



UNIVERSITÀ DI PARMA

ARCHIVIO DELLA RICERCA

University of Parma Research Repository

Stability of Lamellar Configurations in a Nonlocal Sharp Interface Model

This is the peer reviewed version of the following article:

Original

Stability of Lamellar Configurations in a Nonlocal Sharp Interface Model / Acerbi, Emilio Daniele Giovanni; Chen, CHAO NIEN; Choi, YUNG SZE. - In: SIAM JOURNAL ON MATHEMATICAL ANALYSIS. - ISSN 1095-7154. - 54:1(2022), pp. 558-594.

Availability:

This version is available at: 11381/2902918 since: 2024-10-31T09:21:30Z

Publisher:

Society for Industrial and Applied Mathematics

Published

DOI:

Terms of use:

Anyone can freely access the full text of works made available as "Open Access". Works made available

Publisher copyright

note finali coverpage

(Article begins on next page)

10 February 2025

1 **STABILITY OF LAMELLAR CONFIGURATIONS IN A NONLOCAL**
2 **SHARP INTERFACE MODEL***

3 EMILIO ACERBI [†], CHAO-NIEN CHEN [‡], AND YUNG-SZE CHOI [§]

4 **Abstract.** Equilibrium models based on a free energy functional deserve special interest in
5 recent investigations, as their critical points exhibit various pattern structures. These systems are
6 characterized by the presence of coexisting phases, whose distribution results from the competition
7 between short and long-range interactions. This article deals with an energy-driven sharp interface
8 model with long-range interaction being governed by a screened Coulomb kernel. We investigate a
9 number of criteria for the stability of lamellar configurations to ensure that they are indeed strict
10 local minimizers. We also give a sufficient condition to ensure a nontrivial periodic 2D minimal
11 energy configuration.

12 **Key words.** Nonlocal geometric variational problem, Sharp interface model, Stability, Lamella

13 **AMS subject classifications.** 49J20 49K20 49Q10 92C15 35K57

14 **1. Introduction.** The mechanisms responsible for pattern formation have been
15 extensively studied in a number of fields of science [5, 6, 21, 23, 24, 25, 28, 30, 31,
16 32, 36]; for instance, ferroelectric and ferromagnetic films, diblock copolymers and
17 degenerate ferromagnetic semiconductors. Equilibrium models based on a free energy
18 functional deserve special interest in recent investigations, see e.g. [4, 15, 16, 17, 26,
19 33, 34] and the references therein. A typical form of this free energy functional is

20 (1.1)
$$\mathcal{J}_\epsilon(u) = \int_\Omega \left(\frac{\epsilon}{2} |\nabla u|^2 + \epsilon^{-1} F(u) \right) dx + \frac{\sigma}{2} \int_\Omega \int_\Omega \psi(u(x)) G(x, \xi) \psi(u(\xi)) d\xi dx,$$

21 where u is a scalar function, F is a double-well potential, G is a positive kernel, ψ
22 is a given smooth function, ϵ is a small parameter and $\Omega \subset \mathbb{R}^N$ is a given bounded
23 domain. These systems are characterized by the presence of coexisting phases induced
24 by the two wells; the resulting structure of sharp transition interfaces defines the pat-
25 tern. A well-known example of G is the Green's function associated with a uniformly
26 elliptic operator. This turns (1.1) into a competition between short and long-range
27 interactions; who is winning depends on the precise tuning of the control parameters.
28 The short-range ramification, represented by the term with single integral, leads to
29 congregation, favoring large domains of pure phases with boundary shape that min-
30 imizes surface area. The long-range effect, depicted by the double integral term, is
31 repulsive in nature biasing towards small domains.

32 A diblock copolymer is a linear-chain molecule consisting of two subchains joined
33 covalently to each other. Depending on the material properties of the diblock macro-
34 molecules, the observed mesoscopic domains are highly regular periodic structures

*Submitted to the Editors January 24, 2021

Funding: Chen was supported by Ministry of Science and Technology (Taiwan) grant MOST 106-2115-M-007-011, 108-2115-M-007-009; Acerbi is a member of GNAMPA-INDAM and was supported by PRIN 2015PA5MP7 Calcolo delle variazioni

[†]Department of Mathematical, Physical and Computer Sciences, Università degli studi di Parma, Italy (emilio.acerbi@unipr.it).

[‡]Department of Mathematics, National Tsing Hua University, Hsinchu, Taiwan, ROC (chen@math.nthu.edu.tw).

[§]Department of Mathematics, UCONN University of Connecticut, Storrs, CT 06269-3009, USA (choi@math.uconn.edu).

35 that include lamellae, spheres, cylindrical tubes, and double-gyroids [6]. It is a com-
 36 mon belief that these patterns are metastable in certain ranges of the parameters
 37 and that they can undergo morphological instabilities leading to the formation of
 38 more complex patterns. In a model of microphase separation for diblock copoly-
 39 mer melts [32], it was proposed to study the critical points of a functional like (1.1)
 40 with G being the Green function for the Laplace operator subject to the homoge-
 41 neous Neumann boundary conditions or periodic boundary conditions. By setting
 42 $\psi(u) = u - \frac{1}{|\Omega|} \int_{\Omega} u \, dx$ and $F(u) = \frac{u^2(u-1)^2}{4}$ (or choosing $F(u) = \frac{(u+1)^2(u-1)^2}{4}$ in some
 43 articles) in (1.1), several authors [4, 15, 17, 18, 22, 31, 33, 34] investigated the patterns
 44 generated by

$$45 \quad (1.2) \quad \int_{\Omega} \left(\frac{\epsilon}{2} |\nabla u|^2 + \frac{u^2(u-1)^2}{4\epsilon} \right) dx + \frac{\sigma}{2} \int_{\Omega} \int_{\Omega} (u(x) - m)G(x, \xi)(u(\xi) - m) d\xi dx$$

46 with prescribed mass constraint $\frac{1}{|\Omega|} \int_{\Omega} u \, dx = m$ and small ϵ . A derivation of (1.2)
 47 based on the statistical physics of interacting block copolymers can be found in [18].
 48 We refer to a pioneer work of Nishiura and Ohnishi [31] for earlier results of this
 49 model.

50 As $\epsilon \rightarrow 0$ the L^1 norm Γ -limit of the functional (1.2) goes to (except for a
 51 multiplicative constant)

$$52 \quad (1.3) \quad \int_{\Omega} \left(|\nabla \chi| + \frac{\sigma}{2} |\nabla v|^2 \right) dx,$$

53 where χ is a characteristic function and

$$54 \quad (1.4) \quad v(x) = \int_{\Omega} G(x, \xi)(\chi(\xi) - m) d\xi .$$

55 When Ω is a very large domain, one expects that the effect of boundary condition on
 56 v diminishes in its interior and the minimizer may settle down into a natural minimal
 57 energy periodic configuration. Indeed in one space dimension, minimizers of (1.2) and
 58 (1.3) are periodic [15, 34]. To address the fundamental questions, namely to what
 59 extent periodicity holds in higher space dimensions and what effect the nonlocal term
 60 has on the stability of such periodic patterns, Alberti, Choksi and Otto [4] studied
 61 the sharp interface model (1.3)-(1.4) when Ω was a N -dimensional square box $\mathbb{T} =$
 62 $[-T/2, T/2]^N \subset \mathbb{R}^N$ with homogeneous Neumann boundary condition. Using a direct
 63 method in the calculus of variations, they showed uniform energy distribution for the
 64 minimizers in the interior of a large torus; indeed the boundary condition influence
 65 did diminish as far as energy was concerned. On the other hand one still could not tell
 66 if a genuine multi-dimensional periodic minimal energy periodic configuration existed
 67 and if so, what its structure was.

68 From now on in this paper we regard \mathbb{T} as a torus by imposing periodic boundary
 69 condition. We recall a local stability result: Acerbi, Fusco and Morini [3] proved that
 70 any critical configuration of (1.3)-(1.4) in \mathbb{T} , with positive definite second variation
 71 is a strict local minimizer with respect to small L^1 -perturbations. In [19, 33, 34, 35]
 72 the authors constructed several examples of lamellar, spherical and cylindrical critical
 73 configurations and found related conditions under which they are stable. On the
 74 other hand, it remains open if the global minimizers of (1.3)-(1.4) are one dimensional
 75 lamellar configurations. We study this last question for the model (1.5) below.

76 There are spatial patterns resulting from the competition between thermodynamic
 77 forces operating on different length scales. In the derivation of the energy-driven
 78 model, the Green's function G associated with $-\Delta + \kappa^2$ represents a screened Coulomb
 79 kernel, while it is called unscreened Coulomb kernel when $\kappa = 0$. The constant κ has
 80 the physical meaning of the inverse of the Debye screening length [28, 29].

81 In this paper we are interested in the following energy-driven model:

$$82 \quad (1.5) \quad \int_{\mathbb{T}} \left(\frac{\epsilon}{2} |\nabla u|^2 + F(u) \right) dx + \frac{\sigma}{2} \int_{\mathbb{T}} \int_{\mathbb{T}} u(x) (-\Delta + 1)^{-1} u(\xi) d\xi dx ;$$

83 With a screened Coulomb kernel, we seek the critical points of (1.5) with no volume (or
 84 mass) constraint. Instead, the appearance of a volume term gets into the competition
 85 process if the potential wells are slightly imbalanced; for instance

$$86 \quad (1.6) \quad F(u) = \frac{u^2(u-1)^2}{4\epsilon} + \frac{\alpha}{\sqrt{2}} \left(\frac{u^3}{3} - \frac{u^2}{2} \right)$$

87 for small ϵ . Through the Γ -convergence the *sharp interface model* associated with
 88 (1.5) is

$$89 \quad (1.7) \quad J(E) = \mathcal{P}_{\mathbb{T}}(E) - \alpha|E| + \frac{\sigma}{2} \int_E \mathcal{N}_E dx .$$

90 Here $|E|$ is the Lebesgue measure of E and \mathcal{N} is an operator that assigns a measurable
 91 subset E of \mathbb{T} the solution of the following modified Helmholtz equation:

$$92 \quad (1.8) \quad -\Delta \mathcal{N}_E + \mathcal{N}_E = \chi_E \text{ in } \mathbb{T}, \quad \mathcal{N}_E \text{ is periodic in } \mathbb{T} ;$$

93 as known to be the unique \mathbb{T} -periodic minimizer of

$$94 \quad (1.9) \quad v \mapsto \int_{\mathbb{T}} \left(\frac{|Dv|^2}{2} + \frac{v^2}{2} - v\chi_E \right) dx .$$

95 The admissible set of J is

$$96 \quad (1.10) \quad \mathcal{A} = \{E \subset \mathbb{T} : E \text{ is Lebesgue measurable}\} .$$

97 The (possibly infinite) perimeter of E in \mathbb{T} is denoted by $\mathcal{P}_{\mathbb{T}}(E)$. If E is of class C^1 ,
 98 $\mathcal{P}_{\mathbb{T}}(E)$ is the surface measure of the boundary of $\partial E \cap \mathbb{T}$. A classical stationary set
 99 of J has a C^2 interface that satisfies the Euler-Lagrange equation

$$100 \quad (1.11) \quad \mathcal{K}(\partial E \cap \mathbb{T}) - \alpha + \sigma \mathcal{N}_E = 0 \text{ on } \partial E \cap \mathbb{T},$$

101 where as known in [12, 13], \mathcal{K} denotes **the sum of principal curvatures**, which equals
 102 $(N - 1)$ times the mean curvature.

103 In recent years (1.5) has been extensively studied as a paradigmatic activator-
 104 inhibitor system, like the FitzHugh-Nagumo equations, for patterns generated from
 105 homogeneous media destabilized by a spatial modulation. Not only serving as a
 106 prototype model for patterns like stripes and spots, variants of (1.5) preserve rich
 107 structures in systems exhibiting dissipative soliton phenomena [8, 9, 10, 14, 15, 25, 41].
 108 Following similar asymptotic analysis on the Ohta-Kawasaki model [16, 17, 32, 33] as

109 a certain physical parameter going to zero, a Γ -convergence treatment leads to the
 110 geometric variational functional (1.7) as a sharp interface model, which provides an
 111 effective setting for studying localized patterns and waves. The extra volumetric term
 112 $\alpha|E|$ is a result of the imbalance in energy wells due to the nonlinearity F . Depending
 113 on the system parameters, the competitions among the perimeter, the volume and the
 114 nonlocal interactions in this functional give rise to localized structures which may stay
 115 at rest or propagate with a dynamically stabilized velocity. See [12, 13] for studying
 116 pattern formation and [11] in dealing with traveling waves.

Our goal in this paper is to investigate the stability of lamellar configurations
 of (1.7). The structure of global and local minimizers of (1.7) has recently been
 investigated [2]. By minimality one sees that necessarily $\mathcal{N}_E \geq 0$, and since $\mathcal{N}_{\mathbb{T} \setminus E} =$
 $1 - \mathcal{N}_E$ also that $\mathcal{N}_E \leq 1$. From (1.8), by the divergence theorem one gets

$$\int_{\mathbb{T}} \mathcal{N}_E dx = |E| .$$

117 Writing E' for the complement $\mathbb{T} \setminus E$ of E , we thus have
 (1.12)

$$118 \int_E \mathcal{N}_E dx = \int_{\mathbb{T}} \mathcal{N}_E dx - \int_{E'} \mathcal{N}_E dx = |E| - \int_{E'} (1 - \mathcal{N}_{E'}) dx = |E| - |E'| + \int_{E'} \mathcal{N}_{E'} dx .$$

119 This implies

$$120 (1.13) \quad J(E) = J(E') + \left(\frac{\sigma}{2} - \alpha\right)(|E| - |E'|) .$$

121 The nonlocal interaction term of (1.7) containing a positive parameter σ . Its effect
 122 favors an identically zero solution as a minimizer. On the other hand the positive
 123 parameter α measures the driving force towards a non-zero state.

124 Partially motivated by (1.13), we introduce a parameter

$$125 (1.14) \quad c = c(\alpha, \sigma) := 1 - \frac{2\alpha}{\sigma} .$$

126 Clearly the empty state $E = \emptyset$ and the full state $E = \mathbb{T}$ satisfy

$$127 (1.15) \quad J(\emptyset) = 0 , \quad J(\mathbb{T}) = \frac{\sigma}{2} c T^N ;$$

128 the sign of the “fullness parameter” c determines whether the empty torus is more
 129 (when $c > 0$) or less ($c < 0$) energetically favorable than the full torus, and not only
 130 that, as when $c > 0$ global minimizers of J all have measure less than $|\mathbb{T}|/2$, and the
 131 reverse is true if $c < 0$, see [2, Remark 1.3]. It is also true [2, Corollaries 1.6 and
 132 1.7] that the empty (resp. full) state is a global minimizer iff $0 \leq \alpha \leq \alpha_\emptyset$ (resp. iff
 133 $\alpha_{\mathbb{T}} \leq \alpha \leq \sigma$) for some $0 < \alpha_\emptyset < \alpha_{\mathbb{T}} < \sigma$. As a remark, of the three terms composing
 134 $J(E)$, only the volumetric term is nonpositive. Since both the empty state and the
 135 full state have no phase boundary, their competitive advantages depend only on the
 136 volumetric and the nonlocal terms, which is determined by the ratio α/σ .

137 As been demonstrated in [2], there can be multiple laminar configurations in a
 138 fixed torus with the same physical parameters. Among these configurations there is
 139 a lamella with the lowest energy. For this new concept of minimal lamella we showed
 140 that with suitable parameters α, σ in a large torus, a lamella has a lower energy than

141 both the empty set and the full torus (thus in particular there can be global minimizers
 142 other than both trivial states). Under this circumstance a periodic extension of the
 143 minimal lamella is a global minimizer in one space dimension; we will address the
 144 question if global minimizers in a two dimensional torus have lamella structures. The
 145 main results of [2] together with some relevant properties will be given in Section 2
 146 (see Remark 2.3); we will need them in such an investigation.

147 The central issue of this paper is the stability of lamellar configurations of (1.11),
 148 that is, sets E which beside being \mathbb{T} -periodic are also invariant by translations or-
 149 thogonal to a certain direction \mathbf{v} . Without loss of generality, we take \mathbf{v} as the first
 150 axis, and use $(x, x') \in [0, T] \times [0, T]^{N-1}$ as coordinates. Next we fix the notation for a
 151 single lamella and a k -lamella. Let $0 < x_0 < T$ and let $E = L_{x_0} = [0, x_0] \times [0, T]^{N-1}$
 152 be a single lamella with a thickness x_0 in the torus \mathbb{T} . A k -lamellar configuration
 153 \mathbb{L} is composed of k “vertical” lamellae (where $\chi_{\mathbb{L}} = 1$) separated by wedges (where
 154 $\chi_{\mathbb{L}} = 0$) with the first lamella beginning at the left side of \mathbb{T} , i.e. at $x = 0$, and the
 155 total widths of all k lamellae being x_0 . It has been shown [2] that, in every station-
 156 ary k -lamellar configuration, all lamellae have the same width x_0/k and are equally
 157 spaced; so this configuration is not only T -periodic, but has a smaller period T/k .
 158 Moreover for fixed T and k , x_0/k is determined by the ratios α/σ and T/k only (see
 159 (2.5) for the precise formula). This observation helps our investigation later on. In
 160 what follows, k will be referred to as the (lamellar) tightness.

161 In general it is (relatively) easy to check that a candidate E satisfies the Euler-
 162 Lagrange equation of J , i.e., $J'(E) = 0$; much, much harder is the task of proving
 163 that the candidate is a local minimizer of J . As an intermediate step to eliminate
 164 translation modes, one may prove that in some suitable sense $J''(E) > 0$, a property
 165 which we call stability (see Definition 3.4 for the precise meaning of stability), and
 166 then proceed to prove that all stable critical points are local minimizers indeed.

167 It is not difficult to show that for every given α, σ, T , the global minimizer of (1.7)
 168 always exists. Below is a general result for the stability of lamellar configurations on
 169 a N -dimensional torus.

170 THEOREM 1.1. *Let \mathbb{L} be a lamellar configuration of (1.11).*

171 (i) *Stable lamellae are isolated local minimizers of (1.7).*

172 (ii) *Given σ and α , \mathbb{L} is a stable solution on a N -dimensional torus $[-T/2, T/2]^N$
 173 if T is sufficiently small.*

174 To dig into more delicate stability results, we focus on the case $\mathbb{T} = [-T/2, T/2] \times$
 175 $[-T/2, T/2]$ in the investigation of the dependence of J on the parameter c defined by
 176 (1.14). Although we are confident that some of the results hold in the general cases,
 177 the delicate techniques employed here do not seem to extend for free to more than
 178 two dimensions. The next theorem indicates how stability of lamellar configurations
 179 is affected by the physical parameters α and σ , and the disturbance Fourier modes
 180 $m \in \mathbb{N} \cup \{0\}$ on each individual lamellar interface; in particular we work out good
 181 comparison associated with the value c , the tightness k and the disturbance mode m .
 182 It turns out that the mode $m = 0$ is always stable.

183 THEOREM 1.2. *Let $\mathbb{T} = [-T/2, T/2] \times [-T/2, T/2]$ and $\mathbb{L}_k(c)$ denote a k -lamellar
 184 stationary point of (1.11) with c being the measure of physical parameter.*

185 (i) *$\mathbb{L}_k(c)$ is stable if and only if the disturbance mode $m = 1$ is stable. In addition,
 186 if $\mathbb{L}_k(c_1)$ is stable and $|c_2| \geq |c_1|$, then $\mathbb{L}_k(c_2)$ is stable.*

187 (ii) *If $\mathbb{L}_k(0)$ is stable then $\mathbb{L}_j(c)$ is stable for all $j \geq k$ and $|c| < 1$.*

(iii) A necessary and sufficient condition for all stationary k -lamellae to be stable for every value of c and k , is that

$$\sigma < 8\pi^2 \left[T^3 \left(\frac{\tanh(T/4)}{T} - \frac{\tanh(\sqrt{T^2 + 4\pi^2}/4)}{\sqrt{T^2 + 4\pi^2}} \right) \right]^{-1}.$$

Without loss of generality, we only carry out the proof for the case $c \geq 0$; in this case $x_0 \in (0, T/2k]$. In the whole space \mathbb{R}^N (an infinite torus), stationary 1-lamella will occupy the whole space as $c \rightarrow 0^+$, see [13, equations (1.18) and (1.19)]. This lamellar solution disappears for $c \leq 0$. Thus bifurcation from infinity occurs at $c = 0$ in \mathbb{R}^N . It is interesting to note that for radially symmetric solutions in infinite domains, it has also been demonstrated that the line $\sigma = 2\alpha$ in the (α, σ) plane, equivalently $c = 0$, is a boundary where bifurcation occurs; see [13, Figure 2], [12, Figure 2]. In this case an infinitely large bubble disappears once c turns negative. A further study in this regard is underway.

From (2.5) it is observed that a stationary k -lamella in a torus of size T is a stationary 1-lamella in a torus of size T/k ; by Proposition 2.2 the corresponding v_0 and d_0 stay the same. They therefore possess the same stability properties with respect to (T/k) -periodic perturbations. Since T -periodic disturbance is allowed in the T -torus but not in the (T/k) -torus, the extra modes may induce instability in the larger torus. In other words, in a torus a 1-lamella is always unstable whenever k -lamellae are unstable.

As a further exploration, we introduced a function

$$(1.16) \quad \Gamma(c) = |c| - 1 - |c| \log |c|, \quad |c| \leq 1,$$

extended by continuity at $c = 0$ by $\Gamma(0) = -1$. This function is a term derived [2] from an asymptotic formula of the energy for extremely large tori; i.e. as $T \rightarrow \infty$. More detailed properties of $\Gamma(c)$ will be given in Section 2, in particular see Remark 2.3. Not only $\Gamma(c)$ provides a guide to select out a lamellar configuration with least energy (density), it points out a threshold of stability exchange as follows.

THEOREM 1.3. *The following stability results hold:*

- (i) *When $4 + \sigma\Gamma(c) > 0$, stationary lamellae are stable for all T .*
- (ii) *If $4 + \sigma\Gamma(c) < 0$, stationary lamellae are unstable when T is sufficiently large. Moreover the global minimizer of (1.7) has a genuine (non-lamellar) 2D structure if $0 < c < 1$.*
- (iii) *In particular if $c = 0$ and $\sigma > 4$, there exists a $T_k = T_k(\sigma)$ such that the k -lamella is stable if $T < T_k$ and unstable if $T > T_k$.*

Even though x_0 (i.e. the lamellar configuration) is completely determined by c , we note that σ can change its stability while keeping a fixed c . As a consequence of statement (ii), if periodicity were to hold in 2D, the mesoscopic structure has to be a genuine 2D finite size minimal energy configuration when $4 + \sigma\Gamma(c) < 0$. Though not the subject in this paper, knowing its structure will be extremely interesting. For (iii), the same result may still be valid for any c , but the calculation complexity prevents us from drawing a concrete conclusion. Numerical validation [38, 39, 40] has been successfully worked out in certain problems of pattern formation (e.g. the original Ohta-Kawasaki model). It should be equally interesting to have analogous development for studying the geometric variational functional.

228 Section 2 begins with a list of known facts for minimal lamellae. Section 3 works
 229 on first and second variation, as the preliminary for studying the stability of lamellar
 230 configurations. Theorem 1.1(i) follows from Theorem 3.5, which ensures that stable
 231 critical points of (1.7) are isolated local minimizers. That the situation is not trivial is
 232 made evident by the instability result in Proposition 5.5 in some parameter regimes.
 233 The proof of Theorem 3.5 is lengthy, and since it is similar to that of [3, Theorem
 234 1.1], we highlight the relevant differences only (see Appendix A). Theorem 1.1(ii) is
 235 an immediate cosequence of Poincaré inequality as to be seen in Theorem 5.1.

236 For a critical point E of (1.7), its local stability can be investigated through the
 237 second variation calculated by imposing various flows generated by (smooth) velocity
 238 vector fields X , detailed at the beginning of Section 3. The idea is that the critical
 239 set is stable if the functional increases under the perturbation through every such
 240 vector field over a short time interval. If E is a critical lamellar configuration \mathbb{L} , only
 241 the normal component $\eta := X \cdot \nu$ matters, where ν is the unit outward normal to
 242 \mathbb{L} . We decompose $\eta = \mu + \zeta$ where on each connected component of \mathbb{L} the term μ
 243 is a constant and the integral of ζ is zero; μ and ζ are called the mean part and
 244 the zero-average part, respectively. One motivation for this decomposition is that
 245 the rigid body translation mode resides only in the mean part; moreover both parts
 246 are independent of one another in stability analysis as will be seen in expressions
 247 (3.12) and (3.13), which make up the second variation formula. As a by-product,
 248 our analysis on the mean part indicates that all stationary lamellae are stable with
 249 respect to 1D perturbation, see Corollary 4.5.

250 The proof of stability naturally divides into two steps: the mean value part in
 251 Section 4 for checking the stability against 1D periodic perturbations, and then the
 252 zero-average part in Section 5 to draw complete conclusion. We recall that this ap-
 253 proach was also used in a recent paper of Morini and Sternberg [27] who dealt with
 254 the stability of lamellar configurations of the Ohta-Kawasaki model (or a nonlocal
 255 isoperimetric problem) in a thin domain $[0, \epsilon] \times [0, 1]$. There the long-range inter-
 256 action is governed by the Green function associated with the Laplace operator, so
 257 a k -lamellar can be constructed by multiple repeated reflection of a single lamellar
 258 in small interval. In our case the length rescaling argument does not work when
 259 the Helmholtz operator replaces the Laplace operator, even the existence of minimal
 260 lamella is not a simple process in the calculation of energy density. When ϵ is small
 261 enough, the 1D stable periodic configuration remains stable on $[0, \epsilon] \times [0, 1]$ because
 262 the stabilizing effect resulted from the Poincaré inequality on the zero-average part
 263 dominates anything else.

264 Our stability analysis quantitatively calculates for the first time the energy contri-
 265 bution of the nonlocal term, without which an instability result cannot be formulated.
 266 In addition to making extensive use of non-trivial properties of convex functions, we
 267 rely on the explicit computation of the eigenvalues of symmetric block circulant Her-
 268 mitian matrices in the investigation of the mean value part. Examining the similarity
 269 of the structures of the stability matrices, we obtain a simple criterion (5.12) for
 270 stability of zero-average part. The bulk of the paper is devoted to proving that in
 271 dimension $N = 2$ the worst case for stability is when $c = 0$, depicted in Theorem 5.13,
 272 and that stability is most delicate for 1-lamellae, Theorem 5.15. These give rise to
 273 the main consequence, Corollary 5.20, that precisely describes the stability range as
 274 been summarized in Theorem 1.2.

275 Stability of lamellar solutions in a Ohta-Kawasaki model has been studied in [35].

276 Computing the spectrum of the linearized governing equation, the authors obtained
 277 good estimates for the eigenvalues with the help of a Γ -limit as $\epsilon \rightarrow 0$. This calcula-
 278 tion determines the sign of all eigenvalues if the number of interfaces is large. As a
 279 conclusion [35, p.26], 1D local minimizers with higher lamellar tightness are likely to
 280 be stable while those with lower tightness are likely to be unstable in three dimen-
 281 sions. Similar phenomena happen in our study as laid out in Theorem 1.2(ii). On the
 282 other hand, our results indicate a sharp threshold governed by the sign of $4 + \sigma\Gamma(c)$.
 283 The calculation of spectrum in both studies employed the technique of separation of
 284 variables.

285 **2. Known facts on minimal lamellae.** In this section we first prove the exist-
 286 tence of global minimizer of (1.7) and then state certain properties of minimal lamellae
 287 for the convenience of readers.

288 **THEOREM 2.1.** *There always exists a global minimizer of (1.7) for all positive*
 289 *α, σ, T .*

Proof. First we recall that for a \mathbb{T} -periodic set E

$$\mathcal{P}_{\mathbb{T}}(E) = \|D\chi_E\|_{\text{per}} =: \sup\left\{\int_{\mathbb{T}} \chi_E \operatorname{div} \varphi \, dz : \varphi \in C^1(\mathbb{T}), \varphi \text{ is } \mathbb{T}\text{-periodic}, |\varphi| \leq 1\right\}$$

which represents the variation measure of χ_E in a periodic setting. As $J(E) \geq -\alpha T^N$
 for any measurable $E \subset \mathbb{T}$, there exists a minimizing sequence $\{E_j\}_{j=1}^{\infty}$ such that
 $1 + \inf J \geq J(E_j) \rightarrow \inf J$, which leads to a uniform upper bound

$$\mathcal{P}_{\mathbb{T}}(E_j) \leq 1 + \inf J + \alpha T^N.$$

290 By compactness there exists a \mathbb{T} -periodic $E_0 \subset \mathbb{T}$ and a subsequence, still designated
 291 by $\{E_j\}$, such that $\chi_{E_j} \rightarrow \chi_{E_0}$ in $L^1(\mathbb{T})$ and pointwise a.e.; moreover $\liminf \mathcal{P}_{\mathbb{T}}(E_j) \geq$
 292 $\mathcal{P}_{\mathbb{T}}(E_0)$. As the L^∞ norm of characteristic functions are 1, it follows that $\chi_{E_j} \rightarrow$
 293 χ_{E_0} in $L^2(\mathbb{T})$; this immediately gives $\mathcal{N}_{E_j} \rightarrow \mathcal{N}_{E_0}$ in $H_{\text{per}}^1(\mathbb{T})$ so that $\int_{E_j} \mathcal{N}_{E_j} dx \rightarrow$
 294 $\int_{E_0} \mathcal{N}_{E_0} dx$. Hence E_0 is a global minimizer. \square

295 For a while we denote by L the projection of a lamella \mathbb{L} on the x -axis; we also
 296 denote the total thickness of the k -lamella by $x_0 := |L|$. The function $\mathcal{N}_{\mathbb{L}}$ appearing
 297 in the nonlocal term of (1.7) is the unique \mathbb{T} -periodic minimizer of the strictly convex
 298 energy (1.9). But replacing $\mathcal{N}_{\mathbb{L}}$ with its average in the x' directions, by strict convexity
 299 we deduce that $\mathcal{N}_{\mathbb{L}}$ depends only on x . Since not only \mathbb{L} , but also $\mathcal{N}_{\mathbb{L}}$ has a one-
 300 dimensional structure, it will be sometimes useful to drop all but the first variable
 301 and work in one dimension; using the simpler notation $u(x)$ in place of $\mathcal{N}_{\mathbb{L}}(x, x')$, it is
 302 useful to introduce the one-dimensional analogues of (1.8) and (1.9), that is, equation

$$303 \quad (2.1) \quad -v'' + v = \chi_L$$

304 (with periodic boundary conditions in $[0, T]$) and energy

$$305 \quad (2.2) \quad \frac{1}{2} \int_0^T (|v'(x)|^2 + |v(x)|^2) dx - \int_L v(x) dx, \quad v \text{ is } T\text{-periodic}.$$

306 We collect some facts which will be useful in our stability analysis, all references being
 307 to [2].

308 PROPOSITION 2.2. Suppose that the k -lamella \mathbb{L} is a stationary point of the energy
 309 (1.7) and let v be the 1-dimensional function introduced above. Set $v_0 = v(0)$ and
 310 $d_0 = v'(0)$. Then (Proposition 2.6) all lamellae have the same size and are equally
 311 spaced; (Lemma 2.4) the function v is symmetric inside each lamella and inside each
 312 wedge, and in particular v takes the value v_0 at all sides of the lamellae, whereas v'
 313 takes value $+d_0$ (resp. $-d_0$) at each left (resp. right) side of the lamellae. If x_0 is the
 314 total width of the lamellae then (equations 2.6 and 2.7)

$$315 \quad (2.3) \quad v_0 = \frac{1}{\sinh \frac{T}{2k}} \cosh \frac{T-x_0}{2k} \sinh \frac{x_0}{2k} = \frac{1}{2 \sinh \frac{T}{2k}} \left(\sinh \frac{T}{2k} - \sinh \frac{T-2x_0}{2k} \right),$$

316

$$317 \quad (2.4) \quad d_0 = \frac{1}{\sinh \frac{T}{2k}} \sinh \frac{T-x_0}{2k} \sinh \frac{x_0}{2k}.$$

318 Moreover (Theorem 2.9) necessarily $\alpha \leq \sigma$ (which is equivalent to $|c| \leq 1$), the total
 319 thickness x_0 satisfies

$$320 \quad (2.5) \quad \frac{x_0}{k} = \frac{T}{2k} - \operatorname{arcsinh} \left(c \sinh \frac{T}{2k} \right)$$

321 and the corresponding energy is

$$322 \quad (2.6) \quad J(\mathbb{L}) = kT^{N-1} \left\{ 2 + c \frac{\sigma}{2} \left[\frac{T}{2k} - \operatorname{arcsinh} \left(c \sinh \frac{T}{2k} \right) \right] \right. \\ \left. - \frac{\sigma}{2 \sinh \frac{T}{2k}} \left(\cosh \frac{T}{2k} - \sqrt{1 + c^2 \sinh^2 \frac{T}{2k}} \right) \right\}.$$

323 Equation (2.5) concretely justifies the name given to the fullness parameter c : for
 324 stationary k -lamellae, when $c > 0$ lamellae are thinner than wedges, and the opposite
 325 is true when $c < 0$.

We now specialize to minimal lamellae, i.e., k -lamellae in a torus which are optimal
 among all multi-lamellar configurations (the focus is on the best choice of k). Given
 (2.6) it is convenient to set

$$\mathcal{A}(c, t) = \operatorname{arcsinh}(c \sinh(t)), \quad \mathcal{B}(c, t) = \frac{\cosh t - \sqrt{1 + c^2 \sinh^2 t}}{\sinh t},$$

$$\mathcal{L}(c, t) = c(t - \mathcal{A}(c, t)) - \mathcal{B}(c, t)$$

and

$$\mathcal{E}(\sigma, c, t) = \frac{1}{t} \left(2 + \frac{\sigma}{2} \mathcal{L}(c, t) \right),$$

so that (2.6) reads

$$J_{\mathbb{T}}(\mathbb{L}) = \frac{T^N}{2} \mathcal{E} \left(\sigma, c, \frac{T}{2k} \right).$$

326 Many properties of these functions are investigated in [2, Section 3], but here we will
 327 only need to know that

$$328 \quad (2.7) \quad t - \mathcal{A}(c, t) = \begin{cases} -\log c + \omega_t & \text{if } c > 0, \\ 2t + \log |c| + \omega_t & \text{if } c < 0, \end{cases}$$

329 where ω_t designates a function that vanishes as $t \rightarrow \infty$.

A relevant property of the function $t\mathcal{E}(\sigma, c, t) = 2 + (\sigma/2)\mathcal{L}$, see [2, Proposition 3.4], is that if $c > 0$ its limit as $t \rightarrow +\infty$ is

$$2 + \frac{\sigma}{2}(c - 1 - c \log c);$$

whereas if $c < 0$ it has as an asymptote as $t \rightarrow +\infty$ the function

$$\sigma ct + \left[2 + \frac{\sigma}{2}(|c| - 1 - |c| \log |c|) \right].$$

330

331 **REMARK 2.3.** *The threshold function $\Gamma(c)$ plays a crucial role to distinguish the*
 332 *best lamellar configuration. In particular from [2, Theorem 3.5, Remark 3.7] when*
 333 *$2 + \sigma\Gamma(c)/2 \geq 0$, a finer lamella partition of the torus results in a higher energy*
 334 *configuration; thus 1-lamella is the best, but this configuration is always beaten by*
 335 *either trivial state); but if $2 + \sigma\Gamma(c)/2 < 0$ then there is a unique point $t_0 = t_0(c, \sigma) >$
 336 0 such that $\mathcal{E}(\sigma, c, t)$ is strictly decreasing for $0 < t \leq t_0$ and strictly increasing*
 337 *afterwards, thus the best lamellar configuration divides the torus in approximately*
 338 *$T/2t_0$ bands, i.e. when $T/2t_0$ is not an integer, then the optimal number of bands is*
 339 *either the integer just above or just below $T/2t_0$.*

340 **3. First and second variation, and preliminaries to stability.** For the rest
 341 of this paper, all functions defined on \mathbb{T} are understood to be \mathbb{T} -periodic, and those
 342 defined on a face S of a lamella are S -periodic.

We first recall the definition of the variations of our functional J at a set $E \subset \mathbb{T}$ of class \mathcal{C}^2 . Let $X : \mathbb{T} \rightarrow \mathbb{R}^N$ be a \mathcal{C}^2 vector field and consider the associated flow $\Psi : \mathbb{T} \times (-1, 1) \rightarrow \mathbb{T}$ defined by $\Psi_t = X(\Psi)$, $\Psi(x, 0) = x$ and set

$$E_t := \Psi(E, t).$$

343 The first and second variations of J at E with respect to the flow associated with the
 344 field X are defined as the first and second derivatives at $t = 0$ of $J(E_t)$. Computing the
 345 first and second variation of the energy (1.7) is a lengthy exercise, already carried out
 346 in similar settings, see for example [20, Theorem 2.6], [7, Theorem 3.6], [3, Theorem
 347 3.1]. We highlight only the major differences as follows:

- 348 1. these papers use characteristic functions, denoted by u or U , with values in
 349 $\{-1, 1\}$ instead of our $\{0, 1\}$ -valued χ . Some factors of 2's will disappear,
 350 in particular each time when a boundary integral appears in the derivation;
 351 also, with respect to [20] which contains the bulk of the computation one may
 352 dismiss the integrals on the complementary set (where $U = -1$), which cause
 353 all the 2's;
- 354 2. in place of a volumetric constraint on E , we have an extra term which is
 355 proportional to the volume of E ;
- 356 3. our potential function \mathcal{N}_E (as opposed to the notation v or V in the other pa-
 357 pers) is governed by the (modified) Helmholtz operator instead of the Lapla-
 358 cian.

The only likely dangerous point seems to be the last remark; but if G_H and G_L denote the Green's functions for the modified Helmholtz and the Laplacian operators, respectively, in both instances one has

$$\mathcal{N}_E(x) = \int G_H(x, y)\chi_E(y) dy, \quad v(x) = \int G_L(x, y)u(y) dy$$

and the nonlocal terms in their governing functionals are given by

$$\int \mathcal{N}_E(x) \chi_E(x) dx, \quad \int v(x) u(x) dx,$$

359 respectively. Then throughout the derivation all calculations are the same, since the
 360 derivation in [20] uses this form as a starting point. Thus the variations coming from
 361 the nonlocal term can be directly taken from [20], not forgetting to drop the extra 2's
 362 and stopping at formula (2.67) since after this the authors deal with the necessary
 363 corrections due to the volume constraint.

364 The second variation of volume may be found in [20, formula (2.30)], and the
 365 second variation of the perimeter is computed at every regular set E and not only
 366 at critical points in [3, Theorem 3.1]. Neither in the derivation of the nonlocal term
 367 nor in that of the perimeter term the infinitesimal volume preservation condition
 368 $\int_{\partial E} (X \cdot \nu) d\mathcal{H}^{N-1} = 0$ is used, thus in the end one has the following result.

PROPOSITION 3.1. *The first variation of (1.7) with respect to the flow associated with any (regular) vector field $X : \mathbb{T} \rightarrow \mathbb{R}^N$ defined near the boundary of a regular set E , of class C^2 in a torus \mathbb{T} , is*

$$dJ(E)X = \int_{\partial E} \left(\mathcal{K}(\partial E) - \alpha + \sigma \mathcal{N}_E \right) (X \cdot \nu) d\mathcal{H}^{N-1}$$

369 and the second variation is

$$\begin{aligned} 370 \quad d^2 J(E)[X] &= \int_{\partial E} \left(|\nabla_{\tau}(X \cdot \nu)|^2 - \|B_{\partial E}\|^2 (X \cdot \nu)^2 \right) d\mathcal{H}^{N-1} \\ 371 \quad &+ \sigma \int_{\partial E} \int_{\partial E} G(x, y) (X \cdot \nu)(x) (X \cdot \nu)(y) d\mathcal{H}_x^{N-1} d\mathcal{H}_y^{N-1} \\ 372 \quad &+ \sigma \int_{\partial E} (\nabla \mathcal{N}_E \cdot \nu) (X \cdot \nu)^2 d\mathcal{H}^{N-1} \\ 373 \quad &+ \int_{\partial E} \left(\mathcal{K}(\partial E) - \alpha + \sigma \mathcal{N}_E \right) (\operatorname{div} X) (X \cdot \nu) d\mathcal{H}^{N-1} \\ 374 \quad &- \int_{\partial E} \left(\mathcal{K}(\partial E) + \sigma \mathcal{N}_E \right) \operatorname{div}_{\tau}(X_{\tau}(X \cdot \nu)) d\mathcal{H}^{N-1}. \end{aligned}$$

375 Here $\|B_{\partial E}\|^2$ is the sum of the squares of the principal curvatures of ∂E ; G is the
 376 Green's function for the Helmholtz operator in \mathbb{T} with periodic boundary conditions; ν
 377 is the unit outward normal on ∂E ; $\mathcal{K}(\partial E)$ is the sum of principal curvatures of ∂E ;
 378 ∇_{τ} is the gradient on ∂E ; and X_{τ} is the tangential component of X .

DEFINITION 3.2. *A regular subset E of \mathbb{T} is a stationary (or critical) point for (1.7) if*

$$\mathcal{K}(\partial E) - \alpha + \sigma \mathcal{N}_E = 0 \quad \text{on } \partial E.$$

379 REMARK 3.3. *Since \mathcal{N}_E is of class $W^{2,p}$ for any $p > 1$, standard regularity theory*
 380 *and Schauder estimates imply that any regular critical set is of class $C^{3,\alpha}(\mathbb{T})$ for any*
 381 *$0 < \alpha < 1$.*

We remark that we may add to the last integral in Proposition 3.1 a harmless

$$\int_{\partial E} -\alpha \operatorname{div}_{\tau}(X_{\tau}(X \cdot \nu)) d\mathcal{H}^{N-1}$$

(which vanishes by the tangential divergence theorem) so that the last two integrals may be grouped into

$$\int_{\partial E} \left(\mathcal{K}(\partial E) - \alpha + \sigma \mathcal{N}_E \right) \cdot (\dots) d\mathcal{H}^{N-1}$$

382 which vanishes if E was stationary. As all other terms for $d^2J(E)[X]$ only depend on
383 the normal component of X , it is convenient to introduce a function defined on all
384 $\eta \in H^1(\partial E)$ as

$$385 \quad J''(E)[\eta] = \int_{\partial E} \left(|\nabla_{\tau} \eta|^2 - \|B_{\partial E}\|^2 \eta^2 \right) d\mathcal{H}^{N-1}$$

$$386 \quad + \sigma \int_{\partial E} \int_{\partial E} G(x, y) \eta(x) \eta(y) d\mathcal{H}_x^{N-1} d\mathcal{H}_y^{N-1} + \sigma \int_{\partial E} (\nabla \mathcal{N}_E \cdot \nu) \eta^2 d\mathcal{H}^{N-1} .$$

Since $J(E) = J(E + \tau)$ for any translation τ , no set (beside the empty and full states) may be a strict minimum point, so following [3, formula (1.3)] we consider as equivalent any two sets one of which is a translation of the other and define a distance between sets modulo translations as

$$\delta(E, F) := \min_{\tau} |E \Delta (F + \tau)| .$$

387 Invariance by translation implies that the second derivative of $J(E + t\tau)$ always vanishes.
388 In particular on a critical point E the second variation is zero for every constant
389 vector field $X = e_i$ along the coordinate axes resulting in $\eta = X \cdot \nu = \nu_E^i$ for
390 $i = 1, 2, \dots, N$ (the i -th component of the normal ν). There is thus a linear subspace
391 of $H^1(\partial E)$, spanned by the components of the normal, on which $J''(E)$ vanishes. We
392 remark that this subspace can have a dimension less than N , as in the case for lamellar
393 sets. Using $\mathcal{L}\{\dots\}$ to denote the vector space spanned by the functions inside the
394 brackets and $W_{\text{per}}^{1,2}(\partial E)$ for periodic $W^{1,2}(\partial E)$ functions, we set

$$395 \quad \mathcal{T}(\partial E) = \mathcal{L}\{\nu_E^1, \dots, \nu_E^N\}$$

$$396 \quad \mathcal{T}^{\perp}(\partial E) = \{\eta \in W_{\text{per}}^{1,2}(\partial E) : \int_{\partial E} \eta \nu_E^i d\mathcal{H}^{N-1} = 0, i = 1, \dots, N\} .$$

397 DEFINITION 3.4. *A regular critical point E of J is stable if*

$$398 \quad (3.1) \quad J''(E)[\eta] > 0 \quad \text{for all } \eta \in \mathcal{T}^{\perp}(\partial E) \setminus \{0\} .$$

399 The notion of stability of a stationary point E is crucial in the applications, since
400 as we will see it implies that E is a strict local minimizer of J , isolated in the δ
401 distance sense (which measures the norm in L^1 modulo translations). In the spirit of
402 [3, Theorem 1.1] we have

THEOREM 3.5. *Let $E \subset \mathbb{T}$ be a regular critical set of J such that*

$$J''(E)[\eta] > 0 \quad \text{for all } \eta \in \mathcal{T}^{\perp}(\partial E) \setminus \{0\} .$$

Then there exist $\varepsilon, C > 0$ such that

$$J(F) \geq J(E) + C\delta^2(E, F)$$

403 *for all $F \subset \mathbb{T}$ with $\delta(E, F) < \varepsilon$.*

404 The proof closely follows that of [3, Theorem 1.1], which takes up 25 pages, so we only
 405 highlight the relevant differences in the Appendix; the crucial estimate of [3, Lemma
 406 2.6] is replaced by an easier readable version for the Helmholtz operator:

LEMMA 3.6. *If $E, F \subset \mathbb{T}$ are measurable then*

$$\left| \int_{\mathbb{T}} (|DN_E|^2 + \mathcal{N}_E^2) dx - \int_{\mathbb{T}} (|DN_F|^2 + \mathcal{N}_F^2) dx \right| \leq 2|E\Delta F|.$$

407 *Proof.* We write

$$\begin{aligned} 408 & \int_{\mathbb{T}} (|DN_E|^2 + \mathcal{N}_E^2) dx - \int_{\mathbb{T}} (|DN_F|^2 + \mathcal{N}_F^2) dx \\ 409 &= \int_{\mathbb{T}} [(DN_E + DN_F)(DN_E - DN_F) + (\mathcal{N}_E + \mathcal{N}_F)(\mathcal{N}_E - \mathcal{N}_F)] dx \\ 410 &= \int_{\mathbb{T}} (\mathcal{N}_E + \mathcal{N}_F)[(-\Delta\mathcal{N}_E + \mathcal{N}_E) - (-\Delta\mathcal{N}_F + \mathcal{N}_F)] dx \\ 411 &= \int_{\mathbb{T}} (\mathcal{N}_E + \mathcal{N}_F)(\chi_E - \chi_F) dx \end{aligned}$$

412 by (1.8), and the result follows since $\|\chi_E - \chi_F\|_{L^1} = |E\Delta F|$ and $0 \leq \mathcal{N}_{E,F} \leq 1$. \square

413 Stationary points for the area functional have constant mean curvature; they are more
 414 or less easily classified. A nonlocal perturbation of the area functional has been studied
 415 in the Ohta-Kawasaki model; it gives rise to a series of interesting stationary surfaces
 416 (the boundaries of lamellae and, in the Neumann case, also of cylinders, spheres and
 417 some 3D-structures called gyroids) which have been proven to be stable under certain
 418 assumptions on the parameters. Their shapes are easy to handle, the Laplacian scales
 419 well and is well understood, so the proof of their stability requires some effort but is
 420 quite general. Equation (1.11), which is another nonlocal perturbation, is less neat,
 421 and the only known solution in the periodic setting is given by lamellae [2] (in the
 422 entire space there are bubble solutions, see [12, 13]).

423 We now examine k -lamellar stationary points, in order to establish their stability
 424 in certain parameter regimes. The second variation for stationary lamellae \mathbb{L} takes a
 425 simplified form and reads

$$\begin{aligned} 426 & J''(\mathbb{L})[\eta] = \int_{\partial\mathbb{L}} |\nabla\eta|^2 d\mathcal{H}^{N-1} \\ 427 & (3.2) \quad + \sigma \int_{\partial\mathbb{L}} \int_{\partial\mathbb{L}} G(x, y)\eta(x)\eta(y) d\mathcal{H}_x^{N-1} d\mathcal{H}_y^{N-1} \\ 428 & + \sigma \int_{\partial\mathbb{L}} (\nabla\mathcal{N}_{\mathbb{L}} \cdot \nu)\eta^2 d\mathcal{H}^{N-1}. \end{aligned}$$

429 We recall that by Proposition 2.2 k -lamellae which are stationary points of J are
 430 of equal size and spacing, and that the outward normal derivative of the function
 431 $\mathcal{N}_{\mathbb{L}}$ takes value $-d_0$ on both sides of each lamella, with d_0 given by (2.4), since the
 432 outward normal points backwards on left sides of lamellae.

433 We now fix some notation, some of which we already employed. As a coordinate
 434 system we use $z := (x, x') \in \mathbb{T}$, where $x \in [0, T]$; we consider a stationary k -lamella \mathbb{L}
 435 with all k lamellae having a total width $0 < x_0 < T$, orthogonal to the x -axis, with
 436 the first lamella starting at $x = 0$, and we sequentially label ℓ_i , with $i = 1, \dots, 2k + 1$,

437 the x coordinates of the sides of the lamellae (the last is a duplicate of the first side,
438 but is included for convenience), so that

$$439 \quad (3.3) \quad \ell_1 = 0, \quad \ell_2 = \frac{x_0}{k}, \quad \ell_3 = \frac{T}{k}, \quad \ell_4 = \frac{T}{k} + \frac{x_0}{k}, \quad \ell_5 = 2\frac{T}{k},$$

$$440 \quad \dots \quad \ell_{2k} = (k-1)\frac{T}{k} + \frac{x_0}{k} = T - \frac{T-x_0}{k}, \quad \ell_{2k+1} = T.$$

We also name the corresponding faces, which are $(N-1)$ -dimensional squares orthogonal to the x axis, as L_1, \dots, L_{2k+1} . We easily identify the space $\mathcal{T}^\perp(\partial\mathbb{L})$: since the only non-zero component of the outward normal field to \mathbb{L} is the first one, and it takes value -1 on odd sides (i.e., on L_i with i odd) and $+1$ on even sides, a periodic function $\eta \in W_{\text{per}}^{1,2}(\partial\mathbb{L})$ belongs to \mathcal{T}^\perp if

$$\sum_{j=1}^k \int_{L_{2j}} \eta d\mathcal{H}^{N-1} - \sum_{j=1}^k \int_{L_{2j-1}} \eta d\mathcal{H}^{N-1} = 0.$$

Following a reduction method introduced in [27, section 4], for any $\eta \in W_{\text{per}}^{1,2}(\partial\mathbb{L})$ we call η_i the function which coincides with η on L_i and vanishes on all other L_j and we further split η_i as its mean value μ_i on L_i plus a zero-average term ζ_i :

$$\mu_i = \frac{1}{T^{N-1}} \int_{L_i} \eta_i(z) d\mathcal{H}^{N-1}, \quad \zeta_i(z) = \eta_i(z) - \mu_i,$$

441 so in particular $\int_{L_i} \zeta_i d\mathcal{H}^{N-1} = 0$. We remark that

$$442 \quad (3.4) \quad \eta \in \mathcal{T}^\perp(\partial\mathbb{L}) \iff \sum_{j=1}^k \mu_{2j} - \sum_{j=1}^k \mu_{2j-1} = 0$$

443 which is independent of ζ . For subsequent use we denote $\mu := \sum_{j=1}^{2k} \mu_j$ and $\zeta :=$
444 $\sum_{j=1}^{2k} \zeta_j$ so that $\eta = \mu + \zeta$. We now examine the various components of $J''(\mathbb{L})$; for
445 the first we immediately have

$$446 \quad (3.5) \quad \int_{\partial\mathbb{L}} |\nabla\eta|^2 d\mathcal{H}^{N-1} = \sum_{i=1}^{2k} \int_{L_i} |\nabla\zeta_i|^2 d\mathcal{H}^{N-1}.$$

447 We have for all i

$$448 \quad \int_{L_i} |\eta^2| d\mathcal{H}^{N-1} = \int_{L_i} |\eta_i^2| d\mathcal{H}^{N-1} = T^{N-1} \mu_i^2 + \int_{L_i} |\zeta_i^2| d\mathcal{H}^{N-1} + 2\mu_i \int_{L_i} \zeta_i d\mathcal{H}^{N-1}$$

$$449 \quad = T^{N-1} \mu_i^2 + \int_{L_i} |\zeta_i^2| d\mathcal{H}^{N-1}.$$

450 At the same time $\nabla\mathcal{N}_{\mathbb{L}} \cdot \nu = -d_0$ at all L_i so that the last term in (3.2) becomes

$$451 \quad (3.6) \quad -\sigma d_0 \int_{\partial\mathbb{L}} \eta^2 d\mathcal{H}^{N-1} = -\sigma d_0 T^{N-1} \sum_{i=1}^{2k} \mu_i^2 - \sigma d_0 \sum_{i=1}^{2k} \int_{L_i} |\zeta_i^2| d\mathcal{H}^{N-1}.$$

452 Next comes the Green's function term which, upon setting aside the factor σ , we copy
453 as

$$454 \quad \int_{\partial\mathbb{L}} \int_{\partial\mathbb{L}} G(X, Y) \eta(X) \eta(Y) d\mathcal{H}_X^{N-1} d\mathcal{H}_Y^{N-1}$$

$$455 \quad = \int_{\partial\mathbb{L}} \int_{\partial\mathbb{L}} G(X, Y) ((\mu + \zeta)(X)) ((\mu + \zeta)(Y)) d\mathcal{H}_X^{N-1} d\mathcal{H}_Y^{N-1}$$

456 where $X = (x, x')$ and $Y = (y, y')$. We define two (arrays of) measures on \mathbb{T} and one
457 on $[0, T]$ as

$$458 \quad (3.7) \quad M^i = \mu_i \mathcal{H}^{N-1} \llcorner L_i, \quad Z^i = \zeta_i \mathcal{H}^{N-1} \llcorner L_i, \quad m^i = \mu_i \delta_{\ell_i},$$

459 and we solve Helmholtz equation (twice in \mathbb{T} and once in $[0, T]$), thus defining V_M^i ,
460 V_Z^i and v_m^i as the weak solutions of

$$461 \quad (3.8) \quad -\Delta V_M^i + V_M^i = M^i, \quad -\Delta V_Z^i + V_Z^i = Z^i, \quad -(v_m^i)'' + v_m^i = m^i$$

462 with periodic boundary conditions. Notice that if we extend each of the functions
463 $v_m^i(x)$ to \mathbb{T} as $\tilde{v}_m^i(x, x') = v_m^i(x)$, then \tilde{v}_m^i is \mathbb{T} -periodic and satisfies the same
464 Helmholtz equation as V_M^i , thus it coincides with V_M^i , which means that each V_M^i
465 only depends on x but not on x' . In particular this implies

$$466 \quad (3.9) \quad \int_{L_i} G(X, Y) d\mathcal{H}_Y^{N-1} = G_{1D}(x, \ell_i)$$

where $G_{1D} : [0, T] \times [0, T] \rightarrow \mathbb{R}$ is the Green's function of $-\frac{d^2}{dx^2} + 1$ in 1D with periodic
boundary condition on $[0, T]$. For latter purpose we explicitly compute it: to begin
with, if $\mathcal{G}(x)$ is the $[0, T]$ -periodic solution of

$$-\mathcal{G}'' + \mathcal{G} = \delta_0,$$

467 a direct computation yields

$$468 \quad (3.10) \quad \mathcal{G}(x) = \frac{1}{2 \sinh(T/2)} \cosh\left(x - \frac{T}{2}\right) \quad \text{in } [0, T],$$

and we view it as periodically repeated on \mathbb{R} . It is readily checked that $G_{1D}(x, y) =$
 $\mathcal{G}(|x - y|_T)$ where $|x - y|_T \leq T/2$ represents the closest distance of $x, y \in [0, T]$ in the
torus, i.e. $|x - y|_T = \min_{m \in \mathbb{Z}} |x + mT - y|$. Other easy properties are

$$\mathcal{G}(x) = \mathcal{G}(|x|) = \mathcal{G}(x + T) = \mathcal{G}(T - x)$$

469 and from these we deduce

$$470 \quad (3.11) \quad 0 \leq x \leq y \leq T \Rightarrow G_{1D}(x, y) = \frac{1}{2 \sinh(T/2)} \cosh\left(y - x - \frac{T}{2}\right)$$

$$471 \quad 0 \leq y < x \leq T \Rightarrow G_{1D}(x, y) = \frac{1}{2 \sinh(T/2)} \cosh\left((y + T) - x - \frac{T}{2}\right),$$

472 which will be useful since in general $x, y \in [0, T]$.

By linearity when setting

$$V_M = \sum_{i=1}^{2k} V_M^i, \quad V_Z = \sum_{i=1}^{2k} V_Z^i, \quad v_m = \sum_{i=1}^{2k} v_m^i,$$

these functions solve with periodic boundary conditions the Helmholtz equations

$$-\Delta V_M + V_M = M, \quad -\Delta V_Z + V_Z = Z, \quad -v_m'' + v_m = m,$$

473 and V_M only depends on x . Now

$$\begin{aligned}
474 \quad & \int_{\partial\mathbb{L}} \int_{\partial\mathbb{L}} G(X, Y) \mu(X) \zeta(Y) d\mathcal{H}_X^{N-1} d\mathcal{H}_Y^{N-1} \\
475 \quad & = \int_{\mathbb{T}} \int_{\mathbb{T}} G(X, Y) dM(X) dZ(Y) = \int_{\mathbb{T}} v_m(y) dZ(y, y') \\
476 \quad & = \sum_{i=1}^{2k} v_m(\ell_i) \int_{L_i} \zeta_i(\ell_i, y') d\mathcal{H}_{y'}^{N-1} = 0
\end{aligned}$$

477 so using (3.9)

$$\begin{aligned}
478 \quad & \int_{\partial\mathbb{L}} \int_{\partial\mathbb{L}} G(X, Y) \eta(X) \eta(Y) d\mathcal{H}_X^{N-1} d\mathcal{H}_Y^{N-1} \\
479 \quad & = \int_{\partial\mathbb{L}} \int_{\partial\mathbb{L}} G(X, Y) \mu(X) \mu(Y) d\mathcal{H}_X^{N-1} d\mathcal{H}_Y^{N-1} \\
480 \quad & \quad + \int_{\partial\mathbb{L}} \int_{\partial\mathbb{L}} G(X, Y) \zeta(X) \zeta(Y) d\mathcal{H}_X^{N-1} d\mathcal{H}_Y^{N-1} \\
481 \quad & = \sum_{i,j=1}^{2k} \mu_i \mu_j \int_{\partial L_i} \int_{\partial L_j} G(X, Y) d\mathcal{H}_X^{N-1} d\mathcal{H}_Y^{N-1} \\
482 \quad & \quad + \int_{\partial\mathbb{L}} \int_{\partial\mathbb{L}} G(X, Y) \zeta(X) \zeta(Y) d\mathcal{H}_X^{N-1} d\mathcal{H}_Y^{N-1} \\
483 \quad & = \sum_{i,j=1}^{2k} \mu_i \mu_j \int_{\partial L_i} G_{1D}(x, \ell_j) d\mathcal{H}_X^{N-1} + \int_{\mathbb{T}} (|\nabla V_Z|^2 + |V_Z|^2) dX \\
484 \quad & = T^{N-1} \sum_{i,j=1}^{2k} \mu_i \mu_j \mathcal{G}(|\ell_i - \ell_j|_T) + \int_{\mathbb{T}} (|\nabla V_Z|^2 + |V_Z|^2) dX . \\
485 \quad &
\end{aligned}$$

486 This equality, together with (3.5) and (3.6), may be put into the expression (3.2) for
487 J'' , thus obtaining for any stationary lamella

$$\begin{aligned}
488 \quad & J''(\mathbb{L})[\mu + \zeta] \\
489 \quad (3.12) \quad & = \sigma T^{N-1} \left(\sum_{i,j=1}^{2k} \mu_i \mu_j \mathcal{G}(|\ell_i - \ell_j|_T) - d_0 \sum_{i=1}^{2k} \mu_i^2 \right)
\end{aligned}$$

$$\begin{aligned}
490 \quad (3.13) \quad & + \sum_{i=1}^{2k} \left(\int_{L_i} (|\nabla \zeta_i|^2 - \sigma d_0 |\zeta_i^2|) d\mathcal{H}^{N-1} \right) + \sigma \int_{\mathbb{T}} (|\nabla V_Z|^2 + |V_Z|^2) dX .
\end{aligned}$$

491 As a reminder, we impose only translation-free perturbation $\eta = \mu + \zeta$ for stability
492 consideration; this amounts to requiring that $\mu \in \mathbb{R}^{2k}$ satisfies (3.4). Remark that the
493 two lines on the right hand side of the above equation are entirely independent: then
494 it is easy to see that a necessary and sufficient condition for a stationary k -lamella
495 to be stable is to establish that the first line (3.12) on the right hand side is positive
496 for all $\mu \in \mathbb{R}^{2k} \setminus \{0\}$ satisfying (3.4), for ζ may well be zero; and that the second line
497 (3.13) is positive for all not identically vanishing ζ such that each ζ_i is periodic and
498 with zero average on L_i , because positivity of J'' must be attained also at $0\mu + \zeta$.

4. Stability, mean value part. We study the mean part μ in (3.12). To prove positivity of (3.12) it suffices to show

$$\sum_{i,j=1}^{2k} \mu_i \mu_j \mathcal{G}(|\ell_j - \ell_i|_T) - d_0 \sum_{i=1}^{2k} \mu_i^2 > 0$$

499 for all non-zero $\mu \in \mathbb{R}^{2k}$ satisfying (3.4). Defining the (symmetric) matrix

500 (4.1)
$$\mathcal{A}_{i,j} = \mathcal{G}(|\ell_j - \ell_i|_T)$$

and considering the vector in \mathbb{R}^{2k}

$$E = (-1, 1, -1, 1, \dots)$$

501 (so that (3.4) reads $\mu \cdot E = 0$) the above may be rewritten as

502 (4.2)
$$\langle (\mathcal{A} - d_0 \mathcal{I})\mu, \mu \rangle > 0 \quad \text{for all } \mu \perp E, \mu \neq 0$$

503 where \mathcal{I} is the identity matrix. We prove in this section the following

504 **THEOREM 4.1.** *The matrix \mathcal{A} has one simple eigenvalue d_0 , corresponding to the*
 505 *eigenvector E , and all other eigenvalues are strictly larger than d_0 . In particular (4.2)*
 506 *holds, so (3.12) is positive for all $\mu \in \mathbb{R}^{2k}$ satisfying (3.4).*

We highlight some properties of \mathcal{A} . The matrix \mathcal{A} is symmetric because \mathcal{G} is even. Next, since the distance from L_i to L_j is the same as the distance of the sides we get by shifting both in the same direction by T/k , i.e. $|\ell_i - \ell_j|_T = |\ell_{i+2} - \ell_{j+2}|_T$, we have

$$\mathcal{A}_{i+2,j+2} = \mathcal{A}_{i,j},$$

thus all entries in \mathcal{A} repeat themselves if we shift (modulo $2k$) by 2 columns right and 2 rows down. It is convenient to think of \mathcal{A} as made of 2×2 blocks B_0, B_1, \dots, B_{k-1} for a k -lamella: the structure of \mathcal{A} is then

$$\mathcal{A} = \begin{pmatrix} B_0 & B_1 & B_2 & \cdots & B_{k-1} \\ B_{k-1} & B_0 & B_1 & \cdots & B_{k-2} \\ B_{k-2} & B_{k-1} & B_0 & \cdots & B_{k-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_1 & B_2 & B_3 & \cdots & B_0 \end{pmatrix}.$$

507 Due to symmetry of \mathcal{A} , we have $B_j = B_{k-j}^T$ for $j = 0, 1, 2, \dots, k-1$. This means that
 508 \mathcal{A} is a *block circulant symmetric matrix*, which has interesting properties regarding
 509 its eigenvalues: let all k distinct complex roots of the unity be denoted by

510 (4.3)
$$\rho_p = e^{i\phi_p}, \quad \phi_p = p \frac{2\pi}{k}, \quad p = 0, \dots, k-1.$$

511 With $p = 0, \dots, k-1$ define the 2×2 matrices

512 (4.4)
$$H_p = B_0 \rho_p^0 + B_1 \rho_p^1 + \cdots + B_{k-1} \rho_p^{k-1} :$$

513 each has two (if we count multiplicity) eigenvalues λ'_p, λ''_p , and we have, see [37, Section
 514 3.1]:

515 PROPOSITION 4.2. *The eigenvalues of \mathcal{A} are all the numbers λ'_p and λ''_p for $p =$
516 $0, 1, \dots, k-1$.*

Now recall $B_{k-j} = B_j^T$ for $j = 0, 1, 2, \dots, k-1$. In particular B_0 is symmetric and, if k is even, also the middle one $B_{k/2}$ is symmetric. We remark that ρ_p^j is the conjugate of ρ_p^{k-j} . Therefore in the sum (4.4) we may group terms $\rho^j B_j$ in pairs, excluding the first one and also $B_{k/2}$ if k was even, to get

$$\rho_p^j B_j + \rho_p^{k-j} B_{k-j} = \rho_p^j B_j + \bar{\rho}_p^j B_j^T$$

517 for $j = 1, 2, \dots, k/2 - 1$ when k is even or for $j = 1, 2, \dots, (k-1)/2$ when j is odd.
518 Each pair forms a Hermitian matrix, thus H_p in (4.4) is a Hermitian matrix since the
519 first term B_0 is real symmetric and so is the middle term $(-1)^p B_{k/2}$ for even k .

520 Finally we remark that for every p , the entry $[B_p]_{1,1}$ comes from evaluating \tilde{v} with
521 an input equal to the distance between the left sides of some two lamellae, and $[B_p]_{2,2}$
522 relates to the distance between the right sides of the same lamellae. Since these two
523 distances are the same, the diagonal elements in each matrix on the right hand side
524 in (4.4) equal one another, thus the same is true for H_p . We combine the above facts
525 to obtain that each matrix H_p has the form

$$526 \quad (4.5) \quad H_p = \begin{pmatrix} a_p & b_p \\ \bar{b}_p & a_p \end{pmatrix}$$

for some a_p, b_p . As H_p is Hermitian, a_p has to be real. Its eigenvalues are

$$\lambda'_p = a_p - |b_p|, \quad \lambda''_p = a_p + |b_p|.$$

527 Since $\lambda''_p \geq \lambda'_p$, to prove Theorem 4.1, in view of Proposition 4.2 we will show that

528 PROPOSITION 4.3. *The number b_0 is not zero. Moreover $\lambda'_0 = d_0$ and $\lambda'_p > d_0$
529 for all $p > 0$.*

530 *Proof.* In the course of the proof we will also see that E is the eigenvector corre-
531 sponding to d_0 . We are about to compute a_p and b_p . Only the first row of the matrix
532 \mathcal{A} needs to be considered in computing H_p . We write it in full using (3.11),(4.1):
533 since $\ell_1 = 0$, the odd elements are for $p = 0, \dots, k-1$

$$534 \quad (4.6) \quad a_{1,2p+1} = \frac{1}{2 \sinh(T/2)} \cosh\left(\frac{T}{2} - p\frac{T}{k}\right),$$

535 (so for e.g. $p = 3$ we get $[B_3]_{1,1}$) whereas the even elements are

$$536 \quad (4.7) \quad a_{1,2p+2} = \frac{1}{2 \sinh(T/2)} \cosh\left(\frac{T}{2} - \frac{x_0}{k} - p\frac{T}{k}\right).$$

537 To proceed further we first establish the following lemma.

LEMMA 4.4. *If $e^{i\phi}$ is any k -th root of 1 and $\delta \in \mathbb{R}$ then*

$$\sum_{n=0}^{k-1} e^{in\phi} \cosh\left(\frac{T}{2} + \delta - n\frac{T}{k}\right) = \frac{\sinh(\delta + T/k) - e^{-i\phi} \sinh \delta}{\cosh(T/k) - \cos \phi} \sinh \frac{T}{2}.$$

Proof. We expand the hyperbolic cosine so that

$$e^{in\phi} \cosh\left(\frac{T}{2} + \delta - n\frac{T}{k}\right) = \frac{1}{2} (e^{\delta+T/2} e^{n(i\phi-T/k)} + e^{-\delta-T/2} e^{n(i\phi+T/k)})$$

538 and thus (recalling in the second equality below that $k\phi$ is a multiple of 2π)

$$\begin{aligned}
 539 \quad \sum_{n=0}^{k-1} e^{in\phi} \cosh\left(\frac{T}{2} + \delta - n\frac{T}{k}\right) &= \frac{1}{2}e^{\delta+T/2} \frac{1 - e^{-T+ik\phi}}{1 - e^{i\phi-T/k}} + \frac{1}{2}e^{-\delta-T/2} \frac{1 - e^{T+ik\phi}}{1 - e^{i\phi+T/k}} \\
 540 \quad &= \frac{1}{2}e^{\delta} \frac{e^{T/2} - e^{-T/2}}{1 - e^{i\phi-T/k}} - \frac{1}{2}e^{-\delta} \frac{e^{T/2} - e^{-T/2}}{1 - e^{i\phi+T/k}} \\
 541 \quad &= \sinh\frac{T}{2} \cdot \left(\frac{e^{\delta}}{1 - e^{i\phi-T/k}} - \frac{e^{-\delta}}{1 - e^{i\phi+T/k}} \right) \\
 542 \quad &= \sinh\frac{T}{2} \cdot \frac{e^{\delta} - e^{-\delta} - e^{\delta+i\phi+T/k} + e^{-\delta+i\phi-T/k}}{(1 - e^{i\phi-T/k})(1 - e^{i\phi+T/k})}.
 \end{aligned}$$

543 But

$$\begin{aligned}
 544 \quad 1 - e^{i\phi-T/k} &= (e^{(T/2k)-i(\phi/2)} - e^{-(T/2k)+i(\phi/2)})e^{-T/2k}e^{i\phi/2} \\
 545 \quad 1 - e^{i\phi+T/k} &= (e^{-(T/2k)-i(\phi/2)} - e^{(T/2k)+i(\phi/2)})e^{T/2k}e^{i\phi/2}
 \end{aligned}$$

546 so using hyperbolic function identities

$$\begin{aligned}
 547 \quad (1 - e^{i\phi-T/k})(1 - e^{i\phi+T/k}) &= -4e^{i\phi} \sinh\left(\frac{T}{2k} - i\frac{\phi}{2}\right) \sinh\left(\frac{T}{2k} + i\frac{\phi}{2}\right) \\
 548 \quad &= -2e^{i\phi} \left(\cosh\frac{T}{k} - \cosh i\phi \right) = -2e^{i\phi} \left(\cosh\frac{T}{k} - \cos\phi \right)
 \end{aligned}$$

549 since $\cos z = \cosh(iz)$, as well as $i \sin z = \sinh(iz)$. We may thus resume by writing

$$\begin{aligned}
 550 \quad \sum_{n=0}^{k-1} e^{in\phi} \cosh\left(\frac{T}{2} + \delta - n\frac{T}{k}\right) &= -\sinh\frac{T}{2} \cdot \frac{e^{\delta} - e^{-\delta} - e^{\delta+i\phi+T/k} + e^{-\delta+i\phi-T/k}}{2e^{i\phi}(\cosh(T/k) - \cos\phi)} \\
 551 \quad &= \sinh\frac{T}{2} \cdot \frac{-e^{-i\phi} \sinh\delta + \sinh(\delta + T/k)}{\cosh(T/k) - \cos\phi}
 \end{aligned}$$

552 which concludes the proof. \square

553 Returning now to the proof of Proposition 4.3, we apply this formula to compute the
 554 coefficients in the matrices H_p : let $\rho_p = e^{i\phi_p}$, recall (3.10),(4.1),(4.5),(4.6) and we
 555 have

$$556 \quad (4.8) \quad a_p = [H_p]_{1,1} = \sum_{n=0}^{k-1} \rho_p^n \frac{1}{2 \sinh(T/2)} \cosh\left(\frac{T}{2} - n\frac{T}{k}\right).$$

557 Analogously

$$558 \quad (4.9) \quad b_p = [H_p]_{1,2} = \sum_{n=0}^{k-1} \rho_p^n \frac{1}{2 \sinh(T/2)} \cosh\left(\frac{T}{2} - \frac{x_0}{k} - n\frac{T}{k}\right).$$

Lemma 4.4 then implies

$$a_p = \frac{\sinh(T/k)}{2(\cosh(T/k) - \cos\phi_p)}, \quad b_p = \frac{\sinh(T/k - x_0/k) + e^{-i\phi_p} \sinh(x_0/k)}{2(\cosh(T/k) - \cos\phi_p)}.$$

We are now ready to conclude the proof of Proposition 4.3:

Case $p = 0$: we first consider the case $p = 0$ in (4.3). This gives

$$a_0 = \frac{\sinh(T/k)}{2(\cosh(T/k) - 1)}, \quad b_0 = \frac{\sinh(T/k - x_0/k) + \sinh(x_0/k)}{2(\cosh(T/k) - 1)},$$

559 the number b_0 is strictly positive so the two eigenvalues of H_0 are distinct, and the
560 lower one is $\lambda'_0 = a_0 - b_0$. We now

561 (4.10) claim: $a_0 - b_0 = d_0$,

562 thus d_0 will be a simple eigenvalue of H_0 and therefore also an eigenvalue of \mathcal{A} . We
563 remark that a_0 is the sum of the odd elements in the first row of \mathcal{A} and b_0 is the sum
564 of even elements, so our claim, when proved, will show that their difference is d_0 .

Assume the validity of the claim for the time being; by the symmetry of all matrices B_j , $j = 0, 1, \dots, k-1$, for $p = 0$, the second row of $\mathcal{A} - d_0\mathcal{I}$ has the same entries as the first, only interchanging the pair of consecutive odd and even places starting from the first entry; thus the difference between the sum-of-odd and the sum-of-even entries of the second row is also zero. These facts may be rewritten as: the first two entries of $(\mathcal{A} - d_0\mathcal{I})E$ are zero. But as all subsequent rows of $\mathcal{A} - d_0\mathcal{I}$ are just shifted copies of the first two, we get

$$(\mathcal{A} - d_0\mathcal{I})E = 0,$$

565 so E will be an eigenvector corresponding to the eigenvalue d_0 . All we have to do is
566 to prove our claim which we rewrite as

567 (4.11) $a_0 - b_0 = d_0 \quad \Leftrightarrow \quad \sinh \frac{T}{k} - \sinh \frac{x_0}{k} - \sinh \frac{T - x_0}{k} = 2d_0(\cosh(T/k) - 1).$

We now make extensive use of identities associated with hyperbolic functions [1, Chapter 4, section 5] in this paper without further reference. At the left hand side

$$\sinh \frac{T}{k} - \sinh \frac{x_0}{k} = 2 \cosh \frac{T + x_0}{2k} \sinh \frac{T - x_0}{2k}$$

and observe

$$\sinh \frac{T - x_0}{k} = 2 \sinh \frac{T - x_0}{2k} \cosh \frac{T - x_0}{2k},$$

so the left hand side of (4.11) is equal to

$$2 \sinh \frac{T - x_0}{2k} \left(\cosh \frac{T + x_0}{2k} - \cosh \frac{T - x_0}{2k} \right) = 4 \sinh \frac{T - x_0}{2k} \sinh \frac{T}{2k} \sinh \frac{x_0}{2k}.$$

On the other hand, using the expression (2.4) of d_0 and applying hyperbolic function identity to $[\cosh(T/k) - \cosh 0]$ the right hand side of (4.11) is equal to

$$2 \frac{\sinh(x_0/2k) \sinh((T - x_0)/2k)}{\sinh(T/2k)} \cdot 2 \sinh^2 \frac{T}{2k} = 4 \sinh \frac{T - x_0}{2k} \sinh \frac{T}{2k} \sinh \frac{x_0}{2k}$$

568 and claim (4.10) is proved.

569 **Case $p \neq 0$:** the eigenvalues of H_p are now $a_p \pm |b_p|$, and we

570 (4.12) claim: $\lambda'_p = a_p - |b_p| > d_0$,

571 which would conclude the proof of Proposition 4.3 and therefore also of Theorem 4.1.
 572 We write the inequality as

$$\begin{aligned}
 573 \quad \sinh \frac{T}{k} - \sqrt{\left(\sinh \frac{T-x_0}{k} + \sinh \frac{x_0}{k} \cos \phi_p\right)^2 + \left(\sinh \frac{x_0}{k} \sin \phi_p\right)^2} \\
 574 \quad \quad \quad > 2d_0 \left(\cosh \frac{T}{k} - \cos \phi_p\right).
 \end{aligned}$$

We make use of what we proved in the case $p = 0$ by subtracting

$$\sinh \frac{T}{k} - \sinh \frac{x_0}{k} - \sinh \frac{T-x_0}{k}$$

575 from the left hand side and $2d_0(\cosh(T/k) - 1)$, which is the same by (4.11), from
 576 the right hand side. The claim now reads

$$\begin{aligned}
 577 \quad \sinh \frac{T-x_0}{k} + \sinh \frac{x_0}{k} - \sqrt{\left(\sinh \frac{T-x_0}{k} + \sinh \frac{x_0}{k} \cos \phi_p\right)^2 + \left(\sinh \frac{x_0}{k} \sin \phi_p\right)^2} \\
 578 \quad (4.13) \quad \quad \quad > 2d_0(1 - \cos \phi_p).
 \end{aligned}$$

579 We rewrite the argument of the square root, which is

$$\begin{aligned}
 580 \quad \sinh^2 \frac{T-x_0}{k} + \sinh^2 \frac{x_0}{k} \cos^2 \phi_p + 2 \sinh \frac{T-x_0}{k} \sinh \frac{x_0}{k} \cos \phi_p + \sinh^2 \frac{x_0}{k} \sin^2 \phi_p \\
 581 \quad = \sinh^2 \frac{T-x_0}{k} + \sinh^2 \frac{x_0}{k} + 2 \sinh \frac{T-x_0}{k} \sinh \frac{x_0}{k} \cos \phi_p \\
 582 \quad = \left(\sinh \frac{T-x_0}{k} + \sinh \frac{x_0}{k}\right)^2 - 2 \sinh \frac{T-x_0}{k} \sinh \frac{x_0}{k} (1 - \cos \phi_p).
 \end{aligned}$$

Now (4.13) may be rewritten

$$a - \sqrt{a^2 - 2bt} > 2d_0t$$

where we have put

$$a = \sinh \frac{T-x_0}{k} + \sinh \frac{x_0}{k}, \quad b = \sinh \frac{T-x_0}{k} \sinh \frac{x_0}{k}, \quad t = 1 - \cos \phi_p.$$

We remark that $a, b, d_0 > 0$ and that $0 < t \leq 2$ because $\cos \phi_p$ is not equal to 1 in the case $p \neq 0$. Set

$$f(t) = a - \sqrt{a^2 - 2bt} - 2d_0t$$

so that $f(0) = 0$; all we have to prove is that $f(t) > 0$ for $0 < t \leq 2$. We first remark that

$$a^2 - 2bt \geq a^2 - 4b = \left(\sinh \frac{T-x_0}{k} - \sinh \frac{x_0}{k}\right)^2 \geq 0$$

and we note that for $0 \leq t < 2$

$$f'(t) = \frac{b}{\sqrt{a^2 - 2bt}} - 2d_0$$

583 which is a strictly increasing function of t , so f is strictly convex in $[0, 2]$. We now
 584 prove that $f'(0) \geq 0$: we have $f'(0) = (b/a) - 2d_0$ so we have to prove that $b/a \geq 2d_0$.

585 We use hyperbolic function identities at both the numerator and the denominator to
586 write

$$\begin{aligned}
587 \quad \frac{b}{a} &= \frac{\sinh((T-x_0)/k) \sinh(x_0/k)}{\sinh((T-x_0)/k) + \sinh(x_0/k)} \\
588 \quad &= \frac{4 \sinh((T-x_0)/2k) \sinh(x_0/2k) \cosh((T-x_0)/2k) \cosh(x_0/2k)}{2 \sinh(T/2k) \cosh((T-2x_0)/2k)} \\
589 \quad &= 2d_0 \frac{\cosh((T-x_0)/2k) \cosh(x_0/2k)}{\cosh((T-2x_0)/2k)},
\end{aligned}$$

590 so $b/a \geq 2d_0$ provided

$$\begin{aligned}
(4.14) \\
591 \quad \frac{\cosh((T-x_0)/2k) \cosh(x_0/2k)}{\cosh((T-2x_0)/2k)} \geq 1 \quad \Leftrightarrow \quad \cosh \frac{T-x_0}{2k} \cosh \frac{x_0}{2k} \geq \cosh \frac{T-2x_0}{2k}.
\end{aligned}$$

By hyperbolic function identity

$$\cosh \frac{T-x_0}{2k} \cosh \frac{x_0}{2k} = \frac{1}{2} \left(\cosh \frac{T-2x_0}{2k} + \cosh \frac{T}{2k} \right)$$

so (4.14) becomes

$$\cosh \frac{T}{2k} \geq \cosh \frac{T-2x_0}{2k} = \cosh \frac{|T-2x_0|}{2k},$$

592 which is true because from $0 \leq x_0 \leq T$ we deduce that $|T-2x_0| \leq T$. This concludes
593 the proof that $f'(0) \geq 0$, consequently the convex function f is strictly increasing
594 in $[0, 2]$. As $f(0) = 0$ this implies that $f(t) > 0$ for $t > 0$, as desired, and the
595 proof of (4.12) is concluded, thus ending the proof of Proposition 4.3, and also of
596 Theorem 4.1. \square

597 Global minimizers in 1D (which may be the empty set, the full torus or the minimal
598 lamella) are stable when subjected to 1D perturbation. The above Theorem 4.1
599 yields a related strong result.

600 **COROLLARY 4.5.** *All stationary periodic lamellae are stable with respect to 1D*
601 *periodic perturbations.*

602 For use in the next section, we need an important

603 **REMARK 4.6.** *Throughout this section we did not use the explicit value (2.5) of*
604 *x_0 for minimal lamellae, but only the fact that $0 \leq x_0 \leq T$ and the expression (2.4) of*
605 *d_0 in terms of the numbers T and x_0 , so in particular Propositions 4.2 and 4.3 hold*
606 *for any numbers $0 \leq x_0 \leq T$ and d_0 linked by (2.4), provided the coefficients of the*
607 *matrix \mathcal{A} are defined through (4.1) and (3.10).*

608 **5. Stability, zero-average part and conclusion.** To conclude the stability
609 analysis for stationary lamellar configurations we have to prove that the sum of the
610 two terms appearing in (3.13) is non-negative for periodic functions defined on all
611 sides of the lamellae, with zero average on each side. We begin with a general (easy)
612 result, then we specialize to a k -lamella in dimension 2, to get some results which to
613 our knowledge are in an entirely new spirit.

Let $C_{P,N-1}$ denote the Poincaré constant in the unit torus \mathbb{T}_1 of \mathbb{R}^{N-1} with periodic boundary conditions (and zero mean), i.e.,

$$\int_{\mathbb{T}_1} |\nabla \zeta|^2 d\mathcal{H}^{N-1} \geq C_{P,N-1} \int_{\mathbb{T}_1} \zeta^2 d\mathcal{H}^{N-1} \quad \forall \zeta \in H_{\text{per}}^1(\mathbb{T}_1) \text{ s.t. } \int_{\mathbb{T}_1} \zeta d\mathcal{H}^{N-1} = 0;$$

then

$$\int_{L_i} |\nabla \zeta_i|^2 d\mathcal{H}^{N-1} - \sigma d_0 \int_{L_i} |\zeta_i^2| d\mathcal{H}^{N-1} \geq \left(\frac{C_{P,N-1}}{T^2} - \sigma d_0 \right) \int_{L_i} |\zeta_i^2| d\mathcal{H}^{N-1}.$$

614

615 **THEOREM 5.1.** *Let \mathbb{L} be a stationary k -lamella, and assume*

$$616 \quad (5.1) \quad \frac{C_{P,N-1}}{T^2} - \sigma d_0 > 0.$$

617 *Then \mathbb{L} is stable in the sense of (3.1).*

618 The proof is just a check: the first part of (3.13) is non-negative due to assumption
619 (5.1), whereas the last part of (3.13), which contains the contribution of Green's
620 function term, is obviously non-negative.

621 **REMARK 5.2.** *If the original torus \mathbb{T} was not a cube but had length T in the x
622 direction and sides of length T' in the orthogonal direction, the factor T^2 appearing
623 in (5.1) should be $(T')^2$ instead. Thus, the smaller is T' the easier it is to obtain
624 stability, as e.g. in [27].*

625 We now focus only on a stationary k -lamella in a two dimensional torus \mathbb{T} , so that
626 $N = 2$, and let $X = (x, x') \in \mathbb{T}$. First we recall
(5.2)

$$627 \quad J''(\mathbb{L})[\zeta] = \sum_{i=1}^{2k} \left(\int_{L_i} |\zeta'_i(x')|^2 dx' - \sigma d_0 \int_{L_i} |\zeta_i(x')|^2 dx' \right) + \sigma \int_{\mathbb{T}} (|\nabla V_Z|^2 + |V_Z|^2) dX$$

on zero-average functions ζ . For $r = 1, 2, \dots$, define $\rho_{2r-1} = \rho_{2r} := (\frac{2\pi r}{T})^2$ and

$$\varphi_{2r-1}(x') := \sin \frac{2\pi r x'}{T}, \quad \varphi_{2r}(x') := \cos \frac{2\pi r x'}{T}.$$

628 The eigenvalues for the operator $-d^2/dx'^2$ for zero-average functions with periodic
629 boundary condition on each L_i are then the numbers ρ_m with corresponding eigen-
630 functions φ_m , for $m = 1, 2, \dots$. Moreover

$$631 \quad (5.3) \quad \int_{L_i} \varphi_m(z) \varphi_r(z) dz = \begin{cases} 0, & \text{if } m \neq r, \\ T/2, & \text{if } m = r, \end{cases}$$

$$632 \quad (5.4) \quad \int_{L_i} \varphi'_m(z) \varphi'_r(z) dz = \begin{cases} 0, & \text{if } m \neq r, \\ \rho_m T/2, & \text{if } m = r. \end{cases}$$

634 We keep the notation in Section 3, and in particular we label ℓ_i the x coordinates of
635 the sides of lamellae as in (3.3) where x_0/k is the thickness of each lamella. Suppose
636 $\zeta_i(x') = \sum_m \alpha_m^i \varphi_m(x')$, where henceforth all sums run for $m \geq 1$ unless otherwise
637 noted; then

$$638 \quad (5.5) \quad \sum_{i=1}^{2k} \int_{L_i} |\zeta_i|^2 dx' = \frac{T}{2} \sum_{i=1}^{2k} \sum_m (\alpha_m^i)^2, \quad \sum_{i=1}^{2k} \int_{L_i} |\zeta'_i|^2 dx' = \frac{T}{2} \sum_{i=1}^{2k} \sum_m \rho_m (\alpha_m^i)^2.$$

With slight abuse, regard $\zeta_i(x, x') = \sum_m \alpha_m^i \varphi_m(x') dx' \llcorner L_i$ as a measure in the equation $-\Delta V_i + V_i = \zeta_i$ on the torus \mathbb{T} , analogously to what we did in (3.7); it is easily verified that $V_i(x, x') = \sum_m u_m^i(x) \varphi_m(x')$ is the unique solution provided u_m^i satisfies

$$-(u_m^i)''(x) + (1 + \rho_m)u_m^i(x) = \alpha_m^i \delta_{\ell_i}(x)$$

and the periodic boundary condition on $[0, T]$. This yields

$$u_m^i(x) = \alpha_m^i C_m \cosh(\sqrt{1 + \rho_m}(|x - \ell_i|_T - T/2))$$

639 for $0 \leq x \leq T$ when we set

$$640 \quad (5.6) \quad C_m = \frac{1}{2\sqrt{1 + \rho_m} \sinh\left(\frac{T}{2}\sqrt{1 + \rho_m}\right)}.$$

641 In other words

$$642 \quad (5.7) \quad V_i(x, x') = \sum_m \alpha_m^i C_m \cosh(\sqrt{1 + \rho_m}(|x - \ell_i|_T - T/2)) \varphi_m(x').$$

643 As the functions φ_m are orthogonal to one another, we obtain (again we treat the
644 functions ζ_i as measures)

$$\begin{aligned} 645 \quad \sigma \int_{\mathbb{T}} V_i d\zeta_j &= \sigma \sum_m \int_{L_j} V_i(\ell_j, x') \alpha_m^j \varphi_m(x') dx' \\ 646 \quad &= \sigma \sum_m \alpha_m^i \alpha_m^j C_m \cosh(\sqrt{1 + \rho_m}(|\ell_j - \ell_i|_T - T/2)) \int_{L_j} \varphi_m^2(x') dx' \\ 647 \quad &= \frac{\sigma T}{2} \sum_m \alpha_m^i \alpha_m^j C_m \cosh(\sqrt{1 + \rho_m}(|\ell_j - \ell_i|_T - T/2)). \end{aligned}$$

648 Making use of self-adjointness of the Green's function G , and grouping terms by
649 oscillation mode m , the last term in (5.2) becomes

$$\begin{aligned} 650 \quad (5.8) \quad &\sigma \int_{\mathbb{T}} (|\nabla V_Z|^2 + |V_Z|^2) dX \\ 651 \quad &= \sigma \int_{\mathbb{T}} \left(\sum_{i=1}^{2k} V_i \right) d \left(\sum_{j=1}^{2k} \zeta_j \right) = \sigma \sum_{i,j=1}^{2k} \int_{\mathbb{T}} V_i d\zeta_j \\ 652 \quad &= \frac{\sigma T}{2} \sum_m C_m \sum_{i,j=1}^{2k} \alpha_m^i \alpha_m^j \cosh(\sqrt{1 + \rho_m}(|\ell_j - \ell_i|_T - T/2)). \end{aligned}$$

653 Putting (5.5), (5.8) to (5.2), we obtain

$$\begin{aligned} 654 \quad (5.9) \quad \frac{2}{T} J''(\mathbb{L})[\zeta] &= \sum_m \left\{ (\rho_m - \sigma d_0) \sum_{i=1}^{2k} (\alpha_m^i)^2 \right. \\ 655 \quad &\quad \left. + \sigma C_m \sum_{i,j=1}^{2k} \alpha_m^i \alpha_m^j \cosh(\sqrt{1 + \rho_m}(|\ell_j - \ell_i|_T - T/2)) \right\}. \end{aligned}$$

656 Since the function ζ may well exhibit just one mode, for the k -lamella to be stable it
 657 is necessary and sufficient to show that

$$658 \quad (\rho_m - \sigma d_0) \sum_{i=1}^{2k} (\alpha_m^i)^2 + \sigma C_m \sum_{i,j=1}^{2k} \alpha_m^i \alpha_m^j \cosh(\sqrt{1 + \rho_m}(|\ell_j - \ell_i|_T - T/2)) > 0$$

$$659 \quad (5.10) \quad \forall (\alpha_m^1, \dots, \alpha_m^{2k}) \neq 0$$

for each m . We study the last term and we rewrite it as

$$\sigma C_m \sum_{i,j=1}^{2k} \alpha_m^i \alpha_m^j \cosh(\sqrt{1 + \rho_m}(|\ell_j - \ell_i|_T - T/2)) = \frac{\sigma}{\sqrt{1 + \rho_m}} \sum_{i,j=1}^{2k} (\mathcal{A}^{(m)})_{i,j} \alpha_m^i \alpha_m^j$$

where we set, according to (5.6),

$$(\mathcal{A}^{(m)})_{i,j} := \frac{1}{2 \sinh\left(\frac{T}{2} \sqrt{1 + \rho_m}\right)} \cosh(\sqrt{1 + \rho_m}(|\ell_j - \ell_i|_T - T/2)).$$

We now define

$$T^{(m)} := T \sqrt{1 + \rho_m}, \quad x_0^{(m)} := x_0 \sqrt{1 + \rho_m}$$

so that the numbers

$$\ell_i^{(m)} := \ell_i \sqrt{1 + \rho_m}$$

660 have the same definition in terms of $T^{(m)}$ and $x_0^{(m)}$ as the numbers ℓ_i had in terms
 661 of T and x_0 in (3.3); it is convenient to put

$$662 \quad (5.11) \quad a := x_0/T$$

663 (and we remark in particular that $a = x_0^{(m)}/T^{(m)}$), and finally

$$664 \quad d_0^{(m)} = \frac{1}{\sinh(T^{(m)}/2k)} \sinh \frac{T^{(m)} - x_0^{(m)}}{2k} \sinh \frac{x_0^{(m)}}{2k}$$

$$665 \quad = \frac{1}{\sinh(T^{(m)}/2k)} \sinh \frac{(1-a)T^{(m)}}{2k} \sinh \frac{aT^{(m)}}{2k}.$$

Then we may rewrite

$$(\mathcal{A}^{(m)})_{i,j} = \frac{1}{2 \sinh(T^{(m)}/2)} \cosh(|\ell_j^{(m)} - \ell_i^{(m)}|_{T^{(m)}} - T^{(m)}/2).$$

666 Comparing this with (4.1),(3.10), by Remark 4.6 we may apply the first part of Theo-
 667 rem 4.1 and obtain that the least eigenvalue of $\mathcal{A}^{(m)}$ is $d_0^{(m)}$. Recalling the coefficient
 668 in front of $\mathcal{A}^{(m)}$, we have that (5.10) is equivalent to proving that

$$669 \quad (5.12) \quad \rho_m - \sigma d_0 + \frac{\sigma}{\sqrt{1 + \rho_m}} d_0^{(m)} > 0$$

670 for each $m \geq 1$.

We may now precise the result of Theorem 5.1 to obtain a somewhat generic stability result: as the Poincaré constant on the segment $[0, 1]$ with periodic boundary conditions is $C_{P,1} = 4\pi^2$, equation (5.1) turns into

$$\sigma < \frac{4\pi^2}{d_0 T^2};$$

if we want to get a result which is independent of the fullness parameter $c = 1 - 2\alpha/\sigma$ of (1.14), and therefore of the ratio of x_0 to T as seen from (2.5), we may remark that

$$d_0 \leq \max_{0 \leq y \leq T} \frac{\sinh((T-y)/2k) \sinh y/2k}{\sinh(T/2k)},$$

671 which is attained at $y = T/2$. Hence

$$672 \quad (5.13) \quad d_0 \leq \frac{\sinh^2 T/4k}{\sinh(T/2k)} = \frac{1}{2} \tanh(T/4k),$$

673 thus a sufficient condition for any (i.e. for any fullness parameter c) stationary k -
674 lamella to be stable is the following.

675 **COROLLARY 5.3.** *When $\sigma < 8\pi^2/[T^2 \tanh(T/4k)]$, the stationary k -lamella is sta-*
676 *ble for any value of the fullness parameter c .*

677 Remark that the most delicate case (thus the worst for stability) is $k = 1$, and small
678 values of T contribute to stability; also remark that the worst (i.e., the maximum)
679 value of d_0 was obtained for $x_0 = T/2$, thus for $c = 0$. Theorem 5.1 was obtained by
680 disregarding the positive contribution of the Green's function term, and we may now
681 show that when instead we take it into account this corollary becomes much stronger,
682 see Corollary 5.17.

Recall that $\rho_{2r-1} = \rho_{2r}$ so the same happens with all the various quantities depending on the eigenvalues, such as $C_m, T^{(m)}, x_0^{(m)}, d_0^{(m)}$; it therefore suffices to prove (5.12) only for even m . With slight abuse, we redefine

$$\rho_m = \frac{4\pi^2 m^2}{T^2} \quad \text{for } m = 0, 1, \dots$$

683 and study (5.12) for all $m = 1, 2, \dots$. We set for $m = 0, 1, \dots$

$$684 \quad (5.14) \quad \theta_m := \frac{T^2}{4k^2} + \frac{\pi^2 m^2}{k^2} = \frac{T^2}{4k^2} (1 + \rho_m) = \left(\frac{T^{(m)}}{2k} \right)^2$$

so that

$$\rho_m = \frac{4k^2}{T^2} \theta_m - 1 = \frac{4k^2}{T^2} (\theta_m - \theta_0)$$

as $\theta_0 = (T/2k)^2$; we remark that (since $x_0^{(m)}/T^{(m)} = x_0/T = a$)

$$d_0^{(m)} = \frac{1}{\sinh \sqrt{\theta_m}} \sinh((1-a)\sqrt{\theta_m}) \sinh(a\sqrt{\theta_m})$$

thus

$$\frac{d_0^{(m)}}{\sqrt{1 + \rho_m}} = \frac{T}{2k} \frac{\sinh((1-a)\sqrt{\theta_m}) \sinh(a\sqrt{\theta_m})}{\sqrt{\theta_m} \sinh \sqrt{\theta_m}}.$$

685 Since $d_0^{(0)} = d_0$, it is useful to introduce the function

$$686 \quad (5.15) \quad h(x) := \frac{4k^2}{T^2} x + \frac{\sigma T}{2k} \frac{\sinh((1-a)\sqrt{x}) \sinh(a\sqrt{x})}{\sqrt{x} \sinh \sqrt{x}}$$

687 so that we may rewrite (5.12) as

$$688 \quad (5.16) \quad h(\theta_m) - h(\theta_0) > 0 \quad \forall m \geq 1.$$

Now define for every real $x \geq 0$

$$\theta(x) := \frac{T^2}{4k^2} + \frac{\pi^2}{k^2}x,$$

so that $\theta_m = \theta(m^2)$. A sufficient condition for (5.16) is to check that

$$(5.17) \quad h(\theta(x)) - h(\theta(0)) > 0 \quad \forall x \geq 1.$$

691

692 **REMARK 5.4.** *Although we did not stress dependence on the various quantities*
 693 *involved, not to overburden the notation, from (2.5) and (2.4), both x_0 and d_0 depend*
 694 *only on T and c , but not on σ . In addition, changing sign of c converts x_0 into $T - x_0$;*
 695 *this in turn converts a to $1 - a$. However this change will not affect h , so it suffices to*
 696 *study stability only for $c \geq 0$. The cases $x_0 = 0$ (empty set) and $x_0 = T$ (full torus)*
 697 *corresponding to $c = \pm 1$ are trivially stable, we therefore focus only on $0 < x_0 < T$,*
 698 *equivalently $0 < a < 1$, so that the last term in the definition (5.15) of h is positive.*

699 We see that an unrestricted stability statement, such as Theorem 4.1, cannot be
 700 attained, through the negative result underneath with Γ as defined in (1.16).

701 **PROPOSITION 5.5.** *If $0 < |c| < 1$ and $\sigma > -4/\Gamma(c)$, then for any sufficiently large*
 702 *T the stationary 1-lamella is unstable.*

703 *Proof.* From Remark 5.4 and the observation $\Gamma(c) = \Gamma(-c)$, it suffices to study
 704 the case $0 < c < 1$. We will show $m = 1$ is an unstable mode for (5.12). Let $T \gg 1$
 705 and denote by ω_T all terms which are exponentially small in T (we need to keep track
 706 of algebraic small quantities). Then (2.7) still holds, so $x_0 = -\log c + \omega_T$ from (2.5)
 707 (see [2, Proposition 3.2 (vi)]) and $d_0 = \frac{1-c}{2} + \omega_T$ from (2.4); moreover

$$\begin{aligned} \sqrt{\theta_1} &= \sqrt{\frac{T^2}{4} + \pi^2} = \frac{T}{2} \left(1 + \frac{2\pi^2}{T^2} + O\left(\frac{1}{T^4}\right) \right), \\ \sqrt{1 + \rho_1} &= 2\sqrt{\theta_1}/T, \\ T^{(1)} &= 2\sqrt{\theta_1}, \quad x_0^{(1)} = 2x_0\sqrt{\theta_1}/T. \end{aligned}$$

712 Thus computing directly from the left side of (5.12), we obtain

$$\begin{aligned} h(\theta_1) - h(\theta_0) &= \frac{4\pi^2}{T^2} - \sigma \left(\frac{1-c}{2} + \omega_T \right) + \frac{\sigma T}{4\sqrt{\theta_1}} \left(1 - c + \frac{2\pi^2 c x_0}{T^2} + O\left(\frac{1}{T^4}\right) \right) \\ &= \frac{4\pi^2}{T^2} - \sigma \left(\frac{1-c}{2} + \omega_T \right) \\ &\quad + \frac{\sigma}{2} \left(1 - \frac{2\pi^2}{T^2} + O\left(\frac{1}{T^4}\right) \right) \left(1 - c + \frac{2\pi^2 c x_0}{T^2} + O\left(\frac{1}{T^4}\right) \right) \\ &= \frac{\pi^2}{T^2} (4 + \sigma\Gamma(c)) + O\left(\frac{1}{T^4}\right) \\ &< 0 \end{aligned}$$

719 for T large. □

720 **REMARK 5.6.** *The condition $\sigma > 4/|\Gamma(c)|$ imposed in Proposition 5.5 turns out to*
 721 *be both necessary and sufficient for instability of all k -lamellae in a sufficiently large*
 722 *torus. Indeed in the above proof we only treat the mode $m = 1$; but by Theorem 5.11*

723 (which will be proved later) we do not discard any generality for stability studies.
 724 Second, if we carry out the above proof on a k -lamella, then $x_0/k = -\log c + \omega_T$;
 725 however the same final condition, which is independent of k , results.

726 In view of Remark 2.3, when $c > 0$ and we pick a large square torus with side $T = 2t_0$,
 727 then $J(\mathbb{L}) < J(\emptyset) = 0 < J(\mathbb{T})$. This gives

728 COROLLARY 5.7. *Let $0 < c < 1$ and $4 + \sigma\Gamma(c) < 0$. Then for some sufficiently*
 729 *large torus there exists an unstable minimal lamella \mathbb{L} such that $J(\mathbb{L}) < J(\emptyset) =$*
 730 *$0 < J(\mathbb{T})$. Hence in this parameter regime global minimizers (which always exist by*
 731 *Theorem 2.1), being neither the trivial states nor the lamellae, has to have a genuine*
 732 *2D structure.*

733 We now collect the necessary preliminaries to prove the main results. We begin
 734 with easy properties of convex functions.

LEMMA 5.8. *If f is (strictly) convex then so is e^f ; if f is (strictly) convex, so is*
 $f(a + bx)$ for $b \neq 0$; if f is convex on $[0, +\infty)$, then for $0 < a < 1$

$$f(a) + f(1 - a) \leq f(0) + f(1) ,$$

735 and the inequality is strict if f is strictly convex.

736 *Proof.* We only care about the last assertion; convexity of f implies $f(a) \leq (1 -$
 737 $a)f(0) + af(1)$. Replace a by $1 - a$ to obtain a similar inequality and sum the two
 738 inequalities. \square

739 LEMMA 5.9. *The function $P(t) := \frac{t}{\tanh t} + \frac{t^2}{\sinh^2 t} - 2$, continuously extended by*
 740 *$P(0) = 0$, is increasing and strictly convex on $[0, \infty)$, thus positive for $t > 0$.*

741 *Proof.* We have

$$742 \quad P' = \frac{1}{\tanh t} + \frac{t}{\sinh^2 t} - \frac{2t^2 \cosh t}{\sinh^3 t} = \frac{1}{\sinh^2 t} \left(\sinh t \cosh t + t - 2t^2 \coth t \right)$$

$$743 \quad = \frac{1}{\sinh^2 t} \left(\frac{\sinh 2t}{2} + t - 2t^2 \coth t \right) =: \frac{1}{\sinh^2 t} g(t) .$$

745 It is clear that $g(0) = 0$. A direct calculation gives

$$746 \quad g'(t) = \cosh 2t + 1 + \frac{2t^2}{\sinh^2 t} - \frac{4t \cosh t}{\sinh t} = 2 \cosh^2 t + \frac{2t^2}{\sinh^2 t} - \frac{4t \cosh t}{\sinh t}$$

$$747 \quad = 2 \left(\cosh t - \frac{t}{\sinh t} \right)^2 > 0$$

749 for $t \in (0, \infty)$. Hence $g > 0$ and we conclude that P is strictly increasing. Moreover

$$\begin{aligned}
 750 \quad P'' &= \frac{1}{\sinh^2 t} g'(t) - \frac{2}{\sinh^3 t} \cosh t \, g(t) \\
 751 \quad &= \frac{2}{\sinh^2 t} \left(\left(\cosh t - \frac{t}{\sinh t} \right)^2 - \frac{\cosh t}{\sinh t} (\sinh t \cosh t + t - 2t^2 \coth t) \right) \\
 752 \quad &= \frac{2}{\sinh^2 t} \left(\frac{t^2}{\sinh^2 t} - \frac{3t \cosh t}{\sinh t} + \frac{2t^2 \cosh^2 t}{\sinh^2 t} \right) \\
 753 \quad &= \frac{2t}{\sinh^4 t} (t - 3 \cosh t \sinh t + 2t \cosh^2 t) = \frac{2t}{\sinh^4 t} \left(2t - \frac{3}{2} \sinh 2t + t \cosh 2t \right) \\
 754 \quad &= \frac{2t}{\sinh^4 t} \left(-\frac{3}{2} \sum_{n=1}^{\infty} \frac{(2t)^{2n+1}}{(2n+1)!} + t \sum_{n=1}^{\infty} \frac{(2t)^{2n}}{(2n)!} \right) \\
 755 \quad &= \frac{t}{\sinh^4 t} \sum_{n=1}^{\infty} \frac{(2t)^{2n+1}}{(2n+1)!} (2n+1-3) \\
 756 \quad &> 0. \quad \square
 \end{aligned}$$

758 The key tool is the following result.

759 LEMMA 5.10. *The functions h and $h \circ \theta$ are strictly convex.*

Proof. By Lemma 5.8, since θ is an affine function of x it is enough to prove h is strictly convex, which we will do for $x > 0$ or, extending h at 0 by continuity, for $x \geq 0$; we remark that this precision will not be needed, since $\theta(0) = \theta_0 = T^2/4k^2$ will be the least value of the argument of h we will be interested in. As the first term in the definition (5.15) of h is linear, we are only concerned with the second (which, we recall, is positive), and in view of Lemma 5.8 again we may just prove that its logarithm is strictly convex. Disregarding the coefficient $\sigma T/2k$ we set

$$u(x) := \log \frac{\sinh((1-a)\sqrt{x}) \sinh(a\sqrt{x})}{\sqrt{x} \sinh \sqrt{x}};$$

760 then

(5.18)

$$\begin{aligned}
 761 \quad 2u'(x) &= \frac{a}{\sqrt{x} \tanh(a\sqrt{x})} + \frac{1-a}{\sqrt{x} \tanh((1-a)\sqrt{x})} - \frac{1}{\sqrt{x} \tanh \sqrt{x}} - \frac{1}{x}, \\
 762 \quad 4x^2 u''(x) &= -\frac{a\sqrt{x}}{\tanh(a\sqrt{x})} - \frac{a^2 x}{\sinh^2(a\sqrt{x})} - \frac{(1-a)\sqrt{x}}{\tanh((1-a)\sqrt{x})} - \frac{(1-a)^2 x}{\sinh^2((1-a)\sqrt{x})} \\
 763 \quad (5.19) \quad &+ \frac{\sqrt{x}}{\tanh \sqrt{x}} + \frac{x}{\sinh^2 \sqrt{x}} + 2. \\
 764
 \end{aligned}$$

Now let x be fixed and define

$$Q(t) := \frac{t\sqrt{x}}{\tanh(t\sqrt{x})} + \frac{t^2 x}{\sinh^2(t\sqrt{x})} - 2.$$

With P as denoted in Lemma 5.9, it is clear that $Q(t) = P(t\sqrt{x})$; moreover

$$4x^2 u''(x) = Q(1) - Q(a) - Q(1-a).$$

765 Using Lemma 5.9 we see that Q is non-negative and vanishing at 0, strictly convex
 766 and increasing, and applying the last part of Lemma 5.8 we obtain $4x^2 u''(x) > 0$. \square

767 We may now examine the function h and the necessary and sufficient condition (5.16).

768 THEOREM 5.11. *It is necessary and sufficient for the k -lamella to be stable that*
 769 *the first mode is stable, that is, $h(\theta_1) > h(\theta_0)$.*

Proof. The necessity of a stable first mode is clear. On the other hand suppose $h(\theta_1) > h(\theta_0)$. From the strict convexity of h and the fact that θ_m is strictly increasing with respect to m ,

$$\frac{h(\theta_m) - h(\theta_1)}{\theta_m - \theta_1} > \frac{h(\theta_1) - h(\theta_0)}{\theta_1 - \theta_0} > 0,$$

770 hence $h(\theta_m) > h(\theta_1) > h(\theta_0)$ for all $m = 2, 3, \dots$; this immediately gives (5.16). \square

771 REMARK 5.12. *Whenever $h(\theta_1) > h(\theta_0)$, a slight modification of the above argu-*
 772 *ment gives $h(\theta_{m+1}) > h(\theta_m)$ for $m = 0, 1, 2, \dots$.*

773 We saw right after Corollary 5.3 that $c = 0$ and $k = 1$ seemed the most delicate cases;
 774 we are now going to substantiate the claim.

775 THEOREM 5.13. *Stability is increasing with $|c|$, in the sense that if the stationary*
 776 *k -lamella with $|c| = c_0 < 1$ is stable, then it is stable also for $c_0 < |c| \leq 1$.*

777 COROLLARY 5.14. *A necessary and sufficient condition for the stationary k -la-*
 778 *mella to be stable for all values of c is that it is stable for $c = 0$.*

779 *Proof.* By Remark 5.4 we may confine ourselves to the case $c \geq 0$, that is $0 \leq$
 780 $a \leq 1/2$ keeping the notation introduced in (5.11). By (5.15) and hyperbolic function
 781 identities we may rewrite

$$\begin{aligned} 782 \quad h(x) &= \frac{4k^2}{T^2}x + \frac{\sigma T \sinh((1-a)\sqrt{x}) \sinh(a\sqrt{x})}{2k \sqrt{x} \sinh \sqrt{x}} \\ 783 \quad &= \frac{4k^2}{T^2}x + \frac{\sigma T \cosh \sqrt{x} - \cosh((1-2a)\sqrt{x})}{4k \sqrt{x} \sinh \sqrt{x}}, \end{aligned}$$

784 so it is convenient to set $\lambda = 1 - 2a$ and remark that, as x_0 is decreasing with c , the
 785 parameter λ is increasing with c . We will prove that the function

$$\begin{aligned} 786 \quad h(\theta_1) - h(\theta_0) &= \frac{4k^2}{T^2}(\theta_1 - \theta_0) + \frac{\sigma T}{4k} \left(\frac{\cosh \sqrt{\theta_1}}{\sqrt{\theta_1} \sinh \sqrt{\theta_1}} - \frac{\cosh \sqrt{\theta_0}}{\sqrt{\theta_0} \sinh \sqrt{\theta_0}} \right) \\ 787 \quad &\quad - \frac{\sigma T}{4k} \left(\frac{\cosh(\lambda\sqrt{\theta_1})}{\sqrt{\theta_1} \sinh \sqrt{\theta_1}} - \frac{\cosh(\lambda\sqrt{\theta_0})}{\sqrt{\theta_0} \sinh \sqrt{\theta_0}} \right) \end{aligned}$$

is increasing with respect to λ , and therefore to c , thus if it is non-negative for a
 certain value of $c \geq 0$ (which by Theorem 5.11 is equivalent to stability) it is positive
 for all larger values of c : this claim would prove the result. We set

$$\phi(\lambda) = \frac{\cosh(\lambda\sqrt{\theta_1})}{\sqrt{\theta_1} \sinh \sqrt{\theta_1}} - \frac{\cosh(\lambda\sqrt{\theta_0})}{\sqrt{\theta_0} \sinh \sqrt{\theta_0}};$$

it suffices to show that ϕ is decreasing. Indeed (writing for simplicity $A = \sqrt{\theta_0}$ and
 $B = \sqrt{\theta_1}$ and remarking that $A < B$)

$$\phi'(\lambda) = \frac{\sinh(\lambda B)}{\sinh B} - \frac{\sinh(\lambda A)}{\sinh A}$$

and to prove that $\phi' < 0$ for $0 < \lambda < 1$ (which is enough) we establish that

$$\psi(x) = \frac{\sinh(\lambda x)}{\sinh x}$$

is decreasing for $x > 0$:

$$\psi'(x) = \frac{\lambda \cosh(\lambda x) \sinh x - \sinh(\lambda x) \cosh x}{\sinh^2 x} = \frac{\cosh(\lambda x) \cosh x}{\sinh^2 x} (\lambda \tanh x - \tanh(\lambda x)) .$$

The function

$$\omega(x) = \lambda \tanh x - \tanh(\lambda x)$$

vanishes at $x = 0$ and its derivative is

$$\omega'(x) = \frac{\lambda}{\cosh^2 x} - \frac{\lambda}{\cosh^2(\lambda x)} < 0$$

788 because $0 < \lambda < 1$, therefore $\omega < 0$ which concludes the proof. \square

789 Now that we proved the worst case for stability is $c = 0$ we turn our attention to k .

790 **THEOREM 5.15.** *In the case $c = 0$, stability is increasing with k , in the sense that*
 791 *if the stationary k_0 -lamella with $c = 0$ is stable, then all k -lamellae with $k \geq k_0$ and*
 792 *$c = 0$ are stable, which implies they are stable also for every c .*

793 **COROLLARY 5.16.** *A necessary and sufficient condition for the stationary k -la-*
 794 *mella to be stable for all values of c and all values of k is that the stationary 1-lamella*
 795 *is stable for $c = 0$.*

Proof. We take $c = 0$ (correspondingly $a = 1/2$); recalling the definition (5.14) of the numbers θ_m , we introduce the quantities

$$\vartheta_1 := k^2 \theta_1 = \frac{T^2}{4} + \pi^2, \quad \vartheta_0 := k^2 \theta_0 = \frac{T^2}{4}$$

796 so they are independent of k , and we rewrite the left hand side of the stability in-
 797 equality $h(\theta_1) - h(\theta_0) \geq 0$ as

$$\begin{aligned} 798 \quad h(\theta_1) - h(\theta_0) &= \frac{4k^2}{T^2} (\theta_1 - \theta_0) + \frac{\sigma T}{4k} \left(\frac{\cosh \sqrt{\theta_1} - 1}{\sqrt{\theta_1} \sinh \sqrt{\theta_1}} - \frac{\cosh \sqrt{\theta_0} - 1}{\sqrt{\theta_0} \sinh \sqrt{\theta_0}} \right) \\ 799 \quad &= \frac{4}{T^2} (\vartheta_1 - \vartheta_0) + \frac{\sigma T}{4} \left(\frac{\cosh \sqrt{\theta_1} - 1}{\sqrt{\vartheta_1} \sinh \sqrt{\theta_1}} - \frac{\cosh \sqrt{\theta_0} - 1}{\sqrt{\vartheta_0} \sinh \sqrt{\theta_0}} \right) \\ 800 \quad &= \frac{4}{T^2} (\vartheta_1 - \vartheta_0) + \frac{\sigma T}{4} \left(\frac{\tanh(\sqrt{\theta_1}/2)}{\sqrt{\vartheta_1}} - \frac{\tanh(\sqrt{\theta_0}/2)}{\sqrt{\vartheta_0}} \right) \\ 801 \quad (5.20) \quad &= \frac{4}{T^2} (\vartheta_1 - \vartheta_0) + \frac{\sigma T}{8} \left(\frac{\tanh(\sqrt{\vartheta_1}/2k)}{\sqrt{\vartheta_1}/2} - \frac{\tanh(\sqrt{\vartheta_0}/2k)}{\sqrt{\vartheta_0}/2} \right). \end{aligned} \quad \square$$

The first term is independent of k , and to prove the assertion we will show that the second term is increasing with respect to k . We set

$$A = \sqrt{\vartheta_0}/2, \quad B = \sqrt{\vartheta_1}/2, \quad x = 1/k$$

so we have to show that if $A < B$ the function

$$\phi(x) = \frac{\tanh(Bx)}{B} - \frac{\tanh(Ax)}{A}$$

is decreasing. But

$$\phi'(x) = \frac{1}{\cosh^2(Bx)} - \frac{1}{\cosh^2(Ax)} < 0 .$$

802 We now see how taking the Green's function term into consideration dramatically
 803 improves the estimate of Corollary 5.3. According to Theorem 5.11, the worst case of
 804 all, that is, $c = 0$ and $k = 1$, is stable if and only if

$$805 \quad 0 < h(\theta_1) - h(\theta_0) = \frac{4}{T^2}(\vartheta_1 - \vartheta_0) + \frac{\sigma T}{4} \left(\frac{\tanh(\sqrt{\vartheta_1}/2)}{\sqrt{\vartheta_1}} - \frac{\tanh(\sqrt{\vartheta_0}/2)}{\sqrt{\vartheta_0}} \right)$$

$$806 \quad = \frac{4\pi^2}{T^2} - \frac{\sigma T}{2} \left(\frac{\tanh(T/4)}{T} - \frac{\tanh(\sqrt{T^2 + 4\pi^2}/4)}{\sqrt{T^2 + 4\pi^2}} \right).$$

807 Immediately we deduce that

COROLLARY 5.17. *A necessary and sufficient condition for all stationary k -lamellae to be stable, for every value of c and k , is that*

$$\sigma < 8\pi^2 \left/ \left[T^3 \left(\frac{\tanh(T/4)}{T} - \frac{\tanh(\sqrt{T^2 + 4\pi^2}/4)}{\sqrt{T^2 + 4\pi^2}} \right) \right] \right.$$

To compare this result (which is a sharp condition) with Corollary 5.3 we recall that $\tanh t = 1 - O(e^{-2T})$ as $T \rightarrow +\infty$, so that

$$\frac{\tanh(T/4)}{T} \quad \begin{cases} \rightarrow \frac{1}{4} = 0.25 & \text{as } T \rightarrow 0 \\ \sim \frac{1}{T} & \text{as } T \rightarrow +\infty \end{cases}$$

whereas

$$\frac{\tanh(T/4)}{T} - \frac{\tanh(\sqrt{T^2 + 4\pi^2}/4)}{\sqrt{T^2 + 4\pi^2}} \quad \begin{cases} \rightarrow \frac{1}{4} - \frac{1}{2\pi} \tanh \frac{\pi}{2} \sim 0.10 & \text{as } T \rightarrow 0 \\ \sim 2\pi^2/T^3 & \text{as } T \rightarrow +\infty. \end{cases}$$

To leading order accuracy, the estimate of the easier Corollary 5.3 reads

$$\sigma \leq \frac{32\pi^2}{T^3} \quad \text{as } T \rightarrow 0, \quad \sigma \leq \frac{8\pi^2}{T^2} \quad \text{as } T \rightarrow +\infty$$

whereas Corollary 5.17 gives (the numerical figure at 0 is an approximation only)

$$\sigma < \frac{77\pi^2}{T^3} \quad \text{as } T \rightarrow 0, \quad \sigma \leq 4 \quad \text{as } T \rightarrow +\infty.$$

808 We do an independent check for the case $T \rightarrow \infty$. By Remark 5.6 all lamellae
 809 are stable when $\sigma < 4/|\Gamma(c)|$ and the torus is large. If we insist on stability for all
 810 $|c| < 1$, then $\sigma < \inf_c \frac{4}{|\Gamma(c)|} = 4$.

In the sequel we set

$$\eta(x) = \frac{\tanh x}{x}, \quad G(x) = \frac{\tanh \sqrt{x}}{\sqrt{x}}.$$

Referring to the calculation in the proof of Corollary 5.16, for $c = 0$ the condition $h(\theta_1) - h(\theta_0) > 0$ may be rewritten as

$$\frac{4\pi^2}{T^2} + \frac{\sigma T}{8k} [G(\vartheta_1/4k^2) - G(\vartheta_0/4k^2)] > 0,$$

811 so we investigate some properties of G .

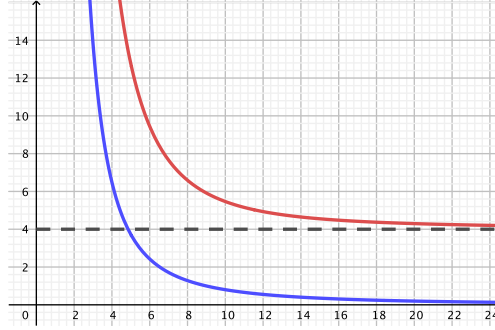


FIG. 5.1. The critical σ as a function of T for Corollaries 5.3 (lower) and 5.17 (upper).

812 LEMMA 5.18. The function G , continuously extended by $G(0) = 1$, is decreasing
 813 and strictly convex for $x \geq 0$. Moreover

814 (5.21)
$$G'(x) \sim -\frac{1}{2x^{3/2}} \quad \text{for large } x.$$

815 Finally as $x \rightarrow +\infty$, for any $\alpha > 0$

816 (5.22)
$$G(x + \alpha) - G(x) = -\frac{\alpha}{2x^{3/2}} + o(x^{-3/2}).$$

817 Before proving the result, we note that instead, η is not convex near the origin.

Proof. Taking logarithmic differentiation for $x > 0$ we see that

$$\frac{G'}{G} = \frac{1}{2\sqrt{x}} \frac{(\cosh^2 \sqrt{x} - \sinh^2 \sqrt{x})}{\sinh \sqrt{x} \cosh \sqrt{x}} - \frac{1}{2x}$$

818 leading by hyperbolic function identity to

819 (5.23)
$$G' = \left(\frac{1}{\sqrt{x} \sinh 2\sqrt{x}} - \frac{1}{2x} \right) G := p(x)G(x),$$

820 which immediately gives monotonicity of G and (5.21). Taking another derivative and
 821 replacing G' with pG we have $G'' = (p' + p^2)G$. It is clear now

822
$$G'' > 0 \iff p' + p^2 > 0 \iff 1 - \left(\frac{1}{p}\right)' > 0$$

823
$$\iff 1 - \frac{d}{dx} \left(-2x + \frac{2x}{1 - \frac{\sinh 2\sqrt{x}}{2\sqrt{x}}} \right) > 0 \iff 3 - 2 \frac{d}{dx} \left(\frac{x}{1 - \frac{\sinh 2\sqrt{x}}{2\sqrt{x}}} \right) > 0$$

824
$$\iff 3 + 2 \frac{d}{dx} \left(\left(\sum_{n=1}^{\infty} \frac{2^{2n} x^{n-1}}{(2n+1)!} \right)^{-1} \right) > 0.$$

825

826 A direct computation yields

$$\begin{aligned}
827 \quad & \left| \frac{d}{dx} \left(\left(\sum_{n=1}^{\infty} \frac{2^{2n} x^{n-1}}{(2n+1)!} \right)^{-1} \right) \right| \\
828 \quad & = \left| - \left(\sum_{n=1}^{\infty} \frac{2^{2n} x^{n-1}}{(2n+1)!} \right)^{-2} \sum_{m=1}^{\infty} \frac{4m}{(2m+2)(2m+3)} \frac{2^{2m} x^{m-1}}{(2m+1)!} \right| \\
829 \quad & < \left| \left(\sum_{n=1}^{\infty} \frac{2^{2n} x^{n-1}}{(2n+1)!} \right)^{-1} \right| \\
830 \quad & < 3/2 \quad \text{by taking only the first term,}
\end{aligned}$$

832 thus $G'' > 0$. Next, the behavior at infinity is an exercise, since $1 - \tanh x$ decays
833 exponentially fast. \square

Now we set

$$H_k(T, \sigma) = h(\theta_1) - h(\theta_0) = \frac{4\pi^2}{T^2} + \frac{\sigma T}{8k} [G(\vartheta_1/4k^2) - G(\vartheta_0/4k^2)];$$

to begin with, since G is decreasing the difference enclosed by the brackets is negative,
so for any $T > 0$ there exists a unique

$$\sigma_k(T) = \frac{32k\pi^2}{T^3} / [G(\vartheta_0/4k^2) - G(\vartheta_1/4k^2)]$$

at which $H_k(T, \sigma) = 0$ with H_k being positive for $\sigma < \sigma_k(T)$. We remark that

$$\frac{\vartheta_0}{4k^2} = \frac{T^2}{16k^2}, \quad \frac{\vartheta_1}{4k^2} = \frac{\vartheta_0}{4k^2} + \frac{\pi^2}{4k^2}$$

and using (5.22) we see that

$$\lim_{T \rightarrow +\infty} \sigma_k(T) = 4$$

834 whereas $\sigma_k(T) \rightarrow +\infty$ as $T \rightarrow 0^+$. We will now prove

835 **PROPOSITION 5.19.** *The function $\sigma_k(T)$ is injective, thus strictly decreasing from*
836 $]0, +\infty[$ to $]4, +\infty[$.

Proof. We begin by remarking that by (5.22)

$$\lim_{T \rightarrow 0^+} H_k(T, \sigma) = +\infty, \quad \lim_{T \rightarrow +\infty} T^2 H_k(T, \sigma) = (4 - \sigma)\pi^2 < 0$$

837 for any $\sigma > 4$, thus for any $\hat{\sigma} > 4$ there is at least one value \hat{T} of T such that
838 $H_k(\hat{T}, \hat{\sigma}) = 0$, i.e. $\sigma_k(\hat{T}) = \hat{\sigma}$. The result will be proved if we show that such \hat{T} is
839 unique; to this aim, we remark that

$$\begin{aligned}
840 \quad & H_k(T, \hat{\sigma}) = 0 \iff \frac{4\pi^2}{T^2} + \frac{\hat{\sigma}T}{8k} [G(\vartheta_1/4k^2) - G(\vartheta_0/4k^2)] = 0 \\
841 \quad & \iff \left(\frac{T}{k} \right)^3 [G(\vartheta_1/4k^2) - G(\vartheta_0/4k^2)] = -\frac{32\pi^2}{\hat{\sigma}k^2},
\end{aligned}$$

and uniqueness of \hat{T} will be proved if we show that the function at the left hand side
in the last line is strictly decreasing with respect to T . Now we rewrite this function
as

$$64 \cdot (T/4k)^3 [G((T/4k)^2 + \pi^2/4k^2) - G((T/4k)^2)]$$

and we prove that

$$x \mapsto x^3 [G(x^2 + \pi^2/4k^2) - G(x^2)]$$

is strictly decreasing. We have

$$x^3 [G(x^2 + \pi^2/4k^2) - G(x^2)] = \int_0^{\pi^2/4k^2} x^3 G'(x^2 + s) ds$$

and the claim will be proved if we show that

$$\frac{\partial}{\partial x} [x^3 G'(x^2 + s)] < 0 \quad \text{for all } s > 0 .$$

842 But

$$\begin{aligned} 843 \quad & \frac{\partial}{\partial x} [x^3 G'(x^2 + s)] = x^2 [3G'(x^2 + s) + 2x^2 G''(x^2 + s)] < 0 \\ 844 \quad & \iff 3G'(x^2 + s) + 2x^2 G''(x^2 + s) < 0 \\ 845 \quad & \iff 3G'(x^2 + s) + 2(x^2 + s)G''(x^2 + s) < 2sG''(x^2 + s) . \end{aligned}$$

846 We prove the left hand side is strictly negative, so the conclusion follows by the
847 convexity of G proved in Lemma 5.18: it is enough to show that for any $X > 0$

$$848 \quad (5.24) \quad 3G'(X) + 2XG''(X) < 0 , \quad \square$$

but recalling that $G(X) = \eta(\sqrt{X})$ we compute

$$G'(X) = \eta'(\sqrt{X}) \cdot \frac{1}{2\sqrt{X}} , \quad G''(X) = \eta''(\sqrt{X}) \cdot \frac{1}{4X} - \frac{1}{4X\sqrt{X}} \eta'(\sqrt{X})$$

so that

$$3G'(X) + 2XG''(X) = \frac{\eta'(\sqrt{X})}{\sqrt{X}} + \frac{1}{2}\eta''(\sqrt{X})$$

and (5.24) is equivalent to

$$\frac{\eta'(t)}{t} + \frac{1}{2}\eta''(t) < 0 \quad \forall t > 0 .$$

A direct computation yields

$$\frac{\eta'(t)}{t} + \frac{1}{2}\eta''(t) = -\frac{(\tanh t)(1 - \tanh^2 t)}{t} < 0 .$$

849 We call $T_k(\sigma)$ the inverse function of $\sigma_k(T)$.

850 **COROLLARY 5.20.** *In the case $c = 0$, for every $\sigma > 4$ the k -lamella is stable for*
851 *$T < T_k(\sigma)$ and unstable for $T \geq T_k(\sigma)$.*

852 **Appendix A. Road map to prove Theorem 3.5.** Throughout this Appendix
853 we refer to statements, formulas and pages of [3], and highlight the changes and
854 focal points needed to adapt the proof of [3, Theorem 1.1] for our Theorem 3.5 in
855 this paper. The proof in [3] needs to resolve a major technicality: the volumetric
856 constraint. Addressing this issue requires lots of efforts to reduce the problem to
857 an unconstrained one, to keep track of the inequalities needed, then to tackle the
858 Lagrange multiplier (and a sequence of them, too).

1. The Euler-Lagrange equation [3, formula (2.8)], which contains a Lagrange multiplier, takes a new form (see Proposition 3.1)

$$\mathcal{K}(\partial E) - \alpha + \sigma \mathcal{N}_E = 0 .$$

859 The corresponding new weak formulation drops the volumetric constraint in
860 [3, Definition 2.2], but adds a term $-\alpha \zeta \cdot \nu$ in its integrands.

- 861 2. The key [3, Lemma 2.6] for the Laplacian is replaced by the (stronger) Lem-
862 ma 3.6 for the Hemholtz operator.
- 863 3. We do not need [3, Proposition 2.7], which is used to weaken volume con-
864 straint.
- 865 4. The slight changes to the derivation of the second variation formula [3, The-
866 orem 3.1] have already been summarized at the beginning of Section 3.
- 867 5. The definition [3, formula (3.4)] of $\partial^2 J$, which is our J'' , acts on all of $H^1(\partial E)$;
868 there is no need to only specify volume preserving vector field X , see (3.2).
6. The very convenient equality [3, formula (3.5)] regarding Green's function for
the Laplacian (and zero average) is replaced by the equally versatile

$$\int_{\partial E} \int_{\partial E} G(x, y) \phi(x) \phi(y) d\mathcal{H}^{N-1}(x) d\mathcal{H}^{N-1}(y) = \int_{\mathbb{T}} (|\nabla V|^2 + |V|^2) dx$$

869 where V is the unique weak solution to the equation $-\Delta V + V = \phi \mathcal{H}^{N-1} \llcorner \partial E$
870 with periodic boundary conditions on \mathbb{T} ; we use this e.g. in Lemma 3.6.

7. The field X in [3, Corollary 3.4] is to be chosen as the gradient of the solution
 u of

$$-\Delta u = \frac{1}{|\partial E|} \int_{\partial E} \phi d\mathcal{H}^{N-1} ;$$

871 ours has no such restriction.

- 872 8. The function spaces and vector fields with tilde, introduced on page 528 of
873 [3] and afterwards, are not needed. Our ambient space is all of H^1 .
874 Both [3, formula (3.9) and Lemma 3.6] still hold, whereas in [3, Theorem 3.7]
875 the last assertion does not, but is not needed in our case (again, it relates to
876 volume preservation).
- 877 9. The proof of the tricky [3, Lemma 3.8], used to control and later remove the
878 translation part, is not related to energies or equations, so it still holds.
- 879 10. The trouble after [3, formula (3.39)] to keep track of the zero average condition
880 is not necessary, thus a_h is not needed and $\tilde{\phi}_h$ is simply $\phi_h \circ \Phi_h$, that is ϕ_h
881 acts on ∂E .

882 In [3, formula (3.40)] we use that the full H^1 product of $(v_h - v)$ and ϕ is
883 $\leq c_\epsilon \|\phi\|$.

884 In [3, formula (3.43)] we also have the difference of $z_h^2 - \tilde{z}_h^2$, but next equation
885 contains the Helmholtz operator and not only the Laplacian, so convergence
886 of $\mu_h - \tilde{\mu}_h$ to zero is preserved.

887 After [3, formula (3.46)] we also have the volume term α and another term
888 appears, but it is not dangerous because the full (not only tangential) diver-
889 gence of X is zero.

- 890 11. The volume penalization after [3, formula (4.2)] is not needed; on the other
891 hand in the chain of inequalities after [3, formula (4.7)] we also have a
892 $-\alpha(|F| - |K_h|) \geq -\alpha|F \Delta K_h|$ so the number Λ chosen in [3, formula (4.6)]
893 must be increased by α .

12. We have no Lagrange multipliers, so the choice of f_h in [3, formula (4.9)] is

$$f_h := \begin{cases} \alpha - \sigma v_{F_h} \\ \alpha - \sigma v_E + \rho_h \end{cases}$$

894 and the rest of the proof becomes silly.

895 **Acknowledgments.** The authors warmly acknowledge the hospitality and nice
896 atmosphere of the National Center for Theoretical Sciences, Taiwan, of the University
897 of Parma and of the University of Connecticut at Storrs.

898

REFERENCES

- 899 [1] M. ABRAMOWITZ AND I. A. STEGUN, *Handbook of mathematical functions with formulas,*
900 *graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Ser-
901 *ies*, No. 55, U. S. Government Printing Office, Washington, D. C., 1964. For sale by the
902 Superintendent of Documents.
- 903 [2] E. ACERBI, C.-N. CHEN, AND Y.-S. CHOI, *Minimal lamellar structures in a periodic FitzHugh-*
904 *Nagumo system*, *Nonlinear Analysis*, 194 (2020), 111436 (13 pages).
- 905 [3] E. ACERBI, N. FUSCO, AND M. MORINI, *Minimality via second variation for a nonlocal isoperi-*
906 *metric problem*, *Comm. Math. Phys.*, 322 (2013), pp. 515–557.
- 907 [4] G. ALBERTI, R. CHOKSI, AND F. OTTO, *Uniform energy distribution for an isoperimetric prob-*
908 *lem with long-range interactions*, *J. Amer. Math. Soc.*, 22 (2009), pp. 569–605.
- 909 [5] S. ALLEN AND J. CAHN, *A microscopic theory for antiphase boundary motion and its application*
910 *to antiphase domain coarsening*, *Acta Metall.*, 27 (1979), pp. 1085–1095.
- 911 [6] F. BATES AND G. FREDRICKSON, *Block copolymers-designer soft materials*, *Phys. Today*, 52
912 (1999), pp. 32–38.
- 913 [7] F. CAGNETTI, M. G. MORA, AND M. MORINI, *A second order minimality condition for the*
914 *Mumford-Shah functional*, *Calc. Var. Partial Differential Equations*, 33 (2008), pp. 37–74.
- 915 [8] C.-N. CHEN, C.-C. CHEN, AND C.-C. HUANG, *Traveling waves for the FitzHugh-Nagumo system*
916 *on an infinite channel*, *J. Differential Equations*, 261 (2016), pp. 3010–3041.
- 917 [9] C.-N. CHEN AND Y.-S. CHOI, *Standing pulse solutions to FitzHugh-Nagumo equations*, *Arch.*
918 *Rational Mech. Anal.*, 206 (2012), pp. 741–777.
- 919 [10] C.-N. CHEN AND Y.-S. CHOI, *Traveling pulse solutions to FitzHugh-Nagumo equations*, *Calc.*
920 *Var. Partial Differential Equations*, 54 (2015), pp. 1–45.
- 921 [11] C.-N. CHEN, Y.-S. CHOI, AND N. FUSCO, *The Γ -limit of traveling waves in the FitzHugh-*
922 *Nagumo system*, *J. Differential Equations*, 267 (2019), pp. 1805–1835.
- 923 [12] C.-N. CHEN, Y.-S. CHOI, Y. HU, AND X. REN, *Higher dimensional bubble profiles in a sharp*
924 *interface limit of the FitzHugh-Nagumo system*, *SIAM J. Math. Anal.*, 50 (2018), pp. 5072–
925 5095.
- 926 [13] C.-N. CHEN, Y.-S. CHOI, AND X. REN, *Bubbles and droplets in a singular limit of the FitzHugh-*
927 *Nagumo system*, *Interfaces Free Bound.*, 20 (2018), pp. 165–210.
- 928 [14] C.-N. CHEN, Y. MORITA, AND S.-Y. KUNG, *Planar standing wavefronts in the FitzHugh-*
929 *Nagumo equations*, *SIAM J. Math. Anal.*, 46 (2014), pp. 657–690.
- 930 [15] X. CHEN AND Y. OSHITA, *Periodicity and uniqueness of global minimizers of an energy func-*
931 *tional containing a long-range interaction*, *SIAM J. Math. Anal.*, 37 (2005), pp. 299–1332.
- 932 [16] R. CHOKSI AND M. PELETIER, *Small volume fraction limit of the diblock copolymer problem:*
933 *I. Sharp-Interface Functional*, *SIAM J. Math. Anal.*, 42 (2010), pp. 1334–1370.
- 934 [17] R. CHOKSI AND M. PELETIER, *Small volume fraction limit of the diblock copolymer problem:*
935 *II. Diffuse-Interface Functional*, *SIAM J. Math. Anal.*, 43 (2011), pp. 739–763.
- 936 [18] R. CHOKSI AND X. REN, *On the derivation of a density functional theory for microphase*
937 *separation of diblock copolymers*, *J. Statist. Phys.*, 113 (2003), pp. 151–176.
- 938 [19] R. CHOKSI AND P. STERNBERG, *Periodic phase separation: the periodic Cahn-Hilliard and*
939 *isoperimetric problems*, *Interfaces Free Bound.*, 8 (2006), pp. 371–392.
- 940 [20] R. CHOKSI AND P. STERNBERG, *On the first and second variations of a nonlocal isoperimetric*
941 *problem*, *J. Reine Angew. Math.*, 611 (2007), pp. 75–108.
- 942 [21] A. DOELMAN, P. VAN HEIJSTER, AND T. KAPER, *Pulse dynamics in a three-component system:*
943 *existence analysis*, *J. Dynam. Differential Equations*, 21 (2008), pp. 73–115.
- 944 [22] P. FIFE AND D. HILHORST, *The Nishiura-Ohnishi free boundary problem in the 1D case*, *SIAM*
945 *J. Math. Anal.*, 33 (2001), pp. 589–606.

- 946 [23] R. KAPRAL AND K. SHOWALTER, *Chemical Waves and Patterns*, Kluwer, 1995.
- 947 [24] L. LEIBLER, *Theory of microphase separation in block copolymers*, *Macromolecules*, 13 (1980),
948 pp. 1602–1617.
- 949 [25] A. LIEHR, *Dissipative Solitons in Reaction-Diffusion Systems*. Springer Series in Synergetics,
950 2013.
- 951 [26] L. MODICA, *The gradient theory of phase transitions and minimal interface criterion*, *Arch.*
952 *Rational Mech. Anal.*, 98 (1987), pp. 123–142.
- 953 [27] M. MORINI AND P. STERNBERG, *Cascade of minimizers for a nonlocal isoperimetric problem*
954 *in thin domains*, *SIAM J. Math. Anal.*, 46 (2014), pp. 2033–2051.
- 955 [28] C. MURATOV, *Theory of domain patterns in systems with long-range interactions of Coulomb*
956 *type*, *Phys. Rev. E*, 66 (2002), 066108.
- 957 [29] C. MURATOV, *Droplet phases in non-local Ginzburg-Landau models with Coulomb repulsion in*
958 *two dimensions*, *Commun. Math. Phys.*, 299 (2010), pp. 45–87.
- 959 [30] F. NIEDERNOSTHEIDE, *Nonlinear Dynamics and Pattern Formation in Semiconductors and*
960 *Devices*, Springer, 1994.
- 961 [31] Y. NISHIURA AND I. OHNISHI, *Some mathematical aspects of the micro-phase separation in*
962 *diblock copolymers*, *Phys. D*, 84 (1984), pp. 31–39.
- 963 [32] T. OHTA AND K. KAWASAKI, *Equilibrium morphology of block copolymer melts*, *Macromolecules*,
964 19 (1986), pp. 2621–2632.
- 965 [33] X. REN AND J. WEI, *On the multiplicity of solutions of two nonlocal variational problems*,
966 *SIAM J. Math. Anal.*, 31 (2000), pp. 909–924.
- 967 [34] X. REN AND J. WEI, *On energy minimizers of the diblock copolymer problem*, *Interfaces Free*
968 *Bound.*, 5 (2003), pp. 193–238.
- 969 [35] X. REN AND J. WEI, *On the spectra of three dimensional lamellar solutions of the diblock*
970 *copolymer problem*, *SIAM J. Math. Anal.*, 35 (2003), pp. 1–32.
- 971 [36] A. SCHEEL, *Coarsening fronts*, *Arch. Rational Mech. Anal.*, 181 (2006), pp. 505–534.
- 972 [37] G. TEE, *Eigenvectors of block circulant and alternating circulant matrices*, *New Zealand J.*
973 *Math.*, 36 (2007), pp. 195–211.
- 974 [38] J. B. VAN DEN BERG AND J. F. WILLIAMS, *Validation of the bifurcation diagram in the 2D*
975 *Ohta-Kawasaki problem*, *Nonlinearity*, 30 (2017), pp. 1584–1638.
- 976 [39] J. B. VAN DEN BERG AND J. F. WILLIAMS, *Rigorously computing symmetric stationary states of*
977 *the Ohta-Kawasaki problem in three dimensions*, *SIAM J. Math. Anal.*, 51 (2019), pp. 131–
978 158.
- 979 [40] J. B. VAN DEN BERG AND J. F. WILLIAMS, *Optimal periodic structures with general space group*
980 *symmetries in the Ohta-Kawasaki problem*, *Phys. D*, 415 (2021), 132732.
- 981 [41] P. VAN HEIJSTER, C.-N. CHEN, Y. NISHIURA, AND T. TERAMOTO, *Localized patterns in a*
982 *three-component FitzHugh-Nagumo model revisited via an action functional*, *J. Dynam.*
983 *Differential Equations*, 30 (2018), pp. 521–555.