A fuzzy reverse logistics inventory system integrating economic order/production quantity models

A fuzzy reverse logistics inventory system integrating economic order/production quantity models

Ehsan Shekarian*, Ezutah Udoncy Olugu, Salwa Hanim Abdul-Rashid

Centre for Product Design and Manufacturing, Department of Mechanical Engineering, Faculty of Engineering, University of Malaya, Lembah Pantai, 50603, Kuala Lumpur, Malaysia

Abstract

This paper develops a reverse inventory model where the recoverable manufacturing process is affected by the learning theory. We propose the inclusion of the fuzzy demand rate of the serviceable products and the fuzzy collection rate of the recoverable products from customers in the total cost function of the model. Two popular defuzzification methods, namely the signed distance technique, a ranking method for fuzzy numbers, and the graded mean integration representation (GMIR) method are employed to find the estimate of the total cost function per unit time in the fuzzy sense. We provide a comprehensive numerical example to illustrate and compare the results obtained by the two mentioned defuzzification methods. This is one of the only few attempts in the related literature comparing the performance of these methods with the effect of the fuzziness of both of the demand and the collection rate in the presence of the learning simultaneously. The results indicate that deciding on which method could be used depends on the target strategy that could focus on the total cost, ordering lot size, or recovery lot size.

Keywords: Fuzzy set theory; Signed distance; Graded mean integration representation; Reverse logistics; Inventory management; Economic order/production quantity

*Corresponding author: Ehsan Shekarian, Tel: +60 011 2315 8808
E-mail: eshokryan@gmail.com, e.shekarian@siswa.um.edu.my
Ezutah Udoncy Olugu (olugu@um.edu.my) Salwa Hanim Abdul-Rashid (salwa_hanim@um.edu.my)

A fuzzy reverse logistics inventory system integrating economic order/production quantity models

Abstract

This paper develops a reverse inventory model where the recoverable manufacturing process is affected by the learning theory. We propose the inclusion of the fuzzy demand rate of the serviceable products and the fuzzy collection rate of the recoverable products from customers in the total cost function of the model. Two popular defuzzification methods, namely the signed distance technique, a ranking method for fuzzy numbers, and the graded mean integration representation (GMIR) method are employed to find the estimate of the total cost function per unit time in the fuzzy sense. We provide a comprehensive numerical example to illustrate and compare the results obtained by the two mentioned defuzzification methods. This is one of the only few attempts in the related literature comparing the performance of these methods with the effect of the fuzziness of both of the demand and the collection rate in the presence of the learning simultaneously. The results indicate that deciding on which method could be used depends on the target strategy that could focus on the total cost, ordering lot size, or recovery lot size.

Keywords: Fuzzy set theory; Signed distance; Graded mean integration representation; Reverse logistics; Inventory management; Economic order/production quantity

1. Introduction

The topics of reverse supply chain (RSC) and reverse logistics activities have been of great interest for more than two decades both in academia and practice. In the literature, RSC activities are defined as "the process of planning, implementing, and controlling the efficient, cost effective flow of raw materials, in-process inventory, finished goods and related information from the point of consumption to the point of origin for the purpose of recapturing value or proper disposal" [1]. This attention is evident by the vast number of publications in the scientific journals, which have been published in recent years. In this context, authors investigated many aspects such as distribution planning, inventory control, and production planning [2–4], managerial remanufacturing [5], empirical [6], inventory management [7], mathematical modeling [8], business perspective [9], quantitative models [10], strategic issues [11, 12], strategic level decisions [13], social sides [14], and pricing models [15]. However, compared with its counterpart, forward logistics, research in reverse logistics is still in its infancy [16].

Previous literature has described a broad range of inventory issues on the RSC structures and analyzed a variety of problems. Inventory management studies play a major role in the operational level of the supply chain by investigating the optimal order quantities and other inventory related decisions regarding remanufacturing effects and return products [17]. The objective of inventory management is to control the external component orders and the internal component recovery process to guarantee a required service level and to minimize the fixed and variable costs, or interchangeably, to maximize the total profit [2]. Research on 'production planning and inventory management', in accord with its quantitative profile, tends to be analytical in content and favors techniques such as simulation, optimization and mathematical programming [3]. Since the first work by Schrady [18], who developed an economic order quantity (EOQ) model for repaired items, a surge of valuable studies has been conducted in this area and many aspects of inventory management and control systems are arguably discussed and analyzed in the reverse and closed-loop models.

These studies have some deficiencies. Although some factors in the problems discussed in the inventory management are inherently uncertain, most of the related works in the reverse inventory literature suppose that input parameters and variables such as the demand and the return rate are precise. In fact, they are usually uncertain due to the lack of historical data or flexible situation, which is an inherent part of the business environment. In this regard, one of the most powerful tools to deal with the mentioned factors in uncertain situations is the fuzzy set theory introduced by Zadeh [19]. Another shortcoming in the earlier reverse inventory models is ignoring the effects of learning phenomena based on the learning theory. However, the learning procedure could affect the inventory policies. According to this theory, when an operator performs a process repetitively, his/her knowledge and experience naturally increase. In a long time horizon, this leads to an improvement in the performance and learning occurs [20]. Only a handful of references dealing with the learning theory in the reverse inventory models are available [21, 22].

To fill the identified gaps in the literature, this paper extends a reverse inventory model with the fuzzy demand rate of the serviceable products and the fuzzy collection rate of the recoverable products, which are receiving from customers. Building upon the research of Tsai [22], who addressed the effect of learning considering an optimal integrated economic order quantity (EOQ) and economic production quantity (EPQ) policy, we compare the performance of two defuzzification methods, namely the signed distance and the graded mean integration representation, to find the estimation of the total cost function per unit time in the fuzzy sense and the corresponding optimal values. To the best of our knowledge, this study is the first to explain the difference between different defuzzification methods in a reverse inventory system considering the mentioned parameters being fuzzified as triangular fuzzy numbers (TFNs).

The paper is organized as follows. In the next Section, we review some previous literature. Section 3 is devoted to explain the notations, assumptions, and crisp model. Required preliminaries are described in Section 4. We develop fuzzy mathematical models in Section 5. A comprehensive numerical example is investigated in Section 6. The applicability of the model is shown in Section 7. In the last Section, the paper is concluded.

2. Literature review

We review the EOQ/EPQ models in the reverse inventory literature. Then, the fuzzy EOQ/EPQ models are reviewed. For a complete discussion, we refer the interested readers to the works that thoroughly reviewed many inventory problems on the RSC [4, 7, 15, 23]. In addition, Bushuev et al. [24] have recently provided a review of the previously published papers on the inventory models.

2.1. Review of the EOQ/EPQ models in the RSC

After the first work by Schrady [18], researchers paid much attention to develop the EOQ/EPQ models in the reverse logistics context. Nahmias and Rivera [25] extended Schrady's work with a finite recycling/repair rate. Considering capital budget restriction, a multi-product model, which is another extension of the EOQ-type reverse logistics model of Schrady was developed by Mabini et al. [26]. By incorporating the collection point into the model, Richter [27, 28] considered modified versions of the model of Schrady with the constant disposal and used-product collection rates by assuming multiple production/repair cycles within a time interval. He obtained a formula for the total average cost without deriving a simple one for the optimal lot size. Along the same line of research, Richter [29] extended his earlier works for the case which the return rate was a decision variable. He concluded that the optimal policy has an external property that is, "dispose all" or "recover all". Furthermore, Richter and Dobos [30] developed a waste disposal model where the return rate is a decision variable.

By assuming different holding cost rates for produced and recovered serviceable items, Teunter [31] developed a deterministic EOQ inventory model. Koh et al. [32] proposed a model with finite manufacturing/remanufacturing rates. Other models similar to those of Schrady and Richter, but with different assumptions could be found in [33–39].

Assuming deterministic demand and return fraction with zero production lead times, Oh and Hwang [40] suggested an optimal policy for a recycling model. A periodic review inventory model with finite horizon and manufacturing/remanufacturing options was studied by Konstantaras and Papachristos [41]. In their subsequent works, Konstantaras and Papachristos [42, 43], extended works of Koh et al. [32] and Teunter [34], respectively. Besides, Jaber and El Saadany [44] developed a model by assuming that the newly produced and remanufactured items are perceived differently by customers. Konstantaras and Skouri [45] studied a production–remanufacturing inventory system with a cost structure consists of the EOQ-type setup, holding, and shortage costs. Deterministic mathematical models were presented by El Saadany and Jaber [46] for multiple remanufacturing and production cycles. In a follow-up study, Konstantaras et al. [47] introduced the inspection and the sorting of returned items.

Alinovi et al. [48] formulated a stochastic EOQ-based inventory control model for a mixed manufacturing/remanufacturing system. El Saadany and Jaber [49] considered a production-remanufacturing inventory model that a subassembly has its own inventory control policy. Alamri [50] presented a unified general inventory model for an integrated production. Regarding an EOQ model and remanufacturing, Hasanov et al. [51] modeled some stock-out situations. Widyadana and Wee [52] developed an EPQ model for deteriorating items with rework. Moreover, an EOQ-based production, remanufacturing and waste disposal model for a two-level chain with the consignment stock policy as a coordination mechanism was extended by Jaber et al. [53]. Matar et al. [54] formulated a reverse logistics inventory model for the production-recycling-reuse of plastic beverage bottles. They indicated that the outcome of the total system unit time cost is mainly affected by the amount of bottles collected during the recycling process. Recently, Nonaka and Fujii [55] have extended the work of Dobos and Richter [56] by proposing a new EOQ model for the reuse and recycling, which introduces a sequentially accumulated marginal reuse rate as a parameter to keep the balance of product demand and supply. Zouadi et al. [57] investigated a lot-sizing problem by considering two types of inventories in which the demand for the items could be satisfied by both the new and the remanufactured products. Taking the effect of learning into account, Singh, Rathore [58, 59] developed two different reverse logistics inventory models for deteriorating items.

2.2. Review of the fuzzy EOQ/EPQ models

One of the underlying assumptions in the basic EOQ model and its extensions, especially the EPQ model, is that all the input parameters and decision variables are certain and known in the inventory system. However, these quantities have little deviations from their exact values. In the uncertain situations of the real world, we can deal with the uncertainty in an inventory system applying the fuzzy set concept. In the following, we review some related papers.

Lee and Yao [60] analyzed a fuzzy EOQ model with fuzzifying the demand and the production quantities as TFNs. Hsieh [61] employed trapezoidal fuzzy numbers in a fuzzy production inventory model. A fuzzy EPQ model considering imperfect quality items was studied by Chang [62]. An extension of the fuzzy inventory model proposed by Ouyang and Yao [63] was formulated by Chang et al. [64]. In addition, Vijayan and Kumaran [65] applied the signed distance method for obtaining the crisp estimation of the fuzzy total cost function in an EOQ model. For a review of the related papers in this area till 2008, Guiffrida [66] conducted a survey to gather and categorize studies including fuzzy inventory models.

Björk [67] addressed an inventory model assuming both demand and lead-time as fuzzy numbers. He also presented a fuzzy inventory model with a finite production rate and without shortage where the model was fuzzified using symmetric TFN [68]. Shekarian et al. [69] considered a fuzzy EPQ model that generates defective items in a single stage production system with planned backorders. Also, Shekarian et al. [70] developed a fuzzy lot size model for a single-stage production system producing defective items that need to be reworked. They considered the rate of defectives and the demand rate as TFNs. Guchhait et al. [71] formulated a retailer's profit maximization problem for both crisp and fuzzy inventory costs.

Recently some fuzzy inventory models have been investigated by researchers. Sharifi et al. [72] developed an EOQ model for items with imperfect quality and partial backordered shortage under screening error. Pal et al. [73] considered an EPQ model for deteriorating items with ramp type demand rate under the effect of inflation, shortages and fuzziness. Kumar and Goswami [74] extended an EPQ model considering the learning effect, and used fuzzy expectation and signed distance methods to transform the fuzzy random cost function into an equivalent crisp one. Guchhait et al. [75] discussed an inventory model for deteriorating items assuming that some inventory costs such as the purchase, holding and setup costs are fuzzy triangular type in nature. A periodic review inventory model with a fuzzy demand was discussed by Sarkar and Mahapatra [76] with the aim of minimizing the expected total annual cost. Yadav et al. [77] studied a fuzzy inventory system by taking the opportunity cost, interest earned/paid rates as TFNs. They used the function principle as arithmetic operation and the signed distance method to defuzzify the fuzzy profit function. Mahata [78] addressed the learning effect of the unit production time on the optimal lot size for an imperfect production process where the setup, holding, backorder, raw material and labor costs were characterized as fuzzy variables. Moreover, Kazemi et al. [79] developed an EOQ model for imperfect quality items using the learning effect on fuzzy parameters. Finally, the fuzzy EOQ model proposed by Björk [67] was modified by Kazemi et al. [80] to incorporate human learning in an uncertain environment.

3. Reverse inventory model

3.1. Notations and assumptions

To develop the proposed model, the following notations are defined:

y	Recovery lot size for each production run (unit/run) (Decision variable)
n	Number of orders for the newly purchased products during a cycle (Decision variable)
Q	Ordering lot size for the newly purchased products (unit/order)
k	Demand rate of the serviceable products (unit/time) (Fuzzified parameter)
r	Collection rate of the recoverable products from customers (unit/time) (Fuzzified parameter)
C_s	Setup cost for the recovery process ($/setup)
C_o	Ordering cost for the newly purchased products ($/order)
H_r	Inventory holding cost for the collected products ($/unit/time)
H_s	Inventory holding cost for the serviceable products ($/unit/time)
L_r	Learning rate in recovery production
b	Learning exponent
C_l	Labor production cost per unit time ($/time)
C_p	Unit purchase cost for the newly purchased products ($/unit)
C_b	Unit buyback cost for the recovered products ($/unit)
$TCU(y,n)$	Total cost function per unit time ($)
$\tilde{V}(y,n)$	Fuzzified total cost function per unit time ($)

The assumptions made in this paper are as follows:

1. Shortages are not allowed.
2. All of the collected products can be recovered and made acceptable to customers.
3. The ordering lots are of equal size through the time.
4. The demand rate is greater than the collection rate of the recoverable products.
5. The time period is infinite.
6. The demand rate for the serviceable products and the collection rate of the recoverable products from customers are treated as fuzzy numbers and shown by TFNs.

According to the above assumptions and notations, Tsai [22] obtained the total cost function per unit time in a proposed reverse inventory system with a single setup for recovery and multi-order policy which is termed as $(1, n)$ policy.

$$TCU(y,n) = \frac{r(nC_o + C_s)}{y} + H_r\left(\frac{y}{2} - \frac{ary^{b+1}}{b+2}\right) + H_s\left[\frac{y(k-r)^2}{2nkr} + \frac{ry}{2k} - \frac{ary^{b+1}}{(b+1)(b+2)}\right] + \left(\frac{ary^b}{b+1}\right)C_l + (k-r)C_p + rC_b \tag{1}$$

It should be noted that the demand is satisfied using both the recovered and the newly purchased products. Due to the learning effect, the unit production time for the recovered products decreases when the number of units produced increases. Wright's learning curve [20] is considered as below:

$$U(x) = Tx^b \tag{2}$$

where $U(x)$ is the time to produce the xth unit, T is the time to produce the first unit, and x is the production account. b can be calculated as $b = \log L_r / \log 2$. In this study, it is assumed that $-1 < b \leq 0$ [81]. Furthermore, the ordering lot size for the newly purchased products is given as below:

$$Q = \frac{y(k-r)}{nr} \tag{3}$$

4. Preliminaries

4.1. Fundamental definitions

In order to treat the fuzziness of the reverse logistics inventory system for the demand rate of the serviceable products and the collection rate of the recoverable products from customers, some related definitions and propositions are required to state as follows [82, 83]:

Definition 1. *Triangular fuzzy number (TFN)* $\tilde{B}$ is a special type of fuzzy numbers denoted by triplet (b_1, b_2, b_3) where satisfies $b_1 < b_2 < b_3$, and b_1, b_2 and b_3 are defined on R. The membership function of $\tilde{B}$ is defined as:

$$\mu_{\tilde{B}}(x) = \begin{cases} (x - b_1)/(b_2 - b_1), & b_1 \leq x \leq b_2, \\ (b_3 - x)/(b_3 - b_2), & b_2 \leq x \leq b_3, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Function principle: Let $\tilde{B} = (b_1, b_2, b_3)$, and $\tilde{C} = (c_1, c_2, c_3)$ be two TFNs, and let k be a real number. Based on Chen and Chang [84], we have operations of the function principle as follows:

$$\tilde{B} \oplus \tilde{C} = (b_1 + c_1, b_2 + c_2, b_3 + c_3), \tag{5}$$
$$\tilde{B} \ominus \tilde{C} = (b_1 - c_3, b_2 - c_2, b_3 - c_1), \tag{6}$$

If b_1, b_2, b_3, c_1, c_2 and c_3 are all nonzero positive real numbers, then

$$\tilde{B} \otimes \tilde{C} = (b_1c_1, b_2c_2, b_3c_3), \tag{7}$$

$$\tilde{B} \oslash \tilde{C} = (b_1/c_3, b_2/c_2, b_3/c_1), \tag{8}$$

$$\begin{cases} k \otimes \tilde{B} = (kb_1, kb_2, kb_3), & k \geq 0, \\ k \otimes \tilde{B} = (kb_3, kb_2, kb_1), & k < 0. \end{cases} \tag{9}$$

Definition 2. Fuzzy set $\tilde{a}_\alpha$ for $0 \leq \alpha \leq 1$ and a range of $x \in R$ is called an α-*level fuzzy point* whose membership function has a form

$$\mu_{\tilde{a}_\alpha}(x) = \begin{cases} \alpha, & x = a, \\ 0, & x \neq a. \end{cases} \tag{10}$$

Remarks 1. If $\alpha = 1$, the membership function of the 1-*level fuzzy point* $\tilde{a}_1$ becomes the characteristic function, i.e.,

$$\mu_{\tilde{a}_1}(x) = \begin{cases} 1, & x = a, \\ 0, & x \neq a. \end{cases} \tag{11}$$

In this case, the fuzzy point $\tilde{a}_1$ and the real number $a \in R$ are similar except for their representation.

Remarks 2. When $b_1 = b_2 = b_3 = b'$, then the TFN $\tilde{B} = (b', b', b')$ is identical to the 1-*level fuzzy point* $\tilde{a}_1$.

Definition 3. For $0 \leq \alpha \leq 1$ and $a < b$, the fuzzy set $[a_\alpha, b_\alpha]$ defined on R is called an α-*level fuzzy interval* if its membership function is given by

$$\mu_{[a_\alpha, b_\alpha]}(x) = \begin{cases} \alpha, & a \leq x \leq b, \\ 0, & \text{otherwise}. \end{cases} \tag{12}$$

Definition 4. Let $\tilde{B}$ be a fuzzy set on R, and $0 \leq \alpha \leq 1$, then the α-*cut* of $\tilde{B}$ (i.e. $B(\alpha)$) includes points x such that $\mu_{\tilde{B}}(x) \geq \alpha$; that is $B(\alpha) = \{x | \mu_{\tilde{B}}(x) \geq \alpha\}$.

Property 1. For $\alpha \in [0,1]$**,** the α-*cut* of TFN $\tilde{B} = (b_1, b_2, b_3)$ is $B(\alpha) = [B_L(\alpha), B_R(\alpha)]$, where $B_L(\alpha) = b_1 + (b_2 - b_1)\alpha$ and $B_R(\alpha) = b_3 - (b_3 - b_2)\alpha$.

Principle of decomposition theory: Suppose that $\tilde{B}$ is a fuzzy set on R, $0 \leq \alpha \leq 1$, and its α-*cut* is $B(\alpha) = [B_L(\alpha), B_R(\alpha)]$ which is a closed interval, then we have:

$$\tilde{B} = \cup_{0 \leq \alpha \leq 1} [B_L(\alpha)_\alpha, B_R(\alpha)_\alpha] = \cup_{0 \leq \alpha \leq 1} \alpha B(\alpha) \tag{13}$$

or

$$\mu_{\tilde{B}}(x) = \vee_{0 \leq \alpha \leq 1} \mu_{[B_L(\alpha)_\alpha, B_R(\alpha)_\alpha]}(x) = \vee_{0 \leq \alpha \leq 1} \alpha C_{B(\alpha)}(x) \tag{14}$$

where

(i) $\alpha B(\alpha)$ is a fuzzy set with the membership function:

$$\mu_{\alpha B(\alpha)}(x) = \begin{cases} \alpha, & x \in B(\alpha), \\ 0, & otherwise. \end{cases}$$

(ii) $C_{B(\alpha)}(x)$ is a characteristic function of $B(\alpha)$; that is

$$C_{B(\alpha)}(x) = \begin{cases} 1, & x \in B(\alpha), \\ 0, & x \notin B(\alpha). \end{cases}$$

For any a_1, a_2, b_1, b_2, $k \in R$, $a_1 < a_2$, and $b_1 < b_2$, the interval operations are as follows:

$$[a_1, a_2](+)[b_1, b_2] = [a_1 + b_1, a_2 + b_2], \tag{15.1}$$

$$[a_1, a_2](-)[b_1, b_2] = [a_1 - b_2, a_2 - b_1]. \tag{15.2}$$

$$k(\cdot)[a_1, a_2] = \begin{cases} [ka_1, ka_2], & k > 0, \\ [ka_2, ka_1], & k < 0. \end{cases} \quad (15.3)$$

Besides, if $0 < a_1$, and $0 < b_1$, then

$$[a_1, a_2](\cdot)[b_1, b_2] = [a_1 b_1, a_2 b_2] \quad (15.4)$$

$$[a_1, a_2](\div)[b_1, b_2] = \left[\frac{a_1}{b_2}, \frac{a_2}{b_1}\right]. \quad (15.5)$$

4.2. Defuzzification methods

In the following, the concepts of the signed distance and the graded mean integration representation for the fuzzy sets are introduced [85, 86].

4.2.1. The signed distance (SD) method

Definition 5. For any $a \in R$, the signed distance from a to 0 is defined as $d_0(a, 0) = a$. If a is positive, then the distance from a to 0 is $a = d_0(a, 0)$; if a is negative, the distance from a to 0 is $a = -d_0(a, 0)$. This is the reason why $d_0(a, 0)$ is referred as the distance from a to 0.

Assume that Ψ be the family of all fuzzy sets $\tilde{B}$ defined on R with which the *α-cut* $B(\alpha) = [B_L(\alpha), B_R(\alpha)]$ exists for every $\alpha \in [0,1]$, and both $B_L(\alpha)$, and $B_R(\alpha)$ are continuous functions on $0 \leq \alpha \leq 1$. Then, for any $\tilde{B} \in \Psi$ from (13), we have

$$\tilde{B} = \cup_{0 \leq \alpha \leq 1} [B_L(\alpha)_\alpha, B_R(\alpha)_\alpha] \quad (16)$$

According to Definition 5, the signed distance of two end points $B_L(\alpha)$, and $B_R(\alpha)$ of the *α-cut* of $\tilde{B}$ (i.e. $B(\alpha) = [B_L(\alpha), B_R(\alpha)]$) to the origin 0 is $d_0(B_L(\alpha), 0) = B_L(\alpha)$, and $d_0(B_R(\alpha), 0) = B_R(\alpha)$, respectively.

Definition 6. $d_0([B_L(\alpha), B_R(\alpha)], 0) = [d_0(B_L(\alpha), 0) + d_0(B_R(\alpha), 0)]/2 = [B_L(\alpha) + B_R(\alpha)]/2$.

We have the following one-to-one mapping relationship between *α-level fuzzy interval* $[B_L(\alpha)_\alpha, B_R(\alpha)_\alpha]$, and the real interval $[B_L(\alpha), B_R(\alpha)]$, that is

$$[B_L(\alpha)_\alpha, B_R(\alpha)_\alpha] \leftrightarrow [B_L(\alpha), B_R(\alpha)] \quad (17)$$

Because the 1-*level fuzzy point* $\tilde{0}_1$ has a one-to-one correspondence with the real number 0, the signed distance of $[B_L(\alpha)_\alpha, B_R(\alpha)_\alpha]$ to $\tilde{0}_1$ can be give as:

$$d\big([B_L(\alpha)_\alpha, B_R(\alpha)_\alpha], \tilde{0}_1\big) = d_0([B_L(\alpha), B_R(\alpha)], 0) = [B_L(\alpha) + B_R(\alpha)]/2 \quad (18)$$

Furthermore, for $\tilde{B} \in \Psi$, since the above function is continuous on $0 \leq \alpha \leq 1$, the integration can be applied to obtain the mean value of the signed distance as follows:

$$\int_0^1 d\big([B_L(\alpha)_\alpha, B_R(\alpha)_\alpha], \tilde{0}_1\big) d\alpha = \frac{1}{2}\int_0^1 [B_L(\alpha) + B_R(\alpha)] d\alpha \quad (19)$$

Definition 7. For $\tilde{B} \in \Psi$, we can define the signed distance of $\tilde{B}$ to $\tilde{0}$ as:

$$d(\tilde{B}, \tilde{0}_1) = \int_0^1 d\big([B_L(\alpha)_\alpha, B_R(\alpha)_\alpha], \tilde{0}_1\big) d\alpha = \frac{1}{2}\int_0^1 [B_L(\alpha) + B_R(\alpha)] d\alpha \quad (20)$$

Property 2. For the TFN $\tilde{B} = (b_1, b_2, b_3)$, the signed distance from $\tilde{B}$ to $\tilde{0}$ is given as:

$$d(\tilde{B}, \tilde{0}_1) = \int_0^1 d\big([B_L(\alpha)_\alpha, B_R(\alpha)_\alpha], \tilde{0}_1\big) d\alpha = \frac{1}{4}(b_1 + 2b_2 + b_3) \quad (21)$$

Considering (15.1)-(15.5) and (17), for two fuzzy sets $\tilde{B}, \tilde{C} \in \Psi$ where $\tilde{B} = \cup_{0\le\alpha\le1}[B_L(\alpha)_\alpha, B_R(\alpha)_\alpha]$ and $\tilde{C} = \cup_{0\le\alpha\le1}[C_L(\alpha)_\alpha, C_R(\alpha)_\alpha]$, and $k \in R$, we have

$$\tilde{B}(+)\tilde{C} = \cup_{0\le\alpha\le1}[(B_L(\alpha) + C_L(\alpha))_\alpha, (B_R(\alpha) + C_R(\alpha))_\alpha], \quad (22.1)$$

$$\tilde{B}(-)\tilde{C} = \cup_{0\le\alpha\le1}[(B_L(\alpha) - C_R(\alpha))_\alpha, (B_R(\alpha) - C_L(\alpha))_\alpha], \quad (22.2)$$

$$\tilde{k}_1(\cdot)\tilde{B} = \begin{cases} \cup_{0\le\alpha\le1}[(kB_L(\alpha))_\alpha, (kB_R(\alpha))_\alpha], & k > 0, \\ \cup_{0\le\alpha\le1}[(kB_R(\alpha))_\alpha, (kB_L(\alpha))_\alpha], & k < 0, \\ \tilde{0}_1, & k = 0. \end{cases} \quad (22.3)$$

Property 3. For two fuzzy sets $\tilde{B}, \tilde{C} \in \Psi$ and $k \in R$,

$$d\big(\tilde{B}(+)\tilde{C}, \tilde{0}_1\big) = d\big(\tilde{B}, \tilde{0}_1\big) + d(\tilde{C}, \tilde{0}_1), \quad (23.1)$$

$$d\big(\tilde{B}(-)\tilde{C}, \tilde{0}_1\big) = d\big(\tilde{B}, \tilde{0}_1\big) - d(\tilde{C}, \tilde{0}_1), \quad (23.2)$$

$$d(\tilde{k}_1(\cdot)\tilde{B}, \tilde{0}_1) = kd(\tilde{B}, \tilde{0}_1). \quad (23.3)$$

4.2.2. Graded mean integration representation (GMIR) method

If we consider $\tilde{B} = (b_1, b_2, b_3)$ as a TFN with α-*cut:* $[B_L(\alpha) + B_U(\alpha)]$, according to Chen and Hsieh [86], its GMIR is given by

$$\Phi\big(\tilde{B}\big) = \frac{\int_0^1 \alpha[B_L(\alpha) + B_U(\alpha)]\,d\alpha}{2\int_0^1 \alpha d\alpha} = \frac{\int_0^1 \alpha[b_1 + b_3 + (2b_2 - b_1 - b_3)\alpha]\,d\alpha}{2\int_0^1 \alpha d\alpha} = \frac{(b_1 + 4b_2 + b_3)}{6} \quad (24)$$

where $B_L(\alpha)$ and $B_R(\alpha)$ could be determined from Property 1.

5. Fuzzy reverse inventory model

In this section, the reverse inventory model presented in Section 3 is modified by incorporating the fuzziness of the demand rate of the serviceable products k and the collection rate of the recoverable products r. To do so, we fuzzify r and k to be two TFNs $\tilde{r}$ and $\tilde{k}$, respectively, where $\tilde{r} = (r - \theta_1, r, r + \theta_2)$, $0 < \theta_1 < r$, $\theta_2 > 0$, and $\tilde{k} = (k - \theta_3, k, k + \theta_4)$, $0 < \theta_3 < k$, $\theta_4 > 0$. It should be noted that θ_1, θ_2, θ_3, and θ_4 could be determined by decision makers. By fuzzifying the mentioned parameters, the total cost per unit time which is also a fuzzy function can be expressed as

$$\tilde{V} \equiv \tilde{V}(y, n) = \frac{\tilde{r}(nC_o + C_s)}{y} + H_r\left(\frac{y}{2} - \frac{a\tilde{r}y^{b+1}}{b+2}\right) + H_s\left[\frac{y\big(\tilde{k} - \tilde{r}\big)^2}{2n\tilde{k}\tilde{r}} + \frac{\tilde{r}y}{2\tilde{k}} - \frac{a\tilde{r}y^{b+1}}{(b+1)(b+2)}\right] + \left(\frac{a\tilde{r}y^b}{b+1}\right)C_l + \big(\tilde{k} - \tilde{r}\big)C_p + \tilde{r}C_b \quad (25)$$

In the next sections, we defuzzify the $\tilde{V}(y, n)$ by using the GMIR and the SD method.

5.1. Defuzzification by the SD method

From Property 3, the signed distance of $\tilde{V}$ to $\tilde{0}_1$ is given by

$$d\big(\tilde{V}, \tilde{0}\big) = \frac{(nC_o + C_s)}{y} d\big(\tilde{r}, \tilde{0}_1\big) + \frac{y}{2}H_r - \frac{H_r a y^{b+1}}{b+2} d\big(\tilde{r}, \tilde{0}_1\big) + \frac{yH_s}{2n} d\left(\frac{\tilde{k}}{\tilde{r}}, \tilde{0}_1\right) + \frac{yH_s}{2n} d\left(\frac{\tilde{r}}{\tilde{k}}, \tilde{0}_1\right) - \frac{yH_s}{n} + \frac{yH_s}{2} d\left(\frac{\tilde{r}}{\tilde{k}}, \tilde{0}_1\right) - \frac{a y^{b+1} H_s}{(b+1)(b+2)} d\big(\tilde{r}, \tilde{0}_1\big)$$

$$+\left(\frac{ay^b}{b+1}\right)C_l d(\tilde{r},\tilde{0}_1)+C_p d((\tilde{k}-\tilde{r}),\tilde{0}_1)+C_b d(\tilde{r},\tilde{0}_1) \tag{26}$$

where $d(\tilde{r},\tilde{0}_1)$, $d(\tilde{k}/\tilde{r},\tilde{0}_1)$, and $d((\tilde{k}-\tilde{r}),\tilde{0}_1)$ are measured as follows. From Property 2, the signed distance of fuzzy number $\tilde{r}$ to $\tilde{0}_1$ is

$$d(\tilde{r},\tilde{0}_1)=\frac{1}{4}[(r-\theta_1)+2r+(r+\theta_2)]=r+\frac{1}{4}(\theta_2-\theta_1) \tag{27}$$

The left and right end points of the *α-cut* of $\tilde{r}$, and $\tilde{k}$ $(0\le\alpha\le 1)$ are $r_L(\alpha)=(r-\theta_1)+\theta_1\alpha$, $r_R(\alpha)=(r+\theta_2)-\theta_2\alpha$, $k_L(\alpha)=(k-\theta_3)+\theta_3\alpha$, and $k_R(\alpha)=(k+\theta_4)-\theta_4\alpha$, respectively. Since $0<r_L(\alpha)<r_R(\alpha)$, $0<k_L(\alpha)<k_R(\alpha)$, from (15.1)-(15.5), the left and right end points of the *α-cut* of $\tilde{k}/\tilde{r}$, $\tilde{k}-\tilde{r}$, and $\tilde{r}/\tilde{k}$ are

$$\left(\frac{k}{r}\right)_L(\alpha)=\frac{k_L(\alpha)}{r_R(\alpha)}=\frac{(k-\theta_3)+\theta_3\alpha}{(r+\theta_2)-\theta_2\alpha} \tag{28}$$

$$\left(\frac{k}{r}\right)_R(\alpha)=\frac{k_R(\alpha)}{r_L(\alpha)}=\frac{(k+\theta_4)-\theta_4\alpha}{(r-\theta_1)+\theta_1\alpha} \tag{29}$$

$$(k-r)_L(\alpha)=k_L(\alpha)-r_R(\alpha)=(k-\theta_3)-(r+\theta_2)+(\theta_2+\theta_3)\alpha \tag{30}$$

$$(k-r)_R(\alpha)=k_R(\alpha)-r_L(\alpha)=(k+\theta_4)-(r-\theta_1)-(\theta_1+\theta_4)\alpha \tag{31}$$

$$\left(\frac{r}{k}\right)_L(\alpha)=\frac{r_L(\alpha)}{k_R(\alpha)}=\frac{(r-\theta_1)+\theta_1\alpha}{(k+\theta_4)-\theta_4\alpha} \tag{32}$$

$$\left(\frac{r}{k}\right)_R(\alpha)=\frac{r_R(\alpha)}{k_L(\alpha)}=\frac{(r+\theta_2)-\theta_2\alpha}{(k-\theta_3)+\theta_3\alpha} \tag{33}$$

respectively. Thus, from Definition 7, the signed distance of $\tilde{k}/\tilde{r}$, $\tilde{k}-\tilde{r}$, and $\tilde{r}/\tilde{k}$ to $\tilde{0}_1$ are

$$\begin{aligned}d\left(\frac{\tilde{k}}{\tilde{r}},\tilde{0}_1\right)&=\frac{1}{2}\int_0^1\left[\left(\frac{k}{r}\right)_L(\alpha)+\left(\frac{k}{r}\right)_U(\alpha)\right]d\alpha\\&=\frac{1}{2}\left[\frac{r\theta_4+k\theta_1}{{\theta_1}^2}Ln\frac{r}{r-\theta_1}-\frac{\theta_4}{\theta_1}+\frac{r\theta_3+k\theta_2}{{\theta_2}^2}Ln\frac{r+\theta_2}{r}-\frac{\theta_3}{\theta_2}\right]\end{aligned} \tag{34}$$

$$\begin{aligned}d\left(\frac{\tilde{r}}{\tilde{k}},\tilde{0}_1\right)&=\frac{1}{2}\int_0^1\left[\left(\frac{r}{k}\right)_L(\alpha)+\left(\frac{r}{k}\right)_U(\alpha)\right]d\alpha\\&=\frac{1}{2}\left[\frac{k\theta_2+r\theta_3}{{\theta_3}^2}Ln\frac{k}{k-\theta_3}-\frac{\theta_2}{\theta_3}+\frac{k\theta_1+r\theta_4}{{\theta_4}^2}Ln\frac{k+\theta_4}{k}-\frac{\theta_1}{\theta_4}\right]\end{aligned} \tag{35}$$

$$\begin{aligned}d\left((\tilde{k}-\tilde{r}),\tilde{0}_1\right)&=\frac{1}{2}\int_0^1[(k-r)_L(\alpha)+(k-r)_U(\alpha)]\,d\alpha\\&=\frac{1}{2}\left([(\theta_2+\theta_3)-(\theta_1+\theta_4)]\frac{1}{2}+(k-\theta_3)+(k+\theta_4)-(r+\theta_2)-(r-\theta_1)\right)\end{aligned} \tag{36}$$

respectively. Substituting the results obtained by (34)-(36) into (26), we have

$$\begin{aligned}V(n,y)&\equiv d(\tilde{V},\tilde{0})\\&=d(\tilde{r},\tilde{0}_1)\left[\frac{(nC_o+C_s)}{y}-\frac{H_r ay^{b+1}}{b+2}-\frac{ay^{b+1}H_s}{(b+1)(b+2)}+\left(\frac{ay^b}{b+1}\right)C_l+C_b\right]\\&\quad+d\left(\frac{\tilde{k}}{\tilde{r}},\tilde{0}_1\right)\frac{yH_s}{2n}+d\left(\frac{\tilde{r}}{\tilde{k}},\tilde{0}_1\right)\left[\frac{yH_s}{2n}+\frac{yH_s}{2}\right]+d((\tilde{k}-\tilde{r}),\tilde{0}_1)C_p-\frac{yH_s}{n}+\frac{y}{2}H_r\end{aligned} \tag{37}$$

$$\begin{aligned}V(n,y)&\equiv d(\tilde{V},\tilde{0})\\&=\left[r+\frac{1}{4}(\theta_2-\theta_1)\right]\left[\frac{(nC_o+C_s)}{y}-\frac{H_r ay^{b+1}}{b+2}-\frac{ay^{b+1}H_s}{(b+1)(b+2)}+\left(\frac{ay^b}{b+1}\right)C_l+C_b\right]\\&\quad+\frac{yH_s}{4n}\left[\frac{r\theta_4+k\theta_1}{{\theta_1}^2}Ln\frac{r}{r-\theta_1}-\frac{\theta_4}{\theta_1}+\frac{r\theta_3+k\theta_2}{{\theta_2}^2}Ln\frac{r+\theta_2}{r}-\frac{\theta_3}{\theta_2}\right]\end{aligned}$$

$$+\frac{(1+n)yH_s}{4n}\left[\frac{k\theta_2+r\theta_3}{{\theta_3}^2}Ln\frac{k}{k-\theta_3}-\frac{\theta_2}{\theta_3}+\frac{k\theta_1+r\theta_4}{{\theta_4}^2}Ln\frac{k+\theta_4}{k}-\frac{\theta_1}{\theta_4}\right]$$
$$+\left(k-r-\frac{\theta_2+\theta_3-\theta_1-\theta_4}{4}\right)C_p-\frac{yH_s}{n}+\frac{y}{2}H_r \quad (38)$$

$V(n,y)$ is considered as the estimate of the total cost per unit time in fuzzy situation. The next step is to determine the optimal recovery lot size to minimize the total cost function $V(n,y)$. By setting the first derivative of $d(\tilde{V},\tilde{0})$ with respect to n equal to zero, we can obtain y as follow

$$\frac{\partial d(\tilde{V},\tilde{0})}{\partial n}=0\rightarrow y=n\sqrt{\frac{2C_o d(\tilde{r},\tilde{0}_1)}{H_s\Delta}}=n\rho \quad (39)$$

where

$$\Delta= d\left(\frac{\tilde{k}}{\tilde{r}},\tilde{0}_1\right)+d\left(\frac{\tilde{r}}{\tilde{k}},\tilde{0}_1\right)-2 \quad \text{and} \quad \rho=\sqrt{\frac{2C_o d(\tilde{r},\tilde{0}_1)}{H_s\Delta}}$$

By setting the first derivative of $d(\tilde{V},\tilde{0})$ with respect to y, we have

$$\frac{\partial d(\tilde{V},\tilde{0})}{\partial y}=-\frac{(nC_o+C_s)}{y^2}d(\tilde{r},\tilde{0}_1)+\frac{1}{2}H_r-\frac{(b+1)H_r a y^b}{b+2}d(\tilde{r},\tilde{0}_1)+\frac{H_s}{2n}d\left(\frac{\tilde{k}}{\tilde{r}},\tilde{0}_1\right)$$
$$+\frac{H_s}{2n}d\left(\frac{\tilde{r}}{\tilde{k}},\tilde{0}_1\right)-\frac{H_s}{n}+\frac{H_s}{2}d\left(\frac{\tilde{r}}{\tilde{k}},\tilde{0}_1\right)-\frac{ay^bH_s}{(b+2)}d(\tilde{r},\tilde{0}_1)$$
$$+\left(\frac{aby^{b-1}}{b+1}\right)C_l d(\tilde{r},\tilde{0}_1)=0 \quad (40)$$

and after substituting y from (39) into (40), $\tilde{g}(n)$ can be derived as

$$\tilde{g}(n)=-\frac{(nC_o+C_s)}{n^2\rho^2}d(\tilde{r},\tilde{0}_1)+\frac{1}{2}H_r-\frac{(b+1)H_r a n^b\rho^b}{b+2}d(\tilde{r},\tilde{0}_1)+\frac{H_s}{2n}$$
$$+\frac{H_s}{2n}\left(d\left(\frac{\tilde{k}}{\tilde{r}},\tilde{0}_1\right)+d\left(\frac{\tilde{r}}{\tilde{k}},\tilde{0}_1\right)-2\right)+\frac{H_s}{2}d\left(\frac{\tilde{r}}{\tilde{k}},\tilde{0}_1\right)-\frac{an^b\rho^bH_s}{(b+2)}d(\tilde{r},\tilde{0}_1)$$
$$+\left(\frac{abn^{b-1}\rho^{b-1}}{b+1}\right)C_l d(\tilde{r},\tilde{0}_1) \quad (41)$$

Letting (y^*,n^*) shows the solution for the considered problem. To prove that y^* and n^* are the optimal recovery lot size and the optimal number of orders, respectively, Theorem 1 and 2 are necessary.

Theorem 1: The optimal solution of (y^*,n^*) not only exists, but is also unique. It is clear that it should satisfy $\tilde{g}(n)=0$, and $y-n\rho=0$, simultaneously.

Proof. See Appendix A. □

Theorem 2: $d(\tilde{V},\tilde{0})$ has a global minimum at (y^*,n^*), where this point is the solution for $\tilde{g}(n)=0$, and $y-n\rho=0$.

Proof. See Appendix B. □

Solution procedure. Finding a closed-form solution is not possible for $\tilde{g}(n)=0$. Instead of a direct method, we can find n^* by a one-dimensional search procedure. When n^* can be found, y^* can be obtained by Eq. (40). As n^* is a positive integer, the proposed solution algorithm in the next section can be applied to find the optimal values.

Remarks 3. From Eqs. (3) and (34), we can estimate the ordering lot size for the newly purchased products Q as $d\left(\tilde{Q},\tilde{0}\right)=\frac{y}{n}d(\frac{\tilde{k}}{\tilde{r}},\tilde{0})-\frac{y}{n}$.

5.2. Defuzzification by the GMIR method

By applying the fuzzy arithmetic operations of the function principle method described in Section 4.1 and Definition 1, the fuzzy total cost per unit time in Eq. (25), can be written as follows:

$$\tilde{V}(y,n)=(\chi_1,\chi_2,\chi_3) \tag{42}$$

where

$$\begin{aligned}\chi_1 =& \frac{(r-\theta_1)(nC_o+C_s)}{y}+H_r\left[\frac{y}{2}-\frac{a(r+\theta_2)y^{b+1}}{b+2}\right]+C_l\frac{a(r-\theta_1)y^b}{b+1}\\ &+H_s\left(\frac{y[(k-\theta_3)-(r+\theta_2)]^2}{2n(k+\theta_4)(r+\theta_2)}+\frac{y(r-\theta_1)}{2(k+\theta_4)}-\frac{a(r+\theta_2)y^{b+1}}{(b+1)(b+2)}\right)\\ &+[(k-\theta_3)-(r+\theta_2)]C_p+(r-\theta_1)C_b\end{aligned} \tag{43}$$

$$\begin{aligned}\chi_2 =& \frac{r(nC_o+C_s)}{y}+H_r\left[\frac{y}{2}-\frac{ary^{b+1}}{b+2}\right]+H_s\left(\frac{y[k-r]^2}{2nkr}+\frac{yr}{2k}-\frac{ary^{b+1}}{(b+1)(b+2)}\right)\\ &+C_l\frac{ary^b}{b+1}+(k-r)C_p+rC_b\end{aligned} \tag{44}$$

$$\begin{aligned}\chi_3 =& \frac{(r+\theta_2)(nC_o+C_s)}{y}+H_r\left[\frac{y}{2}-\frac{a(r-\theta_1)y^{b+1}}{b+2}\right]+C_l\frac{a(r+\theta_2)y^b}{b+1}\\ &+H_s\left(\frac{y[(k+\theta_4)-(r-\theta_1)]^2}{2n(k-\theta_3)(r-\theta_1)}+\frac{y(r+\theta_2)}{2(k-\theta_3)}-\frac{a(r-\theta_1)y^{b+1}}{(b+1)(b+2)}\right)\\ &+[(k+\theta_4)-(r-\theta_1)]C_p+(r+\theta_2)C_b\end{aligned} \tag{45}$$

According to the GMIR method explained in Section 4.2.2, the defuzzified value of $\tilde{V}$ can be given as below:

$$\Phi\left(\tilde{V}(y,n)\right)=\frac{1}{6}(\chi_1+4\chi_2+\chi_3) \tag{46}$$

The next step is to determine the optimal recovery lot size to minimize the total cost function $\Phi(\tilde{V}(y,n))$. By setting the first derivative of $\Phi(\tilde{V}(y,n))$ with respect to n equal to zero, we can obtain y as follows:

$$\frac{\partial\Phi(\tilde{V}(y,n))}{\partial n}=0\rightarrow y=n\pi \tag{47}$$

where

$$\pi=\sqrt{\frac{C_o[(r-\theta_1)+4r+(r+\theta_2)]}{H_s\left(\frac{[(k-\theta_3)-(r+\theta_2)]^2}{2(k+\theta_4)(r+\theta_2)}+\frac{2(k-r)^2}{kr}+\frac{[(k+\theta_4)-(r-\theta_1)]^2}{2(k-\theta_3)(r-\theta_1)}\right)}}$$

By setting the first derivative of $\Phi(\tilde{V}(y,n))$ with respect to y, and after substituting $y=n\pi$, $\tilde{f}(n)$ can be derived as

$$\tilde{f}(n)=-\frac{\gamma C_s}{6n^2C_o}+\frac{(\beta+\delta)n^b+\zeta n^{b-1}}{6}+\varepsilon \tag{48}$$

where

$$\alpha = -\frac{(nC_o + C_s)\gamma}{C_o} \quad , \quad \beta = -\frac{a(b+1)H_r[(r+\theta_2) + 4r + (r-\theta_1)]\pi^b}{b+2} \quad ,$$
$$\gamma = H_s\left(\frac{[(k-\theta_3)-(r+\theta_2)]^2}{2(k+\theta_4)(r+\theta_2)} + \frac{2(k-r)^2}{kr} + \frac{[(k+\theta_4)-(r-\theta_1)]^2}{2(k-\theta_3)(r-\theta_1)}\right) \quad ,$$
$$\delta = -\frac{aH_s\pi^b}{(b+2)}((r-\theta_1) + 4r + (r+\theta_2)) \quad , \quad \zeta = abC_l\pi^{b-1}\left[\frac{(r-\theta_1) + 4r + (r+\theta_2)}{b+1}\right] \quad ,$$
$$\varepsilon = \frac{1}{2}H_r + \frac{1}{6}H_s\left[\frac{(r-\theta_1)}{2(k+\theta_4)} + \frac{2r}{k} + \frac{(r+\theta_2)}{2(k-\theta_3)}\right]$$

Letting (y^*, n^*) shows the solution for the considered problem. To prove that y^* and n^* are the optimal recovery lot size and the optimal number of orders, respectively, Theorem 3 and 4 are required.

Theorem 3: The optimal solution of (y^*, n^*) not only exists, but is also unique. It is clear that it should satisfy $\tilde{f}(n) = 0$, and $y - n\pi = 0$, simultaneously.

Proof. See Appendix C. □

Theorem 4: $\Phi(\tilde{V}(y,n))$ has a global minimum at (y^*, n^*), where this point is the solution for $\tilde{f}(n) = 0$, and $y - n\pi = 0$.

Proof. See Appendix D. □

Solution procedure. Finding a closed-form solution is not possible for $\tilde{f}(n) = 0$. Instead of a direct method, we can find n^* by a one-dimensional search procedure. When n^* can be found, y^* can be obtained by $\partial d(\tilde{V}, \tilde{0})/\partial y = 0$. As n^* is a positive integer, the following solution algorithm can also be applied to find the optimal values.

Solution algorithm. Consider a pre-determined error value $\tau > 0$. Set n_l and n_u as suggested guesses of the root such that $f(n_u) > 0$ and $f(n_l) < 0$. The optimal values could be found by the proposed flowchart in Fig. 1. In this flowchart, $\lfloor n \rfloor$ and $\lceil n \rceil$ show the nearest integers smaller and larger than n^*.

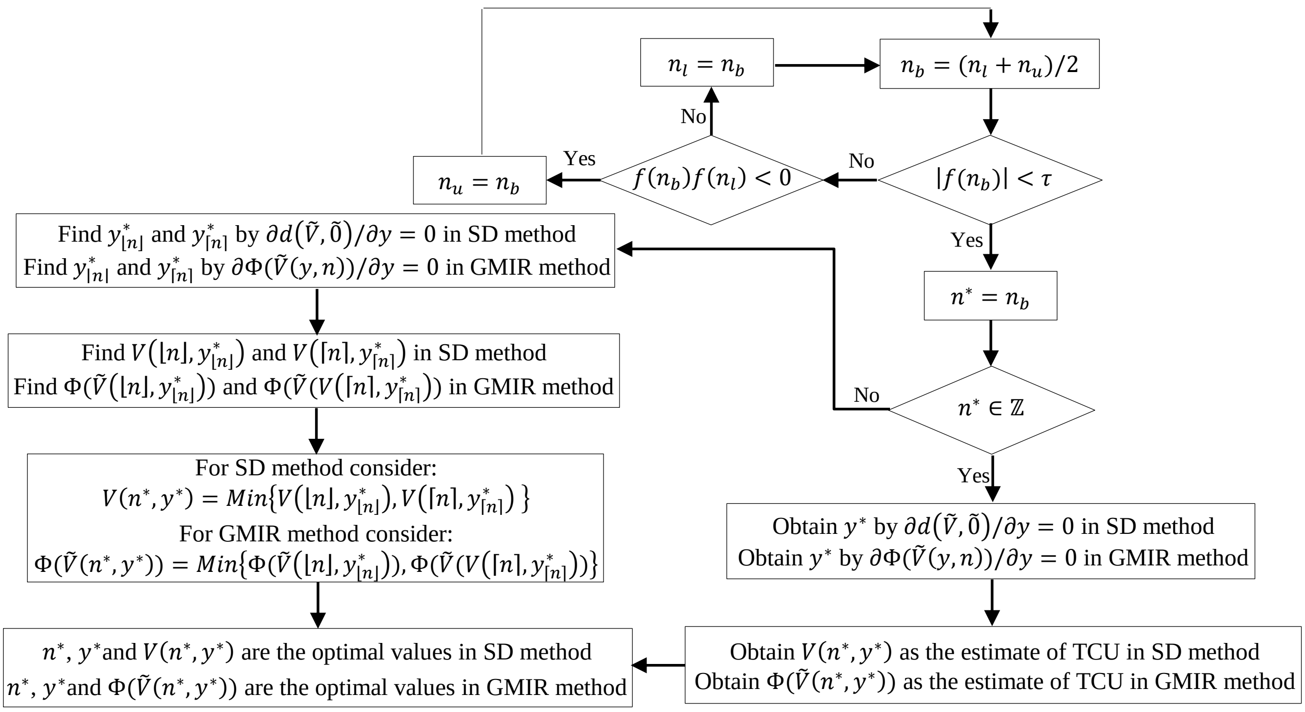

Fig. 1. Proposed flowchart to find the optimal recovery lot size and the number of orders

Remarks 4. From Eqs. (3) and (24), we can estimate ordering lot size for the newly purchased products Q as $\Phi(\tilde{Q}) = \frac{1}{6}(q_1 + 4q_2 + q_3) = \frac{1}{6}(\frac{y[(k-\theta_3)-(r+\theta_2)]}{n(r+\theta_2)} + \frac{4y(k-r)}{nr} + \frac{y[(k+\theta_4)-(r-\theta_1)]}{n(r-\theta_1)})$.

6. Comprehensive numerical example

In this section, we analyze and explain the effects of fuzzification on the developed model through a comprehensive numerical example. We expound the results of each model, separately. Besides, defining some comparative criteria, these results will be compared with each other simultaneously.

Table 1. Considered fuzzy numbers

θ_1	θ_2	$\tilde{r}$	θ_3	θ_4	$\tilde{k}$	θ_3	θ_4	$\tilde{k}$
5	55	(95,100,155)	20	14	(230,250,264)	40	50	(210,250,300)
25	50	(75,100,150)	20	20	(230,250,270)	60	40	(190,250,290)
40	40	(60,100,140)	20	26	(230,250,276)	60	60	(190,250,310)
50	25	(50,100,125)	40	30	(210,250,280)	60	80	(190,250,330)
55	5	(45,100,105)	40	40	(210,250,290)			

Table 2. The results of effecting the crisp model by the GMIR method

No.	θ_1	θ_2	θ_3	θ_4	n^*	y^*	Q^*	TCU^*	$\Phi(\tilde{r})$	$\Phi(\tilde{k}/\tilde{r})$	$\Phi(\tilde{r}/\tilde{k})$	$\Phi(\tilde{k}-\tilde{r})$
1	5	55	20	14	4	678	233	41469.90	108.333	2.377	0.439	140.667
2				20	4	677	235	41686.67	108.333	2.388	0.438	141.667
3				26	4	676	236	41904.77	108.333	2.398	0.436	142.667
4			40	30	4	668	231	41480.96	108.333	2.384	0.446	140.000
5				40	5	712	200	41849.07	108.333	2.401	0.444	141.667
6				50	5	709	201	42216.42	108.333	2.419	0.442	143.333
7			60	40	5	702	194	41311.76	108.333	2.380	0.457	138.330
8				60	5	696	197	42059.24	108.333	2.415	0.454	141.667
9				80	5	689	200	42819.22	108.333	2.450	0.451	145.000
10	25	50	20	14	5	701	212	42096.73	104.167	2.509	0.423	144.833
11				20	5	700	213	42319.71	104.167	2.522	0.422	145.833
12				26	5	698	214	42543.94	104.167	2.536	0.421	146.833
13			40	30	5	688	209	42137.25	104.167	2.522	0.430	144.167
14				40	5	684	211	42520.33	104.167	2.544	0.429	145.833
15				50	5	681	213	42906.78	104.167	2.567	0.427	147.500
16			60	40	5	674	205	41998.27	104.167	2.522	0.441	142.500
17				60	5	665	208	42786.72	104.167	2.567	0.439	145.833
18				80	5	656	211	43588.19	104.167	2.611	0.436	149.167
19	40	40	20	14	5	675	226	42744.49	100.000	2.674	0.406	149.000
20				20	5	672	227	42978.14	100.000	2.690	0.405	150.000
21				26	5	670	229	43213.12	100.000	2.707	0.404	151.000
22			40	30	5	659	223	42823.76	100.000	2.694	0.413	148.333
23				40	6	698	200	43225.10	100.000	2.722	0.412	150.000
24				50	6	693	202	43621.69	100.000	2.750	0.411	151.667
25			60	40	6	686	194	42712.89	100.000	2.698	0.424	146.667
26				60	6	676	198	43522.93	100.000	2.754	0.422	150.000
27				80	6	665	201	44345.71	100.000	2.810	0.420	153.333
28	50	25	20	14	6	692	214	43384.83	95.833	2.853	0.389	153.167
29				20	6	689	215	43621.97	95.833	2.873	0.388	154.167
30				26	6	686	216	43860.42	95.833	2.893	0.387	155.167
31			40	30	6	675	212	43474.18	95.833	2.880	0.396	152.500
32				40	6	669	213	43882.84	95.833	2.913	0.395	154.167
33				50	6	664	215	44294.95	95.833	2.947	0.394	155.833
34			60	40	6	658	207	43376.64	95.833	2.887	0.405	150.833
35				60	6	646	210	44220.43	95.833	2.953	0.403	154.167
36				80	7	674	194	45061.16	95.833	3.020	0.402	157.500
37	55	5	20	14	6	669	224	43949.34	91.667	3.010	0.371	157.333
38				20	6	666	226	44190.54	91.667	3.032	0.371	158.333
39				26	6	663	227	44433.11	91.667	3.054	0.370	159.333
40			40	30	6	653	222	44040.65	91.667	3.037	0.377	156.667
41				40	6	647	224	44457.32	91.667	3.074	0.376	158.333
42				50	7	681	205	44871.33	91.667	3.111	0.375	160.000
43			60	40	7	676	197	43940.99	91.667	3.042	0.385	155.000
44				60	7	663	200	44784.16	91.667	3.116	0.383	158.333
45				80	7	650	203	45640.21	91.667	3.190	0.381	161.667

In order to compare the results of the investigated model with those of the crisp one, let us consider the data in Tsai [22]. These data include $k = 250$ units/day, $r = 100$ units/day, $C_s =$ \$20,000/setup, $C_o =$ \$2,000/order, $C_p =$ \$200/unit, $C_b =$ \$40/unit, $H_r =$ \$4/unit/day, $H_s =$ \$20/unit/day, $C_l =$ \$1000/day, $a = 0.003$ day/unit, $b = -0.089$. Furthermore, as it is shown in Table 1, assuming some arbitrary sets for $\theta_i, i = 1,2,3,4$, the behaviour of the fuzzified models is examined. These parameters are selected such that $0 < \theta_1 < r$, $0 < \theta_3 < k$, and $0 < \theta_2, \theta_4$. Due to the uncertainties inherent in the data and lack of existing knowledge about the whole of the inventory system, these parameters are usually determined according to the experiences of experts as decision makers. We combined the mentioned fuzzy parameters to build 45 iterations. For the fuzzified parameters k and r, five and nine levels of fuzziness are assumed, respectively.

Table 2 presents the results of the GMIR method. From Table 2, it is clear that for the fixed values of (θ_1, θ_2), and constant values of the optimal number of orders n^*, when the level of fuzziness increases by varying the values of (θ_3, θ_4), the optimal recovery lot size y^* decreases, but the optimal ordering lot size and the optimal total cost function per unit time increase.

Table 3. The results of effecting the crisp model by the SD method

No.	θ_1	θ_2	θ_3	θ_4	n^*	y^*	Q^*	TCU^*	$d(\tilde{r}, \tilde{0}_1)$	$d(\tilde{k}/\tilde{r}, \tilde{0}_1)$	$d(\tilde{r}/\tilde{k}, \tilde{0}_1)$	$d(\tilde{k}-\tilde{r}, \tilde{0}_1)$
1	5	55	20	14	4	700	224	40758.79	112.500	2.278	0.456	136.000
2				20	4	699	226	41067.34	112.500	2.293	0.454	137.500
3				26	4	698	228	41376.43	112.500	2.309	0.452	139.000
4			40	30	4	694	222	40631.95	112.500	2.282	0.464	135.000
5				40	4	693	227	41148.65	112.500	2.308	0.461	137.500
6				50	4	692	231	41666.59	112.500	2.334	0.458	140.000
7			60	40	4	688	219	40213.02	112.500	2.271	.0475	132.500
8				60	4	685	227	41249.79	112.500	2.323	0.469	137.500
9				80	4	682	234	42290.74	112.500	2.374	0.464	142.500
10	25	50	20	14	4	668	243	41669.98	106.250	2.456	0.432	142.250
11				20	5	714	211	41980.99	106.250	2.475	0.430	143.750
12				26	5	713	213	42291.53	106.250	2.493	0.428	145.250
13			40	30	4	662	243	41553.24	106.250	2.467	0.440	141.250
14				40	5	707	212	42072.27	106.250	2.497	0.437	143.750
15				50	5	705	215	42592.09	106.250	2.527	0.434	146.250
16			60	40	4	656	239	41136.63	106.250	2.459	0.451	138.750
17				60	5	698	212	42181.57	106.250	2.520	0.446	143.750
18				80	5	695	220	43225.26	106.250	2.580	0.441	148.750
19	40	40	20	14	5	682	226	42545.29	100.000	2.657	0.407	148.500
20				20	5	681	228	42860.38	100.000	2.677	0.405	150.000
21				26	5	680	231	43175.82	100.000	2.698	0.404	151.500
22			40	30	5	676	226	42430.64	100.000	2.672	0.415	147.500
23				40	5	674	230	42957.30	100.000	2.707	0.412	150.000
24				50	5	672	234	43484.75	100.000	2.742	0.410	152.500
25			60	40	5	670	223	42014.63	100.000	2.667	0.426	145.000
26				60	5	666	231	43069.90	100.000	2.736	0.421	150.000
27				80	5	661	239	44127.79	100.000	2.806	0.417	155.000
28	50	25	20	14	5	651	242	43400.39	93.750	2.860	0.381	154.750
29				20	6	692	217	43716.64	93.750	2.883	0.380	156.250
30				26	6	691	220	44031.79	93.750	2.906	0.378	157.750
31			40	30	5	645	242	43281.22	93.750	2.879	0.389	153.750
32				40	6	685	219	43807.99	93.750	2.917	0.386	156.250
33				50	6	683	223	44334.75	93.750	2.956	0.384	158.750
34			60	40	5	639	240	42855.40	93.750	2.874	0.399	151.250
35				60	6	677	220	43913.43	93.750	2.951	0.395	156.250
36				80	6	673	228	44969.57	93.750	3.029	0.391	161.250
37	55	5	20	14	6	664	226	44180.64	87.500	3.044	0.355	161.000
38				20	6	663	229	44497.05	87.500	3.068	0.354	162.500
39				26	6	662	231	44813.73	87.500	3.093	0.353	164.000
40			40	30	6	659	226	44047.28	87.500	3.061	0.361	160.000
41				40	6	657	230	44575.80	87.500	3.102	0.359	162.500
42				50	6	655	234	45104.92	87.500	3.143	0.357	165.000
43			60	40	6	654	224	43608.03	87.500	3.054	0.370	157.500
44				60	6	650	231	44666.59	87.500	3.136	0.366	162.500
45				80	6	646	239	45727.15	87.500	3.218	0.363	167.500

In Table 2, for the fixed values of (θ_3, θ_4), as the estimate of the collection rate of the recoverable products $\Phi(\tilde{r})$ decreases by varying (θ_1, θ_2), the optimal recovery lot size y^* decreases for fixed n^*, but the optimal ordering lot size and the optimal total cost function per unit time increase. However, when n^* increases, y^* and TCU^* increase, and Q^*decreases. Besides, when $(\theta_1, \theta_2, \theta_3)$ are fixed, the optimal ordering lot size, and the optimal total cost function per unit time increase for fixed n^*.

Table 3 shows the results of the signed distance method. Based on the columns 7, 8 and 9 of the Table 3, for the fixed values of (θ_1, θ_2), and the constant values of the optimal number of orders n^*, the behavior of the optimal recovery lot size y^*, the optimal ordering lot size Q^*, and the optimal total cost function per unit time TCU^* is similar to that one which is explained in the GMIR method. However, one of the major differences between them is that as θ_1 increases, the decrease in y^* is higher for the GMIR method as compared to the signed distance method. Moreover, for the fixed values of (θ_3, θ_4), as the estimate of the collection rate of the recoverable products $d(\tilde{r}, \tilde{0}_1)$ decreases by varying (θ_1, θ_2), the reduction in the optimal recovery lot size y^* is lower than that of the GMIR method for fixed n^*, and therewith, the optimal ordering lot size Q^*, and the optimal total cost function per unit time TCU^* have the similar trends similar to the GMIR method.

Table 4. Comparing the results of the GMIR and SD methods with the crisp ones

No.	n^*	y^*_{SD}%	Q^*_{SD}%	TCU^*_{SD}%	n^*	y^*_{GMIR}%	Q^*_{GMIR}%	TCU^*_{GMIR}%
1	4	-0.709	5.660	-4.263	4	-3.830	9.906	-2.593
2	4	-0.851	6.604	-3.538	4	-3.972	10.849	-2.084
3	4	-0.993	7.547	-2.812	4	-4.113	11.321	-1.571
4	4	-1.560	4.717	-4.561	4	-5.248	8.962	-2.567
5	4	-1.702	7.075	-3.347	5	0.993	-5.660	-1.702
6	4	-1.844	8.962	-2.131	5	0.567	-5.189	-0.839
7	4	-2.411	3.302	-5.545	5	-0.426	-8.491	-2.964
8	4	-2.837	7.075	-3.110	5	-1.277	-7.075	-1.208
9	4	-3.262	10.377	-0.665	5	-2.270	-5.660	0.577
10	4	-5.248	14.623	-2.123	5	-0.567	0.000	-1.120
11	5	1.277	-0.472	-1.392	5	-0.709	0.472	-0.597
12	5	1.135	0.472	-0.663	5	-0.993	0.943	-0.070
13	4	-6.099	14.623	-2.397	5	-2.411	-1.415	-1.025
14	5	0.284	0.000	-1.178	5	-2.979	-0.472	-0.125
15	5	0.000	1.415	0.043	5	-3.404	0.472	0.782
16	4	-6.950	12.736	-3.375	5	-4.397	-3.302	-1.352
17	5	-0.993	0.000	-0.921	5	-5.674	-1.887	0.500
18	5	-1.418	3.774	1.530	5	-6.950	-0.472	0.401
19	5	-3.262	6.604	-0.067	5	-4.255	6.604	0.401
20	5	-3.404	7.547	0.673	5	-4.681	7.075	0.950
21	5	-3.546	8.962	1.414	5	-4.965	8.019	1.502
22	5	-4.113	6.604	-0.336	5	-6.525	5.189	0.587
23	5	-4.397	8.491	0.901	6	-0.993	-5.660	1.530
24	5	-4.681	10.377	2.140	6	-1.702	-4.717	2.462
25	5	-4.965	5.189	-1.313	6	-2.695	-8.491	0.327
26	5	-5.532	8.962	1.166	6	-4.113	-6.604	2.230
27	5	-6.241	12.736	3.650	6	-5.674	-5.189	4.162
28	5	-7.660	14.151	1.942	6	-1.844	0.943	1.905
29	6	-1.844	2.358	2.685	6	-2.270	1.415	2.462
30	6	-1.986	3.774	3.425	6	-2.695	1.887	3.022
31	5	-8.511	14.151	1.662	6	-4.255	0.000	2.115
32	6	-2.837	3.302	2.899	6	-5.106	0.472	3.075
33	6	-3.121	5.189	4.136	6	-5.816	1.415	4.043
34	5	-9.362	13.208	0.662	6	-6.667	-2.358	1.886
35	6	-3.972	3.774	3.147	6	-8.369	-0.943	3.868
36	6	-4.539	7.547	5.628	7	-4.397	-8.491	5.843
37	6	-5.816	6.604	3.774	6	-5.106	5.660	3.231
38	6	-5.957	8.019	4.518	6	-5.532	6.604	3.798
39	6	-6.099	8.962	5.262	6	-5.957	7.075	4.368
40	6	-6.525	6.604	3.461	6	-7.376	4.717	3.446
41	6	-6.809	8.491	4.703	6	-8.227	5.660	4.424
42	6	-7.092	10.377	5.946	7	-3.404	-3.302	5.397
43	6	-7.234	5.660	2.430	7	-4.113	-7.075	3.212
44	6	-7.801	8.962	4.916	7	-5.957	-5.660	5.192
45	6	-8.369	12.736	7.407	7	-7.801	-4.245	7.203
Average		-3.997	7.285	0.809		-3.959	0.073	1.446

The collection rate of the recoverable products from customers is an important factor in the reverse logistics literature. For positive levels of fuzziness, the estimations of this factor by the signed distance method are higher than those that are obtained applying the GMIR method. Besides, it is observed that, for negative levels of fuzziness, the estimations of the mentioned factor using the GMIR method returns higher value than the signed distance method. These interesting results should be taken into consideration in practical situations. In both methods, the optimal number of orders n^* increases by decreasing θ_2. In other words, the more the estimation of the difference between the demand rate of the serviceable products and the collection rate of the recoverable products $(k - r)$, the higher the optimal number of orders n^* will be.

Regarding a criterion defined in Eq. (49), the values of percentage changes for the optimal recovery lot size Q^*, and the optimal total cost function per unit time TCU^* compared to the crisp ones are calculated in Table 4. For example, the 3th column of Table 4 shows the percentage changes of optimal recovery lot size using the signed distance method ($y^*_{SD}\%$).

$$\left(\frac{\text{Optimal Fuzzy Value} - \text{Optimal Crisp Value}}{\text{Optimal Crisp Value}}\right) \times 100 \tag{49}$$

In order to have a better comparison, based on this criterion, the behavior of the fuzzified model by both methods is compared for the mentioned optimal values in Figure 2 and 3, simultaneously. In Figure 2, the white stars show the situations that the optimal number of orders n^* increase one unit in the GMIR method, while the black stars show the similar increase in the signed distance method.

According to the Figure 2, in states that the levels of fuzziness are similar, and also, the optimal number of orders n^* are equal, the percentage changes of the optimal recovery lot size in the GMIR method are negative $y^*_{GMIR}\% < 0$, and moreover, in these conditions, the difference between $y^*_{SD}\%$ and $y^*_{GMIR}\%$ is always positive ($y^*_{SD}\% - y^*_{GMIR}\% > 0$). The average of percentage changes for the signed distance and the GMIR method is -3.997, and -3.959, respectively.

Figure 3 indicates that taking the percentage changes of the optimal total cost function per unit time into account, in general, there are similar increasing trends for both methods. When $\theta_2 > \theta_1$, except for two cases, the percentage changes of the optimal total cost by the signed distance method $TCU^*_{SD}\%$ are negative. Moreover, when $\theta_2 < \theta_1$, those are positive in all cases. Generally, the GMIR method takes priority than the signed distance method in adopting positive value for the percentage changes of the optimal total cost. Besides, the average percentage change of the total cost function for the GMIR method (1.446) is greater than the similar one for the signed distance method (0.809).

Table 5 presents some descriptive statistics for the optimal values in each level of fuzziness for the collection rate of the recoverable products r, separately. In Table 5, "level of fuzziness" is a measure defined as the percentage deviation from the crisp value in each level of fuzziness. It is clear that the mentioned measure for the GMIR method is smaller than the similar one by the signed distance method. Although the overall average value of the total cost by the signed distance method is smaller than the calculated one by the GMIR method, its standard deviation in the GMIR method (1053) is smaller than the similar value (1345) by the signed distance method. It indicates that the GMIR method is more stable than the signed distance regarding the total cost. Considering the average for all levels, both methods lead to the same optimal value (677) for the recovering lot size. However, in this situation, the standard deviation by the signed distance is higher than the one obtained by the GMIR. Therefore, deciding on which method could be used depends on the target strategy that could focus on the total cost, the ordering lot size or the recovery lot size.

Besides, Table 6 shows the results of the descriptive statistics for the optimal values according to the criterion defined in Eq. (49) by varying θ_1 and θ_2, regardless of whether the percentage change is positive or negative. Unlike Table 5, in Table 6, these two methods are compared based on the optimal crisp values.

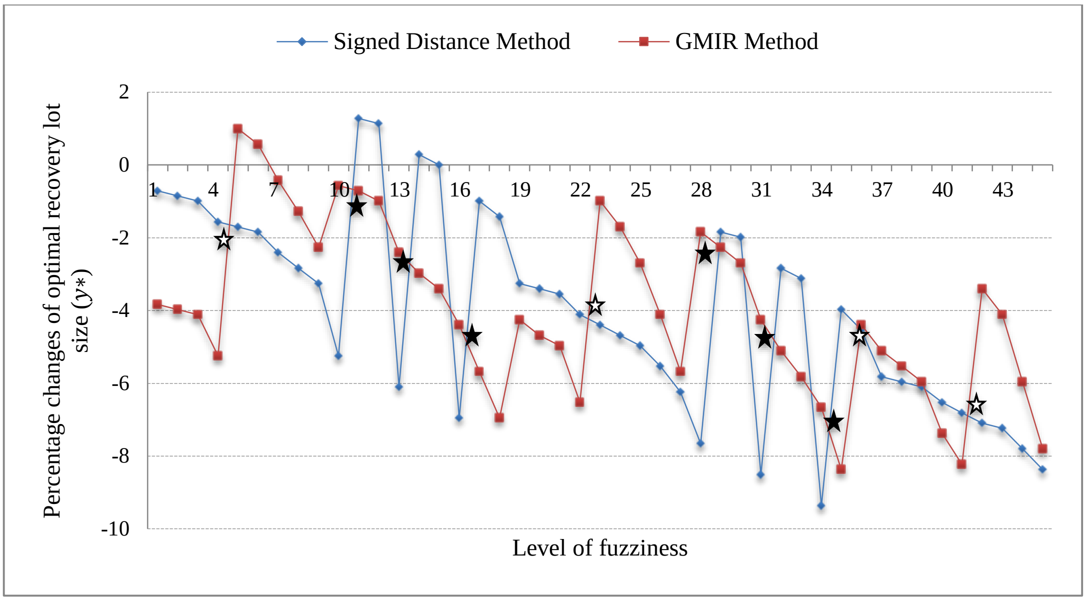

Fig. 2. Comparing the SD and the GMIR method simultaneously considering the relative variation between the crisp and fuzzy situation for the optimal recovery lot size

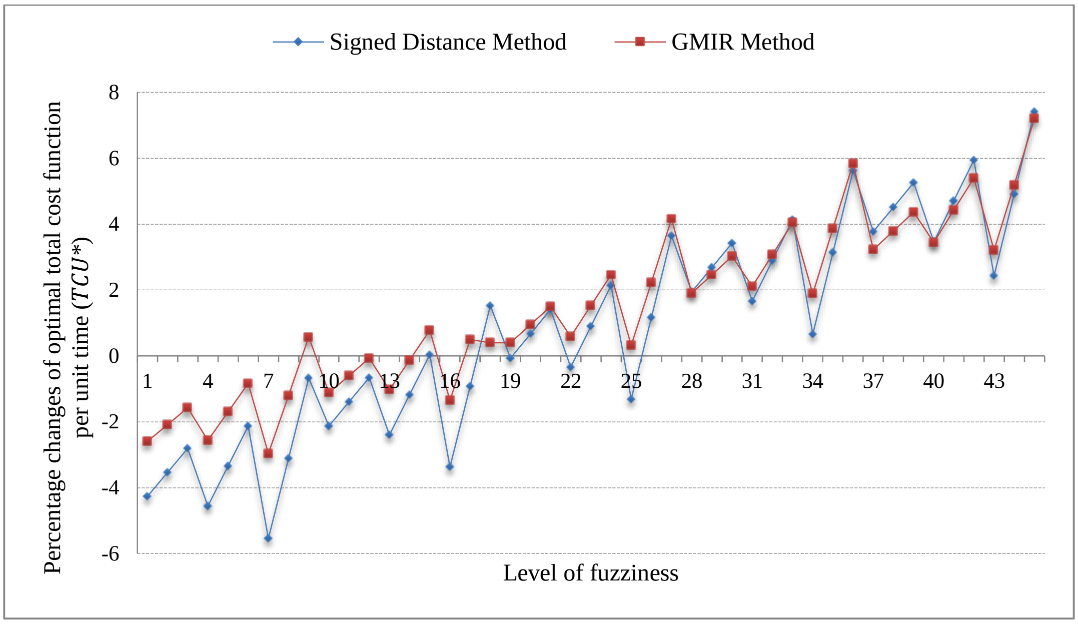

Fig. 3. Comparing the SD and the GMIR method simultaneously considering the relative variation between the crisp and fuzzy situation for the optimal total cost function per unit time

Table 5. Comparing the difference between the results of the GMIR and the SD method

	GMIR method				SD method			
(θ_1,θ_2)	*Level of fuzziness*	y^* (μ,σ)*	Q^* (μ,σ)*	TCU^* (μ,σ)*	*Level of fuzziness*	y^* (μ,σ)*	Q^* (μ,σ)*	TCU^* (μ,σ)*
(5,55)	8.33%	(690,15)	(214,18)	(41866.45,437)	12.50%	(692,6)	(226,4)	(41155.92,571)
(25,50)	4.17%	(683,15)	(211,3)	(42544.21,471)	6.25%	(691,21)	(223,13)	(42078.17,575)
(40,40)	0%	(677,12)	(211,14)	(43243.09,496)	0%	(674,7)	(230,5)	(42962.94,582)
(50,25)	– 4.17%	(673,14)	(211,6)	(43908.60,517)	– 6.25%	(671,19)	(228,10)	(43812.35,582)
(55,5)	– 8.33%	(663,11)	(214,12)	(44478.63,521)	– 12.50%	(657,6)	(230,4)	(44580.13,582)
FOR ALL LEVELS		(677,16)	(212,12)	(43208.20,1053)		(677,19)	(227,9)	(42917.91,1345)

*μ, σ stands for mean, and standard deviation of the optimal values, respectively.

Table 6. Comparing the difference between the results of the GMIR and SD method based on the crisp values

	GMIR method			SD method		
	y^*	Q^*	TCU^*	y^*	Q^*	TCU^*
(θ_1,θ_2)	(μ,σ)*	(μ,σ)*	(μ,σ)*	(μ,σ)*	(μ,σ)*	(μ,σ)*
(5,55)	(2.52%, 1.70%)	(8.12%, 2.20%)	(1.79%, 0.78%)	(1.80%, 0.84%)	(6.81%, 2.00%)	(3.33%, 1.34%)
(25,50)	(3.12%, 2.12%)	(1.05%, 0.96%)	(0.66%, 0.42%)	(2.60%, 2.54%)	(5.35%, 6.23%)	(1.51%, 0.94%)
(40,40)	(3.96%, 1.72%)	(6.39%, 1.24%)	(1.57%, 1.17%)	(4.46%, 0.95%)	(8.39%, 2.13%)	(1.30%, 1.01%)
(50,25)	(4.60%, 2.03%)	(1.99%, 2.39%)	(3.14%, 1.21%)	(4.87%, 2.72%)	(7.49%, 4.69%)	(2.91%, 1.37%)
(55,5)	(5.94%, 1.54%)	(5.56%, 1.21%)	(4.47%, 1.22%)	(6.86%, 0.81%)	(8.49%, 2.04%)	(4.71%, 1.37%)
Overall	(4.03%, 2.19%)	(4.62%, 3.18%)	(2.33%, 1.67%)	(4.12%, 2.53%)	(7.31%, 4.01%)	(2.75%, 1.75%)

*μ, σ stands for mean, and standard deviation of the optimal values, respectively.

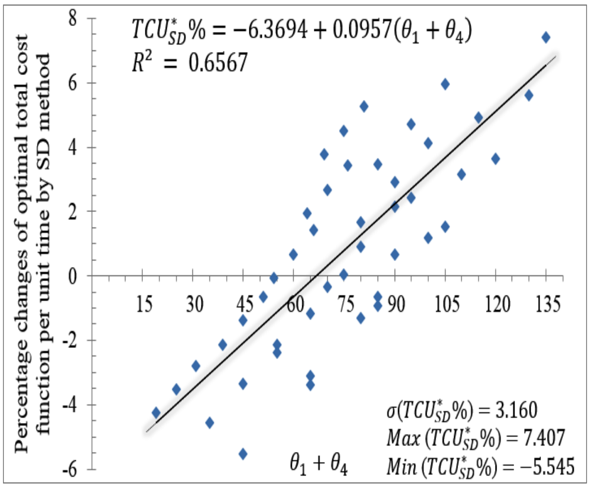

Fig. 4. Estimating a simple regression relationship between the values obtained by $TCU^*_{SD}\%$ and summation of θ_1 and θ_4.

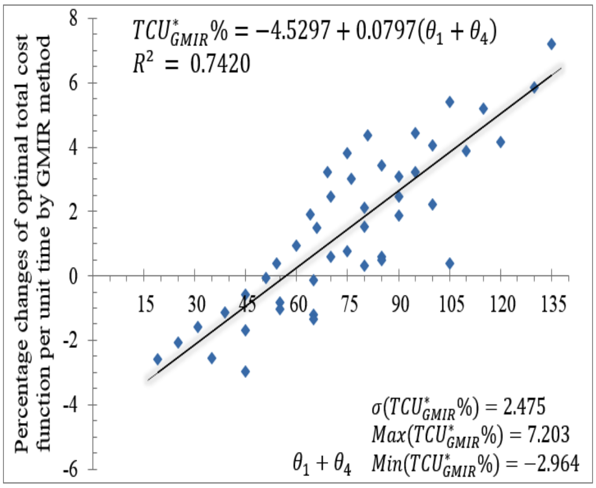

Fig. 5. Estimating a simple regression relationship between the values obtained by $TCU^*_{GMIR}\%$ and summation of θ_1 and θ_4.

There is an interesting statistical relationship between the summations of the first and second deviation values of the collection rate of the recoverable products from customers and demand rate of the serviceable products in the fuzzy situation, respectively, and the percentage changes of the optimal total cost function by the signed distance and GMIR method. They are calculated with a simple regression in Figures 4 and 5 for the signed distance and GMIR method, respectively. As it is depicted, we have considered $(\theta_1 + \theta_4)$ as independent variable; and $TCU^*_{SD}\%$ and $TCU^*_{GMIR}\%$ as dependent variable. The estimation of the percentage changes of the optimal total cost in the GMIR regression is about 10 percent better than the one obtained by signed distance regression because the value of R-square in the first one is about 10 units greater than the second one.

7. Practical implications and application

The applicability of the model is investigated through a real supply chain where a Milk Distribution Company buys the raw milk from the hundreds of local farms, and then, sells the gathered milk in gallons made of plastic polyethylene terephthalate (PET) to other Dairy Factories. These gallons can be bought from a supplier, and at the same time, the factories are motivated to take advantage of some discounts if they return the defective containers. Because of the recyclable characteristics of the defective gallons, the company can do some processes to renew them. Since production of the milk has flexibility in each season, there is uncertainty in the demand and the collection rate of the mentioned gallons. According to the provided data, we have the following information:

$k = 1500$units/month, $r = 300$units/month, $C_s = \$10{,}000$/setup, $C_o = \$1100$/order, $C_p = 80\$$/unit, $C_b = \$20$/unit, $H_r = \$4$/unit/day, $H_s = \$10$/unit/day, $C_l = \$5000$/day, $a = 0.008$day/unit, $b = -0.152$.

Table 7. The results of affecting the inventory system by two defuzzification methods

(Parameter/Level of fuzziness)	GMIR method				SD method			
	n^*	y^*	Q^*	TCU^*	n^*	y^*	Q^*	TCU^*
($\boldsymbol{k}$/10%; $\boldsymbol{r}$/10%)	10	244	113	7497.04	7	265	160	7593.89
($\boldsymbol{k}$/10%; $\boldsymbol{r}$/30%)	10	293	152	9002.76	6	303	227	9116.66
($\boldsymbol{k}$/30%; $\boldsymbol{r}$/10%)	22	234	75	11508.65	10	262	201	10666.23
($\boldsymbol{k}$/30%; $\boldsymbol{r}$/30%)	19	252	110	14705.84	8	283	305	12795.70

Table 7 shows the results for two different levels of fuzziness. As it is clear, in this case when there is the lowest level of fuzziness in the system, the GMIR method gives the lower total cost per unit time. On the other hand, for the maximum level of fuzziness, the total cost per unit time derived by the SD method is lower.

8. Conclusion and future research

One of the most important issues in the reverse inventory models is the lack of historical data for the demand and return (collection) rate. We are accordingly unable to estimate the probability distributions of such parameters. Therefore, these parameters are not determined, and usually it is not logical to decide based on the crisp values while the situation is uncertain. With these perspectives, it is worthwhile to reconsider the reverse inventory system with the learning effect presented by Tsai [22] and provide an alternative approach.

In this research, two fuzzy models were proposed for a reverse inventory problem with the learning effect. In both models, the demand rate of the serviceable products and the collection rate of the recoverable products from customers were presented as fuzzy numbers. To estimate the total cost function per unit time in the fuzzy sense, and then the corresponding optimal recovery lot size and the number of orders for the newly purchased products, in the first model, the signed distance method was employed for the defuzzification, while the GMIR was used in the second one. These models were explained and solved by a comprehensive numerical example. We compared the results of both methods. Besides, we concluded that it is important to decide which method should be chosen regarding the considered strategies.

It is noteworthy that although there are some papers in the forward supply chain literature, which considered fuzzy EOQ/EPQ, there is no similar work in the reverse supply chain literature that compares the performance of the well-known defuzzification methods such as the GMIR and the signed distance. In this study, we compared the performance of these methods in the presence of learning in a reverse inventory model. Future works can apply these methods on the other reverse EOQ/EPQ models. Besides, the other defuzzification methods such as the centroid method could be used [87].

References

1. Tibben-Lembke, R.S., Rogers, D.S.: Going backwards: reverse logistics trends and practices. University of Nevada, Reno Center for Logistics Management. (1998)
2. Fleischmann, M., Bloemhof-Ruwaard, J.M., Dekker, R., van der Laan, E., van Nunen, J.A.E.E., Van Wassenhove, L.N.: Quantitative models for reverse logistics: A review. Eur. J. Oper. Res. **103**(1), 1-17 (1997)
3. Rubio, S., Chamorro, A., Miranda, F.J.: Characteristics of the research on reverse logistics (1995–2005). Int. J. Prod. Res. **46**(4), 1099-1120 (2007)
4. Junior, M.L., Filho, M.G.: Production planning and control for remanufacturing: literature review and analysis. Prod. Plan. Control. **23**(6), 419-435 (2011)

5. Guide Jr, V.D.R.: Production planning and control for remanufacturing: industry practice and research needs. J. Oper Manag. **18**(4), 467-483 (2000)
6. Prahinski, C., Kocabasoglu, C.: Empirical research opportunities in reverse supply chains. Omega-Int. J. Manage. S. **34**(6), 519-532 (2006)
7. Sasikumar, P., Kannan, G.: Issues in reverse supply chains, part I: end-of-life product recovery and inventory management–an overview. Int. J. Sustainable. Eng. **1**(3), 154-172 (2008)
8. Pokharel, S., Mutha, A.: Perspectives in reverse logistics: A review. Resour. Conserv. Recy. **53**(4), 175-182 (2009)
9. Guide Jr, V.D.R., Van Wassenhove, L.N.: The evolution of closed-loop supply chain research. Oper. Res. **57**(1), 10-18 (2009).
10. Akçalı, E., Çetinkaya, S.: Quantitative models for inventory and production planning in closed-loop supply chains. Int. J. Prod. Res. **49**(8), 2373-2407 (2010)
11. Hazen, B.T.: Strategic reverse logistics disposition decisions: from theory to practice. Int. J. Logist. Syst. Manag. **10**(3), 275-292 (2011)
12. Souza, G.C.: Closed-loop supply chains: a critical review, and future research. Decision. Sci **44**(1), 7-38 (2013)
13. Sheriff, K.M., Gunasekaran, A., Nachiappan, S.: Reverse logistics network design: a review on strategic perspective. Int. J. Logist. Syst. Manag. **12**(2), 171-194 (2012)
14. Seuring, S.: A review of modeling approaches for sustainable supply chain management. Decis. Support. Syst. **54**(4), 1513-1520 (2013)
15. Steeneck, D.W., Sarin, S.C.: Pricing and production planning for reverse supply chain: a review. Int. J. Prod. Res. **51**(23-24), 6972-6989 (2013)
16. Chan, F.T., Chan, H.K.: A survey on reverse logistics system of mobile phone industry in Hong Kong. Manage. Decis. **46**(5), 702-708 (2008)
17. Govindan, K., Soleimani, H., Kannan, D.: Reverse logistics and closed-loop supply chain: a comprehensive review to explore the future. Eur. J. Oper. Res. **240**(3), 603-626 (2015)
18. Schrady, D.A.: A deterministic inventory model for reparable items. Nav. Res. Logist. Q. **14**(3), 391-398 (1967)
19. Zadeh, L.A.: Fuzzy sets. Inform Control **8**(3), 338-353 (1965)
20. Wright, T.P.: Factors affecting the cost of airplanes. J. Aeronaut. Sci. (Institute of the Aeronautical Sciences) **3**(4), 122-128 (1936)
21. Jaber, M.Y., El Saadany, A.M.A.: An economic production and remanufacturing model with learning effects. Int. J. Prod. Econ. **131**(1), 115-127 (2011)
22. Tsai, D.-M.: Optimal ordering and production policy for a recoverable item inventory system with learning effect. Int. J. Syst. Sci. **43**(2), 349-367 (2012)
23. Ilgin, M.A., Gupta, S.M.: Environmentally conscious manufacturing and product recovery (ECMPRO): a review of the state of the art. J. Environ. Manage. **91**(3), 563-591 (2010)
24. Bushuev, M.A., Guiffrida, A., Jaber, M.Y., Khan, M., Sarkis, J., Sarkis, J.: A review of inventory lot sizing review papers. Management Research Review **38**(3), 283-298 (2015)
25. Nahmias, S., Rivera, H.: A deterministic model for a repairable item inventory system with a finite repair rate. Int. J. Prod. Res. **17**(3), 215-221 (1979)
26. Mabini, M.C., Pintelon, L.M., Gelders, L.F.: EOQ type formulations for controlling repairable inventories. Int. J. Prod. Econ. **28**(1), 21-33 (1992)
27. Richter, K.: The EOQ repair and waste disposal model with variable setup numbers. Eur. J. Oper. Res. **95**(2), 313-324 (1996)
28. Richter, K.: The extended EOQ repair and waste disposal model. Int. J. Prod. Econ. **45**(1–3), 443-447 (1996)
29. Richter, K.: Pure and mixed strategies for the EOQ repair and waste disposal problem. OR. Spektrum. **19**(2), 123-129 (1997)
30. Richter, K., Dobos, I.: Analysis of the EOQ repair and waste disposal problem with integer setup numbers. Int. J. Prod. Econ. **59**(1–3), 463-467 (1999)
31. Teunter, R.H.: Economic ordering quantities for recoverable item inventory systems. Nav. Res. Log. **48**(6), 484-495 (2001).
32. Koh, S.-G., Hwang, H., Sohn, K.-I., Ko, C.-S.: An optimal ordering and recovery policy for reusable items. Comput. Ind. Eng. **43**(1–2), 59-73 (2002)
33. Teunter, R.H.: Economic order quantities for stochastic discounted cost inventory systems with remanufacturing. Int. J. Logist. **5**(2), 161-175 (2002)
34. Teunter, R.: Lot-sizing for inventory systems with product recovery. Comput. Ind. Eng. **46**(3), 431-441 (2004)
35. Inderfurth, K., Lindner, G., Rachaniotis, N.: Lot sizing in a production system with rework and product deterioration. Int. J. Prod. Res. **43**(7), 1355-1374 (2005)
36. Dobos, I., Richter, K.: A production/recycling model with stationary demand and return rates. Cent. Eur. J. Oper. Res. **11**, 35-46 (2003)
37. Dobos, I., Richter, K.: A production/recycling model with quality consideration. Int. J. Prod. Econ. **104**(2), 571-579 (2006)
38. Konstantaras, I., Papachristos, S.: Lot-sizing for a single-product recovery system with backordering. Int. J. Prod. Res. **44**(10), 2031-2045 (2006)
39. Jaber, M.Y., Rosen, M.A.: The economic order quantity repair and waste disposal model with entropy cost. Eur. J. Oper. Res. **188**(1), 109-120 (2008)
40. Oh, Y., Hwang, H.: Deterministic inventory model for recycling system. J. Intell. Manuf. **17**(4), 423-428 (2006)
41. Konstantaras, I., Papachristos, S.: Optimal policy and holding cost stability regions in a periodic review inventory system with manufacturing and remanufacturing options. Eur. J. Oper. Res. **178**(2), 433-448 (2007)

42. Konstantaras, I., Papachristos, S.: A note on: Developing an exact solution for an inventory system with product recovery. Int. J. Prod. Econ. **111**(2), 707-712 (2008)
43. Konstantaras, I., Papachristos, S.: Note on: An optimal ordering and recovery policy for reusable items. Comput. Ind. Eng. **55**(3), 729-734 (2008)
44. Jaber, M.Y., El Saadany, A.M.A.: The production, remanufacture and waste disposal model with lost sales. Int. J. Prod. Econ. **120**(1), 115-124 (2009)
45. Konstantaras, I., Skouri, K.: Lot sizing for a single product recovery system with variable setup numbers. Eur. J. Oper. Res. **203**(2), 326-335 (2010)
46. El Saadany, A.M.A., Jaber, M.Y.: A production/remanufacturing inventory model with price and quality dependant return rate. Comput. Ind. Eng. **58**(3), 352-362 (2010)
47. Konstantaras, I., Skouri, K., Jaber, M.Y.: Lot sizing for a recoverable product with inspection and sorting. Comput. Ind. Eng. **58**(3), 452-462 (2010)
48. Alinovi, A., Bottani, E., Montanari, R.: Reverse Logistics: a stochastic EOQ-based inventory control model for mixed manufacturing/remanufacturing systems with return policies. Int. J. Prod. Res. **50**(5), 1243-1264 (2012)
49. El Saadany, A.M.A., Jaber, M.Y.: A production/remanufacture model with returns' subassemblies managed differently. Int. J. Prod. Econ. **133**(1), 119-126 (2011)
50. Alamri, A.A.: Theory and methodology on the global optimal solution to a general reverse logistics inventory model for deteriorating items and time-varying rates. Comput. Ind. Eng. **60**(2), 236-247 (2011)
51. Hasanov, P., Jaber, M.Y., Zolfaghari, S.: Production, remanufacturing and waste disposal models for the cases of pure and partial backordering. Appl. Math. Model. **36**(11), 5249-5261 (2012)
52. Widyadana, G.A., Wee, H.M.: An economic production quantity model for deteriorating items with multiple production setups and rework. Int. J. Prod. Econ. **138**(1), 62-67 (2012)
53. Jaber, M.Y., Zanoni, S., Zavanella, L.E.: A consignment stock coordination scheme for the production, remanufacturing and waste disposal problem. Int. J. Prod. Res. **52**(1), 50-65 (2014)
54. Matar, N., Jaber, M.Y., Searcy, C.: A reverse logistics inventory model for plastic bottles. Int. J. Logist. Manag. **25**(2), 315-333 (2014)
55. Nonaka, T., Fujii, N.: An EOQ model for reuse and recycling considering the balance of supply and demand. International journal of automation technology. **9**(3), 303-311 (2015)
56. Dobos, I., Richter, K.: An extended production/recycling model with stationary demand and return rates. Int. J. Prod. Econ. **90**(3), 311-323 (2004)
57. Zouadi, T., Yalaoui, A., Reghioui, M., El Kadiri, K.E.: Lot-sizing for production planning in a recovery system with returns. Rairo-Oper. Res. **49**(1), 123-142 (2015)
58. Singh, S.R., Rathore, H.: Two-warehouse reverse logistic inventory model for deteriorating item under learning effect. In: Das, K.N., Deep, K., Pant, M., Bansal, J.C., Nagar, A. (eds.) Proceedings of Fourth International Conference on Soft Computing for Problem Solving, vol. 335. Advances in Intelligent Systems and Computing, pp. 45-57. Springer India, (2015)
59. Singh, S.R., Rathore, H.: Reverse logistic model for deteriorating items with non-instantaneous deterioration and learning effect. In: Mandal, J.K., Satapathy, S.C., Kumar Sanyal, M., Sarkar, P.P., Mukhopadhyay, A. (eds.) Information Systems Design and Intelligent Applications, vol. 339. Advances in Intelligent Systems and Computing, pp. 435-445. Springer India, (2015)
60. Lee, H.-M., Yao, J.-S.: Economic order quantity in fuzzy sense for inventory without backorder model. Fuzzy. Set. Syst. **105**(1), 13-31 (1999)
61. Hsieh, C.H.: Optimization of fuzzy production inventory models. Inform Sciences. **146**(1), 29-40 (2002)
62. Chang, H.-C.: An application of fuzzy sets theory to the EOQ model with imperfect quality items. Comput. Oper. Res. **31**(12), 2079-2092 (2004)
63. Ouyang, L.-Y., Yao, J.-S.: A minimax distribution free procedure for mixed inventory model involving variable lead time with fuzzy demand. Comput. Oper. Res. **29**(5), 471-487 (2002)
64. Chang, H.-C., Yao, J.-S., Ouyang, L.-Y.: Fuzzy mixture inventory model involving fuzzy random variable lead time demand and fuzzy total demand. Eur. J. Oper. Res. **169**(1), 65-80 (2006)
65. Vijayan, T., Kumaran, M.: Inventory models with a mixture of backorders and lost sales under fuzzy cost. Eur. J. Oper. Res. **189**(1), 105-119 (2008)
66. Guiffrida, A.: Fuzzy inventory models. In: Jaber, M.Y. (ed.) Inventory Management: Non-Classical Views. pp. 173-198. Boca Raton: CRC Press, (2009)
67. Björk, K.-M.: An analytical solution to a fuzzy economic order quantity problem. Int. J. Approx. Reason. **50**(3), 485-493 (2009)
68. Björk, K.-M.: A multi-item fuzzy economic production quantity problem with a finite production rate. Int. J. Prod. Econ. **135**(2), 702-707 (2012)
69. Shekarian, E., Jaber, M.Y., Kazemi, N., Ehsani, E.: A fuzzified version of the economic production quantity (EPQ) model with backorders and rework for a single–stage system. Eur. J. Ind. Eng. **8**(3), 291-324 (2014)
70. Shekarian, E., Glock, C.H., Amiri, S.M.P., Schwindl, K.: Optimal manufacturing lot size for a single-stage production system with rework in a fuzzy environment. J. Intell. Fuzzy. Syst. **27**(6), 3067-3080 (2014)
71. Guchhait, P., Maiti, M.K., Maiti, M.: Inventory policy of a deteriorating item with variable demand under trade credit period. Comput. Ind. Eng. **76**, 75-88 (2014)
72. Sharifi, E., Shabani, S., Sobhanallahi, M.A., Mirzazadeh, A.: A fuzzy economic order quantity model for items with imperfect quality and partial backordered shortage under screening errors. International Journal of Applied Decision Sciences **8**(1), 109-126 (2015)

73. Pal, S., Mahapatra, G.S., Samanta, G.P.: A production inventory model for deteriorating item with ramp type demand allowing inflation and shortages under fuzziness. Econ Model **46**, 334-345 (2015)
74. Kumar, R.S., Goswami, A.: EPQ model with learning consideration, imperfect production and partial backlogging in fuzzy random environment. Int. J. Syst. Sci. **46**(8), 1486-1497 (2015)
75. Guchhait, P., Maiti, M.K., Maiti, M.: An EOQ model of deteriorating item in imprecise environment with dynamic deterioration and credit linked demand. Appl. Math. Model (2015). doi:http://dx.doi.org/10.1016/j.apm.2015.02.003
76. Sarkar, B., Mahapatra, A.S.: Periodic review fuzzy inventory model with variable lead time and fuzzy demand. Int. Trans. Oper. Res. (2015). doi:http://doi: 10.1111/itor.12177
77. Yadav, D., Singh, S., Kumari, R.: Retailer's optimal policy under inflation in fuzzy environment with trade credit. Int. J. Syst. Sci. **46**(4), 754-762 (2015)
78. Mahata, G.C.: A production-inventory model with imperfect production process and partial backlogging under learning considerations in fuzzy random environments. J Intell Manuf, 1-15 (2014)
79. Kazemi, N., Olugu, E.U., Salwa Hanim, A.-R., Ghazilla, R.A.B.R.: Development of a fuzzy economic order quantity model for imperfect quality items using the learning effect on fuzzy parameters. J. Intell. Fuzzy. Syst. **28**(5), 2377-2389 (2015)
80. Kazemi, N., Shekarian, E., Cárdenas-Barrón, L.E., Olugu, E.U.: Incorporating human learning into a fuzzy EOQ inventory model with backorders. Comput. Ind. Eng. **87**, 540-542 (2015)
81. Jaber, M.Y., Guiffrida, A.L.: Learning curves for processes generating defects requiring reworks. Eur. J. Oper. Res. **159**(3), 663-672 (2004)
82. Kauffman, A., Gupta, M.M.: Introduction to fuzzy arithmetic: theory and application. New York: VanNostrand Reinhold (1991)
83. Zimmermann, H.-J.: Fuzzy set theory—and its applications. Springer Science & Business Media, (2001)
84. Chen, S.H., Chang, S.M.: Optimization of fuzzy production inventory model with unrepairable defective products. Int. J. Prod. Econ. **113**(2), 887-894 (2008)
85. Yao, J.-S., Wu, K.: Ranking fuzzy numbers based on decomposition principle and signed distance. Fuzzy. Set. Syst. **116**(2), 275-288 (2000)
86. Chen, S.H., Hsieh, C.H.: Graded mean integration representation of generalized fuzzy number. Journal of Chinese Fuzzy Systems **5**(2), 1-7 (1999)
87. Huang, T.T.: Fuzzy multilevel lot-sizing problem based on signed distance and centroid. Int. J. Fuzzy. Syst. **13**(2), 98-110 (2011)

Appendix A

By taking the first derivative of the $\tilde{g}(n)$ with respect to n, we have

$$\frac{\partial \tilde{g}(n)}{\partial n} = b(\beta' + \delta')n^{b-1} - \frac{2\alpha'}{n^3} + (b-1)\zeta' n^{b-2} - \frac{\gamma'}{n^2} > 0 \tag{A.1}$$

where

$$\alpha' = -\frac{(nC_o + C_s)H_s\Delta}{2C_o} \quad , \quad \beta' = -\frac{(b+1)H_r a\rho^b}{b+2} d(\tilde{r}, \tilde{0}_1) \quad , \quad \gamma' = \frac{H_s}{2}\Delta \quad ,$$

$$\delta' = -\frac{a\rho^b H_s}{(b+2)} d(\tilde{r}, \tilde{0}_1) \quad , \quad \zeta' = \left(\frac{ab\rho^{b-1}}{b+1}\right) C_l d(\tilde{r}, \tilde{0}_1) \quad , \quad \varepsilon' = \frac{1}{2}H_r + \frac{H_s}{2} d\left(\frac{\tilde{r}}{\tilde{k}}, \tilde{0}_1\right)$$

It is positive for all value of $n > 0$. Hence, $\tilde{g}(n)$ is a strictly increasing function for $0 < n < \infty$. Moreover, we have the following limitations

$$\lim_{n \to +\infty} \tilde{g}(n) = \varepsilon' > 0 \tag{A.2}$$

$$\lim_{n \to 0^+} \tilde{g}(n) = -\infty \tag{A.3}$$

Thus, by the Intermediate Value Theorem, there exists a unique $0 < n^* < \infty$ such that $\tilde{g}(n^*) = 0$.

Appendix B

From Theorem 1, it is clear that (y^*, n^*) is the only critical point. Therefore, to prove the Theorem 2, we should firstly calculate the Hessian Matrix of $d(\tilde{V}, \tilde{0})$ as follows

$$\frac{\partial^2 d(\tilde{V}, \tilde{0})}{\partial y^2} = \frac{\partial^2 V(n, y)}{\partial y^2} = \frac{2(nC_o + C_s)}{y^3} d(\tilde{r}, \tilde{0}_1) - \frac{b(b+1)H_r a y^{b-1}}{b+2} d(\tilde{r}, \tilde{0}_1)$$

$$-\frac{bay^{b-1}H_s}{(b+2)}d(\tilde{r},\tilde{0}_1)+\left(\frac{(b-1)aby^{b-2}}{b+1}\right)C_l d(\tilde{r},\tilde{0}_1) \tag{B.1}$$

For all $y > 0$, $n > 0$, $\partial^2 d(\tilde{V},\tilde{0})/\partial y^2 > 0$

$$\frac{\partial^2 d(\tilde{V},\tilde{0})}{\partial n^2}=\frac{yH_s}{n^3}\Delta \tag{B.2}$$

For all $y > 0$, $n > 0$, $\partial^2 d(\tilde{V},\tilde{0})/\partial n^2 > 0$.

$$\frac{\partial^2 V(n,y)}{\partial y \partial n}=-\frac{C_o d(\tilde{r},\tilde{0}_1)}{y^2}-\frac{H_s}{2n^2}\Delta \tag{B.3}$$

Substituting $y^*=\rho n^*$ into Eqs. (B.1)-(B.3), and after some simplifications, determinant of the Hessian Matrix of $d(\tilde{V},\tilde{0})$ at (y^*,n^*) could be given as below:

$$\begin{vmatrix} \frac{\partial^2 V(n,y)}{\partial y^2}|_{(y^*,n^*)} & \frac{\partial^2 V(n,y)}{\partial y \partial n}|_{(y^*,n^*)} \\ \frac{\partial^2 V(n,y)}{\partial y \partial n}|_{(y^*,n^*)} & \frac{\partial^2 V(n,y)}{\partial n^2}|_{(y^*,n^*)} \end{vmatrix} = \frac{y^*H_s}{n^{*3}}\Delta\left[-\frac{b(b+1)H_r ay^{b-1}}{b+2}d(\tilde{r},\tilde{0}_1)-\frac{bay^{b-1}H_s}{(b+2)}d(\tilde{r},\tilde{0}_1) + \left(\frac{(b-1)aby^{b-2}}{b+1}\right)C_l d(\tilde{r},\tilde{0}_1)\right]+\frac{{H_s}^2\Delta^2 C_s}{n^{*5}C_o}>0 \tag{B.4}$$

Hessian Matrix of $d(\tilde{V},\tilde{0})$ is positive. Hence, $d(\tilde{V},\tilde{0})$ has a global minimum at point (y^*,n^*).

Appendix C

The first derivative of the $\tilde{f}(n)$ is positive for all value of $n > 0$.

$$\frac{\partial \tilde{f}(n)}{\partial n}=\frac{2\gamma C_s}{6n^3 C_o}+\frac{b(\beta+\delta)n^{b-1}+\zeta(b-1)n^{b-2}}{6}>0 \tag{C.1}$$

Hence, $\tilde{f}(n)$ is a strictly increasing function for $0 < n < \infty$. Moreover, we have the following limitations

$$\lim_{n\to+\infty}\tilde{f}(n)=\epsilon>0 \tag{C.2}$$
$$\lim_{n\to 0^+}\tilde{f}(n)=-\infty \tag{C.3}$$

Thus, by the Intermediate Value Theorem, there exists a unique $0 < n^* < \infty$ such that $\tilde{f}(n^*) = 0$.

Appendix D

From Theorem 3, it is clear that (y^*,n^*) is the only critical point. Therefore, to prove the Theorem 4, we should firstly calculate the Hessian Matrix of $\Phi(\tilde{V}(y,n))$ as follows

$$\frac{\partial^2\Phi(\tilde{V}(y,n))}{\partial y^2}=\frac{2(nC_o+C_s)(6r+\theta_2-\theta_1)}{y^3}-\frac{ba(b+1)H_r(6r+\theta_2-\theta_1)y^{b-1}}{b+2} -\frac{bay^{b-1}H_s(6r+\theta_2-\theta_1)}{(b+2)}+\frac{abC_l(b-1)(6r+\theta_2-\theta_1)y^{b-2}}{b+1} \tag{D.1}$$

For all $y > 0$, $n > 0$, $\partial^2\Phi(\tilde{V}(y,n))/\partial y^2 > 0$

$$\frac{\partial^2\Phi(\tilde{V}(y,n))}{\partial n^2}=\frac{yH_s}{3n^3}\left[\frac{(k-r-\theta_3-\theta_2)^2}{2(k+\theta_4)(r+\theta_2)}+\frac{2(k-r)^2}{kr}+\frac{(k-r+\theta_4+\theta_1)^2}{2(k-\theta_3)(r-\theta_1)}\right]$$

$$= \frac{yC_o[(r-\theta_1)+4r+(r+\theta_2)]}{3n^3\pi^2} \tag{D.2}$$

For all $y > 0$, $n > 0$, $\partial^2\Phi(\tilde{V}(y,n))/\partial n^2 > 0$.

$$\frac{\partial^2 V(n,y)}{\partial y \partial n} = -\frac{C_o(6r+\theta_2-\theta_1)}{6y^2} - \frac{H_s}{n^2 6}\left[\frac{(k-r-\theta_3-\theta_2)^2}{2(k+\theta_4)(r+\theta_2)} + \frac{2(k-r)^2}{kr} + \frac{(k-r+\theta_4+\theta_1)^2}{2(k-\theta_3)(r-\theta_1)}\right]$$
$$= -\frac{C_o(6r+\theta_2-\theta_1)}{6y^2} - \frac{C_o[(r-\theta_1)+4r+(r+\theta_2)]}{6n^2\pi^2} \tag{D.3}$$

Substituting $y^* = \pi n^*$ into Eqs. (D.1)-(D.3), and after some manipulations, determinant of the Hessian Matrix of $\Phi(\tilde{V}(y,n))$ at (y^*, n^*) could be given as below:

$$\begin{vmatrix} \frac{\partial^2\Phi(\tilde{V}(y,n))}{\partial y^2}\big|_{(y^*,n^*)} & \frac{\partial^2\Phi(\tilde{V}(y,n))}{\partial y \partial n}\big|_{(y^*,n^*)} \\ \frac{\partial^2\Phi(\tilde{V}(y,n))}{\partial y \partial n}\big|_{(y^*,n^*)} & \frac{\partial^2\Phi(\tilde{V}(y,n))}{\partial n^2}\big|_{(y^*,n^*)} \end{vmatrix}$$
$$= \frac{y^* C_o(6r+\theta_2-\theta_1)}{3n^{*3}\pi^2}\left[-\frac{ba(b+1)H_r(6r+\theta_2-\theta_1)y^{b-1}}{b+2} - \frac{bay^{b-1}H_s(6r+\theta_2-\theta_1)}{(b+2)} + \frac{abC_l(b-1)(6r+\theta_2-\theta_1)y^{b-2}}{b+1}\right]$$
$$+ \frac{C_o(6r+\theta_2-\theta_1)^2(5nC_o+6C_s)}{9n^{*5}\pi^4} > 0 \tag{D.4}$$

Hessian Matrix of $\Phi(\tilde{V}(y,n))$ is positive. Hence, $\Phi(\tilde{V}(y,n))$ has a global minimum at point (y^*, n^*).