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*Original*
Instability of free interfaces in premixed flame propagation / Brauner, C. -M.; Lorenzi, L.. - In: DISCRETE AND CONTINUOUS DYNAMICAL SYSTEMS. SERIES S. - ISSN 1937-1632. - 14:2(2021), pp. 575-596. [10.3934/dcdss.2020363]

*Availability:*
This version is available at: 11381/2888438 since: 2022-01-17T10:30:17Z

*Publisher:*
American Institute of Mathematical Sciences

*Published*
DOI:10.3934/dcdss.2020363

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24 April 2024

DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES S

doi:10.3934/dcdss.2020363

# INSTABILITY OF FREE INTERFACES IN PREMIXED FLAME PROPAGATION

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*Dedicated to Michel Pierre on his 70th birthday, in friendship.*

Abstract. In this survey, we are interested in the instability of flame fronts regarded as free interfaces. We successively consider a classical Arrhenius kinetics (*thin flame*) and a stepwise ignition-temperature kinetics (*thick flame*) with two free interfaces. A general method initially developed for thin flame problems subject to interface jump conditions is proving to be an effective strategy for smoother thick flame systems. It relies on the elimination of the free interface(s) and reduction to a fully nonlinear parabolic problem. The theory of analytic semigroups is a key tool to study the linearized operators.

1. **Introduction.** Stability analysis of free boundary problems, or equivalently free interface problems, have been for long a challenging issue (see, e.g., [27, 18]). To mention a few examples, stability or instability questions related to the Stefan problem in all its forms have generated considerable interest since the pioneering work [15] (see, e.g., [19, 20] and the references therein). On the other hand, variational inequalities is an important class of free boundary problems, that includes the obstacle problem (see [3, 28]). Spectral stability in nonlinear variational inequalities has been addressed via conical linearization techniques (see [17, 16]).

In combustion theory, instability of propagating premixed flames is a complex and difficult phenomenon. The basic propagation mode exhibits two main mechanisms of destabilization: one due to the thermal expansion of the gas known as the hydrodynamic instability, and the thermal-diffusive instability which is a result of the competition between the exothermic reaction and the heat diffusion.

2010 *Mathematics Subject Classification.* Primary: 35R35; Secondary 35B35, 35K40, 80A25.

*Key words and phrases.* Free boundary problems with one or two free interfaces, traveling wave solutions, instability, fully nonlinear parabolic problems, analytic semigroups, combustion.

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The thermal-diffusive instability manifests itself by generating a cellular structure, which in turn exhibits chaotic dynamics (see [29, 30]).

The propagation of premixed flames is usually described by the conventional diffusional-thermal model with standard Arrhenius kinetics. Formal asymptotic methods based on large activation energy have allowed simpler descriptions, especially when the *thin flame* zone is replaced by a free interface, called the flame front, which separates burned and unburned gases. At the flame front, the temperature and mass fraction gradients are discontinuous. In the paper [8] and related works (see in particular [6, 7, 10, 12, 22, 23, 24]), we presented a method by which the flame front can be eliminated and, mutatis mutandis, the system reformulated as a *fully nonlinear problems* (see [25]). This new formulation has proved effective for local existence and stability analysis (see above references), and also numerical simulation (see [2]).

On the other hand, models describing dynamics of *thick flames* with stepwise ignition-temperature kinetics have recently received considerable attention (see [4]). There are differences with the Arrhenius kinetics: for example in the case of zero-order stepwise kinetics there are two free interfaces. Moreover, at the free interface(s), the temperature and mass fraction gradients are this time continuous. In this survey, we point out that the general method of [8], which was developed initially for solving *thin flame* problems, works equally well on *thick flame* models with ignition-temperature kinetics, see [1, 5, 11]. In this respect, the method is quite general and suitable for a wide range of free interface problems.

Finally, we note that both free interface problems (Arrhenius and ignition-temperature kinetics) do not fall within the class of Stefan problems, as there is no specific condition on the velocity of the interface(s). However, at least near planar fronts, we are able to associate the velocity with a combination of spatial derivatives up to the second order (*second-order Stefan condition*, see [9] for a general remark).

The paper is organized as follows: Sections 2 and 3 are respectively devoted to the Arrhenius kinetics (*thin flame*) and stepwise ignition-temperature kinetics with zero-order reaction (*thick flame*), that we treat in parallel ways, identifying common ground and differences. As has been said, a main difference is that the stepwise ignition-temperature model presents two free interfaces, the ignition and the trailing fronts. Both models admit one-dimensional, planar traveling wave solutions, unique up to translation. Then, we introduce perturbations of the planar solutions and interfaces. Following the method of [8], we derive in both cases a fully nonlinear parabolic problem of the form:

$$\begin{cases} D_t\boldsymbol{u}(t,\cdot) = \mathscr{L}\boldsymbol{u} + \mathscr{F}(\boldsymbol{u}(t,\cdot)), \quad t>0, \\ \mathscr{B}\mathbf{u}(t,\cdot) = \mathscr{G}(\boldsymbol{u}(t,\cdot)). \end{cases} \tag{1.1}$$

Here, the stepwise kinetics problem presents a substantial difficulty: specifically, the trailing interface does not satisfy the non-degeneracy condition of [8].

The local existence of a solution to problem (1.1) is obtained in Section 4 through a fixed point argument which requires to first solve the linearized version of such a problem. In order to fix the function spaces, one has to take into account the particular nature of the nonlinearities $\mathscr{F}$: due to the dependence on traces of second-order derivatives, if one is interested in classical solutions, then optimal Hölder regularity is needed.

In Section 5, we study the stability of the null solution of (1.1). In the two problems under investigation, the spectrum splits into two parts, namely the continuous spectrum which consists of a parabola in the left halfplane tangent to the imaginary axis at the origin, and the point spectrum which is the set of all complex numbers solutions of the so-called *dispersion relation*. Here too, the stepwise kinetics problem presents additional difficulties, because the associated dispersion relation has no algebraic solutions, see Theorem 5.2. Finally, instability of the zero solution of Equation (1.1), and thus of the traveling waves solutions, is established for both problems; the results are summarized in Theorem 4.2. An important tool is a result of [21] adapted in Theorem 5.3. However, Theorem 4.2 does not give any information about the instability of the front. The latter issue is the subject of the final Section 6, especially Theorem 6.3.

**Notation.** By $\mathbb{R}^2_-$, we denote the subset of $\mathbb{R}^2$ with negative first component. Similarly, for a given $\ell > 0$, we denote by $S_\ell$ the strip $\mathbb{R} \times (-\ell/2, \ell/2)$ and by $S_\ell^+$ (resp. $S_\ell^-$) the subset of $S_\ell$ of elements with positive (resp. negative) first component. Finally, $S_\ell^R$ denotes the set $(R, +\infty) \times (-\ell/2, \ell/2)$.

For $i, j, k \in \mathbb{N}$, we write, respectively, $D_t^i$, $D_x^j$ and $D_y^k$ to denote the derivative $\partial^i/\partial t^i$, $\partial^j/\partial x^j$ and $\partial^k/\partial y^k$. We also use the subscripts $t$, $x$ and $y$ to denote derivatives with respect to $t$, $x$ and $y$. For instance, $u_{txxy}$ denotes the derivative $D_t D_x^2 D_y u$. We use bold style to denote vector valued functions. If $\boldsymbol{u} : D \subset \mathbb{R}^d \to \mathbb{R}^m$, we denote by $u_1, \ldots, u_m$ its components. If $\Omega$ is an open subset of $\mathbb{R}^d$, then we denote by $C_b^\alpha(\Omega; \mathbb{R}^m)$, $\alpha \in (0,1)$, the usual space of bounded and $\alpha$-Hölder continuous functions over $D$ and denote by $\|\cdot\|_{C^\alpha(\Omega;\mathbb{R}^m)}$ the classical norm defined as the sum of the sup-norm and the Hölder seminorm. We use the same notation when $\alpha > 1$ to denote the set of functions which are continuously differentiable up to the $[\alpha]$-th-order such that the derivatives of order $[\alpha]$ are $(\alpha - [\alpha])$-Hölder continuous over $\Omega$. Here (and just here), $[\alpha]$ stands for the integer part of $\alpha$. The norm of a function $\boldsymbol{u}$ in this space is defined as the sum of sup-norms of the function and its derivatives up to the order $[\alpha]$ plus the sum of the $(\alpha - [\alpha])$-seminorms of all the derivatives of order $[\alpha]$ of $\boldsymbol{u}$. If $I \subset \mathbb{R}$ is an interval and $\Omega \subset \mathbb{R}^d$ is an open set, $C_b^{\alpha/2,\alpha}(I \times \Omega; \mathbb{R}^k)$ denotes the set of all bounded functions $\boldsymbol{u} : I \times \Omega \times \mathbb{R}^k$ which are $\alpha$-Hölder continuous with respect to the parabolic distance of $\mathbb{R}^{d+1}$, which is defined by $d((t,x),(s,y)) = \sqrt{|t-s| + |x-y|^2}$. Its norm is the sum of the sup-norm and the $\alpha$-Hölder norm of $\boldsymbol{u}$. Similarly, $C_b^{1+\alpha/2,2+\alpha}(I \times \Omega; \mathbb{R}^k)$ is the set of functions $\boldsymbol{u}$ which admit the classical derivatives $D_t \boldsymbol{u}$ and $D_x^{\gamma_1} D_y^{\gamma_2} \boldsymbol{u}$ in $I \times \Omega$ for $\gamma_1 + \gamma_2 \le 2$ such that the derivatives $D_t \boldsymbol{u}$ and $D_x^{\gamma_1} D_y^{\gamma_2} \boldsymbol{u}$, when $\gamma_1 + \gamma_2 = 2$ belong to $C^{\alpha/2,\alpha}(I \times \Omega; \mathbb{R}^k)$. The norm of $\boldsymbol{u} \in C_b^{1+\alpha/2,2+\alpha}(I \times \Omega; \mathbb{R}^k)$ is the sum of the sup-norm of $\boldsymbol{u}$ and all its derivatives plus the sum of the Hölder seminorms of $D_t \boldsymbol{u}$ and $D_x^{\gamma_1} D_y^{\gamma_2} \boldsymbol{u}$, when $\gamma_1 + \gamma_2 = 2$. Given a scalar function $v : I \to \mathbb{R}$, where $I$ is an interval and $x_0$ is an interior point of $I$, we denote by $[v]_{x_0}$ the jump of $v$ at $x_0$, i.e., provided the limits exist,

$$[v]_{x_0} = \lim_{x \to x_0^+} v(x) - \lim_{x \to x_0^-} v(x).$$

## 2. Flame propagation with Arrhenius kinetics (*thin flames*).

### 2.1. The diffusional-thermal model.

Flames constitute a complex physical system involving fluid dynamics, multistep chemical kinetics, as well as molecular and radiative transfer. An important parameter is the Lewis number Le, corresponding to the ratio of thermal and mass diffusivities. The laminar flames of low-Lewis-number premixtures are known to display diffusive-thermal instability, responsible for the formation of a non-steady cellular structure (see [30]). However, the cellular instability may be successfully captured by a model involving only two equations: the heat equation for the system's temperature and the diffusion equation for the deficient reactant's concentration. In suitably chosen units, the so-called diffusional-thermal model reads (see, e.g., [13]):

$$T_t = T_{xx} + T_{yy} + \omega(T, Y), \tag{2.1}$$

$$Y_t = \mathrm{Le}^{-1}(Y_{xx} + Y_{yy}) - \omega(T, Y). \tag{2.2}$$

Here, $T$ is the scaled temperature and $Y$ the scaled concentration of the deficient reactant. The scaled reaction rate $\omega(T, Y)$ is given by the Arrhenius law (see [13])

$$\omega(T, Y) = \frac{1}{2}\mathrm{Le}^{-1}\beta^2 Y \exp\left(\frac{\beta(T-1)}{\sigma + (1-\sigma)T}\right), \tag{2.3}$$

where $\beta$ is the dimensionless Zeldovich number, assumed to be large, and $\sigma$ is the thermal expansion coefficient. The normalizing pre-exponential factor ensures that the planar flame propagates at speed close to unity when $\beta \gg 1$.

Due to the distributed nature of the reaction rate $\omega$, it is still difficult to theoretically explore the system (2.1)-(2.3). One, therefore, turns to the conventional high activation energy limit $\beta \to +\infty$, which converts the reaction rate term into a localized source distributed over a free interface, $x = \xi(t, y)$, the flame front. The study of a *thin flame* propagation is thus reduced asymptotically to a free interface problem.

2.2. **Near-equidiffusional flames.** To ensure that the free interface model does not involve large parameters, one combines the limit of large activation energy $\beta \to +\infty$ with the requirement that the product

$$\gamma = \frac{1}{2}\beta(1 - \mathrm{Le}) \tag{2.4}$$

remains finite, i.e., the Lewis number Le should be closed to unity. This is the Near-Equidiffusive Flames model, in short NEF, introduced in [26]. Here, we consider only the case where $\gamma$ is positive, i.e., the case of high mobility of the deficient reactant. Expanding $T$ and $Y$ in a series of powers of $\beta^{-1}$, where $\beta$ is the Zeldovich number, one ends up after some recombinations with the following free interface problem for temperature $\Theta$ and enthalpy $S$ (see [13, 14] for further details about the NEF theory)

$$\begin{cases} \Theta_t(t,x,y) = \Delta\Theta(t,x,y), & t > 0, \quad x < F(t,y), \quad y \in \mathbb{R}, \\ \Theta(t,x,y) = 1, & t > 0, \quad x \geq F(t,y), \quad y \in \mathbb{R}, \\ S_t(t,x,y) = \Delta S(t,x,y) - \gamma\Delta\Theta(t,x,y), & t > 0, \quad x \neq F(t,y) \quad y \in \mathbb{R}. \end{cases} \tag{2.5}$$

The functions $\Theta$ and $S$ are continuous at the front, whereas their normal derivatives (say, $D_n\Theta$ and $D_nS$) satisfy the following jump conditions at the interface

$$\begin{cases} \displaystyle\lim_{x\to F(t,y)^+} D_n\Theta(t,x,y) - \lim_{x\to F(t,y)^-} D_n\Theta(t,x,y) = -\exp(S(t,F(t,y),y)), \\ \displaystyle\lim_{x\to F(t,y)^+} D_n[S(t,x,y) - \gamma\Theta(t,x,y)] - \lim_{x\to F(t,y)^-} D_n[S(t,x,y) - \gamma\Theta(t,x,y)]. \end{cases} \tag{2.6}$$

Further, as $x$ tends to $\pm\infty$, the following conditions are prescribed

$$\Theta(t,-\infty,y)=S(t,-\infty,y)=S(t,+\infty,y)=0. \tag{2.7}$$

As it is easily verified, this system admits a planar traveling wave solution, with velocity $-1$, which reads in the coordinate $z=x+t$:

$$\Theta^0(z)=\begin{cases} e^z, & z\le 0,\\ 1, & z>0,\end{cases}\qquad S^0(z)=\begin{cases}\gamma z e^z, & z\le 0,\\ 0, & z>0.\end{cases} \tag{2.8}$$

2.3. **Derivation of the fully nonlinear problem (Arrhenius kinetics).** It is standard to fix the interface at the origin by setting $F(t,y)=-t+s(t,y)$, $\xi=x-F(t,y)=z-s(t,y)$. In this new framework:

$$\begin{cases}\Theta_t+(1-s_t)\Theta_\xi=\Delta_s\Theta, & \text{in } (0,+\infty)\times(-\infty,0)\times\mathbb{R},\\ \Theta=1, & \text{in } (0,+\infty)\times(0,+\infty)\times\mathbb{R},\\ S_t+(1-s_t)S_\xi=\Delta_s S-\gamma\Delta_s\Theta, & \text{in } (0,+\infty)\times\mathbb{R}\setminus\{0\}\times\mathbb{R},\end{cases} \tag{2.9}$$

where

$$\Delta_s=[1+(s_y)^2]D_{\xi\xi}+D_{yy}-s_{yy}D_\xi-2s_yD_{\xi y}.$$

The jump conditions (computed at $\xi=0$) are $[\Theta]_0=[S]_0=0$ and

$$\sqrt{1+(s_y)^2}\,[\Theta_\xi]_0=-\exp(S),\qquad [S_\xi]_0=\gamma[\Theta_\xi]_0, \tag{2.10}$$

which follow from (2.6). The main step now is the ansatz (see [8, 23]):

$$\Theta=\Theta^0+s\Theta^0_x+u_1,\qquad S=S^0+sS^0_x+u_2,$$

which, taking advantage of the boundary conditions

$$[\Theta]_0=[\Theta^0]_0=0,\qquad [\Theta^0_x]_0=[\Theta^0_x]_0=-1,$$

enables us to express the interface $s$ in terms of the trace of $u_2$ at $\xi=0^-$:

$$s(t,y)=[u_2(t,\cdot,y)]_0=-u_2(t,0^-,y),\qquad t>0,\ \ y\in\mathbb{R}. \tag{2.11}$$

Replacing (2.11) in (2.9) and (2.10), we obtain a system in the only unknowns $u_1$, $u_2$. However, it is convenient to rewrite it in the standard form of a system in $\mathbb{R}^2_-$, setting $\boldsymbol{u}=(u_1,u_2,u_3)$ where $u_3(t,\xi,y)=u_2(t,-\xi,y)$ for $\xi<0$ and $y\in\mathbb{R}$. We get

$$\begin{cases}\boldsymbol{u}_t=\mathscr{L}\boldsymbol{u}+\mathscr{F}_0(\boldsymbol{u})-v_t(\cdot,0,\cdot)\Psi(\boldsymbol{u}), & \text{in } (0,+\infty)\times\mathbb{R}^2_-,\\ \mathscr{B}\boldsymbol{u}=\mathscr{G}(\boldsymbol{u}), & \text{in } (0,+\infty)\times\mathbb{R},\end{cases} \tag{2.12}$$

where the linear operator $\mathscr{L}$ is given by

$$\mathscr{L}\boldsymbol{v}=\mathscr{L}(v_1,v_2,v_3)=(\Delta v_1-D_\xi v_1,\ \Delta v_2-D_\xi v_2-\gamma\Delta v_1,\ \Delta v_3+D_\xi v_3), \tag{2.13}$$

the linear boundary operator $\mathscr{B}$ has three components $\mathscr{B}_1$, $\mathscr{B}_2$ and $\mathscr{B}_3$, defined by

$$\begin{cases}\mathscr{B}_1\boldsymbol{v}=\gamma v_1(0,\cdot)-v_2(0,\cdot)+v_3(0,\cdot),\\ \mathscr{B}_2\boldsymbol{v}=\gamma v_1(0,\cdot)+\gamma D_xv_1(0,\cdot)-D_xv_2(0,\cdot)-D_xv_3(0,\cdot),\\ \mathscr{B}_3\boldsymbol{v}=v_1(0,\cdot)+v_3(0,\cdot)-D_xv_1(0,\cdot),\end{cases} \tag{2.14}$$

$\mathscr{F}_0(\boldsymbol{v}) = (f_1(\boldsymbol{v}), f_2(\boldsymbol{v}), f_3(\boldsymbol{v}))$ with

$$\begin{aligned}
f_1(\boldsymbol{v}) =&(D_y v_1(0,\cdot))^2[\Theta^0_{\xi\xi} - v_1(0,\cdot)\Theta^0_{\xi\xi\xi} + D_{\xi\xi}v_1] + D_{yy}v_1(0,\cdot)[D_\xi v_1 - v_1(0,\cdot)\Theta^0_{\xi\xi}]\\
&+ 2D_y v_1(0,\cdot)[D_{\xi y}v_1 - D_y v_1(0,\cdot)\Theta^0_{\xi\xi}],\\
f_2(\boldsymbol{v}) =&(D_y v_1(0,\cdot))^2[S^0_{\xi\xi} - v(0,\cdot)S^0_{\xi\xi\xi} + D_{\xi\xi}v_2] + D_{yy}v_1(0,\cdot)[D_\xi v_2 - v_1(0,\cdot)S^0_{\xi\xi}]\\
&+ 2D_y v_1(0,\cdot)[D_{\xi y}v_2 - D_y v_1(0,\cdot)S^0_{\xi\xi}] - \gamma f_1(\boldsymbol{v}),\\
f_3(\boldsymbol{v}) =&(D_y v_1(0,\cdot))^2 D_{\xi\xi}v_3 - 2D_y v_1(0,\cdot)D_{\xi y}v_3 - D_{yy}v_1(0,\cdot)D_x v_3,
\end{aligned}$$

on smooth enough functions $\boldsymbol{v} : \mathbb{R}^2_- \to \mathbb{R}^3$. Finally,

$$\Psi(\boldsymbol{v}) = (-v_1(0,\cdot)\Theta^0_{\xi\xi} + D_\xi v_1,\ -v_1(0,\cdot)S^0_{\xi\xi} + D_\xi v_2,\ -D_\xi v_3),$$

and

$$\mathscr{G}(\boldsymbol{v}) = (0,0,g(\boldsymbol{v})), \qquad g(\boldsymbol{v}) = 1 + h(0,\cdot) - \frac{e^{v_3(0,\cdot)}}{\sqrt{1+(D_y v_1(0,\cdot))^2}}.$$

However, the differential system in (2.12) contains $D_t u_1(t,0,y)$ in the right-hand side. The main point is that Equation (2.11) yields $D_t u_1(t,0,y) = -s_t(t,y)$. The first equation in (2.12) reads for $u_1$ and $D_x u_1$ small enough:

$$\begin{aligned}
D_t u_1(t,\xi,y) =&\Delta v(t,\xi,y) - D_\xi u_1(t,\xi,y) + (f_1(\boldsymbol{u}(t\cdot,\cdot))(\xi,y)\\
&- D_t u_1(t,0,y)[-u_1(t,0,y)e^\xi + D_\xi u_1(t,\xi,y)],
\end{aligned}$$

so that if we evaluate it at $\xi = 0$ then we get the formula:

$$s_t = -\frac{\Delta u_1(\cdot,0,\cdot) - D_\xi u_1(\cdot,0,\cdot) + f_1(\boldsymbol{u}(t,\cdot,\cdot))}{1 - u_1(\cdot,0,\cdot) + D_x u_1(\cdot,0,\cdot)}. \tag{2.15}$$

Therefore, the velocity of the interface $s$ is expressed in terms of the trace of first- and second-order derivatives of $\boldsymbol{u}$ at the interface itself (see [9]).

Plugging (2.15) in (2.12), we get the following fully nonlinear parabolic problem for $\boldsymbol{u}$:

$$\begin{cases}
\boldsymbol{u}_t(t,\xi,y) = \mathscr{L}\boldsymbol{u}(t,\xi,y) + (\mathscr{F}(\boldsymbol{u}(t,\cdot,\cdot)))(\xi,y), & t \ge 0, \quad \xi < 0, \quad y \in \mathbb{R},\\
(\mathscr{B}\boldsymbol{u}(t,\cdot))(y) = \mathscr{G}(\boldsymbol{u}(t,\cdot))(y), & t \ge 0, \qquad\qquad y \in \mathbb{R},
\end{cases} \tag{2.16}$$

with

$$\mathscr{F}(\boldsymbol{v}) = \mathscr{F}_0(\boldsymbol{v}) - \frac{\Delta v_1(0,\cdot) - D_\xi v_1(0,\cdot) + (f_1(\boldsymbol{v})(0,\cdot))}{1 - v_1(0,\cdot) + D_\xi v_1(0,\cdot)}\Psi(\boldsymbol{v})$$

on smooth enough functions $\boldsymbol{v} : \mathbb{R}^2_- \to \mathbb{R}^3$.

## 3. Flame propagation with stepwise temperature kinetics (*thick flames*).

Models with stepwise ignition-temperature kinetics (see [4]) are substantially different from those arising in conventional thermo-diffusive combustion with the standard Arrhenius kinetics at large Zeldovich number. Here, we are going to focus on a zero-order stepwise kinetics model, see [11] for a model with stepwise ignition-temperature kinetics and a first-order reaction.

3.1. **Zero-order stepwise kinetics model.** For the zero-order stepwise kinetics (see [1, 4, 5]), the model reads (compare to (2.1)-(2.3)):

$$\begin{cases} T_t = T_{xx} + T_{yy} + W(T,Y), \\ Y_t = \mathrm{Le}^{-1}(Y_{xx} + Y_{yy}) - W(T,Y), \end{cases} \tag{3.1}$$

where the reaction rate $W(T,Y)$ is given by

$$W(T,Y) = \begin{cases} A, & \text{if} \quad T \geq \theta_i \quad \text{and} \qquad Y > 0, \\ 0, & \text{if} \quad T < \theta_i \quad \text{and/or} \quad Y = 0. \end{cases} \tag{3.2}$$

Here, $0 < \theta_i < 1$ is the ignition temperature and $A > 0$ is a normalizing factor. For the first-order stepwise kinetics, the reaction rate is more standard and reads $W(T,Y) = AYH(T - \theta_i)$, where $H$ stands for the Heaviside function (see [4],[11]).

There are two principal differences with Arrhenius kinetics. The first one is that the reaction zone is of order unity, while in the case of Arrhenius kinetics the reaction zone is infinitely thin. This fact suggests to refer to traveling fronts for stepwise temperature kinetics as *thick flames*, in contrast to *thin flames* arising in Arrhenius kinetics. The second, even more important difference, is the following. In the case of Arrhenius kinetics (see Section 2), there is a single interface separating burned and unburned gases. In contrast to that, in case of zero-order ignition-temperature kinetics given by (3.2), there are two interfaces: the *ignition interface* where $T = \theta_i$ and the *trailing interface* being defined as a largest value of $x$ where the concentration is equal to zero.

Denoting by $x = F(t,y)$ the ignition interface and by $x = G(t,y)$ the trailing interface, the model that we consider in this section, set in the strip $S_\ell = \mathbb{R} \times (-\ell/2, \ell/2)$ of $\mathbb{R}^2$, is the following one:

$$\begin{cases} T_t(t,x,y) = \Delta T(t,x,y), & x < G(t,y), \\ Y(t,x,y) = 0, & x < G(t,y), \\ T_t(t,x,y) = \Delta T(t,x,y) + A, & G(t,y) < x < F(t,y), \\ Y_t(t,x,y) = \mathrm{Le}^{-1}\Delta Y(t,x,y) - A, & G(t,y) < x < F(t,y), \\ T_t(t,x,y) = \Delta T(t,x,y), & x > F(t,y), \\ Y_t(t,x,y) = \mathrm{Le}^{-1}\Delta Y(t,x,y), & x > F(t,y), \end{cases} \tag{3.3}$$

where the functions $T$ and $Y$ are continuous across the interfaces for $t > 0$ and their normal derivatives are continuous as well at the interfaces.

This system admits a one-dimensional traveling wave (planar) solution $(T^0, Y^0)$ which propagates with constant positive velocity $V$ (see [4, Section 4]).[1] It is convenient to choose the normalizing factor $A = 1/R$ in such a way that $V = 1$, where the positive number $R = R(\theta_i)$ is given by:

$$\theta_i R = 1 - e^{-R}, \quad 0 < \theta_i < 1.$$

[1] The attentive reader will have noticed that the flame front moves here from $-\infty$ to $+\infty$, while in Section 2 it propagates from $+\infty$ to $-\infty$; it is just a matter of convention.

With this choice, the traveling wave solution reads as follows in the coordinate $z = x - t$:

$$T^0(z) = \begin{cases} 1, & z \le 0, \\ 1 + \dfrac{1 - z - e^{-z}}{R}, & 0 < z < R, \\ \theta_i e^{R-z}, & x \ge R. \end{cases}$$

$$Y^0(z) = \begin{cases} 0, & z \le 0, \\ \dfrac{e^{-\mathrm{Le} z} - 1 + \mathrm{Le} z}{\mathrm{Le}\, R}, & 0 < z < R, \\ 1 + \dfrac{1 - e^{\mathrm{Le} R}}{\mathrm{Le}\, R e^{\mathrm{Le} z}}, & z \ge R. \end{cases}$$

3.2. **Derivation of the fully nonlinear problem (stepwise kinetics).** As in Subsection 2.3, we look for solutions close to the traveling wave solution and we transform system (3.3) into an equivalent system set in a fixed domain. There are some differences and some additional difficulties, one of those is the presence of two moving boundaries as already outlined. We list here below the steps to be followed to get to the final system.

(1) *Free interfaces as small perturbations of the interfaces of the traveling wave:* we write $F$ and $G$ in the form

$$G(t, y) = g(t, y), \qquad F(t, y) = R + f(t, y),$$

with $f$ and $g$ smooth and small enough.
(2) *Cut-off function:* we introduce a smooth function $\beta : \mathbb{R} \to \mathbb{R}$, which is compactly supported in $(-2\delta, 2\delta)$ and equals one in $(-\delta, \delta)$ for some $\delta > 0$.
(3) *New coordinates*: we replace the $x$ variable with the new variable $\xi$ defined by $x = t + \xi + \varrho(t, \xi, y)$, where

$$\varrho(t, \xi, y) = \beta(\xi) g(t, y) + \beta(\xi - R) f(t, y).$$

In the new systems of variables $(t, \xi, y)$, the trailing front is fixed at $\xi = 0$, whereas the ignition front is fixed at $\xi = R$.
(4) *New unknowns:* in the spirit of Subsection 2.3, we introduce the ansatz:

$$T(t, \xi, y) = T^0(\xi) + \varrho(t, \xi, y) T^0_\xi(\xi) + u(t, \xi, y), \tag{3.4}$$
$$Y(t, \xi, y) = Y^0(\xi) + \varrho(t, \xi, y) Y^0_\xi(\xi) + v(t, \xi, y), \tag{3.5}$$

which can be interpret of a sort of Taylor expansion of $(T, Y)$ around the traveling wave solution $(T^0, Y^0)$. Functions $u$ and $v$ play the role of a remainder and for stability issues we can assume that $u$ and $v$ are "sufficiently small" in a sense still to be made precise.
Expanding $(1 + \varrho_\xi)^{-1} = 1 - \varrho_\xi + (\varrho_\xi)^2 (1 + \varrho_\xi)^{-1}$, after a long but rather straightforward computation, we can determine the differential equations for the unknowns

$u$ and $v$ in the new variables $(t, \xi, y)$. They read as follows:

$$\begin{aligned}
u_t =& u_\xi + \Delta u + \varrho_t (1+\varrho_\xi)^{-1} (\varrho T^0_{\xi\xi} + u_\xi) - (1+\varrho_\xi)^{-3} \varrho_{\xi\xi} (1+\varrho_y^2)(\varrho T^0_{\xi\xi} + u_\xi) \\
&- (1+\varrho_\xi)^{-1} \big[ (\varrho_\xi + \varrho_{yy})(\varrho T^0_{\xi\xi} + u_\xi) + 2\varrho_y (\varrho_y T^0_{\xi\xi} + u_{\xi y}) \big] \\
&+ (1+\varrho_\xi)^{-2} \big[ 2\varrho_y \varrho_{\xi y} (\varrho T^0_{\xi\xi} + u_\xi) + (\varrho_y^2 - \varrho_\xi^2)(\varrho T^0_{\xi\xi\xi} + T^0_{\xi\xi} + u_{\xi\xi}) \\
&\qquad - 2\varrho_\xi (\varrho T^0_{\xi\xi\xi} - \varrho_y^2 T^0_{\xi\xi} + u_{\xi\xi}) \big],
\end{aligned} \tag{3.6}$$

in $(0, +\infty) \times (\mathbb{R} \setminus \{0, R\}) \times (-\ell/2, \ell/2)$,

$$\begin{aligned}
v_t =& v_\xi + \mathrm{Le}^{-1} \Delta v + \varrho_t (1+\varrho_\xi)^{-1} (\varrho Y^0_{\xi\xi} + v_\xi) \\
&- \mathrm{Le}^{-1} (1+\varrho_\xi)^{-3} \varrho_{\xi\xi} (1+\varrho_y^2)(\varrho Y^0_{\xi\xi} + v_\xi) \\
&- \mathrm{Le}^{-1} (1+\varrho_\xi)^{-1} \big[ (\mathrm{Le}\, \varrho_\xi + \varrho_{yy})(\varrho Y^0_{\xi\xi} + v_\xi) + 2\varrho_y (\varrho_y Y^0_{\xi\xi} + v_{\xi y}) \big] \\
&+ \mathrm{Le}^{-1} (1+\varrho_\xi)^{-2} \big[ 2\varrho_y \varrho_{\xi y} (\varrho Y^0_{\xi\xi} + v_\xi) + (\varrho_y^2 - \varrho_\xi^2)(\varrho Y^0_{\xi\xi\xi} + Y^0_{\xi\xi} + v_{\xi\xi}) \\
&\qquad - 2\varrho_\xi (\varrho Y^0_{\xi\xi\xi} - \varrho_y^2 Y^0_{\xi\xi} + v_{\xi\xi}) \big],
\end{aligned} \tag{3.7}$$

in $(0, +\infty) \times [(0, R) \cup (R, +\infty)] \times (-\ell/2, \ell/2)$,

$$v = 0 \qquad \text{in } (0, +\infty) \times (-\infty, 0) \times (-\ell/2, \ell/2). \tag{3.8}$$

The right-hand sides of the previous two equations contain the function $\varrho$, so that they depend on the functions $f$ and $g$. To get rid of these terms, we argue as follows.

(6) *Writing $\varrho$ in terms of $u$ and (the $\xi$-derivative of) $v$:* the derivative $T^0_\xi(R)$ does not vanish at the interface $x = R$ and gives rise to a kind of transversality or non-degeneracy condition (see [8]). In particular, since $T^0_\xi(R) = -\theta_i$, evaluating Equation (3.4) at $\xi = R$, we deduce that

$$f(t, y) = \theta_i^{-1} u(t, R, y), \qquad t \in (0, +\infty), \;\; y \in (-\ell/2, \ell/2). \tag{3.9}$$

The trailing interface has a different nature with respect to the ignition interface. Indeed, since $T^0_x(0) = Y^0_x(0) = 0$, the non-degeneracy condition of [8] is not verified and this prevents us from writing $g$ in terms of $u$ or $v$. On the other hand, $T^0_\xi(0^+) = -R^{-1}$ and $Y^0_{\xi\xi}(0^+) = R^{-1}\mathrm{Le}$, so that they do not vanish. So, we can write

$$g(t, y) = -R\, \mathrm{Le}^{-1} v_\xi(t, 0^+, y), \qquad t \in (0, +\infty), \;\; y \in (-\ell/2, \ell/2). \tag{3.10}$$

It thus follows that

$$\varrho(t, \xi, y) = {\theta_i}^{-1} \beta(\xi - R) u(t, R, y) - R\, \mathrm{Le}^{-1} \beta(\xi) w(t, 0^+, y). \tag{3.11}$$

Although the front $g$ could be eliminated, the method used in Subsection 2.3, which has been introduced in [8], is not applicable since $g$ is related to the derivative of $v$ and not to $v$. To overcome this difficulty, we look at the problem satisfied by $u$ and $w = v_\xi$. Differentiating equation (3.7), at least at a formal level so far, is not complicated, due to the fact that $\varrho$ is independent of $\xi$ and, in fact, it turns out that $w$ solves the equation

$$\begin{aligned}
w_t =& w_\xi + \mathrm{Le}^{-1} \Delta w + \varrho_t (1+\varrho_\xi)^{-1} (\varrho Y^0_{\xi\xi\xi} + w_\xi) \\
&- \mathrm{Le}^{-1} (1+\varrho_\xi)^{-3} \varrho_{\xi\xi} (1+\varrho_y^2)(\varrho Y^0_{\xi\xi\xi} + w_\xi) \\
&- \mathrm{Le}^{-1} (1+\varrho_\xi)^{-1} \big[ (\mathrm{Le}\, \varrho_\xi + \varrho_{yy})(\varrho Y^0_{\xi\xi\xi} + w_\xi) + 2\varrho_y (\varrho_y Y^0_{\xi\xi\xi} + w_{\xi y}) \big] \\
&+ \mathrm{Le}^{-1} (1+\varrho_\xi)^{-2} \big[ 2\varrho_y \varrho_{\xi y} (\varrho Y^0_{\xi\xi\xi} + w_\xi) + (\varrho_y^2 - \varrho_\xi^2)(\varrho Y^0_{\xi\xi\xi\xi} + Y^0_{\xi\xi\xi} + v_{\xi\xi\xi}) \\
&\qquad - 2\varrho_\xi (\varrho Y^0_{\xi\xi\xi\xi} - \varrho_y^2 Y^0_{\xi\xi\xi} + w_{\xi\xi}) \big]
\end{aligned}$$

in $(0,+\infty)\times[(0,R)\cup(R,+\infty)]\times(-\ell/2,\ell/2)$,

(7) *Elimination of $\varrho$ from the right-hand side of the equation for $u$ and $w$:* differentiating (3.11) with respect to $\xi$ and $y$ is easy, so we skip the details. On the other hand, the right-hand sides of (3.6) and (3.7) depend also on $\varrho_t$, which in its turn depends on the traces at $\xi = R$ and $\xi = 0$ of the $t$-derivative of $v$ and $w$, respectively. To get rid of the $t$-derivative of $u$, we evaluate (3.6) at $\xi = R^+$. Since all the derivatives of $\varrho$ with respect of $\xi$ vanish and taking (3.11) into account, we get

$$\begin{aligned}
u_t(t,R,y) =& u_\xi(t,R^+,y) + \Delta u(t,R^+,y) - \theta_i^{-1}u_{yy}(t,R,y)u_\xi(t,R^+,y)\\
&+ \theta_i^{-1}u_t(t,R,y)[u(t,R,y)+u_\xi(t,R^+,y)]\\
&- 2\theta_i^{-1}u_y(t,R,y)u_{\xi y}(t,R^+,y)\\
&+ \theta_i^{-2}(u_y(t,R,y))^2[u_{\xi\xi}(t,R^+,y) - u(t,R,y) - \theta_i]\\
&- \theta_i^{-1}u(t,R,y)u_{yy}(t,R,y).
\end{aligned}$$

Since $u$ and $v$ are small perturbations of the traveling wave solutions, we can assume that $1-\theta_i{}^{-1}(u(t,R,y)+u_\xi(t,R^+,y))$ is positive, so that

$$\begin{aligned}
u_t(t,R,y) =& [1-\theta_i{}^{-1}(u(t,R,y)+u_\xi(t,R^+,y))]^{-1}\\
&\times\{u_\xi(t,R^+,y) + \Delta u(t,R^+,y) - \theta_i^{-1}u_{yy}(t,R,y)u_\xi(t,R^+,y)\\
&\quad - \theta_i^{-1}u(t,R,y)u_{yy}(t,R,y) - 2\theta_i^{-1}u_y(t,R,y)u_{\xi y}(t,R^+,y)\\
&\quad + \theta_i^{-2}(u_y(t,R,y))^2[u_{\xi\xi}(t,R^+,y) - u(t,R,y) - \theta_i]\}.
\end{aligned}\tag{3.12}$$

Arguing similarly, differentiating and evaluating (3.7) at $x = 0^+$, we get

$$\begin{aligned}
w_t(t,0^+,y) = \{&\mathrm{Le}\, w_\xi(t,0^+,y) + \Delta w(t,0^+,y)\\
&+ R\mathrm{Le}^{-1}\big[w_{yy}(t,0^+,y)(\mathrm{Le}\, w(t,0^+,y) + w_\xi(t,0^+,y))\\
&\qquad + 2w_y(t,0^+,y)w_{\xi y}(t,0^+,y)\big]\\
&+ R^2\mathrm{Le}^{-2}(w_y(t,0^+,y))^2\\
&\qquad \times[-\mathrm{Le}^2 w(t,0^+,y) + R^{-1}\mathrm{Le}^2 + w_{\xi\xi}(t,0^+,y)]\}\\
&\qquad\quad \times[\mathrm{Le} + R(\mathrm{Le}\, w(t,0^+,y) + w_\xi(t,0^+,y))]^{-1}.
\end{aligned}\tag{3.13}$$

(8) *Interface conditions for $u$:* since $T^0$, $Y^0$ belong to $C^1(\mathbb{R})$ and $T$ and $Y$, in the original variables $t$, $x$ and $y$, are continuous at the ignition and trailing fronts, with continuous normal derivatives, it turns out that in the new unknowns $t,\xi,y$, the derivatives $T_\xi$ and $Y_\xi$ are continuous at $\xi = 0$ and $\xi = R$. Thus, from (3.4) we deduce that

$$[Y_\xi(t,\cdot,y)]_{\xi_0} = [T^0_\xi(t,\cdot)]_{\xi_0} + \rho(t,\xi_0,y)[T^0_\xi(t,\cdot)]_{\xi_0} + [u(t,\cdot,y)]_{\xi_0},$$

where $\xi_0 \in \{0,R\}$, i.e., $[u(t,\cdot,y)]_0 = [u(t,\cdot,y)]_R = 0$ for $t\in(0,+\infty)$ and $y \in [-\ell/2,\ell/2]$.

Differentiating (3.4) and (3.5) for $\xi \neq R$ and taking the jumps across $\xi = R$, it can be easily shown that

$$[u_\xi(t,\cdot,y)]_R = -R^{-1}f(t,y), \qquad [w(t,\cdot,y)]_R = R^{-1}\mathrm{Le}\, f(t,y)$$

for $t \in (0,+\infty)$ and $y \in (-\ell/2, \ell/2)$. Using (3.9) and (3.10), we obtain the two jump conditions for $u_\xi$ at the fronts, which are

$$u(t,R,y) + \theta_i R[u_\xi(t,\cdot,y)]_R = 0, \qquad \mathrm{Le}[u_\xi(t,\cdot,y)]_R + [w(t,\cdot,y)]_R = 0$$

for $t \in (0,+\infty)$ and $y \in (-\ell/2, \ell/2)$.

(9) *The missing jump conditions:* to recover the last two missing conditions at the trailing and ignition fronts, we differentiate (3.5) twice in a neighborhood of $x=0$ and take the trace at $x=0^+$. Condition (3.10) allows us to get

$$Y_{\xi\xi}(0^+,\xi,y) = \mathrm{Le}\, R^{-1} + \mathrm{Le}\, w(t,0^+,y) + w_\xi(t,0^+,y). \tag{3.14}$$

We now eliminate $\Phi_{\xi\xi}$ from the left-hand side of (3.14). For this purpose, we observe that, for $\xi$ positive and sufficiently small, the equation for $Y$ in the variables $t$, $\xi$ and $y$ reduces to

$$\begin{aligned} Y_t =& Y_\xi + \mathrm{Le}^{-1}\Delta Y + \mathrm{Le}^{-1} g_y^2 Y_{\xi\xi} - 2\mathrm{Le}^{-1} g_y Y_{\xi y} - R^{-1} \\ &+ (g_t - \mathrm{Le}^{-1} g_{yy}) Y_\xi \end{aligned}$$

Computing the limit as $\xi$ tends to $0^+$ gives

$$Y_{\xi\xi}(t,0^+,R)[1+(g_y(t,y))^2] = \mathrm{Le}\, R^{-1}.$$

Finally, taking advantage of (3.10) and (3.14) we get the additional interface condition at the trailing interface

$$\mathrm{Le}\, w(t,0^+,y) + w_\xi(t,0^+,y) = \mathrm{Le}\, R^{-1}\{[1+R^2\mathrm{Le}^{-2}(w_y(t,0^+,y))^2]^{-1} - 1\}.$$

The condition at the ignition interface $\xi = R$ can be obtained in a similar way. More precisely, (i) one differentiates (3.5) twice with respect to $\xi$ and takes the jump at $x = R$ (taking (3.9) into account), (ii) then, one computes directly the jump at $\xi = R$ of $Y_{\xi\xi}$. Putting everything together, in the end one gets the condition

$$\mathrm{Le}\,[w(t,\cdot,y)]_R + [w_\xi(t,\cdot,y)]_R = -\mathrm{Le}\, R^{-1}\{[1+\theta_i^{-2}(u_y(t,R,y))^2]^{-1} - 1\}.$$

Combining all the previous steps, we conclude that the pair $\boldsymbol{u} = (u,w)$ satisfies the nonlinear problem

$$\begin{cases} D_t\boldsymbol{u}(t,\cdot) = \mathscr{L}\boldsymbol{u} + \mathscr{F}(\boldsymbol{u}(t,\cdot)), & t>0, \\ \mathscr{B}\boldsymbol{u}(t,\cdot) = \mathscr{G}(\boldsymbol{u}(t,\cdot)), \end{cases} \tag{3.15}$$

where

$$\mathscr{L}\boldsymbol{v} = (\Delta v_1 + D_x v_1, \mathrm{Le}^{-1}\Delta v_2 + D_x v_2), \tag{3.16}$$

$$\mathscr{B}\boldsymbol{v} = \begin{pmatrix} v_1(0^+,\cdot) - v_1(0^-,\cdot) \\ v_1(R^+,\cdot) - v_1(R^-,\cdot) \\ \mathrm{Le}[D_x v_1(0^+,\cdot) - D_x v_1(0^-,\cdot)] + v_2(0^+,\cdot) \\ \mathrm{Le}\, v_2(0^+,\cdot) + D_x v_2(0^+,\cdot) \\ \dfrac{1}{2}(v_2(R^+,\cdot) + v_2(R^-,\cdot)) + \theta_i R[D_x v_2(R^+,\cdot) - D_x v_2(R^-,\cdot)] \\ \mathrm{Le}[D_x v_1(R^+,\cdot) - D_x v_1(R^-,\cdot)] + v_2(R^+,\cdot) - v_2(R^-,\cdot) \\ \mathrm{Le}[v_2(R^+,\cdot) - v_2(R^-,\cdot)] + D_x v_2(R^+,\cdot) - D_x v_2(R^-,\cdot) \end{pmatrix}, \tag{3.17}$$

on smooth functions $\boldsymbol{v} = (v_1, v_2)$. We denote by $\mathscr{B}_j$ ($j = 1,\dots,7$) the seven components of the operator $\mathscr{B}$.

The nonlinear functions $\mathscr{F}$ and $\mathscr{G}$ have the same structure as the corresponding operators in Subsection 2.3, even if their expressions are much more complicated (we refer the reader to [1] for the expression of the such nonlinearities). The two main features of such operators are the following:

(i) they are quadratic at zero;
(ii) function $\mathscr{F}(\boldsymbol{v})$ depends also on the traces of second-order derivatives of $\boldsymbol{v}$ at $\xi = 0$ and $\xi = R$.

**Remark 3.1.** In the same way as in Subsection 2.3, in view of Equation (3.11) for $\varrho_t$ together with formulae (3.12)-(3.13), the velocities of interfaces $f$ and $g$ are expressed in terms of traces of first- and second-order derivatives of $\boldsymbol{u}$ (see [9]).

4. **Local existence and function spaces.** The local existence of a solution to problems (2.16) (Arrhenius kinetics) and (3.15) (stepwise kinetics) is obtained through a fixed point argument which requires to first solve the linearized version (at zero) of the above problems. In order to fix the function spaces where to study such linearized problems, one has to take into account the particular nature of the nonlinearities $\mathscr{F}$. Working with classical solutions to problems (2.16) and (3.15), it comes out that optimal Hölder regularity is needed, due to the dependence of the previous nonlinearity on traces of second-order derivative of the unknown. Thus, for problem (2.16), one deals with the Hölder spaces $X_0$, $X_\alpha$, $X_1$ and $X_{2+\alpha}$ ($\alpha \in (0,1)$), which are defined as follows:

> $X_0$ is the set of all functions $\boldsymbol{u} \in C_b(\overline{\mathbb{R}^2_-};\mathbb{R}^3)$ such that $\boldsymbol{u}(\cdot,y)$ vanishes as $x$ tends to $-\infty$, for all $y \in \mathbb{R}$;
> $X_\alpha$ is the set of all functions $\boldsymbol{u} \in C^\alpha_b(\mathbb{R}^2_-;\mathbb{R}^3)$ such that $\boldsymbol{u}(\cdot,y)$ vanishes as $x$ tends to $-\infty$, for all $y \in \mathbb{R}$;
> $X_1$ is the set of all functions $\boldsymbol{u} \in C^1_b(\overline{\mathbb{R}^2_-};\mathbb{R}^3)$ such that $\boldsymbol{u}(\cdot,y)$ vanishes as $x$ tends to $-\infty$, for all $y \in \mathbb{R}$;
> $X_{2+\alpha}$ is the set of all functions $\boldsymbol{u} \in C^{2+\alpha}_b(\mathbb{R}^2_-;\mathbb{R}^3)$ such that the components of $\boldsymbol{u}(\cdot,y)$ and its first- and second-order derivatives vanish as $x$ tends to $-\infty$ for each $y \in \mathbb{R}$.

Here, $X_0$ and $X_\alpha$ are endowed with the norm of $C_b(\overline{\mathbb{R}^2_-};\mathbb{R}^3)$ and $C^\alpha_b(\mathbb{R}^2_-;\mathbb{R}^3)$, respectively, whereas $X_1$ and $X_{2+\alpha}$ are endowed with the norm of $C^1_b(\overline{\mathbb{R}^2_-};\mathbb{R}^3)$ and $C^{2+\alpha}_b(\mathbb{R}^2_-;\mathbb{R}^3)$, respectively.

On the other hand, when one deals with problem (3.15), the spaces $X_0$, $X_\alpha$, $X_1$ and $X_{2+\alpha}$ are defined as follows:

> $X_0$ is the set of all pairs $\boldsymbol{f} = (f_1, f_2)$ such that (i) $f_1 \in C_b(\overline{S^-_\ell};\mathbb{R}) \cap C([0,R] \times [-\ell/2,\ell/2];\mathbb{R}) \cap C_b(\overline{S^R_\ell};\mathbb{R})$, (ii) $f_2 \in C_b([0,R] \times [-\ell/2,\ell/2];\mathbb{R}) \cap C_b(\overline{S^R_\ell};\mathbb{R})$, (iii) $f_j(\cdot,-\ell/2) = f_j(\cdot,\ell/2)$ for $j = 1,2$;
> $X_\alpha$ is the set of all pairs $\boldsymbol{f} = (f_1, f_2)$ such that (i) $f_1 \in C^\alpha_b(S^-_\ell;\mathbb{R}) \cap C^\alpha((0,R) \times (-\ell/2,\ell/2);\mathbb{R}) \cap C^\alpha_b(S^R_\ell;\mathbb{R})$, (ii) $f_2 \in C^\alpha_b((0,R) \times (-\ell/2,\ell/2);\mathbb{R}) \cap C^\alpha_b(S^R_\ell;\mathbb{R})$, (iii) $f_j(\cdot,-\ell/2) = f_j(\cdot,\ell/2)$ for $j = 1,2$;
> $X_1$ is the set of all pairs $\boldsymbol{f} = (f_1, f_2)$ such that (i) $f_1 \in C^1_b(\overline{S^-_\ell};\mathbb{R}) \cap C^1_b([0,R] \times [-\ell/2,\ell/2];\mathbb{R}) \cap C^1_b(\overline{S^R_\ell};\mathbb{R})$, (ii) $f_2 \in C^1([0,R] \times [-\ell/2,\ell/2];\mathbb{R}) \cap C^1_b(\overline{S^R_\ell};\mathbb{R})$, (iii) $f_j(\cdot,-\ell/2) = f_j(\cdot,\ell/2)$ and $\nabla f_j(\cdot,-\ell/2) = \nabla f_j(\cdot,\ell/2)$ for $j = 1,2$;
> $X_{2+\alpha}$ denotes the set of all pairs $\boldsymbol{f} = (f_1, f_2)$ such that $D^\beta \boldsymbol{f} = (D^\gamma f_1, D^\gamma f_2) \in X_\alpha$, $D^\gamma f_j(\cdot,-\ell/2) = D^\gamma f_j(\cdot,\ell/2)$ for each $|\gamma| \le k$, $j = 1,2$.

The previous spaces are endowed with the norms

$$\|\boldsymbol{f}\|_{X_0} = \|f_1\|_{C_b(\overline{S_\ell^-};\mathbb{R})} + \sum_{j=1}^{2}(\|f_j\|_{C([0,R]\times[-\ell/2,\ell/2];\mathbb{R})} + \|f_j\|_{C_b(\overline{S_\ell^+};\mathbb{R})}),$$

$$\|\boldsymbol{f}\|_{X_\alpha} = \|f_1\|_{C_b^\alpha(S_\ell^-;\mathbb{R})} + \sum_{j=1}^{2}(\|f_j\|_{C^\alpha((0,R)\times(-\ell/2,\ell/2);\mathbb{R})} + \|f_j\|_{C_b^\alpha(S_\ell^+;\mathbb{R})}),$$

$$\|\boldsymbol{f}\|_{X_1} = \|f_1\|_{C_b^1(\overline{S_\ell^-};\mathbb{R})} + \sum_{j=1}^{2}(\|f_j\|_{C^1([0,R]\times[-\ell/2,\ell/2];\mathbb{R})} + \|f_j\|_{C_b^1(\overline{S_\ell^+};\mathbb{R})}),$$

$$\|\boldsymbol{f}\|_{X_{2+\alpha}} = \sum_{|\gamma|<2}\|D^\gamma \boldsymbol{f}\|_\infty + \sum_{|\gamma|=2}\|D^\gamma\boldsymbol{f}\|_{X_\alpha}.$$

Also, some parabolic Hölder spaces are needed. In the case of problem (2.16), for $T \in (0,+\infty]$ they are defined as follows:

$\mathscr{X}_{\alpha/2,\alpha}(T)$ is the set of functions $\boldsymbol{u} \in C_b^{\alpha/2,\alpha}((0,T)\times\mathbb{R}^2_-;\mathbb{R}^3)$ such that $\boldsymbol{u}(t,\xi,y)=0$ vanishes as $\xi$ tends to $-\infty$ for all $t\in[0,T]$ and $y\in\mathbb{R}$;

$\mathscr{X}_{1+\alpha/2,2+\alpha}(T)$ is the set of all functions $\boldsymbol{u} \in C_b^{1+\alpha/2,2+\alpha}((0,T)\times\mathbb{R}^2_-;\mathbb{R}^3)$ such that $D_t^{\gamma_1}D_\xi^{\gamma_2}D_y^{\gamma_3}\boldsymbol{u}(t,\xi,y)$ vanishes as $\xi$ tends to $-\infty$ for every $t\in[0,T]$, $y\in\mathbb{R}$ and $2\gamma_1+\gamma_2+\gamma_3\le 2$.

Such spaces are endowed, respectively, with the norm of $C_b^{\alpha/2,\alpha}((0,T)\times\mathbb{R}^2_-;\mathbb{R}^3)$ and $C_b^{1+\alpha/2,2+\alpha}((0,T)\times\mathbb{R}^2_-;\mathbb{R}^3)$.

The corresponding spaces in the case of problem (3.15) are defined as follows:

$\mathscr{X}_{\alpha/2,\alpha}(T)$ is the set of all pairs $\boldsymbol{f}=(f_1,f_2)$ such that $f_1:[0,T]\times\overline{S_\ell}\to\mathbb{R}$, $f_2:[0,T]\times\overline{S_\ell^+}\to\mathbb{R}$ and $\sup_{t\in(0,T)}\|\boldsymbol{f}(t,\cdot,\cdot)\|_{X_\alpha}$, $\sup_{(\xi,y)\in S_\ell}\|f_1(\cdot,\xi,y)\|_{C^{\alpha/2}((0,T))}$ and $\sup_{(\xi,y)\in S_\ell^+}\|f_2(\cdot,\xi,y)\|_{C^{\alpha/2}((0,T))}$ are all finite;

$\mathscr{X}_{1+\alpha/2,2+\alpha}(T)$ denotes the space of all the pairs $\boldsymbol{f}$ such that $D_t^{\gamma_1}D_\xi^{\gamma_2}D_y^{\gamma_3}\boldsymbol{f}$ belongs to $\mathscr{X}_{\alpha/2,\alpha}(T)$ for every $\gamma_1,\gamma_2,\gamma_3\ge 0$ such that $2\gamma_1+\gamma_2+\gamma_3\le 2$.

These are Banach spaces with the norms

$$\begin{aligned}\|\boldsymbol{f}\|_{\mathscr{X}_{\alpha/2,\alpha}(T)} = &\sup_{t\in(0,T)}\|\boldsymbol{f}(t,\cdot,\cdot)\|_{X_\alpha} + \sup_{(\xi,y)\in S_\ell}\|f_1(\cdot,\xi,y)\|_{C^{\alpha/2}((0,T))}\\ &+\sup_{(\xi,y)\in S_\ell^+}\|f_2(\cdot,\xi,y)\|_{C^{\alpha/2}((0,T))},\end{aligned}$$

$$\|\boldsymbol{f}\|_{\mathscr{X}_{1+\alpha/2,2+\alpha}(T)} = \sum_{2\gamma_1+\gamma_2+\gamma_3\le 2}\|D_t^{\gamma_1}D_\xi^{\gamma_2}D_y^{\gamma_3}\boldsymbol{f}\|_{\mathscr{X}_{\alpha/2,\alpha}(T)}.$$

The theory of analytic semigroup is a very useful tool to study the linearized problems associated to (2.16) and (3.15), and prove optimal Schauder estimates for the solution of those problems.

The main steps in this direction are the following:

(1) one proves that in both the two problems under consideration, a suitable realization $L$ of the linear operator $\mathscr{L}$ (defined in (2.13) and (3.16)), whose domain contains functions $\boldsymbol{u}$ such that $\mathscr{B}\boldsymbol{u}=\boldsymbol{0}$ (where the operator $\mathscr{B}$ is defined in (2.14) and (3.17)), generates an analytic semigroup in $X_0$ (where now we need to consider complex-valued functions);

(2) one characterizes the interpolation spaces of order $\alpha/2$ and $1+\alpha/2$, i.e., the spaces $D_L(\alpha/2,\infty)$ and $D_L(1+\alpha/2,\infty)$, as spaces of Hölder continuous functions.

Such realizations are defined as follows. In the case of problem (2.12),[2]

$$\begin{cases} D(L)=\{\boldsymbol{u}\in X_1\cap W^{2,p}_{\rm loc}(\mathbb{R}^2_-;\mathbb{C}^3):\ \mathscr{L}\boldsymbol{u}\in X_0,\ \mathscr{B}\boldsymbol{u}=0 \text{ at } x=0\},\\ L\boldsymbol{u}=\mathscr{L}\boldsymbol{u},\ \ \boldsymbol{u}\in D(L),\end{cases}$$

whereas, in the case of problem (3.15), $D(L)$ is the set of all functions $\boldsymbol{u}\in X$ such that (i) $u_j(\cdot,-\ell/2)=u_j(\cdot,\ell/2)$ for $j=1,2$, (ii) denoting by $u_1^\sharp$ and $u_2^\sharp$ the periodic extension, with respect to the variable $y$, of the functions $u_1$ and $u_2$, it holds that

$u_1^\sharp\in C^1_b((-\infty,0]\times\mathbb{R};\mathbb{C})\cap\bigcap_{p<+\infty}W^{2,p}_{\rm loc}((-\infty,0]\times\mathbb{R};\mathbb{C})$;
$u_1^\sharp,u_2^\sharp\in C^1([0,R]\times\mathbb{R};\mathbb{C})\cap C^1_b([R,+\infty)\times\mathbb{R};\mathbb{C})\cap\bigcap_{p<+\infty}W^{2,p}_{\rm loc}((\mathbb{R}_+\setminus\{R\})\times\mathbb{R};\mathbb{C})$,

(iii) $\mathscr{L}\boldsymbol{u}\in X_0$ and $\mathscr{B}\boldsymbol{u}=\boldsymbol{0}$. Moreover, $L\boldsymbol{u}=\mathscr{L}\boldsymbol{u}$ for every $\boldsymbol{u}\in D(L)$.

Then, the theory of analytic semigroup applies and allows to show the following result.

**Proposition 4.1.** *Fix $\alpha\in(0,1)$ and $T>0$. Then the following properties are satisfied:*

**Linearized problem (2.16):** *for every $\boldsymbol{f}\in\mathscr{X}_{\alpha/2,\alpha}$, $\psi\in C_b^{(1+\alpha)/2,1+\alpha}([0,T]\times\mathbb{R})$ and $\boldsymbol{u}_0\in X_{2+\alpha}$, satisfying the compatibility conditions*

$$\mathscr{B}_1\boldsymbol{u}_0=\mathscr{B}_2\boldsymbol{u}_0=0,\qquad \mathscr{B}_3\boldsymbol{u}_0=\psi(0,\cdot),$$

*the Cauchy problem*

$$\begin{cases} D_t\boldsymbol{u}(t,\cdot,\cdot)=\mathscr{L}\boldsymbol{u}(t,\cdot,\cdot)+\boldsymbol{f}(t,\cdot,\cdot), & t\in[0,T],\\ \mathscr{B}_1(\boldsymbol{u}(t,\cdot,\cdot))=0, & t\in[0,T],\\ \mathscr{B}_2(\boldsymbol{u}(t,\cdot,\cdot))=0, & t\in[0,T],\\ \mathscr{B}_3(\boldsymbol{u}(t,\cdot,\cdot))=\psi(t,\cdot), & t\in[0,T],\\ \boldsymbol{u}(0,\cdot)=\boldsymbol{u}_0, \end{cases}\tag{4.1}$$

*admits a unique solution $\boldsymbol{u}\in\mathscr{X}_{1+\alpha/2,2+\alpha}(T)$. Moreover, there exists a positive constant $C$, independent of data and $\boldsymbol{u}$, such that*

$$\|\boldsymbol{u}\|_{\mathscr{X}_{1+\alpha/2,2+\alpha}(T)}\le C\big(\|\boldsymbol{f}\|_{\mathscr{X}_{\alpha/2,\alpha}(T)}+\|\boldsymbol{u}_0\|_{X_{2+\alpha}}+\|\psi\|_{C^{(1+\alpha)/2,1+\alpha}((0,T)\times\mathbb{R})}\big).\tag{4.2}$$

**Linearized problem (3.15):** *for every $\boldsymbol{f}\in\mathscr{X}_{\alpha/2,\alpha}$, $\psi_1,\psi_2\in C^{(1+\alpha)/2,1+\alpha}((0,T)\times(-\ell/2,\ell/2))$ and $\boldsymbol{u}_0\in X_{2+\alpha}$, which satisfy the compatibility conditions*

$$\mathscr{B}\boldsymbol{u}_0=(0,0,0,\psi_1,0,0,\psi_2)\qquad \mathscr{B}_j(\mathscr{L}\boldsymbol{u}_0(0,\cdot)+\boldsymbol{f}(0,\cdot))=0,\ \ j=1,2$$

*and the conditions $\boldsymbol{f}(0,\cdot,-\ell/2)=\boldsymbol{f}(0,\cdot,\ell/2)$, $D^\gamma\boldsymbol{u}_0(\cdot,-\ell/2)=D^\gamma\boldsymbol{u}_0(\cdot,\ell/2)$, $D_y^{(j)}\psi_1(\cdot,-\ell/2)=D_y^{(j)}\psi_1(\cdot,\ell/2)$ and $D_y^{(j)}\psi_2(\cdot,-\ell/2)=D_y^{(j)}\psi_2(\cdot,\ell/2)$ for every multi-index $\gamma$ with length at most two and $j=0,1$, the Cauchy problem*

$$\begin{cases} D_t\boldsymbol{u}(t,\cdot,\cdot)=\mathscr{L}\boldsymbol{u}(t,\cdot,\cdot)+\boldsymbol{f}(t,\cdot,\cdot), & t\in[0,T],\\ \mathscr{B}_j(\boldsymbol{u}(t,\cdot,\cdot))=0, & t\in[0,T],\ \ j=0,1,2,3,5,6,\\ \mathscr{B}_4(\boldsymbol{u}(t,\cdot,\cdot))=\psi_1(t,\cdot), & t\in[0,T],\\ \mathscr{B}_7(\boldsymbol{u}(t,\cdot,\cdot))=\psi_2(t,\cdot), & t\in[0,T],\\ \boldsymbol{u}(0,\cdot)=\boldsymbol{u}_0, \end{cases}\tag{4.3}$$

[2]Here, $X_1$ consists of complex-valued functions.

*admits a unique solution* $\boldsymbol{u} \in \mathscr{X}_{1+\alpha/2,2+\alpha}(T)$ *such that*

$$D_\xi^{\gamma_1} D_y^{\gamma_2} \boldsymbol{u}(t,\cdot,-\ell/2) = D_\xi^{\gamma_1} D_y^{\gamma_2} \boldsymbol{u}(t,\cdot,\ell/2), \qquad t \in [0,T], \quad \gamma_1+\gamma_2 \le 2.$$

*Moreover, there exists a positive constant* $C$*, independent of data and* $\boldsymbol{u}$*, such that*

$$\begin{aligned}&\|\boldsymbol{u}\|_{\mathscr{X}_{1+\alpha/2,2+\alpha}(T)}\\ \le & C\bigg(\|\boldsymbol{f}\|_{\mathscr{X}_{\alpha/2,\alpha}(T)} + \|\boldsymbol{u}_0\|_{X_{2+\alpha}} + \sum_{j=1}^{2} \|\psi_j\|_{C^{(1+\alpha)/2,1+\alpha}((0,T)\times(-\ell/2,\ell/2))}\bigg).\end{aligned} \tag{4.4}$$

To face the nonhomogeneous boundary conditions, one needs to introduce some suitable so-called "lifting operators", i.e. suitable operators $\mathcal{N}$ with the following properties:

- in the case of problem (2.16), such operator maps $C_b^\alpha(\mathbb{R})$ into $X_{\alpha+1}$ and
$$\mathscr{B}_1\mathcal{N}\psi = \mathscr{B}_1\mathcal{N}\psi = 0, \qquad \mathscr{B}_2\mathcal{N}\psi = \psi$$
for each function $\psi \in C_b^\alpha(\mathbb{R})$;
- in the case of problem (3.15), such operator maps $C([-\ell/2,\ell/2];\mathbb{R}^2)$ into $X_{2+\alpha}$ and
$$\mathscr{B}\mathcal{N}\boldsymbol{\psi} = (0,0,0,\psi_1,0,0,\psi_2).$$

Using these lifting operators, one can write the solution $\boldsymbol{u}$ to problem (4.1) in the form

$$\boldsymbol{u}(t,\cdot) = e^{tL}\boldsymbol{u}_0 + \int_0^t e^{(t-s)L}[\boldsymbol{f}(s,\cdot) + \mathscr{L}\mathcal{N}\psi(s,\cdot)]ds - L\int_0^t e^{(t-s)L}\mathcal{N}\psi(s,\cdot)ds \tag{4.5}$$

and the solution to problem (4.3) in the form

$$\boldsymbol{u}(t,\cdot) = e^{tL}\boldsymbol{u}_0 + \int_0^t e^{(t-s)L}[\boldsymbol{f}(s,\cdot,\cdot) + \mathscr{L}\mathcal{N}\boldsymbol{\psi}(s,\cdot)]ds - L\int_0^t e^{(t-s)L}\mathcal{N}\boldsymbol{\psi}(s,\cdot)ds, \tag{4.6}$$

for $t \in [0,T]$. These two formulae are a variant of the well-known Balakrishnan formula used to write the solution of a homogeneous (at the boundary) problem using the semigroup $e^{tL}$ generated by the realization $L$ of the operator $\mathscr{L}$ mentioned above. Such formulae will be extremely important in the analysis of the stability of the traveling wave solutions, which will be addressed in Section 5.

In view of Proposition 4.1 and using in particular estimates (4.2) and (4.4) together with the fact that $\mathscr{F}$ and $\mathscr{G}$ are quadratic at zero, one can quite easily prove the following result.

**Theorem 4.2** (Theorem 3.1 in [12] and Theorem 5.1 in [1])**.** *Fix any* $T > 0$ *and* $\alpha \in (0,1)$*. There exist* $\rho$, $\rho_0 > 0$ *such that the following properties are satisfied.*

**Problem** (2.16)**:** *for every* $\boldsymbol{u}_0 \in X_{2+\alpha}$ *with* $\|\mathbf{u_0}\|_{X_{2+\alpha}} \le \rho_0$ *and satisfying the compatibility conditions*

$$\mathscr{B}_1\boldsymbol{u}_0 = \mathscr{B}_2\boldsymbol{u}_0 = 0, \qquad \mathscr{B}_3\boldsymbol{u}_0 = g(\boldsymbol{u}_0), \qquad \mathscr{B}_1(\mathscr{L}\boldsymbol{u}_0 + \mathscr{F}(\boldsymbol{u}_0)) = 0,$$

*problem* (2.16) *admits a unique solution* $\boldsymbol{u} \in \mathscr{X}_{1+\alpha/2,2+\alpha}(0,T)$ *such that* $\boldsymbol{u}(0) = \boldsymbol{u}_0$ *and* $\|\boldsymbol{u}\|_{\mathscr{X}_{1+\alpha/2,2+\alpha}(0,T)} \le \rho$.

**Problem (3.15):** *for each $\boldsymbol{u}_0 \in X_{2+\alpha}$ with $\|\mathbf{u_0}\|_{X_{2+\alpha}} \le \rho_0$ and satisfying the compatibility conditions*

$$\mathscr{B}\boldsymbol{u}_0 = \mathscr{G}(\boldsymbol{u}_0), \qquad \mathscr{B}_1(\mathscr{L}\boldsymbol{u}_0 + \mathscr{F}(\boldsymbol{u}_0)) = \mathscr{B}_2(\mathscr{L}\boldsymbol{u}_0 + \mathscr{F}(\boldsymbol{u}_0)) = \mathbf{0},$$
$$D^\gamma \boldsymbol{u}_0(\cdot, -\ell/2) = D^\gamma \boldsymbol{u}_0(\cdot, \ell/2)$$

*for each multi-index $\gamma$ with length at most two, problem (3.15) admits a unique solution $\boldsymbol{u} \in \mathscr{X}_{1+\alpha/2,2+\alpha}(T)$ with $\boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0$. Moreover, $\|\boldsymbol{u}\|_{\mathscr{X}_{1+\alpha/2,2+\alpha}(T)} \le \rho$.*

5. **Instability of the traveling wave solutions.** The change of variables and unknowns that we have performed in Subsections 2.3 and 3.2 changed the traveling wave solutions to problems (2.5)-(2.6) and (3.3) into the null solution to problems (2.16) and (3.15), respectively. So, to study the stability of the traveling wave solution to problem (2.5)-(2.6) (resp. (3.3)), it suffices to study the stability of the null solution to problem (2.16) (resp. (3.15)). This latter issue is strongly related to the location of the spectrum of the operator $L$. So, a deep analysis of $\sigma(L)$ is required. In both the two problems under investigation, the spectrum splits into two parts: the so-called continuous spectrum and the point spectrum. The former consists of a parabola in the left-hand plane which is tangent to the imaginary axis at the origin, the latter is the set of all the admissible complex numbers $\lambda$, roots of the so-called *dispersion relation* (see, e.g., [29, Section 5]), which combines the wavenumber $k$, $\lambda$ and a real parameter, hereafter $\gamma$ or Le.

The dispersion relation associated with problem (2.16) is not difficult to set and to analyze. In fact, it reads:

$$D(k,\lambda,\gamma) = r_2(1-2r_1) - \frac{\gamma(r_2^2 - k^2)}{1-2r_1} = 0,$$

where

$$r_1 = r_1(k,\lambda) = \frac{1+\sqrt{1+4\lambda+4k^2}}{2}, \quad r_2 = r_2(k,\lambda) = 1 - r_1, \tag{5.1}$$

$k \in \mathbb{R}$ and $\gamma$ is a physico-chemical parameter (see (2.4)).

It can be checked that, for $\gamma > 1$, it defines implicitly a real-valued function $k \mapsto \lambda(k)$, defined in the interval $[0, 2^{-1}\sqrt{\gamma-1}]$ with $K = K(\gamma) > 0$, it is positive in $(0, 2^{-1}\sqrt{\gamma-1})$ and vanishes at 0 and at $2^{-1}\sqrt{\gamma-1}$. Moreover, $k \mapsto \lambda(k)$ is increasing in $(0, k_c)$ and decreasing in $(k_c, 2^{-1}\sqrt{\gamma-1})$, where $k_c$ is defined by

$$k_c = \frac{1}{\sqrt{2\gamma}}\left[\left(\frac{1+\sqrt{1+3\gamma}}{3}\right)^3 + \left(\frac{\gamma}{2}-1\right)\left(\frac{1+\sqrt{1+3\gamma}}{3}\right)^2 - \gamma\left(\frac{1+\sqrt{1+3\gamma}}{3}\right) + \frac{\gamma}{2}\right]^{\frac{1}{2}}.$$

In particular, the following result holds true.

**Theorem 5.1** (Theorem 4.1 of [12]). *For $\gamma > 1$, let $\lambda_c = \lambda(k_c)$. Then, the interval $[0, \lambda_c]$ consists of eigenvalues of $L$. Moreover, the halfplane $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > \lambda_c\}$ is contained in the resolvent set of the operator $L$.*

On the other hand, the (reduced) dispersion relation associated with problem (3.15) is more involved; we have infinitely many functions

$$\mathcal{D}(k,\lambda,\mathrm{Le}) = \exp\left(\frac{R}{2}(\mathrm{Le} - 1 - X_k(\lambda) - Y_k(\lambda,\mathrm{Le}))\right) - 1 + \theta_i R X_k(\lambda) = 0, \tag{5.2}$$

where

$$\begin{cases} X_k(\lambda) = \sqrt{1 + 4\lambda + 4\lambda_k}, \\ Y_k(\lambda, \mathrm{Le}) = \sqrt{\mathrm{Le}^2 + 4\lambda\mathrm{Le} + 4\lambda_k}, \\ \lambda_k = 4\pi^2 k^2 \ell^{-2}, \end{cases} \tag{5.3}$$

for each $k \in \mathbb{N} \cup \{0\}$. Here, the real parameter is the Lewis number $\mathrm{Le} \in (0, 1)$. Any root $\lambda$ of the equation $\mathcal{D}(k, \lambda, \mathrm{Le}) = 0$ defines an eigenvalue of operator $L$.

The analysis of (5.2) is not easy at all and its solutions can not be determined explicitly. The strategy to overcome such a difficulty relies on the use of the implicit function theorem and leads to the following theorem.

**Theorem 5.2** (Corollary 6.4 in [1]). *For fixed ignition temperature $\theta_i \in (0, 1)$ and width $\ell$ sufficiently large, there exists a critical value of the Lewis number $\mathrm{Le}_c \in (0, 1)$ such that, whenever $0 < \mathrm{Le} < \mathrm{Le}_c$, the spectrum of the operator $L$ contains elements with positive real parts. Moreover, the part of $\sigma(L)$ in the right halfplane $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \geq 0\}$ consist of $0$ and a finite number of eigenvalues.*

We sketch here below the main points to obtain the proof of such a theorem.

(1) As a first step, one proves that there exists $\ell_0(\theta_i)$ such that, for all $\ell > \ell_0(\theta_i)$ there exists a maximal integer $K$ such that the equation $\mathcal{D}(k, \lambda, \mathrm{Le}) = 0$ has a unique root $\mathrm{Le}_c = \mathrm{Le}_c(k) \in (0, 1)$ for every $k \in \{1, \ldots, K\}$. Moreover,

$$0 < \mathrm{Le}_c(K) \leq \cdots \leq \mathrm{Le}_c(2) \leq \mathrm{Le}_c(1).$$

(2) Under the assumptions of the previous point, one then proves that there exist $\lambda_* \in (0, \sqrt{\lambda_1})$ and a decreasing, continuously differentiable function $\widetilde{\varphi} : (0, \mathrm{Le}_c) \to (0, \lambda_*)$ such that $\mathcal{D}(1, \widetilde{\varphi}(\mathrm{Le}), \mathrm{Le}) = 0$ for all $\mathrm{Le} \in (0, \mathrm{Le}_c(1))$. This is the point where the implicit function theorem is used, thanks to the fact that the function $\mathcal{D}(1, \cdot, \cdot)$ is smooth in $[0, \sqrt{\lambda_1}] \times [0, \mathrm{Le}_c(1)]$.
(3) Finally, we set $\mathrm{Le}_c = \mathrm{Le}_c(1)$.

To prove the pointwise instability result for both the two problems, the following adaption of a result in [21, p.105] plays a crucial role.

**Theorem 5.3** (Lemma 6.5 in [1]). *Let $X$ be a complex Banach space, $r > 0$ and, for every $n \in \mathbb{N}$, let $S_n : B(0, r) \subset X \to X$ be a bounded operator. Then the following properties are satisfied.*

(i) *If $S_n(x) = Mx + O(\|x\|^p)$ as $\|x\| \to 0$, for some $p > 1$ and some bounded linear operator $M$ on $X$ with spectral radius $\rho > 1$, then the origin is unstable, i.e., there exist $c > 0$ and, for any $\delta > 0$, $x_0 \in B(0, \delta)$ and $n_0 \in \mathbb{N}$ (depending on $\delta$) such that the sequence $x_0, \ldots, x_{n_0}$, where $x_n = S_n(x_{n-1})$ for any $n = 1, \ldots, n_0$, is well defined and $\|x_{n_0}\| \geq C$.*
(ii) *In addition to the assumptions in (i), assume that there exists an eigenvector $u$ of $M$ with eigenvalue $\lambda \in \mathbb{C}$ such that $|\lambda|^p > \rho$ and that there exists $x' \in X'$ such that $x'(u) \neq 0$. Then, there exist $c > 0$ and, for any $\delta > 0$, $x_0 \in B(0, \delta)$ and $n_0 \in \mathbb{N}$ (depending on $\delta$) such that the sequence $x_0, \ldots, x_{n_0}$, where $x_n = S_n(x_{n-1})$ for any $n = 1, \ldots, n_0$, is well defined and $|x'(x_{n_0})| \geq c|x'(u)|$.*

The idea would be to apply such a theorem with $X$ being the set of all admissible initial data for problems (2.12) and (3.15), i.e., the sets

$$\{\boldsymbol{u} \in X_{2+\alpha} : v(0,y) - v_\xi(0,y) \neq 1 \text{ for } y \in \mathbb{R},$$
$$\mathscr{B}_1\boldsymbol{u}_0 = \mathscr{B}_2\boldsymbol{u}_0 = 0,\ \mathscr{B}_3\boldsymbol{u}_0 = \mathcal{G}(\boldsymbol{u}_0),\ B_1(\mathscr{L}\boldsymbol{u}_0 + \mathscr{F}\boldsymbol{u}_0) = 0\}$$

$$\{\boldsymbol{u} \in X_{2+\alpha} : u(t,R,y) + u_\xi(t,R^+,y) \neq \theta_i,\ R(\mathrm{Le}\, w(t,0^+,y) + w_\xi(t,0^+,y)) \neq -\mathrm{Le},$$
$$\text{for } \xi \in [-\ell/2,\ell/2],\quad \mathscr{G}\boldsymbol{u}_0 = \mathcal{G}(\boldsymbol{u}_0),\ \mathscr{B}_j(\mathscr{L}\boldsymbol{u}_0 + \mathscr{F}\boldsymbol{u}_0) = 0,\ \ j = 1,2\}$$

and the operator $S_n$, defined by $S_n(\boldsymbol{u}_0) = \boldsymbol{u}(n,\boldsymbol{u}_0,n-1)$, where $\boldsymbol{u}(n,\cdot,\boldsymbol{u}_0,n-1)$ denotes the solution to problem

$$\begin{cases} D_t\boldsymbol{u}(t,\cdot) = \mathscr{L}(\boldsymbol{u}(t,\cdot)) + \mathscr{F}(u(t,\cdot)), & t > n-1,\\ \mathscr{B}(\boldsymbol{u}(t,\cdot)) = \mathscr{G}(\boldsymbol{u}(t,\cdot)), & t > n-1,\\ \boldsymbol{u}(n-1,\cdot) = \boldsymbol{u}_0. \end{cases}$$

Since the problem is *autonomous*, by Theorem 4.2 this problem has a solution defined in a time-interval $[n-1, n-1+T_n]$ for every $n \in \mathbb{N}$ and the infimum of the sequence $(T_n)$ is positive.

However this choice is not admissible since both the two previous sets (let us denote them by $\mathcal{Y}$) are not Banach spaces due to the compatibility conditions which are of nonlinear type. The trick to overcome this problem consists in showing that the intersection of $\mathcal{Y}$ with a sufficiently small neighborhood of the origin (in $X_{2+\alpha}$) is the graph of a smooth function defined in a neighborhood of 0 of the interpolation space $D_L(1+\alpha/2,\infty)$. Then, Theorem 5.3 will be applied taking as $X$ the interpolation space $D_L(1+\alpha/2,\infty)$.

To prove that $\mathcal{Y}$ is the graph of a smooth function, the crucial step is the definition of a suitable projection on the space $X_{2+\alpha}$. Such a projection is defined through a right-inverse of the operator $\boldsymbol{u} \mapsto \mathscr{C}\boldsymbol{u} = (\mathscr{B}_1\boldsymbol{u}, \mathscr{B}_2\boldsymbol{u}, \mathscr{B}_3\boldsymbol{u}, \mathscr{B}_1\mathscr{L}\boldsymbol{u})$ (resp. of the operator $\boldsymbol{u} \mapsto \mathscr{C}\boldsymbol{u} = (\mathscr{B}\boldsymbol{u}, \mathscr{B}_1\boldsymbol{u}, \mathscr{B}_2\boldsymbol{u}))$ defined on $X_{2+\alpha}$, let us denote it by $\mathscr{M}$. Setting $P = I - \mathcal{M}\mathscr{C}$, it turns out that $P$ projects onto the kernel of the operator $\mathscr{C}$, which, in fact, coincides with the interpolation spaces $D_L(1+\alpha/2,\infty)$.

**Lemma 5.4** (Lemma 4.4 in [12] and Lemma 4.7 in [1])**.** *For $0 < \alpha < 1$ there exists a neighborhood $\Omega$ of $0$ in $X_{2+\alpha}$ such that $\mathscr{I} \cap \Omega$ is the graph of a smooth function $\Phi : B(0,\rho) \subset D(L_\alpha) \to (I-P)(X_{2+\alpha})$ for a suitable $\rho > 0$. Moreover $\Phi'(0) = 0$.*

To prove the lemma it suffices to observe that the nonlinear function $\mathscr{H}$ : $B(0,r) \subset X_{2+\alpha} \mapsto Y_\alpha$, defined by

$$\mathscr{H}(\boldsymbol{u}) = (\mathscr{B}\boldsymbol{u} - \mathscr{G}(\boldsymbol{u}), B_1(\mathscr{L}\boldsymbol{u} + \mathscr{F}(\boldsymbol{u})),$$

if we are dealing with problem (2.12), and

$$\mathscr{H}(\boldsymbol{u}) = (\mathscr{B}\boldsymbol{u} - \mathscr{G}(\boldsymbol{u}), B_1(\mathscr{L}\boldsymbol{u} + \mathscr{F}(\boldsymbol{u})), B_2(\mathscr{L}\boldsymbol{u} + \mathscr{F}(\boldsymbol{u}))),$$

otherwise, where $r > 0$ is chosen sufficiently small such that $\mathscr{F}$ is well defined in $B(0,r)$. Then $F$ is smooth and $F'(0) = \mathscr{C}$ is an isomorphism from $(I-P)(X_{2+\alpha})$ to $Y_\alpha$. Applying the implicit function theorem one can conclude the proof.

Now, for each $n \in \mathbb{N}$ we can apply the operator $S_n : B(0,\rho) \subset D_L(1+\alpha/2,\infty) \to D_L(1+\alpha/2,\infty)$ by setting

$$S_n\boldsymbol{u}_0 = P\boldsymbol{u}(n,\cdot,\boldsymbol{u}_0 + \Phi(\boldsymbol{u}_0), n-1).$$

Note that $\mathscr{L}$, $\mathscr{B}$ are independent of $t$ and the nonlinear operators $\mathscr{F}$ and $\mathscr{G}$ depend on $t$ only through the unknown function $\boldsymbol{u}$. Therefore, the uniqueness of the solution of the initial value problem, associated with problem (2.16) (resp. (3.15)), implies that $\boldsymbol{u}(n,\cdot,\boldsymbol{u}_0+\Phi(\boldsymbol{u}_0),n-1)=\mathbf{u}(1,\cdot,\boldsymbol{u}_0+\Phi(\boldsymbol{u}_0),0)$ (see Theorem 4.2). Thus, the function $S_n\boldsymbol{u}_0$ is the projection along $P(X_{2+\alpha})$ of the value at $t=1$ of the solution to problem (2.12) (resp. (3.15)) with initial condition (at $t=0$) $\boldsymbol{u}(0,\cdot)=\boldsymbol{u}_0$, i.e., $S_n\boldsymbol{u}_0=P\boldsymbol{u}(1,\cdot,\boldsymbol{u}_0+\Phi(\boldsymbol{u}_0),0):=T_0\boldsymbol{u}$. To apply Theorem 5.3(i) one needs to show that there exist a linear operator $M$ and an exponent $p>1$ such that $S_n(x)+Mx+O(\|x\|^p)$ as $x$ tends to 0. In fact, in our case, we can take $p=2$. Indeed, using formulae (4.5) and (4.6) it is immediate to check that

$$\begin{aligned}T_n\boldsymbol{u}_0=T_0\boldsymbol{u}_0=&e^L\boldsymbol{u}_0+P\int_0^1 e^{(1-s)L}[\mathscr{F}(\boldsymbol{u}(s,\cdot))+\mathscr{L}\mathscr{N}g(\boldsymbol{u}(s,\cdot)]ds\\&-PL\int_0^1 e^{(1-s)L}\mathscr{N}g(\boldsymbol{u}(s,\cdot))ds\end{aligned}$$

if we are dealing with problem (2.12) and

$$\begin{aligned}T_n\boldsymbol{u}_0=T_0\boldsymbol{u}_0=&e^L\boldsymbol{u}_0+P\int_0^1 e^{(1-s)L}[\mathscr{F}(\boldsymbol{u}(s,\cdot))+\mathscr{L}\mathscr{N}(g_1(\boldsymbol{u}(s,\cdot)),g_2(\boldsymbol{u}(s,\cdot)))]ds\\&-L\int_0^1 e^{(1-s)L}\mathscr{N}(g_1(\boldsymbol{u}(s,\cdot)),g_2(\boldsymbol{u}(s,\cdot)))ds\end{aligned}$$

otherwise. It has to be noticed that $Pe^L(\boldsymbol{u}_0+\Phi(\boldsymbol{u}_0))=e^L\boldsymbol{u}_0$ since $\boldsymbol{u}_0$ belongs to $D_L(1+\alpha/2,\infty)$, which is invariant under the action of the operator $L$ since $P$ commutes with $e^L$, and $\Phi(\boldsymbol{u}_0)\in(I-P)(D_L(1+\alpha/2,\infty))$, so that $Pe^L\Phi(\boldsymbol{u}_0)=0$.

Since the functions $\mathscr{F}$, $g$, $g_4$ and $g_7$ are quadratic at zero, a direct computation reveals that the integral terms in the previous two formulae are quadratic at zero as well. It turns out that the splitting $S_n\boldsymbol{u}_0=M\boldsymbol{u}_0+O(\|\boldsymbol{u}_0\|^2)$ holds true if one takes $M=e^L$.

We summarize the result so far obtained in the following theorem.

**Theorem 5.5** (Corollary 4.5 in [12] and Theorem 6.6 in [1])**.** *Fix $\alpha\in(0,1)$ in Lemma 5.4. Then, the following properties are satisfied.*

**Problem** (2.16)**:** *for $\gamma>1$, the null solution to problem* (2.16) *is unstable in the $X_{2+\alpha}$-norm.*

**Problem** (3.15)**:** *under the assumptions of Theorem 5.2, for each* $\mathrm{Le}\in(0,\mathrm{Le}_c)$ *the null solution of problem* (3.15) *is unstable in the $X_{2+\alpha}$-norm.*

6. **Instability of the fronts.** The instability result of Theorem 5.5 is rather weak because the $C^{2+\alpha}$-norm of the space $X_{2+\alpha}$ is a very heavy norm. In particular, it does not give any information about the instability of the front: it could happen that $\|\boldsymbol{u}(t,\cdot)\|_{X_{2+\alpha}}$ is far from 0 for some $t$ but $s(t,\cdot)=-v(t,0,\cdot)$ stays small for every $t$.

To prove that also the front is pointwise unstable, we take advantage of the second part of Theorem 5.3. For this purpose, a deeper analysis of the part of the spectrum of $L$ which lies in the right-halfplane is required. In the case of problem (2.12), things are a bit easier and one can show the following result.

**Proposition 6.1** (Theorem 4.1 in [12])**.** *Fix $\gamma>1$. Then, for each $\lambda=\lambda(k)\in[0,\lambda_c]$ (see Theorem 5.1), there exists a corresponding eigenfunction of the form $\boldsymbol{u}(\xi,y)=\boldsymbol{u}_*(\xi)g(y)$ for every $(x,y)\in\mathbb{R}^2_-$, where function $g$ is any solution to the* *ordinary differential equation $g''=-k^2g$, whereas the function $\boldsymbol{u}_*=(u_{*,1},u_{*,2},u_{*,3})$ is defined by*

$$\begin{cases} u_{*,1}(\xi)=e^{r_1\xi},\\ u_{*,2}(\xi)=-\dfrac{\gamma}{1-2r_1}\xi e^{r_1\xi}(r_1^2-k^2)+(\gamma-r_2)e^{r_1\xi},\\ u_{*,3}(\xi)=-r_2e^{-r_2\xi},\end{cases}$$

*for every $\xi\le 0$, where $r_j=r_j(k,\omega)$, $j=1,2$, are defined in (5.1).*

In the case of problem (3.15), one can prove the following.

**Proposition 6.2** (Theorem 6.6 in [1])**.** *Under the assumptions of Theorem 5.2, there exists an eigenvalue $\lambda$ of operator $L$, whose modulus equals the spectral radius of the operator $M=e^L$. In particular, for every $y_0\in\mathbb{R}$, there exists an eigenfunctions $\boldsymbol{u}_*$ such that $u_{*,1}(R^+,y_0)\neq 0$ and $u_{*,2}(0^+,y_0)\neq 0$. It suffices to take $\boldsymbol{u}_*=(u_{*,1}e_1(\cdot-2\pi\ell^{-1}y_0),u_{*,2}e_1(\cdot-2\pi\ell^{-1}y_0))$, where*

$$u_{*,1}(\xi)=e^{\nu_1^+\xi}\chi_{(-\infty,0]}(\xi)+\left(\frac{e^{(X_1+\mu_1^+)R}(\theta_iRX_1-1)}{e^{\mu_1^+R}-e^{\nu_1^+R}}e^{\nu_1^-\xi}+\frac{e^{\mu_1^+R}}{e^{\mu_1^+R}-e^{\nu_1^+R}}e^{\nu_1^+\xi}\right)\chi_{(0,R)}(\xi)$$
$$+\frac{\theta_iRe^{(X_1+\mu_1^+)R}X_1}{e^{\mu_1^+R}-e^{\nu_1^+R}}e^{\nu_1^-\xi}\chi_{[R,+\infty)}(\xi),$$

$$u_{*,2}(\xi)=-\frac{\mathrm{Le}(\mathrm{Le}+\mu_1^+)e^{\nu_1^+R}X_1}{(e^{\mu_1^+R}-e^{\nu_1^+R})Y_1}\left(e^{\mu_1^-\xi}-(\mathrm{Le}+\mu_1^-)e^{\mu_1^+\xi}\right)\chi_{[0,R)}(\xi)$$
$$+(1-e^{Y_1R})d_1e^{\mu_1^-\xi}\chi_{[R,+\infty)}(\xi),$$

*for every $\xi\in\mathbb{R}$, where, $X_1$ and $Y_1$ are defined in (5.3), whereas*

$$\nu_1^\pm=-\frac{1}{2}\pm\sqrt{1+4\lambda+\frac{16\pi^2}{\ell^2}},\qquad \mu_1^\pm=-\frac{\mathrm{Le}}{2}\pm\sqrt{\mathrm{Le}^2+4\mathrm{Le}\lambda+\frac{16\pi^2}{\ell^2}}.$$

Fix $y_0\in\mathbb{R}$. Applying Theorem 5.3, with

**Problem** (2.12)**:** $x'(\boldsymbol{u})=-u_2(0,y_0)$, for every $\boldsymbol{u}$ and taking as $\boldsymbol{u}$ an eigenfunction corresponding to the eigenvalue $r=e^{\omega_c}$ of the operator $e^L$ such that $g(y_0)\neq 0$;

**Problem** (3.15)**:** $x'(\boldsymbol{u})=-u_2(0,y_0)$, for every $\boldsymbol{u}$ and taking as $\boldsymbol{u}$ the eigenfunction in Proposition 6.2, corresponding to an eigenvalue of $L$ such that $|\lambda|$ equals the spectrum radius of $L$, to prove the instability of the trailing interface; $x'(\boldsymbol{u})=-u_1(R,y_0)$, for every $\boldsymbol{u}$ and taking as $\boldsymbol{u}$ the eigenfunction in Proposition 6.2, corresponding to an eigenvalue of $L$ such that $|\lambda|$ equals the spectrum radius of $L$, to prove the instability of the ignition interface,

we can prove the instability of the front for both the two problems. We summarize such a result in the following theorem.

**Theorem 6.3** (Corollary 4.8 in [12] and Theorem 6.6 in [1])**.** *The following properties are satisfied.*

**Problem** (2.16)**:** *fix $\gamma>1$. Then the front of the planar travelling wave solution of problem (2.5)-(2.7) is pointwise unstable, i.e., there exists a positive constant $C'$ such that for every $y_0\in\mathbb{R}$ and for every $\rho>0$ there exist $\boldsymbol{u}_0\in X_{2+\alpha}$ with $\|\boldsymbol{u}_0\|_{X_{2+\alpha}}\le\rho$, and $n_0\in\mathbb{N}$ such that, $|s(n_0,\boldsymbol{u}_0)(y_0)|\ge C'$.*

**Problem** (3.15)**:** *fix* $0 < \theta_i < 1$ *and the* $\ell$ *sufficiently large. Then, for each* $\mathrm{Le} \in (0, \mathrm{Le}_c)$ *both the trailing and the ignition interfaces of the planar travelling wave solution to problem* (3.3) *is unstable, i.e., there exists a positive constant* $C'$ *such that for each* $y_0 \in \mathbb{R}$ *and* $\delta > 0$ *there exist* $\boldsymbol{u}_0, \boldsymbol{u}_0^* \in B(0,\delta) \subset X_{2+\alpha}$ *and* $n_0, n_0^* \in \mathbb{N}$ *depending on* $\delta$ *such that* $\min\{|f(n_0, y_0)|, |g(n_0^*, y_0)|\} \geq C'$.

7. **Conclusion.** In this paper, we have considered two classes of free interface problems in combustion theory describing the propagation of premixed flames:

(i) the conventional diffusional-thermal models with standard Arrhenius kinetics (see [13]): at the flame front, i.e. the free interface, the temperature and mass fraction gradients are discontinuous (*thin flame*);

(ii) models describing dynamics of *thick flames* with stepwise ignition-temperature kinetics that have recently received considerable attention (see [4]). There are differences with the Arrhenius kinetics: in the case of zero-order stepwise kinetics there are two free interfaces; the temperature and mass fraction gradients are this time continuous at the free interfaces.

We have shown that in both classes the instability of the traveling wave solution can be addressed by the method of [8] initially developed for solving problems with discontinuous gradient at the interface. The velocity of the front is associated with a combination of spatial derivatives up to the second-order. Subsequently, the system is reformulated as a *fully nonlinear problem* (see [25]) and the theory of analytic semigroups is then a key tool to study the linearized operators. We also observed that the non-degeneracy (or transversality) condition in [8] may be circumvented by differentiating at least partially the system.

Finally, we point out that this method is quite general and may apply to other gamuts of problems that involve a finite number of free interfaces or free boundaries.

## REFERENCES


[1] D. Addona, C.-M. Brauner, L. Lorenzi and W. Zhang, Instabilities in a combustion model with two free interfaces, *J. Differential Equations*, **268** (2020), 396–4016.

[2] O. Baconneau, C.-M. Brauner and A. Lunardi, Computation of bifurcated branches in a free boundary problem arising in combustion theory, *ESAIM Math. Model. Numer. Anal.*, **34** (2000), 223–239.

[3] I. Blank, Sharp results for the regularity and stability of the free boundary in the obstacle problem, *Indiana Univ. Math. J.*, **50** (2001), 1077–1112.

[4] I. Brailovsky, P. V. Gordon, L. Kagan and G. I. Sivashinsky, Diffusive-thermal instabilities in premixed flames: Stepwise ignition-temperature kinetics, *Combustion and Flame* **162** (2015), 2077–2086.

[5] C.-M Brauner, P. V. Gordon and W. Zhang, An ignition-temperature model with two free interfaces in premixed flames, *Combust. Theory Model.*, **20** (2016), 976–994.

[6] C.-M. Brauner, L. Hu and L. Lorenzi, Asymptotic analysis in a gas-solid combustion model with pattern formation, *Chin. Ann. Math. Ser. B*, **34** (2013), 65–88.

[7] C.-M. Brauner, J. Hulshof, L. Lorenzi and G. I. Sivashinsky, A fully nonlinear equation for the flame front in a quasi-steady combustion model, *Discrete Contin. Dyn. Syst. Ser. A*, **27** (2010), 1415–1446.

[8] C.-M. Brauner, J. Hulshof and A. Lunardi, A general approach to stability in free boundary problems, *J. Differential Equations*, **164** (2000), 16–48.

[9] C.-M. Brauner and L. Lorenzi, Local existence in free interface problems with underlying second-order Stefan condition, *Rev. Roumaine Math. Pures Appl.*, **63** (2018), 339–359.

[10] C.-M. Brauner, L. Lorenzi, G.I. Sivashinsky and C.-J. Xu, On a strongly damped wave equation for the flame front, *Chin. Ann. Math. Ser. B*, **31** (2010), 819–840.

[11] C.-M. Brauner, L. Lorenzi and M. Zhang, Stability analysis and Hopf bifurcation for large Lewis number in a combustion model with free interface, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, (2020), in press.

[12] C.-M. Brauner and A. Lunardi, Instabilities in a two-dimensional combustion model with free boundary, *Arch. Ration. Mech. Anal.*, **154** (2000), 157–182.

[13] J. D. Buckmaster and G. S. S. Ludford, *Theory of Laminar Flames*, Cambridge University Press, Cambridge-New York, 1982.

[14] J. D. Buckmaster and G. S. S. Ludford, Lectures on Mathematical Combustion, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1983.

[15] J. R. Cannon and J. Douglas Jr., The stability of the boundary in a Stefan problem, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **21** (1967), 83–91.

[16] F. Conrad and F. Issard-Roch, Loss of stability at turning points in nonlinear elliptic variational inequalities, *Nonlinear Anal.*, **14** (1990), 329–356.

[17] F. Conrad, F. Issard-Roch, C.-M. Brauner and B. Nicolaenko, Nonlinear eigenvalue problems in elliptic variational inequalities: A local study, *Comm. Partial Differential Equations*, **10** (1985), 151–190.

[18] C. Elliot and J. R. Ockendon, *Weak and Variational Methods for Moving Boundary Problems*, Pitman, Boston, Mass.-London, 1982.

[19] M. Hadžić and S. Shkoller, Global stability and decay for the classical Stefan problem, *Comm. Pure Appl. Math.*, **68** (2015), 689–757.

[20] M. Hadžić, G. Navarro and S. Shkoller, Local well-posedness and global stability of the two-phase Stefan problem, *SIAM J. Math. Anal.*, **49** (2017), 4942–5006.

[21] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lect. Notes in Math., Vol. 840, Springer-Verlag, Berlin-New York, 1981.

[22] L. Lorenzi, Regularity and analyticity in a two-dimensional combustion model, *Adv. Differential Equations*, **7** (2002), 1343–1376.

[23] L. Lorenzi, A free boundary problem stemmed from combustion theory. Part I: Existence, uniqueness and regularity results, *J. Math. Anal. Appl.*, **274** (2002), 505–535.

[24] L. Lorenzi, A free boundary problem stemmed from combustion theory. Part II: Stability, instability and bifurcation results, *J. Math. Anal. Appl.*, **275** (2002), 131–160.

[25] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems, Birkhäuser Verlag, Basel, 1995.

[26] B. J. Matkowsky and G. I. Sivashinsky, An asymptotic derivation of two models in flame theory associated with the constant density approximation, *SIAM J. Appl. Math.*, **37** (1979), 686–699.

[27] J. R. Ockendon, Linear and nonlinear stability of a class of moving boundary problems, in *Free Boundary Problems: Volume II*, Ist. Naz. Alta Mat. Francesco Severi, Rome, 1980, 443–478.

[28] S. Serfaty and J. Serra, Quantitative stability of the free boundary in the obstacle problem, *Anal. PDE*, **11** (2018), 1803–1839.

[29] G. I. Sivashinsky, On flame propagation under condition of stoichiometry, *SIAM J. Appl. Math.*, **39** (1980), 67–82.

[30] G. I. Sivashinsky, Instabilities, pattern formation and turbulence in flames, *Ann. Rev. Fluid Mech.*, **15** (1983), 179–199.

Received December 2019; revised February 2020.

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