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A polyatomic gas with slow relaxation of the internal modes in contact with a solid boundary is considered. In a previous paper [K. Aoki *et al.*, Phys. Rev. E **102**, 023104 (2020)], the two-temperature Navier–Stokes system, i.e., a set of compressible Navier–Stokes equations with the translational and internal temperatures, was derived from the ellipsoidal-statistical (ES) model of the Boltzmann equation for a polyatomic gas under the assumption that the Knudsen number is small and the ratio of the collisional mean free time to the relaxation time of the internal modes is as small as the Knudsen number. In the present study, the appropriate boundary conditions for the two-temperature Navier–Stokes system are derived by the analysis of the Knudsen layer on the basis of the ES model for a polyatomic gas and the Maxwell-type diffuse-specular reflection condition on the boundary. The resulting boundary conditions, which are of the type of slip boundary conditions, are summarized, together with the two-temperature Navier–Stokes equations, in a form that is applicable to practical applications immediately.

I. INTRODUCTION

Rarefied polyatomic gas flows play increasingly important roles in non-equilibrium gas dynamics and its applications. For these flows, one needs kinetic theory based on the Boltzmann equation, which can be written in an abstract form using the transition probabilities of microscopic states of molecules during molecular collisions [1–9]. However, the transition probabilities, which depend on the detailed structure of molecules and thus on individual gases, are not known for many polyatomic gases. Therefore, it is hard to directly apply the Boltzmann equation for practical flow problems.

To avoid this difficulty, two different approaches are often taken, in addition to the direct simulation Monte Carlo (DSMC) method [10], which will not be discussed in this paper. One approach is to use kinetic model equations, such as the models of the Bhatnagar–Gross–Krook (BGK) type, that have dramatically simplified collision integrals and satisfy some basic properties of the original Boltzmann equation [11–25]. The other is to use macroscopic or fluid equations that are simpler than the kinetic models but are expected to be accurate when the state of the gas is close to a local equilibrium. In the present study, we consider the latter approach.

There have been many attempts to construct macroscopic equations on the basis of kinetic theory or purely macroscopic considerations [6, 8, 21, 26–34]. One of the standard approaches is to derive equations of Euler and Navier–Stokes types using the Chapman–Enskog procedure [35] from the Boltzmann equation [6, 8, 28, 33, 34]. In the case of a polyatomic gas, the standard Chapman–Enskog expansion leads to the ordinary Navier–Stokes equations with a single temperature and with bulk viscosity. However, for a gas in which the characteristic (or relaxation) times of different internal modes of a molecule differ significantly, the ordinary Navier–Stokes equations with a single temperature are not sufficient to describe flow properties [28, 31, 33].

In order to reduce the difficulty, some authors have derived the Euler- or Navier–Stokes-type equations with multi temperatures associated with the translational motion and with the internal modes of a molecule on the basis of the Boltzmann equation, taking into account the differences in the relaxation times of different internal modes [1, 28, 33, 34]. However, although accurate theoretically, these approaches require a large amount of information on the molecular structure, containing numerical and empirical formulas and some assumptions, for individual gases. Therefore, for practical applications, it was desirable to construct handy and overall fluid-dynamic models that do not depend on the detailed molecular structure but contain only overall information.

To answer this problem, four of the present authors proposed a handy set of macroscopic equations of Navier–Stokes type with two temperatures, which they called the *two-temperature* Navier–Stokes equations (or system) [36]. Unlike the previous studies [1, 28, 33], the starting point was not the original Boltzmann equation but the polyatomic version of the ellipsoidal-statistical (ES) model, which is one of the widely used kinetic models proposed in [16] and re-derived in a systematic way in [37]. The ES model contains a parameter that is of the order of the ratio of the mean free time of the gas molecules to the relaxation time of the internal modes. Under the assumption that this parameter is as small as the Knudsen number, the Chapman–Enskog expansion [35] was carried out to derive the two-temperature Navier–Stokes equations. Since the number of parameters contained in the ES model is much less than the original Boltzmann equation, the transport properties of the resulting two-temperature Navier–Stokes system in terms of the parameters are much simpler and perfectly explicit. Therefore, it has a wide applicability to practical flow problems. In fact, the system was successfully applied to the problem of the structure of a stationary shock wave in CO_2 gas [36, 38].

However, since most of practical flow problems contain solid boundaries, we need appropriate boundary conditions in order to enlarge the applicability of the two-temperature Navier–Stokes equations. The appropriate boundary conditions for the standard Navier–Stokes equations with a single temperature have been obtained for a polyatomic gas in [39] on the basis of the ES model, following the procedure in [40] for a monatomic gas. In the present study, we will derive the appropriate boundary conditions for the two-temperature Navier–Stokes equations [36], starting from the ES model, together with the condition of diffuse-specular reflection on the boundaries (the so-called Maxwell-type condition), and following the method in [39]. The essence of the procedure lies in the analysis of the Knudsen layer adjacent to the boundary. The adoption of the ES model, which is consistent with the two-temperature Navier–Stokes equations, facilitates the analysis of the Knudsen layer and enables to obtain the explicit form of the boundary conditions, which are of the form of slip conditions, as we will see in Sec. VII. It should be emphasized that it is a great advantage of the two-temperature Navier–Stokes equations to have clear boundary conditions, compared with other types of moment equations.

Here, the following remark is in order. Let us consider the case of a monatomic gas. The (compressible) Navier–Stokes equations correspond to the first-order Chapman–Enskog solution, which formally satisfies the Boltzmann equation up to the order of the Knudsen number (Kn). Therefore, the boundary conditions for the Navier–Stokes equations should be constructed in such a manner that the kinetic boundary condition for the Boltzmann equation is satisfied up to the order of Kn, and the resulting conditions are of the form of slip conditions, as pointed out in [40]. In this sense, the usual no-slip conditions, which correspond to satisfying the kinetic boundary condition only at the zeroth order in Kn, are not consistent, and use is to be made of the slip boundary conditions for the Navier–Stokes equations. The same remark applies to the case of a polyatomic gas, including the boundary conditions for the two-temperature Navier–Stokes equations.

The paper is organized as follows. After this introduction, the slip boundary conditions for the two-temperature Navier–Stokes equations, which are the main results of the paper, are summarized in Sec. II, where the original kinetic problem and the assumptions are stated, and the two-temperature Navier–Stokes equations are also summarized. The ES model and its initial and boundary conditions are mentioned in Sec. III, and their dimensionless forms are presented in Sec. IV. In Sec. V, the parameter setting for a polyatomic gas with slow relaxation of the internal modes is explained, and the first-order Chapman–Enskog solution, which corresponds to the two-temperature Navier–Stokes system, is summarized. The Knudsen-layer is introduced in Sec. VI, and its analysis is carried out to derive the slip boundary conditions in Secs. VI and VII. The two-temperature Navier–Stokes equations and the derived slip boundary conditions are summarized in dimensional form in Sec. VIII. Section IX is devoted to brief remarks. In addition, the main text is supplemented by four appendices.

II. SUMMARY OF MAIN RESULTS

Our aim is to construct the boundary conditions for the two-temperature Navier–Stokes equations derived in [36]. For this purpose, we need to start with the description of the problem in the framework of kinetic theory. One will see how the analysis of the kinetic problem provides

the desired boundary conditions in the following sections. However, since the analysis takes several pages, we will summarize the main results, together with some necessary information, in this section.

A. Problem and assumptions

The basic kinetic problem is described as follows. Let us consider a polyatomic (or diatomic) ideal gas in contact with solid boundaries of arbitrary but smooth shape. The gas may extend to infinity, and no external force acts on the gas molecules. We investigate the unsteady behavior of the gas under the following assumptions:

(i) The behavior of the gas is described by the ES model of the Boltzmann equation for a polyatomic gas [16, 37].

(ii) The boundaries do not deform and undergo a rigid-body motion, and the gas-surface interaction is described by the Maxwell-type diffuse-specular reflection.

(iii) The Knudsen number, which is the ratio of the mean free path (or the mean free time) of the gas molecules at the reference equilibrium state at rest to the characteristic length (or the characteristic time) of the system, is sufficiently small.

(iv) The gas is such that the internal modes relax much slower than the translational mode. To be more precise, the ratio of the mean free time of the gas molecules to the characteristic (or relaxation) time of the internal modes is as small as the Knudsen number.

(v) At the initial time, the boundaries are at rest and have a uniform temperature, and the gas is in the equilibrium state at rest with the same temperature. After the initial time, the boundaries may start moving smoothly, and their temperature may change smoothly in time and position. (For the problems including infinities, the corresponding initial state and slow variations should be assumed at infinities.)

We put assumption (v) to avoid the occurrence of the initial layer and that of the interaction between the initial layer and the Knudsen layer during the initial stage for the sake of theoretical rigor (cf. [40]). Assumption (v) may be relaxed if we admit the inaccuracy during the initial stage with the duration of the order of the mean free time.

B. Notation and parameters

Let us denote by δ the number of the internal degrees of freedom of the gas molecule, where δ is a constant such that $\delta \geq 2$. Then, the specific heat at constant volume c_{v}, that at constant pressure c_{p}, and the ratio of the specific heats $\gamma = c_{\mathrm{p}}/c_{\mathrm{v}}$ are all constant and are expressed as

$$c_{\mathrm{v}} = \frac{\delta + 3}{2} R, \qquad c_{\mathrm{p}} = \frac{\delta + 5}{2} R, \qquad \gamma = \frac{\delta + 5}{\delta + 3}. \tag{1}$$

Here, R is the gas constant per unit mass and is related to the Boltzmann constant k_{B} and the mass of a molecule m by $R = k_{\mathrm{B}}/m$.

Let t be the time variable and $\boldsymbol{X}$ (or X_i) be the position vector in the physical space. Let ρ denote the density, $\boldsymbol{v}$ (or v_i) the flow velocity, T_{tr} the temperature associated with the translational energy, T_{int} the temperature associated with the energy of the internal modes, and T the temperature. The kinetic definitions of these macroscopic quantities are given in Eqs. (18d), (18e), and (18g)–(18i) in Sec. III A, respectively.

The ES model, which will be detailed in Sec. III A and Appendix A, contains two adjustable parameters $\nu \in [-1/2, 1)$ and $\theta \in [0, 1]$, as well as a function $A_c(T)$ of the temperature T such that $A_c(T)\rho$ indicates the collision frequency of the gas molecules. These quantities are related to the viscosity $\mu(T)$, the bulk viscosity $\mu_b(T)$, and the thermal conductivity $\lambda(T)$ as shown by Eq. (A7a)–(A7c) in Appendix A 2. As the result, the Prandtl number $\mathrm{Pr} = c_{\mathrm{p}}\mu/\lambda$ is expressed in terms of ν and θ by Eq. (A8).

C. Two-temperature Navier–Stokes equations

In this subsection, we summarize the two-temperature Navier–Stokes equations, derived in [36], in the dimensionless form. Their dimensional form is shown in Sec. VIII.

Let us denote by L the reference length, t_0 the reference time, ρ_0 the reference density, and T_0 the reference temperature. In the present study, we choose t_0 as

$$t_0 = L/(2RT_0)^{1/2}, \tag{2}$$

which corresponds to the so-called fluid-dynamic scaling. Now we introduce the dimensionless quantities $[\hat{t},\ x_i,\ \hat{\rho},\ \hat{v}_i,\ \hat{T}_{\rm tr},\ \hat{T}_{\rm int},\ \hat{T},\ \hat{A}_c(\hat{T})]$, which correspond to the original dimensional quantities $[t,\ X_i,\ \rho,\ v_i,\ T_{\rm tr},\ T_{\rm int},\ T,\ A_c(T)]$, by the following relations:

$$\begin{aligned} &\hat{t} = t/t_0, \qquad x_i = X_i/L, \qquad \hat{\rho} = \rho/\rho_0, \qquad \hat{v}_i = v_i/(2RT_0)^{1/2}, \\ &(\hat{T}_{\rm tr},\, \hat{T}_{\rm int},\, \hat{T}) = (T_{\rm tr},\, T_{\rm int},\, T)/T_0, \qquad \hat{A}_c(\hat{T}) = A_c(T)/A_c(T_0). \end{aligned} \tag{3}$$

According to assumption (iii) in Sec. II A, the Knudsen number Kn, defined by ${\rm Kn} = l_0/L$, is small, where l_0 is the mean free path of the gas molecules at the reference equilibrium state at rest with density ρ_0 and temperature T_0 [cf. Eq. (A5) in Appendix A 1]. In the present paper, we use the small parameter ϵ:

$$\epsilon = \frac{\sqrt{\pi}}{2}{\rm Kn} = \frac{\sqrt{\pi}}{2}\frac{l_0}{L} \ll 1, \tag{4}$$

in place of Kn. As will be explained in Sec. V A, the ratio of the mean free time of the gas molecules to the characteristic (or relaxation) time of the internal modes is represented by the parameter θ included in the ES model. Therefore, assumption (iv) in Sec. II A indicates the following setting:

$$\theta = \alpha\epsilon \ll 1, \tag{5}$$

where α is a positive constant (parameter) of the order of unity. It follows from Eq. (A9) that small values of θ indicate large values of the ratio μ_b/μ of the bulk viscosity to the viscosity. Therefore, we can also say that we are considering gases with large bulk viscosities.

The two-temperature Navier–Stokes equations, which have been derived from the ES model by the Chapman–Enskog expansion under the condition (5), have the following form [36]:

$$\frac{\partial\hat{\rho}}{\partial\hat{t}} + \frac{\partial(\hat{\rho}\hat{v}_j)}{\partial x_j} = 0, \tag{6a}$$

$$\frac{\partial(\hat{\rho}\hat{v}_i)}{\partial\hat{t}} + \frac{\partial(\hat{\rho}\hat{v}_i\hat{v}_j)}{\partial x_j} + \frac{1}{2}\frac{\partial(\hat{\rho}\hat{T}_{\rm tr})}{\partial x_i} = \frac{1}{2}\,\epsilon\frac{\partial}{\partial x_j}\left[\Gamma_\mu(\hat{T},\,\hat{T}_{\rm tr})\left(\frac{\partial\hat{v}_i}{\partial x_j} + \frac{\partial\hat{v}_j}{\partial x_i} - \frac{2}{3}\frac{\partial\hat{v}_k}{\partial x_k}\delta_{ij}\right)\right], \tag{6b}$$

$$\begin{aligned} &\frac{\partial}{\partial\hat{t}}\left[\hat{\rho}\left(\frac{3}{2}\hat{T}_{\rm tr} + \hat{v}_i^2\right)\right] + \frac{\partial}{\partial x_j}\left[\hat{\rho}\hat{v}_j\left(\frac{5}{2}\hat{T}_{\rm tr} + \hat{v}_i^2\right)\right] - \frac{3}{2}\alpha\hat{A}_c(\hat{T})\hat{\rho}^2(\hat{T} - \hat{T}_{\rm tr}) \\ &\quad = \frac{5}{4}\,\epsilon\frac{\partial}{\partial x_j}\left[\Gamma_\lambda(\hat{T},\,\hat{T}_{\rm tr})\frac{\partial\hat{T}_{\rm tr}}{\partial x_j}\right] + \epsilon\frac{\partial}{\partial x_j}\left[\Gamma_\mu(\hat{T},\,\hat{T}_{\rm tr})\hat{v}_i\left(\frac{\partial\hat{v}_i}{\partial x_j} + \frac{\partial\hat{v}_j}{\partial x_i} - \frac{2}{3}\frac{\partial\hat{v}_k}{\partial x_k}\delta_{ij}\right)\right], \end{aligned} \tag{6c}$$

$$\frac{\partial(\hat{\rho}\hat{T}_{\rm int})}{\partial\hat{t}} + \frac{\partial(\hat{\rho}\hat{v}_j\hat{T}_{\rm int})}{\partial x_j} - \alpha\hat{A}_c(\hat{T})\hat{\rho}^2(\hat{T} - \hat{T}_{\rm int}) = \frac{1}{2}\,\epsilon\frac{\partial}{\partial x_j}\left[\Gamma_\lambda(\hat{T},\,\hat{T}_{\rm tr})\frac{\partial\hat{T}_{\rm int}}{\partial x_j}\right], \tag{6d}$$

where

$$\Gamma_\mu(\hat{T},\,\hat{T}_{\rm tr}) = \frac{\hat{T}_{\rm tr}}{(1-\nu)\hat{A}_c(\hat{T})}, \qquad \Gamma_\lambda(\hat{T},\,\hat{T}_{\rm tr}) = \frac{\hat{T}_{\rm tr}}{\hat{A}_c(\hat{T})}, \tag{7}$$

and $\hat{T} = (3\hat{T}_{\rm tr} + \delta\hat{T}_{\rm int})/(3+\delta)$. These are the equations for $\hat{\rho}$, $\hat{v}_i$, $\hat{T}_{\rm tr}$, and $\hat{T}_{\rm int}$ and contain the parameters ϵ, ν, and α. Here and in what follows, we basically use the summation convention, i.e., $a_ib_i = \sum_{i=1}^3 a_ib_i$, $a_i^2 = \sum_{i=1}^3 a_i^2$, etc. It is shown in Appendix B that the ordinary Navier–Stokes equations with a single temperature can be recovered from Eq. (6).

D. Slip boundary conditions: Main results

The slip boundary conditions for the two-temperature Navier–Stokes equations, which are obtained by the analysis of the Knudsen layer detailed in Secs. VI and VII, are the main results in this paper. Their dimensionless form is summarized below. See Sec. VIII for their dimensional form.

Let $\boldsymbol{X}_{\rm w}$ (or $X_{{\rm w}i}$) be the position of a point on the boundary, $\boldsymbol{v}_{\rm w}$ (or $v_{{\rm w}i}$) and $T_{\rm w}$ be, respectively, the velocity and temperature of the boundary at the point $\boldsymbol{X}_{\rm w}$. Let us denote by $\boldsymbol{n}$ (or n_i) the unit normal vector to the boundary, pointing into the gas, at $\boldsymbol{X}_{\rm w}$ and by $\boldsymbol{t}$ (or t_i) an arbitrary unit tangential vector to the boundary at the same point. Following the notion of fields, we understand that the arguments of $\boldsymbol{v}_{\rm w}$, $T_{\rm w}$, $\boldsymbol{n}$, and $\boldsymbol{t}$ are $(t,\, \boldsymbol{X}_{\rm w})$. Then, we introduce the dimensionless quantities $(x_{{\rm w}i},\, \hat{v}_{{\rm w}i},\, \hat{T}_{\rm w})$ corresponding to $(X_{{\rm w}i},\, v_{{\rm w}i},\, T_{\rm w})$ by

$$x_{{\rm w}i} = X_{{\rm w}i}/L, \qquad \hat{v}_{{\rm w}i} = v_{{\rm w}i}/(2RT_0)^{1/2}, \qquad \hat{T}_{\rm w} = T_{\rm w}/T_0. \tag{8}$$

The kinetic boundary condition for the ES model is the Maxwell-type condition, which is a linear combination of the diffuse reflection, with coefficient a_c $(0 \le a_c \le 1)$, and the specular reflection, with coefficient $1 - a_c$ [see assumption (ii) in Sec. II A]. The coefficient a_c is the so-called accommodation coefficient, giving the specular reflection when $a_c = 0$ and the diffuse reflection when $a_c = 1$. In the present paper, we exclude the case of specular reflection assuming that a_c is strictly positive. The explicit form of the kinetic boundary condition is given by Eq. (25) in Sec. III B in the dimensional form and by Eq. (41) in Sec. IV B in the dimensionless form.

Under the Maxwell-type condition, the ES model gives the following slip boundary conditions for Eq. (6):

$$(\hat{v}_i - \hat{v}_{{\rm w}i})n_i = 0, \tag{9a}$$

$$(\hat{v}_i - \hat{v}_{{\rm w}i})t_i = \epsilon c_v^I \frac{\hat{T}_{\rm w}^{1/2}}{\hat{A}_c(\hat{T}_{\rm w})} \frac{1}{\hat{\rho}} \left(\frac{\partial \hat{v}_i}{\partial x_j} + \frac{\partial \hat{v}_j}{\partial x_i} \right) n_i t_j + \epsilon c_T^I \frac{1}{\hat{A}_c(\hat{T}_{\rm w})} \frac{1}{\hat{\rho}} \frac{\partial \hat{T}_{\rm w}}{\partial x_i} t_i, \tag{9b}$$

$$\hat{T}_{\rm tr} - \hat{T}_{\rm w} = \epsilon c_v^{II} \frac{\hat{T}_{\rm w}}{\hat{A}_c(\hat{T}_{\rm w})} \frac{1}{\hat{\rho}} \frac{\partial \hat{v}_i}{\partial x_j} n_i n_j + \epsilon c_T^{II} \frac{\hat{T}_{\rm w}^{1/2}}{\hat{A}_c(\hat{T}_{\rm w})} \frac{1}{\hat{\rho}} \frac{\partial \hat{T}_{\rm tr}}{\partial x_i} n_i, \tag{9c}$$

$$\hat{T}_{\rm int} - \hat{T}_{\rm w} = \epsilon \tilde{c}_T^{II} \frac{\hat{T}_{\rm w}^{1/2}}{\hat{A}_c(\hat{T}_{\rm w})} \frac{1}{\hat{\rho}} \frac{\partial \hat{T}_{\rm int}}{\partial x_i} n_i, \tag{9d}$$

where the quantities $\hat{\rho}$, $\hat{v}_i$, $\hat{T}_{\rm tr}$, and $\hat{T}_{\rm int}$ as well as their derivatives are all evaluated at the boundary $\boldsymbol{x} = \boldsymbol{x}_{\rm w}$. The coefficients c_v^I, c_T^I, c_v^{II}, c_T^{II}, and $\tilde{c}_T^{II}$, which are called the slip coefficients, depend on the parameters ν (except c_T^I and $\tilde{c}_T^{II}$) and a_c. Leaving the details of the analysis determining the slip coefficients to Sec. VII, we show only their resulting numerical values here.

It is noted that the coefficients c_v^I, c_T^I, and $\tilde{c}_T^{II}$ satisfy the following relations:

$$c_v^I = \frac{1}{1-\nu}\, c_{v{\rm BGK}}^I, \qquad c_T^I = c_{T{\rm BGK}}^I, \qquad \tilde{c}_T^{II} = c_{v{\rm BGK}}^I, \tag{10}$$

where $c_{v{\rm BGK}}^I$ and $c_{T{\rm BGK}}^I$ are, respectively, the shear-slip and thermal-slip (creep) coefficients for the BGK model [41, 42] for a monatomic gas (see Sec. VII C for the details). Therefore, the coefficients c_v^I, c_T^I, and $\tilde{c}_T^{II}$ are obtained immediately from the known values in the literature. The numerical values of $c_{v{\rm BGK}}^I$ and $c_{T{\rm BGK}}^I$, taken from [43], are tabulated in Table I for various values of a_c.

In contrast, a new numerical analysis is required to obtain the numerical values of c_v^{II} and c_T^{II} (see Sec. VII C and Appendix D). We show the resulting numerical values in Tables II and III for various values of the parameter ν and the accommodation coefficient a_c. The range $-1/2 \le \nu \le 1/2$ corresponds to the range $2/(3-\theta) \le {\rm Pr} \le 2/(1+\theta)$ of the Prandtl number, and $\nu = 0$ (i.e., ${\rm Pr} = 1$) corresponds to the BGK model. It is noted that the values of c_v^{II} and c_T^{II} for $a_c = 1$ and $\nu = -0.5$ and 0 had been obtained in [44, 45] and that the values in Tables II and III agree with the corresponding values (i.e., $c_5^{(0)}$ and $c_1^{(0)}$, respectively) in [44, 45] up to four to five decimal places. With the numerical data in Tables II and III, one can easily obtain the values of c_v^{II} and c_T^{II} at arbitrary ν and a_c by appropriate interpolation.

TABLE I. Values of $c^I_{v\mathrm{BGK}}$ and $c^I_{T\mathrm{BGK}}$ [43].

a_c	$c^I_{v\mathrm{BGK}}$	$c^I_{T\mathrm{BGK}}$
0.1	17.10313	0.2641783
0.2	8.224902	0.2781510
0.3	5.255112	0.2919238
0.4	3.762619	0.3055019
0.5	2.861190	0.3188906
0.6	2.255410	0.3320949
0.7	1.818667	0.3451195
0.8	1.487654	0.3579692
0.9	1.227198	0.3706483
1.0	1.016191	0.3831612

TABLE II. Values of c_v^{II}.

a_c	$\nu=-0.5$	$\nu=-0.4$	$\nu=-0.3$	$\nu=-0.2$	$\nu=-0.1$
0.1	0.229056	0.245551	0.264600	0.286847	0.313171
0.2	0.235862	0.252963	0.272726	0.295826	0.323182
0.3	0.242643	0.260334	0.280791	0.304717	0.333070
0.4	0.249400	0.267666	0.288797	0.313522	0.342836
0.5	0.256135	0.274961	0.296747	0.322244	0.352485
0.6	0.262849	0.282221	0.304641	0.330886	0.362018
0.7	0.269545	0.289446	0.312481	0.339448	0.371439
0.8	0.276224	0.296640	0.320270	0.347934	0.380750
0.9	0.282887	0.303802	0.328010	0.356345	0.389955
1.0	0.289536	0.310936	0.335700	0.364684	0.399056
a_c	$\nu=0$	$\nu=0.1$	$\nu=0.3$	$\nu=0.5$	
0.1	0.344801	0.383520	0.494486	0.695347	
0.2	0.356088	0.396415	0.512244	0.722726	
0.3	0.367200	0.409062	0.529488	0.748894	
0.4	0.378141	0.421470	0.546243	0.773934	
0.5	0.388917	0.433645	0.562528	0.797920	
0.6	0.399532	0.445596	0.578365	0.820919	
0.7	0.409989	0.457328	0.593771	0.842994	
0.8	0.420294	0.468848	0.608765	0.864202	
0.9	0.430449	0.480163	0.623364	0.884595	
1.0	0.440460	0.491277	0.637583	0.904222	

If we plot c_v^{II} versus a_c for fixed values of ν, we find that c_v^{II} is almost linear with respect to a_c (the plot is omitted). Similarly, if we plot c_v^{II} versus $1/(1-\nu)$, we can observe that c_v^{II} is almost proportional to $1/(1-\nu)$ (the plot is omitted). These facts suggest an approximate numerical fit of the following simple form:

$$c_v^{II} \approx \frac{k_1 a_c + k_2}{1-\nu}, \qquad k_1 = 0.1046, \quad k_2 = 0.3358. \tag{11}$$

This formula can reproduce the values in Table II within the error of 1 percent for $-0.3 \le \nu \le 0.1$; 1.6 percent for $\nu = -0.5$, -0.4, and 0.3; and 3 percent for $\nu = 0.5$. On the other hand, it is seen from Table III that c_T^{II} is almost independent of ν. Therefore, neglecting the dependence of c_T^{II} on ν, one can propose the following simple numerical fit for c_T^{II}:

$$c_T^{II} \approx \frac{k_3 - a_c}{a_c}, \qquad k_3 = 2.302. \tag{12}$$

TABLE III. Values of c_T^{II}.

a_c	$\nu=-0.5$	$\nu=-0.4$	$\nu=-0.3$	$\nu=-0.2$	$\nu=-0.1$
0.1	21.4458	21.4465	21.4472	21.4480	21.4490
0.2	10.3436	10.3442	10.3449	10.3456	10.3465
0.3	6.62711	6.62763	6.62822	6.62888	6.62964
0.4	4.75734	4.75780	4.75832	4.75891	4.75957
0.5	3.62652	3.62692	3.62738	3.62789	3.62846
0.6	2.86536	2.86571	2.86611	2.86655	2.86705
0.7	2.31560	2.31591	2.31625	2.31663	2.31705
0.8	1.89811	1.89837	1.89866	1.89898	1.89933
0.9	1.56890	1.56912	1.56936	1.56963	1.56993
1.0	1.30160	1.30178	1.30198	1.30220	1.30244
a_c	$\nu=0$	$\nu=0.1$	$\nu=0.3$	$\nu=0.5$	
0.1	21.4501	21.4515	21.4550	21.4607	
0.2	10.3475	10.3486	10.3517	10.3566	
0.3	6.63052	6.63154	6.63423	6.63841	
0.4	4.76033	4.76122	4.76354	4.76711	
0.5	3.62913	3.62989	3.63188	3.63489	
0.6	2.86762	2.86827	2.86996	2.87248	
0.7	2.31753	2.31809	2.31951	2.32161	
0.8	1.89974	1.90021	1.90139	1.90311	
0.9	1.57026	1.57065	1.57162	1.57301	
1.0	1.30272	1.30303	1.30381	1.30492	

This formula can reproduce the values in Table III within the error of 1 percent for $a_c = 0.3$, 0.4, 0.5, 0.9, and 1; 1.5 percent for $a_c = 0.6$, 0.7, and 0.8; 1.7 percent for $a_c = 0.2$; and 2.7 percent for $a_c = 0.1$.

The boundary conditions (9) should be supplemented by the initial condition. If we admit the inaccuracy in the initial stage $0 < \hat{t} < O(\text{mean free time})$ in practical applications, we may ignore assumption (v) in Sec. II A and assume the following initial conditions:

$$\hat{\rho} = \hat{\rho}^{\text{in}}(\boldsymbol{x}), \qquad \hat{\boldsymbol{v}} = \hat{\boldsymbol{v}}^{\text{in}}(\boldsymbol{x}), \qquad \hat{T}_{\text{tr}} = \hat{T}_{\text{tr}}^{\text{in}}(\boldsymbol{x}), \qquad \hat{T}_{\text{int}} = \hat{T}_{\text{int}}^{\text{in}}(\boldsymbol{x}), \qquad \text{at } \hat{t} = 0, \tag{13}$$

where $\hat{\rho}^{\text{in}}(\boldsymbol{x})$, $\hat{\boldsymbol{v}}^{\text{in}}(\boldsymbol{x})$, $\hat{T}_{\text{tr}}^{\text{in}}(\boldsymbol{x})$, and $\hat{T}_{\text{int}}^{\text{in}}(\boldsymbol{x})$ are appropriately chosen functions (see Sec. VII D for the discussion about this point).

E. Remarks on applications

We now have the complete system consisting of the two-temperature Navier–Stokes equations (6), the slip boundary conditions (9), and the initial conditions (13). However, in order to apply the system to practical problems, we have to identify the parameters ϵ, ν, and θ (or α) [cf. Eq. (5)] and the function $\hat{A}_c(\hat{T})$ from the properties of the gas under consideration. The accommodation coefficient a_c, which depends also on the property of the boundary, is excluded in this discussion. A conventional way to identify the parameters and $\hat{A}_c(\hat{T})$ is to use the data of transport coefficients of the gas, since the data of shear and bulk viscosities and the thermal conductivity (or the Prandtl number) are often available. Here, we should note that these transport coefficients make sense under the ordinary Navier–Stokes constitutive laws (A6). Therefore, we use the expressions (A7a)–(A9) of the transport coefficients in terms of ν, θ, and $A_c(T)$. To be more specific, we first determine ν and θ from the data of μ_b/μ and Pr, which are often available, using Eqs. (A8) and (A9). The parameter θ thus obtained should be small, otherwise the two-temperature Navier–Stokes equations are not valid. Then, knowing the temperature dependence of the viscosity μ from the data, we determine $A_c(T)$

from Eq. (A7a). Once $A_c(T)$ is determined, its dimensionless counterpart $\hat{A}_c(\hat{T})$ is determined by Eq. (3), and then ϵ by Eqs. (4) and (A5).

F. Examples of real gases

Now we provide some examples of real gases. We consider the gases with large bulk viscosities listed in Table IV. The data for the viscosity μ, the ratio of the specific heats γ, and the Prandtl number Pr, which are at $T = 300$K and $p = 101$kPa, are taken from [46], whereas those for the ratio μ_b/μ, which are at $T = 300$K, are taken from [47]. Note that for the ES model, μ does not depend on p. From these data, one obtains the values of θ and ν in Table IV by using Eqs. (A8) and (A9).

TABLE IV. Thermophysical properties of some gases with large bulk viscosities.

	μ [10^{-6}Pa · s]	s	γ	Pr	μ_b/μ	$\theta \times 10^3$	ν
H_2	8.955[a]	0.67	1.406[a]	0.7072[a]	28[b]	13.2	−0.420
CO_2	14.91[a]	0.83	1.293[a]	0.767[a]	3849[b]	0.127	−0.304
SF_6	15.24[a]	0.90	1.097[a]	0.806[a]	320[b]	2.21	−0.241
CH_4	11.43[a]	0.83	1.305[a]	0.7630[a]	240[b]	1.98	−0.311
C_2H_4	10.30[a]	0.97	1.245[a]	0.7718[a]	130[b]	4.20	−0.297
C_3H_8	8.219[a]	0.97	1.136[a]	0.6853[a]	240[b]	3.23	−0.461

[a] Value at 300K, 101kPa in [46].
[b] Value at 300K in [47].

With ν in Table IV, we can obtain the values of the slip coefficients for each gas. The values of c_v^I, c_T^I, and $\tilde{c}_T^{II}$ are obtained immediately from Eq. (10) and Table I. The values of c_v^{II} and c_T^{II} are obtained from Tables II and III or Eqs. (11) and (12). The results for $a_c = 0.2$, 0.5, and 1 are shown in Table V, where "interpolation" indicates the values obtained by a simple linear interpolation, with respect to $1/(1-\nu)$, using the two neighboring values in Tables II and III, and "numerical" indicates the numerical result based on the method outlined in Appendix D. It is seen that the interpolation based on Tables II and III is sufficient to obtain accurate values.

In order to determine $A_c(T)$, we assume the following power law with respect to T for the viscosity μ:

$$\mu(T) = \mu(300\,\mathrm{K}) \times (T/300\,\mathrm{K})^s, \tag{14}$$

and determine the exponent s using the method of least squares on the basis of the data provided in [46]. The obtained value of s for each gas is also shown in Table IV. Equation (14) with these values of s reproduces the data of μ given in [48] quite well in the temperature range 250–1300 K for CO_2 and 250–600 K for SF_6, CH_4, and C_2H_4. Equations (A7a) and (A8) then give

$$A_c(T) = A_c(300\,\mathrm{K}) \times (T/300\,\mathrm{K})^{1-s}, \qquad A_c(300\,\mathrm{K}) = (R\mathrm{Pr}) \times [300\,\mathrm{K}/\mu(300\,\mathrm{K})], \tag{15}$$

and thus $\hat{A}_c(\hat{T}) = \hat{T}^{1-s}$. With this $A_c(T)$, the reference mean free path l_0 is obtained by Eq. (A5). Then, the parameters ϵ and α are determined by Eqs. (4) and (5), respectively.

III. ES MODEL AND ITS INITIAL AND BOUNDARY CONDITIONS

In this and the following sections (Secs. III–VII), we will describe the steps that have led to our results summarized in Sec. II. First, the basic kinetic problem, which is described in Sec. II A, is formulated in this section.

TABLE V. Values of c_v^{II} and c_T^{II} for some gases.

		c_v^{II}			c_T^{II}		
	a_c	Eq. (11)	interpolation	numerical	Eq. (12)	interpolation	numerical
H_2	0.2	0.251211	0.249350	0.249348	10.5100	10.3441	10.3441
	0.5	0.273310	0.270984	0.270979	3.60400	3.62684	3.62684
	1.0	0.310141	0.306415	0.306410	1.30200	1.30174	1.30174
CO_2	0.2	0.273558	0.271877	0.271877	10.5100	10.3449	10.3448
	0.5	0.297623	0.295811	0.295810	3.60400	3.62736	3.62736
	1.0	0.337730	0.334637	0.334635	1.30200	1.30197	1.30197
SF_6	0.2	0.287446	0.285905	0.285899	10.5100	10.3453	10.3453
	0.5	0.312732	0.311293	0.311283	3.60400	3.62767	3.62767
	1.0	0.354875	0.352236	0.352224	1.30200	1.30211	1.30210
CH_4	0.2	0.272098	0.270404	0.270403	10.5100	10.3448	10.3448
	0.5	0.296034	0.294188	0.294184	3.60400	3.62733	3.62732
	1.0	0.335927	0.332791	0.332787	1.30200	1.30196	1.30195
C_2H_4	0.2	0.275035	0.273367	0.273367	10.5100	10.3449	10.3449
	0.5	0.299229	0.297455	0.297453	3.60400	3.62739	3.62739
	1.0	0.339553	0.336504	0.336503	1.30200	1.30199	1.30198
C_3H_8	0.2	0.244162	0.242253	0.242250	10.5100	10.3438	10.3438
	0.5	0.265640	0.263171	0.263164	3.60400	3.62667	3.62667
	1.0	0.301437	0.297534	0.297527	1.30200	1.30167	1.30167

A. ES model

We first describe the ES model for a polyatomic gas that was proposed in [16] and re-derived in a systematic way in [37]. Its basic properties are shown in Appendix A.

We have introduced the number of the internal degrees of freedom δ, the time variable t, and the space position vector $\boldsymbol{X}$ in Sec. II B. In addition, let $\boldsymbol{\xi}$ (or ξ_i) be the molecular velocity and $\mathcal{E}$ be the energy per unit mass associated with the internal modes (i.e., the combined energy for the δ modes), which is continuous ranging from 0 to ∞. We denote the number of the gas molecules, at time t, contained in an infinitesimal volume $d\boldsymbol{X}d\boldsymbol{\xi}d\mathcal{E}$ around a point $(\boldsymbol{X}, \boldsymbol{\xi}, \mathcal{E})$ in the seven-dimensional (extended) phase space consisting of $\boldsymbol{X}$, $\boldsymbol{\xi}$, and $\mathcal{E}$ by

$$\frac{1}{m} f(t, \boldsymbol{X}, \boldsymbol{\xi}, \mathcal{E}) d\boldsymbol{X} d\boldsymbol{\xi} d\mathcal{E}. \tag{16}$$

Therefore, $f(t, \boldsymbol{X}, \boldsymbol{\xi}, \mathcal{E})$ is the mass density in the seven-dimensional phase space. We call $f(t, \boldsymbol{X}, \boldsymbol{\xi}, \mathcal{E})$ the velocity-energy distribution function of the gas molecules. It is governed by the ES model of the Boltzmann equation for a polyatomic gas [16, 37], which can be written in the following form:

$$\frac{\partial f}{\partial t} + \xi_i \frac{\partial f}{\partial X_i} = Q(f), \tag{17}$$

where

$$Q(f) = A_c(T)\rho(\mathcal{G} - f), \tag{18a}$$

$$\mathcal{G} = \frac{\rho \mathcal{E}^{\delta/2-1}}{(2\pi)^{3/2}(\det \mathsf{T})^{1/2}(RT_{\rm rel})^{\delta/2}\Gamma(\delta/2)}$$
$$\times \exp\left(-\frac{1}{2}(\mathsf{T}^{-1})_{ij}(\xi_i - v_i)(\xi_j - v_j) - \frac{\mathcal{E}}{RT_{\rm rel}}\right), \tag{18b}$$

$$(\mathsf{T})_{ij} = (1-\theta)[(1-\nu)RT_{\rm tr}\delta_{ij} + \nu p_{ij}/\rho] + \theta RT\delta_{ij}, \tag{18c}$$

$$\rho = \iint_0^\infty f d\mathcal{E} d\boldsymbol{\xi}, \tag{18d}$$

$$v_i = \frac{1}{\rho}\iint_0^\infty \xi_i f d\mathcal{E} d\boldsymbol{\xi}, \tag{18e}$$

$$p_{ij} = \iint_0^\infty (\xi_i - v_i)(\xi_j - v_j) f d\mathcal{E} d\boldsymbol{\xi}, \tag{18f}$$

$$T_{\rm tr} = \frac{1}{3R\rho}\iint_0^\infty |\boldsymbol{\xi} - \boldsymbol{v}|^2 f d\mathcal{E} d\boldsymbol{\xi}, \tag{18g}$$

$$T_{\rm int} = \frac{2}{\delta R\rho}\iint_0^\infty \mathcal{E} f d\mathcal{E} d\boldsymbol{\xi}, \tag{18h}$$

$$T = \frac{3T_{\rm tr} + \delta T_{\rm int}}{3+\delta}, \tag{18i}$$

$$T_{\rm rel} = \theta T + (1-\theta)T_{\rm int}. \tag{18j}$$

Here, ρ, v_i, $T_{\rm tr}$, $T_{\rm int}$, and T are the macroscopic quantities already appeared in Sec. II B, p_{ij} is the stress tensor, $d\boldsymbol{\xi} = d\xi_1 d\xi_2 d\xi_3$, and the domain of integration with respect to $\boldsymbol{\xi}$ is its whole space $\mathbb{R}^3$. The symbol δ_{ij} indicates the Kronecker delta, $\nu \in [-1/2, 1)$ and $\theta \in [0, 1]$ are the adjustable parameters mentioned in Sec. II B, and $A_c(T)$, also appeared in Sec. II B, is a function of T such that $A_c(T)\rho$ is the collision frequency of the gas molecules. In addition, $\Gamma(z)$ is the gamma function defined by

$$\Gamma(z) = \int_0^\infty s^{z-1}e^{-s} ds, \tag{19}$$

T is the 3×3 matrix with its (i, j) component defined by Eq. (18c), and $\det\mathsf{T}$ and T^{-1} are, respectively, its determinant and inverse.

The other important macroscopic quantities, the pressure p and the heat-flow vector q_i, are defined by

$$p = R\rho T, \tag{20}$$

and

$$q_i = q_{({\rm tr})i} + q_{({\rm int})i}, \tag{21a}$$
$$q_{({\rm tr})i} = \frac{1}{2}\iint_0^\infty (\xi_i - v_i)|\boldsymbol{\xi} - \boldsymbol{v}|^2 f d\mathcal{E} d\boldsymbol{\xi}, \tag{21b}$$
$$q_{({\rm int})i} = \iint_0^\infty (\xi_i - v_i)\mathcal{E} f d\mathcal{E} d\boldsymbol{\xi}, \tag{21c}$$

where Eq. (20) is the equation of state.

It should be noted that in [16], the variable I, which is related to our $\mathcal{E}$ as $\mathcal{E} = I^{2/\delta}$, is used as an independent variable instead of $\mathcal{E}$. See [49] or Appendix A in [50] for the relation between the notation in [16] and that of the present paper (see also [24]). In addition, the case with $\theta = 0$, which is excluded in [16], is included here, and it plays an important role in the present analysis. In the ES model (17), the energy associated with the internal modes is expressed by a single continuous variable $\mathcal{E}$. Some models also use a continuous energy variable (e.g.,

[13, 17, 24]), whereas the others use a discrete energy variable (e.g., [11, 12, 14, 15]). However, the corresponding macroscopic equations are analogous in both cases (see, e.g., [20]). Some mathematical studies of the ES model for a polyatomic gas are found in [51, 52].

Finally, we mention how the ES model for a monatomic gas is recovered from Eq. (17) (see Sec. 7 in [52]). Let us introduce the marginal distribution function $F(t, \boldsymbol{X}, \boldsymbol{\xi})$:

$$F(t, \boldsymbol{X}, \boldsymbol{\xi}) = \int_0^\infty f(t, \boldsymbol{X}, \boldsymbol{\xi}, \mathcal{E}) d\mathcal{E}. \tag{22}$$

If we integrate both sides of Eq. (17) with respect to $\mathcal{E}$ from 0 to ∞, let $\theta = 0$, and interpret $T_{\rm tr}$ as the temperature T, then, we obtain the equation for F, which is exactly the same as the ES model for a monatomic gas [16]. The linearized version of this property will be used in the Knudsen-layer analysis later.

B. Initial and boundary conditions

The local equilibrium for Eq. (17) is shown in Eq. (A1). Correspondingly, the global equilibrium distribution f_0 with the uniform density ρ_0 (reference density) and the uniform temperature T_0 (reference temperature) is given by

$$f_0 = \frac{\rho_0 \mathcal{E}^{\delta/2-1}}{(2\pi R T_0)^{3/2} (RT_0)^{\delta/2} \Gamma(\delta/2)} \exp\left(-\frac{|\boldsymbol{\xi}|^2}{2RT_0} - \frac{\mathcal{E}}{RT_0} \right). \tag{23}$$

According to assumption (v) in Sec. II A, the initial condition for f is given at time $t = 0$ by

$$f(0, \boldsymbol{X}, \boldsymbol{\xi}, \mathcal{E}) = f_0. \tag{24}$$

We will relax this condition later for practical applications.

From the form of the local equilibrium distribution (A1), the Maxwell-type boundary condition is written as follows:

$$\begin{aligned} f(t, \boldsymbol{X}_{\rm w}, \boldsymbol{\xi}, \mathcal{E}) &= (1 - a_c) \mathcal{R} f(t, \boldsymbol{X}_{\rm w}, \boldsymbol{\xi}, \mathcal{E}) \\ &\quad + a_c \frac{\rho_{\rm w} \mathcal{E}^{\delta/2-1}}{(2\pi R T_{\rm w})^{3/2} (RT_{\rm w})^{\delta/2} \Gamma(\delta/2)} \exp\left(-\frac{|\boldsymbol{\xi} - \boldsymbol{v}_{\rm w}|^2}{2RT_{\rm w}} - \frac{\mathcal{E}}{RT_{\rm w}} \right), \\ &\qquad\qquad \text{for} \quad (\boldsymbol{\xi} - \boldsymbol{v}_{\rm w}) \cdot \boldsymbol{n} > 0, \end{aligned} \tag{25a}$$

$$\rho_{\rm w} = -\left(\frac{2\pi}{RT_{\rm w}} \right)^{1/2} \int_{(\boldsymbol{\xi} - \boldsymbol{v}_{\rm w}) \cdot \boldsymbol{n} < 0} \int_0^\infty (\boldsymbol{\xi} - \boldsymbol{v}_{\rm w}) \cdot \boldsymbol{n} f(t, \boldsymbol{X}_{\rm w}, \boldsymbol{\xi}, \mathcal{E}) d\mathcal{E} d\boldsymbol{\xi}, \tag{25b}$$

where $\mathcal{R}$ indicates the reflection operator defined by

$$\mathcal{R} g(\xi_i) = g(\xi_i - 2(\xi_j - v_{{\rm w}j}) n_j n_i), \tag{26}$$

with an arbitrary function $g(\boldsymbol{\xi})$ of $\boldsymbol{\xi}$. It should be recalled that the quantities $\boldsymbol{X}_{\rm w}$, $\boldsymbol{v}_{\rm w}$, $T_{\rm w}$, and $\boldsymbol{n}$, as well as the accommodation coefficient a_c ($0 < a_c \le 1$), have already appeared in Sec. II D.

Note that this boundary condition satisfies the condition that there is no instantaneous mass flow across the boundary, i.e.,

$$\iint_0^\infty (\boldsymbol{\xi} - \boldsymbol{v}_{\rm w}) \cdot \boldsymbol{n} f(t, \boldsymbol{X}_{\rm w}, \boldsymbol{\xi}, \mathcal{E}) d\mathcal{E} d\boldsymbol{\xi} = 0. \tag{27}$$

To be consistent with assumption (v), $\boldsymbol{v}_{\rm w} = 0$ and $T_{\rm w} = T_0$ should hold at $t = 0$, and $\boldsymbol{X}_{\rm w}$ (thus, $\boldsymbol{v}_{\rm w}$), $T_{\rm w}$, and $\boldsymbol{n}$ are assumed to change smoothly with t. In practical applications, however, this condition may be relaxed occasionally.

IV. DIMENSIONLESS SYSTEM

We should recall that the dimensionless quantities $[\hat{t},\, x_i,\, \hat{\rho},\, \hat{v}_i,\, \hat{T}_{\rm tr},\, \hat{T}_{\rm int},\, \hat{T},\, \hat{A}_c(\hat{T}),\, x_{{\rm w}i},\, \hat{v}_{{\rm w}i},\, \hat{T}_{\rm w}]$, as well as the reference quantities L, t_0, ρ_0, and T_0 with t_0 being chosen as Eq. (2), have been introduced in Secs. II C and II D [cf. Eqs. (3) and (8)]. In addition, we let $p_0 = R\rho_0 T_0$ be the reference pressure.

Now we introduce the additional dimensionless quantities $[\zeta_i,\, \hat{\mathcal{E}},\, \hat{f},\, \hat{\mathcal{G}},\, \hat{T}_{\rm rel},\, \hat{p}_{ij},\, \hat{p},\, \hat{q}_{({\rm tr})i},\, \hat{q}_{({\rm int})i},\, \hat{q}_i]$, which correspond to the original dimensional quantities $[\xi_i,\, \mathcal{E},\, f,\, \mathcal{G},\, T_{\rm rel},\, p_{ij},\, p,\, q_{({\rm tr})i},\, q_{({\rm int})i},\, q_i]$, by the following relations:

$$\begin{aligned}
&\zeta_i = \xi_i/(2RT_0)^{1/2}, \qquad \hat{\mathcal{E}} = \mathcal{E}/RT_0, \qquad (\hat{f},\, \hat{\mathcal{G}}) = (f,\, \mathcal{G})/2\rho_0(2RT_0)^{-5/2},\\
&\hat{T}_{\rm rel} = T_{\rm rel}/T_0, \qquad \hat{p}_{ij} = p_{ij}/p_0, \qquad \hat{p} = p/p_0,\\
&(\hat{q}_{({\rm tr})i},\, \hat{q}_{({\rm int})i},\, \hat{q}_i) = (q_{({\rm tr})i},\, q_{({\rm int})i},\, q_i)/p_0(2RT_0)^{1/2}.
\end{aligned} \tag{28}$$

We occasionally use the bold-faced letters $\boldsymbol{x}$, $\boldsymbol{\zeta}$, $\hat{\boldsymbol{v}}$, $\hat{\boldsymbol{q}}$, $\boldsymbol{x}_{\rm w}$, and $\hat{\boldsymbol{v}}_{\rm w}$ in place of x_i, ζ_i, $\hat{v}_i$, $\hat{q}_i$, $x_{{\rm w}i}$, and $\hat{v}_{{\rm w}i}$, respectively.

A. Dimensionless form of ES model

With Eqs. (3) and (28), the ES model (17) is transformed into the following dimensionless form:

$$\frac{\partial \hat{f}}{\partial \hat{t}} + \zeta_i \frac{\partial \hat{f}}{\partial x_i} = \frac{1}{\epsilon}\hat{Q}(\hat{f}), \tag{29}$$

where

$$\hat{Q}(\hat{f}) = \hat{A}_c(\hat{T})\hat{\rho}(\hat{\mathcal{G}} - \hat{f}), \tag{30a}$$

$$\hat{\mathcal{G}} = \frac{\hat{\rho}}{\pi^{3/2}(\det\hat{\mathsf{T}})^{1/2}\,\hat{T}_{\rm rel}^{\delta/2}\Gamma(\delta/2)}\,\hat{\mathcal{E}}^{\delta/2-1}\exp\left(-(\hat{\mathsf{T}}^{-1})_{ij}(\zeta_i - \hat{v}_i)(\zeta_j - \hat{v}_j) - \frac{\hat{\mathcal{E}}}{\hat{T}_{\rm rel}}\right), \tag{30b}$$

$$(\hat{\mathsf{T}})_{ij} = (1-\theta)[(1-\nu)\hat{T}_{\rm tr}\delta_{ij} + \nu\hat{p}_{ij}/\hat{\rho}] + \theta\hat{T}\delta_{ij}, \tag{30c}$$

$$\hat{\rho} = \iint_0^\infty \hat{f}\, d\hat{\mathcal{E}} d\boldsymbol{\zeta}, \tag{30d}$$

$$\hat{v}_i = \frac{1}{\hat{\rho}}\iint_0^\infty \zeta_i \hat{f}\, d\hat{\mathcal{E}} d\boldsymbol{\zeta}, \tag{30e}$$

$$\hat{p}_{ij} = 2\iint_0^\infty (\zeta_i - \hat{v}_i)(\zeta_j - \hat{v}_j)\hat{f}\, d\hat{\mathcal{E}} d\boldsymbol{\zeta}, \tag{30f}$$

$$\hat{T}_{\rm tr} = \frac{2}{3\hat{\rho}}\iint_0^\infty (\zeta_k - \hat{v}_k)^2 \hat{f}\, d\hat{\mathcal{E}} d\boldsymbol{\zeta}, \tag{30g}$$

$$\hat{T}_{\rm int} = \frac{2}{\delta\hat{\rho}}\iint_0^\infty \hat{\mathcal{E}}\hat{f}\, d\hat{\mathcal{E}} d\boldsymbol{\zeta}, \tag{30h}$$

$$\hat{T} = \frac{3\hat{T}_{\rm tr} + \delta\hat{T}_{\rm int}}{3+\delta}, \tag{30i}$$

$$\hat{T}_{\rm rel} = \theta\hat{T} + (1-\theta)\hat{T}_{\rm int}. \tag{30j}$$

Here, ϵ is a small parameter of the order of the Knudsen number defined by Eq. (4), $d\boldsymbol{\zeta} = d\zeta_1 d\zeta_2 d\zeta_3$, and the domain of integration with respect to $\boldsymbol{\zeta}$ is the whole space ($\mathbb{R}^3$) of $\boldsymbol{\zeta}$.

The dimensionless pressure $\hat{p}$ and heat-flow vector $\hat{q}_i$, corresponding to Eqs. (20) and (21), are given by

$$\hat{p} = \hat{\rho}\hat{T}, \tag{31}$$

and

$$\hat{q}_i = \hat{q}_{(\mathrm{tr})i} + \hat{q}_{(\mathrm{int})i}, \tag{32a}$$

$$\hat{q}_{(\mathrm{tr})i} = \iint_0^\infty (\zeta_i - \hat{v}_i)|\boldsymbol{\zeta} - \hat{\boldsymbol{v}}|^2 \hat{f} d\hat{\mathcal{E}} d\boldsymbol{\zeta}, \tag{32b}$$

$$\hat{q}_{(\mathrm{int})i} = \iint_0^\infty (\zeta_i - \hat{v}_i)\hat{\mathcal{E}} \hat{f} d\hat{\mathcal{E}} d\boldsymbol{\zeta}. \tag{32c}$$

The dimensionless form of the first two basic properties of the ES model shown in Appendix A 1 is described as follows:

Equilibrium: $\hat{Q}(\hat{f}) = 0$ is equivalent to the fact that $\hat{f}$ is the dimensionless local equilibrium given by

$$\hat{f}_{\mathrm{eq}} = \frac{\hat{\rho}\hat{\mathcal{E}}^{\delta/2-1}}{(\pi\hat{T})^{3/2}\hat{T}^{\delta/2}\Gamma(\delta/2)} \exp\left(-\frac{|\boldsymbol{\zeta} - \hat{\boldsymbol{v}}|^2}{\hat{T}} - \frac{\hat{\mathcal{E}}}{\hat{T}}\right), \tag{33}$$

where $\hat{\rho}$, $\hat{\boldsymbol{v}}$, and $\hat{T}$ are arbitrary functions of $\hat{t}$ and $\boldsymbol{x}$.

Conservations: For an arbitrary function $\hat{g}(\hat{t}, \boldsymbol{x}, \boldsymbol{\zeta}, \hat{\mathcal{E}})$, the relation

$$\iint_0^\infty \hat{\varphi}_r \hat{Q}(\hat{g}) d\hat{\mathcal{E}} d\boldsymbol{\zeta} = 0, \tag{34}$$

holds, where $\hat{\varphi}_r$ $(r = 0, ..., 4)$ are the dimensionless collision invariants, i.e.,

$$\hat{\varphi}_0 = 1, \qquad \hat{\varphi}_i = \zeta_i \quad (i = 1, 2, 3), \qquad \hat{\varphi}_4 = |\boldsymbol{\zeta}|^2 + \hat{\mathcal{E}}. \tag{35}$$

Here, we should note that when the parameter θ vanishes, these equilibrium and conservation properties take slightly different forms. Their dimensional versions are shown in Appendix A 3 in [53], and the dimensionless versions are stated as follows (see Sec. III B in [36]):

Equilibrium: $\hat{Q}(\hat{f})|_{\theta=0} = 0$ is equivalent to the fact that $\hat{f}$ is the (dimensionless) local equilibrium of the form

$$\hat{f}_{\mathrm{eq}} = \frac{\hat{\rho}\,\hat{\mathcal{E}}^{\delta/2-1}}{(\pi\hat{T}_{\mathrm{tr}})^{3/2}\hat{T}_{\mathrm{int}}^{\delta/2}\Gamma(\delta/2)} \exp\left(-\frac{|\boldsymbol{\zeta} - \hat{\boldsymbol{v}}|^2}{\hat{T}_{\mathrm{tr}}} - \frac{\hat{\mathcal{E}}}{\hat{T}_{\mathrm{int}}}\right), \tag{36}$$

where $\hat{\rho}$, $\hat{\boldsymbol{v}}$, $\hat{T}_{\mathrm{tr}}$, and $\hat{T}_{\mathrm{int}}$ are arbitrary dimensionless functions of $\hat{t}$ and $\boldsymbol{x}$.

Conservations: For an arbitrary function $\hat{g}(\hat{t}, \boldsymbol{x}, \boldsymbol{\zeta}, \hat{\mathcal{E}})$, the relation

$$\iint_0^\infty \hat{\phi}_r \hat{Q}(\hat{g})|_{\theta=0} d\hat{\mathcal{E}} d\boldsymbol{\zeta} = 0, \tag{37}$$

holds, where $\hat{\phi}_r$ $(r = 0, ..., 5)$ are the (dimensionless) collision invariants, i.e.,

$$\hat{\phi}_0 = 1, \qquad \hat{\phi}_i = \zeta_i \quad (i = 1, 2, 3), \qquad \hat{\phi}_4 = |\boldsymbol{\zeta}|^2, \qquad \hat{\phi}_5 = \hat{\mathcal{E}}. \tag{38}$$

These properties for $\theta = 0$ play important roles in the present study.

B. Dimensionless form of initial and boundary conditions

The dimensionless form of the initial condition (24) can be written in the following form:

$$\hat{f}(0, \boldsymbol{x}, \boldsymbol{\zeta}, \hat{\mathcal{E}}) = \hat{f}_0, \tag{39}$$

where

$$\hat{f}_0 = [\Gamma(\delta/2)]^{-1} E(\zeta) \hat{\mathcal{E}}^{\delta/2-1} e^{-\hat{\mathcal{E}}}, \qquad \zeta = |\boldsymbol{\zeta}| = (\zeta_i^2)^{1/2}, \qquad E(\zeta) = \pi^{-3/2} \exp(-\zeta^2). \tag{40}$$

With the dimensionless quantities defined by Eq. (8), the boundary condition (25) is non-dimensionalized as follows:

$$\begin{aligned} \hat{f}(\hat{t},\, \boldsymbol{x}_{\mathrm{w}},\, \boldsymbol{\zeta},\, \hat{\mathcal{E}}) &= (1-a_c)\hat{\mathcal{R}}\hat{f}(\hat{t},\, \boldsymbol{x}_{\mathrm{w}},\, \boldsymbol{\zeta},\, \hat{\mathcal{E}}) \\ &\quad + a_c \frac{\hat{\rho}_{\mathrm{w}} \hat{\mathcal{E}}^{\delta/2-1}}{(\pi \hat{T}_{\mathrm{w}})^{3/2} \hat{T}_{\mathrm{w}}^{\delta/2} \Gamma(\delta/2)} \exp\left(-\frac{|\boldsymbol{\zeta} - \hat{\boldsymbol{v}}_{\mathrm{w}}|^2}{\hat{T}_{\mathrm{w}}} - \frac{\hat{\mathcal{E}}}{\hat{T}_{\mathrm{w}}} \right), \\ &\qquad\qquad \text{for} \quad (\boldsymbol{\zeta} - \hat{\boldsymbol{v}}_{\mathrm{w}}) \cdot \boldsymbol{n} > 0, \end{aligned} \tag{41a}$$

$$\hat{\rho}_{\mathrm{w}} = -2 \left(\frac{\pi}{\hat{T}_{\mathrm{w}}} \right)^{1/2} \int_{(\boldsymbol{\zeta} - \hat{\boldsymbol{v}}_{\mathrm{w}}) \cdot \boldsymbol{n} < 0} \int_0^\infty (\boldsymbol{\zeta} - \hat{\boldsymbol{v}}_{\mathrm{w}}) \cdot \boldsymbol{n} \hat{f}(\hat{t},\, \boldsymbol{x}_{\mathrm{w}},\, \boldsymbol{\zeta},\, \hat{\mathcal{E}}) d\hat{\mathcal{E}} d\boldsymbol{\zeta}, \tag{41b}$$

where $\hat{\mathcal{R}}$ is the dimensionless reflection operator, corresponding to Eq. (26), acting on any function $\hat{g}$ of ζ_i, i.e.,

$$\hat{\mathcal{R}}\hat{g}(\zeta_i) = \hat{g}(\zeta_i - 2(\zeta_j - \hat{v}_{\mathrm{w}j}) n_j n_i). \tag{42}$$

Corresponding to Eq. (27), the following condition holds on the boundary:

$$\iint_0^\infty (\boldsymbol{\zeta} - \hat{\boldsymbol{v}}_{\mathrm{w}}) \cdot \boldsymbol{n} \hat{f}(\hat{t},\, \boldsymbol{x}_{\mathrm{w}},\, \boldsymbol{\zeta},\, \hat{\mathcal{E}}) d\hat{\mathcal{E}} d\boldsymbol{\zeta} = 0. \tag{43}$$

We note that $\hat{\boldsymbol{v}}_{\mathrm{w}} = 0$ and $\hat{T}_{\mathrm{w}} = 1$ at $\hat{t} = 0$, and $\boldsymbol{x}_{\mathrm{w}}$ (thus $\hat{\boldsymbol{v}}_{\mathrm{w}}$), $\hat{T}_{\mathrm{w}}$, and $\boldsymbol{n}$ are assumed to change smoothly in $\hat{t}$ though this restriction may be relaxed occasionally in practical applications.

V. TWO-TEMPERATURE NAVIER–STOKES EQUATIONS

A. Preliminary remarks

In [36], the two-temperature Navier–Stokes equations have been derived from the ES model in the case where the parameter θ, as well as the Knudsen number Kn, is small, that is, under the setting (5). The background of this assumption is explained in Sec. II E in [36]. However, a brief description of its essence is given below.

As can be seen from Eq. (A9) in Appendix A 2, small θ indicates large values of the ratio μ_b/μ, where μ is the (shear) viscosity and μ_b the bulk viscosity [cf. Eqs. (A6) and (A7) in Appendix A 2]. On the other hand, as is seen from Eq. (A10) in Appendix A 2, small θ corresponds to slow relaxation of the internal modes. These two statements are consistent because it is a common understanding that large bulk viscosity is related to the slow relaxation of the internal modes [47, 54]. Therefore, the condition (5) targets the behavior in the near continuum regime of a gas with slow relaxation of the internal modes or with large bulk viscosity. The two-temperature Navier–Stokes equations are the consequence of the Chapman–Enskog expansion under the condition (5).

As mentioned in Sec. II F, some gases have large bulk viscosities, more precisely, large values of the ratio μ_b/μ of the bulk viscosity to the viscosity (see Table IV in Sec. II F). The reader is referred to [47] and p. 30 in [55] concerning gases with large bulk viscosities. It is noted that the impact of large bulk viscosity for H_2 gas flows is investigated in [56]. Here, it should also be mentioned that some authors are doubtful about the large values of μ_b/μ for CO_2 gas [33, 57]. However, as in [36], we here follow the view that μ_b/μ is large for CO_2 gas.

The standard Chapman–Enskog expansion for small ϵ applied to the ES model leads to the ordinary Navier–Stokes equations with the single temperature and with the bulk viscosity [16] (cf. Appendix A 2). In this case, it is implicitly assumed that the parameter θ is of the order of unity. When θ is small, therefore, one expects that the ordinary Navier–Stokes equations are not valid and a new system of equations is required. For this reason, we considered the parameter setting (5) and, as the result, derived the two-temperature Navier–Stokes equations, which have been summarized in Sec. II C.

B. Chapman–Enskog solution and two-temperature Navier–Stokes equations

In this subsection, we summarize the result of the Chapman–Enskog expansion carried out in [36] to derive the two-temperature Navier–Stokes equations (6). Let us put aside the initial and boundary conditions (39) and (41).

If the Chapman–Enskog procedure is applied to the ES model (29) under the assumption (5), the solution $\hat{f}$ is expressed in the expansion of the following form:

$$\hat{f} = \hat{f}^{(0)} + \hat{f}^{(1)}\epsilon + \hat{f}^{(2)}\epsilon^2 + \cdots . \tag{44}$$

Here, the leading-order term $\hat{f}^{(0)}$ is given by the local equilibrium distribution (36) for $\theta = 0$, i.e.,

$$\hat{f}^{(0)} = \frac{\hat{\rho}\,\hat{\mathcal{E}}^{\delta/2-1}}{(\pi\hat{T}_{\rm tr})^{3/2}\hat{T}_{\rm int}^{\delta/2}\Gamma(\delta/2)} \exp\left(-\frac{|\boldsymbol{\zeta}-\hat{\boldsymbol{v}}|^2}{\hat{T}_{\rm tr}} - \frac{\hat{\mathcal{E}}}{\hat{T}_{\rm int}}\right), \tag{45}$$

and the first-order term $\hat{f}^{(1)}$ is given as

$$\begin{aligned}\hat{f}^{(1)} = &-\frac{1}{\hat{A}_c(\hat{T})\hat{\rho}}\hat{f}^{(0)}\left\{\frac{1}{1-\nu}\left[\frac{(\zeta_i-\hat{v}_i)(\zeta_j-\hat{v}_j)}{\hat{T}_{\rm tr}} - \frac{1}{3}\frac{(\zeta_k-\hat{v}_k)^2}{\hat{T}_{\rm tr}}\delta_{ij}\right]\left(\frac{\partial\hat{v}_i}{\partial x_j}+\frac{\partial\hat{v}_j}{\partial x_i}\right)\right.\\ &\left.+\frac{(\zeta_j-\hat{v}_j)}{\hat{T}_{\rm tr}}\left[\frac{(\zeta_k-\hat{v}_k)^2}{\hat{T}_{\rm tr}}-\frac{5}{2}\right]\frac{\partial\hat{T}_{\rm tr}}{\partial x_j} + \frac{(\zeta_j-\hat{v}_j)}{\hat{T}_{\rm int}}\left(\frac{\hat{\mathcal{E}}}{\hat{T}_{\rm int}}-\frac{\delta}{2}\right)\frac{\partial\hat{T}_{\rm int}}{\partial x_j}\right\}.\end{aligned} \tag{46}$$

The expansion is designed in such a way that the macroscopic quantities $\hat{\rho}$, $\hat{\boldsymbol{v}}$, $\hat{T}_{\rm tr}$, and $\hat{T}_{\rm int}$ in Eqs. (30d), (30e), (30g), and (30h) are not expanded and are generated by the leading-order term $\hat{f}^{(0)}$, that is,

$$\hat{\rho} = \iint_0^\infty \hat{f}^{(0)} d\hat{\mathcal{E}} d\boldsymbol{\zeta}, \tag{47a}$$

$$\hat{v}_i = \frac{1}{\hat{\rho}}\iint_0^\infty \zeta_i \hat{f}^{(0)} d\hat{\mathcal{E}} d\boldsymbol{\zeta}, \tag{47b}$$

$$\hat{T}_{\rm tr} = \frac{2}{3\hat{\rho}}\iint_0^\infty (\zeta_k-\hat{v}_k)^2 \hat{f}^{(0)} d\hat{\mathcal{E}} d\boldsymbol{\zeta}, \tag{47c}$$

$$\hat{T}_{\rm int} = \frac{2}{\delta\hat{\rho}}\iint_0^\infty \hat{\mathcal{E}}\hat{f}^{(0)} d\hat{\mathcal{E}} d\boldsymbol{\zeta}. \tag{47d}$$

This is equivalent to imposing the following condition for the higher-order terms $\hat{f}^{(1)}$, $\hat{f}^{(2)}$, ...:

$$\iint_0^\infty \hat{\phi}_r \hat{f}^{(m)} d\hat{\mathcal{E}} d\boldsymbol{\zeta} = 0, \qquad (r = 0, \dots, 5;\ m = 1, 2, \dots), \tag{48}$$

where $\hat{\phi}_r$ ($r = 0$, ..., 5) are defined by Eq. (38).

In accordance with Eq. (44), other macroscopic quantities $\hat{p}_{ij}$, $\hat{q}_i$, $\hat{q}_{({\rm tr})i}$, $\hat{q}_{({\rm int})i}$, and $\hat{T}_{\rm rel}$ are expanded as

$$\hat{h} = \hat{h}^{(0)} + \hat{h}^{(1)}\epsilon + \cdots, \qquad (\hat{h} = \hat{p}_{ij},\ \hat{q}_i,\ \hat{q}_{({\rm tr})i},\ \hat{q}_{({\rm int})i}, \text{ and } \hat{T}_{\rm rel}). \tag{49}$$

Here, the coefficients for $\hat{p}_{ij}$, $\hat{q}_i$, $\hat{q}_{({\rm tr})i}$, and $\hat{q}_{({\rm int})i}$ are obtained by substituting Eqs. (44) and (49) into Eqs. (30f) and (32), i.e.,

$$\hat{p}_{ij}^{(m)} = 2\iint_0^\infty (\zeta_i-\hat{v}_i)(\zeta_j-\hat{v}_j)\hat{f}^{(m)} d\hat{\mathcal{E}} d\boldsymbol{\zeta}, \qquad (m = 0, 1, \dots), \tag{50}$$

and

$$\hat{q}_i^{(m)} = \hat{q}_{({\rm tr})i}^{(m)} + \hat{q}_{({\rm int})i}^{(m)}, \qquad (m = 0, 1, \dots), \tag{51a}$$

$$\hat{q}_{({\rm tr})i}^{(m)} = \iint_0^\infty (\zeta_i-\hat{v}_i)|\boldsymbol{\zeta}-\hat{\boldsymbol{v}}|^2 \hat{f}^{(m)} d\hat{\mathcal{E}} d\boldsymbol{\zeta}, \tag{51b}$$

$$\hat{q}_{({\rm int})i}^{(m)} = \iint_0^\infty (\zeta_i-\hat{v}_i)\hat{\mathcal{E}}\hat{f}^{(m)} d\hat{\mathcal{E}} d\boldsymbol{\zeta}, \tag{51c}$$

and those for $\hat{T}_{\rm rel}$ are obtained from Eqs. (30j) and (5), i.e.,

$$\hat{T}_{\rm rel}^{(0)} = \hat{T}_{\rm int}, \qquad \hat{T}_{\rm rel}^{(1)} = \alpha(\hat{T} - \hat{T}_{\rm int}), \qquad \hat{T}_{\rm rel}^{(m+2)} = 0, \qquad (m = 0,\, 1,\, \dots), \tag{52}$$

where $\hat{T}$ is determined by Eq. (30i) and thus is not expanded in ϵ.

If we use $\hat{f} = \hat{f}^{(0)} + \hat{f}^{(1)}\epsilon + O(\epsilon^2)$ with Eqs. (45) and (46) in Eqs. (49), (50), and (51), we obtain the following expressions of $\hat{p}_{ij}$, $\hat{q}_{({\rm tr})i}$, $\hat{q}_{({\rm int})i}$, and $\hat{q}_i$:

$$\hat{p}_{ij} = \hat{\rho}\hat{T}_{\rm tr}\delta_{ij} - \Gamma_\mu(\hat{T},\, \hat{T}_{\rm tr}) \left(\frac{\partial \hat{v}_i}{\partial x_j} + \frac{\partial \hat{v}_j}{\partial x_i} - \frac{2}{3}\frac{\partial \hat{v}_k}{\partial x_k}\delta_{ij} \right) \epsilon + O(\epsilon^2), \tag{53a}$$

$$\hat{q}_{({\rm tr})i} = -\frac{5}{4}\Gamma_\lambda(\hat{T},\, \hat{T}_{\rm tr}) \frac{\partial \hat{T}_{\rm tr}}{\partial x_i}\epsilon + O(\epsilon^2), \tag{53b}$$

$$\hat{q}_{({\rm int})i} = -\frac{\delta}{4}\Gamma_\lambda(\hat{T},\, \hat{T}_{\rm tr}) \frac{\partial \hat{T}_{\rm int}}{\partial x_i}\epsilon + O(\epsilon^2), \tag{53c}$$

$$\hat{q}_i = \hat{q}_{({\rm tr})i} + \hat{q}_{({\rm int})i}, \tag{53d}$$

where $\Gamma_\mu(\hat{T},\, \hat{T}_{\rm tr})$ and $\Gamma_\lambda(\hat{T},\, \hat{T}_{\rm tr})$ are defined by Eq. (7).

The two-temperature Navier–Stokes equations, which correspond to the first-order solution $\hat{f} = \hat{f}^{(0)} + \hat{f}^{(1)}\epsilon$ and whose constitutive laws are given by Eq. (53) with $O(\epsilon^2)$ terms being neglected, are the equations for $\hat{\rho}$, $\hat{v}_i$, $\hat{T}_{\rm tr}$, and $\hat{T}_{\rm int}$ and take the form of Eq. (6). One can derive the equation for the conservation of the total energy from Eqs. (6c) and (6d), that is,

$$\begin{aligned}
&\frac{\partial}{\partial \hat{t}} \left[\hat{\rho} \left(\frac{3+\delta}{2}\hat{T} + \hat{v}_i^2 \right) \right] + \frac{\partial}{\partial x_j} \left[\hat{\rho}\hat{v}_j \left(\frac{3+\delta}{2}\hat{T} + \hat{T}_{\rm tr} + \hat{v}_i^2 \right) \right] \\
&\quad = \frac{1}{2}\epsilon \frac{\partial}{\partial x_j} \left[\Gamma_\lambda(\hat{T},\, \hat{T}_{\rm tr}) \left(\frac{3+\delta}{2}\frac{\partial \hat{T}}{\partial x_j} + \frac{\partial \hat{T}_{\rm tr}}{\partial x_j} \right) \right] \\
&\qquad + \epsilon \frac{\partial}{\partial x_j} \left[\Gamma_\mu(\hat{T},\, \hat{T}_{\rm tr})\hat{v}_i \left(\frac{\partial \hat{v}_i}{\partial x_j} + \frac{\partial \hat{v}_j}{\partial x_i} - \frac{2}{3}\frac{\partial \hat{v}_k}{\partial x_k}\delta_{ij} \right) \right].
\end{aligned} \tag{54}$$

Equation (54) may be used in place of Eq. (6c) or (6d). The two-temperature Navier–Stokes equations can be reduced to the ordinary Navier–Stokes equations with a single temperature by considering the case when α is large. The procedure is explained in Appendix B.

C. No-slip conditions and their inconsistency

Now, let us consider the initial and boundary conditions (39) and (41). If we assume

$$\hat{\rho} = 1, \qquad \hat{\boldsymbol{v}} = 0, \qquad \hat{T}_{\rm tr} = \hat{T}_{\rm int} = 1, \qquad \text{at } \hat{t} = 0, \tag{55}$$

then the Chapman–Enskog solution, Eq. (44) with Eqs. (45) and (46), satisfies Eq. (39) up to $O(\epsilon)$ because $\partial \hat{v}_i/\partial x_j = 0$, $\partial \hat{T}_{\rm tr}/\partial x_i = 0$, and $\partial \hat{T}_{\rm int}/\partial x_i = 0$ hold. Therefore, under assumption (v) in Sec. II A, Eq. (55) is the correct initial condition for Eq. (6).

Next, we consider the boundary condition (41). Since the leading-order term $\hat{f}^{(0)}$ of the Chapman–Enskog solution is a local equilibrium distribution with two temperatures [Eq. (45)], it can be made to satisfy Eq. (41) by assuming that

$$\hat{\boldsymbol{v}} = \hat{\boldsymbol{v}}_{\rm w}, \qquad \hat{T}_{\rm tr} = \hat{T}_{\rm int} = \hat{T}_{\rm w}, \qquad \text{at } \boldsymbol{x} = \boldsymbol{x}_{\rm w}. \tag{56}$$

In this way, we are able to satisfy the boundary condition (41) at the zeroth order of ϵ with the choice (56). Equation (56) is the so-called no-slip boundary conditions for the two-temperature Navier–Stokes equations (6).

However, to be consistent with the fact that the first-order Chapman–Enskog solution $\hat{f} = \hat{f}^{(0)} + \hat{f}^{(1)}\epsilon$, which corresponds to the two-temperature Navier–Stokes equations, satisfies the ES model (29) formally up to $O(\epsilon)$, we need to satisfy the boundary condition (41) also up

to $O(\epsilon)$. If we try to do so with $\hat{f} = \hat{f}^{(0)} + \hat{f}^{(1)}\epsilon$, we must impose the following conditions in addition to Eq. (56):

$$\frac{\partial \hat{v}_i}{\partial x_j} + \frac{\partial \hat{v}_j}{\partial x_i} = 0, \qquad \frac{\partial \hat{T}_{\rm tr}}{\partial x_j} = 0, \qquad \frac{\partial \hat{T}_{\rm int}}{\partial x_j} = 0, \qquad \text{at } \boldsymbol{x} = \boldsymbol{x}_{\rm w}. \tag{57}$$

However, these conditions are too many for Eq. (6), so that this scheme does not work at the order of ϵ. This difficulty can be resolved by introducing the Knudsen layer, as is well known (cf. [58, 59]). As we will see in the next section, the correction of $O(\epsilon)$ to Eq. (56) is obtained by the analysis of the Knudsen layer.

The above argument indicates that the no-slip boundary conditions (56) are not consistent with the two-temperature Navier–Stokes equations. In classical gasdynamics, the no-slip boundary conditions are usually used for the standard compressible Navier–Stokes equations (with the single temperature). As pointed out in [40], the no-slip conditions are inconsistent also in this case.

VI. KNUDSEN LAYER

A. Introduction of Knudsen layer

In Sec. V C, we have seen that the Chapman–Enskog solution, Eq. (44) with Eqs. (45) and (46), cannot be made to satisfy the kinetic boundary condition (41) at the first order of ϵ. In order to obtain the solution satisfying the boundary condition, one has to introduce the kinetic boundary layer, the so-called Knudsen layer, with thickness of the order of ϵ (of the order of the mean free path in the dimensional physical space) adjacent to the boundary [58, 59].

Let us denote the Chapman–Enskog solution, Eq. (44) with Eqs. (45) and (46), by $\hat{f}_{\rm CE}$, the correction term inside the Knudsen layer by $\hat{f}_{\rm K}$, and the total solution that satisfies the boundary condition by $\hat{f}_{\rm tot}$. Then, we write

$$\hat{f}_{\rm tot} = \hat{f}_{\rm CE} + \hat{f}_{\rm K}. \tag{58}$$

Correspondingly, we denote the macroscopic quantities by

$$\hat{h}_{\rm tot} = \hat{h}_{\rm CE} + \hat{h}_{\rm K}, \tag{59}$$

where $\hat{h}$ stands for any of the dimensionless macroscopic quantities, $\hat{\rho}$, $\hat{v}_i$, $\hat{p}_{ij}$, $\hat{T}_{\rm tr}$, etc., appeared in Eqs. (30d)–(30j), (31), and (32), and $\hat{h}_{\rm CE}$ and $\hat{h}_{\rm K}$ indicate these macroscopic quantities associated with the Chapman–Enskog solution and the Knudsen-layer correction, respectively. Note that the macroscopic quantities appeared in Sec. V belong to $\hat{h}_{\rm CE}$ although the subscript "CE" was not used there.

We assume the following properties for the correction term $\hat{f}_{\rm K}$:

(a) $\hat{f}_{\rm K}$ is appreciable only in the Knudsen layer and vanishes rapidly away from the boundary.

(b) $\hat{f}_{\rm K}$ has the length scale of variation of the order of ϵ (i.e., of the order of the mean free path l_0 in the dimensional physical space) in the direction normal to the boundary, that is, $n_j \partial \hat{f}_{\rm K}/\partial x_j = O(\hat{f}_{\rm K}/\epsilon)$.

(c) $\hat{f}_{\rm K}$ has the length scale of variation of the order of 1 (i.e., of the order of the reference length L in the dimensional physical space) in the direction along the boundary.

(d) $\hat{f}_{\rm K}$ has the time scale of variation of the order of 1 [i.e., of the order of $t_0 = L/(2RT_0)^{1/2}$ in the dimensional time], i.e., $\partial \hat{f}_{\rm K}/\partial \hat{t} = O(\hat{f}_{\rm K})$.

These assumptions can be justified if such a solution is obtained consistently.

The fact that the Chapman–Enskog solution, Eq. (44) with Eqs. (45) and (46), can be made to satisfy the boundary condition (41) at the zeroth order in ϵ by the choice (56) indicates that

the differences $\hat{\boldsymbol{v}}-\hat{\boldsymbol{v}}_{\rm w}$, $\hat{T}_{\rm tr}-\hat{T}_{\rm w}$, and $\hat{T}_{\rm int}-\hat{T}_{\rm w}$ are small and of the order of ϵ on the boundary. Therefore, we put

$$\hat{\boldsymbol{v}}-\hat{\boldsymbol{v}}_{\rm w}=\bar{\bar{\boldsymbol{v}}}\epsilon, \qquad \hat{T}_{\rm tr}-\hat{T}_{\rm w}=\bar{\bar{T}}_{\rm tr}\epsilon, \qquad \hat{T}_{\rm int}-\hat{T}_{\rm w}=\bar{\bar{T}}_{\rm int}\epsilon, \qquad \text{at } \boldsymbol{x}=\boldsymbol{x}_{\rm w}, \tag{60}$$

where $\bar{\bar{\boldsymbol{v}}}$, $\bar{\bar{T}}_{\rm tr}$, and $\bar{\bar{T}}_{\rm int}$ are the quantities of $O(1)$. This fact also indicates that $\hat{f}_{\rm K}$ starts at the order of ϵ, so that we let

$$\hat{f}_{\rm K}=\hat{f}_{\rm K}^{(1)}\epsilon+R_f\epsilon^2, \tag{61}$$

where $R_f\epsilon^2$ is the remainder, and R_f is of $O(1)$ and has the properties (a)–(d). Correspondingly, we put

$$\hat{h}_{\rm K}=\hat{h}_{\rm K}^{(1)}\epsilon+R_h\epsilon^2, \tag{62}$$

where $R_h\epsilon^2$ is the remainder corresponding to $R_f\epsilon^2$.

We insert Eqs. (58) and (59) with Eqs. (61) and (62) into Eqs. (30d)–(30j), (31), and (32) (with $\hat{f}=\hat{f}_{\rm tot}$ and $\hat{h}=\hat{h}_{\rm tot}$) and note that $\hat{f}_{\rm CE}$ and $\hat{h}_{\rm CE}$ satisfy the same relations as Eqs. (30d)–(30j), (31), and (32) (with $\hat{f}=\hat{f}_{\rm CE}$ and $\hat{h}=\hat{h}_{\rm CE}$). In this process, we have to note the following. From Eqs. (30j) and (5), we have

$$\begin{aligned}(\hat{T}_{\rm rel})_{\rm tot}&=\alpha\epsilon\hat{T}_{\rm tot}+(1-\alpha\epsilon)(\hat{T}_{\rm int})_{\rm tot}\\&=\alpha\epsilon[\hat{T}+\hat{T}_{\rm K}^{(1)}\epsilon+O(R_h\epsilon^2)]+(1-\alpha\epsilon)[\hat{T}_{\rm int}+\hat{T}_{\rm intK}^{(1)}\epsilon+O(R_h\epsilon^2)]\\&=\hat{T}_{\rm int}+\alpha\epsilon(\hat{T}-\hat{T}_{\rm int})+\hat{T}_{\rm intK}^{(1)}\epsilon+O(R_h\epsilon^2),\end{aligned} \tag{63}$$

where $\hat{T}$ and $\hat{T}_{\rm int}$ are the Chapman–Enskog quantities. On the other hand, $(\hat{T}_{\rm rel})_{\rm tot}=\hat{T}_{\rm rel}+\hat{T}_{\rm relK}^{(1)}\epsilon+O(R_h\epsilon^2)$ by definition, and $\hat{T}_{\rm rel}=\hat{T}_{\rm rel}^{(0)}+\hat{T}_{\rm rel}^{(1)}\epsilon=\hat{T}_{\rm int}+\alpha(\hat{T}-\hat{T}_{\rm int})\,\epsilon$ from Eq. (52). Therefore, $\hat{T}_{\rm relK}^{(1)}$ is identified as

$$\hat{T}_{\rm relK}^{(1)}=\hat{T}_{\rm intK}^{(1)}. \tag{64}$$

Then, picking up the terms of $O(\epsilon)$ for $\hat{h}_{\rm K}^{(1)}$ and putting the $O(\epsilon^2)$ terms in $R_h\epsilon^2$, we obtain the following expressions of $\hat{h}_{\rm K}^{(1)}$:

$$\hat{\rho}_{\rm K}^{(1)}=\iint_0^\infty \hat{f}_{\rm K}^{(1)}d\hat{\mathcal{E}}d\boldsymbol{\zeta}, \tag{65a}$$

$$\hat{v}_{{\rm K}i}^{(1)}=\frac{1}{\hat{\rho}}\iint_0^\infty(\zeta_i-\hat{v}_i)\hat{f}_{\rm K}^{(1)}d\hat{\mathcal{E}}d\boldsymbol{\zeta}, \tag{65b}$$

$$\hat{p}_{{\rm K}ij}^{(1)}=2\iint_0^\infty(\zeta_i-\hat{v}_i)(\zeta_j-\hat{v}_j)\hat{f}_{\rm K}^{(1)}d\hat{\mathcal{E}}d\boldsymbol{\zeta}, \tag{65c}$$

$$\hat{T}_{\rm trK}^{(1)}=\frac{2}{3\hat{\rho}}\iint_0^\infty\left[(\zeta_k-\hat{v}_k)^2-\frac{3}{2}\hat{T}_{\rm tr}\right]\hat{f}_{\rm K}^{(1)}d\hat{\mathcal{E}}d\boldsymbol{\zeta}, \tag{65d}$$

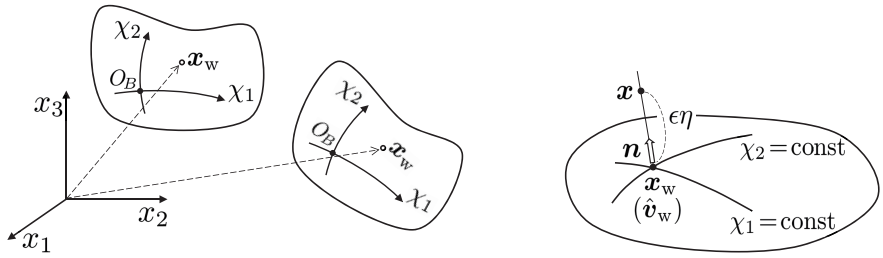
$$\hat{T}_{\rm intK}^{(1)}=\frac{2}{\delta\hat{\rho}}\iint_0^\infty\left(\hat{\mathcal{E}}-\frac{\delta}{2}\hat{T}_{\rm int}\right)\hat{f}_{\rm K}^{(1)}d\hat{\mathcal{E}}d\boldsymbol{\zeta}, \tag{65e}$$

$$\hat{T}_{\rm K}^{(1)}=\frac{3\hat{T}_{\rm trK}^{(1)}+\delta\hat{T}_{\rm intK}^{(1)}}{3+\delta}, \tag{65f}$$

$$\hat{T}_{\rm relK}^{(1)}=\hat{T}_{\rm intK}^{(1)}, \tag{65g}$$

$$\hat{p}_{\rm K}^{(1)}=\hat{\rho}\hat{T}_{\rm K}^{(1)}+\hat{\rho}_{\rm K}^{(1)}\hat{T}, \tag{65h}$$

$$\left.\begin{aligned}&\hat{q}_{({\rm tr}){\rm K}i}^{(1)}=\iint_0^\infty(\zeta_i-\hat{v}_i)\left[(\zeta_k-\hat{v}_k)^2-\frac{5}{2}\hat{T}_{\rm tr}\right]\hat{f}_{\rm K}^{(1)}d\hat{\mathcal{E}}d\boldsymbol{\zeta},\\&\hat{q}_{({\rm int}){\rm K}i}^{(1)}=\iint_0^\infty(\zeta_i-\hat{v}_i)\left(\hat{\mathcal{E}}-\frac{\delta}{2}\hat{T}_{\rm int}\right)\hat{f}_{\rm K}^{(1)}d\hat{\mathcal{E}}d\boldsymbol{\zeta},\\&\hat{q}_{{\rm K}i}^{(1)}=\hat{q}_{({\rm tr}){\rm K}i}^{(1)}+\hat{q}_{({\rm int}){\rm K}i}^{(1)}.\end{aligned}\right\} \tag{65i}$$



(a) (b)

FIG. 1. Coordinate systems. (a) Coordinate system on the boundary, (b) coordinate system for the Knudsen layer.

Note again that $\hat{\rho}$, $\hat{v}_i$, $\hat{T}_{\rm tr}$, and $\hat{T}_{\rm int}$ here are the macroscopic quantities associated with the Chapman–Enskog solution though the subscript CE is not attached. It should also be mentioned that use has been made of the fact that $\hat{p}_{ij} = \hat{\rho}\hat{T}_{\rm tr}\delta_{ij} + O(\epsilon)$ [Eq. (53a)] in the derivation of Eq. (65).

If we substitute Eq. (58) with Eq. (61) into Eq. (29) and take into account the fact that $\hat{f}_{\rm CE}$ is also the solution of Eq. (29), we obtain the following equation for $\hat{f}_{\rm K}^{(1)}$ (see Appendix C for the outline of the derivation):

$$\epsilon\frac{\partial \hat{f}_{\rm K}^{(1)}}{\partial \hat{t}} + \epsilon\zeta_i\frac{\partial \hat{f}_{\rm K}^{(1)}}{\partial x_i} = \hat{A}_c(\hat{T})\hat{\rho}(\hat{\mathcal{G}}_{\rm K}^{(1)} - \hat{f}_{\rm K}^{(1)}) + O(R_f\epsilon), \tag{66}$$

where

$$\begin{aligned}\hat{\mathcal{G}}_{\rm K}^{(1)} = \hat{f}^{(0)}\Bigg\{ &\frac{\hat{\rho}_{\rm K}^{(1)}}{\hat{\rho}} + 2\frac{(\zeta_i - \hat{v}_i)}{\hat{T}_{\rm tr}}\hat{v}_{{\rm K}i}^{(1)} + \left[\frac{(\zeta_i - \hat{v}_i)^2}{\hat{T}_{\rm tr}} - \frac{3}{2}\right]\frac{\hat{T}_{\rm trK}^{(1)}}{\hat{T}_{\rm tr}} \\ &+\nu\left[\frac{(\zeta_i - \hat{v}_i)(\zeta_j - \hat{v}_j)}{\hat{T}_{\rm tr}} - \frac{1}{3}\frac{(\zeta_k - \hat{v}_k)^2}{\hat{T}_{\rm tr}}\delta_{ij}\right]\frac{\hat{p}_{{\rm K}ij}^{(1)}}{\hat{\rho}\hat{T}_{\rm tr}} \\ &+\left(\frac{\hat{\mathcal{E}}}{\hat{T}_{\rm int}} - \frac{\delta}{2}\right)\frac{\hat{T}_{\rm intK}^{(1)}}{\hat{T}_{\rm int}}\Bigg\},\end{aligned} \tag{67}$$

and note that the function $\hat{A}_c(\hat{T})$ is unexpanded because $\hat{T}$ is not expanded. Equation (66) is the starting point of the analysis of the Knudsen layer in the following.

B. Knudsen-layer equation

The analysis of this subsection follows that in Sec. V A 3 in [39], so that we omit the details and only show the main points referring to [39] occasionally. However, in order to explain the essence of the coordinate system to describe the Knudsen layer, we need to start with repeating the explanation in [39].

We first express a point $\boldsymbol{x}_{\rm w}$ on the boundary as a function of coordinates χ_1 and χ_2 fixed on the surface of the boundary and of time $\hat{t}$ [see Fig. 1(a)], i.e.,

$$\boldsymbol{x}_{\rm w} = \boldsymbol{x}_{\rm w}(\hat{t},\, \chi_1,\, \chi_2). \tag{68}$$

When χ_1 and χ_2 are fixed, the function $\boldsymbol{x}_{\rm w}(\hat{t},\, \chi_1,\, \chi_2)$ of $\hat{t}$ gives the trajectory of a fixed point on the boundary, and when $\hat{t}$ is fixed, the function $\boldsymbol{x}_{\rm w}(\hat{t},\, \chi_1,\, \chi_2)$ of χ_1 and χ_2 gives the parameter representation of the boundary surface at time $\hat{t}$. The velocity of the boundary $\hat{\boldsymbol{v}}_{\rm w}$ and the unit normal vector to the boundary $\boldsymbol{n}$, which are also the functions of $\hat{t}$, χ_1, and χ_2, are expressed as

$$\hat{\boldsymbol{v}}_{\rm w}(\hat{t},\, \chi_1,\, \chi_2) = \frac{\partial \boldsymbol{x}_{\rm w}}{\partial \hat{t}}, \tag{69a}$$

$$\boldsymbol{n}(\hat{t},\,\chi_1,\,\chi_2) = \pm\left(\frac{\partial \boldsymbol{x}_{\rm w}}{\partial\chi_1}\times\frac{\partial \boldsymbol{x}_{\rm w}}{\partial\chi_2}\right)\left|\frac{\partial \boldsymbol{x}_{\rm w}}{\partial\chi_1}\times\frac{\partial \boldsymbol{x}_{\rm w}}{\partial\chi_2}\right|^{-1}, \tag{69b}$$

where $\times$ indicates the vector product, and $+$ sign or $-$ sign is chosen in such a way that $\boldsymbol{n}$ points into the gas region.

In order to analyze the Knudsen layer, we introduce a new coordinate system that is local near the boundary and appropriate to describe the rapid change of the physical quantities in the direction normal to the boundary. We introduce the new variables $\tilde{t}$, η, and $\boldsymbol{\zeta}_{\rm w}$ by the following relations [see Fig. 1(b)]:

$$\hat{t} = \tilde{t}, \tag{70a}$$

$$\boldsymbol{x} = \epsilon\,\eta\,\boldsymbol{n}(\tilde{t},\,\chi_1,\,\chi_2) + \boldsymbol{x}_{\rm w}(\tilde{t},\,\chi_1,\,\chi_2), \tag{70b}$$

$$\boldsymbol{\zeta} = \boldsymbol{\zeta}_{\rm w} + \hat{\boldsymbol{v}}_{\rm w}(\tilde{t},\,\chi_1,\,\chi_2). \tag{70c}$$

Here, η is a stretched normal coordinate, and $\boldsymbol{\zeta}_{\rm w}$ is the molecular velocity relative to the velocity of the boundary. In accordance with the properties (a)–(d) in Sec. VI A, we assume that $\hat{f}_{\rm K}$ is a function of $(\tilde{t},\,\eta,\,\chi_1,\,\chi_2,\,\boldsymbol{\zeta}_{\rm w},\,\hat{\mathcal{E}})$ and vanishes rapidly as $\eta\to\infty$:

$$\hat{f}_{\rm K} = \hat{f}_{\rm K}(\tilde{t},\,\eta,\,\chi_1,\,\chi_2,\,\boldsymbol{\zeta}_{\rm w},\,\hat{\mathcal{E}}), \tag{71a}$$

$$\hat{f}_{\rm K} \to 0, \qquad \text{as } \eta\to\infty. \tag{71b}$$

Therefore, Eq. (71) also holds for $\hat{f}_{\rm K}^{(1)}$ and R_f in Eq. (61).

We now consider Eq. (66) inside the Knudsen layer, i.e., $\eta = O(1)$ or $(\boldsymbol{x}-\boldsymbol{x}_{\rm w})\cdot\boldsymbol{n} = O(\epsilon)$. The $\boldsymbol{x}$-dependence of $\hat{f}^{(0)}$ is through $\hat{\rho}$, $\hat{\boldsymbol{v}}$, $\hat{T}_{\rm tr}$, and $\hat{T}_{\rm int}$, whose length scale is of $O(1)$. Therefore, inside the Knudsen layer, they can be Taylor expanded around $\boldsymbol{x} = \boldsymbol{x}_{\rm w}$, that is,

$$\hat{\rho} = \hat{\rho}_{\rm B} + O(\epsilon\eta), \quad \hat{\boldsymbol{v}} = \hat{\boldsymbol{v}}_{\rm B} + O(\epsilon\eta), \quad \hat{T}_{\rm tr} = (\hat{T}_{\rm tr})_{\rm B} + O(\epsilon\eta), \quad \hat{T}_{\rm int} = (\hat{T}_{\rm int})_{\rm B} + O(\epsilon\eta), \tag{72}$$

where the subscript B indicates the value on the boundary $\boldsymbol{x} = \boldsymbol{x}_{\rm w}$ or $\eta = 0$. Because $\hat{\boldsymbol{v}}_{\rm B} = \hat{\boldsymbol{v}}_{\rm w} + O(\epsilon)$, $(\hat{T}_{\rm tr})_{\rm B} = \hat{T}_{\rm w} + O(\epsilon)$, and $(\hat{T}_{\rm int})_{\rm B} = \hat{T}_{\rm w} + O(\epsilon)$ [Eq. (60)], we can write

$$\begin{aligned}
&\hat{\rho} = \hat{\rho}_{\rm B} + O(\epsilon\eta), \qquad \hat{\boldsymbol{v}} = \hat{\boldsymbol{v}}_{\rm w} + O(\epsilon(\eta+1)),\\
&\hat{T}_{\rm tr} = \hat{T}_{\rm w} + O(\epsilon(\eta+1)), \qquad \hat{T}_{\rm int} = \hat{T}_{\rm w} + O(\epsilon(\eta+1)), \qquad \hat{T} = \hat{T}_{\rm w} + O(\epsilon(\eta+1)).
\end{aligned} \tag{73}$$

Here, the last equation is the consequence of Eq. (30i).

If Eq. (73) is substituted into Eq. (65), the $O(\epsilon\eta)$ and $O(\epsilon(\eta+1))$ terms in Eq. (73) produce the terms of the order of $\epsilon(\eta+1)$ times a moment of $\hat{f}_{\rm K}^{(1)}$, which vanish rapidly as $\eta\to\infty$. Therefore, we put these terms (times ϵ) into $R_h\epsilon^2$ in Eq. (62) to simplify the expressions of $\hat{h}_{\rm K}^{(1)}$. To be more specific, we can transform Eq. (65) in the following way:

$$\text{[Eq. (65)]} \implies \text{[Eq. (65) with } \hat{\rho} = \hat{\rho}_{\rm B},\ \hat{\boldsymbol{v}} = \hat{\boldsymbol{v}}_{\rm w}, \text{ and } \hat{T}_{\rm tr} = \hat{T}_{\rm int} = \hat{T} = \hat{T}_{\rm w}\text{]}. \tag{74}$$

The right-hand side of Eq. (74) is the same as Eq. (63) in [39] except that Eq. (63g) there has been replaced by the equation $\hat{T}_{\rm relK}^{(1)} = \hat{T}_{\rm intK}^{(1)}$.

With the help of Eq. (73), $\hat{f}^{(0)}$ in Eq. (45) inside the Knudsen layer is expanded as

$$\hat{f}^{(0)} = \hat{f}_{\rm w}[1 + O(\epsilon(\eta+1))], \tag{75}$$

where

$$\hat{f}_{\rm w} = \frac{\hat{\rho}_{\rm B}\hat{\mathcal{E}}^{\delta/2-1}}{(\pi\hat{T}_{\rm w})^{3/2}\hat{T}_{\rm w}^{\delta/2}\Gamma(\delta/2)}\exp\left(-\frac{(\zeta_j-\hat{v}_{{\rm w}j})^2}{\hat{T}_{\rm w}} - \frac{\hat{\mathcal{E}}}{\hat{T}_{\rm w}}\right). \tag{76}$$

Here, we follow the procedure that was used in [39] in the derivation of Eqs. (66)–(68) there. That is, we use Eqs. (73) and (75) in Eqs. (66) and (67) and put the terms of $O(\epsilon\eta)$ and $O(\epsilon(\eta+1))$, which are produced by the terms of $O(\epsilon\eta)$ and $O(\epsilon(\eta+1))$ in Eqs. (73) and (75) and vanish rapidly as $\eta\to\infty$, into the remainder ϵR_f in Eq. (66). In addition, we take into account

the fact that the left-hand side of Eq. (66), in terms of the new variables ($\tilde{t}$, η, χ_1, χ_2, $\boldsymbol{\zeta}_{\rm w}$), reduces to (see Sec. 5.2.2 in [40])

$$\zeta_{{\rm w}i} n_i \frac{\partial \hat{f}_{\rm K}^{(1)}}{\partial \eta} + O(R_f \epsilon). \tag{77}$$

Then, we obtain the following equations corresponding to Eqs. (68), (66b), and (66c) in [39]:

$$\zeta_{{\rm w}i} n_i \frac{\partial \hat{f}_{\rm K}^{(1)}}{\partial \eta} = \hat{A}_c(\hat{T}_{\rm w}) \hat{\rho}_{\rm B} (\hat{\mathcal{G}}_{\rm K}^{(1)} - \hat{f}_{\rm K}^{(1)}) + O(\epsilon R_f), \tag{78a}$$

$$\begin{aligned} \hat{\mathcal{G}}_{\rm K}^{(1)} = \hat{f}_{\rm w} \Bigg\{ & \frac{\hat{\rho}_{\rm K}^{(1)}}{\hat{\rho}_{\rm B}} + 2 \frac{(\zeta_i - \hat{v}_{{\rm w}i})}{\hat{T}_{\rm w}} \hat{v}_{{\rm K}i}^{(1)} + \left[\frac{(\zeta_i - \hat{v}_{{\rm w}i})^2}{\hat{T}_{\rm w}} - \frac{3}{2} \right] \frac{\hat{T}_{\rm trK}^{(1)}}{\hat{T}_{\rm w}} \\ & + \nu \left[\frac{(\zeta_i - \hat{v}_{{\rm w}i})(\zeta_j - \hat{v}_{{\rm w}j})}{\hat{T}_{\rm w}} - \frac{1}{3} \frac{(\zeta_k - \hat{v}_{{\rm w}k})^2}{\hat{T}_{\rm w}} \delta_{ij} \right] \frac{\hat{p}_{{\rm K}ij}^{(1)}}{\hat{\rho}_{\rm B} \hat{T}_{\rm w}} \\ & + \left(\frac{\hat{\mathcal{E}}}{\hat{T}_{\rm w}} - \frac{\delta}{2} \right) \frac{\hat{T}_{\rm intK}^{(1)}}{\hat{T}_{\rm w}} \Bigg\}. \end{aligned} \tag{78b}$$

Here, we should mention that Eq. (78) is the same as Eqs. (68), (66b), and (66c) in [39] with $\theta = 0$ (note that $\hat{T}_{\rm relK}^{(1)} = \hat{T}_{\rm intK}^{(1)}$ for $\theta = 0$). Therefore, we can utilize the transformation from Eqs. (68), (66b), and (66c) in [39] to their final form, Eq. (79) there. That is, the final equation transformed from Eq. (78) is nothing but Eq. (79) in [39] with $\theta = 0$. The equation is summarized below.

We first introduce new variables $\boldsymbol{\mathcal{C}}_{\rm w}$ and $\bar{\mathcal{E}}_{\rm w}$ by

$$\boldsymbol{\mathcal{C}}_{\rm w} = \frac{\boldsymbol{\zeta}_{\rm w}}{\hat{T}_{\rm w}^{1/2}} = \frac{\boldsymbol{\zeta} - \hat{\boldsymbol{v}}_{\rm w}}{\hat{T}_{\rm w}^{1/2}}, \qquad \bar{\mathcal{E}}_{\rm w} = \frac{\hat{\mathcal{E}}}{\hat{T}_{\rm w}}, \tag{79}$$

and denote the normal component and magnitude of $\boldsymbol{\mathcal{C}}_{\rm w}$ by $\mathcal{C}_{\rm wn}$ and $\mathcal{C}_{\rm w}$, respectively, i.e.,

$$\mathcal{C}_{\rm wn} = \mathcal{C}_{{\rm w}j} n_j = \boldsymbol{\mathcal{C}}_{\rm w} \cdot \boldsymbol{n}, \qquad \mathcal{C}_{\rm w} = (\mathcal{C}_{{\rm w}j}^2)^{1/2} = |\boldsymbol{\mathcal{C}}_{\rm w}|. \tag{80}$$

Then, $\hat{f}_{\rm w}$ can be expressed as

$$\hat{f}_{\rm w} = \frac{\hat{\rho}_{\rm B}}{\hat{T}_{\rm w}^{5/2} \Gamma(\delta/2)} E(\mathcal{C}_{\rm w}) \bar{\mathcal{E}}_{\rm w}^{\delta/2 - 1} e^{-\bar{\mathcal{E}}_{\rm w}}, \tag{81}$$

where $E(\mathcal{C}_{\rm w}) = \pi^{-3/2} \exp(-\mathcal{C}_{\rm w}^2)$ as defined in Eq. (40). It should be noted that $\hat{\rho}_{\rm B}$ and $\hat{T}_{\rm w}$ are functions of ($\tilde{t}$, χ_1, χ_2), so that $\hat{f}_{\rm w}$ is a function of ($\tilde{t}$, χ_1, χ_2, $\mathcal{C}_{\rm w}$, $\bar{\mathcal{E}}_{\rm w}$).

We further introduce the following new normal coordinate y in place of η:

$$y = \hat{\rho}_{\rm B} \frac{\hat{A}_c(\hat{T}_{\rm w})}{\hat{T}_{\rm w}^{1/2}} \eta, \tag{82}$$

and change the independent variables from ($\tilde{t}$, η, χ_1, χ_2, $\boldsymbol{\zeta}_{\rm w}$, $\hat{\mathcal{E}}$) to ($\tilde{t}$, y, χ_1, χ_2, $\boldsymbol{\mathcal{C}}_{\rm w}$, $\bar{\mathcal{E}}_{\rm w}$) by letting

$$\begin{aligned} & \hat{f}_{\rm K}^{(1)} \left(\tilde{t},\, [\hat{\rho}_{\rm B} \hat{A}_c(\hat{T}_{\rm w})]^{-1} \hat{T}_{\rm w}^{1/2} y,\, \chi_1,\, \chi_2,\, \hat{T}_{\rm w}^{1/2} \boldsymbol{\mathcal{C}}_{\rm w},\, \hat{T}_{\rm w} \bar{\mathcal{E}}_{\rm w} \right) \\ & \quad = \hat{f}_{\rm w}(\tilde{t},\, \chi_1,\, \chi_2,\, \mathcal{C}_{\rm w},\, \bar{\mathcal{E}}_{\rm w})\, \phi(\tilde{t},\, y,\, \chi_1,\, \chi_2,\, \boldsymbol{\mathcal{C}}_{\rm w},\, \bar{\mathcal{E}}_{\rm w}). \end{aligned} \tag{83}$$

Then, Eq. (78) is transformed into the following equation for ϕ:

$$\mathcal{C}_{\rm wn} \frac{\partial \phi}{\partial y} = \mathcal{L}_0(\phi) + O(\epsilon R_f / \hat{f}_{\rm w}). \tag{84}$$

Here the linear integral operator $\mathcal{L}_0$, which is equal to the linearized collision operator $\mathcal{L}$ of the ES model [Eq. (32) in [39]] with $\theta = 0$, is defined as

$$\mathcal{L}_0[\phi(\boldsymbol{\mathcal{C}}_{\rm w}, \bar{\mathcal{E}}_{\rm w})](\boldsymbol{\mathcal{C}}_{\rm w}, \bar{\mathcal{E}}_{\rm w}) = \omega + 2\mathcal{C}_{{\rm w}i}u_i + \left(\mathcal{C}_{\rm w}^2 - \frac{3}{2}\right)\tau_{\rm tr} + \nu\left(\mathcal{C}_{{\rm w}i}\mathcal{C}_{{\rm w}j} - \frac{1}{3}\mathcal{C}_{\rm w}^2\delta_{ij}\right)P_{ij} + \left(\bar{\mathcal{E}}_{\rm w} - \frac{\delta}{2}\right)\tau_{\rm int} - \phi, \tag{85}$$

where

$$\omega = \langle\!\langle\phi\rangle\!\rangle, \qquad u_i = \langle\!\langle\mathcal{C}_{{\rm w}i}\phi\rangle\!\rangle, \qquad P_{ij} = 2\langle\!\langle\mathcal{C}_{{\rm w}i}\mathcal{C}_{{\rm w}j}\phi\rangle\!\rangle, \tag{86a}$$

$$\tau_{\rm tr} = \frac{2}{3}\left\langle\!\!\left\langle\left(\mathcal{C}_{\rm w}^2 - \frac{3}{2}\right)\phi\right\rangle\!\!\right\rangle, \qquad \tau_{\rm int} = \frac{2}{\delta}\left\langle\!\!\left\langle\left(\bar{\mathcal{E}}_{\rm w} - \frac{\delta}{2}\right)\phi\right\rangle\!\!\right\rangle, \tag{86b}$$

and $\langle\!\langle\,\cdot\,\rangle\!\rangle$ is defined, with an arbitrary function $\hat{g}(\boldsymbol{\mathcal{C}}_{\rm w}, \bar{\mathcal{E}}_{\rm w})$ of $\boldsymbol{\mathcal{C}}_{\rm w}$ and $\bar{\mathcal{E}}_{\rm w}$, as

$$\langle\!\langle\,\hat{g}(\boldsymbol{\mathcal{C}}_{\rm w}, \bar{\mathcal{E}}_{\rm w})\,\rangle\!\rangle = [\Gamma(\delta/2)]^{-1}\iint_0^\infty \hat{g}(\boldsymbol{\mathcal{C}}_{\rm w}, \bar{\mathcal{E}}_{\rm w})E(\mathcal{C}_{\rm w})\bar{\mathcal{E}}_{\rm w}^{\delta/2-1}e^{-\bar{\mathcal{E}}_{\rm w}}d\bar{\mathcal{E}}_{\rm w}d\boldsymbol{\mathcal{C}}_{\rm w}. \tag{87}$$

On the left-hand side of Eq. (85), the arguments $\boldsymbol{\mathcal{C}}_{\rm w}$ and $\bar{\mathcal{E}}_{\rm w}$ of ϕ and those of $\mathcal{L}_0(\phi)$ are shown explicitly, the other arguments $\tilde{t}$, y, χ_1, and χ_2 being omitted. If we neglect the terms of $O(\epsilon R_f/\hat{f}_{\rm w})$ in Eq. (84), we obtain the equation for ϕ, i.e., that for $\hat{f}_{\rm K}^{(1)}$.

Here, it is noted that the macroscopic quantities $\hat{\rho}_{\rm K}^{(1)}$, $\hat{v}_{{\rm K}i}^{(1)}$, $\hat{p}_{{\rm K}ij}^{(1)}$, $\hat{T}_{\rm trK}^{(1)}$, and $\hat{T}_{\rm intK}^{(1)}$, which are given by Eqs. (65a)–(65e) with $\hat{\rho} = \hat{\rho}_{\rm B}$, $\hat{\boldsymbol{v}} = \hat{\boldsymbol{v}}_{\rm w}$, and $\hat{T}_{\rm tr} = \hat{T}_{\rm int} = \hat{T}_{\rm w}$ [cf. Eq. (74)] and are the functions of $(\tilde{t}, \eta, \chi_1, \chi_2)$, are related to ω, u_i, P_{ij}, $\tau_{\rm tr}$, and $\tau_{\rm int}$ in Eqs. (86a) and (86b), which are the functions of $(\tilde{t}, y, \chi_1, \chi_2)$, by the following relations:

$$\frac{\hat{\rho}_{\rm K}^{(1)}}{\hat{\rho}_{\rm B}} = \omega, \quad \frac{\hat{v}_{{\rm K}i}^{(1)}}{\hat{T}_{\rm w}^{1/2}} = u_i, \quad \frac{\hat{p}_{{\rm K}ij}^{(1)}}{\hat{\rho}_{\rm B}\hat{T}_{\rm w}} = P_{ij}, \quad \frac{\hat{T}_{\rm trK}^{(1)}}{\hat{T}_{\rm w}} = \tau_{\rm tr}, \quad \frac{\hat{T}_{\rm intK}^{(1)}}{\hat{T}_{\rm w}} = \tau_{\rm int}. \tag{88}$$

In addition, $\hat{q}_{({\rm tr}){\rm K}i}^{(1)}$ and $\hat{q}_{({\rm int}){\rm K}i}^{(1)}$, which are given by Eq. (65i) with $\hat{\boldsymbol{v}} = \hat{\boldsymbol{v}}_{\rm w}$ and $\hat{T}_{\rm tr} = \hat{T}_{\rm int} = \hat{T}_{\rm w}$ [cf. Eq. (74)] and are the functions of $(\tilde{t}, \eta, \chi_1, \chi_2)$, are expressed as

$$\frac{\hat{q}_{({\rm tr}){\rm K}i}^{(1)}}{\hat{\rho}_{\rm B}\hat{T}_{\rm w}^{3/2}} = \left\langle\!\!\left\langle\,\mathcal{C}_{{\rm w}i}\left(\mathcal{C}_{\rm w}^2 - \frac{5}{2}\right)\phi\right\rangle\!\!\right\rangle, \qquad \frac{\hat{q}_{({\rm int}){\rm K}i}^{(1)}}{\hat{\rho}_{\rm B}\hat{T}_{\rm w}^{3/2}} = \left\langle\!\!\left\langle\,\mathcal{C}_{{\rm w}i}\left(\bar{\mathcal{E}}_{\rm w} - \frac{\delta}{2}\right)\phi\right\rangle\!\!\right\rangle, \tag{89}$$

where the right-hand sides are the functions of $(\tilde{t}, y, \chi_1, \chi_2)$.

The operator $\mathcal{L}_0(\,\cdot\,)$ is the linearized collision operator of the ES collision operator $\hat{Q}(\,\cdot\,)$ in Eq. (30a) [or $Q(\,\cdot\,)$ in Eq. (18a)] when $\theta = 0$ and is equal to $\mathcal{L}(\,\cdot\,)$ defined by Eq. (32) in [39] with $\theta = 0$. The solution of $\mathcal{L}_0(\phi) = 0$ (equilibrium solution) is given by the six-parameter family of the form

$$\phi = c_0 + c_1\mathcal{C}_{\rm w1} + c_2\mathcal{C}_{\rm w2} + c_3\mathcal{C}_{\rm w3} + c_4\mathcal{C}_{\rm w}^2 + c_5\bar{\mathcal{E}}_{\rm w}, \tag{90}$$

where c_0, c_2, ... are parameters. This corresponds to Eq. (36). The operator $\mathcal{L}_0$ also satisfies the relation

$$\langle\!\langle\hat{\phi}_r\mathcal{L}_0(\hat{g})\rangle\!\rangle = 0, \tag{91}$$

which corresponds to Eq. (37); here, $\hat{\phi}_r$ ($r = 0$, ..., 5) are given by Eq. (38) with $\zeta_i \to \mathcal{C}_{{\rm w}i}$ and $\hat{\mathcal{E}} \to \bar{\mathcal{E}}_{\rm w}$.

Multiplying Eq. (84) by $(1, \mathcal{C}_{{\rm w}i}, |\boldsymbol{\mathcal{C}}_{\rm w}|^2, \bar{\mathcal{E}}_{\rm w})$, taking $\langle\!\langle\,\cdot\,\rangle\!\rangle$ of the respective equations, and using the property (91), we obtain $\partial\langle\!\langle\mathcal{C}_{\rm wn}\phi\rangle\!\rangle/\partial y = \partial\langle\!\langle\mathcal{C}_{\rm wn}\mathcal{C}_{{\rm w}i}\phi\rangle\!\rangle/\partial y = \partial\langle\!\langle\mathcal{C}_{\rm wn}|\boldsymbol{\mathcal{C}}_{\rm w}|^2\phi\rangle\!\rangle/\partial y = \partial\langle\!\langle\mathcal{C}_{\rm wn}\bar{\mathcal{E}}_{\rm w}\phi\rangle\!\rangle/\partial y = O(R_h\epsilon)$. Since $\phi \to 0$ as $y \to \infty$, it follows that

$$\langle\!\langle\mathcal{C}_{\rm wn}\phi\rangle\!\rangle = \langle\!\langle\mathcal{C}_{\rm wn}\mathcal{C}_{{\rm w}i}\phi\rangle\!\rangle = \langle\!\langle\mathcal{C}_{\rm wn}|\boldsymbol{\mathcal{C}}_{\rm w}|^2\phi\rangle\!\rangle = \langle\!\langle\mathcal{C}_{\rm wn}\bar{\mathcal{E}}_{\rm w}\phi\rangle\!\rangle = O(R_h\epsilon). \tag{92}$$

C. Knudsen-layer boundary condition

Now we consider the boundary condition. We impose the boundary condition (41) to the total solution $\hat{f}_{\rm tot}$ [Eq. (58)]. Using Eq. (58) with $\hat{f}_{\rm CE} = \hat{f}^{(0)} + \hat{f}^{(1)}\epsilon + O(\epsilon^2)$ [Eq. (44)] and with $\hat{f}_{\rm K} = \hat{f}_{\rm K}^{(1)}\epsilon + O(R_f\epsilon^2)$ [Eq. (61)] in Eq. (41), we obtain the following relation at $\eta = 0$ (or $\boldsymbol{x} = \boldsymbol{x}_{\rm w}$):

$$\epsilon\hat{f}_{\rm K}^{(1)} = (1-a_c)\epsilon\widehat{\mathcal{R}}\hat{f}_{\rm K}^{(1)} - \hat{f}^{(0)} - \epsilon\hat{f}^{(1)} + (1-a_c)\widehat{\mathcal{R}}(\hat{f}^{(0)} + \epsilon\hat{f}^{(1)}) + a_c\frac{\hat{\rho}_{\rm w}}{\hat{\rho}_{\rm B}}\hat{f}_{\rm w} + O(\epsilon^2 R_{\rm w}),$$
$$\text{for} \quad (\boldsymbol{\zeta} - \hat{\boldsymbol{v}}_{\rm w})\cdot\boldsymbol{n} > 0, \tag{93a}$$

$$\hat{\rho}_{\rm w} = -2\left(\frac{\pi}{\hat{T}_{\rm w}}\right)^{1/2}\int_{(\boldsymbol{\zeta}-\hat{\boldsymbol{v}}_{\rm w})\cdot\boldsymbol{n}<0}\int_0^\infty (\boldsymbol{\zeta} - \hat{\boldsymbol{v}}_{\rm w})\cdot\boldsymbol{n}(\hat{f}^{(0)} + \epsilon\hat{f}^{(1)} + \epsilon\hat{f}_{\rm K}^{(1)})d\hat{\mathcal{E}}d\boldsymbol{\zeta} + O(\epsilon^2), \tag{93b}$$

where $R_{\rm w}$ is a function of $O(1)$ vanishing rapidly as $|\boldsymbol{\zeta}|$ (or $|\boldsymbol{\mathcal{C}}_{\rm w}|$) $\to\infty$ and as $\hat{\mathcal{E}}$ (or $\bar{\mathcal{E}}_{\rm w}$) $\to\infty$. For instance, $\hat{f}_{\rm w}$ [Eq. (76)] belongs to the class of $R_{\rm w}$.

Noting that $\hat{f}^{(0)}$ and $\hat{f}^{(1)}$ in Eq. (93) are evaluated at the boundary and following the procedure in Appendix B in [40] for a monatomic gas, we obtain the following expressions of $\hat{f}^{(0)}$, $\hat{f}^{(1)}$, and $\hat{\rho}_{\rm w}$ contained in Eq. (93):

$$\hat{f}^{(0)} = \hat{f}_{\rm w}\left\{1 + \epsilon\left[2\mathcal{C}_{{\rm w}i}\frac{\bar{\bar{v}}_i}{\hat{T}_{\rm w}^{1/2}} + \left(\mathcal{C}_{\rm w}^2 - \frac{3}{2}\right)\frac{\bar{\bar{T}}_{\rm tr}}{\hat{T}_{\rm w}} + \left(\bar{\mathcal{E}}_{\rm w} - \frac{\delta}{2}\right)\frac{\bar{\bar{T}}_{\rm int}}{\hat{T}_{\rm w}}\right] + O(\epsilon^2)\right\}, \tag{94a}$$

$$\hat{f}^{(1)} = -\frac{1}{\hat{A}_c(\hat{T}_{\rm w})\hat{\rho}_{\rm B}}\hat{f}_{\rm w}\Bigg\{\frac{1}{1-\nu}\left(\mathcal{C}_{{\rm w}i}\mathcal{C}_{{\rm w}j} - \frac{1}{3}\mathcal{C}_{\rm w}^2\delta_{ij}\right)\left[\left(\frac{\partial\hat{v}_i}{\partial x_j}\right)_{\rm B} + \left(\frac{\partial\hat{v}_j}{\partial x_i}\right)_{\rm B}\right]$$
$$+ \mathcal{C}_{{\rm w}i}\left(\mathcal{C}_{\rm w}^2 - \frac{5}{2}\right)\frac{1}{\hat{T}_{\rm w}^{1/2}}\left(\frac{\partial\hat{T}_{\rm tr}}{\partial x_i}\right)_{\rm B} + \mathcal{C}_{{\rm w}i}\left(\bar{\mathcal{E}}_{\rm w} - \frac{\delta}{2}\right)\frac{1}{\hat{T}_{\rm w}^{1/2}}\left(\frac{\partial\hat{T}_{\rm int}}{\partial x_i}\right)_{\rm B} + O(\epsilon)\Bigg\}, \tag{94b}$$

$$\frac{\hat{\rho}_{\rm w}}{\hat{\rho}_{\rm B}} = 1 + \epsilon\Bigg\{-\sqrt{\pi}\frac{\bar{\bar{v}}_i}{\hat{T}_{\rm w}^{1/2}}n_i + \frac{1}{2}\frac{\bar{\bar{T}}_{\rm tr}}{\hat{T}_{\rm w}}$$
$$- \frac{1}{3}\frac{1}{\hat{A}_c(\hat{T}_{\rm w})}\frac{1}{1-\nu}\frac{1}{\hat{\rho}_{\rm B}}\left[\left(\frac{\partial\hat{v}_i}{\partial x_j}\right)_{\rm B} + \left(\frac{\partial\hat{v}_j}{\partial x_i}\right)_{\rm B}\right]\left[n_i n_j - \frac{1}{2}(\delta_{ij} - n_i n_j)\right]$$
$$- 2\sqrt{\pi}\frac{1}{\Gamma(\delta/2)}\int_{\mathcal{C}_{\rm wn}<0}\int_0^\infty \mathcal{C}_{\rm wn}\phi E(\mathcal{C}_{\rm w})\bar{\mathcal{E}}_{\rm w}^{\delta/2-1}e^{-\bar{\mathcal{E}}_{\rm w}}d\bar{\mathcal{E}}_{\rm w}d\boldsymbol{\mathcal{C}}_{\rm w}\Bigg\} + O(\epsilon^2). \tag{94c}$$

Here, use has been made of Eq. (83) and the new variables $\boldsymbol{\mathcal{C}}_{\rm w}$ and $\bar{\mathcal{E}}_{\rm w}$ introduced in Eq. (79), together with the symbols $\mathcal{C}_{\rm w}$ and $\mathcal{C}_{\rm wn}$ defined by Eq. (80). For the new velocity variable $\boldsymbol{\mathcal{C}}_{\rm w}$, the reflection operator (42) is replaced by the following $\widetilde{\mathcal{R}}$:

$$\widetilde{\mathcal{R}}\hat{g}(\mathcal{C}_{{\rm w}i}) = \hat{g}(\mathcal{C}_{{\rm w}i} - 2\mathcal{C}_{{\rm w}j}n_j n_i), \tag{95}$$

where $\hat{g}(\mathcal{C}_{{\rm w}i})$ is a function of $\mathcal{C}_{{\rm w}i}$. It should be noted that $\widetilde{\mathcal{R}}\hat{f}_{\rm w} = \hat{f}_{\rm w}$ holds.

Then, following the procedure that derived Eq. (93) in [39] from Eqs. (80)–(83) there for the standard (one-temperature) Navier–Stokes equations, we transform Eqs. (93) and (94) further. In particular, one can show that

$$\bar{\bar{\boldsymbol{v}}}\cdot\boldsymbol{n} = O(\epsilon), \tag{96}$$

in the same way as the derivation of Eq. (90) in [39]. In consequence, Eq. (93) is reduced to the following form in terms of ϕ [cf. Eq. (83)]:

$$\phi = (1-a_c)\widetilde{\mathcal{R}}\phi - a_c\left(\mathcal{C}_{\rm w}^2 - 2\right)\frac{\bar{\bar{T}}_{\rm tr}}{\hat{T}_{\rm w}} - a_c\left(\bar{\mathcal{E}}_{\rm w} - \frac{\delta}{2}\right)\frac{\bar{\bar{T}}_{\rm int}}{\hat{T}_{\rm w}} - 2a_c\mathcal{C}_{{\rm w}i}(\delta_{ij} - n_i n_j)\frac{\bar{\bar{v}}_j}{\hat{T}_{\rm w}^{1/2}}$$
$$- 2\sqrt{\pi}a_c\frac{1}{\Gamma(\delta/2)}\int_{\mathcal{C}_{\rm wn}<0}\int_0^\infty \mathcal{C}_{\rm wn}\phi E(\mathcal{C}_{\rm w})\bar{\mathcal{E}}_{\rm w}^{\delta/2-1}e^{-\bar{\mathcal{E}}_{\rm w}}d\bar{\mathcal{E}}_{\rm w}d\boldsymbol{\mathcal{C}}_{\rm w}$$

$$
\begin{aligned}
&+ (2-a_c)\frac{1}{\hat{A}_c(\hat{T}_{\rm w})\hat{\rho}_{\rm B}\hat{T}_{\rm w}^{1/2}}\mathcal{C}_{\rm wn}\left[\left(\mathcal{C}_{\rm w}^2-\frac{5}{2}\right)\left(\frac{\partial\hat{T}_{\rm tr}}{\partial x_i}\right)_{\rm B}n_i+\left(\bar{\mathcal{E}}_{\rm w}-\frac{\delta}{2}\right)\left(\frac{\partial\hat{T}_{\rm int}}{\partial x_i}\right)_{\rm B}n_i\right]\\
&+ a_c\frac{1}{\hat{A}_c(\hat{T}_{\rm w})\hat{\rho}_{\rm B}}\frac{2}{1-\nu}\left(\mathcal{C}_{\rm wn}^2-\frac{1}{3}\mathcal{C}_{\rm w}^2-\frac{1}{3}\right)\left(\frac{\partial\hat{v}_i}{\partial x_j}\right)_{\rm B}n_in_j\\
&+ (2-a_c)\frac{1}{\hat{A}_c(\hat{T}_{\rm w})\hat{\rho}_{\rm B}}\frac{2}{1-\nu}\mathcal{C}_{\rm wn}\mathcal{C}_{{\rm w}i}n_l(\delta_{ij}-n_in_j)\left[\left(\frac{\partial\hat{v}_l}{\partial x_j}\right)_{\rm B}+\left(\frac{\partial\hat{v}_j}{\partial x_l}\right)_{\rm B}\right]\\
&+ a_c\frac{1}{\hat{A}_c(\hat{T}_{\rm w})\hat{\rho}_{\rm B}\hat{T}_{\rm w}^{1/2}}\mathcal{C}_{{\rm w}i}(\delta_{ij}-n_in_j)\left(\mathcal{C}_{\rm w}^2+\bar{\mathcal{E}}_{\rm w}-\frac{5+\delta}{2}\right)\frac{\partial\hat{T}_{\rm w}}{\partial x_j}+O(\epsilon),\\
&\qquad\qquad (y=0,\ \mathcal{C}_{\rm wn}>0). \qquad (97)
\end{aligned}
$$

In the derivation of Eq. (97), use has been made of the formulas (91a) and (91c) in [39], which are also valid in the present paper, as well as the expressions $\mathcal{C}_{{\rm w}i}=\mathcal{C}_{\rm wn}n_i+\mathcal{C}_{{\rm w}j}(\delta_{ij}-n_in_j)$ and $\widetilde{\mathcal{R}}\mathcal{C}_{{\rm w}i}=-\mathcal{C}_{\rm wn}n_i+\mathcal{C}_{{\rm w}j}(\delta_{ij}-n_in_j)$. In addition, because of the relations $\hat{T}_{\rm tr}=\hat{T}_{\rm w}+O(\epsilon)$ and $\hat{T}_{\rm int}=\hat{T}_{\rm w}+O(\epsilon)$ on the boundary [cf. Eq. (60)] and of the fact that $(\delta_{ij}-n_in_j)(\partial S/\partial x_j)_{\rm B}$ consists of tangential derivatives of S on the boundary, we can write

$$
(\delta_{ij}-n_in_j)\left(\frac{\partial\hat{T}_{\rm tr}}{\partial x_j}\right)_{\rm B}=(\delta_{ij}-n_in_j)\frac{\partial\hat{T}_{\rm w}}{\partial x_j}+O(\epsilon), \tag{98a}
$$

$$
(\delta_{ij}-n_in_j)\left(\frac{\partial\hat{T}_{\rm int}}{\partial x_j}\right)_{\rm B}=(\delta_{ij}-n_in_j)\frac{\partial\hat{T}_{\rm w}}{\partial x_j}+O(\epsilon). \tag{98b}
$$

Note that $(\delta_{ij}-n_in_j)\partial\hat{T}_{\rm w}/\partial x_j$ is defined only on the boundary. Equation (98) has been used in the derivation of the term containing $(\delta_{ij}-n_in_j)\partial\hat{T}_{\rm w}/\partial x_j$ in Eq. (97).

D. Summary

If we omit the terms of $O(\epsilon)$ in Eqs. (84) and (97) and take into account Eq. (71b), then we obtain the problem for ϕ. In order to avoid cumbersome notations, we change the names of the variables from $(\boldsymbol{\mathcal{C}}_{\rm w},\bar{\mathcal{E}}_{\rm w})$ to $(\boldsymbol{\zeta},\hat{\mathcal{E}})$ and denote ϕ as a function of $(\tilde{t},\,y,\,\chi_1,\,\chi_2,\,\boldsymbol{\zeta},\,\hat{\mathcal{E}})$, that is,

$$
\boldsymbol{\mathcal{C}}_{\rm w}\to\boldsymbol{\zeta}\ \ (\text{thus}\ \ \mathcal{C}_{\rm wn}\to\zeta_{\rm n}\ \text{and}\ \mathcal{C}_{\rm w}\to\zeta),\qquad \bar{\mathcal{E}}_{\rm w}\to\hat{\mathcal{E}}, \tag{99a}
$$

$$
\phi(\tilde{t},\,y,\,\chi_1,\,\chi_2,\,\boldsymbol{\mathcal{C}}_{\rm w},\,\bar{\mathcal{E}}_{\rm w})\to\phi(\tilde{t},\,y,\,\chi_1,\,\chi_2,\,\boldsymbol{\zeta},\,\hat{\mathcal{E}}). \tag{99b}
$$

Here, $(\boldsymbol{\zeta},\,\hat{\mathcal{E}})$ should not be confused with $(\boldsymbol{\zeta},\,\hat{\mathcal{E}})$ used until Sec. VI C. Then, the equation and the boundary condition for ϕ become as follows:

$$
\zeta_{\rm n}\frac{\partial\phi}{\partial y}=\mathcal{L}_0(\phi),\qquad (y>0), \tag{100a}
$$

$$
\begin{aligned}
\phi=\ &(1-a_c)\widetilde{\mathcal{R}}\phi-a_c\left(\zeta^2-2\right)\frac{\bar{\bar{T}}_{\rm tr}}{\hat{T}_{\rm w}}-a_c\left(\hat{\mathcal{E}}-\frac{\delta}{2}\right)\frac{\bar{\bar{T}}_{\rm int}}{\hat{T}_{\rm w}}-2a_c\zeta_i(\delta_{ij}-n_in_j)\frac{\bar{\bar{v}}_j}{\hat{T}_{\rm w}^{1/2}}\\
&-2\sqrt{\pi}a_c\frac{1}{\Gamma(\delta/2)}\int_{\zeta_{\rm n}<0}\int_0^\infty\zeta_{\rm n}\phi E(\zeta)\hat{\mathcal{E}}^{\delta/2-1}e^{-\hat{\mathcal{E}}}d\hat{\mathcal{E}}d\boldsymbol{\zeta}\\
&+(2-a_c)\frac{1}{\hat{A}_c(\hat{T}_{\rm w})\hat{\rho}_{\rm B}\hat{T}_{\rm w}^{1/2}}\zeta_{\rm n}\left[\left(\zeta^2-\frac{5}{2}\right)\left(\frac{\partial\hat{T}_{\rm tr}}{\partial x_i}\right)_{\rm B}n_i+\left(\hat{\mathcal{E}}-\frac{\delta}{2}\right)\left(\frac{\partial\hat{T}_{\rm int}}{\partial x_i}\right)_{\rm B}n_i\right]\\
&+a_c\frac{1}{\hat{A}_c(\hat{T}_{\rm w})\hat{\rho}_{\rm B}}\frac{2}{1-\nu}\left(\zeta_{\rm n}^2-\frac{1}{3}\zeta^2-\frac{1}{3}\right)\left(\frac{\partial\hat{v}_i}{\partial x_j}\right)_{\rm B}n_in_j\\
&+(2-a_c)\frac{1}{\hat{A}_c(\hat{T}_{\rm w})\hat{\rho}_{\rm B}}\frac{2}{1-\nu}\zeta_{\rm n}\zeta_in_l(\delta_{ij}-n_in_j)\left[\left(\frac{\partial\hat{v}_l}{\partial x_j}\right)_{\rm B}+\left(\frac{\partial\hat{v}_j}{\partial x_l}\right)_{\rm B}\right]
\end{aligned}
$$

$$+ a_c \frac{1}{\hat{A}_c(\hat{T}_\mathrm{w})\hat{\rho}_\mathrm{B}\hat{T}_\mathrm{w}^{1/2}} \zeta_i(\delta_{ij} - n_i n_j) \left(\zeta^2 + \hat{\mathcal{E}} - \frac{5+\delta}{2}\right) \frac{\partial \hat{T}_\mathrm{w}}{\partial x_j}, \qquad (y = 0,\ \zeta_\mathrm{n} > 0), \quad (100\mathrm{b})$$

$$\phi \to 0, \qquad (y \to \infty), \quad (100\mathrm{c})$$

where $\mathcal{L}_0(\phi)$ is defined by Eqs. (85)–(87) with the change of notations (99) being applied. Here and in what follows, the reflection operator $\widetilde{\mathcal{R}}$ indicates $\widetilde{\mathcal{R}}\hat{g}(\zeta_i) = \hat{g}(\zeta_i - 2\zeta_j n_j n_i)$ because of Eqs. (95) and (99).

The problem (100) is a steady boundary-value problem of the linearized ES model for a polyatomic gas with $\theta = 0$ in the half space $y > 0$. It looks similar to the corresponding problem in [39] for the ES model with $\theta = O(1)$ [Eq. (95) in [39]]. However, there is a significant difference. In the latter case, the boundary condition contains two quantities $\bar{\bar{T}}$ and $\bar{\boldsymbol{v}}$, instead of the three quantities $\bar{\bar{T}}_\mathrm{tr}$, $\bar{\bar{T}}_\mathrm{int}$, and $\bar{\boldsymbol{v}}$ in Eq. (100b), and the problem has the unique solution only when $\bar{\bar{T}}$ and $\bar{\boldsymbol{v}}$ are related to the derivatives $(\partial\hat{T}/\partial x_j)_\mathrm{B}$ and $(\partial\hat{v}_i/\partial x_j)_\mathrm{B}$ appropriately. These relations provided the so-called slip boundary conditions for the standard compressible Navier–Stokes equations with the single temperature and with the bulk viscosity [39]. This structure is analogous to the corresponding half-space problem of the linearized Boltzmann equation for a monatomic gas, which has been studied mathematically [60–64] and whose mathematical structure, such as the existence and uniqueness of the solution, has been well understood. Numerical analysis of some relevant problems can also be found in the literature (e.g., [65, 66]).

As mentioned above, the present problem, Eq. (100), contains the three quantities $\bar{\bar{T}}_\mathrm{tr}$, $\bar{\bar{T}}_\mathrm{int}$, and $\bar{\boldsymbol{v}}$ in the boundary condition. This difference from the corresponding problem in [39] is due to the following fact: The equilibrium solution $\mathcal{L}_0(\phi) = 0$ is the six-parameter family given by $\phi = c_0 + c_1\zeta_1 + c_2\zeta_2 + c_3\zeta_3 + c_4\zeta^2 + c_5\hat{\mathcal{E}}$ [cf. Eq. (90)], whereas the equilibrium solution $\mathcal{L}(\phi) = 0$, where $\mathcal{L}(\,\cdot\,)$ is the linearized collision operator of the ES model defined by Eq. (32) in [39] [recall that $\mathcal{L}_0(\,\cdot\,) = \mathcal{L}(\,\cdot\,)|_{\theta=0}$], is the five-parameter family $\phi = c_0 + c_1\zeta_1 + c_2\zeta_2 + c_3\zeta_3 + c_4(\zeta^2 + \hat{\mathcal{E}})$. In analogy with the case of [39], we can expect that the problem (100) has the unique solution only when $\bar{\bar{T}}_\mathrm{tr}$, $\bar{\bar{T}}_\mathrm{int}$, and $\bar{\boldsymbol{v}}$ are related to the derivatives $(\partial\hat{T}_\mathrm{tr}/\partial x_j)_\mathrm{B}$, $(\partial\hat{T}_\mathrm{int}/\partial x_j)_\mathrm{B}$, $(\partial\hat{v}_i/\partial x_j)_\mathrm{B}$, and $(\delta_{ij} - n_i n_j)\partial\hat{T}_\mathrm{w}/\partial x_j$, appropriately. These relations provide the desired boundary conditions for the two-temperature Navier–Stokes equations, as we will see in the following section.

We have not mentioned the initial condition for the Knudsen-layer equation so far. Since the time-derivative term is not contained in Eq. (100a), we cannot impose the initial condition to this equation. However, we can show that the problem (100) is consistent with the initial condition (55) for the two-temperature Navier–Stokes equations and assumption (v) in Sec. II A. For the detailed discussion on this point, the reader is referred to Sec. 5.2.4 in [40].

VII. SLIP BOUNDARY CONDITIONS

In this section, we analyze the Knudsen-layer problem (100) to establish the slip boundary conditions for the two-temperature Navier–Stokes equations (6) basically following the descriptions in [39].

A. Decomposition of Knudsen-layer problem

We first introduce the following reduced velocity distribution functions:

$$\begin{bmatrix} \Phi(\tilde{t},\, y,\, \chi_1,\, \chi_2,\, \boldsymbol{\zeta}) \\ \Psi(\tilde{t},\, y,\, \chi_1,\, \chi_2,\, \boldsymbol{\zeta}) \end{bmatrix} = \frac{1}{\Gamma(\delta/2)} \int_0^\infty \begin{bmatrix} \hat{\mathcal{E}}^{\delta/2-1} \\ (2/\delta)\hat{\mathcal{E}}^{\delta/2} \end{bmatrix} \phi(\tilde{t},\, y,\, \chi_1,\, \chi_2,\, \boldsymbol{\zeta},\, \hat{\mathcal{E}}) e^{-\hat{\mathcal{E}}} d\hat{\mathcal{E}}. \qquad (101)$$

Let us integrate each of Eqs. (100a), (100b), and (100c), multiplied by $[\Gamma(\delta/2)]^{-1}\hat{\mathcal{E}}^{\delta/2-1}e^{-\hat{\mathcal{E}}}$, with respect to $\hat{\mathcal{E}}$ from 0 to ∞ and make use of Eq. (101). Then, we obtain the closed set of

equation and boundary conditions for Φ of the following form:

$$\zeta_{\rm n}\frac{\partial\Phi}{\partial y}=\mathcal{L}_{\rm ES}(\Phi),\qquad (y>0), \tag{102a}$$

$$\begin{aligned}\Phi=&(1-a_c)\widetilde{\mathcal{R}}\Phi-a_c\left(\zeta^2-2\right)\frac{\bar{\bar{T}}_{\rm tr}}{\hat{T}_{\rm w}}-2a_c\zeta_i(\delta_{ij}-n_in_j)\frac{\bar{\bar{v}}_j}{\hat{T}_{\rm w}^{1/2}}\\&-2\sqrt{\pi}a_c\int_{\zeta_{\rm n}<0}\zeta_{\rm n}\Phi E(\zeta)d\boldsymbol{\zeta}\\&+(2-a_c)\frac{1}{\hat{A}_c(\hat{T}_{\rm w})\hat{\rho}_{\rm B}\hat{T}_{\rm w}^{1/2}}\zeta_{\rm n}\left(\zeta^2-\frac{5}{2}\right)\left(\frac{\partial\hat{T}_{\rm tr}}{\partial x_i}\right)_{\rm B}n_i\\&+a_c\frac{1}{\hat{A}_c(\hat{T}_{\rm w})\hat{\rho}_{\rm B}}\frac{2}{1-\nu}\left(\zeta_{\rm n}^2-\frac{1}{3}\zeta^2-\frac{1}{3}\right)\left(\frac{\partial\hat{v}_i}{\partial x_j}\right)_{\rm B}n_in_j\\&+(2-a_c)\frac{1}{\hat{A}_c(\hat{T}_{\rm w})\hat{\rho}_{\rm B}}\frac{2}{1-\nu}\zeta_{\rm n}\zeta_in_l(\delta_{ij}-n_in_j)\left[\left(\frac{\partial\hat{v}_l}{\partial x_j}\right)_{\rm B}+\left(\frac{\partial\hat{v}_j}{\partial x_l}\right)_{\rm B}\right]\\&+a_c\frac{1}{\hat{A}_c(\hat{T}_{\rm w})\hat{\rho}_{\rm B}\hat{T}_{\rm w}^{1/2}}\zeta_i(\delta_{ij}-n_in_j)\left(\zeta^2-\frac{5}{2}\right)\frac{\partial\hat{T}_{\rm w}}{\partial x_j},\qquad (y=0,\ \zeta_{\rm n}>0),\end{aligned} \tag{102b}$$

$$\Phi\to 0,\qquad (y\to\infty). \tag{102c}$$

Here,

$$\begin{aligned}\mathcal{L}_{\rm ES}(\Phi)&=\frac{1}{\Gamma(\delta/2)}\int_0^\infty\mathcal{L}_0(\phi)\hat{\mathcal{E}}^{\delta/2-1}e^{-\hat{\mathcal{E}}}d\hat{\mathcal{E}}\\&=\omega+2\zeta_iu_i+\left(\zeta^2-\frac{3}{2}\right)\tau_{\rm tr}+\nu\left(\zeta_i\zeta_j-\frac{1}{3}\zeta^2\delta_{ij}\right)P_{ij}-\Phi,\end{aligned} \tag{103}$$

where

$$\omega=\langle\Phi\rangle,\qquad u_i=\langle\zeta_i\Phi\rangle,\qquad \tau_{\rm tr}=\frac{2}{3}\left\langle\left(\zeta^2-\frac{3}{2}\right)\Phi\right\rangle,\qquad P_{ij}=2\langle\zeta_i\zeta_j\Phi\rangle, \tag{104}$$

and $\langle\,\cdot\,\rangle$ is defined, for an arbitrary function $\hat{g}(\boldsymbol{\zeta})$ of $\boldsymbol{\zeta}$, by

$$\langle\,\hat{g}(\boldsymbol{\zeta})\,\rangle=\int_{\mathbb{R}^3}\hat{g}(\boldsymbol{\zeta})E(\zeta)d\boldsymbol{\zeta}. \tag{105}$$

It should be emphasized here that $\mathcal{L}_{\rm ES}(\,\cdot\,)$ is nothing but the linearized collision operator of the ES model for a monatomic gas. It follows from Eq. (91) [with the replacement (99)] that, for an arbitrary function $g(\zeta_i)$,

$$\langle\mathcal{L}_{\rm ES}(g)\rangle=\langle\zeta_i\mathcal{L}_{\rm ES}(g)\rangle=\langle\zeta^2\mathcal{L}_{\rm ES}(g)\rangle=0. \tag{106}$$

In [40], the slip boundary conditions for the compressible Navier–Stokes equations were derived for a monatomic gas on the basis of the Boltzmann equation including the BGK model. Although the ES model was not considered there, it is not difficult to see that the problem (102) is the Knudsen-layer problem for the ES model for a monatomic gas if $\hat{T}_{\rm tr}$ is regarded as the temperature $\hat{T}$ [cf. Eqs. (100) and (101) in [40]; note that for a monatomic gas, the ES model reduces to the BGK model when $\nu=0$]. Therefore, the problem (102) can be handled in the same way as in [40].

Next, we integrate each of Eqs. (100a), (100b), and (100c), multiplied by $(2/\delta)[\Gamma(\delta/2)]^{-1}\hat{\mathcal{E}}^{\delta/2}e^{-\hat{\mathcal{E}}}$, with respect to $\hat{\mathcal{E}}$ from 0 to ∞ and make use of Eq. (101). Then, from the resulting equations, we subtract Eqs. (102a)–(102c), respectively. As the result, we obtain the problem for the

difference $\Psi - \Phi$, i.e.,

$$\zeta_{\rm n}\frac{\partial \Upsilon}{\partial y} = \langle \Upsilon \rangle - \Upsilon, \qquad (y > 0), \tag{107a}$$

$$\Upsilon = (1-a_c)\widetilde{\mathcal{R}}\Upsilon - a_c\frac{\bar{\bar{T}}_{\rm int}}{\hat{T}_{\rm w}} + (2-a_c)\frac{1}{\hat{A}_c(\hat{T}_{\rm w})\hat{\rho}_{\rm B}\hat{T}_{\rm w}^{1/2}}\zeta_{\rm n}\left(\frac{\partial \hat{T}_{\rm int}}{\partial x_i}\right)_{\rm B} n_i$$
$$+ a_c\frac{1}{\hat{A}_c(\hat{T}_{\rm w})\hat{\rho}_{\rm B}\hat{T}_{\rm w}^{1/2}}\zeta_i(\delta_{ij}-n_in_j)\frac{\partial \hat{T}_{\rm w}}{\partial x_j}, \qquad (y=0,\ \zeta_{\rm n}>0), \tag{107b}$$

$$\Upsilon \to 0, \qquad (y \to \infty), \tag{107c}$$

where Υ is defined by the difference

$$\Upsilon(\tilde{t},\, y,\, \chi_1,\, \chi_2,\, \boldsymbol{\zeta}) = \Psi(\tilde{t},\, y,\, \chi_1,\, \chi_2,\, \boldsymbol{\zeta}) - \Phi(\tilde{t},\, y,\, \chi_1,\, \chi_2,\, \boldsymbol{\zeta}). \tag{108}$$

Let us consider the problem (102). As discussed in Sec. 5.3.1 in [40], if the terms containing the boundary values of the derivatives $(\partial \hat{T}_{\rm tr}/\partial x_i)_{\rm B}$ and $(\partial \hat{v}_i/\partial x_j)_{\rm B}$ and the tangential derivative of the boundary temperature $(\delta_{ij}-n_in_j)(\partial \hat{T}_{\rm w}/\partial x_j)$ in Eq. (102b) are all set to be zero, then the problem (102) has a trivial solution $\Phi = 0$, $\bar{\bar{T}}_{\rm tr} = 0$, and $\bar{\bar{v}}_i = 0$, which should be unique in analogy with the case of the linearized Boltzmann equation for a monatomic gas [61]. Therefore, these terms are regarded as the inhomogeneous terms, and $\bar{\bar{T}}_{\rm tr}$ and $\bar{\bar{v}}_i$ are a part of the solution. That is, Φ as well as $\bar{\bar{T}}_{\rm tr}$ and $\bar{\bar{v}}_i$ is determined depending on the inhomogeneous terms. Because of its linearity, the problem (102) can be decomposed in accordance with the form of the inhomogeneous terms.

Here, we note that $\zeta_i(\delta_{ij}-n_in_j)$ in the last two lines of Eq. (102b) indicates the tangential component of $\boldsymbol{\zeta}$, i.e., the projection of $\boldsymbol{\zeta}$ onto the plane tangent to the boundary. From the form of the inhomogeneous terms in Eq. (102b), we assume the solution Φ in the following form:

$$\begin{aligned}
&\Phi(\tilde{t},\, y, \chi_1,\, \chi_2,\, \boldsymbol{\zeta}) \\
&\quad = \frac{1}{\hat{A}_c(\hat{T}_{\rm w})\hat{\rho}_{\rm B}}\left[\left(\frac{\partial \hat{v}_l}{\partial x_j}\right)_{\rm B} + \left(\frac{\partial \hat{v}_j}{\partial x_l}\right)_{\rm B}\right]\zeta_i n_l(\delta_{ij}-n_in_j)\,\Phi_v^I(y,\, \zeta_{\rm n},\, \zeta) \\
&\qquad + \frac{1}{\hat{A}_c(\hat{T}_{\rm w})\hat{\rho}_{\rm B}\hat{T}_{\rm w}^{1/2}}\frac{\partial \hat{T}_{\rm w}}{\partial x_j}\,\zeta_i(\delta_{ij}-n_in_j)\,\Phi_T^I(y,\, \zeta_{\rm n},\, \zeta) \\
&\qquad + \frac{1}{\hat{A}_c(\hat{T}_{\rm w})\hat{\rho}_{\rm B}}\left(\frac{\partial \hat{v}_i}{\partial x_j}\right)_{\rm B} n_in_j\,\Phi_v^{II}(y,\, \zeta_{\rm n},\, \zeta) \\
&\qquad + \frac{1}{\hat{A}_c(\hat{T}_{\rm w})\hat{\rho}_{\rm B}\hat{T}_{\rm w}^{1/2}}\left(\frac{\partial \hat{T}_{\rm tr}}{\partial x_i}\right)_{\rm B} n_i\Phi_T^{II}(y,\, \zeta_{\rm n},\, \zeta).
\end{aligned} \tag{109}$$

Then, we also set the unknown parameters $\bar{\bar{v}}_j$ and $\bar{\bar{T}}_{\rm tr}$ in the form consistent with the form of the macroscopic quantities in the inhomogeneous terms. To be more specific, the tangential vector $(\delta_{ij}-n_in_j)\bar{\bar{v}}_j$ should be related to the (macroscopic) tangential vectors in the inhomogeneous terms, i.e., $(\delta_{ij}-n_in_j)[(\partial \hat{v}_l/\partial x_j)_{\rm B} + (\partial \hat{v}_j/\partial x_l)_{\rm B}]n_l$ and $(\delta_{ij}-n_in_j)(\partial \hat{T}_{\rm w}/\partial x_j)$, whereas the scalar $\bar{\bar{T}}_{\rm tr}$ should be related to the (macroscopic) scalars in the inhomogeneous terms, i.e., $(\partial \hat{v}_i/\partial x_j)_{\rm B}n_in_j$ and $(\partial \hat{T}_{\rm tr}/\partial x_i)_{\rm B}n_i$. To summarize, we let

$$\frac{\bar{\bar{v}}_j}{\hat{T}_{\rm w}^{1/2}}(\delta_{ij}-n_in_j) = c_v^I\frac{1}{\hat{A}_c(\hat{T}_{\rm w})\hat{\rho}_{\rm B}}\left[\left(\frac{\partial \hat{v}_l}{\partial x_j}\right)_{\rm B} + \left(\frac{\partial \hat{v}_j}{\partial x_l}\right)_{\rm B}\right]n_l(\delta_{ij}-n_in_j)$$
$$+ c_T^I\frac{1}{\hat{A}_c(\hat{T}_{\rm w})\hat{\rho}_{\rm B}\hat{T}_{\rm w}^{1/2}}\frac{\partial \hat{T}_{\rm w}}{\partial x_j}\,(\delta_{ij}-n_in_j), \tag{110a}$$

$$\frac{\bar{\bar{T}}_{\rm tr}}{\hat{T}_{\rm w}} = c_v^{II}\frac{1}{\hat{A}_c(\hat{T}_{\rm w})\hat{\rho}_{\rm B}}\left(\frac{\partial \hat{v}_i}{\partial x_j}\right)_{\rm B} n_in_j + c_T^{II}\frac{1}{\hat{A}_c(\hat{T}_{\rm w})\hat{\rho}_{\rm B}\hat{T}_{\rm w}^{1/2}}\left(\frac{\partial \hat{T}_{\rm tr}}{\partial x_i}\right)_{\rm B} n_i, \tag{110b}$$

where c_v^I, c_T^I, c_v^{II}, and c_T^{II} are undetermined constants that depend on the properties of the gas as well as on the accommodation coefficient a_c and are determined together with the solutions Φ_v^I, Φ_T^I, Φ_v^{II}, and Φ_T^{II}.

Similarly, the problem (107) can be decomposed by letting

$$\begin{aligned}\Upsilon(\tilde{t},\, y, \chi_1,\, \chi_2,\, \boldsymbol{\zeta})&\\ =&\frac{1}{\hat{A}_c(\hat{T}_\mathrm{w})\hat{\rho}_\mathrm{B}\hat{T}_\mathrm{w}^{1/2}}\frac{\partial \hat{T}_\mathrm{w}}{\partial x_j}\,\zeta_i(\delta_{ij}-n_in_j)\,\Upsilon_T^I(y,\,\zeta_\mathrm{n},\,\zeta)\\ &+\frac{1}{\hat{A}_c(\hat{T}_\mathrm{w})\hat{\rho}_\mathrm{B}\hat{T}_\mathrm{w}^{1/2}}\left(\frac{\partial \hat{T}_\mathrm{int}}{\partial x_i}\right)_\mathrm{B} n_i\Upsilon_T^{II}(y,\,\zeta_\mathrm{n},\,\zeta),\end{aligned}\tag{111}$$

and

$$\frac{\bar{\hat{T}}_\mathrm{int}}{\hat{T}_\mathrm{w}}=\tilde{c}_T^{II}\frac{1}{\hat{A}_c(\hat{T}_\mathrm{w})\hat{\rho}_\mathrm{B}\hat{T}_\mathrm{w}^{1/2}}\left(\frac{\partial \hat{T}_\mathrm{int}}{\partial x_i}\right)_\mathrm{B} n_i,\tag{112}$$

where $\tilde{c}_T^{II}$ is an undetermined constant depending on the properties of the gas as well as on the accommodation coefficient a_c and is determined together with the solution Υ_T^{II}. Since $\bar{\hat{T}}_\mathrm{int}$ is a scalar, it cannot be related to the tangential vector $(\delta_{ij}-n_in_j)(\partial\hat{T}_\mathrm{w}/\partial x_j)$. Therefore, $\bar{\hat{T}}_\mathrm{int}$ must have the form of Eq. (112). In other words, the problem for Υ_T^I does not contain any undetermined constant.

If we substitute Eqs. (109) and (110) into Eq. (102) and substitute Eqs. (111) and (112) into Eq. (107), we obtain six decomposed problems for Φ_v^I, Φ_T^I, Φ_v^{II}, Φ_T^{II}, Υ_T^I, and Υ_T^{II}. Once they have been determined together with the constants c_v^I, c_T^I, c_v^{II}, c_T^{II}, and $\tilde{c}_T^{II}$, Eqs. (110) and (112) provide the desired slip boundary conditions for the two temperature Navier–Stokes equations. The assumption that Φ_v^I, Φ_T^I, Φ_v^{II}, Φ_T^{II}, Υ_T^I, and Υ_T^{II} are all functions of y, ζ_n, and ζ will turn out to be consistent.

Before presenting the decomposed problems, we need a small preparation. Let us introduce two unit vectors $\boldsymbol{s}$ and $\boldsymbol{t}$ tangent to the boundary and orthogonal to each other, i.e., $\boldsymbol{n}\cdot\boldsymbol{s}=\boldsymbol{n}\cdot\boldsymbol{t}=\boldsymbol{s}\cdot\boldsymbol{t}=0$, and denote the two orthogonal tangential components of $\boldsymbol{\zeta}$ by $\zeta_\mathrm{s}=\boldsymbol{\zeta}\cdot\boldsymbol{s}$ and $\zeta_\mathrm{t}=\boldsymbol{\zeta}\cdot\boldsymbol{t}$. Then, we have the expressions, such as $\zeta_i=\zeta_\mathrm{n}n_i+\zeta_\mathrm{s}s_i+\zeta_\mathrm{t}t_i$, $\zeta^2=\zeta_\mathrm{n}^2+\zeta_\mathrm{s}^2+\zeta_\mathrm{t}^2$, and $n_in_j+s_is_j+t_it_j=\delta_{ij}$. With these relations, it is easy to verify the following relation for any function $g=g(\zeta_\mathrm{n},\,\zeta)$:

$$\mathcal{L}_\mathrm{ES}[\zeta_i(\delta_{ij}-n_in_j)g(\zeta_\mathrm{n},\,\zeta)]=\zeta_i(\delta_{ij}-n_in_j)\mathcal{L}_\mathrm{ES}^S[g(\zeta_\mathrm{n},\,\zeta)],\tag{113}$$

where

$$\mathcal{L}_\mathrm{ES}^S[g(\zeta_\mathrm{n},\,\zeta)]=\langle(\zeta^2-\zeta_\mathrm{n}^2)g\rangle+2\nu\zeta_\mathrm{n}\langle\zeta_\mathrm{n}(\zeta^2-\zeta_\mathrm{n}^2)g\rangle-g.\tag{114}$$

Now we list the resulting six decomposed problems for Φ_v^I, Φ_T^I, Φ_v^{II}, Φ_T^{II}, Υ_T^I, and Υ_T^{II}.

(i) Problems for $(\Phi_\kappa^I;\, c_\kappa^I)$ $(\kappa=v,\,T)$:

$$\zeta_\mathrm{n}\frac{\partial\Phi_\kappa^I}{\partial y}=\mathcal{L}_\mathrm{ES}^S(\Phi_\kappa^I),\qquad (y>0),\tag{115a}$$

$$\Phi_\kappa^I=(1-a_c)\widetilde{\mathcal{R}}\Phi_\kappa^I-2a_cc_\kappa^I+H_\kappa^I,\qquad (y=0,\,\zeta_\mathrm{n}>0),\tag{115b}$$

$$\Phi_\kappa^I\to 0,\qquad (y\to\infty),\tag{115c}$$

where

$$H_v^I=(2-a_c)\,\frac{2}{1-\nu}\zeta_\mathrm{n},\qquad H_T^I=a_c\left(\zeta^2-\frac{5}{2}\right).\tag{116}$$

If we take $\langle\cdot\rangle$ of Eq. (115a) multiplied by $\zeta^2-\zeta_\mathrm{n}^2$ and note that $\langle\zeta^2-\zeta_\mathrm{n}^2\rangle=1$, we have $d\langle\zeta_\mathrm{n}(\zeta^2-\zeta_\mathrm{n}^2)\Phi_\kappa^I\rangle/dy=0$. Then, it follows from Eq. (115c) that

$$\langle\zeta_\mathrm{n}(\zeta^2-\zeta_\mathrm{n}^2)\Phi_\kappa^I\rangle=0.\tag{117}$$

(ii) Problems for $(\Phi_\kappa^{II};\, c_\kappa^{II})$ $(\kappa = v,\, T)$:

$$\zeta_{\rm n}\frac{\partial \Phi_\kappa^{II}}{\partial y} = \mathcal{L}_{\rm ES}(\Phi_\kappa^{II}), \qquad (y>0), \tag{118a}$$

$$\Phi_\kappa^{II} = (1-a_c)\widetilde{\mathcal{R}}\Phi_\kappa^{II} - 2\sqrt{\pi}\,a_c\int_{\zeta_{\rm n}<0}\zeta_{\rm n}\Phi_\kappa^{II}E(\zeta)d\boldsymbol{\zeta} - a_c\left(\zeta^2-2\right)c_\kappa^{II} + H_\kappa^{II}, \quad (y=0,\ \zeta_{\rm n}>0), \tag{118b}$$

$$\Phi_\kappa^{II} \to 0, \qquad (y\to\infty), \tag{118c}$$

where

$$H_v^{II} = a_c\,\frac{2}{1-\nu}\left(\zeta_{\rm n}^2 - \frac{1}{3}\zeta^2 - \frac{1}{3}\right), \qquad H_T^{II} = (2-a_c)\zeta_{\rm n}\left(\zeta^2-\frac{5}{2}\right). \tag{119}$$

Taking $\langle\,\cdot\,\rangle$ of Eq. (118a) multiplied by $(1,\,\zeta_{\rm n},\,\zeta^2)$ and taking account of Eq. (106), we have $d\langle\zeta_{\rm n}\Phi_\kappa^{II}\rangle/dy = d\langle\zeta_{\rm n}^2\Phi_\kappa^{II}\rangle/dy = d\langle\zeta_{\rm n}\zeta^2\Phi_\kappa^{II}\rangle/dy = 0$. Then, from Eq. (118c), we obtain

$$\langle\zeta_{\rm n}\Phi_\kappa^{II}\rangle = \langle\zeta_{\rm n}^2\Phi_\kappa^{II}\rangle = \langle\zeta_{\rm n}\zeta^2\Phi_\kappa^{II}\rangle = 0. \tag{120}$$

(iii) Problem for Υ_T^I:

$$\zeta_{\rm n}\frac{\partial \Upsilon_T^I}{\partial y} = -\Upsilon_T^I, \qquad (y>0), \tag{121a}$$

$$\Upsilon_T^I = (1-a_c)\widetilde{\mathcal{R}}\Upsilon_T^I + a_c, \qquad (y=0,\ \zeta_{\rm n}>0), \tag{121b}$$

$$\Upsilon_T^I \to 0, \qquad (y\to\infty). \tag{121c}$$

(iv) Problem for $(\Upsilon_T^{II};\, \tilde{c}_T^{II})$:

$$\zeta_{\rm n}\frac{\partial \Upsilon_T^{II}}{\partial y} = \langle\Upsilon_T^{II}\rangle - \Upsilon_T^{II}, \qquad (y>0), \tag{122a}$$

$$\Upsilon_T^{II} = (1-a_c)\widetilde{\mathcal{R}}\Upsilon_T^{II} - a_c\tilde{c}_T^{II} + (2-a_c)\zeta_{\rm n}, \qquad (y=0,\ \zeta_{\rm n}>0), \tag{122b}$$

$$\Upsilon_T^{II} \to 0, \qquad (y\to\infty). \tag{122c}$$

Taking $\langle\,\cdot\,\rangle$ of Eq. (122a) and taking Eq. (122c) into account, we have

$$\langle\zeta_{\rm n}\Upsilon_T^{II}\rangle = 0. \tag{123}$$

B. Further reduction of decomposed problems

The independent variables in the decomposed problems listed in Sec. VII A are y, $\zeta_{\rm n}$, and ζ. In the present section, we will further reduce the independent variables to y and $\zeta_{\rm n}$ using the conventional procedure first introduced for the BGK model [67]. For this purpose, we introduce the following reduced velocity distribution functions $(\varphi_v^{II},\ \varphi_T^{II})$, $(\psi_v^I,\ \psi_T^I,\ \psi_v^{II},\ \psi_T^{II})$, and $(\upsilon_T^I,\ \upsilon_T^{II})$:

$$\varphi_\kappa^{II}(y,\,\zeta_{\rm n}) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi_\kappa^{II}(y,\,\zeta_{\rm n},\,\zeta)e^{-\zeta_{\rm s}^2-\zeta_{\rm t}^2}d\zeta_{\rm t}d\zeta_{\rm s}, \tag{124a}$$

$$\psi_\kappa^{N}(y,\,\zeta_{\rm n}) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\zeta_{\rm s}^2+\zeta_{\rm t}^2)\Phi_\kappa^{N}(y,\,\zeta_{\rm n},\,\zeta)e^{-\zeta_{\rm s}^2-\zeta_{\rm t}^2}d\zeta_{\rm t}d\zeta_{\rm s}, \tag{124b}$$

$$\upsilon_T^{N}(y,\,\zeta_{\rm n}) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Upsilon_T^{N}(y,\,\zeta_{\rm n},\,\zeta)e^{-\zeta_{\rm s}^2-\zeta_{\rm t}^2}d\zeta_{\rm t}d\zeta_{\rm s}, \tag{124c}$$

$$(\kappa = v,\,T,\ \text{and}\ N = I,\,II).$$

Here, we have expressed $\boldsymbol{\zeta}$ using the components $\zeta_{\rm n}$, $\zeta_{\rm s}$, and $\zeta_{\rm t}$, i.e., $\boldsymbol{\zeta} = \zeta_{\rm n}\boldsymbol{n} + \zeta_{\rm s}\boldsymbol{s} + \zeta_{\rm t}\boldsymbol{t}$, so that $\zeta^2 - \zeta_{\rm n}^2 = \zeta_{\rm s}^2 + \zeta_{\rm t}^2$ holds.

Equations (117), (120), and (123) are, respectively, expressed in terms of the reduced velocity distribution functions as follows:

$$\int_{-\infty}^{\infty} \zeta_{\rm n} \psi_\kappa^I e^{-\zeta_{\rm n}^2} d\zeta_{\rm n} = 0, \tag{125a}$$

$$\int_{-\infty}^{\infty} \zeta_{\rm n} \begin{bmatrix} \varphi_\kappa^{II} \\ \zeta_{\rm n} \varphi_\kappa^{II} \\ \zeta_{\rm n}^2 \varphi_\kappa^{II} + \psi_\kappa^{II} \end{bmatrix} e^{-\zeta_{\rm n}^2} d\zeta_{\rm n} = 0, \tag{125b}$$

$$\int_{-\infty}^{\infty} \zeta_{\rm n} \upsilon_T^{II} e^{-\zeta_{\rm n}^2} d\zeta_{\rm n} = 0, \tag{125c}$$

$$(\kappa = v,\, T).$$

Then, by the use of Eq. (125a), $\mathcal{L}_{\rm ES}^S(\Phi_\kappa^I)$ $(\kappa = v,\, T)$ [cf. Eq. (114)] is expressed as

$$\mathcal{L}_{\rm ES}^S(\Phi_\kappa^I) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \psi_\kappa^I e^{-\zeta_{\rm n}^2} d\zeta_{\rm n} - \Phi_\kappa^I, \qquad (\kappa = v,\, T), \tag{126}$$

and with the help of Eq. (125b), $\mathcal{L}_{\rm ES}(\Phi_\kappa^{II})$ $(\kappa = v,\, T)$ [cf. Eq. (103)] is expressed as follows:

$$\mathcal{L}_{\rm ES}(\Phi_\kappa^{II}) = \omega + \left(\zeta^2 - \frac{3}{2}\right) \tau_{\rm tr} + \nu \left(\zeta_i \zeta_j - \frac{1}{3} \zeta^2 \delta_{ij}\right) P_{ij} - \Phi_\kappa^{II}, \qquad (\kappa = v,\, T), \tag{127}$$

with

$$\omega = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi_\kappa^{II} e^{-\zeta_{\rm n}^2} d\zeta_{\rm n}, \tag{128a}$$

$$\tau_{\rm tr} = -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi_\kappa^{II} e^{-\zeta_{\rm n}^2} d\zeta_{\rm n} + \frac{2}{3} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \psi_\kappa^{II} e^{-\zeta_{\rm n}^2} d\zeta_{\rm n}, \tag{128b}$$

$$P_{ij} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \psi_\kappa^{II} e^{-\zeta_{\rm n}^2} d\zeta_{\rm n} \, (\delta_{ij} - n_i n_j). \tag{128c}$$

Here, we have omitted the subscript κ and superscript II for the macroscopic quantities ω, P_{ij}, and $\tau_{\rm tr}$ to avoid cumbersome notation. In addition, $\langle \Upsilon_T^{II} \rangle$ in Eq. (122a) is reduced to

$$\langle \Upsilon_T^{II} \rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \upsilon_T^{II} e^{-\zeta_{\rm n}^2} d\zeta_{\rm n}. \tag{129}$$

With these preparations, we consider the problems for $(\Phi_\kappa^I;\, c_\kappa^I)$ and $(\Phi_\kappa^{II};\, c_\kappa^{II})$ $(\kappa = v,\, T)$ as well as those for Υ_T^I and $(\Upsilon_T^{II};\, \tilde{c}_T^{II})$ listed in Sec. VII A.

(i) Problems for $(\Phi_\kappa^I;\, c_\kappa^I)$ $(\kappa = v,\, T)$:

Let us multiply each of Eqs. (115a) [with Eq. (126)], (115b), and (115c) by $(1/\pi)(\zeta_{\rm s}^2 + \zeta_{\rm t}^2)\, e^{-\zeta_{\rm s}^2 - \zeta_{\rm t}^2}$ and integrate the resulting equations with respect to $\zeta_{\rm s}$ and $\zeta_{\rm t}$ from $-\infty$ to ∞ for both variables. In addition, we let

$$(\psi_v^I,\, c_v^I) = \frac{1}{1-\nu} (\bar{\psi}_v^I,\, \bar{c}_v^I), \qquad (\psi_T^I,\, c_T^I) = (\bar{\psi}_T^I,\, \bar{c}_T^I). \tag{130}$$

Then, we obtain the following half-space problems for $(\bar{\psi}_\kappa^I;\, \bar{c}_\kappa^I)$ $(\kappa = v,\, T)$:

$$\zeta_{\rm n} \frac{\partial \bar{\psi}_\kappa^I}{\partial y} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \bar{\psi}_\kappa^I e^{-\zeta_{\rm n}^2} d\zeta_{\rm n} - \bar{\psi}_\kappa^I, \qquad (y > 0), \tag{131a}$$

$$\bar{\psi}_\kappa^I = (1 - a_c) \widetilde{\mathcal{R}} \bar{\psi}_\kappa^I - 2 a_c \bar{c}_\kappa^I + G_\kappa^I, \qquad (y = 0,\, \zeta_{\rm n} > 0), \tag{131b}$$

$$\bar{\psi}_\kappa^I \to 0, \qquad (y \to \infty), \tag{131c}$$

where

$$G_v^I = 2(2 - a_c)\, \zeta_{\rm n}, \qquad G_T^I = a_c \left(\zeta_{\rm n}^2 - \frac{1}{2}\right), \tag{132}$$

and $\widetilde{\mathcal{R}} \bar{\psi}_\kappa^I(0,\, \zeta_{\rm n}) = \bar{\psi}_\kappa^I(0,\, -\zeta_{\rm n})$.

(ii) Problems for $(\Phi_\kappa^{II};\, c_\kappa^{II})$ $(\kappa = v,\, T)$:

We multiply each of Eqs. (118a) [with Eq. (127)], (118b), and (118c) by $(1/\pi)(1,\, \zeta_{\rm s}^2 + \zeta_{\rm t}^2)\, e^{-\zeta_{\rm s}^2-\zeta_{\rm t}^2}$ and integrate the resulting equations with respect to $\zeta_{\rm s}$ and $\zeta_{\rm t}$ from $-\infty$ to ∞ for both variables. Then, we obtain the following half-space problems for $(\varphi_\kappa^{II},\, \psi_\kappa^{II};\, c_\kappa^{II})$ $(\kappa = v,\, T)$:

$$\zeta_{\rm n}\frac{\partial}{\partial y}\begin{bmatrix}\varphi_\kappa^{II}\\ \psi_\kappa^{II}\end{bmatrix} = \omega\begin{bmatrix}1\\ 1\end{bmatrix} + \tau_{\rm tr}\begin{bmatrix}\zeta_{\rm n}^2-\frac{1}{2}\\ \zeta_{\rm n}^2+\frac{1}{2}\end{bmatrix} - \frac{1}{3}\nu P_{ii}\begin{bmatrix}\zeta_{\rm n}^2-\frac{1}{2}\\ \zeta_{\rm n}^2-1\end{bmatrix} - \begin{bmatrix}\varphi_\kappa^{II}\\ \psi_\kappa^{II}\end{bmatrix}, \qquad (y>0), \tag{133a}$$

$$\begin{bmatrix}\varphi_\kappa^{II}\\ \psi_\kappa^{II}\end{bmatrix} = (1-a_c)\widetilde{\mathcal{R}}\begin{bmatrix}\varphi_\kappa^{II}\\ \psi_\kappa^{II}\end{bmatrix} - 2a_c\int_{-\infty}^{0}\zeta_{\rm n}\varphi_\kappa^{II}e^{-\zeta_{\rm n}^2}d\zeta_{\rm n}\begin{bmatrix}1\\ 1\end{bmatrix} - a_c c_\kappa^{II}\begin{bmatrix}\zeta_{\rm n}^2-1\\ \zeta_{\rm n}^2\end{bmatrix} + \mathbf{G}_\kappa^{II}, \qquad (y=0,\ \zeta_{\rm n}>0), \tag{133b}$$

$$\begin{bmatrix}\varphi_\kappa^{II}\\ \psi_\kappa^{II}\end{bmatrix} \to 0, \qquad (y\to\infty), \tag{133c}$$

where

$$\mathbf{G}_v^{II} = \frac{4}{3}a_c\frac{1}{1-\nu}\begin{bmatrix}\zeta_{\rm n}^2-1\\ \zeta_{\rm n}^2-\frac{3}{2}\end{bmatrix}, \qquad \mathbf{G}_T^{II} = (2-a_c)\zeta_{\rm n}\begin{bmatrix}\zeta_{\rm n}^2-\frac{3}{2}\\ \zeta_{\rm n}^2-\frac{1}{2}\end{bmatrix}, \tag{134}$$

and ω, $\tau_{\rm tr}$, and P_{ii} $(= P_{ij}\delta_{ij})$ contained in Eq. (133a) are defined by Eqs. (128a)–(128c).

(iii) Problem for Υ_T^I:

Integrating each of Eqs. (121a)–(121c), multiplied by $(1/\pi)e^{-\zeta_{\rm s}^2-\zeta_{\rm t}^2}$, with respect to $\zeta_{\rm s}$ and $\zeta_{\rm t}$ from $-\infty$ to ∞ for both variables, we obtain the following problem for υ_T^I:

$$\zeta_{\rm n}\frac{\partial \upsilon_T^I}{\partial y} = -\upsilon_T^I, \qquad (y>0), \tag{135a}$$

$$\upsilon_T^I = (1-a_c)\widetilde{\mathcal{R}}\upsilon_T^I + a_c, \qquad (y=0,\ \zeta_{\rm n}>0), \tag{135b}$$

$$\upsilon_T^I \to 0, \qquad (y\to\infty). \tag{135c}$$

(iv) Problem for $(\Upsilon_T^{II};\, \tilde{c}_T^{II})$:

Integrating each of Eqs. (122a)–(122c), multiplied by $(1/\pi)e^{-\zeta_{\rm s}^2-\zeta_{\rm t}^2}$, with respect to $\zeta_{\rm s}$ and $\zeta_{\rm t}$ from $-\infty$ to ∞ for both variables, we obtain the following problem for $(\upsilon_T^{II},\, \tilde{c}_T^{II})$:

$$\zeta_{\rm n}\frac{\partial \upsilon_T^{II}}{\partial y} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\upsilon_T^{II}e^{-\zeta_{\rm n}^2}d\zeta_{\rm n} - \upsilon_T^{II}, \qquad (y>0), \tag{136a}$$

$$\upsilon_T^{II} = (1-a_c)\widetilde{\mathcal{R}}\upsilon_T^{II} - a_c\tilde{c}_T^{II} + (2-a_c)\zeta_{\rm n}, \qquad (y=0,\ \zeta_{\rm n}>0), \tag{136b}$$

$$\upsilon_T^{II} \to 0, \qquad (y\to\infty). \tag{136c}$$

C. Some remarks on reduced problems

As the result of the analysis in the preceding sections, the Knudsen-layer problem to determine the slip boundary conditions has been reduced to the half-space problems for (i) $(\bar{\psi}_\kappa^I;\, \bar{c}_\kappa^I)$ $(\kappa = v,\, T)$, i.e., Eqs. (131) and (132); (ii) $(\varphi_\kappa^{II},\, \psi_\kappa^{II};\, c_\kappa^{II})$ $(\kappa = v,\, T)$, i.e., Eqs. (133) and (134); (iii)

υ_T^I, i.e., Eq. (135); and (iv) $(\upsilon_T^{II};\, \tilde{c}_T^{II})$, i.e., Eq. (136). At this point, we give some remarks on these problems.

We first note that the problem of $(\bar{\psi}_v^I;\, \bar{c}_v^I)$, i.e., Eqs. (131) and (132) with $\kappa = v$, is exactly the same as the problem of the Knudsen layer for the shear slip based on the linearized BGK model for a monatomic gas [41, 42] and the Maxwell-type boundary condition. It is one of the fundamental classical problems in kinetic theory studied by various authors [43, 68–71] [there are some earlier results [72, 73] for $a_c = 1$ (diffuse reflection)]. In fact, $\bar{c}_v^I$ is the slip coefficient of the problem, which is equal to κ in [69] and ζ_P in [43]. Therefore, we can easily find the value of $\bar{c}_v^I$ in the literature and can, in principle, recover the solution $\bar{\psi}_v^I$ from the data in the literature. Consequently, we can obtain the reduced distribution function ψ_v^I of Φ_v^I and the value of c_v^I of the original problem, Eqs. (115) and (116), immediately from Eq. (130), that is,

$$\psi_v^I = \frac{1}{1-\nu}\psi_{v\mathrm{BGK}}^I, \qquad c_v^I = \frac{1}{1-\nu}c_{v\mathrm{BGK}}^I, \tag{137}$$

where and in what follows the subscript BGK indicates the corresponding quantities for the BGK model for a monatomic gas. This reduction from the ES model to the BGK model has been used in [44] in connection with the analysis of the Knudsen layer in the framework of the generalized slip flow theory based on the linearized ES model for a monatomic gas.

We next note that the problem of $(\bar{\psi}_T^I;\, \bar{c}_T^I)$, i.e., Eqs. (131) and (132) with $\kappa = T$, is exactly the same as the problem of the Knudsen layer for the thermal creep based on the linearized BGK model for a monatomic gas and the Maxwell-type boundary condition. It is also a fundamental classical problem that has been investigated in several papers [43, 70, 74, 75] (there is an earlier work [76] for $a_c = 1$), and $\bar{c}_T^I$ corresponds to the slip coefficient of the problem, which is equal to $d/2$ in [75] and ζ_T in [43]. Therefore, the numerical value of $\bar{c}_T^I$ is available, and the reduced distribution $\bar{\psi}_T^I$ associated with Φ_T^I can, in principle, be obtained from the literature. From Eq. (130), this fact can be summarized as

$$\psi_T^I = \psi_{T\mathrm{BGK}}^I, \qquad c_T^I = c_{T\mathrm{BGK}}^I. \tag{138}$$

This equivalence between the ES model and the BGK model for a monatomic gas is also pointed out in [44].

Now let us consider the problem for $(\varphi_v^{II},\, \psi_v^{II};\, c_v^{II})$, i.e., Eqs. (133) and (134) with $\kappa = v$. This problem is exactly the same as that of the Knudsen layer for the temperature jump, caused by the normal gradient of the normal component of the flow velocity, based on the linearized ES model for a monatomic gas and the Maxwell-type boundary condition [cf. the sentences following Eq. (106)]. It has been studied in [44], and the numerical results for $a_c = 1$ (diffuse reflection) and $\nu = -0.5$ and 0 (BGK model) are found there (see also [45]), e.g., $c_v^{II}(\text{here}) = c_5^{(0)}$ (in [44, 45]). However, since no result is available for the general case, a new numerical analysis of the problem is required.

The problem for $(\varphi_T^{II},\, \psi_T^{II};\, c_T^{II})$, i.e., Eqs. (133) and (134) with $\kappa = T$ is the same as that of the Knudsen layer for the standard temperature jump based on the linearized ES model for a monatomic gas and the Maxwell-type boundary condition [cf. the sentences following Eq. (106)]. It has been studied in [44], and the numerical results for $a_c = 1$ and $\nu = -0.5$ and 0 (BGK model) are found there (see also [45]), e.g., $c_T^{II}(\text{here}) = c_1^{(0)}$ (in [44, 45]). However, since no result is available for the general case, we need new numerical computation for this problem.

Finally, we consider the problem for υ_T^I and that for $(\upsilon_T^{II};\, \tilde{c}_T^{II})$, i.e., Eqs. (135) and (136), respectively. The problem (135) can be solved immediately, that is,

$$\upsilon_T^I = \begin{cases} 0, & (\zeta_{\mathrm{n}} < 0), \\ a_c e^{-y/\zeta_{\mathrm{n}}}, & (\zeta_{\mathrm{n}} > 0). \end{cases} \tag{139}$$

By comparing Eq. (136) and Eq. (131) with $\kappa = v$, one immediately finds that

$$\upsilon_T^{II} = \frac{1}{2}\bar{\psi}_v^I = \frac{1}{2}\psi_{v\mathrm{BGK}}^I, \qquad \tilde{c}_T^{II} = \bar{c}_v^I = c_{v\mathrm{BGK}}^I. \tag{140}$$

In this way, the problem for $(\upsilon_T^{II};\, \tilde{c}_T^{II})$ has been solved.

In summary, what we only need is to solve the temperature jump problems, i.e., Eq. (133) with $\kappa = v, T$, for specified ν.

D. Summary and additional remarks

We note that ν and the accommodation coefficient a_c are the only parameters that enter the solution of the problem for $(\psi_v^I,\, c_v^I)$ [Eq. (137)] and the solutions of the problems (133) for $(\varphi_\kappa^{II},\, \psi_\kappa^{II};\, c_\kappa^{II})$ $(\kappa = v,\, T)$. In addition, the solution of the problem for $(\psi_T^I,\, c_T^I)$ [Eq. (138)] and that of the problem for $(\upsilon_T^{II},\, \tilde{c}_T^{II})$ [Eq. (140)] are independent of ν. It follows from Eq. (A8) that $\mathrm{Pr} = 1/(1-\nu+\theta\nu) \approx 1/(1-\nu)$ when $\theta \ll 1$. Therefore, ν is basically determined by the Prandtl number.

According to Eqs. (137), (138), and (140), the coefficients c_v^I, c_T^I, and $\tilde{c}_T^{II}$ are obtained immediately from the slip coefficients for the BGK model for a monatomic gas. The result is shown in Eq. (10) with Table I.

By contrast, we need a new numerical analysis of the problem (133) (with $\kappa = v,\, T$) to obtain the numerical values of c_v^{II} and c_T^{II}. Since the analysis, which is based on a finite-difference method, is straightforward, we show only its outline in Appendix D. As mentioned in Sec. II D, the resulting numerical values are shown in Tables II and III. The dependence of c_v^{II} and c_T^{II} on the parameters ν and a_c is discussed in Sec. II D.

With these numerical values of c_v^I, c_T^I, c_v^{II}, c_T^{II}, and $\tilde{c}_T^{II}$, the slip boundary conditions for the two-temperature Navier–Stokes equations (6) follow immediately from Eqs. (60), (96), (110), and (112). The result is summarized in Eq. (9) in Sec. II D. As noted in [40], Eq. (9) forms two-dimensional fields on the boundary at each time and is independent of the trajectory of the points on the boundary.

The initial conditions for Eq. (6) are given by Eq. (55) under assumption (v) in Sec. II A. However, if we are interested only in the behavior of the gas in the fluid-dynamic time scale that is much longer than the mean free time and admit the inaccuracy in the initial stage $0 < \hat{t} < O(\text{mean free time})$, we may ignore assumption (v) in Sec. II A and assume the more general initial conditions of the form of Eq. (13), where $\hat{\rho}^{\mathrm{in}}(\boldsymbol{x})$, $\hat{\boldsymbol{v}}^{\mathrm{in}}(\boldsymbol{x})$, $\hat{T}_{\mathrm{tr}}^{\mathrm{in}}(\boldsymbol{x})$, and $\hat{T}_{\mathrm{int}}^{\mathrm{in}}(\boldsymbol{x})$ are appropriately chosen functions and are related to the initial condition for the ES model specified in the problem under consideration, say

$$\hat{f}(0,\, \boldsymbol{x},\, \boldsymbol{\zeta},\, \hat{\mathcal{E}}) = \hat{f}^{\mathrm{in}}(\boldsymbol{x},\, \boldsymbol{\zeta},\, \hat{\mathcal{E}}), \tag{141}$$

which may be more general than Eq. (39). The reader is referred to Sec. 5.2.4 in [40] for more detailed discussion about the initial conditions.

E. Macroscopic quantities inside Knudsen layer

We consider the Knudsen-layer parts of the macroscopic quantities $\hat{h}_{\mathrm{K}}^{(1)}$ in Eq. (62) or more specifically Eq. (65), on the basis of the expressions (88) [with Eq. (86)] and (89). Equation (92) indicates that $\hat{v}_{\mathrm{K}i}^{(1)} n_i$, $\hat{p}_{\mathrm{K}ij}^{(1)} n_j$, $\hat{q}_{(\mathrm{tr})\mathrm{K}i}^{(1)} n_i$, and $\hat{q}_{(\mathrm{int})\mathrm{K}i}^{(1)} n_i$ are all of $O(R_h\epsilon)$. Other components of the macroscopic quantities $\hat{h}_{\mathrm{K}}^{(1)}$ can be obtained by using Eqs. (101), (108), (109), and (111) in Eqs. (88) [with Eq. (86)] and (89) and by noting that the change of the names of the variables (99) has been made in Eqs. (101), (108), (109), and (111). In this process, use is also made of the relations derived in Sec. VII C. We summarize the results of the Knudsen-layer corrections of the macroscopic quantities, i.e., $\hat{h}_{\mathrm{K}} = \hat{h}_{\mathrm{K}}^{(1)}\epsilon + O(R_h\epsilon^2)$ [Eq. (62)], neglecting the terms of $O(R_h\epsilon^2)$. That is,

$$\hat{v}_{\mathrm{K}i} n_i = 0, \tag{142a}$$

$$\hat{v}_{\mathrm{K}i} t_i = \epsilon Y_v(y) \frac{\hat{T}_{\mathrm{w}}^{1/2}}{\hat{A}_c(\hat{T}_{\mathrm{w}})} \frac{1}{\hat{\rho}} \left(\frac{\partial \hat{v}_i}{\partial x_j} + \frac{\partial \hat{v}_j}{\partial x_i} \right) n_i t_j + \epsilon Y_T(y) \frac{1}{\hat{A}_c(\hat{T}_{\mathrm{w}})} \frac{1}{\hat{\rho}} \frac{\partial \hat{T}_{\mathrm{w}}}{\partial x_i} t_i, \tag{142b}$$

$$\hat{\rho}_{\mathrm{K}} = \epsilon \Omega_v(y) \frac{1}{\hat{A}_c(\hat{T}_{\mathrm{w}})} \frac{\partial \hat{v}_i}{\partial x_j} n_i n_j + \epsilon \Omega_T(y) \frac{1}{\hat{A}_c(\hat{T}_{\mathrm{w}}) \hat{T}_{\mathrm{w}}^{1/2}} \frac{\partial \hat{T}_{\mathrm{tr}}}{\partial x_i} n_i, \tag{142c}$$

$$\hat{T}_{\mathrm{trK}} = \epsilon \Theta_v(y) \frac{\hat{T}_{\mathrm{w}}}{\hat{A}_c(\hat{T}_{\mathrm{w}})} \frac{1}{\hat{\rho}} \frac{\partial \hat{v}_i}{\partial x_j} n_i n_j + \epsilon \Theta_T(y) \frac{\hat{T}_{\mathrm{w}}^{1/2}}{\hat{A}_c(\hat{T}_{\mathrm{w}})} \frac{1}{\hat{\rho}} \frac{\partial \hat{T}_{\mathrm{tr}}}{\partial x_i} n_i, \tag{142d}$$

$$\hat{T}_{\rm intK} = \epsilon\widetilde{\Theta}_T(y)\frac{\hat{T}_{\rm w}^{1/2}}{\hat{A}_c(\hat{T}_{\rm w})}\frac{1}{\hat{\rho}}\frac{\partial\hat{T}_{\rm int}}{\partial x_i}n_i, \tag{142e}$$

$$\hat{T}_{\rm K} = \frac{3\hat{T}_{\rm trK} + \delta\hat{T}_{\rm intK}}{3+\delta}, \tag{142f}$$

$$\hat{p}_{{\rm K}ij}n_j = 0, \tag{142g}$$

$$\hat{p}_{{\rm K}ij}t_j = \epsilon\Pi_v(y)\frac{\hat{T}_{\rm w}}{\hat{A}_c(\hat{T}_{\rm w})}\frac{\partial\hat{v}_j}{\partial x_k}n_jn_kt_i + \epsilon\Pi_T(y)\frac{\hat{T}_{\rm w}^{1/2}}{\hat{A}_c(\hat{T}_{\rm w})}\frac{\partial\hat{T}_{\rm tr}}{\partial x_j}n_jt_i, \tag{142h}$$

$$\hat{q}_{({\rm tr}){\rm K}i}n_i = 0, \tag{142i}$$

$$\hat{q}_{({\rm tr}){\rm K}i}t_i = \epsilon H_v(y)\frac{\hat{T}_{\rm w}^{3/2}}{\hat{A}_c(\hat{T}_{\rm w})}\left(\frac{\partial\hat{v}_i}{\partial x_j} + \frac{\partial\hat{v}_j}{\partial x_i}\right)n_it_j + \epsilon H_T(y)\frac{\hat{T}_{\rm w}}{\hat{A}_c(\hat{T}_{\rm w})}\frac{\partial\hat{T}_{\rm w}}{\partial x_i}t_i, \tag{142j}$$

$$\hat{q}_{({\rm int}){\rm K}i}n_i = 0, \tag{142k}$$

$$\hat{q}_{({\rm int}){\rm K}i}t_i = \epsilon\widetilde{H}_T(y)\frac{\hat{T}_{\rm w}}{\hat{A}_c(\hat{T}_{\rm w})}\frac{\partial\hat{T}_{\rm w}}{\partial x_i}t_i, \tag{142l}$$

$$\hat{q}_{{\rm K}i} = \hat{q}_{({\rm tr}){\rm K}i} + \hat{q}_{({\rm int}){\rm K}i}, \tag{142m}$$

where, with $\kappa = v$ and T,

$$Y_\kappa(y) = \frac{1}{2}\left\langle(\zeta^2-\zeta_{\rm n}^2)\Phi_\kappa^I\right\rangle = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\psi_\kappa^I e^{-\zeta_{\rm n}^2}d\zeta_{\rm n}, \tag{143a}$$

$$\Omega_\kappa(y) = \left\langle\Phi_\kappa^{II}\right\rangle = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\varphi_\kappa^{II}e^{-\zeta_{\rm n}^2}d\zeta_{\rm n}, \tag{143b}$$

$$\Theta_\kappa(y) = \frac{2}{3}\left\langle\left(\zeta^2-\frac{3}{2}\right)\Phi_\kappa^{II}\right\rangle = -\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\left(\varphi_\kappa^{II}-\frac{2}{3}\psi_\kappa^{II}\right)e^{-\zeta_{\rm n}^2}d\zeta_{\rm n}, \tag{143c}$$

$$\widetilde{\Theta}_T(y) = \left\langle\Upsilon_T^{II}\right\rangle = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\upsilon_T^{II}e^{-\zeta_{\rm n}^2}d\zeta_{\rm n} = \frac{1-\nu}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\psi_v^I e^{-\zeta_{\rm n}^2}d\zeta_{\rm n} = (1-\nu)Y_v(y), \tag{143d}$$

$$\Pi_\kappa(y) = \frac{3}{2}[\Omega_\kappa(y)+\Theta_\kappa(y)], \tag{143e}$$

$$H_\kappa(y) = \frac{1}{2}\left\langle(\zeta^2-\zeta_{\rm n}^2)\left(\zeta^2-\frac{5}{2}\right)\Phi_\kappa^I\right\rangle, \tag{143f}$$

$$\widetilde{H}_T(y) = \frac{\delta}{4}\left\langle(\zeta^2-\zeta_{\rm n}^2)\Upsilon_T^I\right\rangle. \tag{143g}$$

Here, $\langle\cdots\rangle$ is defined by Eq. (105). In Eq. (142), the quantities $\hat{\rho}$, $\hat{v}_i$, $\hat{T}_{\rm tr}$, and $\hat{T}_{\rm int}$ belong to the Chapman–Enskog solution, and $\hat{\rho}$ and the derivatives of $\hat{v}_i$, $\hat{T}_{\rm tr}$, and $\hat{T}_{\rm int}$ are all evaluated on the boundary. By the use of Eqs. (137) and (138) and by a similar procedure to that in Appendix C in [39], we obtain the following relations (the details are omitted here):

$$Y_v(y) = \frac{1}{1-\nu}Y_{v{\rm BGK}}(y), \qquad Y_T(y) = Y_{T{\rm BGK}}(y), \tag{144a}$$

$$H_v(y) = \frac{1}{1-\nu}H_{v{\rm BGK}}(y), \qquad H_T(y) = H_{T{\rm BGK}}(y), \tag{144b}$$

$$\widetilde{H}_T(y) = \frac{1}{4\sqrt{\pi}}a_c\delta J_0(y), \tag{144c}$$

where $Y_{v{\rm BGK}}(y)$, $Y_{T{\rm BGK}}(y)$, $H_{v{\rm BGK}}(y)$, and $H_{T{\rm BGK}}(y)$ are the corresponding functions for the BGK model for a monatomic gas, and $J_n(y)$ is the so-called Abramowitz function [77] defined by

$$J_n(y) = \int_0^\infty z^n e^{-z^2-\frac{y}{z}}dz, \qquad (y\geq 0). \tag{145}$$

The basic functions $Y_{v{\rm BGK}}(y)$, $Y_{T{\rm BGK}}(y)$, $H_{v{\rm BGK}}(y)$, and $H_{T{\rm BGK}}(y)$ versus y are plotted in Fig. 2 in [39] for three different values of the accommodation coefficient a_c [note

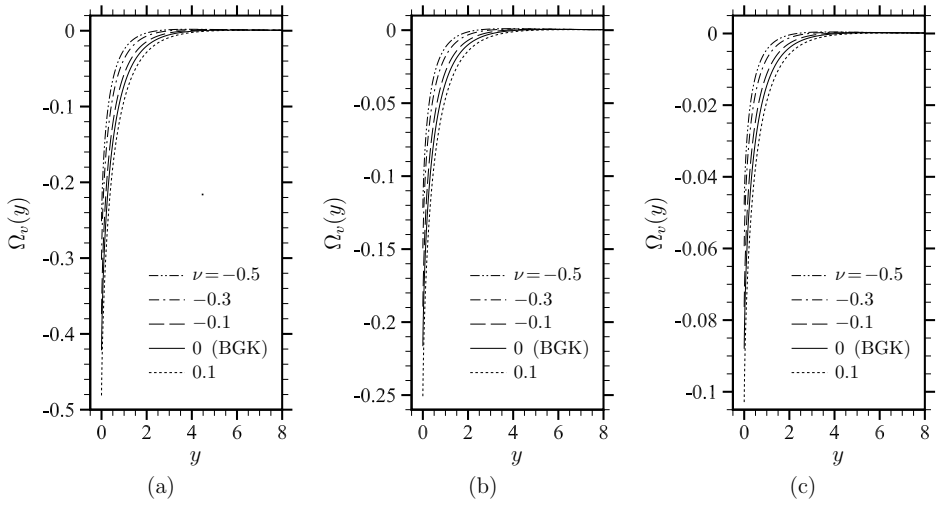


FIG. 2. Profile of $\Omega_v(y)$. (a) $a_c = 1$, (b) $a_c = 0.5$, (c) $a_c = 0.2$.

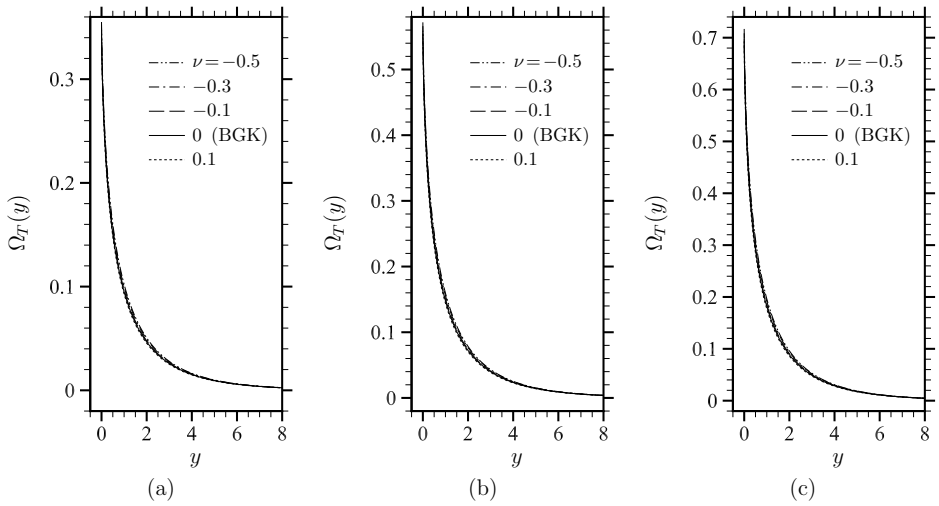


FIG. 3. Profile of $\Omega_T(y)$. (a) $a_c = 1$, (b) $a_c = 0.5$, (c) $a_c = 0.2$.

that a_c (here) = α (in [39])], i.e., a_c = 1.0, 0.5, and 0.2. These functions vanish rapidly as $y \to \infty$. Here, it should be noted that $[Y_{v\mathrm{BGK}}(y), Y_{T\mathrm{BGK}}(y), H_{v\mathrm{BGK}}(y), H_{T\mathrm{BGK}}(y)]$ (here) = $[Y_1^{(1)}(y), Y_2^{(1)}(y), H_1^{(1)}(y), H_2^{(1)}(y)]$ (in [44, 45]) and the latter functions for $a_c = 1$ are tabulated in Table 6 in [45]. In addition, the function $J_0(y)$ is tabulated in Table V in [39]. Therefore, these data are omitted here. The functions $Y_v(y)$, $Y_T(y)$, $H_v(y)$, $H_T(y)$, and $\widetilde{H}_T(y)$ are recovered from these data once the value of ν is known for the gas under consideration.

In Figs. 2–5, we show the profiles of the functions $\Omega_v(y)$, $\Omega_T(y)$, $\Theta_v(y)$, and $\Theta_T(y)$, respectively, in the case of a_c = 1.0, 0.5, and 0.2 for $\nu = -0.5$, -0.3, -0.1, and 0.1. In these figures, the corresponding profiles for the BGK model for a monatomic gas ($\nu = 0$) are also shown. We should recall that the functions $\Omega_v(y)$, $\Omega_T(y)$, $\Theta_v(y)$, and $\Theta_T(y)$ are the same as the corresponding functions for the ES model for a monatomic gas. To be more specific, $[\Omega_v(y), \Omega_T(y), \Theta_v(y), \Theta_T(y)]$ (here) = $[\Omega_5^{(0)}(y), \Omega_1^{(0)}(y), \Theta_5^{(0)}(y), \Theta_1^{(0)}(y)]$ (in [44, 45]), and the latter functions for $a_c = 1$ and $\nu = -0.5$ (Pr = 2/3) and 0 (BGK model) are tabulated in Tables 4 and 6 in [45]. Corresponding to the fact that c_T^{II} is almost constant with respect to ν, the functions Ω_T and Θ_T are almost independent of ν.

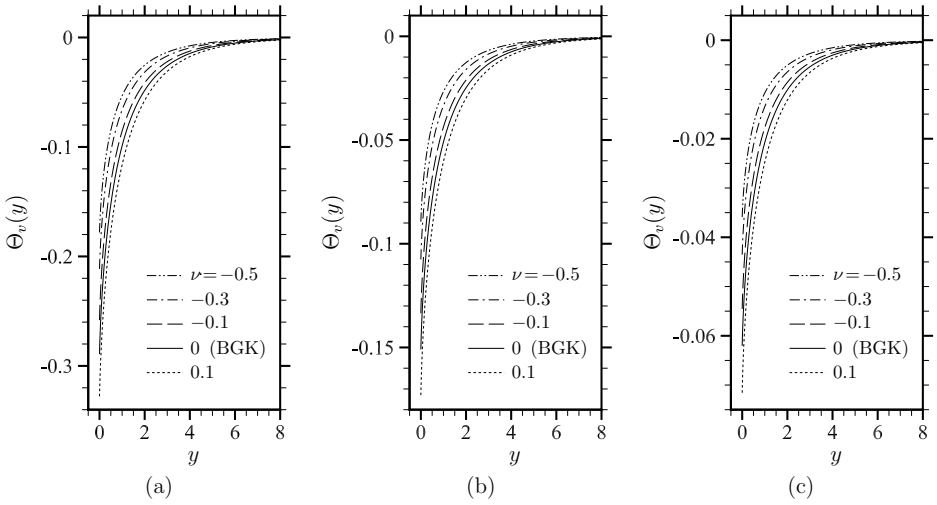


FIG. 4. Profile of $\Theta_v(y)$. (a) $a_c = 1$, (b) $a_c = 0.5$, (c) $a_c = 0.2$.

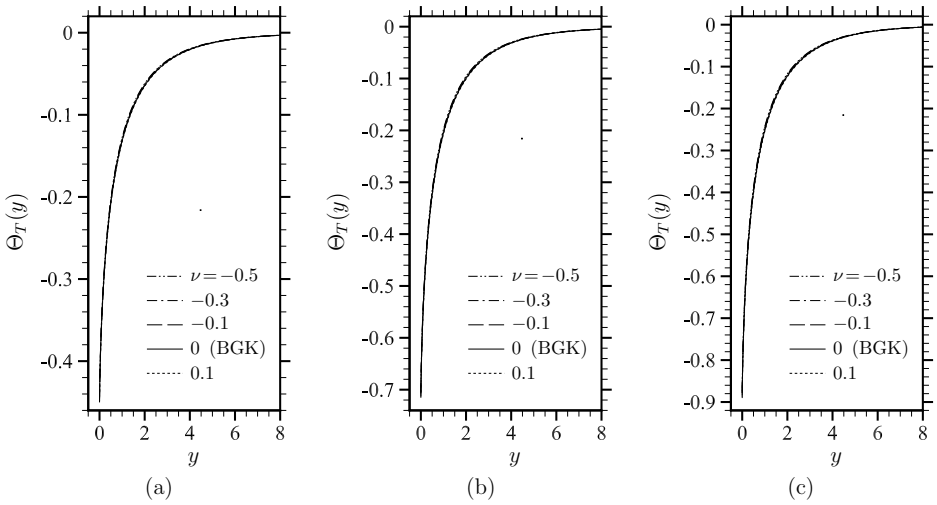


FIG. 5. Profile of $\Theta_T(y)$. (a) $a_c = 1$, (b) $a_c = 0.5$, (c) $a_c = 0.2$.

VIII. TWO-TEMPERATURE NAVIER–STOKES EQUATIONS AND SLIP BOUNDARY CONDITIONS IN DIMENSIONAL FORM

We first summarize the dimensional form of the two-temperature Navier–Stokes equations following Sec. III C in [36]. The stress tensor p_{ij} and the heat-flow vector q_i, which are the dimensional version of Eq. (53) with the $O(\epsilon^2)$ terms being neglected, are given by

$$p_{ij} = \rho R T_{\rm tr}\delta_{ij} - \mu_{\rm tr}(T,\, T_{\rm tr})\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3}\frac{\partial v_k}{\partial X_k}\delta_{ij}\right), \tag{146a}$$

$$q_{({\rm tr})i} = -\lambda_{\rm tr}(T,\, T_{\rm tr})\frac{\partial T_{\rm tr}}{\partial X_i}, \qquad q_{({\rm int})i} = -\lambda_{\rm int}(T,\, T_{\rm tr})\frac{\partial T_{\rm int}}{\partial X_i}, \tag{146b}$$

$$q_i = q_{({\rm tr})i} + q_{({\rm int})i}, \tag{146c}$$

where

$$\mu_{\rm tr}(T,\, T_{\rm tr}) = \frac{1}{1-\nu}\frac{RT_{\rm tr}}{A_c(T)}, \quad \lambda_{\rm tr}(T,\, T_{\rm tr}) = \frac{5}{2}R\frac{RT_{\rm tr}}{A_c(T)}, \quad \lambda_{\rm int}(T,\, T_{\rm tr}) = \frac{\delta}{2}R\frac{RT_{\rm tr}}{A_c(T)}, \tag{147}$$

and $T = (3T_{\rm tr} + \delta T_{\rm int})/(3+\delta)$ [Eq. (18i)]. Correspondingly, the dimensional two-temperature Navier–Stokes equations, which are the dimensional version of Eq. (6), are given by the following

equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_j)}{\partial X_j} = 0, \tag{148a}$$

$$\frac{\partial (\rho v_i)}{\partial t} + \frac{\partial (\rho v_i v_j)}{\partial X_j} + \frac{\partial (\rho R T_{\rm tr})}{\partial X_i} = \frac{\partial}{\partial X_j}\left[\mu_{\rm tr}(T,\, T_{\rm tr})\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3}\frac{\partial v_k}{\partial X_k}\delta_{ij}\right)\right], \tag{148b}$$

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\rho\left(\frac{3}{2}RT_{\rm tr} + \frac{1}{2}v_i^2\right)\right] + \frac{\partial}{\partial X_j}\left[\rho v_j\left(\frac{5}{2}RT_{\rm tr} + \frac{1}{2}v_i^2\right)\right] - \frac{3}{2}\theta A_c(T)\rho^2 R(T - T_{\rm tr})\\ &\quad = \frac{\partial}{\partial X_j}\left[\lambda_{\rm tr}(T,\, T_{\rm tr})\frac{\partial T_{\rm tr}}{\partial X_j}\right] + \frac{\partial}{\partial X_j}\left[\mu_{\rm tr}(T,\, T_{\rm tr})v_i\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3}\frac{\partial v_k}{\partial X_k}\delta_{ij}\right)\right],\end{aligned} \tag{148c}$$

$$\frac{\partial (\rho T_{\rm int})}{\partial t} + \frac{\partial (\rho v_j T_{\rm int})}{\partial X_j} - \theta A_c(T)\rho^2(T - T_{\rm int}) = \frac{2}{\delta}\frac{1}{R}\frac{\partial}{\partial X_j}\left[\lambda_{\rm int}(T,\, T_{\rm tr})\frac{\partial T_{\rm int}}{\partial X_j}\right]. \tag{148d}$$

The $A_c(T)$ in Eqs. (148c) and (148d) can be replaced by an expression in terms of $\mu_{\rm tr}$, $\lambda_{\rm tr}$, or $\lambda_{\rm int}$ with the help of Eq. (147). However, the terms containing $A_c(T)$ indicate the relaxation of the translational temperature and that of the internal temperature. Since $[\theta A_c(T)\rho]^{-1}$ is the time scale of the relaxation of the translational and internal temperatures [cf. Eq. (A10)], it would be more natural to keep $\theta A_c(T)$ in Eqs. (148c) and (148d). The dimensional version of Eq. (54) is omitted here [cf. Eq. (70) in [36]].

Next, we transform the slip boundary conditions (9) into their dimensional form. To be more specific, we use Eqs. (3) and (8) and eliminate ϵ with the help of Eqs. (4), (A5), and (147) (with $T = T_{\rm w}$ and $T_{\rm tr} = T_{\rm w}$) in Eq. (9). Then, we obtain the following dimensional form of the slip boundary conditions:

$$(v_i - v_{{\rm w}i})n_i = 0, \tag{149a}$$

$$(v_i - v_{{\rm w}i})t_i = \frac{\sqrt{2}}{R^{1/2}}a_v^I\frac{\mu_{\rm tr}(T_{\rm w},\, T_{\rm w})}{\rho T_{\rm w}^{1/2}}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i}\right)n_i t_j + \frac{4}{5R}a_T^I\frac{\lambda_{\rm tr}(T_{\rm w},\, T_{\rm w})}{\rho T_{\rm w}}\frac{\partial T_{\rm w}}{\partial X_i}t_i, \tag{149b}$$

$$T_{\rm tr} - T_{\rm w} = \frac{1}{R}a_v^{II}\frac{\mu_{\rm tr}(T_{\rm w},\, T_{\rm w})}{\rho}\frac{\partial v_i}{\partial X_j}n_i n_j + \frac{2\sqrt{2}}{5R^{3/2}}a_T^{II}\frac{\lambda_{\rm tr}(T_{\rm w},\, T_{\rm w})}{\rho T_{\rm w}^{1/2}}\frac{\partial T_{\rm tr}}{\partial X_i}n_i, \tag{149c}$$

$$T_{\rm int} - T_{\rm w} = \frac{2\sqrt{2}}{\delta R^{3/2}}\tilde{a}_T^{II}\frac{\lambda_{\rm int}(T_{\rm w},\, T_{\rm w})}{\rho T_{\rm w}^{1/2}}\frac{\partial T_{\rm int}}{\partial X_i}n_i, \tag{149d}$$

where

$$\begin{aligned}&a_v^I = (1-\nu)c_v^I = c_{v{\rm BGK}}^I, \qquad a_T^I = c_T^I = c_{T{\rm BGK}}^I,\\ &a_v^{II} = (1-\nu)c_v^{II}, \qquad a_T^{II} = c_T^{II},\\ &\tilde{a}_T^{II} = \tilde{c}_T^{II} = c_{v{\rm BGK}}^I.\end{aligned} \tag{150}$$

Here, Eq. (10) has been used. In Eq. (149), $RT_{\rm w}/A_c(T_{\rm w})$ has been eliminated by the use of the relation $RT_{\rm w}/A_c(T_{\rm w}) = (1-\nu)\mu_{\rm tr}(T_{\rm w},\, T_{\rm w}) = (2/5R)\lambda_{\rm tr}(T_{\rm w},\, T_{\rm w}) = (2/\delta R)\lambda_{\rm int}(T_{\rm w},\, T_{\rm w})$ [cf. Eq. (147)]. More specifically, $\mu_{\rm tr}$ is used in the terms containing the derivative of v_i, $\lambda_{\rm tr}$ is used in the terms containing the derivative of $T_{\rm w}$ or $T_{\rm tr}$, and $\lambda_{\rm int}$ is used in the term containing the derivative of $T_{\rm int}$. This choice follows the analogous choice in the slip boundary conditions for the ordinary Navier–Stokes equations (with a single temperature) for a polyatomic gas [see Eq. (139) in [39]]. In the case of the ordinary Navier–Stokes equations in [39], the equations as well as the slip boundary conditions are expressed in terms of the viscosity μ, bulk viscosity μ_b, and thermal conductivity λ [cf. Eqs. (138) and (139) in [39]], the data of which are available for many gases. Therefore, these data can be input directly in the equations and the boundary conditions without identifying the function $A_c(T)$ in practical applications. In contrast, the two-temperature Navier–Stokes equations (148) contains $A_c(T)$. In addition, the transport coefficients $\mu_{\rm tr}$, $\lambda_{\rm tr}$, and $\lambda_{\rm int}$ contained in the equations (148) and in the boundary conditions (149) are the quantities whose direct measurements would be difficult. Therefore, we need to identify the function $A_c(T)$ by the procedure described in Sec. II E. An example of the procedure is shown in the last paragraphs in Sec. II F.

The initial condition for Eq. (148) is given by

$$\rho = \rho_0, \qquad \boldsymbol{v} = 0, \qquad T_{\rm tr} = T_{\rm int} = T_0, \qquad \text{at } t = 0, \tag{151}$$

which corresponds to Eq. (55) under assumption (v) in Sec. II A, or

$$\rho = \rho^{\rm in}(\boldsymbol{X}), \qquad \boldsymbol{v} = \boldsymbol{v}^{\rm in}(\boldsymbol{X}), \qquad T_{\rm tr} = T_{\rm tr}^{\rm in}(\boldsymbol{X}), \qquad T_{\rm int} = T_{\rm int}^{\rm in}(\boldsymbol{X}), \qquad \text{at } t = 0, \tag{152}$$

which corresponds to Eq. (13) in more general case without assumption (v) and in the case when we ignore the accuracy for short time (within the scale of the mean free time). Here $\rho^{\rm in}$, $\boldsymbol{v}^{\rm in}$, $T_{\rm tr}^{\rm in}$, and $T_{\rm int}^{\rm in}$ are the density, flow velocity, translational temperature, and internal temperature obtained from the initial distribution $f^{\rm in}$ corresponding to Eq. (141) (see Sec. 5.2.4 in [40]).

IX. CONCLUDING REMARKS

In [36], four of the present authors derived the two-temperature Navier–Stokes equations from the ES model for a polyatomic gas under the assumption that the time scale of the relaxation of the internal modes is much longer than the collisional mean free time. Then, the equations have successfully been applied to the problem of the structure of a stationary shock wave in CO_2 gas (see also [38]). Incidentally, it is generally understood that gases with slow relaxation of the internal modes have large bulk viscosities.

In the present study, we tried, as the next step, to derive the appropriate boundary conditions for the two-temperature Navier–Stokes equations. Since most of practical flow problems contain solid boundaries, the applicability of the equations will be dramatically enlarged with the boundary conditions.

As is well known [58, 59, 78, 79], the appropriate boundary conditions, which are in the form of the slip boundary conditions, for the ordinary Navier–Stokes equations can be obtained only by the analysis of the Knudsen layer, which is a thin layer with thickness of the order of the mean free path of the gas molecules adjacent to the solid boundary. The reader is referred to [40] for the slip boundary conditions for a monatomic gas and [39] for those for a polyatomic gas.

In the present study, following these references, we have carried out a precise analysis of the Knudsen layer, in the case where the time scale of the relaxation of the internal modes is much longer than the collisional mean free time, on the basis of the ES model for a polyatomic gas and the Maxwell-type boundary condition. As the result, we have derived the slip boundary conditions for the two-temperature Navier–Stokes equations, together with the explicit numerical values of the coefficients included in the conditions (the so-called slip coefficients).

The boundary conditions for the flow velocity and the translational temperature are essentially the same as the slip boundary conditions for the ES model for a monatomic gas, and the internal temperature does not appear there. The boundary condition for the internal temperature only contains the normal derivative of itself and is free from the translational temperature as well as the flow velocity.

In this way, we have established the handy system consisting of the two-temperature Navier–Stokes equations and their slip boundary conditions and presented it in the form that is explicit and immediately applicable to practical flow problems of a polyatomic gas with slow relaxation of the internal modes (or large bulk viscosity) [see Eqs. (6) and (9) or Eqs. (148) and (149)]. It is a great advantage of the two-temperature Navier–Stokes equations to have clear boundary conditions, compared with other macroscopic moment equations. The application of the new system to some fundamental flow problems will be our forthcoming project. It would also be possible to extend the present approach, as well as that of [36], to more sophisticated models than the present ES model, such as the model proposed recently by [25], to derive multi-temperature Navier–Stokes equations and their slip boundary conditions.

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Appendix A: Basic properties of ES model

1. Basic properties

The ES model (17) has the basic properties listed in the following.

Equilibrium: The vanishing of the collision term $Q(f) = 0$ is equivalent to the fact that f is the following local equilibrium distribution [16] (see also Appendix A in [53]):

$$f_{\rm eq} = \frac{\rho \mathcal{E}^{\delta/2-1}}{(2\pi RT)^{3/2}(RT)^{\delta/2}\Gamma(\delta/2)} \exp\left(-\frac{|\boldsymbol{\xi}-\boldsymbol{v}|^2}{2RT} - \frac{\mathcal{E}}{RT}\right), \tag{A1}$$

where ρ, $\boldsymbol{v}$, and T are arbitrary functions of t and $\boldsymbol{X}$.

Conservations: For an arbitrary function $g(t, \boldsymbol{X}, \boldsymbol{\xi}, \mathcal{E})$, the following relation holds [16] (see also Appendix A in [53]):

$$\iint_0^\infty \varphi_r Q(g) d\mathcal{E} d\boldsymbol{\xi} = 0, \tag{A2}$$

where φ_r ($r = 0, ..., 4$) are the so-called collision invariants, i.e.,

$$\varphi_0 = 1, \qquad \varphi_i = \xi_i \quad (i = 1, 2, 3), \qquad \varphi_4 = \frac{1}{2}|\boldsymbol{\xi}|^2 + \mathcal{E}. \tag{A3}$$

Entropy inequality: For an arbitrary function $g(t, \boldsymbol{X}, \boldsymbol{\xi}, \mathcal{E})$, the following inequality holds [16]:

$$\iint_0^\infty \left(\ln \frac{g}{\mathcal{E}^{\delta/2-1}}\right) Q(g) d\mathcal{E} d\boldsymbol{\xi} \le 0, \tag{A4}$$

and the equality sign holds if and only if $g = f_{\rm eq}$ in Eq. (A1).

Mean free path: The mean free path l_0 of the gas molecules in the equilibrium state at rest at density ρ_0 and temperature T_0 is given by

$$l_0 = \frac{2}{\sqrt{\pi}} \frac{(2RT_0)^{1/2}}{A_c(T_0)\rho_0}, \tag{A5}$$

for Eq. (17), since $A_c(T_0)\rho_0$ is the collision frequency at this equilibrium state.

2. Transport and relaxation properties

When the mean free path of the gas molecules l_0 is small compared with the characteristic length of the system, we can formally derive the ordinary (compressible) Navier–Stokes equations for a polyatomic gas from the ES model (17) by the Chapman–Enskog method [16]. The Navier–Stokes constitutive laws thus obtained are as follows (see Sec. VI of [39] for these forms and for the entire Navier–Stokes equations):

$$p_{ij} = p\delta_{ij} - \mu(T)\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3}\frac{\partial v_k}{\partial X_k}\delta_{ij}\right) - \mu_b(T)\frac{\partial v_k}{\partial X_k}\delta_{ij}, \tag{A6a}$$

$$q_i = -\lambda(T)\frac{\partial T}{\partial X_i}, \tag{A6b}$$

where $\mu(T)$, $\mu_b(T)$, and $\lambda(T)$ are, respectively, the viscosity, the bulk viscosity, and the thermal conductivity and are expressed as follows:

$$\mu(T) = \frac{1}{1-\nu+\theta\nu}\frac{RT}{A_c(T)}, \tag{A7a}$$

$$\mu_b(T) = \frac{2}{3}\frac{\delta}{\theta(\delta+3)}\frac{RT}{A_c(T)}, \tag{A7b}$$

$$\lambda(T) = \frac{\delta+5}{2}\frac{R^2T}{A_c(T)}. \tag{A7c}$$

From Eqs. (A7a), (A7c), and (1), the Prandtl number $\mathrm{Pr} = c_\mathrm{p}\mu/\lambda$ is obtained as

$$\mathrm{Pr} = 1/(1-\nu+\theta\nu), \tag{A8}$$

and from Eqs. (A7a), (A7b), (A8), and (1), the ratio μ_b/μ is expressed as

$$\frac{\mu_b}{\mu} = \frac{2}{3}\frac{\delta}{\theta(\delta+3)}\frac{1}{\mathrm{Pr}} = \frac{1}{\theta}\left(\frac{5}{3}-\gamma\right)\frac{1}{\mathrm{Pr}}. \tag{A9}$$

Here, it should be noted that the ratio μ_b/μ does not depend on T and is inversely proportional to the parameter θ contained in the ES model.

In [36], the relaxation of the internal modes is examined in the space homogeneous case where f does not depend on $\boldsymbol{X}$, i.e., $f = f(t, \boldsymbol{\xi}, \mathcal{E})$. As shown in Sec. II D in [36], the temperature T_tr associated with the translational energy and the temperature T_int associated with the energy of the internal modes evolve with time as

$$T_\mathrm{tr} = T + (T_{\mathrm{tr}*} - T)\, e^{-\theta A_c(T)\rho t}, \tag{A10a}$$

$$T_\mathrm{int} = T + (T_{\mathrm{int}*} - T)\, e^{-\theta A_c(T)\rho t}, \tag{A10b}$$

where T is the temperature that is constant, and $T_{\mathrm{tr}*}$ and $T_{\mathrm{int}*}$ are, respectively, the initial value of T_tr and that of T_int at $t = 0$, which satisfy the relation $(3T_{\mathrm{tr}*} + \delta T_{\mathrm{int}*})/(3+\delta) = T$. That is, T_tr and T_int approach the total temperature T with the time scale $1/[\theta A_c(T)\rho]$. Since $A_c(T)\rho$ is the collision frequency of the gas molecules, $1/[A_c(T)\rho]$ is the mean free time. Therefore, the time scale of relaxation of the internal modes is (mean free time)$/\theta$.

In summary, small values of the parameter θ correspond to the case of large μ_b/μ and to the case of slow relaxation of the internal modes.

Appendix B: Reduction of two-temperature Navier–Stokes equations to ordinary Navier–Stokes equations

In this appendix, we try to recover the ordinary Navier–Stokes equations with a single temperature from the two-temperature Navier–Stokes equations (6) following the procedure in [28]. The basic assumption to derive the latter equations is Eq. (5), i.e., $\alpha = \theta/\epsilon = O(1)$, whereas the ordinary Navier–Stokes equations are based on the assumption that θ is of the order of unity [39], which corresponds to $\alpha = O(1/\epsilon) \gg 1$. In order to consider the situation close to the case of $\alpha = O(1/\epsilon) \gg 1$ in the two-temperature Navier–Stokes equations that are valid for $\alpha = O(1)$, we let

$$1 \ll \alpha \ll 1/\epsilon, \quad \text{i.e.,} \quad \epsilon \ll \theta \ll 1. \tag{B1}$$

It is seen from the order of magnitude of each term in Eqs. (6c) and (6d) that

$$\hat{T}_\mathrm{tr} - \hat{T} = O(1/\alpha), \qquad \hat{T}_\mathrm{int} - \hat{T} = O(1/\alpha). \tag{B2}$$

Subtracting Eq. (6b) multiplied by $2\hat{v}_i$ from Eq. (6c), using Eq. (6a) occasionally, and dividing by $\hat{\rho}$, we obtain

$$\begin{aligned}\frac{\partial \hat{T}_\mathrm{tr}}{\partial \hat{t}} + \hat{v}_j\frac{\partial \hat{T}_\mathrm{tr}}{\partial x_j} =& -\frac{2}{3}\hat{T}_\mathrm{tr}\frac{\partial \hat{v}_j}{\partial x_j} + \alpha\hat{A}_c(\hat{T})\hat{\rho}(\hat{T}-\hat{T}_\mathrm{tr}) + \frac{5}{6}\,\epsilon\,\frac{1}{\hat{\rho}}\frac{\partial}{\partial x_j}\left[\Gamma_\lambda(\hat{T}, \hat{T}_\mathrm{tr})\frac{\partial \hat{T}_\mathrm{tr}}{\partial x_j}\right] \\ &+ \frac{2}{3}\,\epsilon\,\frac{1}{\rho}\Gamma_\mu(\hat{T}, \hat{T}_\mathrm{tr})\frac{\partial \hat{v}_i}{\partial x_j}\left(\frac{\partial \hat{v}_i}{\partial x_j} + \frac{\partial \hat{v}_j}{\partial x_i} - \frac{2}{3}\frac{\partial \hat{v}_k}{\partial x_k}\delta_{ij}\right).\end{aligned} \tag{B3}$$

On the other hand, with the help of Eq. (6a), Eq. (6d) is transformed to the following form:

$$\frac{\partial \hat{T}_{\rm int}}{\partial \hat{t}} + \hat{v}_j \frac{\partial \hat{T}_{\rm int}}{\partial x_j} = \alpha \hat{A}_c(\hat{T})\hat{\rho}(\hat{T} - \hat{T}_{\rm int}) + \frac{1}{2}\,\epsilon\,\frac{1}{\hat{\rho}}\frac{\partial}{\partial x_j}\left[\Gamma_\lambda(\hat{T},\,\hat{T}_{\rm tr})\frac{\partial \hat{T}_{\rm int}}{\partial x_j}\right]. \tag{B4}$$

If we neglect the small terms of the order of ϵ in Eqs. (B3) and (B4) and subtract the latter from the former, then we have

$$\frac{\partial}{\partial \hat{t}}(\hat{T}_{\rm tr} - \hat{T}_{\rm int}) + \hat{v}_j \frac{\partial}{\partial x_j}(\hat{T}_{\rm tr} - \hat{T}_{\rm int}) = -\frac{2}{3}\hat{T}_{\rm tr}\frac{\partial \hat{v}_j}{\partial x_j} - \alpha \hat{A}_c(\hat{T})\hat{\rho}(\hat{T}_{\rm tr} - \hat{T}_{\rm int}). \tag{B5}$$

Since $\hat{T}_{\rm tr} - \hat{T}_{\rm int} = O(1/\alpha)$ because of Eq. (B2), the left-hand side of Eq. (B5) is of $O(1/\alpha)$, while its right-hand side is of $O(1)$. Neglecting the left-hand side leads to the following expression of $\hat{T}_{\rm tr} - \hat{T}_{\rm int}$:

$$\hat{T}_{\rm tr} - \hat{T}_{\rm int} = -\frac{2}{3}\frac{1}{\alpha}\frac{\hat{T}_{\rm tr}}{\hat{A}_c(\hat{T})\hat{\rho}}\frac{\partial \hat{v}_j}{\partial x_j} = -\frac{2}{3}\frac{1}{\alpha}\frac{\hat{T}}{\hat{A}_c(\hat{T})\hat{\rho}}\frac{\partial \hat{v}_j}{\partial x_j} + O\left(\frac{1}{\alpha^2}\right), \tag{B6}$$

where use has been made of Eq. (B2). One can see that the degree of nonequilibrium $\hat{T}_{\rm tr} - \hat{T}_{\rm int}$ is directly expressed in terms of the divergence of the velocity field.

Using the definition (30i) of $\hat{T}$ and Eq. (B6), we have

$$\hat{T}_{\rm tr} - \hat{T} = \frac{\delta}{3+\delta}(\hat{T}_{\rm tr} - \hat{T}_{\rm int}) = -\frac{2\delta}{3(3+\delta)}\frac{1}{\alpha}\frac{\hat{T}}{\hat{A}_c(\hat{T})\hat{\rho}}\frac{\partial \hat{v}_j}{\partial x_j} + O\left(\frac{1}{\alpha^2}\right), \tag{B7a}$$

$$\hat{T}_{\rm int} - \hat{T} = -\frac{3}{3+\delta}(\hat{T}_{\rm tr} - \hat{T}_{\rm int}) = \frac{2}{3+\delta}\frac{1}{\alpha}\frac{\hat{T}}{\hat{A}_c(\hat{T})\hat{\rho}}\frac{\partial \hat{v}_j}{\partial x_j} + O\left(\frac{1}{\alpha^2}\right). \tag{B7b}$$

Let us substitute Eq. (B7a) into Eqs. (6b) and (54) to eliminate $\hat{T}_{\rm tr}$, neglect the terms of $O(1/\alpha^2)$, and replace $1/\alpha$ with ϵ/θ. Then, we obtain the following equations:

$$\begin{aligned}\frac{\partial(\hat{\rho}\hat{v}_i)}{\partial \hat{t}} + \frac{\partial(\hat{\rho}\hat{v}_i\hat{v}_j)}{\partial x_j} = &-\frac{1}{2}\frac{\partial \hat{p}}{\partial x_i} + \frac{\epsilon}{2}\frac{\partial}{\partial x_j}\left[\widetilde{\Gamma}_1(\hat{T})\left(\frac{\partial \hat{v}_i}{\partial x_j} + \frac{\partial \hat{v}_j}{\partial x_i} - \frac{2}{3}\frac{\partial \hat{v}_k}{\partial x_k}\delta_{ij}\right)\right] \\ &+ \frac{\epsilon}{2}\frac{\partial}{\partial x_i}\left[\Gamma_b(\hat{T})\frac{\partial \hat{v}_j}{\partial x_j}\right],\end{aligned} \tag{B8a}$$

$$\begin{aligned}&\frac{\partial}{\partial \hat{t}}\left[\hat{\rho}\left(\frac{3+\delta}{2}\hat{T} + \hat{v}_i^2\right)\right] + \frac{\partial}{\partial x_j}\left[\hat{\rho}\hat{v}_j\left(\frac{5+\delta}{2}\hat{T} + \hat{v}_i^2\right)\right] \\ &\quad = \frac{5}{4}\,\epsilon\frac{\partial}{\partial x_j}\left[\Gamma_2(\hat{T})\frac{\partial \hat{T}}{\partial x_j}\right] + \epsilon\frac{\partial}{\partial x_j}\left[\widetilde{\Gamma}_1(\hat{T})\hat{v}_i\left(\frac{\partial \hat{v}_i}{\partial x_j} + \frac{\partial \hat{v}_j}{\partial x_i} - \frac{2}{3}\frac{\partial \hat{v}_k}{\partial x_k}\delta_{ij}\right)\right] \\ &\qquad + \epsilon\,\frac{\partial}{\partial x_j}\left[\Gamma_b(\hat{T})\hat{v}_j\frac{\partial \hat{v}_k}{\partial x_k}\right],\end{aligned} \tag{B8b}$$

where $\hat{p} = \hat{\rho}\hat{T}$ [Eq. (31)] and

$$\widetilde{\Gamma}_1(\hat{T}) = \frac{1}{1-\nu}\frac{\hat{T}}{\hat{A}_c(\hat{T})}, \qquad \Gamma_2(\hat{T}) = \left(1 + \frac{\delta}{5}\right)\frac{\hat{T}}{\hat{A}_c(\hat{T})}, \qquad \Gamma_b(\hat{T}) = \frac{1}{\theta}\frac{2\delta}{3(3+\delta)}\frac{\hat{T}}{\hat{A}_c(\hat{T})}. \tag{B9}$$

Equations (6a), (B8a), and (B8b) coincide with the ordinary Navier–Stokes equations with a single temperature derived from the ES model, i.e., Eq. (46) in [39], except that in Eqs. (B8a) and (B8b), $\widetilde{\Gamma}_1(\hat{T})$ appears in place of

$$\Gamma_1(\hat{T}) = \frac{1}{1-\nu+\theta\nu}\frac{\hat{T}}{\hat{A}_c(\hat{T})}. \tag{B10}$$

However, since $\epsilon \ll \theta \ll 1$, it holds that $\Gamma_1(\hat{T}) = \widetilde{\Gamma}_1(\hat{T}) + O(\theta) \approx \widetilde{\Gamma}_1(\hat{T})$. In this way, the ordinary Navier–Stokes equations can be recovered from the two-temperature Navier–Stokes equations. The quantity $\epsilon\Gamma_b(\hat{T})$ corresponds to the bulk viscosity. It should be noted that the bulk viscosity terms in Eqs. (B8a) and (B8b) originate from the relation (B6).

Appendix C: Outline of derivation of Eqs. (66) and (67)

The derivation of Eqs. (66) and (67) is basically the same as the corresponding analysis in Appendix A in [39], where $\theta = O(1)$ and $\epsilon \ll 1$ are assumed instead of Eq. (5). Therefore, we give its outline referring occasionally to [39].

We note that the Chapman–Enskog solution $\hat{f}_{\rm CE}$ satisfies Eq. (29). If we substitute Eq. (58) into Eq. (29) and subtract Eq. (29) with $\hat{f} = \hat{f}_{\rm CE}$, then we have

$$\epsilon \frac{\partial \hat{f}_{\rm K}^{(1)}}{\partial \hat{t}} + \epsilon \zeta_i \frac{\partial \hat{f}_{\rm K}^{(1)}}{\partial x_i} + O(\epsilon^2 R_f) = \frac{1}{\epsilon}[\hat{Q}(\hat{f}_{\rm CE} + \hat{f}_{\rm K}) - \hat{Q}(\hat{f}_{\rm CE})], \tag{C1}$$

where Eq. (61) has been used on the left-hand side.

In order to calculate the right-hand side of Eq. (C1), we first consider $(\hat{\mathsf{T}})_{ij}$ in Eq. (30c). Let us use Eq. (59) with Eq. (62) in Eq. (30c) and recall that $\hat{h}_{\rm CE}$ indicates the Chapman–Enskog macroscopic quantities appeared in Sec. V B [i.e., $\hat{h}$ in Eq. (49)] though the subscript CE is not attached there. This operation has been done in Appendix A in [39] for $\theta = O(1)$ and led to the following expression [cf. Eqs. (A2) and (A3) in [39]]:

$$\left(\hat{\mathsf{T}}|_{\hat{h}=\hat{h}_{\rm tot}}\right)_{ij} = (\mathsf{A})_{ij} + \epsilon(\mathsf{B})_{ij} + O(\epsilon^2 R_h), \tag{C2}$$

where

$$(\mathsf{A})_{ij} = \left(\hat{\mathsf{T}}|_{\hat{h}=\hat{h}_{\rm CE}}\right)_{ij} = (1-\theta)\left[(1-\nu)\hat{T}_{\rm tr}\delta_{ij} + \nu\frac{\hat{p}_{ij}}{\hat{\rho}}\right] + \theta\hat{T}\delta_{ij}, \tag{C3a}$$

$$(\mathsf{B})_{ij} = (1-\theta)\left[(1-\nu)\hat{T}_{\rm trK}^{(1)}\delta_{ij} + \nu\frac{1}{\hat{\rho}}\left(\hat{p}_{{\rm K}ij}^{(1)} - \hat{p}_{ij}\frac{\hat{\rho}_{\rm K}^{(1)}}{\hat{\rho}}\right)\right] + \theta\hat{T}_{\rm K}^{(1)}\delta_{ij}, \tag{C3b}$$

and $\hat{\mathsf{T}}|_{\hat{h}=\hat{h}_{\rm CE}}$ indicates $\hat{\mathsf{T}}$ evaluated with $\hat{h} = \hat{h}_{\rm CE}$. Note that $\hat{\rho}$, $\hat{p}_{ij}$, $\hat{T}_{\rm tr}$, and $\hat{T}$ in Eq. (C3) are the Chapman–Enskog macroscopic quantities. Equation (C2) is a decomposition of $\hat{\mathsf{T}}|_{\hat{h}=\hat{h}_{\rm tot}}$ into the Chapman–Enskog part $(\mathsf{A})_{ij}$, Knudsen-layer part $(\mathsf{B})_{ij}$ up to $O(\epsilon)$, which contains the Chapman–Enskog macroscopic quantities $\hat{\rho}$ and $\hat{p}_{ij}$ in the coefficients, and the remainder of $O(R_h\epsilon^2)$ originating from the remainder $R_h\epsilon^2$ in Eq. (62). Note that $(\mathsf{A})_{ij}$ and $(\mathsf{B})_{ij}$, which are of $O(1)$, depend on ϵ and contain higher-order terms in ϵ. For instance, since $\hat{p}_{ij}$ takes the form of expansion in ϵ, it contains in general higher order terms in ϵ. Equation (C2) with Eq. (C3) is one of the possible expressions that are correct within the error of $O(R_h\epsilon^2)$.

Since Eq. (C2) with Eq. (C3) does not contain the explicit form of the Chapman–Enskog solution, it also holds in the present setting $\theta = \alpha\epsilon \ll 1$ [Eq. (5)]. Therefore, Eqs. (A4)–(A11) in Appendix A in [39] are also the same in the present case.

To be more specific, Eq. (A11) in [39] gives the following expression of the difference appearing in Eq. (C1):

$$\hat{Q}(\hat{f}_{\rm CE} + \hat{f}_{\rm K}) - \hat{Q}(\hat{f}_{\rm CE}) = \hat{A}_{\rm c}(\hat{T})\,\hat{\rho}\,(\hat{\mathcal{G}}^{(1)} - \hat{f}_{\rm K}^{(1)})\epsilon + O(R_f\epsilon^2), \tag{C4}$$

where $\hat{\mathcal{G}}^{(1)}$ is given by Eq. (A10b) in [39], i.e.,

$$\hat{\mathcal{G}}^{(1)} = \hat{\mathcal{G}}^{(0)}\left(\frac{\hat{\rho}_{\rm K}^{(1)}}{\hat{\rho}} - \frac{1}{2}\frac{D^{(1)}}{D^{(0)}} - \frac{\delta}{2}\frac{\hat{T}_{\rm relK}^{(1)}}{\hat{T}_{\rm rel}} + \mathcal{P}^{(1)}\right), \tag{C5}$$

and $\hat{\mathcal{G}}^{(0)}$, $D^{(0)}$, $D^{(1)}$, and $\mathcal{P}^{(1)}$ are defined by Eqs. (A10a), (A6a), (A6b), and (A8b) in [39], respectively (the explicit expressions are omitted here).

The difference between [39] and the present case arises from now on. Using the relations $\hat{T}_{\rm rel} = \hat{T}_{\rm int} + O(\epsilon)$, $\hat{T}_{\rm relK}^{(1)} = \hat{T}_{\rm intK}^{(1)}$ [cf. Eq. (64)], and $\hat{p}_{ij} = \hat{\rho}\hat{T}_{\rm tr}\delta_{ij} + O(\epsilon)$ [cf. Eq. (53a)], we obtain the following expressions in place of Eqs. (A12) and (A13) in [39]:

$$(\mathsf{A})_{ij} = \hat{T}_{\rm tr}\delta_{ij} + O(\epsilon), \tag{C6a}$$

$$(\mathsf{A}^{-1})_{ij} = \hat{T}_{\rm tr}^{-1}\delta_{ij} + O(\epsilon), \tag{C6b}$$

$$(\mathsf{B})_{ij} = \hat{T}_{\rm tr} d_{{\rm K}ij} + O(R_h\epsilon), \tag{C6c}$$

$$(\mathsf{A}^{-1}\mathsf{B}\,\mathsf{A}^{-1})_{ij} = \hat{T}_{\rm tr}^{-1} d_{{\rm K}ij} + O(R_h\epsilon), \tag{C6d}$$

and

$$d_{{\rm K}ij} = -\nu\frac{\hat{\rho}_{\rm K}^{(1)}}{\hat{\rho}}\delta_{ij} + (1-\nu)\frac{\hat{T}_{\rm trK}^{(1)}}{\hat{T}_{\rm tr}}\delta_{ij} + \nu\frac{\hat{p}_{{\rm K}ij}^{(1)}}{\hat{\rho}\hat{T}_{\rm tr}}. \tag{C7}$$

If these relations are used in Eqs. (A6a), (A6b), and (A8b) in [39], they give the following expressions of $D^{(0)}$, $D^{(1)}$, and $\mathcal{P}^{(1)}$ in place of Eq. (A14) in [39]:

$$D^{(0)} = \hat{T}_{\rm tr}^3 + O(\epsilon), \tag{C8a}$$

$$D^{(1)} = \hat{T}_{\rm tr}^3 d_{{\rm K}ii} + O(R_h\epsilon), \tag{C8b}$$

$$\mathcal{P}^{(1)} = \frac{(\zeta_i - \hat{v}_i)(\zeta_j - \hat{v}_j)}{\hat{T}_{\rm tr}} d_{{\rm K}ij} + 2\,\frac{\zeta_i - \hat{v}_i}{\hat{T}_{\rm tr}}\hat{v}_{{\rm K}i}^{(1)} + \frac{\hat{\mathcal{E}}}{\hat{T}_{\rm int}^2}\hat{T}_{\rm intK}^{(1)} + O(R_h S\epsilon), \tag{C8c}$$

where S indicates an appropriate function of $\boldsymbol{\zeta}$ and $\hat{\mathcal{E}}$ that decays fast enough when multiplied by a rapidly decaying function of $\boldsymbol{\zeta}$ and $\hat{\mathcal{E}}$. Using these expressions in Eq. (C5) and noting that $\hat{\mathcal{G}}^{(0)} = \hat{f}^{(0)} + O(\epsilon\hat{f}_{\rm CE})$, we have the following expression of $\hat{\mathcal{G}}^{(1)}$ in place of Eq. (A15) in [39]:

$$\begin{aligned}
\hat{\mathcal{G}}^{(1)} = \hat{f}^{(0)} &\left[\frac{\hat{\rho}_{\rm K}^{(1)}}{\hat{\rho}} - \frac{1}{2}d_{{\rm K}ii} - \frac{\delta}{2}\frac{\hat{T}_{\rm intK}^{(1)}}{\hat{T}_{\rm int}} + \frac{(\zeta_i - \hat{v}_i)(\zeta_j - \hat{v}_j)}{\hat{T}_{\rm tr}} d_{{\rm K}ij}\right.\\
&\left. + 2\frac{(\zeta_i - \hat{v}_i)}{\hat{T}_{\rm tr}}\hat{v}_{{\rm K}i}^{(1)} + \frac{\hat{\mathcal{E}}}{\hat{T}_{\rm int}^2}\hat{T}_{\rm intK}^{(1)}\right] + O(R_f\epsilon)\\
= \hat{f}^{(0)} &\left\{\frac{\hat{\rho}_{\rm K}^{(1)}}{\hat{\rho}} + 2\frac{(\zeta_i - \hat{v}_i)}{\hat{T}_{\rm tr}}\,\hat{v}_{{\rm K}i}^{(1)} + \left[\frac{(\zeta_i - \hat{v}_i)^2}{\hat{T}_{\rm tr}} - \frac{3}{2}\right]\frac{\hat{T}_{\rm trK}^{(1)}}{\hat{T}_{\rm tr}}\right.\\
&+\nu\left[\frac{(\zeta_i - \hat{v}_i)(\zeta_j - \hat{v}_j)}{\hat{T}_{\rm tr}} - \frac{1}{3}\frac{(\zeta_k - \hat{v}_k)^2}{\hat{T}_{\rm tr}}\delta_{ij}\right]\frac{\hat{p}_{{\rm K}ij}^{(1)}}{\hat{\rho}\hat{T}_{\rm tr}}\\
&\left.+\left(\frac{\hat{\mathcal{E}}}{\hat{T}_{\rm int}} - \frac{\delta}{2}\right)\frac{\hat{T}_{\rm intK}^{(1)}}{\hat{T}_{\rm int}}\right\} + O(R_f\epsilon).
\end{aligned} \tag{C9}$$

Equations (C1), (C4), and (C9) lead to Eqs. (66) and (67).

Appendix D: Outline of numerical analysis

In this appendix, we show the outline of numerical analysis of the half-space problem (133).

1. Strategy

We start with the problem (118), which is the original problem of Eq. (133). The boundary condition (118b) contains an unknown constant c_κ^{II} as well as the integral $\int_{\zeta_{\rm n}<0}\zeta_{\rm n}\Phi_\kappa^{II}E(\zeta)d\boldsymbol{\zeta}$, which is also an unknown constant. It is not preferable to handle the boundary condition containing unknown quantities in the numerical analysis. Therefore, we will convert the problem to another form that is more convenient for numerical analysis, following the strategy in [80].

Let us put

$$\widetilde{\Phi}_\kappa^{II} = \Phi_\kappa^{II} + \beta_1 + \beta_2\left(\zeta^2 - \frac{3}{2}\right), \tag{D1a}$$

$$b_\kappa^{II} = 2\sqrt{\pi}\int_{\zeta_{\rm n}<0}\zeta_{\rm n}\Phi_\kappa^{II}E(\zeta)d\boldsymbol{\zeta} - \frac{1}{2}c_\kappa^{II}, \tag{D1b}$$

where β_1 and β_2 are undetermined constants. The b_κ^{II} is also interpreted as an undetermined constant because the integral is not known until the solution Φ_κ^{II} is determined. Then, since $\mathcal{L}_{\rm ES}(1) = \mathcal{L}_{\rm ES}(\zeta^2) = 0$, the problem (118) is recast as

$$\zeta_{\rm n}\frac{\partial\widetilde{\Phi}_\kappa^{II}}{\partial y} = \mathcal{L}_{\rm ES}(\widetilde{\Phi}_\kappa^{II}), \tag{D2a}$$

$$\widetilde{\Phi}_\kappa^{II} = (1-a_c)\widetilde{\mathcal{R}}\widetilde{\Phi}_\kappa^{II} - a_c\left[b_\kappa^* + c_\kappa^*\left(\zeta^2 - \frac{3}{2}\right)\right] + H_\kappa^{II}, \qquad (y=0,\ \zeta_{\rm n}>0), \tag{D2b}$$

$$\widetilde{\Phi}_\kappa^{II} \to \beta_1 + \beta_2\left(\zeta^2 - \frac{3}{2}\right), \qquad (y\to\infty), \tag{D2c}$$

where

$$b_\kappa^* = b_\kappa^{II} - \beta_1, \qquad c_\kappa^* = c_\kappa^{II} - \beta_2. \tag{D3}$$

Since Φ_κ^{II} vanishes rapidly as $y\to\infty$ (this can be confirmed by the numerical result), the following relations hold approximately for a large positive number d:

$$\widetilde{\Phi}_\kappa^{II}(d,\,\zeta_{\rm n},\,\zeta) = \beta_1 + \beta_2\left(\zeta^2 - \frac{3}{2}\right), \tag{D4a}$$

$$\widetilde{\omega}(d) \coloneqq \langle\widetilde{\Phi}_\kappa^{II}(d,\,\zeta_{\rm n},\,\zeta)\rangle = \beta_1, \qquad \widetilde{\tau}_{\rm tr}(d) \coloneqq \frac{2}{3}\left\langle\left(\zeta^2 - \frac{3}{2}\right)\widetilde{\Phi}_\kappa^{II}(d,\,\zeta_{\rm n},\,\zeta)\right\rangle = \beta_2. \tag{D4b}$$

Therefore, the following reflection condition holds at $y=d$:

$$\widetilde{\Phi}_\kappa^{II}(d,\,\zeta_{\rm n},\,\zeta) = \widetilde{\Phi}_\kappa^{II}(d,\,-\zeta_{\rm n},\,\zeta). \tag{D5}$$

Now we consider the boundary-value problem of Eq. (D2a) in a finite domain $0\le y\le d$ ($d\gg 1$) with the boundary conditions (D2b) and (D5), which is expected to have a unique solution for specified constants b_κ^* and c_κ^* [81]. Once the solution $\widetilde{\Phi}_\kappa^{II}$ is obtained numerically, the constants β_1 and β_2 are determined by Eq. (D4b). Then the original solution Φ_κ^{II} and the slip coefficient c_κ^{II} are obtained from Eq. (D1a) and the second of Eq. (D3), respectively. The b_κ^{II}, which is determined by the first of Eq. (D3), gives the value of the integral in Eq. (D1b), which is not relevant to the slip boundary conditions. This is the procedure to obtain the solution Φ_κ^{II} and the slip coefficient c_κ^{II} of the problem (118).

2. Reduced equations

In the actual computation, we solve the version of the problem (D2a), (D2b), and (D5) that corresponds to the problem (133). This version can be derived by the procedure used in the derivation of the problem (133) from that (118). Let us define $(\widetilde{\varphi}_\kappa^{II},\,\widetilde{\psi}_\kappa^{II})$ by Eqs. (124a) and (124b) with $\Phi_\kappa^{II} = \widetilde{\Phi}_\kappa^{II}$, i.e.,

$$\widetilde{\varphi}_\kappa^{II}(y,\,\zeta_{\rm n}) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\widetilde{\Phi}_\kappa^{II}(y,\,\zeta_{\rm n},\,\zeta)e^{-\zeta_{\rm s}^2-\zeta_{\rm t}^2}d\zeta_{\rm t}d\zeta_{\rm s}, \tag{D6a}$$

$$\widetilde{\psi}_\kappa^{II}(y,\,\zeta_{\rm n}) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\zeta_{\rm s}^2+\zeta_{\rm t}^2)\widetilde{\Phi}_\kappa^{II}(y,\,\zeta_{\rm n},\,\zeta)e^{-\zeta_{\rm s}^2-\zeta_{\rm t}^2}d\zeta_{\rm t}d\zeta_{\rm s}. \tag{D6b}$$

Then, the problem to be solved in place of Eq. (133) is obtained as follows:

$$\zeta_{\rm n}\frac{\partial}{\partial y}\begin{bmatrix}\widetilde{\varphi}_\kappa^{II}\\ \widetilde{\psi}_\kappa^{II}\end{bmatrix}=\widetilde{\omega}\begin{bmatrix}1\\ 1\end{bmatrix}+\widetilde{\tau}_{\rm tr}\begin{bmatrix}\zeta_{\rm n}^2-\frac{1}{2}\\ \zeta_{\rm n}^2+\frac{1}{2}\end{bmatrix}$$
$$+\nu\widetilde{P}_{ij}\left(n_in_j-\frac{1}{3}\delta_{ij}\right)\begin{bmatrix}\zeta_{\rm n}^2-\frac{1}{2}\\ \zeta_{\rm n}^2-1\end{bmatrix}-\begin{bmatrix}\widetilde{\varphi}_\kappa^{II}\\ \widetilde{\psi}_\kappa^{II}\end{bmatrix},\quad (0<y<d), \tag{D7a}$$

$$\begin{bmatrix}\widetilde{\varphi}_\kappa^{II}(0,\,\zeta_{\rm n})\\ \widetilde{\psi}_\kappa^{II}(0,\,\zeta_{\rm n})\end{bmatrix}=(1-a_c)\begin{bmatrix}\widetilde{\varphi}_\kappa^{II}(0,\,-\zeta_{\rm n})\\ \widetilde{\psi}_\kappa^{II}(0,\,-\zeta_{\rm n})\end{bmatrix}-a_cb_\kappa^*\begin{bmatrix}1\\ 1\end{bmatrix}-a_cc_\kappa^*\begin{bmatrix}\zeta_{\rm n}^2-\frac{1}{2}\\ \zeta_{\rm n}^2+\frac{1}{2}\end{bmatrix}+\mathbf{G}_\kappa^{II},\quad (\zeta_{\rm n}>0), \tag{D7b}$$

$$\begin{bmatrix}\widetilde{\varphi}_\kappa^{II}(d,\,\zeta_{\rm n})\\ \widetilde{\psi}_\kappa^{II}(d,\,\zeta_{\rm n})\end{bmatrix}=\begin{bmatrix}\widetilde{\varphi}_\kappa^{II}(d,\,-\zeta_{\rm n})\\ \widetilde{\psi}_\kappa^{II}(d,\,-\zeta_{\rm n})\end{bmatrix},\qquad (\zeta_{\rm n}<0), \tag{D7c}$$

where

$$\begin{aligned}&\widetilde{\omega}=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\widetilde{\varphi}_\kappa^{II}e^{-\zeta_{\rm n}^2}d\zeta_{\rm n},\qquad \widetilde{\tau}_{\rm tr}=\frac{2}{3}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\left[\left(\zeta_{\rm n}^2-\frac{3}{2}\right)\widetilde{\varphi}_\kappa^{II}+\widetilde{\psi}_\kappa^{II}\right]e^{-\zeta_{\rm n}^2}d\zeta_{\rm n},\\ &\widetilde{P}_{ij}=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\left[2\zeta_{\rm n}^2\widetilde{\varphi}_\kappa^{II}n_in_j+\widetilde{\psi}_\kappa^{II}(\delta_{ij}-n_in_j)\right]e^{-\zeta_{\rm n}^2}d\zeta_{\rm n},\end{aligned} \tag{D8}$$

and $\mathbf{G}_\kappa^{II}$ is defined by Eq. (134). In contrast to Eq. (125b), $\int_{-\infty}^{\infty}\zeta_{\rm n}^2\widetilde{\varphi}_\kappa^{II}e^{-\zeta_{\rm n}^2}d\zeta_{\rm n}$ is a non-zero constant in general, so that Eq. (D8) differs from Eq. (128).

In summary, we solve the boundary-value problem in the domain $0\le y\le d$, i.e., Eq. (D7), numerically for specified b_κ^* and c_κ^*. Once $\widetilde{\varphi}_\kappa^{II}$ and $\widetilde{\psi}_\kappa^{II}$ are obtained, β_1 and β_2 are determined by Eq. (D4b), that is, $\beta_1=\widetilde{\omega}(d)$ and $\beta_2=\widetilde{\tau}_{\rm tr}(d)$. Then, the solution $(\varphi_\kappa^{II},\,\psi_\kappa^{II};c_\kappa^{II})$ of the original problem (133) is obtained immediately with the help of Eqs. (D1a) and (D3), that is, $\varphi_\kappa^{II}=\widetilde{\varphi}_\kappa^{II}-\beta_1-\beta_2(\zeta_{\rm n}^2-1/2)$, $\psi_\kappa^{II}=\widetilde{\psi}_\kappa^{II}-\beta_1-\beta_2(\zeta_{\rm n}^2+1/2)$, and $c_\kappa^{II}=c_\kappa^*+\beta_2$.

The $\widetilde{P}_{ij}n_in_j$ calculated by the third of Eq. (D8) with $\widetilde{\varphi}_\kappa^{II}=\varphi_\kappa^{II}$ should be zero theoretically [cf. Eq. (125b)]. However, it does not vanish exactly with the numerical solution φ_κ^{II} and takes small values because of the numerical error. These values provide a measure of accuracy of the numerical solution (see Appendix D 5).

3. Finite-difference scheme

We solve Eq. (D7) or its variants, which are essentially the same as the former, by a finite-difference method [82, 83]. We write the problem (D7) symbolically as

$$\begin{aligned}&\zeta_{\rm n}\frac{\partial f}{\partial y}=L(y,\,\zeta_{\rm n})-f,\qquad (0<y<d),\\ &f(0,\,\zeta_{\rm n})=(1-a_c)f(0,\,-\zeta_{\rm n})+G(\zeta_{\rm n}),\qquad (\zeta_{\rm n}>0),\\ &f(d,\,\zeta_{\rm n})=f(d,\,-\zeta_{\rm n}),\qquad (\zeta_{\rm n}<0),\end{aligned} \tag{D9}$$

where $f(y,\,\zeta_{\rm n})$, $L(y,\,\zeta_{\rm n})$, and $G(y,\,\zeta_{\rm n})$ are two-component vectors, and L and G are known [L in Eq. (D7a) is not known as the function of y; however, it is regarded as a known function in the scheme shown below].

Let us denote the grid points in y by $y^{(i)}$ $(i=0,1,\cdots,2N;\ y^{(0)}=0,\ y^{(2N)}=d)$, restrict the range of $\zeta_{\rm n}$ to a finite interval $-Z\le\zeta_{\rm n}\le Z$, and denote the grid points in $\zeta_{\rm n}$ by $\zeta_{\rm n}^{(j)}$ $(j=-2M,-2M+1,\cdots,0,\cdots,2M;\ \zeta_{\rm n}^{(0)}=0,\ \zeta_{\rm n}^{(2M)}=Z,\ \zeta_{\rm n}^{(-j)}=-\zeta_{\rm n}^{(j)})$. We denote the

values of f, L, etc. at the grid points at the nth step of iteration by

$$\begin{aligned} &f_{ij}^{(n)} = f(y^{(i)}, \zeta_{\rm n}^{(j)}), \qquad L_{ij}^{(n)} = L(y^{(i)}, \zeta_{\rm n}^{(j)}), \qquad G_j = G(\zeta_{\rm n}^{(j)}), \\ &h_i^{(n)} = h(y^{(i)}) \quad (h = \widetilde{\omega},\, \widetilde{\tau}_{\rm tr},\, \widetilde{P}_{ij}), \end{aligned} \tag{D10}$$

where the right-hand side of each equation indicates the value at the nth iteration. For $i = 0$, we specially define $f_{0,\pm 0}^{(n)} = f(0,\, \pm 0)$ at the nth step of iteration.

The following finite-difference scheme is applied for Eq. (D9):

$$\zeta_{\rm n}^{(j)} \nabla f_{ij}^{(n+1)} = L_{ij}^{(n)} - f_{ij}^{(n+1)}, \tag{D11}$$

where $\nabla f_{ij}^{(n)}$ indicates the second-order upwind difference for $\partial f/\partial y$ defined by

(a) $\zeta_{\rm n}^{(j)} > 0$

$$\nabla f_{ij}^{(n)} = \begin{cases} \left(f_{1,j}^{(n)} - f_{0,j}^{(n)} \right) / d_1, & (i = 1), \\ w_0(d_i, d_{i-1}) f_{ij}^{(n)} - w_1(d_i, d_{i-1}) f_{i-1,j}^{(n)} + w_2(d_i, d_{i-1}) f_{i-2,j}^{(n)}, & (2 \le i \le 2N), \end{cases} \tag{D12a}$$

(b) $\zeta_{\rm n}^{(j)} < 0$

$$\nabla f_{ij}^{(n)} = \begin{cases} \left(-f_{2N-1,j}^{(n)} + f_{2N,j}^{(n)} \right) / d_{2N}, & (i = 2N - 1), \\ -w_0(d_{i+1}, d_{i+2}) f_{ij}^{(n)} + w_1(d_{i+1}, d_{i+2}) f_{i+1,j}^{(n)} - w_2(d_{i+1}, d_{i+2}) f_{i+2,j}^{(n)}, & \\ \qquad\qquad (0 \le i \le 2N - 2), \end{cases} \tag{D12b}$$

where

$$d_i = y^{(i)} - y^{(i-1)}, \quad w_0(a,b) = \frac{2a+b}{a(a+b)}, \quad w_1(a,b) = \frac{a+b}{ab}, \quad w_2(a,b) = \frac{a}{b(a+b)}.$$

4. Process of computation

Suppose that the macroscopic quantities $h_i^{(n)}$ (thus $L_{ij}^{(n)}$) at the nth step and $f_{2N,j}^{(n)}$ ($j = 1,\, 2,\, \cdots,\, 2M$) at $y = d$ for $\zeta_{\rm n} > 0$ are known. Then, the physical quantities at the $(n+1)$th step are obtained by the following procedure:

(i) From the condition at $y = d$, we set $f_{2N,j}^{(n+1)} = f_{2N,-j}^{(n)}$ ($j = -2M, -2M+1, \cdots, -1$).

(ii) For $\zeta_{\rm n}^{(j)} = 0$, we let $f_{i,0}^{(n+1)} = L_{i,0}^{(n)}$ ($i = 1, 2, \cdots, 2N$) and $f_{0,-0}^{(n+1)} = L_{0,0}^{(n)}$.

(iii) For $\zeta_{\rm n}^{(j)} < 0$, we obtain $f_{ij}^{(n+1)}$ ($j = -2M, -2M+1, \cdots, -1$) for $i = 2N-1, 2N-2, \cdots, 0$ successively using the finite-difference scheme (D11).

(iv) From the boundary condition at $y = 0$, we set $f_{0,j}^{(n+1)} = (1 - a_c) f_{0,-j}^{(n+1)} + G_j$ ($j = 1, 2, \cdots, 2M$) and $f_{0,+0}^{(n+1)} = (1 - a_c) f_{0,-0}^{(n+1)} + G_0$.

(v) For $\zeta_{\rm n}^{(j)} > 0$, we obtain $f_{ij}^{(n+1)}$ ($j = 1, 2, \cdots, 2M$) for $i = 1, 2, \cdots, 2N$ successively using the finite-difference scheme (D11).

(vi) Integrating the obtained $f_{ij}^{(n+1)}$ by Simpson's rule, we obtain the macroscopic quantities $h_i^{(n+1)}$.

We repeat the processes (i) to (vi) until the following convergence criteria are fulfilled: For sufficiently small $\varepsilon_{\rm e}$ (> 0)

$$\max_i \left| h_i^{(n+1)} - h_i^{(n)} \right| < \varepsilon_{\rm e}, \qquad (h = \widetilde{\omega},\, \widetilde{P} \coloneqq \widetilde{\tau}_{\rm tr} + \widetilde{\omega}). \tag{D13}$$

We have set $\varepsilon_{\rm e} = 10^{-10}$ in the present computation.

5. Grid systems and accuracy tests

The following grid systems have been used in the computation.

$$y^{(i)} = \frac{40}{1+c}\left[\frac{2(i/2N_0)^a}{1+(i/2N_0)^{a-1}} + c\,\frac{i}{2N_0}\right], \quad (i = 0, 1, \cdots, 2N), \tag{D14a}$$

$$\zeta_{\rm n}^{(j)} = \frac{5}{1+c_\zeta}\left[\left(\frac{j}{2M_0}\right)^3 + c_\zeta\,\frac{j}{2M_0}\right], \quad (j = -2M, -2M+1, \cdots, 0, \cdots, 2M), \tag{D14b}$$

where $c = 2.5 \times 10^{-6}$ and $c_\zeta > 0$, and a, N_0, N, M_0, and M are positive integers. Different sets of values of the parameters in Eq. (D14), which are listed in Tables VI and VII, have been tried for test computations, and (S1, M1) system has been used to obtain the results shown in Secs. II D, II F, and VII E.

TABLE VI. Parameters for y-grid.

	a	N_0	N	$d\,[= y^{(2N)}]$	$y^{(1)}$	$y^{(2N)} - y^{(2N-1)}$
S1	3	400	420	44.05	2.81(−7)[a]	1.02(−1)
S2	3	200	210	44.05	1.50(−6)	2.05(−1)
S3	3	400	410	42.01	2.81(−7)	1.01(−1)
S4	4	400	420	45.07	1.25(−7)	1.28(−1)

[a] Read as 2.81×10^{-7}.

TABLE VII. Parameters for $\zeta_{\rm n}$-grid.

	c_ζ	M_0	M	$Z\,[= \zeta_{\rm n}^{(2M)}]$	$\zeta_{\rm n}^{(1)}$	$\zeta_{\rm n}^{(2M)} - \zeta_{\rm n}^{(2M-1)}$
M1	0.01	100	104	5.62	2.48(−4)[a]	8.02(−2)
M2	0.01	50	52	5.62	5.00(−4)	1.60(−1)
M3	0.01	100	102	5.30	2.48(−4)	7.71(−2)
M4	0.005	100	104	5.62	1.25(−4)	8.05(−2)

[a] Read as 2.48×10^{-4}.

Our reference grid system is (S1, M1), and the test computation has been carried out by using the following six different systems: (S2, M1) where the number of y-grid is reduced by half; (S3, M1) where the range of y is reduced slightly; (S4, M1) where y-grid is more concentrated around $y = 0$; (S1, M2) where the number of $\zeta_{\rm n}$-grid is reduced by half; (S1, M3) where the range of $\zeta_{\rm n}$ is reduced slightly; (S1, M4) where $\zeta_{\rm n}$-grid is more concentrated around $\zeta_{\rm n} = 0$. Let

$$\begin{aligned} &\max|P_{\rm nn}| = \max_{0\le y\le d}\{|\widetilde{P}_{ij} n_i n_j| \text{ with } \widetilde{\varphi}_\kappa^{II} = \varphi_\kappa^{II}\}, \\ &A_\kappa = \max\left\{[\widetilde{\varphi}_\kappa^{II}]_{y=d},\ [\widetilde{\psi}_\kappa^{II}]_{y=d}\right\}, \qquad B_\kappa = \max\left\{[\widetilde{\varphi}_\kappa^{II}]_{|\zeta_{\rm n}|=Z},\ [\widetilde{\psi}_\kappa^{II}]_{|\zeta_{\rm n}|=Z}\right\}, \end{aligned} \tag{D15}$$

where

$$[f]_{y=d} = \frac{\max\limits_{|\zeta_{\rm n}|\le Z}\left|f(d,\zeta_{\rm n})e^{-\zeta_{\rm n}^2}\right|}{\max\limits_{0\le y\le d,\,|\zeta_{\rm n}|\le Z}\left|f(y,\zeta_{\rm n})e^{-\zeta_{\rm n}^2}\right|}, \qquad [f]_{|\zeta_{\rm n}|=Z} = \frac{\max\limits_{0\le y\le d}\left|f(y,\pm Z)e^{-Z^2}\right|}{\max\limits_{0\le y\le d,\,|\zeta_{\rm n}|\le Z}\left|f(y,\zeta_{\rm n})e^{-\zeta_{\rm n}^2}\right|}. \tag{D16}$$

See the last paragraph in Appendix D 2 about $\max|P_{\rm nn}|$. The quantities $\max|P_{\rm nn}|$, A_κ, and B_κ should be small enough. We have checked the values of c_κ^{II}, $\max|P_{\rm nn}|$, A_κ, and B_κ with (S1, M1) system plus the above six systems for various values of the parameters (ν, a_c). As examples, the results for (ν, a_c) = (−0.5, 0.1) and (0.5, 1) are shown in Tables VIII and IX. The results for other values of (ν, a_c) are more or less of the same order of magnitude. Considering all these results, we have used (S1, M1) system for the main computation.

TABLE VIII. Test for $(\nu,\, a_c) = (-0.5,\, 0.1)$.

	$\kappa = v$				$\kappa = T$			
grid system	c_v^{II}	$\max\|P_{\rm nn}\|$	A_v	B_v	c_T^{II}	$\max\|P_{\rm nn}\|$	A_T	B_T
(S1, M1)	0.22905585	1.9(−7)[a]	3.4(−9)	5.1(−13)	21.445809	3.6(−7)	4.4(−9)	7.4(−12)
(S2, M1)	0.22905639	3.0(−7)	3.4(−9)	5.1(−13)	21.445867	4.0(−6)	4.3(−9)	7.4(−12)
(S3, M1)	0.22905586	1.7(−7)	3.4(−9)	5.1(−13)	21.445809	3.8(−7)	8.0(−9)	7.4(−12)
(S4, M1)	0.22905584	2.0(−7)	3.4(−9)	5.1(−13)	21.445808	4.1(−7)	3.2(−9)	7.4(−12)
(S1, M2)	0.22905585	1.9(−7)	3.4(−9)	5.1(−13)	21.445809	3.6(−7)	4.4(−9)	7.4(−12)
(S1, M3)	0.22905585	1.9(−7)	3.4(−9)	1.4(−11)	21.445809	3.6(−7)	4.3(−9)	1.9(−10)
(S1, M4)	0.22905585	1.9(−7)	3.4(−9)	5.0(−13)	21.445809	3.6(−7)	4.4(−9)	7.3(−12)

[a] Read as 1.9×10^{-7}.

TABLE IX. Test for $(\nu,\, a_c) = (0.5,\, 1)$.

	$\kappa = v$				$\kappa = T$			
grid system	c_v^{II}	$\max\|P_{\rm nn}\|$	A_v	B_v	c_T^{II}	$\max\|P_{\rm nn}\|$	A_T	B_T
(S1, M1)	0.90422215	1.2(−6)[a]	1.9(−9)	4.6(−13)	1.3049195	1.0(−7)	4.9(−9)	7.5(−12)
(S2, M1)	0.90422374	8.4(−6)	1.9(−9)	4.6(−13)	1.3049232	1.6(−6)	4.8(−9)	7.5(−12)
(S3, M1)	0.90422213	1.2(−6)	3.8(−9)	4.6(−13)	1.3049194	9.1(−8)	9.4(−9)	7.5(−12)
(S4, M1)	0.90422225	1.4(−6)	1.3(−9)	4.6(−13)	1.3049195	1.0(−7)	3.4(−9)	7.5(−12)
(S1, M2)	0.90422215	1.2(−6)	1.9(−9)	4.6(−13)	1.3049195	1.0(−7)	4.9(−9)	7.5(−12)
(S1, M3)	0.90422215	1.2(−6)	1.9(−9)	1.3(−11)	1.3049194	9.9(−8)	4.8(−9)	1.9(−10)
(S1, M4)	0.90422215	1.2(−6)	1.9(−9)	4.5(−13)	1.3049195	1.0(−7)	4.9(−9)	7.3(−12)

[a] Read as 1.2×10^{-6}.

[1] C. S. Wang Chang and G. E. Uhlenbeck, Transport phenomena in polyatomic gases, Univ. of Michigan, Ann Arbor, Engineering Research Institute Report CM-681 (1951).

[2] J. H. Ferziger and H. G. Kaper, *Mathematical Theory of Transport Processes in Gases* (North Holland, Amsterdam, 1972).

[3] F. R. W. McCourt, J. J. M. Beenakker, W. E. Köhler, and I. Kuščer, *Nonequilibrium Phenomena in Polyatomic Gases, Volume 1: Dilute Gases* (Clarendon, Oxford, 1990).

[4] J.-F. Bourgat, L. Desvillettes, P. Le Tallec, and B. Perthame, Microreversible collisions for polyatomic gases and Boltzmann's theorem, Eur. J. Mech. B/Fluids **13**, 237–254 (1994).

[5] A. Rossani and G. Spiga, Kinetic theory with inelastic interactions, Transp. Theor. Stat. Phys. **27**, 273–287 (1998).

[6] V. Giovangigli, *Multicomponent Flow Modeling* (Birkhäuser, Boston, 1999).

[7] L. Desvillettes, R. Monaco, and F. Salvarani, A kinetic model allowing to obtain the energy law of polytropic gases in the presence of chemical reactions, Eur. J. Mech. B/Fluids **24**, 219–236 (2005).

[8] E. Nagnibeda and E. Kustova, *Non-Equilibrium Reacting Gas Flows: Kinetic Theory of Transport and Relaxation Processes* (Springer, Berlin, 2009).

[9] G. M. Kremer, *An Introduction to the Boltzmann Equation and Transport Processes in Gases* (Springer, Berlin, 2010).

[10] G. A. Bird, *Molecular Gas Dynamics and the Direct Simulation of Gas Flows* (Oxford University Press, Oxford, 1994).

[11] T. F. Morse, Kinetic model for gases with internal degrees of freedom, Phys. Fluids **7**, 159–169 (1964).

[12] L. H. Holway, Jr., New statistical models for kinetic theory: Methods of construction, Phys. Fluids **9**, 1658–1673 (1966).

[13] V. A. Rykov, A model kinetic equation for a gas with rotational degrees of freedom, Fluid Dyn. **10**, No. 6, 959–966 (1975).

[14] M. Groppi and G. Spiga, Kinetic approach to chemical reactions and inelastic transitions in a rarefied gas, J. Math. Chem. **26**, 197–219 (1999).

[15] H. Struchtrup, The BGK model for an ideal gas with an internal degree of freedom, Transp. Theor. Stat. Phys. **28**, 369–385 (1999).

[16] P. Andries, P. Le Tallec, J.-P. Perlat, and B. Perthame, The Gaussian-BGK model of Boltzmann equation with small Prandtl number, Eur. J. Mech. B/Fluids **19**, 813–830 (2000).

[17] B. Rahimi and H. Struchtrup, Macroscopic and kinetic modelling of rarefied polyatomic gases, J. Fluid Mech. **806**, 437–505 (2016).

[18] M. Bisi and M. J. Cáceres, A BGK relaxation model for polyatomic gas mixtures, Commun. Math. Sci. **14**, 297–325 (2016).

[19] M. Bisi and G. Spiga, On kinetic models for polyatomic gases and their hydrodynamic limits, Ricerche Mat. **66**, 113–124 (2017).

[20] M. Bisi, T. Ruggeri, and G. Spiga, Dynamical pressure in a polyatomic gas: Interplay between kinetic theory and extended thermodynamics, Kinet. Relat. Mod. **11**, 71–95 (2018).

[21] T. Arima, T. Ruggeri, and M. Sugiyama, Extended thermodynamics of rarefied polyatomic gases: 15-field theory incorporating relaxation processes of molecular rotation and vibration, Entropy **20**, 301 (2018).

[22] M. Bisi, R. Monaco, and A. J. Soares, A BGK model for reactive mixtures of polyatomic gases with continuous internal energy, J. Phys. A - Math. Theor. **51**, 125501 (2018).

[23] S. Kosuge, H.-W. Kuo, and K. Aoki, A kinetic model for a polyatomic gas with temperature-dependent specific heats and its application to shock-wave structure, J. Stat. Phys. **177**, 209–251 (2019).

[24] C. Baranger, Y. Dauvois, G. Marois, J. Mathé, J. Mathiaud, and L. Mieussens, A BGK model for high temperature rarefied gas flows, Eur. J. Mech. B/Fluids **80**, 1–12 (2020).

[25] Y. Dauvois, J. Mathiaud, and L. Mieussens, An ES-BGK model for polyatomic gases in rotational and vibrational nonequilibrium, Eur. J. Mech. B/Fluids **88**, 1–16 (2021).

[26] R. S. Myong, A generalized hydrodynamic computational model for rarefied and microscale diatomic gas flows, J. Comput. Phys. **195**, 655–676 (2004).

[27] R. S. Myong, Coupled nonlinear constitutive models for rarefied and microscale gas flows: Subtle interplay of kinematics and dissipation effects, Continuum Mech. Thermodyn. **21**, 389–399 (2009).

[28] D. Bruno and V. Giovangigli, Relaxation of internal temperature and volume viscosity, Phys. Fluids **23**, 093104 (2011); Erratum, Phys. Fluids **25**, 039902 (2013).

[29] T. Arima, S. Taniguchi, T. Ruggeri, and M. Sugiyama, Extended thermodynamics of real gases with dynamic pressure: an extension of Meixner's theory, Phys. Lett. A **376**, 2799–2803 (2012).

[30] T. Arima, T. Ruggeri, M. Sugiyama, and S. Taniguchi, Non-linear extended thermodynamics of real gases with 6 fields, Int. J. Non-Linear Mech. **72**, 6–15 (2015).

[31] T. Ruggeri and M. Sugiyama, *Rational Extended Thermodynamics beyond the Monatomic Gas* (Springer International Publishing Switzerland, 2015).

[32] T. Ruggeri, Non-linear maximum entropy principle for a polyatomic gas subject to the dynamic pressure, Bulletin of the Institute of Mathematics, Academia Sinica (New Series), **11**, 1–22 (2016).

[33] E. Kustova, M. Mekhonoshina, and A. Kosareva, Relaxation processes in carbon dioxide, Phys. Fluids **31**, 046104 (2019).

[34] E. Kustova and M. Mekhonoshina, Multi-temperature vibrational energy relaxation rates in CO_2, Phys. Fluids **32**, 096101 (2020).

[35] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-uniform Gases*, 3rd edition (Cambridge Univ. Press, Cambridge, 1991).

[36] K. Aoki, M. Bisi, M. Groppi, and S. Kosuge, Two-temperature Navier–Stokes equations for a polyatomic gas derived from kinetic theory, Phys. Rev. E **102**, 023104 (2020).

[37] S. Brull and J. Schneider, On the ellipsoidal statistical model for polyatomic gases, Continuum Mech. Thermodyn. **20**, 489–508 (2009).

[38] K. Aoki, M. Bisi, M. Groppi, and S. Kosuge, A note on the steady Navier–Stokes equations derived from an ES-BGK model for a polyatomic gas, Fluids **6**, 32 (2021).

[39] M. Hattori, S. Kosuge, and K. Aoki, Slip boundary conditions for the compressible Navier–Stokes equations for a polyatomic gas, Phys. Rev. Fluids **3**, 063401 (2018).

[40] K. Aoki, C. Baranger, M. Hattori, S. Kosuge, G. Martalò, J. Mathiaud, and L. Mieussens, Slip boundary conditions for the compressible Navier–Stokes equations, J. Stat. Phys. **169**, 744–781 (2017).

[41] P. L. Bhatnagar, E. P. Gross, and M. Krook, A model for collision processes in gases. I. Small amplitude processes in charged and neutral one-component systems, Phys. Rev. **94**, 511–525 (1954).

[42] P. Welander, On the temperature jump in a rarefied gas, Ark. Fys. **7**, 507–553 (1954).

[43] C. E. Siewert and F. Sharipov, Model equations in rarefied gas dynamics: Viscous-slip and thermal-slip coefficients, Phys. Fluids **14**, 4123–4129 (2002).

[44] S. Takata, M. Hattori, and T. Hasebe, Slip/jump coefficients and Knudsen-layer corrections for the ES model in the generalized slip-flow theory, in *30th International Symposium on Rarefied Gas Dynamics*, AIP Conf. Proc. 1786 (AIP, Merville, 2016), 040004.

[45] S. Takata and M. Hattori, Numerical data for the generalized slip-flow theory, Kyoto Univ. Res. Inform. Repository: http://hdl.handle.net/2433/199811.

[46] Jpn. Soc. Thermophys. Properties (ed.), *Thermophysical Properties Handbook* (Yokendo, Tokyo, 1990) (in Japanese).

[47] M. S. Cramer, Numerical estimates for the bulk viscosity of ideal gases, Phys. Fluids **24**, 066102

(2012).
[48] A. Boushehri, J. Bzowski, J. Kestin, and E. A. Mason, Equilibrium and transport properties of eleven polyatomic gases at low density, J. Phys. Chem. Ref. Data **16**, 445–466 (1987); Erratum, J. Phys. Chem. Ref. Data **17**, 255 (1988).
[49] S. Takata, H. Funagane, and K. Aoki, Fluid modeling for the Knudsen compressor: Case of polyatomic gases, Kinet. Relat. Mod. **3**, 353–372 (2010).
[50] H. Funagane, S. Takata, K. Aoki, and K. Kugimoto, Poiseuille flow and thermal transpiration of a rarefied polyatomic gas through a circular tube with applications to microflows, Bollettino dell'Unione Matematica Italiana Ser. 9, **4**, 19–46 (2011).
[51] S. Park and S.-B. Yun, Entropy production estimates for the polyatomic ellipsoidal BGK model, Appl. Math. Lett. **58**, 26–33 (2016).
[52] S. Park and S.-B. Yun, Cauchy problem for the ellipsoidal BGK model for polyatomic particles, J. Differ. Equ. **266**, 7678–7708 (2019).
[53] S. Kosuge and K. Aoki, Shock-wave structure for a polyatomic gas with large bulk viscosity, Phys. Rev. Fluids **3**, 023401 (2018).
[54] G. Emanuel, Bulk viscosity of a dilute polyatomic gas, Phys. Fluids A **2**, 2252–2254 (1990).
[55] P. A. Thompson, *Compressible-Fluid Dynamics* (McGraw-Hill, New York, 1972).
[56] G. Billet, V. Giovangigli, and G. de Gassowski, Impact of volume viscosity on a shock-hydrogen-bubble interaction, Combust. Theory Model **12**, 221–248 (2008).
[57] W. E. Meador, G. A. Miner, and L. W. Townsend, Bulk viscosity as a relaxation parameter: Fact or fiction?, Phys. Fluids **8**, 258–261 (1996).
[58] Y. Sone, *Kinetic Theory and Fluid Dynamics* (Birkhäuser, Boston, 2002); Supplementary Notes and Errata: http://hdl.handle.net/2433/66099.
[59] Y. Sone, *Molecular Gas Dynamics: Theory, Techniques, and Applications* (Birkhäuser, Boston, 2007); Supplementary Notes and Errata: http://hdl.handle.net/2433/66098.
[60] H. Grad, Singular and nonuniform limits of solutions of the Boltzmann equation, in *Transport Theory*, edited by R. Bellman, G. Birkhoff, and I. Abu-Shumays (American Mathematical Society, Providence, 1969), pp. 269–308.
[61] C. Bardos, R. E. Caflisch, and B. Nicolaenko, The Milne and Kramers problems for the Boltzmann equation of a hard sphere gas, Commun. Pure Appl. Math. **39**, 323–352 (1986).
[62] C. Cercignani, Half-space problems in the kinetic theory of gases, in *Trends in Applications of Pure Mathematics to Mechanics*, Lecture Notes in Physics 249, edited by E. Kröner and K. Kirchgässner (Springer, Berlin, 1986), pp. 35–50.
[63] F. Golse and F. Poupaud, Stationary solutions of the linearized Boltzmann equation in a half-space, Math. Methods Appl. Sci. **11**, 483–502 (1989).
[64] K. Aoki, Y.-C. Lin, and K.-C. Wu, Milne problem for the linear and linearized Boltzmann equations relevant to a binary gas mixture, J. Differ. Equ. **269**, 257–287 (2020).
[65] Y. Sone, T. Ohwada, and K. Aoki, Temperature jump and Knudsen layer in a rarefied gas over a plane wall: Numerical analysis of the linearized Boltzmann equation for hard-sphere molecules, Phys. Fluids A **1**, 363–370 (1989).
[66] T. Ohwada, Y. Sone, and K. Aoki, Numerical analysis of the shear and thermal creep flows of a rarefied gas over a plane wall on the basis of the linearized Boltzmann equation for hard-sphere molecules, Phys. Fluids A **1**, 1588–1599 (1989).
[67] C. K. Chu, Kinetic-theoretic description of the formation of a shock wave, Phys. Fluids **8**, 12–22 (1965).
[68] C. Cercignani, The Kramers problem for a not completely diffusing wall, J. Math. Anal. Appl. **10**, 568–586 (1965).
[69] Y. Onishi, Effects of accommodation coefficient on shear flow of rarefied gas, Trans. Japan Soc. Aero. Space Sci. **17**, 93–98 (1974).
[70] S. K. Loyalka, N. Petrellis, and T. S. Storvick, Some numerical results for the BGK model: Thermal creep and viscous slip problems with arbitrary accomodation at the surface, Phys. Fluids **18**, 1094–1099 (1975).
[71] C. E. Siewert, Kramers' problem for a variable collision frequency model, Euro. J. Appl. Math. **12**, 179–191 (2001).
[72] C. Cercignani, Elementary solutions of the linearized gas-dynamics Boltzmann equation and their application to the slip-flow problem, Ann. Phys. (NY) **20**, 219–233 (1962).
[73] Y. Sone, Kinetic theory analysis of linearized Rayleigh problem, J. Phys. Soc. Jpn. **19**, 1463–1473 (1964).
[74] Y. Sone, A note on thermal creep in rarefied gas, J. Phys. Soc. Jpn. **29**, 1655 (1970).
[75] Y. Onishi, Effect of accommodation coefficient on thermal creep flow of rarefied gas, Trans. Japan Soc. Aero. Space Sci. **15**, 117–123 (1972).
[76] Y. Sone, Thermal creep in rarefied gas, J. Phys. Soc. Jpn. **21**, 1836–1837 (1966).
[77] M. Abramowitz and I. A. Stegun (eds.), *Handbook of Mathematical Functions* (Dover, New York, 1970), p. 1001.

[78] M. N. Kogan, *Rarefied Gas Dynamics* (Plenum, New York, 1969).
[79] C. Cercignani, *The Boltzmann Equation and Its Applications* (Springer, Berlin, 1988).
[80] Y. Sone, T. Ohwada, and K. Aoki, Evaporation and condensation on a plane condensed phase: Numerical analysis of the linearized Boltzmann equation for hard-sphere molecules, Phys. Fluids A **1**, 1398–1405 (1989).
[81] C. Cercignani, Existence and uniqueness in the large for boundary value problems in kinetic theory, J. Math. Phys. **8**, 1653–1656 (1967).
[82] K. Aoki, Y. Sone, and T. Yamada, Numerical analysis of gas flows condensing on its plane condensed phase on the basis of kinetic theory, Phys. Fluids A **2**, 1867–1878 (1990).
[83] K. Aoki, K. Nishino, Y. Sone, and H. Sugimoto, Numerical analysis of steady flows of a gas condensing on or evaporating from its plane condensed phase on the basis of kinetic theory: Effect of gas motion along the condensed phase, Phys. Fluids A **3**, 2260–2275 (1991).