



Higher order Levi forms on homogeneous CR manifolds

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Abstract

We investigate the nondegeneracy of higher order Levi forms on weakly nondegenerate homogeneous CR manifolds. Improving previous results, we prove that general orbits of real forms in complex flag manifolds have order less or equal than 3 and the compact ones less or equal 2. Finally we construct by Lie extensions weakly nondegenerate CR vector bundles with arbitrary orders of nondegeneracy.

Keywords Lie pair · CR algebra · Lie algebra extension · Levi degeneracy

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Introduction

The Levi form is a basic invariant of CR geometry (see e.g. [9]). It is a hermitian symmetric form on the space of tangent holomorphic vector fields, which, when the CR codimension is larger than one, is vector valued. Its nondegeneracy was shown in [25] to be a sufficient condition to apply Cartan's method to investigate equivalence and automorphisms of CR structures and is an obvious obstruction for locally representing the manifold as a product of a CR manifold of smaller dimension and of a nontrivial complex manifold. Sufficient more general conditions preventing a CR manifold M from being foliated by complex leaves of positive dimension or from having an infinite dimensional group of local CR automorphisms can be expressed by the nondegeneracy of higher order Levi forms (see e.g. [12,13]). In the case of homogeneous CR manifolds these properties can be rephrased in terms of their associated CR algebras and lead to the notions of *weak nondegeneracy* and *ideal nondegeneracy* in [21]. The last one was renamed *contact nondegeneracy* and proved sufficient for the finite dimensionality of the group of CR automorphisms in [18].

Iterations of the Levi forms can be described by building descending chains of algebras of vector fields, whose lengths can be taken as a measure of nondegeneracy (see Sect. 1.1). One of these numbers, that we call here *Levi order*, and relates to weak nondegeneracy, is the main topic of this paper. The real submanifolds M of a complex flag manifold F of a semisimple complex group S that are orbits of its real form $S_{\mathbb{R}}$ form an interesting class of homogeneous CR manifolds, that has been studied e.g. in [1,3]. In [10] G. Fels showed that when the isotropy Q of F is a maximal parabolic subgroup, and M is weakly nondegenerate, then its Levi order is at most 3 and found an example where it is in fact equal to 3. In Sect. 2 we prove that this bound is valid for general weakly nondegenerate real orbits, dropping the maximality assumption on Q and give further examples of weakly nondegenerate real orbits having Levi order 3. Moreover we show that the minimal orbit (the single one which is compact, cf. [27]) cannot have a finite Levi order larger than 2 and that the same result is valid for a larger class of orbits, that we name *of the minimal type*. Orbits which are not of the minimal type may have any finite order 1, 2, 3. Our methods are illustrated by several examples. We point out that, together with the new results obtained here, those in [1], where descriptions in terms of cross-marked Satake diagrams are emphasised, would allow to list all minimal orbits of Levi orders 1 and 2.

In [10] Fels posed the question of the existence of weakly nondegenerate homogeneous CR manifolds with Levi order larger than 3. In Sect. 3 we exhibit, by constructing some CR vector bundles over $\mathbb{C}P^1$, weakly nondegenerate homogeneous CR manifolds having Levi order q , for every positive integer q .

1 Nondegeneracy conditions

1.1 Abstract CR manifolds

In this subsection we discuss some notions of nondegeneracy for general smooth abstract CR manifolds of type (n, k) . We will eventually be interested in the locally homogeneous case and therefore, in the rest of this section, in their reformulation in the framework of Lie algebras theory.

We recall that an abstract CR manifold of type (n, k) is defined by the datum, on a smooth manifold M of real dimension $2n+k$, of a rank n smooth complex linear subbundle $T^{0,1}M$ of its complexified tangent bundle $T^{\mathbb{C}}M$, satisfying

$$T^{0,1}M \cap \overline{T^{0,1}M} = \{0\} \tag{1.1}$$

and the *formal integrability condition*

$$[\Gamma^\infty(M, T^{0,1}M), \Gamma^\infty(M, \overline{T^{0,1}M})] \subseteq \Gamma^\infty(M, T^{0,1}M). \tag{1.2}$$

Set

$$T^{1,0}M = \overline{T^{0,1}M}, \quad H^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M, \quad HM = H^{\mathbb{C}}M \cap TM. \tag{1.3}$$

The rank $2n$ real subbundle HM of TM is the *real contact distribution* underlying the CR structure of M .

A smooth \mathbb{R} -linear bundle map $J_M : HM \rightarrow HM$ is defined by the equation

$$T^{0,1}M = \{v + iJ_M v \mid v \in HM\}. \tag{1.4}$$

The map J_M squares to $-I_H$ and is the *partial complex structure* of M .

An equivalent definition of the CR structure can be given by assigning first an even dimensional distribution HM and then a smooth partial complex structure J_M on HM in such a way that the complex distribution (1.4) satisfies (1.2).

Let us denote by \mathcal{H} (resp. \mathcal{T} , $\mathcal{H}^{\mathbb{C}}$, $\mathcal{T}^{0,1}$, $\mathcal{T}^{1,0}$) the sheaf of germs of smooth sections of HM (resp. TM , $H^{\mathbb{C}}M$, $T^{0,1}M$, $T^{1,0}M$).

Definition 1.1 A CR manifold M is called *fundamental* at its point x if \mathcal{H}_x generates the Lie algebra \mathcal{T}_x .

We define recursively a nested sequence of sheaves of germs of smooth complex valued vector fields on M

$$\mathcal{T}_0^{0,1} \supseteq \mathcal{T}_1^{0,1} \supseteq \dots \supseteq \mathcal{T}_p^{0,1} \supseteq \mathcal{T}_{p+1}^{0,1} \supseteq \dots \tag{1.5}$$

by setting

$$\begin{cases} \mathcal{T}_0^{0,1} = \mathcal{T}^{0,1}, \\ \mathcal{T}_p^{0,1} = \bigsqcup_{x \in M} \left\{ Z \in \mathcal{T}_{p-1}^{0,1} \mid [Z, \mathcal{T}_x^{1,0}] \subseteq \mathcal{T}_{p-1}^{0,1} + \mathcal{T}_x^{1,0} \right\}, \text{ for } p \geq 1. \end{cases} \tag{1.6}$$

By conjugation we obtain another nested sequence of sheaves

$$\mathcal{T}_0^{1,0} \supseteq \mathcal{T}_1^{1,0} \supseteq \dots \supseteq \mathcal{T}_p^{1,0} \supseteq \mathcal{T}_{p+1}^{1,0} \supseteq \dots \tag{1.7}$$

where

$$\begin{cases} \mathcal{T}_0^{1,0} = \mathcal{T}^{1,0}, \\ \mathcal{T}_p^{1,0} = \bigsqcup_{x \in M} \left\{ Z \in \mathcal{T}_{p-1,x}^{1,0} \mid [Z, \mathcal{T}_x^{0,1}] \subseteq \mathcal{T}_{p-1,x}^{1,0} + \mathcal{T}_x^{0,1} \right\}, \text{ for } p \geq 1. \end{cases} \tag{1.8}$$

These sequences, considered by Freeman in [12, Thm.3.1], correspond to the chain (1.11) that we construct in the locally homogeneous case.

In the same paper (cf. [12, Remarks 4.5]) also another sequence, introduced before in [15,16], was considered, which will correspond to (1.15), namely

$$\mathcal{H} \supseteq \mathcal{H}^1 \supseteq \dots \supseteq \mathcal{H}^p \supseteq \mathcal{H}^{p+1} \supseteq \dots \tag{1.9}$$

with

$$\begin{cases} \mathcal{H}^0 = \mathcal{H}^{\mathbb{C}}, \\ \mathcal{H}^p = \bigsqcup_{x \in M} \left\{ X \in \mathcal{H}_x^{p-1} \mid [X, \mathcal{H}_x] \subseteq \mathcal{H}_x^{p-1} \right\}, \text{ for } p > 0. \end{cases} \tag{1.10}$$

Definition 1.2 The CR manifold M has, at its point x ,

- Levi order q if $\mathcal{T}_{q-1,x}^{0,1} \neq \mathcal{T}_q^{0,1} = \{0\}$;
- contact order q if $\mathcal{H}_x^{q-1} \neq \mathcal{H}_x^q = \{0\}$.

We say that M is, at its point x

- *Levi (resp. contact) nondegenerate* if it has Levi (resp. contact) order 1;
- *weakly (resp. contact) nondegenerate* if it has finite Levi (resp. contact) order $p \geq 1$;
- *holomorphically (resp. contact) degenerate* if it is not weakly (resp. contact) nondegenerate.

The Levi order at x is the smallest q for which, given any nonzero germ $\bar{Z} \in \mathcal{T}_x^{0,1}$, we can find a $p \leq q$ and $Z_1, \dots, Z_p \in \mathcal{T}_x^{1,0}$ such that

$$[Z_1, [Z_2, \dots, [Z_p, \bar{Z}]]] \notin \mathcal{H}_x^0. \tag{*}$$

The contact order can be defined in the same way, but with Z_1, \dots, Z_p taken in \mathcal{H}_x^0 . We have therefore

Proposition 1.1 *Let us keep the notation introduced above. Fix a point x in M. Then:*

- M has Levi order 1 at x if and only if it has contact order 1 at x .
- If M has finite Levi order $q \geq 2$ at x , then it has also finite contact order q' , with $2 \leq q' \leq q$ at x . □

1.2 Homogeneous CR manifolds and CR algebras

Let $\mathbf{G}_{\mathbb{R}}$ be a Lie group of CR diffeomorphisms acting transitively on a CR manifold M. Fix a point x of M and let $\pi : \mathbf{G}_{\mathbb{R}} \ni g \rightarrow g \cdot x \in M$ be the natural projection. The differential at x defines a map $\pi_* : \mathfrak{g}_{\mathbb{R}} \rightarrow T_x M$ of the Lie algebra $\mathfrak{g}_{\mathbb{R}}$ of $\mathbf{G}_{\mathbb{R}}$ onto the tangent space to M at x . By the formal integrability of the partial complex structure of M, the pullback $\mathfrak{q} = (\pi_*^{\mathbb{C}})^{-1}(T_x^{0,1} M)$ of the space of tangent vectors of type (0, 1) at x by the complexification of the differential is a complex Lie subalgebra \mathfrak{q} of the complexification $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{R}}$.

Vice versa, the assignment of a complex Lie subalgebra \mathfrak{q} of \mathfrak{g} yields a formally integrable $\mathbf{G}_{\mathbb{R}}$ -equivariant partial complex structure on a locally homogeneous space M of $\mathbf{G}_{\mathbb{R}}$ by the requirement that $T_x^{0,1}M = \pi_*^{\mathbb{C}}(\mathfrak{q})$ (see e.g. [1,21]). These considerations led to the following definition.

Definition 1.1 A *CR algebra* is a pair $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{q})$, consisting of a real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ and a complex Lie subalgebra \mathfrak{q} of its complexification $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}}$, such that the quotient $\mathfrak{g}_{\mathbb{R}} / (\mathfrak{g}_{\mathbb{R}} \cap \mathfrak{q})$ is a finite dimensional real vector space.

We call the intersection $\mathfrak{q} \cap \mathfrak{g}_{\mathbb{R}}$ its *isotropy subalgebra* and say that $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{q})$ is *effective* when $\mathfrak{q} \cap \mathfrak{g}_{\mathbb{R}}$ does not contain any nontrivial ideal of $\mathfrak{g}_{\mathbb{R}}$.

If $\mathbf{G}_{\mathbb{R}}$ is a real form of a complex Lie algebra \mathbf{G} and \mathfrak{q} the Lie algebra of its closed subgroup \mathbf{Q} , then M is locally *CR* diffeomorphic to the orbit of $\mathbf{G}_{\mathbb{R}}$ in the complex homogeneous space \mathbf{G}/\mathbf{Q} and its *CR* structure is *induced* by the complex structure of \mathbf{G}/\mathbf{Q} . These considerations can be generalized to *locally homogeneous CR manifolds* (see e.g. [1]).

The *CR*-dimension and codimension of M are expressed in terms of its associated *CR* algebra $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{q})$ by

$$\begin{cases} CR - \dim_{\mathbb{C}} M = \dim_{\mathbb{C}} \mathfrak{q} - \dim_{\mathbb{C}}(\mathfrak{q} \cap \bar{\mathfrak{q}}), \\ CR - \text{codim} M = \dim_{\mathbb{C}} \mathfrak{g} - \dim_{\mathbb{C}}(\mathfrak{q} + \bar{\mathfrak{q}}). \end{cases}$$

Definition 1.3 We call *fundamental* a *CR* algebra $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{q})$ such that $\mathfrak{q} + \bar{\mathfrak{q}}$ generates \mathfrak{g} as a Lie algebra and we say that it is

- of *complex type* if $\mathfrak{q} + \bar{\mathfrak{q}} = \mathfrak{g}$,
- of *contact type* if $\mathfrak{q} + \bar{\mathfrak{q}} \subsetneq \mathfrak{g}$.

A corresponding *CR* manifold M is in the first case a complex manifold by Newlander-Nirenberg theorem (cf. [4,22]), while contact type is equivalent to the fact that its *CR distribution* is strongly non-integrable.

1.3 Levi-order of weak nondegeneracy

The *Levi form* is a basic invariant of *CR* geometry. When M is locally homogeneous, it can be computed by using its associated *CR* algebra $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{q})$ (for definitions and basic properties, cf. e.g. [9]). Nondegeneracy of the Levi form can be stated by

$$\forall Z \in \mathfrak{q} \setminus \bar{\mathfrak{q}}, \exists Z' \in \bar{\mathfrak{q}} \text{ such that } [Z, Z'] \notin \mathfrak{q} + \bar{\mathfrak{q}}.$$

This is equivalent to

$$\mathfrak{q}^{(1)} := \{Z \in \mathfrak{q} \mid [Z, \bar{\mathfrak{q}}] \subseteq \mathfrak{q} + \bar{\mathfrak{q}}\} = \mathfrak{q} \cap \bar{\mathfrak{q}}.$$

When this condition is not satisfied, we say that $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{q})$ is *Levi-degenerate*. To measure the degeneracy of the Levi form, one can consider its *iterations*: in the homogeneous case this means, given a $Z \in \mathfrak{q} \setminus (\mathfrak{q} \cap \bar{\mathfrak{q}})$, to seek whether it is possible to find $L_1, \dots, L_p \in \bar{\mathfrak{q}}$ such that $[L_1, \dots, L_p, Z] \notin \mathfrak{q} + \bar{\mathfrak{q}}$. To this aim, it is convenient to consider the descending chain (see e.g. [10,12,13,18,21])

$$\begin{cases} \mathfrak{q}^{(0)} \supseteq \mathfrak{q}^{(1)} \supseteq \dots \supseteq \mathfrak{q}^{(p-1)} \supseteq \mathfrak{q}^{(p)} \supseteq \mathfrak{q}^{(p+1)} \supseteq \dots, & \text{with} \\ \mathfrak{q}^{(0)} = \mathfrak{q}, \quad \mathfrak{q}^{(p)} = \{Z \in \mathfrak{q}^{(p-1)} \mid [Z, \bar{\mathfrak{q}}] \subseteq \mathfrak{q}^{(p-1)} + \bar{\mathfrak{q}}\} & \text{for } p \geq 1. \end{cases} \tag{1.11}$$

Note that $\mathfrak{q} \cap \bar{\mathfrak{q}} \subseteq \mathfrak{q}^{(p)}$ for all integers $p \geq 0$. Since by assumption $\mathfrak{q}/(\mathfrak{q} \cap \bar{\mathfrak{q}})$ is finite dimensional, there is a smallest nonnegative integer q such that $\mathfrak{q}^{(p)} = \mathfrak{q}^{(q)}$ for all $p \geq q$.

Definition 1.4 We call (1.11) the *descending Levi chain* of $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{q})$.

Let q be a positive integer. The CR algebra $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{q})$ is said to be

- *weakly nondegenerate* of Levi order q if $\mathfrak{q}^{(q-1)} \supsetneq \mathfrak{q}^{(q)} = \mathfrak{q} \cap \bar{\mathfrak{q}}$.
- *strictly nondegenerate* if it is weakly nondegenerate of Levi order 1.
- *holomorphically degenerate* if $\mathfrak{q}^{(q)} \neq \mathfrak{q} \cap \bar{\mathfrak{q}}$ for all integers $q > 0$

Proposition 1.2 The terms $\mathfrak{q}^{(p)}$ of (1.11) are Lie subalgebras of \mathfrak{q} .

Proof By definition, $\mathfrak{q}^{(0)} = \mathfrak{q}$ is a Lie subalgebra of \mathfrak{q} . If $Z_1, Z_2 \in \mathfrak{q}^{(1)}$, then

$$[[Z_1, Z_2], \bar{\mathfrak{q}}] \subseteq [Z_1, [Z_2, \bar{\mathfrak{q}}]] + [Z_2, [Z_1, \bar{\mathfrak{q}}]] \subseteq [Z_1 + Z_2, \mathfrak{q} + \bar{\mathfrak{q}}] \subseteq \mathfrak{q} + \bar{\mathfrak{q}}$$

because $[Z_i, \mathfrak{q}] \subseteq [\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{q}$, and $[Z_i, \bar{\mathfrak{q}}] \subseteq \mathfrak{q} + \bar{\mathfrak{q}}$ by the definition of $\mathfrak{q}^{(1)}$. This shows that $\mathfrak{q}^{(1)}$ is a Lie subalgebra of \mathfrak{q} .

Next we argue by recurrence. Let $p \geq 1$ and assume that $\mathfrak{q}^{(p)}$ is a Lie subalgebra of \mathfrak{q} . If $Z_1, Z_2 \in \mathfrak{q}^{(p+1)}$, then $[Z_1, Z_2] \in \mathfrak{q}^{(p)}$ by the inductive assumption that $\mathfrak{q}^{(p)}$ is a Lie subalgebra and

$$[[Z_1, Z_2], \bar{\mathfrak{q}}] \subseteq [Z_1, [Z_2, \bar{\mathfrak{q}}]] + [Z_2, [Z_1, \bar{\mathfrak{q}}]] \subseteq [Z_1 + Z_2, \mathfrak{q}^{(p)} + \bar{\mathfrak{q}}] \subseteq \mathfrak{q}^{(p)} + \bar{\mathfrak{q}},$$

showing that also $[Z_1, Z_2] \in \mathfrak{q}^{(p+1)}$. This completes the proof. □

Let us introduce the notation

$$\mathfrak{h} = \mathfrak{q} + \bar{\mathfrak{q}}, \quad \mathfrak{h}_{\mathbb{R}} = \mathfrak{h} \cap \mathfrak{g}_{\mathbb{R}}. \tag{1.12}$$

The *weak nondegeneracy* defined here is equivalent to the notion of [21], consisting in the requirement that, for a complex Lie subalgebra \mathfrak{f} of \mathfrak{g} ,

$$\mathfrak{q} \subseteq \mathfrak{f} \subseteq \mathfrak{h} \implies \mathfrak{f} = \mathfrak{q}. \tag{1.13}$$

Indeed, it easily follows from [21, Lemma 6.1] that

$$\mathfrak{f} = \mathfrak{q} + \bar{\mathfrak{q}}^{(\infty)}, \quad \text{with } \mathfrak{q}^{(\infty)} = \bigcap_{p \geq 0} \mathfrak{q}^{(p)} \tag{1.14}$$

is the largest complex Lie subalgebra \mathfrak{f} of \mathfrak{g} with $\mathfrak{q} \subseteq \mathfrak{f} \subseteq \mathfrak{h}$.

1.4 Contact nondegeneracy

A less restrictive nondegeneracy condition in terms of iterations of the Levi form can be expressed by requiring that, given $Z \in \mathfrak{q} \setminus (\mathfrak{q} \cap \bar{\mathfrak{q}})$ there are $L_1, \dots, L_p \in \mathfrak{h}$ such that $[L_1, \dots, L_p, Z] \notin \mathfrak{h}$. For a CR algebra of the contact type this is equivalent to the fact that any ideal \mathfrak{a} of $\mathfrak{g}_{\mathbb{R}}$ that is contained in $\mathfrak{h}_{\mathbb{R}}$ is contained in $\mathfrak{q} \cap \mathfrak{g}_{\mathbb{R}}$. Thus this property was called *ideal nondegeneracy* in [21].

When $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{q})$ is the CR algebra at x of a (locally) homogeneous CR manifold M , the subspace $\mathfrak{h}_{\mathbb{R}}$ is the pullback to $\mathfrak{g}_{\mathbb{R}}$ of the real contact distribution associated to the CR structure of M . Thus this notion was renamed *contact nondegeneracy* in [18]. It was shown in [21, Lemma 7.2] that, for the complexification \mathfrak{a} of the largest ideal of $\mathfrak{g}_{\mathbb{R}}$ contained in $\mathfrak{h}_{\mathbb{R}}$, the sum $\mathfrak{a} + \mathfrak{q} \cap \bar{\mathfrak{q}}$ is the limit of the descending chain

$$\begin{cases} \mathfrak{q}^{[0]} \supseteq \mathfrak{q}^{[1]} \supseteq \dots \supseteq \mathfrak{q}^{[p-1]} \supseteq \mathfrak{q}^{[p]} \supseteq \mathfrak{q}^{[p+1]} \supseteq \dots, & \text{with} \\ \mathfrak{q}^{[0]} = \mathfrak{h}, \quad \mathfrak{q}^{[p]} = \{Z \in \mathfrak{h} \mid [Z, \mathfrak{h}] \subseteq \mathfrak{q}^{[p-1]}\}, & \text{for } p \geq 1. \end{cases} \tag{1.15}$$

Since $\mathfrak{q} \cap \bar{\mathfrak{q}} \subseteq \mathfrak{q}^{[p]}$ for all integers $p \geq 0$, the chain (1.15) stabilizes.

Remark 1.3 Note that, for $p \geq 1$,

$$\mathfrak{q}^{[p]} = \{Z \in \mathfrak{q}^{[p-1]} \mid [Z, \mathfrak{q}^{[0]}] \subseteq \mathfrak{q}^{[p-1]}\}. \tag{1.16}$$

This is true in fact for $p=1$ and for $p>2$ follows from $\mathfrak{q}^{[p-1]} \subseteq \mathfrak{q}^{[p-2]}$.

Definition 1.5 We call (1.15) the *descending contact chain*.

Let q be a positive integer. The CR algebra $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{q})$ is said to have

- *finite contact order* q if $\mathfrak{q}^{[q-1]} \supsetneq \mathfrak{q}^{[q]} = \mathfrak{q} \cap \bar{\mathfrak{q}}$.

If $\mathfrak{q}^{[q]} \neq \mathfrak{q} \cap \bar{\mathfrak{q}}$ for all integers $q > 0$, we say that $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{q})$ is *contact degenerate*.

We note that we can equivalently use the descending chain

$$\begin{cases} \mathfrak{a}_{\mathbb{R}}^{(0)} \supseteq \mathfrak{a}_{\mathbb{R}}^{(1)} \supseteq \dots \supseteq \mathfrak{a}_{\mathbb{R}}^{(p-1)} \supseteq \mathfrak{a}_{\mathbb{R}}^{(p)} \supseteq \mathfrak{a}_{\mathbb{R}}^{(p+1)} \supseteq \dots, & \text{with} \\ \mathfrak{a}_{\mathbb{R}}^{(0)} = \mathfrak{h}_{\mathbb{R}}, \quad \mathfrak{a}_{\mathbb{R}}^{(p)} = \{X \in \mathfrak{a}_{\mathbb{R}}^{(p-1)} \mid [X, \mathfrak{h}_{\mathbb{R}}] \subseteq \mathfrak{a}_{\mathbb{R}}^{(p-1)}\}, & \text{for } p \geq 1 \end{cases} \tag{1.17}$$

of [21]. This follows from

Lemma 1.4 *With the notation introduced above, we have:*

1. For each $p \geq 1$, $\mathfrak{a}_{\mathbb{R}}^{(p)}$ is a Lie algebra and, for $p > 1$ an ideal of $\mathfrak{a}_{\mathbb{R}}^{(1)}$;
2. Let $\mathfrak{a}^{(p)}$ be the complexification of $\mathfrak{a}_{\mathbb{R}}^{(p)}$. Then

$$\mathfrak{q}^{[p]} = \mathfrak{q} \cap \bar{\mathfrak{q}} + \mathfrak{a}^{(p)} \quad \text{for all } p \geq 0.$$

Proof The first statement is trivial. We can check the second one by recurrence. This is in fact true for $p=0$, since $\mathfrak{q} + \bar{\mathfrak{q}}$ is the complexification of \mathfrak{h} . □

We already considered two descending chains whose length defines the order of contact nondegeneracy. It is in fact convenient to consider a third one, which is easier to deal with (see Sect. 2 below), namely:

$$\begin{aligned} & \mathfrak{q}^{(0)} \supseteq \mathfrak{q}^{(1)} \supseteq \dots \supseteq \mathfrak{q}^{(p)} \subseteq \mathfrak{q}^{(p+1)} \supseteq \dots \\ \text{with } & \begin{cases} \mathfrak{q}^{(0)} = \mathfrak{q}^{(0)} = \{Z \in \mathfrak{q} \mid [Z, \bar{\mathfrak{q}}] \subseteq \mathfrak{q} + \bar{\mathfrak{q}}\}, \\ \mathfrak{q}^{(p)} = \{Z \in \mathfrak{q}^{(p-1)} \mid [Z, \mathfrak{q} + \bar{\mathfrak{q}}] \subseteq \mathfrak{q}^{(p-1)} + \bar{\mathfrak{q}}\}, & \text{for } p > 0. \end{cases} \end{aligned} \tag{1.18}$$

The equivalence is a consequence of

Proposition 1.5 *With the notation above:*

- $\mathfrak{q}^{[p]} = (\mathfrak{q} \cap \bar{\mathfrak{q}} + \mathfrak{a}^{(p)}) \cap \mathfrak{q}$ for all integers $p \geq 0$.
- $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{q})$ is contact nondegenerate of order $q \geq 1$ if and only if

$$\mathfrak{q}^{[p-1]} \neq \mathfrak{q}^{[q]} = \mathfrak{q} \cap \bar{\mathfrak{q}}.$$

□

Since we obviously have the inclusion

$$\mathfrak{q}^{[p]} \subseteq \mathfrak{q}^{(p)} \quad \forall p \geq 0, \tag{1.19}$$

we obtain

Proposition 1.6 *If the CR algebra $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{q})$ has Levi order $q < \infty$, then $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{q})$ has contact order $q' \leq q$. A contact degenerate $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{q})$ is also holomorphically degenerate.* □

2 Orbits of real forms in complex flag manifolds

2.1 Complex flag manifolds

A complex flag manifold F is a smooth compact algebraic variety that can be described as the quotient of a complex semisimple Lie group S by a parabolic subgroup Q ; according to Wolf [27], a real form $S_{\mathbb{R}}$ of S has finitely many orbits in F . Only one of them, having minimal dimension, is compact. With the partial complex structures induced by F , these orbits make a class of homogeneous CR manifolds that were studied by many authors (see e.g. [1–3,8,10,11,14,17,19]).

Cross-marked Dynkin diagrams Being connected and simply connected, a complex flag manifold $F = S/Q$ is completely described by the Lie pair $(\mathfrak{s}, \mathfrak{q})$ consisting of the Lie algebras of S and of Q and vice versa to any Lie pair $(\mathfrak{s}, \mathfrak{q})$ of a complex semisimple Lie algebra and its parabolic subalgebra \mathfrak{q} corresponds a unique flag manifold F . Therefore the classification of complex flag manifolds reduces to that of parabolic subalgebras of semisimple complex Lie algebras. Parabolic subalgebras \mathfrak{q} of \mathfrak{s} are classified, modulo automorphisms, by a finite set of parameters. In fact, after fixing any Cartan subalgebra \mathfrak{h} of \mathfrak{s} , their equivalence classes are in one to one correspondence with the subsets of a basis \mathcal{B} of simple roots of the root system \mathcal{R} of $(\mathfrak{s}, \mathfrak{h})$ (see e.g. [7, Ch.VIII,§3.4]).

We recall that the Dynkin diagram $\Delta_{\mathcal{B}}$ is a graph with no loops, whose nodes are the roots in \mathcal{B} and in which two nodes may be joined by at most 3 edges. Each root β in \mathcal{R} can be written in a unique way as a nontrivial linear combination

$$\beta = \sum_{\alpha \in \mathcal{B}} k_{\beta, \alpha} \alpha, \quad (2.1)$$

with integral coefficients $k_{\beta, \alpha}$ which are either all ≥ 0 , or all ≤ 0 and we set

$$\text{supp}(\beta) = \{\alpha \in \mathcal{B} \mid k_{\beta, \alpha} \neq 0\}. \quad (2.2)$$

The parabolic subalgebras \mathfrak{q} are parametrized, modulo isomorphisms, by subsets Φ of \mathcal{B} : to a $\Phi \subseteq \mathcal{B}$ we associate

$$\begin{cases} \mathcal{Q}_{\Phi} = \{\beta \in \mathcal{R} \mid k_{\beta, \alpha} \leq 0, \forall \alpha \in \Phi\}, \\ \mathfrak{q}_{\Phi} = \mathfrak{h} \oplus \sum_{\beta \in \mathcal{Q}_{\Phi}} \mathfrak{s}^{\beta}, \quad \text{with } \mathfrak{s}^{\beta} = \{Z \in \mathfrak{s} \mid [H, Z] = \beta(H)Z, \forall H \in \mathfrak{h}\}. \end{cases} \quad (2.3)$$

The set \mathcal{Q}_{Φ} is a *parabolic* set of roots, i.e.

$$(\mathcal{Q}_{\Phi} + \mathcal{Q}_{\Phi}) \cap \mathcal{R} \subseteq \mathcal{Q}_{\Phi} \quad \text{and} \quad \mathcal{Q}_{\Phi} \cup (-\mathcal{Q}_{\Phi}) = \mathcal{R}.$$

To specify the \mathfrak{q}_{Φ} of (2.3) we can cross the nodes corresponding to the roots in Φ . In this way each cross-marked Dynkin diagram encodes a specific complex flag manifold F_{Φ} .

Notation 2.1 Let ξ_{Φ} be the linear functional on the linear span of \mathcal{R} which equals one on the roots in Φ and zero on those in $\mathcal{B} \setminus \Phi$. Then

$$\mathcal{Q}_{\Phi} = \{\beta \in \mathcal{R} \mid \xi_{\Phi}(\beta) \leq 0\} \quad (2.4)$$

and we get partitions

$$\left\{ \begin{aligned} \mathcal{Q}_\Phi &= \mathcal{Q}'_\Phi \cup \mathcal{Q}''_\Phi, \quad \mathcal{R} = \mathcal{Q}'_\Phi \cup \mathcal{Q}''_\Phi \cup \mathcal{Q}^c_\Phi, \quad \text{with} \\ \mathcal{Q}'_\Phi &= \{\beta \in \mathcal{Q}_\Phi \mid -\beta \in \mathcal{Q}_\Phi\} = \{\beta \in \mathcal{R} \mid \xi_\Phi(\beta) = 0\}, \\ \mathcal{Q}''_\Phi &= \{\beta \in \mathcal{Q}_\Phi \mid -\beta \notin \mathcal{Q}_\Phi\} = \{\beta \in \mathcal{R} \mid \xi_\Phi(\beta) < 0\}, \\ \mathcal{Q}^c_\Phi &= \{\beta \in \mathcal{R} \mid -\beta \in \mathcal{Q}''_\Phi\} = \{\beta \in \mathcal{R} \mid \xi_\Phi(\beta) > 0\}. \end{aligned} \right. \tag{2.5}$$

We recall (see e.g. [7, Ch.VIII,§3]):

- $\mathfrak{q}^r_\Phi = \mathfrak{h} \oplus \sum_{\beta \in \mathcal{Q}'_\Phi} \mathfrak{s}^\beta$ is a reductive complex Lie algebra;
- $\mathfrak{q}^n_\Phi = \sum_{\beta \in \mathcal{Q}''_\Phi} \mathfrak{s}^\beta$ is the nilradical of \mathfrak{q}_Φ ;
- $\mathfrak{q}_\Phi = \mathfrak{q}^r_\Phi \oplus \mathfrak{q}^n_\Phi$ is the Levi-Chevalley decomposition of \mathfrak{q}_Φ ;
- $\mathfrak{q}^c_\Phi = \sum_{\beta \in \mathcal{Q}^c_\Phi} \mathfrak{s}^\beta$ is a Lie subalgebra of \mathfrak{s} consisting of $\text{ad}_\mathfrak{s}$ -nilpotent elements;
- $\mathfrak{q}^\vee_\Phi = \mathfrak{q}^r_\Phi \oplus \mathfrak{q}^c_\Phi$ is the parabolic Lie subalgebra of \mathfrak{s} opposite of \mathfrak{q}_Φ , decomposed into the direct sum of its reductive subalgebra \mathfrak{q}^r_Φ and its nilradical \mathfrak{q}^c_Φ .

2.2 Real forms

Let us take, as we can, \mathbf{S} connected and simply connected. Then real automorphisms of its Lie algebra \mathfrak{s} lift to automorphisms of the Lie group \mathbf{S} , so that real forms $\mathbf{S}_\mathbb{R}$ of \mathbf{S} are in one-to-one correspondence with the anti- \mathbb{C} -linear involutions σ of \mathfrak{s} . We will denote by \mathfrak{s}_σ the real Lie subalgebra consisting of the fixed points of σ : it is the Lie algebra of the real form \mathbf{S}_σ of fixed points of the lift $\tilde{\sigma}$ of σ to \mathbf{S} . Its orbits are CR submanifolds $M_{\Phi,\sigma}$ of F_Φ whose CR algebra at the base point \mathbf{Q} is the pair $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$.

Definition 2.1 (cf. [1, §5]) A *parabolic CR algebra* is a pair $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ consisting of a real semisimple Lie algebra \mathfrak{s}_σ and a parabolic complex Lie subalgebra \mathfrak{q}_Φ of its complexification \mathfrak{s} . We say that $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is *minimal* if $M_{\Phi,\sigma}$ is the minimal orbit in F_Φ of the real form \mathbf{S}_σ of \mathbf{S} .

When \mathfrak{s}_σ is not simple, the corresponding orbits $M_{\Phi,\sigma}$ are CR diffeomorphic to a cartesian product of orbits of simple real Lie groups (see e.g. [3]). If

$$M_{\Phi,\sigma} \simeq M_{\Phi_1,\sigma_1} \times \cdots \times M_{\Phi_k,\sigma_k}, \tag{2.6}$$

we call each M_{Φ_i,σ_i} a *factor* of $M_{\Phi,\sigma}$.

A simple \mathfrak{s}_σ is of the *real type* if also \mathfrak{s} is simple; otherwise, \mathfrak{s} is the direct sum of two complex simple Lie algebras $\mathfrak{s}', \mathfrak{s}''$, which are \mathbb{R} -isomorphic to \mathfrak{s}_σ , and we say in this case that \mathfrak{s}_σ is of the *complex type*.

To list all the orbits of a real form, one can use the fact that the *isotropy subalgebra* $\mathfrak{s}_\sigma \cap \mathfrak{q}$ contains a Cartan subalgebra $\mathfrak{h}_\mathbb{R}$ of \mathfrak{s}_σ (see e.g. [3]). By choosing \mathfrak{h} equal to its complexification, we obtain on \mathcal{R} a conjugation which is compatible with the one defined on \mathfrak{s} by its real form \mathfrak{s}_σ (and which, for simplicity, we still denote by σ). Vice versa, an orthogonal involution σ of \mathcal{R} lifts, although in general not in a unique way, to a conjugation of \mathfrak{s} . The conjugation on \mathfrak{s} depends indeed also on the description of which roots in $\mathcal{R}^\sigma = \{\beta \in \mathcal{R} \mid \sigma(\beta) = -\beta\}$ are *compact*. This is determined by the choice of a *Cartan involution* θ on \mathfrak{s} , with $\theta(\mathfrak{h}) = \mathfrak{h}$ and $\sigma \circ \theta = \theta \circ \sigma$, which induces a map, that we will denote by the same symbol,

$$\theta : \mathcal{R} \ni \alpha \rightarrow -\sigma(\alpha) \in \mathcal{R}. \tag{2.7}$$

We will write for simplicity $\bar{\alpha}$ instead of $\sigma(\alpha)$ and \mathcal{R}_\bullet for $\mathcal{R}_\bullet^\sigma$ when this will not cause confusion. We recall that $\kappa_\sigma = \{X \in \mathfrak{s}_\sigma \mid \theta(X)=X\}$ is a maximal compact Lie subalgebra of \mathfrak{s}_σ and that we have the *Cartan decomposition*

$$\mathfrak{s}_\sigma = \kappa_\sigma \oplus \mathfrak{p}_\sigma, \quad \text{with } \mathfrak{p}_\sigma = \{X \in \mathfrak{s}_\sigma \mid \theta(X)=-X\}$$

of \mathfrak{s}_σ . When $\theta(\alpha) = \alpha$, then either \mathfrak{s}^α is contained in the complexification κ of κ_σ (*compact root*) or in the complexification \mathfrak{p} of \mathfrak{p}_σ (*hermitian root*).

The subalgebras $\mathfrak{q}_\Phi \cap \bar{\mathfrak{q}}_\Phi$, \mathfrak{q}_Φ and $\bar{\mathfrak{q}}_\Phi$ turn out to be direct sums of \mathfrak{h} and root subspaces \mathfrak{s}^α ; in particular $\mathfrak{q}_\Phi \cap \bar{\mathfrak{q}}_\Phi$ is the direct sum of \mathfrak{h} and the root subspaces \mathfrak{s}^α with $\mathfrak{s}^\alpha + \mathfrak{s}^{\bar{\alpha}} \subset \mathfrak{q}_\Phi$.

We note that $\mathfrak{q}_\Phi \cap \bar{\mathfrak{q}}_\Phi$ is a Lie subalgebra of \mathfrak{s} and $(\mathfrak{q}_\Phi + \bar{\mathfrak{q}}_\Phi)$ is a $(\mathfrak{q}_\Phi \cap \bar{\mathfrak{q}}_\Phi)$ -module.

Having fixed a base \mathcal{B} of simple roots of the root system \mathcal{R} associated to $(\mathfrak{s}, \mathfrak{h})$, the orbit of the real form is determined by the data of:

- a subset Φ of \mathcal{B} specifying the parabolic subalgebra \mathfrak{q}_Φ ;
- a conjugation σ of \mathcal{R} ;
- a splitting $\mathcal{R}_\bullet^\sigma = \mathcal{R}_{\bullet,+}^\sigma \cup \mathcal{R}_{\bullet,-}^\sigma$ of $\mathcal{R}_\bullet^\sigma$ into a first set consisting of the *compact* and a second of the *hermitian* roots.

We point out that different choices of σ may yield the same *CR* submanifold $M_{\Phi,\sigma}$. In particular, we can conjugate σ by any element of the subgroup of the Weyl group generated by reflections with respect to roots in $\mathcal{B} \setminus \Phi$.

2.3 Contact nondegeneracy for parabolic CR algebras

Since by definition simple Lie algebra have no proper nontrivial ideals, we obtain

Proposition 2.1 *A real orbit $M_{\Phi,\sigma}$ which is fundamental and does not have a totally complex factor is contact nondegenerate.*

Proof We can indeed reduce to the case where $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is effective and \mathfrak{s}_σ is simple, in which the proof is straightforward. □

We have the following criterion

Proposition 2.2 *A real orbit $M_{\Phi,\sigma}$ is fundamental iff its CR algebra $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is fundamental.*

Let $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ be a parabolic CR algebra and set

$$\Phi_\sigma^\circ = \{\alpha \in \Phi \mid \sigma(\alpha) > 0\}.$$

If $\Phi_\sigma^\circ = \emptyset$, then $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is fundamental. When $\Phi_\sigma^\circ \neq \emptyset$, we have

- $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is fundamental if and only if $(\mathfrak{s}_\sigma, \mathfrak{q}_{\Phi_\sigma^\circ})$ is fundamental;
- $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ and $(\mathfrak{s}_\sigma, \mathfrak{q}_{\Phi_\sigma^\circ})$ are fundamental if and only if

$$\bar{Q}_{\Phi_\sigma^\circ}^c \cap \Phi_\sigma^\circ = \emptyset. \tag{2.8}$$

Proof If $\Phi_\sigma^\circ = \emptyset$, then $\mathcal{B} \subseteq \mathcal{Q}_\Phi \cup \bar{\mathcal{Q}}_\Phi$ and hence $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is trivially fundamental. Let us consider next the case where $\Phi_\sigma^\circ \neq \emptyset$.

Since $\Phi_\sigma^\circ \subseteq \Phi$, we have $\mathfrak{q}_\Phi \subseteq \mathfrak{q}_{\Phi_\sigma^\circ}$ and therefore $(\mathfrak{s}_\sigma, \mathfrak{q}_{\Phi_\sigma^\circ})$ is fundamental when $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is fundamental. To show the vice versa, we note that any Lie subalgebra of \mathfrak{s} containing \mathfrak{q}_Φ is of the form \mathfrak{q}_Ψ for some $\Psi \subseteq \Phi$. If it contains $\mathfrak{q}_\Phi + \bar{\mathfrak{q}}_\Phi$, then $\Psi \subseteq \Phi_\sigma^\circ$. This proves the first item.

Definition 2.2 For any root $\beta \in Q_\Phi \setminus \bar{Q}_\Phi$ we denote by $q_\Phi^\sigma(\beta)$ and call its *Levi order* the smallest number k for which (2.11) is valid. We put $q_\Phi^\sigma(\beta) = +\infty$ when (2.11) is not valid for any positive integer k .

Lemma 2.6 Assume that $\beta \in Q_\Phi \setminus \bar{Q}_\Phi$ has finite Levi order $q_\Phi^\sigma(\beta) = q$ and (2.11) is satisfied for a sequence $\alpha_1, \dots, \alpha_q$. Then

- (i) $\alpha_i \in \bar{Q}_\Phi \setminus Q_\Phi$ for all $1 \leq i \leq q$;
- (ii) $\beta + \sum_{i \leq h} \alpha_i \in Q_\Phi \setminus \bar{Q}_\Phi$ for all $h < q$;
- (iii) (2.11) is satisfied by all permutations of $\alpha_1, \dots, \alpha_q$;
- (iv) $\alpha_i + \alpha_j \notin \mathcal{R}$ for all $1 \leq i < j \leq q$.

Proof Let us first prove (ii). With the notation in (2.11), we observe that $\gamma_h \notin \bar{Q}_\Phi$ for $h < q$, because, otherwise, $\gamma_q \in \bar{Q}_\Phi$.

Next we prove (iii). Let $\{Z_\alpha\}_{\alpha \in \mathcal{R}} \cup \{H_i \in \mathfrak{h} \mid 1 \leq i \leq \ell\}$ be a Chevalley basis for $(\mathfrak{s}, \mathfrak{h})$. Then (2.11) is equivalent to the fact that

$$[Z_{\alpha_q}, \dots, Z_{\alpha_{q-1}}, \dots, Z_{\alpha_1}, Z_\beta] := [Z_{\alpha_q}, [Z_{\alpha_{q-1}}, [\dots, [Z_{\alpha_1}, Z_\beta] \dots]]] \notin \mathfrak{q}_\Phi + \bar{\mathfrak{q}}_\Phi.$$

The item (iii) follows because

$$\begin{aligned} [Z_{\alpha_q}, \dots, Z_{\alpha_{i+1}}, Z_{\alpha_i}, \dots, Z_{\alpha_1}, Z_\beta] - [Z_{\alpha_q}, \dots, Z_{\alpha_i}, Z_{\alpha_{i+1}}, \dots, Z_{\alpha_1}, Z_\beta] \\ = [Z_{\alpha_q}, \dots, [Z_{\alpha_{i+1}}, Z_{\alpha_i}], \dots, Z_{\alpha_1}, Z_\beta] \end{aligned}$$

and, by the minimality assumption, the right hand side belongs to $\mathfrak{q}_\Phi + \bar{\mathfrak{q}}_\Phi$.

Let us prove (i) by contradiction. If $\alpha_i \in Q_\Phi \cap \bar{Q}_\Phi$ for some $1 \leq i \leq q$, then we could assume by (iii) that it was α_q . Then

$$[Z_{\alpha_{q-1}}, \dots, Z_{\alpha_1}, Z_\beta] \in \mathfrak{q}_\Phi + \bar{\mathfrak{q}}_\Phi \implies [Z_{\alpha_q}, Z_{\alpha_{q-1}}, \dots, Z_{\alpha_1}, Z_\beta] \in \mathfrak{q}_\Phi + \bar{\mathfrak{q}}_\Phi$$

yields the contradiction. Also (iv) is an easy consequence of (iii), because if $\alpha_i + \alpha_j$ ($1 \leq i, j \leq q$) is a root, then it would belong to $\bar{Q}_\Phi \cap Q_\Phi^c$ and, by substituting to the two roots α_i, α_j the single root $\alpha_i + \alpha_j$ we would obtain a sequence satisfying (2.11) and containing $q-1$ terms.

The proof is complete. □

Remark 2.7 Since $\xi_\Phi(\alpha) \geq 1$ for all $\alpha \in Q_\Phi^c$, if $\beta \in Q_\Phi \cap \bar{Q}_\Phi^c$ and $q_\Phi^\sigma(\beta) < +\infty$, then

$$q_\Phi^\sigma(\beta) \leq 1 - \xi_\Phi(\beta). \tag{2.12}$$

Corollary 2.8 If $\beta \in Q_\Phi^r \setminus \bar{Q}_\Phi$, then its Levi order is either one or $+\infty$. □

We obtain also a useful criterion of weak nondegeneracy (cf. [3, Thm.6.4])

Proposition 2.9 The parabolic CR algebra $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is weakly nondegenerate if and only if

$$\forall \beta \in Q_\Phi \cap \bar{Q}_\Phi^c \exists \alpha \in \bar{Q}_\Phi \cap Q_\Phi^c \text{ such that } \beta + \alpha \in \bar{Q}_\Phi^c. \tag{2.13}$$

Proof By Lemma 2.6 the condition is necessary. To prove that it is also sufficient, we can argue by contradiction: if we could find $\beta \in Q \cap \bar{Q}_\Phi^c$ with $q_\Phi^\sigma(\beta) = +\infty$, then by (2.13) we could construct an infinite sequence $(\alpha_i)_{i \geq 1}$ in $\bar{Q}_\Phi \cap Q_\Phi^c$ with

$$\gamma_h = \beta + \sum_{i=1}^h \alpha_i \in Q_\Phi \cap \bar{Q}_\Phi^c, \quad \forall h = 1, 2, \dots$$

Since $\xi_\Phi(\gamma_h) \geq \xi_\Phi(\beta) + h$ and ξ_Φ is bounded, we get a contradiction. □

2.5 Levi order of general orbits

To discuss Levi order of weakly nondegenerate real orbits $M_{\phi, \sigma}$ in F_{ϕ} by employing Lemma 2.6, we introduce:

Definition 2.3 If $\beta \in \mathcal{R}$, we denote by $q(\beta)$ the largest positive integer q for which

$$\exists \alpha_1, \dots, \alpha_q \in \mathcal{R} \text{ s.t. } \begin{cases} \alpha_i + \alpha_j \notin \mathcal{R} \cup \{0\}, \forall 1 \leq i, j \leq q, \\ \gamma_{i_1, \dots, i_h} = \beta + \alpha_{i_1} + \dots + \alpha_{i_h} \in \mathcal{R}, \\ \text{for all distinct } i_1, \dots, i_h \text{ in } \{1, \dots, q\}. \end{cases} \tag{2.14}$$

Proposition 2.10 Let $\beta \in \mathcal{R}$ belong to a simple root system containing more than two elements. Then $q(\beta) \leq 4$ and, if $q(\beta) = 4$ and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is a sequence satisfying (2.14), then

$$\beta + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -\beta. \tag{2.15}$$

More precisely we obtain:

$$\left\{ \begin{array}{l} q(\beta)=1, \text{ if } \beta \text{ belongs to a root system of type } A_2; \\ \qquad \qquad \qquad \text{or is a long root of a root system of type } B_2; \\ q(\beta)=2, \text{ if } \beta \text{ belongs to a root system of type } A_{\geq 3}, C, \\ \qquad \qquad \qquad \text{or is a short root of a system of type } B_2, G; \\ q(\beta)=3, \text{ if } \beta \text{ is a short root of a root system of type } B_{\geq 3}, F; \\ q(\beta)=4, \text{ if } \beta \text{ belongs to a root system of type } D, E, \\ \qquad \qquad \qquad \text{or is a long root of a root system of type } B_{\geq 3}, F, G. \end{array} \right. \tag{2.16}$$

Proof For short we will call *admissible* a sequence (α_i) for which (2.14) is valid. Let us set

$$\mathcal{R}^{\text{add}}(\beta) = \{\alpha \in \mathcal{R} \mid \beta + \alpha \in \mathcal{R}\}.$$

We consider the different cases using for root systems the notation of [6].

Type A We have $\mathcal{R} = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n\}$ where e_1, \dots, e_n is an orthonormal basis of \mathbb{R}^n . We can take $\beta = e_2 - e_1$. Then

$$\mathcal{R}^{\text{add}}(e_2 - e_1) = \{e_1 - e_i \mid i > 2\} \cup \{e_i - e_2 \mid i > 2\}. \tag{*A}$$

An admissible sequence (α_i) can contain at most one element from each of the two sets in the right hand side of (*A).

If $n = 3$, then $\mathcal{R}^{\text{add}}(\beta) = \{e_3 - e_2, e_1 - e_3\}$ contains two elements, whose sum is still a root and therefore $q(\beta) = 1$.

If $n > 3$, then the only possible choice is that of a couple of roots $e_i - e_2, e_1 - e_j$ with $3 \leq i \neq j \leq n$ and hence $q(\beta) = 2$.

Type B We have $\mathcal{R} = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm e_i \mid 1 \leq i \leq n\}$, for an orthonormal basis e_1, \dots, e_n of \mathbb{R}^n ($n \geq 2$).

If β is a short root, we can take $\beta = -e_1$. Then

$$\mathcal{R}^{\text{add}}(-e_1) = \{\pm e_i \mid 2 \leq i \leq n\} \cup \{e_1 \pm e_j \mid 2 \leq j \leq n\}. \tag{*B}$$

An admissible sequence contains at most one root from the first and two from the second set in the right hand side of (*B). Thus $q(-e_1) \leq 3$. The sequence $e_1 - e_2, e_1 + e_2$ satisfies (2.14) and therefore $q(-e_1) \geq 2$.

We have equality if $n=2$, because in this case $\mathcal{R}^{\text{add}}(-e_1) = \{\pm e_2, e_1 \pm e_2\}$ and the maximal admissible sequences are then $(e_2), (-e_2), (e_1+e_2, e_1-e_2)$.

If $n > 2$ the admissible sequence

$$(e_1+e_2, e_1-e_2, e_3)$$

shows that $q(-e_1)=3$. All admissible maximal sequences are of this form.

If β is a long root, we can assume that $\beta = -e_1-e_2$. Then

$$\mathcal{R}^{\text{add}}(-e_1-e_2) = \{e_1, e_2\} \cup \{e_1 \pm e_j \mid j > 2\} \cup \{e_2 \pm e_j \mid j > 2\}. \tag{**B}$$

An admissible sequence contains at most two equal terms from the first and two from each of the second and third on the right hand side of (**B). Moreover, if one term is taken from the first, we can take at most one from each one of the other two. This implies that $q(-e_1-e_2) \leq 4$ and in fact $q(-e_1-e_2)=4$, with maximal sequences isomorphic to one of

$$\begin{aligned} &e_1 + e_3, e_1 - e_3, e_2 + e_4, e_2 - e_4, \\ &e_1, e_1, e_1 - e_3, e_1 + e_3, \end{aligned}$$

which, summed up to $(-e_1-e_2)$, gives e_1+e_2 .

Type C We can take $\mathcal{R} = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm 2e_i \mid 1 \leq i \leq n\}$, for an orthonormal basis e_1, \dots, e_n of \mathbb{R}^n ($n \geq 3$).

If β is a short root, we can assume that $\beta = (-e_1-e_2)$. Then

$$\mathcal{R}^{\text{add}}(-e_1-e_2) = \{2e_1, 2e_2\} \cup \{e_1 \pm e_j \mid j \geq 3\} \cup \{e_2 \pm e_j \mid j \geq 3\}. \tag{*C}$$

An admissible sequence may contain both roots of the first, but at most one root from each the second and third sets on the right hand side of (*C). Moreover, a term in one of the last two forbids the corresponding term in the first one. This yields $q(-e_1-e_2)=2$, with maximal sequences isomorphic to (the third one should be omitted if $n=3$)

$$(2e_1, 2e_2), (2e_1, e_2+e_3), (e_1+e_3, e_2+e_4)$$

If β is a long root, we can assume that $\beta = -2e_1$. Then

$$\mathcal{R}^{\text{add}}(-2e_1) = \{e_1 \pm e_i \mid i > 1\}. \tag{**C}$$

We note that $q(-2e_1) \leq 4$. We cannot take in an admissible sequence both the element e_1+e_i and e_1-e_i , because they add up to the root $2e_i$. Hence in fact $q(-2e_1)=2$, with maximal sequence isomorphic to

$$e_1+e_2, e_1+e_3.$$

Type D We can take $\mathcal{R} = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\}$, where e_1, \dots, e_n is an orthonormal basis of \mathbb{R}^n ($n \geq 4$).

We can assume that $\beta = -e_1-e_2$. We have

$$\mathcal{R}^{\text{add}}(-e_1-e_2) = \{e_1 \pm e_j \mid j \geq 3\} \cup \{e_2 \pm e_j \mid j \geq 3\}. \tag{*D}$$

An admissible sequence contains at most two elements from each set in the right hand side of (*D). Therefore $q(-e_1-e_2) \leq 4$ and in fact we have equality with maximal admissible sequences isomorphic to

$$e_1+e_3, e_1-e_3, e_2+e_4, e_2-e_4,$$

which, summed up to $(-e_1-e_2)$, give e_1+e_2 .

Type E Since the root systems E_6 and E_7 can be considered as subsystems of E_8 , we will restrain to this case. We consider, for an orthonormal basis e_1, \dots, e_8 of \mathbb{R}^8 ,

$$\mathcal{R} = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 8\} \cup \left\{ \frac{1}{2} \sum_{i=1}^8 (-1)^{h_i} e_i \mid h_i \in \mathbb{Z}, \sum_{i=1}^8 h_i \in 2\mathbb{Z} \right\}.$$

We can take $\beta = (-e_1 - e_2)$. Then

$$\begin{cases} \mathcal{R}^{\text{add}}(-e_1 - e_2) = \{e_1 \pm e_i \mid 3 \leq i \leq 8\} \cup \{e_2 \pm e_i \mid 3 \leq i \leq 8\} \\ \cup \left\{ \frac{1}{2} \left(e_1 + e_2 + \sum_{i=3}^8 (-1)^{h_i} e_i \right) \mid h_i \in \mathbb{Z}, \sum_{i=3}^8 h_i \in 2\mathbb{Z} \right\} \end{cases} \quad (**E)$$

An admissible sequence may contain at most two roots from each set on the right hand side of $(**E)$ and no more than four terms. Clearly we can take the maximal sequence

$$e_1 + e_3, e_1 - e_3, e_2 + e_4, e_2 - e_4,$$

showing that $q(-e_1 - e_2) = 4$. Moreover, any admissible sequence containing four terms sums up to $(-e_1 - e_2)$ to yield $e_1 + e_2$.

Type F For an orthonormal basis e_1, e_2, e_3, e_4 of \mathbb{R}^4 we take

$$\mathcal{R} = \{\pm e_i \mid 1 \leq i \leq 4\} \cup \{\pm e_1 \pm e_j \mid 1 \leq i < j \leq 4\} \cup \left\{ \frac{1}{2} (\pm e_1 \pm e_2 \pm e_3 \pm e_4) \right\}.$$

If β is a short root, we can take $\beta = -e_1$. Then

$$\mathcal{R}^{\text{add}}(-e_1) = \{\pm e_i \mid 2 \leq i \leq 4\} \cup \{e_1 \pm e_i \mid 2 \leq i \leq 4\} \cup \left\{ \frac{1}{2} (e_1 \pm e_2 \pm e_3 \pm e_4) \right\}. \quad (**F)$$

To build an an admissible sequence we can take at most one element from the first, two from the second and from the third set in the right hand side of $(**F)$. Indeed two roots of the form $\frac{1}{2} (\pm e_1 \pm e_2 \pm e_3 \pm e_4)$ do not add up to a root if and only if they differ by only one sign. Moreover, no root can be taken from the first if one is taken from the last set. These considerations imply that $q(-e_1) \leq 3$ and in fact equality holds, as $(-e_1)$ is contained in a subsystem of type B_3 .

If β is a long root, we can assume $\beta = (-e_1 - e_2)$. We have

$$\begin{cases} \mathcal{R}^{\text{add}}(-e_1 - e_2) = \{e_1, e_2\} \cup \{e_1 \pm e_i \mid 3 \leq i \leq 4\} \\ \cup \{e_2 \pm e_i \mid 3 \leq i \leq 4\} \cup \left\{ \frac{1}{2} (e_1 + e_2 \pm e_3 \pm e_4) \right\}. \end{cases} \quad (**F)$$

We note that the sum of four terms of $\mathcal{R}^{\text{add}}(-e_1 - e_2)$ is a linear combination $\beta + k_1 e_1 + k_2 e_2 + k_3 e_3 + k_4 e_4$ with $k_1 + k_2 \geq 2$ and therefore, if they form an admissible sequence, is equal to $e_1 + e_2$. Since \mathcal{R} contains subsystems of type B_3 , there are indeed admissible sequences with four elements.

Type G For an orthonormal basis e_1, e_2, e_3 of \mathbb{R}^3 we set

$$\mathcal{R} = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq 3\} \cup \{\pm(2e_i - e_j - e_k) \mid \{i, j, k\} = \{1, 2, 3\}\}.$$

We consider first the case of a short root. We can take $\beta = e_2 - e_1$. Then

$$\mathcal{R}^{\text{add}}(e_2 - e_1) = \{e_3 - e_2, e_1 - e_3\} \cup \{2e_1 - e_2 - e_3, e_1 + e_3 - 2e_2\}. \quad (**G)$$

Maximal admissible sequences have a root from the first and one from the second set, hence $q(e_2 - e_1) = 2$ and, moreover, summed up to $e_2 - e_1$, give $e_1 - e_2$.

As a long root we take $\beta = (e_2 + e_3 - 2e_1)$. Then

$$\mathcal{R}^{\text{add}}(e_2 + e_3 - 2e_1) = \{e_1 - e_2, e_1 - e_3\} \cup \{e_1 + e_2 - 2e_3, e_1 + e_3 - 2e_2\}. \quad (**G)$$

One checks that in this case $q(e_2+e_3-2e_1) = 4$, with a maximal admissible sequence

$$e_1 - e_2, e_1 - e_2, e_1 - e_2, e_1 + e_2 - 2e_3$$

which indeed sums up to the opposite root $2e_1 - e_2 - e_3$.

The proof is complete. □

As an easy consequence we obtain:

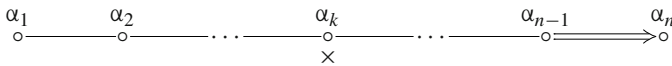
Theorem 2.11 *Let $M_{\Phi, \sigma}$ be a real orbit which is fundamental and weakly nondegenerate. Then its Levi order is less or equal to 3.*

Proof This is a consequence of Prop. 2.10 and the fact that, if β does not belong to \bar{Q}_Φ , then $-\beta \in \bar{Q}_\Phi$ because \bar{Q}_Φ is a parabolic set of roots. □

Example 2.12 ([10, §7]) Let n be an integer ≥ 3 and fix a symmetric \mathbb{C} -bilinear form \mathfrak{b} on \mathbb{C}^{2n+1} . The Lie algebra of the group of \mathbb{C} -linear transformations of \mathbb{C}^{2n+1} that keep \mathfrak{b} invariant is a simple complex Lie algebra $\mathfrak{o}_{2n+1}(\mathbb{C})$ of type B_n , with root system

$$\mathcal{R} = \{\pm e_i \mid 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\}$$

for an orthonormal basis e_1, \dots, e_n of \mathbb{R}^n . Fix k with $1 < k < n$. The cross-marked Dynkin diagram



represents the grassmannian of totally \mathfrak{b} -isotropic k -planes in \mathbb{C}^{2n+1} . Here $\alpha_i = e_i - e_{i+1}$ for $1 \leq i < n$ and $\alpha_n = e_n$. We have $\Phi = \{\alpha_k\}$ and

$$\xi_\Phi(e_i) = \begin{cases} 1, & \text{if } 1 \leq i \leq k, \\ 0, & \text{if } k < i \leq n. \end{cases}$$

Real forms are obtained by fixing a conjugation σ on \mathbb{C}^{2n+1} . Then

$$\mathfrak{b}_\sigma(v, w) = \mathfrak{b}(v, \sigma(w)), \quad \forall v, w \in \mathbb{C}^{2n+1}$$

is hermitian symmetric and nondegenerate, of signature (p, q) for a pair of nonnegative integers with $p+q=2n+1$. The Lie algebra of the group of \mathbb{C} -linear transformations which keep fixed both \mathfrak{b} and \mathfrak{b}_σ is a real form \mathfrak{s}_σ of $\mathfrak{s} \simeq \mathfrak{o}_{2n+1}(\mathbb{C})$, which is isomorphic to the real simple Lie algebra $\mathfrak{o}(p, q)$.

We define a conjugation σ on \mathcal{R} by

$$\sigma(e_1) = e_n, \quad \sigma(e_i) = -e_i, \text{ if } 1 < i < n, \quad \sigma(e_n) = e_1.$$

According to the number of compact roots between e_2, \dots, e_{n-1} , this conjugation corresponds to any of the Lie algebras $\mathfrak{o}(p, 2n+1-p)$ with $1 \leq p \leq 2n$. The orbit $M_{\Phi, \sigma}$ consists of \mathfrak{b}_σ -isotropic k -spaces ℓ_k with $\dim(\ell_k \cap \sigma(\ell_k)) = k-1$. By (1.13), since q_Φ is maximal, if the parabolic CR algebra $(\mathfrak{s}_\sigma, q_\Phi)$ is fundamental and not totally complex, then it is also weakly nondegenerate. To compute its Levi order we observe that

$$\begin{cases} Q_\Phi^c \cap \bar{Q}_\Phi^c = \{e_1 + e_n\}, \\ Q_\Phi^c \cap \bar{Q}_\Phi = \{e_i \mid 1 \leq i \leq k\} \cup \{e_i \pm e_j \mid 1 \leq i \leq k < j \leq n\}, \\ Q_\Phi \cap \bar{Q}_\Phi^c = \{-e_i \mid 2 \leq i \leq k\} \cup \{e_n\} \cup \{e_n \pm e_j \mid k < j < n\} \\ \quad \cup \{-e_i \pm e_j \mid 2 \leq i \leq k < j < n\} \cup \{\pm e_1 - e_i \mid 2 \leq i \leq k\} \end{cases}$$

The fact that $Q_\Phi^c \cap \bar{Q}_\Phi^c \neq \emptyset$ shows that $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is not totally complex.

Since $e_n \in Q_\Phi \cap \bar{Q}_\Phi^c$ and $e_1 \in \bar{Q}_\Phi \cap Q_\Phi^c$ add up to $e_1 + e_n$, the CR algebra $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is fundamental and therefore, as we noticed above, weakly nondegenerate.

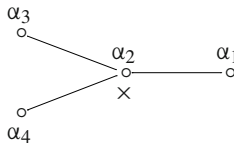
The roots $\beta_i = -(e_1 + e_i)$, for $2 \leq i \leq k$ belong to $Q_\Phi \cap \bar{Q}_\Phi^c$ and have $\xi_\Phi(\beta_i) = -2$. Since $\xi_\Phi(e_1 + e_2) = 1$ and $\xi_\Phi(\alpha) \leq 1$ for all $\alpha \in \bar{Q}_\Phi \cap Q_\Phi^c$, no chain (2.11) that added up to β_i yields $e_1 + e_n$ contains less than three elements. By Theorem 2.11 this shows that $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ has Levi order 3. We have indeed

$$(-e_1 - e_i) + e_1 + e_1 + (e_i + e_n) = e_1 + e_n.$$

Example 2.13 Consider a simple complex Lie algebra \mathfrak{g} of type D_4 . Its root system is described, by using an orthonormal basis e_1, e_2, e_3, e_4 of \mathbb{R}^4 , by

$$\mathcal{R} = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 4\}.$$

Consider the complex flag manifold F_Φ corresponding to the cross-marked Dynkin diagram



with $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_3 = e_3 - e_4, \alpha_4 = e_3 + e_4$ and $\Phi = \{\alpha_2\}$.

It is the grassmannian of projective lines contained in the nondegenerate quadric complex hypersurface in $\mathbb{C}P^7$. The grading functional is

$$\xi_\Phi(e_i) = \begin{cases} 1, & i = 1, 2, \\ 0, & i = 3, 4. \end{cases}$$

Take the conjugation

$$\sigma(e_1) = e_4, \quad \sigma(e_2) = -e_2, \quad \sigma(e_3) = -e_3, \quad \sigma(e_4) = e_1.$$

We have

$$Q_\Phi^c \cap \bar{Q}_\Phi^c = \{e_1 + e_4\}.$$

This shows that $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is not totally complex and therefore, since \mathfrak{q}_Φ is maximal parabolic, this CR algebra is weakly nondegenerate iff it is fundamental. We have

$$\begin{cases} Q_\Phi^c \cap \bar{Q}_\Phi = \{e_1 - e_3, e_1 - e_4, e_2 - e_3, e_2 - e_4, e_3 - e_4, e_1 + e_2, e_1 + e_3, e_2 + e_3, e_2 + e_4\} \\ Q_\Phi \cap \bar{Q}_\Phi^c = \{e_3 + e_4, e_4 - e_1, e_3 - e_2, -e_1 - e_2, -e_1 - e_3, e_4 - e_2, e_4 - e_3, \\ \qquad \qquad \qquad -e_2 - e_3, e_1 - e_2\}. \end{cases}$$

Note that $e_3 + e_4 \in Q_\Phi \cap \bar{Q}_\Phi^c$ and $e_1 - e_3 \in \bar{Q}_\Phi \cap Q_\Phi^c$ sum up to $e_1 + e_4$. This shows that $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is not totally complex and fundamental. Since \mathfrak{q}_Φ is maximal parabolic, this implies that $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is weakly nondegenerate.

The root $\beta = -e_1 - e_2$ belongs to $Q_\Phi \cap \bar{Q}_\Phi^c$ and $\xi_\Phi(\beta) = -2$. Since all roots α in $\bar{Q}_\Phi \cap Q_\Phi^c$ distinct from $e_1 + e_2$ have $\xi_\Phi(\alpha) = 1$, a sequence satisfying (2.11) and summing up with β to $\{e_1 + e_4\}$ contains at least 3 elements. By Theorem 2.11 this shows that $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ has Levi order 3. An admissible sequence for $-e_1 - e_2$ is $(e_1 - e_3, e_1 + e_3, e_2 + e_4)$.

Example 2.14 Consider a semisimple complex Lie algebra \mathfrak{s} of type G_2 . Having fixed a Cartan subalgebra, we can write its root system in the form

$$\mathcal{R} = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq 3\} \cup \{\pm(2e_{\sigma_1} - e_{\sigma_2} - e_{\sigma_3}) \mid \sigma \in \mathbf{S}_3, \sigma_2 < \sigma_3\},$$

for an orthonormal basis e_1, e_2, e_3 of \mathbb{R}^3 . We consider the complex flag manifold F_Φ corresponding to the cross-marked Dynkin diagram

$$\begin{array}{c} \alpha_2 \xrightarrow{\quad} \alpha_1 \\ \circ \xrightarrow{\quad} \circ \\ \times \end{array} \quad \alpha_1 = e_1 - e_2 \quad \alpha_2 = 2e_2 - e_1 - e_3.$$

It corresponds to the grading functional ξ_Φ with

$$\xi_\Phi(e_1) = 1, \quad \xi_\Phi(e_2) = 1, \quad \xi_\Phi(e_3) = 0.$$

We consider the conjugation defined by

$$\sigma(e_1) = e_3, \quad \sigma(e_2) = e_2, \quad \sigma(e_3) = e_1.$$

Then

$$Q_\Phi^c \cap \bar{Q}_\Phi^c = \{2e_2 - e_1 - e_3\}.$$

Since \mathfrak{q}_Φ is maximal and $Q_\Phi^c \cap \bar{Q}_\Phi^c \neq \emptyset$, then is sufficient to check that $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is weakly nondegenerate to find that it is also fundamental.

The root $\beta = 2e_3 - e_1 - e_2$ belongs to $Q_\Phi \cap \bar{Q}_\Phi^c$. We have

$$Q_\Phi^c \cap \bar{Q}_\Phi = \{e_1 - e_3, e_2 - e_3, 2e_1 - e_2 - e_3\}.$$

Since ξ_Φ equals one on every root of $Q_\Phi^c \cap \bar{Q}_\Phi$, a sequence satisfying (2.11) has at least three roots. We find indeed that

$$e_2 - e_3, e_2 - e_3, e_2 - e_3$$

is a sequence with the desired properties, proving that $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is fundamental and has Levi order three.

2.6 Levi order of orbits of the minimal type

Weak nondegeneracy for minimal orbits was characterized in [1, Thm.11.5] by using their description in terms of cross-marked Satake diagrams (see e.g. [5,24]).

Let $\mathfrak{h}_\mathbb{R}$ be a maximally vectorial Cartan subalgebra of \mathfrak{s}_σ , \mathfrak{h} its complexification and \mathcal{R} the root system of $(\mathfrak{s}, \mathfrak{h})$. Then all roots in \mathcal{R}_\bullet are compact. We can select a basis \mathcal{B} such that the conjugate of any positive noncompact root stays positive. This condition defines an involution $\epsilon : \mathcal{B} \rightarrow \mathcal{B}$, which keeps fixed the elements of $\mathcal{B}_\bullet = \mathcal{B} \cap \mathcal{R}_\bullet$ and such that, for nonnegative $n_{\alpha, \beta} \in \mathbb{Z}$,

$$\begin{cases} \bar{\alpha} = -\alpha, & \forall \alpha \in \mathcal{B}_\bullet, \\ \bar{\alpha} = \epsilon(\alpha) + \sum_{\beta \in \mathcal{B}} n_{\alpha, \beta} \beta, & \forall \alpha \in \mathcal{B} \setminus \mathcal{B}_\bullet. \end{cases} \tag{2.17}$$

The Satake diagram $\Sigma_\mathcal{B}$ is obtained from $\Delta_\mathcal{B}$ by painting black the roots in \mathcal{B}_\bullet and joining by an arch the pairs of distinct simple roots α_1, α_2 with $\epsilon(\alpha_1) = \alpha_2$.

Minimal orbits correspond to *cross-marked Satake diagrams*: they are associated to parabolic \mathfrak{q}_Φ for which all roots in $Q_\Phi^c \cap \bar{Q}_\Phi^n$ are compact.

Let us drop the assumption that $\mathfrak{h}_\mathbb{R}$ is maximally vectorial. The map $\theta : \alpha \mapsto -\bar{\alpha}$ induced on \mathcal{R} by the Cartan involution (see Sect. 2.2) acts on $Q_\Phi^c \cap \bar{Q}_\Phi^n$, which is therefore the union of its fixed points, which are roots in \mathcal{R}_\bullet , and of pairs $(\alpha, -\bar{\alpha})$ of distinct roots.

Definition 2.4 We say that the CR algebra $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ and the corresponding CR manifold $M_{\Phi, \sigma}$ are of the *minimal type* if the roots in $Q_\Phi^c \cap \bar{Q}_\Phi^n$ are fixed by the Cartan involution, i.e. if

$$Q_\Phi^c \cap \bar{Q}_\Phi^n \subseteq \mathcal{R}_\bullet \tag{2.18}$$

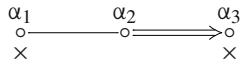
Lemma 2.15 For a parabolic CR algebra $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ the following are equivalent to the fact that it is of the minimal type:

$$\xi_\Phi(\bar{\beta}) \geq 0, \quad \forall \beta \in Q_\Phi^c \setminus \mathcal{R}_\bullet; \tag{2.19}$$

$$\xi_\Phi(\bar{\beta}) = 0, \quad \forall \beta \in (Q_\Phi^c \cap \bar{Q}_\Phi^n) \setminus \mathcal{R}_\bullet. \tag{2.20}$$

Proof (2.19) is equivalent to (2.20). Indeed, since $\xi_\Phi(\bar{\beta}) \leq 0$ for all $\beta \in \bar{Q}_\Phi$, clearly (2.19) is a consequence of (2.20). The two are equivalent because $\xi_\Phi(\bar{\beta}) > 0$ for $\beta \in \bar{Q}_\Phi^c$ and $\mathcal{R} = \bar{Q}_\Phi \cup \bar{Q}_\Phi^c$. The equivalence of (2.18) with (2.19) reduces to the observation that the elements of Q_Φ^c on which $\xi_\Phi \circ \sigma$ is negative make the set $Q_\Phi^c \cap \bar{Q}_\Phi^n$. \square

Example 2.16 Keep the notation of Example 2.12. The cross-marked Dynkin diagram of B_3



corresponds to

$$\Phi = \{\alpha_1, \alpha_3\}, \quad \xi_\Phi(e_i) = \begin{cases} 2, & i=1, \\ 1, & i=2, 3. \end{cases}$$

Consider the conjugation

$$\sigma(e_1) = e_2, \quad \sigma(e_2) = e_1, \quad \sigma(e_3) = -e_3.$$

Since $\Phi \subset \mathcal{R}_\bullet$, by Proposition 2.2 the CR algebra $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is fundamental. We have

$$\begin{aligned} Q_\Phi^c \cap \bar{Q}_\Phi^c &= \{e_1, e_2, e_1+e_2, e_1-e_3, e_2+e_3\}, \\ Q_\Phi^c \cap \bar{Q}_\Phi &= \{e_3, e_1+e_3, e_1-e_2\}, \\ Q_\Phi \cap \bar{Q}_\Phi^c &= \{-e_3, e_2-e_3, e_2-e_1\}. \end{aligned}$$

This $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is of the minimal type, because

$$Q^c = (Q_\Phi^c \cap \bar{Q}_\Phi^c) \cup \{e_1-e_2, e_3\} \subset (Q_\Phi^c \cap \bar{Q}_\Phi^c) \cup \mathcal{R}_\bullet.$$

However, $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is not the CR algebra of the minimal orbit of a real form of $\mathbf{SO}_7(\mathbb{C})$ in F_Φ , because, although $\alpha_1, \alpha_3 \in \mathcal{R}_\bullet$ and $\bar{\alpha}_2 = \alpha_1 + \alpha_2 + 2\alpha_3 > 0$, showing that the basis $\alpha_1, \alpha_2, \alpha_3$ defines an S -chamber according to [3], the diagram obtained by blackening the nodes α_1, α_3 is not Satake. The equalities

$$\begin{cases} (-e_3 + (e_1 + e_3)) = e_1 \in Q_\Phi^c \cap \bar{Q}_\Phi^c, \\ (e_2 - e_3) + e_3 = e_2 \in Q_\Phi^c \cap \bar{Q}_\Phi^c, \\ (e_2 - e_1) + (e_1 + e_3) = e_2 + e_3 \in Q_\Phi^c \cap \bar{Q}_\Phi^c, \end{cases}$$

show that $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is weakly nondegenerate.

We can choose real forms $\mathbf{SO}(2, 5)$ or $\mathbf{SO}(3, 4)$ compatible with the complex symmetric bilinear form \mathfrak{b} used to define $\mathbf{SO}_7(\mathbb{C})$. Then $M_{\Phi, \sigma}$ consists of pairs $(\ell_1 \subset \ell_3)$ with a \mathfrak{b}_σ -isotropic ℓ_1 with $\ell_1 \cap \bar{\ell}_1 = \{0\}$ and an ℓ_3 on which the restriction of \mathfrak{b}_σ has rank 1.

Theorem 2.17 *A real orbit $M_{\Phi, \sigma}$ of the minimal type is either holomorphically degenerate or has Levi order less or equal two.*

Proof Let $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ be a parabolic CR algebra of the minimal type. Keeping the notation used throughout the section, we note that (2.18) can be rewritten in the form

$$\xi_\Phi(\beta) \geq 0, \quad \forall \beta \in \bar{Q}_\Phi^c \setminus \mathcal{R}_\bullet. \tag{*}$$

Let $\beta \in Q_\Phi \cap \bar{Q}_\Phi^c$. If $\beta \notin \mathcal{R}_\bullet$, then $\xi_\Phi(\beta) = 0$ by (*) and hence, by Corollary 2.8, $q_\Phi^\sigma(\beta)$ is either 1 or $+\infty$.

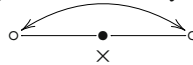
Let us consider now the case where $q_\Phi^\sigma(\beta)$ is an integer $q > 1$. Then $\beta \in \mathcal{R}_\bullet$. Let $(\alpha_1, \dots, \alpha_q)$ be a sequence satisfying (2.11) and thus the conditions in Lemma 2.6. Since $Q_\Phi^c \cap \bar{Q}_\Phi^c \cap \mathcal{R}_\bullet = \emptyset$, there is at least one root α_i which does not belong to \mathcal{R}_\bullet . By the Lemma we can assume it is α_1 . Then $\beta + \alpha_1$ belongs to $(Q_\Phi \cap \bar{Q}_\Phi^c) \setminus \mathcal{R}_\bullet$ and therefore, by the first part of the proof, $q_\Phi^\sigma(\beta + \alpha_1) = 1$. This implies that $q = 2$. The proof is complete. \square

Lemma 2.18 *The parabolic CR algebra $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ of a minimal orbit is of the minimal type.*

Proof Suppose that Φ is the set of crossed roots in a cross-marked Satake diagram. Since all roots β in Q_Φ^c are positive, by (2.17), if $\beta \in Q_\Phi^c \setminus \mathcal{R}_\bullet$, then its conjugate $\bar{\beta}$ is positive, and hence has $\xi_\Phi(\beta) \geq 0$. This shows that (2.18) is valid, i.e. that $M_{\Phi, \sigma}$ is of the minimal type. \square

Corollary 2.19 *A minimal orbit $M_{\Phi, \sigma}$ is either holomorphically degenerate or has Levi order less or equal to two.* \square

Example 2.20 Consider the CR algebra described by the cross-marked Satake diagram



It is associated to the minimal orbit $M_{\Phi, \sigma}$ of $\mathbf{SU}(1, 3)$ is the Grassmannian of isotropic two-planes of \mathbb{C}^4 for a hermitian symmetric form of signature $(1, 3)$.

Here $\mathfrak{s} \simeq \mathfrak{sl}_4(\mathbb{C})$, $\mathcal{R} = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq 4\}$, $\mathcal{B} = \{e_1 - e_2, e_2 - e_3, e_3 - e_4\}$ for an orthonormal basis e_1, e_2, e_3, e_4 of \mathbb{R}^4 , $\Phi = \{e_2 - e_3\}$,

$$\xi(e_i) = \begin{cases} 1, & i=1, 2, \\ 0, & i=3, 4, \end{cases} \quad \begin{cases} \sigma(e_1) = -e_4, & \sigma(e_2) = -e_2, \\ \sigma(e_3) = -e_3, & \sigma(e_4) = -e_1. \end{cases}$$

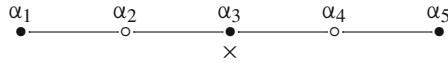
We obtain

$$\begin{aligned} Q_\Phi \cap \bar{Q}_\Phi^c &= \{e_1 - e_4\}, \\ \bar{Q}_\Phi \cap Q_\Phi^c &= \{e_1 - e_3, e_2 - e_3, e_2 - e_4\}, \\ Q_\Phi \cap \bar{Q}_\Phi^c &= \{e_3 - e_4, e_3 - e_2, e_1 - e_2\}. \end{aligned}$$

Since $Q_\Phi^c \cap \bar{Q}_\Phi^c$ is nonempty, $e_1 - e_4 = (e_3 - e_4) + (e_1 - e_3)$ and \mathfrak{q}_Φ is maximal, we obtain that $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is fundamental and weakly nondegenerate. Since $\xi_\Phi(e_3 - e_2) = -1$ and ξ_Φ is 1 on all the elements of $\bar{Q}_\Phi \cap Q_\Phi^c$, the Levi order is at least, and thus equal, by Theorem 2.17, to 2. We have in fact

$$e_1 - e_4 = (e_3 - e_2) + (e_1 - e_3) + (e_2 - e_4), \quad e_1 - e_4 = (e_1 - e_2) + (e_2 - e_4).$$

Example 2.21 Consider the CR algebra described by the cross-marked Satake diagram



Here $\mathcal{R} = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq 6\}$, $\alpha_i = e_i - e_{i+1}$, $\Phi = \{\alpha_3\}$,

$$\xi_{\Phi}(e_i) = \begin{cases} 1, & \text{for } 1 \leq i \leq 3, \\ 0, & \text{for } 4 \leq i \leq 6, \end{cases} \quad \sigma(e_i) = \begin{cases} e_{i+1}, & \text{if } i \text{ is odd,} \\ e_{i-1}, & \text{if } i \text{ is even.} \end{cases}$$

It corresponds to the CR algebra $(\mathfrak{s}_{\sigma}, \mathfrak{q}_{\Phi})$, with $\mathfrak{s}_{\sigma} \simeq \mathfrak{sl}_3(\mathbb{H})$, of the grassmannian $M_{\Phi, \sigma}$ of 3-planes of $\mathbb{C}^6 \simeq \mathbb{H}^3$ containing a quaternionic line. We have

$$\begin{aligned} Q_{\Phi}^c \cap \bar{Q}_{\Phi}^c &= \{e_1 - e_5, e_1 - e_6, e_2 - e_5, e_2 - e_6\}, \\ Q_{\Phi}^c \cap \bar{Q}_{\Phi} &= \{e_1 - e_4, e_2 - e_4, e_3 - e_4, e_3 - e_5, e_3 - e_6\}, \\ Q_{\Phi} \cap \bar{Q}_{\Phi}^c &= \{e_2 - e_3, e_1 - e_3, e_4 - e_3, e_4 - e_6, e_4 - e_5\} \end{aligned}$$

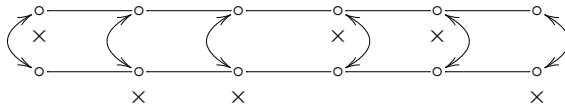
Since \mathfrak{q}_{Φ} is maximal, it suffices to note that $(e_4 - e_5) + (e_1 - e_4) = e_1 - e_5 \in Q_{\Phi}^c \cap \bar{Q}_{\Phi}^c$ to conclude that $(\mathfrak{s}_{\sigma}, \mathfrak{q}_{\Phi})$ is fundamental and weakly nondegenerate.

We have $Q_{\Phi} \cap \bar{Q}_{\Phi}^c \cap \mathcal{R}_{\bullet} = \{e_4 - e_3\}$. Since both

$$(e_4 - e_3) + (e_1 - e_4) = e_1 - e_3 \in Q_{\Phi} \cap \bar{Q}_{\Phi}^c, \quad (e_4 - e_3) + (e_2 - e_4) = e_2 - e_3 \in Q_{\Phi} \cap \bar{Q}_{\Phi}^c$$

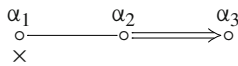
we get $q_{\Phi}^{\sigma}(e_4 - e_3) = 2$, showing that the Levi order of $(\mathfrak{s}_{\sigma}, \mathfrak{q}_{\Phi})$ equals two.

Example 2.22 The CR algebra described by the cross-marked Satake diagram



corresponding to $(\mathfrak{s}_{\sigma}, \mathfrak{q}_{\Phi})$, with $\mathfrak{s}_{\sigma} \simeq \mathfrak{sl}_7(\mathbb{C})$, is fundamental and weakly nondegenerate. This can be proved e.g. by applying the criteria in [1]. Since $\mathcal{R}_{\bullet} = \emptyset$, its Levi order is one.

Example 2.23 Keep the notation of Example 2.12 and consider the cross-marked Dynkin diagram of B_3



corresponding to

$$\Phi = \{\alpha_1\}, \quad \xi_{\Phi}(e_i) = \begin{cases} 1, & i=1, \\ 0, & i=2, 3. \end{cases}$$

Consider the conjugation

$$\sigma(e_1) = -e_2, \quad \sigma(e_2) = -e_1, \quad \sigma(e_3) = e_3.$$

Then, for the corresponding CR algebra $(\mathfrak{s}_{\sigma}, \mathfrak{q}_{\Phi})$, we have

$$\begin{aligned} Q_{\Phi}^c \cap \bar{Q}_{\Phi}^c &= \{e_1 - e_2\}, \\ Q_{\Phi}^c \cap \bar{Q}_{\Phi} &= \{e_1, e_1 + e_2, e_1 + e_3, e_1 - e_3\}, \end{aligned}$$

$$Q_\Phi \cap \bar{Q}_\Phi^c = \{-e_2, -e_1 - e_2, -e_2 + e_3, -e_2 - e_3\}.$$

Since $\xi_\Phi(\gamma) = 1$ for all $\gamma \in Q_\Phi^c \cap \bar{Q}_\Phi$, and $\xi_\Phi(\gamma) \geq -1$ for all $\gamma \in Q_\Phi \cap \bar{Q}_\Phi^c$, the Levi order of $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is two. Then $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is a CR algebra of the minimal type, although it is not the CR algebra of a minimal orbit.

Remark 2.24 It was observed in [3] that a parabolic CR algebra $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ can always be described by using a base \mathcal{B} associated to an *S-chamber*: this means one with $\bar{\alpha} > 0$ for all $\alpha \in \mathcal{B} \setminus (\Phi \cup \mathcal{B}_\bullet)$. The condition of being of the *minimal type* translates for this choice of \mathcal{B} into the fact that $\bar{\alpha} > 0$ also for the elements in $\Phi \setminus \mathcal{B}_\bullet$. The real dimension of $M_{\Phi, \sigma}$ is the difference $\dim_{\mathbb{C}}(\mathfrak{s}) - \dim_{\mathbb{C}}(\mathfrak{q}_\Phi \cap \bar{\mathfrak{q}}_\Phi)$, i.e. $\#\mathcal{R} - \#(Q_\Phi \cap \bar{Q}_\Phi^c)$. Thus, in case Φ contains a root $\alpha \notin \mathcal{B}_\bullet$ with $\bar{\alpha} < 0$, the symmetry with respect to α yields a new basis \mathcal{B}' that, with the crosses in the same positions, describes a new real orbit whose dimension is smaller by one unit. Then, parametrizing the real orbits that we can describe, after having made a fixed choice of $\mathfrak{h}_{\mathbb{R}}$, by using the Weyl chambers of \mathcal{R} , those of the minimal type are a sort of *local minima* with respect to dimension. One has to be cautious because, unless $\mathfrak{h}_{\mathbb{R}}$ is maximally vectorial, there can be several inequivalent choices of \mathcal{B} such that $\bar{\alpha} > 0$ for all $\alpha \in \mathcal{B} \setminus \mathcal{B}_\bullet$ that we can look at as yielding different *local minima* for the dimension of a class of real orbits.

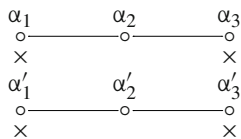
2.7 Further examples

We already showed that there are weakly nondegenerate CR algebras $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ of Levi order 3, which, by Theorem 2.17, are not of the minimal type. In this subsection we exhibit examples of weakly nondegenerate parabolic CR algebras which are not of the minimal type and have Levi orders 1, 2.

Example 2.25 Consider $\mathfrak{sl}_3(\mathbb{C})$ as a simple real Lie algebra. Its complexification is the direct sum of two copies of $\mathfrak{sl}_3(\mathbb{C})$. Its root system can be described, after fixing orthogonal basis e_1, e_2, e_3, e_4 and e'_1, e'_2, e'_3, e'_4 of two copies of \mathbb{R}^4 , by

$$\mathcal{R} = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq 4\} \cup \{\pm(e'_i - e'_j) \mid 1 \leq i < j \leq 4\}.$$

Let us consider the cross-marked Dynkin diagram



where $\alpha_i = (e_i - e_{i+1})$ and $\alpha'_i = (e'_i - e'_{i+1})$, with $\Phi = \{\alpha_1, \alpha_3\} \cup \{\alpha'_1, \alpha'_3\}$.

Let us fix the conjugation

$$\sigma(e_i) = \begin{cases} e'_{i+1}, & i = 1, 3, \\ e'_{i-1}, & i = 2, 4, \end{cases} \quad \sigma(e'_i) = \begin{cases} e_{i+1}, & i = 1, 3, \\ e_{i-1}, & i = 2, 4, \end{cases}$$

Then

$$\begin{aligned}
 Q_\Phi^c \cap \bar{Q}_\Phi^c &= \{e_1 - e_3, e_2 - e_4\} \cup \{e'_1 - e'_3, e'_2 - e'_4\}, \\
 Q_\Phi^c \cap \bar{Q}_\Phi &= \{e_1 - e_2, e_1 - e_4, e_3 - e_4\} \cup \{e'_1 - e'_2, e'_1 - e'_4, e'_3 - e'_4\}, \\
 Q_\Phi \cap \bar{Q}_\Phi^c &= \{e_2 - e_1, e_2 - e_3, e_4 - e_3\} \cup \{e'_2 - e'_1, e'_2 - e'_3, e'_4 - e'_3\}.
 \end{aligned}$$

Since all roots in Φ have a negative conjugate, the parabolic CR algebra $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is fundamental. It is not of the minimal type because $\mathcal{R}_\bullet = \emptyset$ and

$$Q_\Phi^n \cap \bar{Q}_\Phi^c = \{e_2 - e_1, e_4 - e_3\} \cup \{e'_2 - e'_1, e'_4 - e'_3\} \neq \emptyset.$$

Let us check that $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ has Levi order 1. We get indeed

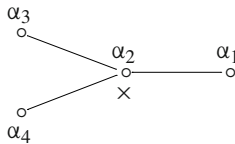
$$\begin{aligned} (e_2 - e_1) + (e_1 - e_4) &= (e_2 - e_4), & (e'_2 - e'_1) + (e'_1 - e'_4) &= (e'_2 - e'_4), \\ (e_2 - e_3) + (e_3 - e_4) &= (e_2 - e_4), & (e'_2 - e'_3) + (e'_3 - e'_4) &= (e'_2 - e'_4), \\ (e_2 - e_3) + (e_1 - e_2) &= (e_1 - e_3), & (e'_2 - e'_3) + (e'_1 - e'_2) &= (e'_1 - e'_3), \\ (e_4 - e_3) + (e_1 - e_4) &= (e_1 - e_3), & (e'_4 - e'_3) + (e'_1 - e'_3) &= (e'_1 - e'_3). \end{aligned}$$

This also shows that $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is weakly nondegenerate. The orbit $M_{\Phi, \sigma}$ is a CR manifold of CR dimension 6 and CR codimension 4. Its points are quadruples $(\ell_1, \ell'_1, \ell_3, \ell'_3)$ of linear subspaces of a $\mathbb{C}^4 \simeq \mathbb{H}^2$ with ℓ_1, ℓ'_1 complex lines such that $\ell_1 + \ell'_1$ is a quaternionic line and ℓ_3, ℓ'_3 complex hypersurfaces with $\ell_3 \cap \ell'_3 = \ell_1 + \ell'_1$.

Example 2.26 Consider a root system

$$\mathcal{R} = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 4\}$$

of type D_4 and the maximal parabolic \mathfrak{q}_Φ described by the cross-marked Dynkin diagram



Here $\alpha_i = e_i - e_{i+1}$, for $1 \leq i \leq 3$ and $\alpha_4 = e_3 + e_4$, $\Phi = \{\alpha_2\}$,

$$\xi_\Phi(e_i) = \begin{cases} 1, & i = 1, 2, \\ 0, & i = 3, 4. \end{cases}$$

With the conjugation

$$\sigma(e_1) = e_4, \sigma(e_2) = -e_3, \sigma(e_3) = -e_2, \sigma(e_4) = e_1,$$

we obtain

$$\begin{aligned} Q_\Phi^c \cap \bar{Q}_\Phi^c &= \{e_1 + e_4, e_2 + e_4, e_1 - e_3, e_2 - e_3\} \\ Q_\Phi^c \cap \bar{Q}_\Phi &= \{e_1 + e_2, e_1 + e_3, e_2 + e_3, e_1 - e_4, e_2 - e_4\} \\ Q_\Phi \cap \bar{Q}_\Phi^c &= \{e_4 - e_3, e_4 - e_2, -e_2 - e_3, e_4 - e_1, -e_3 - e_1\}, \\ (Q_\Phi^n \cap \bar{Q}_\Phi^c) \setminus \mathcal{R}_\bullet &= \{e_4 - e_2, -e_3 - e_1\}. \end{aligned}$$

It is easy to check, using the fact that \mathfrak{q}_Φ is maximal, that $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is fundamental and weakly nondegenerate; moreover the last line of the equalities above shows that $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is not of the minimal type. To check that $(\mathfrak{s}_\sigma, \mathfrak{q}_\Phi)$ is Levi nondegenerate (i.e. has Levi order 1) we observe that

$$\begin{aligned} (e_4 - e_3) + (e_1 - e_4) &= (e_1 - e_3), \\ (e_4 - e_2) + (e_1 + e_2) &= (e_1 + e_4), \\ (-e_2 - e_3) + (e_1 + e_2) &= (e_1 - e_3), \end{aligned}$$

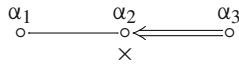
$$(e_4 - e_1) + (e_1 + e_2) = (e_2 + e_4),$$

$$(-e_1 - e_3) + (e_1 + e_2) = (e_2 - e_3).$$

Example 2.27 Consider a root system

$$\mathcal{R} = \{\pm(e_i + e_j) \mid 1 \leq i \leq j \leq 3\} \cup \{\pm(e_i - e_j) \mid 1 \leq i < j \leq 3\}$$

of type C_3 and the cross-marked Dynkin diagram



with $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, $\alpha_3 = 2e_3$ and $\Phi = \{\alpha_2\}$ so that

$$\xi_{\Phi}(e_i) = \begin{cases} 1, & i = 1, 2, \\ 0, & i = 3. \end{cases}$$

Consider the conjugation

$$\sigma(e_1) = e_3, \quad \sigma(e_2) = -e_2, \quad \sigma(e_3) = e_1.$$

We obtain

$$\begin{aligned} \mathcal{Q}_{\Phi}^c \cap \bar{\mathcal{Q}}_{\Phi}^c &= \{e_1 + e_3\}, \\ \mathcal{Q}_{\Phi}^c \cap \bar{\mathcal{Q}}_{\Phi} &= \{2e_1, 2e_2, e_1 + e_2, e_2 + e_3, e_1 - e_3, e_2 - e_3\} \\ \mathcal{Q}_{\Phi} \cap \bar{\mathcal{Q}}_{\Phi}^c &= \{2e_3, -2e_2, e_3 - e_2, e_1 - e_2, e_3 - e_1, -e_1 - e_2\}. \end{aligned}$$

Since the parabolic \mathfrak{q}_{Φ} is maximal, it is easy to check that $(\mathfrak{s}_{\sigma}, \mathfrak{q}_{\Phi})$ is fundamental and weakly nondegenerate. It has Levi order two, as one can check from

$$\begin{aligned} 2e_3 + (e_1 - e_3) &= (e_1 + e_3), & -2e_2 + (e_1 + e_2) + (e_2 + e_3) &= (e_1 + e_3) \\ (e_3 - e_2) + (e_1 + e_2) &= (e_1 + e_3), & (e_1 - e_2) + (e_2 + e_3) &= (e_1 + e_3), \\ (e_3 - e_1) + 2e_1 &= (e_1 + e_3), & (-e_1 - e_2) + 2e_1 + (e_2 + e_3) &= (e_1 + e_3). \end{aligned}$$

3 Weakly nondegenerate CR manifolds with larger Levi orders

Fix any integer $q \geq 1$. In this last section we discuss in detail the example of a homogeneous CR manifold M of CR dimension $q+1$ and CR codimension 1 which is fundamental and weakly nondegenerate of Levi order q .

The compact group $SU(2)$ acts transitively on the complex projective line $\mathbb{C}P^1$. The homogeneous complex structure of $\mathbb{C}P^1$ can be defined by the totally complex CR algebra $(\mathfrak{su}(2), \mathfrak{b})$, where $\mathfrak{su}(2)$ is the real Lie algebra of anti-Hermitian 2×2 matrices and \mathfrak{b} a Borel subalgebra of its complexification $\mathfrak{sl}_2(\mathbb{C})$. This CR algebra corresponds to the simple cross-marked Satake diagram



The root system of the complexification $\mathfrak{sl}_2(\mathbb{C})$ is $\mathcal{R} = \{\pm(e_1 - e_2)\}$, and we take $\alpha = (e_1 - e_2)$, with fundamental weight $\omega = \alpha/2$.

With our usual notation, $\Phi = \{\alpha\}$, so that $\mathfrak{b} = \mathfrak{q}_\Phi$; moreover $\xi_\Phi(e_i) = (-1)^{i+1}/2$ and $\mathfrak{s}_\sigma = \mathfrak{su}_2$, with conjugation $\sigma(e_1) = e_2, \sigma(e_2) = e_1$.

The irreducible finite dimensional complex linear representations of $\mathfrak{sl}_2(\mathbb{C})$ are indexed by the nonnegative integral multiples $k \cdot \omega$ of ω and the corresponding irreducible $\mathfrak{sl}_2(\mathbb{C})$ -module $V_{k \cdot \omega}$ can be identified with the space of complex homogeneous polynomials of degree k in two indeterminates

$$V_{k \cdot \omega} = \left\{ \sum_{h=0}^k a_h z^h w^{k-h} \mid a_h \in \mathbb{C} \right\}.$$

We have

$$V_{k \cdot \omega} = \bigoplus_{h=0}^k V_{k \cdot \omega}^{(k-2h) \cdot \omega},$$

where, for a diagonal H in the canonical Cartan subalgebra of $\mathfrak{sl}_2(\mathbb{C})$,

$$V_{k \cdot \omega}^{(k-2h) \cdot \omega} = \{v \in V_{k \cdot \omega} \mid H \cdot v = (k-2h) \omega(H)v\} = \{a \cdot z^h w^{k-h} \mid a \in \mathbb{C}\}, \quad 0 \leq h \leq k,$$

are the one-dimensional weight spaces contained in $V_{k \cdot \omega}$.

Since $\bar{\omega} = -\omega$, we have $\bar{V}_{k \cdot \omega} = V_{k \cdot \omega}$. The anti- \mathbb{C} -linear automorphism $\theta_{k \cdot \omega}$ of $V_{k \cdot \omega}$ defined by the conjugation σ comes from $(z, w) \mapsto (-\bar{w}, \bar{z})$ and therefore

$$\theta \left(\sum_{h=0}^k a_h z^h w^{k-h} \right) = \sum_{h=0}^k (-1)^h \bar{a}_h w^h z^{k-h}$$

Then $\theta_{k \cdot \omega}^2$ equals $\text{id}_{V_{k \cdot \omega}}$ for k even and $-\text{id}_{V_{k \cdot \omega}}$ for k odd. Accordingly, for k even $V_{k \cdot \omega}$ is the complexification of an irreducible $(k+1)$ -dimensional representation of the real type, that we will denote by $V_{k \cdot \omega}^{\mathbb{R}}$; for k odd is isomorphic to a $2(k+1)$ -dimensional irreducible representation of the quaternionic type of \mathfrak{su}_2 (see e.g. [7, Ch.IX, App.II, Prop.2]).

Remark 3.1 Studying irreducible representation of \mathfrak{su}_2 turns out to be of some interest in quantum physics, as they arise when considering rotations on fermionic and bosonic systems (for more details see [26, Ch.5, §5]).

The subspace

$$V_{k \cdot \omega}^- = \bigoplus_{k < 2h \leq 2k} V_{k \cdot \omega}^{(k-2h) \cdot \omega}$$

is a \mathfrak{b} -submodule of $V_{k \cdot \omega}$ and we can consider the semidirect sum $\mathfrak{b} \oplus V_{k \cdot \omega}^-$ as a subalgebra of the abelian extension $\mathfrak{sl}_2(\mathbb{C}) \oplus V_{k \cdot \omega}$ (cf. e.g. [23, Ch.VII, §3]). We may consider the map $\mathbf{SL}_2(\mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^1$ associated to our choice of a Borel subalgebra \mathbf{B} as a principal bundle with structure group \mathbf{B} . Then the Lie pair $(\mathfrak{sl}_2(\mathbb{C}) \oplus V_{k \cdot \omega}, \mathfrak{b} \oplus V_{k \cdot \omega}^-)$ defines a complex holomorphic vector bundle E_k with base $\mathbb{C}\mathbb{P}^1$ and typical fiber $V_{k \cdot \omega} / V_{k \cdot \omega}^- \simeq \bigoplus_{2h \leq k} V_{k \cdot \omega}^{(k-2h) \cdot \omega}$ (this is an example of Mostow fibration, see e.g. [20] and the bibliography there in).

Proposition 3.2 *Let q be any positive integer. Then*

$$(\mathfrak{g}_{\mathbb{R}}, \mathfrak{q}'_\Phi) = (\mathfrak{su}_2 \oplus V_{2q \cdot \omega}^{\mathbb{R}}, \mathfrak{b} \oplus V_{2q \cdot \omega}^-) \tag{3.1}$$

is the CR algebra of a CR manifold E_{2q} , of CR dimension $q+1$ and CR codimension 1, which is fundamental and weakly nondegenerate of Levi order q .

Proof We have

$$\bar{V}_{2q\omega}^- = V_{2q\omega}^+ = \bigoplus_{h=1}^q V_{2q\omega}^{2h\omega} \quad \text{and} \quad V_{2q\omega} = V_{2q\omega}^- \oplus V_{2q\omega}^0 \oplus V_{2q\omega}^+.$$

If $Z_\alpha, Z_{-\alpha}, H$ is the canonical basis of $\mathfrak{sl}_2(\mathbb{C})$ and w a nonzero vector of $V_{2q\omega}^{-2q\omega}$, then the images of $X_{-\alpha}, w, X_\alpha w, \dots, X_\alpha^{q-1} w$ generate $\mathfrak{q}'_\Phi / (\mathfrak{q}'_\Phi \cap \bar{\mathfrak{q}}'_\Phi)$. Since

$$\underbrace{[X_\alpha, \dots, X_\alpha, X_\alpha^{q-h} w]}_{h \text{ times}} = X_\alpha^q w \in V_{2q\omega}^0 \setminus \{0\}, \quad [X_\alpha^{q+1} w, X_{-\alpha}] = -2X_\alpha^q w \in V_{2q\omega}^0 \setminus \{0\}$$

we obtain that E_{2q} is fundamental and weakly nondegenerate. With the notation of the previous section, we have $\mathcal{Q}_\Phi^c \cap \bar{\mathcal{Q}}_\Phi = \{\alpha\}$, with $\xi_\Phi(\alpha) = 1$ and $\xi_\Phi(-2j\omega) = -j$. Since $\mathfrak{g}/(\mathfrak{q}'_\Phi + \bar{\mathfrak{q}}'_\Phi)$ is generated by the image of $V_{2q\omega}^0$, by the above considerations the Levi order of an element of $V_{2q\omega}^{-2j\omega}$ equals j . This shows that the Levi order of $(\mathfrak{g}_\mathbb{R}, \mathfrak{q}'_\Phi)$ is q . \square

In an analogous way we can also prove

Proposition 3.3 *For each positive integer q , the homogeneous CR manifold E_{2q} is contact nondegenerate of order q .* \square

Remark 3.4 Representations $V_{k\omega}$ with an odd k are canonically associated with complex holomorphic vector bundles E_k , of rank $(k+1)$, with base $\mathbb{C}\mathbb{P}^1$.

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