

Let $g : \Delta \rightarrow \Delta$ and $g(0) = p, g'(0) = \mu\xi$. By Schwarz Lemma, we know that $|g(rz)| < |rz| < r$ for $|z| < 1$. Then we choose $f(z) = g(rz) \in \Omega_r$ with $f'(0) = cg'(0)$. Therefore, we have

$$cK_\Omega(p, \xi) \geq K_{\Omega_R}(p, \xi) \geq K_\Omega(p, \xi).$$

□

Hence, $d_\Omega((0, w_0), (z_j, w_j)) \approx d_{D_{n+1}}((0, w_0), (z_j, w_j)) \rightarrow \infty$ for general a . Thus, there exists a point $(z_0, w_0) = (0, 1 - \delta) \in \Omega$, where $\delta \rightarrow 0$, so that for any $(\tilde{z}, \tilde{w}) \in \cup_k^\infty \{F^{-k}(\varepsilon, 0)\}, k \geq 0$, the Kobayashi distance $d_\Omega((z_0, w_0), (\tilde{z}, \tilde{w})) \geq C$.

□

$$F(z, w) = (P(z), Q(z, w))$$

IV.4.2 Dynamics of $F(z, w) = (z^2 + az, w^2 + cw + bz), 0 < |a|, |b|, |c| \ll 1$

Theorem IV.4.4. *Suppose $F(z, w) = (az + z^2, w^2 + cw + bz), 0 < |a|, |b|, |c| \ll 1$, and $|a| \gg |c|, |a| \gg |b|, |c| \gg |ab|$. Let Ω be the immediate attracting basin of $(0, 0)$. We choose an arbitrary constant $C > 0$. Then there exists a point $(z_0, w_0) \in \Omega$ so that for any $(\tilde{z}, \tilde{w}) \in \cup_k^\infty \{F^{-k}(0, 0)\}, k \geq 0$, the Kobayashi distance $d_\Omega((z_0, w_0), (\tilde{z}, \tilde{w})) \geq C$.*

Note that if we first fix a, c and let b be chosen smaller and smaller, then Theorem IV.3.2 always fails, but in the limit case when $b = 0$, Theorem IV.3.2 is valid. This shows that the situation is very unstable.

Proof. To prove this theorem, we first prove the following lemma:

Lemma IV.4.5. *Let $\Omega_{2/3} = \{(z, w), z \in \Omega_P, |w| < 2/3\}$, then $F(\Omega_{2/3}) \subseteq \Omega_{2/3}$. Moreover, $\Omega_{2/3} \subseteq \Omega$.*

Proof. We know $\Omega_P \subseteq \{z, |z| < 2\}$. If $|w| < 2/3$, we obtain $|w^2 + cw + bz| \leq |w^2| + |c||w| + |b||z| < 4/9 + 2c/3 + |b||z| < 4/9 + |c| + 2|b| < 2/3$ for $0 < |c|, |b| \ll 1$. Thus, $F(\Omega_{2/3}) \subseteq \Omega_{2/3}$.

Let $(z_1, w_1) \in \Omega_{2/3}$ and (z_n, w_n) be the orbit of (z_1, w_1) . We know $z_n \rightarrow 0$, and $|w_n| < 2/3$ since $F(\Omega_{2/3}) \subseteq \Omega_{2/3}$. Then $|w_{n+1}| \leq |w_n|^2 + |c||w_n| + |b||z_n| \leq (2/3 + |c|)|w_n| + |b||z_n|$. Hence $w_n \rightarrow 0$. Therefore, $\Omega_{2/3} \subseteq \Omega$. □

Lemma IV.4.6. *Let Ω_z be the slice of Ω at z and $\Omega_{z,2/3} = \{(z, w) \in \Omega_z, |w| < 2/3\}$. Then, each Ω_z is connected and simply connected.*

Proof. Since $Q(w) = w^2 + cw + bz$ is a two-to-one function, every point $w \in \Omega_{P(z)}$ has two preimages inside Ω_z counting multiplicity. Hence $F : \Omega_z \rightarrow \Omega_{P(z)}$ is a double covering and it has a critical point $w = -\frac{c}{2}$ in the w -coordinate.

Suppose there exists at least two disjoint connected components Ω_z^1, Ω_z^2 inside Ω_z , and $(z, 0) \in \Omega_z^1$. Then $\Omega_{z,2/3} \subseteq \Omega_z^1$. By Lemma IV.4.5, we know $F(\Omega_{z,2/3}) \subseteq \Omega_{P(z),2/3}$. In addition, F sends Ω_z to $\Omega_{P(z)}$, then we know $F(\Omega_z^1) \subseteq \Omega_{P(z)}$.

Furthermore, we know that every point inside $\Omega_{P(z),2/3}$ has two preimages inside Ω_z^1 since F is a double covering, and it has a critical point $w = -\frac{c}{2}$, which is very close to 0 in the w -coordinate. Hence $F(\Omega_z^2) \cap \Omega_{P(z),2/3} = \emptyset$. Inductively, we know $F^n(\Omega_z^2) \cap \Omega_{P^n(z),2/3} = \emptyset$. Thus, $F^n(\Omega_z^2)$ cannot converge to 0. Therefore, Ω_z^2 is not inside Ω_z .

By the maximum principle, we know that Ω_z is simply connected. □

Lemma IV.4.7. *Let z_N be a preimage of $-a \in P^{-1}(0)$, i.e., $z_N \in P^{-N}(-a)$, where N is large enough. Then we have $(0, 0) \notin F^n(\Omega_{z_N, 2/3})$ for any integer $n \geq 1$.*

Proof. Let us take a point $(z_N, w_N) \in \Omega_{z_N, 2/3}$ where N is sufficiently large, then $F(z_N, w_N) = (z_{N-1}, w_{N-1}) \in \Omega_{z_{N-1}, 1/2}$, $F^2(z_N, w_N) = (z_{N-2}, w_{N-2}) \in \Omega_{z_{N-2}, 1/3}$, $F^3(z_N, w_N) = (z_{N-3}, w_{N-3}) \in \Omega_{z_{N-3}, 1/4}$. If $4|b| < |w| < 1/4$, we know that

$$|w^2 + cw + 2b| < |w|(|w| + |c| + 1/2) < 7/8|w|.$$

This shows w will shrink to 0 very fast. Thus, for some uniformly large $L \gg 4$, we will have $|w_{N-l}| < 4|b|$ for all $L \leq l \leq N$. Then inductively, $F^{N-1}(z_N, w_N) = (-a, w_1) \in \Omega_{z_1, 4b}$, i.e., $|w_1| \leq 4|b|$. And $F^N(z_N, w_N) = (0, w_0) = (0, w_1^2 + cw_1 + bz_1)$. Then

$$0 < \frac{1}{2}|ab| \leq |ab| - 16|b|^2 - 4|cb| \leq |w_0| = |w_1^2 + cw_1 + bz_1|$$

and

$$|w_1^2 + cw_1 + bz_1| < 16|b|^2 + 4|cb| + |ab| \leq 2|ab| < |c|,$$

since $|a| \gg |c|, |a| \gg |b|, |c| \gg |ab|$.

Therefore, $(0, 0) \notin F^n(z_N, \Omega_{2/3})$ for any $n \leq N$. However, for $n > N$, we use that F restricted to the w -axis is just $w \rightarrow w^2 + cw \approx cw$, so w_n goes to 0 but never lands on 0. □

Lemma IV.4.8. *Let $\Omega_n = F^{-n}(\Omega_{2/3})$. Then for $(\tilde{z}_n, \tilde{w}_n) \in \Omega_n$, the Euclidean distance $d_E(\tilde{w}_n, \partial\Omega_{\tilde{z}_n}) \rightarrow 0$ in $\Omega_{\tilde{z}_n}$ when $n \rightarrow \infty$.*

Proof. Suppose that for some $\varepsilon > 0$, there exists arbitrarily large N_1 such that, in $\Omega_{\tilde{z}_{N_1}}$, the Euclidean distance $d_E(\tilde{w}_{N_1}, \partial\Omega_{\tilde{z}_{N_1}}) > \varepsilon$. Then since $\left|\frac{\partial Q}{\partial w}\right| > 2|w| - |c| > \frac{7}{6} > 1$ for $|w| > \frac{2}{3}$, it follows that the distance $d_E(\tilde{w}_{N_1-1}, \partial\Omega_{\tilde{z}_{N_1-1}}) > \frac{7}{6}\varepsilon$. Repeating this for large l times, we have $d_E(\tilde{w}_{N_1-l}, \partial\Omega_{\tilde{z}_{N_1-l}}) > \frac{7}{6}\varepsilon^l \geq 4$. It will get a contradiction to $d_E(\tilde{w}_{N_1-l}, \partial\Omega_{\tilde{z}_{N_1-l}})$ bounded by 4. □

Lemma IV.4.9. *Let $D := \partial\Omega \cap \{(z, w); z \in P(z)\}$. Then D is laminated by holomorphic graphs $w = f_\alpha(z)$. Moreover, the Kobayashi distance $d_\Omega((z_N, 0), (\tilde{z}, \tilde{w})) \geq C$ for any $(z_N, 0) \in \Omega_{z_N}$, $(\tilde{z}, \tilde{w}) \in \cup_k^\infty \{F^{-k}(0, 0)\} \subset \Omega_{z_k}, k \geq 0, N \geq N(C)$.*

Proof. The set $\partial\Omega_{2/3}$ is laminated by graphs $w = \frac{2}{3}e^{i\theta}$. Then we take $F^{-1}(w = \{\frac{2}{3}e^{i\theta}\}) = \frac{-c \pm \sqrt{c^2 - 4bz - \frac{8}{3}e^{i\theta}}}{2} := f_1^{1,2}$. It is obvious that $c^2 - 4bz - \frac{8}{3}e^{i\theta} \neq 0$ since $0 < |b|, |c| \ll 1, |z| < 2$. Hence $F^{-1}(w = \{\frac{2}{3}e^{i\theta}\})$ always has two disjoint preimages. Then we can use $f_1^{1,2}$ to laminate $F^{-1}(\partial\Omega_{2/3})$.

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$$F(z, w) = (P(z), Q(z, w))$$

Inductively, we can use $f_j^{1,2}$ to laminate $F^{-j}(\partial\Omega_{2/3})$, $j \geq 2$. We know that $F^{-(j-1)}(\partial\Omega_{2/3})$ is laminated by $f_{j-1}^{1,2}$, hence $w = f_{j-1}^{1,2}(z)$ are graphs inside $F^{-(j-1)}(\partial\Omega_{2/3})$. Then we calculate $F^{-1}(w = f_{j-1}^{1,2}(z))$: let

$$(Z, W) \in F^{-1}(w = f_{j-1}^{1,2}(z)) \text{ i.e., } F(Z, W) \in (w = f_{j-1}^{1,2}(z)).$$

Hence

$$Z^2 + aZ = z, W^2 + cW + bZ = w,$$

then

$$\begin{aligned} W^2 + cW + bZ &= f_j(z) = f_j(Z^2 + aZ), \\ W^2 + cW + bZ - f_j(Z^2 + aZ) &= 0, \end{aligned}$$

thus,

$$W = \frac{-c \pm \sqrt{c^2 - 4bZ + 4f_j(Z^2 + aZ)}}{2} = f_{j+1}^{1,2}(Z).$$

Then let $f_\alpha(z) := \lim_{j \rightarrow \infty} F^{-j}(w = \{\frac{2}{3}e^{i\theta}\})$. Therefore, D is laminated by graphs $w = f_\alpha(z)$.

Next, we need to show in any different slices Ω_{z_N} , and any $(\tilde{z}, \tilde{w})$ in any slices Ω_{z_k} , we always have $d_\Omega((z_N, 0), (\tilde{z}, \tilde{w})) \geq C$ as long as $(\tilde{z}, \tilde{w}) \rightarrow \partial\Omega$. Here we choose N sufficiently large. Note that the Kobayashi distance $d_{\Omega_P}(z_k, z_N) \geq C$, if $z_N \rightarrow \partial\Omega_P$ for a fixed k (See [16]). So we can assume both k and N are very large. However, by Lemma IV.4.8, $\tilde{w}$ is very close to some point denoted by $(\tilde{z}, \eta)$ in $\partial\Omega_{\tilde{z}}$.

Let $H : \Omega \rightarrow \Delta(0, R) \setminus \{0\}$, $H(z, w) = w - f_\alpha(z)$. Here we choose α so that $f_\alpha(\tilde{z}) = \eta$. Then $H(z, w)$ is holomorphic on Ω . And we know that $H(z, w) = w - f_\alpha(z)$, then $\lim_{w \rightarrow f_\alpha(z)} H(z, w) = 0$. Hence $H(z, w)$ is vanishing at $(\tilde{z}, \eta)$ and $H(\Omega) \subset \Delta(0, R) \setminus \{0\}$ for some constant radius $R > 1$ since Ω is a bounded set.

Then let $x' := H(z, w)$ and $y' := H(\tilde{z}, \tilde{w})$. Then

$$\begin{aligned} d_\Omega((z, w), (\tilde{z}, \tilde{w})) &\geq d_{\Delta(0, R) \setminus \{0\}}(x', y') \\ &= \left| \int_{x'}^{y'} \frac{1}{|z| \ln \frac{|z|}{R}} dz \right| \\ &= |\ln(|\ln y'/R|) - \ln(|\ln x'/R|)| \\ &\rightarrow \infty, \end{aligned}$$

as $x' \rightarrow 0$, i.e., $w \rightarrow f_\alpha(z)$.

□

Now we continue to prove Theorem IV.4.4 using the same method as in Theorem IV.4.1. We take $(z_0, w_0) = (z_N, 0) \in \Omega_{z_N, 2/3}$. Then $H(\Omega_{2/3}) \subset \Delta(0, R) \setminus \{0\}$, $H(\tilde{z}, \tilde{w}) \rightarrow 0$ as $k \rightarrow \infty$.

Therefore, we know that there exists a point $(z_0, w_0) = (z_N, 0)$ so that for any $(\tilde{z}, \tilde{w}) \in \cup_k^\infty \{F^{-k}(0, 0)\}$, $k \geq 0$, the Kobayashi distance $d_\Omega((z_0, w_0), (\tilde{z}, \tilde{w})) = d_\Omega((z_N, 0), (P^{-i}(0), Q^{-i}(0, 0))) \geq d_{\Delta(0, R) \setminus \{0\}}(x', y') \geq C$ for all $i \in \mathbb{N}$.

□

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