On Taming and Compatible Symplectic Forms

ON TAMING AND COMPATIBLE SYMPLECTIC FORMS

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Abstract. Let (X, J) be an almost complex manifold. The almost complex structure J acts on the space of 2-forms on X as an involution. A 2-form α is *J-anti-invariant* if $J\alpha = -\alpha$. We investigate the anti-invariant forms and their relation to taming and compatible symplectic forms. For every closed almost complex manifold, in contrast to invariant forms, we show that the space of closed anti-invariant forms has finite dimension.

If X is a closed almost-complex manifold with a taming symplectic form then we show that there are no non trivial exact anti-invariant forms. On the other hand we construct many examples of almost-complex manifolds with exact anti-invariant forms, which are therefore not tamed by any symplectic form. In particular we use our analysis to give an explicit example of an almost-complex structure which is locally almost-Kähler but not globally tamed.

The non-existence of exact anti-invariant forms however does not in itself imply that there exists a taming symplectic form. We show how to construct examples in all dimensions.

Introduction

Almost-complex structures on a manifold X can be categorized according to whether or not there exist taming or compatible symplectic forms. We recall that a symplectic form ω *tames* an almost-complex structure J if $\omega(v, Jv) > 0$ for all nonzero tangent vectors v, and ω is *compatible* with J if the formula $g(v, w) = \omega(v, Jw)$ defines a Riemannian metric g on X. If ω is compatible with J then the triple (X, J, ω) is sometimes called an almost-Kähler manifold. A Kähler manifold is an almost-Kähler manifold with J integrable.

Let $\mathcal{J} = \mathcal{J}(X)$ be the set of almost-complex structures on X, then we can define subsets $\mathcal{J}_{tame} = \mathcal{J}_{tame}(X)$ and $\mathcal{J}_{comp} = \mathcal{J}_{comp}(X)$ of $\mathcal{J}$ to be the almost-complex structures for which there exists a taming or compatible symplectic form respectively. We also define a subset $\mathcal{J}_{loc.tame}$ of $\mathcal{J}_{tame}$ which consists of locally tame almost-complex structures, that is, $J \in \mathcal{J}_{loc.tame}$ if there exists an open cover $\{U_i\}$ of X such that $J|_{U_i} \in \mathcal{J}_{tame}(U_i)$ for all i. Similarly we can define locally compatible almost-complex structures $\mathcal{J}_{loc.comp} \subset \mathcal{J}_{comp}$. It is immediate that $\mathcal{J}_{loc.tame} = \mathcal{J}$ and that the set of (integrable) complex structures $\mathcal{I} \subset \mathcal{J}_{loc.comp}$.

In summary we have the following diagram of inclusions.

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$$
\begin{array}{ccccccc}
\mathcal{I} & \subset & \mathcal{J}_{loc.comp} & \subset_j & \mathcal{J}_{loc.tame} & = & \mathcal{J} \\
 & & \bigcup_{k_1} & & \bigcup_{k_2} & & \\
 & & \mathcal{J}_{comp} & \subset_i & \mathcal{J}_{tame} & &
\end{array} \tag{1}
$$

When our manifold X has dimension 4 the map j is actually a surjection, in other words $\mathcal{J} = \mathcal{J}_{loc.comp}$. For a complete proof of this see Lejmi [13, Theorem 1]. It is a question of Donaldson [4, question 2] as to whether the map i is also a surjection.

In this paper a key observation is the following.

Proposition 0.1. *If X is closed (compact without boundary) and $J \in \mathcal{J}_{tame}$ then there are no non-zero exact J-anti-invariant 2 forms.*

Recall that a 2-form α is anti-invariant if $J\alpha = -\alpha$, where $J\alpha(v,w) = \alpha(Jv, Jw)$. In dimension 4 there are no non-zero exact anti-invariant forms with respect to any J, see Corollary 1.2, but in higher dimensions the existence of an exact anti-invariant form is an obstruction to the existence of a taming symplectic form.

It is in fact quite easy to find examples of almost-complex structures admitting exact anti-invariant forms. The following is a consequence of Theorem 1.4.

Theorem 0.2. *Suppose that W^{4n} is a $4n$ dimensional manifold with trivial tangent bundle. Then $X = W \times S^1 \times S^1$ has an almost-complex structure J for which there exist non-zero exact anti-invariant 2-forms.*

The methods used to establish Theorem 0.2 are very topological, they rely on Gromov's h-principle. Therefore we have little control on the almost-complex structure, in particular it is difficult in this way to find examples which lie in $\mathcal{J}_{loc.comp}$. This issue is addressed in section 3. For example, in section 3.3 we explicitly construct a nonintegrable almost-complex structure on a 6-dimensional manifold which is locally compatible yet admits a non-zero exact anti-invariant form, and so lies in $\mathcal{J}_{loc.comp} \setminus (\mathcal{J}_{tame} \cup \mathcal{I})$.

We remark however that the non-existence of exact anti-invariant forms is not a sufficient condition for an almost-complex structure to be tamed by a symplectic form. In dimension 4, since we never have any exact anti-invariant forms, examples are given by any almost-complex manifolds which are not symplectic. In higher dimensions, we can use a theorem of Peternell [17, Theorem 1.4] to imply the following.

Theorem 0.3. *A non-Kähler Moišezon manifold has no non-zero exact anti-invariant forms but no taming symplectic form.*

In dimension 6, a simpler concrete example is the following.

Theorem 0.4. *The product of the Hopf surface and $\mathbb{C}P^1$ does not have non-zero exact anti-invariant forms or any symplectic forms at all.*

Cohomology properties can also be used to categorize almost-complex structures. Following [5] we can define subspaces $H_J^+(X), H_J^-(X) \subset H^2(X, \mathbb{R})$, the second de Rham cohomology of X, as follows. A class $\mathfrak{a} \in H_J^+(X)$ if there exists a 2 form α with $[\alpha] = \mathfrak{a}$ and $J\alpha = \alpha$. Similarly a class $\mathfrak{a} \in H_J^-(X)$ if it has a representative α which is anti-invariant with respect to J. The almost-complex manifold (X, J) is called $\mathcal{C}^\infty$-*pure* if

$H_J^+(X) \cap H_J^-(X) = \{0\}$ and $\mathcal{C}^\infty$-*full* if $H_J^+(X) + H_J^-(X) = H^2(X, \mathbb{R})$. In [5, Theorem 2.3], Drahjici, Li and Zhang show that an almost complex structure on a compact 4-dimensional manifold is $\mathcal{C}^\infty$-pure-and-full. Furthermore, in [14, Theorem 1.3], Li and Zhang proved that if J is $\mathcal{C}^\infty$-full and if the compatible cone

$$\mathcal{K}_J^c = \left\{[\omega] \in H^2(X; \mathbb{R}) \mid \omega \text{ is compatible with } J\right\}$$

is non-empty, then

$$\mathcal{K}_J^t \;=\; \mathcal{K}_J^c \;+\; H_J^-(X).$$

Here we focus on $H_J^-(X)$ and study $\mathcal{Z}_J^-(X)$, the real vector space of closed anti-invariant 2-forms. We have already seen that if X is closed and $J \in \mathcal{J}_{tame}$ then there are no nonzero exact anti-invariant forms and so the map

$$\mathcal{Z}_J^-(X) \to H_J^-(X) \subset H^2(X, \mathbb{R})$$

is an injection. This can be contrasted with the case of invariant forms $\mathcal{Z}_J^+(X)$. At least if J is integrable then $\mathcal{Z}_J^+(X)$ is always infinite dimensional.

In the case when $J \in \mathcal{J}_{comp}$ we can be more precise. Let g be the Riemannian metric associated to a compatible symplectic form. Then we show in Proposition 2.2 that $\mathcal{Z}_J^-(X) \subset \mathcal{H}_g(X)$, the set of harmonic 2-forms with respect to g. In other words, we have the following.

Proposition 0.5. *J-anti-invariant forms are harmonic with respect to any Riemannian metric associated to a compatible symplectic form.*

It turns out that even if $J \notin \mathcal{J}_{comp}$ the closed anti-invariant forms $\mathcal{Z}_J^-$ lie in the kernel of a second order elliptic operator, see Proposition 2.4. Hence we have an alternative proof of a theorem from [10] saying that anti-invariant forms satisfy a unique continuation principle, see Proposition 2.6.

The paper is organized as follows. After fixing some notation, we establish some basic facts about anti-invariant forms in section 1 and prove Proposition 0.1 and Theorem 0.2. To complement these results we also derive the examples of Theorems 0.3 and 0.4. In section 2 we discuss the relation between anti-invariant forms and harmonic forms and in particular prove the unique continuation theorem for anti-invariant forms. Finally in section 3 we construct our explicit examples.

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1. Anti-invariant forms on almost-complex manifolds.

We start by fixing some notation. Let (X, J) be a $2n$-dimensional almost complex manifold and denote by $\Lambda^2(X)$ the space of 2-forms on X. The almost complex structure acts on $\Lambda^2(X)$ as an involution by setting $J\alpha(u, v) = \alpha(Ju, Jv)$. Following [5], $\alpha \in \Lambda^2(X)$ is said to be *J-invariant* or *invariant*, if $J\alpha = \alpha$ and *J-anti-invariant* or *anti-invariant* if $J\alpha = -\alpha$. We denote by $\Lambda_J^+(X)$ and by $\Lambda_J^-(X)$ the space of J-invariant, J-anti-invariant forms respectively. Let $\mathcal{Z}(X)$ be the space of closed 2-forms on X. We set $\mathcal{Z}_J^\pm(X) = \Lambda_J^\pm(X) \cap \mathcal{Z}(X)$ and

$$H_J^\pm(X) = \{\mathfrak{a} \in H^2(X; \mathbb{R}) \mid \mathfrak{a} = [\alpha],\; \alpha \in \mathcal{Z}_J^\pm(X)\}\,.$$

According to [5], an almost complex structure J is said to be $\mathcal{C}^\infty$-*pure* if $H^+_J(X)\cap H^-_J(X) = \{0\}$, $\mathcal{C}^\infty$-*full* if $H^2(X;\mathbb{R}) = H^+_J(X) + H^-_J(X)$.

We begin our study of anti-invariant forms in dimension 4.

Lemma 1.1. *Let (X, J, g) be a 4-dimensional almost Hermitian manifold. Let $\alpha \in \Lambda^2_-(X)$. Then $\alpha^2 = f\,\mathrm{Vol}_g$, where $f : X \to \mathbb{R}$ is a smooth non-negative function on X and Vol_g denotes the Riemannian volume form.*

Proof. Let $p \in X$ and let $\{v_1, Jv_1, v_2, Jv_2\}$ be a g-orthonormal positive basis of T_pX. Then, if $\{v_1^*, Jv_1^*, v_2^*, Jv_2^*\}$ denotes the dual basis of $\{v_1, Jv_1, v_2, Jv_2\}$, any J-anti-invariant 2-form at p can be written as

$$\alpha(p) = \lambda(v_1^* \wedge v_2^* - Jv_1^* \wedge Jv_2^*) + \mu(v_1^* \wedge Jv_2^* + Jv_1^* \wedge v_2^*)\,.$$

Hence,

$$\alpha^2(p) = 2(\lambda^2 + \mu^2)(v_1^* \wedge Jv_1^* \wedge v_2^* \wedge Jv_2^*)\,.$$

□

Corollary 1.2. *Let (X, J, g) be a compact 4-dimensional almost Hermitian manifold. Then there are no non-trivial exact anti-invariant forms on X.*

Proof. Let $\alpha \in \mathcal{Z}^-_J(X)$. By assumption $\alpha \neq 0$; assume by contradiction that $\alpha = d\beta$. Then,

$$0 = \int_X d(\beta \wedge d\beta) = \int_X \alpha^2 = \int_X f\,\mathrm{Vol}_g > 0\,,$$

and this is absurd. □

We will see in Theorem 1.4 and in section 3 that non-zero exact anti-invariant forms can exist on higher dimensional almost-complex manifolds, but the following proposition rules this out if the almost-complex structure $J \in \mathcal{J}_{tame}$.

Proposition 1.3. *Let (X, J) be a compact $2n$-dimensional almost complex manifold. If ω is a symplectic form taming J, then there are no non-zero exact J-anti-invariant forms.*

Proof. By contradiction. Let ω be a symplectic form taming J. Let $\alpha \in \mathcal{Z}^-_J(X)$ be exact, $\alpha = d\beta$. Let $2k$ be the maximal rank of $\alpha(p)$, for $p \in X$. The following claim from linear algebra is useful.

Claim. A skew-symmetric anti-invariant 2-form η on a complex vector space V has rank $2r$ divisible by 4. Moreover η^r generates the complex orientation on $V/\ker\eta$ (with its induced complex structure) if $r/2$ is even and the opposite of the complex orientation if $r/2$ is odd.

Proof of claim. First note that as η is anti-invariant $\ker(\eta)$ is indeed a complex subspace of V and so the quotient $W = V/\ker(\eta)$ inherits a complex structure. Then η^r is a volume form on W which implies that $J\eta^r = \lambda\eta^r$ for some $\lambda > 0$. As η is anti-invariant this in turn implies that r must be even and we can take $\lambda = 1$.

Now we choose a basis of W of the form $e_1, f_1, e_2, f_2, \ldots, e_r, f_r$ such that if i is odd we have $\eta(e_i, f_i) = 1$ and $Je_i = e_{i+1}$ and $Jf_i = f_{i+1}$. Then necessarily if i is even we have $\eta(e_i, f_i) = -1$. We may also assume that $\eta(e_i, e_j) = \eta(f_i, f_j) = 0$ for all i, j and $\eta(e_i, f_j) = 0$ for all $i \neq j$.

Given this we compute

$$\eta^r(e_1, e_2, f_1, f_2, \dots, f_{r-1}, f_r) = (-1)^r \eta(e_1, f_1)\eta(e_2, f_2)\dots\eta(e_r, f_r) = (-1)^{r/2}$$

and the claim follows. $\square$

Returning to the proof, we have that k is even and $\alpha^k \wedge \omega^{n-k} = (-1)^{k/2} f\omega^n$, where f is a non-negative function on X. This is because ω gives the complex orientation on any complex subspace. The function f is positive exactly when α has maximum rank. Therefore

$$0 = (-1)^{k/2} \int_X d(\beta \wedge (d\beta)^{k-1} \wedge \omega^{n-k}) = (-1)^{k/2} \int_X \alpha^k \wedge \omega^{n-k} = \int_X f\, \omega^n > 0$$

and this is absurd. $\square$

To complement the above proposition, the following theorem shows that almost-complex structures admitting exact anti-invariant forms can be constructed under fairly general hypotheses.

Theorem 1.4. *Suppose that an orientable manifold M^{4n+1} admits a 2 form $\tilde{\alpha}$ of everywhere maximal rank $4n$ such that the quotient bundle $TM/\ker\tilde{\alpha} \to M$ has an almost-complex structure for which $\tilde{\alpha}$ is anti-invariant. Then there exists an almost-complex structure J on $M \times S^1$ which admits an exact nonzero anti-invariant 2 form.*

We emphasize that the hypotheses of the theorem are purely topological, in particular we do not need to assume that $\tilde{\alpha}$ is closed. The proof does not use the hypothesis that M has dimension $4n+1$, only that the dimension is odd. However we have seen above that the rank of an anti-invariant form is necessarily a multiple of 4.

Proof. The result is a consequence of a theorem of McDuff, see [15] and [7, Thm 10.4.1], which states (in a simple form) that a 2-form of maximal rank on an odd dimensional manifold can be deformed through forms of maximal rank to an exact form. Hence we can find maximal rank 2-forms α_t on M such that $\alpha_0 = \tilde{\alpha}$ and α_1 is exact. Fixing a Riemannian metric on M the $4n$ dimensional subbundles $(\ker\tilde{\alpha})^\perp$ and $(\ker\alpha_1)^\perp$ are isomorphic as symplectic vector bundles with forms $\tilde{\alpha}$ and α_1 respectively. Hence $(\ker\alpha_1)^\perp$ also admits an almost-complex structure J anti-invariant with respect to α_1. The corresponding orientation on $(\ker\alpha_1)^\perp$ together with one on M determines a trivialization of $\ker\alpha$. Hence we can extend J to an almost-complex structure on $M \times S^1$ such that J maps $\ker\alpha_1$ onto TS^1. Let us pull back α_1 to a 2-form α on $M \times S^1$ using the natural projection. Then α is also nonzero and exact. Finally since α vanishes on the complex planes spanned by $\ker\alpha_1$ and TS^1 it is anti-invariant as required. $\square$

To close this section we discuss our examples of complex manifolds which have no exact anti-invariant 2-forms but still have no taming symplectic forms.

First let X be a Moišezon manifold, that is, a compact complex manifold which admits a proper modification from a projective manifold. Then the following result holds, see Peternell [17, Thm.1.4]

Theorem 1.5. *Let X be a Moišezon manifold. Assume there exists a real $(1,1)$-form ω and a real 2-form φ on X such that*

i) *ω is positive definite,*

ii) $d(\omega - \varphi) = 0$,
iii) $\int_C \varphi = 0$ *for all curves* $C \subset X$.
Then X *is projective.*

This directly implies Theorem 0.3 as follows.

Proposition 1.6. *Any non-Kähler Moišezon manifold* X *has no non-trivial* d*-exact anti-invariant* 2*-forms and no taming symplectic forms.*

Proof. First, if α is a d-exact anti-invariant 2-form α on X then its pull back to a projective manifold is also exact and anti-invariant. By Proposition 1.3 this implies that α must be identically zero.

The fact that X has no taming symplectic form has already been pointed out by Draghici and Zhang, [6], but we give the argument here for completeness. Arguing by contradiction, suppose that η is a taming symplectic form. We can write $\eta = \omega - \psi_1 - \bar{\psi}_2$ where ω is a real $(1,1)$-form, ψ_1 is a real $(2,0)$-form and $\bar{\psi}_2$ is a real $(0,2)$-form. Then ψ_1 and $\bar{\psi}_2$ vanish on complex lines and so since η is taming the form ω is positive definite. Setting $\varphi = \psi_1 + \bar{\psi}_2$ the remaining two conditions of Theorem 1.5 are clearly satisfied and so X must be projective, a contradiction. □

Finally we give a proof of Theorem 0.4. Let Y be the Hopf surface, that is, $Y = (\mathbb{C}^2 \setminus 0)/z \sim 2z$ with its induced complex structure.

Proposition 1.7. *The product* $X = \mathbb{C}P^1 \times Y$ *does not have exact anti-invariant forms or any symplectic forms at all.*

Proof. The 6-manifold X is diffeomorphic to $S^2 \times S^3 \times S^1$ and so has no cohomology classes $\mathfrak{a}$ with $\mathfrak{a}^3 \neq 0$. Therefore it admits no symplectic forms at all.

There are two projections $p_1, p_2 : X \to \mathbb{C}P^1$. The first is just projection onto the first factor, the second is induced by projection onto Y and then quotienting by $\mathbb{C}^*$ to get $Y/\mathbb{C}^* = (\mathbb{C}^2 \setminus 0)/\mathbb{C}^* = \mathbb{C}P^1$. Therefore we can pull-back the Fubini-Study form using p_1 and p_2 to get invariant 2-forms ω_1 and ω_2 on X.

Suppose that there exists a non-zero exact anti-invariant 2-form α on X. As we are working in dimension 6 we have that $\alpha(x)$ has rank 0 or 4 at all points $x \in X$. Observe that applying Stokes' Theorem as in Proposition 1.3 gives a contradiction if there exists a closed 2-form Ω on X which satisfies $\alpha^2 \wedge \Omega \geq 0$ and $\alpha^2 \wedge \Omega(x) > 0$ at least for some $x \in X$. When $\alpha \neq 0$ its kernel is a complex line. Therefore as ω_1 and ω_2 are invariant we have $\alpha^2 \wedge (\omega_1 + \omega_2) \geq 0$ (for the complex orientation on X) and hence by the argument above we must have $\alpha^2 \wedge (\omega_1 + \omega_2) \equiv 0$.

This implies that when $\alpha(x) \neq 0$ it's kernel is generated by r and ir, where r is the radial, or S^1, direction in Y (coming from a suitably scaled radial vector in $\mathbb{C}^2$) and ir is parallel to the Hopf fibration. Indeed, if the kernel were transverse to this plane the form $\omega_1 + \omega_2$ would evaluate nontrivially. Hence r and ir lie in $\ker(\alpha(x))$ for all $x \in X$ and α is invariant under the vectorfields r and ir. These vectorfields generate a torus action on X whose projection onto the orbit space is just the holomorphic projection $(p_1, p_2) : X \to \mathbb{C}P^1 \times \mathbb{C}P^1$. Hence α is a pull-back of a form α' on $\mathbb{C}P^1 \times \mathbb{C}P^1$. As α is a closed anti-invariant form so is α'. Furthermore, as α' is anti-invariant it must vanish when restricted to both $\mathbb{C}P^1$ factors. Therefore it's cohomology class is trivial and so

α' is exact. But by Corollary 1.2 the only exact anti-invariant forms on a 4-dimensional manifold are identically 0, and this completes our proof. □

2. Hodge star operator for anti-invariant forms

Let (X, J, g) be an almost Hermitian manifold of dimension $2n$ and denote by ω the fundamental form of g. Then we have the following

Proposition 2.1. *Let α be J-anti-invariant 2-form on (X, J, g). Then*

$$*\alpha = \frac{1}{(n-2)!}\alpha \wedge \omega^{n-2}\,. \tag{2}$$

For the sake of completeness we give the proof of (2).

Proof. Let α be any J-anti-invariant form on (X, J). Then $*\alpha$ is a $(2n-2)$-form. The Lefschetz decomposition applied to $\Lambda^{2n-2}(X)$ yields to

$$\Lambda^{2n-2}(X) = \bigoplus_{i\geq 0} L^i(P^{2(n-i)-2}(X))\,,$$

where $L : \Lambda^k(X) \to \Lambda^{k+2}$, $L(\gamma) = \gamma\wedge\omega$ is the Lefschetz operator and $P^k(X)$ is the space of primitive forms, which can be identified with $\ker L^{n-k+1}|_{\Lambda^k(X)}$ (see e.g., [11, Prop.1.2.30]). Therefore,

$$*\alpha = f\omega^{n-1} + L^{n-2}(\gamma)\,,$$

where f is a smooth function and $\gamma \in P^2(X)$. Then, taking $*$ in the last formula, by [11, Prop.1.2.30], we get

$$\alpha = f\omega - (n-2)!J\gamma\,.$$

Since α is J-anti-invariant, by the last formula, $f = 0$ and $\gamma = \frac{1}{(n-2)!}\alpha$. Then (2) is proved. □

As a consequence, we obtain the following

Proposition 2.2. *Let (X, J, g, ω) be a $2n$-dimensional almost Kähler manifold. Then $\mathcal{Z}_J^-(X) \subset \mathcal{H}^2(X)$, where $\mathcal{H}^2(X)$ denotes the space of 2-harmonic forms on X with respect to the Hermitian metric g.*

Proof. Let $\alpha \in \mathcal{Z}_J^-(X)$. Then by formula (2), since α and ω are closed, we get:

$$d^*\alpha = - * d * (\alpha) = -\frac{1}{(n-2)!} * d(\alpha \wedge \omega^{n-2}) = 0\,,$$

that is α co-closed. Since α is closed by assumption, then α is harmonic. □

We record the following corollary, which of course also follows from Proposition 1.3.

Corollary 2.3. *If (X, J, g, ω) is a compact $2n$-dimensional almost Kähler manifold, then the natural map*

$$\mathcal{Z}_J^-(X) \hookrightarrow H^2_{dR}(X; \mathbb{R})$$

is an injection. In particular $\dim_{\mathbb{R}}(\mathcal{Z}_J^-(X)) \leq b_2(X)$. Furthermore, the map is an isomorphism if and only if $H_J^-(X) = H^2_{dR}(X; \mathbb{R})$.

In general, on an almost Hermitian manifold (X, J, g) of dimension $2n$, define a generalized co-differential on the space of 2-forms $\Gamma(\Lambda^2(X))$, $d^*_- : \Gamma(\Lambda^2(X)) \to \Gamma(\Lambda^1(X))$, by setting

$$d^*_-(\alpha) = d^*(\alpha) + \frac{1}{(n-2)!} * (\alpha \wedge d(\omega^{n-2}))\,,$$

where d^* denotes the usual co-differential on (X, g). By formula (2), it follows that d^*_- vanishes on $\mathcal{Z}^2_-(X)$. Let E be the differential operator on $\Gamma(\Lambda^2(X))$ defined as

$$\mathbb{E} = \Delta(\alpha) + \frac{1}{(n-2)!} d(*(\alpha \wedge d(\omega^{n-2})))$$

Proposition 2.4. *The differential operator $\mathbb{E}$ is a second order elliptic operator, the principal part is the Hodge-de Rham laplacian Δ and $\mathcal{Z}^-_J(X) \subset \ker(\mathbb{E})$.*

Proof. By the definition of $\mathbb{E}$, for any $\alpha \in \mathcal{Z}^-_J(X)$, we have:

$$\begin{aligned}
\mathbb{E}(\alpha) &= \Delta(\alpha) + \frac{1}{(n-2)!} d(*(\alpha \wedge d(\omega^{n-2}))) = dd^*(\alpha) + d^*d(\alpha) + \\
&\quad \frac{1}{(n-2)!} d(*(\alpha \wedge d(\omega^{n-2}))) \\
&= dd^*(\alpha) + \frac{1}{(n-2)!} d(*(\alpha \wedge d(\omega^{n-2}))) = dd^*_-(\alpha) = 0.
\end{aligned}$$

□

Corollary 2.5. *If (X, J) is a compact $2n$-dimensional almost complex manifold, then $\dim \mathcal{Z}^-_J(X) < +\infty$.*

In contrast, $\mathcal{Z}^+_J(X)$ has infinite dimension if J is integrable, because for any smooth function $f : X \to \mathbb{R}$ we have $dd^c f \in \mathcal{Z}^+_J(X)$.

We can now give another proof of the analytic continuation property for closed anti-invariant 2-forms (see [10, Thm.4.1])

Proposition 2.6. *Let X be a $2n$-dimensional connected almost complex manifold. Let $\alpha \in \mathcal{Z}^-_J(X)$ be vanishing at infinite order at some point $p \in X$. Then α is identically zero.*

Proof. By Proposition 2.4, α is a solution of an elliptic PDE, whose leading term is the Laplacian. Hence by [1] (see also [12]), the form α has strong unique continuation. □

In contrast, this is false for $\mathcal{Z}^+_J(X)$ for the same reason as before.

3. Computations of $\mathcal{Z}^2_-(X)$

In this section we will do some explicit computations on the space of anti-invariant forms on complex manifolds, to contrast with the indirect Theorem 1.4. In section 3.1 we give an example of a complex manifold with $\dim_{\mathbb{R}} \mathcal{Z}^-_J(X) > \dim_{\mathbb{R}} H^-_J(X)$. By Corollary 2.3 this implies that the manifold is not almost Kähler. Indeed by Proposition 1.3 there is not even a taming symplectic form. Another such example is given in section 3.2. Finally in section 3.3 we construct an almost-complex manifold for which we can write down explicitly a compatible symplectic form on small open sets. However it also admits a non-zero exact anti-invariant form and so by Proposition 1.3 has no globally defined taming symplectic form.

3.1. **Iwasawa manifold.** On $\mathbb{C}^3$, consider the product $*$ defined as

$$(z_1,\ z_2,\ z_3)\ *\ (w_1,\ w_2,\ w_3) = (z_1 + w_1,\ z_2 + w_2,\ z_3 + z_1 w_2 + w_3)\ .$$

It is immediate to check that $\left(\mathbb{C}^3,\, *\right)$ is a nilpotent Lie group isomorphic to

$$\mathbb{H}(3) = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \in GL\,(3;\, \mathbb{C}) \ \mid\ z_1, z_2, z_3 \in \mathbb{C} \right\}\ .$$

We have that $(\mathbb{Z}\,[\mathrm{i}])^3 \subset \mathbb{C}^3$ is a cocompact discrete subgroup of $\left(\mathbb{C}^3,\, *\right)$. The *Iwasawa manifold* X is defined as the manifold

$$X = (\mathbb{Z}\,[\mathrm{i}])^3 \big\backslash \left(\mathbb{C}^3,\, *\right)\ .$$

It is a compact complex 3-dimensional nilmanifold; by [8], it follows that X is not formal; hence, it has no Kähler metrics, see [3, Main Theorem]; nevertheless, there exists a balanced metric on X. Let $\left(z^i\right)_{i\in\{1,2,3\}}$ be the standard complex coordinate system on $\mathbb{C}^3$; the following $(1,0)$-forms on $\mathbb{C}^3$ are invariant for the action (on the left) of $(\mathbb{Z}\,[\mathrm{i}])^3$, so they give rise to a global coframe for $T^{*\,1,0}X$:

$$\begin{cases} \varphi^1 = \mathrm{d}\, z^1, \\ \varphi^2 = \mathrm{d}\, z^2, \\ \varphi^3 = \mathrm{d}\, z^3 - z^1\, \mathrm{d}\, z^2. \end{cases}$$

The structure equations are therefore

$$\begin{cases} \mathrm{d}\,\varphi^1 &=& 0, \\ \mathrm{d}\,\varphi^2 &=& 0, \\ \mathrm{d}\,\varphi^3 &=& -\varphi^1 \wedge \varphi^2. \end{cases}$$

By Hattori-Nomizu theorem, we compute the real cohomology group $H^2_{dR}(X;\mathbb{R})$ of X (for simplicity, we list the harmonic representative instead of its class and write $\varphi^{A\bar{B}}$ for $\varphi^A \wedge \bar{\varphi}^B$):

$$\begin{aligned} H^2_{dR}(X;\mathbb{R}) \ &= \ \mathrm{span}_{\mathbb{R}} \left\{ \varphi^{13} + \varphi^{\bar{1}\bar{3}},\ \mathrm{i}\left(\varphi^{13} - \varphi^{\bar{1}\bar{3}}\right),\ \varphi^{23} + \varphi^{\bar{2}\bar{3}}, \right. \\ &\qquad \left. \mathrm{i}\left(\varphi^{23} - \varphi^{\bar{2}\bar{3}}\right),\ \varphi^{1\bar{2}} - \varphi^{2\bar{1}},\ \mathrm{i}\left(\varphi^{1\bar{2}} + \varphi^{2\bar{1}}\right),\ \mathrm{i}\,\varphi^{1\bar{1}},\ \mathrm{i}\,\varphi^{2\bar{2}} \right\}, \end{aligned}$$

Note that each harmonic representative is of pure degree and hence the complex structure is $\mathcal{C}^\infty$-pure and full. The Betti numbers of X are

$$b^0 \ = \ 1\ , \quad b^1 \ = \ 4\ , \quad b^2 \ = \ 8\ , \quad b^3 \ = \ 10\ .$$

Then

$$\begin{gathered} \frac{1}{2}(\varphi^2 \wedge \varphi^3 + \overline{\varphi}^2 \wedge \overline{\varphi}^3), \frac{1}{2i}(\varphi^2 \wedge \varphi^3 - \overline{\varphi}^2 \wedge \overline{\varphi}^3), \frac{1}{2}(\varphi^1 \wedge \varphi^2 + \overline{\varphi}^1 \wedge \overline{\varphi}^2), \\ \frac{1}{2i}(\varphi^1 \wedge \varphi^2 - \overline{\varphi}^1 \wedge \overline{\varphi}^2), \frac{1}{2}(\varphi^1 \wedge \varphi^3 + \overline{\varphi}^1 \wedge \overline{\varphi}^3), \frac{1}{2i}(\varphi^1 \wedge \varphi^3 - \overline{\varphi}^1 \wedge \overline{\varphi}^3), \end{gathered}$$

are J-anti-invariant closed 2-forms on X and consequently $\dim_{\mathbb{R}} \mathcal{Z}^-_J(X) > \dim_{\mathbb{R}} H^-_J(X)$.

3.2. **Nakamura manifold.** The *Nakamura manifold* is the compact quotient $X = \Gamma\backslash G$ of G by a uniform discrete subgroup Γ.

By [2, Corollary 4.2] we have

$$\begin{aligned} H^2_{dR}(X;\mathbb{R}) &= \mathrm{span}_{\mathbb{R}}\Big\{[e^{14}], [e^{26}-e^{35}], [e^{23}-e^{56}], [\cos(2x_4)(e^{23}+e^{56})-\sin(2x_4)(e^{26}+e^{35})], \\ &\qquad [\sin(2x_4)(e^{23}+e^{56})-\cos(2x_4)(e^{26}+e^{35})]\Big\}, \end{aligned}$$

i.e. in this case the de Rham cohomology of M is not isomorphic to $H^*(\mathfrak{g})$. The previous representatives are all harmonic forms. The complex structure on the solvmanifold X can be defined in term of $(1,0)$-forms as follows:

$$\varphi^1 = e^1 + ie^4\,, \quad \varphi^2 = e^2 + ie^5\,, \quad \varphi^3 = e^3 + ie^6$$

We have that the real forms

$$\frac{1}{2}(\varphi^2\wedge\varphi^3+\overline{\varphi}^2\wedge\overline{\varphi}^3), \frac{1}{2i}(\varphi^2\wedge\varphi^3-\overline{\varphi}^2\wedge\overline{\varphi}^3), \frac{1}{2}(\varphi^1\wedge\varphi^2+\overline{\varphi}^1\wedge\overline{\varphi}^2),$$

$$\frac{1}{2i}(\varphi^1\wedge\varphi^2-\overline{\varphi}^1\wedge\overline{\varphi}^2), \frac{1}{2}(\varphi^1\wedge\varphi^3+\overline{\varphi}^1\wedge\overline{\varphi}^3), \frac{1}{2i}(\varphi^1\wedge\varphi^3-\overline{\varphi}^1\wedge\overline{\varphi}^3),$$

are anti-invariant and closed. Therefore, $\dim \mathcal{Z}^-_J(X) > b_2(X)$ and by Corollary 2.3, the complex structure J does not admit any compatible Kähler metric.
This can be also derived by complex Hodge theory, since φ^2 is a non closed holomorphic 1-form.
It has also to be remarked that as a consequence of a result by Hasegawa (see [9, main theorem]) X does not admit any Kähler structure.

3.3. **Locally almost-Kähler non globally almost Kähler manifold.** In this section we will provide a family of 6-dimensional almost complex (non complex) manifolds (N, J) which are locally almost Kähler but not globally. We first recall the construction of N (see [16] and [2]). Let $A \in \mathrm{SL}(2,\mathbb{Z})$ with two distinct real eigenvalues e^{λ} and $e^{-\lambda}$, where $\lambda > 0$. Let $Q \in \mathrm{GL}(2,\mathbb{R})$ such that

$$QAQ^{-1} = \Lambda = \begin{pmatrix} e^{-\lambda} & 0 \\ 0 & e^{\lambda} \end{pmatrix}.$$

On $\mathbb{C}^2$, with coordinates (z,w), let $\sim$ be defined by

$$\begin{pmatrix} z' \\ w' \end{pmatrix} \sim \begin{pmatrix} z \\ w \end{pmatrix} \iff \begin{pmatrix} z' \\ w' \end{pmatrix} = \begin{pmatrix} z \\ w \end{pmatrix} + Q\begin{pmatrix} m_1 + 2\pi i n_1 \\ m_2 + 2\pi i n_2 \end{pmatrix},$$

where $m_1, m_2, n_1, n_2 \in \mathbb{Z}$. Then $\mathbb{C}^2/\sim$ is a complex torus $\mathbb{T}^2_{\mathbb{C}}$ and

$$\Lambda\left[\begin{pmatrix} z \\ w \end{pmatrix}\right] = \left[\Lambda\begin{pmatrix} z \\ w \end{pmatrix}\right]$$

is a well defined automorphism of $\mathbb{T}^2_{\mathbb{C}}$. Indeed, if $\begin{pmatrix} z' \\ w' \end{pmatrix} \sim \begin{pmatrix} z \\ w \end{pmatrix}$, then

$$\Lambda \begin{pmatrix} z' \\ w' \end{pmatrix} = \Lambda \begin{pmatrix} z \\ w \end{pmatrix} + \Lambda Q \begin{pmatrix} m_1 + 2\pi i n_1 \\ m_2 + 2\pi n_2 \end{pmatrix} =$$

$$= \Lambda \begin{pmatrix} z \\ w \end{pmatrix} + QA \begin{pmatrix} m_1 + 2\pi i n_1 \\ m_2 + 2\pi n_2 \end{pmatrix} = \Lambda \begin{pmatrix} z \\ w \end{pmatrix} + Q \begin{pmatrix} m_1 + 2\pi i n_1 \\ m_2 + 2\pi n_2 \end{pmatrix}$$

so that $\Lambda \begin{pmatrix} z' \\ w' \end{pmatrix} \sim \Lambda \begin{pmatrix} z \\ w \end{pmatrix}$.
For example, take

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} .$$

Then $\lambda = \log \frac{3+\sqrt{5}}{2}$ and we can choose

$$P = \begin{pmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{\sqrt{5}-1}{2} \end{pmatrix} . \tag{3}$$

Set

$$\lambda = \log \frac{3+\sqrt{5}}{2} \,, \;\; \mu = \frac{\sqrt{5}-1}{2} \,.$$

Let x_1, x_3, x_4, x_5, x_6 denote coordinates on $\mathbb{R}^6$ and, according to the previous notation, set $z = x_3 + ix_5$, $w = x_4 + ix_6$. Consider the following transformation of $\mathbb{R}^5$:

$$T_1(x_1, x_3, x_4, x_5, x_6) \;=\; \left(x_1 + \lambda, e^{\lambda} x_3, e^{-\lambda} x_4, e^{\lambda} x_5, e^{-\lambda} x_6\right) .$$

We set

$$N \;=\; \frac{\mathbb{R}_{x_2}}{2\pi\mathbb{Z}} \times \frac{\mathbb{R}_{x_1} \times \mathbb{R}^4_{x_3,x_4,x_5,x_6}/\Gamma}{\langle T_1(x) \rangle}$$

where

$$\Gamma \;=\; \mathrm{Span}_{\mathbb{Z}} \left\langle (1,\mu,0,0)^t, (-\mu,1,0,0)^t, (0,0,2\pi,2\pi\mu)^t, (0,0,-2\pi\mu,2\pi)^t \right\rangle$$

and $\langle T_1(x) \rangle$ denotes the subgroup of transformations generated by $T_1(x)$, so that $\mathbb{T}^2_{\mathbb{C}} \simeq \mathbb{R}^4_{x_3,x_4,x_5,x_6}/\Gamma$. Then N is a compact 6-dimensional manifold. The following six 1-forms on $\mathbb{R}^6$

$$\begin{cases} e^1 &= dx_1 \,, \\ e^2 &= dx_2 \,, \\ e^3 &= \exp(-x_1) dx_3 \,, \\ e^4 &= \exp(x_1) dx_4 \,, \\ e^5 &= \exp(-x_1)\, dx_5 \,, \\ e^6 &= \exp(x_1)\, dx_6 \,, \end{cases}$$

induce 1-forms on the manifold N. Therefore, we immediately get

$$
\begin{cases}
de^1 &= 0\,, \\
de^2 &= 0\,, \\
de^3 &= -e^1\wedge e^3\,, \\
de^4 &= e^1\wedge e^4\,, \\
de^5 &= -e^1\wedge e^5\,, \\
de^6 &= e^1\wedge e^6\,.
\end{cases} \tag{4}
$$

The dual global frame $\{e_1,\ldots,e_6\}$ on N is given by

$$
\begin{array}{lll}
e_1 = \frac{\partial}{\partial x_1}\,, & e_2 = \frac{\partial}{\partial x_2}\,, & e_3 = \exp(x_1)\frac{\partial}{\partial x_3} \\
e_4 = \exp(-x_1)\frac{\partial}{\partial x_4}\,, & e_5 = \exp(x_1)\frac{\partial}{\partial x_5}\,, & e_6 = \exp(-x_1)\frac{\partial}{\partial x_6}
\end{array}
$$

Let $f = f(x_2)$ be a never vanishing $\mathbb{Z}$-periodic function; let us define the almost complex structure J on N as

$$
Je_1 = e_2\,, \quad Je_2 = -e_1\,, \quad Je_3 = f(x_2)e_5\,, \quad Je_4 = e_6\,, \quad Je_5 = -\frac{1}{f(x_2)}e_3\,, \quad Je_6 = -e_4\,.
$$

Then it can be checked that J is integrable if and only if f is constant. We show that J is locally almost Kähler. Indeed, let ω be the local non degenerate and closed 2-form defined as

$$
\omega = dx_1\wedge dx_2 + dx_3\wedge dx_5 + dx_4\wedge dx_6\,;
$$

then, since

$$
\begin{array}{lll}
J\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_2}\,, & J\frac{\partial}{\partial x_2} = -\frac{\partial}{\partial x_1}\,, & J\frac{\partial}{\partial x_3} = f(x_2)\frac{\partial}{\partial x_5}\,, \\
J\frac{\partial}{\partial x_4} = \frac{\partial}{\partial x_6}\,, & J\frac{\partial}{\partial x_5} = -\frac{1}{f(x_2)}\frac{\partial}{\partial x_3}\,, & J\frac{\partial}{\partial x_6} = -\frac{\partial}{\partial x_4}\,,
\end{array}
$$

we immediately get that $J\omega = \omega$ and $\omega(J\cdot,\cdot) > 0$ for any non-zero tangent vector, i.e., J is locally almost Kähler. Now we prove that J cannot be globally Kähler, and more generally that there is no global taming symplectic form. In view of Proposition 1.3, it is sufficient to find a nonzero J-anti-invariant exact form. To this purpose, let

$$
\alpha = \cos(x_2)e^2\wedge e^4 + \sin(x_2)e^1\wedge e^4 - \sin(x_2)e^2\wedge e^6 + \cos(x_2)e^1\wedge e^6\,;
$$

then, according to (4) and to definition of J, we have that $\alpha = d(\sin(x_2)e^4 + \cos(x_2)e^6)$ and that $J\alpha = -\alpha$, i.e., α is a J-anti-invariant exact 2-form.

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