Reaching Nirvana with a defaultable asset?

Original
Reaching Nirvana with a defaultable asset? / Battauz, Anna; DE DONNO, Marzia; Sbuelz, Alessandro. - In: DECISIONS IN ECONOMICS AND FINANCE. - ISSN 1129-6569. - 40:(2017), pp. 31-52. [10.1007/s10203-017-0192-x]

Availability:
This version is available at: 11381/2825920 since: 2021-11-09T15:54:36Z

Publisher:
Springer-Verlag Italia s.r.l.

Published
DOI:10.1007/s10203-017-0192-x

Reaching nirvana with a defaultable asset?

Anna Battauz[*], Marzia De Donno[†], and Alessandro Sbuelz[‡]

This version: May 2017

ABSTRACT

We study the optimal dynamic portfolio exposure to predictable default risk, taking inspiration from the search for yield by means of defaultable assets observed before the 2007-2008 crisis and in its aftermath. Under no arbitrage, default risk is compensated by an 'yield pickup' that can strongly attract aggressive investors via an investment-horizon effect in their optimal non-myopic portfolios. We show it by stating the optimal dynamic portfolio problem of Kim and Omberg (1996) for a defaultable risky asset and by rigorously proving the existence of nirvana-type solutions. We achieve such a contribution to the portfolio optimization literature by means of a careful, closed-form-yielding adaptation to our defaultable-asset setting of the general convex duality approach of Kramkov and Schachermayer (1999, 2003).

MSC: 91G10, 91G80.

JEL: G01, G10, G11, G12, D84, D90, C61.

Keywords: dynamic asset allocation, duality-based optimal portfolio solutions, convex duality, non-myopic speculation, leverage, investment horizon, Sharpe ratio risk, reaching for yield, predictable default risk.

[*]Department of Finance and IGIER, Bocconi University, Milan, Italy. (anna.battauz@unibocconi.it).

[†]Department of Economics, University of Parma, Italy. (marzia.dedonno@unipr.it).

[‡]Department of Quantitative Finance and Econometrics, Catholic University of Milan, Milan, Italy. (alessandro.sbuelz@unicatt.it). Corresponding author, CAREFIN Research Fellow. Full address: Catholic University of Milan, Largo Gemelli, 1, 20123, Milan, Italy. Tel: +39-02-7234-2345. Fax: +39-02-7234-232. Email: alessandro.sbuelz@unicatt.it

1 Introduction

The massive risk taking observed before the 2007-2008 financial crisis did not shun defaultable assets. A chief example is offered by the banking industry's course of action during the bullish years up to early 2007. A large quantity of assets exposed to default risk did find its way into commercial and investment banks' portfolios, the corresponding escalation of which was mainly backed by short-term wholesale funding. More broadly, before and after the market turmoil of 2007-2008, sophisticated investors with possibly long investment horizons have been quite sensitive to the higher yield provided by defaultable assets (e.g. Rajan (2005), Diamond and Rajan (2009), and Gennaioli, Martin, and Rossi (2014)). Such a reaching-for-yield behavior involving defaultable assets has been especially pronounced at times of low volatility and of surging markets. What does optimal dynamic portfolio theory have to say about the search for yield carried out by means of defaultable assets?

We offer a novel closed-form optimal portfolio analysis of non-myopic speculation on predictable default risk, which is inspired by the reaching-for-yield behavior that has been involving assets exposed on predictable default risk in the years preceding and following the 2007-2008 crisis. We show that the ability of a default-prone asset to provide an 'yield pickup' (an excess expected return that endures at times of subdued volatility and of high market value) can lead an aggressive investor (he/she is less averse to risk than a log-utility agent) to take significant long geared positions in the asset through an investment-horizon effect. We assume a Cox and Ross (1976) defaultable asset value process, for which no-arbitrage comes from the balance between 'yield-pickup' and predictable default risk. Such a balance is rooted in the inverse relationship between asset returns and their subsequent volatility: upward asset-value paths that enjoy tremendous Sharpe ratios (paths characterized by shrinking volatility) are counteracted by downward asset-value paths that end up with predictable default (paths characterized by swelling volatility). In our model, Sharpe-ratio risk is intimately linked to predictable default risk. The aggressive investor non-myopically speculates on Sharpe-ratio risk by going long the asset (the investor seeks more wealth in the states coupled with a high productivity of wealth). The aggressive investor places a non-myopic (possibly geared) bet on the upward asset-value paths to unfold before his/her investment maturity. A clear investment-horizon effect emerges. The longer the maturity is, the more muscular and geared non-myopic speculation becomes: the bet has more time to make good.

Our paper is related to the search for yield literature, as we use optimal dynamic portfolio theory to assess why non-conservative long-term financial institutions like banks could engage in conspicuous levered reaching-for-yield activities that expose them to default risk. We contribute to the important debate on banks' risk taking and search for yield (e.g Dell'Ariccia, Laeven, and Marquez (2014), Buch, Eickmeier, Prieto (2014), Jiménez, Ongena, Peydró, and Saurina (2014), and Ioannidou, Ongena and Peydró (2015)) by emphasizing that banks' long investment horizon is a distinct optimal-portfolio channel of momentous exposure to default risk. On the other hand, our optimal portfolio analysis offers a precise market-timing rationale for the yield-seeking efforts observed even for less aggressive investors. Kacperczyk and Schnabl (2013) show that relevant segments of the shadow banking system like the money funds have been reaching for yield by increasing their portfolio exposure to default-prone commercial paper and bank obligations in the run-up to the crisis. Di Maggio (2013), Chodorow-Reich (2014), and Di Maggio and Kacperczyk (2016)) show a similar behavior for the money funds in the years following the crisis. Consistently, our optimal portfolio results imply that conservative financial institutions with short investment horizons increase their optimal exposure to defaultable assets at times of low volatility and of surging market value while their hedging demand is minor. We also show that conservative investors with longer horizons exhibit the same optimal portfolio response to low volatility and high valuations while their hedging demand remains bounded albeit stronger (Becker and Ivashina (2015) document insurance companies' portfolio tilt toward higher yield bonds exposed to corporate default risk).

Our paper is linked to the dynamic asset allocation literature that examines the optimal non-myopic portfolio implications of Sharpe-ratio risk. Related studies include e.g. Merton (1971), Kim and Omberg (1996), Bekaert and Ang (2002), Wachter (2002), Lioui and Poncet (2003), Guidolin and Timmermann (2007), Liu (2007), Guidolin and Hyde (2012), Della Corte, Sarno, and Tsiakas (2012), Larsen and Munk (2012), and Branger, Larsen, and Munk (2013), who have however focused on non-defaultable risky assets only. Our analysis of non-myopic speculation on predictable default risk is associated with the so-called nirvana solution to optimal dynamic portfolio problems. Nirvana optimal portfolio solutions grow to large levels over suitably long investment horizons and they have been originally conjectured by Kim and Omberg (1996) for dynamic asset allocation problems characterized by subdued levels of risk aversion and by feeble mean-reversion in the Sharpe-ratio process. Kim and Omberg (1996) consider a non-defaultable asset and assume its Sharpe ratio to be a stationary Gaussian process. Battauz, De Donno, and Sbuelz (2015) provide a convex-duality based derivation of the nirvana solutions in the original Kim and Omberg setting. In this paper, we prove the existence of nirvana solutions in the presence of a defaultable asset and of a non-stationary non-Gaussian Sharpe ratio. We argue that frail/absent mean reversion makes the unfolding of paths with swelling Sharpe ratios a plausible event to punt on for non-myopic aggressive investors.

Our paper is also related to the literature on the analytical/numerical solution techniques for optimal dynamic portfolio problems. Kim and Omberg (1996) do not provide verification theorems for their Hamilton-Jacobi-Bellman approach results and in particular for their nirvana-case conjecture. Such a challenge for the Hamilton-Jacobi-Bellman approach remains open because the unbounded nirvana-case value function lacks the usual differentiability requirements (see e.g. Gozzi and Russo (2006)). An alternative route is the use of duality-based solution methods. Since our financial market is complete, one would be tempted to employ the standard martingale method of Cox and Huang (1989). However, our setting does not meet the Cox and Huang (1989) price-system assumptions[1]. We overcome this difficulty by using the more general convex duality approach of Kramkov and Schachermayer (1999, 2003), which is expressly designed to solve optimal dynamic asset allocation problems for a very broad class of abitrage-free risky markets. Our closed-form optimal portfolio results contribute a significant example of the flexibility of such a general approach, thus introducing a useful technical toolkit to the financial-economics readership. Numerical martingale methods for optimal dynamic portfolio problems are discussed by e.g. Detemple, Garcia, and Rindisbacher (2003, 2005).

The paper is organized as follows. Section 2 details the features of the defaultable asset value. Section 3 shows that large long geared positions can be the rational outcome of the dynamic portfolio problem. Section 4 draws the conclusions and an Appendix collects the proofs of the propositions in Section 3.

2 The defaultable asset

Given a terminal investment date T $(0 < T < +\infty)$, there are essentially two properties we want the value process $(P_t)_{0 \leq t \leq T}$ of the risky defaultable asset to have: (i) it must be arbitrage-free; (ii) it must support an 'yield pickup' (a positive excess expected return that endures at times of subdued volatility). Property (i) is meant to rule out the possible emergence of extreme portfolios due to the existence of free lunches. Property (ii) is meant to bestow the asset with a glamor similar to the one defaultable assets seemed to possess in the years straddling the crisis.

The Cox and Ross (1976) value process parsimoniously meets the two properties of interest. Its

[1] Cox and Huang (1989) require the global Lipschitz continuity of the diffusive coefficient for the risky asset value process (see Conditions A and B at p. 46 in Cox and Huang (1989)). By contrast, the diffusive coefficient in our setting is the square root of the risky asset value.

dynamics is

$$dP_t = P_t\left(r + \sqrt{Y_t}\sigma_t\right)dt + P_t\sigma_t dZ_t, \quad \sigma_t \equiv P_t^{-\frac{1}{2}}, \quad P_0 = p \geq 0, \tag{2.1}$$

$$Y_t \equiv \left(\frac{\xi}{\sigma_t}\right)^2 = \xi^2 P_t \quad \text{(squared Sharpe ratio)}, \tag{2.2}$$

where the excess expected return on the asset is $\xi > 0$ and the riskfree rate is $r > 0$. $(Z_t)_{0\leq t\leq T}$ is a Wiener process under the objective probability measure $\mathbb{P}$. The riskless security with value $B_t = e^{rt}$ is the money market account. From the boundary classification, the point 0 is an attainable state for the process $(P_t)_{0\leq t\leq T}$. The point 0 is an an exit boundary[2] (bankruptcy) and is consistent with zero recovery at default[3].

Predictable default becomes possible only if the asset returns' local volatility inflates as the asset value deteriorates. Importantly, what makes predictable default possible also engenders what we refer to as the asset's 'yield pickup': the excess expected return ξ is positive and constant no matter how small, along upward asset-value paths, the local volatility $P^{-\frac{1}{2}}$ is. The Cox and Ross (1976) value process naturally possesses property (ii).

The Cox and Ross (1976) value process does enjoy property (i), as the following proposition maintains.

Proposition 2.1 *The value process with dynamics described by (2.1) complies with the no-arbitrage assumption. In particular, there exists a unique equivalent martingale measure, with Radon-Nikodym density*

$$\eta = \exp\left(-\int_0^T \sqrt{Y_s}dZ_s - \frac{1}{2}\int_0^T Y_s ds\right). \tag{2.3}$$

Proof. Theorem 2.3 in Delbaen and Shirakawa (2002) holds. ■

Under no arbitrage, predictable default risk balances the presence of bullish asset-value paths along which the Sharpe ratio $\sqrt{Y}$ bloats. If no free lunches are to emerge, predictable default risk must counteract the 'yield pickup' offered by the defaultable asset. This is borne out by the following proposition.

Proposition 2.2 *Consider the positive value process with dynamics described by (2.1), that is* $(P_t)_{0\leq t\leq T}$ *under the conditional objective probability measure* $\mathbb{P}[\ \cdot \mid P_T > 0\ ]$. *For such a positive value process there always exist arbitrage opportunities.*

[2] See for instance Davidov and Linetsky (2001), p. 952, first paragraph, with $S_t = P_t$ and $p = \frac{1}{2}$.

[3] The objective probability of the asset defaulting within the date $T > 0$ is $\mathbb{P}[P_h = 0,\ 0 \leq h \leq T \mid P_0 = p\ ] = \Gamma\left(\frac{2(r+\xi)p}{1-e^{-(r+\xi)T}}, 1\right)$, where $\Gamma(k,l) = \int_k^{+\infty} u^{l-1}e^{-u}du$, $k \geq 0$ is the incomplete gamma function (see e.g. the Proposition 1 in Campi and Sbuelz (2005)).

Proof. Theorem 4.2 of Delbaen and Shirakawa (2002) holds. ■

Weitzman (1998, 2009), Gollier (2002), and Martin (2012) show that the no-arbitrage value of a long-dated asset may be dictated by extreme outcomes. In our Cox and Ross (1976) setting the no-arbitrage value of the defaultable asset comes from the balance between the predictable default event and the extreme outcome represented by exploding Sharpe ratios. In the next section, we show that the fair pricing of such an extreme outcome does not deter rational aggressive investors with a sufficiently long investment maturity from massively gambling on it. The allure of the asset-value paths along which the Sharpe ratio $\sqrt{Y}$ inflates can become supreme for aggressive investors, making them 'rationally forget' about the balancing force of default risk.

3 A duality approach to dynamic asset allocation

The investor seeks to maximize the expected utility from his/her terminal wealth by allocating his/her capital to two assets, the riskfree asset and the defaultable asset introduced in Section 2. There is no intermediate consumption or income. The investor has Constant-Relative-Risk-Aversion (CRRA) utility from terminal wealth,

$$U(z) = \begin{cases} z^{1-\phi}/(1-\phi) & \text{for} \quad z > 0 \quad , \\ -\infty & \text{for} \quad z \leq 0 \quad , \end{cases}$$

where the level of relative risk aversion equals the parameter $\phi > 0$.

By construction (see Definition 2.2), the squared Sharpe ratio $(Y_t)_{0 \leq t \leq T}$ of the defaultable asset is always non-negative and has square-root-type dynamics deprived of mean-reversion,

$$dY_t = Y_t (r + \xi) \, dt + \xi \sqrt{Y_t} dZ_t \ , \qquad Y_0 = y \geq 0 \ . \tag{3.1}$$

The correlation between its innovations and the defaultable-asset value innovations is 1. The initial squared Sharpe ratio y supplies all the available information on the investment opportunities. Indeed, Nielsen and Vassalou (2006) show that, in typical continuous-time portfolio problems, the only time-variation that matters for portfolio choice is the time-variation in the slope (the Sharpe ratio) and the intercept (the riskfree rate) of the instantaneous capital market line.

Let $W = (W_t)_{0 \leq t \leq T}$ be the value process of a self-financing portfolio, given the the investor's initial wealth w. The discounted process is given by

$$\tilde{W}_t = e^{-rt} W_t = w + \int_0^t H_s d\tilde{P}_s$$

where $\tilde{P}_t = P_t e^{-rt}$ and the adapted portfolio quantity H_t represents the units of the defaultable asset held at time t. We call the strategy $H = (H_t)_{0 \leq t \leq T}$ admissible if there exists some constant $C > 0$ such that $\int_0^t H_s d\tilde{P}_s \geq -C$ almost surely for any $t \in [0, T]$. Admissibility rules out doubling strategies. The discounted monetary investment in the defaultable asset at time t is

$$\psi_t = H_t \tilde{P}_t.$$

We collect in the set $W(w)$ all the non-negative self-financing portfolios[4] with initial value w, and denote with $\tilde{\mathcal{W}}(w)$ the set of the corresponding discounted portfolios, namely

$$\tilde{W} \in \tilde{\mathcal{W}}(w) \qquad \Longleftrightarrow \qquad \tilde{W}_t = w + \int_0^t H_s d\tilde{P}_s \geq 0 \;\; \text{almost surely for any } t \in [0, T] \,.$$

Kramkov and Schachermayer (1999, 2003) consider the general problem of expected terminal-wealth utility maximization in a market where arbitrage-free asset prices are semimartingales. They solve it by employing a flexible duality approach based only on the finiteness of the value function and on the non-emptiness of the set of martingale measures, which is implied by the absence of arbitrage opportunities. The following proposition uses the duality approach of Kramkov and Schachermayer (1999, 2003) to express the investor's value function in our Cox and Ross (1976) setting[5]. The filtration representing the investor's information meets the usual assumptions, so that the conditional expectation at time $t = 0$ coincides with the unconditional expectation.

Proposition 3.1 *If $E\left[\eta^{1-\frac{1}{\phi}}\right] < +\infty$, the investor's value function is*

$$J(w, T, y) = \left(e^{rT}\right)^{1-\phi} \sup_{\tilde{W} \in \tilde{\mathcal{W}}(w)} E[U(\tilde{W}_T)]$$

and admits the representation

$$J(w, T, y) = U(we^{rT}) F(T, y) \,,$$

$$F(T, y) = \left(E\left[\exp\left(\frac{1-\phi}{\phi}\int_0^T \sqrt{Y_t} dZ_t + \frac{1-\phi}{2\phi}\int_0^T Y_t dt\right)\right]\right)^{\phi} \quad .$$

Proof. See the appendix. ■

[4]The non-negativity requirement is innocuous in our setting as the utility function U is $-\infty$ for negative wealth levels.

[5]Battauz, De Donno, and Sbuelz (2015) apply the Kramkov and Schachermayer (1999, 2003) approach to the standard Kim and Omberg (1996) portfolio problem with a non-defaultable risky asset.

$F(T,y)$ equals $\left(E\left[\eta^{1-\frac{1}{\phi}}\right]\right)^{\phi}$ and is the key ingredient of the value function as it summarizes the dependence on the state variable y of the investor's indirect utility. The next proposition makes sure that the assumption $E\left[\eta^{1-\frac{1}{\phi}}\right] < +\infty$ in Proposition 3.1 holds true for any risk-averse investor $(\phi > 0)$ and provides an explicit characterization of $F(T,y)$.

Proposition 3.2 *Assume*

$$T < T^{*} = \frac{1}{\sqrt{q}} \ln\left(\frac{b+\sqrt{q}}{b-\sqrt{q}}\right) \qquad \text{for} \qquad 0 < \phi < 1.$$

One has

$$F(T,y) = \exp\left(y \, \frac{a\left(e^{\sqrt{q}T}-1\right)}{\sqrt{q}+b+\left(\sqrt{q}-b\right)\, e^{\sqrt{q}T}} \right),$$

with

$$\begin{aligned} a &= \frac{1}{\phi}-1, \\ b &= (r+\xi)+a\xi = r+\frac{\xi}{\phi} > 0, \\ c &= \frac{\xi^2}{4\phi} > 0, \\ q &= b^2-4ac = r^2+\frac{1}{\phi}\left((r+\xi)^2-r^2\right) > 0. \end{aligned}$$

Proof. See the appendix. ■

Proposition 3.2 implies that, for investors less risk averse than the log-utility agent $(0 < \phi < 1)$, the value function $U\left(we^{rT}\right)F(T,Y)$ is a nirvana solution in the sense of Kim and Omberg (1996) since it diverges to $+\infty$ as the investment horizon T tends to the positive finite time T^{*} from the left. Notice that the value function for the investors with $0 < \phi < 1$ remains bounded if $T < T^{*}$, whereas the value function for the investors with $\phi > 1$ remains bounded for any T.

$F(T,y)$ enters the investor's marginal indirect utility of wealth and plays a major role in the optimal dynamic portfolio, which is qualified in detail by the following proposition.

Proposition 3.3 *The optimal monetary investment in the defaultable asset is* $\psi_0^{*} = \psi^{*}(w,T,y)$, *where*

$$\psi^{*}(w,T,y) = w\left(\frac{y}{\phi\xi}+\frac{\ln F(T,y)}{\phi}\right).$$

Proof. See the appendix. ■

Proposition 3.3 states that the optimal fraction of wealth invested in the defaultable asset contains two components. The first component,

$$\frac{y}{\phi\xi} \ ,$$

is the myopic demand for the defaultable asset. It is the allocation that an investor optimally holds if the investment maturity T shrinks to zero-the investor does not care about future investment opportunities. The second component,

$$\frac{\ln F(T,y)}{\phi} \ ,$$

is the intertemporal non-myopic demand (see for instance Merton (1971)). The reason the investor forms non-myopic demands is to deal optimally with changes in future investment opportunities.

Since an unexpected drop in the asset value implies a deterioration in the investment opportunities offered by the asset (the Sharpe ratio drops), a non-log-utility and sufficiently risk-averse investor hedges against such an adverse effect by shorting the asset to profit from the unexpected drop in the asset value[6]. The following proposition highlights a standard investment-horizon effect: the longer the maturity T is, the more energetic the hedging act becomes[7].

Proposition 3.4 *Given a conservative investor ($\phi > 1$), the non-myopic component*

$$\frac{w \ln F(T,y)}{\phi}$$

of $\psi^*(w,x,T)$ in Proposition 3.3 is negative and strictly decreasing in the investment maturity T.

Proof. If $\phi > 1$, then $a < 0$ and $\sqrt{q} > b$. ■

Conversely, when the investor is less averse to risk than the log-utility agent, optimal non-myopic speculation on the Sharpe-ratio risk ensues.

Proposition 3.5 *Given an aggressive investor ($0 < \phi < 1$), the non-myopic component*

$$\frac{w \ln F(T,y)}{\phi}$$

of $\psi^*(w,x,T)$ in Proposition 3.3 is positive and strictly increasing in the investment maturity T.

[6] There are parameter values that make the investor with $\phi > 1$ take a net short position in the risky defaultable asset.

[7] Similar investment-horizon effects have been found in the literature on dynamic portfolio choice with a risky non-defaultable asset characterized by a mean-reverting drift and a constant volatility (see for example Koijen, Rodriguez, and Sbuelz (2009)).

Proof. If $0 < \phi < 1$, then $a > 0$ and $\sqrt{q} < b$. ■

Proposition 3.5 shows that the aggressive investor non-myopically speculates on Sharpe-ratio risk by buying the defaultable asset, in contrast with the conservative investor who non-myopically hedges against Sharpe-ratio risk by shorting the asset. Importantly, the conservative investor's optimal non-myopic demand remains bounded in the investment maturity T, whereas this is not the case for $0 < \phi < 1$. The aggressive investor uses the defaultable asset to seek profits in the states that come with a high productivity of wealth, that is, with high Sharpe ratios. Hence, it is optimal for the aggressive investor to make a non-myopic gamble on the upward asset-value paths to unfold before his/her investment maturity T (upward asset-value paths go along with swelling Sharpe ratios). If T stretches out, non-myopic gambling becomes increasingly hefty and levered: the bet has more time to succeed.

[Figures 1 and 2 about here]

Figures 1 and 2 visualize the optimal portfolio results in Proposition 3.5 for an aggressive investor with a coefficient of relative risk aversion of $\phi = 0.5$. Given a riskfree rate of 1% and a risk premium of 4%, a 15-year-horizon investor borrows to invest in the defaultable asset 172% of his/her initial capital when the Sharpe ratio $\sqrt{y}$ is about 14% (100% is the myopic allocation), as shown by the left-hand panel of Figure 1. He/she assigns 86% of his/her capital to the defaultable asset when $\sqrt{y}$ is 10% (50% is the myopic allocation).

The right-hand panel of Figure 1 illustrates the impact of a higher riskfree rate ($r = 4\%$). There is a drift effect, which is the drop in the probability of default caused by a higher drift $r + \xi$. The drift effect results stronger and non-myopic risk taking is encouraged. The 15-year-horizon investor allocates 200% of his/her initial capital to the defaultable asset when $\sqrt{y}$ is about 14%. Figure 2 shows the optimal portfolio impact of changing the risk premium (r is fixed at 1%). An increase in ξ fosters non-myopic risk taking more than an increase in r does. There is a stronger drift effect that decidedly strengthens the incentive of placing the non-myopic bet on the upward asset-value paths. A higher ξ tends to lift every Sharpe-ratio path while shrinking the probability of default. Given $\xi = 5\%$, the 15-year-horizon investor devotes 178% of his/her initial capital to the defaultable asset when $\sqrt{y}$ is about 14% (80% is the myopic allocation).

Figures 1 and 2 show that a surging market for the defaultable asset does entice the aggressive investor via his/her optimal tactical and strategic market timing. A mounting Sharpe ratio pushes up quadratically the optimal total exposure to the defaultable asset. Importantly such an optimal market-timing activity involves also the conservative investors, as it can be seen in Figures 3 and 4 for which the coefficient of relative risk aversion is $\phi = 3$. Consistently with the observed reaching-for-yield behavior, our optimal portfolio results imply that aggressive as well as conservative investors increase their optimal exposure to the defaultable asset at times of low volatility and of surging market value.

[Figures 3 and 4 about here]

Figures 5 and 6 show the optimal non-myopic exposure to predictable default risk vis-à-vis the myopic benchmark, as the ratio $\frac{\psi^*(w,T,y)}{\psi^*(w,0,y)}$ is plotted against the relative risk aversion parameter ϕ for a 10-year-horizon investor. The figures confirm our discussion of the optimal portfolios choosed by non-log-utility, non-myopic investors.

[Figures 5 and 6 about here]

4 Conclusions

There is extensive evidence that, before the 2007-2008 crisis and in its aftermath over the following years, financial institutions have been embarking in reaching-for-yield behavior by taking conspicuous long positions in defaultable assets. What does optimal dynamic portfolio theory have to say about the search for yield accomplished by means of defaultable assets? Our answer to this important question is a novel closed-form optimal portfolio analysis of non-myopic speculation on predictable default risk.

Our paper takes inspiration from the search for yield literature. We employ optimal dynamic portfolio theory to draw attention to a possible reason why long-term financial intermediaries like banks may take on major levered reaching-for-yield acts that load predictable default risk. Our analysis is related to the debate on the sources of banks' risk taking and search for yield as we highlight that banks' long investment horizon may largely amplify the optimal portfolio exposure to default risk. On the other hand, our optimal portfolio analysis goes towards offering a market-timing rationale for the search for yield observed even among less aggressive investors like money funds and insurance companies.

Our analysis of optimal dynamic portfolio exposure to predictable default risk contributes to the dynamic asset allocation literature that deals with Sharpe-ratio risk. We prove the existence of nirvana optimal-portfolio solutions in the case of a defaultable risky asset and of a non-stationary non-Gaussian Sharpe ratio. We point out that frail/absent mean reversion makes the unfolding of paths with swelling Sharpe ratios a plausible event to punt on for non-myopic aggressive investors.

Our paper also contributes to the literature on the solution methods for optimal dynamic portfolio problems. Our technical focus is the explicit solution of an optimal dynamic asset allocation problem in an abitrage-free defaultable risky market. As the viability of the Hamilton-Jacobi-Bellman and Cox and Huang (1989) approaches is impaired in our setting, we use the convex duality theory of Kramkov and Schachermayer (1999, 2003) who aim at solving optimal dynamic portfolio problems in very general arbitrage-free risky markets. Our closed-form optimal portfolio findings are a significant illustration of how flexible and effective the Kramkov and Schachermayer (1999, 2003) approach is, thus presenting a valuable technical toolkit to the financial-economics readership.

In this paper we have sought the most parsimonious model to study the non-myopic bets on predictable default risk made by rational aggressive investors and to examine the market-timing portfolio decisions expressed by them and by more conservative investors. We expect the introduction of an additional non-defaultable risky asset (say a lognormal security) to leave our core results unchanged. Aggressive investors are very likely to keep making non-myopic bets on the only asset that offers the default-compensating prospect of a rising Sharpe ratio. However, a careful study of the optimal portfolio composition and of its time variation in a multiple risky asset setting are surely interesting avenues of future research.

Acknowledgements. We would like to thank the editors and two anonymous referees for their insightful comments and suggestions.

REFERENCES

Battauz, A., De Donno, M., and A. Sbuelz, 2015, Kim and Omberg Revisited: The Duality Approach. Journal of Probability and Statistics, Volume 2015, Article ID 581854, 6 pages, http://dx.doi.org/10.1155/2015/581854.

Becker, B., and V. Ivashina, 2015, Reaching for yield in the bond market, Journal of Finance, 70, 1863-1902.

Bekaert, G., and A. Ang, 2002, International Asset Allocation with Regime Shifts, Review of Financial Studies, 15(4), 1137-1187.

Branger, N., Larsen, L.S., and C. Munk, 2013, Robust portfolio choice with ambiguity and learning about return predictability, Journal of Banking and Finance, 37 (5), 1397-1411.

Buch, C., Eickmeier, S., and E. Prieto, 2014, Macroeconomic Factors and Microlevel Bank Behavior, Journal of Money, Credit and Banking, 46(4), 715–751.

Campi, L. and A. Sbuelz, 2005, Closed-Form Pricing of Benchmark Equity Default Swaps under The CEV Assumption, Risk Letters, vol. 1, ISSN: 1740-9551.

Chodorow-Reich, G., 2014, Effects of unconventional monetary policy on financial institutions, Brookings Papers on Economic Activity (Spring), 155–204

Cox, J.C. and C.F. Huang, 1989. Optimal Consumption and Portfolio Policies when Asset Prices Follow a Diffusion Process, Journal of Economic Theory, 49,33-83.

Cox, J. C. and S. Ross, 1976, The Valuation of Options for Alternative Stochastic Processes, Journal of Financial Economics, 3, 145-166.

Cvitanic, J., I. Karatzas, 1992, Convex duality in constrained portfolio optimization, Annals of Applied Probabability, 2, 767–818.

Cvitanić, J., V. Polimenis, and F. Zapatero, 2008, Optimal Portfolio Allocation with Higher Moments, Annals of Finance, 4(1), 1-28.

Delbaen, F., and H. Shirakawa, 2002, A Note on Option Pricing for the Constant Elasticity of Variance Model, Financial Engineering and the Japanese Markets, 9(2), 85-99.

Della Corte, P., Sarno, L., and I. Tsiakas, 2012, Volatility and Correlation Timing in Active Currency Management, Handbook of Exchange Rates, edited by J. James, I. Marsh, and L. Sarno, John Wiley & Sons.

Dell'Ariccia, G., Laeven, L., and R. Marquez, 2014, Monetary policy, leverage, and bank risk-taking, Journal of Economic Theory 149, 65-99.

Detemple, J.B., Garcia, R., and M. Rindisbacher, 2003, A Monte-Carlo method for optimal portfolios, Journal of Finance, 58, 401–446.

Detemple, J.B., Garcia, R., and M. Rindisbacher, 2005, Intertemporal asset allocation: A comparison of methods, Journal of Banking and Finance, 29(11), 2821-2848.

Diamond, D. W., and R. G. Rajan, 2009, The Credit Crisis: Conjectures about Causes and Remedies, American Economic Review, 99(2), 606-610.

Di Maggio, M., 2013, Market Turmoil and Destabilizing Speculation, Columbia Working Paper.

Di Maggio, M., and M.T. Kacperczyk, 2016, The unintended consequences of the zero lower bound policy, Journal of Financial Economics, forthcoming.

Gennaioli, N., Martin, A., and S. Rossi, 2014, Sovereign Default, Domestic Banks, and Financial Institutions, Journal of Finance, 69(2), 819-866.

Gollier, C., 2002, Discounting an Uncertain Future, Journal of Public Economics, 85, 149-166.

Gozzi, F., and F. Russo, 2006, Verification theorems for stochastic optimal control problems via a time dependent Fukushima–Dirichlet decomposition, Stochastic Processes and their Applications, 116(11), 1530-1562.

Guidolin, M., and S. Hyde, 2012, Can VAR models capture regime shifts in asset returns? A long-horizon strategic asset allocation perspective, Journal of Banking and Finance, 36(3), 695-716.

Guidolin, M., and A. Timmermann, 2007, Asset allocation under multivariate regime switching, Journal of Economic Dynamics and Control, 31, 3503–3544.

Jiménez G., Ongena, S., Peydró, J.L., and J. Saurina, 2014, Hazardous times for monetary policy: What do twenty-three million bank loans say about the effects of monetary policy on credit risk?, Econometrica, 82(2), 463-505.

Kacperczyk, M., and P. Schnabl, 2013, How Safe are Money Market Funds?, Quarterly Journal of Economics 128, 1073-1122.

Kim, T. S., and E. Omberg, 1996, Dynamic Nonmyopic Portfolio Behavior, Review of Financial Studies, 9, 141–161.

Klebaner, F. and R. Lipster, 2014, When a Stochastic Exponential is a True Martingale. Extension of a Method of Benes, *Theory of Probability and its Applications*, Vol. 58, No. 1, 38–62.

Koijen, R.S.J., Rodríguez, J.C., and A. Sbuelz, 2009, Momentum and Mean Reversion in Strategic Asset Allocation, Management Science 55, 1199-1213.

Kramkov, D. and W. Schachermayer, 1999, The Asymptotic Elasticity of Utility functions and Optimal Investment in Incomplete Market, Annals of Applied Probability, 9 (3), 904–950.

Kramkov, D. and W. Schachermayer, 2003, Necessary and sufficient confitions in the problem of optimal investment in incomplete markets, Annals of Applied Probability, 13 (4), 1504–1516.

Larsen, L.S., and C. Munk, 2012, The costs of suboptimal dynamic asset allocation: General results and applications to interest rate risk, stock volatility risk, and growth/value tilts, Journal of Economic Dynamics & Control 36 (2012) 266–293.

Lioui, A., and P. Poncet, 2003, International asset allocation: A new perspective, Journal of Banking and Finance, 27(11), 2203-2230.

Liu, J., 2007, Portfolio selection in stochastic environments, Review of Financial Studies 20, 1–39.

Martin, I., 2012, On the Valuation of Long-Dated Assets, Journal of Political Economy, 120(2), 346-358.

Merton, R. C., 1971, Optimum Consumption and Portfolio Rules in a Continuous-Time Model, Journal of Economic Theory, 3, 373–413.

Nielsen, L. T., and M. Vassalou, 2006, The instantaneous capital market line, Economic Theory 28, 651–664.

Rajan, R., 2005, “Has Financial Development Made the World Riskier?” Proceedings, Federal Reserve Bank of Kansas City, August, pp. 313-69.

Schachermayer, W., 2001, Optimal investment in incomplete markets when wealth may become negative, Annals of Applied Probability, 11, 694–734.

Vasso, I., Ongena, S., and J.L. Peydró, 2015, Monetary policy, risk-taking and pricing: Evidence from a quasi-natural experiment, Review of Finance, 19 (1), 95-144.

Wachter, J., 2002, Portfolio and consumption decisions under mean-reverting returns: An exact solution for complete markets, Journal of Financial and Quantitative Analysis, 37, 63–91.

Weitzman, M. L., 1998, Why the Far-Distant Future Should Be Discounted At Its Lowest Possible Rate, Journal of Environmental Economics and Management, 36, 201-208.

Weitzman, M. L., 2009, On Modeling and Interpreting the Economics of Catastrophic Climate Change, Review of Economics and Statistics, 91, 1-19.

A Appendix

A.1 Proof of Proposition 3.1

The problem of utility maximization can be written as $J(w) = \left(e^{rT}\right)^{1-\phi} u(w)$ where u is defined as

$$u(w) = \sup_{\tilde{W} \in \tilde{\mathcal{W}}(w)} E[U(\tilde{W}_T)]. \tag{A.1}$$

We apply to problem $(A.1)$ the duality approach developed by Kramkov and Schachermayer (1999, 2003). To this aim, we observe that the utility function U satisfies Inada conditions (equation (2.4) in Kramkov and Schachermayer (1999)). Let V denote the conjugate function[8] of U, that is $V(y) = \frac{\phi}{1-\phi} y^{-\frac{1-\phi}{\phi}}$, and define

$$v(y) = E\left[V\left(y\eta\right)\right]$$

where η is given by (2.3). Kramkov and Schachermayer (2003) show that if $v(y) < \infty$ for all $y > 0$, then $u(w) < \infty$ for all $w > 0$ and u and v are conjugate. They also prove that the optimal solution $\tilde{W}^* \in \tilde{W}(w)$ to (A.1) exists and is unique. Moreover, taking $y = u'(w)$ (or equivalently $w = -v'(y)$), they provide the dual relation for the optimizer $\tilde{W}^* = -V'(y\eta)$ (see Theorems 1,2 and Note 3).

Assuming that $E\left[\eta^{-\frac{1-\phi}{\phi}}\right] < +\infty$, from the condition $w = -v'(y)$, we get

$$y = \left(\frac{w}{E\left[\eta^{-\frac{1-\phi}{\phi}}\right]}\right)^{-\phi}$$

and

$$\begin{aligned} \tilde{W}^* = -V'(y\eta) &= \left(\left(\frac{w}{E\left[\eta^{-\frac{1-\phi}{\phi}}\right]}\right)^{-\phi} \eta\right)^{-\frac{1}{\phi}} \\ &= w\frac{\eta^{-\frac{1}{\phi}}}{E\left[\eta^{-\frac{1-\phi}{\phi}}\right]}. \end{aligned}$$

[8] The functions U and V are conjugate if and only if $U(w) = \inf_{y>0}(V(y) + wy)$ and $V(y) = \sup_{w>0}(U(w) - wy)$.

The value function is then

$$J(w) = \left(e^{rT}\right)^{1-\phi} E\left[U(\tilde{W}^*)\right] = U(we^{rT})E\left[\left(\frac{\eta^{-\frac{1}{\phi}}}{E\left[\eta^{-\frac{1-\phi}{\phi}}\right]}\right)^{1-\phi}\right]$$
$$= U(we^{rT})\left(E\left[\eta^{-\frac{1-\phi}{\phi}}\right]\right)^{\phi}$$
$$= U(we^{rT})\left(E\left[\exp\left(\frac{1-\phi}{\phi}\int_0^T \sqrt{Y_t}dZ_t + \frac{1-\phi}{2\phi}\int_0^T Y_t dt\right)\right]\right)^{\phi}$$
$$= U(we^{rT})F(T,y). \qquad \square$$

A.2 Proof of Proposition 3.2

Since $a = \frac{1-\phi}{\phi}$, then

$$E\left[\exp\left(\frac{1-\phi}{\phi}\int_0^T \sqrt{Y_t}dZ_t + \frac{1-\phi}{2\phi}\int_0^T Y_t dt\right)\right] = E\left[\exp\left(a\int_0^T \sqrt{Y_t}dZ_t + \frac{a}{2}\int_0^T Y_t dt\right)\right]$$

We can write:

$$E\left[\exp\left(a\int_0^T \sqrt{Y_t}dZ_t + \frac{a}{2}\int_0^T Y_t dt\right)\right] = E\left[L_T \exp\left(\frac{a^2+a}{2}\int_0^T Y_t dt\right)\right]$$

where

$$L_T = \exp\left(a\int_0^T \sqrt{Y_t}dZ_t - \frac{a^2}{2}\int_0^T Y_t dt\right). \tag{A.2}$$

The random variable L_T in $(A.2)$ is the Radon-Nikodym density of a probability measure equivalent to $\mathbb{P}$. In fact, Theorem[9] 2.3 in Delbaen and Shirakawa (2002) applied to $S^{DS} = Y$, $\rho^{DS} = 0.5$, $r^{DS} = b > 0$, $\sigma^{DS} = 2\sqrt{c\phi}$, $\mu^{DS} = b + 2a\sqrt{c\phi}$, and $\theta^{DS} = a$ implies that $\eta_T^{DS} = L_T$ is the Radon-Nikodym density of an equivalent probability measure $\hat{\mathbb{Q}}$ equivalent to $\mathbb{P}$. Girsanov's theorem implies then that

$$\hat{Z}_t = Z_t - \int_0^t a\sqrt{Y_s}ds \tag{A.3}$$

is a $\hat{\mathbb{Q}}-$Brownian motion. Thus

$$F(T,y) = \left(E^{\hat{\mathbb{Q}}}\left[\exp\left(\frac{a^2+a}{2}\int_0^T Y_t dt\right)\right]\right)^{\phi}, \tag{A.4}$$

where Y_t has the following dynamics under $\hat{\mathbb{Q}}$

$$dY_t = bY_t dt + 2\sqrt{c\phi Y_t}d\hat{Z}_t, \tag{A.5}$$

[9]The superscript $(\cdot)^{DS}$ refers to the notations of Delbaen and Shirakawa, 2002, whereas a, b, c are defined in the statement of Proposition 3.2.

with $Y_0 = y$. We specify that, if we define the default time $\tau_y = \inf\{t \geq 0 : Y_t = 0\}$, we have $Y_t \equiv 0$ on $\{\tau_y \leq t\}$ and Y satisfies the stochastic differential equation (A.5) on the whole time interval $[0, T]$. To compute the expectation in $(A.4)$, we define the process

$$M_t = \exp\left(\frac{a^2+a}{2}\int_0^t Y_s ds\right) G(T-t, Y_t) \tag{A.6}$$

where $G(t, y)$ is a $\mathcal{C}^{1,2}$ function to be determined in such a way that $G(0, y) = 1$ and M is a $\hat{\mathbb{Q}}$-martingale. In particular, equation $(A.6)$ implies for $t = 0$ that $M_0 = G(T, y)$ and for $t = T$

$$M_T = \exp\left(\frac{a^2+a}{2}\int_0^T Y_t dt\right) G(0, Y_T) = \exp\left(\frac{a^2+a}{2}\int_0^T Y_t dt\right),$$

since $G(0, \cdot) = 1$. The martingality condition $M_0 = E^{\hat{\mathbb{Q}}}[M_T]$ yields then

$$G(T, y) = E^{\hat{\mathbb{Q}}}\left[\exp\left(\frac{a^2+a}{2}\int_0^T Y_s dt\right)\right],$$

that allows us to find $(A.4)$. By imposing 0 drift on the Ito decomposition under $\hat{\mathbb{Q}}$ of the process M_t we get the partial differential equation for G

$$\begin{cases} G_t = \frac{\xi^2}{2} y G_{yy} + by G_y + \frac{a^2+a}{2} y G \\ G(0, y) = 1. \end{cases} \tag{A.7}$$

We guess a solution of the form $G(t, y) = e^{yg(t)}$ and we obtain the following differential equation for g:

$$\begin{cases} g'(t) = \frac{\xi^2}{2} g^2(t) + bg(t) + \frac{a^2+a}{2} \\ g(0) = 0. \end{cases} \tag{A.8}$$

Equation $(A.8)$ is a Riccati equation whose solution is

$$g(t) = \frac{(a^2+a)(e^{\sqrt{q}t} - 1)}{\sqrt{q} + b + e^{\sqrt{q}t}\left(\sqrt{q} - b\right)}.$$

Since M_t has 0 drift in the Ito decomposition under $\hat{\mathbb{Q}}$, M_t is a $\hat{\mathbb{Q}}$ local martingale. To conclude that M_t is a martingale we define

$$\mathfrak{z}_t = \frac{M_t}{M_0},$$

which is a $\hat{\mathbb{Q}}$ *local* martingale as well, and show that $\mathfrak{z}_t$ is a $\hat{\mathbb{Q}}$ martingale. To this aim, we first observe that process $\mathfrak{z}_t$ is a stochastic exponential. In fact Ito formula implies that

$$dM_t = e^{\frac{a^2+a}{2}\int_0^t Y_s ds} \frac{\partial}{\partial y} G(T-t, Y_t) 2\sqrt{c\phi Y_t} d\hat{Z}_t$$

from the dynamics of Y with respect to $\hat{\mathbb{Q}}$ in Equation $(A.5)$. Since $\frac{\partial}{\partial y}G(T-t,Y_t) = e^{Y_t g(T-t)} \cdot g(T-t) = G(T-t,Y_t)\cdot g(T-t)$, we obtain

$$dM_t = e^{\frac{a^2+a}{2}\int_0^t Y_s ds} G(T-t,Y_t)\cdot g(T-t) 2\sqrt{c\phi Y_t} d\hat{Z}_t$$

leading to

$$dM_t = 2\sqrt{c\phi Y_t} M_t g(T-t) d\hat{Z}_t. \tag{A.9}$$

Therefore

$$\mathfrak{z}_t = 1 + \int_0^t \mathfrak{z}_s dm_s$$

with

$$m_t = \int_0^t 2\sqrt{c\phi} g(T-s)\sqrt{Y_s} d\hat{Z}_s.$$

We apply Theorem 4.1 in Klebaner and Lipster (2014) to conclude that $\mathfrak{z}_t$ is a true martingale. In particular with Klebaner and Lipster notations (4.2) at page 44

$$\begin{aligned} &a_s(y) = by \qquad b_s(y) = 2\sqrt{c\phi}\sqrt{y} && \text{from (A.5)} \qquad \text{and} \\ &\sigma_s(y) = 2\sqrt{c\phi} g(T-s)\sqrt{y} && \text{from our def. of } m_t, \end{aligned}$$

we get

$$\begin{aligned} L_s(y) &= 2y a_s(y) + (b_s(y))^2 \\ &= 2by^2 + 4c\phi y \\ \mathcal{L}_s(y) &= 2y\left[a_s(y) + b_s(y)\sigma_s(y) + (b_s(y))^2\right] \\ &= 2by^2 + 8c\phi y^2 g(T-s) + 8c\phi y^2 \end{aligned}$$

Since g is bounded, it follows that $(\sigma_s(y))^2$, $L_s(y)$, and $\mathcal{L}_s(y)$ are all dominated by a quadratic polynomial in y, and therefore assumptions (1)-(2)-(3) of Theorem 4.1 are satisfied. This allows us to conclude that $\mathfrak{z}_t = \frac{M_t}{M_0}$ is a martingale and therefore M_t is a martingale as well.Hence we can write (A.4) as

$$\begin{aligned} F(T,y) &= \left(\exp\left(y\frac{(a^2+a)(e^{\sqrt{q}T}-1)}{\sqrt{q}+b+e^{\sqrt{q}T}\left(\sqrt{q}-b\right)}\right)\right)^{\phi} \\ &= \exp\left(y\frac{a(e^{\sqrt{q}T}-1)}{\sqrt{q}+b+e^{\sqrt{q}T}\left(\sqrt{q}-b\right)}\right), \end{aligned}$$

since

$$\phi(a^2+a) = a. \qquad \square$$

A.3 Proof of Proposition 3.3

In what follows, we will mainly work under the martingale measure $\mathbb{Q}$, whose density with respect to $\mathbb{P}$ is η in Equation (2.3). We denote with $Z_t^{\mathbb{Q}}$ the $\mathbb{Q}$-brownian motion

$$Z_t^{\mathbb{Q}} = Z_t + \int_0^t \sqrt{Y_s}ds. \tag{A.10}$$

Before proving the result, we first list some technical lemmas.

Lemma A.1 *Let $L^* = \frac{L_T}{\eta}$ where L_T is given by* (A.2)*. Then $L_t^* = E^{\mathbb{Q}}[L^*|\mathcal{F}_t]$ satisfies the stochastic differential equation*

$$dL_t^* = (a+1)L_t^*\sqrt{Y_t}dZ_t^{\mathbb{Q}} = \frac{1}{\phi}L_t^*\sqrt{Y_t}dZ_t^{\mathbb{Q}} \tag{A.11}$$

with the initial condition $L_0^* = 1$. In particular, L^* is the Radon-Nikodym density of the probability measure $\hat{\mathbb{Q}}$ (whose density with respect to $\mathbb{P}$ is L in $(A.2)$) with respect to $\mathbb{Q}$ and $\hat{Z}_t$ defined in $(A.3)$ can be written as

$$\hat{Z}_t = Z_t^{\mathbb{Q}} - \frac{1}{\phi}\int_0^t \sqrt{Y_s}ds.$$

Proof. It is easy to observe that

$$L^* = \frac{L_T}{\eta} = \frac{\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}}{\frac{d\mathbb{Q}}{d\mathbb{P}}} = \frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}},$$

and from the definitions of η in (2.3) and of L in $(A.2)$ that

$$\begin{aligned} L_T^* = \frac{L_T}{\eta} &= \exp\left(a\int_0^T \sqrt{Y_s}dZ_s - \frac{a^2}{2}\int_0^T Y_s ds + \int_0^T \sqrt{Y_s}dZ_s + \frac{1}{2}\int_0^T Y_s ds\right) \\ &= \exp\left((a+1)\int_0^T \sqrt{Y_s}dZ_s + \frac{1-a^2}{2}\int_0^T Y_s ds\right). \end{aligned}$$

From the definition of $Z_s^{\mathbb{Q}}$ in Equation $(A.10)$ we get

$$\begin{aligned} L_T^* &= \exp\left((a+1)\int_0^T \sqrt{Y_s}\left(dZ_s^{\mathbb{Q}} - \sqrt{Y_s}ds\right) + \frac{1-a^2}{2}\int_0^T Y_s ds\right) \\ &= \exp\left((a+1)\int_0^T \sqrt{Y_s}dZ_s^{\mathbb{Q}} - \frac{(a+1)^2}{2}\int_0^T Y_s ds\right). \end{aligned}$$

This is equivalent to

$$dL_t^* = (a+1)L_t^*\sqrt{Y_t}dZ_t^{\mathbb{Q}}.$$

Moreover, from the definition of $\hat{Z}_t$ in $(A.3)$ we get

$$\begin{aligned}\hat{Z}_t &= Z_t - \int_0^t a\sqrt{Y_s}ds \\ &= Z_t^{\mathbb{Q}} - \int_0^t \sqrt{Y_s}ds - \int_0^t a\sqrt{Y_s}ds \qquad \text{(from } (A.10) \text{)} \\ &= Z_t^{\mathbb{Q}} - (a+1)\int_0^t \sqrt{Y_s}ds \qquad \text{(with } a+1 = \frac{1}{\phi} \text{),}\end{aligned}$$

that proves the lemma.

Lemma A.2 *Let M_t be the $\hat{\mathbb{Q}}$-martingale defined in* (A.6)*, namely*

$$M_t = e^{\frac{a^2+a}{2}\int_0^t Y_s ds} e^{Y_t g(T-t)}.$$

Then we have

$$\begin{aligned}dM_t &= 2\sqrt{c\phi Y_t} M_t g(T-t) d\hat{Z}_t \\ &= 2\sqrt{c\phi Y_t} M_t g(T-t) \left(dZ_t^{\mathbb{Q}} - \frac{1}{\phi}\sqrt{Y_t}dt \right)\end{aligned} \tag{A.12}$$

where $\hat{Z}_t$ is defined in $(A.3)$ and $Z_t^{\mathbb{Q}}$ in Equation $(A.10)$.

Proof of the Lemma. The first line in equation $(A.12)$ is equation $(A.9)$, that leads to $(A.12)$ by recalling that

$$\hat{Z}_t = Z_t^{\mathbb{Q}} - \frac{1}{\phi}\int_0^t \sqrt{Y_s}ds. \qquad \square$$

Proof of Proposition 3.3. The discounted optimizer

$$\tilde{W}^* = w\frac{\eta^{-\frac{1}{\phi}}}{E\left[\eta^{-\frac{1-\phi}{\phi}}\right]}$$

is the value at time T of a self-financing discounted portfolio, which admits the following representation under $\mathbb{Q}$

$$\tilde{W}^* = w + \int_0^T \psi_t^* \frac{d\tilde{P}_t}{\tilde{P}_t} = w + \int_0^T \frac{\xi\psi_t^*}{\sqrt{Y_t}} dZ_t^{\mathbb{Q}} \tag{A.13}$$

since $d\tilde{P}_t = \tilde{P}_t\sqrt{Y_t}\sigma_t dt + \tilde{P}_t\sigma_t dZ_t = \tilde{P}_t\frac{\xi}{\sqrt{Y_t}}dZ_t^{\mathbb{Q}}$ from (2.1) and $(A.10)$.

Therefore we look for the Ito representation of $\tilde{W}^*(t) = E^{\mathbb{Q}}[\tilde{W}^*|\mathcal{F}_t]$ to derive ψ^*. Denoting with $L_t = E\left[L_T\middle|\mathcal{F}_t\right]$, and $\eta_t = E\left[\eta\middle|\mathcal{F}_t\right]$, we have

$$
\begin{aligned}
E^{\mathbb{Q}}\left[\eta^{-\frac{1}{\phi}}\middle|\mathcal{F}_t\right] &= \frac{E\left[\eta^{1-\frac{1}{\phi}}\middle|\mathcal{F}_t\right]}{\eta_t} \qquad \text{(by Bayes' rule)}\\
&= \frac{E\left[\exp\left(a\int_0^T \sqrt{Y_t}dZ_t + \frac{a}{2}\int_0^T Y_t dt\right)\middle|\mathcal{F}_t\right]}{\eta_t}\\
&= \frac{E\left[L_T \exp\left(\frac{a^2+a}{2}\int_0^T Y_t dt\right)\middle|\mathcal{F}_t\right]}{\eta_t} \qquad \text{(by } (A.2) \text{)}\\
&= \frac{L_t E^{\hat{\mathbb{Q}}}\left[\exp\left(\frac{a^2+a}{2}\int_0^T Y_t dt\right)\middle|\mathcal{F}_t\right]}{\eta_t} \qquad \text{(by Bayes' rule)}\\
&= L_t^* E^{\hat{\mathbb{Q}}}\left[\exp\left(\frac{a^2+a}{2}\int_0^T Y_t dt\right)\middle|\mathcal{F}_t\right]\\
&= L_t^* E^{\hat{\mathbb{Q}}}\left[M_T\middle|\mathcal{F}_t\right] = L_t^* M_t \qquad \text{(by the definition of } M \text{ in } (A.6) \text{).}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\tilde{W}_t^* &= E^{\mathbb{Q}}[\tilde{W}^*|\mathcal{F}_t] =\\
&= E^{\mathbb{Q}}\left[w\frac{\eta^{-\frac{1}{\phi}}}{E\left[\eta^{-\frac{1-\phi}{\phi}}\right]}\middle|\mathcal{F}_t\right]\\
&= \frac{w}{E\left[\eta^{-\frac{1-\phi}{\phi}}\right]}E^{\mathbb{Q}}\left[\eta^{-\frac{1}{\phi}}\middle|\mathcal{F}_t\right]\\
&= \frac{w}{E\left[\eta^{-\frac{1-\phi}{\phi}}\right]}L_t^* M_t =\\
&= \frac{w}{G(T,y)}L_t^* M_t
\end{aligned}
$$

because $G(T,y) = (F(T,y))^{\frac{1}{\phi}} = E\left[\eta^{-\frac{1-\phi}{\phi}}\right]$. It follows that the differential of the $\mathbb{Q}-$martingale $\tilde{W}_t^*$ is given by

$$
\begin{aligned}
d\tilde{W}_t^* &= \frac{w}{G(T,y)}d\left(L_t^* M_t\right)\\
&= \frac{w}{G(T,y)}L_t^* M_t\left[\frac{1}{\phi} + 2\sqrt{c\phi}g(T-t)\right]\sqrt{Y_t}dZ_t^{\mathbb{Q}} \qquad \text{by } (A.11) \text{ and } (A.12)\\
&= \tilde{W}_t^*\left[\frac{1}{\phi} + 2\sqrt{c\phi}g(T-t)\right]\sqrt{Y_t}dZ_t^{\mathbb{Q}}
\end{aligned}
$$

Comparing this equation with equation (A.13), we obtain

$$\frac{\xi \psi_t^*}{\sqrt{Y_t}} = W^*(t) \left[\frac{1}{\phi} + 2\sqrt{c\phi} g(T-t) \right] \sqrt{Y_t}$$

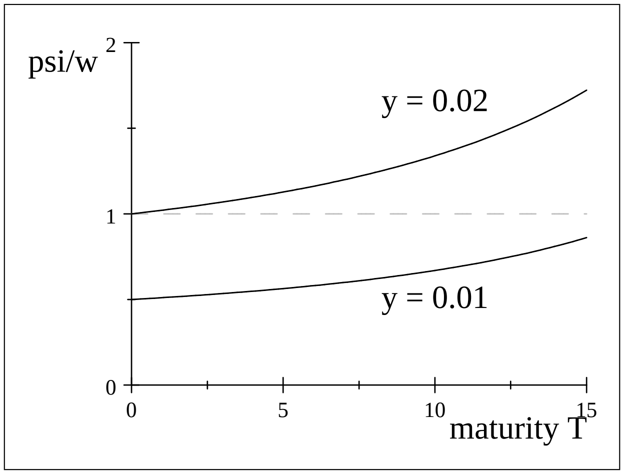
hence, recalling that $2\sqrt{c\phi} = \xi$ and $g(T-t) = \frac{1}{Y_t} \ln G(T-t, Y_t) = \frac{1}{\phi Y_t} \ln F(T-t, Y_t)$, we have:

$$\begin{aligned} \psi_t^* &= \frac{\tilde{W}_t^*}{\xi} \left[\frac{1}{\phi} + 2\sqrt{c\phi} g(T-t) \right] Y_t \\ &= \tilde{W}_t^* \left[\frac{Y_t}{\phi \xi} + \frac{\ln F(T-t, Y_t)}{\phi} \right]. \end{aligned}$$

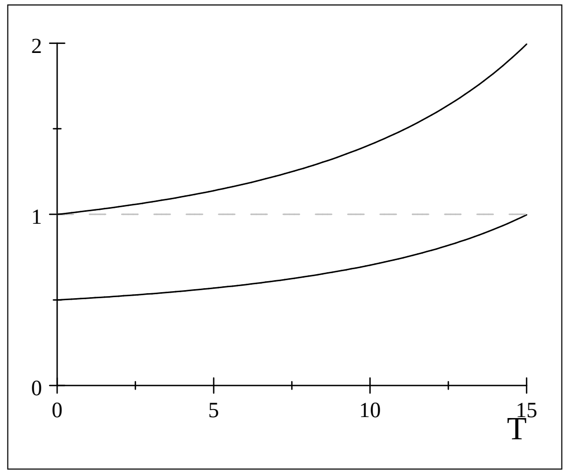
In particular, at $t = 0$ we obtain $\psi_0^* = w \left[\frac{y}{\phi \xi} + \frac{\ln F(T,y)}{\phi} \right]$, as in the statement of the proposition.

Figure 1

Optimal fractional investment ψ_0^*/w in the defaultable asset versus T $(\phi = 0.5, \quad \xi = 4\%)$



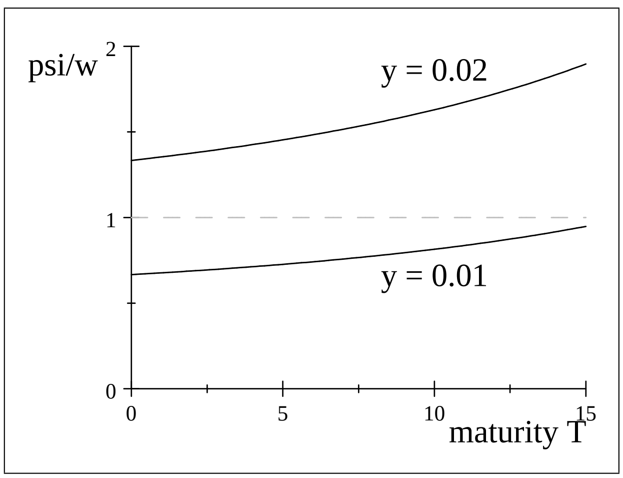
$r = 1\%$



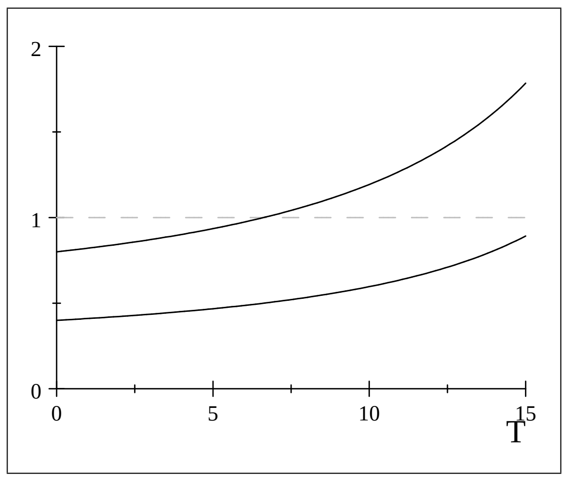
$r = 4\%$

Figure 2

Optimal fractional investment ψ_0^*/w in the defaultable asset versus T $(\phi = 0.5, \quad r = 1\%)$



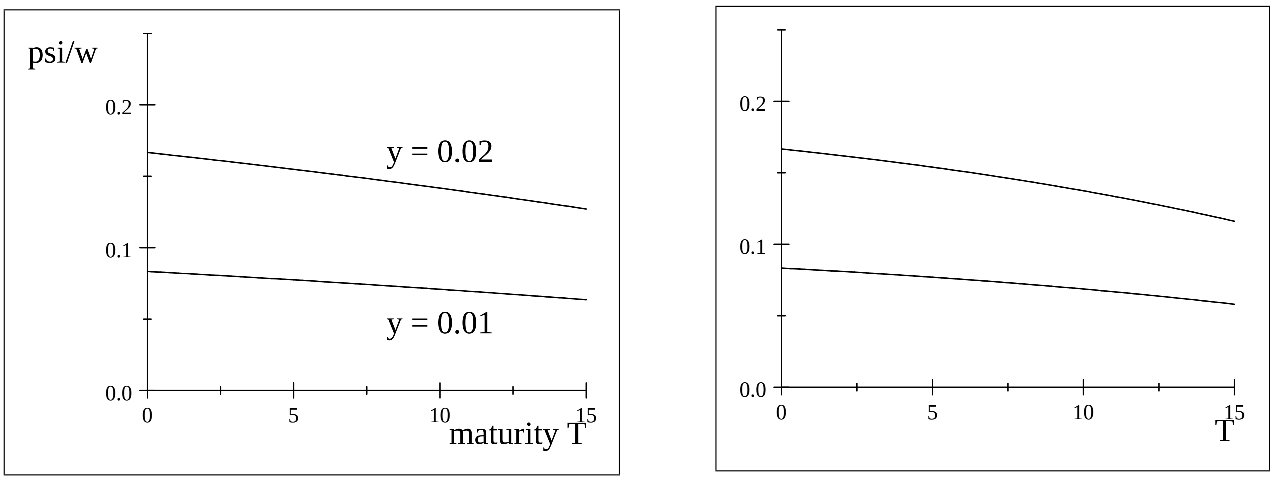
$\xi = 3\%$



$\xi = 5\%$

Figure 3

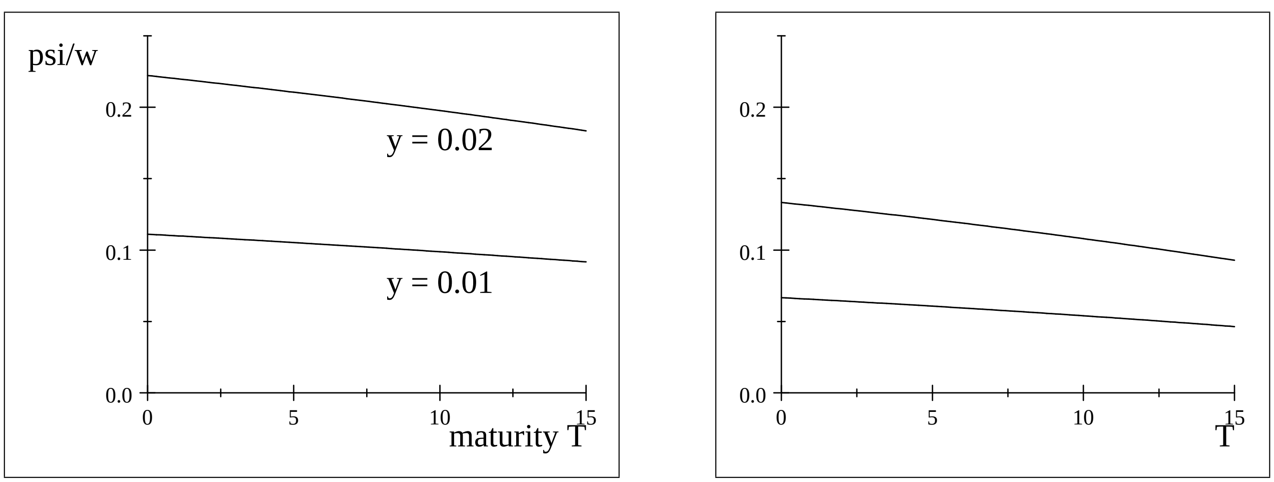
Optimal fractional investment ψ_0^*/w in the defaultable asset versus T $(\phi = 3, \quad \xi = 4\%)$



$r = 1\%$ $\qquad$ $r = 4\%$

Figure 4

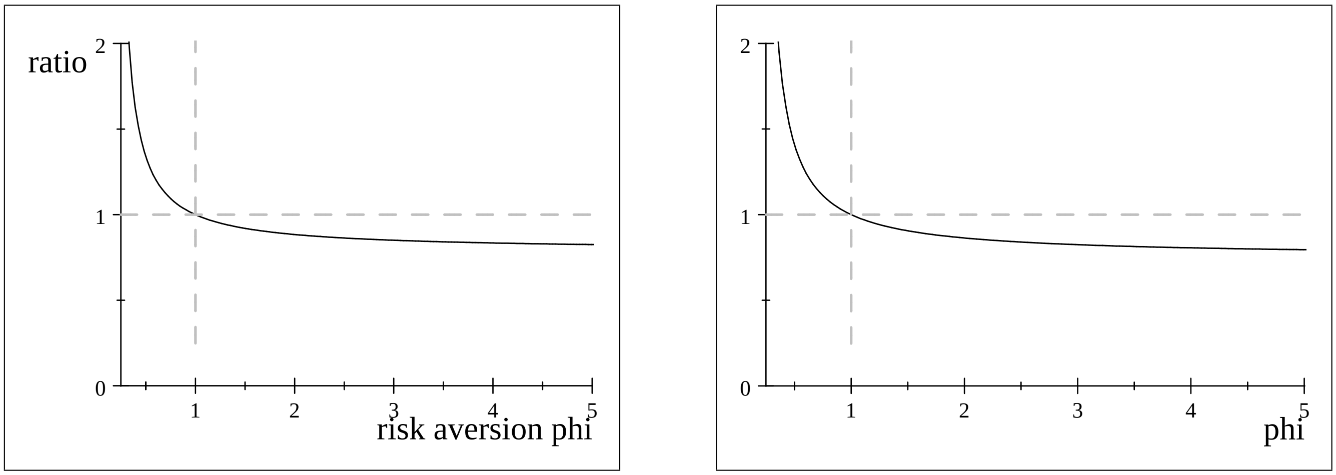
Optimal fractional investment ψ_0^*/w in the defaultable asset versus T $(\phi = 3, \quad r = 1\%)$



$\xi = 3\%$ $\qquad$ $\xi = 5\%$

Figure 5

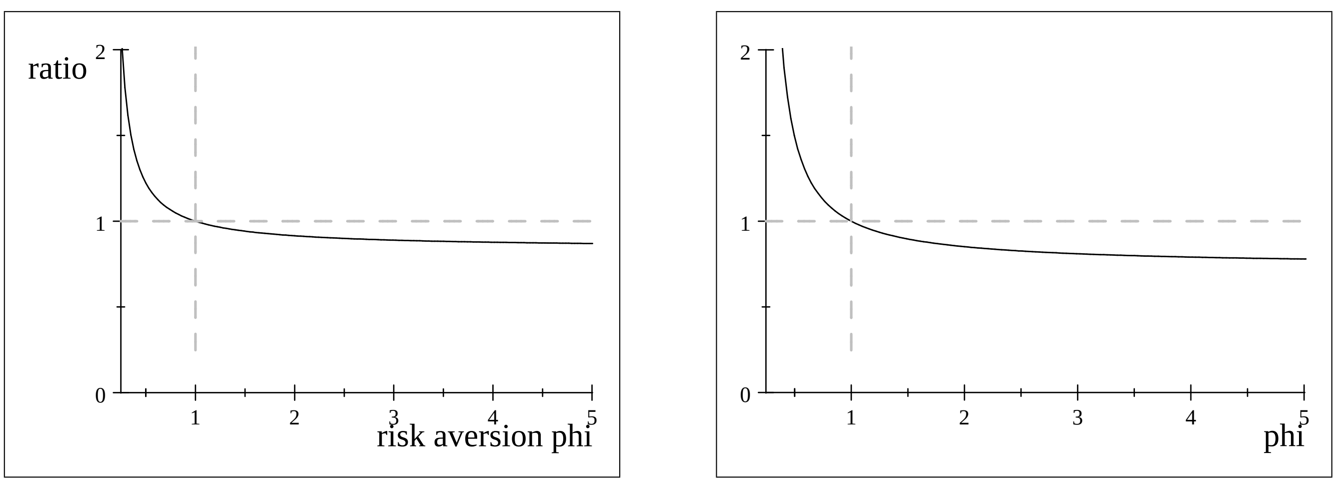
Ratio $\frac{\psi^*(w,T,y)}{\psi^*(w,0,y)}$ versus ϕ $(T = 10 < T^*, \quad \xi = 4\%)$



$r = 1\%$ $r = 4\%$

Figure 6

Ratio $\frac{\psi^*(w,T,y)}{\psi^*(w,0,y)}$ versus ϕ $(T = 10 < T^*, \quad r = 1\%)$



$\xi = 3\%$ $\xi = 5\%$