Partial regularity for non-autonomous degenerate quasi-convex functionals with general growth

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Abstract

We study partial $C^{1,\alpha}$ – regularity of minimizers of quasi–convex variational integrals with non–standard growth. We assume in particular that the relevant integrands satisfy an Orlicz's type growth condition, i.e. a so–called general growth condition. Moreover, the functionals are supposed to be non–autonomous and possibly degenerate.

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1. Introduction

In this paper we study partial $C^{1,\alpha}$ – regularity of minimizers of non–autonomous variational integrals of the form

$$\mathcal{F}(u) = \int_{\Omega} f(x, Du(x))\, dx,$$

where Ω is a bounded open set with smooth boundary in $\mathbb{R}^n$ $(n \geq 2)$, $u(x) \in \mathbb{R}^N$ $(N \geq 1)$ and f is a possibly degenerate Caratheodory function featuring non–standard growth. The non–standard growth condition we consider in this paper is of Orlicz's type and we assume that f satisfies a Hölder continuity condition for the x variable.

Partial regularity of solutions of nonlinear elliptic systems or minimizers of variational integrals with vector–valued admissible functions is a classical and still active topic in the fields of partial differential equations and calculus of variations. In view of various examples (see for instance [35, 42] and the survey paper [34]), only partial regularity of minimizers of $\mathcal{F}$ in the vectorial case $(N > 1)$ is naturally expected if the integrand f dose not have the so–called Uhlenbeck's

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structure: $f(x,A) \equiv g(x,|A|)$. For instance, solutions to systems of the type $-\mathrm{div}(a(x)|Du|^{p-2}Du) = \mu$ are everywhere regular, provided data $a(x)$ and μ are regular enough; see for instance [43, 31]. As for partial regularity in the general quasi–convex case, we refer to [23, 1, 8] as far as functionals with standard p – growth are concerned. We note that the main approach in these papers is the blow–up technique, see [9, 2] for its origin. After then, a different technique based on the $\mathcal{A}$ – harmonic approximation was adopted in [16, 18, 30, 17]. The $\mathcal{A}$ – harmonic approximation was introduced in [22] where the approximation is carried out in L^2 by using a compactness argument in the Sobolev space $W^{1,2}$. In the same spirit, the p – harmonic approximation was obtained in [19]. On the other hand, using the Lipschitz truncation argument, the $\mathcal{A}$ – harmonic approximation in the Orlicz space and the G – harmonic approximation were proved in [12] and [14] respectively.
We note that the above results consider autonomous integrands f, i.e. $f(x,A) \equiv f(A)$, satisfying a non–degeneracy condition. For degenerate quasi–convex functionals with p – growth, Duzaar and Mingione in [20] proved partial $C^{1,\alpha}$ – regularity under the assumption that $f(x,A) \to |A|^p$ as $A \to \mathbf{0}$ formally. The corresponding parabolic result has been obtained in [5]. On the other hand, non–autonomous quasi–convex functionals with p – growth were systematically investigated by Foss and Mingione in [24], see also [6, 21, 39], and we also refer to [4] for degenerate non–autonomous quasi–convex functionals with p – growth, to [41] for non–degenerate quasi–convex functionals with (p,q) – growth and finally to [7] for non–degenerate quasi–convex functionals with (φ,ψ) – growth. Finally, classical papers on non-standard growth conditions featuring everywhere regularity results are those of Marcellini [32, 33]; for the non–autonomous case we instead mention [3].
We point out that all quasi–convex functionals with non–standard growth considered in the papers mentioned above are autonomous and non–degenerate. We note that the paper [12] considers degenerate quasi-convex functionals with general growth. However, in this paper, partial regularity is obtained only in a non–degenerate circumstance (which is connected with the inequality 43a) and still consider autonomous functionals. This leads us to study a degenerate non–autonomous problem. We also mention that in recent years, partial regularity results for non–autonomous elliptic systems or convex functionalss with non–standard growth have been obtained in [15, 25, 26, 27, 29, 36, 37, 38].

Statement of the main result. We now turn to the hypotheses on the integral functional that we are going to consider throughout the paper. We refer to the next Section 2 for the notation.
Let $G\colon [0,+\infty) \to \mathbb{R}$ be a function such that

(G1) $G \in C^1([0,+\infty)) \cap C^2((0,+\infty))$;

(G2) $G(0) = 0$, $G'(0) = 0$, $G'(t) > 0$ for $t > 0$ and $\lim_{t\to+\infty} G'(t) = +\infty$;

(G3) $0 < g_1 - 1 \le \inf_{t>0} \dfrac{tG''(t)}{G'(t)} \le \sup_{t>0} \dfrac{tG''(t)}{G'(t)} \le g_2 - 1$;

for suitable constants $1 < g_1 \le g_2$. Without loss of generality we can assume that

$$1 < g_1 < 2 < g_2.$$

In the sequel we shall refer to this set of assumptions as hypotheses (G).
Every function G satisfying the hypotheses (G) is an N – function and, given a bounded open set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, we denote by $W^{1,G}(\Omega, \mathbb{R}^N)$ and $W^{1,G}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ the corresponding Orlicz–Sobolev spaces endowed with the usual norm and seminorms. We denote also by $W^{1,G}_0(\Omega, \mathbb{R}^N)$ the closure of $\mathcal{D}(\Omega, \mathbb{R}^N)$ in the $W^{1,G}$ – norm.
With this function G, we associate a Caratheodory function $f \colon \Omega \times \mathbb{M}^{N\times n} \to \mathbb{R}$ with the following properties:

(A0) **differentiability**: for every $x \in \Omega$, the function

$$A \in \mathbb{M}^{N\times n} \mapsto f(x, A)$$

is of class $C^1(\mathbb{M}^{N\times n}) \cap C^2(\mathbb{M}^{N\times n} \setminus \{\mathbf{0}\})$;
and throughout the paper we agree to write $Df := D_A f$ and $D^2 f := D^2_A f$ for the first and second gradients of the mapping $A \in \mathbb{M}^{N\times n} \mapsto f(x, A)$ for fixed $x \in \Omega$. The other assumptions on f are the following:

(A1) **coercivity**: $x \in \Omega \mapsto f(x, \mathbf{0})$ is integrable and there exists $c_0 > 0$ such that

$$c_0 G(|A|) \le f(x, A) - f(x, \mathbf{0})$$

holds for every $x \in \Omega$ and $A \in \mathbb{M}^{N\times n}$;

(A2) **growth condition**: there exists $\Lambda > 0$ such that

$$|Df(x, A)| \le \Lambda G'(|A|) \quad \text{and} \quad |D^2 f(x, A)| \le \Lambda G''(|A|)$$

hold for every $x \in \Omega$ and $A \in \mathbb{M}^{N\times n}$ with $A \ne 0$;

(A3) **strict $W^{1,G}$ – quasiconvexity**: there exists $\lambda > 0$ such that

$$\int_B \big[f(y, A + D\varphi(x)) - f(y, A)\big]\, dx \ge \lambda \int_B G''(|A| + |D\varphi(x)|)|D\varphi(x)|^2\, dx$$

holds for every $y \in \Omega$ and $A \in \mathbb{M}^{N\times n}$ and for every open ball $B \subset \mathbb{R}^n$ and every test function $\varphi \in \mathcal{D}(B, \mathbb{R}^N)$;

(A4) **Hölder continuity assumption with respect to x**: there exist $\beta_0 \in (0, 1)$ and a continuous, concave modulus of continuity $\omega \colon [0, +\infty) \to [0, +\infty)$ with

$$\omega(r) \le c_\omega r^{\beta_0} \qquad \text{for every } r \ge 0 \tag{1}$$

and $c_\omega \ge 0$ such that

$$|f(x_1, A) - f(x_2, A)| \le \omega(|x_1 - x_2|)\, G(|A|),$$

holds for every $x_i \in \Omega$ $(i = 1, 2)$ and $A \in \mathbb{M}^{N\times n}$;

(A5) **assumption for the non–degenerate case**: for the same $\beta_0 \in (0, 1)$ in (A4) there exists $c_1 > 0$ such that

$$|D^2 f(x, A) - D^2 f(x, A + B)| \leq c_1\, G''(|A|) \left(\frac{|B|}{|A|}\right)^{\beta_0}$$

holds for every $x \in \Omega$ and for every $A, B \in \mathbb{M}^{N\times n}$ with $0 < |B| \leq |A|/2$;

(A6) **assumption for the degenerate case**: for every $x \in \Omega$, the limit

$$\lim_{t\to 0^+} \frac{Df(x, tA)}{G'(t)} = A$$

exists uniformly with respect to $A \in \mathbb{M}^{N\times n}$ with $|A| = 1$ and for the same $\beta_0 \in (0, 1)$ and $c_1 > 0$ in (A4) and (A5), the inequality

$$|G''(s) - G''(s + t)| \leq c_1\, G''(s) \left(\frac{t}{s}\right)^{\beta_0} \tag{2}$$

holds for every $0 < t \leq s/2$.

Without loss of generality in (A4) we can assume that

$$c_\omega = 1 \quad \text{and} \quad \omega(r) \leq 1 \quad \text{for every } r \geq 0.$$

We note also that the growth condition (A2) implies that

$$f(x, A) - f(x, \mathbf{0}) \leq |A| \int_0^1 |Df(x, tA)|\, dt \leq \Lambda\, |A| \int_0^1 G'(t|A|)\, dt = \Lambda\, G(|A|). \tag{3}$$

Moreover, the hypothesis of strict $W^{1,G}$–quasiconvexity (A3) implies that the following Legendre–Hadamard condition

$$\langle D^2 f(x, A)\, \eta \otimes \xi \,|\, \eta \otimes \xi\rangle \geq \lambda'\, G''(|A|)|\eta|^2|\xi|^2, \qquad \eta \in \mathbb{R}^N \;\text{ and }\; \xi \in \mathbb{R}^n,$$

holds for every $x \in \Omega$ and $A \in \mathbb{M}^{N\times n}$ for some constant $\lambda' = \lambda'(\lambda, g_1, g_2) > 0$. Finally, in view of (A4), we see that $f(x_1, \mathbf{0}) = f(x_2, \mathbf{0})$ for all $x_1, x_2 \in \Omega$ which means that $f(x, \mathbf{0}) = a$ for every x and for some constant $a \in \mathbb{R}$. Therefore, since the minimization of $\mathcal{F}$ is unaffected by adding a constant to f, without loss of generality we always assume that

$$f(x, \mathbf{0}) = 0, \qquad x \in \Omega.$$

To every function $f\colon \Omega \times \mathbb{M}^{N\times n} \to \mathbb{R}$ satisfying (A0)–(A6) we associate the corresponding variational integral

$$\mathcal{F}(u, \Omega) := \int_\Omega f(x, Du(x))\, dx, \qquad u \in W^{1,G}(\Omega, \mathbb{R}^N), \tag{4}$$

which is well defined for all functions $u \in W^{1,G}(\Omega, \mathbb{R}^N)$ because of (A1) and (3). We can then state the main result of the paper.

Theorem 1.1. *Let G satisfy the hypotheses (G) with $1 < g_1 < 2 < g_2$, $f\colon \Omega \times \mathbb{M}^{N\times n} \to \mathbb{R}$ be a Caratheodory function such that the hypotheses* (A0) – (A6) *hold and let $u \in W^{1,G}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a local minimizer of $\mathcal{F}$. Then, there exist $\gamma = \gamma(n, N, g_1, g_2, c_0, c_1, \Lambda, \beta_0) \in (0, 1)$ and an open set $\Omega_u \subset \Omega$ with $|\Omega \backslash \Omega_u| = 0$ such that $V(Du) \in C^{\gamma}(\Omega_u, \mathbb{M}^{N\times n})$ and so $Du \in C^{2\gamma/g_2}(\Omega_u, \mathbb{M}^{N\times n})$.*

It turns out that the singular set $\Omega \setminus \Omega_u$ is contained in the set $\Sigma_+ \cup \Sigma_\infty$ where

$$\Sigma_+ = \left\{ x_0 \in \Omega : \ \liminf_{r\to 0^+} \fint_{B_r(x_0)} |V(Du) - (V(Du))_{x_0,r}|^2 \, dx > 0 \right\};$$

$$\Sigma_\infty = \left\{ x_0 \in \Omega : \ \limsup_{r\to 0^+} \fint_{B_r(x_0)} |V(Du)|^2 \, dx = +\infty \right\};$$

and the function $V\colon \mathbb{M}^{N\times n} \to \mathbb{M}^{N\times n}$ is defined by (11).

2. Notation and preliminary results

In this section we introduce the notation that we are going to use throughout the paper and we recall some preliminary results.

Notation. We denote the norm of a vector $x \in \mathbb{R}^n$ by $|x|$ and the open ball in $\mathbb{R}^n$ with center at $x_0 \in \mathbb{R}^n$ and radius $r > 0$ by $B_r(x_0)$ and we briefly write B_r instead of $B_r(x_0)$ when the center x_0 is immaterial or evident by the context. We also write $A \Subset B$ to mean that the closure $\overline{A}$ of A is compact and contained in B.
We denote the space of $N\times n$ matrices by $\mathbb{M}^{N\times n}$ and denote by $C^{k,\alpha}(\Omega, \mathbb{R}^N)$ ($k \in \mathbb{N} \cup \{0\}$ and $\alpha \in (0, 1]$) the spaces of functions which are α – Hölder continuous (when $k = 0$) or have α – Hölder continuous derivatives of order k on Ω (when $k > 0$). We denote also by $\mathcal{D}(\Omega, \mathbb{R}^N)$ the spaces of vector valued test functions on Ω respectively.
We denote the (Lebesgue) measure of a measurable set E in $\mathbb{R}^n$ by $|E|$ and for an integrable scalar or vector valued function $u\colon E \to \mathbb{R}^N$ with $|E| > 0$ we denote the average of u over E by

$$(u)_E := \fint_E u \, dx = \frac{1}{|E|} \int_E u \, dx.$$

We briefly write $(u)_{x_0,r}$ or even $(u)_r$ when $E = B_r(x_0)$.
Finally, given two functions $\varphi, \psi\colon A \to \mathbb{R}$, we write $\varphi \simeq \psi$ to mean that

$$L^{-1}\varphi(t) \le \psi(t) \le L\varphi(t), \qquad t \in A,$$

for suitable constants $L \ge 1$. If this is the case, we say that the functions φ and ψ are equivalent.

Orlicz functions. We begin by recalling the notion of Orlicz N – functions. We refer to [40] for details and proofs.

Let $G\colon [0,+\infty) \to [0,+\infty)$ be an N – function, i.e. G is defined by

$$G(t) = \int_0^t g(s)\,ds, \qquad t \geq 0,$$

for some right–continuous and increasing function $g\colon [0,+\infty) \to [0,+\infty)$ such that $g(0) = 0$, $g(s) > 0$ for $s > 0$ and $g(s) \to +\infty$ as $s \to +\infty$. Thus, G is convex, superlinear and has a right derivative $G'_+(t) = g(t)$ at every point $t \geq 0$. The conjugate function $G^*\colon [0,+\infty) \to [0,+\infty)$ of N – function G is the function defined by

$$G^*(t) := \sup_{s\geq 0} \left[st - G(s) \right], \qquad t \geq 0.$$

Then G^* is also an N – function, and G and G^* together satisfy the following Young's inequality

$$st \leq G(s) + G^*(t), \qquad s,t \geq 0.$$

The N – function G satisfies the so-called Δ_2 – *condition* if

$$G(2t) \leq c\,G(t), \qquad t \geq 0,$$

holds for some constant $c \geq 1$ in which case we write $G \in \Delta_2$ and $c = \Delta_2(G)$ for the optimal constant. As is well known, the Δ_2 – condition holds if and only if the inequality

$$G(at) \leq c\,G(t), \qquad t \geq 0,$$

holds for every $a > 0$ for some constant $c = c(a) > 0$. Moreover, the N – function G satisfies the ∇_2 – *condition* when $G^* \in \Delta_2$ in which case we write $G \in \nabla_2$. If $G \in \Delta_2 \cap \nabla_2$, Young's inequality can be written as

$$st \leq \varepsilon\, G(s) + c(\varepsilon)\, G^*(t) \quad \text{or} \quad st \leq \tilde{c}(\varepsilon)\, G(s) + \varepsilon\, G^*(t), \qquad s,t \geq 0,$$

for every $\varepsilon > 0$ and suitable constants $c(\varepsilon), \tilde{c}(\varepsilon) > 0$.
The following proposition examines the relation between N – functions and the hypotheses (G). The proof is elementary and well known.

Proposition 2.1. *Let $G\colon [0,+\infty) \to \mathbb{R}$ be a function such that the hypotheses* (G) *hold. Then,*

(a) G is an N – function and

$$g_1 \leq \inf_{t>0} \frac{tG'(t)}{G(t)} \leq \sup_{t>0} \frac{tG'(t)}{G(t)} \leq g_2; \tag{5}$$

(b) the mappings

$$t \in (0,+\infty) \mapsto \frac{G'(t)}{t^{g_1-1}},\ \frac{G(t)}{t^{g_1}} \quad \textit{and} \quad t \in (0,+\infty) \mapsto \frac{G'(t)}{t^{g_2-1}},\ \frac{G(t)}{t^{g_2}}$$

are increasing and decreasing respectively;

(c) *the following inequalities hold for every $t \geq 0$:*

$$a^{g_2}G(t) \leq G(at) \leq a^{g_1}G(t) \quad \textit{and} \quad a^{g_2-1}G'(t) \leq G'(at) \leq a^{g_1-1}G'(t) \quad \textit{if} \;\; 0 < a \leq 1;$$
$$a^{g_1}G(t) \leq G(at) \leq a^{g_2}G(t) \quad \textit{and} \quad a^{g_1-1}G'(t) \leq G'(at) \leq a^{g_2-1}G'(t) \quad \textit{if} \;\; a \geq 1.$$

In particular, it follows from (c) that both G and G^* satisfy the Δ_2 – condition with constants $\Delta_2(G)$ and $\Delta_2(G^*)$ determined by g_1 and g_2. Moreover, we have

$$G(t) \simeq tG'(t); \quad G(t) \simeq t^2G''(t); \quad G^*(G'(t)) \simeq G^*(G(t)/t)) \simeq G(t); \tag{6}$$

for $t > 0$ and for the inverse function G^{-1} and for $G' \circ G^{-1}$ the following inequalities hold:

$$a^{1/g_1}G^{-1}(t) \leq G^{-1}(at) \leq a^{1/g_2}G^{-1}(t); \tag{7a}$$
$$(g_1/g_2)a^{1-1/g_2}G'(G^{-1}(t)) \leq G'(G^{-1}(at)) \leq (g_2/g_1)a^{1-1/g_1}G'(G^{-1}(t)) \tag{7b}$$

for every $t \geq 0$ with $0 < a \leq 1$. By exchanging the role of g_1 and g_2 the same inequalities hold for $a \geq 1$.
Then, we present (reversed) Jensen's and Sobolev–Poincaré's type inequalities for the N – functions satisfying the hypotheses (G). In fact, the following estimates still hold for N – functions satisfying Δ_2 – and ∇_2 – conditions.

Lemma 2.2. *Let $G\colon [0, +\infty) \to [0, +\infty)$ be an N – function satisfying satisfying the hypotheses* (G)*.*

(a) *If $u \in L^1(B_r, \mathbb{R}^N)$, then*

$$\fint_{B_r} \big[G(|u|)\big]^{1/g_2} \, dx \leq 2 \left[G\left(\fint_{B_r} |u| \, dx\right)\right]^{1/g_2}.$$

(b) *There exist $\theta = \theta(n, g_1, g_2) \in (0, 1)$ and $c = c(n, N, g_1, g_2) > 0$ such that the inequality*

$$\fint_{B_r} G\left(\frac{|u - (u)_r|}{r}\right) dx \leq c \left(\fint_{B_r} \big[G(|Du|)\big]^{\theta} \, dx\right)^{1/\theta}$$

holds for every function $u \in W^{1,1}(B_r, \mathbb{R}^N)$.

Proof. (a) Let $H(t)\colon [0, +\infty) \to [0, +\infty)$ be the function defined by

$$H(t) = [G(t)]^{1/g_2}, \qquad t \geq 0.$$

Then, H is increasing whereas the function $t \in (0, +\infty) \mapsto H(t)/t$ is decreasing by Proposition 2.1 (b). Therefore, [37, Lemma 2.2] yields the existence of a concave function $K\colon [0, +\infty) \to [0, +\infty)$ such that

$$\frac{1}{2}K(t) \leq H(t) \leq K(t), \qquad t \geq 0,$$

and by Jensen's inequality we get

$$\fint_{B_r} H(|u|)\,dx \le \fint_{B_r} K(|u|)\,dx \le K\left(\fint_{B_r} |u|\,dx\right) \le 2H\left(\fint_{B_r} |u|\,dx\right).$$

(b) It follows from [10, Theorem 7]. $\square$

Auxiliary functions. Let G be an N – function. Following [10], for $a \ge 0$ we define the shifted function

$$G_a(t) := \int_0^t \frac{G'(a+s)}{a+s} s\,ds, \qquad t \ge 0. \tag{8}$$

Note that all shifted functions are also N – functions such that G_a and $G_a^* := (G_a)^*$ satisfy the Δ_2 – condition uniformly with respect to $a \ge 0$. In particular, if G satisfies the hypotheses (G) we have the following properties.

Proposition 2.3. *Suppose that the N – function G satisfies the hypotheses* (G) *with $1 < g_1 < 2 < g_2$ and let G_a $(a \ge 0)$ be the shifted functions of G defined in* (8)*. Then, for $b \ge 1$ we have*

$$b^{g_1} G_a(t) \le G_a(bt) \le b^{g_2} G_a(t), \qquad t \ge 0, \tag{9}$$

and the following relations

$$G_a(t) \simeq G_a'(t)t; \tag{10a}$$

$$G_a(t) \simeq G''(a+t)t^2 \simeq \frac{G(a+t)}{(a+t)^2}t^2 \simeq \frac{G'(a+t)}{a+t}t^2; \tag{10b}$$

$$G(a+t) \simeq [G_a(t) + G(a)]; \tag{10c}$$

hold uniformly with respect to $a \ge 0$. Here relevant constants depend only on g_1 and g_2.

Proof. Though these properties have been already used in [10, 13, 12], we give detailed proofs for the sake of completeness. In particular, we assume that $1 < g_1 < 2 < g_2$.
We first observe that

$$\left(\frac{a+bs}{a+s}\right)^{g_2-2} = \left(b - \frac{(b-1)a}{a+s}\right)^{g_2-2} \le b^{g_2-2} \qquad \text{and} \qquad \left(\frac{a+bs}{a+s}\right)^{g_1-2} \ge b^{g_1-2}$$

for every $s > 0$. Then, we see by Proposition 2.1 (b) that

$$G_a(bt) = b^2 \int_0^t \frac{G'(a+bs)}{a+bs} s\,ds \le b^2 \int_0^t \frac{G'(a+s)}{a+s}\left(\frac{a+bs}{a+s}\right)^{g_2-2} s\,ds \le b^{g_2} \int_0^t \frac{G'(a+s)}{a+s} s\,ds \le b^{g_2} G_a(t)$$

which proves the second inequality in (9). The first one is similar.
The inequalities in (9) imply that the mappings $t \mapsto G_a(t)/t^{g_1}$ and $t \mapsto G_a(t)/t^{g_2}$ are increasing and decreasing respectively. Therefore, by differentiating these

mappings, we obtain that $g_1 G_a(t) \le G'_a(t)t \le g_2 G_a(t)$ for $t > 0$ and this proves (10a). From (10a), the definition of G_a and (G3) we get

$$G_a(t) \simeq G'_a(t)t = \frac{G'(a+t)}{a+t} t^2 \simeq G''(a+t)t^2$$

which is the first equivalence in (10b). The others follow from (G3) again. We are thus left to prove (10c). By the very definition of G_a we have

$$G_a(t) \le G(a+t) - G(a), \qquad t \ge 0.$$

On the one hand, if $0 \le a \le t$, we have $t \le a+t \le 2t$ and from Proposition 2.1 (c) and (G3) we get

$$G_a(t) \ge \frac{1}{2t} \int_{t/2}^{t} G'(s)s\, ds \ge \frac{t}{8} G'(t/2) \ge \frac{g_1}{4^{g_2+1}} G(2t) \ge \frac{g_1}{4^{g_2+1}} G(a+t)$$

which gives

$$G(a+t) \simeq G_a(t) + G(a)$$

for $0 \le a \le t$. On the other hand, if $a \ge t$, we have $a \le a+t \le 2a$ and hence from Proposition 2.1 (c) again we get

$$G(a) \ge \frac{1}{2^{g_2}} G(2a) \ge \frac{1}{2^{g_2}} G(a+t)$$

and this completes the proof of (10c). □

To the function G, we associate the matrix–valued function $V\colon \mathbb{M}^{N\times n} \to \mathbb{M}^{N\times n}$ defined by

$$V(A) := \sqrt{\frac{G'(|A|)}{|A|}} A, \qquad A \in \mathbb{M}^{N\times n}. \tag{11}$$

Then,

$$|V(A)|^2 \simeq \sqrt{G''(|A|)}|A| \simeq G(|A|),$$

and the following relations between V and G

$$|V(A) - V(B)|^2 \simeq G_{|A|}(|A-B|) \tag{12}$$

hold for every $A, B \in \mathbb{M}^{N\times n}$ (see [12, Lemma 7]). Moreover, recalling f in the preceding section with the second inequality in (A2), we also have

$$|Df(x,A) - Df(x,B)| \le c\, G'_{|A|}(|A-B|) = c\, \frac{G'(|A| + |A-B|)}{|A| + |A-B|}|A-B|. \tag{13}$$

for every x and every $A, B \in \mathbb{M}^{N\times n}$ (see [12, (2.14)]).

Basic estimates. In this part we recall Caccioppoli inequality and local and global higher integrability results for local minimizers of the integral functional $\mathcal{F}$ defined by (4) where G satisfies hypotheses (G) and f is a Caratheodory function satisfying (A0) and (3) (with $f(x,\mathbf{0}) \equiv 0$) only.

Theorem 2.4. *Let $u \in W^{1,G}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a local minimizer of $\mathcal{F}$. Then, the following inequality*

$$\fint_{B_\rho} G(|Du|)\, dx \le c \fint_{B_r} G\left(\frac{|u-\xi|}{r-\rho}\right) dx$$

holds for every $\xi \in \mathbb{R}^N$ and for every pair of concentric balls $B_\rho \Subset B_r \Subset \Omega$ with some constant $c = c\,(n, N, g_1, g_2, c_0, \Lambda) > 0$.

Theorem 2.5. *Let $u \in W^{1,G}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a local minimizer of $\mathcal{F}$. There exists $\kappa_1 = \kappa_1(n, N, g_1, g_2, c_0, \Lambda) > 0$ such that $G(|Du|) \in L^{1+\kappa_1}_{\mathrm{loc}}(\Omega)$ and the inequality*

$$\fint_{B_\rho} \big[G(|Du|)\big]^{1+\kappa}\, dx \le c \left(\frac{r}{r-\rho}\right)^{n\kappa} \left(\fint_{B_r} G\left(|Du|\right)\, dx\right)^{1+\kappa} \tag{14}$$

holds for every $\kappa \in [0, \kappa_1]$ and for every pair of concentric balls $B_\rho \Subset B_r \Subset \Omega$ with some constant $c = c\,(n, N, g_1, g_2, c_0, \Lambda) > 0$.

The proofs of Theorem 2.4 and Theorem 2.5 when $\rho = r/2$ are essentially the same as Step 1 in the proof of Corollary 3.4 below. In addition, Theorem 2.5 for general ρ can follow from the case $\rho = r/2$ by using a standard covering argument. Hence, we omit the proofs of these two theorems here.
The next result gives global higher integrability on balls of minimizers of $\mathcal{F}$. This can be shown by an argument similar to the one used in the above interior result, see for instance [28, Theorem 6.8].

Theorem 2.6. *Let $u_0 \in W^{1,G}(B_r, \mathbb{R}^N)$ and $\kappa_1 > 0$ be such that*

$$G(|Du_0|) \in L^{1+\kappa_1}(B_r)$$

and let the function $u \in W^{1,G}(B_r, \mathbb{R}^N)$ be a minimizer of $\mathcal{F}$ with $\Omega = B_r$ such that $u = u_0$ on ∂B_r. Then, there exists $\kappa_2 = \kappa_2(n, N, g_1, g_2, c_0, \Lambda, \kappa_1) \in (0, \kappa_1)$ such that the inequality

$$\int_{B_r} \big[G(|Du|)\big]^{1+\kappa}\, dx \le c \int_{B_r} \big[G(|Du_0|)\big]^{1+\kappa}\, dx \tag{15}$$

holds for every $\kappa \in [0, \kappa_2]$ with some constant $c = c\,(n, N, g_1, g_2, c_0, \Lambda) > 0$.

Harmonic approximation results. In this part we recall some harmonic approximation results in the setting of Orlicz functions.
We consider first a bilinear form $\mathcal{A}$ on $\mathbb{M}^{N\times n}$ which we assume to be strongly elliptic in the sense of Legendre–Hadamard, i.e.

$$\lambda_0|\eta|^2|\xi|^2 \le \langle \mathcal{A}(\eta\otimes\xi)\,|\,(\eta\otimes\xi)\rangle \le \Lambda_0|\eta|^2|\xi|^2, \qquad \eta \in \mathbb{R}^N \ \text{ and } \ \xi \in \mathbb{R}^n,$$

holds for some constants $\Lambda_0 \ge \lambda_0 > 0$. Then, for a given Sobolev function $v \in W^{1,1}(B_r, \mathbb{R}^N)$ on some open ball B_r, we let h be the $\mathcal{A}$–harmonic function which agrees with v on ∂B_r, i.e. $h \in W^{1,1}(B_r, \mathbb{R}^N)$ is the unique weak solution to the Dirichlet problem

$$\begin{cases} -\mathrm{div}\,(\mathcal{A}Dh) = 0 & \text{in } B_r \\ h \in v + W_0^{1,1}(B_r, \mathbb{R}^N). \end{cases} \tag{16}$$

As is well known, the solution h is smooth. Then, the following $\mathcal{A}$ – harmonic approximation result holds in the setting of Orlicz space.

Lemma 2.7. (*Modified version of [12, Theorem 14]*) *Let $\mathcal{A}$ be a bilinear form on $\mathbb{M}^{N\times n}$ as above and let $H\colon [0,+\infty) \to [0,\infty)$ be an N – function such that $H, H^* \in \Delta_2$ and let $\mu > 0$ and $p > 1$. Then, for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n, N, \Delta_2(H), \Delta_2(H^*), \lambda_0, \Lambda_0, p) > 0$ such that the following holds: if $v \in W^{1,H}(B_r, \mathbb{R}^N)$ satisfies*

$$\fint_{B_r} H(|Dv|)\, dx \le \left(\fint_{B_r} [H(|Dv|)]^p\, dx \right)^{\frac{1}{p}} \le H(\mu)$$

and the following almost $\mathcal{A}$ – harmonic condition

$$\left| \fint_{B_r} \langle \mathcal{A}Dv \mid D\varphi \rangle\, dx \right| \le \delta\mu \|D\varphi\|_\infty, \qquad \forall\, \varphi \in \mathcal{D}(B_r, \mathbb{R}^N),$$

the (unique) weak solution h to (16) is in $W^{1,H}(B_r, \mathbb{R}^N)$ and satisfies

$$\fint_{B_r} H\left(\frac{|h-v|}{r} \right) dx + \fint_{B_r} H(|Dh - Dv|)\, dx \le \varepsilon H(\mu). \tag{17}$$

The proof is exactly same as the proof of [12, Theorem 14] with $\psi = H$ and with

$$s, \quad \fint_{\tilde{B}} |Du|\, dx, \quad \fint_{\tilde{B}} H(|Du|)\, dx, \quad \fint_{B} [H(|Du|)]^s\, dx$$

replaced by p, μ, $H(\mu)$ and $[H(\mu)]^p$ respectively. We note also that if H satisfies the hypotheses (G) with constants g_1 and g_2, then in the above lemma δ actually depends on g_1 and g_2 instead of $\Delta_2(H)$ and $\Delta_2(H^*)$.
Then, we turn to the G – harmonic approximation. Let G satisfy the set of hypotheses (G) and let $g \in W^{1,G}(B_r, \mathbb{R}^N)$ be a G – harmonic map in some open ball B_r, i.e. g is a weak solution to

$$-\mathrm{div} \left(G'(|Dg|) \frac{Dg}{|Dg|} \right) = 0 \tag{18}$$

in B_r. Then, its gradient Du and $V(Du)$ are Hölder continuous due to the following decay estimate.

Lemma 2.8. *[13, Theorem 6.4] Let G satisfy the hypotheses (G) and (2) and let $g \in W^{1,G}(B_r, \mathbb{R}^N)$ be a G – harmonic map in the open ball B_r. Then, there exists $\gamma_0 = \gamma_0(n, N, g_1, g_2, c_1, \beta_0) > 0$ such that*

$$\fint_{B_{\tau r}} |V(Dg) - (V(Dg))_{B_{\tau r}}|^2\, dx \le c\, \tau^{2\gamma_0} \fint_{B_r} |V(Dg) - (V(Dg))_{B_r}|^2\, dx$$

holds for every $\tau \in (0,1)$ with some constant $c = c\,(n, N, g_1, g_2, c_1) > 0$.

The next lemma is a $G-$harmonic approximation result.

Lemma 2.9. *[14, Lemma 1.1] Let G satisfy the hypotheses* (G). *For every $\varepsilon \in (0,1)$ and $\theta \in (0,1)$, there exists $\delta = \delta(n, N, g_1, g_2, \varepsilon, \theta) > 0$ such that the following holds: if $v \in W^{1,G}(B_{4r}, \mathbb{R}^N)$ satisfies the following almost $G-$harmonic condition*

$$\left| \fint_{B_r} \langle G'(|Dv|) \frac{Dv}{|Dv|} \mid D\varphi \rangle \, dx \right| \leq \delta \left(\fint_{B_{4r}} G(|Dv|) \, dx + G(\|D\varphi\|_\infty) \right) \tag{19}$$

for all functions $\varphi \in \mathcal{D}(B_r, \mathbb{R}^N)$, then the (unique) weak solution $g \in W^{1,G}(B_r, \mathbb{R}^N)$ of (18) *subject to the Dirichlet boundary condition $g = v$ on ∂B_r satisfies*

$$\left(\fint_{B_r} |V(Dv) - V(Dg)|^{2\theta} dx \right)^{1/\theta} \leq \varepsilon \fint_{B_{4r}} G(|Dv|) \, dx. \tag{20}$$

We note that the estimate (20) can be improved when $G(|Dv|)$ satisfies a reverse Hölder inequality.

Corollary 2.10. *Let G satisfy the hypotheses* (G) *and let $v \in W^{1,G}(B_{4r}, \mathbb{R}^N)$ be such that*

$$\left(\fint_{B_r} [G(|Dv|)]^{1+\kappa_1} dx \right)^{\frac{1}{1+\kappa_1}} \leq \tilde{c}_0 \fint_{B_{4r}} G(|Dv|) \, dx \tag{21}$$

for $\kappa_1, \tilde{c}_0 > 0$. Then, for every $\varepsilon \in (0,1)$ there exists $\delta_0 = \delta_0(n, N, g_1, g_2, \kappa_1, \tilde{c}_0, \varepsilon) > 0$ such that the following holds: if u satisfies the almost $G-$harmonic condition (19) *with δ replaced by δ_0, then the (unique) weak solution $g \in W^{1,G}(B_r, \mathbb{R}^N)$ of* (18) *subject to the Dirichlet boundary condition $g = v$ on ∂B_r satisfies*

$$\fint_{B_r} |V(Dv) - V(Dg)|^2 \, dx \leq \varepsilon \fint_{B_{4r}} G(|Dv|) \, dx. \tag{22}$$

Proof. Since g is a minimizer of $\mathcal{F}$ with $f(A) = G(|A|)$, from Theorem 2.6 we have that

$$\fint_{B_r} [G(|Dg|)]^{1+\kappa_2} dx \leq c \fint_{B_r} [G(|Dv|)]^{1+\kappa_2} dx \tag{23}$$

where $\kappa_2 \in (0, \kappa_1)$ depends on n, N, g_1, g_2 and κ_1. Then, for $\tau \in (0,1)$ defined by

$$1 = \frac{1-\tau}{2} + (1+\kappa_2)\tau, \tag{24}$$

applying Hölder inequality and Lemma 2.8 with $\theta = 1/2$, we have

$$\begin{aligned} &\fint_{B_r} |V(Dv) - V(Dg)|^2 dx \\ &\quad \leq \left(\fint_{B_r} |V(Dv) - V(Dg)| \, dx \right)^{(1-\tau)} \left(\fint_{B_r} |V(Dv) - V(Dg)|^{2(1+\kappa_2)} dx \right)^{\tau} \\ &\quad \leq \left(\varepsilon \fint_{B_{2r}} G(|Dv|) \, dx \right)^{(1-\tau)/2} \left(\fint_{B_r} |V(Dv) - V(Dg)|^{2(1+\kappa_2)} dx \right)^{\tau}. \end{aligned}$$

Since $|V(A)|^2 \simeq G(|A|)$, we have from (23) that

$$\fint_{B_r} |V(Dv)-V(Dg)|^{2(1+\kappa_2)} dx \le c \fint_{B_r} [G(|Dv|)]^{1+\kappa_2} dx \le c \left(\fint_{B_{4r}} G(|Dv|)\, dx \right)^{1+\kappa_2}.$$

Therefore, combining the last two estimates and using (24) we obtain (22). □

3. Caccioppoli's inequality and Ekeland's variational principle

We derive in this section special versions of Caccioppoli's inequality and Ekeland's variational principle which take into account the dependence of the integrand f on the x variable.
Throughout this section we assume that G and f satisfy the hypotheses (G) and (A0) – (A6) respectively, and that $\mathcal{F}$ is the integral functional defined by (4).

Caccioppoli's inequality and consequences. Let us first prove Caccioppoli's type inequality for local minimizers of $\mathcal{F}$ involving affine functions. This result is the x – dependent version of [12, Theorem 11].

Theorem 3.1. *Let $u \in W^{1,G}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a local minimizer of $\mathcal{F}$. Then, for every ball $B_{2r}(x_0) \Subset \Omega$ and for every affine function $L : \mathbb{R}^n \to \mathbb{R}^N$ defined by*

$$Lx = Q(x - x_0) + y_0, \qquad x \in \mathbb{R}^n,$$

with $Q \in \mathbb{M}^{N\times n}$ and $y_0 \in \mathbb{R}^N$, the following inequality holds:

$$\fint_{B_r(x_0)} G_{|Q|}(|Du-Q|)\, dx \le c \fint_{B_{2r}(x_0)} G_{|Q|}\left(\frac{|u-L|}{r}\right) dx + c\,\omega(2r) G(|Q|) \tag{25}$$

for some $c = c(n, N, g_1, g_2, c_0, \Lambda, \lambda) > 0$.

Proof. We assume that the center of balls is the origin and set $B_\rho = B_\rho(0)$ for $\rho > 0$. For $0 < r \le r_1 < r_2 \le 3r/2$ and $r_3 = (r_1+r_2)/2$, let $\eta \in \mathcal{D}(B_{r_3})$ be a cut–off function with $0 \le \eta \le 1$, $\eta = 1$ in B_{r_1} and $|D\eta| \le c/(r_3 - r_1) = 2c/(r_2 - r_1)$ and set $\varphi = \eta(u - L)$ and $\psi = (1 - \eta)(u - L)$ on Ω so that $\varphi + \psi = u - L$ and $D\varphi + D\psi = Du - Q$ a.e. on Ω. Then, in view of (10b) and of the strict $W^{1,G}$ – quasiconvexity assumption (A3) of f with φ as above, we have

$$\int_{B_{r_3}} G_{|Q|}(|D\varphi|)\, dx \le c \int_{B_{r_3}} G''(|Q| + |D\varphi|)|D\varphi|^2\, dx \le c \int_{B_{r_3}} [f(0, Q + D\varphi) - f(0, Q)]\, dx$$

and we write

$$\begin{aligned} f(0, Q + D\varphi(x)) - f(0, Q) \le\; & [f(0, Q + D\varphi(x)) - f(0, Q + D\varphi(x) + D\psi(x))] \\ & + [f(0, Du(x)) - f(x, Du(x))] \\ & + [f(x, Du(x)) - f(x, Du(x) - D\varphi(x))] \\ & + [f(x, Q + D\psi(x)) - f(0, Q + D\psi(x))] \\ & + [f(0, Q + D\psi(x)) - f(0, Q)] \end{aligned}$$

for a.e. $x \in \Omega$ so that the following estimate holds

$$\int_{B_{r_3}} G_{|Q|}(|D\varphi|)\,dx \le c\,(I_1 + I_2 + I_3 + I_4 + I_5)$$

with obvious meaning of $I_1, \dots, I_5$.
(i) Estimate of $I_1 + I_5$*.* We have from (13) that

$$\begin{aligned}
f(0,&Q + D\varphi) - f(0, Q + D\varphi + D\psi) + f(0, Q + D\psi) - f(0, Q)\\
&= -\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\,[f(0, Q + D\varphi + tD\psi) - f(0, Q + tD\psi)]\,dt\\
&= \int_0^1 \langle [Df(0, Q + tD\psi) - Df(0, Q)] - [Df(0, Q + D\varphi + tD\psi) - Df(0, Q)] \,|\, D\psi\rangle\,dt\\
&\le c\int_0^1 \Big[G'_{|Q|}(t|D\psi|) + G'_{|Q|}(|D\varphi| + t|D\psi|)\Big]\,|D\psi|\,dt\\
&\le c\,G_{|Q|}(|D\psi|) + c\,G'_{|Q|}(|D\varphi| + |D\psi|)|D\psi|.
\end{aligned}$$

By Young's inequality and (6), we have $sG'_a(t) \le c(\delta)G_a(s) + \delta G_a(t)$ for every $0 < \delta < 1$ uniformly with respect to a and hence we get

$$G'_{|Q|}(|D\varphi| + |D\psi|)|D\psi| \le \delta G_{|Q|}(|D\varphi| + |D\psi|) + c(\delta)G_{|Q|}(|D\varphi|) \le c\,\delta G_{|Q|}(|D\varphi|) + c(\delta)G_{|Q|}(|D\psi|).$$

Choosing $\delta > 0$ small enough, we have

$$I_1 + I_5 \le c\int_{B_{r_3}} G_{|Q|}(|D\psi|)\,dx + \frac{1}{2}\int_{B_{r_3}} G_{|Q|}(|D\varphi|)\,dx.$$

(ii) Estimate of $I_2 + I_4$*.* By (A4) and (10c) we have

$$\begin{aligned}
I_2 + I_4 &\le c\,\omega(2r)\int_{B_{r_3}} G(|Du|)\,dx + c\,\omega(2r)\int_{B_{r_3}} G(|Q| + |D\psi|)\,dx\\
&\le c\,\omega(2r)\int_{B_{r_3}} G(|Du|)\,dx + c\int_{B_{r_3}} G_{|Q|}(|D\psi|)\,dx + c\,\omega(2r)r^n G(|Q|).
\end{aligned}$$

(iii) Estimate of I_3*.* The minimality of u yields $I_3 \le 0$.
Combining the previous estimates we get

$$\int_{B_{r_3}} G_{|Q|}(|D\varphi|)\,dx \le c\int_{B_{r_3}} G_{|Q|}(|D\psi|)\,dx + c\,\omega(2r)\int_{B_{r_3}} G(|Du|)\,dx + c\,\omega(2r)r^n G(|Q|)$$

which, in view of the definition of φ and ψ, yields

$$\begin{aligned}
\int_{B_{r_1}} G_{|Q|}(|Du - Q|)\,dx \le c\int_{B_{r_3}\setminus B_{r_1}} G_{|Q|}(|Du - Q|)\,dx + c\int_{B_{r_3}\setminus B_{r_1}} G_{|Q|}\left(\frac{|u - L|}{r_2 - r_1}\right)dx\\
+ c\,\omega(2r)\int_{B_{r_3}} G(|Du|)\,dx + c\,\omega(2r)r^n G(|Q|).
\end{aligned}$$

In addition, exploiting Hölder's inequality and the higher integrability property (14) of u, we have

$$\int_{B_{r_3}} G(|Du|)\,dx \le c\left(\frac{r}{r_2-r_1}\right)^{\frac{n\kappa_1}{1+\kappa_1}} \int_{B_{r_2}} G(|Du|)\,dx.$$

Then, inserting this into the previous estimate, letting $r_1=\tau r$ and $r_2=tr$ with $1\le\tau<t\le 3/2$ and taking into account (9), we obtain

$$\begin{aligned}\int_{B_{\tau r}} G_{|Q|}(|Du-Q|)\,dx \le c_* \int_{B_{tr}\setminus B_{\tau r}} G_{|Q|}(|Du-Q|)\,dx + \frac{c}{(t-\tau)^{g_2}}\int_{B_{3r/2}} G_{|Q|}\left(\frac{|u-L|}{r}\right)dx\\ + \frac{c\,\omega(2r)}{(t-\tau)^{\frac{n\kappa_1}{1+\kappa_1}}}\int_{B_{3r/2}} G(|Du|)\,dx + c\,\omega(2r)r^n G(|Q|)\end{aligned}$$

for every t and τ as above which, by filling the hole, yields

$$\begin{aligned}\int_{B_{\tau r}} G_{|Q|}(|Du-Q|)\,dx \le \frac{c_*}{1+c_*} \int_{B_{tr}} G_{|Q|}(|Du-Q|)\,dx + \frac{c}{(t-\tau)^{g_2}}\int_{B_{3r/2}} G_{|Q|}\left(\frac{|u-L|}{r}\right)dx\\ + \frac{c\,\omega(2r)}{(t-\tau)^{\frac{n\kappa_1}{1+\kappa_1}}}\int_{B_{3r/2}} G(|Du|)\,dx + c\,\omega(2r)r^n G(|Q|).\end{aligned}$$

Therefore, in view of a standard iteration argument ([28, Lemma 6.1]) we get

$$\int_{B_r} G_{|Q|}(|Du-Q|)\,dx \le c\int_{B_{3r/2}} G_{|Q|}\left(\frac{|u-L|}{r}\right)dx + c\,\omega(2r)\int_{B_{3r/2}} G(|Du|)\,dx + c\,\omega(2r)r^n G(|Q|)$$

and we are left to get rid of the integral of $G(|Du|)$ at the right hand side. Exploiting the standard Caccioppoli's inequality (Theorem 2.4) with $\xi=L(0)=y_0$ and $3r/2$ and $2r$ in place of ρ and r and taking into account (10c) that

$$G\left(\frac{|u-y_0|}{r}\right) \le G\left(\frac{|u-L|}{r}+2|Q|\right) \le c\left[G_{|Q|}\left(\frac{|u-L|}{r}\right)+G(|Q|)\right], \qquad x\in B_{2r},$$

we get

$$\int_{B_{3r/2}} G(|Du|)\,dx \le c\int_{B_{2r}} G_{|Q|}\left(\frac{|u-L|}{r}\right)dx + c\,r^n G(|Q|).$$

Inserting this into the above estimate and recalling that $\omega(r)\le 1$, we get the desired estimate. □

We next exploit Gehring's lemma to obtain a reversed Hölder inequality for the local minimizer u.

Corollary 3.2. *Let $u\in W^{1,G}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ be a local minimizer of $\mathcal{F}$. There exists $\kappa\in(0,1)$ and $c>0$ depending on n, N, g_1, g_2, c_0, Λ and λ such that the* *following inequality*

$$\left(\fint_{B_r(x_0)} \left[G_{|Q|}(|Du-Q|)\right]^{1+\kappa} dx\right)^{1/(1+\kappa)} \leq c\, G_{|Q|}\left(\fint_{B_{2r}(x_0)} |Du-Q|\, dx\right) + c\,\omega(2r)G(|Q|) \tag{26}$$

holds for every ball $B_{2r}(x_0) \Subset \Omega$ *and for every matrix* $Q \in \mathbb{M}^{N\times n}$.

Proof. Let $B_{2\rho}(y) \subset B_{2r}(x_0)$ and $L : \mathbb{R}^n \to \mathbb{R}^N$ be the affine function defined by

$$Lx = Q(x-x_0) + (u)_{y,2\rho}, \qquad x \in \mathbb{R}^n.$$

Since $(u-L)_{y,2r} = 0$, Poincaré's inequality (Theorem 2.2 (b)) applied to $u-L$ gives

$$\fint_{B_{2\rho}(y)} G_{|Q|}\left(\frac{|u-L|}{\rho}\right) dx \leq c\left(\fint_{B_{2\rho}(y)} \left[G_{|Q|}(|Du-Q|)\right]^{\theta} dx\right)^{1/\theta}$$

for some $\theta \in (0,1)$ and so, from (25), we have

$$\fint_{B_\rho(y)} G_{|Q|}(|Du-Q|)\, dx \leq c\left(\fint_{B_{2\rho}(y)} \left[G_{|Q|}(|Du-Q|)\right]^{\theta} dx\right)^{1/\theta} + c\,\omega(2\rho)G(|Q|).$$

Hence, the inequality

$$\fint_{B_\rho(y)} G_{|Q|}(|Du-Q|)\, dx \leq c\left(\fint_{B_{2\rho}(y)} \left[G_{|Q|}(|Du-Q|)\right]^{\theta} dx\right)^{1/\theta} + c\,\omega(2r)G(|Q|)$$

holds on every ball $B_{2\rho}(y) \subset B_{2r}(x_0)$ and therefore, being the last summand on the right independent of the choice of the ball, Gehring's lemma ([28, Theorem 6.7] for instance) yields the existence of $\kappa > 0$ such that

$$\left(\fint_{B_\rho(y)} \left[G_{|Q|}(|Du-Q|)\right]^{1+\kappa} dx\right)^{1/(1+\kappa)} \leq c \fint_{B_{2\rho}(y)} G_{|Q|}(|Du-Q|)\, dx + c\,\omega(2r)G(|Q|)$$

holds on every ball $B_{2\rho}(y) \subset B_{2r}(x_0)$. Hence applying the same argument of [28, Remark 6.12] we have in particular that

$$\left(\fint_{B_r(x_0)} \left[G_{|Q|}(|Du-Q|)\right]^{1+\kappa} dx\right)^{1/(1+\kappa)} \leq c\left(\fint_{B_{2r}(x_0)} \left[G_{|Q|}(|Du-Q|)\right]^{1/g_2} dx\right)^{g_2} + c\,\omega(2r)G(|Q|)$$

Finally, applying Lemma 2.2 (a), we obtain formula (26). □

From now on, we define

$$E(x_0,\rho,Q) := \fint_{B_\rho(x_0)} G_{|Q|}(|Du-Q|)\, dx;$$

$$\Psi(x_0,\rho) := \fint_{B_\rho(x_0)} G\left(\frac{|u-(u)_{x_0,\rho}|}{\rho}\right) dx; \tag{27}$$

for every $x_0 \in \Omega$ and $\rho > 0$ such that $B_\rho(x_0) \Subset \Omega$ and we note that following relations

$$E(x_0, \rho, Q) \simeq \fint_{B_\rho(x_0)} |V(Du) - V(Q)|^2 \, dx; \tag{28a}$$

$$\Psi(x_0, \rho) \leq c \fint_{B_\rho(x_0)} G(|Du|) \, dx; \tag{28b}$$

hold by (12) and by Poincaré's inequality respectively.

Ekeland's variational principle. The x – dependence of f can be dealt with by a freezing argument based on Ekeland's variational principle ([28, Theorem 5.6]).
In this part, we fix a ball $B_\rho(x_0) \Subset \Omega$ and we set

$$f_0(A) = f(x_0, A), \qquad A \in \mathbb{M}^{N \times n}, \tag{29}$$

and

$$K(x_0, \rho) = \omega(\rho)\Psi(x_0, \rho) \qquad \text{and} \qquad K_0(x_0, \rho) = \frac{K(x_0, \rho)}{G^{-1}(K(x_0, \rho))}. \tag{30}$$

Theorem 3.3. *Let $u \in W^{1,G}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a local minimizer of $\mathcal{F}$. Then, for every ball $B_\rho(x_0) \Subset \Omega$ there exists $v \in u + W^{1,G}_0(B_{\rho/2}(x_0), \mathbb{R}^N)$ such that*

$$\fint_{B_{\rho/2}(x_0)} f_0(Dv) \, dx \leq \fint_{B_{\rho/2}(x_0)} f_0(Du) \, dx, \tag{31}$$

$$\fint_{B_{\rho/2}(x_0)} |Du - Dv| \, dx \leq c \, G^{-1}(K(x_0, \rho)) \tag{32}$$

for some $c = c\,(n, N, g_1, g_2, c_0, \Lambda, \lambda) > 0$ and the following inequality

$$\fint_{B_{\rho/2}(x_0)} f_0(Dv) \, dx \leq \fint_{B_{\rho/2}(x_0)} f_0(Dw) \, dx + K_0(x_0, \rho) \fint_{B_{\rho/2}(x_0)} |Dw - Dv| \, dx \tag{33}$$

holds for every function $w \in u + W^{1,G}_0(B_{\rho/2}(x_0), \mathbb{R}^N)$ with $K_0(x_0, \rho)$ defined by (30)*. Moreover, the following inequality*

$$\left| \fint_{B_{\rho/2}(x_0)} \langle Df_0(Dv) \,|\, D\varphi \rangle \, dx \right| \leq K_0(x_0, \rho) \fint_{B_{\rho/2}(x_0)} |D\varphi| \, dx, \tag{34}$$

holds for every $\varphi \in W^{1,G}_0(B_{\rho/2}(x_0), \mathbb{R}^N)$.

Proof. Since the point x_0 is fixed throughout the proof, we briefly write B_ρ, $(u)_\rho$, $\Psi(\rho)$ and so on omitting the dependence on x_0 and we let $\tilde{v} \in u + W^{1,G}_0(B_{\rho/2}, \mathbb{R}^N)$ be a minimizer of

$$\mathcal{F}_0(w) = \mathcal{F}_0(w, B_{\rho/2}) := \fint_{B_{\rho/2}} f_0(Dw) \, dx, \qquad w \in u + W^{1,G}_0(B_{\rho/2}, \mathbb{R}^N).$$

We want to estimate $\mathcal{F}_0(u) - \mathcal{F}_0(\tilde{v})$ and to this aim we write

$$\mathcal{F}_0(u) - \mathcal{F}_0(\tilde{v}) = \left[\mathcal{F}_0(u) - \frac{1}{|B_{\rho/2}|}\mathcal{F}(u, B_{\rho/2})\right] + \left[\frac{1}{|B_{\rho/2}|}\mathcal{F}(u, B_{\rho/2}) - \mathcal{F}_0(\tilde{v})\right].$$

As to the first summand, by (A4) and the standard Caccioppoli inequality (Theorem 2.4) we have

$$\mathcal{F}_0(u) - \frac{1}{|B_{\rho/2}|}\mathcal{F}(u, B_{\rho/2}) = \fint_{B_{\rho/2}} \big[f(x_0, Du) - f(x, Du)\big]\, dx$$
$$\leq c\,\omega(\rho) \fint_{B_{\rho/2}} G(|Du|)\, dx \leq c\,\omega(\rho) \fint_{B_\rho} G\left(\frac{|u-(u)_\rho|}{\rho}\right) dx = c\,\omega(\rho)\Psi(\rho).$$

We then turn to the second summand. Since u is a local minimizer of $\mathcal{F}$, we have

$$\frac{1}{|B_{\rho/2}|}\mathcal{F}(u, B_{\rho/2}) - \mathcal{F}_0(\tilde{v}) \leq \frac{1}{|B_{\rho/2}|}\mathcal{F}(\tilde{v}, B_{\rho/2}) - \mathcal{F}_0(\tilde{v}) = \fint_{B_{\rho/2}} [f(x, D\tilde{v}) - f(x_0, D\tilde{v})]\, dx$$
$$\leq c\,\omega(\rho) \fint_{B_{\rho/2}} G(|D\tilde{v}|)\, dx.$$

by (A4) again. Then, a standard energy estimate which exploits (A0) and (3) gives

$$\fint_{B_{\rho/2}} G(|D\tilde{v}|)\, dx \leq c \fint_{B_{\rho/2}} G(|Du|)\, dx$$

and hence by the same argument used above we conclude that

$$\frac{1}{|B_{\rho/2}|}\mathcal{F}(u, B_{\rho/2}) - \mathcal{F}_0(\tilde{v}) \leq c\,\omega(\rho)\Psi(\rho).$$

We have thus proved that

$$\mathcal{F}_0(u) \leq \mathcal{F}_0(\tilde{v}) + K(\rho) = \min\left\{\mathcal{F}_0(w) :\ w \in u + W_0^{1,G}(B_{\rho/2}, \mathbb{R}^N)\right\} + c_* K(\rho)$$

and finally, choosing the distance defined by

$$d(v_1, v_2) := \frac{1}{c_* G^{-1}(K(\rho))} \fint_{B_{\rho/2}} |Dv_1 - Dv_2|\, dx, \qquad v_1, v_2 \in u + W_0^{1,G}(B_{\rho/2}, \mathbb{R}^N),$$

Ekeland's variational principle yields a function $v \in u + W_0^{1,G}(B_{\rho/2}(x_0), \mathbb{R}^N)$ which satisfies (31), (32) and (33). (It is clear that $(u + W^{1,G_0}(B_{2\rho}), d)$ is a complete metric space and that $\mathcal{F}_0 : u + W^{1,G_0}(B_{2\rho}) \to \mathbb{R}$ is lower semicontinuous in this metric topology.) Moreover, since v is a minimizer of the functional

$$\mathcal{F}_d(w) = \fint_{B_{\rho/2}(x_0)} f_0(Dw)\, dx + K_0(x_0, \rho) \fint_{B_{\rho/2}(x_0)} |Dw - Dv|\, dx \tag{35}$$

defined for every function $w \in u + W_0^{1,G}(B_{\rho/2}(x_0), \mathbb{R}^N)$, it is a solution of the Euler–Lagrange system for the functional $\mathcal{F}_d$ whence (34) follows. □

Corollary 3.4. *Let $v \in u + W_0^{1,G}(B_{\rho/2}(x_0), \mathbb{R}^N)$ be as in Theorem 3.3. Then, for $\tau_1 = \tau_1(n, N, g_1, g_2, c_0, \Lambda) \in (0,1)$ defined by*

$$\frac{\tau_1}{1+\kappa_3} + (1-\tau_1) g_2 = 1, \tag{36}$$

where $\kappa_3 > 0$ is determined by (38) below, we have

$$\fint_{B_{\rho/4}(x_0)} G(|Du - Dv|)\, dx \le c\, [\omega(\rho)]^{1-\tau_1} \Psi(x_0, \rho) \tag{37}$$

for some $c = c\,(n, N, g_1, g_2, c_0, \Lambda, \lambda) > 0$.

Proof. We set $\Psi(\rho)$, $K(\rho)$ and so on as in the proof of Theorem 3.3.

Step 1. Higher integrability of Dv. We first prove that

$$\left(\fint_{B_{\rho/4}(x_0)} [G(|Dv|)]^{1+\kappa_3}\, dx \right)^{1/(1+\kappa_3)} \le c \fint_{B_{\rho/2}(x_0)} G(|Dv|)\, dx + cK(\rho) \tag{38}$$

for some $\kappa_3 = \kappa_3(n, N, \lambda, \Lambda, g_1, g_2, c_0, c_1) > 0$. Without loss of generality, we may assume that $\kappa_3 \le \kappa_1$ where κ_1 is the exponent determined in Theorem 2.5. Since v is a minimizer of $\mathcal{F}_d$ defined by (35), we have for every ball $B_{2s} = B_{2s}(y)$ with $B_{2s} \subset B_{\rho/2}(x_0)$ and for every $1 \le \tau < t \le 2$,

$$\begin{aligned}
\int_{B_{ts}} f_0(Dv)\, dx &\le \int_{B_{ts}} f_0(D[v - \eta(v - (v)_{B_{2s}})])\, dx + K_0(\rho) \int_{B_{ts}} |D[v - \eta(v - (v)_{B_{2s}})] - Dv|\, dx \\
&\le \int_{B_{ts}} f_0((1-\eta)Dv - (v - (v)_{B_{2s}})D\eta)\, dx + K_0(\rho) \int_{B_{ts}} |\eta Dv + (v - (v)_{B_{2s}})D\eta|\, dx,
\end{aligned}$$

where $\eta \in \mathcal{D}(B_{ts})$ with $0 \le \eta \le 1$ and $\eta = 1$ in $B_{\tau s}$. Then by (A1), (3) and Young's inequality with (6) and (30),

$$\begin{aligned}
\int_{B_{\tau s}} G(|Dv|)\, dx &\le c \int_{B_{ts}} \left[G((1-\eta)|Dv|) + G\left(\frac{|v - (v)_{B_{2s}}|}{(t-\tau)s} \right) \right] dx \\
&\qquad + c \int_{B_{ts}} \left[G(|Dv|) + G\left(\frac{|v - (v)_{B_{2s}}|}{(t-\tau)s} \right) + G^*(K_0(\rho)) \right] dx \\
&\le c_* \int_{B_{ts} \setminus B_{\tau s}} G(|Dv|)\, dx + \frac{c}{(t-\tau)^{g_2}} \int_{B_{2s}} G\left(\frac{|v - (v)_{B_{2s}}|}{s} \right) dx \\
&\qquad + c \int_{B_{2s}} G(|Dv|) + c\, K(\rho) |B_{ts}|
\end{aligned}$$

and so

$$\begin{aligned}
\int_{B_{\tau s}} G(|Dv|)\, dx &\le \frac{c_*}{1+c^*} \int_{B_{ts}} G(|Dv|)\, dx + \frac{c}{(t-\tau)^{g_2}} \int_{B_{2s}} G\left(\frac{|v - (v)_{B_{2s}}|}{s} \right) dx \\
&\qquad + c \int_{B_{2s}} G(|Dv|)\, dx + c\, K(\rho) |B_{2s}|.
\end{aligned}$$

Here we used the fact that

$$G^*(K_0(\rho)) = G^*\left(\frac{G(G^{-1}(K(\rho)))}{G^{-1}(K(\rho))}\right) \le c\, G(G^{-1}(K(\rho))) = c\, K(\rho).$$

Therefore, in view of a standard iteration argument ([28, Lemma 6.1]) and Poincaré's inequality (Lemma 2.2) we get

$$\begin{aligned}\fint_{B_s} G(|Dv|)\, dx &\le c \fint_{B_{2s}} G\left(\frac{|v-(v)_{B_{2s}}|}{s}\right) dx + c\fint_{B_{2s}} G(|Dv|)\, dx + cK(\rho)\\ &\le c\left(\fint_{B_{2s}} G(|Du|)^\theta\, dx\right)^{\frac{1}{\theta}} + c\fint_{B_{2s}} G(|Dv|)\, dx + cK(\rho)\end{aligned}$$

for every ball $B_{2s} \subset B_{\rho/2}(x_0)$. Therefore, in view of Gehring's lemma (see [28, Theorem 6.7]), we obtain (38).

Step 2. Proof of (37). We we omit the dependence on x_0. In view of the definition of τ_1, Hölder's inequality gives

$$\fint_{B_{\rho/4}} G(|Du-Dv|)\, dx \le \left(\fint_{B_{\rho/4}} [G(|Du-Dv|)]^{1+\kappa_3}\, dx\right)^{\frac{\tau_1}{1+\kappa_3}} \left(\fint_{B_{\rho/4}} [G(|Du-Dv|)]^{\frac{1}{g_2}}\, dx\right)^{(1-\tau_1)g_2}.$$

By Lemma 2.2 (a) and (32), the second factor on the right hand side above can be estimated such that

$$\left(\fint_{B_{\rho/4}} [G(|Du-Dv|)]^{\frac{1}{g_2}}\, dx\right)^{(1-\tau_1)g_2} \le c\left[G\left(\fint_{B_{\rho/4}} |Du-Dv|\, dx\right)\right]^{1-\tau_1} \le c\,[K(\rho)]^{1-\tau_1}.$$

As to the first factor, we have

$$\begin{aligned}&\left(\fint_{B_{\rho/4}} [G(|Du-Dv|)]^{1+\kappa_3}\, dx\right)^{\frac{\tau_1}{1+\kappa_3}}\\ &\quad\le c\left(\fint_{B_{\rho/4}} [G(|Du|)]^{1+\kappa_3}\, dx\right)^{\frac{\tau_1}{1+\kappa_3}} + c\left(\fint_{B_{\rho/4}} [G(|Dv|)]^{1+\kappa_3}\, dx\right)^{\frac{\tau_1}{1+\kappa_3}}\end{aligned}$$

For the first summand, the higher integrability inequality (14) with $\kappa_3 \le \kappa_1$ and Caccioppoli's inequality yield

$$\begin{aligned}\left(\fint_{B_{\rho/4}} [G(|Du|)]^{1+\kappa_3}\, dx\right)^{\frac{1}{1+\kappa_3}} &\le c\fint_{B_{\rho/2}} G(|Du|)\, dx\\ &\le c\fint_{B_\rho} G\left(\frac{|u-(u)_\rho|}{\rho}\right) dx = c\,\Psi(\rho).\end{aligned}$$

As to the second summand, since

$$\fint_{B_{\rho/2}} G(|Dv|)\, dx \le c\fint_{B_{\rho/2}} f_0(Dv)\, dx \le c\fint_{B_{\rho/2}} f_0(Du)\, dx \le c\fint_{B_{\rho/2}} G(|Du|)\, dx$$

by (A1), (3) and (31), we have from (38), Caccioppoli's inequality and the definitions of $K(\rho)$ and $\Psi(\rho)$ and Theorem 2.4 that

$$\begin{aligned}\left(\fint_{B_{\rho/4}} [G(|Dv|)]^{1+\kappa_3}\, dx\right)^{\frac{1}{1+\kappa}} &\le c \fint_{B_{\rho/2}} G(|Du|)\, dx + c\, K(\rho) \\ &\le c \fint_{B_\rho} G\left(\frac{|u-(u)_\rho|}{\rho}\right) dx + c\,\Psi(\rho) \le c\,\Psi(\rho).\end{aligned}$$

Finally, combining all the previous estimates we obtain

$$\fint_{B_{\rho/4}(x_0)} G(|Du-Dv|)\, dx \le c\,[\Psi(\rho)]^{\tau_1}[K(\rho)]^{1-\tau_1} = c\,[\omega(\rho)]^{1-\tau_1}\Psi(\rho)$$

and this completes the proof. □

4. Decay estimates via harmonic approximations

In this section we prove decay estimates for local minimizers via harmonic approximation in both non–degenerate and degenerate cases. These cases are distinguished by the inequalities (43a) and (61a) and, in order to obtain the decay estimates, we exploit the assumption (A5) for the non–degenerate case and (A6) for the degenerate case.

Throughout this section we assume that G and f satisfy the hypotheses (G) and (A0) – (A6) respectively and that $\mathcal{F}$ is the integral functional defined by (4). Moreover, we fix a ball $B_\rho(x_0) \Subset \Omega$, we let the functions f_0 be defined by (29) and $u \in W^{1,G}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a local minimizer of $\mathcal{F}$.

Non–degenerate case. For $Q \in \mathbb{M}^{N\times n}$, $Q \neq 0$, let $L\colon \mathbb{R}^n \to \mathbb{R}^N$ be the affine function defined by

$$Lx = Q(x-x_0) + (u)_{x_0,\rho}, \qquad x \in \mathbb{R}^n$$

and $\mathcal{A}\colon \mathbb{M}^{N\times n} \times \mathbb{M}^{N\times n} \to \mathbb{R}$ be the bilinear form defined by

$$\mathcal{A}(Q) := \frac{D^2 f_0(Q)}{G''(|Q|)}. \tag{39}$$

Then the following result holds.

Lemma 4.1. *Suppose that*

$$E(x_0,\rho,Q) \le G(|Q|). \tag{40}$$

Then, there exists $\beta_1 = \beta_1(n,N,g_1,g_2,c_0,\Lambda,\beta_0) \in (0,1/2)$ *such that*

$$\fint_{B_{\rho/4}(x_0)} \langle \mathcal{A}(Q)(Du-Q) \,|\, D\varphi\rangle\, dx \le c\left\{[\omega(\rho)]^{\beta_1} + \frac{E(x_0,\rho,Q)}{G(|Q|)} + \left(\frac{E(x_0,\rho,Q)}{G(|Q|)}\right)^{\frac{1+\beta_0}{2}}\right\}|Q|\|D\varphi\|_\infty$$

holds for every $\varphi \in \mathcal{D}(B_{\rho/4}(x_0), \mathbb{R}^N)$ *for some* $c = c(n,N,g_1,g_2,c_0,c_1,\Lambda,\lambda) >$ 0.

The exponent β_1 is given by

$$\beta_1 := \min\left\{\frac{1+\beta_0}{2g_2}, 1-\frac{1}{g_1}, \frac{(1-\tau_1)(1+\beta_0)}{2}\right\} < \frac{1}{2} \tag{41}$$

where τ_1 is defined by (36).

Proof. Since the point x_0 is fixed throughout the proof, we omit the dependence on x_0 and we write $B_{\rho/4}$, $\Psi(\rho)$, $K(\rho)$, $E(\rho)$ and so on as we did before.
We first observe that, since $L(x_0) = (u)_\rho$, the definition of $\Psi(\rho)$ and (10c) give

$$\Psi(\rho) = \fint_{B_\rho} G\left(\frac{|u-L(x_0)|}{\rho}\right) dx \le \fint_{B_\rho} G\left(|Q| + \frac{|u-L|}{\rho}\right) dx \le c\left(\fint_{B_\rho} G_{|Q|}\left(\frac{|u-L|}{\rho}\right) dx + G(|Q|)\right).$$

and hence, since $(u-L)_\rho = 0$, Poincaré's inequality and (40) yield

$$\fint_{B_\rho} G_{|Q|}\left(\frac{|u-L|}{\rho}\right) dx \le c \fint_{B_\rho} G_{|Q|}(|Du-Q|)\, dx = c\, E(\rho, Q) \le c\, G(|Q|).$$

Therefore, we have

$$\Psi(\rho) \le c\, G(|Q|) \qquad \text{and} \qquad K(\rho) \le c\, \omega(\rho) G(|Q|). \tag{42}$$

Then, we consider the function $v \in u + W_0^{1,G}(B_{\rho/2}, \mathbb{R}^N)$ associated to u by Theorem 3.3 and choose $\varphi \in \mathcal{D}(B_{\rho/4}, \mathbb{R}^N)$ with $\|D\varphi\|_\infty \le 1$. Then, taking into account that

$$\fint_{B_{\rho/4}} \langle \frac{Df_0(Q)}{G''(|Q|)} \,|\, D\varphi\rangle\, dx = 0,$$

we compute

$$\begin{aligned}
I &= \fint_{B_{\rho/4}} \langle \mathcal{A}(Q)(Du-Q) \,|\, D\varphi\rangle\, dx \\
&= \fint_{B_{\rho/4}} \langle \mathcal{A}(Q)(Du-Dv) \,|\, D\varphi\rangle\, dx + \fint_{B_{\rho/4}} \langle \mathcal{A}(Q)(Dv-Q) \,|\, D\varphi\rangle\, dx \\
&= \fint_{B_{\rho/4}} \langle \mathcal{A}(Q)(Du-Dv) \,|\, D\varphi\rangle\, dx + \fint_{B_{\rho/4}} \langle \frac{Df_0(Dv)}{G''(|Q|)} \,|\, D\varphi\rangle\, dx \\
&\quad + \fint_{B_{\rho/4}} \left\{\langle \frac{D^2 f_0(Q)}{G''(|Q|)}(Dv-Q) - \frac{Df_0(Dv) - Df_0(Q)}{G''(|Q|)} \,|\, D\varphi\rangle\right\} \\
&=: I_1 + I_2 + I_3
\end{aligned}$$

and we estimate the three terms thus obtained.
As to the first term I_1, since $|\mathcal{A}| \le \Lambda$ by (A2), $\|D\varphi\|_\infty \le 1$ and $\omega(\rho) \le 1$, from (32), (42) and (7a) we get

$$|I_1| \le c \fint_{B_{\rho/4}} |Du - Dv|\, dx \le c\, G^{-1}(K(\rho)) \le c\, [\omega(\rho)]^{1/g_2} |Q|.$$

We then turn to the second term I_2. Since $\|D\varphi\|_\infty \le 1$, by (34), (30), (7a), (42) and (6) we have

$$G''(|Q|)|I_2| = \left| \fint_{B_{\rho/4}} \langle Df_0(Dv) \,|\, D\varphi \rangle \, dx \right| \le c\, K_0(\rho) = c\, [\omega(\rho)]^{1-1/g_1} \frac{G(|Q|)}{|Q} \le c\, [\omega(\rho)]^{1-1/g_1} G''(|Q|)|Q|,$$

whence

$$|I_2| \le c\, [\omega(\rho)]^{1-1/g_1} |Q|$$

follows.

We are thus left to estimate I_3. Since

$$Df_0(Dv) - Df_0(Q) = \int_0^1 \langle D^2 f_0(Q + t(Dv - Q)) \,|\, Dv - Q \rangle \, dt,$$

we have

$$\begin{aligned} I_3 &= \frac{1}{G''(|Q|)} \fint_{B_{\rho/4}} \int_0^1 \langle (D^2 f_0(Q) - D^2 f_0(Q + t(Dv - Q)))(Dv - Q) \,|\, D\varphi \rangle \, dt \, dx \\ &= \frac{1}{G''(|Q|)} \fint_{B_{\rho/4}} 1_E(x) \int_0^1 \ldots dt \, dx + \frac{1}{G''(|Q|)} \fint_{B_{\rho/4}} 1_F(x) \int_0^1 \ldots dt \, dx \\ &=: I_{3a} + I_{3b} \end{aligned}$$

where the sets E and F are defined by

$$E := \{2|Dv - Q| \ge |Q|\} \cap B_{\rho/4} \quad \text{and} \quad F := \{2|Dv - Q| < |Q|\} \cap B_{\rho/4}.$$

We first estimate I_{3a}. Recalling that

$$\int_0^1 \frac{G'(|tA + (1-t)B|)}{|tA + (1-t)B|} \, dt \simeq \frac{G'(|A| + |B|)}{|A| + |B|}$$

holds uniformly with respect to $A, B \in \mathbb{M}^{N \times n}$ with $|A|+|B| > 0$ ([10, Lemma 20]), from (A2) and (6) we obtain that

$$\begin{aligned} |I_{3a}| &\le \frac{c}{G''(|Q|)} \fint_{B_{\rho/4}} 1_E \int_0^1 [G''(|Q|) + G''(|tDv + (1-t)Q|)] \, dt |Dv - Q| \, dx \\ &\le \frac{c}{G''(|Q|)} \fint_{B_{\rho/4}} 1_E \left(\frac{G'(|Q|)}{|Q|} + \frac{G'(|Dv| + |Q|)}{|Dv| + |Q|} \right) |Dv - Q| \, dx \\ &\le \frac{c}{G''(|Q|)|Q|} \fint_{B_{\rho/4}} 1_E \, G'(|Q| + |Dv|)|Dv - Q| \, dx \\ &\le c \, \frac{|Q|}{G(|Q|)} \fint_{B_{\rho/4}} 1_E \, G'(|Q| + |Dv|)|Dv - Q| \, dx. \end{aligned}$$

Noting that $|Q| + |Dv| \le 2|Q| + |Dv - Q| \le 5|Dv - Q|$ holds a.e. in E and recalling that $G(a + t) \le c\, G_a(t)$ holds for $t \ge a$ because of (10b), we obtain that

$$G'(|Q|+|Dv|)|Dv-Q| \le c\, G(|Dv-Q|) \le c\, G(2|Dv-Q|+|Q|) \le c\, G_{|Q|}(|Dv-Q|)$$

holds a.e. in E and this implies that

$$|I_{3a}| \le c \left(\frac{1}{G(|Q|)} \fint_{B_{\rho/4}} G_{|Q|}(|Dv - Q|)\, dx \right) |Q|.$$

We next estimate I_{3b}. Because of (A5) the inequality

$$|D^2 f_0(Q) - D^2 f_0(Q + t(Dv - Q))| \le c\, G''(|Q|) \left(\frac{|Dv - Q|}{|Q|} \right)^{\beta_0}$$

holds a.e. in F for every $t \in [0, 1]$ and from this we obtain

$$|I_{3b}| \le c\,|Q| \fint_{B_{\rho/4}} 1_F \left(\frac{|Dv - Q|}{|Q|} \right)^{1+\beta_0} dx \le c\,|Q| \fint_{B_{\rho/4}} 1_F \left(\frac{G'(|Q|)|Dv - Q|^2}{G'(|Q|)|Q|^2} \right)^{\frac{1+\beta_0}{2}} dx.$$

Then, taking into account that G' is increasing, that $|Q| + |Dv - Q| < 3|Q|/2$ a.e. in F and that $G'(t) \simeq G(t)t$, we obtain

$$|I_{3b}| \le c\,|Q| \fint_{B_{\rho/4}} 1_F \left(\frac{G'(|Q| + |Dv - Q|)|Dv - Q|^2}{G(|Q|)(|Q| + |Dv - Q|} \right)^{\frac{1+\beta_0}{2}} dx$$

and hence, using once more (10b) and that $G'(t)/t \simeq G''(t)$ and exploiting Jensen's inequality with $(1 + \beta_0)/2 < 1$, we finally get

$$|I_{3b}| \le c \left(\frac{1}{G(|Q|)} \fint_{B_{\rho/4}} G_{|Q|}(|Dv - Q|)\, dx \right)^{\frac{1+\beta_0}{2}} |Q|.$$

Therefore, combining the inequalities obtained for I_{3a} and I_{3b} we have that

$$|I_3| \le \frac{c}{G(|Q|)} \fint_{B_{\rho/4}} G_{|Q|}(|Dv - Q|)\, dx + c \left(\frac{1}{G(|Q|)} \fint_{B_{\rho/4}} G_{|Q|}(|Dv - Q|)\, dx \right)^{\frac{1+\beta_0}{2}}.$$

Moreover, since

$$\fint_{B_{\rho/4}} G_{|Q|}(|Dv - Q|)\, dx \le c \fint_{B_{\rho/4}} G_{|Q|}(|Du - Dv|)\, dx + c\, E(\rho)$$

and, from $G_a(t) \le c\,[G(t) + G(a)t/a]$ with $a = |Q|$ and $t = |Du - Dv|$,

$$\fint_{B_{\rho/4}} G_{|Q|}(|Du - Dv|)\, dx \le c \fint_{B_{\rho/4}} G(|Du - Dv|)\, dx + c\, \frac{G(|Q|)}{|Q|} \fint_{B_{\rho/4}} |Du - Dv|\, dx$$

we have

$$\fint_{B_{\rho/4}} G_{|Q|}(|Dv - Q|)\, dx \le c \fint_{B_{\rho/4}} G(|Du - Dv|)\, dx + c\, \frac{G(|Q|)}{|Q|} \fint_{B_{\rho/4}} |Du - Dv|\, dx + c\, E(\rho).$$

Hence, taking into account formulas (32), (37) and (42) and recalling that $G^{-1}(K(\rho)) \le c\,[\omega(\rho)]^{1/g_2}|Q|$ as in the estimate of I_1, we obtain that

$$\frac{1}{G(|Q|)} \fint_{B_{\rho/4}} G_{|Q|}(|Dv - Q|)\, dx \le c\,[\omega(\rho)]^{1-\tau_1} + c\,[\omega(\rho)]^{1/g_2} + c\,\frac{E(\rho)}{G(|Q|)}.$$

Combining the results for I_1, I_2 and I_3 and choosing β_1 as in (41), we finally get the desired estimate. □

Lemma 4.2. *For every* $\varepsilon \in (0,1)$ *there exist* $\delta_i = \delta_i(n, N, g_1, g_2, c_0, c_1, \Lambda, \lambda, \beta_0, \varepsilon) > 0$ $(i = 1, 2)$ *with the following property: if*

$$\fint_{B_\rho(x_0)} |V(Du) - (V(Du))_{x_0,\rho}|^2\, dx \le \delta_1 \fint_{B_\rho(x_0)} |V(Du)|^2\, dx; \tag{43a}$$

$$[\omega(\rho)]^{\beta_1} \le \delta_2; \tag{43b}$$

then the following inequality

$$\fint_{B_{\tau\rho}(x_0)} |V(Du) - (V(Du))_{\tau\rho,x_0}|^2\, dx \\ \le c\,\tau^2 \left(1 + \frac{\varepsilon}{\tau^{n+2}}\right)\left(\fint_{B_\rho(x_0)} |V(Du) - (V(Du))_{\rho,x_0}|^2\, dx + [\omega(\rho)]^{\beta_1} \fint_{B_\rho(x_0)} |V(Du)|^2\, dx\right). \tag{44}$$

holds for every $\tau \in (0,1)$ *for some* $c = c\,(n, N, g_1, g_2, c_0, c_1, \Lambda, \lambda) > 0$.

Proof. We fix $\varepsilon \in (0,1)$ and, as usual, we omit the dependence on the point x_0 which is fixed throughout the proof.
We first note from (28a) and [11, Lemma A.2] that

$$E(\rho, (Du)_\rho) \simeq \fint_{B_\rho} |V(Du) - V((Du)_\rho)|^2\, dx \tag{45}$$

and from [12, Lemma 23 and 25] and (12) that

$$\fint_{B_\rho} |V(Du)|^2\, dx \le 4|V((Du)_\rho)|^2 \simeq G(|(Du)_\rho|). \tag{46}$$

Throughout the rest of the proof we set

$$Q = (Du)_\rho$$

and from (28a), the previous estimates and (43a) we obtain

$$E(\rho, Q) \le c \fint_{B_\rho} |V(Du) - V(Q)|^2\, dx \le \tilde{c}_0\, \delta_1 G(|Q|) \tag{47}$$

for a suitable constant $\tilde{c}_0 > 0$. Then, for sufficiently small $\delta_1 > 0$, we see that (40) holds and, letting $\mathcal{A} = \mathcal{A}(Q)$ be defined by (39) with $Q = (Du)_\rho$, in view of Lemma 4.1, for every $\varphi \in \mathcal{D}(B_{\rho/4}, \mathbb{R}^N)$ with $\|D\varphi\|_\infty \leq 1$ we have

$$\fint_{B_{\rho/4}} \langle \mathcal{A}\left(\frac{Du - Q}{|Q|}\right) \mid D\varphi \rangle \, dx \leq \tilde{c}_1 \left\{ [\omega(\rho)]^{\frac{\beta_1}{2}} + \left[\frac{E(\rho, Q)}{G(|Q|)}\right]^{\frac{1}{2}} + \left[\frac{E(\rho, Q)}{G(|Q|)}\right]^{\frac{\beta_0}{2}} \right\} \left[\frac{E^*(\rho, Q)}{G(|Q|)}\right]^{\frac{1}{2}}, \tag{48}$$

for some $\tilde{c}_1 > 0$ where

$$E^*(\rho, Q) := E(\rho, Q) + [\omega(\rho)]^{\beta_1} G(|Q|). \tag{49}$$

In view of from (47) and (43b), we note that

$$\frac{E^*(\rho, Q)}{G(|Q|)} \leq \tilde{c}_0 \, \delta_1 + \delta_2.$$

We next define

$$H(t) := (\widetilde{G})_1(t) \qquad \text{where} \qquad \widetilde{G}(t) := \frac{G(|Q|t)}{G(|Q|)}. \tag{50}$$

(Here $(\widetilde{G})_1$ is the shifted function of $\widetilde{G}$ with $a = 1$. Then $\widetilde{G}$ satisfies the hypotheses (G) and it is easy to check that

$$H(t) = \frac{G_{|Q|}(|Q|t)}{G(|Q|)}, \qquad t \geq 0.$$

Then, recalling (10b) and that G is increasing, we have

$$H(t) = (\widetilde{G})_1(t) \geq c \, \frac{\widetilde{G}(1+t)}{(1+t)^2} t^2 = c \, \frac{G(|Q| + |Q|t)}{G(|Q|)(1+t)^2} t^2 \geq c \, \frac{t^2}{(1+t)^2} \geq c \, t^2/4, \qquad t \in [0, 1],$$

and so the inequality

$$t^2 \leq \tilde{c}_2 \, H(t), \qquad t \in [0, 1],$$

follows for a suitable constant $\tilde{c}_2 > 0$. Moerover, we observe also from Corollary 3.2 that

$$\begin{aligned} \left(\fint_{B_{\rho/4}} \left[H\left(\frac{|Du - Q|}{|Q|}\right)\right]^{1+\kappa} dx\right)^{\frac{1}{1+\kappa}} &= \left(\fint_{B_{\rho/4}} \left[\frac{G_{|Q|}(|Du - Q|)}{G(|Q|)}\right]^{1+\kappa} dx\right)^{\frac{1}{1+\kappa}} \\ &\leq \frac{c}{G(|Q|)} \fint_{B_{\rho/2}} G_{|Q|}(|Du - Q|) \, dx + c \, \omega(\rho) \\ &\leq \tilde{c}_3 \, \frac{E^*(\rho, Q)}{G(|Q|)} \end{aligned}$$

holds for some constant $\tilde{c}_3 > 0$.

Then, with $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3 > 0$ determined above, by choosing δ_i $(i = 1, 2)$ sufficiently small we see that

$$\mu := \max\left\{\tilde{c}_1, \sqrt{\tilde{c}_2\tilde{c}_3}\right\}\left[\frac{E^*(\rho)}{G(|Q|)}\right]^{\frac{1}{2}} \leq \max\left\{\tilde{c}_1, \sqrt{\tilde{c}_2\tilde{c}_3}\right\}(\tilde{c}_0\delta_1 + \delta_2)^{\frac{1}{2}} < 1. \quad (51)$$

Therefore, combining the previous estimates, we obtain

$$\left(\fint_{B_{\rho/4}}\left[H\left(\frac{|Du-Q|}{|Q|}\right)\right]^{1+\kappa} dx\right)^{\frac{1}{1+\kappa}} \leq \tilde{c}_3\,\frac{E^*(\rho,Q)}{G(|Q|)} = \tilde{c}_3\,\frac{\mu^2}{\left(\max\left\{\tilde{c}_1, \sqrt{\tilde{c}_2\tilde{c}_3}\right\}\right)^2} \leq \frac{\mu^2}{\tilde{c}_2} \leq H(\mu) \quad (52)$$

and, inserting (47) and (51) into (48),

$$\fint_{B_{\rho/4}}\left\langle \mathcal{A}\left(\frac{Du-Q}{|Q|}\right) \mid D\varphi\right\rangle dx \leq \frac{\tilde{c}_1(\delta_2^{\frac{1}{2}} + [\tilde{c}_0\,\delta_1]^{\frac{1}{2}} + [\tilde{c}_0\,\delta_1]^{\frac{\beta_0}{2}})}{\max\left\{\tilde{c}_1, \sqrt{\tilde{c}_2\tilde{c}_3}\right\}}\mu.$$

Therefore, we can apply Lemma 2.7 to the function

$$v = \frac{u-L}{|Q|} \qquad \text{where} \qquad Lx = Q(x-x_0) + (u)_\rho \qquad \text{and} \qquad p = 1+\kappa,$$

by choosing δ_i $(i = 1, 2)$ sufficiently small, so that we have

$$\frac{1}{G(|Q|)}\fint_{B_{\rho/4}} G_{|Q|}(|Du - Q - |Q|Dh)\,dx = \fint_{B_{\rho/4}} H\left(|Dv - Dh|\right)\,dx \leq \varepsilon H(\mu). \quad (53)$$

Here h is the $\mathcal{A}-$harmonic function in $B_{\rho/4}$ with $h = v$ on $\partial B_{\rho/4}$. In addition, since

$$H(\mu) \leq c\,\frac{G(|Q|(1+\mu))}{G(|Q|)(1+\mu)^2}\mu^2 \leq c\,\mu^2,$$

we finally obtain

$$\fint_{B_{\rho/2}} G_{|Q|}(|Du - Q - |Q|Dh)\,dx \leq \tilde{c}_4\,\varepsilon E^*(\rho,Q) \quad (54)$$

for a suitable constant $\tilde{c}_4 > 0$. Next, we choose $\tau \in (0,1)$ and we note that it suffices to consider $\tau < 1/16$. Then, from (12) we get

$$\begin{aligned}\fint_{B_{\tau\rho}} |V(Du) - (V(Du))_{\tau\rho}|^2 dx &\leq 4\fint_{B_{\tau\rho}} |V(Du) - V(Q + |Q|(Dh)_{\tau\rho})|^2\,dx \\ &\leq c\fint_{B_{\tau\rho}} G_{|Q+|Q|(Dh)_{\tau\rho}|}(|Du - Q - |Q|(Dh)_{\tau\rho}|)\,dx\end{aligned}$$

and we note from (52) and (53) that

$$\fint_{B_{\rho/4}} H(|Dh|)\,dx \leq c\fint_{B_{\rho/4}} H(|Dv|)\,dx + c\fint_{B_{\rho/4}} H(|Dv-Dh|)\,dx \leq c\,(1+\varepsilon)H(\mu) \leq c\,H(\mu)$$

so that basic regularity properties of $\mathcal{A}$ – harmonic functions, Jensen's inequality and (51) give

$$\sup_{B_{\rho/8}} |Dh| \le c \fint_{B_{\rho/4}} |Dh| \, dx \le c\, H^{-1}\left(\fint_{B_{\rho/4}} H(|Dh|)\, dx \right) \le c \, \max \left\{ \tilde{c}_1, \sqrt{\tilde{c}_2 \tilde{c}_3} \right\} (\tilde{c}_0 \delta_1 + \delta_2)^{\frac{1}{2}} \le \frac{1}{2}$$

for sufficiently small $\delta_i > 0$ $(i = 1, 2)$. This yields

$$\frac{|Q|}{2} \le |Q| \left(1 - |(Dh)_{\tau\rho}|\right) \le |Q + |Q|(Dh)_{\tau\rho}| \le |Q| \left(1 + |(Dh)_{\tau\rho}|\right) \le \frac{3|Q|}{2} \quad (55)$$

and

$$|Q| \left(1 + |(Dh)_{\rho/8}|\right) \ge |Q| \left(1 + \tau |(Dh)_{\rho/8}|\right) \ge |Q| \ge \frac{2}{3} |Q| \left(1 + |(Dh)_{\rho/8}|\right). \quad (56)$$

Therefore, we have from (55) that

$$G_{|Q + |Q|(Dh)_{\tau\rho}|}(t) \simeq G_{|Q|}(t)$$

and from (10b), (56) and Jensen's inequality that

$$\begin{aligned} G_{|Q|}(\tau |Q| |(Dh)_{\rho/8}|) &\simeq \frac{G'(|Q| + \tau |Q| |(Dh)_{\rho/8}|)}{|Q| + \tau |Q| |(Dh)_{\rho/8}|} \left(\tau |Q| |(Dh)_{\rho/8}|\right)^2 \simeq \tau^2 G_{|Q|}(|Q| |(Dh)_{\rho/8}|) \\ &\le \tau^2 \fint_{B_{\rho/8}} G_{|Q|}(|Q| |Dh|) \, dx. \end{aligned}$$

In addition, by regularity results for the harmonic maps, we also have

$$\sup_{B_{\tau\rho}} |Dh - (Dh)_{\tau\rho}| \le c\, \tau \fint_{\rho/8} |Dh - (Dh)_{\rho/8}| \, dx \le c\, \tau (Dh)_{\rho/8}.$$

Therefore, using the above results we have

$$\begin{aligned} \fint_{B_{\tau\rho}} |V(Du) - (V(Du))_{\tau\rho}|^2 \, dx &\le c \fint_{B_{\tau\rho}} G_{|Q|}(|Du - Q - |Q|(Dh)_{\tau\rho}|) \, dx \\ &\le c \fint_{B_{\tau\rho}} G_{|Q|}(|Du - Q - |Q| Dh|) \, dx + c \fint_{B_{\tau\rho}} G_{|Q|}(|Q| |Dh - (Dh)_{\tau\rho}|) \, dx \\ &\le c\, \tau^{-n} \fint_{B_\rho} G_{|Q|}(|Du - Q - |Q| Dh|) \, dx + c\, G_{|Q|}(\tau |Q| |(Dh)_{\rho/8}|) \, dx \\ &\le c\, \tau^{-n} \fint_{B_\rho} G_{|Q|}(|Du - Q - |Q| Dh|) \, dx + c\, \tau^2 \fint_{B_{\rho/4}} G_{|Q|}(|Q| |Dh|) \, dx \\ &\le c\, \tau^{-n} \fint_{B_\rho} G_{|Q|}(|Du - Q - |Q| Dh|) \, dx + c\, \tau^2 E(\rho, Q) \end{aligned}$$

where the last inequality follows from

$$\begin{aligned} \fint_{B_{\rho/4}} \frac{G_{|Q|}(|Q| |Dh|)}{G(|Q|)} \, dx &= \fint_{B_{\rho/4}} H(|Dh|) \, dx \\ &\le c \fint_{B_{\rho/4}} H(|Dv|) \, dx = \fint_{B_{\rho/4}} \frac{G_{|Q|}(|Du - Q|)}{G(|Q|)} \, dx \le 4^n \frac{E(\rho, Q)}{G(|Q|)} \end{aligned}$$

with H defined by (50), which is the Calderón–Zygmund estimate for $\mathcal{A}$–harmonic equation (16) in Orlicz spaces (see [12, Theorem 18]). Therefore, by (54) and (49), we have

$$\fint_{B_{\tau\rho}} |V(Du) - (V(Du))_{\tau\rho}|^2\,dx \le c\,\tau^2 \left[\frac{\varepsilon}{\tau^{n+2}} + 1\right] \left(E(\rho, Q) + [\omega(\rho)]^{\beta_1} G(|Q|)\right).$$

Finally, recalling that $Q = (Du)_\rho$ and using (45) and (46) we get the desired estimate. □

Degenerate case. From the assumption (A6), we have the following fact: for every $\delta > 0$ there exists $\sigma = \sigma(\delta) > 0$ such that

$$\begin{cases} 0 < t \le \sigma \\ A \in \mathbb{M}^{N\times n} \text{ and } |A| = 1 \end{cases} \qquad \Longrightarrow \qquad \left|\frac{Df_0(tA)}{G'(t)} - A\right| \le \delta. \tag{57}$$

Putting $A = B/|B|$ and $t = |B|$ we have

$$\left|Df_0(B) - \frac{B}{|B|}G'(|B|)\right| \le \delta G'(|B|) \tag{58}$$

for every matrix $B \in \mathbb{M}^{N\times n}$ with $0 < |B| \le \sigma$. We recall $\Psi(x_0, \rho)$ defined by (27).

Lemma 4.3. *There exists $\beta_2 = \beta_2(n, N, g_1, g_2, c_0, \Lambda) > 0$ such that, for every $\delta > 0$ and for $\sigma = \sigma(\delta) > 0$ given by (57), the inequality*

$$\begin{aligned} &\left|\fint_{B_{\rho/4}(x_0)} \langle G'(|Du|)\frac{Du}{|Du|} : D\varphi\rangle\,dx\right| \\ &\qquad \le c\left(\delta + [\omega(\rho)]^{\beta_2} + \frac{G^{-1}(\Psi(x_0,\rho))}{\sigma}\right)\left(\fint_{B_\rho(x_0)} G(|Du|)\,dx + G(\|D\varphi\|_\infty)\right) \end{aligned}$$

holds for every $\varphi \in \mathcal{D}(B_{\rho/4}(x_0), \mathbb{R}^N)$ for some $c = c\,(n, N, g_1, g_2, c_0, \Lambda, \lambda) > 0$.

The exponent β_2 is actually given by

$$\beta_2 := (1-\tau_1)\min\left\{1 - \frac{1}{g_1}, \frac{\tau_2}{g_2}\right\} \quad \text{with} \quad \tau_2 := \min\left\{\frac{g_1 - 1}{2}, \frac{g_1}{g_2 - g_1}\right\} \le \frac{1}{2} \tag{59}$$

where τ_1 is defined by (36).

Proof. As in the previous proofs, we write B_ρ, $\Psi(\rho)$ and so on omitting the dependence on the point x_0.
We first note that $(G' \circ G^{-1})(t) \simeq t/G^{-1}(t)$ and that the function $H(t) = t/G^{-1}(t)$ satisfies the assumptions of [37, Lemma 2.2]. Therefore we have

$$\fint_U (G' \circ G^{-1})(|w|)\,dx \le c(G' \circ G^{-1})\left(\fint_U |w|\,dx\right) \tag{60}$$

for every measurable set U with positive measure and for every function w integrable over U.
Let $\varphi \in \mathcal{D}(B_{\rho/4}(x_0), \mathbb{R}^N)$ be such that $\|D\varphi\|_\infty \le 1$ and let $v \in u + W_0^{1,G}(B_{\rho/2}, \mathbb{R}^N)$ be the function associated to u by Theorem 3.3. Then,

$$\fint_{B_{\rho/4}} \langle G'(|Du|)\frac{Du}{|Du|} : D\varphi\rangle \, dx = \fint_{B_{\rho/4}} \langle G'(|Du|)\frac{Du}{|Du|} - Df_0(Du) \,|\, D\varphi\rangle \, dx + \fint_{B_{\rho/4}} \langle Df_0(Du) - Df_0(Dv) \,|\, D\varphi\rangle \, dx + \fint_{B_{\rho/4}} \langle Df_0(Dv) \,|\, D\varphi\rangle \, dx =: I_1 + I_2 + I_3.$$

(i) Estimate of I_1. Let $\sigma = \sigma(\delta) > 0$ be defined by (57), set

$$E := \{|Du| \le \sigma\} \cap B_{\rho/4} \quad \text{and} \quad F := \{|Du| > \sigma\} \cap B_{\rho/4}$$

and denote by $I_{1,E}$ and $I_{1,F}$ the integrals I_1 over the sets E and F respectively.
We first estimate $I_{1,E}$. From (58), (60) with $U = B_{\rho/4}$ and $w = Du$ and from the standard Caccioppoli's inequality (Theorem 2.4) we have

$$|I_{1,E}| \le \frac{1}{|B_{\rho/4}|} \int_E \left| G'(|Du|)\frac{Du}{|Du|} - Df_0(Du) \right| dx \le \delta \fint_{B_{\rho/4}} G'(|Du|)\, dx \le c\,\delta (G' \circ G^{-1}) \left(\fint_{B_{\rho/4}} G(|Du|)\, dx \right) \le c\,\delta (G' \circ G^{-1})(\Psi(\rho)).$$

We then turn to the estimate of $I_{1,F}$. From (A2) and (60) with $U = F$ and $w = G(|Du|)$ we have

$$|I_{1,F}| \le c\,\frac{|F|}{|B_{\rho/4}|} \fint_{B_F} G'(|Du|)\, dx \le c\,\frac{|F|}{|B_{\rho/4}|} (G' \circ G^{-1}) \left(\fint_{B_F} G(|Du|)\, dx \right)$$

and, noting that

$$1 \le \frac{1}{\sigma} G^{-1} \left(\fint_F G(|Du|)\, dx \right),$$

and recalling that $G^{-1}(t)(G' \circ G^{-1})(t) \simeq t$, from the inequality above and the standard Caccioppoli's inequality (Theorem 2.4) we obtain

$$|I_{1,F}| \le c\,\frac{|F|}{|B_{\rho/4}|} \frac{1}{\sigma} \fint_F G(|Du|)\, dx \le \frac{c}{\sigma} \fint_{B_{\rho/4}} G(|Du|)\, dx \le \frac{c}{\sigma} \Psi(\rho) \le c\, \frac{G^{-1}(\Psi(\rho))}{\sigma} (G' \circ G^{-1})(\Psi(\rho)).$$

Combining the previous inequalities for $I_{1,E}$ and $I_{1,F}$, we conclude that

$$|I_1| \le c \left(\delta + \frac{G^{-1}(\Psi(\rho))}{\sigma} \right) (G' \circ G^{-1})(\Psi(\rho)).$$

(ii) Estimate of I_2. Let $\beta_2 > 0$ and $0 < \tau_2 < 1/2$ be defined by (59), set

$$E := \{|Du| < |Du - Dv|\} \cap B_{\rho/4} \quad \text{and} \quad F := \{|Du| \ge |Du - Dv|\} \cap B_{\rho/4}$$

and define by $I_{2,E}$ and $I_{2,F}$ the integrals I_2 over the sets E and F respectively. As to $I_{2,E}$, recalling that $\|D\varphi\|_\infty \le 1$, from (13) and (60) we have

$$|I_{2,E}| \le c\frac{1}{|B_{\rho/4}|}\int_E \frac{G'(|Du|+|Du-Dv|)}{|Du|+|Du-Dv|}|Du-Dv|\,dx$$
$$\le c\fint_{B_{\rho/4}} G'(|Du-Dv|)\,dx \le c\,(G'\circ G^{-1})\left(\fint_{B_{\rho/4}} G(|Du-Dv|)\,dx\right).$$

As to $I_{2,F}$, exploiting (13) as before and the fact that $0<\tau_2<1/2$, from Hölder inequality with conjugate exponents $g_2/[g_2-(1+\tau_2)]$ and $g_2/(1+\tau_2)$ we get

$$|I_{2,F}| \le c\frac{1}{|B_{\rho/4}|}\int_F \frac{G'(|Du|+|Du-Dv|)}{|Du|+|Du-Dv|}|Du-Dv|\,dx$$
$$\le c\frac{1}{|B_{\rho/4}|}\int_F \frac{G'(|Du|)}{|Du|^{\tau_2}}|Du-Dv|^{\tau_2}\,dx$$
$$\le c\frac{1}{|B_{\rho/4}|}\int_F \frac{G(|Du|)}{|Du|^{1+\tau_2}}|Du-Dv|^{\tau_2}\,dx$$
$$\le c\left(\fint_{B_{\rho/4}}\left[\frac{G(|Du|)}{|Du|^{1+\tau_2}}\right]^{\frac{g_2}{g_2-(1+\tau_2)}}dx\right)^{\frac{g_2-(1+\tau_2)}{g_2}}\left(\fint_{B_{\rho/4}}|Du-Dv|^{\frac{g_2\tau_2}{1+\tau_2}}\,dx\right)^{\frac{1+\tau_2}{g_2}}.$$

Then, we define

$$H(t) := \left[(\widetilde{G}\circ G^{-1})(t)\right]^{\frac{g_2}{g_2-(1+\tau_2)}} \qquad \text{and} \qquad \tilde{H}(t) := \left[G^{-1}(t)\right]^{\frac{\tau_2 g_2}{1+\tau_2}},$$

where $\widetilde{G}(t) := G(t)/t^{1+\tau_2}$ which is an increasing function because of Proposition 2.1 (b) and $1+\tau_2<g_1$. Since

$$\frac{H(t)}{t} = \left[\frac{G(G^{-1}(t))}{[G^{-1}(t)]^{g_2}}\right]^{\frac{1+\tau_2}{g_2-(1+\tau_2)}} \qquad \text{and} \qquad \frac{\tilde{H}(t)}{t} = \left[\frac{G(G^{-1}(t))}{[G^{-1}(t)]^{\frac{\tau_2 g_2}{1+\tau_2}}}\right]^{-1},$$

the functions $H(t)/t$ and $\tilde{H}(t)/t$ turn out to be decreasing too because of Proposition 2.1 (b) and $\tau_2 g_2/(1+\tau_2)\le g_1$ (see (59)). Therefore, H and $\tilde{H}$ satisfy the assumptions of [37, Lemma 2.2] and hence we have

$$\left(\fint_{B_{\rho/4}}\left[\frac{G(|Du|)}{|Du|^{1+\tau_2}}\right]^{\frac{g_2}{g_2-(1+\tau_2)}}dx\right)^{\frac{g_2-(1+\tau_2)}{g_2}} \le c\,(\widetilde{G}\circ G^{-1})\left(\fint_{B_{\rho/4}} G(|Du|)\,dx\right);$$
$$\left(\fint_{B_{\rho/4}}|Du-Dv|^{\frac{g_2\tau_2}{1+\tau_2}}\,dx\right)^{\frac{1+\tau_2}{g_2}} \le c\left[G^{-1}\left(\fint_{B_{\rho/4}} G(|Du-Dv|)\,dx\right)\right]^{\tau_2}.$$

Hence, in wiew of the previous inequalities we have

$$|I_2| \le c\,(G' \circ G^{-1}) \left(\fint_{B_{\rho/4}} G(|Du - Dv|)\,dx \right) \\ + c\,(\widetilde{G} \circ G^{-1}) \left(\fint_{B_{\rho/4}} G(|Du|)\,dx \right) \left[G^{-1} \left(\fint_{B_{\rho/4}} G(|Du - Dv|)\,dx \right) \right]^{\tau_2}.$$

Then, on account (37) and (7b), for the first summand on the right we have

$$(G' \circ G^{-1}) \left(\fint_{B_{\rho/4}} G(|Du - Dv|)\,dx \right) \le c\,[\omega(\rho)]^{(1-\tau_1)(1-1/g_1)} (G' \circ G^{-1})(\Psi(\rho))$$

and for the second, from Caccioppoli's inequality (Theorem 2.4), the definition of $\widetilde{G}$ and (7a), we have

$$(\widetilde{G} \circ G^{-1}) \left(\fint_{B_{\rho/4}} G(|Du|)\,dx \right) \le c\,\frac{\Psi(\rho)}{[G^{-1}(\Psi(\rho))]^{1+\tau_2}};$$

$$\left[G^{-1} \left(\fint_{B_{\rho/4}} G(|Du - Dv|)\,dx \right) \right]^{\tau_2} \le c\,[\omega(\rho)]^{(1-\tau_1)\tau_2/g_2} [G^{-1}(\Psi(\rho))]^{\tau_2}.$$

Finally, combining these inequalities we conclude that

$$|I_2| \le c\,[\omega(\rho)]^{\beta_2} \frac{\Psi(\rho)}{G^{-1}(\Psi(\rho))} \le c\,[\omega(\rho)]^{\beta_2} (G' \circ G^{-1})(\Psi(\rho))$$

with $\beta_2 > 0$ defined by (59).

(iii) Estimate of I_3. Since $\|D\varphi\|_\infty \le 1$, from (34), the definition of $K(\rho)$, (30) and (7b) we have

$$|I_3| \le \frac{K(\rho)}{G^{-1}(K(\rho))} \le c\,(G' \circ G^{-1})(\omega(\rho)\Psi(\rho)) \le c\,[\omega(\rho)]^{1-1/g_1} (G' \circ G^{-1})(\Psi(\rho)).$$

Therefore, combining the previous results and noting that $0 < \beta_2 < 1 - 1/g_1$, we get

$$\left| \fint_{B_{\rho/4}(x_0)} \langle G'(|Du|) \frac{Du}{|Du|} : D\varphi \rangle\,dx \right| \le c\,(G' \circ G^{-1})(\Psi(\rho)) \left(\delta + [\omega(\rho)]^{\beta_2} + \frac{G^{-1}(\Psi(\rho))}{\sigma} \right) \|D\varphi\|_\infty$$

for every $\varphi \in \mathcal{D}(B_{\rho/4}, \mathbb{R}^N)$. Finally, using Young's inequality and (6), we have

$$(G' \circ G^{-1})(\Psi(\rho)) \|D\varphi\|_\infty \le G^* \left((G' \circ G^{-1})(\Psi(\rho)) \right) + G(\|D\varphi\|_\infty) \le c\,\Psi(\rho) + G(\|D\varphi\|_\infty)$$

which gives the desired estimate. □

Lemma 4.4. *Let $\gamma_0 > 0$ be defined by Lemma 2.8. Then for every $0 < \gamma < \gamma_0$ and for every small $\chi \in (0,1)$, there exist $\varepsilon_i > 0$ $(i = 1,2)$ and $\tau \in (0,1)$ depending on n, N, g_1, g_2, c_0, c_1, Λ, λ, γ_0, γ and χ (ε_1 depends also on $\sigma(\delta)$ where δ satisfies (62) below) with the following property: if*

$$\chi \fint_{B_\rho(x_0)} |V(Du)|^2\,dx \le \fint_{B_\rho(x_0)} |V(Du) - (V(Du))_{x_0,\rho}|^2\,dx; \tag{61a}$$

$$\fint_{B_\rho(x_0)} |V(Du) - (V(Du))_{x_0,\rho}|^2\,dx \le \varepsilon_1; \tag{61b}$$

$$[\omega(\rho)]^{\beta_2} \le \varepsilon_2; \tag{61c}$$

where $\beta_2 > 0$ is defined in (59), then

$$\fint_{B_{\tau\rho}(x_0)} |V(Du) - (V(Du))_{x_0,\tau\rho}|^2\,dx \le \tau^{2\gamma} \fint_{B_\rho(x_0)} |V(Du) - (V(Du))_{x_0,\rho}|^2\,dx$$

Proof. As usual, throughout the proof we omit the dependence on the point x_0. First, we fix γ and χ as in the statement, we choose $\tau = \tau(\gamma,\gamma_0,\chi) \in (0,1/4)$ such that

$$\tilde{c}_1 \tau^{2\gamma_0}\chi^{-1} \le \tau^{2\gamma} \qquad \Longleftrightarrow \qquad \tau \le \left(\tilde{c}_1^{-1}\chi\right)^{\frac{1}{2(\gamma_0-\gamma)}}$$

where $\tilde{c}_1 > 0$ is an absolute constant (depending only on g_1, g_2, n, N and c_1) to be specified below and we set

$$\varepsilon = \tau^{2\gamma_0+n} > 0.$$

Next, we let $\delta_0 = \delta_0(\varepsilon) > 0$ be associated to $\varepsilon > 0$ by Corollary 2.10 where g is the $G-$harmonic function in $B_{\rho/4}$ such that $g = u$ on $\partial B_{\rho/4}$ and the assumption (21) is replaced by (14). Then, we choose $\delta = \delta(\varepsilon) > 0$ such that

$$c_*\delta \le \frac{\delta_0}{2} \tag{62}$$

where $c_* > 0$ denotes the constant in Lemma 4.3. This choice of δ determines $\sigma = \sigma(\delta) > 0$ by (57). Next, on account of (28b), (61a) and (61b), we note that

$$\Psi(\rho) \le c \fint_{B_\rho} G(|Du|)\,dx \le c \fint_{B_\rho} |V(Du)|^2\,dx \le c\,\chi^{-1} \fint_{B_\rho} |V(Du) - (V(Du))_\rho|^2\,dx \le c\,\chi^{-1}\varepsilon_1$$

and hence from the assumptions we obtain that

$$c_*\left(\delta + [\omega(\rho)]^{\beta_2} + \frac{G^{-1}(\Psi(\rho))}{\sigma}\right) \le \frac{\delta_0}{2} + c\left(\varepsilon_2 + \frac{\chi^{-\frac{1}{g_1}} G^{-1}(\varepsilon_1)}{\sigma}\right) \le \delta_0$$

for suitable choices of $\varepsilon_i > 0$ $(i = 1,2)$. Therefore, in view of Lemma 4.3, the function u satisfies the almost $G-$harmonic condition

$$\left|\fint_{B_{\rho/4}} \langle G'(|Du|)\frac{Du}{|Du|} \mid D\varphi\rangle\,dx\right| \le \delta_0\left(\fint_{B_\rho} G(|Du|)\,dx + G(\|D\varphi\|_\infty)\right)$$

for every $\varphi \in \mathcal{D}(B_{\rho/4}, \mathbb{R}^N)$ which gives

$$\fint_{B_{\rho/4}} |V(Du) - V(Dg)|^2\, dx \leq \varepsilon \fint_{B_\rho} G(|Du|)\, dx \leq c\,\varepsilon \fint_{B_\rho} |V(Du)|^2\, dx \tag{63}$$

by Corollary 2.10. Then, by a standard energy estimate we have

$$\fint_{B_{\rho/4}} |V(Dg)|^2\, dx \leq c \fint_{B_{\rho/4}} G(|Dg|)\, dx \leq c \fint_{B_{\rho/4}} G(|Du|)\, dx \leq c \fint_{B_\rho} |V(Du)|^2\, dx$$

and, since $\tau \in (0, 1/4)$, by Lemma 2.8 we get

$$\fint_{B_{\tau\rho}} |V(Dg) - (V(Dg))_{\tau\rho}|^2\, dx \leq c\,\tau^{2\gamma_0} \fint_{B_{\rho/4}} |V(Dg) - (V(Dg))_{\rho/4}|^2\, dx. \tag{64}$$

Therefore, in view of the previous inequalities (63) and (64) and of the choice of ε we have

$$\begin{aligned}
\fint_{B_{\tau\rho}} |V(Du) - (V(Du))_{\tau\rho}|^2\, dx &\leq 4 \fint_{B_{\tau\rho}} |V(Du) - (V(Dg))_{\tau\rho}|^2\, dx \\
&\leq 8 \fint_{B_{\tau\rho}} |V(Du) - V(Dg)|^2\, dx + 8 \fint_{B_{\tau\rho}} |V(Dg) - (V(Dg))_{\tau\rho}|^2\, dx \\
&\leq c\,\tau^{-n}\varepsilon \fint_{B_\rho} |V(Du)|^2\, dx + c\,\tau^{2\gamma_0} \fint_{B_{\rho/4}} |V(Dg) - (V(Dg))_{\tau\rho}|^2\, dx \\
&\leq \tilde{c}_1\, \tau^{2\gamma_0} \fint_{B_\rho} |V(Du)|^2\, dx
\end{aligned}$$

for some $\tilde{c}_1 = \tilde{c}_1(n, N, g_1, g_2, c_1) > 0$. Finally, on account of (61a), we conclude that

$$\fint_{B_{\tau\rho}} |V(Du) - (V(Du))_{\tau\rho}|^2\, dx \leq \tilde{c}_1\, \tau^{2\gamma_0} \chi^{-1} \fint_{B_\rho} |V(Du) - (V(Du))_\rho|^2$$

which is the desired inequality because of the choice of τ. □

5. Iteration: proof of $C^{1,\alpha}$ – regularity

In this final part, we set up the iteration scheme which proves the partial regularity of minimizer u of the functional $\mathcal{F}$ defined by (4). we assume that G and f satisfy the hypotheses (G) and (A0) – (A6) respectively. First we consider the non–degenerate case and, from Lemma 4.2, we prove the following result.

Lemma 5.1. *Let $\beta \in (0, 1)$ and $B_R(x_0) \Subset \Omega$ with $0 < R < 1$, and let $\beta_0, \beta_1 \in (0, 1)$ be given by (1) and (41) respectively. Then, there exist $\delta_3, \delta_4 > 0$ depending only on n, N, g_1, g_2, c_0, c_1, Λ, λ, β_0 and β with the following property: if*

$$\omega(R)^{\beta_1} \leq R^{\beta_0\beta_1} \leq \delta_4; \tag{65a}$$

$$\fint_{B_R(x_0)} |V(Du) - (V(Du))_{x_0,R}|^2\, dx \leq \delta_3 \fint_{B_R(x_0)} |V(Du)|^2\, dx, \tag{65b}$$

then we have

$$\fint_{B_r(x_0)} |V(Du) - (V(Du))_{x_0,r}|^2 \, dx$$

$$\leq c\left(\frac{r}{R}\right)^{2\tilde{\beta}} \fint_{B_R(x_0} |V(Du) - (V(Du))_{x_0,R}|^2 \, dx + c\, r^{2\tilde{\beta}} \fint_{B_R(x_0} |V(Du)|^2 \, dx \tag{66}$$

for every $r \in (0,R)$ where

$$\tilde{\beta} := \min\left\{\beta, \frac{\beta_0\beta_1}{2}\right\}. \tag{67}$$

Proof. As usual, throughout the proof we omit the dependence on the point x_0 and without loss of generality we assume that $R \in (0,1)$.

Step 1. Choice of parameters. We let $c^* > 0$ by the constant given in (44) and we choose the parameters τ and ε in Lemma 4.2 as follows:

$$\tau := \min\left\{\left(\frac{1}{2c^*}\right)^{\frac{1}{1-\beta}}, \left(\frac{1}{16}\right)^{\frac{1}{1-\beta}}\right\} \qquad \text{and} \qquad \varepsilon := \frac{\tau^{n+1+\beta}}{2c^*}. \tag{68}$$

This determines δ_1 and δ_2 in Lemma 4.2. We next choose δ_3 and δ_4 as follows:

$$\delta_3 := \min\left\{\delta_1, \frac{1}{8(1+\tau^{-n})}, \frac{(\sqrt{2}-1)^2(1-\tau^{\tilde{\beta}})^2\tau^n}{2}\right\} \qquad \text{and} \qquad \delta_4 := \min\{\delta_2, \delta_3\}. \tag{69}$$

Step 2. Induction. We prove by induction that the following inequalities hold

$$\fint_{B_{\tau^k R}} |V(Du) - (V(Du))_{\tau^k R}|^2 \, dx \leq \tau^{2\tilde{\beta}k}\delta_3 \fint_{B_{\tau^k R}} |V(Du)|^2 \, dx; \tag{70a}$$

$$\fint_{B_{\tau^k R}} |V(Du) - (V(Du))_{\tau^k R}|^2 \, dx \leq \tau^{(1+\tilde{\beta})k} \fint_{B_R} |V(Du) - (V(Du))_R|^2 \, dx + 2\frac{1-\tau^{(1-\tilde{\beta})k}}{1-\tau^{1-\tilde{\beta}}}(\tau^k R)^{2\tilde{\beta}} \fint_{B_R} |V(Du)|^2 \, dx; \tag{70b}$$

$$\fint_{B_{\tau^k R}} |V(Du)|^2 \, dx \leq 2 \fint_{B_R} |V(Du)|^2 \, dx \tag{70c}$$

for every $k \geq 0$.

For convenience, in the sequel we shall write $(70a)_k$, $(70b)_k$ and $(70c)_k$ to denote (70a), (70b) and (70c) for a specific value of k. Clearly, (70a), (70b) and (70c) hold for $k = 0$. We next suppose that $(70a)_h$, $(70b)_h$ and $(70c)_h$ hold for $h = 0, 1, 2, \ldots, k-1$ for some $k \geq 1$ and then prove $(70a)_k$, $(70b)_k$ and $(70c)_k$. By (65a), $(70a)_{k-1}$ and (69), we see that (43a) and (43b) hold for $\rho = \tau^{k-1}R$.

Hence, we can apply Lemma 4.2 for $\rho = \tau^{k-1}R$ to get

$$\begin{aligned}
&\fint_{B_{\tau^k R}} |V(Du) - (V(Du))_{\tau^k R}|^2 \, dx \\
&\quad \le c^* \tau^2 (1 + \varepsilon \tau^{-n-2}) \left(\fint_{B_{\tau^{k-1} R}} |V(Du) - (V(Du))_{\tau^{k-1} R}|^2 \, dx + (\tau^{k-1}R)^{\beta_0 \beta_1} \fint_{B_{\tau^{k-1} R}} |V(Du)|^2 \, dx \right)
\end{aligned}$$

and from (68) we see that $c^* \tau^{1-\beta} \le 1/2$ and $c^* \varepsilon \tau^{-\beta-n-1} \le 1/2$ which yield

$$c^* \tau^2 (1 + \varepsilon \tau^{-n-2}) = \tau^{1+\beta} (c^* \tau^{1-\beta} + c^* \varepsilon \tau^{-\beta-n-1}) \le \tau^{1+\beta}.$$

Hence, recalling (67), we have

$$\begin{aligned}
&\fint_{B_{\tau^k R}} |V(Du) - (V(Du))_{\tau^k R}|^2 \, dx \\
&\quad \le \tau^{1+\beta} \left(\fint_{B_{\tau^{k-1} R}} |V(Du) - (V(Du))_{\tau^{k-1} R}|^2 \, dx + (\tau^{k-1}R)^{\beta_0 \beta_1} \fint_{B_{\tau^{k-1} R}} V(Du)|^2 \, dx \right) \\
&\quad \le \tau^{1+\tilde{\beta}} \fint_{B_{\tau^{k-1} R}} |V(Du) - (V(Du))_{\tau^{k-1} R}|^2 \, dx + \tau^{1-\tilde{\beta}} (\tau^k R)^{2\tilde{\beta}} \fint_{B_{\tau^{k-1} R}} |V(Du)|^2 \, dx.
\end{aligned} \tag{71}$$

Using the first inequality in (71), $(70\text{a})_{k-1}$, (65a) and the facts that $\tau^{1-\beta} \le 1/(16)$ by (68) and $\delta_4 \le \delta_3$ by (69), we see that

$$\begin{aligned}
&\fint_{B_{\tau^k R}} |V(Du) - (V(Du))_{\tau^k R}|^2 \, dx \\
&\quad \le \tau^{1+\beta} \left(\fint_{B_{\tau^{k-1} R}} |V(Du) - (V(Du))_{\tau^{k-1} R}|^2 \, dx + (\tau^{k-1}R)^{\beta_0 \beta_1} \fint_{B_{\tau^{k-1} R}} |V(Du)|^2 \, dx \right) \\
&\quad \le \tau^{1-\beta} \tau^{2\tilde{\beta}} \left(\tau^{2\tilde{\beta}(k-1)} \delta_3 \fint_{B_{\tau^{k-1} R}} |V(Du)|^2 \, dx + \tau^{2\tilde{\beta}(k-1)} \delta_4 \fint_{B_{\tau^{k-1} R}} |V(Du)|^2 \, dx \right) \\
&\quad \le \frac{1}{8} \tau^{2\tilde{\beta}k} \delta_3 \fint_{B_{\tau^{k-1} R}} |V(Du)|^2 \, dx.
\end{aligned} \tag{72}$$

On the other hand, by $(70\text{a})_{k-1}$ and the fact that $4(1 + \tau^{-n})\delta_3 \le 1/2$ by (69), we have

$$
\begin{aligned}
\fint_{B_{\tau^{k-1}R}} |V(Du)|^2\,dx &\le 4\fint_{B_{\tau^{k-1}R}} |V(Du)-(V(Du))_{\tau^{k-1}R}|^2\,dx \\
&\qquad + 4|(V(Du))_{\tau^{k-1}R}-(V(Du))_{\tau^k R}|^2 + 4\fint_{B_{\tau^k R}} |V(Du)|^2\,dx \\
&\le 4\left(1+\tau^{-n}\right)\fint_{B_{\tau^{k-1}R}} |V(Du)-(V(Du))_{\tau^{k-1}R}|^2\,dx + 4\fint_{B_{\tau^k R}} |V(Du)|^2\,dx \\
&\le 4\left(1+\tau^{-n}\right)\delta_3\fint_{B_{\tau^{k-1}R}} |V(Du)|^2\,dx + 4\fint_{B_{\tau^k R}} |V(Du)|^2\,dx \\
&\le \frac{1}{2}\fint_{B_{\tau^{k-1}R}} |V(Du)|^2\,dx + 4\fint_{B_{\tau^k R}} |V(Du)|^2\,dx
\end{aligned}
$$

which implies that

$$
\fint_{B_{\tau^{k-1}R}} |V(Du)|^2\,dx \le 8\fint_{B_{\tau^k R}} |V(Du)|^2\,dx.
$$

Inserting this into (72), we obtain $(70\text{a})_k$.
We next show that $(70\text{b})_k$ holds. From the second inequality in (71) and from $(70\text{b})_{k-1}$ and $(70\text{c})_{k-1}$, we have

$$
\begin{aligned}
&\fint_{B_{\tau^k R}} |V(Du)-(V(Du))_{\tau^k R}|^2\,dx \\
&\quad\le \tau^{1+\tilde\beta}\fint_{B_{\tau^{k-1}R}} |V(Du)-(V(Du))_{\tau^{k-1}R}|^2\,dx + \tau^{1-\tilde\beta}(\tau^k R)^{2\tilde\beta}\fint_{B_{\tau^{k-1}R}} |V(Du)|^2\,dx \\
&\quad\le \tau^{(1+\tilde\beta)k}\fint_{B_R} |V(Du)-(V(Du))_R|^2\,dx \\
&\qquad\quad + 2\tau^{1+\tilde\beta}\frac{1-\tau^{(1-\tilde\beta)(k-1)}}{1-\tau^{1-\tilde\beta}}(\tau^{k-1}R)^{2\tilde\beta}\fint_{B_R} |V(Du)|^2\,dx + 2(\tau^k R)^{2\tilde\beta}\fint_{B_R} |V(Du)|^2\,dx \\
&\quad= \tau^{(1+\tilde\beta)k}\fint_{B_R} |V(Du)-(V(Du))_R|^2\,dx + 2\frac{1-\tau^{(1-\tilde\beta)k}}{1-\tau^{1-\tilde\beta}}(\tau^k R)^{2\tilde\beta}\fint_{B_R} |V(Du)|^2\,dx
\end{aligned}
$$

which is $(70\text{b})_k$.
Finally, by $(70\text{a})_h$ and $(70\text{c})_h$ with $h = 0, 1, 2, \ldots, k-1$ and the fact that

$\tau^{-\frac{n}{2}}(2\delta_3)^{\frac{1}{2}}\frac{1}{1-\tau^{\tilde\beta}} \le \sqrt{2}-1$ by (69), we obtain

$$\begin{aligned}\left(\fint_{B_{\tau^k R}} |V(Du)|^2\,dx\right)^{\frac{1}{2}} &\le \tau^{-\frac{n}{2}}\sum_{h=0}^{k-1}\left(\fint_{B_{\tau^h R}} |V(Du)-(V(Du))_{\tau^h R}|^2\,dx\right)^{\frac{1}{2}} + \left(\fint_{B_R}|V(Du)|^2\,dx\right)^{\frac{1}{2}}\\ &\le \tau^{-\frac{n}{2}}\delta_3^{\frac{1}{2}}\sum_{h=0}^{k-1}\tau^{\tilde\beta h}\left(\fint_{B_{\tau^h R}}|V(Du)|^2\,dx\right)^{\frac{1}{2}} + \left(\fint_{B_R}|V(Du)|^2\,dx\right)^{\frac{1}{2}}\\ &\le \left(\tau^{-\frac{n}{2}}(2\delta_3)^{\frac{1}{2}}\frac{1}{1-\tau^{\tilde\beta}}+1\right)\left(\fint_{B_R}|V(Du)|^2\,dx\right)^{\frac{1}{2}}\\ &\le \left(2\fint_{B_R}|V(Du)|^2\,dx\right)^{\frac{1}{2}}\end{aligned}$$

which implies $(70\mathrm{c})_k$.

Step 3. Decay estimates. Let $r \in (0,R)$. Then $\tau^{k+1}R \le r < \tau^k R$ for some $k \ge 0$. Therefore, by $(70\mathrm{b})_k$ we have

$$\begin{aligned}&\fint_{B_r} |V(Du)-(V(Du))_r|^2\,dx\\ &\quad\le 4\tau^{-n}\fint_{B_{\tau^k R}}|V(Du)-(V(Du))_{\tau^k R}|^2\,dx\\ &\quad\le 4\tau^{-n}\tau^{(1+\tilde\beta)k}\fint_{B_R}|V(Du)-(V(Du))_R|^2\,dx + 8\tau^{-n}\frac{1-\tau^{(1-\tilde\beta)k}}{1-\tau^{1-\tilde\beta}}(\tau^k R)^{2\tilde\beta}\fint_{B_R}|V(Du)|^2\,dx\\ &\quad\le 4\tau^{-n-1-\tilde\beta}\left(\frac{r}{R}\right)^{2\tilde\beta}\fint_{B_R}|V(Du)-(V(Du))_R|^2\,dx + \frac{8\tau^{-n}}{1-\tau^{1-\tilde\beta}}\left(\frac{r}{\tau}\right)^{2\tilde\beta}\fint_{B_R}|V(Du)|^2\,dx.\end{aligned}$$

Consequently, recalling τ denoted by (68), we have (66). □

We can finally give the proof of the main theorem.

Proof of Theorem 1.1. Let γ_0 be defined by Lemma 2.8. We fix $\gamma \in (0,\gamma_1)$ where

$$\gamma_1 := \min\left\{\gamma_0, \frac{\beta_0\beta_1}{2}\right\}$$

and we let δ_3 and δ_4 be associated to $\beta=\gamma$ by Lemma 5.1. This implies that $\tilde\beta=\beta=\gamma$. We also set

$$\chi = \delta_3.$$

Consequently, δ_3 and δ_4 in Lemma 5.1 and ε_1, ε_2 and τ in Lemma 4.4 are determined and depend only on the structure constants and on γ and γ_0.
Now, suppose that a point $x_0 \in \Omega$ satisfies

$$\liminf_{r\to 0^+}\fint_{B_r(x_0)}|V(Du)-(V(Du))_{x_0,r}|^2\,dx = 0$$

and

$$M := \limsup_{r\to 0^+} \fint_{B_r(x_0)} |V(Du)|^2\,dx < +\infty.$$

Then, there exists $R_0 > 0$ with $B_{2R_0}(x_0) \Subset \Omega$ such that

$$\fint_{B_{R_0}(x_0)} |V(Du)|^2\,dx \le M+1; \qquad \fint_{B_{R_0}(x_0)} |V(Du) - (V(Du))_{x_0,R_0}|^2\,dx \le \frac{\varepsilon_1}{4};$$

and moreover

$$R_0^{\beta_0\beta_1} \le \frac{\varepsilon_1}{4(M+1)}, \qquad R_0^{\beta_0\beta_1} \le \delta_4, \qquad \text{and} \qquad R_0^{\beta_0\beta_2} \le \varepsilon_2. \tag{73}$$

Therefore, by the continuity of the integrals above with respect to the translation of the domain of integration, there exists $R_1 > 0$ with $R_1 < R_0$ such that for every $y \in B_{R_1}(x_0)$ we have

$$\fint_{B_{R_0}(y)} |V(Du)|^2\,dx \le 2(M+1) \qquad \text{and} \qquad \fint_{B_{R_0}(y)} |V(Du)-(V(Du))_{y,R_0}|^2\,dx \le \frac{\varepsilon_1}{2}. \tag{74}$$

Now we fix an arbitrary point $y \in B_{R_1}(x_0)$. We first suppose that

$$\delta_3 \fint_{B_{\tau^k R_0}(y)} |V(Du)|^2\,dx \le \fint_{B_{\tau^k R_0}(y)} |V(Du)-(V(Du))_{y,\tau^k R_0}|^2\,dx \qquad \text{for every } k \ge 0. \tag{75}$$

In view of (74) and of the third inequality in (73) with $\omega(r) \le r^{\beta_0}$, applying Lemma 4.4 inductively with $B_\rho(x_0)$ replaced by $B_{\tau^k R_0}(x_0)$, we have that

$$\begin{aligned}\fint_{B_{\tau^k R_0}(y)} |V(Du) - (V(Du))_{y,\tau^k R_0}|^2\,dx &\le \tau^{2\gamma} \fint_{B_{\tau^{k-1} R_0}(y)} |V(Du) - (V(Du))_{y,\tau^{k-1} R_0}|^2\,dx \\ &\le \ldots \le \tau^{2k\gamma} \fint_{B_{R_0}(y)} |V(Du) - (V(Du))_{y,R_0}|^2\,dx \\ &\le \fint_{B_{R_0}(y)} |V(Du) - (V(Du))_{y,R_0}|^2\,dx \le \frac{\varepsilon_1}{2}\end{aligned} \tag{76}$$

holds for every $k \ge 0$. Therefore, for $r \in (0, R_0)$ there exists $k \ge 0$ such that $\tau^{k+1} R_0 \le r < \tau^k R_0$ and so

$$\begin{aligned}\fint_{B_r(y)} |V(Du) - (V(Du))_{y,r}|^2\,dx &\le 4\tau^{-n} \fint_{B_{\tau^k R_0}(y)} |V(Du) - (V(Du))_{y,\tau^k R_0}|^2\,dx \\ &\le 4\tau^{-n-2\gamma} \left(\frac{r}{R_0}\right)^{2\gamma} \fint_{B_{R_0}(y)} |V(Du) - (V(Du))_{y,R_0}|^2\,dx.\end{aligned}$$

Therefore, by (74) we have

$$\fint_{B_r(y)} \frac{|V(Du) - (V(Du))_{y,r}|^2}{r^{2\gamma}}\,dx \le \frac{2\varepsilon_1}{\tau^{n+2\gamma} R_0^{2\gamma}}. \tag{77}$$

We next suppose that (75) does not hold. Then there exists $k_0 \geq 0$ such that

$$\delta_3 \fint_{B_{\tau^k R_0}(y)} |V(Du)|^2 \, dx \leq \fint_{B_{\tau^k R_0}(y)} |V(Du) - (V(Du))_{y,\tau^k R_0}|^2 \, dx \qquad (78)$$

for every $k = 0, \ldots, k_0 - 1$ (this is meaningless when $k_0 = 0$) and

$$\fint_{B_{\tau^{k_0} R_0}(y)} |V(Du) - (V(Du))_{y,\tau^{k_0} R_0}|^2 \, dx < \delta_3 \fint_{B_{\tau^{k_0} R_0}(y)} |V(Du)|^2 \, dx. \qquad (79)$$

If $k_0 = 0$, in view of Lemma 5.1 with $R = R_0$, the equality $\tilde{\beta} = \gamma$ and (74), for every $r \in (0, R_0)$ we have

$$\begin{aligned}\fint_{B_r(y)} |V(Du) - (V(Du))_{y,r}|^2 \, dx &\leq c \left(\frac{r}{R_0}\right)^{2\gamma} \fint_{B_{R_0}(y)} |V(Du) - (V(Du))_{y,R_0}|^2 \, dx \\ &\quad + c\, r^{2\gamma} \fint_{B_{R_0}(y)} |V(Du)|^2 \, dx \\ &\leq c\, \varepsilon_1 \left(\frac{r}{R_0}\right)^{2\gamma} + c\, r^{2\gamma}(M+1)\end{aligned}$$

and so

$$\fint_{B_r(y)} \frac{|V(Du) - (V(Du))_{y,r}|^2}{r^{2\gamma}} \, dx \leq c \left(\frac{\epsilon_1}{R_0^{2\gamma}} + M + 1\right). \qquad (80)$$

It remains the case when (78) and (79) hold for some $k_0 \geq 1$. For $r \in [\tau^{k_0} R_0, R_0)$, we obtain (77) by the very same argument already used when (75) holds. On the other hand, if $r \in (0, \tau^{k_0} R_0)$, by Lemma 5.1 with $R = \tau^{k_0} R_0$ and (77) with $r = \tau^{k_0} R_0$, we have

$$\begin{aligned}&\fint_{B_r(y)} |V(Du) - (V(Du))_{y,r}|^2 \, dx \\ &\quad \leq c \left(\frac{r}{\tau^{k_0} R_0}\right)^{2\gamma} \fint_{B_{\tau^{k_0} R_0}(y)} |V(Du) - (V(Du))_{y,\tau^{k_0} R_0}|^2 \, dx + c\, r^{2\gamma} \fint_{B_{\tau^{k_0} R_0}(y)} |V(Du)|^2 \, dx \\ &\quad \leq c\, \frac{\varepsilon_1}{2\tau^{n+2\gamma}} \left(\frac{r}{R_0}\right)^{2\gamma} + c\, r^{2\gamma} \fint_{B_{\tau^{k_0} R_0}(y)} |V(Du)|^2 \, dx.\end{aligned}$$

Moreover, arguing as in (76), in view of (78) and Lemma 4.4, we have that

$$\fint_{B_{\tau^{k_0-1} R_0}(y)} |V(Du) - (V(Du))_{y,\tau^{k_0-1} R_0}|^2 \, dx \leq \cdots \leq \fint_{B_{R_0}(y)} |V(Du) - (V(Du))_{y,R_0}|^2 \, dx \leq \frac{\varepsilon_1}{2}$$

and

$$\fint_{B_{\tau^{k_0} R_0}(y)} |V(Du)|^2 \, dx \leq 2\tau^{-n} \fint_{B_{\tau^{k_0-1} R_0}(y)} |V(Du) - (V(Du))_{y,\tau^{k_0-1}R_0}|^2 \, dx$$
$$+ 2 \fint_{B_{\tau^{k_0-1} R_0}(y)} |V(Du)|^2 \, dx$$
$$\leq 2\left(\tau^{-n} + \delta_3^{-1}\right) \fint_{B_{\tau^{k_0-1} R_0}(y)} |V(Du) - (V(Du))_{y,\tau^{k_0-1}R_0}|^2 \, dx$$
$$\leq \left(\tau^{-n} + \delta_3^{-1}\right) \varepsilon_1.$$

Therefore, for every $r \in (0, R_0)$ we have

$$\fint_{B_r(y)} \frac{|V(Du) - (V(Du))_{y,r}|^2}{r^{2\gamma}} \, dx \leq \frac{c\,\varepsilon_1}{\tau^{n+2\gamma} R_0^{2\gamma}} + c\,\left(\tau^{-n} + \delta_3^{-1}\right) \varepsilon_1. \tag{81}$$

Consequently, by (77), (80) and (81) we have that the inequality

$$\fint_{B_r(y)} \frac{|V(Du) - (V(Du))_{y,r}|^2}{r^{2\gamma}} \, dx \leq C$$

holds for every ball $B_r(y)$ with $y \in B_{R_1}(x_0)$ and for every $r \in (0, R_0)$ and this implies that $V(Du)$ is in $C^\gamma(B_{R_1}(x_0))$ and so $u \in C^\alpha(B_{R_1}(x_0))$ for some $\alpha = \alpha()$ □

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