A variational characterization of complex submanifolds

A VARIATIONAL CHARACTERIZATION OF COMPLEX SUBMANIFOLDS

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ABSTRACT. In this note, we generalize our results in [6] to integer p-currents of any degree. We prove that if the mass of a current, as a functional of the ambient metric, has a critical or stable point in some special directions, then the current is complex. This holds for any dimension and codimension. We also study a natural functional on the space of currents representing a fixed homology class, closely related to the first derivative of the Mass in our new approach, detecting the deviation of a surface from being holomorphic.

Mathematics Subject Classification (2010): 53A10 (primary), 53D05 (secondary).

1. Introduction

In this paper we expand in various directions the study started in [6] about the relationship between volume minimizers and holomorphic submanifolds of Kähler manifolds. Let us recall that, while classically known that *positively oriented* chains of holomorphic submanifolds are volume minimizers in their homology class thanks to Wirtinger's Inequality, the converse is by now known to be largely false (see e.g. [5], [3], [4] and [18]). On top of this, and in fact not unrelated, the limitation about the positive orientation of volume minimizers (which appears clearly when looking for example at two parallel flat discs in $\mathbf{R}^4$) prevents this classical approach to be of much use in attacking various natural problems in Algebraic Geometry.

This has indicated the need for the search for more refined functionals, more capable to detect the holomorphic properties of their minimizers and at the same time to get rid of this orientation problem so that any *integral* chain of holomorphic submanifolds becomes a minimum among its competitors.

In [6] we proposed the following construction: consider a *fixed* immersion F of a surface Σ inside $(M^{2n}, \bar{\omega}, J_M)$, a compact symplectic manifold with compatible almost complex structure J_M, and look at the space of potentials $\mathcal{H} = \{\rho \in \mathcal{C}^\infty(M, \mathbf{R}) \mid \bar{\omega}_\rho := \bar{\omega} + dd^c\rho \text{ tames } J_M\}$, which is clearly a nonempty open subset of $\mathcal{C}^\infty(M, \mathbf{R})$.

Given $\rho \in \mathcal{H}$ and $\bar{\omega}_\rho(t) = \bar{\omega}_\rho + dd^c\varphi(t)$ which tames J_M, we can associate a family of Riemannian metrics $\bar{g}_\rho(t)$ on M given by

$$\bar{g}_\rho(t)(X, Y) = \frac{1}{2}\left(\bar{\omega}_\rho(t)(X, J_M Y) + \bar{\omega}_\rho(t)(Y, J_M X)\right). \tag{1.1}$$

(denote $\bar{g}_\rho = \bar{g}_\rho(0)$) and we then define

$$\mathcal{A}(\rho) = \mathrm{Area}(F(\Sigma), F^*(\bar{g}_\rho)) = \int_\Sigma d\mu_\rho \ , \tag{1.2}$$

Key words and phrases. current, stationary, stable, complex.

where $d\mu_\rho$ is the volume form of the induced metric $g_\rho := F^*(\bar{g}_\rho)$.
We are then looking at the area functional *not* on the space of immersions but on the space of metrics generated by potentials in $\mathcal{H}$ in the ambient manifold.

One of the main results in [6] was then

Theorem 1.1. *Let $(M^{2n}, \bar{\omega}, J)$ be a compact symplectic manifold with compatible almost complex structure J and $F : \Sigma^2 \to M$ be an injective immersion. Set $d\colon M \to \boldsymbol{R}$ any smooth extension from a tubular neighborhood of $F(\Sigma)$ to M of the distance function from $F(\Sigma)$, i.e. $d(Q) = dist(Q, F(\Sigma))$ for Q sufficiently near $F(\Sigma)$. If*

$$\frac{d}{dt}|_{t=0}\mathcal{A}(\bar{\omega}_\rho + tdd^c(\frac{d^2}{2})) = 0$$

for some $\rho \in \mathcal{H}$, then the immersion is J-holomorphic. In particular, if the area functional $\mathcal{A}$ has a critical point in $\mathcal{H}$, then the immersion is J-holomorphic. Moreover this holds also when F is not injective and has branch points (but one need more than one function to test the critical property).

In fact the proof of this result shows that the regular part of a union of injectively immersed surfaces is a chain of holomorphic submanifolds with possibly different orientations, and indeed it is easy to check that fixing such an object the Area is constant on the set of potentials (hence it has infinitely many critical points).

The first aim of this paper is to extend the above Theorem to higher dimensional submanifolds. But equally important is to extend the setup described above to much less regular objects, building in this way an existence problem in Geometric Measure Theory with some hope of having a positive solution, very much in the spirit of the classical volume-minimizing problem which led to Almgren's celebrated Big Theorem ([2], [9], [10], [11]). The area functional above gets then substituted by the Mass (again for a fixed object and moving metric!) and Σ by an integer multiplicity p-current.
Recall that an integer current $\mathcal{S}$ is called *complex*, if $\mu_{\mathcal{S}}$-almost all tangent planes of $\mathcal{S}$ are complex (see Definition 3.1). Since $\mu_{\mathcal{S}}(SingS) = 0$, in order to prove that an integer current is complex, we only need to prove that the tangent space at each regular point is complex.

The main result of this paper is then the following

Theorem 1.2. *Let $(M^{2n}, \bar{\omega}, J_M)$ be a compact symplectic manifold with compatible almost complex structure J_M and $\mathcal{S} \in \mathcal{R}_p(M)$ be an integer p-current in M with $p < 2n$. If the Mass has a critical point $\rho \in \mathcal{H}$, then any embedded C^2 component S_j of $Reg(\mathcal{S})$ is complex.*

We pay the price of allowing singular competitors in our generalized setting by loosing the possibility of studying deformations of metrics in one specific direction (given by the distance square function in Theorem 1.1). We believe that the C^2 assumption is not necessary in the above result in that even general C^1 components will satisfy the same property, but it naturally arises in our proof to construct some special test variations.

Thanks to Harvey-Shiffman ([16]) and Alexander's results ([1]) in the case of integrable complex structures, we immediately get the following

Corollary 1.3. *If $(M^{2n}, \bar{\omega}, J_M)$ is Kähler, $Reg(\mathcal{S})$ has all C^2-components and the Mass has a critical point in $\mathcal{H}$, then $p = 2k$ and $\mathcal{S}$ is a holomorphic k-chain, i.e. it is the current of integration over a finite integral combination of holomorphic submanifolds.*

Of course, the integrability of the ambient complex structure is crucial in applying Harvey-Shiffman-Alexander's Theorem and the analogue questions in the non-integrable case are subject of intensive and deep research (see e.g. Tian-Riviere [22]).
Almgren's Big Theorem on the other hand easily implies the following

Corollary 1.4. *Let $(M^{2n}, \bar{\omega}, J_M, \bar{g})$ be a compact symplectic manifold with compatible almost complex structure J_M and $\mathcal{S} \in \mathcal{R}_p(M)$ be an integer p-current in M with $p < 2n$. Suppose the mass has a critical point $\rho \in \mathcal{H}$ and that $\mathcal{S}$ is area-minimizing in $(M^{2n}, \bar{g}_\rho)$ in the usual sense, then $p = 2k$ and $\mathcal{S}$ is a holomorphic k-chain.*

The above results show that the first variation of the Mass with varying metrics does detect J-holomorphicity, but again as we proved for surfaces in our previous work, even the second variation (without assuming to be at a critical point of course) does the same job:

Theorem 1.5. *Let $(M^{2n}, \bar{\omega}, J_M, \bar{g})$ be a compact symplectic manifold with compatible almost complex structure J_M and $\mathcal{S} \in \mathcal{R}_p(M)$ be an integer p-current in M with $p < 2n$. If the mass has a stable point $\rho \in \mathcal{H}$, then any embedded C^2 component S_j of $Reg(\mathcal{S})$ is complex. In particular, if $(M^{2n}, \bar{\omega}, J_M)$ is Kähler, $Reg(\mathcal{S})$ has all C^2-components and the Mass has a stable point in $\mathcal{H}$, then $p = 2k$ and $\mathcal{S}$ is a holomorphic k-chain.*

As explained in our previous work [6], this approach is inspired by a classical work of Lawson-Simons [17], where the ambient manifold is assumed to be projective and the class of deformations of metrics where restricted to families coming from the action of the automorphism group of the projective space. In the Section 4 of this paper we extend, in analogy with the results obtained in [6] for regular 2-dimensional objects, these results to this much more general setting. We believe these results explain, in connection with Tian celebrated approximation Theorem ([21]), the naturality of our approach.

All this suggests to study a new type functionals $\mathcal{F}_c$ defined on the space of immersions, which come essentially from the integration of $|J^\perp|^2$, which is the first derivative in the direction of the distance squared of the Mass functional studied up to now. Thus these functionals can be used to detect the deviation of a submanifold from being holomorphic. In the surface case, we compute the Euler-Lagrangian equation for $\mathcal{F}_c$, and prove that similar to minimal surface system, the equation with $c > 1$ is weakly elliptic, with null directions coming from those directions tangential to the surface, i.e. the kernel of the principle symbol arises from the diffeomorphisms of the submanfold. We also conclude that any symplectic $\mathcal{F}_c$-critical surface with $c \geq 1$ in a Kähler-Einstein surface with nonnegative scalar curvature must be holomorphic. One interesting and challenging problem is whether Almgren's Big Theorem is true for these functionals.

Aknowledgments: The first author wishes to thank C. De Lellis for pointing out reference [1] to our attention and him, G. De Philippis and E. Spadaro for many important discussions. The first author was partially supported by FIRB Project RBFR08B2HY, and wishes to thank CIRM-FBK (Trento) for providing an ideal working atmosphere.

2. Variational formulas for the Mass in a symplectic manifold

Let $(M^{2n}, \bar{\omega}, J_M)$ be a compact symplectic manifold with compatible almost complex structure J_M. As in [6], let $\mathcal{H} = \{\rho \in \mathcal{C}^\infty(M, \mathbf{R}) \mid \bar{\omega}_\rho := \bar{\omega} + dd^c\rho \text{ tames } J_M\}$, which is clearly a nonempty open subset of $\mathcal{C}^\infty(M, \mathbf{R})$. Given $\rho \in \mathcal{H}$ and $\bar{\omega}_\rho(t) = \bar{\omega}_\rho + dd^c\varphi(t)$ which tames J_M, we can associated a family of Riemanian metrics $\bar{g}_\rho(t)$ on M given by

$$\bar{g}_\rho(t)(X,Y) = \frac{1}{2}\left(\bar{\omega}_\rho(t)(X, J_M Y) + \bar{\omega}_\rho(t)(Y, J_M X)\right). \tag{2.1}$$

Denote $\bar{g}_\rho = \bar{g}_\rho(0)$.

Let S be an $\mathcal{H}^p$-measurable countably p-rectifiable set in M. Then we know that the approximate tangent space T_xS exists for $\mathcal{H}^p$-a.e. $x \in S$. Actually, we can express S as the disjoint union $\cup_{j=0}^\infty S_j$ ([13], [20]), where $\mathcal{H}^p(S_0) = 0$, S_j is $\mathcal{H}^p$-measurable, and $S_j \subset N_j$, with N_j an embedded p-dimensional C^1 submanifold of M. We have

$$T_xS = T_xN_j, \quad \mathcal{H}^p - a.e.\ x \in S_j.$$

We will denote $SingS = S_0$ and $RegS = \cup_{j=1}^\infty S_j$. Then $\mathcal{H}^p(SingS) = 0$, and $RegS$ is the disjoint union of pieces, each of which is a part of an embedded p-dimensional C^1 submanifold of M.
Let $\mathcal{S}$ be an integer multiplicity p-current in $(M, \bar{g}_\rho)$ (27.1 of [20]). Namely, it can be represented as

$$\mathcal{S}(\omega) = \int_S \langle \omega(x), \xi(x)\rangle \theta(x) d\mathcal{H}^p(x), \quad \omega \in \Lambda^p M, \tag{2.2}$$

where S is an $\mathcal{H}^p$-measurable countably p-rectifiable subset of M, θ is a locally $\mathcal{H}^p$-integrable positive integer-valued function on S, and $\xi : S \to \Lambda^p(M)$ is an $\mathcal{H}^p$-measurable function such that for $\mathcal{H}^p$-a.e. point $x \in S$, $\xi(x)$ can be represented in the form $\tau_1 \wedge \cdots \wedge \tau_p$, where $\tau_1, \cdots, \tau_p$ form an orthonormal basis for the approximate tangent space T_xS with respect to the metric $\bar{g}_\rho$. Furthermore, denote $\mu_{\mathcal{S}}$ the Radon measure associated with the current $\mathcal{S}$, then we see that (26.7 and 27.1 of [20])

$$d\mu_{\mathcal{S}} = \theta d\mathcal{H}^p, \tag{2.3}$$

and (2.2) can be written as

$$\mathcal{S}(\omega) = \int_S \langle \omega(x), \xi(x)\rangle d\mu_{\mathcal{S}}(x), \quad \omega \in \Lambda^p M. \tag{2.4}$$

We plan to compute the first and second variation formulas for the mass of the current when the target metric varies by $\bar{\omega}_\rho(t) = \bar{\omega}_\rho + dd^c\varphi(t)$. When the variation of the target metric is induced by a vector field on M, the formulas are well-known. (See, for example, Theorem 1 of [17].) In our case, $\bar{g}_\rho(t)$ are not induced by a vector field on M. So we need to modify the argument. By Nash Embedding Theorem, we know that there exists a family of isometric embeddings

$$i_\rho(t) : (M^{2n}, \bar{g}_\rho(t)) \to (\mathbf{R}^N, g_{euc}^N), \tag{2.5}$$

i.e., $i_\rho(t)^* g_{euc}^N = \bar{g}_\rho(t)$. Here, g_{euc}^N is the standard Euclidean metric on $\mathbf{R}^N$. (Actually, we can take $N = n(6n+11)$ if M is compact and $N = n(2n+1)(6n+11)$ if M in noncompact.) It is obvious that $i(t)$ is smooth in t if $\varphi(t)$ is. Then the mass of the current $\mathcal{S}$ with respect to $\bar{g}_\rho(t)$ is given by (27.2 of [20])

$$\mathbf{M}_\rho(t) = \mathbf{M}_\rho(i_\rho(t)_\sharp \mathcal{S}) = \int_S J_S i_\rho(t) d\mu_{\mathcal{S}}, \tag{2.6}$$

where $J_S i_\rho(t)$ is the Jacobian of $i_\rho(t)$ relative to S, that is,

$$J_S i_\rho(t)(x) = \sqrt{det(d^S i_\rho(t)_x)^* \circ d^S i_\rho(t)_x}. \tag{2.7}$$

Here, $d^S i_\rho(t)_x : T_x S \to \mathbf{R}^N$ is the gradient of $i_\rho(t)$ restricting on S, which is well-defined $\mathcal{H}^p$-a.e. on S and $(d^S i_\rho(t)_x)^* : \mathbf{R}^N \to T_x S$ is its adjoint. (See section 12 of [20].) From (2.6), we see that

$$\frac{d}{dt}|_{t=0}\mathbf{M}_\rho(t) = \int_S \frac{d}{dt}|_{t=0} J_S i_\rho(t) d\mu_{\mathcal{S}}, \tag{2.8}$$

and

$$\frac{d^2}{dt^2}|_{t=0}\mathbf{M}_\rho(t) = \int_S \frac{d^2}{dt^2}|_{t=0} J_S i_\rho(t) d\mu_{\mathcal{S}}. \tag{2.9}$$

We will compute the integrand at the point $x \in S_j$ for $j \geq 1$, where S_j is a piece of a C^1 submanifold of M. Then $J_S i(t)$ is well-defined near x. We take a local coordinate around x. Namely, let $W \subset \mathbf{R}^p$ be an open set, and the coordinate on W is given by $\{x_1, \cdots, x_p\}$. Let $\Psi : W \to M$ be a C^1 immersion such that $\Psi(0) = x$, $\Psi(W) = U \cap S_j$, for some open set $U \subset M$ containing x. Then $T_x S_j$ is spanned by $\{\frac{\partial \Psi}{\partial x_i}(0)\}_{i=1}^p$. We further assume that, the coordinate $\{x_i\}$ is chosen so that $\{e_i = \frac{\partial \Psi}{\partial x_i}(0)\}$ is an orthonormal basis of $T_x S_j = T_x S$ with the induced metric by $i_\rho(0)$ (thus orthonormal by the induced metric from $(M, \bar{g}_\rho)$). Note that $(d^S i_\rho(t)_x)^* \circ d^S i_\rho(t)_x : T_x S \to T_x S$ can be represented as a $p \times p$ matrix. It is easy to check that

$$\left((d^S i(t)_x)^* \circ d^S i(t)_x\right)_{ij} = g^N_{euc}\left(\frac{\partial(i(t)\circ\Psi)}{\partial x^i}, \frac{\partial(i(t)\circ\Psi)}{\partial x^j}\right) = \bar{g}_\rho(t)(\frac{\partial \Psi}{\partial x_i}, \frac{\partial \Psi}{\partial x_j}).$$

Note that we have $\bar{g}_\rho(0)(\frac{\partial \Psi}{\partial x_i}(0), \frac{\partial \Psi}{\partial x_j}(0)) = \delta_{ij}$. Therefore, we have at x:

$$\frac{d}{dt}|_{t=0} J_T i_\rho(t)(x) = \frac{1}{2}\sum_{i=1}^p \bar{g}'_\rho(0)(e_i, e_i)$$

and, as $\mu_{\mathcal{S}}(S_0) = 0$, by (2.3), we have

$$\frac{d}{dt}|_{t=0}\mathbf{M}_\rho(t) = \frac{1}{2}\sum_{i=1}^p \int_S \bar{g}'_\rho(0)(e_i, e_i) d\mu_{\mathcal{S}}. \tag{2.10}$$

Here, the integrand is an $\mathcal{H}^p$-measurable function, and $\{e_i\}_{i=1}^p$ is any orthonormal basis of $T_x S$ with respect to the metric induced from $\bar{g}_\rho$ for $\mathcal{H}^p$-a.e. $x \in S$.

If $\bar{g}_\rho(t)$ is given by (2.1), then we have

$$\frac{d}{dt}|_{t=0}\mathbf{M}_\rho(t) = \frac{1}{2}\sum_{i=1}^p \int_S \bar{\omega}'_\rho(0)(e_i, Je_i) d\mu_{\mathcal{S}}. \tag{2.11}$$

If furthermore, we assume $\bar{\omega}_\rho(t) = \bar{\omega}_\rho + dd^c\varphi(t)$ for a family of C^2 functions $\varphi(t)$ on M with $\varphi(0) \equiv 0$, then we have

$$\frac{d}{dt}|_{t=0}\mathbf{M}_\rho(t) = \frac{1}{2}\sum_{i=1}^p \int_S (dd^c\psi)(e_i, Je_i) d\mu_{\mathcal{S}}, \tag{2.12}$$

where $\psi = \frac{\partial\varphi(t)}{\partial t}|_{t=0}$.

Similarly, by computing on the regular part of S and proceeding in the same way as for smooth case (see [6] for $p = 2$), if

$$\frac{\partial\varphi}{\partial t}|_{t=0} = \psi, \quad \frac{\partial^2\varphi}{\partial t^2}|_{t=0} = \eta,$$

then we have

$$\begin{aligned}\frac{d^2}{dt^2}|_{t=0}\mathbf{M}_\rho(t) \quad = \quad & \frac{1}{2}\sum_{i=1}^{p}\int_S [(dd^c\eta)(e_i, Je_i)]\, d\mu_{\mathcal{S}} \\ & -\frac{1}{4}\sum_{1\le i<j\le p}\int_S [(dd^c\psi)(e_i, Je_j) + (dd^c\psi)(e_j, Je_i)]^2\, d\mu_{\mathcal{S}} \\ & -\frac{1}{4}\sum_{1\le i<j\le p}\int_S [(dd^c\psi)(e_i, Je_i) - (dd^c\psi)(e_j, Je_j)]^2\, d\mu_{\mathcal{S}} \\ & +\frac{p-2}{4}\sum_{i=1}^{p}\int_S [(dd^c\psi)(e_i, Je_i)]^2\, d\mu_{\mathcal{S}}.\end{aligned} \tag{2.13}$$

For our later use, let's recall the following simple facts:

Lemma 2.1. (1) *For any smooth function ψ on M, we have*

$$d^c\psi = -d\psi \circ J. \tag{2.14}$$

(2) *For any C^2 function ψ on M and any tangent vector fields X, Y on M, we have*

$$(dd^c\psi)(X,Y) = -(\overline{\nabla}^2\psi)(X, JY) + (\overline{\nabla}^2\psi)(Y, JX) + \langle\overline{\nabla}\psi, (\overline{\nabla}_Y J)X - (\overline{\nabla}_X J)Y\rangle. \tag{2.15}$$

Here, $\langle\cdot,\cdot\rangle$ is any Riemannian metric on M and $\overline{\nabla}$ is its Levi-Civita connection.

3. Proof of the main results

In this section, we will prove that, each C^2 component of an integer current in a symplectic manifold for which the mass has a critical point or stable point is complex. In the following, we will denote $\mathcal{R}_p(M)$ the space of integer multiplicity p-currents in M. Let us first recall the definition of complex current.

Definition 3.1. *Let $(M^{2n}, \bar{\omega}, J_M)$ be a compact symplectic manifold with compatible almost complex structure J_M. Then an integer p-current $\mathcal{S}$ is said to be* ***complex*** *if $\mu_{\mathcal{S}}$-almost all tangent planes of $\mathcal{S}$ are complex, i.e., for $\mu_{\mathcal{S}}$-a.e. $x \in S$, $(J_M)_x$ maps T_xS onto itself.*

Recall that when $p = 2$ and S is a smooth submanifold of $(M^{2n}, \bar{\omega}, J, \bar{g})$, we can define the Kähler angle of the surface ([8]). In the current case, we can also define this similarly. The **cosine of the Kähler angle** of a rectifiable 2-current $\mathcal{S} = (S, \theta, \xi)$ is a $\mu_{\mathcal{S}}$-measurable function $\cos\alpha : S \to \mathbf{R}$ such that for $\mu_{\mathcal{S}}$-almost all $x \in S$ with $\xi_x = e_1 \wedge e_2$, $\cos\alpha = \bar{\omega}(e_1, e_2)$. Here, $\{e_1, e_2\}$ is any orthonormal basis of T_xS.

Similar to the smooth case, we can easily see that

Proposition 3.1. *Let $(M^{2n}, \bar{\omega}, J, \bar{g})$ be a compact symplectic manifold with compatible almost complex structure J_M. A current $\mathcal{S} \in \mathcal{R}_2(M)$ is complex if and only if $\sin\alpha$ vanishes as a measurable function, namely, $\sin\alpha(x) = 0$ for $\mu_{\mathcal{S}}$-a.e. $x \in S$.*

Now can now give the following

Definition 3.2. *Given a current $\mathcal{S} \in \mathcal{R}_p(M)$ in M, we say that the mass* ***M has a critical point*** *$\rho \in \mathcal{H}$ if for any $\varphi(t) \in \mathcal{H}$ with $\varphi(0) = \rho$*

$$\frac{d}{dt}|_{t=0}\boldsymbol{M}(t) = 0.$$

Definition 3.3. *Given a current $\mathcal{S} \in \mathcal{R}_p(M)$ in M, we say that the mass* ***M has a stable point*** *$\rho \in \mathcal{H}$ if*

$$\frac{d^2}{dt^2}|_{t=0}\boldsymbol{M}(t) \geq 0$$

for any $\varphi(t) \in \mathcal{H}$, $\varphi(0) = \rho$.

As before, let $(M^{2n}, \bar{\omega}_\rho, J_M, \bar{g}_\rho)$ be a compact symplectic manifold with compatible almost complex structure J_M and $\mathcal{S} = (S, \theta, \xi)$ be an integer p-current in M. We have shown that, for $\bar{\omega}_\rho(t) = \bar{\omega}_\rho + dd^c\varphi(t)$ with $\frac{\partial\varphi}{\partial t}|_{t=0} = \psi$, the first variation formula is given by (2.12). We already know that $\mathcal{H}^p(SingS) = 0$, and $RegS$ can be expressed as disjoint unions $RegS = \cup_{j=1}^{\infty} S_j$, where each component S_j $(j \geq 1)$ is contained in an embedded p-dimensional C^1 submanifold of M. Our main result in this section is as follows:

Theorem 3.2. *Let $(M^{2n}, \bar{\omega}, J_M)$ be a compact symplectic manifold with compatible almost complex structure J_M and $\mathcal{S} \in \mathcal{R}_p(M)$ be an integer p-current in M with $p < 2n$. If the mass has a critical point $\rho \in \mathcal{H}$, then any embedded C^2 component S_j of $RegS$ is complex.*

Proof: By our assumption, for any $x \in S_j$, there exists a ball $B_{3r}(x) \subset M$, such that $B_{3r}(x) \cap S = B_{3r}(x) \cap S_j$ is a C^2 submanifold of M, and $d(y, S) = d(y, B_{3r}(x) \cap S_j)$ for $y \in B_{2r}(x)$. Here, the distance is measured by the metric $\bar{g}_\rho$, and we will denote $d(y) = d(y, S)$. Then it is known that $\xi = \frac{1}{2}d^2$ is a C^2 function in $B_{2r}(x)$ for r small. Taking a cutoff function $\zeta \in C_0^\infty(B_{2r}(x))$ on M, so that $\zeta \equiv 1$ in $B_r(x)$. Then $\psi = \zeta\xi$ is a C^2 function on M with the property that: $supp\psi \subset B_{2r}(x)$ and $\psi = \xi = \frac{1}{2}d^2$ in $B_r(x)$. By (2.15) and (2.12), we have

$$\begin{aligned}\frac{d}{dt}|_{t=0}\mathbf{M}_\rho(t) &= \frac{1}{2}\sum_{i=1}^{p}\int_S \left[(\overline{\nabla}^2(\zeta\xi))(e_i, e_i) + (\overline{\nabla}^2(\zeta\xi))(J_M e_i, J_M e_i)\right] d\mu_{\mathcal{S}} \\ &\quad + \frac{1}{2}\sum_{i=1}^{p}\int_S \langle\overline{\nabla}(\zeta\xi), (\overline{\nabla}_{J_M e_i} J_M)e_i - (\overline{\nabla}_{e_i} J_M)J_M e_i\rangle d\mu_{\mathcal{S}}.\end{aligned} \tag{3.1}$$

Note that by the choice of ξ, we have $\xi = 0$ and $\overline{\nabla}\xi = 0$ on $B_{2r}(x) \cap S_j$. Furthermore, $\zeta = 0$ outside $B_{2r}(x)$. Therefore,

$$\frac{d}{dt}|_{t=0}\mathbf{M}_\rho(t) = \frac{1}{2}\sum_{i=1}^{p}\int_{S_j}\left[\zeta(\overline{\nabla}^2\xi)(e_i, e_i) + \zeta(\overline{\nabla}^2\xi)(J_M e_i, J_M e_i)\right] d\mu_{\mathcal{S}}.$$

Recall that (Proposition 2.5 of [6]) for any $x_0 \in S_j$, $Hess(\xi)(x_0)$ represents the orthogonal projection on the normal space to S_j at x_0. Namely, for each $U, V \in T_{x_0}M$ and $x_0 \in S_j$, we have

$$(\overline{\nabla}^2 \xi)(U, V)(x_0) = \langle U^\perp, V^\perp \rangle, \tag{3.2}$$

where $T_{x_0}M = T_{x_0}S_j \oplus N_{x_0}S_j$ and $U^\perp$ is the projection of U onto $N_{x_0}S_j$. With ψ chosen as above, we have

$$\frac{d}{dt}|_{t=0}\mathbf{M}_\rho(t)) = \frac{1}{2}\sum_{i=1}^{p}\int_{S_j} \zeta \left|(J_M e_i)^\perp\right|^2 d\mu_{\mathcal{S}}.$$

In particular, by the definition of critical point, we have that $(J_M)^\perp = 0$ on $B_r(x) \cap S_j$. In particular, J_M maps $T_x S_j$ onto itself. As $x \in S_j$ is arbitrary, by Definition 3.1, we see that S_j is complex. Q.E.D.

Remark 3.3. *In the proof of Theorem 3.2, we see that we actually only need the mass to have a critical point along some special directions at each regular point. More precisely, at each regular point, if the mass has a critical point in the direction $\frac{d^2}{2}$ locally, then the tangent space at this point is complex. When S is a closed C^2 embedded submanifold of M, we can define the function d^2 globally in a neighborhood of Σ in M. In this case, we only need one special direction $\frac{d^2}{2}$, and Theorem 3.2 reduces to a higher dimensional generalization of Theorem 1.1.*

Let us now recall the following definition due to Harvey and Shiffman (Definition 1.7 of [16]):

Definition 3.4. *Let $(M^{2n}, \bar{\omega}, J_M, \bar{g})$ be a compact Kähler manifold. A current $\mathcal{T}$ is said to be a* ***holomorphic k-chain*** *in M, if it can be written as a finite sum $\mathcal{T} = \sum n_j[V_j]$, where each $n_j \in \boldsymbol{Z}$ and $V = \cup V_j$ is a pure p-dimensional subvariety of M with irreducible components $\{V_j\}$.*

Roughly speaking, a holomorphic k-chain is a locally finite integral combination of complex subvariaties. It is known that (Proposition 3.1 of [16]), a positive holomorphic current is homologically area-minimizing, while a holomorphic k-chain is stable in the usual sense. It is obvious that a holomorphic k-chain is a complex $2k$-current. The main result of Harvey-Shiffman (Theorem 2.1 of [16]) says that a complex $2k$-current $\mathcal{S}$ with $d\mathcal{S} = 0$ and $H^{2k+1}(supp\mathcal{S}) = 0$ is a holomorphic k-chain and later Alexander [1] removed the support hypothesis.
Corollaries 1.3 and 1.4 follow then immediately (in the second case applying Almgren's Big Theorem) from our main result.

The case of a stable point can be easily handled thanks to

Proposition 3.4. *If $\rho \in \mathcal{H}$ is a stable point of the mass $\boldsymbol{M}$, then it is also a critical point of the mass $\boldsymbol{M}$.*

Proof: To this end, we consider special path in $\mathcal{H}$, which is given by $\varphi(t) = \rho + \frac{t^2}{2}\eta$ with $\eta \in \mathcal{C}^\infty(M, \mathbf{R})$. In this case, we have $\varphi'(0) = \psi = 0$ and $\varphi''(0) = \eta$. By (2.13), we have

$$\frac{d^2}{dt^2}|_{t=0}\mathbf{M}_\rho(t) = \frac{1}{2}\sum_{i=1}^{p}\int_{S} [(dd^c\eta)(e_i, Je_i)]\, d\mu_{\mathcal{S}}$$

Suppose $\rho \in \mathcal{H}$ is a stable point of the mass $\mathbf{M}$, then by definition, $\frac{d^2}{dt^2}|_{t=0}\mathbf{M}_\rho(t) \geq 0$ for any $\varphi(t) \in \mathcal{H}$, $\varphi(0) = \rho$. In particular, for $\varphi_1(t) = \rho + \frac{t^2}{2}\eta$ and $\varphi_2(t) = \rho - \frac{t^2}{2}\eta$, we have

$$\frac{1}{2}\sum_{i=1}^{p}\int_S [(dd^c\eta)(e_i, Je_i)]\, d\mu_{\mathcal{S}} \geq 0$$

and

$$-\frac{1}{2}\sum_{i=1}^{p}\int_S [(dd^c\eta)(e_i, Je_i)]\, d\mu_{\mathcal{S}} \geq 0.$$

In particular, we have

$$\frac{1}{2}\sum_{i=1}^{p}\int_S [(dd^c\eta)(e_i, Je_i)]\, d\mu_{\mathcal{S}} = 0$$

for every $\eta \in \mathcal{C}^\infty(M, \mathbf{R})$. By the first variation formula (2.12) and Definition 3.2, we see that ρ is a critical point. Q.E.D.

Combining Proposition 3.4 and Theorem 3.2, we obtain:

Theorem 3.5. *Let $(M^{2n}, \bar{\omega}, J_M, \bar{g})$ be a compact symplectic manifold with compatible almost complex structure J_M and $\mathcal{S} \in \mathcal{R}_p(M)$ be an integer p-current in M with $p < 2n$. If the mass has a stable point $\rho \in \mathcal{H}$, then any embedded C^2 component S_j of $RegS$ is complex.*

Remark 3.6. *As in Remark 3.3, to obtain the conclusion of Theorem 3.5, we only need to ask for the mass to have a stable point in the directions $\pm\psi$ around each regular point, where ψ is defined in the proof of Theorem 3.2. ψ is essentially $\frac{d^2}{2}$ locally.*

Corollary 3.7. *Let $(M^{2n}, \bar{\omega}, J_M, \bar{g})$ be a compact symplectic manifold with compatible almost complex structure J_M and $\mathcal{S} \in \mathcal{R}_p(M)$ be an integer p-current in M with $p < 2n$. Suppose the mass has a stable point $\rho \in \mathcal{H}$ and that $\mathcal{S}$ is area-minimizing $(M^{2n}, \bar{g}_\rho)$ in the usual sense, then $p = 2k$ and $\mathcal{S}$ is a holomorphic k-chain.*

In particular, by Remark 3.3 and Remark 3.6, when Σ is a smooth manifold and $F : \Sigma \to M$ is an injective immersion, Theorem 3.2 and Theorem 3.5 generalize the first two theorems of [6] to arbitrary dimension and codimension. Note that, by definition, an immersion $F : \Sigma \to M$ is $\pm J_M$-holomorphic if and only if $F(S)$ is a complex current.

Corollary 3.8. *Let $(M^{2n}, \bar{\omega}, J_M)$ be a compact symplectic manifold with compatible almost complex structure J and $F : \Sigma^p \to M$ be an injective immersion. Set $d\colon M \to$ **R** any smooth extension from a tubular neighborhood of $F(\Sigma)$ to M of the distance function from $F(\Sigma)$, i.e. $d(Q) = dist(Q, F(\Sigma))$ for Q sufficiently near $F(\Sigma)$. If*

$$\frac{d}{dt}|_{t=0}\mathcal{A}(\bar{\omega} + tdd^c(\frac{d^2}{2})) = 0,$$

or

$$\frac{d^2}{dt^2}|_{t=0}\mathcal{A}(\bar{\omega} \pm t^2dd^c(\frac{d^2}{2})) \geq 0$$

for some $\rho \in \mathcal{H}$, then the immersion is $\pm J_M$-holomorphic.

Remark 3.9. *Comparing with Theorem 3.2 of* [6] *(with $p = 2$), we even do not need the stable point to be compatible with respect to the almost complex structure J_M here. Moreover, for any immersion (without injectivity assumption), existence of critical points* *or stable points is enough to guarantee that the immersion is $\pm J_M$-holomorphic. In this case we can not find one special direction as in the injective case.*

Corollary 3.10. *Let $(M^{2n}, \bar{\omega}, J_M, \bar{g})$ be a compact symplectic manifold with compatible almost complex structure J_M and $F : \Sigma^p \to M^{2n}$ be an immersion with $p < 2n$. If p is odd, then the area functional $\mathcal{A}$ does not have any critical point or stable point in $\mathcal{H}$.*

4. Approximation results

In order to understand the nature of the new stability previously introduced, we take any holomorphic vector field V on a Kähler manifold M. Then V will generate a family of holomorphic diffeomorphisms of M, denoted by Φ_t. We know that $\Phi_t^* \bar{\omega} = \bar{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi(t)$ for a family of smooth functions $\varphi(t)$ on M. Furthermore, as V is a holomorphic vector field, we know that if we denote $\bar{g}(t) = \Phi_t^* \bar{g}$, then $(\bar{\omega}(t), \bar{g}(t), J)$ is a compatible triple for each t. Note that the former (namely, $\bar{g}(t) = \Phi_t^* \bar{g}$) is in the classical category, while the latter is in our category. In particular, if the area functional is stable in our sense, then the second variation of the area functional in the classical sense is nonnegative when the variation is induced by Φ_t.

In fact, we can say more about this, relating the classical case to our case. If we denote $\dot{\varphi} = \psi$, and $\ddot{\varphi} = \eta$, then in our language, the second variation formula is given by (2.13), where ψ and η are two independent functions. However, when $\varphi(t)$ is induced by a holomorphic vector field V as above, we know that both ψ and η are determined by V. In fact, we have

$$\sqrt{-1}\partial\bar{\partial}\psi = L_V \bar{\omega}, \quad \sqrt{-1}\partial\bar{\partial}\eta = L_V(L_V \bar{\omega}) = L_V(\sqrt{-1}\partial\bar{\partial}\psi),$$

which shows that ψ and η are not independent in this case. Actually, we can give more precise relation between ψ and η. By Moser's trick, it is easy to see that, if we take $X(t) = -\frac{1}{2}\overline{\nabla}^t \dot{\varphi}(t)$ and Ψ_t the family of diffeomorphisms generated by $X(t)$, then we have $\Psi_t^* \bar{\omega}(t) = \bar{\omega}$. Here, $\overline{\nabla}^t$ is the gradient taken with respect to the metric $\bar{g}(t)$. In particular, combining with the choice of $\bar{\omega}(t)$, we see that we have $\Psi_t = \Phi_t^{-1}$. It is easy to see that

$$V = -(\Phi_t)_* X(t) = \frac{1}{2}(\Phi_t)_* \overline{\nabla}^t \dot{\varphi}(t). \tag{4.1}$$

Then we have $V = -X(0) = \frac{1}{2}\overline{\nabla}\psi$. Using this fact and taking derivative with respect to t on both side of (4.1), we can obtain

$$\frac{d}{dt}|_{t=0} \overline{\nabla}^t \dot{\varphi}(t) = 0. \tag{4.2}$$

Using the fact that $\bar{g}(t)(U, V) = \bar{\omega}(t)(U, JV)$, we can finally get that

$$d\eta = -(\sqrt{-1}\partial\bar{\partial}\psi)(J\overline{\nabla}\psi, \cdot). \tag{4.3}$$

The point we want to explore now is that one should not restrict only to holomorphic vector fields on M, but to the effect any holomorphic vector field of an ambient projective space. In fact, in the projective case looking at the space of metrics in a given cohomology class induced by an embedding into projective spaces of increasing dimension (the so-called *Bergman space* $\mathcal{B}_k$), thanks to Tian's celebrated approximation result ([21])) we know we can approximate any potential in $\mathcal{H}$ and moreover such approximation is sufficiently strong that we can also approximate tangent directions and accelerations of curves in $\mathcal{H}$ with

corresponding objects in the Bergman spaces (this boils down to the uniformity of the Tian-Yau-Zelditch expansion as noted in [12], Proposition 6).
This immediately implies that our stability can be thought as the limit of the stability of the volume functional of the triple ($\Sigma \subset M \subset \mathbf{CP}^{N_k}$), i.e. when restricted to the Bergmann space of degree k. This gives the following

Theorem 4.1. *If for any k sufficiently big there exists a function $\rho_k \in \mathcal{B}_k$ s.t. ρ_k is a stable point for $\boldsymbol{M}_{|\mathcal{B}_k}$ and ρ_k converges to ρ is $\mathcal{H}$, then ρ is a stable point for $\boldsymbol{M}$.*

It is then natural to ask whether the existence of a stable point of $\mathbf{M}_{|\mathcal{B}_k}$ for a given fixed k is enough to guarantee our conclusion. That's the problem we address in the next subsections under various conditions (for $p = 2$).

4.1. **Algebraic case (Lawson-Simons** [17]**).** Let us now assume that the target manifold is an algebraic manifold that embeds into some complex projective space $\mathbf{CP}^N$ holomorphically and isometrically, namely that there is an embedding

$$\iota : (M, \bar{\omega}, J, \bar{g}) \to (\mathbf{CP}^N, \omega_{FS}, J_{FS}, g_{FS}),$$

which is holomorphic, such that

$$\iota^*\omega_{FS} = \bar{\omega}, \quad \iota^* g_{FS} = \bar{g}. \tag{4.4}$$

Denote by $\mathcal{H}_N$ and $\mathcal{K}_N$ the space of holomorphic vector fields and Killing vector fields on $\mathbf{CP}^N$. Then it is well-known that $\mathcal{H}_N = \mathcal{K}_N \oplus J\mathcal{K}_N$. Given any $W \in J\mathcal{K}_N$, it will generate a one parameter family of diffeomorphisms Φ_t of $\mathbf{CP}^N$. It is known that there exists a family of smooth functions $\phi(t)$ on $\mathbf{CP}^N$, such that $\tilde{\omega}(t) = \Phi_t^*\omega_{FS} = \omega_{FS} + dd^c\phi(t)$. Set $\varphi(t) = \phi(t) \circ \iota$, which is a family of smooth functions on M. Set $\dot{\varphi} = \frac{d}{dt}|_{t=0}\varphi(t)$.

Definition 4.1. *Given a current $\mathcal{S} \in \mathcal{R}_p(M)$ in M, we say that the mass **$\boldsymbol{M}$ has a linearly projectively stable point at** $\rho \in \mathcal{H}$ if $\bar{\omega}_\rho$ is projectively induced and*

$$\frac{d^2}{dt^2}|_{t=0}\boldsymbol{M}(t) \geq 0$$

for any $\bar{\omega}_\rho(t) = \bar{\omega}_\rho + tdd^c\dot{\varphi}$, where $\varphi(t)$ is defined with $\bar{\omega}$ replaced by $\bar{\omega}_\rho$ as above.

We can then give a new more geometric proof of the following result of [17] (in fact they proved it without restrictions on p) :

Theorem 4.2. *Let $(M, \bar{\omega}, J, \bar{g})$ be an algebraic manifold with all structures induced by the projective space as above and $\mathcal{S} \in \mathcal{R}_2(M)$ be a current in M. If the mass has a linearly projectively stable point, then the current $\mathcal{S}$ is a holomorphic 1-chain.*

Proof: As J is compatible with any Kähler metric in $[\bar{\omega}]$, without loss of generality, we assume that $\rho \equiv 0$ so that $\bar{\omega}_\rho = \bar{\omega}$. Recall that for $\bar{\omega}(t) = \bar{\omega} + tdd^c\psi$, the second variation formula is given by (see (2.13))

$$\frac{d^2}{dt^2}|_{t=0}\mathbf{M}(t) = -\frac{1}{4}\int_S D_1^2 d\mu_{\mathcal{S}} - \frac{1}{4}\int_S D_2^2 d\mu_{\mathcal{S}} \geq 0,$$

where

$$D_1 = \sin\alpha[-(dd^c\psi)(e_1, e_4) + (dd^c\psi)(e_2, e_3)], \quad D_2 = \sin\alpha\left[(dd^c\psi)(e_1, e_3) + (dd^c\psi)(e_2, e_4)\right].$$

By the choice of ψ, we can see that (see Section 4 of [6])

$$D_2(W) = -2\sin\alpha\left[\langle\overline{\nabla}^N_{\tilde{e}_1}V, \tilde{e}_3\rangle + \langle\overline{\nabla}^N_{\tilde{e}_2}V, \tilde{e}_4\rangle\right].$$

We know that $\cos\alpha$ is well-defined and continuous on $RegS$. Now, we by our assumption, we see that

$$D_2(W) = 0, \quad on\ M_j, \quad j \geq 1.$$

Using Lemma 4.2 of [6], similar to the proof for smooth case, we see that we must have $\sin\alpha = 0$ on $RegS$. As $\mathcal{H}^2(SingS) = 0$, we have $\mu_{\mathcal{S}}(SingS) = 0$. Therefore, we see that $\sin\alpha = 0$ $\mu_{\mathcal{S}}$-a.e. on S. By Proposition 3.1, we see that the current is complex. Then the conclusion follows from Harvey-Shiffman-Alexander's Theorem. Q.E.D.

4.2. **Symplectic case with rational class.** Let $(M^{2n}, \bar{\omega}, \bar{g}, J_M)$ be a compact symplectic manifold with symplectic form $\bar{\omega}$, compatible almost complex structure J_M and associated Riemannian metric $\bar{g}$, such that for any $X, Y \in TM$,

$$\bar{g}(X, Y) = \bar{\omega}(X, J_M Y). \tag{4.5}$$

Since $\bar{\omega}$ defines a rational cohomology class, by a Theorem of Borthwick and Uribe (Theorem 1.1 of [7]), we known that there exists a sequence of embeddings

$$\iota_k : M \to (\mathbf{CP}^{N_k}, \omega_{FS}, g_{FS}, J_{FS}), \tag{4.6}$$

such that, if we put

$$\bar{\omega}_k = \iota_k^*\omega_{FS}, \quad \bar{g}_k = \iota_k^* g_{FS}, \tag{4.7}$$

then for $k \geq k_0$

$$\left|\left|\frac{1}{k}\bar{\omega}_k - \bar{\omega}\right|\right|_{C^0} \leq \frac{C_1}{k}, \tag{4.8}$$

and

$$\left|\left|\frac{1}{k}\bar{g}_k - \bar{g}\right|\right|_{C^0} \leq \frac{C_2}{k}, \tag{4.9}$$

for some constants C_1 and C_2 and large integer k_0.

Let $\mathcal{S}$ be an integer multiplicity p-current in M. Denote α and α_k the Kähler angle of $RegS$ in $(M^{2n}, \bar{\omega}, \bar{g}, J_M)$ and $(\mathbf{CP}^{N_k}, \omega_{FS}, g_{FS}, J_{FS})$, respectively. More precisely, for $x \in RegS$, let $\{e_1, e_2\}$ be an orthonormal basis of T_xS with respect to the induced metric from $\bar{g}$, and $\{e_{1,k}, e_{2,k}\}$ be any orthonormal basis of T_xS with respect to the induced metric from $\bar{g}_k = \iota_k^* g^k_{FS}$, then

$$\cos\alpha(x) = \bar{\omega}(e_1, e_2)(x), \quad \cos\alpha_k(x) = \bar{\omega}_k(e_{1,k}, e_{2,k})(x).$$

We can take

$$e_{1,k} = \frac{e_1}{|e_1|_{\bar{g}_k}}, \quad e_{2,k} = \frac{e_2 - \bar{g}_k(e_2, e_{1,k})e_{1,k}}{|e_2 - \bar{g}_k(e_2, e_{1,k})e_{1,k}|_{\bar{g}_k}}.$$

By (4.8) and (4.9), we see that

$$\cos\alpha_k(x) \to \cos\alpha(x), \quad \sin\alpha_k(x) \to \sin\alpha(x) \quad for \quad x \in regS. \tag{4.10}$$

Set $\mathcal{K}_k$ the space of Killing vector fields on $\mathbf{CP}^{N_k}$. Given any holomorphic vector field $W \in J_{FS}\mathcal{K}_k$, let Φ_t be the one-parameter family of diffeomorphisms generated by W. Set $\omega_k(t) = \Phi_t^*\omega_{FS} = \omega_{FS} + dd^c_{FS}\varphi(t)$ for a family of smooth functions $\varphi(t)$ on $\mathbf{CP}^{N_k}$.

Note that $\frac{1}{k}\bar{\omega}_k$ and $\bar{\omega}$ are in the same cohomology class. Thus, there exists a smooth one form γ_k on M, such that $\bar{\omega} = \frac{1}{k}\bar{\omega}_k + d\gamma_k$. We consider a family of projectively induced symplectic forms on M given by

$$\bar{\omega}(t) = \frac{1}{k}\iota_k^*\omega_k(t) = \frac{1}{k}\iota_k^*\Phi_t^*\omega_{FS} = \frac{1}{k}\bar{\omega}_k + d(\frac{1}{k}\iota_k^* d_{FS}^c\varphi(t)) \equiv \bar{\omega} + d\beta_k(t),$$

where $\beta_k(t) = \frac{1}{k}\iota_k^* d_{FS}^c\varphi(t) - \gamma_k$ is a family of smooth 1-forms on M.

Definition 4.2. *Given a current $\mathcal{S} \in \mathcal{R}_p(M)$ in M, we say that **$\boldsymbol{M}$ has a compatible linearly $\mathcal{M}^k$-stable point at** $\rho \in \mathcal{H}$ if $\bar{\omega}_\rho$ is compatible with J and*

$$\frac{d^2}{dt^2}|_{t=0}\boldsymbol{M}(t) \geq 0$$

for any $\bar{\omega}(t) = \bar{\omega} + td\dot{\beta}_k$, where $\beta_k(t)$ is defined with $\bar{\omega}$ replaced by $\bar{\omega}_\rho$ in the above construction.

Similar to the proof of Theorem 5.1 in [6] and the proof of Theorem 4.2 above, we can show that:

Theorem 4.3. *Let $(M^{2n}, \bar{\omega}, J_M, \bar{g})$ be a symplectic manifold as above and $\mathcal{S} \in \mathcal{R}_2(M)$ be a current in M. There exists an integer K_1, such that if the mass has a compatible linearly $\mathcal{M}^k$-stable point for some $k \geq K_1$, then the current $\mathcal{S}$ is holomorphic 1-chain.*

4.3. **Kähler case with possibly non rational Kähler class.** We now assume that (M, J) is an algebraic manifold, that is, a submanifold of some complex projective space. When $[\bar{\omega}]$ is a rational class and $\bar{g}$ is the metric induced by the Fubini-Study metric, we showed in Sections 4.1 that, existence of linearly projectively stable point also implies holomorphicity. In this subsection we allow $[\bar{\omega}]$ to be any real Kähler class and $\bar{g}$ any J-induced metric. Take any Kähler metric $\bar{\omega}$ on M with $[\bar{\omega}] \in H^2(M, \mathbf{R}) \cap H^{1,1}(M, \mathbf{C})$. Let $\bar{g}$ be the Riemannian metric associated to $\bar{\omega}$ and J.

As (M, J) is an algebraic manifold it is easy to see that there exists a sequence of Kähler forms τ_m with $[\tau_m] \in H^2(M, \mathbf{Q}) \cap H^{1,1}(M, \mathbf{C})$, such that

$$||\tau_m - \bar{\omega}||_{C^2} \leq \varepsilon_m, \tag{4.11}$$

with $\varepsilon_m \to 0$ as $m \to \infty$. Here, the C^2 norm is taken with respect to the metric $\bar{\omega}$. Since $[\tau_m]$ is rational, there exists, for every $m \in \mathbf{N}$, a holomorphic line bundle $(L_m, h_m) \to M$ carrying a hermitian connection D_m of curvature $\frac{\sqrt{-1}}{2\pi}D_m^2 = \tau_m$. In particular, $c_1(L_m) = [\tau_m]$. For each positive integer $k > 0$, the hermitian metric h_m induces a hermitian metric h_m^k on L_m^k. Choose an orthonormal basis $\{S_{m,0}^k, \cdots, S_{m,N_{m,k}}^k\}$ of the space $H^0(M, L_m^k)$ of all holomorphic global sections of L_m^k. Here, the inner product on $H^0(M, L_m^k)$ is the natural one induced by the Kähler metric τ_m and the hermitian metric h_m^k on L_m^k. By Kodaira embedding theorem, there exists an integer $k_{m,0}$ such that if $k \geq k_{m,0}$, then such a basis induces a holomorphic embedding $\Psi_{m,k}$ of M into $\mathbf{CP}^{N_{m,k}}$ given by

$$\Psi_{m,k} : M \to \mathbf{CP}^{N_{m,k}}, \quad \Psi_{m,k}(z) := [S_{k,0}^k(z) : \cdots : S_{m,N_{k,m}}^k(z)]. \tag{4.12}$$

Let ω_{FS} be the standard Fubini-Study metric on $\mathbf{CP}^{N_{m,k}}$. Then $\frac{1}{k}\Psi^*_{m,k}\omega_{FS}$ is a Kähler form on M which lies in the same Kähler class as τ_m. We call $\frac{1}{k}\Psi^*_{m,k}\omega_{FS}$ the Bergman metric. A famous Theorem proved by Tian ([21]) tells us that

$$\left|\left|\frac{1}{k}\Psi^*_{m,k}\omega_{FS} - \tau_m\right|\right|_{C^2} \leq \frac{C}{\sqrt{k}}. \tag{4.13}$$

Here the C^2 norm is taken with respect to the metric τ_m and the constant C depends on τ_m. Because of (4.11), we can assume that the constant is uniformly bounded with respect to m. Although the Bergman metric $\frac{1}{k}\Psi^*_{m,k}\omega_{FS}$ depends on the Kähler metric τ_m, the set of Bergman metrics

$$\mathcal{P}_{m,k} := \left\{\frac{1}{k}\Psi^*_{m,k}\sigma^*(\omega_{FS})|\sigma \in Aut(\mathbf{CP}^{N_{m,k}})\right\}, \tag{4.14}$$

is independent of the choice of τ_m in $[\tau_m]$ and $\mathcal{P}_m := \cup_{k=1}^{\infty}\mathcal{P}_{k,m}$ is dense in $[\tau_m] \cap Ka(M)$ in the C^2-topology induced by the one on $\Lambda^2 M$. Here, $Ka(M)$ is the space of Kähler metrics on M. It is known that $\mathcal{P}_{m,k}$ has finite dimension for each k and m. Set

$$\mathcal{Q}_m := \left\{\frac{1}{k(m)}\Psi^*_{m,k(m)}\sigma^*(\omega_{FS})|\sigma \in Aut(\mathbf{CP}^{N_{m,k(m)}})\right\}, \tag{4.15}$$

where $k(m) \geq k_{m,0}$ is a sequence of integers such that $k(m) \to \infty$ as $m \to \infty$.

Define

$$B_m := \{\bar{\omega}\} - \{\tau_m\} + \mathcal{Q}_m = \left\{\bar{\omega} - \tau_m + \frac{1}{k(m)}\Psi^*_{m,k(m)}\sigma^*(\omega_{FS})|\sigma \in Aut(\mathbf{CP}^{N_{m,k(m)}})\right\}$$

Then B_m is a finitely dimensional submanifold of $[\bar{\omega}]$. In particular, for any $\sigma(t) \subset Aut(\mathbf{CP}^{N_{m,k(m)}})$, there exists a smooth function $\varphi(t)$ on M, such that

$$\bar{\omega}(t) := \bar{\omega} - \tau_m + \frac{1}{k(m)}\Psi^*_{m,k(m)}\sigma(t)^*(\omega_{FS}) = \bar{\omega} + 2\sqrt{-1}\partial\bar{\partial}\varphi(t) = \bar{\omega} + dd^c\varphi(t). \tag{4.16}$$

Definition 4.3. *Given a current $\mathcal{S} \in \mathcal{R}_p(M)$ in M, we call the mass **$\boldsymbol{M}$ has an** m-**linearly projectively stable point at** $\rho \in \mathcal{H}$ if there exists a smooth function ρ on M, such that $\bar{\omega}_\rho \in Ka(M)$ and*

$$\frac{d^2}{dt^2}|_{t=0}\boldsymbol{M}(t) \geq 0$$

for any $\bar{\omega}(t) = \bar{\omega} + tdd^c\dot{\varphi}$, where $\varphi(t)$ is given with $\sigma(0) = id$ and $\bar{\omega}$ replaced by $\bar{\omega}_\rho$ in the above construction.

Similar to the proof of Theorem 6.1 in [6] and the proof of Theorem 4.2 above, we can show that:

Theorem 4.4. *Let (M, J) be an algebraic manifold, $\bar{\omega}$ be any Kähler metric and $\mathcal{S} \in \mathcal{R}_2(M)$ be a current in M. Then there exists an integer K, such that if the mass has an m-linearly projectively stable point at $\rho \in \mathcal{H}$ for some $m \geq K$, then the current $\mathcal{S}$ is holomorphic 1-chain.*

5. $\mathcal{F}_c$-FUNCTIONAL

All we have seen up to now naturally induces to study a new type of functionals which measure the deviation of a surface from a holomorphic curve. We will carry out this analysis which resembles what Han-Li have done in [14] and [15] for a different type of functionals defined on the space of symplectic surfaces in a 4-manifold.

Let M^{2n} be a compact Kähler manifold with Kähler form $\bar{\omega}$, complex structure J, and compatible Kähler metric $\bar{g}$, such that for any $U, V \in TM$,

$$\bar{g}(U, V) = \bar{\omega}(U, JV). \tag{5.1}$$

Let Σ be a compact real surface. Fix an immersion

$$F : \Sigma \to (M, \bar{g}).$$

We consider the functional

$$\mathcal{F}_0(\Sigma) := \frac{1}{2}\int_{\Sigma_t} |J^{\perp}|^2 d\mu. \tag{5.2}$$

Notice that this precisely (up to a multiple) the functional associated to every embedded Σ when computing the first derivative of the Mass in the direction of the distance squared. Fix a point $x \in \Sigma$, it is easy to see that we can choose a $\bar{g}$-orthonormal frame $\{e_1, e_2, e_3, \cdots, e_{2n}\}$ of T_xM, such that $\{e_1, e_2\}$ spans the tangent space of Σ, $\{e_3, \cdots, e_{2n}\}$ spans the normal space of Σ, and the complex structure takes the form

$$J = \begin{pmatrix} (J_1)_{4\times 4} & 0_{4\times(2n-4)} \\ 0_{(2n-4)\times 4} & (J_2)_{(2n-4)\times(2n-4)} \end{pmatrix}, \tag{5.3}$$

where

$$J_1 = \begin{pmatrix} 0 & \cos\alpha & \sin\alpha & 0 \\ -\cos\alpha & 0 & 0 & -\sin\alpha \\ -\sin\alpha & 0 & 0 & \cos\alpha \\ 0 & \sin\alpha & -\cos\alpha & 0 \end{pmatrix}, \tag{5.4}$$

and $J_2 = diag\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$. From (5.3), we can easily see that

$$|J^{\perp}|^2 = \sum_{\alpha=3}^{2n}\sum_{i=1}^{2} (\bar{g}(Je_i, e_\alpha))^2 = 2\sin^2\alpha. \tag{5.5}$$

Therefore, actually we have

$$\mathcal{F}_0 = \int_{\Sigma} \sin^2\alpha d\mu. \tag{5.6}$$

For our later use, it natural and does not matter to add a constant in the integrand and we will consider the functional

$$\mathcal{F}_c(\Sigma) = \int_{\Sigma}(c + \sin^2\alpha)d\mu = \int_{\Sigma}(c + 1 - \cos^2\alpha)d\mu, \quad c \in \mathbf{R}. \tag{5.7}$$

5.1. **The first variation formula.** In this subsection, we first compute the first variation formula. Given a family of immersions

$$F_t : \Sigma \times (-\delta, \delta) \to M.$$

At a fixe point $x \in \Sigma$, let $\{x_i\}$ be the normal coordinate on Σ around x. The induced metric on $F_t(\Sigma)$ is

$$g_{ij}(t) = \langle \frac{\partial F_t}{\partial x_i}, \frac{\partial F_t}{\partial x_j} \rangle.$$

For simplicity, we denote $\frac{\partial F_0}{\partial x_i}$ by e_i, $g_{ij}(t)$ by g_{ij} and F_t by F. Suppose $\mathbf{V} = \frac{\partial F_t}{\partial t} |_{t=0}$ is the variational vector field. Then it is easy to see that

$$\frac{\partial}{\partial t} |_{t=0} \, g_{ij} = \langle \overline{\nabla}_{e_i} \mathbf{V}, e_j \rangle + \langle e_i, \overline{\nabla}_{e_j} \mathbf{V} \rangle. \tag{5.8}$$

Since

$$\cos \alpha_t = \frac{\bar{\omega}(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2})}{\sqrt{\det(g_{ij})}}, \tag{5.9}$$

we have

$$\mathcal{F}_c(F_t) = \int_\Sigma \frac{(c+1)\det(g_{ij}) - \bar{\omega}^2(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2})}{\sqrt{\det(g_{ij})}} dx^1 \wedge dx^2. \tag{5.10}$$

Denote

$$I_c = \frac{(c+1)\det(g_{ij}) - \bar{\omega}^2(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2})}{\sqrt{\det(g_{ij})}}. \tag{5.11}$$

Then using (5.8), we can easily get

$$\frac{\partial I_c}{\partial t} |_{t=0} = (c+1+\cos^2\alpha)\langle \overline{\nabla}_{e_i} \mathbf{V}, e_i \rangle - 2\cos\alpha(\bar{\omega}(\overline{\nabla}_{e_1}\mathbf{V}, e_2) + \bar{\omega}(e_1, \overline{\nabla}_{e_2}\mathbf{V})). \tag{5.12}$$

Therefore, we have

$$\mathcal{F}_c'(0) = \int_\Sigma [(c+1+\cos^2\alpha)\langle \overline{\nabla}_{e_i} \mathbf{V}, e_i \rangle - 2\cos\alpha(\bar{\omega}(\overline{\nabla}_{e_1}\mathbf{V}, e_2) + \bar{\omega}(e_1, \overline{\nabla}_{e_2}\mathbf{V}))] d\mu. \tag{5.13}$$

In order to obtain Euler-Lagrangian equation for the functional $\mathcal{V}$, we suppose the variational vector field $\mathbf{V}$ is a normal vector field. Then we have by (5.13)

$$\begin{aligned}
\mathcal{F}_c'(0) &= -\int_\Sigma (c+1+\cos^2\alpha)\langle \mathbf{V}, \mathbf{H}\rangle d\mu - 2\int_\Sigma \cos\alpha(\bar{\omega}(\overline{\nabla}_{e_1}\mathbf{V}, e_2) + \bar{\omega}(e_1, \overline{\nabla}_{e_2}\mathbf{V})) d\mu \\
&= -\int_\Sigma (c+1+\cos^2\alpha)\langle \mathbf{V}, \mathbf{H}\rangle d\mu \\
&\quad -2\int_\Sigma \cos\alpha \left(e_1[\bar{\omega}(\mathbf{V}, e_2)] + e_2[\bar{\omega}(e_1, \mathbf{V})] + \bar{\omega}(\mathbf{V}, \overline{\nabla}_{e_2} e_1 - \overline{\nabla}_{e_1} e_2) \right) d\mu \\
&= -\int_\Sigma (c+1+\cos^2\alpha)\langle \mathbf{V}, \mathbf{H}\rangle d\mu + 2\int_\Sigma [\bar{\omega}(\mathbf{V}, e_2)\nabla_{e_1}\cos\alpha + \bar{\omega}(e_1, \mathbf{V})\nabla_{e_2}\cos\alpha] d\mu \\
&= \int_\Sigma \langle \mathbf{V}, 2J(\nabla_{e_2}\cos\alpha e_1 - \nabla_{e_1}\cos\alpha e_2) - (c+1+\cos^2\alpha)\mathbf{H}\rangle d\mu. \qquad (5.14)
\end{aligned}$$

Therefore, the Euler-Lagrangian equation is given by

$$(c+1+\cos^2\alpha)\mathbf{H} + 2(J(\nabla_{e_1}\cos\alpha e_2 - \nabla_{e_2}\cos\alpha e_1))^\perp = 0. \tag{5.15}$$

We call a surface satisfying (5.15) an $\mathcal{F}_c$**-critical surface**.

Using (5.3), we can easily obtain that ([14])

$$(J\nabla\cos\alpha)^T = (\nabla_{e_1}\cos\alpha e_2 - \nabla_{e_2}\cos\alpha e_1)\cos\alpha. \tag{5.16}$$

If we further assume that Σ is simplectic, i.e., $\cos\alpha > 0$, then we see from (5.16) that (5.15) is equivalent to

$$\cos\alpha(c+1+\cos^2\alpha)\mathbf{H} + 2(J(J\nabla\cos\alpha)^T)^\perp = 0. \tag{5.17}$$

It is known that the minimal surface equation $\mathbf{H} = 0$ is a weakly elliptic system, where the kernel of the principle symbol arises from the diffeomorphisms of Σ. By computing the principle symbol of the equation (5.15), we can obtain the following for the $\mathcal{F}_c$-critical equation:

Proposition 5.1. *The equation (5.15) is an elliptic system modulo the diffeomorphisms of Σ for $c > 1$.*

We will present the proof of the proposition in the appendix.

5.2. **Elliptic equation of Kähler angle on $\mathcal{F}_c$-critical surfaces.** In this subsection, we will compute the elliptic equation satisfied by $\cos\alpha$ on a $\mathcal{F}_c$-critical surface. We assume that M is a Kähler surface, i.e., $n = 2$. Let's first recall the following result proved in [14]:

Proposition 5.2. *Let M be a Kähler surface with Kähler form $\bar\omega$ and let J be the complex structure compatible with ω on M. If Σ is a surface which is smoothly immersed in M with Kähler angle α, then*

$$\begin{aligned}\Delta\cos\alpha &= \cos\alpha(-|h^3_{1k} - h^4_{2k}|^2 - |h^4_{1k} + h^3_{2k}|^2)\\ &\quad + \sin\alpha(H^4_{,1} + H^3_{,2}) - Ric(Je_1, e_2)\sin^2\alpha,\end{aligned} \tag{5.18}$$

where Ric is the Ricci curvature tensor of $(M, \bar g)$ and $H^\alpha_{,i} = \langle\bar\nabla^N_{e_i}\boldsymbol{H}, v_\alpha\rangle$.

The main result in this subsection is as follows:

Theorem 5.3. *Suppose that M is a Kähler surface and Σ is an $\mathcal{F}_c$-critical surface in M with Kähler angle α. Then we have*

$$\begin{aligned}(3\cos^2\alpha + c - 1)\Delta\cos\alpha &= -\frac{2(c^2+2c+3)\cos\alpha + 4(c-1)\cos^3\alpha + 6\cos^5\alpha}{c+1+\cos^2\alpha}|\nabla\alpha|^2\\ &\quad - \sin^2\alpha(c+1+\cos^2\alpha)Ric(Je_1, e_2).\end{aligned} \tag{5.19}$$

In particular, if M is a Kähler-Einstein surface with scalar curvature R, then

$$\begin{aligned}(3\cos^2\alpha + c - 1)\Delta\cos\alpha &= -\frac{2(c^2+2c+3)\cos\alpha + 4(c-1)\cos^3\alpha + 6\cos^5\alpha}{c+1+\cos^2\alpha}|\nabla\alpha|^2\\ &\quad - \frac{R}{4}\sin^2\alpha\cos\alpha(c+1+\cos^2\alpha).\end{aligned} \tag{5.20}$$

Proof: Note that the we can choose local coordinate around the fix point p such that at p, the complex structure J takes the form of (5.4). However, we can not assure that it is of this form in a neighborhood of p. Since we will take derivatives with respect to the components of J, so around p, we assume that J takes the form

$$J = \begin{pmatrix} 0 & x & y & z \\ -x & 0 & z & -y \\ -y & -z & 0 & x \\ -z & y & -x & 0 \end{pmatrix}, \tag{5.21}$$

where $x^2 + y^2 + z^2 = 1$. By definition of the Kähler angle, we know that

$$x = \cos\alpha = \omega(e_1, e_2) = \langle Je_1, e_2\rangle.$$

Note also that at the fixed point p, we have that $y = \sin\alpha$ and $z = 0$. Now we have around p that

$$\begin{aligned}(J(\nabla_{e_1}\cos\alpha e_2 - \nabla_{e_2}\cos\alpha e_1))^\perp &= \partial_1\cos\alpha(Je_2)^\perp - \partial_2\cos\alpha(Je_1)^\perp \\ &= -\sin\alpha\partial_1\alpha(ze_3 - ye_4) + \sin\alpha\partial_2\alpha(ye_3 + ze_4) \\ &= \sin\alpha(y\partial_2\alpha - z\partial_1\alpha)e_3 + \sin\alpha(y\partial_1\alpha + z\partial_2\alpha)e_4.\end{aligned}$$

Combining this with (5.15), we finally get that

$$H^3 = -\frac{2\sin\alpha}{c+1+\cos^2\alpha}(y\partial_2\alpha - z\partial_1\alpha), \quad H^4 = -\frac{2\sin\alpha}{c+1+\cos^2\alpha}(y\partial_1\alpha + z\partial_2\alpha). \tag{5.22}$$

Furthermore,

$$\begin{aligned}\partial_1\cos\alpha &= \omega(\bar\nabla_{e_1}e_1, e_2) + \omega(e_1, \bar\nabla_{e_1}e_2) \\ &= h_{11}^\beta\langle Je_\beta, e_2\rangle + h_{12}^\beta\langle Je_1, e_\beta\rangle \\ &= (h_{11}^4 + h_{12}^3)y + (h_{12}^4 - h_{11}^3)z.\end{aligned} \tag{5.23}$$

Similarly, we can get that,

$$\partial_2\cos\alpha = (h_{22}^3 + h_{12}^4)y + (h_{22}^4 - h_{12}^3)z. \tag{5.24}$$

In particular, at p, we have

$$\partial_1\alpha = -(h_{11}^4 + h_{12}^3), \quad \partial_2\alpha = -(h_{22}^3 + h_{12}^4). \tag{5.25}$$

If we set $\mathbf{V} = \partial_2\alpha e_3 + \partial_1\alpha e_4$, then by direct computation, we have at p

$$\begin{aligned}|h_{1k}^3 - h_{2k}^4|^2 + |h_{1k}^4 + h_{2k}^3|^2 &= |\mathbf{H}|^2 + 2|\mathbf{V}|^2 + 2\mathbf{H}\cdot\mathbf{V} \\ &= \left(\frac{4\sin^4\alpha}{(c+1+\cos^2\alpha)^2} + 2 - \frac{4\sin^2\alpha}{c+1+\cos^2\alpha}\right)|\nabla\alpha|^2 \\ &= \frac{2(c^2+1) + 4(2c-1)\cos^2\alpha + 10\cos^4\alpha}{(c+1+\cos^2\alpha)^2}|\nabla\alpha|^2.\end{aligned} \tag{5.26}$$

Furthermore, using (5.22), we have at p,

$$\begin{aligned}\sin\alpha(H_{,1}^4 + H_{,2}^3) &= \sin\alpha(\langle\bar\nabla_{e_1}\mathbf{H}, e_4\rangle + \langle\bar\nabla_{e_2}\mathbf{H}, e_3\rangle) \\ &= \sin\alpha(\partial_1(H^4) + H^3\langle\bar\nabla_{e_1}e_3, e_4\rangle + \partial_2(H^3) + H^4\langle\bar\nabla_{e_2}e_4, e_3\rangle) \\ &= -\sin\alpha\left\{\partial_1[\frac{2\sin\alpha}{c+1+\cos^2\alpha}(y\partial_1\alpha + z\partial_2\alpha)]\right. \\ &\qquad \left. +\partial_2[\frac{2\sin\alpha}{c+1+\cos^2\alpha}(y\partial_2\alpha - z\partial_1\alpha)]\right\} \\ &\quad +\sin\alpha(H^3\langle\bar\nabla_{e_1}e_3, e_4\rangle + H^4\langle\bar\nabla_{e_2}e_4, e_3\rangle)\end{aligned}$$

$$
\begin{aligned}
&= -\sin^2\alpha\left[\partial_1\Big(\frac{2\sin\alpha}{c+1+\cos^2\alpha}\partial_1\alpha\Big)+\partial_2\Big(\frac{2\sin\alpha}{c+1+\cos^2\alpha}\partial_2\alpha\Big)\right]\\
&\quad -\frac{2\sin^2\alpha}{c+1+\cos^2\alpha}(\partial_1\alpha\partial_1 y+\partial_2\alpha\partial_2 y)-\frac{2\sin^2\alpha}{c+1+\cos^2\alpha}(\partial_2\alpha\partial_1 z-\partial_1\alpha\partial_2 z)\\
&\quad +\sin\alpha(H^3\langle\bar\nabla_{e_1}e_3,e_4\rangle+H^4\langle\bar\nabla_{e_2}e_4,e_3\rangle)\\
&= -\frac{2\sin^3\alpha}{c+1+\cos^2\alpha}\Delta\alpha-\frac{2(c+3)\cos\alpha-2(c+4)\cos^3\alpha+2\cos^5\alpha}{(c+1+\cos^2\alpha)^2}|\nabla\alpha|^2\\
&\quad -\frac{2\sin^2\alpha}{c+1+\cos^2\alpha}(\partial_1\alpha\partial_1 y+\partial_2\alpha\partial_2 y)-\frac{2\sin^2\alpha}{c+1+\cos^2\alpha}(\partial_2\alpha\partial_1 z-\partial_1\alpha\partial_2 z)\\
&\quad +\sin\alpha(H^3\langle\bar\nabla_{e_1}e_3,e_4\rangle+H^4\langle\bar\nabla_{e_2}e_4,e_3\rangle). \qquad (5.27)
\end{aligned}
$$

From $y=\langle Je_1,e_3\rangle$, we have at p,

$$
\begin{aligned}
\partial_1\alpha\partial_1 y+\partial_2\alpha\partial_2 y &= \partial_1\alpha(\langle J\bar\nabla_{e_1}e_1,e_3\rangle+\langle Je_1,\bar\nabla_{e_1}e_3\rangle)+\partial_2\alpha(\langle J\bar\nabla_{e_2}e_1,e_3\rangle+\langle Je_1,\bar\nabla_{e_2}e_3\rangle)\\
&= \partial_1\alpha(h_{11}^{\beta}\langle Je_\beta,e_3\rangle+\langle\cos\alpha e_2+\sin\alpha e_3,\bar\nabla_{e_1}e_3\rangle)\\
&\quad +\partial_2\alpha(h_{12}^{\beta}\langle Je_\beta,e_3\rangle+\langle\cos\alpha e_2+\sin\alpha e_3,\bar\nabla_{e_2}e_3\rangle)\\
&= -\cos\alpha(h_{11}^4+h_{12}^3)\partial_1\alpha-\cos\alpha(h_{12}^4+h_{22}^3)\partial_2\alpha\\
&= \cos\alpha|\nabla\alpha|^2. \qquad (5.28)
\end{aligned}
$$

Here, we have used (5.25). Similarly, from $z=\langle Je_1,e_4\rangle$, we have at p,

$$
\begin{aligned}
&\partial_2\alpha\partial_1 z-\partial_1\alpha\partial_2 z\\
&= \frac{(c-1)\cos\alpha+3\cos^3\alpha}{c+1+\cos^2\alpha}|\nabla\alpha|^2-\sin\alpha(\langle\bar\nabla_{e_1}e_3,e_4\rangle\partial_2\alpha+\langle\bar\nabla_{e_2}e_4,e_3\rangle\partial_1\alpha). \qquad (5.29)
\end{aligned}
$$

Putting (5.28) and (5.29) into (5.27) and using (5.22) yields

$$
\begin{aligned}
&\sin\alpha(H^4_{,1}+H^3_{,2})\\
&= \frac{2\sin^2\alpha}{c+1+\cos^2\alpha}\Delta\cos\alpha+\frac{-4(c+1)\cos\alpha+4c\cos^3\alpha+4\cos^5\alpha}{(c+1+\cos^2\alpha)^2}|\nabla\alpha|^2. \qquad (5.30)
\end{aligned}
$$

Then (5.19) follows from (5.26), (5.30) and Proposition 5.2. Q.E.D.

Applying the maximum principle to (5.20), we have

Corollary 5.4. *Suppose that M is a Kähler-Einstein surface with positive scalar curvature. Then any symplectic $\mathcal{F}_c$-critical surface with $c\geq 1$ in M is a holomorphic curve.*

By a standard computation as above and in [19], [14], we can obtain the second variation formula:

Proposition 5.5. *Let M be a Kähler-Einstein surface with scalar curvature R. If we choose $\boldsymbol{X}=x_3e_3+x_4e_4$ and $\boldsymbol{Y}=-J_\nu\boldsymbol{X}=x_4e_3-x_3e_4$, then the second variation formula of the functional $\mathcal{F}_c$ on a $\mathcal{F}_c$-critical surface is*

$$
\begin{aligned}
II_c(\boldsymbol{X})+II_c(\boldsymbol{Y}) &= -2(c+1)\int_\Sigma|\boldsymbol{X}|^2K_{1234}d\mu+2(c+1)\int_\Sigma|\overline{\nabla}^{\perp}\boldsymbol{X}|^2d\mu\\
&\quad -\frac{1}{2}\int_\Sigma(c+1+\cos^2\alpha)|\boldsymbol{X}|^2R\sin^2\alpha d\mu+\int_\Sigma(2\cos^2\alpha-1)|\bar\partial\boldsymbol{X}|^2
\end{aligned}
$$

$$-2\int_{\Sigma}|\boldsymbol{X}|^2\frac{(2\cos^2\alpha+c)(3\cos^2\alpha+c-1)}{c+1+\cos^2\alpha}|\nabla\alpha|^2d\mu. \tag{5.31}$$

Appendix A

In this appendix, we will prove Proposition 5.1. Before we prove the proposition, we first recall some basic facts on principle symbols.

Let Σ be a smooth manifold and suppose E is a smooth vector bundle over M. To a linear differential operator $P:\Gamma(E)\to\Gamma(E)$ of order k, at every point $x\in M$ and for every $\xi\in T_x^*M$ one can associated an algebraic object, the **principle symbol** $\sigma_\xi(P;x)$, often written simply by $\sigma_\xi(P)$. If, in local coordinate,

$$Pu=\sum_{|\alpha|\le k}a_\alpha(x)\partial^\alpha u, \tag{A.1}$$

where a_α are $dimE\times dimE$ matrices, then $\sigma_\xi(P;x)$ is the matrix

$$\sigma_\xi(P;x)=\sum_{|\alpha|=k}a_\alpha(x)\xi^\alpha. \tag{A.2}$$

Here, $\xi^\alpha=\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$.

Definition A.1. *A linear differential operator $P:\Gamma(E)\to\Gamma(E)$ is* ***(strictly) elliptic*** *if there exists $\lambda>0$ such that*

$$\langle\sigma_\xi(P;x)v,v\rangle\ge\lambda|\xi|^2|v|^2, \tag{A.3}$$

for all $(x,\xi)\in T^*(M)$ and $v\in\Gamma(E)$.

For a nonlinear differential operator $P(x,\partial^k u)$, its **linearization at u** is the linear operator

$$DP(u)v=\frac{d}{dt}P(x,\partial^k(u+tv))|_{t=0}. \tag{A.4}$$

The nonlinear equation $P(x,\partial^k u)=0$ is **elliptic at** u, if its linearization at u is elliptic in the sense of Definition A.1.

Now we can prove the following proposition:

Proposition A.1. *The equation (5.15) is a weakly elliptic system for $c>1$. The kernel of the principle symbol arises from the tangential directions of Σ.*

Proof: For simplicity, we suppose that Σ is a surface in $\mathbf{C}^n$. The general case is similar. In local coordinate, we can express the surface as

$$\begin{aligned} F:\Sigma &\longrightarrow \mathbf{C}^n=\mathbf{R}^{2n}\\ (x_1,x_2) &\longmapsto F(x_1,x_2)=(F^1(x_1,x_2),\cdots,F^{2n}(x_1,x_2)). \end{aligned}$$

We will use the following conventions:

$$1\le i,j,\cdots\le 2,\quad 3\le\alpha,\beta,\cdots\le 2n,\quad 1\le A,B,\cdots\le 2n.$$

The tangent space of Σ at a fixed point $x \in \Sigma$ is spanned by $\{e_1, e_2\}$ given by

$$e_1 = \frac{\partial F}{\partial x_1} = \frac{\partial F^A}{\partial x_1} E_A, \quad e_2 = \frac{\partial F}{\partial x_2} = \frac{\partial F^A}{\partial x_2} E_A, \tag{A.5}$$

where $\{E_1, \cdots, E_{2n}\}$ is the standard orthonormal basis of $\mathbf{R}^{2n}$. Therefore, the induced metric on Σ is given by

$$g_{ij} = \langle e_i, e_j \rangle = \frac{\partial F^A}{\partial x_i} \frac{\partial F^A}{\partial x_j}. \tag{A.6}$$

We can take the coordinate so that at the fixed point $x \in \Sigma$, we have $g_{ij}(x) = \delta_{ij}$. We will also take the standard complex structure J on $\mathbf{C}^n$ given by

$$J = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}. \tag{A.7}$$

Then we have

$$\begin{cases} JE_{2k-1} = E_{2k}, \\ JE_{2k} = E_{2k-1}. \end{cases} \tag{A.8}$$

Furthermore, we choose any orthonormal basis $\{n_\alpha\}_{\alpha=3}^{2n}$ of the normal space. Denote

$$P = (c + 1 + \cos^2 \alpha)\mathbf{H} + 2(J(\nabla_{e_1} \cos \alpha e_2 - \nabla_{e_2} \cos \alpha e_1))^{\perp}. \tag{A.9}$$

We will compute the principle symbol of P.

First we consider the principal part of $\mathbf{H}$. Note that by (A.6), we can easily see that the Christoffel symbol of the induced metric is:

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left\{ \frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l} \right\} = g^{kl} \frac{\partial^2 F^B}{\partial x_i \partial x_j} \frac{\partial F^B}{\partial x_l}.$$

Therefore, we have

$$\begin{aligned} \mathbf{H} = \Delta_\Sigma F &= g^{ij} \left(\frac{\partial^2 F}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial F}{\partial x_k} \right) \\ &= g^{ij} \left(\frac{\partial^2 F}{\partial x_i \partial x_j} - g^{kl} \frac{\partial^2 F^B}{\partial x_i \partial x_j} \frac{\partial F^B}{\partial x_l} \frac{\partial F}{\partial x_k} \right) \\ &= g^{ij} \left(\frac{\partial^2 F^A}{\partial x_i \partial x_j} - g^{kl} \frac{\partial F^A}{\partial x_k} \frac{\partial F^B}{\partial x_l} \frac{\partial^2 F^B}{\partial x_i \partial x_j} \right) E_A. \end{aligned} \tag{A.10}$$

The linearization of the operator at F in the direction G is:

$$D(\mathbf{H})(F)G = g^{ij} \left(\frac{\partial^2 G^A}{\partial x_i \partial x_j} - g^{kl} \frac{\partial F^A}{\partial x_k} \frac{\partial F^B}{\partial x_l} \frac{\partial^2 G^B}{\partial x_i \partial x_j} \right) E_A + first\ order\ terms. \tag{A.11}$$

Next, we will consider the second part of P. By definition,

$$\cos \alpha = \frac{\bar{\omega}(e_1, e_2)}{\sqrt{det(g_{ij})}} = \frac{\langle Je_1, e_2 \rangle}{\sqrt{det(g_{ij})}}. \tag{A.12}$$

By (A.5) and (A.7), we have:

$$\langle Je_1, e_2\rangle = \sum_{k=1}^{n}\left(\frac{\partial F^{2k-1}}{\partial x_1}\frac{\partial F^{2k}}{\partial x_2} - \frac{\partial F^{2k}}{\partial x_1}\frac{\partial F^{2k-1}}{\partial x_2}\right). \tag{A.13}$$

Therefore, we have

$$\cos\alpha = \frac{\sum_{k=1}^{n}\left(\frac{\partial F^{2k-1}}{\partial x_1}\frac{\partial F^{2k}}{\partial x_2} - \frac{\partial F^{2k}}{\partial x_1}\frac{\partial F^{2k-1}}{\partial x_2}\right)}{\sqrt{det(g_{ij})}},$$

$$\begin{aligned}
&\frac{\partial \cos\alpha}{\partial x_1} = \\
= \;& \frac{1}{\sqrt{det(g_{ij})}}\left\{\sum_{k=1}^{n}\left(\frac{\partial^2 F^{2k-1}}{\partial x_1^2}\frac{\partial F^{2k}}{\partial x_2} + \frac{\partial F^{2k-1}}{\partial x_1}\frac{\partial^2 F^{2k}}{\partial x_1\partial x_2} - \frac{\partial^2 F^{2k}}{\partial x_1^2}\frac{\partial F^{2k-1}}{\partial x_2} - \frac{\partial F^{2k}}{\partial x_1}\frac{\partial^2 F^{2k-1}}{\partial x_1\partial x_2}\right)\right.\\
&\left. -\frac{1}{2}\langle Je_1, e_2\rangle g^{ij}\left(\frac{\partial^2 F^A}{\partial x_1\partial x_i}\frac{\partial F^A}{\partial x_j} + \frac{\partial F^A}{\partial x_i}\frac{\partial^2 F^A}{\partial x_1\partial x_j}\right)\right\},
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial \cos\alpha}{\partial x_2} = \\
= \;& \frac{1}{\sqrt{det(g_{ij})}}\left\{\sum_{k=1}^{n}\left(\frac{\partial^2 F^{2k-1}}{\partial x_1\partial x_2}\frac{\partial F^{2k}}{\partial x_2} + \frac{\partial F^{2k-1}}{\partial x_1}\frac{\partial^2 F^{2k}}{\partial x_2^2} - \frac{\partial^2 F^{2k}}{\partial x_1\partial x_2}\frac{\partial F^{2k-1}}{\partial x_2} - \frac{\partial F^{2k}}{\partial x_1}\frac{\partial^2 F^{2k-1}}{\partial x_2^2}\right)\right.\\
&\left. -\frac{1}{2}\langle Je_1, e_2\rangle g^{ij}\left(\frac{\partial^2 F^A}{\partial x_2\partial x_i}\frac{\partial F^A}{\partial x_j} + \frac{\partial F^A}{\partial x_i}\frac{\partial^2 F^A}{\partial x_2\partial x_j}\right)\right\}.
\end{aligned}$$

By our choice of the frame, at the fixed point x, we have

$$\begin{aligned}
(J(\nabla_{e_1}\cos\alpha e_2 - \nabla_{e_2}\cos\alpha e_1))^{\perp} &= \frac{\partial\cos\alpha}{\partial x_1}(Je_2)^{\perp} - \frac{\partial\cos\alpha}{\partial x_2}(Je_1)^{\perp}\\
&= \frac{\partial\cos\alpha}{\partial x_1}\langle Je_2, n_\alpha\rangle n_\alpha - \frac{\partial\cos\alpha}{\partial x_2}\langle Je_1, n_\alpha\rangle n_\alpha.
\end{aligned} \tag{A.14}$$

Notice that $\cos\alpha$, g_{ij}, e_i and n_α only involve first order derivative of the immersion F. Therefore, by (A.11) and (A.14), we know that the linearization of the operator P at F in the direction G (computed at the point x) is:

$$\begin{aligned}
D(\mathbf{P})(F)G \;=\;& (c+1+\cos^2\alpha)g^{ij}\left(\frac{\partial^2 G^A}{\partial x_i\partial x_j} - g^{kl}\frac{\partial F^A}{\partial x_k}\frac{\partial F^B}{\partial x_l}\frac{\partial^2 G^B}{\partial x_i\partial x_j}\right)E_A\\
&+2\left\{\sum_{k=1}^{n}\left(\frac{\partial^2 G^{2k-1}}{\partial x_1^2}\frac{\partial F^{2k}}{\partial x_2} + \frac{\partial F^{2k-1}}{\partial x_1}\frac{\partial^2 G^{2k}}{\partial x_1\partial x_2} - \frac{\partial^2 G^{2k}}{\partial x_1^2}\frac{\partial F^{2k-1}}{\partial x_2} - \frac{\partial F^{2k}}{\partial x_1}\frac{\partial^2 G^{2k-1}}{\partial x_1\partial x_2}\right)\right.\\
&\left. -\frac{1}{2}\langle Je_1, e_2\rangle g^{ij}\left(\frac{\partial^2 G^A}{\partial x_1\partial x_i}\frac{\partial F^A}{\partial x_j} + \frac{\partial F^A}{\partial x_i}\frac{\partial^2 G^A}{\partial x_1\partial x_j}\right)\right\}\langle Je_2, n_\alpha\rangle n_\alpha\\
&-2\left\{\sum_{k=1}^{n}\left(\frac{\partial^2 G^{2k-1}}{\partial x_1\partial x_2}\frac{\partial F^{2k}}{\partial x_2} + \frac{\partial F^{2k-1}}{\partial x_1}\frac{\partial^2 G^{2k}}{\partial x_2^2} - \frac{\partial^2 G^{2k}}{\partial x_1\partial x_2}\frac{\partial F^{2k-1}}{\partial x_2} - \frac{\partial F^{2k}}{\partial x_1}\frac{\partial^2 G^{2k-1}}{\partial x_2^2}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\langle Je_1, e_2\rangle g^{ij}\left(\frac{\partial^2 G^A}{\partial x_2\partial x_i}\frac{\partial F^A}{\partial x_j}+\frac{\partial F^A}{\partial x_i}\frac{\partial^2 G^A}{\partial x_2\partial x_j}\right)\Bigg\}\langle Je_1, n_\alpha\rangle n_\alpha \\
&+ \textit{first order terms.}
\end{aligned}
\tag{A.15}
$$

We will denote G^T and $G^\perp$ the projection of $G\in \mathbf{R}^{2n}$ on the tangent bundle and normal bundle of Σ, respectively. It is easy to see that

$$
|G^T|^2 = g^{kl}\langle G, \frac{\partial F}{\partial x_k}\rangle\langle G, \frac{\partial F}{\partial x_l}\rangle = g^{kl}G^A G^B\frac{\partial F^A}{\partial x_k}\frac{\partial F^B}{\partial x_l}.
$$

Then we see that the principle symbol of P is given by:

$$
\begin{aligned}
\langle \sigma(D(P))(x,\xi)G, G\rangle &= (c+1+\cos^2\alpha)g^{ij}\left(\xi_i\xi_j|G|^2 - g^{kl}\frac{\partial F^A}{\partial x_k}\frac{\partial F^B}{\partial x_l}\xi_i\xi_j G^A G^B\right)\\
&\quad +2\left\{\sum_{k=1}^n\left(G^{2k-1}\frac{\partial F^{2k}}{\partial x_2}-G^{2k}\frac{\partial F^{2k-1}}{\partial x_2}\right)\langle Je_2, G^\perp\rangle\xi_1^2\right.\\
&\quad +\sum_{k=1}^n\left(G^{2k-1}\frac{\partial F^{2k}}{\partial x_1}-G^{2k}\frac{\partial F^{2k-1}}{\partial x_1}\right)\langle Je_1, G^\perp\rangle\xi_2^2\\
&\quad -\sum_{k=1}^n\left[\left(G^{2k-1}\frac{\partial F^{2k}}{\partial x_1}-G^{2k}\frac{\partial F^{2k-1}}{\partial x_1}\right)\langle Je_2, G^\perp\rangle\right.\\
&\quad +\left.\left(G^{2k-1}\frac{\partial F^{2k}}{\partial x_2}-G^{2k}\frac{\partial F^{2k-1}}{\partial x_2}\right)\langle Je_1, G^\perp\rangle\right]\xi_1\xi_2\\
&\quad \left.+\cos\alpha g^{ij}G^A\left(\frac{\partial F^A}{\partial x_j}\langle Je_1, G^\perp\rangle\xi_2\xi_i - \frac{\partial F^A}{\partial x_j}\langle Je_2, G^\perp\rangle\xi_1\xi_i\right)\right\}.
\end{aligned}
\tag{A.16}
$$

By (A.8), we have

$$
\begin{aligned}
\langle \sigma(D(P))(x,\xi)G, G\rangle &= (c+1+\cos^2\alpha)|\xi|^2|G^\perp|^2\\
&\quad +2\Big\{\Big(-\langle Je_2, G^\perp\rangle\langle Je_2, G\rangle - \langle Je_1, e_2\rangle\langle G, e_1\rangle\langle Je_2, G^\perp\rangle\Big)\xi_1^2\\
&\quad +\Big(-\langle Je_1, G^\perp\rangle\langle Je_1, G\rangle + \langle Je_1, e_2\rangle\langle G, e_2\rangle\langle Je_1, G^\perp\rangle\Big)\xi_2^2\\
&\quad +\Big(\langle Je_2, G^\perp\rangle\langle Je_1, G\rangle + \langle Je_1, G^\perp\rangle\langle Je_2, G\rangle\\
&\quad +\langle Je_1, e_2\rangle\langle G, e_1\rangle\langle Je_1, G^\perp\rangle - \langle Je_1, e_2\rangle\langle G, e_2\rangle\langle Je_2, G^\perp\rangle\Big)\xi_1\xi_2\Big\}.
\end{aligned}
\tag{A.17}
$$

Note that $(Je_1)^T = \langle Je_1, e_2\rangle e_2$ and $(Je_2)^T = -\langle Je_1, e_2\rangle e_1$. Thus we have

$$
\begin{aligned}
\langle \sigma(D(P))(x,\xi)G, G\rangle &= (c+1+\cos^2\alpha)|\xi|^2|G^\perp|^2\\
&\quad -2\left(\langle G^\perp, Je_2\rangle^2\xi_1^2 - 2\langle G^\perp, Je_1\rangle\langle G^\perp, Je_2\rangle\xi_1\xi_2 + \langle G^\perp, Je_1\rangle^2\xi_2^2\right)\\
&= \left((c+1+\cos^2\alpha)|G^\perp|^2 - 2\langle G^\perp, Je_2\rangle^2\right)\xi_1^2\\
&\quad +4\langle G^\perp, Je_1\rangle\langle G^\perp, Je_2\rangle\xi_1\xi_2\\
&\quad +\left((c+1+\cos^2\alpha)|G^\perp|^2 - 2\langle G^\perp, Je_1\rangle^2\right)\xi_2^2.
\end{aligned}
\tag{A.18}
$$

The coefficient matrix is given by
(A.19)

$$O = \left(\begin{array}{cc} (c+1+\cos^2\alpha)|G^\perp|^2 - 2\langle G^\perp, Je_2\rangle^2 & 2\langle G^\perp, Je_1\rangle\langle G^\perp, Je_2\rangle \\ 2\langle G^\perp, Je_1\rangle\langle G^\perp, Je_2\rangle & (c+1+\cos^2\alpha)|G^\perp|^2 - 2\langle G^\perp, Je_1\rangle^2 \end{array} \right).$$

We have

$$\begin{aligned} detA &= (c+1+\cos^2\alpha)|G^\perp|^2\Big((c+1+\cos^2\alpha)|G^\perp|^2 - 2(\langle G^\perp, Je_1\rangle^2 + \langle G^\perp, Je_2\rangle^2)\Big) \\ &\geq (c-1+\cos^2\alpha)(c+1+\cos^2\alpha)|G^\perp|^4. \end{aligned} \tag{A.20}$$

From (A.19) and (A.20), we see that if $c > 1$, then for any $(\xi_1, \xi_2) \neq (0,0)$ and $G \in \mathbf{R}^{2n}$, we have

$$\langle \sigma(D(P))(x,\xi)G, G\rangle \geq 0,$$

and the inequality is strict unless $G^\perp = 0$, i.e., G is tangential to Σ. This finishes the proof of the Proposition. Q.E.D.

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