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Kinetic models for systems of interacting agents with multiple microscopic states

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ABSTRACT

We propose and investigate general kinetic models with transition probabilities that can describe the simultaneous change of multiple microscopic states of the interacting agents. These models can be applied to many problems in socio-economic sciences, where individuals may change both their compartment and their characteristic microscopic variable, as for instance kinetic models for epidemic diffusion or for international trade with possible transfers of agents. Mathematical properties of the kinetic model are proved, as existence and uniqueness of a solution for the Cauchy problem in suitable Wasserstein spaces. The quasi-invariant asymptotic regime, leading to simpler kinetic Fokker–Planck-type equations, is investigated and commented on in comparison with other existing models. Some numerical tests are performed in order to show the time evolution of distribution functions and of meaningful macroscopic fields, even in case of non-constant interaction rates and transfer probabilities.

1. Introduction

In the literature of kinetic models for multi-agent systems, there is an increasing interest in phenomena where the agents are characterized by a multiple microscopic state and they are, in particular, divided into subpopulations.

Kinetic theory for multi-agent systems has its roots in the classical kinetic theory related to the Boltzmann equation for the description of a rarefied gas [1], in which individuals are molecules identified by a microscopic state v, that is the velocity, and that changes because of *binary collisions*. The classical kinetic theory for gas dynamics has been generalized to various kinds of interacting systems, where the microscopic state is not necessarily the velocity, providing reasonable mathematical models for many socio-economic problems, as the evolution of wealth distribution [2,3], the opinion formation [4,5], the pedestrian or vehicular traffic dynamics [6,7], birth and death processes [8,9] and many others. In this context, as the agents do not collide but more appropriately interact, we may refer generally to *binary interactions*, while the kinetic equation is often referred to as *collision-like Boltzmann equation*. Of course these social interactions, which rely essentially on human behavior, are generally modeled heuristically, since there is not a background physical theory providing a fundamental model for them. However, some classical tools of the kinetic theory for gas dynamics are extremely efficient to understand the deep links between individual and collective behavior. In [10], an extended overview of the parallelism between the classical Boltzmann approach and kinetic models for socio-economic sciences is presented. A more historical perspective on the attempts of modeling society by means of laws of statistical physics may be found in [11,12]. Concerning multiple interactions, in many kinetic models for social sciences, each interaction involving simultaneously more than two individuals is separated in a chain of binary interactions: the problem is then seen as a huge amount of small binary exchanges, leading to the possibility of mathematically investigating suitable quasi-invariant asymptotic limits. Some generalizations of Boltzmann-type equations to the case of multiple interactions (involving more than two individuals) have been recently proposed [3,13]. Another context in which multiple interactions arise is the one of collective motion, e.g. flocks of birds and schools of fishes. In particular, it has been shown experimentally that birds and fishes interact in a certain perception range [14], and, in particular, birds in flocks interact with six or seven birds in a given neighborhood [15,16]. At a kinetic level, this has been described by *nonlocal* binary interactions, i.e. among individuals having a certain distance between them, for example by means of the Boltzmann–Povzner equation [17]. Another example is given by *active particle methods*, in which nonlinearly additive outputs from interactions that can happen in a sensitivity perception range are taken into account (see e.g. [18,19] and references therein).

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The classical Boltzmann equation has been extended some decades ago to mixtures of different gaseous species [20,21], even in presence of chemical reactions [22], and also consistent BGK approximations for mesoscopic models of gas particles (originally introduced by Bhatnagar, Gross, and Krook, replacing the Boltzmann-type kernel with a relaxation towards the equilibrium distribution of the full kinetic equation) have been proposed and investigated [23–25]. In these models, when describing bimolecular chemical reactions, a given binary interaction between molecules may simultaneously cause both the change of molecular velocities and transfers of the involved molecules to different species. Also models in which the molecule is characterized by the belonging to a species, a molecular velocity and another inner variable modeling the internal energy have been proposed [26].

In the field of kinetic equations for multi-agent systems applied to socio-economic phenomena, a very interesting class of models is the one in which the total population is divided into subgroups and each agent is also described by a physical quantity (wealth, opinion, viral load, etc.). Each group is then characterized by a distribution function depending on the given microscopic physical quantity that can be exchanged both with individuals of the same subgroup and with individuals of a different subgroup, according to suitable interaction rules. We mention, as examples, [27,28] for wealth exchanges, [29] for opinion formation in presence of leaders, and [30] for multilane traffic models. This can be related to multi-agent systems in which agents are characterized by a multiple microscopic state: in fact, in this case the agents are characterized by a physical quantity (wealth, opinion, velocity) and by a label denoting the subgroup. For simplicity, interactions causing exchanges of the physical quantity and the ones giving rise to a change of subgroup of one agent are often modeled separately, by means of different kinetic operators [27–29,31–33]. Formally, models belonging to this class have been derived in [9], where the authors describe multi-agent systems in which the agents are characterized by a double microscopic state: a physical quantity v, changing because of binary interactions, and a label x denoting the subgroup of the agent, changing as a consequence of a Markovian process. The two stochastic processes for the evolution of v and x are *independent* and occur with different frequencies, giving thus rise to different operators, where the one relevant to the variation of v may be the sum of *inter-* or *intra-* species interactions. There are other possible formal approaches allowing to consider similar problems of interacting particles in the presence of multiple populations and switching mechanisms. In particular, following [34], in [35] the authors propose a mean-field analysis of a multi-population agent-based dynamics as introduced in [36], gaining a system of coupled drift-reaction equations, where the drift contribution embodies the interactions, while the reaction implements the switching mechanism. In [37,38] by adding a stochasticity to the microscopic dynamics describing the interactions, a Fokker–Planck equation is retrieved, in the same line to the quasi-invariant limit in [9].

However, a class of models worthy to be investigated is the one in which the agent changes the microscopic quantity v and the subgroup simultaneously as a consequence of the *same* binary interaction. An example is given by the aforementioned bimolecular chemical reactions in gas mixtures. The classical kinetic description for chemically reacting gases was introduced in the pioneering work [39], where each gas species has a different distribution function of the microscopic velocity of its molecules. A pair of molecules belonging to potentially different species may interact both exchanging the velocity and changing the species. In the literature of kinetic theory for socio-economic sciences, the recent paper [40] describes the trade among different subpopulations, living in different countries, taking into account also possible transfers of individuals from one country to another, by means of suitable Boltzmann-type operators similar to the ones modeling bimolecular chemical reactions in gas mixtures. In this model, then, binary interactions between individuals may lead to both an exchange of wealth and to a transfer to another subpopulation as a consequence of the *same* binary interaction, i.e. the microscopic state identifying the wealth and the one related to the label, that denotes the subpopulation, change *simultaneously*.

Another type of kinetic models allowing transfers of individuals from a compartment to another involves suitable Markov-type jump processes governed by suitable transition probabilities appearing in the kernel of the Boltzmann operators. This approach is usual in kinetic models for the spread of epidemics, where each agent is characterized by the microscopic viral load v and by a label x denoting its compartment, and the microscopic dynamics is thus described through discrete-time stochastic processes [41,42]. This kind of formulation of the Boltzmann equation is well known also in kinetic theory for a single gas. Indeed, besides the classical Boltzmann operator [1], other different forms have been used in the literature. The most common one involving transition probabilities is the so-called Waldmann representation [43]. In the literature of kinetic equations for socio-economic sciences, the authors in [44] show the equivalence between the collision-like Boltzmann equation and Markovian jump-processes described by transition probabilities that can be related to the Waldmann (probabilistic) representation of the Boltzmann equation.

In this paper we will present kinetic models for socio-economic problems in which agents have a multiple microscopic state. We will start from a microscopic stochastic process ruled by transition probabilities that allow to describe the simultaneous change of all the microscopic variables and by a microscopic state-dependent interaction frequency. In the case in which the microscopic state of the agent is given by a microscopic physical quantity v and by a label x denoting the subgroup, we will show that this approach has some advantages with respect to the classical collision-like model [40]. Indeed, as explained also in [40], the construction of the gain term of the Boltzmann operators requires the invertibility of the interaction process, that obviously holds in gas-dynamics (because of conservation of total momentum and energy in each interaction), but not in human interactions, that are also influenced by non-deterministic (random) effects. This invertibility property is not needed in the operator with a transition probability in the kernel, because each single interaction has its own probability, that is not related to the reverse process. Moreover, realistic situations with non-constant interaction probabilities are more easily manageable in the proposed framework, therefore this approach could have many applications in kinetic modeling of social sciences. For these reasons in this paper we present a formal and organic treatment of kinetic equations implementing a microscopic stochastic process that simultaneously changes several internal states of the interacting agents: typically, these will be given by the label denoting the subgroup and a continuous variable describing a physical quantity. In more detail, the paper is organized as follows.

In Section 2 we formally derive the general form of a kinetic model implementing a microscopic stochastic dynamics in which each agent is characterized by a set $\boldsymbol{z} \in \Omega \subset \mathbb{R}^d$ of microscopic states which may change simultaneously in each binary interaction, that is described by a transition probability and ruled by a microscopic state-dependent frequency. Even though the procedure is quite classical, stating the discrete-time stochastic process will be useful for writing the Nanbu–Babovski algorithm for simulating the microscopic dynamics. The relation with some well-known models such as the collision-like Boltzmann equation and the kinetic equations describing transfers among different groups due to binary interactions is treated in Appendix. Finally, we explicitly derive the kinetic equation for a multi-agent system in which an agent is characterized by $\boldsymbol{z} = (x, v)$, and in which a binary interaction causes simultaneously a transfer (a change of the label x) and an exchange of a microscopic physical quantity v. In Section 3, mathematical properties of the Cauchy problem associated to the derived kinetic equation are discussed, proving existence and uniqueness of a solution in suitable Wasserstein spaces. Then, the quasi-invariant limit commonly used to investigate socio-economic kinetic models is adapted to our general frame, allowing to derive suitable Fokker–Planck equations with additional terms taking into account transfers of agents. Section 4 is devoted to the investigation of a specific kinetic model fitting into our general framework, describing international trade

with possible transfers of individuals among different countries. The evolution equations of the number density and of the mean wealth of each country are derived from the kinetic model, the quasi-invariant limit is performed, and analogies and differences with respect to analogous models for a single population [2] are discussed, with particular reference to the Pareto index of the steady state distributions. In Section 5 we show some numerical tests, simulating the kinetic equations by means of a Nanbu–Babovski Monte Carlo algorithm implementing the discrete-time stochastic process presented in Section 2: the evolution of distribution functions and of macroscopic quantities are commented on for several different sets of parameters. Section 6 contains some concluding remarks and perspectives.

2. Kinetic models for binary interactions processes

In this section, we provide a formal derivation of kinetic equations implementing binary interaction processes among agents having a multiple microscopic state, i.e. whose microscopic state is a vector $\boldsymbol{z} \in \mathbb{R}^d$. The outcome of the binary interaction is described by means of Markovian processes whose frequency depends on the microscopic state of the interacting agents. This allows one to describe binary interactions in which agents change all the components of their microscopic state simultaneously as a consequence of a single binary interaction and not as a consequence of independent interaction processes. We will use this in order to introduce a general framework for describing, through kinetic equations, multi-agent systems in which the agents are divided into subgroups and are also characterized by a physical quantity. Hence, the agents will be typically characterized by a microscopic state $\boldsymbol{z} = (x, v)$, where x is a *label* denoting the subgroup and v is a *physical quantity*. These agents perform binary interactions leading, simultaneously, to both exchanges of the physical quantity and migrations between different groups. Other more complicated options could also fit into our framework, as for instance $\boldsymbol{z} = (x, v, k) \in \mathbb{R}^3$, where agents of group x are characterized by two individual quantities (for example wealth v and level of knowledge k).

2.1. Kinetic models with transition probabilities

Let us consider a large system of agents described by a microscopic state $\boldsymbol{z} \in \Omega \subset \mathbb{R}^d$. We shall suppose that the change of the microscopic state (of all its components simultaneously) is due to stochastic *binary interactions*. A probabilistic description of such interactions may be given by means of *transition probability* functions

$$T(\boldsymbol{z}'|\boldsymbol{z}, \boldsymbol{y}) > 0, \qquad \tilde{T}(\boldsymbol{y}'|\boldsymbol{z}, \boldsymbol{y}) > 0 \qquad \forall \boldsymbol{z}, \boldsymbol{y} \in \Omega,\ t > 0, \tag{1}$$

namely the *conditional* probabilities that, given a binary interaction between an agent $\boldsymbol{z}$ and an agent $\boldsymbol{y}$, the first changes into $\boldsymbol{z}'$ while the second into $\boldsymbol{y}'$, respectively. Such a microscopic description may be assimilated to a *Markov-type jump process*. In order for $T(\boldsymbol{z}'|\boldsymbol{z}, \boldsymbol{y}), \tilde{T}(\boldsymbol{y}'|\boldsymbol{z}, \boldsymbol{y})$ to be conditional probability densities, they have to satisfy the following further property:

$$\int_\Omega T(\boldsymbol{z}'|\boldsymbol{z}, \boldsymbol{y})\, d\boldsymbol{z}' = 1, \quad \int_\Omega \tilde{T}(\boldsymbol{y}'|\boldsymbol{z}, \boldsymbol{y})\, d\boldsymbol{y}' = 1 \qquad \forall \boldsymbol{z}, \boldsymbol{y} \in \Omega,\ t > 0. \tag{2}$$

The binary interactions may happen with a frequency $\lambda_{\boldsymbol{z}\boldsymbol{y}}$, namely the frequency of the binary interactions between two agents having microscopic states $\boldsymbol{z}, \boldsymbol{y}$ depends on the microscopic states themselves. We remark that the two transition probabilities T and $\tilde{T}$ are given in order to take into account possible asymmetries in the binary interactions. The symmetry of the binary interactions is here recovered when

$$T(\boldsymbol{z}'|\boldsymbol{z}, \boldsymbol{y}) = \tilde{T}(\boldsymbol{z}'|\boldsymbol{y}, \boldsymbol{z}), \tag{3}$$

and $\lambda_{\boldsymbol{z}\boldsymbol{y}} = \lambda_{\boldsymbol{y}\boldsymbol{z}}$. As classically done [45], a kinetic description of the multi-agent system can be derived by introducing discrete-time stochastic processes. Let $\boldsymbol{Z}_t, \boldsymbol{Y}_t \in \Omega$ be random variables describing the microscopic state of two agents at time $t > 0$. Let $f = f(\boldsymbol{z}, t)$ be the probability density function associated to the multi-agent system, i.e. the probability density function of the random variable of a given agent $\boldsymbol{Z}_t$, thus satisfying

$$\int_\Omega f(\boldsymbol{z}, t)\, d\boldsymbol{z} = 1. \tag{4}$$

During a sufficiently small time $\Delta t > 0$ the agents may or may not change their state $\boldsymbol{Z}_t, \boldsymbol{Y}_t$ depending on whether a binary interaction takes place or not. We express this discrete-time random process as

$$\begin{aligned} \boldsymbol{Z}_{t+\Delta t} &= (1-\Theta)\boldsymbol{Z}_t + \Theta \boldsymbol{Z}'_t, \\ \boldsymbol{Y}_{t+\Delta t} &= (1-\Theta)\boldsymbol{Y}_t + \Theta \boldsymbol{Y}'_t, \end{aligned} \tag{5}$$

where $\boldsymbol{Z}'_t, \boldsymbol{Y}'_t$ are random variables describing the new microscopic state of $\boldsymbol{Z}_t$ and $\boldsymbol{Y}_t$ respectively after a binary interaction and having *joint* probability density functions $g = g(\boldsymbol{Z}'_t = \boldsymbol{z}'; \boldsymbol{Z}_t = \boldsymbol{z}, \boldsymbol{Y}_t = \boldsymbol{y}), \tilde{g} = \tilde{g}(\boldsymbol{Y}'_t = \boldsymbol{y}'; \boldsymbol{Z}_t = \boldsymbol{z}, \boldsymbol{Y}_t = \boldsymbol{y})$, while $\Theta \in \{0, 1\}$ is a Bernoulli random variable, which we assume to be independent of all the other random variables appearing in (5), discriminating whether a binary interaction takes place ($\Theta = 1$) or not ($\Theta = 0$) during the time interval Δt. In particular, we set the probability to change the microscopic state

$$\mathrm{Prob}(\Theta = 1) = \lambda_{\boldsymbol{Z}_t\boldsymbol{Y}_t}\Delta t, \tag{6}$$

where $\lambda_{\boldsymbol{Z}_t\boldsymbol{Y}_t}$ is the interaction frequency between agents with microscopic states $\boldsymbol{Z}_t$ and $\boldsymbol{Y}_t$. Notice that, for consistency, we need $\lambda_{\boldsymbol{Z}_t\boldsymbol{Y}_t}\Delta t \le 1$.

Let now $\phi = \phi(\boldsymbol{z})$ be an observable quantity defined on $\boldsymbol{z} \in \Omega$. From (5)–(6), together with the assumed independence of Θ, we see that the mean variation rate of ϕ in the time interval Δt satisfies

$$\begin{aligned} &\frac{\langle \phi(\boldsymbol{Z}_{t+\Delta t})\rangle - \langle \phi(\boldsymbol{Z}_t)\rangle}{\Delta t} + \frac{\langle \phi(\boldsymbol{Y}_{t+\Delta t})\rangle - \langle \phi(\boldsymbol{Y}_t)\rangle}{\Delta t} \\ &= \frac{\langle (1-\lambda_{\boldsymbol{Z}_t\boldsymbol{Y}_t}\Delta t)\phi(\boldsymbol{Z}_t)\rangle + \Delta t\langle \lambda_{\boldsymbol{Z}_t\boldsymbol{Y}_t}\phi(\boldsymbol{Z}'_t)\rangle - \langle \phi(\boldsymbol{Z}_t)\rangle}{\Delta t} \\ &+ \frac{\langle (1-\lambda_{\boldsymbol{Z}_t\boldsymbol{Y}_t}\Delta t)\phi(\boldsymbol{Y}_t)\rangle + \Delta t\langle \lambda_{\boldsymbol{Z}_t\boldsymbol{Y}_t}\phi(\boldsymbol{Y}'_t)\rangle - \langle \phi(\boldsymbol{Y}_t)\rangle}{\Delta t}, \end{aligned}$$

where $\langle C_t \rangle$ denotes the average of the random variable C_t with respect to its probability density function. Whence, we deduce the instantaneous time variation of the average of ϕ in the limit $\Delta t \to 0^+$ as

$$\frac{d}{dt}\langle \phi(\boldsymbol{Z}_t)\rangle = \frac{1}{2}\Big(\langle \lambda_{\boldsymbol{Z}_t\boldsymbol{Y}_t}\phi(\boldsymbol{Z}'_t)\rangle + \langle \lambda_{\boldsymbol{Z}_t\boldsymbol{Y}_t}\phi(\boldsymbol{Y}'_t)\rangle - \langle \lambda_{\boldsymbol{Z}_t\boldsymbol{Y}_t}\phi(\boldsymbol{Z}_t)\rangle - \langle \lambda_{\boldsymbol{Z}_t\boldsymbol{Y}_t}\phi(\boldsymbol{Y}_t)\rangle\Big), \tag{7}$$

where we have used the fact that $\langle \phi(\boldsymbol{Z}_t)\rangle = \langle \phi(\boldsymbol{Y}_t)\rangle$ which implies $\langle \phi(\boldsymbol{Z}_t)\rangle + \langle \phi(\boldsymbol{Y}_t)\rangle = 2\langle \phi(\boldsymbol{Z}_t)\rangle$.

We now specify the gain terms as

$$\langle \lambda_{\boldsymbol{Z}_t\boldsymbol{Y}_t}\phi(\boldsymbol{Z}'_t)\rangle = \int_\Omega\int_{\Omega^2} \phi(\boldsymbol{z}')\lambda_{\boldsymbol{z}\boldsymbol{y}} g(\boldsymbol{z}';\boldsymbol{z},\boldsymbol{y})\,d\boldsymbol{z}d\boldsymbol{y}\,d\boldsymbol{z}',$$
$$\langle \lambda_{\boldsymbol{Z}_t\boldsymbol{Y}_t}\phi(\boldsymbol{Y}'_t)\rangle = \int_\Omega\int_{\Omega^2} \phi(\boldsymbol{y}')\lambda_{\boldsymbol{z}\boldsymbol{y}} \tilde{g}(\boldsymbol{y}';\boldsymbol{z},\boldsymbol{y})\,d\boldsymbol{z}d\boldsymbol{y}\,d\boldsymbol{y}',$$

where, as already mentioned, g and $\tilde{g}$ are the joint probability density functions of $\boldsymbol{Z}'_t$ and $\boldsymbol{Y}'_t$, respectively, and of the samples of the random variables $\boldsymbol{Z}_t = \boldsymbol{z}, \boldsymbol{Y}_t = \boldsymbol{y}$ at time t. The probability density functions g and $\tilde{g}$ are defined as

$$g(\boldsymbol{z}';\boldsymbol{z},\boldsymbol{y}) = T(\boldsymbol{z}'|\boldsymbol{z},\boldsymbol{y})\,f_2(\boldsymbol{z},\boldsymbol{y},t), \qquad \tilde{g}(\boldsymbol{y}';\boldsymbol{z},\boldsymbol{y}) = \tilde{T}(\boldsymbol{y}'|\boldsymbol{z},\boldsymbol{y})\,f_2(\boldsymbol{z},\boldsymbol{y},t), \tag{8}$$

where $f_2(\boldsymbol{z},\boldsymbol{y},t)$ is the joint distribution of the couple $(\boldsymbol{z},\boldsymbol{y})$ at time t. As typically done in kinetic theory, we assume *propagation of chaos*, i.e. we assume that there is stochastic independence of two random agents as the number of agents is large. Hence, $\boldsymbol{z}$ and $\boldsymbol{y}$ are independently distributed, which allows us to perform the factorization $f_2(\boldsymbol{z},\boldsymbol{y},t) = f(\boldsymbol{z},t)f(\boldsymbol{y},t)$, so that we can write

$$g(\boldsymbol{z}';\boldsymbol{z},\boldsymbol{y}) = T(\boldsymbol{z}'|\boldsymbol{z},\boldsymbol{y})\,f(\boldsymbol{z},t)f(\boldsymbol{y},t), \qquad \tilde{g}(\boldsymbol{y}';\boldsymbol{z},\boldsymbol{y}) = \tilde{T}(\boldsymbol{y}'|\boldsymbol{z},\boldsymbol{y})\,f(\boldsymbol{z},t)f(\boldsymbol{y},t).$$

It is immediate to verify that g and $\tilde{g}$ are probability density functions thanks to (2) and (4). Analogously, the loss terms can be naturally written as

$$\langle \lambda_{\boldsymbol{Z}_t\boldsymbol{Y}_t}\phi(\boldsymbol{Z}_t)\rangle = \int_{\Omega^2} \phi(\boldsymbol{z})\lambda_{\boldsymbol{z}\boldsymbol{y}} f(\boldsymbol{z},t)f(\boldsymbol{y},t)\,d\boldsymbol{z}d\boldsymbol{y},$$
$$\langle \lambda_{\boldsymbol{Z}_t\boldsymbol{Y}_t}\phi(\boldsymbol{Y}_t)\rangle = \int_{\Omega^2} \phi(\boldsymbol{y})\lambda_{\boldsymbol{z}\boldsymbol{y}} f(\boldsymbol{z},t)f(\boldsymbol{y},t)\,d\boldsymbol{z}d\boldsymbol{y}.$$

Therefore (7) can be stated as

$$\begin{aligned}\frac{d}{dt}\int_\Omega f(\boldsymbol{z},t)\phi(\boldsymbol{z})\,d\boldsymbol{z} = {}& \frac{1}{2}\int_\Omega\int_{\Omega^2} \lambda_{\boldsymbol{z}\boldsymbol{y}} T(\boldsymbol{z}'|\boldsymbol{z},\boldsymbol{y})\big(\phi(\boldsymbol{z}') - \phi(\boldsymbol{z})\big)\,f(\boldsymbol{z},t)f(\boldsymbol{y},t)\,d\boldsymbol{z}d\boldsymbol{y}d\boldsymbol{z}' \\ &+ \frac{1}{2}\int_\Omega\int_{\Omega^2} \lambda_{\boldsymbol{z}\boldsymbol{y}} \tilde{T}(\boldsymbol{y}'|\boldsymbol{z},\boldsymbol{y})\big(\phi(\boldsymbol{y}') - \phi(\boldsymbol{y})\big)\,f(\boldsymbol{z},t)f(\boldsymbol{y},t)\,d\boldsymbol{z}d\boldsymbol{y}d\boldsymbol{y}',\end{aligned} \tag{9}$$

where we have used (2) in order to write the loss terms.

2.2. Interacting agents with label switch and exchange of physical quantities

Let us now consider the case in which the agents of the system are characterized by an exchangeable physical quantity and they are also divided into subpopulations. Therefore, an agent is characterized by $v \in \mathbb{R}_+$ and by a discrete variable $x \in \mathcal{I}_n = \{1,\dots,n\}$. This discrete variable $x \in \mathcal{I}_n$ is regarded as a *label*, that may denote the belonging of the agent to a certain subgroup or subpopulation. We assume that the label changes because of *label switches*, i.e. migrations across subpopulations or transfers. A binary interaction between agents may cause a transfer of (potentially) both of the agents, but in such a way that the total mass in the system is conserved. We say that this process is formally a Markov-type one because the probability to switch from the current labels x, y to new labels x', y' does not depend on how the agents reached previously the labels x, y. In particular we denote

$$P_{xy}^{x'y'} := P(x',y'|x,y) \tag{10}$$

the conditional probability density function of switching to the groups x', y' given the pre-interaction labels x, y.

Remark 2.1. Since the variables x, y are discrete, the mapping $(x',y') \mapsto P(x',y'|x,y)$ is a discrete probability measure. Consequently, we actually have

$$\int_{\mathcal{I}_n^2} P(x',y'|x,y)\,dx'dy' = \sum_{x',y'\in\mathcal{I}_n} P(x',y'|x,y) = 1. \tag{11}$$

The physical quantity v changes as a consequence of binary interaction rules, as classically done in kinetic theory [45]. Here, agents *within the same group*, i.e. with the same label, are assumed to be indistinguishable. Therefore, given a couple of interacting agents with their label $(x,v),(y,w)$, their post-interaction physical quantities v', w' will be given by

$$v' = I_{xy}(v,w) + D_{xy}(v,w)\eta, \qquad w' = \tilde{I}_{xy}(v,w) + \tilde{D}_{xy}(v,w)\eta_*, \tag{12}$$

where η and η_* are independent random variables satisfying $\langle\eta\rangle = \langle\eta_*\rangle = 0, \langle\eta^2\rangle = \langle\eta_*^2\rangle = 1$, namely with zero average and unitary variance, while $I_{x,y}, \tilde{I}_{xy}$ and $D_{xy}, \tilde{D}_{xy}$ model the deterministic and stochastic contribution of the binary interaction rule, respectively. We remark that we can take into account the possibility that the interactions among agents with the same label differ from those among agents with different labels.

Hence, now, referring to the notation of the previous section, the microscopic state is $\boldsymbol{z} = (x,v) \in \Omega = \mathcal{I}_n \times \mathbb{R}_+$. The present framework allows one to describe a situation in which an agent, as a consequence of a single binary interaction, changes (simultaneously) both the microscopic quantity v and the label x. We now want to derive a kinetic equation for the joint distribution function $f = f(x,v,t) \geq 0$, such that $f(x,v,t)dv$ gives the

proportion of agents labeled by $x \in \mathcal{I}_n$ and having microscopic state comprised between v and $v + dv$ at time t. The discreteness of x allows us to represent f as [9]

$$f(x, v, t) = \sum_{i=1}^{n} f_i(v, t)\delta(x - i), \tag{13}$$

where $f_i = f_i(v, t) \geq 0$ is the distribution function of the microscopic state v of the agents with label i and, in particular, $f_i(v, t)dv$ is the proportion of agents with label i whose microscopic state is comprised between v and $v + dv$ at time t.

Since both the interactions and the label switching conserve the total mass of the system, we may assume that $f(x, v, t)$ is a probability distribution, namely:

$$\int_{\mathbb{R}_+} \int_{\mathcal{I}_n} f(x, v, t)\, dx\, dv = \sum_{i=1}^{n} \int_{\mathbb{R}_+} f_i(v, t)\, dv = 1 \qquad \forall t > 0. \tag{14}$$

Notice, however, that the f_i's are in general not probability density functions because their v-integral varies in time due to the label switching. We denote by

$$\rho_i(t) := \int_{\mathbb{R}_+} f_i(v, t)\, dv \tag{15}$$

the mass of the group of agents with label i, thus $0 \leq \rho_i(t) \leq 1$ and

$$\sum_{i=1}^{n} \rho_i(t) = 1 \qquad \forall t > 0.$$

Let us also define the first statistical moment of f_i

$$M_i(t) := \int_{\mathbb{R}_+} v\, f_i(v, t)\, dv$$

so that the average of the ith group is

$$m_i(t) := \frac{M_i(t)}{\rho_i(t)}.$$

The second statistical moment of f_i, related to the variance of the distribution (see the Appendix), is defined as

$$E_i(t) := \int_{\mathbb{R}_+} v^2\, f_i(v, t)\, dv.$$

The kinetic evolution equation for $f(x, v, t)$, expressed as in (13), is given by (9), where now $\boldsymbol{z} = (x, v)$, which has to hold for every $\phi = \phi(x, v) : \mathcal{I}_n \times \mathbb{R}_+ \to \mathbb{R}$. Hence the evolution equation for f is

$$\begin{aligned} &\frac{d}{dt} \sum_{i=1}^{n} \int_{\mathbb{R}_+} \phi(i, v) f_i(v, t)\, dv \\ &\quad = \frac{1}{2} \int_{\mathbb{R}_+^2} \sum_{i,j,k=1}^{n} \lambda_{jk} \int_{\mathbb{R}_+} T((i, v')|(j, v), (k, w))\big(\phi(i, v') - \phi(j, v)\big) f_j(v, t) f_k(w, t)\, dv dw dv' \\ &\quad + \frac{1}{2} \int_{\mathbb{R}_+^2} \sum_{l,j,k=1}^{n} \lambda_{jk} \int_{\mathbb{R}_+} \tilde{T}((l, w')|(j, v), (k, w))\big(\phi(l, w') - \phi(k, w)\big) f_j(v, t) f_k(w, t)\, dv dw dw'. \end{aligned} \tag{16}$$

Choosing $\phi(x, v) = \psi(x)\varphi(v)$, with ψ such that $\psi(s) = 1$ for a certain $s \in \mathcal{I}_n$ and $\psi(x) = 0$ for all $x \in \mathcal{I}_n \setminus \{s\}$, we finally obtain the following system of equations for each one of the subgroup distributions f_s

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}_+} \varphi(v) f_s(v, t)\, dv \\ &\quad = \frac{1}{2} \int_{\mathbb{R}_+^2} \sum_{j,k=1}^{n} \int_{\mathbb{R}_+} \Big(\lambda_{jk} \varphi(v') T((s, v')|(j, v), (k, w)) f_j(v, t) \\ &\quad - \lambda_{sk} \varphi(v) \sum_{i=1}^{n} T((i, v')|(s, v), (k, w)) f_s(v, t) \Big) f_k(w, t)\, dv dw dv' \\ &\quad + \frac{1}{2} \int_{\mathbb{R}_+^2} \sum_{j,k=1}^{n} \int_{\mathbb{R}_+} \Big(\lambda_{jk} \varphi(w') \tilde{T}((s, w')|(j, v), (k, w)) f_k(w, t) \\ &\quad - \lambda_{js} \varphi(w) \sum_{l=1}^{n} \tilde{T}((l, w')|(j, v), (s, w)) f_s(w, t) \Big) f_j(v, t) dv dw dw'. \end{aligned} \tag{17}$$

In order to implement the microscopic processes (10)–(12), we choose as transition probability for the microscopic state $\boldsymbol{z} = (x, v)$ the following

$$\begin{aligned} T((x', v')|(x, v), (y, w)) &= \langle \int_{\mathcal{I}_n} P_{xy}^{x'y'}(v, w)\delta(v' - (I_{xy}(v, w) + D_{xy}(v, w)\eta))dy' \rangle, \\ \tilde{T}((y', w')|(x, v), (y, w)) &= \langle \int_{\mathcal{I}_n} P_{xy}^{x'y'}(v, w)\delta(w' - (\tilde{I}_{xy}(v, w) + \tilde{D}_{xy}(v, w)\eta_*))dx' \rangle. \end{aligned} \tag{18}$$

The former transition probability T describes the fact that an agent (x, v), given a binary interaction with another agent (y, w), migrates to x' according to (10) and changes its physical quantity into v' according to (12). The symmetric statement holds for the interacting agent (y, w) whose

evolution is described by $\tilde{T}$. Considering (18), (17) becomes

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}_+}\varphi(v)f_s(v,t)\,dv &= \frac{1}{2}\langle\int_{\mathbb{R}_+^2}\sum_{j,k,l=1}^{n}\Big(\beta_{jk}^{sl}\varphi(I_{xy}(v,w)+D_{xy}(v,w)\eta)f_j(v,t)\\ &\qquad -\beta_{sk}^{jl}\varphi(v)f_s(v,t)\Big)f_k(w,t)\,dv\,dw\rangle\\ &+\frac{1}{2}\langle\int_{\mathbb{R}_+^2}\sum_{j,k,i=1}^{n}\Big(\beta_{jk}^{is}\varphi(\tilde{I}_{xy}(v,w)+\tilde{D}_{xy}(v,w)\eta_*)f_k(w,t)\\ &\qquad -\beta_{js}^{ik}\varphi(w)f_s(w,t)\Big)f_j(v,t)dv\,dw\rangle.\end{aligned} \tag{19}$$

Here, we have defined

$$\beta_{ij}^{kl} := \lambda_{ij}P_{ij}^{kl} \tag{20}$$

the *rate* of transfer for a couple from the subgroups (i,j) to (k,l) as a consequence of a binary interaction between two agents labeled i and j. We remark that β_{ij}^{kl} may depend on the microscopic variables v and w through P_{ij}^{kl}. Moreover, as done in [46] in a linear case, also the interaction frequency may depend on the microscopic state v.

We remark that in this case where we consider both exchanges (12) and transfers (10), asymmetric binary interactions arise quite commonly, even if the two processes (12) and (10), separately, are symmetric. This is mainly due to the fact that the microscopic rule (12) depends on the label of the agents. Indeed, if we consider a transfer $(i,j)\to(k,l)$, the reverse transfer $(k,l)\to(i,j)$ may occur with a different probability and with a different interaction law, losing, thus, the microscopic reversibility of the process, usually assumed in classical Boltzmann descriptions. Specifically, in gas mixtures the microscopic reversibility is guaranteed by the conservation of momentum and total energy, and the post-collision velocities may be uniquely determined in terms of the pre-collision velocities and of the impact angles [20,21], even in presence of chemical reactions [39]. The break of symmetry between the direct and the reverse collision is known to occur in presence of inelastic collisions (for instance in granular media [47]) causing a decay in time of the kinetic energy of the system. It is well known that the Boltzmann H-theorem (namely the entropy dissipation in the collision process) holds even in presence of microscopic reversibility, and it is due to molecular chaos, namely to randomness of collisions. An extensive discussion on the concept of global irreversibility in classical kinetic theory may be found in [48,49]. For interactions involving human beings, the kinetic approach is much more complex (even under simplistic assumptions), and both conservativity and reversibility may be lost even at the microscopic level. Just to give an example, in socio-economic problems, the model may be not pointwise conservative (the average or the variance is not preserved in each interaction) due to the choice of the deterministic part of the exchange rule, while reversibility may be lost essentially due to random variables appearing in the general interaction rule (12); moreover, the fraction of the own wealth that each agent is willing to exchange may depend also on the proper amount of wealth [50], and a transfer from a poor country to a rich one might be much more probable than the reverse transfer [40]. This is why the general approach with generally different transition probabilities T and $\tilde{T}$ provides a useful tool for a correct description of this kind of processes. Moreover, it allows one to build more easily exchange and transfer operators for a generic number n of subpopulations. Indeed, the usual way of extending the Boltzmann theory to a set of $n>1$ constituents consists in building up a set of n Boltzmann equations, each one for the distribution function of the ith constituent, with $i=1,\dots,n$ [22,27,39]. Then, in the case of chemically reactive gases, the bimolecular reaction is described by means of a collision operator that involves both distributions of reactants and products, and the change of the velocity is included by means of a transformation with unit Jacobian like in the classical strong Boltzmann equation. On the other hand, our transition probability approach includes the label of the individual compartment into the set of microscopic states characterizing the individual, dealing thus with only one kinetic equation, that could be separated into n different equations only when one needs to compute the pertinent moments of each compartment, by choosing the appropriate test function as done in (17). This turns out to be a great advantage even from the computational point of view. As we will see in numerical tests shown in Section 5, in the present approach it is also straightforward to consider transition rates β_{ij}^{kl} explicitly dependent on the microscopic states, both through the binary interaction frequency and /or through the transfer probability P_{ij}^{kl}, investigating thus more realistic cases with respect to classical kinetic descriptions that, for the sake of simplicity, assume constant interaction probabilities in the kernel of the Boltzmann operators. Furthermore, this approach allows one to include the stochastic contributions η and η_* in the binary interaction rules, as invertibility is not required in the construction of the operators.

3. Formal study of the kinetic equation with transition probabilities

In this section we intend to revise and illustrate some analytical tools that are useful for formally studying Eq. (9). After briefly stating some results on the existence and uniqueness of the solution, we consider the quasi-invariant limit in various regimes of Eq. (19) involving label switching and exchange of physical quantities.

3.1. Basic theory of kinetic models with transition probabilities in Wasserstein spaces

The strong form of (9) coupled with an initial condition $f_0(\boldsymbol{z})$ defines the following Cauchy problem

$$\begin{cases}\dfrac{\partial}{\partial t}f(\boldsymbol{z},t)=Q^+(f,f)-f(\boldsymbol{z},t)\displaystyle\int_\Omega\lambda_{\boldsymbol{zy}}f(\boldsymbol{y},t)\,d\boldsymbol{y}, & t>0,\quad \boldsymbol{z}\in\Omega,\\ f(\boldsymbol{z},0)=f_0(\boldsymbol{z}), & \boldsymbol{z}\in\Omega,\end{cases} \tag{21}$$

being

$$Q^+(f,f)=\frac{1}{2}\int_{\Omega^2}\lambda_{\prime\boldsymbol{z}'\boldsymbol{y}}T(\boldsymbol{z}|'\boldsymbol{z},'\boldsymbol{y})f('\boldsymbol{z},t)f('\boldsymbol{y},t)\,d'\boldsymbol{z}d'\boldsymbol{y}+\frac{1}{2}\int_{\Omega^2}\lambda_{\prime\boldsymbol{z}'\boldsymbol{y}}\tilde{T}(\boldsymbol{z}|'\boldsymbol{y},'\boldsymbol{z})f('\boldsymbol{z},t)f('\boldsymbol{y},t)\,d'\boldsymbol{z}d'\boldsymbol{y}, \tag{22}$$

where $'\boldsymbol{z}, '\boldsymbol{y}$ are the pre-interaction states, with the compatibility condition $\int_\Omega f_0(\boldsymbol{z})\,d\boldsymbol{z} = 1$ as (4) holds true. We remark that everything could be written for a generic mass $\rho > 0$. Let us now define

$$\bar{\lambda} := \int_\Omega \lambda_{\boldsymbol{zy}} f(\boldsymbol{y}, t)\,d\boldsymbol{y}$$

that we assume to be throughout the whole text independent of $\boldsymbol{y}$ (this assumption includes the case of a constant $\lambda_{\boldsymbol{zy}}$). If we multiply both sides of the equation by $e^{\bar{\lambda}t}$ and we integrate in time we get

$$\begin{aligned} f(\boldsymbol{z}, t) = e^{-\bar{\lambda}t} f_0(\boldsymbol{z}) + \int_0^t e^{\bar{\lambda}(s-t)} \Bigg[\frac{1}{2} \int_{\Omega^2} \lambda_{'\boldsymbol{z}'\boldsymbol{y}} T(\boldsymbol{z}|'\boldsymbol{z}, '\boldsymbol{y}) f('\boldsymbol{z}, s) f('\boldsymbol{y}, s)\,d'\boldsymbol{z}d'\boldsymbol{y} \\ + \frac{1}{2} \int_{\Omega^2} \lambda_{'\boldsymbol{z}'\boldsymbol{y}} \tilde{T}(\boldsymbol{z}|'\boldsymbol{y}, '\boldsymbol{z}) f('\boldsymbol{z}, s) f('\boldsymbol{y}, s)\,d'\boldsymbol{z}d'\boldsymbol{y} \Bigg] ds, \end{aligned} \tag{23}$$

where we have used (4). Let us define (Ω, d) a polish space. Analogously to what has been done in [7], we see that an appropriate space in which (23) can be studied is $X := C([0, \bar{t}]; \mathcal{M}_+(\Omega))$, where $\bar{t} > 0$ is a final time and $\mathcal{M}_+(\Omega)$ is the space of positive measures on Ω having unitary mass. Therefore $f \in X$ is a continuous mapping as a function of time over $[0, \bar{t}]$ and it is a positive measure satisfying (4) as a function of the microscopic state $\boldsymbol{z} \in \Omega$. In particular, X is a complete state with the distance

$$\sup_{t \in [0, \bar{t}]} W_1(f(\boldsymbol{z}, t), g(\boldsymbol{y}, t))$$

where

$$W_1(f(\cdot, t), g(\cdot, t)) = \inf_{\underline{\mu} \in \Gamma(f,g)} \int_{\Omega^2} d(\boldsymbol{z}, \boldsymbol{y}) \underline{\mu}(t, \boldsymbol{z}, \boldsymbol{y})\,d\boldsymbol{z}d\boldsymbol{y} \tag{24}$$

is the 1-Wasserstein distance between $f(\cdot, t)$ and $g(\cdot, t) \in \mathcal{M}_+(\Omega)$, being $\Gamma(f, g)$ the space of the probability density functions defined on Ω^2 having marginals f and g. In particular, we shall consider the dual form called Kantorovich–Rubinstein distance, that is defined as

$$W_1(f(\cdot, t), g(\cdot, t)) := \sup_{\varphi \in \mathrm{Lip}_1(\Omega)} \int_\Omega \varphi(\boldsymbol{z}) \left(f(\boldsymbol{z}, t) - g(\boldsymbol{z}, t) \right) d\boldsymbol{z}, \tag{25}$$

where Lip_1 is the space of Lipschitz functions with unitary Lipschitz constant, i.e.

$$\mathrm{Lip}_1 := \{\varphi \ : \ \|\varphi\|_{\mathrm{Lip}} := \sup_{\boldsymbol{z} \neq \boldsymbol{y}} \frac{|\varphi(\boldsymbol{z}) - \varphi(\boldsymbol{y})|}{d(\boldsymbol{z}, \boldsymbol{y})} \leq 1\}.$$

We remark that d must not be the distance defining the polish space ([51], Chapter 1) . If d is bounded, then W_1 defines a metric on $\mathscr{P}(\Omega)$, where $\mathscr{P}(\Omega)$ is the space of probability measures on Ω, but this can be exploited also in the case in which d is unbounded, by replacing it with the distance $\tilde{d} = \min\{d, 1\}$ ([51], Chapter 7).

As done in [7], we shall always assume that the transition probabilities T and $\tilde{T}$ satisfy the following Lipschitz continuity property.

Assumption 3.1. Let $T(\boldsymbol{z}|'\boldsymbol{z}, '\boldsymbol{y}), \tilde{T}(\boldsymbol{y}|'\boldsymbol{z}, '\boldsymbol{y}) \in \mathscr{P}(\Omega)$ for all $'\boldsymbol{z}, '\boldsymbol{y} \in \Omega$. We assume that there exists $Lip(T) > 0$, such that

$$W_1(T(\cdot|'\boldsymbol{z}, '\boldsymbol{y}), T(\cdot|'\boldsymbol{z}_*, '\boldsymbol{y}_*)) \leq \mathrm{Lip}(T)(d('\boldsymbol{z}, '\boldsymbol{z}_*) + d('\boldsymbol{y}, '\boldsymbol{y}_*))$$

for all $'\boldsymbol{z}, '\boldsymbol{y}, '\boldsymbol{z}_*, '\boldsymbol{y}_* \in \Omega$ and that the same holds for $\tilde{T}$.

The following result holds.

Theorem 3.2. *Let $f_0 \in \mathcal{M}_+(\Omega)$ and let us assume that*

- *T, $\tilde{T}$ satisfy Assumption 3.1;*
- *$\bar{\lambda}$ is constant and $\lambda_{\boldsymbol{zy}}$ is lower and upper bounded, i.e. $\exists\, \lambda_m, \lambda_M > 0$ such that $0 < \lambda_m < \lambda_{\boldsymbol{zy}} < \lambda_M < \infty$, $\forall \boldsymbol{z}, \boldsymbol{y} \in \Omega$.*

Then, there exists a unique $f \in X$ which solves (21), and (21) exhibits continuous dependence on the initial data. Moreover, if $Lip(T), Lip(\tilde{T}) < \frac{1}{2}\frac{\lambda_m}{\lambda_M}$, then (21) admits a unique equilibrium distribution f^∞, which is a probability measure on Ω and which is also globally attractive, i.e.

$$\lim_{t \to \infty} W_1(f(\cdot, t), f^\infty) = 0$$

for every solution f to (21).

Proof. The proof follows the same steps as done in [7], Appendix, where the authors prove the results in the case of a constant $\lambda_{\boldsymbol{zy}}$ and in the case of a bounded Ω, and, thus, use the dual form of the 1-Wasserstein distance (25), but, also, the dual form (25) where $\varphi \in C_b(\Omega) \cap \mathrm{Lip}_1(\Omega)$, so that φ is bounded in the L_∞ norm. In the present case, as Ω is arbitrary, we can use in the proof the definition (25) with φ only Lipschitz. We report a sketch of the proof in Appendix. □

The former Theorem applies to the case of Section 2.2 and can be written as

Theorem 3.3. *Let $\Omega = \mathcal{I}_n \times \mathbb{R}_+$ and let the transition probability distributions have the form*

$$T((x, v)|('x, 'v), ('y, 'w)) = \sum_{i=1}^n T_i(v|('x, 'v), ('y, 'w))\delta(x - i) \tag{26}$$

where T_i, as a function of $v \in \mathbb{R}_+$, satisfies

$$W_1(T_i(\cdot|('x, 'v), ('y, 'w))), T_i(\cdot|('x^*, 'v^*), ('y^*, 'w^*)) < Lip(T_i) \left(d('x, 'x^*) + d('y, 'y^*) + d('v, 'v^*) + d('w, 'w^*) \right)$$

and let the analogous property hold for $\tilde{T}$ and $\tilde{T}_i$. Let moreover

$$f_0(x,v) = \sum_{i=1}^{n} f_i^{(0)}(v)\,\delta(x-i)$$

be a prescribed kinetic distribution function at time $t=0$ over the space of microscopic states $(x,v) \in \mathcal{I}_n \times \mathbb{R}_+$ such that $f_i \geq 0$, $\sum_{i=1}^{n} \int_{\mathbb{R}_+} f_i(v,t)\,dv = 1$, $\forall t > 0$. Then the unique solution to (21) is of the form

$$f(x,v,t) = \sum_{i=1}^{n} f_i(v,t)\,\delta(x-i)$$

(analogous to (13)), with coefficients $f_i(v,t)$ given by (19) along with the initial conditions $f_i(v,0) = f_i^{(0)}(v)$. In addition, it depends continuously on the initial datum as stated by Theorem 3.2.

Proof. Thanks to Theorem 3.2, it is sufficient to prove that 3.1 holds true. In fact, we have that:

$$\begin{aligned}
&W_1(T(\cdot|('x,'v),('y,'w)), T(\cdot|('x^*,'v^*),('y^*,'w^*)))\\
&= \sup_{\varphi \in \mathrm{Lip}_1(\Omega)} \int_{\Omega} \varphi((x,v))(T((x,v)|('x,'v),('y,'w)) - T((x,v)|('x^*,'v^*),('y^*,'w^*)))\,dx\,dv\\
&= \sum_{i=1}^{n} \sup_{\phi \in \mathrm{Lip}_1(\mathbb{R}_+)} \int_{\mathbb{R}_+} \phi(v)(T_i(v|('x,'v),('y,'w) - T_i(v|('x^*,'v^*),('y^*,'w^*)))\,dv
\end{aligned}$$

(where we have used (26) and $\varphi((x,v)) = \psi(x)\phi(v)$ with the choice $\psi = 1$ as we are now considering the W_1 distance for T_i as a distribution in v)

$$\begin{aligned}
&= \sum_{i=1}^{n} W_1(T_i(\cdot|('x,'v),('y,'w)), T_i(\cdot|('x^*,'v^*),('y^*,'w^*)))\\
&\leq \sum_{i=1}^{n} \mathrm{Lip}(T_i)\left(d('x,'x^*) + d('y,'y^*) + d('v,'v^*) + d('w,'w^*)\right).
\end{aligned}$$

□

3.2. Quasi-invariant limit

One of the most interesting issues in the study of kinetic models is the characterization of the stationary distributions arising asymptotically for $t \to +\infty$, which depict the emergent behavior of the system. The jump process model (9) does not allow in general to determine explicitly the profile of the stationary distributions, and the explicit expression of the steady state f^∞ can be inferred only in particular cases [7,52,53]. It is widely known that the classic collision-like Boltzmann equation (reported for readers convenience in Appendix, in (68)–(69)) offers several analytical tools, which often permit to explicitly recover accurate approximations of f^∞ by means of suitable asymptotic procedures. The basic idea of such procedures is to approximate the integro-differential Boltzmann equation with an appropriate partial differential equation, more amenable to analytical investigations, at least in some regimes of the parameters of the microscopic interactions. A prominent framework in which this type of asymptotic analysis is successfully applied to (69) is that of the *quasi-invariant interactions*. This concept was first introduced in the kinetic literature on multi-agent systems in [2,4] as a reminiscence of the *grazing collisions* studied in the classical kinetic theory, see [54]. This corresponds to introducing a small parameter ϵ such that the microscopic interaction rule can be written as

$$v' \approx v + \mathcal{O}(\epsilon) \tag{27}$$

and analyzing the dynamics on a longer time scale, setting a new time variable

$$\tau := \epsilon t \tag{28}$$

in order to compensate for the smallness of the interactions.

In this spirit, in [44], that concerns the investigation of a parallelism between the model (68)–(69) and (70), the authors propose a way to translate the concept of quasi-invariance, typically used in the context of collision-like Boltzmann equations (68)–(69), to the language of transition probabilities. The idea is the following. Let $'\boldsymbol{Z}, \boldsymbol{Z} \in \mathbb{R}_+$ be the random variables representing the pre- and post-interaction states, respectively, of an agent, and $'\boldsymbol{Z}_* \in \mathbb{R}_+$ the one representing the pre-interaction state of the other agent involved in the interaction. In the probabilistic description via the transition probabilities, we say that interactions are quasi-invariant if, given $0 < \epsilon \ll 1$,

$$\mathrm{Prob}(\left|\boldsymbol{Z} - '\boldsymbol{Z}\right| > \epsilon \,|\, '\boldsymbol{Z}, '\boldsymbol{Z}_*) \leq \epsilon; \tag{29}$$

in other words, if the post-interaction state is, in probability, close to the pre-interaction state, so that the interactions produce a small transfer of microscopic state between the interacting agents.

In the present framework, in order to have a *quasi-invariant transition probability*, we can introduce rescaled transition probabilities defined by the following transform

$$T_\epsilon(\boldsymbol{z}|'\boldsymbol{z},'\boldsymbol{y}) = \mathcal{F}_\epsilon[T](\boldsymbol{z}|'\boldsymbol{z},'\boldsymbol{y}), \qquad \tilde{T}_\epsilon(\boldsymbol{y}|'\boldsymbol{z},'\boldsymbol{y}) = \mathcal{F}_\epsilon[\tilde{T}](\boldsymbol{y}|'\boldsymbol{z},'\boldsymbol{y}) \tag{30}$$

where

$$\mathcal{F}_\epsilon[T] : \mathscr{P}(\Omega) \longmapsto \mathscr{P}(\Omega)$$

is a family of operators (for $\epsilon > 0$) defined on the space of the probability measures defined on Ω. We require that $\mathcal{F}_\epsilon$ satisfies the following three properties:

$F1)$ $\mathcal{F}_1$ is the identity;

$F2)$

$$\lim_{\epsilon\to 0} W_1(\mathcal{F}_\epsilon[T], \delta(\boldsymbol{z}'-\boldsymbol{z})) = 0, \qquad \lim_{\epsilon\to 0} W_1(\mathcal{F}_\epsilon[\tilde{T}], \delta(\boldsymbol{z}'-\boldsymbol{z})) = 0; \tag{31}$$

$F3)$

$$\lim_{\epsilon\to 0} W_1(\frac{\mathcal{F}_\epsilon[T]}{\epsilon}, T) = 0, \qquad \lim_{\epsilon\to 0} W_1(\frac{\mathcal{F}_\epsilon[\tilde{T}]}{\epsilon}, \tilde{T}) = 0. \tag{32}$$

These three properties mean that $\epsilon = 1$ corresponds to the basic regime $(F1)$, that for small values of ϵ the microscopic state tends not to change $(F2)$, and that on the long time scale (28) the dynamics is ruled by T and $\tilde{T}$ $(F3)$ [44]. An example of properly rescaled transition probabilities is

$$T_\epsilon(\boldsymbol{z}'|\boldsymbol{z},\boldsymbol{y}) = (1-\epsilon)\delta(\boldsymbol{z}'-\boldsymbol{z}) + \epsilon T(\boldsymbol{z}'|\boldsymbol{z},\boldsymbol{y}), \qquad \tilde{T}_\epsilon(\boldsymbol{y}'|\boldsymbol{z},\boldsymbol{y}) = (1-\epsilon)\delta(\boldsymbol{y}'-\boldsymbol{y}) + \epsilon \tilde{T}(\boldsymbol{y}'|\boldsymbol{z},\boldsymbol{y}), \tag{33}$$

as introduced in [44], satisfying the properties $F1$, $F2$, $F3$.

In the following we shall investigate the quasi-invariant limit in the examples previously illustrated.

3.2.1. *Interacting agents with label switch and exchange of physical quantities*

Let us now consider the case in which the binary interactions lead both to a transfer and to an exchange of the physical quantity v. Therefore, for the two processes we shall consider a quasi-invariant regime given by (30)–(35). In this case the transition probability (18) may be rescaled as

$$T_\epsilon((x',v')|(x,v),(y,w)) = P^{x'y'}_{\epsilon_{xy}}\,\delta\Big(v' - (v + \epsilon(I_{xy}-v) + \sqrt{\epsilon(1-\epsilon)(I_{xy}-v)^2 + \epsilon D^2_{xy}}\,\eta)\Big) \tag{34}$$

and analogously for $\tilde{T}_\epsilon$. Here, we have defined a quasi-invariant switching probability as

$$P^{x'y'}_{\epsilon_{xy}} = \epsilon P^{x'y'}_{xy} \quad \text{if} \quad (x,y)\neq(x',y'), \qquad P^{x'y'}_{\epsilon_{xy}} = 1-\epsilon \quad \text{if} \quad (x,y)=(x',y') \tag{35}$$

where we are considering $P^{xy}_{xy} = 0$ (that is what we find for $\epsilon = 1$). Let us consider the symmetric case for simplicity of notation, bearing in mind that the asymmetric case can be treated analogously. Plugging the latter in (19) and considering the re-scaling (28) and reminding (35), if we let $\epsilon \to 0^+$ we obtain

$$\begin{aligned}&\frac{d}{d\tau}\int_{\mathbb{R}_+}\varphi(v)f_s(v,\tau)\,dv\\ &= \int_{\mathbb{R}^2_+}\sum_{j,k,l=1,j\neq s\vee k\neq l}^{n}\varphi(v)\big(\beta^{sl}_{jk}f_j(v,\tau) - \beta^{jl}_{sk}f_s(v,\tau)\big)f_k(w,\tau)\,dv\,dw\end{aligned} \tag{36}$$

that means that the dynamics is ruled by the label switches and gives no further information. In case of symmetry, it is the symmetric form of (19).

Let us now consider a different regime and, in particular, let us only consider a quasi-invariant exchange rule, i.e. (27). Let us then rescale (19) with (28) and let us consider the quasi-invariant transition probability

$$T_\epsilon((x',v')|(x,v),(y,w)) = P^{x'y'}_{xy}\,\delta\Big(v' - (v + \epsilon(I_{xy}-v) + \sqrt{\epsilon(1-\epsilon)(I_{xy}-v)^2 + \epsilon D^2_{xy}}\,\eta)\Big), \tag{37}$$

i.e. the exchange of the physical quantity is actually quasi-invariant, whilst the label-switch process is not. We obtain

$$\begin{aligned}&\frac{d}{d\tau}\int_{\mathbb{R}_+}\varphi(v)f_s(v,\tau)\,dv\\ &= \langle\frac{1}{\epsilon}\int_{\mathbb{R}^2_+}\sum_{j,k,l=1}^{n}\Big(\beta^{sl}_{jk}\varphi(v+\epsilon(I_{jk}-v) + \sqrt{\epsilon(1-\epsilon)(I_{jk}-v)^2+\epsilon D^2_{jk}}\eta)f_j(v,\tau)\\ &\qquad - \beta^{jl}_{sk}\varphi(v)f_s(\tau,v)\Big)\,f_k(w,\tau)\,dv\,dw\rangle.\end{aligned} \tag{38}$$

As ϵ is small, we can Taylor expand

$$\begin{aligned}&\langle\varphi(v+\epsilon(I_{jk}(v,w)-v) + \sqrt{\epsilon(1-\epsilon)(I_{jk}-v)^2+\epsilon D^2_{jk}}\eta)\rangle\\ &= \varphi(v) + \epsilon(I_{jk}(v,w)-v)\varphi'(v) + \frac{1}{2}\varphi''(v)\epsilon\left((1-\epsilon)(I_{jk}-v)^2 + D^2_{jk}\right) + \mathcal{O}(\epsilon^2)\end{aligned}$$

and plugging the latter in (38) we have that

$$\begin{aligned}&\frac{d}{d\tau}\int_{\mathbb{R}_+}\varphi(v)f_s(v,\tau)\,dv\\ &\qquad= \int_{\mathbb{R}^2_+}\sum_{j,k,l=1}^{n}\beta^{sl}_{jk}\Big(\varphi'(v)(I_{jk}-v) + \frac{1}{2}((1-\epsilon)(I_{jk}-v)^2\\ &\qquad\qquad + D^2_{jk})\varphi''(v)\Big)\,f_j(v,\tau)f_k(w,\tau)\,dv\,dw\\ &\qquad + \frac{1}{\epsilon}\int_{\mathbb{R}_+}\sum_{j,k,l=1,j\neq s}^{n}\varphi(v)\big(\beta^{sl}_{jk}f_j(v,\tau) - \beta^{jl}_{sk}f_s(v,\tau)\big)\rho_k(\tau)\,dv\end{aligned}$$

where we remind that ρ_k is the mass of the kth population. Let us now consider an expansion for the probability density function of the whole population f

$$f(x,v,\tau) = f^{(0)}(x,v,\tau) + \epsilon f^{(1)}(x,v,\tau) + \mathcal{O}(\epsilon^2) \tag{39}$$

where the zero-th and first order moments satisfy

$$\begin{aligned}
&\int_\Omega f(x,v,\tau)dxdv = \rho^{(0)}(\tau) := \int_\Omega f^{(0)}(x,v,\tau)\,dxdv,\\
&\rho^{(1)}(\tau) := \int_\Omega f^{(1)}(x,v,\tau)\,dxdv = 0,\\
&\int_\Omega v\,f(x,v,\tau)\,dxdv = M^{(0)}(\tau) := \int_\Omega v\,f^{(0)}(x,v,\tau)\,dxdv,\\
&M^{(1)}(\tau) := \int_\Omega v\,f^{(1)}(x,v,\tau)\,dxdv = 0,
\end{aligned} \tag{40}$$

where we remind that now $\Omega = \mathcal{I}_n \times \mathbb{R}_+$. For each f_i this translates into

$$f_i(v,\tau) = f_i^{(0)}(v,\tau) + \epsilon f_i^{(1)}(v,\tau) + \mathcal{O}(\epsilon^2)$$

and (40) translates to

$$\rho^{(0)} = \sum_{i=1}^n \rho_i^{(0)} = \int_\Omega f\,dxdv, \qquad M^{(1)} = \sum_{i=1}^n M_i^{(1)} = \sum_{i=1}^n \rho_i^{(1)} = 0. \tag{41}$$

Comparing equal orders of ϵ and supposing symmetry, we obtain

$$\frac{1}{\epsilon}\int_{\mathbb{R}_+} \sum_{j,k,l=1,j\neq s}^n \varphi(v)\big(\beta_{jk}^{sl} f_j^{(0)}(v,\tau) - \beta_{sk}^{jl} f_s^{(0)}(v,\tau)\big)\rho_k^{(0)}(\tau)dv = 0, \tag{42}$$

so that

$$f_s^{(0)}(v,\tau) = \frac{\sum_{j,k,l=1,j\neq s}^n \beta_{jk}^{sl} f_j^{(0)}(v,\tau)\rho_k^{(0)}(\tau)}{\sum_{j,k,l=1,j\neq s}^n \beta_{sk}^{jl}\rho_k^{(0)}(\tau)}, \tag{43}$$

while, at the first order

$$\begin{aligned}
&\frac{d}{d\tau}\int_{\mathbb{R}_+} \varphi(v) f_s^{(0)}(v,\tau)\,dv\\
&\qquad = \int_{\mathbb{R}_+^2} \sum_{j,k,l=1}^n \beta_{jk}^{sl}\Big(\varphi'(v)(I_{jk}(v,w)-v) + \frac{1}{2}((I_{jk}(v,w)-v)^2\\
&\qquad\qquad + D_{jk}^2)\varphi''(v)\Big) f_j^{(0)}(v,\tau) f_k^{(0)}(w,\tau)\,dvdw\\
&\qquad + \int_{\mathbb{R}_+} \sum_{j,k,l=1,j\neq s}^n \varphi(v)\big(\beta_{jk}^{sl} f_j^{(0)}(v,\tau) - \beta_{sk}^{jl} f_s^{(0)}(v,\tau)\big)\rho_k^{(1)}(\tau)dv\\
&\qquad + \int_{\mathbb{R}_+} \sum_{j,k,l=1,j\neq s}^n \varphi(v)\big(\beta_{jk}^{sl} f_j^{(1)}(v,\tau) - \beta_{sk}^{jl} f_s^{(1)}(v,\tau)\big)\rho_k^{(0)}(\tau)dv,
\end{aligned} \tag{44}$$

from which we can obtain, resorting to the strong form thanks to integration by parts, a Fokker–Planck-type equation with a reaction term for each $f_s^{(0)}$ that is

$$\begin{aligned}
&\partial_\tau f_s^{(0)}(v,\tau)\\
&\qquad = -\partial_v \sum_{j,k,l=1}^n \beta_{jk}^{sl}\int_{\mathbb{R}_+} (I_{jk}(v,w)-v) f_k^{(0)}(w,\tau)\,dw f_j^{(0)}(v,\tau)\\
&\qquad + \partial_{vv}^2 \frac{1}{2}\sum_{j,k,l=1}^n \beta_{jk}^{sl}\int_{\mathbb{R}_+} ((I_{jk}(v,w)-v)^2 + D_{jk}^2) f_k^{(0)}(w,\tau)\,dw f_j^{(0)}(v,\tau)\\
&\qquad + \sum_{j,k,l=1,j\neq s}^n \big(\beta_{jk}^{sl} f_j^{(0)}(v,\tau) - \beta_{sk}^{jl} f_s^{(0)}(v,\tau)\big)\rho_k^{(1)}(\tau)\\
&\qquad + \sum_{j,k,l=1,j\neq s}^n \big(\beta_{jk}^{sl} f_j^{(1)}(v,\tau) - \beta_{sk}^{jl} f_s^{(1)}(v,\tau)\big)\rho_k^{(0)}(\tau).
\end{aligned} \tag{45}$$

The latter reaction terms also involve the first order corrections $f_i^{(1)}$, $i = 1,\dots,n$. In order to find univocally the solutions $f_s^{(0)}, f_s^{(1)}$ to (43) and (45) satisfying (41), we need a number of conditions (to be looked for example in conserved quantities) that is equal to the number of degrees of freedom.

4. Kinetic model for international trade allowing transfer of individuals

In this section, we are going to rephrase with the current framework the kinetic model for international trade allowing transfers of individuals investigated in [40]. In [40] the author presents a model of interacting individuals divided into two subpopulations and that are allowed, by means of binary interactions, to exchange wealth and to migrate to the other subgroup. Here, then, we have that $n = 2$, the physical quantity $v \in \mathbb{R}_+$ is the wealth, while the label $x \in \mathcal{I}_2$ denotes the subgroup. Note that, in the present work, we are only going to consider binary interactions giving rise to both exchanges of the wealth and transfer simultaneously.

4.1. From the microscopic to the macroscopic model

As described in the review [55], the first explicit description of a binary wealth exchange model dates back to Angle [56], and more precise relations with laws of statistical mechanics have been described about one decade later [57,58]: in each binary interaction the winner and the loser are randomly chosen, and the loser yields a fraction of his wealth to the winner. From here, Chakraborti and Chakrabarti [59] developed a class of strictly conservative binary interactions, where each trader gives to the partner a small fraction of his wealth, determined by a proper saving parameter; this model describes a completely socialistic society, whose long time behavior provides a Dirac delta distribution, with all individuals sharing the same amount of wealth. To overcome this non-realistic result, Cordier, Pareschi and Toscani in [2] took into account the fact that a trade occurs because agents intend to invest money in some asset, property, etc., and such investments bear some risk, that either provides buyer and/or seller with some additional wealth, or leads to the loss of wealth in a non-deterministic way. This model turns out to be able to explain from a mathematical point of view the formation of *Pareto tails*, i.e. *fat tails* that describe the fact that the total wealth is owned by a small fraction of the population [60,61], as usually observed in Western societies [58,62]. Formally, we can introduce the *Pareto Index* that is defined in order to describe the Pareto tails of a stationary probability density function f^∞ that can be characterized as

$$F^\infty(v) = \int_v^\infty f^\infty(w)\,dw.$$

In particular, the Pareto Index is the exponent μ such that

$$F^\infty(v) \sim \frac{1}{v^\mu}.$$

We can observe that a small value of μ means that the distribution f^∞ has fat tails or Pareto tails. Moreover, suitable choices for the random variables appearing in [2] also reproduce an increase of total wealth versus time, consistent with historical data. For these reasons, for what concerns the exchange of the physical quantity v, we are going to consider simple linear microscopic rules as done in [40]. In particular, we are going to consider (12) with

$$I_{xy}(v,w) = (1-\omega_x)v + \omega_y w, \qquad D_{xy} = \zeta_{xy} v, \tag{46}$$

where $\omega_i \in [0,1]$, $i \in \mathcal{I}_2$ and we are dealing with symmetric interactions. We consider possible transfers given by

$$\begin{aligned} &\text{(a)} \quad 1+1 \to 1+2, \qquad \text{(b)} \quad 2+2 \to 1+2,\\ &\text{(c)} \quad 1+2 \to 1+1, \qquad \text{(d)} \quad 1+2 \to 2+2, \end{aligned} \tag{47}$$

therefore only one of the two interacting agents moves to the other subgroup. The latter implies that the only non-vanishing values of P_{ij}^{kl}, that is the probability that agents labeled with i,j move to the subgroups k,l, correspond to the 4-plets

$$(i,j,k,l) \in \{(1,1,1,2),(1,1,2,1),(2,2,1,2),(2,2,2,1),(1,2,1,1),(1,2,2,2),(2,1,1,1),(2,1,2,2)\}. \tag{48}$$

The kinetic equation describing this microscopic dynamics is (17) with (18), where P_{ij}^{kl} is defined according to (48) and the microscopic exchange dynamics by (46). The evolution of the mass of each subpopulation is given by setting $\varphi = 1$ in (17), with the prescribed dynamics, for $s = 1,2$, and it results in

$$\begin{aligned} \partial_t \rho_1 &= (\beta_{22}^{12}\rho_2 + \beta_{12}^{11}\rho_1)\rho_2 - (\beta_{11}^{12}\rho_1 + \beta_{12}^{22}\rho_2)\rho_1,\\ \partial_t \rho_2 &= (\beta_{11}^{12}\rho_1 + \beta_{12}^{22}\rho_2)\rho_1 - (\beta_{22}^{12}\rho_2 + \beta_{12}^{11}\rho_1)\rho_2, \end{aligned} \tag{49}$$

while setting $\varphi = v$ in (17) gives the evolution of the first moments for $s = 1,2$

$$\begin{aligned} \partial_t M_1 &= \left(\beta_{12}^{11}\rho_1 + \beta_{22}^{12}\rho_2\right) M_2 - \left(\beta_{12}^{22}\rho_2 + \beta_{11}^{12}\rho_1\right) M_1,\\ \partial_t M_2 &= \left(\beta_{12}^{22}\rho_2 + \beta_{11}^{12}\rho_1\right) M_1 - \left(\beta_{12}^{11}\rho_1 + \beta_{22}^{12}\rho_2\right) M_2. \end{aligned} \tag{50}$$

As a consequence, the averages of the wealth of population 1 and 2 evolve as

$$\begin{aligned} \partial_t m_1 &= \frac{\rho_2}{\rho_1}\left(\beta_{22}^{12}\rho_2 + \beta_{12}^{11}\rho_1\right)(m_2 - m_1),\\ \partial_t m_2 &= \frac{\rho_1}{\rho_2}\left(\beta_{11}^{12}\rho_1 + \beta_{12}^{22}\rho_2\right)(m_1 - m_2). \end{aligned} \tag{51}$$

We remark that we have assumed symmetry in the interaction rates, i.e.

$$\beta_{ii}^{12} = \beta_{ii}^{21}, \quad \beta_{12}^{ii} = \beta_{21}^{ii}, \quad \forall i = 1,2.$$

We observe that the total mass and average

$$\bar\rho := \rho_1 + \rho_2 = 1, \qquad \bar M := M_1 + M_2$$

are conserved in time. Regarding the stationary states of the masses, we have that

$$\rho_2^\infty = \alpha \rho_1^\infty, \tag{52}$$

where

$$\alpha = \frac{-(\beta_{12}^{11} - \beta_{12}^{22}) + \sqrt{(\beta_{12}^{11} - \beta_{12}^{22})^2 + 4\beta_{11}^{12}\beta_{22}^{12}}}{2\beta_{22}^{12}}.$$

Relation (52) quantifies the ratio between the densities of the subgroups 1 and 2, whose final stationary value may be greater or less than one according to the values of the rates β_{ij}^{kl}. In particular, we may remark that, taking into account the fact that the sum of the two densities is constant,

$$\rho_1^\infty = \frac{\bar\rho}{1+\alpha}, \qquad \rho_2^\infty = \frac{\alpha\bar\rho}{1+\alpha},$$

and, therefore

$$\rho_1^\infty = \rho_2^\infty = \frac{\bar\rho}{2} \qquad \text{if and only if} \qquad \alpha = 1.$$

We remind the reader that the interaction rate is defined in (20) as $\beta_{ij}^{kl} = P_{ij}^{kl}\lambda_{ij}$, where P_{ij}^{kl} is the probability for the interacting agents labeled with i and j to migrate to the subgroups labeled with k and l respectively. Therefore, the latter condition $\alpha = 1$ is satisfied if $P_{12}^{22} = P_{12}^{11} = 0.5$ and $\beta_{11}^{12} = \beta_{22}^{12}$, which means that the probability for interacting agents with different labels of going to the same subgroup is the same for groups 1 and 2, and also the rate of interacting agents of the same subgroup is the same for all subgroups. On the other hand, if $\beta_{11}^{12} = \beta_{22}^{12}$, $P_{12}^{22} \lesseqgtr P_{12}^{11}$ implies $\alpha \lesseqgtr 1$, since the probability for an agent labeled 1 of going to subgroup 2 is larger (smaller, respectively) than the probability of going from 2 to 1. To this regard, we observe that, as the stationary state only depends on α, there may be a switch in the population size $((\rho_2^\infty - \rho_1^\infty)(\rho_2(0) - \rho_1(0)) < 0)$ if

$$(\rho_2(0) - \rho_1(0))(\alpha - 1) < 0. \tag{53}$$

For what concerns the average, the sufficient and necessary condition to be met at the stationary state is

$$m_1^\infty = m_2^\infty =: m^\infty \tag{54}$$

for every choice of the parameters. The latter implies that $M_2^\infty > M_1^\infty$ if and only if $\alpha > 1$. Moreover, because of conservation of mass and total momentum, we have that

$$m^\infty = \rho_1(0)m_1(0) + \rho_2(0)m_2(0) \tag{55}$$

that implies that the final average wealth is closer to the initial average wealth of the subgroup that was more populated at $t = 0$.

4.2. Quasi-invariant limit

If we assume, for simplicity of notation, that the probability of transfer towards the ith subgroup is independent of the countries of the interacting agents, i.e.

$$\beta_1^2 := \beta_{11}^{12} = \beta_{12}^{22}, \qquad \beta_2^1 := \beta_{22}^{12} = \beta_{12}^{11}, \tag{56}$$

we have that

$$\rho_2^\infty = \alpha\rho_1^\infty, \qquad M_2^\infty = \alpha M_1^\infty, \tag{57}$$

where $\alpha = \beta_1^2/\beta_2^1$, and, as $\bar M$ and $\bar\rho = 1$ are conserved quantities, we have that

$$\rho_1^\infty = \frac{\bar\rho}{1+\alpha}, \qquad M_1^\infty = \frac{\bar M}{1+\alpha}.$$

Let us now consider the quasi-invariant regime defined by the transition probability

$$T_\epsilon((x',v')|(x,v),(y,w)) = P_{xy}^{x'y'}\,\delta\Big(v' - (v + \epsilon(I_{xy}(v,w) - v) + \sqrt{\epsilon D_{xy}(v,w)^2}\eta)\Big). \tag{58}$$

The latter, even if it satisfies the requirements $F1$, $F2$, $F3$ prescribed in Section 3.2, differently from (37), does not guarantee the same evolution of the second statistical moment in the quasi-invariant regime. By expanding the distribution functions in powers of ϵ, imposing that the globally invariant quantities (zero-th and first order moments) remain unexpanded, we get that the constraints (41) are

$$\rho_1^{(1)} + \rho_2^{(1)} = 0, \qquad M_1^{(1)} + M_2^{(1)} = 0. \tag{59}$$

Therefore, we have two degrees of freedom and we can determine the values of $\rho_1^{(1)}, M_1^{(1)}$ as we have two conserved quantities. Following the same procedure as before, we find that (43) now is

$$f_1^{(0)} = \frac{1}{\alpha} f_2^{(0)}. \tag{60}$$

Then we have that $\rho_2^{(0)} = \alpha\rho_1^{(0)}$ and $M_2^{(0)} = \alpha M_1^{(0)}$, i.e. $\rho_i^{(0)} = \rho_i^\infty, M_i^{(0)} = M_i^\infty$, $i = 1,2$, which means that the masses and averages of order zero correspond to the equilibrium ones. Therefore

$$\rho_1^{(0)} = \frac{\bar\rho}{1+\alpha}, \qquad M_1^{(0)} = \frac{\bar M}{1+\alpha}. \tag{61}$$

At the first order (45) for $s = 1$ (for $s = 2$ an analogous result applies) specializes into

$$\begin{aligned}
\partial_\tau f_1^{(0)} = &-\partial_v \Big(\beta_1^2(\omega_1 M_1^{(0)} - \omega_2 v\rho_1^{(0)})f_1^{(0)} + \beta_1^2(\omega_2 M_2^{(0)} - \omega_2 v\rho_2^{(0)})f_1^{(0)} \\
&\qquad + \beta_1^2(\omega_1 M_1^{(0)} - \omega_1 v\rho_1^{(0)})f_1^{(0)} + \beta_2^1(\omega_2 M_2^{(0)} - \omega_1 v\rho_2^{(0)})f_1^{(0)} \Big) \\
&+ \partial_{v^2}^2 \frac{1}{2} \Big(\beta_1^2 \left[\zeta_{12}^2 v^2\right] f_1^{(0)}\rho_1 + \beta_1^2 \left[\zeta_{22}^2 v^2\right] f_1^{(0)}\rho_2 \\
&\qquad + \beta_1^2 \left[\zeta_{11}^2 v^2\right] f_1^{(0)}\rho_1^{(0)} + \beta_2^1 \left[\zeta_{12}^2 v^2\right] f_1^{(0)}\rho_2^{(0)} \Big) \\
&+ \Big(f_2^{(1)}\beta_2^1 - f_1^{(1)}\beta_1^2 \Big)\bar\rho,
\end{aligned} \tag{62}$$

where use of (60) has been made. Integrating (62) over $\mathbb{R}_+$ and (62) multiplied by v over $\mathbb{R}_+$, along with the conditions $f_1(0) = 0$ and $\lim_{v\to+\infty} f_1(v) = 0$, and remembering (59), we discover that both $\rho_1^{(1)} = \rho_2^{(1)} = 0$ and $M_1^{(1)} = M_2^{(1)} = 0$. This implies the fact that both the masses and

the averages of f_1 and f_2 are at the equilibrium even at $\mathcal{O}(\epsilon)$ accuracy. Using relations (60)–(61) in (62), we obtain

$$\begin{aligned}\partial_\tau f_1^{(0)} &= -\frac{\beta_1^2}{1+\alpha}\partial_v\left[\left((\omega_1\bar{M}-\omega_2 v\bar{\rho})+\alpha(\omega_2\bar{M}-\omega_2 v\bar{\rho})+(\omega_1\bar{M}-\omega_1 v\bar{\rho})+(\omega_2\bar{M}-\omega_1 v\bar{\rho})\right)f_1^{(0)}\right]\\ &+\frac{\zeta^2\beta_1^2\bar{\rho}}{2(1+\alpha)}\partial^2_{v^2}\left(v^2(3+\alpha)f_1^{(0)}\right)\\ &+\left(f_2^{(1)}\beta_2^1-f_1^{(1)}\beta_1^2\right)\bar{\rho},\end{aligned} \tag{63}$$

where we have also assumed that the stochastic fluctuations are the same in each kind of interaction, i.e.

$$\zeta_{ij}=\zeta, \qquad \forall i,j\in\mathcal{I}_2.$$

Eq. (63) is a Fokker–Planck equation with reaction term. In (63), the first and second line in the right-hand side represent the drift and diffusion contributions to the evolution equation. We remark that these terms only involve $f_1^{(0)}$ and its (known) mass and average (61). The third line in the right-hand side of (63) accounts for a reaction term that only depends on $f_1^{(1)}, f_2^{(1)}$. As $\rho_1^{(1)}=\rho_2^{(1)}=0$ and $M_1^{(1)}=M_2^{(1)}=0$, the reaction term does not influence the mass and average of $f_1^{(0)}$. It is therefore reasonable to look for the stationary solution to the Fokker–Planck equation without reaction term, i.e.

$$\begin{aligned}\partial_\tau \tilde{f}_1^{(0)} &= -\frac{\beta_1^2}{1+\alpha}\partial_v\left[\left((\omega_1\bar{M}-\omega_2 v\bar{\rho})+\alpha(\omega_2\bar{M}-\omega_2 v\bar{\rho})+(\omega_1\bar{M}-\omega_1 v\bar{\rho})+(\omega_2\bar{M}-\omega_1 v\bar{\rho})\right)\tilde{f}_1^{(0)}\right]\\ &+\frac{\zeta^2\beta_1^2\bar{\rho}}{2(1+\alpha)}\partial^2_{v^2}\left(v^2(3+\alpha)\tilde{f}_1^{(0)}\right),\end{aligned} \tag{64}$$

that is (63) where we neglect the third term on the right hand side, as this does not contribute to a variation of mass and average of $f_1^{(0)}$. Therefore we obtain

$$\tilde{f}_1^{(0)}=\frac{\bar{\rho}}{1+\alpha}v^{-2\left(1+\frac{\gamma}{2}\right)}\exp^{-\frac{\bar{M}}{\bar{\rho}}\frac{\gamma}{v}}, \qquad \gamma=\frac{B}{D}, \quad B=2\omega_1+\omega_2(1+\alpha), \quad D=\zeta^2\frac{3+\alpha}{2}. \tag{65}$$

The mass and average can be verified to be $\frac{\bar{\rho}}{1+\alpha}$ and $\frac{\bar{M}}{1+\alpha}$ respectively, while the second statistical moment is $\frac{\bar{M}}{(1+\alpha)(\gamma-1)}$. Moreover, we can determine the *Pareto index* of the first population. The Pareto Index of population 1 is then (approximated by)

$$\mu_1=\gamma+1=\frac{2\omega_1+\omega_2(1+\alpha)}{\frac{\zeta^2}{2}(3+\alpha)}+1 \tag{66}$$

that depends on the trading propensity of both populations ω_1,ω_2, on the ratio α that involves the rates β_i^j and on the stochasticity ζ^2. Since, according to (60), $\tilde{f}_2^{(0)}=\alpha\,\tilde{f}_1^{(0)}$, both populations have the same (approximated) Pareto index.

We remark that considering only one population corresponds to setting $\omega_1=\omega_2=\omega$ and $\beta_1^2=\beta_2^1$. If $\beta_2^1=\beta_1^2$, then there is no reaction term in (63) so that the stationary state (65) is exact and $\alpha=1$ that implies $f_1^{(0)}=f_2^{(0)}$. Moreover, the Pareto index (now exact) is

$$\frac{2\omega}{\zeta^2}+1,$$

that coincides with the one commonly obtained from a kinetic model for a single population [2]. Note that, even keeping $\alpha\neq 1$, in the case $\omega_1=\omega_2$ the Pareto index of each group coincides with the one relative to a single population.

5. Numerical tests

In this section we present some numerical tests that illustrate the dynamics of the model that we have introduced in the previous section. We integrate the microscopic model by directly implementing the discrete-time stochastic process (5)–(6)–(8) with (18) as illustrated in Section 2.2 for N agents using a modified version of the Nanbu–Babovski Monte Carlo algorithm (see Algorithm 1 in the Appendix). We remark that in the limit $N\to\infty$ and $\Delta t\to 0^+$ this produces the kinetic Eq. (19). In particular, we shall consider the microscopic rules (46)–(48), with $n=2$. We perform an empirical statistics of the N simulated agents with the Monte Carlo method and define the distribution functions f_1^{MC}, f_2^{MC}, their masses ρ_1^{MC},ρ_2^{MC} and first moments M_1^{MC}, M_2^{MC}. The apex MC refers to the numerical solutions obtained by integration of the microscopic model with the Monte Carlo algorithm.

5.1. Interacting agents with label switch and exchange of physical quantities

In all numerical tests we consider $\rho_1(0)=0.9$ and $\rho_2(0)=0.1$, i.e. the subgroup labeled with $x=1$ is initially more populated. We remark that the assumed symmetry in the process (48) implies $P_{11}^{11}=P_{11}^{22}=P_{22}^{22}=P_{22}^{11}=0$ and then $P_{11}^{12}=P_{11}^{21}=0.5$, being P_{ij}^{kl} a conditional probability. In the following numerical tests, we shall always consider $\omega_1=\omega_2=0.5$, as the values of ω_1 and ω_2 do not affect the averages' evolution and stationary state (see (51) and (54)). Moreover, we fix

$$\lambda_{11}=1$$

and we vary the other parameters.

In the first set of simulations (**Test 1** in the following), we choose $\lambda_{22}=10, \lambda_{12}=1$, $P_{12}^{11}=0.5, P_{12}^{22}=0.5$ and we consider two different initial conditions for the distributions f_1 and f_2. In **Case A** we have that the first population is poorer than the second population at time $t=0$, i.e.

$$f_1(v,0)=\rho_1(0)\mathbb{1}_{[0,1]}(v), \qquad f_2(v,0)=\rho_2(0)\frac{1}{10}\mathbb{1}_{[5,15]}(v) \tag{67}$$

while in **Case B** we invert the initial wealths

$$f_1(v,0) = \rho_1(0)\frac{1}{10}\mathbb{1}_{[5,15]}(v), \qquad f_2(v,0) = \rho_2(0)\mathbb{1}_{[0,1]}(v),$$

i.e. the second population is poorer than the first population at time $t = 0$. We report the results in Fig. 1. First of all, we observe that this choice of parameters prescribes $\alpha < 1$, that, as showed by the macroscopic equations (49), implies $\rho_1^\infty > \rho_2^\infty$ (see Fig. 1(a)). As forecast by theoretical results, the final average wealth $m^\infty = m_1 = m_2$ is closer to the initial wealth of the initially more populated subgroup: then m^∞ is smaller in case (A) and larger in case (B) (see Fig. 1(b)). This implies a different behavior of the first moment of both f_1 and f_2 (see Fig. 1(c)): while in case (B) the first population remains the richer one as it is the one that is initially more populated, in scenario (A) the mean wealth is inverted as the first population becomes richer. In each case we compare the evolution of the macroscopic quantities as prescribed by the microscopic model (5)–(6)–(8)–(18) with (46)–(48) and the ones whose evolution is given by the derived Eqs. (49)–(50)–(51). We denote $\rho_1^{MC}, \rho_2^{MC}, M_1^{MC}, M_2^{MC}, m_1^{MC}, m_2^{MC}$ the macroscopic quantities prescribed by the microscopic model that we integrate with a Monte Carlo scheme, while we denote $\rho_1, \rho_2, M_1, M_2, m_1, m_2$ the macroscopic quantities that are solution of the macroscopic model (49)–(50)–(51). Being $\Delta t = 10^{-2}$ and $N = 10^6$, we observe a very good agreement between the solution of the microscopic model and the one of the macroscopic model. The integration of the kinetic equations also allows one to approximate numerically the distribution functions f_1 and f_2 that we report at the equilibrium in Fig. 1(d) in both cases (A) and (B). In Fig. 1(e)–(f) we report the time evolution of f_1^{MC} and f_2^{MC} in case (A) with initial condition (67).

In the second set of simulations (**Test 2**) we choose $\lambda_{11} = 1, \lambda_{22} = 1, \lambda_{12} = 10$, i.e. intra-species interactions have the same frequency in the two populations, while the inter-group interactions have a higher frequency. Moreover, we consider three cases for the inter-group interactions: in case (i) $P_{12}^{11} = 0.5, P_{12}^{22} = 0.5$, i.e. given an inter-group interaction, the probability for both agents of transferring is the same, while in case (ii) $P_{12}^{11} = 0.2, P_{12}^{22} = 0.8$, i.e. the probability of transferring to the subgroup 2 is higher and in case (iii) $P_{12}^{11} = 0.8, P_{12}^{22} = 0.2$, i.e. the probability of transferring to the subgroup 1 is higher. The initial condition is set as in (67). We observe that in the three cases (i), (ii) and (iii) we have respectively $\alpha = 1, \alpha > 1, \alpha < 1$, that imply, see (49), $\rho_1^\infty = \rho_2^\infty, \rho_1^\infty < \rho_2^\infty, \rho_2^\infty < \rho_1^\infty$, respectively. In Fig. 2 we report the numerical integration of the microscopic model (markers) and of the macroscopic model (49)–(50) (solid lines). In particular, because of (53), in case (ii) we have a switch in the trend of the populations, as population 1 becomes the less populated (see Fig. 2(a)). This also implies a different (non-monotone) trend of the first moments as reported in Fig. 2(b).

In **Test 3**, we consider a switching probability depending on the microscopic wealth. In this case, it is not immediate, in general, to derive equations for the macroscopic quantities, unless in special cases, for example P_{ij}^{kl} having a linear dependence on v, or by imposing a monokinetic closure [45]. In Fig. 3 we have that $\lambda_{11} = 0.1, \lambda_{22} = 1, \lambda_{12} = 10$ and $P_{12}^{11} = 0.2\frac{1}{4}\left[1 - \exp^{-v} + 1 - \exp^{-w}\right], P_{12}^{22} = 0.8\frac{1}{4}\left[1 - \exp^{-v} + 1 - \exp^{-w}\right]$. We also present a comparison with the solution of macroscopic equations (49)–(51)–(50) (solid lines) where we consider constant switching probabilities $P_{12}^{11} = 0.2, P_{12}^{22} = 0.8$. We observe that both the microscopic model with v-dependent switching probabilities and the macroscopic model with constant switching probabilities forecast a similar behavior of the macroscopic quantities in the long run. More precisely, the microscopic model with a v-dependent switching probability forecasts the same behavior but with a delay, and this is due to the fact that the switching probabilities defined as $P_{12}^{11} = 0.2\frac{1}{4}\left[1 - \exp^{-v} + 1 - \exp^{-w}\right], P_{12}^{22} = 0.8\frac{1}{4}\left[1 - \exp^{-v} + 1 - \exp^{-w}\right]$ are smaller than the constant ones that are chosen as $P_{12}^{11} = 0.2, P_{12}^{22} = 0.8$. In particular, the v-dependent and the constant transition probabilities coincide only for very large values of the wealth, i.e. $v \to +\infty$. In magenta and green we also present the results of the simulation of the microscopic model in case $P_{12}^{11} = 0.2\frac{1}{2}\left[\exp^{-v} + \exp^{-w}\right], P_{12}^{22} = 0.8\frac{1}{2}\left[\exp^{-v} + \exp^{-w}\right]$. In this case we can observe that the convergence is even slower. This is due to the fact that for large values of the wealth v, the switching probability is very small, while it is equal to the constant case only if $v = 0$.

5.2. Quasi-invariant regime and Fokker–Planck equation

In this section, we consider the quasi-invariant regime (58), i.e. we analyze the dynamics on a long time-scale by considering small exchanges of wealth, while the switching probability is not rescaled. In this framework, we have seen that it is possible to approximate the leading order of the stationary solution $f_1^{(0)}$ through (65). Here, we compare the stationary state (65) with the solution f_1^{MC} obtained by the numerical integration of the microscopic process (5)–(6)–(8)–(58) with the microscopic rules (46)–(48), where in the quasi-invariant regime (58) we have chosen $\epsilon = 10^{-3}$. In Fig. 4 we represent the analytical $\tilde{f}_1^{(0)}$ as given in (65) and the approximation of f_1^{MC}. We can remark that, despite the fact that f_1^{MC} is obtained through a Monte Carlo simulation and $\tilde{f}_1^{(0)}$ is an approximation, the agreement is quite good. In the right panel, we also represent the numerical approximation of the distribution function g_1^{MC} of the first population, in case we consider a quasi-invariant transition probability defined by (37). With this choice it is granted that the evolution of both the average and the second statistical moment in the quasi-invariant regime $\epsilon \ll 1$ is the same as in the standard regime defined by $\epsilon = 1$. In this case it was not possible to determine the stationary state explicitly like in the quasi-invariant regime leading to (65) that, on the other hand, only grants that the average is the same for each ϵ.

6. Conclusions

In this paper we have presented a general framework for modeling systems of interacting agents with multiple microscopic states changing simultaneously according to a given dynamics. In particular, the microscopic description relies on Markovian processes described by transition probabilities, as they depend on the pre-interaction states, and the interaction frequency depends on the microscopic states. The fact of starting from the microscopic stochastic process allows one to describe in more detail the dynamics, by including parameters and quantities related to the phenomenon under study that can be observed. The derivation, through kinetic equations, of macroscopic equations allows one to obtain also at the aggregate level a higher level of detail that is inherited from the underlying microscopic dynamics.

Under some assumptions, general results concerning well-posedness, existence and uniqueness of a solution for the Cauchy problem associated to our kinetic equation have been shown. We have also rephrased the concept of quasi-invariant limit in the present framework, leading to evolution equations of Fokker–Planck type.

We have applied the present modeling framework in order to describe systems of binarily interacting agents characterized by a physical quantity v (representing wealth, or opinion, or viral load, etc.) and by a label x denoting the belonging to a given subgroup. The physical quantity changes according to binary interaction rules, while the label changes, simultaneously with the physical quantity, through a switch process caused by the

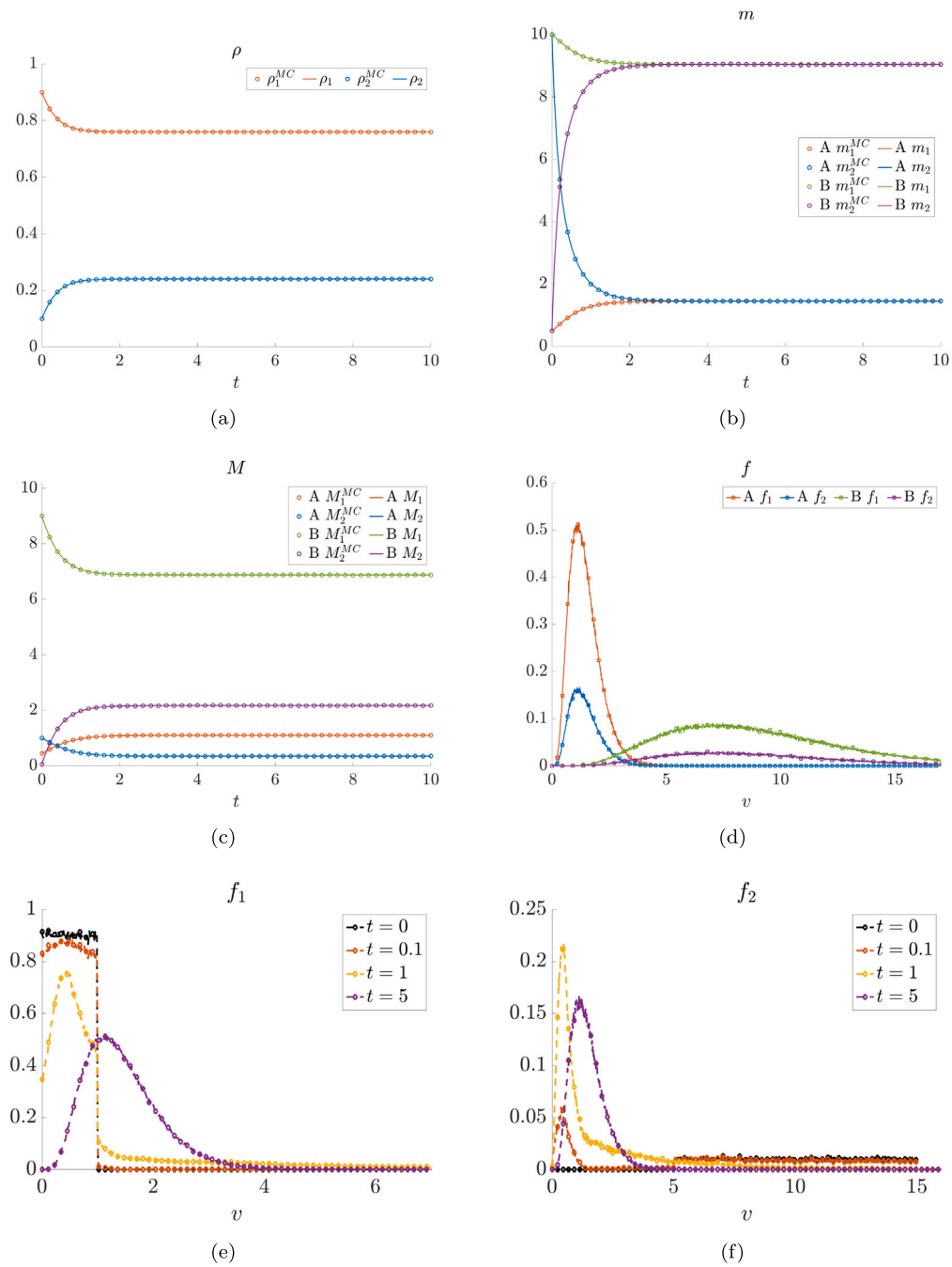

Fig. 1. Test 1: $\lambda_{11} = 1, \lambda_{22} = 10, \lambda_{12} = 1,\ P_{12}^{11} = 0.5, P_{12}^{22} = 0.5$. In all figures we report the time evolution of the macroscopic quantities as prescribed by the kinetic model and by the derived macroscopic equations: masses (a), averages (b), first moments (c). Solid lines correspond to the solution to the macroscopic equations (49)–(51)–(50), while circles correspond to the solution of the kinetic equation that we obtain by simulating with the Monte Carlo algorithm 1 the microscopic model (5)–(6)–(8)–(18) as illustrated in Section 2.2 with the microscopic rules (46)–(48). In all figures we compare the results of the solutions given the two different initial conditions (A) and (B). In figure (d) we show the steady states of distributions f_1, f_2 in both test cases (A) and (B), while in figures (e), (f) we report time evolution of distributions functions in the test case (A).

same binary interaction. In this context, we have seen that the description of the microscopic process by means of transition probabilities allows us to remove the reversibility assumption on the interaction rule, modeling thus also stochasticity in the binary encounters giving rise to transfers (not present in the paper [40] using classical Boltzmann operators analogous to the reactive ones). Moreover, in our framework it is easier to consider a non-constant switching probability, i.e. depending on the microscopic physical quantity. We have analyzed and discussed various quasi-invariant regimes and performed some numerical tests showing a very good agreement between the microscopic Monte Carlo simulations and the derived macroscopic equations.

The modeling framework investigated in this paper is worth to be applied and generalized to many other problems. As first, different microscopic rules for the exchange of v could be considered. In fact, the modeling framework is proposed for general microscopic rules involving a deterministic and a stochastic part. The possible hurdles coming from a choice of nonlinear rules in v could be the derivation of closed macroscopic equations as well as the quasi-invariant limit technique. Moreover, the model for international trade with transfers presented in Section 4 could be extended

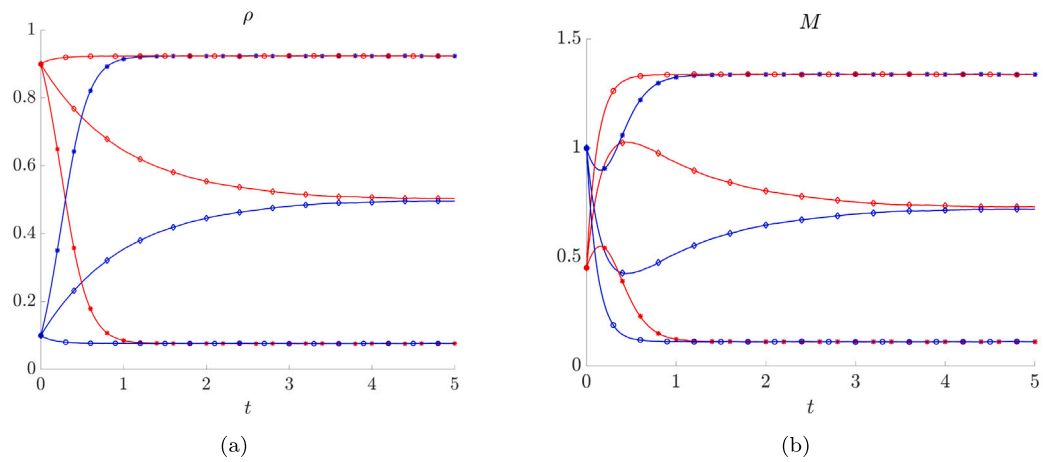

Fig. 2. Test 2. Solutions to the macroscopic model (49)–(50) (solid lines) and microscopic model (markers). In (a) we report the masses ρ_1, ρ_2 and in (b) the first moments M_1, M_2. Population 1 is in red, while population 2 is in black. The parameters are $\lambda_{11} = 1, \lambda_{22} = 1, \lambda_{12} = 10$ and we show three cases: (i) ($\diamond$ marker) $P_{12}^{11} = 0.5, P_{12}^{22} = 0.5$, (ii) (* marker) $P_{12}^{11} = 0.2, P_{12}^{22} = 0.8$ and (iii) (o marker) $P_{12}^{11} = 0.8, P_{12}^{22} = 0.2$.

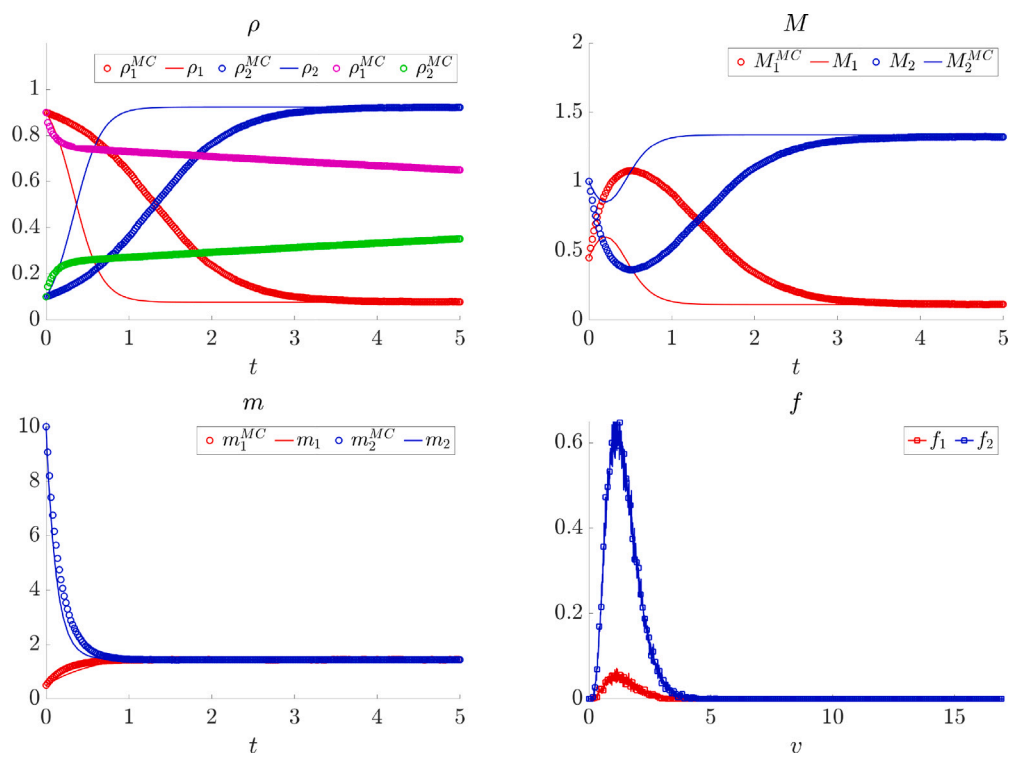

Fig. 3. Test 3. Solution to the microscopic model (5)–(6)–(8)–(18) as illustrated in Section 2.2 with the microscopic rules (46)–(48) with the Monte Carlo algorithm 1 (circles). Here $\lambda_{11} = 1, \lambda_{22} = 0.1, \lambda_{12} = 10$ and $P_{12}^{11} = 0.2\frac{1}{2}\left[1 - \exp^{-v} + 1 - \exp^{-w}\right], P_{12}^{22} = 0.8\frac{1}{2}\left[1 - \exp^{-v} + 1 - \exp^{-w}\right]$. We also present a comparison with the solution of macroscopic equations (49)–(51)–(50) (solid lines) where we consider constant switching probabilities $P_{12}^{11} = 0.2, P_{12}^{22} = 0.8$. In magenta and green we also present the results of the simulation of the microscopic model in case $P_{12}^{11} = 0.2\frac{1}{2}\left[\exp^{-v} + \exp^{-w}\right], P_{12}^{22} = 0.8\frac{1}{2}\left[\exp^{-v} + \exp^{-w}\right]$.

by adding an extra independent microscopic process for v, describing the exchange of goods without transfers; this would make the model even more similar to the kinetic description of gaseous mixtures, where elastic collisions (which do not change the nature of the particles) coexist with chemical reactions (changing the species of the reacting particles). Epidemic models based on a kinetic approach could be improved owing to our stochastic framework with multiple states as it allows one to start from a microscopic description and to consider independent or simultaneous microscopic stochastic dynamics for the different variables of the microscopic state. For example, the so-called "non–conservative" interactions giving rise to the switch from one compartment to another could be made more realistic taking into account also the simultaneous change of viral load (of individuals) as done in [41,42] or of the internal activity (of cells). Moreover, the relation with kinetic models with label switching and gradient descent could be established [63]. Eventually, applications to situations with many internal states is the final scope of our framework. It could provide for instance a physically reasonable description of mixtures of polyatomic gases, with each molecule characterized by its species label i, its velocity $\mathbf{v} \in \mathbb{R}^3$, and its internal energy that could also be separated into the vibrational part (typically described by a discrete variable)

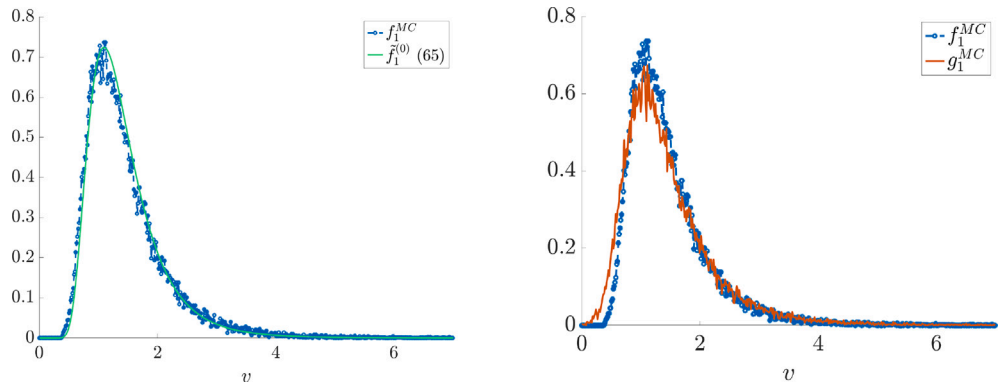

Fig. 4. Solution of the Fokker–Planck equation (63). Left: comparison between the approximated stationary state $\tilde{f}_1^{(0)}$ given by (65) and the solution f_1^{MC} of the microscopic model (5)–(6)–(8)–(58) with the microscopic rules (46)–(48), with $\epsilon = 10^{-3}$ in the quasi-invariant regime (58) obtained with $N = 10^6, \Delta t = 10^{-3}$. Right: comparison between f_1^{MC} and g_1^{MC} obtained integrating the same, with the quasi-invariant regime (37).

and the rotational part (typically approximated by a continuous variable) [26,64]. Even in econophysics, the possible influence of the personal knowledge of the market on the strategy adopted in the trades (as sketched in [65] for a single population) could be described considering the individual knowledge as an additional microscopic state, besides the population label and the individual amount of wealth. Suitable quasi-invariant limits and properties of steady states of such non-standard kinetic descriptions of various interacting populations are completely open problems worth to be investigated in future research.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix

A.1. Boltzmann-type description of classical binary interaction dynamics

Let us consider the case in which the microscopic state of the agent is a non-negative physical quantity $z = v \in \Omega = \mathbb{R}_+$. Extensions to negative and possibly also bounded microscopic states are mostly a matter of technicalities. In general, as classically done in kinetic theory [45], if $v, w \in \mathbb{R}_+$ denote the pre-interaction states of any two interacting agents, their post-interaction states v', w' will be given by general interaction rules in the form

$$v' = I(v,w) + D(v,w)\eta, \qquad w' = \tilde{I}(v,w) + \tilde{D}(v,w)\eta_* \tag{68}$$

where η and η_* are independent random variables satisfying $\langle \eta \rangle = \langle \eta_* \rangle = 0, \langle \eta^2 \rangle = \langle \eta_*^2 \rangle = 1$, namely with zero average and unitary variance. It is known that an aggregate description of the (sole) binary interaction dynamics inspired by the principles of statistical mechanics can be obtained by introducing a probability density function $f = f(v,t) \geq 0$ such that $f(v,t)dv$ gives the proportion of agents having at time t a microscopic state comprised between v and $v + dv$. Such a probability density function satisfies a Boltzmann-type kinetic equation, which in weak form reads

$$\frac{d}{dt}\int_{\mathbb{R}_+} f(v,t)\phi(v)\,dv = \frac{\lambda}{2}\Big\langle \int_{\mathbb{R}_+^2} \big(\phi(v') + \phi(w') - \phi(v) - \phi(w)\big) f(v,t) f(w,t)\,dv dw \Big\rangle \tag{69}$$

where λ is the interaction frequency λ_{zy} that we here assume to be independent of the microscopic states of the agents. On the other hand, if we want to describe the binary interactions through transition probabilities (1) with $z' = v', y' = w'$, we have that (9) can be rewritten as

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}_+} f(v,t)\phi(v)\,dv = \frac{\lambda}{2}\int_{\mathbb{R}_+^2}\left(\int_{\mathbb{R}_+}\phi(v')T(v'|v,w)\,dv' - \phi(v)\right) f(v,t)f(w,t)\,dv\,dw \\ + \frac{\lambda}{2}\int_{\mathbb{R}_+^2}\left(\int_{\mathbb{R}_+}\phi(w')\tilde{T}(w'|v,w)\,dw' - \phi(w)\right) f(v,t)f(w,t)\,dv\,dw.\end{aligned} \tag{70}$$

We can define the first two statistical moments of the distribution function f as:

$$M(t) := \int_{\mathbb{R}_+} v f(t,v)\,dv, \qquad E(t) := \int_{\mathbb{R}_+} v^2 f(t,v)\,dv, \qquad e(t) := \int_{\mathbb{R}_+} (v-M)^2 f(t,v)\,dv$$

that represent the average, the second statistical moment and the variance, respectively. In particular, we remark that $E(t) = M(t)^2 + e(t)$. As done in [44] in the symmetric case, in order to establish a relation between (68)–(69) and (70), we investigate primarily the trend of the statistical moments as prescribed by the two different models. Setting $\phi(v) = v, v^2$ both in (70) and in (68)–(69) yields the evolution equations for M, E for a system of agents obeying the microscopic dynamics expressed in terms of transition probabilities (1) or interaction rules (68), respectively. By comparing the evolution equations of M and E prescribed by the two different kinetic models, we see that the evolution is the same if we choose

$$\begin{aligned} I(v,w) &= V_T(v,w), \quad D(v,w) = \sqrt{E_T(v,w) - V_T^2(v,w)} =: D_T(v,w), \\ \tilde{I}(v,w) &= V_{\tilde{T}}(v,w), \quad \tilde{D}(v,w) = \sqrt{E_{\tilde{T}}(v,w) - V_{\tilde{T}}^2(v,w)} =: D_{\tilde{T}}(v,w), \end{aligned}$$

where

$$V_T(v,w) := \int_{\mathbb{R}_+} v' T(v' \mid v,w)\,dv', \qquad E_T(v,w) := \int_{\mathbb{R}_+} v'^2 T(v' \mid v,w)\,dv'$$

and

$$V_{\tilde{T}}(v,w) := \int_{\mathbb{R}_+} w' \tilde{T}(w' \mid v,w)\,dw', \qquad E_{\tilde{T}}(v,w) := \int_{\mathbb{R}_+} w'^2 \tilde{T}(w' \mid v,w)\,dw'$$

denote the mean and the second moment, respectively, of T and $\tilde{T}$ for a given pair $(v,w) \in \mathbb{R}_+ \times \mathbb{R}_+$ of pre-interaction states, while $D_T(v,w)$ and $D_{\tilde{T}}(v,w)$ are the standard deviations of T and $\tilde{T}$, respectively. Therefore, if dealing with (69) we can consider the interactions

$$v' = V_T(v,w) + D_T(v,w)\eta, \qquad w' = V_{\tilde{T}}(v,w) + D_{\tilde{T}}(v,w)\eta_*, \tag{71}$$

and this choice makes formulations (69) and (70) equivalent *at the macroscopic level* (at least for the mass, average and second statistical moment). As highlighted in [44], in general, (69) with (68) and (70) are not the same kinetic equation, although with the choice (71) they account for the same evolution of the first and second statistical moments of f. Nevertheless, if in (70) we take

$$T(v'|v,w) = \delta\Big(v' - (V_T(v,w) + D_T(v,w)\eta)\Big), \qquad \tilde{T}(w'|v,w) = \delta\Big(w' - (V_{\tilde{T}}(v,w) + D_{\tilde{T}}(v,w)\eta_*)\Big) \tag{72}$$

where in the right-hand side δ is the Dirac delta, then we can formally show that (70) becomes exactly (68)–(69). Of course, in this case the right hand side (70) is meant to be written in brackets $\langle\cdot\rangle$.

This establishes the parallelism, in the field of kinetic equations for socio-economic sciences [44], between the collision-like Boltzmann equation and Markovian jump-processes described by transition probabilities that can be related to the Waldmann (probabilistic) representation of the Boltzmann equation [43]. In classical kinetic theory, the Waldmann representation shows in the kernel the probability distribution of the collision process transforming pre-collision velocities (v,w) into the post-collision ones (v',w'). In this representation, the Boltzmann integral over the unit sphere is replaced by integrals over the post-collision velocity variables. An analogous scattering kernel formulation replaces the Waldmann kernel by its integral over the velocity of the partner molecule [66]. The equivalence between these kinetic equations has been proved in [67] for microscopic interactions that conserve the average and the energy.

A.1.1. Quasi-invariant limit procedure

The quasi-invariant limit procedure is classically applied to the collision-like Boltzmann equation with microscopic interaction rules (68)–(69) (or (69)–(71)). Let us introduce a small parameter $0 < \epsilon \ll 1$, a time scale $\tau = \epsilon t$ and a corresponding probability density function $f^\epsilon(\tau,v) = f(\tau/\epsilon, v)$. For what we have said in Section 2, as shown in [44], we have that the quasi-invariant microscopic rule for having the same evolution of the average and the second moment of both f and f^ϵ on the t- and τ-time scale, respectively, is

$$v' = V_{T_\epsilon}(v,w) + D_{T_\epsilon}(v,w)\eta,$$

and analogously for the microscopic rule for w'. If we consider the quasi-invariant transition probability (33), we have that $V_{T_\epsilon}(v,w) = v + \epsilon(I(v,w) - v)$ and $D_{T_\epsilon}(v,w) = \sqrt{\epsilon(1-\epsilon)(I(v,w)-v)^2 + \epsilon D(v,w)^2}$ so that the quasi-invariant microscopic rules are

$$\begin{aligned} v' &= v + \epsilon(I(v,w) - v) + \sqrt{\epsilon(1-\epsilon)(I(v,w)-v)^2 + \epsilon D(v,w)^2}\,\eta, \\ w' &= w + \epsilon(\tilde{I}(v,w) - w) + \sqrt{\epsilon(1-\epsilon)(\tilde{I}(v,w)-w)^2 + \epsilon \tilde{D}(v,w)^2}\,\eta_*. \end{aligned} \tag{73}$$

Therefore, in terms of transition probabilities, in order to recover the same evolution of the first two moments in the two models (68)–(69) and (70), we must consider T_ϵ having average $\epsilon(I(v,w) - v)$ and variance $\epsilon(1-\epsilon)(I(v,w)-v)^2 + \epsilon D(v,w)^2$ and analogously for $\tilde{T}_\epsilon$. In particular, in order for (68)–(69) and (70) to be the same equation, an appropriate choice is (33) with $z = v$ or

$$\begin{aligned} T_\epsilon(v'|v,w) &= \delta\Big(v' - (v + \epsilon(I(v,w) - v) + \sqrt{\epsilon(1-\epsilon)(I(v,w)-v)^2 + \epsilon D(v,w)^2}\,\eta)\Big), \\ \tilde{T}_\epsilon(w'|v,w) &= \delta\Big(w' - (w + \epsilon(\tilde{I}(v,w) - w) + \sqrt{\epsilon(1-\epsilon)(\tilde{I}(v,w)-w)^2 + \epsilon \tilde{D}(v,w)^2}\,\eta_*)\Big). \end{aligned} \tag{74}$$

We remark that for $\epsilon = 1$ we recover T and $\tilde{T}$, while for $\epsilon \to 0$ we have that

$$W_1(T_\epsilon(\cdot|v,w), \delta(\cdot - v)) \leq \int_{\mathbb{R}_+^2} |v' - w'| T_\epsilon(v'|v,w)\delta(w' - v)\, dv'\, dw'$$
$$= |\epsilon(I(v,w) - v) + \sqrt{\epsilon(1-\epsilon)(I-v)^2 + \epsilon D^2}\,\eta| \longrightarrow_{\epsilon\to 0} 0$$

and it can also be easily verified that property $F3$ given in (32) holds true for T_ϵ defined by (74), and analogously for $\tilde{T}_\epsilon$. If we consider for a moment the symmetric case (3), for simplicity, we have that plugging (28) into (70) and considering (30) satisfying (32), letting ϵ to 0^+ yields

$$\partial_\tau f = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} T(v\,|\,'v, {}'w) f('v, \tau) f('w, \tau)\, d'v\, d'w - f, \tag{75}$$

which is structurally identical to the very general equation (70) and does not give any further information. Therefore, in spite of the quasi-invariant structure of the interactions, it is in principle not easier to extract from (75) any more detailed information about the asymptotic trends.

Instead, if we consider (70) with (74) and (28), then we can perform the quasi-invariant limit, as in [44], and we obtain letting ϵ to 0^+

$$\begin{aligned}\partial_\tau f = \frac{1}{2}\partial_v^2 &\left\{\left[\int_{\mathbb{R}_+}\left(\left(V_T(v,w) - v\right)^2 + D_T^2(v,w)\right) f(w,\tau)\, dw\right] f\right\}\\ &- \partial_v\left[\left(\int_{\mathbb{R}_+} V_T(v,w) f(w,\tau)\, dw - v\right) f\right],\end{aligned} \tag{76}$$

where V_T, D_T are the average and variance of T defined in Appendix A.1.

A.2. Label switch process caused by binary interactions

Let us now consider the case in which $\Omega = \mathcal{I}_n = \{1, \dots, n\}$ and the microscopic state is given by the discrete label $x \in \mathcal{I}_n$ that may denote the belonging of the agent to a certain group or subpopulation. We assume that label switches, i.e. migrations across subpopulations, can be caused by binary interactions between agents, causing a transfer of (potentially) both of them, but in such a way that the total mass of the agents in the system is conserved. We say that this process is formally a Markov-type one because the probability to switch from the current labels x, y to new labels x', y' does not depend on how the agents reached previously the labels x, y. In particular we denote

$$P_{xy}^{x'y'} := P(x', y'|x, y) \tag{77}$$

the conditional probability density function of switching to the groups x', y' given the pre-interaction labels x, y.

Remark A.1. Since the variables x, y are discrete, the mapping $(x', y') \mapsto P(x', y'|x, y)$ is a discrete probability measure. Consequently, we actually have

$$\int_{\mathcal{I}_n^2} P(x', y'|x, y)\, dx'\, dy' = \sum_{x', y' \in \mathcal{I}_n} P(x', y'|x, y) = 1. \tag{78}$$

If we introduce the probability density function $f = f(x,t) \geq 0$ of the agents with label x at time t, its evolution can be modeled by a kinetic equation describing a Markov-type jump process:

$$\partial_t f(x', t) = \lambda\left(\int_{\mathcal{I}_n^3} P_{xy}^{x'y'} f(x,t) f(y,t)\, dx\, dy\, dy' - f(x', t)\right), \tag{79}$$

where $\lambda > 0$ is the (constant) switch frequency. In weak form (79) reads

$$\frac{d}{dt}\int_{\mathcal{I}_n} \psi(x) f(x,t)\, dx = \lambda \int_{\mathcal{I}_n^2}\int_{\mathcal{I}_n^2} (\psi(x') - \psi(x)) P_{xy}^{x'y'} f(x,t) f(y,t)\, dx'\, dy'\, dx\, dy, \tag{80}$$

where $\psi : \mathcal{I}_n \to \mathbb{R}$ is an observable quantity (test function) defined on $\mathcal{I}_n$. Eq. (80) can be derived by (9) by setting $z = x$ and

$$P_{xy}^{x'y'} = \frac{T(x'|x,y) + \tilde{T}(y'|x,y)}{2}.$$

Since $x \in \mathcal{I}_n$ is discrete, we may conveniently represent the distribution function f as

$$f(x,t) = \sum_{i=1}^n f_i(t)\delta(x - i), \tag{81}$$

where $\delta(x - i)$ is the Dirac distribution centered in $x = i$ and $f_i = f_i(t) \geq 0$ is the probability that an agent is labeled by $x = i$ at time t. In this way, we reconcile the weak form (80) with the convention introduced in Remark 2.1, and (80) actually becomes

$$\sum_{i=1}^n \psi(i) f_i'(t) = \lambda \sum_{i,l=1}^n \sum_{j,k=1}^n (\psi(i) - \psi(j)) P_{jk}^{il} f_j(t) f_k(t). \tag{82}$$

We have conservation of the total mass thanks to (78), as we can verify setting $\psi = 1$ in (80). Setting $\psi = 1$ in (82), this corresponds to

$$\sum_{i=1}^n f_i(t) = 1. \tag{83}$$

Choosing ψ such that $\psi(s) = 1$ for a certain $s \in \mathcal{I}_n$ and $\psi(x) = 0$ for all $x \in \mathcal{I}_n \setminus \{s\}$ we get in particular

$$f_s' = \lambda\left(\sum_{j,k,l=1}^n P_{jk}^{sl} f_j f_k - f_s\right), \qquad s = 1, \dots, n, \tag{84}$$

where we have used (78) and (83). If we allow the interaction frequency λ_{xy} to depend on the labels of the interacting agents, then the equation becomes

$$\sum_{i=1}^{n} \psi(i) f_i'(t) = \sum_{i,l=1}^{n} \sum_{j,k=1}^{n} (\psi(i) - \psi(j)) \lambda_{jk} P_{jk}^{il} f_j(t) f_k(t)$$

so that

$$f_s' = \sum_{j,k,l=1}^{n} \left(\lambda_{jk} P_{jk}^{sl} f_j f_k - \lambda_{sk} P_{sk}^{jl} f_s f_k \right), \qquad s = 1, \dots, n, \tag{85}$$

where we remind that $\beta_{ij}^{kl} := \lambda_{ij} P_{ij}^{kl}$, i.e. the *rate* of transfer for a couple from the subgroups (i, j) to (k, l) as a consequence of a binary interaction between two agents labeled i and j.

A.2.1. Quasi-invariant limit procedure

Let us now consider the transfer process described by the kinetic Eq. (85). In order to write a quasi-invariant regime, we can express the fact that, given a collision, individuals have a small probability of jumping (35). Then, considering a long time scale (28) we have that (85) is

$$\frac{d f_s^\epsilon}{d\tau} = \frac{1}{\epsilon} \sum_{j,k,l=1,\ j\neq s, k\neq l}^{n} \left(P_{\epsilon_{jk}}^{sl} \lambda_{jk} f_j^\epsilon - P_{\epsilon_{sk}}^{jl} \lambda_{sk} f_s^\epsilon \right) f_k^\epsilon, \qquad s = 1, \dots, n. \tag{86}$$

Plugging (35) in (86) we obtain

$$\frac{d f_s^\epsilon}{d\tau} = \sum_{j,k,l=1}^{n} \left(\beta_{jk}^{sl} f_j^\epsilon f_k^\epsilon - \beta_{sk}^{jl} f_s^\epsilon f_k^\epsilon \right), \qquad s = 1, \dots, n$$

which is structurally identical to the very general equation (85) with τ instead of t, meaning that it is the probability of jumping that rules the dynamics on the long time scale.

A.3. Proof of Theorem 3.2

In this Appendix A.3 we show a sketch of the proof of Theorem 3.2 that follows the proof in [7, Appendix]. We will prove, in the order

- existence and uniqueness of the solution;
- continuous dependence on the initial datum;
- existence and uniqueness of the equilibrium;
- attractiveness of the equilibrium.

In particular, we focus on the first step, as the following ones rely on the same arguments as in [7] with the needed adaptations due to the generic Ω and non-constant λ_{zy}. For convenience of notation, we show the proof in the symmetric case (3), i.e. (23) reads

$$f(z,t) = e^{-\bar{\lambda}t} f_0(z) + \int_0^t e^{\bar{\lambda}(s-t)} \int_{\Omega^2} \lambda_{\prime z' y} T(z|'z, 'y) f('z, s) f('y, s)\, d'z d'y ds. \tag{87}$$

We remark that if η is a Lipschitz function of constant K, then

$$\int_\Omega \eta(z) \left(f(z) - g(z) \right) dz = K \int_\Omega \frac{\eta(z)}{K} \left(f(z) - g(z) \right) dz \le K W_1(f, g) \tag{88}$$

due to (24), to the arbitrariness of η and to the fact that $\frac{\eta}{K}$ is a Lipschitz function with constant 1.

Existence and uniqueness. We fix $\tilde{t} > 0$ and define the operator $\mathcal{S}$ on X as

$$\mathcal{S}(f)(z,t) := e^{-\bar{\lambda}t} f_0(z) + \int_0^t e^{\bar{\lambda}(s-t)} \int_{\Omega^2} \lambda_{\prime z' y} T(z|'z, 'y) f('z, s) f('y, s)\, d'z d'y ds. \tag{89}$$

Let us denote

$$\mathcal{I}[f](z,s) := \int_{\Omega^2} \lambda_{\prime z' y} T(z|'z, 'y) f('z, s) f('y, s)\, d'z d'y. \tag{90}$$

We remark that $\mathcal{S}(f)(z,t) := e^{-\bar{\lambda}t} f_0(z) + \int_0^t e^{\bar{\lambda}(s-t)} \mathcal{I}[f](z,s) ds$. Then we state the problem of uniqueness of the solution of problem (21) as $f = \mathcal{S}(f)$, meaning that the solution of (21) are fixed points of $\mathcal{S}$.

- First we prove that $\mathcal{S}(X) \subset X$: let $f \in X$. As $f_0 \in X$ and T is positive, then $\mathcal{S}(f)$ is positive. Moreover it is easy to verify that $\int_\Omega \mathcal{S}(f)\, dz = 1$. Therefore $\mathcal{S}(f) \in \mathcal{M}_+(\Omega)$. Then, we need to prove the continuity of $t \to \mathcal{S}(f)(\cdot, t)$. With respect to [7] where Ω is bounded, we have not that $\varphi \in C_b(\Omega) \cap \mathrm{Lip}_1(\Omega)$. However, if d is bounded by M, we can remark that, without loss of generality we can consider $0 \le \varphi \le M$,

i.e. $\varphi \in \mathrm{Lip}_{1,M} = \{\varphi \text{ lipschitz}, 0 \le \varphi \le M, \|\varphi\|_{\mathrm{Lip}} \le 1\}$ ([51], Chapter 7). Then we take $t_1, t_2 \in [0, \tilde{t}]$ with $t_1 \le t_2$ and we have that

$$\begin{aligned}
W_1(\mathcal{S}(f)(\cdot,t_1), \mathcal{S}(f)(\cdot,t_2)) = \sup_{\varphi\in\mathrm{Lip}_{1,M}} &\Bigg[\left(\exp^{-\bar{\lambda}t_1} - \exp^{-\bar{\lambda}t_2}\right)\int_\Omega \varphi(\boldsymbol{z}) f_0(\boldsymbol{z})\, d\boldsymbol{z} \\
&+ \int_0^{t_2} \exp^{\bar{\lambda}(s-t_2)} \int_\Omega \mathcal{I}(\boldsymbol{z},s)\, d\boldsymbol{z}\, ds - \int_0^{t_1} \exp^{\bar{\lambda}(s-t_1)} \int_\Omega \mathcal{I}(\boldsymbol{z},s)\, d\boldsymbol{z}\, ds\Bigg] \\
&\le M\,|\exp^{-\bar{\lambda}t_1} - \exp^{-\bar{\lambda}t_2}|\left(1 + \lambda_M \frac{\bar{\lambda}}{\bar{\lambda}} \int_0^{t_1} \exp^{\bar{\lambda}s}\, ds\right) \\
&+ M\lambda_M \frac{\bar{\lambda}}{\bar{\lambda}} \exp^{-\bar{\lambda}t_2} \int_{t_1}^{t_2} \exp^{\bar{\lambda}s}\, ds \\
&\le M \frac{\lambda_M}{\lambda_m} |\exp^{-\bar{\lambda}t_1} - \exp^{-\bar{\lambda}t_2}|\left(1 + \bar{\lambda} \int_0^{t_1} \exp^{\bar{\lambda}s}\, ds\right) \\
&+ M \frac{\lambda_M}{\lambda_m} \bar{\lambda} \exp^{-\bar{\lambda}t_2} \int_{t_1}^{t_2} \exp^{\bar{\lambda}s}\, ds \\
&\le 3M\bar{\lambda}\frac{\lambda_M}{\lambda_m}|t_2 - t_1| < 3M\frac{\lambda_M^2}{\lambda_m}|t_2 - t_1|,
\end{aligned}$$

where we have used the hypothesis $\lambda_m < \lambda_{\boldsymbol{z}\boldsymbol{y}} < \lambda_M$ that implies the fact that $\int_\Omega \mathcal{I}(s,\boldsymbol{z})\, d\boldsymbol{z} \le \lambda_M M$, $\lambda_m < \bar{\lambda} < \lambda_M$ and $1 < \dfrac{\lambda_M}{\lambda_m}$ and we have concluded thanks to the Lipschitz continuity of the exponential function.

- Let us now prove that, if $\tilde{t}$ is small, then $\mathcal{S}$ is a contraction on X. Let $f, g \in X$. Then, at this step, we can use, as in [7], the definition (25)

$$\begin{aligned}
W_1(\mathcal{S}(f), \mathcal{S}(g)) &= \int_\Omega \varphi(\boldsymbol{z})\left(\mathcal{S}(f(\boldsymbol{z},t)) - \mathcal{S}(g(\boldsymbol{z},t))\right) d\boldsymbol{z} \\
&= \int_0^t \exp^{\bar{\lambda}(s-t)} \int_\Omega \varphi(\boldsymbol{z}) \int_{\Omega^2} \lambda_{'\boldsymbol{z}'\boldsymbol{y}} T(\boldsymbol{z}|'\boldsymbol{z},'\boldsymbol{y})\left(f('\boldsymbol{z},s)f('\boldsymbol{y},s) - g('\boldsymbol{z},s)g('\boldsymbol{y},s)\right) d'\boldsymbol{z}d'\boldsymbol{y}d\boldsymbol{z}\, ds \\
&= \int_0^t \exp^{\bar{\lambda}(s-t)} \int_\Omega \int_\Omega \int_\Omega \varphi(\boldsymbol{z})\lambda_{'\boldsymbol{z}'\boldsymbol{y}} T(\boldsymbol{z}|'\boldsymbol{z},'\boldsymbol{y}) g('\boldsymbol{z},s)\, d'\boldsymbol{z}\left(g('\boldsymbol{y},s) - f('\boldsymbol{y},s)\right) d'\boldsymbol{y}d\boldsymbol{z}ds \\
&+ \int_0^t \exp^{\bar{\lambda}(s-t)} \int_\Omega \int_\Omega \int_\Omega \varphi(\boldsymbol{z})\lambda_{'\boldsymbol{z}'\boldsymbol{y}} T(\boldsymbol{z}|'\boldsymbol{z},'\boldsymbol{y}) f('\boldsymbol{y},s)\, d'\boldsymbol{y}\left(g('\boldsymbol{z},s) - f('\boldsymbol{z},s)\right) d'\boldsymbol{z}d\boldsymbol{z}ds.
\end{aligned}$$

We remark that, thanks to 3.1

$$\int_\Omega \int_\Omega \varphi(\boldsymbol{z})\lambda_{'\boldsymbol{z}'\boldsymbol{y}_1} T(\boldsymbol{z}|'\boldsymbol{z},'\boldsymbol{y}_1) g('\boldsymbol{z},s) - \varphi(\boldsymbol{z})\lambda_{'\boldsymbol{z}'\boldsymbol{y}_2} T(\boldsymbol{z}|'\boldsymbol{z},'\boldsymbol{y}_2) g('\boldsymbol{z},s)\, d'\boldsymbol{z} \le \lambda_M \mathrm{Lip}(T) d('\boldsymbol{y}_1,'\boldsymbol{y}_2)$$

i.e. the mapping $'\boldsymbol{y} \to \int_\Omega\int_\Omega \varphi(\boldsymbol{z})\lambda_{'\boldsymbol{z}'\boldsymbol{y}} T(\boldsymbol{z}|'\boldsymbol{z},'\boldsymbol{y})\, d\boldsymbol{z} g('\boldsymbol{z},s)\, d'\boldsymbol{z}$ is Lipschitz with constant $\lambda_M \mathrm{Lip}(T)$ and the same holds for the other mapping $'\boldsymbol{z} \to \int_\Omega\int_\Omega \varphi(\boldsymbol{z})\lambda_{'\boldsymbol{z}'\boldsymbol{y}} T(\boldsymbol{z}|'\boldsymbol{z},'\boldsymbol{y})\, d\boldsymbol{z} g('\boldsymbol{y},s)\, d'\boldsymbol{y}$. Then, using (88) we have that

$$\begin{aligned}
W_1(\mathcal{S}(f)(\cdot,t), \mathcal{S}(g)(\cdot,t)) &\le 2\lambda_M \int_0^t \exp^{\bar{\lambda}(s-t)} \mathrm{Lip}(T) W_1(f(\cdot,s), g(\cdot,s))\, ds \\
&\le (1 - \exp^{-\bar{\lambda}t})\mathrm{Lip}(T)\frac{\lambda_M}{\bar{\lambda}} 2 \sup_{t\in[0,\tilde{t}]} W_1(f(\cdot,t), g(\cdot,t)) \\
&\le (1 - \exp^{-\bar{\lambda}t})\mathrm{Lip}(T)\frac{\lambda_M}{\lambda_m} 2 \sup_{t\in[0,\tilde{t}]} W_1(f(\cdot,t), g(\cdot,t)).
\end{aligned}$$

Then

$$\sup_{t\in[0,\tilde{t}]} W_1(\mathcal{S}(f), \mathcal{S}(g)) \le \sup_{t\in[0,\tilde{t}]} (1 - \exp^{-\bar{\lambda}t}) 2\mathrm{Lip}(T)\frac{\lambda_M}{\lambda_m} W_1(f,g).$$

If $\mathrm{Lip}(T) > \dfrac{1}{2}\dfrac{\lambda_m}{\lambda_M}$, then it is possible to take $\tilde{t} = \dfrac{1}{\bar{\lambda}} \log\left(\dfrac{2\frac{\lambda_M}{\lambda_m}\mathrm{Lip}(T)}{2\frac{\lambda_M}{\lambda_m}\mathrm{Lip}(T) - 1}\right)$. If $\mathrm{Lip}(T) < \dfrac{1}{2}\dfrac{\lambda_m}{\lambda_M}$ then it is a contraction for all $\tilde{t}$. Owing to the properties above, Banach fixed-point theorem implies the existence of a unique fixed point $f \in X$. If $\mathrm{Lip}(T) \le \dfrac{1}{2}\dfrac{\lambda_m}{\lambda_M}$ the solution is global in time, whereas if $\mathrm{Lip}(T) > \dfrac{1}{2}\dfrac{\lambda_m}{\lambda_M}$ it is only local. Notwithstanding, a simple continuation argument is sufficient, based on taking $f(\tilde{t}) \in \mathcal{M}_+(\Omega)$ as a new initial condition for $t = \tilde{t}$. Then repeating the procedure above, shows that we can extend it uniquely on the interval $[\tilde{t}, 2\tilde{t}]$. Proceeding in this way, we do the same on all subsequent intervals of the form $[k\tilde{t}, (k+1)\tilde{t}]$, $k = 1, 2, \ldots$ and we obtain also in this case a global-in-time solution.

Continuous dependence on initial data. Let us consider two initial data $f_0^1, f_0^2 \in \mathcal{M}_+(\Omega)$. Let $f_1, f_2, \in X$ be the corresponding solutions. Then, using the same procedure as in [7] and as in the previous paragraph, we can prove that

$$W_1(f_1, f_2) \le \exp^{-2\bar{\lambda}t} W_1(f_0^1, f_0^2) + 2\mathrm{Lip}(T)\frac{\lambda_m}{\lambda_M}(1 - \exp^{-\bar{\lambda}t}) W(f_1, f_2).$$

Then, if $\mathrm{Lip}(T) < \dfrac{1}{2}\dfrac{\lambda_M}{\lambda_m}$ we have that $W_1(f_1, f_2) \to 0$.

Existence and uniqueness of the equilibrium. We now show that if $\mathrm{Lip}(T) < \frac{1}{2}\frac{\lambda_m}{\lambda_M}$, then there exists a unique stationary equilibrium. Equilibria are time independent distribution functions $f^\infty(z) \in \mathcal{M}_+(\Omega)$ such that

$$Q^+(f^\infty, f^\infty) - \bar{\lambda} f^\infty = 0$$

i.e.

$$f^\infty = \frac{1}{\bar{\lambda}} Q^+(f, f).$$

Then, the proof, as in [7] and following the same calculations as before, relies on the Banach fixed point Theorem. It is easy to show that the operator $\frac{1}{\bar{\lambda}} Q^+$ maps $\mathcal{M}_+(\Omega)$ into itself and that

$$W_1(\frac{1}{\bar{\lambda}} Q_+(f, f), \frac{1}{\bar{\lambda}} Q_+(g, g)) \leq 2\mathrm{Lip}(T)\frac{\lambda_M}{\lambda_m} W_1(f, g).$$

Therefore, if $\mathrm{Lip}(T) \leq \frac{1}{2}\frac{\lambda_m}{\lambda_M}$, the operator $\frac{1}{\bar{\lambda}} Q^+$ is a contraction.

Attractiveness of the equilibrium. As a consequence of the continuous dependence on initial data, by choosing $f_1 = f$ and $f_2 = f^\infty$, we have that

$$W_1(f(\cdot, t), f^\infty(\cdot)) \leq \exp^{-\bar{\lambda}(1-2\frac{\lambda_M}{\lambda_m}\mathrm{Lip}(T))t} W_1(f_0, f^\infty).$$

Therefore, if $\mathrm{Lip}(T) \leq \frac{1}{2}\frac{\lambda_m}{\lambda_M}$ any solution $f \in C(X)$ converges to f^∞ in the W_1 norm.

A.4. Nanbu–Babovski algorithm

Algorithm 1: Nanbu–Babovski algorithm with mass transfer for model (5)–(6)–(8)–(46)–(48)

Data:
- $N \in \mathbb{N}$ total number of agents of the system;
- N_1^n, $N_2^n \in \mathbb{N}$ numbers of agents in $x = 1$, $x = 2$, respectively, at time $t^n := n\Delta t$ and v_1^n, v_2^n the microscopic states of agents in $x = 1$, $x = 2$, respectively, at time t^n;

Fix $\Delta t \leq \min\{\frac{1}{\max \lambda_{ij}}\}$;
for $n = 0, 1, 2, \ldots$ **do**
  Compute
  $$\rho_1^{MC,n} = \frac{N_1^n}{N}, \qquad \rho_2^{MC,n} = \frac{N_2^n}{N}, \qquad m_1^{MC,n} = \frac{1}{N_1^n}\sum_{k=1}^{N_1^n} v_k^n, \qquad m_2^{MC,n} = \frac{1}{N_2^n}\sum_{k=1}^{N_2^n} v_k^n;$$
  repeat
    Pick randomly two agents (x_i^n, v_i^n), (x_j^n, v_j^n) with $i \neq j$;
    for $h = i, j$ **do**
      Sample $\Theta \sim \mathrm{Bernoulli}(\lambda_{x_i^n x_j^n}\Delta t)$;
      if $\Theta = 1$ **then**
        for $\{x_i', x_j'\} \in \mathcal{I}_n^2$ **do**
          Sample $J \in \{1, 0\}$ with law $\mathrm{Prob}(J = 1) = P_{x_i^n, x_j^n}^{x_i' x_j'}$, $\mathrm{Prob}(J = 0) = 1 - P_{x_i^n, x_j^n}^{x_i' x_j'}$;
          if $J = 1$ **then**
            Set $(x_i^{n+1}, x_j^{n+1}) = (x_i', x_j')$;
            Set $(v_i^{n+1}, v_j^{n+1}) = (v_i', v_j')$ where (v_i', v_j') is given by (12)–(46) and **break**
      else
        Set $x_h^{n+1} = x_h^n$, $v_h^{n+1} = v_h^n$;
  until *no unused pairs of agents are left*;

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