Letter | Editors' Suggestion

Bootstrapping the half-BPS line defect CFT in $\mathcal{N}=4$ supersymmetric Yang-Mills theory at strong coupling

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We consider the one-dimensional (1D) conformal field theory defined by the half-BPS Wilson line in planar $\mathcal{N}=4$ super Yang-Mills. Using analytic bootstrap methods we derive the four-point function of the superdisplacement operator at fourth order in a strong coupling expansion. Via AdS/CFT, this corresponds to the first three-loop correlator in anti–de Sitter ever computed. To do so we address the operator mixing problem by considering a family of auxiliary correlators. We further extract the anomalous dimension of the lightest nonprotected operator and find agreement with the integrability-based numerical result of Grabner, Gromov, and Julius.

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I. INTRODUCTION

It is hard to overstate the fundamental role played by symmetries and consistency conditions in quantum field theory (QFT). This is especially true for conformal field theories (CFTs) where the latter can be explicitly formulated and used to give concrete predictions for observable quantities. This strategy, called the conformal bootstrap, has produced spectacular results over the past decade or so (see, e.g., [1]).

Here, we focus on a one-parameter family of one-dimensional (1D) CFTs [2] with extended supersymmetry, namely $\mathfrak{osp}(4^*|4)$, that admits two holographically dual realizations: as a line defect in planar $d=4$, $\mathcal{N}=4$ super Yang-Mills, namely as a Wilson line in the fundamental representation, or as a two-dimensional QFT in AdS_2 [3]. This family of CFTs is parametrized by the 't Hooft coupling λ and each description is perturbative in opposite regimes.

The aim of this paper is to show that this 1D CFT can be efficiently and systematically solved perturbatively at strong coupling (large λ) using analytic bootstrap methods introduced in 1D in [4–6]. The power and success of our procedure is established by the determination of the scaling dimension of the lightest nonprotected scalar operator to be [7]

$$\Delta_{\phi^2} = 2 - \frac{5}{\sqrt{\lambda}} + \frac{295}{24}\frac{1}{\lambda} - \frac{305}{16}\frac{1}{\lambda^{3/2}} + \left(\frac{351845}{13824} - \frac{75}{2}\zeta(3)\right)\frac{1}{\lambda^2} + \cdots. \tag{1}$$

Excitingly, this formula agrees with the numerical result obtained in [8] by the completely independent, integrability-based, quantum spectral curve method. We extract (1) from the four-point function of the so-called superdisplacement operator, which we bootstrap up to the same order. Its explicit expression is given in the Supplemental Material [9]. With the current technology it would have been impossible to determine this correlator directly from Witten diagrams as it corresponds to a three-loop computation in anti–de Sitter (AdS).

The basic idea, applied in, e.g., [10–13], is to construct an ansatz for the correlator and to impose consistency with the operator product expansion and Bose symmetry. The main obstacle in implementing this procedure is the problem of operator mixing. This makes it necessary to consider not just one, but a whole family of correlators that is large enough depending on the specific CFT and on the perturbative order one is interested in. An interesting feature of this 1D CFT is that the degeneracies in the spectrum of conformal dimensions at $\lambda=\infty$ are unaffected at the first perturbative order. Thus, mixing plays a noticeable role in the determination of the four-point function of the superdisplacement operator starting at fourth order, when the square of the second order anomalous dimension matrix first appears. Additionally, the knowledge of all four-point functions of half-BPS operators at second order, which will be presented in [14], is not

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enough to take into account this effect, and correlators involving nonprotected external operators need to be included. This goes fundamentally beyond what has been done in previous works, for example [11,15,16].

II. SUPERCONFORMAL SYMMETRY

The symmetry of the 1D super-CFT (SCFT) we are studying is $\mathfrak{osp}(4^*|4)$. Its bosonic subalgebra is $\mathfrak{so}(4^*) \oplus \mathfrak{sp}(4) \simeq \mathfrak{sl}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{sp}(4)$, where the first term corresponds to the 1D conformal group while the remaining two can be thought of as R symmetries. The relevant representations of $\mathfrak{osp}(4^*|4)$ are uniquely specified by the scaling dimensions and R-symmetry representation $\omega = \{\Delta, s, [a,b]\}$ [17] of the superconformal primary. Two types of supermultiplets $\mathcal{R}$ will be relevant in this work: (i) long multiplets $\mathcal{L}^{\Delta}_{s,[a,b]}$ where Δ is subject to the unitarity bound, (ii) short multiplets $\mathcal{D}_k$, with $\omega = \{k, 0, [0,k]\}$. A distinguished role is played by the superdisplacement operator $\mathcal{D}_1$, which is ultrashort and whose decomposition in irreducible representations of the bosonic symmetry is

$$\mathcal{D}_1:\ \phi^{\Delta=1}_{(\mathbf{1},\mathbf{5})} \to \psi^{\Delta=3/2}_{(\mathbf{2},\mathbf{4})} \to f^{\Delta=2}_{(\mathbf{3},\mathbf{1})}, \tag{2}$$

where the arrow refers to the action of supersymmetry generators, while $(\mathbf{m},\mathbf{n})$ denotes the dimensions of the $\mathfrak{su}(2) \oplus \mathfrak{sp}(4)$ representation. In the following we will consider four-point functions of two types,

$$\langle \mathcal{D}_1\mathcal{D}_1\mathcal{D}_k\mathcal{D}_k\rangle, \qquad \langle \mathcal{D}_1\mathcal{D}_1\mathcal{D}_2\mathcal{L}^{\Delta_{\rm ext}}_{0,[0,0]}\rangle. \tag{3}$$

The implications of superconformal symmetry on correlation functions involving only short operators have been analyzed in [5,18] using superspace. They not only imply that the four-point functions of all the members of the short supermultiplet are determined by one of the superprimaries, but that the latter are subject to constraints. In the simplest example of $\langle \mathcal{D}_1\mathcal{D}_1\mathcal{D}_1\mathcal{D}_1\rangle$ these can be solved in terms of a constant and a single function of the bosonic cross ratio $\chi = \frac{t_{12}t_{34}}{t_{13}t_{24}}$, where t is a coordinate on the line and $t_{ij} = t_i - t_j$. The explicit parametrization is

$$\frac{\langle \mathcal{D}_1\mathcal{D}_1\mathcal{D}_1\mathcal{D}_1\rangle}{\langle \mathcal{D}_1\mathcal{D}_1\rangle\langle \mathcal{D}_1\mathcal{D}_1\rangle} = \mathsf{f}\mathfrak{X} + \mathbb{D}f(\chi), \tag{4}$$

where the superconformal invariant $\mathfrak{X}$ and the differential operator $\mathbb{D}$ are given in the Supplemental Material [9]. The number f in (4) is a datum of the topological algebra associated with any 1D CFT with $\mathfrak{osp}(4^*|4)$ symmetry by the cohomological construction of [19,20]; see [18]. If the 1D CFT in question is a Wilson line in $\mathcal{N}=4$ SYM, f can be computed by supersymmetric localization [21–23]. See [5] for more details.

To address the mixing problem we also consider correlators of the second type in (3). Superconformal symmetry implies that each of them is determined by a single function $F(\chi)$; see the Supplemental Material [9,14].

Correlation functions of local operators admit a decomposition in superconformal blocks, defined by the $\mathfrak{osp}(4^*|4)$ Casimir equation supplemented with the appropriate boundary conditions. We parametrize the Casimir eigenvalues as

$$\begin{aligned}\mathfrak{C}_2(\mathcal{R}) = \Delta(\Delta+3) + \frac{s(s+2)}{4} \\ -\frac{a^2}{2} - a(b+2) - b(b+3),\end{aligned} \tag{5}$$

with $\omega_{\mathcal{R}} = \{\Delta, s, [a,b]\}$. Explicit expressions of superconformal blocks are given in the Supplemental Material [9], and their derivation will be presented in [14].

The conformal blocks decomposition of any four-point function follows from the operator product expansion (OPE) rules. In the case of $\mathcal{D}_1$ they take the form

$$\mathcal{D}_1 \times \mathcal{D}_1 = \mathcal{I} + \mathcal{D}_2 + \mathcal{L}^{\Delta}_{0,[0,0]}. \tag{6}$$

The OPE $\mathcal{D}_k \times \mathcal{D}_k$ has the same form plus extra representations that are projected away in the correlator of interest. It follows that the decomposition of (4) in superconformal blocks is

$$f(\chi) = \mathfrak{f}_{\mathcal{I}}(\chi) + \mu^2_{\mathcal{D}_2}\mathfrak{f}_{\mathcal{D}_2}(\chi) + \sum_{\mathcal{O}} \mu^2_{\mathcal{O}}\mathfrak{f}_{\Delta_{\mathcal{O}}}(\chi), \tag{7}$$

and $\mathsf{f} = 1 + \mu^2_{\mathcal{D}_2}$, where $\mathcal{O}$ are superconformal primaries of type $\mathcal{L}^{\Delta}_{0,[0,0]}$. We shall also use the OPE [24]

$$\mathcal{D}_2 \times \mathcal{L}^{\Delta_{\rm ext}}_{0,[0,0]} = \mathcal{D}_2 + \mathcal{L}^{\Delta}_{0,[0,0]} + \cdots, \tag{8}$$

where $\cdots$ indicates representations that do not contribute to (3). The selection rules for the other channel can be found in the Supplemental Material [9].

III. FREE THEORY

The free 1D CFT from which we start the perturbation is easy to describe. Its local operators are built by taking normal ordered products of the fundamental fields $\Phi = (\phi^I(t), \psi^A_\alpha(t), f_{\alpha\beta}(t))$, with $I = 1,\ldots,5$, $A = 1,\ldots,4$, $\alpha,\beta = 1, 2$ [see (2)], and their derivatives. Correlation functions are defined and computed by Wick contractions using the two-point function of Φ. By the state-operator correspondence we can think in terms of the space of states

$$\mathcal{H} = \bigoplus_L \mathcal{H}_L, \qquad \mathcal{H}_L = (\mathbb{V}_\Phi \otimes \ldots \otimes \mathbb{V}_\Phi)^{S_L}, \tag{9}$$

where $\mathbb{V}_\Phi \simeq \mathcal{D}_1$, the symbol S_L indicates that the tensor products are totally graded-symmetrized and the integer

L corresponds to the length of composite operators. Each factor $\mathcal{H}_L$ decomposes into irreducible representations of $\mathfrak{osp}(4^*|4)$ as

$$\mathcal{H}_L = \bigoplus_{\mathcal{R}} \mathsf{d}_L(\mathcal{R}) \otimes \mathcal{R}, \tag{10}$$

where $\mathsf{d}_L(\mathcal{R})$ are multiplicity spaces. Their dimensions can be obtained by expanding the partition function that counts the words made of Φ and its derivatives in characters of $\mathfrak{osp}(4^*|4)$ (see [5]). For length two, the decomposition (10) is multiplicity free and contains only the multiplets $\mathcal{D}_2$ and $\mathcal{L}^{\Delta}_{0,[0,0]}$ with $\Delta = 2, 4, \ldots$. After turning on the perturbation, the corresponding operators will mix with operators in the same representation in $\mathcal{H}_{L>2}$. At the perturbative order considered in this work a fundamental role will be played by length four operators in such representations; their number is given by

$$\dim(\mathsf{d}_4(\Delta)) = \mathtt{Floor}\left[\left(\frac{\Delta}{4}\right)^2\right], \qquad \Delta = 4, 6, \ldots, \tag{11}$$

where we introduced the notation $\mathsf{d}_L(\Delta) \coloneqq \mathsf{d}_L(\mathcal{L}^{\Delta}_{0,[0,0]})$. While this counting gives valuable information, for our purposes we will need to construct the length four operators explicitly.

The finite dimensional multiplicity spaces $\mathsf{d}_L(\mathcal{R})$ are equipped with an inner product g, which is determined by the two-point functions in the free theory and does not mix operators of different lengths. Additionally, three-point functions provide trilinear maps

$$\mathsf{C}^{(0)}\colon\ \mathsf{d}_{L_1}(\mathcal{R}_1) \times \mathsf{d}_{L_2}(\mathcal{R}_2) \times \mathsf{d}_{L_3}(\mathcal{R}_3) \to \mathbb{C}^{\#}, \tag{12}$$

where # denotes the number of invariant structures of type $\langle \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 \rangle$. Only situations with $\# = 1$ will be relevant in this work.

IV. THE BOOTSTRAP PROBLEM

A. The mixing problem

Consider the conformal block decomposition (7) and expand the CFT data in a small parameter $1/\sqrt{\lambda}$, for example,

$$\Delta_{\mathcal{O}} = \Delta^{(0)}_{\mathcal{O}} + \frac{1}{\sqrt{\lambda}} \gamma^{(1)}_{\mathcal{O}} + \frac{1}{\lambda} \gamma^{(2)}_{\mathcal{O}} + \cdots, \tag{13}$$

and similarly for the OPE coefficients. This produces logarithms in the small χ expansion of the correlation function. More precisely, the correlator at order ℓ has the structure

$$f^{(\ell)}(\chi) = \sum_{k=0}^{\ell} f^{(\ell)}_{\log^k}(\chi) (\log \chi)^k, \tag{14}$$

where $f^{(\ell)}_{\log^k}(\chi)$ are analytic at $\chi = 0$. Their explicit expression in terms of CFT data is given in the Supplemental Material [9]. The functions that multiply higher powers of the logarithms (those with $k > 1$) are expressed in terms of CFT data at lower order. What makes the bootstrap problem more complicated, but also more interesting, is that, in general, due to degeneracies in the spectrum of the free theory, these CFT data cannot be obtained from the correlator (4) alone.

From the knowledge of $f^{(0)}(\chi)$ and $f^{(1)}(\chi)$ one can extract, via the decomposition (7), the combinations

$$\langle a^{(0)}_{\Delta} \rangle \coloneqq \sum_{\mathcal{O}|\Delta^{(0)}_{\mathcal{O}}=\Delta} (\mu^{(0)}_{\mathcal{O}})^2, \tag{15}$$

$$\langle a^{(0)}_{\Delta} \gamma^{(1)}_{\Delta} \rangle \coloneqq \sum_{\mathcal{O}|\Delta^{(0)}_{\mathcal{O}}=\Delta} (\mu^{(0)}_{\mathcal{O}})^2 \gamma^{(1)}_{\mathcal{O}}, \tag{16}$$

where $\Delta = 2, 4, 6, \ldots$. To reconstruct the highest logarithm at the next order, namely $f^{(2)}_{\log^2}(\chi)$, one needs to know the quantity $\langle a^{(0)}_{\Delta} (\gamma^{(1)}_{\Delta})^2 \rangle$, but these "averaged moments" cannot be extracted from (15) and (16) when operators are degenerate. The 1D SCFT analyzed here has the following interesting property: at first order, the anomalous dimension of any operator is proportional to the eigenvalue of the quadratic Casimir of $\mathfrak{osp}(4^*|4)$ (5):

$$\gamma^{(1)}_{\mathcal{O}} = -\frac{1}{2} \mathfrak{C}_2(\mathcal{R}_{\mathcal{O}}). \tag{17}$$

This implies that the degeneracy is not lifted at first order, and hence any factor of $\gamma^{(1)}_{\mathcal{O}}$ in averages of the type (16) can be replaced by (17) and pulled out of the sum. A simple proof of (17) based on the properties of the dilatation operator, which can be derived from Witten diagram considerations or directly from the bootstrap, is presented in the Supplemental Material [9]. More details will be presented in [14].

Let us move to higher orders in the perturbative expansion. By looking at the expression of $f^{(\ell)}_{\log^k}(\chi)$ entering (14) in terms of CFT data, it is not hard to realize that the first time an unknown combination of CFT data appears for the higher logarithms $k > 1$ is at fourth order. Specifically, $f^{(4)}_{\log^2}(\chi)$ contains terms of the form

$$\langle a^{(0)}_{\Delta} (\gamma^{(2)}_{\Delta})^2 \rangle \coloneqq \sum_{\mathcal{O}|\Delta^{(0)}_{\mathcal{O}}=\Delta} (\mu^{(0)}_{\mathcal{O}})^2 (\gamma^{(2)}_{\mathcal{O}})^2. \tag{18}$$

The main obstacle to bootstrap the correlator (4) at this order is to determine (18). Luckily, to do this we do not need to find the eigenvalues $\gamma^{(2)}_{\mathcal{O}}$ and eigenvectors $\mathcal{O}$ of the second order dilatation operator. In fact, we can work in an arbitrary (nonorthogonal) basis for the exchanged operators. It is convenient to use a basis in which the length L is a good quantum number, which implies two simplifications. The first is obvious: the three-point function $\mathsf{C}^{(0)}_{\mathcal{D}_1\mathcal{D}_1-}$ [see (12)] is nonvanishing only if the third operator has length two—these are nondegenerate, and we denote the corresponding OPE coefficient by $\mathsf{C}^{(0)}_{11\Delta}$. The second concerns the anomalous dimension matrix $\Gamma^{(2)}_{\Delta}$: its components are nonvanishing only among operators of the same length or whose lengths differ by two units [25]. We denote the corresponding building blocks by $\Gamma^{(2)}_{\Delta,L\to L}$ and $\Gamma^{(2)}_{\Delta,L\to L+2}$, with

$$\Gamma^{(2)}_{\Delta,L_1\to L_2} \in \mathsf{d}_{L_1}(\Delta) \times \mathsf{d}_{L_2}(\Delta), \tag{19}$$

where the degeneracy spaces $\mathsf{d}_L(\Delta)$ were defined in (10). In the Supplemental Material [9] we give explicit examples for $\Delta = 4$, 6. We conclude that in this basis, after normalizing the length-two operators, the expression (18) takes the form

$$\langle a^{(0)}_{\Delta}(\gamma^{(2)}_{\Delta})^2\rangle = \langle a^{(0)}_{\Delta}\rangle[(\Gamma^{(2)}_{\Delta,2\to 2})^2 + \delta\Gamma^{(2)}_{\mathtt{sq}}(\Delta)], \tag{20}$$

where

$$\delta\Gamma^{(2)}_{\mathtt{sq}}(\Delta) \coloneqq \Gamma^{(2)}_{\Delta,2\to 4}\cdot \mathsf{g}_4 \cdot \Gamma^{(2)}_{\Delta,2\to 4}, \tag{21}$$

and g_4 is the metric in the space (11). While the number $\Gamma^{(2)}_{\Delta,2\to 2}$ is obtained from $f^{(2)}(\chi)$, to extract the vector $\Gamma^{(2)}_{\Delta,2\to 4}$ one has to consider a family of correlators at second order. A natural choice is given by

$$\langle \mathcal{D}_1\mathcal{D}_1\mathcal{D}_2\mathcal{O}_{\mathtt{ext}}\rangle, \tag{22}$$

with $\mathcal{O}_{\mathtt{ext}}$ either of type $\mathcal{D}_2$ or $\mathcal{L}^{\Delta_{\mathtt{ext}}}_{0,[0,0]}$. From the superconformal block decomposition of the correlator (22) in the channel (6) and (8) we can extract

$$\langle a^{(0)}_{\Delta}\gamma^{(2)}_{\Delta}\rangle_{112\mathcal{O}_{\mathtt{ext}}} = \langle a^{(0)}_{\Delta}\rangle_{112\mathcal{O}_{\mathtt{ext}}}\Gamma^{(2)}_{\Delta,2\to 2} + \mathsf{C}^{(0)}_{11\Delta}\mathsf{X}_{\Delta,\mathcal{O}_{\mathtt{ext}}}, \tag{23}$$

where [26]

$$\mathsf{X}_{\Delta,\mathcal{O}_{\mathtt{ext}}} \coloneqq \Gamma^{(2)}_{\Delta,2\to 4}\cdot \mathsf{C}^{(0)}_{\mathcal{D}_2\mathcal{O}_{\mathtt{ext}}}. \tag{24}$$

The subscript in $\langle\cdots\rangle_{112\mathcal{O}_{\mathtt{ext}}}$ entering (23) indicates that the average in question is determined by the correlator (22), in contrast with all the averages $\langle\cdots\rangle$ encountered so far which correspond to the four-point function (4).

To extract $\Gamma^{(2)}_{\Delta,2\to 4}$ from (24) we need to consider enough external operators such that the vectors $\mathsf{C}^{(0)}_{\mathcal{D}_2\mathcal{O}_{\mathtt{ext}}}$ span, upon varying $\mathcal{O}_{\mathtt{ext}}$, the whole degeneracy space $\mathsf{d}_4(\Delta)$. As their dimension grows quadratically with Δ [27] [see (11)], the number of auxiliary correlators (22) that we consider should grow accordingly. This is achieved by taking as external operator $\mathcal{D}_2$ together with all the $L = 2$ and $L = 4$ operators of type $\mathcal{L}^{\Delta_{\mathtt{ext}}}_{0,[0,0]}$ where $\Delta_{\mathtt{ext}} = 2, 4, \ldots$, up to some maximum depending on the value of Δ in $\Gamma^{(2)}_{\Delta,2\to 4}$ [28]. The three-point functions in (24) are a crucial input for this procedure. We compute them in the free theory after constructing the operators explicitly using a new method described in [14]. Notice that $\mathcal{D}_k \times \mathcal{D}_k$ probes the same direction in $\mathsf{d}_4(\Delta)$ for any $k > 1$, so half-BPS external operators are insufficient to take mixing into account.

B. The ansatz

To bootstrap perturbative correlators we follow and develop the strategy introduced in [5] and extended to higher orders in [6] (see also [29–31]), which uses a basis of harmonic polylogarithms. In [4,6] it was argued that the correct basis for 1D CFTs contains the "words" that can be built using the symbol map [32] from the two "letters" χ and $1-\chi$. The use of such a basis requires an external input, namely the maximal transcendental weight $\mathtt{t}$ of the harmonic polylogarithms. Because of the structure of the perturbative OPE and the polynomiality in Δ of the first-order anomalous dimensions, the correct choice of basis at the ℓth perturbative order has $\mathtt{t} = \ell$. An explicit basis is given in the Supplemental Material [9] up to $\mathtt{t} = 4$; its dimension is $\sum_{\mathtt{t}=0}^{\ell} 2^{\mathtt{t}} = 2^{\ell+1} - 1$. For a generic ℓth order correlator $G^{(\ell)}(\chi)$, we make the ansatz

$$G^{(\ell)}(\chi) = \sum_{i=1}^{2^{\ell+1}-1} r_i(\chi)\mathcal{T}_i(\chi), \tag{25}$$

where $r_i(\chi)$ are polynomials in χ divided by powers of χ and $(1-\chi)$ [6], while $\mathcal{T}_i(\chi)$ form our basis of harmonic polylogarithms for transcendentality up to $\mathtt{t} = \ell$. The bootstrap problem is then reduced to that of fixing the rational functions $r_i(\chi)$ appearing in (25). In the following we describe the strategy for the two types of correlators introduced in (3).

We can fix the correlator $\langle\mathcal{D}_1\mathcal{D}_1\mathcal{D}_1\mathcal{D}_1\rangle$ completely up to fourth order using the ansatz described above, with the following constraints:

(a) Crossing symmetry, which for the reduced correlator $f(\chi)$ appearing in (4) reads

$$(1-\chi)^2 f(\chi) + \chi^2 f(1-\chi) = 0. \tag{26}$$

(b) As discussed, at every order the highest powers of $\log\chi$ [those with $k > 1$ in Eq. (14)] in the $\chi \to 0$ limit can be obtained from previous order data.

(c) The invariance of the free theory under $\chi \to \frac{\chi}{\chi-1}$ is "weakly" broken by perturbative corrections, but it still constrains correlators at each order, see [5,6].

(d) As discussed in Sec. II, the quantity $\mu^2_{\mathcal{D}_2}$ is known from localization. The first orders read

$$\mu^2_{\mathcal{D}_2} = 2 - \frac{3}{\lambda^{1/2}} + \frac{45}{8\lambda^{3/2}} + \frac{45}{4\lambda^2} + \cdots. \tag{27}$$

This provides the definition of the coupling λ.

This fixes $f(\chi)$ at each order up to polynomial ambiguities in the anomalous dimensions, of degree 6, 10, 14, … in Δ, which we fix by requiring that, at each order, the average anomalous dimension has the mildest possible growth at large Δ:

$$\langle \gamma^{(\ell)}_\Delta \rangle \coloneqq \frac{\langle a^{(0)}_\Delta \gamma^{(\ell)}_\Delta \rangle}{\langle a^{(0)}_\Delta \rangle} \sim \Delta^{\ell+1}, \qquad 1 \leq \ell \leq 4. \tag{28}$$

We now move to correlators (22), with $\mathcal{O}_{\mathtt{ext}} = \mathcal{L}^{\Delta_{\mathtt{ext}}}_{0,[0,0]}$. Given the knowledge of (17), the free theory and first-order results do not present any conceptual novelty. At second order, it is important to realize that not all the correlator is necessary to extract the averages (23). Rather, it is enough to bootstrap the part of the correlator that produces a $\log \chi$ in the OPE around $\chi = 0$, where those averages first appear: we shall refer to this as $F^{(2)}_{\log}(\chi)$. Once the terms of transcendentality 2 in the ansatz are fixed by lower-order data and property (c), $F^{(2)}_{\log}(\chi)$ is known up to a finite number of undetermined coefficients, which we fix with the following recursive procedure.

Imagine knowing all $F^{(2)}_{\log}(\chi)$ up to $\Delta_{\mathtt{ext}} = \hat{\Delta} - 2$: these allow one to extract $\Gamma^{(2)}_{\Delta,2\to4}$ at least for $\Delta = 2, \ldots, \hat{\Delta} + 2$. We use the latter to compute the combination (23) for the same values of Δ, but now averaged over each correlator with $\Delta_{\mathtt{ext}} = \hat{\Delta}$, which in turn fixes $F^{(2)}_{\log}(\chi)$ completely for $\Delta_{\mathtt{ext}} = \hat{\Delta}$. Once that is known, we can compute new entries of $\Gamma^{(2)}_{\Delta,2\to4}$ that we use for the following recursive step. The starting point of the recursion is $\Delta_{\mathtt{ext}} = 2$, which can be fixed by using the averages computed from $\langle \mathcal{D}_1 \mathcal{D}_1 \mathcal{D}_k \mathcal{D}_k \rangle$.

C. Results

Our first important intermediate result is the computation of the average (20), which is necessary to bootstrap $f^{(4)}(\chi)$. While $\Gamma^{(2)}_{\Delta,2\to2}$ can be found in Eq. (6.24) of [5], the newly computed contribution due to mixing is

$$\delta\Gamma^{(2)}_{\mathtt{sq}}(\Delta) = j^2_\Delta \left(\frac{(j^2_\Delta - 2)}{2} S_{-2}(\Delta) + \frac{3j^2_\Delta - 4}{8} H^2_\Delta \right) - \frac{p_1(\Delta)}{(\Delta+1)(\Delta+2)} H_\Delta + \frac{p_2(\Delta)}{(\Delta+2)}, \tag{29}$$

where $j^2_\Delta = \Delta(\Delta+3)$, $S_{-2}(\Delta) = \sum_{n=1}^{\Delta} \frac{(-1)^n}{n^2}$, H_n is the nth harmonic number, and $p_{1,2}(\Delta)$ are polynomials given in the Supplemental Material [9].

Our main result is the determination of $f(\chi)$ up to fourth order at large λ. The explicit expressions are contained in the Supplemental Material [9]. From this correlator we extract the conformal dimension [see (1)] and squared OPE coefficient of the lightest nonprotected operator ϕ^2

$$\mu^2_{\phi^2} = \frac{2}{5} - \frac{43}{30\sqrt{\lambda}} + \frac{5}{6\lambda} + \left(\frac{11195}{1728} + 4\zeta(3) \right) \frac{1}{\lambda^{3/2}} - \left(\frac{1705}{96} + \frac{1613}{24}\zeta(3) \right) \frac{1}{\lambda^2} + \cdots. \tag{30}$$

More averaged CFT data extracted from this correlator are given in the Supplemental Material [9].

V. DISCUSSION

In this paper we have shown how to bootstrap correlation functions in perturbation theory for a special 1D SCFT from the knowledge of the unperturbed theory, symmetries, consistency conditions and some extra physical input. To implement this program we put forward a new strategy to take into account operator degeneracies which we believe can be applied more broadly, e.g., for holographic CFTs [11,15,16] at higher orders and for the ϵ expansion in [12,33]. There are several interesting open questions for the future.

The first direction is to consider higher orders [34] in the perturbative expansion. This is technically more challenging since additional operators will participate in the mixing, but also involves new conceptual problems related to additional ambiguities and the uniqueness of the theory; see discussion above (28). To address this question it will be useful to study the 1D SCFT defined by the Wilson line with different gauge groups, still at large rank, and in different representations; see, e.g., [35,36].

The theory we are considering is supposed to be integrable [37–39]. In this work we used the integrability-based results of [8] only as a check of our procedure. It would be interesting to numerically determine the conformal dimension of other operators in addition to (1) as a function of the coupling using the method of [8] and compare to our findings. How to directly incorporate integrability into the bootstrap remains a crucial open question; see [40] for explorations in this direction.

Finally, the 1D SCFT considered in this work is also an excellent playground to test and use the so-called inversion formula of [41].

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[1] D. Poland, S. Rychkov, and A. Vichi, Rev. Mod. Phys. **91**, 015002 (2019).
[2] J. Qiao and S. Rychkov, J. High Energy Phys. 12 (2017) 119.
[3] S. Giombi, R. Roiban, and A. A. Tseytlin, Nucl. Phys. **B922**, 499 (2017).
[4] L. F. Alday, Unpublished notes (2017).
[5] P. Liendo, C. Meneghelli, and V. Mitev, J. High Energy Phys. 10 (2018) 077.
[6] P. Ferrero, K. Ghosh, A. Sinha, and A. Zahed, J. High Energy Phys. 07 (2020) 170.
[7] The definition of λ is fixed by the OPE coefficient (27).
[8] D. Grabner, N. Gromov, and J. Julius, J. High Energy Phys. 07 (2020) 042.
[9] See Supplemental Material at (http://link.aps.org/supplemental/10.1103/PhysRevD.104.L081703) for several technical details including the conformal blocks, the perturbative expansion of correlators and CFT data and the relevant space of transcendental functions.
[10] O. Aharony, L. F. Alday, A. Bissi, and E. Perlmutter, J. High Energy Phys. 07 (2017) 036.
[11] F. Aprile, J. M. Drummond, P. Heslop, and H. Paul, J. High Energy Phys. 01 (2018) 035.
[12] L. F. Alday, J. Henriksson, and M. van Loon, J. High Energy Phys. 07 (2018) 131.
[13] S. Caron-Huot, L. J. Dixon, J. M. Drummond, F. Dulat, J. Foster, O. Gürdoğan, M. von Hippel, A. J. McLeod, and G. Papathanasiou, Proc. Sci., CORFU2019 (**2020**) 003.
[14] P. Ferrero and C. Meneghelli (to be published).
[15] L. F. Alday and A. Bissi, Phys. Rev. Lett. **119**, 171601 (2017).
[16] L. F. Alday, S. M. Chester, and H. Raj, J. High Energy Phys. 01 (2021) 133.
[17] Here $s \in \mathbb{N}$ corresponds to the $s+1$ dimensional representation of $\mathfrak{su}(2)$ while $[a, b]$ are $\mathfrak{sp}(4)$ Dynkin labels, so that $[1, 0] = \mathbf{4}$ and $[0, 1] = \mathbf{5}$.
[18] P. Liendo and C. Meneghelli, J. High Energy Phys. 01 (2017) 122.
[19] S. M. Chester, J. Lee, S. S. Pufu, and R. Yacoby, J. High Energy Phys. 03 (2015) 130.
[20] C. Beem, W. Peelaers, and L. Rastelli, Commun. Math. Phys. **354**, 345 (2017).
[21] V. Pestun, Commun. Math. Phys. **313**, 71 (2012).
[22] S. Giombi and V. Pestun, J. High Energy Phys. 10 (2010) 033.
[23] S. Giombi and S. Komatsu, J. High Energy Phys. 05 (2018) 109; 11 (2018) 123.
[24] There is a unique super-conformal invariant structure of type $\langle \mathcal{D}_2 \mathcal{L}^{\Delta_{\mathtt{ext}}}_{0,[0,0]} \mathcal{L}^{\Delta}_{0,[0,0]} \rangle$. The relevant OPE coefficient can be extracted from the three-point correlator of the $\mathcal{D}_2$ and $\mathcal{L}^{\Delta_{\mathtt{ext}}}_{0,[0,0]}$ superprimaries with the $Q^4|_{0,[0,2]}$ descendant of the exchanged operator of type $\mathcal{L}^{\Delta}_{0,[0,0]}$.
[25] This can be argued from the structure of the Witten diagrams or directly from the bootstrap, see Ref. [14].
[26] According to Eq. (12), upon specifying the second and third operator in $\mathsf{C}^{(0)}$ we obtain a map $\mathsf{C}^{(0)}_{\mathcal{D}_2 \mathcal{O}_{\mathtt{ext}}} : \mathsf{d}_4(\Delta) \to \mathbb{C}$. This map is composed with $\Gamma^{(2)}_{\Delta, 2\to 4}$, see (19), to produce a number.
[27] This is an important difference compared to Refs. [11,15,16].
[28] With the maximum values $\Delta_{\mathtt{ext}} = 14$ and $\Delta_{\mathtt{ext}} = 22$ for the length-four and length-two operators one can span all directions of $\mathsf{d}_4(\Delta)$ up to $\Delta = 26$.
[29] A. Gimenez-Grau and P. Liendo, J. High Energy Phys. 03 (2020) 121.
[30] L. Bianchi, G. Bliard, V. Forini, L. Griguolo, and D. Seminara, J. High Energy Phys. 08 (2020) 143.
[31] N. Drukker, S. Giombi, A. A. Tseytlin, and X. Zhou, J. High Energy Phys. 07 (2020) 101.
[32] C. Duhr, H. Gangl, and J. R. Rhodes, J. High Energy Phys. 10 (2012) 075.
[33] D. Carmi, J. Penedones, J. A. Silva, and A. Zhiboedov, SciPost Phys. **10**, 145 (2021).
[34] While the perturbative expansion in $\lambda^{-1/2}$ is asymptotic, the expansion of CFT data, like $\mu^2_{\mathcal{D}_2}$, as a function of $\Delta_{\phi^2} - 2$ appears to be convergent.
[35] J. Gomis and F. Passerini, J. High Energy Phys. 08 (2006) 074.
[36] S. Giombi and B. Offertaler, arXiv:2006.10852.
[37] N. Drukker and S. Kawamoto, J. High Energy Phys. 07 (2006) 024.
[38] N. Drukker, J. High Energy Phys. 10 (2013) 135.
[39] D. Correa, J. Maldacena, and A. Sever, J. High Energy Phys. 08 (2012) 134.
[40] N. Kiryu and S. Komatsu, J. High Energy Phys. 02 (2019) 090.
[41] D. Mazáč, J. High Energy Phys. 06 (2019) 082.