REGULARITY FOR MINIMIZERS OF DOUBLE PHASE FUNCTIONALS WITH MILD TRANSITION AND REGULAR COEFFICIENTS

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Abstract. We prove sharp regularity results for minimizers of the functional

$$\mathcal{P}(w,\Omega) := \int_\Omega b(x,w)\left[|Dw|^p + a(x)|Dw|^p \log(e+|Dw|)\right] dx\,,$$

with $w \in W^{1,1}(\Omega)$, $p>1$, $a \in L^\infty(\Omega)$, $a(\cdot) \geq 0$, and $0<\nu \leq b(\cdot,\cdot) \leq L$.

$\mathcal{P}$ is a double phase functional with mild transition between $|Du|^p$ and $|Du|^p \log(e+|Du|)$. First, under suitable conditions on the moduli of continuity of $a(\cdot)$ and $b(\cdot,\cdot)$, we prove that local minimizers are of class $C^{0,\alpha}$ for every $\alpha \in (0,1)$, then that they are of class $C^{1,\alpha}$ for some $\alpha>0$, provided the functions $a(\cdot)$ and $b(\cdot,\cdot)$ are Hölder continuous.

1. Introduction and results

A major topic in Calculus of Variations is the study of the regularity of minimizers of integral functionals as

$$\mathcal{F}(w,\Omega) := \int_\Omega F(x,w,Dw)\,dx\ ,$$

where Ω is an open subset of $\mathbb{R}^n$ and $F\colon \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ a Carathéodory function satisfying the following growth conditions:

$$\frac{1}{c}\,|z|^p \leq F(x,v,z) \leq c\,(|z|^q\,+1) \tag{1.1}$$

for a.e. $x\in\Omega$ and for all $(v,z)\in\mathbb{R}\times\mathbb{R}^n$, with $1<p\leq q$, $c\geq 1$. If $p=q$ in (1.1) we say that the functional has standard growth conditions of order p. On the other hand, if F depends only on the gradient, i.e. $F(x,v,z)\equiv F(z)$, $\mathcal{F}$ is called an autonomous functional. The most natural autonomous functional with p-growth is clearly

$$\mathcal{F}_p(w,\Omega) := \int_\Omega |Dw|^p\,dx\,,$$

whose corresponding Euler-Lagrange equation is the well-known p-Laplace equation

$$\operatorname{div}(|Dw|^{p-2}Dw) = 0\,.$$

The maximal regularity of minimizers of the functional $\mathcal{F}_p$ is $C^{1,\alpha}$ for some exponent $\alpha\in(0,1)$ depending only on p and on the dimension n, as shown by Ural'tseva in 1968 [49]. The corresponding result in the vectorial case has been instead obtained by Uhlenbeck [48]. Subsequently, the issue has by now been widely developed through extensive literature, also in the evolutionary case. We refer to [37, 39] for an overview on the elliptic case, and to [19, 36] in the parabolic case. We refer to [27, 28, 38] for general information about regularity theory. In the case $p<q$ we obtain the so-called functionals with non-standard growth conditions of (p,q)-type, as initially defined and studied by Marcellini [40, 41, 42]; these may be anisotropic, e.g. also $F(Dw)=\sum_{i=1}^n |D_i w|^{p_i}\,dx$, $p_i>1$ may be considered (see [1, 50]).

In his seminal works [51, 52], Zhikov studied some kinds of non-autonomous functionals with non-standard growth, whose integrands change their ellipticity rate according to the position. He was interested in describing the behaviour of strongly anisotropic materials in the context of homogenisation, nonlinear elasticity and Lavrentiev phenomenon. In particular, in [51] Zhikov considered, among others, the following models functionals:

$$\mathcal{F}_{p(x)}(w,\Omega) := \int_\Omega |Dw|^{p(x)}\,dx\,, \qquad 1<p_1\le p(x)\le p_2<\infty\,, \tag{1.2}$$

$$\mathcal{F}_{p,q}(w,\Omega) := \int_\Omega [\,|Dw|^p + a(x)|Dw|^q\,]\,dx \qquad 0\le a(x)\le L \quad 1<p<q\,. \tag{1.3}$$

Functionals like (1.2) and (1.3) can be used to describe the behaviour of strongly anisotropic materials. Similar models appear in the study of non-Newtonian fluids, that change their viscosity in the presence of an electro-magnetic field (see [3]) or in image segmentation problems [31]. When referring to (p,q)-growth conditions, the quantity $q/p>1$ is called the gap (ratio) of the integrand F. In order to ensure the regularity of minima, the gap cannot differ too much from 1, in other words the distance between p and q cannot be too large (see the counterexamples [25, 26, 42] concerning both autonomous and non-autonomous functionals). In the last years there has been a considerable amount of interest in functionals with (p,q)-growth, see for instance [8, 7, 6, 11, 12, 18, 22, 35, 44, 47].

In the autonomous case the bound on the gap depends on the space dimension n. Otherwise, for non-autonomous functionals the presence of x cannot be treated as a perturbation, as in the standard $p=q$ case, since x directly influences the growth of the integrand. The effect of x can thus be very relevant and the regularity of minima is ruled by a subtle interaction between the regularity of F with respect to the x-variable and the gap q/p. Confirming this, in [5] Baroni, Colombo and Mingione provides a complete regularity theory for the functional $\mathcal{F}_{p,q}$ in (1.3) under the assumptions

$$0\le a(\cdot)\in C^{0,\alpha}(\Omega) \quad \textit{and} \quad \frac{q}{p}\le 1+\frac{\alpha}{n}\,, \tag{1.4}$$

where the bound on the gap is sharp by the counterexamples in [25]. Further contributions to the regularity theory of double phase functionals can be found in [10, 17, 21, 31, 33, 45]; we recommend the reader the references [30, 46] for a list of developments on problems with strong anisotropicity.

In the case of the functional $\mathcal{F}_{p(x)}$ in (1.2), the Hölder continuity of minimizers can be proven under a suitable assumption on the logarithmic modulus of continuity $\omega(\cdot)$ on the exponent $p(x)$, which is a non-decreasing continuous function such that

$$\omega(0)=0 \qquad \text{and} \qquad |p(x)-p(y)|\le\omega(|x-y|) \quad \text{holds for every } x,y\in\Omega\,.$$

The log-continuity assumption prescribes that

$$\limsup_{r\to 0}\,\omega(r)\log\left(\frac{1}{r}\right) =: l \;<\; \infty\,; \tag{1.5}$$

assumption (1.5) is crucial: Zhikov proved in [51] that if condition (1.5) is violated, then minimizers can be discontinuous. Moreover, it holds that

- [24] if $l<\infty$, then $u\in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha\in(0,1)$;
- [2] if $l=0$, then $u\in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for every $\alpha\in(0,1)$.

On the other hand, as done in order to prove Hölder continuity of the gradient (the result is proven in [16] by the author and Mingione) it is unavoidable to assume that $p(x)$ is itself Hölder continuous, that is

$$\omega(r)\le L\,r^\alpha\,, \qquad \alpha>0\,,$$

which is obviously stronger than (1.5). This result is sharp in the sense that if $p(x)$ is not Hölder continuous then the gradient is not even continuous in general, as shown in [33]. See also [14] for a further contribution to the issue. In the recent paper [4] another significant model example with non-standard (p, q)-growth is considered:

$$\mathcal{P}_{\log}(w, \Omega) := \int_\Omega \left[|Dw|^p + a(x)|Dw|^p \log(e + |Dw|) \right] dx \,, \tag{1.6}$$

where the non-negative function $a(\cdot)$ is supposed to be bounded. This functional shares features both with $\mathcal{P}_{p(x)}$ and $\mathcal{P}_{p,q}$ introduced in (1.2) and (1.3): the structure resembles that of $\mathcal{P}_{p,q}$, but with a mild phase transition between $|Du|^p$ and $|Du|^p \log(e + |Du|)$. Then, to avoid discontinuity of minima, Hölder continuity of the coefficient $a(\cdot)$ is not needed but, again, a log-continuity assumption on $a(\cdot)$ is required, exactly as for the exponent $p(x)$ of the functional $\mathcal{F}_{p(x)}$. For a contribution to the study of the functional $\mathcal{P}_{\log}$ see also [12]. Indeed, in analogy with the regularity theory of $p(x)$-growth functional, we have the following:

Theorem 1.1 ([4]). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional $\mathcal{P}_{\log}$ defined in* (1.6) *and assume that the function $a(\cdot)$ is non-negative and bounded. Let $\omega(\cdot)$ be a modulus of continuity of $a(\cdot)$ and denote*

$$l := \limsup_{r \to 0} \omega(r) \log\left(\frac{1}{r}\right) \,. \tag{1.7}$$

Then

- *if $l < \infty$, then $u \in C^{0,\beta}_{\mathrm{loc}}(\Omega)$ for some $\beta \in (0, 1)$;*
- *if $l = 0$, then $u \in C^{0,\beta}_{\mathrm{loc}}(\Omega)$ for every $\beta \in (0, 1)$;*
- *if $\omega(r) \lesssim r^\sigma$ with $\sigma \in (0, 1)$, then Du is locally Hölder continuous in Ω .*

In the sequel, to simplify the notation, we will denote

$$H(x, z) := |z|^p + a(x)|z|^p \log(e + |z|) \text{ so that } \mathcal{P}_{\log}(w, \Omega) := \int_\Omega H(x, Dw)\, dx \,. \tag{1.8}$$

The first assertion of Theorem 1.1 holds for a larger family of functionals defined for $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ as

$$\mathcal{F}(w, \Omega) := \int_\Omega F(x, w, Dw)\, dx \,, \tag{1.9}$$

where the energy density $F \colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function satisfying the following growth conditions: $\nu \; H(x, z) \leq F(x, v, z) \leq L \; H(x, z)$ whenever $x \in \Omega$, $v \in \mathbb{R}$ and $z \in \mathbb{R}^n$, with $0 < \nu \leq 1 \leq L$. Indeed, it is proven in [4, Theorem 4.1], that a local minimizer $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ of the functional $\mathcal{F}$ defined in (1.9) is locally Hölder continuous under the only assumption on the modulus of continuity of $a(\cdot)$ that $l < \infty$, with l being defined in (1.7). In the same theorem also local higher integrability of $H(\cdot, Du(\cdot))$ is proven.

Let us recall that, due to the (p, q)-growth conditions satisfied by the integrand F, the following more general definition of local minimizer is usually adopted for the functional $\mathcal{F}$ introduced in (1.9).

Definition 1. A function $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a local minimizer of the functional $\mathcal{F}$, introduced in (1.9), if and only if $F(\cdot, u(\cdot), Du(\cdot)) \in L^1_{\mathrm{loc}}(\Omega)$ and the minimality condition

$$\int_{\mathrm{supp}\, \phi} F(x, u, Du)\, dx \leq \int_{\mathrm{supp}\, \phi} F(x, u + \phi, Du + D\phi)\, dx$$

is satisfied for any variation $\phi \in W^{1,1}_{\mathrm{loc}}(\Omega)$ such that $\mathrm{supp}\, \phi \subset \Omega$.

In this paper we shall consider a family of functionals based on the model functional $\mathcal{P}_{\log}$, defined in (1.6), allowing for dependence on the variables x and w via a coefficient $b(x,w)$ which is assumed to be positive and bounded from above and from below (see (1.11)). The space dimension will always be $n \geq 2$. We shall indeed consider the functional defined by

$$\begin{aligned}(1.10)\qquad \mathcal{P}(w,\Omega) &:= \int_\Omega b(x,w)\ \left[|Dw|^p + a(x)|Dw|^p \log(e+|Dw|)\right] dx \\ &= \int_\Omega b(x,w)\ H(x,Dw)\, dx\,,\end{aligned}$$

with the notation introduced in (1.8). In the rest of the paper the coefficients $a(\cdot)$ and $b(\cdot,\cdot)$ satisfy the following assumption:

Assumption 1

- the function $a\colon \Omega \to \mathbb{R}$ is measurable, non-negative and bounded:
$$0 \leq a(\cdot) \leq ||a||_{L^\infty(\Omega)} < \infty\,,$$
- the function $b\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a positive Carathéodory function bounded from above and from below:

$$(1.11)\qquad 0 < \nu \leq b(\cdot,\cdot) \leq L < \infty \qquad \text{with } 0 < \nu \leq 1 \leq L\,.$$

Notice that, by Definition 1 and the bounds (1.11) on $b(\cdot,\cdot)$, any local minimizer u is in $W^{1,p}_{\rm loc}(\Omega)$ with $H(\cdot, Du(\cdot)) \in L^1_{\rm loc}(\Omega)$. Since all the forthcoming results are local in nature, the global integrability of local minimizers and of their energy will be assumed with no loss of generality and for this reason we shall several times assume that local minimizers u are directly in $W^{1,p}(\Omega)$ with $H(\cdot, Du(\cdot)) \in L^1(\Omega)$.

In order to prove any regularity result for local minimizers of $\mathcal{P}$ we need to assume that the function $a(\cdot)$ is uniformly continuous on Ω. Then let $\omega_a : [0,+\infty[\to [0,+\infty[$ be a modulus of continuity of $a(\cdot)$, that is a non-decreasing continuous function such that $\omega_a(0) = 0$ and

$$(1.12)\qquad |a(x) - a(y)| \leq \omega_a(|x-y|) \qquad \text{for every } x, y \in \Omega\,.$$

We stress that, in order to prove the first assertion of Theorem 1.2, no assumptions on the function $b(\cdot,\cdot)$ are needed, except for Assumption 1.

Otherwise, to prove higher regularity results we require uniform continuity also of the coefficient $b(\cdot,\cdot)$. Thus, let $\omega_b\colon [0,+\infty[\to [0,+\infty[$ be a modulus of continuity of $b(\cdot,\cdot)$, that is a non-decreasing continuous function such that $\omega_b(0) = 0$ and

$$(1.13)\quad |b(x,v) - b(y,w)| \leq \omega_b(|x-y| + |v-w|) \text{ for every } x, y \in \Omega \text{ and } v, w \in \mathbb{R}\,.$$

In both cases, without loss of generality we may assume that $\omega_a(\cdot)$ and $\omega_b(\cdot)$ are concave.

The main result of the paper is the following

Theorem 1.2. *Let $u \in W^{1,1}_{\rm loc}(\Omega)$ be a local minimizer of the functional $\mathcal{P}$ defined in* (1.10)*; let us assume that the functions $a(\cdot)$ and $b(\cdot,\cdot)$ satisfy Assumption 1, that $\omega_a(\cdot)$ is a modulus of continuity of $a(\cdot)$ as defined in* (1.12)*, and denote*

$$(1.14)\qquad l := \limsup_{r\to 0}\, \omega_a(r) \log\left(\frac{1}{r}\right)\,.$$

Then

- *if $l < \infty$, then $u \in C^{0,\alpha}_{\rm loc}(\Omega)$ for some $\alpha \in (0,1)$.*

In addition, if $\omega_b(\cdot)$ is a modulus of continuity of $b(\cdot,\cdot)$ as defined in (1.13)*, we have*

- *if* $l = 0$, *then* $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ *for every* $\alpha \in (0,1)$ *;*
- *if* $\omega_a(r) \lesssim r^{\sigma_1}$ *and* $\omega_b(r) \lesssim r^{\sigma_2}$ *with* $\sigma_1, \sigma_2 \in (0,1)$, *then* Du *is locally Hölder continuous in* Ω *.*

We point out that the continuity of the coefficient $b(\cdot,\cdot)$, together with the condition $l = 0$, is needed to prove the second assertion in Theorem 1.2. We also notice that the real new facts in Theorem 1.2 are the second and the third one, the first one being actually a consequence of [4, Theorem 4.1] and we have included it for the sake of completeness. This has been in fact already remarked after Theorem 1.1.

It is worth remarking that a vectorial version of Theorem 1.2 is obtainable. Specifically, the second and the third statements in Theorem 1.2 still hold in the case of vector valued minimizers, provided $b(\cdot)$ only depends on x, and it is w-independent. The proof of this fact follows along the lines of that given here in the scalar case. The w-independence of $b(\cdot)$ is not a technical assumption as, already in the case of quadratic functionals, minimizers might exhibit singularities in the case of explicit w-dependence of the integrand. For this, we refer to the classical counterexample in [29]. For more results concerning double phase type functionals in the vectorial case, and estimates involving potentially singular minimizers, we refer to [21].

In Section 3 (Theorem 3.1) Hölder continuity of minima is obtained, for some possibly small exponent, under the assumption $l < \infty$, with l being defined in (1.14), as a particular case of [4, Theorem 4.1]. The remaining higher regularity results, including gradient Hölder continuity, are then proven in Section 4. More precisely, the second assertion in Theorem 1.2 will be proven in Paragraph 4.3 while the last one will be proven in Paragraph 4.4. The proofs of the results are based on a comparison lemma (Paragraph 4.1) and a decay estimate (Paragraph 4.2).

2. Preliminaries

2.1. **Notation and elementary properties.** In what follows we denote by c a general positive constant possibly varying from line to line; special occurrences will be denoted by c_0, c_1 or the like. All such constants will always be *larger or equal than one*; moreover relevant dependencies on parameters will be emphasised using parentheses, i.e., $c \equiv c(n, p, \delta)$ means that c depends on n, p, δ. We denote by

$$B_r(x_0) := \{x \in \mathbb{R}^n \, : \, |x - x_0| < r\}$$

the open ball with center x_0 and radius $r > 0$; when not important, or clear from the context, we shall omit denoting the center just denoting as follows: $B_r \equiv B_r(x_0)$. Unless otherwise stated, different balls in the same context will have the same center. With $\mathcal{B} \subset \mathbb{R}^n$ being a measurable set with positive, finite measure $|\mathcal{B}| > 0$, and with $g\colon \mathcal{B} \to \mathbb{R}^k$, $k \geq 1$, being a locally integrable map, we shall denote by

$$(g)_{\mathcal{B}} \equiv \fint_{\mathcal{B}} g(x)\,dx := \frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} g(x)\,dx$$

its integral average. A well-known property is the following: for any $g \in L^p(\mathcal{B}, \mathbb{R}^k)$, $p \geq 1, k \geq 1$, the estimate

$$\fint_{\mathcal{B}} |g(x) - (g)_{\mathcal{B}}|^p\,dx \leq 2^p \fint_{\mathcal{B}} |g(x) - \zeta|^p\,dx \tag{2.1}$$

holds for each $\zeta \in \mathbb{R}^n$.

For $g\colon \mathcal{B} \to \mathbb{R}$ being a measurable bounded map, we denote its oscillation by

$$\operatorname*{osc}_{\mathcal{B}} g = \sup_{\mathcal{B}} g - \inf_{\mathcal{B}} g \; . \tag{2.2}$$

Given a bounded open subset $A \subset \mathbb{R}^n$ and $\alpha \in (0,1]$, a map $g : A \to \mathbb{R}^k$ is said to be α-Hölder continuous if there exists a constant $C_\alpha > 0$ such that

$$|g(x) - g(y)| \le C_\alpha \, |x-y|^\alpha \qquad \text{for every } x, y \in A\,. \tag{2.3}$$

We denote by $C^{0,\alpha}(A, \mathbb{R}^k)$ the space of α-Hölder continuous maps $g : A \to \mathbb{R}^k$; in this space we can consider the Hölder seminorm

$$[g]_{\mathcal{C}^{0,\alpha}(A,\mathbb{R}^k)} := \sup_{\substack{x,y\in A\\ x\neq y}} \frac{|g(x)-g(y)|}{|x-y|^\alpha} \tag{2.4}$$

and the norm $\|g\|_{\mathcal{C}^{0,\alpha}} = [g]_{\mathcal{C}^{0,\alpha}} + \|g\|_{L^\infty}$. In the above definition (2.4) we can fix $d > 0$ and consider only points $x, y \in A$ such that $|x - y| \le d$: indeed if $|x-y| > d$ every quotient is bounded by $2\, d^{-\alpha} \|g\|_{L^\infty}$.

The Sobolev exponent p^* is $np/(n-p)$ if $p < n$ or every number larger than p, in the case $p \ge n$.

It is well known (see for instance [27, Proposition 3.23]) that if a function g belongs to the Sobolev space $W^{1,p}(A)$, then $g^+ = \max\{g, 0\} \in W^{1,p}(A)$ with

$$Dg^+ = \begin{cases} Dg & \text{a.e. on } \{g > 0\} \\ 0 & \text{a.e. on } \{g \le 0\} \end{cases} \tag{2.5}$$

and $Dg = 0$ a.e. on $\{g = 0\}$. Then it is easy to prove that

$$\text{if } g_h \to g \text{ strongly in } W^{1,p}(A)\,, \text{ then } g_h^+ \to g^+ \text{ strongly in } W^{1,p}(A)\,. \tag{2.6}$$

Indeed, the convergence $g_h^+ \to g^+$ in $L^p(A)$ follows from $g_h^+ = \frac{1}{2}(|g_h| + g_h)$, while the fact that $Dg_h^+ \to Dg^+$ in $L^p(A)$ from $|Dg_h^+| \le |Dg_h|$, $Dg_h^+ \to Dg^+$ a.e. on A and a well-known variant of the Lebesgue's dominated convergence theorem.

We denote by $z \otimes z$ the tensor product, i.e., the matrix $z \otimes z = (z_i z_j)_{i,j=1,\ldots,n}$, with $z \in \mathbb{R}^n$. It is easy to verify that the equality

$$\langle (z \otimes z)\lambda\,,\, \lambda \rangle = (\langle z\,,\, \lambda\rangle)^2 \tag{2.7}$$

holds for every $\lambda \in \mathbb{R}^n$, where $\langle \cdot\,,\, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$.

We recall some useful properties of the logarithm function of later frequent use. For every $t \ge 0$ we have

$$\log(e+t) \ge 1\,, \tag{2.8}$$

$$\log(e+t) \le 2\,\log t \qquad \text{for every } t \ge e\,, \tag{2.9}$$

$$\log(e+At) \le A \log(e+t) \qquad \text{for every } A \ge 1\,, \tag{2.10}$$

$$\log(e+At) \ge A \log(e+t) \qquad \text{for every } 0 < A \le 1\,. \tag{2.11}$$

In addition

$$\log(A/R) \le 2\log(1/R) \qquad \text{for every } A > 0 \text{ and every } 0 < R \le 1/A\,. \tag{2.12}$$

2.2. ***N*-functions setting.** In the following we are going to introduce a general class of tools, related to the so-called general class of N-functions. For the results we mention here see for instance [23, 30].

We consider a convex function $\varphi : [0,\infty) \to [0,\infty)$, such that

$$\begin{aligned} &\varphi \in C^1([0,\infty)) \cap C^2((0,\infty)), \qquad \varphi(0) = \varphi'(0) = 0, \\ &\varphi'(t) \text{ is monotone increasing and } \lim_{t\to\infty} \varphi'(t) = \infty\,. \end{aligned} \tag{2.13}$$

In addition we assume that there exists a constant $c_\varphi \geq 1$ such that

$$\frac{1}{c_\varphi} \leq \frac{\varphi''(t)t}{\varphi'(t)} \leq c_\varphi, \qquad \text{for all } t > 0\,. \tag{2.14}$$

If the function φ verifies (2.13) and (2.14), then we call φ as an N-function.

Notice that every non-decreasing function $\varphi : [0,\infty) \to [0,\infty)$ satisfies the following property

$$\varphi(t+s) \leq \varphi(2t) + \varphi(2s)\,. \tag{2.15}$$

We denote by φ^* the conjugate function of φ which is defined, for $s \geq 0$, as

$$\varphi^*(s) = \sup_{t\geq 0}(st - \varphi(t))\,. \tag{2.16}$$

By definition, for every $s,t \geq 0$, the conjugate function φ^* satisfies the Young's inequality

$$t\,s \leq \varphi(t) + \varphi^*(s) \tag{2.17}$$

and also the property (see for instance [30, proof of Theorem 2.4.10])

$$\varphi^*\big(\frac{\varphi(s)}{s}\big) \leq \varphi(s)\,. \tag{2.18}$$

We define the auxiliary vector field $V_\varphi \colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$V_\varphi(z) := \left(\frac{\varphi'(|z|)}{|z|}\right)^{1/2} z\,, \tag{2.19}$$

where V_φ is continuously extended to zero when $z = 0$; V_φ turns out to be a bijection of $\mathbb{R}^n$ by (2.13).

Under the assumption (2.14) V_φ describes the monotonicity properties of the map $[\varphi'(|z|)/|z|]z$ in the sense that for $z_1, z_2 \in \mathbb{R}^n$, $z_1, z_2 \neq 0$ we have

$$\frac{1}{c}\,|V_\varphi(z_1) - V_\varphi(z_2)|^2 \leq \langle \frac{\varphi'(|z_1|)}{|z_1|} z_1 - \frac{\varphi'(|z_2|)}{|z_2|} z_2, z_1 - z_2\rangle \leq c\,|V_\varphi(z_1) - V_\varphi(z_2)|^2\,, \tag{2.20}$$

for a constant $c \geq 1$ depending on c_φ. For another constant $c \equiv c(c_\varphi)$ the following relations (see [23, Lemma 2.4]) hold for every $z, z_1, z_2 \in \mathbb{R}^n$ with z_1 or z_2 different from zero (which means $|z_1| + |z_2| > 0$):

$$\frac{1}{c}\,\varphi(|z|) \leq |V_\varphi(z)|^2 \leq c\,\varphi(|z|)\,, \tag{2.21}$$

$$\frac{1}{c}\,\varphi''(|z_1|+|z_2|)|z_1 - z_2|^2 \leq |V_\varphi(z_1) - V_\varphi(z_2)|^2 \leq c\,\varphi''(|z_1|+|z_2|)|z_1 - z_2|^2\,. \tag{2.22}$$

We are interested in

$$\varphi_p(t) = t^p \qquad \text{and} \qquad \varphi_{\log}(t) = t^p \log(e+t)\,, \tag{2.23}$$

which verify all the assumptions (2.13). In addition

$$t\,\varphi_p''(t) \;=\; (p-1)\varphi_p'(t) \tag{2.24}$$

$$t\,\varphi_{\log}''(t) \leq 2p\,\varphi_{\log}'(t)\,, \qquad t\,\varphi_{\log}''(t) \geq (p-1)\,\varphi_{\log}'(t)\,, \tag{2.25}$$

so (2.14) is satisfied with a constant depending only on p. We also need the following estimates

$$p(p-1)\,t^{p-2}\log(e+t) \leq \varphi_{\log}''(t) \leq p(p+1)\,t^{p-2}\log(e+t)\,, \tag{2.26}$$

which can be easily deduced using also (2.8).

Let us denote by $V_p(\cdot)$ the vector field $V_p(z) := \sqrt{p}\ |z|^{(p-2)/2} z$ generated by φ_p, and by $V_{\log}(\cdot)$ the one generated by $\varphi_{\log}$, that is

$$V_{\log}(z) := \sqrt{p|z|^{p-2}\log(e+|z|) + \frac{|z|^{p-1}}{(e+|z|)}}\ z\,,$$

both continuously extended to zero when $z = 0$. It is easy to verify that
(2.27)
$$|z|^p \le |V_p(z)|^2 = p\,|z|^p\,, \quad |z|^p \log(e+|z|) \le |V_{\log}(z)|^2 \le (p+1)\,|z|^p \log(e+|z|)\,,$$

thus in both cases (2.21) holds for a constant c depending only on p.

In addition, since $\varphi_p''(t) = p(p-1)t^{p-2}$ the estimate in (2.22), which holds whenever $z_1, z_2 \in \mathbb{R}^n$, $|z_1| + |z_2| > 0$, becomes:

$$\frac{1}{c}|z_1 - z_2|^2\,(|z_1| + |z_2|)^{p-2} \le |V_p(z_1) - V_p(z_2)|^2 \le c|z_1 - z_2|^2\,(|z_1| + |z_2|)^{p-2} \tag{2.28}$$

where c depends only on p. In particular, when $p \ge 2$,

$$|z_1 - z_2|^p \le c|V_p(z_1) - V_p(z_2)|^2 \tag{2.29}$$

holds, while for $1 < p < 2$ we will use that

$$|z_1 - z_2|^p \le c\ |V_p(z_1) - V_p(z_2)|^p\,(|z_1| + |z_2|)^{(2-p)p/2}\,, \tag{2.30}$$

both for a suitable constant $c \equiv c(p)$. Instead, using (2.26), from (2.22) we deduce that the vector field $V_{\log}$ satisfies

$$|V_{\log}(z_1) - V_{\log}(z_2)|^2 \le c\,|z_1 - z_2|^2(|z_1| + |z_2|)^{p-2}\log(e + |z_1| + |z_2|) \tag{2.31}$$

for every $z_1, z_2 \in \mathbb{R}^n$, $|z_1| + |z_2| > 0$, and a constant c depending on p.

Recalling (2.23) we adopt the notation

$$f(z) := |z|^p = \varphi_p(|z|)\,, \qquad g(z) := |z|^p\log(e+|z|) = \varphi_{\log}(|z|)\,, \tag{2.32}$$

from which

$$H(x,z) \;=\; f(z) \;+\; a(x)\,g(z) \;=\; \varphi_p(|z|) \;+\; a(x)\,\varphi_{\log}(|z|)\,. \tag{2.33}$$

Then the first inequality in (2.20) (see also (2.43) below) can be rewritten for V_p and $V_{\log}$ with a constant $c \equiv c(p)$ as

$$\frac{1}{c}\ |V_p(z_1) - V_p(z_2)|^2 \le\ \langle \partial f(z_1) - \partial f(z_2), z_1 - z_2\rangle\,, \tag{2.34}$$

$$\frac{1}{c}\ |V_{\log}(z_1) - V_{\log}(z_2)|^2 \le\ \langle \partial g(z_1) - \partial g(z_2), z_1 - z_2\rangle\,. \tag{2.35}$$

2.3. **Preliminary results.** Recalling (1.8) and (2.32) we adopt the notation

$$h(x,v,z) := b(x,v)\,H(x,z) \;=\; b(x,v)\left[f(z) + a(x)g(z)\right]. \tag{2.36}$$

To prove the comparison estimate in Lemma 4.2 we need the forthcoming Lemmata 2.2 and 2.3 and the maximum principle 4.1. Lemma 2.3 can be deduced by Taylor's formula, using estimates on the matrix of the second derivatives of the functions f and g defined in (2.32). Such estimates are the content of the following Lemma 2.1, together with estimates on the gradient of f and g, while Lemma 2.2 contains estimates on the function $H(\cdot,\cdot)$ defined in (1.8), needed to prove the maximum principle and the comparison estimate.

Lemma 2.1. *Let f, $g\colon \mathbb{R}^n \to \mathbb{R}$ be the functions defined in* (2.32). *Then the following relations hold for every z , $\lambda \in \mathbb{R}^n$:*

$$|\partial f(z)| = p|z|^{p-1}\,, \tag{2.37}$$

$$|\partial g(z)| \le (p+1)\,|z|^{p-1}\,\log(e+|z|)\,, \tag{2.38}$$

$$\langle \partial f(z)\,,\, z\rangle = p\,|z|^p\,, \tag{2.39}$$

$$\langle \partial g(z)\,,\, z\rangle \geq p\,|z|^p\,\log(e+|z|)\,, \tag{2.40}$$

$$\varphi_p''(|z|)\,|\lambda|^2 \leq c(p)\,\langle \partial^2 f(z)\,\lambda\,,\,\lambda\rangle\,, \tag{2.41}$$

$$\varphi_{\log}''(|z|)\,|\lambda|^2 \leq c(p)\,\langle \partial^2 g(z)\,\lambda\,,\,\lambda\rangle\,, \tag{2.42}$$

with the condition $z \neq 0$ *in* (2.41) *and* (2.42).

Proof. We calculate

$$\partial f(z) = \varphi_p'(|z|)\frac{z}{|z|} = p\,|z|^{p-2}\,z\,, \tag{2.43}$$

$$\partial g(z) = \varphi_{\log}'(|z|)\frac{z}{|z|} = \big(p|z|^{p-2}\,\log(e+|z|) + \frac{|z|^{p-1}}{e+|z|}\big)\,z\ ;$$

identities (2.37) and (2.39) follow immediately from (2.43), while (2.38) and (2.40) from (2.43) and (2.8).

Now, let us prove (2.42) since with exactly the same calculations, using (2.24) instead of the first inequality in (2.25), we can prove (2.41) with the constant $c(p) = \max\{1, p-1\}$. Using (2.7) we calculate

$$\begin{aligned}\partial^2 g(z) \;&=\; \big[\varphi_{\log}''(|z|) \;-\; \frac{\varphi_{\log}'(|z|)}{|z|}\big]\;\frac{z\otimes z}{|z|^2} \;+\; \frac{\varphi_{\log}'(|z|)}{|z|}\,\mathrm{I}\,,\\ \langle \partial^2 g(z)\,\lambda\,,\,\lambda\rangle \;&=\; \big[\varphi_{\log}''(|z|) \;-\; \frac{\varphi_{\log}'(|z|)}{|z|}\big]\;\frac{(\langle z\,,\,\lambda\rangle)^2}{|z|^2} \;+\; \frac{\varphi_{\log}'(|z|)}{|z|}\,|\lambda|^2\,,\end{aligned}$$

where I denotes the identity matrix in $\mathbb{R}^n$.

If $z \in \mathbb{R}^n \setminus \{0\}$ is such that $\varphi_{\log}''(|z|) \geq \varphi_{\log}'(|z|)/|z|$, from the first inequality in (2.25) we get immediately that

$$\langle \partial^2 g(z)\,\lambda\,,\,\lambda\rangle \geq \frac{1}{2p}\,\varphi_{\log}''(|z|)\,|\lambda|^2\,.$$

Otherwise, if z is such that $\varphi_{\log}''(|z|) \;<\; \varphi_{\log}'(|z|)/|z|$, using Cauchy-Schwartz inequality we obtain

$$\langle \partial^2 g(z)\,\lambda\,,\,\lambda\rangle \geq \big[\varphi_{\log}''(|z|) \;-\; \frac{\varphi_{\log}'(|z|)}{|z|}\big]\;|\lambda|^2 \;+\; \frac{\varphi_{\log}'(|z|)}{|z|}\,|\lambda|^2 \;=\; \varphi_{\log}''(|z|)\,|\lambda|^2$$

and we conclude that (2.42) holds with the constant $c(p) = 2p$. □

By (2.10) the function $H(x,z)$ defined in (1.8) satisfies the following estimate for every $x \in \Omega$, every $A \geq 1$ and every $z \in \mathbb{R}^n$:

$$H(x\,,\,Az) \leq A^{p+1} H(x,z)\,. \tag{2.44}$$

In addition notice that for every $z_1, z_2 \in \mathbb{R}^n$ we have

$$H(x, z_1 \pm z_2) \leq 2^{p+1}(\,H(x,z_1) + H(x,z_2))\,. \tag{2.45}$$

Indeed, using (2.33), (2.15) and (2.44) we get

$$\begin{aligned}H(x, z_1 \pm z_2) &= \varphi_p(|z_1 \pm z_2|) + a(x)\,\varphi_{\log}(|z_1 \pm z_2|)\\ &\leq \varphi_p(|z_1| + |z_2|) + a(x)\,\varphi_{\log}(|z_1| + |z_2|)\\ &\leq \varphi_p(2|z_1|) + a(x)\,\varphi_{\log}(2|z_1|) + \varphi_p(2|z_2|) + a(x)\,\varphi_{\log}(2|z_2|)\\ &\leq 2^{p+1}(H(x,z_1) + H(x,z_2))\,.\end{aligned}$$

In addition, we get the following estimates on the gradient of the function H.

Lemma 2.2. *Let $H\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ be the function defined in* (1.8). *Then the following estimates hold for every $x \in \Omega$ and every $z\,,\,\lambda \in \mathbb{R}^n$:*

$$|\partial_z H(x,z)| \le (p+1)\left[\,|z|^{p-1} \,+\, a(x)\,|z|^{p-1}\,\log(e+|z|)\,\right], \tag{2.46}$$

$$\langle \partial_z H(x,z)\,,\, z\rangle \ge p\left[\,|z|^{p} \,+\, a(x)\,|z|^{p}\log(e+|z|)\,\right], \tag{2.47}$$

$$|\langle \partial_z H(x,z)\,,\, \lambda\rangle| \le (p+1)\left(H(x,z) \,+\, H(x,\lambda)\right). \tag{2.48}$$

Proof. Inequality (2.46) follows immediately from (2.37) and (2.38), while (2.47) from (2.39) and (2.40).

In order to prove (2.48), let us assume that $|z| \neq 0$, otherwise the inequality is trivial. Using Cauchy-Schwartz inequality and (2.46) we obtain

$$\begin{aligned} |\langle \partial_z H(x,z)\,,\, \lambda\rangle| &\le (p+1)\left[\,|z|^{p-1}\,|\lambda| \,+\, a(x)\,|z|^{p-1}\,\log(e+|z|)\,|\lambda|\,\right] \\ &= (p+1)\left[\,I_1 \,+\, a(x)\,I_2\,\right]. \end{aligned} \tag{2.49}$$

Thanks to Young's inequality with conjugate exponents $(p\,,\,p/(p-1))$, we estimate:

$$I_1 = |z|^{p-1}\,|\lambda| \le \frac{p-1}{p}\,|z|^p \,+\, \frac{1}{p}\,|\lambda|^p \le |z|^p + |\lambda|^p\,. \tag{2.50}$$

For the second term, we need to consider the conjugate function $\varphi^*_{\log}(\cdot)$, as defined in (2.16), of the N-function $\varphi_{\log}$: by properties (2.17), applied with $t = |\lambda|$ and $s = |z|^{p-1}\,\log\,(e + |z|)$, and (2.18), we obtain the following chain of inequalities:

$$\begin{aligned} I_2 &= |z|^{p-1}\,\log(e+|z|)\,|\lambda| \\ &\le \varphi^*_{\log}(|z|^{p-1}\,\log(e+|z|)) + \varphi_{\log}(|\lambda|) \\ &= \varphi^*_{\log}\Big(\frac{|z|^p\,\log(e+|z|)}{|z|}\Big) + \varphi_{\log}(|\lambda|) \\ &\le \varphi_{\log}(|z|) + \varphi_{\log}(|\lambda|) \\ &= |z|^p\,\log(e+|z|) + |\lambda|^p\,\log(e+|\lambda|)\,. \end{aligned} \tag{2.51}$$

Putting together (2.49), (2.50) and (2.51), the proof of (2.48) is complete. □

Now we are able to prove the following lemma.

Lemma 2.3. *Let $h\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be the function defined in* (2.36). *Then there exists a positive constant $c = c(p,\nu)$ such that the following inequality holds for every $(x,v) \in \Omega \times \mathbb{R}$ and every $z_1, z_2 \in \mathbb{R}^n$*

$$\begin{aligned} &\frac{1}{c}\,\left[\,|V_p(z_1) - V_p(z_2)|^2 \,+\, a(x)\,|V_{\log}(z_1) - V_{\log}(z_2)|^2\,\right] \\ &\qquad\qquad \le\; h(x,v,z_1) - h(x,v,z_2) - \langle \partial_z h(x,v,z_2), z_1 - z_2\rangle\,. \end{aligned} \tag{2.52}$$

Proof. In the case $z_1 = z_2$ inequality (2.52) is trivial, so we can assume that $z_1 \neq z_2$, which in particular means $|z_1| + |z_2| > 0$.

The first step is to prove that there exists a constant $c \equiv c(p)$ such that

$$|z_1 - z_2|^2(|z_1| + |z_2|)^{p-2} \le c(p)\left[f(z_1) - f(z_2) - \langle \partial f(z_2)\,,\, z_1 - z_2\rangle\right], \tag{2.53}$$

$$\begin{aligned} &|z_1 - z_2|^2(|z_1| + |z_2|)^{p-2}\log(e + |z_1| + |z_2|) \\ &\qquad\qquad \le c(p)\left[g(z_1) - g(z_2) - \langle \partial g(z_2)\,,\, z_1 - z_2\rangle\right]. \end{aligned} \tag{2.54}$$

We postpone the proof of (2.53)-(2.54) and show how it implies (2.52). Indeed, using the bounds (1.11) on $b(\cdot\,,\,\cdot)$, (2.53), (2.54), the second estimate in (2.28) and (2.31) we obtain

$$\begin{aligned} &h(x,v,z_1) - h(x,v,z_2) - \langle \partial_z h(x,v,z_2), z_1 - z_2\rangle \\ &\quad = b(x,v)\,\Big[\,f(z_1) - f(z_2) - \langle \partial f(z_2)\,,\, z_1 - z_2\rangle \end{aligned}$$

$$
\begin{aligned}
&\quad + a(x)\big(g(z_1)-g(z_2)-\langle \partial g(z_2)\,,\,z_1-z_2\rangle\big)\Big]\\
&\ge \frac{\nu}{c(p)}\Big[|z_1-z_2|^2(|z_1|+|z_2|)^{p-2}\\
&\quad + a(x)\,|z_1-z_2|^2\big(\,|z_1|+|z_2|\,\big)^{p-2}\log(e+|z_1|+|z_2|\,)\Big]\\
&\ge \frac{1}{c(p,\nu)}\big[\,|V_p(z_1)-V_p(z_2)|^2\,+\,a(x)\,|V_{\log}(z_1)-V_{\log}(z_2)|^2\big]\,,
\end{aligned}
$$

which proves the lemma.

Now, let us prove (2.54), since, with exactly the same arguments and calculations, we can prove (2.53), without the estimates due to the presence of the logarithmic term. Let us assume for a moment that $t\,z_1+(1-t)z_2\neq 0$ for every $t\in[0,1]$ and consider the function $G\colon[0,1]\to\mathbb{R}$ defined as $G(t):=g(t\,z_1+(1-t)\,z_2)$: since $G\in C^2([0,1])$ we can use Taylor formula with integral remainder obtaining that

$$
G(1)\,=\,G(0)+G'(0)+\int_0^1(1-s)\,G''(s)\,ds\,. \tag{2.55}
$$

As

$$
\begin{aligned}
G'(t)\,&=\,\langle \partial g(t\,z_1+(1-t)\,z_2)\,,\,z_1-z_2\rangle\,,\\
G''(t)\,&=\,\langle \partial^2 g(t\,z_1+(1-t)\,z_2)\;(z_1-z_2)\,,\,z_1-z_2\rangle\,,
\end{aligned}
$$

from (2.55) we obtain

$$
\begin{aligned}
&g(z_1)-g(z_2)-\langle \partial g(z_2)\,,\,z_1-z_2\rangle\,= \\
&\quad=\int_0^1(1-s)\,\langle \partial^2 g(s\,z_1+(1-s)\,z_2)\;(z_1-z_2)\,,\,z_1-z_2\rangle\,ds\,=\,I_g\,.
\end{aligned} \tag{2.56}
$$

Using (2.42) with $\lambda=z_1-z_2$, $z=s\,z_1+(1-s)\,z_2$ and the first inequality in (2.26) we estimate I_g in (2.56) as

$$
c\,I_g\,\ge\,|z_1-z_2|^2\int_0^1(1-s)\,|s\,z_1+(1-s)\,z_2|^{p-2}\,\log(e+|s\,z_1+(1-s)\,z_2|)\,ds\,, \tag{2.57}
$$

for a constant c depending only on p.

Now, if $1<p<2$ we may estimate

$$
|s\,z_1+(1-s)\,z_2|^{p-2}\ge(|z_1|+|z_2|)^{p-2}\,, \tag{2.58}
$$

instead in the case $p\ge 2$, in order to estimate from below $|s\,z_1+(1-s)\,z_2|^{p-2}$ we have to distinguish between $|z_2|\le|z_1|$ and $|z_2|>|z_1|$ on a suitable subinterval of $[0,1]$.

More precisely, if $|z_2|\le|z_1|$ and $s\in[3/4,1]$, then $-1/4\le s-1\le 0$ and

$$
|s\,z_1+(1-s)\,z_2|\ge s\,|z_1|+(s-1)\,|z_2|\ge\frac34|z_1|-\frac14|z_2|\ge\frac14(|z_1|+|z_2|)\,, \tag{2.59}
$$

while, if $|z_2|>|z_1|$ and $s\in[0,1/4]$, then $3/4\le 1-s\le 1$ and

$$
|s\,z_1+(1-s)\,z_2|\ge(1-s)\,|z_2|-s\,|z_1|\ge\frac34|z_2|-\frac14|z_1|\ge\frac14(|z_1|+|z_2|)\,.
$$

Therefore

$$
|s\,z_1+(1-s)\,z_2|^{p-2}\ge 4^{2-p}\,(|z_1|+|z_2|)^{p-2} \tag{2.60}
$$

holds when $p\ge 2$ on a suitable subinterval of $[0,1]$.

For the logarithmic term, by (2.11) we get

$$
\log(e+\frac14(|z_1|+|z_2|))\ge\frac14\log(e+|z_1|+|z_2|)\,. \tag{2.61}
$$

Now, if $1 < p < 2$ and $|z_2| \le |z_1|$, using (2.58), (2.59) together with the monotonicity of the logarithm function and (2.61) we can estimate from below

$$\begin{aligned} &\int_{3/4}^{1} (1-s)\,|s\,z_1 + (1-s)\,z_2|^{p-2}\, \log(e + |s\,z_1 + (1-s)\,z_2|)\, ds \\ &\ge (|z_1| + |z_2|)^{p-2} \int_{3/4}^{1} (1-s)\, \log(e + \frac{1}{4}(|z_1| + |z_2|))\, ds \qquad (2.62) \\ &\ge \frac{1}{128}\, (|z_1| + |z_2|)^{p-2} \log(e + |z_1| + |z_2|)\,. \end{aligned}$$

From (2.56), (2.57) and (2.62) it follows that there exists a constant $c \equiv c(p)$ such that (2.54) holds. By restricting the interval to $[0, 1/4]$ instead of $[3/4, 1]$ we obtain (2.54) also in the case $1 < p < 2$ and $|z_2| > |z_1|$.

Otherwise, if $p \ge 2$ and $|z_2| \le |z_1|$, the same arguments with (2.60) instead of (2.58) allow to estimate

$$\begin{aligned} &\int_{3/4}^{1} (1-s)\,|s\,z_1 + (1-s)\,z_2|^{p-2}\, \log(e + |s\,z_1 + (1-s)\,z_2|)\, ds \\ &\ge \frac{1}{8 \cdot 4^p}\, (|z_1| + |z_2|)^{p-2} \log(e + |z_1| + |z_2|)\,, \end{aligned}$$

so (2.54) holds also in this case; finally the case $p \ge 2$ and $|z_2| > |z_1|$ is obtained by restricting the interval to $[0, 1/4]$.

To conclude the proof of (2.54) we need to consider the possibility that there exists $t_0 \in [0, 1]$ such that $t_0 z_1 + (1 - t_0) z_2 = 0$, where we can assume also that $z_2 \ne 0$, otherwise inequality (2.54) is trivially satisfied by the definition of the function g. In this case the result can be obtained by continuity by applying (2.54) with z_2 and $z_{1,\varepsilon} := z_1 + \varepsilon\, \mathbf{e}_j$ (where $\mathbf{e}_j$ is the unit vector of the j-axis and j is chosen in such a way that $t z_{1,\varepsilon} + (1-t) z_2 \ne 0$ for every $t \in [0, 1]$) and passing to the limit as $\varepsilon \to 0$. □

2.4. **Campanato spaces.** We recall the definition and some basic properties about such spaces, referring the reader to [13, 28, 27] for more details.

Definition 2. Let Ω be a bounded domain of $\mathbb{R}^n$ with Lipschitz boundary. Set $\Omega(x_0, \varrho) := \Omega \cap B_\varrho(x_0)$ and for every $1 \le p \le +\infty$, $k \ge 1$, $\lambda \ge 0$ define the Campanato space

$$\mathcal{L}^{p,\lambda}(\Omega, \mathbb{R}^k) := \left\{ u \in L^p(\Omega, \mathbb{R}^k) : \sup_{\substack{x_0 \in \Omega \\ \varrho > 0}} \varrho^{-\lambda} \int_{\Omega(x_0,\varrho)} |u - (u)_{\Omega(x_0,\varrho)}|^p\, dx < \infty \right\}.$$

In the Campanato space we can consider the seminorm

$$[u]_{\mathcal{L}^{p,\lambda}} := \Big(\sup_{\substack{x_0 \in \Omega \\ \varrho > 0}} \varrho^{-\lambda} \int_{\Omega(x_0,\varrho)} |u - (u)_{\Omega(x_0,\varrho)}|^p\, dx \Big)^{1/p}$$

and the norm $\|u\|_{\mathcal{L}^{p,\lambda}} = [u]_{\mathcal{L}^{p,\lambda}} + \|u\|_{L^p}$.

Remark 2.1. In the above definition only small radii are relevant: we can fix $\varrho_0 > 0$ and consider

$$\sup_{\substack{x_0 \in \Omega \\ 0 < \varrho \le \varrho_0}} \varrho^{-\lambda} \int_{\Omega(x_0,\varrho)} |u - (u)_{\Omega(x_0,\varrho)}|^p\, dx\,;$$

indeed by Jensen's inequality we get easily that

$$\sup_{\substack{x_0\in\Omega \\ \varrho>\varrho_0}} \varrho^{-\lambda} \int_{\Omega(x_0,\varrho)} |u-(u)_{\Omega(x_0,\varrho)}|^p\,dx \le (\varrho_0)^{-\lambda}\,2^p\,\|u\|^p_{L^p(\Omega)}\,. \tag{2.63}$$

For the following result due to Campanato see for instance [28, Theorem 2.9] or [27, Theorem 5.5].

Theorem 2.1. *Let Ω be a bounded domain of $\mathbb{R}^n$ with Lipschitz boundary. For $n<\lambda\le n+p$ and $\alpha=(\lambda-n)/p$ we have $\mathcal{L}^{p,\lambda}(\Omega,\mathbb{R}^k)\cong\mathcal{C}^{0,\alpha}(\overline{\Omega},\mathbb{R}^k)$. Moreover the Hölder seminorm* (2.4) *is equivalent to $[u]_{\mathcal{L}^{p,\lambda}}$.*

Remark 2.2. More precisely the Hölder and Campanato seminorms are equivalent in the following way:

$$[u]_{\mathcal{L}^{p,\lambda}} \le (\omega_n)^{1/p}\,[u]_{\mathcal{C}^{0,\alpha}}\,, \qquad [u]_{\mathcal{C}^{0,\alpha}} \le c(n,p,\lambda,\mathrm{Lip}(\partial\Omega))\,[u]_{\mathcal{L}^{p,\lambda}}\,, \tag{2.64}$$

where ω_n is the measure of the unitary ball in $\mathbb{R}^n$ and $\mathrm{Lip}(\partial\Omega)$ denotes the Lipschitz constant of $\partial\Omega$.

Thus we obtain the following well-known integral characterization of Hölder continuity.

Lemma 2.4. *Let Ω be a domain of $\mathbb{R}^n$, let $\alpha\in(0,1]$ and let $u\in L^p_{\mathrm{loc}}(\Omega,\mathbb{R}^k)$ with $p\ge 1$, $k\ge 1$. If for every $\Omega'\Subset\Omega$ there exist positive constants $c_0=c_0(\Omega')$ and $\varrho_0=\varrho_0(\Omega')<\mathrm{dist}(\Omega',\partial\Omega)$ such that*

$$\int_{B_\varrho(x_0)} |u-(u)_{B_\varrho}|^p\,dx \le c_0\,\varrho^{p\alpha+n} \tag{2.65}$$

for every $x_0\in\Omega'$ and every $0<\varrho\le\varrho_0$, then $u\in\mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\Omega,\mathbb{R}^k)$.

2.5. **An iteration lemma.** In order to prove our regularity results we will use the following iteration lemma which can be easily deduced from [28, Lemma 7.3].

Lemma 2.5. *Let $\phi\colon[0,\bar R]\to[0,\infty)$ be a non-decreasing function, such that the following inequality holds for some $\varepsilon\ge 0$ and whenever $0<\varrho\le R\le\bar R$:*

$$\phi(\varrho)\le c_0\left[\left(\frac{\varrho}{R}\right)^n+\varepsilon\right]\phi(R)\,.$$

For every $\delta\in(0,n)$, there exists $\varepsilon_0\equiv\varepsilon_0(n,\delta,c_0)>0$ such that if $\varepsilon\le\varepsilon_0$, then

$$\phi(\varrho)\le c_1\left(\frac{\varrho}{R}\right)^{n-\delta}\phi(R)$$

holds whenever $0<\varrho\le R\le\bar R$ and for a constant $c_1\equiv c_1(n,\delta,c_0)$.

3. Hölder continuity of local minimizers and approximation by smooth functions

In this section we discuss the first assertion of Theorem 1.2. Further, we prove an approximation result (see Lemma 3.2 below) that allows us to extend the class of admissible test functions for the Euler-Lagrange equations (see Remark 4.3) we will use in the proof of our comparison lemma.

3.1. **Basic regularity.** Let us recall that in order to get Hölder continuity of local minimizers we need to assume that $a(\cdot)$ is a uniformly continuous function with a modulus of continuity $\omega_a(\cdot)$ as defined in (1.12). As pointed out in the introduction, due to the local nature of our results, we may assume that Ω is a bounded domain and that local minimizers u are directly in $W^{1,p}(\Omega)$ with in addition $H(\cdot, Du(\cdot)) \in L^1(\Omega)$. The Hölder continuity of a local minimizer u of the functional $\mathcal{P}$ defined in (1.10) follows immediately from the results in [4] under the only assumption

$$\limsup_{r\to 0} \omega_a(r) \log\left(\frac{1}{r}\right) < \infty \, ,$$

which can be rewritten for a constant $\tilde{L} \geq 1$ as

$$\omega_a(r) \log\left(\frac{1}{r}\right) \leq \tilde{L} \qquad \text{for every } r \leq 1 \, . \tag{3.1}$$

More precisely, thanks to the bounds on $b(\cdot,\cdot)$ in Assumption 1 the following theorem is a particular case of [4, Theorem 4.1].

Theorem 3.1. *Let $u \in W^{1,p}(\Omega)$ be a local minimizer of the functional $\mathcal{P}$ defined in* (1.10) *and let $\omega_a(\cdot)$ be the modulus of continuity of the function $a(\cdot)$, under the only assumption* (3.1)*. Then:*

- *(Gehring's theory) There exists a positive integrability exponent $\delta_g > 0$, depending only on $n, p, \nu, L, \tilde{L}, \|Du\|_{L^p(\Omega)}$, such that*

$$H(x, Du) \in L^{1+\delta_g}_{\mathrm{loc}}(\Omega) \, . \tag{3.2}$$

  More precisely, the local reverse Hölder's inequality

$$\left(\fint_{B_{R/2}} [H(x, Du)]^{1+\delta_g}\, dx\right)^{1/(1+\delta_g)} \leq c \fint_{B_R} H(x, Du)\, dx \tag{3.3}$$

  holds true for every ball $B_R \subset \Omega$ and for a constant c depending only on $n, p, \nu, L, \tilde{L}, \|Du\|_{L^p(\Omega)}$. In particular, if $p > n/(1+\delta_g)$, then u is locally Hölder continuous.
- *(De Giorgi's theory) u is locally bounded. Moreover, when $p \leq n/(1+\delta_g)$, for every open subset $\Omega' \Subset \Omega$ there exists $\alpha \in (0,1)$, depending only on $n, p, \nu, L, \tilde{L}$ and $\|u\|_{L^\infty(\Omega')}$, such that*

$$u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega') \, .$$

Remark 3.1. From the proof of [4, Theorem 4.1] we can see that if the minimizer u is assumed a priori to be bounded, i.e. $u \in L^\infty(\Omega)$, then there exists an exponent $\alpha \in (0,1)$, depending on $n, p, \nu, L, \tilde{L}$ and $\|u\|_{L^\infty(\Omega)}$, such that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$. In addition, on every $\Omega' \Subset \Omega$ the minimizer u satisfies (2.3) with a constant C_α which depends not only on $n, p, \nu, L, \tilde{L}, \|u\|_{L^\infty(\Omega)}$ but also on the distance $\mathrm{dist}(\Omega', \partial\Omega)$.

Remark 3.2. Let us remark that all the constants in the above and forthcoming a priori estimates depend on the starting quantities assigned by the problem, that is on $n, p, \nu, L, \tilde{L}$ and, as in all other non-uniformly elliptic problems, also on the specific minimizer in question. In this paper the dependence is via $\|Du\|_{L^p(\Omega)}$ or $\|u\|_{L^\infty(\Omega)}$ as in Theorem 3.1 and in the comparison estimate (Lemma 4.2), or via $\|H(\cdot, Du(\cdot))\|_{L^1(\Omega)}$ as in the decay and Morrey type estimates (Lemmata 4.3 and 4.4) and in the estimate of the excess (Lemma 4.5).

As explained in [4, Remark 4.3] the only dependence of the constant c in (3.3) on $\|Du\|_{L^p(\Omega)}$ (and therefore the only dependence of δ_g and c in the statement of Theorem 3.1) comes from the estimate in the last line of [4, proof of Theorem 4.1, (4.13)]. In particular we can see that the dependence of the exponent δ_g on $\|Du\|_{L^p(\Omega)}$ is monotonically decreasing, while that of the constant c is monotonically increasing. In addition also the dependence on $\|Du\|_{L^p(\Omega)}$ of the constants appearing in the comparison and decay estimates [4, Lemmata 5.2 and 5.3] and in the estimate of the excess [4, (5.24)] is monotonically increasing, by carefully looking at the proofs.

3.2. **An approximation result.** In this paragraph we prove that, given a ball $B_R \Subset \Omega$, every function $\phi \in W_0^{1,p}(B_R)$ with finite energy can be approximated in $W^{1,p}$ by a sequence $\{\phi_k\} \subset C_0^\infty(B_R)$ with the energy of ϕ_k converging to the energy of ϕ. First we need a result on the continuity in L^1 of the dilatation.

For every $\varepsilon \in (0, 1/2)$ let us consider the dilatation $\tau_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\tau_\varepsilon(x) := \frac{x}{1-2\varepsilon}\,. \tag{3.4}$$

Let us prove the following lemma.

Lemma 3.1. *For $\varepsilon \in (0, \varepsilon_n = \dfrac{\sqrt[n]{2}-1}{2\sqrt[n]{2}}\,]$ let τ_ε be the dilatation defined in* (3.4).

Let $f \in L^1(\mathbb{R}^n)$ be a function such that $\operatorname{supp} f \subset \overline{B_R(0)}$ and let us set $f_\varepsilon := f \circ \tau_\varepsilon$. Then $f_\varepsilon \in L^1(\mathbb{R}^n)$ and $f_\varepsilon \to f$ strongly in $L^1(\mathbb{R}^n)$ as $\varepsilon \to 0$.

Remark 3.3. In the proof of Lemma 3.1, given a measurable set $A \subset \mathbb{R}^n$ with finite measure, we need to control the measure of $\tau_\varepsilon(A)$ in term of the measure of A, so we consider the condition $\varepsilon \le \dfrac{\sqrt[n]{2}-1}{2\sqrt[n]{2}}$ which means $\dfrac{1}{(1-2\varepsilon)^n} \le 2$ and implies $|\tau_\varepsilon(A)| \le 2\,|A|$.

Proof. By changing variables $\int_{\mathbb{R}^n} f_\varepsilon(x)\,dx = (1-2\varepsilon)^n \int_{\mathbb{R}^n} f(y)\,dy$, thus $f_\varepsilon \in L^1(\mathbb{R}^n)$.

By the absolute continuity of the integral of $f \in L^1(\mathbb{R}^n)$, for every $\zeta > 0$ there exists $\delta > 0$ such that $\int_E |f(x)|\,dx < \zeta$ for every measurable set $E \subset \mathbb{R}^n$ with $|E| < \delta$. In addition, using Lusin's theorem with $\delta/4 > 0$ there exist a function $g \in C_0^0(B_R)$, extended to zero outside B_R, and a set $A \subset B_R$ with $|A| < \delta/4$ such that $g = f$ a.e. on $\mathbb{R}^n \setminus A$.

We want to show that for every $\zeta > 0$ there exists $0 < \bar\varepsilon < \varepsilon_n$ sufficiently small such that for every $\varepsilon \in (0, \bar\varepsilon)$ we have

$$\int_{\mathbb{R}^n} |f_\varepsilon(x) - f(x)|\,dx < 3\zeta \tag{3.5}$$

which proves the lemma. Since $f_\varepsilon(x) = g(\tau_\varepsilon(x))$ a.e. on $\mathbb{R}^n \setminus \tau_\varepsilon^{-1}(A)$, we can write

$$\begin{aligned}\int_{\mathbb{R}^n} |f_\varepsilon(x) - f(x)|\,dx &= \\ &= \int_{\mathbb{R}^n \setminus (A \cup \tau_\varepsilon^{-1}(A))} |g(\tau_\varepsilon(x)) - g(x)|\,dx + \int_{A \cup \tau_\varepsilon^{-1}(A)} |f_\varepsilon(x) - f(x)|\,dx \\ &\le \int_{\mathbb{R}^n} |g(\tau_\varepsilon(x)) - g(x)|\,dx + \int_{A \cup \tau_\varepsilon^{-1}(A)} |f_\varepsilon(x) - f(x)|\,dx = I_1 + I_2\,.\end{aligned} \tag{3.6}$$

By Remark 3.3 we have that $|\tau_\varepsilon(A)| < \delta/2$, while $|\tau_\varepsilon^{-1}(A)| \le |A| < \delta/4$, so that $|A \cup \tau_\varepsilon^{-1}(A)| < \delta/2$ and $|\tau_\varepsilon(A \cup \tau_\varepsilon^{-1}(A))| = |\tau_\varepsilon(A) \cup A| < 3\delta/4$. Therefore, by changing variables

$$\begin{aligned} I_2 &\le \int_{A \cup \tau_\varepsilon^{-1}(A)} |f_\varepsilon(x)|\,dx + \int_{A \cup \tau_\varepsilon^{-1}(A)} |f(x)|\,dx \\ &\le \int_{\tau_\varepsilon(A \cup \tau_\varepsilon^{-1}(A))} |f(y)|\,dy + \int_{A \cup \tau_\varepsilon^{-1}(A)} |f(x)|\,dx \le 2\zeta\,.\end{aligned} \tag{3.7}$$

In order to estimate I_1, notice that by continuity $g(\tau_\varepsilon(x)) \to g(x)$ for every $x \in \mathbb{R}^n$, while by changing variables $\int_{\mathbb{R}^n} |g(\tau_\varepsilon(x))|dx = (1-2\varepsilon)^n \int_{\mathbb{R}^n} |g(y)|dy \to \int_{\mathbb{R}^n} |g(x)|\,dx$.

A well-known result on L^1-convergence assures that $g(\tau_\varepsilon(x)) \to g(x)$ strongly in $L^1(\mathbb{R}^n)$, then there exists $0 < \bar{\varepsilon} < \varepsilon_n$ such that $I_1 < \zeta$ for every $\varepsilon \in (0, \bar{\varepsilon})$ which, together with (3.6) and (3.7), proves (3.5). □

To prove the following approximation lemma we adapt the proof of [15, Proposition 3.1] (see also [20] for related results).

Lemma 3.2. *Let us consider a ball $B = B_R(x_0) \Subset \Omega$ and a function $\phi \in W_0^{1,p}(B_R)$ such that $H(x, D\phi) \in L^1(B_R)$, with the function H defined in* (1.8) *and the modulus of continuity $\omega_a(\cdot)$ of the function $a(\cdot)$ satisfying assumption* (3.1).

Then there exists a sequence $\{\phi_k\} \subset C_0^\infty(B_R)$ such that $D\phi_k \to D\phi$ a.e. and

$$H(x, D\phi_k) \to H(x, D\phi) \qquad \text{strongly in } L^1(B_R)\,. \tag{3.8}$$

Remark 3.4. Notice that $\phi_k \to \phi$ strongly in $W^{1,p}(B_R)$: by (2.45) we estimate

$$|D\phi_k - D\phi|^p \le H(x, D\phi_k - D\phi) \le 2^{p+1}\big(\, H(x, D\phi_k) + H(x, D\phi)\,\big)\,,$$

and the conclusion follows from a well-known variant of the Lebesgue's dominated convergence theorem and Poincaré inequality in $W_0^{1,p}$.

Proof. By dilatation and translation we reduce to the case $B = B_1(0) = B_1$. We consider the function $a(x)$ only on B_1 (the modulus of continuity ω_a results modified by a factor R if $R > 1$, otherwise it remains exactly the same): we can extend the function $a(x)$ as a continuous non-negative function on the whole $\mathbb{R}^n$, with the same modulus of continuity $\omega_a(\cdot)$, by taking $a(x) = \inf_{y \in B_1}\{a(y) + \omega_a(|x - y|)\}$.

We first take the null extension of ϕ outside B_1 assuming that $\phi \in W_0^{1,p}(\mathbb{R}^n)$ and $H(x, D\phi) \in L^1(\mathbb{R}^n)$. We then consider a family $\{\eta_\varepsilon\}$ of standard, radially symmetric mollifiers: $\eta_\varepsilon \in C_0^\infty(B_\varepsilon)$ such that $0 \le \eta_\varepsilon \le c(n)/\varepsilon^n$ and $\int_{B_\varepsilon(0)} \eta_\varepsilon \, dx = 1$. Fixed ε_n as in Lemma 3.1, for every $\varepsilon \in (0, \varepsilon_n)$ with in addition ε sufficiently small to satisfy (3.14) let τ_ε be the dilatation as in (3.4); notice that $\varepsilon < 1/6$ and

$$(1 - 2\varepsilon)^{-1} < 3/2\,. \tag{3.9}$$

Let us define $\tilde{\phi}_\varepsilon(x) = \phi(\tau_\varepsilon(x))$: we have $\tilde{\phi}_\varepsilon \in W_0^{1,p}(B_1)$, supp $\tilde{\phi}_\varepsilon \subset \overline{B}_{1-2\varepsilon}$ and

$$\Big(\int_{B_1} |D\tilde{\phi}_\varepsilon(x)|^p \, dx\Big)^{1/p} \le \frac{3}{2}\,\Big(\int_{B_1} |D\phi(y)|^p \, dy\Big)^{1/p}\,. \tag{3.10}$$

Accordingly for $x \in \mathbb{R}^n$ we define $\tilde{a}_\varepsilon(x) = a(\tau_\varepsilon(x))$; by changing variables and (2.10) we obtain that $\tilde{a}_\varepsilon(x)\,|D\tilde{\phi}_\varepsilon(x)|^p \,\log(e + |D\tilde{\phi}_\varepsilon(x)|) \in L^1(B_1)$.

Finally we consider the mollification $\phi_\varepsilon = \tilde{\phi}_\varepsilon * \eta_\varepsilon \in C_0^\infty(B_{1-\varepsilon})$ and introduce the auxiliary functions

$$a_\varepsilon(x) := \inf_{y \in B_\varepsilon(x)} \tilde{a}_\varepsilon(y) \quad \text{and} \quad H_\varepsilon(x, z) := |z|^p + a_\varepsilon(x)\,|z|^p \,\log(e + |z|)\,, \tag{3.11}$$

for every $x \in B_1$ and $z \in \mathbb{R}^n$.

In order to prove (3.8) we want to apply the Lebesgue's dominated convergence theorem to a suitable subsequence of $H(\cdot, D\phi_\varepsilon(\cdot))$. To this aim, for every $x \in B_{1-\epsilon}$, by Jensen inequality, the definition of the function $\tilde{\phi}_\varepsilon$, (2.44) for H_ε, (3.9), the definition (3.11) of the function $a_\epsilon(\cdot)$ and the definition of convolution we estimate

$$\begin{aligned} H_\varepsilon(x, D\phi_\varepsilon(x)) &\le \int_{B_\varepsilon(x)} H_\varepsilon(x, D\tilde{\phi}_\varepsilon(y))\,\eta_\varepsilon(x - y)\,dy \\ &\le (1 - 2\varepsilon)^{-(p+1)} \int_{B_\varepsilon(x)} H_\varepsilon(x, D\phi(\tau_\varepsilon(y)))\,\eta_\varepsilon(x - y)\,dy \\ &\le 2^{p+1} \int_{B_\varepsilon(x)} H(\tau_\varepsilon(y), D\phi(\tau_\varepsilon(y)))\,\eta_\varepsilon(x - y)\,dy \qquad (3.12) \\ &= 2^{p+1}\big[H(\tau_\varepsilon(\cdot), D\phi(\tau_\varepsilon(\cdot))) * \eta_\varepsilon\big](x)\,. \end{aligned}$$

By Hölder's inequality with conjugate exponents p and $p/(p-1)$ and (3.10) we may control the gradient of ϕ_ε:

$$
\begin{aligned}
|D\phi_\varepsilon(x)| &\leq \int_{B_\varepsilon(x)} |D\tilde{\phi}_\varepsilon(y)|\, \eta_\varepsilon(x-y)\, dy \\
&\leq \Big(\int_{B_\varepsilon(x)} |D\tilde{\phi}_\varepsilon(y)|^p\, dy\Big)^{1/p} \Big(\int_{B_\varepsilon(x)} (\eta_\varepsilon(x-y))^{p/(p-1)}\, dy\Big)^{1-1/p} \\
&\leq \Big(\int_{B_1} |D\tilde{\phi}_\varepsilon(y)|^p\, dy\Big)^{1/p} \frac{c(n)}{\varepsilon^n} |B_\varepsilon|^{1-1/p} \leq \frac{c(n,p)\|D\phi\|_{L^p(B_1)}}{\varepsilon^{n/p}}\,. \qquad (3.13)
\end{aligned}
$$

Thus, if ε is sufficiently small to verify

$$
\varepsilon^{n/p} \leq \min\{\frac{c}{e}\,,\, \frac{1}{c}\} \qquad (3.14)
$$

where $c = c(n,p)\|D\phi\|_{L^p(B_1)}$ is the constant in (3.13), using (3.13), (2.9) and (2.12), we obtain that

$$
\begin{aligned}
\log(e + |D\phi_\varepsilon(x)|) &\leq \log(e + \frac{c(n,p)\|D\phi\|_{L^p(B_1)}}{\varepsilon^{n/p}}) \qquad (3.15) \\
&\leq 2\, \log(\frac{c(n,p)\|D\phi\|_{L^p(B_1)}}{\varepsilon^{n/p}}) \\
&\leq 4\, \log(\frac{1}{\varepsilon^{n/p}}) = 4\,\frac{n}{p}\, \log(\frac{1}{\varepsilon})\,.
\end{aligned}
$$

In addition, by the definition (3.11) of the function $a_\varepsilon(\cdot)$, the concavity of $\omega_a(\cdot)$ and (3.9), for every $x \in B_{1-\varepsilon}$ we can prove that

$$
|a(x) - a_\varepsilon(x)| \leq 4\,\omega_a(\varepsilon) \qquad (3.16)
$$

since the distance between x and $\tau_\varepsilon(y)$ is bounded by $4\,\varepsilon$ for every $y \in B_\varepsilon(x)$. Finally, from (3.16) and (3.15) we estimate

$$
\begin{aligned}
H(x, D\phi_\varepsilon(x)) &\leq |a(x) - a_\varepsilon(x)|\; |D\phi_\varepsilon(x)|^p\; \log(e + |D\phi_\varepsilon(x)|) + H_\varepsilon(x, D\phi_\varepsilon(x)) \\
&\leq c(n,p)\; \omega_a(\varepsilon)\; \log(\frac{1}{\varepsilon})\; |D\phi_\varepsilon(x)|^p + H_\varepsilon(x, D\phi_\varepsilon(x)) \\
&\leq c(n,p)\tilde{L}\; H_\varepsilon(x, D\phi_\varepsilon(x))
\end{aligned}
$$

recalling assumption (3.1) on $\omega_a(\cdot)$. Then (3.12) implies that

$$
H(x, D\phi_\varepsilon(x)) \leq c(n,p,\tilde{L})\; \big[H(\tau_\varepsilon(\cdot), D\phi(\tau_\varepsilon(\cdot))) * \eta_\varepsilon\big](x)\,, \qquad (3.17)
$$

where

$$
H(\tau_\varepsilon(\cdot), D\phi(\tau_\varepsilon(\cdot))) * \eta_\varepsilon \to H(\cdot, D\phi(\cdot)) \quad \text{strongly in } L^1(B_1) \text{ as } \varepsilon \to 0\,. \qquad (3.18)
$$

Indeed Lemma 3.1, applied to $f(\cdot) = H(\cdot, D\phi(\cdot)) \in L^1(\mathbb{R}^n)$ with supp $f \subset \overline{B_1(0)}$, guarantees that $H(\tau_\varepsilon(\cdot), D\phi(\tau_\varepsilon(\cdot))) \to H(\cdot, D\phi(\cdot))$ strongly in $L^1(B_1)$ and (3.18) follows by well-known properties of convolution.

It remains to select a suitable subsequence $\{\varepsilon_k\}$ converging to zero such that, setting $\phi_k := \phi_{\varepsilon_k}$, it results $D\phi_k \to D\phi$ a.e., so that

$$
H(x, D\phi_k(x)) \to H(x, D\phi(x)) \quad \text{a.e.}\,; \qquad (3.19)
$$

then (3.17), (3.18) and (3.19) allow to apply a well-known variant of the Lebesgue's dominated convergence theorem to the sequence $H(x, D\phi_k(x))$ obtaining (3.8) and the proof of the lemma is complete. To obtain (3.19), let us show that

$$
D\phi_\varepsilon \to D\phi \quad \text{strongly in } L^1(B_1)\,. \qquad (3.20)
$$

As $D\phi_\varepsilon = D\tilde{\phi}_\varepsilon * \eta_\varepsilon$ by well-known properties of convolution it is enough to prove that $D\tilde{\phi}_\varepsilon \to D\phi$ strongly in $L^1(B_1)$. Since $D\tilde{\phi}_\varepsilon(\cdot) = (1-2\varepsilon)^{-1} D\phi(\tau_\varepsilon(\cdot))$ and

$D\phi(\tau_\varepsilon(\cdot)) \to D\phi(\cdot)$ strongly in $L^1(B_1)$ by Lemma 3.1 now applied to $f = D\phi \in L^1(\mathbb{R}^n)$, by changing variables we estimate

$$\|D\tilde{\phi}_\varepsilon - D\phi\|_{L^1} \leq 2\varepsilon\|D\phi\|_{L^1} + \|D\phi(\tau_\varepsilon(\cdot))) - D\phi(\cdot)\|_{L^1}$$

which proves (3.20). □

4. Gradient Hölder regularity

In this section we prove the last two assertions of Theorem 1.2.

In order to prove higher regularity results, as pointed out in the introduction, from now on let us assume that also the coefficient $b(\cdot,\cdot)$ is uniformly continuous and let $\omega_b(\cdot)$ be a modulus of continuity of $b(\cdot\,,\,\cdot)$ as defined in (1.13). First, we prove, in Paragraph 4.3 below, that $u \in C^{0,\alpha}_{\rm loc}(\Omega)$ for every $\alpha \in (0,1)$ assuming that $l = 0$ (with l defined in (1.14)). Then, in Paragraph 4.4, we shall finally prove the Hölder gradient continuity of minimizers assuming that $a(\cdot)$ and $b(\cdot,\cdot)$ are Hölder continuous.

4.1. **Comparison lemma.** In this paragraph we prove a comparison lemma (see Lemma 4.2 below), where we estimate the distance between a minimizer of $\mathcal{P}$ and a minimizer of a frozen functional obtained by freezing both variables x and w.

Before going on, let us premise a few basic facts on minimizers of frozen functionals obtained by $\mathcal{P}$ and let us prove the validity of the weak formulation of the Euler-Lagrange equation of such frozen functionals, specifying the class of admissible test functions.

Let $u \in W^{1,p}(\Omega)$ be a local minimizer of $\mathcal{P}$ and let $B_R \equiv B_R(x_0) \Subset \Omega$ be a ball with radius $R > 0$. By Theorem 3.1 we have that $u \in C^0(B_R)$ and we can denote $u_0 := u(x_0)$. Finally, let $\bar{v} \in W^{1,p}(B_{R/2})$ and $v \in W^{1,p}(B_{R/4})$ be the solutions of the following Dirichlet problems:

$$\begin{cases} \bar{v} \mapsto \min\limits_{w} \displaystyle\int_{B_{R/2}} b(x,u_0)\ H(x,Dw)\,dx \\ \qquad w \in u + W^{1,p}_0(B_{R/2})\,, \end{cases} \tag{4.1}$$

$$\begin{cases} v \mapsto \min\limits_{w} \displaystyle\int_{B_{R/4}} b(x_0,u_0)\ H(x,Dw)\,dx \\ \qquad w \in \bar{v} + W^{1,p}_0(B_{R/4})\,. \end{cases} \tag{4.2}$$

Remark 4.1. Problems (4.1) and (4.2) are well-posed. The existence of a minimizer follows from the Direct Methods of the Calculus of Variations: the functional $\mathcal{F}$ defined in (1.9) is l.s.c. in the weak topology of $W^{1,p}$ and coercive on the Dirichlet classes $u + W^{1,p}_0(B_{R/2})$ and $\bar{v} + W^{1,p}_0(B_{R/4})$. In addition, thanks to the bounds (1.11) on $b(\cdot,\cdot)$ we get

$$\int_{B_{R/2}} b(x,u_0)\,H(x,Du)\,dx \quad \leq L\int_{B_R} H(x,Du)\,dx < \infty\,,$$

and this fact guarantees that problem (4.1) has a minimizer because the function u belongs to the class of competitors with finite energy in the Dirichlet class $u + W^{1,p}_0(B_{R/2})$. Further, by the minimality of $\bar{v}$ and again (1.11), we obtain

$$\begin{aligned} \int_{B_{R/4}} H(x,D\bar{v})\,dx \quad &\leq \int_{B_{R/2}} H(x,D\bar{v})\,dx \quad \leq \frac{1}{\nu}\int_{B_{R/2}} b(x,u_0)\,H(x,D\bar{v})\,dx \\ &\leq \frac{1}{\nu}\int_{B_{R/2}} b(x,u_0)\,H(x,Du)\,dx \quad \leq \frac{L}{\nu}\int_{B_R} H(x,Du)\,dx < \infty\,, \end{aligned} \tag{4.3}$$

thus also problem (4.2) has a minimizer because the function $\bar{v}$ belongs to the class of competitors with finite energy in the Dirichlet class $\bar{v} + W_0^{1,p}(B_{R/4})$.

Notice that the minimality of v and (4.3) imply that

$$\int_{B_{R/4}} H(x, Dv)\, dx \le \int_{B_{R/2}} H(x, D\bar{v})\, dx \le \frac{L}{\nu} \int_{B_R} H(x, Du)\, dx < \infty\,, \tag{4.4}$$

so both energies of v and $\bar{v}$ can be estimated by the energy of the minimizer u; in particular $H(\cdot, Dv(\cdot)) \in L^1(B_{R/4})$ and $H(\cdot, D\bar{v}(\cdot)) \in L^1(B_{R/2})$.

Remark 4.2. Let us remark that the minimizers $\bar{v}$ and v of the Dirichlet problems (4.1) and (4.2) are in particular local minimizers of the functional $\mathcal{P}$ defined in (1.10) respectively on $\Omega = B_{R/2}$ with $b(x, w(x)) = b(x, u_0)$ and on $\Omega = B_{R/4}$ with $b(x, w(x)) = b(x_0, u_0)$. Thus, by Theorem 3.1 we have that $\bar{v}$ and v are continuous at every point in the interior of $B_{R/2}$ and $B_{R/4}$ respectively. In addition, we point out that the function v is also a local minimizer of the functional $\mathcal{P}_{\log}$ defined in (1.6) on $\Omega = B_{R/4}$ and this fact allows us to apply the results in [4] to the minimizer v.

Remark 4.3. We can show that the Euler-Lagrange equations

$$\int_{B_{R/4}} \langle \partial_z H(x, Dv), D\phi \rangle\, dx = 0 \tag{4.5}$$

$$\int_{B_{R/2}} b(x, u_0) \langle \partial_z H(x, D\bar{v}), D\phi \rangle\, dx = 0 \tag{4.6}$$

are valid for every $\phi \in W_0^{1,p}(B_{R/4})$ with $H(\cdot, D\phi(\cdot)) \in L^1(B_{R/4})$ for (4.5) and for every $\phi \in W_0^{1,p}(B_{R/2})$ with $H(\cdot, D\phi(\cdot)) \in L^1(B_{R/2})$ for (4.6).

Let us prove (4.6) since with exactly the same arguments we can prove (4.5). We argue by approximation since it is well known that the equation holds for every $\phi \in C_0^\infty(B_{R/2})$. Let $\phi \in W_0^{1,p}(B_{R/2})$ such that $H(\cdot, D\phi(\cdot)) \in L^1(B_{R/2})$: by Lemma 3.2 there exists a sequence $\{\phi_k\} \subset C_0^\infty(B_{R/2})$ such that $D\phi_k \to D\phi$ a.e. and

$$H(\cdot, D\phi_k(\cdot)) \to H(\cdot, D\phi(\cdot)) \qquad \text{strongly in } L^1(B_{R/2})\,.$$

Using (1.11) and (2.48), with $z = D\bar{v}$ and $\lambda = D\phi_k$, we estimate on $B_{R/2}$

$$|\, b(x, u_0) \langle \partial_z H(x, D\bar{v}), D\phi_k \rangle \,| \le L\, c(p) \big(H(x, D\bar{v}) + H(x, D\phi_k) \big)$$

and we can conclude the strong convergence in $L^1(B_{R/2})$ of

$$b(\cdot, u_0) \langle \partial_z H(\cdot, D\bar{v}(\cdot)), D\phi_k(\cdot) \rangle \to b(\cdot, u_0) \langle \partial_z H(\cdot, D\bar{v}(\cdot)), D\phi(\cdot) \rangle$$

by a well-known variant of the Lebesgue's dominated convergence theorem. Therefore, since every ϕ_k satisfies (4.6) also ϕ does.

For the solutions $\bar{v}$ and v of the Dirichlet problems (4.1) and (4.2) a maximum principle holds.

Lemma 4.1. *Let $u \in W^{1,p}(\Omega)$ be a local minimizer of $\mathcal{P}$ and let $\bar{v} \in W^{1,p}(B_{R/2})$ and $v \in W^{1,p}(B_{R/4})$ be the solutions of the Dirichlet problems* (4.1) *and* (4.2)*; then*

$$\min_{\partial B_{R/2}} u \le \bar{v}(x) \le \max_{\partial B_{R/2}} u \qquad \text{for every } x \in B_{R/2}\,, \tag{4.7}$$

$$\min_{\partial B_{R/4}} \bar{v} \le v(x) \le \max_{\partial B_{R/4}} \bar{v} \qquad \text{for every } x \in B_{R/4}\,. \tag{4.8}$$

Remark 4.4. Recalling (2.2), from Lemma 4.1 it follows immediately that

$$\operatorname*{osc}_{B_{R/2}} \bar{v} \leq \operatorname*{osc}_{B_{R/2}} u \quad \text{and} \quad |\bar{v}(x) - u(y)| \leq \operatorname*{osc}_{B_{R/2}} u \tag{4.9}$$

for every $x, y \in B_{R/2}$. In addition $\bar{v} \in L^\infty(B_{R/2})$, $v \in L^\infty(B_{R/4})$ and

$$\|v\|_{L^\infty(B_{R/4})} \leq \|\bar{v}\|_{L^\infty(B_{R/2})} \leq \|u\|_{L^\infty(B_{R/2})} \,.$$

Proof. Recalling that u is continuous on $\overline{B}_{R/2}$, let us set $k = \max\limits_{\partial B_{R/2}} u$ and define $\phi := \max\{\bar{v} - k\,,\, 0\}$. By (2.5) we have that $\phi \in W^{1,p}(B_{R/2})$ with

$$D\phi \;=\; \begin{cases} D\bar{v} & \text{a.e. on } \{\bar{v} \,>\, k\} \\ 0 & \text{a.e. on } \{\bar{v} \,\leq\, k\}\,. \end{cases} \tag{4.10}$$

Let us prove that $\phi \in W_0^{1,p}(B_{R/2})$ by constructing a sequence in $W_0^{1,p}(B_{R/2})$ converging to ϕ in $W^{1,p}(B_{R/2})$. By assumption $\bar{v} - u \in W_0^{1,p}(B_{R/2})$, then there exists a sequence $\{u_h\} \subset C_0^\infty(B_{R/2})$ such that $u_h \to \bar{v} - u$ strongly in $W^{1,p}(B_{R/2})$, so the sequence $\{v_h = u_h + u\} \subset C^0(\overline{B}_{R/2}) \cap W^{1,p}(B_{R/2})$ is such that $v_h \to \bar{v}$ strongly in $W^{1,p}(B_{R/2})$ and $v_h(x) \leq k$ for every $x \in \partial B_{R/2}$. Therefore (see for instance [9, Theorem 9.17]), we can consider in $W_0^{1,p}(B_{R/2})$ the sequence $\phi_h = \max\{v_h - k, 0\}$: (2.6) implies that $\phi_h \to \phi$ strongly in $W^{1,p}(B_{R/2})$ and this proves that $\phi \in W_0^{1,p}(B_{R/2})$. In addition, by (4.10) and (4.4) we get immediately that $H(x, D\phi) \in L^1(B_{R/2})$. Testing Euler-Lagrange equation (4.6) with the function ϕ and using (1.11) and (2.47) we obtain

$$\begin{aligned} 0 \;&=\; \int_{B_{R/2}} b(x, u_0)\, \langle \partial_z H(x, D\bar{v}), D\phi \rangle \, dx \\ &=\; \int_{B_{R/2} \cap \{\bar{v} \,>\, k\}} b(x, u_0)\, \langle \partial_z H(x, D\phi), D\phi \rangle \, dx \\ &\geq \nu\, p \int_{B_{R/2}} |D\phi|^p \,+\, a(x)\, |D\phi|^p \log(e + |D\phi|)\, dx \geq \nu\, p \int_{B_{R/2}} |D\phi|^p \, dx \,, \end{aligned}$$

which means that $D\phi = 0$ a.e. on $B_{R/2}$; as $\phi \in W_0^{1,p}(B_{R/2})$ we obtain $\phi = 0$ a.e. on $B_{R/2}$. Since $\bar{v}$ is continuous on $B_{R/2}$ by Remark 4.2, we conclude that $\max\{\bar{v}(x) - k\,,\, 0\} = 0$ for every $x \in B_{R/2}$ and the right-hand estimate in (4.7) is proven. Setting $k = \min\limits_{\partial B_{R/2}} u$ and testing with $\phi := \min\{\bar{v} - k\,,\, 0\}$ analogous calculations conclude the proof of (4.7).

Finally, by Remark 4.2 the function $\bar{v}$ is continuous on $\overline{B}_{R/4}$ and if we set $k = \max\limits_{\partial B_{R/4}} \bar{v}$ $\left(k = \min\limits_{\partial B_{R/4}} \bar{v}\right)$ and define $\phi := \max\{v - k\,,\, 0\}$ $(\phi := \min\{v - k\,,\, 0\})$ with exactly the same arguments we can prove (4.8) and the proof of the lemma is complete. □

In the following $\tilde{L}$ denotes a finite constant such that (3.1) holds.

Every local minimizer $u \in W^{1,p}(\Omega)$ of the functional $\mathcal{P}$ is locally bounded by Theorem 3.1. Since all our results are local in nature, without loss of generality we assume for the rest of the paper that $u \in L^\infty(\Omega)$. Thus, by Theorem 3.1 and Remark 3.1 we have that

$$u \in C^{0,\beta}_{\mathrm{loc}}(\Omega)\,, \tag{4.11}$$

for an exponent $\beta \in (0,1)$ depending on $n, p, \nu, L, \tilde{L}, \|u\|_{L^\infty(\Omega)}$.

Now, let us fix $\Omega' \Subset \Omega$, let us set $\mathrm{d} := \operatorname{dist}(\Omega', \partial\Omega)$ and consider the open set $\Omega'_{\mathrm{d}} := \{\, x \in \Omega \,:\, \operatorname{dist}(x, \Omega') < \mathrm{d}/2\}$. Then $u \in C^{0,\beta}(\Omega'_{\mathrm{d}})$ and by Remark 3.1 we may assume that (2.3) holds with a constant $C_\beta \geq 1$ depending on $n, p, \nu, L, \tilde{L}, \|u\|_{L^\infty(\Omega)}$, $\mathrm{dist}(\Omega', \partial\Omega)$. In particular for every ball $B_R(x_0)$ with center $x_0 \in \Omega'$ and radius $R \leq \mathrm{d}/4$ we have that

$$|u(x) - u(y)| \;\leq\; C_\beta\; |x-y|^\beta \tag{4.12}$$

for every $x, y \in B_R(x_0)$ and a constant C_β independent on the ball we consider.

Now we can state and prove our comparison lemma.

Lemma 4.2 (Comparison)**.** *Let $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a local minimizer of $\mathcal{P}$ such that $H(\cdot, Du(\cdot)) \in L^1(\Omega)$, let $\Omega' \Subset \Omega$, let $\mathrm{d} := \mathrm{dist}(\Omega', \partial\Omega)$ and let $B_R \equiv\; B_R(x_0)$ be a ball with center $x_0 \in \Omega'$ and radius $R \leq \min\{1, \mathrm{d}/4\}$. Let $\omega_a(\cdot)$ and $\omega_b(\cdot)$ be the modulus of continuity of $a(\cdot)$ and $b(\cdot,\cdot)$ respectively as defined in* (1.12) *and* (1.13)*, with the only assumption* (3.1)*, let $\beta \in (0,1)$ be the Hölder exponent of u as in* (4.11) *and let $v \in W^{1,p}(B_{R/4})$ be the solution of the Dirichlet problem* (4.2)*. Then the inequality*

$$\begin{aligned}\int_{B_{R/4}} &(|V_p(Du) - V_p(Dv)|^2 + a(x)|V_{\log}(Du) - V_{\log}(Dv)|^2)\, dx \\ &\leq c\; \omega_b(R^\beta) \int_{B_R} H(x, Du)\, dx \end{aligned}\tag{4.13}$$

holds for a constant $c = c(n, p, \nu, L, \tilde{L}, \|u\|_{L^\infty(\Omega)}, \mathrm{dist}(\Omega', \partial\Omega))$.

Proof. The proof of the comparison lemma consists of two steps, by freezing one variable at time: more precisely we consider the minimizer $\bar{v}$ of the Dirichlet problem (4.1) and we prove two comparison estimates, the first one between v and $\bar{v}$ and the second one between $\bar{v}$ and u. We stress that in Step 1 we make use of the Euler-Lagrange equations of v and $\bar{v}$, while in Step 2 we will use Lemma 2.3 since, due to the lack of differentiability of the functional $\mathcal{P}$, the Euler equation of u cannot even be written.

Step 1 Let $\bar{v} \in W^{1,p}(B_{R/2})$ be the minimizer of the Dirichlet problem (4.1): the following comparison estimate between v and $\bar{v}$ holds:

$$\begin{aligned}\int_{B_{R/4}} &\big[\, |V_p(D\bar{v}) - V_p(Dv)|^2 + a(x)|V_{\log}(D\bar{v}) - V_{\log}(Dv)|^2 \,\big]\, dx \\ &\leq c(p, \nu, L)\, \omega_b(R) \int_{B_R} H(x, Du)\, dx\ . \end{aligned}\tag{4.14}$$

Since both v and $\bar{v}$ are minimizers, we can use the corresponding Euler-Lagrange equations (4.5) and (4.6). We can test with $\phi = \bar{v} - v \in W^{1,p}_0(B_{R/4})$ (extended to 0 on $B_{R/2} \setminus B_{R/4}$) since (2.45) and (4.4) guarantees that $H(x, D\phi) \in L^1(B_{R/4})$:

$$\begin{aligned}&\int_{B_{R/4}} b(x, u_0)\langle \partial_z H(x, D\bar{v}), D\bar{v} - Dv\rangle\, dx \\ &- \int_{B_{R/4}} b(x_0, u_0)\langle \partial_z H(x, Dv), D\bar{v} - Dv\rangle\, dx \;=\; 0\,. \end{aligned}\tag{4.15}$$

Using (2.48) of Lemma 2.2, with $z = D\bar{v}(x)$ and $\lambda = D\bar{v}(x) - Dv(x)$, and (2.45) we may estimate

$$|\langle \partial_z H(x, D\bar{v}), D\bar{v} - Dv\rangle| \leq c(p)\big(H(x, D\bar{v}) + H(x, Dv)\big)\,, \tag{4.16}$$

thus in (4.15) we can add and substract the integral

$$\int_{B_{R/4}} b(x_0, u_0)\langle \partial_z H(x, D\bar{v}), D\bar{v} - Dv\rangle\, dx\,,$$

which is finite by (4.16) and (4.4), obtaining

$$\begin{aligned}\mathcal{D}_1 &:= \int_{B_{R/4}} b(x_0,u_0)\langle \partial_z H(x,D\bar{v}) - \partial_z H(x,Dv), D\bar{v} - Dv\rangle\,dx\\ &= \int_{B_{R/4}} [\,b(x_0,u_0) - b(x,u_0)\,]\,\langle \partial_z H(x,D\bar{v}), D\bar{v} - Dv\rangle\,dx := \mathcal{D}_2\,.\end{aligned}$$

Since, by (2.33), we can compute

$$\begin{aligned}\mathcal{D}_1 = b(x_0,u_0)\int_{B_{R/4}} &\langle \partial f(D\bar{v}) - \partial f(Dv), D\bar{v} - Dv\rangle\\ &+ a(x)\langle \partial g(D\bar{v}) - \partial g(Dv), D\bar{v} - Dv\rangle\,dx\,,\end{aligned}$$

using (2.34) and (2.35), with $z_1 = D\bar{v}(x)$ and $z_2 = Dv(x)$, and the bounds (1.11) on $b(\cdot,\cdot)$, we can estimate $\mathcal{D}_1$ from below obtaining that

(4.17)
$$\frac{\nu}{c(p)}\int_{B_{R/4}} \big[\,|V_p(D\bar{v}) - V_p(Dv)|^2 + a(x)|V_{\log}(D\bar{v}) - V_{\log}(Dv)|^2\,\big]\,dx \le \mathcal{D}_1 = |\mathcal{D}_2|\,.$$

Recalling the definition (1.13) of $\omega_b(\cdot)$, by (4.16) and (4.4) we can estimate $|\mathcal{D}_2|$ as

$$\begin{aligned}|\mathcal{D}_2| &\le \int_{B_{R/4}} \big|b(x_0,u_0) - b(x,u_0)\big|\,\big|\langle \partial_z H(x,D\bar{v}), D\bar{v} - Dv\rangle\big|\,dx\\ &\le c(p)\,\omega_b\,(\,|x-x_0|\,)\int_{B_{R/4}} \big[\,H(x,D\bar{v}) \,+\, H(x,Dv)\,\big]\,dx\\ &\le c(p,L,\nu)\;\omega_b(R)\int_{B_R} H(x,Du)\,dx\,.\end{aligned} \tag{4.18}$$

Putting together (4.17) and (4.18) the proof of the comparison estimate (4.14) is complete for a constant $c \equiv c(p,\nu,L)$.

Step 2 The following comparison estimate between $\bar{v}$ and u holds:

$$\begin{aligned}\int_{B_{R/2}} &\big[\,|V_p(Du) - V_p(D\bar{v})|^2 + a(x)|V_{\log}(Du) - V_{\log}(D\bar{v})|^2\,\big]\,dx\\ &\le c\;\omega_b(R^\beta)\int_{B_R} H(x,Du)\,dx\,,\end{aligned} \tag{4.19}$$

where β is the Hölder exponent of u as in (4.11) and the constant c depends on $n, p, \nu, L, \tilde{L}, \|u\|_{L^\infty(\Omega)}, \mathrm{dist}(\Omega', \partial\Omega)$.

All the integrals we will consider in the following calculations are finite by (4.4). Using Lemma 2.3 with $v = u_0$, $z_1 = Du(x)$, $z_2 = D\bar{v}(x)$, and the Euler-Lagrange equation (4.6) for $\bar{v}$ (tested with $\phi = u - \bar{v} \in W^{1,p}_0(B_{R/2})$ such that $H(x,D\phi) \in L^1(B_{R/2})$ by (2.45) and (4.4)), we obtain that

$$\begin{aligned}\frac{1}{c(p,\nu)}&\int_{B_{R/2}} \big[\,|V_p(Du) - V_p(D\bar{v})|^2 + a(x)|V_{\log}(Du) - V_{\log}(D\bar{v})|^2\,\big]\,dx\\ &\le \int_{B_{R/2}} \big[\,h(x,u_0,Du) - h(x,u_0,D\bar{v}) - \langle \partial_z h(x,u_0,D\bar{v}), Du - D\bar{v}\rangle\,\big]\,dx\\ &= \int_{B_{R/2}} \big[\,b(x,u_0)\,H(x,Du) - b(x,u_0)\,H(x,D\bar{v})\,\big]\,dx\\ &\quad - \int_{B_{R/2}} b(x,u_0)\,\langle \partial_z H(x,D\bar{v}), Du - D\bar{v}\rangle\,dx\\ &= \int_{B_{R/2}} \big[\,b(x,u_0)\,H(x,Du) - b(x,u_0)\,H(x,D\bar{v})\,\big]\,dx \qquad (4.20)\end{aligned}$$

$$
\begin{aligned}
&= \int_{B_{R/2}} [\, b(x,u_0) - b(x,u(x)) \,]\, H(x,Du)\, dx \\
&+ \int_{B_{R/2}} \big[\, b(x,u(x))\, H(x,Du) - b(x,\bar v(x))\, H(x,D\bar v) \,\big]\, dx \\
&+ \int_{B_{R/2}} [\, b(x,\bar v(x)) - b(x,u_0) \,]\, H(x,D\bar v)\, dx \\
&= I_1 + I_2 + I_3 \,.
\end{aligned}
$$

Since u is a local minimizer of the functional $\mathcal{P}$ defined in (1.10) we get $I_2 \le 0$. Using the definition (1.13) of the modulus of continuity $\omega_b(\cdot)$, the maximum principle (4.9) and (4.4) we estimate

$$
\begin{aligned}
I_1 + I_3 \le\ & \int_{B_{R/2}} \big| b(x,u_0) - b(x,u(x)) \big|\, H(x,Du)\, dx \\
& + \int_{B_{R/2}} \big| b(x,\bar v(x)) - b(x,u_0) \big|\, H(x,D\bar v)\, dx \qquad (4.21)\\
\le\ & \int_{B_{R/2}} \omega_b(|u_0 - u(x)|)\, H(x,Du)\, dx + \int_{B_{R/2}} \omega_b(|\bar v(x) - u_0|)\, H(x,D\bar v)\, dx \\
\le\ & \omega_b(\operatorname*{osc}_{B_{R/2}} u) \int_{B_{R/2}} H(x,Du)\, dx + \omega_b(\operatorname*{osc}_{B_{R/2}} u) \int_{B_{R/2}} H(x,D\bar v)\, dx \\
\le\ & (1 + \frac{L}{\nu})\, \omega_b(\operatorname*{osc}_{B_{R/2}} u) \int_{B_R} H(x,Du)\, dx \,.
\end{aligned}
$$

Recalling that (4.12) means that $\operatorname*{osc}_{B_{R/2}} u \le C_\beta\, R^\beta$ for a constant $C_\beta \ge 1$ depending on $n, p, \nu, L, \tilde L, \|u\|_{L^\infty(\Omega)}, \mathrm{dist}(\Omega', \partial\Omega)$, and the concavity of $\omega_b(\cdot)$ we obtain

$$
\omega_b(\operatorname*{osc}_{B_{R/2}} u) \le \omega_b(C_\beta\, R^\beta) \le C_\beta\, \omega_b(R^\beta)\,. \tag{4.22}
$$

From (4.20), (4.21) and (4.22) we deduce that the comparison estimate (4.19) holds with a constant $c = c(n, p, \nu, L, \tilde L, \|u\|_{L^\infty(\Omega)}, \mathrm{dist}(\Omega', \partial\Omega))$.

From (4.14) and (4.19), since $R \le R^\beta$ and $\omega_b(\cdot)$ is increasing, we deduce immediately the comparison estimate (4.13). $\square$

4.2. **Decay estimate.** Thanks to the previous comparison lemma, we deduce a decay lemma for the minimizer of $\mathcal{P}$.

Lemma 4.3 (Decay). *Let $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a local minimizer of $\mathcal{P}$ such that $H(\cdot, Du(\cdot)) \in L^1(\Omega)$, let $\Omega' \Subset \Omega$, let $\mathrm{d} := \mathrm{dist}(\Omega', \partial\Omega)$ and let $B_R \equiv B_R(x_0)$ be a ball with center $x_0 \in \Omega'$ and radius $R \le \min\{1/8, \mathrm{d}/4\}$. Moreover, let $\omega_a(\cdot)$ and $\omega_b(\cdot)$ be as in* (1.12) *and* (1.13) *with the additional assumption* (3.1)*, and let $\beta \in (0,1)$ be the Hölder exponent of u as in* (4.11)*. Then the inequality*

$$
\int_{B_\varrho} H(x,Du)\, dx \le c_d \left[\left(\frac{\varrho}{R}\right)^n + \omega_a(R) \log\left(\frac{1}{R}\right) + \omega_b(R^\beta) \right] \int_{B_R} H(x,Du)\, dx \tag{4.23}
$$

holds for a constant $c_d \equiv c_d(n, p, \nu, L, \tilde L, \|H(\cdot, Du(\cdot))\|_{L^1(\Omega)}, \|u\|_{L^\infty(\Omega)}, \mathrm{dist}(\Omega', \partial\Omega))$ for every $B_\varrho \equiv B_\varrho(x_0)$ with $0 < \varrho \le R$.

Proof. Let us fix a ball $B_R \equiv B_R(x_0)$ with radius $R \le \min\{1/8, \mathrm{d}/4\}$ and center $x_0 \in \Omega'$ and let us denote by v the minimizer of the Dirichlet problem (4.2) on $B_{R/4}$. It suffices to prove (4.23) for $0 < \varrho \le R/8$, indeed for $R/8 < \varrho \le R$ the inequality follows immediately from $8(\varrho/R) > 1$. Using estimates (2.27) from below and from above and the comparison estimate (4.13), for every $0 < \varrho \le R/8$ we deduce the following chain of inequalities:

$$
\begin{aligned}
\int_{B_\varrho} H(x,Du)\,dx &= \int_{B_\varrho} |Du|^p + a(x)|Du|^p \log(e+|Du|)\,dx \\
&\le \int_{B_\varrho} |V_p(Du)|^2 + a(x)|V_{\log}(Du)|^2\,dx \\
&\le 2\int_{B_\varrho} |V_p(Du)-V_p(Dv)|^2 + a(x)|V_{\log}(Du)-V_{\log}(Dv)|^2\,dx \\
&\qquad + 2\int_{B_\varrho} |V_p(Dv)|^2 + a(x)|V_{\log}(Dv)|^2\,dx \\
&\le c\;\omega_b(R^\beta)\int_{B_R} H(x,Du)\,dx + c(p)\int_{B_\varrho} H(x,Dv)\,dx
\end{aligned} \tag{4.24}
$$

for a constant c depending on $n, p, \nu, L, \tilde{L}, \|u\|_{L^\infty(\Omega)}, \operatorname{dist}(\Omega', \partial\Omega)$. To estimate the last term appearing in (4.24), let us recall (Remark 4.2) that the function v is a local minimizer of the functional $\mathcal{P}_{\log}$ defined in (1.6) on $\Omega = B_{R/4}$, thus we can apply the decay estimate in [4, Lemma 5.3] to v on $B_{R/8}$ obtaining that

$$
\int_{B_\varrho} H(x,Dv)\,dx \le c\left[\left(\frac{\varrho}{R/8}\right)^n + \omega_a(R/8)\log\left(\frac{1}{R/8}\right)\right]\int_{B_{R/8}} H(x,Dv)\,dx
$$

for every $0 < \varrho \le R/8$ and for a constant $c \equiv c(n, p, \nu, L, \tilde{L}, \|H(\cdot, Du(\cdot))\|_{L^1(\Omega)})$, where we have assumed the dependence of the constant c on $\nu, L, \|H(\cdot, Du(\cdot))\|_{L^1(\Omega)}$ instead of $\|Dv\|_{L^p(B_{R/4})}$ thanks to Remark 3.2 and (4.4).

By the monotonicity of $\omega_a(\cdot)$, (2.12) with $A = 8$, and (4.4) we conclude that

$$
\int_{B_\varrho} H(x,Dv)\,dx \le c\left[\left(\frac{\varrho}{R}\right)^n + \omega_a(R)\log\left(\frac{1}{R}\right)\right]\;\frac{L}{\nu}\int_{B_R} H(x,Du)\,dx\,. \tag{4.25}
$$

From (4.24) and (4.25) we deduce the decay estimate (4.23) holds with a constant $c_d \equiv c_d(n, p, \nu, L, \tilde{L}, \|H(\cdot, Du(\cdot))\|_{L^1(\Omega)}, \|u\|_{L^\infty(\Omega)}, \operatorname{dist}(\Omega', \partial\Omega))$. □

4.3. **Hölder continuity of minima.** In this paragraph we prove the second assertion in Theorem 1.2. Therefore we consider local minimizers $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ of the functional $\mathcal{P}$ defined in (1.10) with also $H(\cdot, Du(\cdot)) \in L^1(\Omega)$ and we prove that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for every $\alpha \in (0,1)$. To get this result we need the additional assumption

$$
\limsup_{r\to 0}\;\omega_a(r)\log\left(\frac{1}{r}\right) = 0 \tag{4.26}
$$

on the modulus of continuity $\omega_a(\cdot)$ of the coefficient $a(\cdot)$ (see (1.12)). Since by the definition (1.13) of the modulus of continuity $\omega_b(\cdot)$ of the coefficient $b(\cdot,\cdot)$ we have that $\lim_{r\to 0}\omega_b(r) = 0$, it follows that also

$$
\lim_{r\to 0}\;\omega_b(r^\beta) = 0\;, \tag{4.27}
$$

where β is the Hölder exponent of the minimizer u which appears in the comparison and decay estimates.

First, we prove the following Morrey type estimate.

Lemma 4.4. *Let $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a local minimizer of $\mathcal{P}$ such that $H(\cdot, Du(\cdot)) \in L^1(\Omega)$, let the modulus of continuity $\omega_a(\cdot)$ of $a(\cdot)$ satisfy assumption (4.26) and let $\omega_b(\cdot)$ be the modulus of continuity of $b(\cdot,\cdot)$. Then for every $\Omega' \Subset \Omega$ and every $\delta \in (0,n)$, there exist positive constants $c_{\mathrm{dec}} \equiv c_{\mathrm{dec}}(n, p, \nu, L, \tilde{L},$*

$\|H(\cdot, Du(\cdot))\|_{L^1(\Omega)}, \|u\|_{L^\infty(\Omega)}, \mathrm{dist}(\Omega', \partial\Omega), \delta)$ *and* $R_0 \leq \min\{1/8, \mathrm{dist}(\Omega', \partial\Omega)/4\}$ *with*

$$R_0 \equiv R_0(n, p, \nu, L, \tilde{L}, \|H(\cdot, Du(\cdot))\|_{L^1(\Omega)}, \|u\|_{L^\infty(\Omega)}, \mathrm{dist}(\Omega', \partial\Omega), \delta)\ , \tag{4.28}$$

such that the decay estimate

$$\int_{B_\varrho} H(x, Du)\, dx \leq c_{\mathrm{dec}} \left(\frac{\varrho}{R}\right)^{n-\delta} \int_{B_R} H(x, Du)\, dx \tag{4.29}$$

holds whenever $0 < \varrho \leq R \leq R_0$ *and* $B_\varrho \subset B_R$ *are concentric balls with center in* $x_0 \in \Omega'$.

Proof. Let us fix $\Omega' \Subset \Omega$ and $\delta \in (0, n)$; let us set $\bar{R} = \min\{1/8, \mathrm{dist}(\Omega', \partial\Omega)/4\}$ and for any $x_0 \in \Omega'$ consider the function $\phi \colon [0, \bar{R}] \to [0, \infty)$ defined as

$$\phi(\varrho) \equiv \int_{B_\varrho} H(x, Du)\, dx\ .$$

Thanks to (4.23) we can apply Lemma 2.5 to the function ϕ with $c_0 = c_d$, obtaining $\varepsilon_0 > 0$ depending on (n, δ, c_d) that is $\varepsilon_0 \equiv \varepsilon_0(n, p, \nu, L, \tilde{L}, \|H(\cdot, Du(\cdot))\|_{L^1(\Omega)}$, $\|u\|_{L^\infty(\Omega)}, \mathrm{dist}(\Omega', \partial\Omega), \delta)$. Since the conclusion of Lemma 2.5 holds only if $\varepsilon \leq \varepsilon_0$, using (4.26) and (4.27) we have to fix $0 < R_0 \leq \min\{1/8, \mathrm{dist}(\Omega', \partial\Omega)/4\}$ such that

$$\omega_a(R) \log(1/R) + \omega_b(R^\beta) \leq \varepsilon_0 \qquad \text{for every } 0 < R \leq R_0$$

with $R_0 \equiv R_0(n, p, \nu, L, \tilde{L}, \|H(\cdot, Du(\cdot))\|_{L^1(\Omega)}, \|u\|_{L^\infty(\Omega)}, \mathrm{dist}(\Omega', \partial\Omega), \delta)$.

Therefore

$$\phi(\varrho) \leq c_d \left[\left(\frac{\varrho}{R}\right)^n + \varepsilon_0 \right] \phi(R)$$

for every $0 < \varrho \leq R \leq R_0$, so Lemma 2.5, together with the generality of x_0, gives

$$\phi(\varrho) \leq c_1 \left(\frac{\varrho}{R}\right)^{n-\delta} \phi(R) \tag{4.30}$$

whenever $0 < \varrho \leq R \leq R_0$ and $B_\varrho \subset B_R$ are concentric balls with center in Ω', with a constant $c_1 \equiv c_1(n, \delta, c_d)$. Inequality (4.30) is exactly (4.29) with a constant $c_{\mathrm{dec}} \equiv c_{\mathrm{dec}}(n, p, \nu, L, \tilde{L}, \|H(\cdot, Du(\cdot))\|_{L^1(\Omega)}, \|u\|_{L^\infty(\Omega)}, \mathrm{dist}(\Omega', \partial\Omega), \delta)$ and the proof of the lemma is complete. □

The Hölder regularity of u follows from the integral characterization of Hölder continuity due to Campanato (see Lemma 2.4). Indeed, fixed $\Omega' \Subset \Omega$, by Poincaré inequality and the decay estimate (4.29) we obtain

$$\begin{aligned} \int_{B_\varrho(x_0)} \left| \frac{u - (u)_{B_\rho}}{\rho} \right|^p dx &\leq c(n) \int_{B_\varrho(x_0)} |Du|^p\, dx \\ &\leq c(n) \int_{B_\varrho(x_0)} H(x, Du)\, dx \leq\ c \left(\frac{\varrho}{R}\right)^{n-\delta} \int_{B_R(x_0)} H(x, Du)\, dx \end{aligned} \tag{4.31}$$

for every $\delta \in (0, n)$, every $x_0 \in \Omega'$, for every $0 < \varrho \leq R \leq R_0$ and for a constant $c \equiv c(n, p, \nu, L, \tilde{L}, \|H(\cdot, Du(\cdot))\|_{L^1(\Omega)}, \|u\|_{L^\infty(\Omega)}, \mathrm{dist}(\Omega', \partial\Omega), \delta)$. Considering $\delta \in (0, p)$ and $\alpha = 1 - \delta/p$, inequality (4.31) can be rewritten as

$$\int_{B_\varrho(x_0)} \left| u - (u)_{B_\rho} \right|^p dx \leq c\, \frac{1}{R^{n-\delta}}\, \varrho^{p\alpha + n} \int_{B_R(x_0)} H(x, Du)\, dx\,. \tag{4.32}$$

Thus, there exist positive constants $\varrho_0 = R_0$ and c_0 such that, using (4.32) with $R = \varrho_0$

$$\int_{B_\varrho(x_0)} \left| u - (u)_{B_\rho} \right|^p dx \leq \left[c\, \frac{1}{{\varrho_0}^{n-\delta}} \int_\Omega H(x, Du)\, dx \right] \varrho^{p\alpha+n} = c_0\, \varrho^{p\alpha+n} \tag{4.33}$$

for a constant $c_0 \equiv c_0(n, p, \nu, L, \tilde{L}, \|H(\cdot, Du(\cdot))\|_{L^1(\Omega)}, \|u\|_{L^\infty(\Omega)}, \mathrm{dist}(\Omega', \partial\Omega), \delta)$, for every $x_0 \in \Omega'$ and every $0 < \varrho \le \varrho_0$. As $\Omega' \Subset \Omega$ is arbitrary, Lemma 2.4 implies that $u \in \mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\Omega)$; since δ can be taken arbitrarily close to zero we can reach any $\alpha \in (0,1)$ and conclude that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for every $\alpha \in (0,1)$.

In addition, fixed $\Omega' \Subset \Omega$, for every $B_R(x_0) \subset \Omega'$ with $R \le R_0$ and every $\alpha \in (0,1)$ we may estimate the Hölder seminorm of u (see (2.4)) as

$$[u]_{\mathcal{C}^{0,\alpha}(B_{R/2})} \le c \left(R^{p(1-\alpha)} \fint_{B_R} H(x, Du)\, dx \right)^{1/p} \tag{4.34}$$

for a constant $c \equiv c\,(n, p, \alpha, \nu, L, \tilde{L}, \|H(\cdot, Du(\cdot))\|_{L^1(\Omega)}, \|u\|_{L^\infty(\Omega)}, \mathrm{dist}(\Omega', \partial\Omega))$. To prove (4.34) we estimate $[u]_{\mathcal{L}^{p,\lambda}(B_{R/2})}$ with $\lambda = p\alpha + n$ and then use (2.64). Indeed, for every $\bar{x} \in B_{R/2}$ and every $0 < \varrho \le R/2$, using (2.1), (4.32) in $\bar{x}$ with $R/2$ instead of R and the fact that $B_{R/2}(\bar{x}) \subset B_R(x_0)$, we may estimate

$$\begin{aligned} &\varrho^{-\lambda} \int_{B_{R/2}(x_0) \cap B_\varrho(\bar{x})} |u - (u)_{B_{R/2}(x_0) \cap B_\varrho(\bar{x})}|^p\, dx \le \varrho^{-\lambda}\, 2^p \int_{B_\varrho(\bar{x})} |u - (u)_{B_\varrho(\bar{x})}|^p\, dx \\ &\le c\, R^{p(1-\alpha)-n} \int_{B_{R/2}(\bar{x})} H(x, Du)\, dx \le c\, R^{p(1-\alpha)} \fint_{B_R(x_0)} H(x, Du)\, dx\,. \end{aligned}$$

Otherwise, if $\varrho > R/2$, using (2.1) and (4.32) in x_0 with $\varrho = R/2$, we obtain

$$\begin{aligned} &\varrho^{-\lambda} \int_{B_{R/2}(x_0) \cap B_\varrho(\bar{x})} |u - (u)_{B_{R/2}(x_0) \cap B_\varrho(\bar{x})}|^p\, dx \\ &\le 2^p\, (R/2)^{-\lambda} \int_{B_{R/2}(x_0)} |u - (u)_{B_{R/2}(x_0)}|^p\, dx \le c\, R^{p(1-\alpha)} \fint_{B_R(x_0)} H(x, Du)\, dx\,. \end{aligned}$$

Remark 4.5. The fact that the minimizer u belongs to $\mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for every $\alpha \in (0,1)$ allows us to rewrite the comparison estimate (4.13) choosing any $\beta \in (0,1)$.

4.4. **Hölder continuity of the gradient.** Here we complete the proof of Theorem 1.2 demonstrating the validity of the third and last assertion, the one concerning the gradient Hölder continuity of minimizers. Let $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a local minimizer of the functional $\mathcal{P}$ such that $H(\cdot, Du(\cdot)) \in L^1(\Omega)$; we prove that, if

$$\omega_a(R) \le \tilde{L}_1 R^{\sigma_1}\,, \quad \omega_b(R) \le \tilde{L}_2 R^{\sigma_2} \qquad \text{hold for every } R \le 1 \tag{4.35}$$

for some $\sigma_1, \sigma_2 \in (0,1)$ and $\tilde{L}_1, \tilde{L}_2 \ge 1$, then there exists $\alpha \in (0,1)$, depending only on n, p, σ_1, σ_2 such that

$$Du \in C^{0,\alpha}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)\,. \tag{4.36}$$

We start showing the following estimate of the excess.

Lemma 4.5. *Let $\Omega' \Subset \Omega$ and let us set* $\mathrm{d} := \mathrm{dist}(\Omega', \partial\Omega)$.

For every ball $B_R \equiv B_R(x_0)$ with center $x_0 \in \Omega'$ the inequality

$$\fint_{B_\varrho} |Du - (Du)_{B_\varrho}|^p\, dx \le c_e \left[\left(\frac{\varrho}{R}\right)^{\tilde{\alpha}p} + R^\sigma \left(\frac{R}{\varrho}\right)^n \right] \fint_{B_R} H(x, Du)\, dx \tag{4.37}$$

holds whenever $0 < \varrho \le R \le \min\{1, \mathrm{d}/4\}$ and $B_\varrho \subset B_R$ are concentric balls, for an exponent $\tilde{\alpha} \in (0,1)$ depending only on n and p, an exponent $\sigma \equiv \sigma(p, \sigma_1, \sigma_2)$ and for a constant $c_e \equiv c_e(n, p, \nu, L, \tilde{L}_1, \tilde{L}_2, \sigma_1, \|H(\cdot, Du(\cdot))\|_{L^1(\Omega)}, \|u\|_{L^\infty(\Omega)}, \mathrm{dist}(\Omega', \partial\Omega))$.

Proof. Let us fix $\Omega' \Subset \Omega$ and $B_R \equiv B_R(x_0)$ with center $x_0 \in \Omega'$ and radius $R \le \min\{1, \mathrm{d}/4\}$. Let us prove the validity of (4.37) for radii $0 < \varrho \le R/4$; in the case $R/4 < \varrho \le R$ the inequality follows easily and we discuss it at the end.

Let v be the solution of the minimum problem (4.2) on $B_{R/4}(x_0)$. For every $0 < \varrho \leq R/4$, using (2.1) with $z = (Dv)_{B_\varrho}$ we obtain

$$
\begin{aligned}
\fint_{B_\varrho} |Du - (Du)_{B_\varrho}|^p \, dx &\leq 2^p \fint_{B_\varrho} |Du - (Dv)_{B_\varrho}|^p \, dx \\
&\leq 4^p \fint_{B_\varrho} |Dv - (Dv)_{B_\varrho}|^p \, dx + 4^p \fint_{B_\varrho} |Du - Dv|^p \, dx \, . \qquad (4.38)
\end{aligned}
$$

To estimate the first term in the right-hand side of (4.38), let us recall (Remark 4.2) that the function v is a local minimizer of the functional $\mathcal{P}_{\log}$ defined in (1.6) on $\Omega = B_{R/4}(x_0)$. Thanks to the Hölder continuity of $a(\cdot)$ as in the assumption (4.35), we can apply the estimate (5.24) in [4] to the function v on $B_{R/4}(x_0)$ (the assumption $R/4 \leq 1/e$ is satisfied) with $\sigma = \sigma_1$, $\tilde{L} = \tilde{L}_1$, obtaining that there exist positive constants $\tilde{\alpha} \equiv \tilde{\alpha}(n,p) \in (0,1)$ and $c \equiv c(n,p,\nu,L,\tilde{L}_1,\sigma_1,\|H(\cdot,Du(\cdot))\|_{L^1(\Omega)})$ such that

$$
\fint_{B_\varrho} |Dv - (Dv)_{B_\varrho}|^p \, dx \leq c \left[\left(\frac{4\varrho}{R} \right)^{\tilde{\alpha}p} + (R/4)^{\sigma_1/4} \left(\frac{R}{4\varrho} \right)^n \right] \fint_{B_{R/4}} H(x, Dv) \, dx \, ,
$$

where we have assumed the dependence of the constant c on $\nu, L, \|H(\cdot,Du(\cdot))\|_{L^1(\Omega)}$ instead of $\|Dv\|_{L^p(B_{R/4})}$ thanks to Remark 3.2 and (4.4). Using again (4.4) we conclude that

$$
\fint_{B_\varrho} |Dv - (Dv)_{B_\varrho}|^p \, dx \leq c \left[\left(\frac{\varrho}{R} \right)^{\tilde{\alpha}p} + R^{\sigma_1/4} \left(\frac{R}{\varrho} \right)^n \right] \fint_{B_R} H(x, Du) \, dx \qquad (4.39)
$$

for a constant c depending only on $n,p,\nu,L,\tilde{L}_1,\sigma_1,\|H(\cdot,Du(\cdot))\|_{L^1(\Omega)}$. As for the second term in the right-hand side of (4.38), we have two different situations depending on the value of p. If $p \geq 2$, by (2.29), the comparison lemma 4.2 with $\tilde{L} = \tilde{L}_1$ and $\beta = 1/2$ (see Remark 4.5), and the Hölder continuity of $b(\cdot,\cdot)$ as in assumption (4.35), we obtain

$$
\begin{aligned}
\fint_{B_\varrho} |Du - Dv|^p \, dx &\leq 4^{-n} \left(\frac{R}{\varrho} \right)^n \fint_{B_{R/4}} |Du - Dv|^p \, dx \\
&\leq c(n,p) \left(\frac{R}{\varrho} \right)^n \fint_{B_{R/4}} |V_p(Du) - V_p(Dv)|^2 \, dx \\
&\leq c \left(\frac{R}{\varrho} \right)^n \omega_b(R^{1/2}) \fint_{B_R} H(x, Du) \, dx \\
&\leq c \, R^{\frac{1}{2}\sigma_2} \left(\frac{R}{\varrho} \right)^n \fint_{B_R} H(x, Du) \, dx \qquad (4.40)
\end{aligned}
$$

for some constant c depending on $n,p,\nu,L,\tilde{L}_1,\tilde{L}_2,\|u\|_{L^\infty(\Omega)},\operatorname{dist}(\Omega',\partial\Omega)$.

Instead, when $1 < p < 2$, by (2.30), Hölder's inequality applied with conjugate exponents $2/p$ and $2/(2-p)$, the comparison lemma 4.2 with $\beta = 1/2$, again (4.4) and the Hölder continuity of $b(\cdot,\cdot)$, we estimate

(4.41)

$$
\begin{aligned}
&\fint_{B_\varrho} |Du - Dv|^p \, dx \leq 4^{-n} \left(\frac{R}{\varrho} \right)^n \fint_{B_{R/4}} |Du - Dv|^p \, dx \\
&\leq c(n,p) \left(\frac{R}{\varrho} \right)^n \fint_{B_{R/4}} |V_p(Du) - V_p(Dv)|^p (|Du| + |Dv|)^{p(2-p)/2} \, dx \\
&\leq c \left(\frac{R}{\varrho} \right)^n \left(\fint_{B_{R/4}} |V_p(Du) - V_p(Dv)|^2 \, dx \right)^{p/2} \left(\fint_{B_{R/4}} (|Du| + |Dv|)^p \, dx \right)^{(2-p)/2}
\end{aligned}
$$

$$\leq c\left(\frac{R}{\varrho}\right)^n (\omega_b(R^{1/2}))^{p/2}\left(\fint_{B_R} H(x,Du)\,dx\right)^{p/2}\left(\fint_{B_R} H(x,Du)\,dx\right)^{(2-p)/2}$$
$$\leq c\left(\frac{R}{\varrho}\right)^n (\tilde{L}_2\,R^{\frac{1}{2}\sigma_2})^{p/2}\fint_{B_R} H(x,Du)\,dx$$
$$= c\,R^{\frac{1}{4}p\sigma_2}\left(\frac{R}{\varrho}\right)^n\fint_{B_R} H(x,Du)\,dx\,.$$

The above inequality again holds for some constant c depending on $n,p,\nu,L,\tilde{L}_1\tilde{L}_2$, $\|u\|_{L^\infty(\Omega)}, \mathrm{dist}(\Omega',\partial\Omega)$. Defining $\sigma := \min\left\{\frac{1}{4}\sigma_1, \frac{1}{2}\sigma_2, \frac{1}{4}\sigma_2 p\right\}$, from (4.38), (4.39), (4.40) and (4.41), we deduce that (4.37) holds in the case $0<\varrho\leq R/4$.

Since for $R/4<\varrho\leq R$, using Jensen's inequality and $4\varrho/R>1$ we easily obtain

$$\fint_{B_\varrho}|Du-(Du)_{B_\varrho}|^p\,dx \leq 2^p\fint_{B_\varrho}|Du|^p\,dx \leq 2^p\fint_{B_\varrho} H(x,Du)\,dx$$
$$\leq 2^p 4^n\fint_{B_R} H(x,Du)\,dx \leq c(n,p)\left(\frac{\varrho}{R}\right)^{\tilde{\alpha}p}\fint_{B_R} H(x,Du)\,dx$$

the proof of (4.37) is complete. □

Finally let us prove (4.36), that is $Du\in C^{0,\alpha}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$ for some exponent $\alpha\in(0,1)$ depending only on n,p,σ_1,σ_2. We choose

$$\varepsilon = \frac{\sigma}{\tilde{\alpha}p+\sigma+n}\in(0,1) \qquad \text{and} \qquad \alpha=\frac{\varepsilon\tilde{\alpha}}{2}=\frac{\tilde{\alpha}\sigma}{2(\tilde{\alpha}p+\sigma+n)}\in(0,1) \tag{4.42}$$

where σ is defined in Lemma 4.5 and depends on p and on the Hölder exponents σ_1,σ_2 in (4.35). Notice that ε is chosen in such a way that in (4.44) the exponents $[(1-\varepsilon)\sigma-\varepsilon\, n]$ and $(\varepsilon\,\tilde{\alpha}\,p)$ coincide.

For every $\Omega'\Subset\Omega$ we set $\mathrm{d}:=\mathrm{dist}(\Omega',\partial\Omega)$, we choose $\delta=\varepsilon\tilde{\alpha}p/[2(1-\varepsilon)]\in(0,n)$ and we determine $R_0\leq\min\{1/8,\mathrm{d}/4\}$ such that the decay estimate (4.29) holds for every ball $B_R(x_0)$ with center $x_0\in\Omega'$ and $0<\varrho\leq R\leq R_0$. The value of δ is chosen to get $(1-\varepsilon)\delta=\varepsilon\tilde{\alpha}p/2$ in (4.45).

In order to apply Lemma 2.4 to the gradient Du we show that there exist positive constants $\varrho_0=R_0^{1/(1-\varepsilon)}\leq R_0$ and c_0 such that

$$\int_{B_\varrho(x_0)}|Du-(Du)_{B_\varrho}|^p\,dx\leq c_0\;\varrho^{p\alpha+n} \tag{4.43}$$

holds for every $x_0\in\Omega'$ and every $0<\varrho\leq\varrho_0$.

First, notice that $\varrho\in(0,1)$ thus $0<\varrho\leq\varrho^{1-\varepsilon}\leq R_0$, so we can apply (4.37) between the radii ϱ and $R=\varrho^{1-\varepsilon}$ to estimate

$$\begin{aligned}\fint_{B_\varrho}|Du-(Du)_{B_\varrho}|^p\,dx &\leq c_e\left[\varrho^{\varepsilon\tilde{\alpha}p}+\varrho^{(1-\varepsilon)\sigma-\varepsilon n}\right]\fint_{B_{\varrho^{1-\varepsilon}}} H(x,Du)\,dx\\ &= 2\,c_e\,\varrho^{\varepsilon\tilde{\alpha}p}\fint_{B_{\varrho^{1-\varepsilon}}} H(x,Du)\,dx\end{aligned} \tag{4.44}$$

with the constant $c=2\,c_e\equiv c(n,p,\nu,L,\tilde{L}_1,\tilde{L}_2,\sigma_1,\|H(\cdot,Du(\cdot))\|_{L^1(\Omega)},\|u\|_{L^\infty(\Omega)}$, $\mathrm{dist}(\Omega',\partial\Omega))$. Then, by applying (4.29) between the radii $\varrho^{1-\varepsilon}$ and R_0 we deduce that

$$\int_{B_\varrho}|Du-(Du)_{B_\varrho}|^p\,dx\leq 2\,c_e\,\varrho^{\varepsilon\tilde{\alpha}p+n}\varrho^{-(1-\varepsilon)n}\int_{B_{\varrho^{1-\varepsilon}}} H(x,Du)\,dx$$
$$\leq 2\,c_e\,\varrho^{\varepsilon\tilde{\alpha}p+n}\varrho^{-(1-\varepsilon)n}\,c_{\mathrm{dec}}\left(\frac{\varrho^{1-\varepsilon}}{R_0}\right)^{n-\delta}\int_{B_{R_0}} H(x,Du)\,dx$$

$$
\begin{aligned}
(4.45) \qquad & = 2\, c_e\, c_{\mathrm{dec}}\, \frac{1}{R_0^{n-\delta}}\, \varrho^{\varepsilon\tilde{\alpha}p-(1-\varepsilon)\delta+n} \int_{B_{R_0}} H(x,Du)\,dx \\
& \leq \left[2\, c_e\, c_{\mathrm{dec}}\, \frac{1}{R_0^{n-\delta}} \int_{\Omega} H(x,Du)\,dx \right] \varrho^{\varepsilon\tilde{\alpha}p/2+n} = c_0\, \varrho^{p\alpha+n}
\end{aligned}
$$

with $c_0 \equiv c_0(n,p,\nu,L,\tilde{L}_1,\tilde{L}_2,\sigma_1,\sigma_2,\|H(x,Du)\|_{L^1(\Omega)},\|u\|_{L^\infty(\Omega)},\mathrm{dist}(\Omega',\partial\Omega))$ and the exponent α defined in (4.42). As $\Omega' \Subset \Omega$ is arbitrary, by Lemma 2.4 we conclude that $Du \in C^{0,\alpha}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$. This completes the proof of (4.36) and the whole proof of Theorem 1.2 is complete.

4.5. **A result in a Sobolev-Orlicz space.** About Orlicz spaces, Sobolev-Orlicz spaces and $L^p \log L$ spaces we refer to [43, 30]. Given an open bounded set $\Omega \subset \mathbb{R}^n$ and a Young function $\varphi : [0,\infty[\to [0,\infty[$ (φ is convex, strictly monotone increasing, $\lim_{t\to 0} \frac{\varphi(t)}{t} = 0$ and $\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty$) the Orlicz space $L^\varphi(\Omega;\mathbb{R}^k)$ is the set of measurable maps $f : \Omega \to \mathbb{R}^k$ such that $\int_\Omega \varphi(\lambda|f(x)|)\,dx < \infty$ for some $\lambda > 0$. The natural Luxemburg type norm is then defined by

$$
(4.46) \qquad \|f\|_{L^{\varphi(\cdot)}(\Omega)} := \inf\left\{ \lambda > 0 : \fint_\Omega \varphi\left(\frac{|f|}{\lambda}\right)\,dx \leq 1 \right\}.
$$

In particular if $\varphi(t) = t^p \log(e+t)$, $p > 1$, the associated Orlicz space is denoted by $L^p \log L(\Omega)$ and it consists of the measurable functions such that

$$
\int_\Omega |f(x)|^p \, \log(e+|f(x)|)\,dx < \infty .
$$

It is known that

$$
(4.47) \qquad f \in L^q(B_R)\,,\ q > p \quad \Rightarrow \quad f \in L^p \log L(B_R)\ ,
$$

$$
(4.48) \qquad f_h \to f \text{ in } L^q(B_R)\,,\ q > p \quad \Rightarrow \quad f_h \to f \text{ in } L^p \log L(B_R)\,.
$$

If the Young function φ is allowed to depend also on the space variable x we obtain a generalized Orlicz space, which is also called Musielak-Orlicz space. Thus, let us consider $\varphi_H : \Omega \times [0,\infty[\to [0,\infty[$ defined as $\varphi_H(x,t) = t^p + a(x)t^p \log(e+t)$, with $a(x)$ satisfying assumption (3.1) on its modulus of continuity, and consider the Musielak-Orlicz space $L^{\varphi_H(\cdot)}(\Omega;\mathbb{R}^k)$ which is defined as the set of measurable maps $f : \Omega \to \mathbb{R}^k$ such that

$$
\int_\Omega \varphi_H(x,|f(x)|)\,dx = \int_\Omega H(x,f(x))\,dx < \infty .
$$

It results (see [30, Lemma 3.3.3]) that

$$
(4.49) \qquad f_h \to f \text{ in } L^{\varphi_H(\cdot)}(\Omega;\mathbb{R}^k) \quad \Longleftrightarrow \quad \int_\Omega H(x,f_h(x)-f(x))\,dx \to 0 .
$$

The Sobolev-Orlicz space $W^{1,\varphi_H(\cdot)}(\Omega)$ can be defined by prescribing that a function $f \in L^{\varphi_H(\cdot)}(\Omega)$ has its weak gradient $Df \in L^{\varphi_H(\cdot)}(\Omega;\mathbb{R}^n)$. Thanks to (3.1) smooth functions are dense in $W^{1,\varphi_H(\cdot)}(\Omega)$ (see [30, Lemma 6.4.7]), so we denote by $W_0^{1,\varphi_H(\cdot)}(\Omega)$ the closure in $W^{1,\varphi_H(\cdot)}(\Omega)$ of $C_0^\infty(\Omega)$; it results (see [32, Lemma 6.9] or [30, Lemma 6.1.6]) that

$$
(4.50) \qquad W_0^{1,\varphi_H(\cdot)}(\Omega) \subset W_0^{1,p}(\Omega)\,,
$$

since $W^{1,\varphi_H(\cdot)}(\Omega)$ convergence implies $W^{1,p}(\Omega)$ convergence.

Remark 4.6. By Sobolev inequalities every function $f \in W^{1,p}(B_R)$ belongs to $L^{p^*}(B_R)$, thus (4.47) implies that $W^{1,p}(B_R) \subset L^{\varphi_H(\cdot)}(B_R)$ and (4.48)-(4.49) that convergence in $W^{1,p}(B_R)$ implies convergence in $L^{\varphi_H(\cdot)}(B_R)$. We want to stress here that a local minimizer u of the functional $\mathcal{P}$ defined in (1.10) belongs to

$W^{1,\varphi_H(\cdot)}(B_R)$ for every $B_R \Subset \Omega$ since $H(\cdot, Du(\cdot)) \in L^1(B_R)$ means that $Du \in L^{\varphi_H(\cdot)}(B_R;\mathbb{R}^n)$; the Sobolev-Orlicz space $W^{1,\varphi_H(\cdot)}_{\mathrm{loc}}(\Omega)$ shall be the natural space for the functional $\mathcal{P}$.

In this paragraph we want to point out that Lemma 3.2 allows to prove that

$$W^{1,\varphi_H(\cdot)}_0(B_R) = W^{1,p}_0(B_R) \cap W^{1,\varphi_H(\cdot)}(B_R)\,, \tag{4.51}$$

under assumption (3.1) on the modulus of continuity $\omega_a(\cdot)$ of the function $a(\cdot)$. This property does not hold for general domains Ω also in the case of standard Sobolev spaces: it is proven in [34] that for $1 < p < q$ the relation $W^{1,q}_0(\Omega) = W^{1,p}_0(\Omega) \cap W^{1,q}(\Omega)$ holds only for sufficiently smooth domains Ω satisfying a q-density condition; in particular the relation holds for Lipschitz domains. The result of [34] has not been generalized for variable exponent or generalized Orlicz spaces, but (4.51) shows that the result holds for $W^{1,\varphi_H(\cdot)}$ when Ω is a ball.

The inclusion $\subset$ in (4.51) holds on every bounded open set Ω from (4.50); to get the opposite inclusion let us consider $\phi \in W^{1,p}_0(B_R) \cap W^{1,\varphi_H(\cdot)}(B_R)$ and let us prove that $\phi \in W^{1,\varphi_H(\cdot)}_0(B_R)$. As $H(\cdot, D\phi(\cdot)) \in L^1(B_R)$, by Lemma 3.2 we obtain a sequence $\{\phi_k\} \subset C^\infty_0(B_R)$ such that $D\phi_k \to D\phi$ a.e., $\phi_k \to \phi$ in $W^{1,p}(B_R)$ (see Remark 3.4), and $H(\cdot, D\phi_k(\cdot)) \to H(\cdot, D\phi(\cdot))$ strongly in $L^1(B_R)$.

From Remark 4.6 we have that $\phi_k \to \phi$ in $L^{\varphi_H(\cdot)}(B_R)$, while to get $D\phi_k \to D\phi$ in $L^{\varphi_H(\cdot)}(B_R;\mathbb{R}^n)$ by (4.49) it is enough to prove that

$$\int_{B_R} H(x, D\phi_k(x) - D\phi(x))\,dx \to 0\ .$$

Since $H(x, D\phi_k(x)-D\phi(x)) \le 2^{p+1}\big(H(x, D\phi_k(x))+H(x, D\phi(x))\big)$ with $H(x, D\phi_k(x)-D\phi(x)) \to 0$ a.e., the conclusion follows by a well-known variant of Lebesgue dominated convergence theorem and (4.51) is proven.

Acknowledgments. We thank Paolo Baroni for some stimulating discussions.

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