On transitive contact and *CR* algebras

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Abstract. We consider locally homogeneous *CR* manifolds and show that, under a condition only depending on their underlying contact structure, their *CR* automorphisms form a finite dimensional Lie group.

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1. Introduction

In the past years some of the Authors introduced and investigated the notion of *CR* algebra (see [2, 14]) to describe the local structure of homogeneous *CR* manifolds. The understanding of local models is important, *e.g.*, for applying the method of E. Cartan to describe the differential invariants of *CR* structures. A key point is to find under which conditions the infinitesimal automorphisms of the structure form a finite dimensional Lie algebra. A *CR* structure can be defined by the datum of a smooth involutive *complex* distribution. The real parts of its vectors define a *real* distribution. A strong version of the condition that the *CR* manifold is not foliated by *CR* submanifolds of lower dimension is that this real distribution is a (generalised) *contact* distribution, *i.e.* that its iterated commutators span the full tangent space. The strong interplay between *CR* and underlying contact structures was clearly exploited in the work of N. Tanaka (see [23, 24]). He considered, at each point, the nilpotent $\mathbb{Z}$-graded real Lie algebra $\mathfrak{m}$ canonically associated to a contact structure. To describe the infinitesimal automorphisms, one needs to consider *extensions*, or prolongations $\mathfrak{G}$ of $\mathfrak{m}$. In this setting, they can be described recursively in terms of derivations of $\mathfrak{m}$. When a *CR* structure is imposed on the contact distribuiton, finite dimensionality of the maximal prolongation is, for this $\mathbb{Z}$-graded model, equivalent to the fact that the vector valued Levi form has trivial kernel. Thus Cartan's method applies, via Tanaka's theory, to the case where the associated

$\mathbb{Z}$-graded Levi-Tanaka algebras (*cf*. [13]) are isomorphic at all points and the Levi form is nondegenerate.

The idea of introducing *CR* algebras in [14] originated from the observation that many interesting homogeneous examples of *CR* manifolds lead to infinite dimensional Levi-Tanaka algebras, their Levi forms having nontrivial kernels. An obvious generalization of the nondegeneracy condition is to require that the iterated Levi forms have a trivial kernel. This condition, that was called *weak nondegeneracy* in [14] and is equivalent to the notion of *(Levi) k-nondegeneracy* used by other authors (see, *e.g*., [5]), is indeed equivalent to the fact that the corresponding *CR* manifold is not the total space of a *CR* fibration with complex fibres. The differential invariants for *CR* manifolds of hypersurface type in real dimension 5 satisfying the notion of k-nondegenericity have been so far studied by several authors with different techniques (see, *e.g*., [9, 12, 15, 16, 18]). Further developments in higher dimensions appeared in [19, 21]. A theory of invariants for 2-nondegenerate *CR* hypersurfaces in arbitrary dimension, modeled on Tanaka's approach, has been recently developed by Porter and Zelenko in [20].

In this paper we address the question on finite dimensionality of the full group of the automorphisms of *CR* manifolds of arbitrary *CR* dimension and *CR* codimension, whose *CR* structure is locally homogeneous. Weak nondegeneracy is a much more restrictive condition that the one we found to guarantee the finite dimensionality of the Lie algebra of infinitesimal *CR* automorphisms. In fact, our criterion only involves the underlying contact structure. We found this fact very interesting. Indeed, it is preliminary to an approach where this (generalized) contact structure is a priori given as a characteristic of the manifold on which the addition of a *CR* structure is meant to modelling different geometrical or physical situations. Our condition was called *ideal nondegeneracy* in [14], where the fact that it was a sufficient criterion for the finite dimensionality of the maximal extension was correctly stated; however, in the proof given there there was a gap that we fill here in Section 7.

Our proof of the existence of maximal extensions of *CR* algebras relies on a review of the classical work on transitive geometry (see, *e.g*., [6,7,10,11]), allowing us to substitute formal power series to the canonical construction of Tanaka in the $\mathbb{Z}$-graded case. Our discussion is restrained mostly at a purely algebraic level. Thus, for a better understanding of the geometrical significance of our results, we refer the reader to [4] for a thorough introduction to *CR* and homogeneous *CR* manifolds.

Let us briefly describe the contents of the paper.

Section 2 collects some general notions we thought relevant for the exposition. Contact and transitive pairs and triples and *CR* algebras are defined, not restraining to finite dimensionality. We explicitly required that the (possibly infinite dimensional) Lie algebras involved have a topological structure, although this structure is implicitly defined by the requirement that the isotropy subalgebra is closed and has finite codimension. We also list various nondegeneracy conditions that will be investigated in the later sections.

In Section 3 we construct a canonical descending chain of subspaces which is associated to a contact pair to explain *contact nondegeneracy*.

An analogous construction in Section 4, characteristic of *CR* algebras, describes weak *CR*-nondegeneracy. We show by an example that it is in fact a much more restrictive condition than nondegeneracy of the underlying contact structure.

We found convenient to explain in Section 5 the way the abstract contact triples of Section 2 relate to actual homogeneous contact manifolds, to motivate our later use of transitive contact geometry.

In Section 6 we introduce graded Lie algebras and the finiteness criterion of Noboru Tanaka.

By using Tanaka's criterion, we prove in Section 7 our main result, which states that *CR* algebras whose corresponding contact triple is nondegenerate are finite dimensional.

In Section 8 we deal with the general construction of the representation of transitive contact pairs by structures involving vector fields with formal power series coefficients. This is the main tool in the *transitive geometry* of [11]: in this purely algebraic setting a germ of homogeneous space is substituted by a topological Lie algebra in which the isotropy subalgebra is closed and has finite codimension. This describes a situation in which the values of the infinitesimal automorphisms of the structure span the full tangent space at a point.

In the final Section 9 we utilize transitive geometry to construct maximal extensions of *CR* algebras. Then the result of Section 7 yields the theorem that locally homogeneous *CR* manifolds with a nondegenerate underlying contact structure have a finite dimensional Lie algebra of infinitesimal *CR* automorphisms and hence, in particular, their *CR* automorphisms make a finite dimensional Lie group.

2. Definitions and preliminaries

In this section we introduce some notions which are relevant for an *infinitesimal* description of homogeneous (generalised) contact manifolds and of various geometrical structures that can be defined on them. We are particularly interested in partial complex structures, and *CR* algebras (see [14]) fall in this realm.

A *topological Lie algebra* is a Lie algebra $\mathfrak{g}_0$ over a topological field $\Bbbk$ with a fixed structure of topological Hausdorff vector space for which the Lie product is continuous. We say that $\mathfrak{g}_0$ is *linearly compact* if the intersection of any family of affine subspaces of $\mathfrak{g}_0$ having the finite intersection property has a nonempty intersection. (For details, see, *e.g.*, [7]).

In the following we assume that $\mathfrak{g}_0$ is real and denote by $\mathfrak{g}$ its complexification. Conjugation in $\mathfrak{g}$ is always understood with respect to the real form $\mathfrak{g}_0$. For a $\mathbb{C}$-linear subspace $\mathcal{L}$ of $\mathfrak{g}$ we set

$$\tilde{\mathcal{L}}_0 = \{\operatorname{Re}(Z) \mid Z \in \mathcal{L}\}, \qquad \check{\mathcal{L}}_0 = \mathcal{L} \cap \bar{\mathcal{L}} \cap \mathfrak{g}_0. \tag{2.1}$$

Definition 2.1.

- A *contact pair* is the pair $(\mathfrak{g}_0, \mathcal{L}_0)$ consisting of a linearly compact topological real Lie algebra $\mathfrak{g}_0$ and a closed linear subspace $\mathcal{L}_0$ of $\mathfrak{g}_0$ having a finite dimensional complement in $\mathfrak{g}_0$ and spanning $\mathfrak{g}_0$ as a Lie algebra;

- A *contact* $\mathbb{C}$-*pair* is the pair $(\mathfrak{g}_0, \mathcal{L})$ consisting of a linearly compact topological real Lie algebra $\mathfrak{g}_0$ and a closed complex linear subspace $\mathcal{L}$ of $\mathfrak{g}$ such that $(\mathfrak{g}_0, \check{\mathcal{L}}_0)$ is a contact pair;
- A *transitive pair* $(\mathfrak{g}_0, \mathfrak{h}_0)$ consists of a linearly compact topological Lie algebra $\mathfrak{g}_0$ and a closed subalgebra $\mathfrak{h}_0$ having finite codimension in $\mathfrak{g}_0$ and not containing nontrivial ideals of $\mathfrak{g}_0$;
- A *contact triple* is a triple $(\mathfrak{g}_0, \mathfrak{h}_0, \mathcal{L}_0)$ such that $(\mathfrak{g}_0, \mathcal{L}_0)$ is a contact pair, $(\mathfrak{g}_0, \mathfrak{h}_0)$ a transitive pair, $\mathfrak{h}_0 \subseteq \mathcal{L}_0$ and $[\mathfrak{h}_0, \mathcal{L}_0] \subseteq \mathcal{L}_0$;
- A *contact* $\mathbb{C}$-*triple* is a triple $(\mathfrak{g}_0, \mathfrak{h}_0, \mathcal{L})$, such that $(\mathfrak{g}_0, \mathfrak{h}_0)$ is transitive, $(\mathfrak{g}_0, \mathcal{L})$ is a contact $\mathbb{C}$-pair, $\mathfrak{h}_0 \subset \mathcal{L}$ and $[\mathfrak{h}_0, \mathcal{L}] \subseteq \mathcal{L}$;
- A *fundamental almost CR pair* is a contact $\mathbb{C}$-pair $(\mathfrak{g}_0, \mathcal{L})$ for which $\check{\mathcal{L}}_0$ is a Lie subalgebra of $\mathfrak{g}_0$ and $[\check{\mathcal{L}}_0, \mathcal{L}] \subseteq \mathcal{L}$. The contact triple $(\mathfrak{g}_0, \check{\mathcal{L}}_0, \tilde{\mathcal{L}}_0)$ is said to be *associated* to $(\mathfrak{g}_0, \mathcal{L})$;
- A *fundamental CR algebra* is an almost *CR* pair $(\mathfrak{g}_0, \mathfrak{q})$ such that $\mathfrak{q}$ is a complex Lie subalgebra of $\mathfrak{g}$.

Remark 2.2. In the definition of a *CR* algebra $(\mathfrak{g}_0, \mathfrak{q})$ of [14] it was not required that $(\mathfrak{q} + \bar{\mathfrak{q}}) \cap \mathfrak{g}_0$ generates $\mathfrak{g}_0$ as a Lie algebra. When this is not the case and $(\mathfrak{g}_0, \mathfrak{q}_0)$ is a Lie algebra associated to a homogeneous *CR* manifold M_0, then the manifold M_0 can be described, at least locally, as the product $M_0' \times N_0$ of a *CR* manifolds M_0' having the same *CR* dimension of M_0 and a totally real N_0. For many purposes we could reduce to M_0', which is a homogeneous *CR* manifold whose *CR* algebra at a point has the same $\mathfrak{q}$, while $\mathfrak{g}_0$ is substituted by the span of $\tilde{\mathfrak{q}}_0$.

We will use the following notions.

Definition 2.3 (Nondegeneracy conditions). We say that a contact triple $(\mathfrak{g}_0, \mathfrak{h}_0, \mathcal{L}_0)$ is

- *Strictly nondegenerate* if $\{X \in \mathcal{L}_0 \mid [X, \mathcal{L}_0] \subseteq \mathcal{L}_0\} = \mathfrak{h}_0$;
- *Nondegenerate* if any ideal of $\mathfrak{g}_0$ which is contained in $\mathcal{L}_0$ is already contained in $\mathfrak{h}_0$.

A fundamental almost *CR*-pair $(\mathfrak{g}_0, \mathcal{L})$ is

- *Strictly nondegenerate* if $\{Z \in \bar{\mathcal{L}} \mid [Z, \mathcal{L}] \subset \mathcal{L} + \bar{\mathcal{L}}\} = \mathcal{L} \cap \bar{\mathcal{L}}$;
- *Weakly non-degenerate* if there is no almost *CR* pair $(\mathfrak{g}_0, \mathcal{L}')$ with $\mathcal{L} \subsetneqq \mathcal{L}' \subseteq \mathcal{L} + \bar{\mathcal{L}}$;
- *Contact nondegenerate* if the associated contact triple $(\mathfrak{g}_0, \check{\mathcal{L}}_0, \tilde{\mathcal{L}}_0)$ is nondegenerate.

Remark 2.4. For a fundamental almost *CR* pair, strict nondegeneracy is equivalent to strict nondegeneracy of the associated contact triple and implies weak nondegeneracy, which in turn implies contact nondegeneracy.

3. Descending chain of a contact pair

Given a contact pair $(\mathfrak{g}_0, \mathcal{L}_0)$, we construct a descending chain of $\mathbb{R}$-linear subspaces of $\mathfrak{g}_0$

$$\cdots \supseteq \mathcal{G}_{-h} \supseteq \mathcal{G}_{1-h} \supset \cdots \mathcal{G}_{-1} \supseteq \mathcal{G}_0 \supseteq \mathcal{G}_1 \supseteq \cdots \mathcal{G}_h \supseteq \mathcal{G}_{h+1} \supseteq \cdots \qquad (3.1)$$

defining by recurrence

$$\begin{cases} \mathcal{G}_{-1} = \mathcal{L}_0 \\ \mathcal{G}_h = \mathcal{G}_{h+1} + [\mathcal{G}_{h+1}, \mathcal{G}_{-1}] & \text{if } h < -1 \\ \mathcal{G}_h = \{X \in \mathcal{G}_{-1} \mid [X, \mathcal{G}_{-1}] \subseteq \mathcal{G}_{h-1}\} & \text{if } h \geq 0. \end{cases} \qquad (3.2)$$

Since, by assumption, $\mathcal{L}_0$ is a subspace of finite codimension that generates $\mathfrak{g}_0$ as a Lie algebra, there is a nonnegative integer μ such that $\mathcal{G}_{-\mu} = \mathfrak{g}_0$. Indeed the ascending chain of subspaces

$$\{0\} = (\mathcal{G}_{-1}/\mathcal{G}_{-1}) \subseteq (\mathcal{G}_{-2}/\mathcal{G}_{-1}) \subseteq \cdots (\mathcal{G}_{-h}/\mathcal{G}_{-1}) \subseteq (\mathcal{G}_{-h-1}/\mathcal{G}_{-1}) \subseteq \cdots$$

of the finite dimensional vector space $(\mathfrak{g}_0/\mathcal{G}_{-1})$ stabilizes and, by their definition, if $(\mathcal{G}_{-h}/\mathcal{G}_{-1}) = (\mathcal{G}_{-h-1}/\mathcal{G}_{-1})$, then $\mathcal{G}_{-r} = \mathcal{G}_{-h}$ for all $r > h$. For a contact pair $\bigcup_h \mathcal{G}_{-h} = \mathfrak{g}_0$, and hence $\mathcal{G}_{-h} = \mathcal{G}_{-h-1} = \mathfrak{g}_0$ for some $h > 0$.

Definition 3.1. The smallest nonnegative integer μ for which $\mathcal{G}_{-\mu} = \mathfrak{g}_0$ is called the *depth*, or *kind*, of the contact pair $(\mathfrak{g}_0, \mathcal{L}_0)$.

Proposition 3.2. *Let* $(\mathfrak{g}_0, \mathcal{L}_0)$ *be a contact pair,* (3.1) *the associated descending chain. Set*

$$\mathfrak{c}_0 = \bigcap_{h \in \mathbb{Z}} \mathcal{G}_h. \qquad (3.3)$$

Then:

(1) *All* $\mathcal{G}_h$ *are closed subspaces of* $\mathfrak{g}_0$*;*
(2) $\mathfrak{c}_0$ *is the largest ideal of* $\mathfrak{g}_0$ *contained in* $\mathcal{L}_0$*;*
(3) (3.1) *is a filtration of* $\mathfrak{g}_0$*;*
(4) *For all* $h \geq 0$*,* $\mathcal{G}_h$ *is a Lie subalgebra of* $\mathfrak{g}_0$*.*

Proof. (1) The closedness of $\mathcal{G}_{-1}$ was assumed in the definition of a contact pair. For $h \geq 0$ the statement follows by recurrence, because each $\mathcal{G}_h$ is an intersection of the inverse images of the closed subspace $\mathcal{G}_{h-1}$ by the continuous linear maps $\mathcal{G}_{-1} \ni X \to [X, Y] \in \mathfrak{g}_0$, for Y varying in $\mathcal{G}_{-1}$.

For $h < -1$, the statement is true because all $\mathbb{R}$-linear subspaces V with $\mathcal{L}_0 \subseteq V \subseteq \mathfrak{g}_0$ are closed in $\mathfrak{g}_0$, since $\mathcal{L}_0$ is closed and has finite codimension in $\mathfrak{g}_0$. This completes the proof of (1).

(2) and the fact that $\mathcal{G}_0$ is a Lie subalgebra of $\mathfrak{g}_0$ are streighforward consequences of the defintions.

To complete the proof, it suffices to check that (3.1) if a filtration. We begin by checking the commutators of elements belonging to subspaces with negative indices. If $h < -1$, and we assume that $[\mathcal{G}_0, \mathcal{G}_{h+1}] \subseteq \mathcal{G}_{h+1}$, then

$$
\begin{aligned}
[\mathcal{G}_0, \mathcal{G}_h] &= \big[\mathcal{G}_0, \mathcal{G}_{h+1} + [\mathcal{G}_{h+1}, \mathcal{G}_{-1}]\big] \\
&\subseteq [\mathcal{G}_0, \mathcal{G}_{h+1}] + \big[[\mathcal{G}_0, \mathcal{G}_{h+1}], \mathcal{G}_{-1}\big] + [\mathcal{G}_{h+1}, [\mathcal{G}_0, \mathcal{G}_{-1}]] \\
&\subseteq \mathcal{G}_{h+1} + [\mathcal{G}_{h+1}, \mathcal{G}_{-1}] = \mathcal{G}_h.
\end{aligned}
$$

This implies by recurrence that $[\mathcal{G}_0, \mathcal{G}_h] \subseteq \mathcal{G}_h$ for all $h \leq 0$.

Let now $h > 0$ and assume that $[\mathcal{G}_0, \mathcal{G}_{h-1}] \subset \mathcal{G}_{h-1}$. By (3.2) we already have $[\mathcal{G}_h, \mathcal{G}_{-1}] \subseteq \mathcal{G}_{h-1}$. Then

$$
\begin{aligned}
\big[[\mathcal{G}_0, \mathcal{G}_h], \mathcal{G}_{-1}\big] &\subseteq \big[[\mathcal{G}_0, \mathcal{G}_{-1}], \mathcal{G}_h\big] + [\mathcal{G}_0, [\mathcal{G}_h, \mathcal{G}_{-1}]] \\
&\subseteq [\mathcal{G}_h, \mathcal{G}_{-1}] + [\mathcal{G}_0, \mathcal{G}_{h-1}] \subseteq \mathcal{G}_{h-1}
\end{aligned}
$$

shows, by recurrence, that $[\mathcal{G}_0, \mathcal{G}_h] \subseteq \mathcal{G}_h$ for all $h \geq 0$.

By (3.2) we have $[\mathcal{G}_h, \mathcal{G}_{-1}] \subseteq \mathcal{G}_{h-1}$ for all integers h and this implies that $[\mathcal{G}_{h_1}, \mathcal{G}_{h_2}] \subseteq \mathcal{G}_{h_1+h_2}$ when either $h_1 \leq 0$ or $h_2 \leq 0$. When both $h_1, h_2 > 0$, we can argue by recurrence on $h_1 + h_2$. In fact

$$
\begin{aligned}
\big[[\mathcal{G}_{h_1}, \mathcal{G}_{h_2}], \mathcal{G}_{-1}\big] &\subseteq \big[[\mathcal{G}_{h_1}, \mathcal{G}_{-1}], \mathcal{G}_{h_2}\big] + [\mathcal{G}_{h_1}, [\mathcal{G}_{h_2}, \mathcal{G}_{-1}]] \\
&\subseteq [\mathcal{G}_{h_1-1}, \mathcal{G}_{h_2}]][\mathcal{G}_{h_1}, \mathcal{G}_{h_2-1}] \subseteq \mathcal{G}_{h_1+h_2-1}
\end{aligned}
$$

if we assumed that $[\mathcal{G}_{h'}, \mathcal{G}_{h''}] \subseteq \mathcal{G}_{h'+h''}$ when $h' + h'' < h_1 + h_2$. This completes the proof of the fact that (3.1) is a filtration and hence of the proposition. $\qquad\square$

Lemma 3.3. *If $\mathfrak{h}_0$ is any Lie subalgebra of $\mathfrak{g}_0$ such that $[\mathfrak{h}_0, \mathcal{L}_0] \subseteq \mathcal{L}_0$, then*

$$[\mathfrak{h}_0, \mathcal{G}_h] \subseteq \mathcal{G}_h, \quad \forall h \in \mathbb{Z}. \tag{3.4}$$

In particular, if $(\mathfrak{g}_0, \mathfrak{h}_0, \mathcal{L}_0)$ is a contact triple, then all subspaces $\mathcal{G}_h$ of the canonical filtration (3.1) are $\mathfrak{h}_0$-modules.

Lemma 3.4. *Let* (3.1) *be the canonical filtration of the contact pair of a contact triple $(\mathfrak{g}_0, \mathfrak{h}_0, \mathcal{L}_0)$. Then,*

(1) *$(\mathfrak{g}_0, \mathfrak{h}_0, \mathcal{L}_0)$ is strictly nondegenerate if and only if $\mathfrak{h}_0 = \mathcal{G}_0$;*
(2) *$(\mathfrak{g}_0, \mathfrak{h}_0, \mathcal{L}_0)$ is nondegenerate, if and only if there is a positive integer k such that $\mathcal{G}_k \subseteq \mathfrak{h}_0$.*

Proof. Statement (1) follows immediately from the definitions.

To prove (2), we note that, with $c_0 = \bigcap_{h \in \mathbb{Z}} \mathcal{G}_h$ as in (3.3), the nondegeneracy condition can be restated by saying that $c_0 \subseteq \mathfrak{h}_0$. This is equivalent to the fact that the intersection of all subspaces $(\mathcal{G}_h + \mathfrak{h}_0)/\mathfrak{h}_0$ in $\mathfrak{g}_0/\mathfrak{h}_0$ is $\{0\}$. The statement follows because these subspaces form a descending chain of vector subspaces of the finite dimensional vector space $\mathfrak{g}_0/\mathfrak{h}_0$. $\qquad\square$

Definition 3.5. The *order of degeneracy* of a contact triple $(\mathfrak{g}_0, \mathfrak{h}_0, \mathcal{L}_0)$ is the smallest positive integer k for which $\mathcal{G}_k \subseteq \mathfrak{h}_0$.

We observe that 0 is *strict* nondegeneracy, and *degenerate* corresponds to ∞-degenerate.

Example 3.6. To a contact pair $(\mathfrak{g}_0, \mathcal{L}_0)$ we can always associate the triples $(\mathfrak{g}_0, \{0\}, \mathcal{L}_0)$ and $(\mathfrak{g}_0/\mathfrak{a}_0, \mathcal{G}_0/\mathfrak{a}_0, \mathcal{L}_0/\mathfrak{a}_0)$, where $\mathfrak{a}_0$ is the largest ideal of $\mathfrak{g}_0$ which is contained in $\mathcal{G}_0$. The first one is a contact triple iff $\mathfrak{g}_0$ is finite dimensional. The second one is a contact triple provided $\mathfrak{g}_0/\mathcal{G}_0$ is finite dimensional.

Remark 3.7. In Section 5 we will explain how a contact triple is canonically associated to a homogeneous contact manifold, providing in this way a geometrical motivation for Definition 2.1.

It is useful to reformulate the nondegeneracy conditions of Section 2 in terms of iterated Lie brackets. We define by recurrence

$$\begin{cases} [X_1] = X_1, \\ [X_1, X_2] = \text{Lie product of } X_1 \text{ and } X_2 \text{ in } \mathfrak{g}, \\ [X_1, \ldots, X_k, X_{k+1}] = [[X_1, \ldots, X_k], X_{k+1}], \quad \text{for } k \geq 2, \text{ if } X_1, \ldots, X_{k+1} \in \mathfrak{g}. \end{cases}$$

Proposition 3.8. *A necessary and sufficient condition in order that a contact triple* $(\mathfrak{g}_0, \mathfrak{h}_0, \mathcal{L}_0)$ *be k-nondegenerate is that*

$$\forall X \in \mathcal{L}_0 \setminus \mathfrak{h}_0, \ \exists X_0, \ldots, X_k \in \mathcal{L}_0 \ \text{ such that } \ [X, X_0, \ldots, X_k] \notin \mathcal{L}_0. \quad (3.5)$$

A necessary and sufficient condition in order that a CR algebra $(\mathfrak{g}_0, \mathfrak{q})$ *be weakly nondegenerate is that*

$$\forall Z \in \mathfrak{q}, \ \exists Z_0, \ldots, Z_k \in \mathfrak{q} \ \text{such that } [\bar{Z}, Z_0, \ldots, Z_k] \notin (\mathfrak{q} + \bar{\mathfrak{q}}). \quad (3.6)$$

4. Descending chain of a *CR* algebra

Let $(\mathfrak{g}_0, \mathfrak{q})$ be a *CR* algebra. We already noted that strict nondegeneracy implies weak nondegeneracy. It is well known that the two conditions are not equivalent. Set $\tilde{\mathfrak{q}} = (\mathfrak{q} + \bar{\mathfrak{q}})$ and $\tilde{\mathfrak{q}}_0 = \tilde{\mathfrak{q}} \cap \mathfrak{g}_0$. These are not, in general, subalgebras, but only

linear subspaces of $\mathfrak{g}$ and $\mathfrak{g}_0$, respectively. To better understand weak *CR* nondegeneracy, we construct recursively descending chains of complex Lie subalgebras and of complex vector subspaces of $\mathfrak{g}$:

$$\bar{\mathfrak{q}}^{(0)} \supseteq \bar{\mathfrak{q}}^{(1)} \supseteq \cdots \supseteq \bar{\mathfrak{q}}^{(h)} \supseteq \bar{\mathfrak{q}}^{h+1} \supseteq \cdots \qquad (4.1)$$

$$\tilde{\mathfrak{q}}^{(0)} \supseteq \tilde{\mathfrak{q}}^{(1)} \supseteq \cdots \supseteq \tilde{\mathfrak{q}}^{(h)} \supseteq \tilde{\mathfrak{q}}^{h+1} \supseteq \cdots \qquad (4.2)$$

by setting

$$\begin{cases} \bar{\mathfrak{q}}^{(0)} = \bar{\mathfrak{q}} \\ \tilde{\mathfrak{q}}^{(0)} = \mathfrak{q} + \bar{\mathfrak{q}} & \text{for } h = 0 \\ \bar{\mathfrak{q}}^{(h)} = \left\{ Z \in \bar{\mathfrak{q}}^{(h-1)} \mid [Z, \mathfrak{q}] \subseteq \tilde{\mathfrak{q}}^{(h-1)} \right\} \\ \tilde{\mathfrak{q}}^{(h)} = \mathfrak{q} + \bar{\mathfrak{q}}^{(h)} & \text{for } h \geq 1. \end{cases} \qquad (4.3)$$

Lemma 4.1. *We have*

$$\bigcap_{h \geq 0} \bar{\mathfrak{q}}^{(h)} = \bigcap_{h \geq 1} \left\{ Z \in \bar{\mathfrak{q}} \mid [Z, W_1, \ldots, W_h] \in \tilde{\mathfrak{q}}, \ \forall W_1, \ldots, W_h \in \mathfrak{q} \right\}. \qquad (4.4)$$

Proof. Let us denote by $\mathcal{A}$, $\mathcal{B}$ the left- and right-hand side of (4.4), respectively. Since $[\bar{\mathfrak{q}}^{(h)}, \mathfrak{q}] \subset \mathfrak{q} + \bar{\mathfrak{q}}^{(h-1)}$ for all $h > 0$, we obtain that $\mathcal{A} \subseteq \mathcal{B}$.

Vice versa, if $Z \in \bar{\mathfrak{q}}$ does not belong to $\mathcal{A}$, then there is a positive integer h with $Z \notin \bar{\mathfrak{q}}^{(h)}$. This means that there is $W_1 \in \mathfrak{q}$ such that $[Z, W_1] \notin \mathfrak{q} + \bar{\mathfrak{q}}^{(h-1)}$. If $h = 1$, this suffices to show that $Z \notin \mathcal{B}$. If $h > 1$, write $[Z, W_1] = Z_1 + W_1'$ with $Z_1 \in \bar{\mathfrak{q}}$ and $W_1 \in \mathfrak{q}$. Since, by assumption, $Z_1 \notin \bar{\mathfrak{q}}^{(h-1)}$ we can find $W_2 \in \mathfrak{q}$ such that $[Z_1, W_2]$, and hence also $[Z, W_1, W_2]$, does not belong to $\mathfrak{q} + \bar{\mathfrak{q}}^{(h-2)}$. Iterating this argument, we show that there are $W_1, \ldots, W_h \in \mathfrak{q}$ such that $[Z, W_1, \ldots, W_h] \notin \mathfrak{q} + \bar{\mathfrak{q}}$ and therefore Z does not belong to $\mathcal{B}$. This completes the proof. $\qquad \square$

Lemma 4.2. *All $\bar{\mathfrak{q}}^{(h)}$ are complex Lie algebras and*

$$\mathfrak{q}' = \mathfrak{q} + \bigcap_{h \geq 0} \bar{\mathfrak{g}}^{(h)} = \bigcap_{h \geq 0} \tilde{\mathfrak{q}}^{(h)} \qquad (4.5)$$

is the largest complex Lie subalgebra satisfying

$$\mathfrak{q} \subseteq \mathfrak{q}' \subseteq \mathfrak{q} + \bar{\mathfrak{q}}. \qquad (4.6)$$

Proof. Let us show first that the $\bar{\mathfrak{q}}^{(h)}$'s are Lie subalgebras. This holds true for $h = 0$, because the conjugate $\bar{\mathfrak{q}}$ of $\mathfrak{q}$ with respect to the real form $\mathfrak{g}_0$ is a complex Lie subalgebra of $\mathfrak{g}$. Assume that we already know that $\bar{\mathfrak{q}}^{(h)}$ is a Lie algebra for

some $h \geq 0$. If $Z_1, Z_2 \in \bar{\mathfrak{q}}^{(h+1)}$, we have $[Z_1, Z_2] \in \bar{\mathfrak{q}}^{(h)}$ because $\bar{\mathfrak{q}}^{(h+1)} \subseteq \bar{\mathfrak{q}}^{(h)}$ and, by our inductive assumption, $\bar{\mathfrak{q}}^{(h)}$ is a Lie algebra. Moreover,

$$\big[[Z_1, Z_2], \mathfrak{q}\big] \subseteq \big[Z_1, [Z_2, \mathfrak{q}]\big] + \big[Z_2, [Z_1, \mathfrak{q}]\big] \subseteq \big[Z_1, \bar{\mathfrak{q}}^{(h)} + \mathfrak{q}\big] + \big[Z_2, \bar{\mathfrak{q}}^{(h)} + \mathfrak{q}\big]$$

$$\subseteq \big[Z_1, \bar{\mathfrak{q}}^{(h)}\big] + [Z_1, \mathfrak{q}] + \big[Z_2, \bar{\mathfrak{q}}^{(h)}\big] + [Z_2, \mathfrak{q}] \subseteq \bar{\mathfrak{q}}^{(h)} + \mathfrak{q},$$

shows that $[Z_1, Z_2] \in \bar{\mathfrak{q}}^{(h+1)}$. Clearly the right-hand side of (4.5) is a Lie subalgebra of $\mathfrak{g}$ with $\mathfrak{q} \subseteq \mathfrak{q}' \subseteq \mathfrak{q} + \bar{\mathfrak{q}}$. By Lemma 4.1 it is the maximal complex Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{q}$ and contained in $\mathfrak{q} + \bar{\mathfrak{q}}$. In fact it contains $\mathfrak{l} \cap \bar{\mathfrak{q}}$ for all complex Lie subalgebras with $\mathfrak{q} \subseteq \mathfrak{l} \subseteq \mathfrak{q} + \bar{\mathfrak{q}}$. $\qquad\square$

Proposition 4.3. *The following are equivalent:*

(1) $(\mathfrak{g}_0, \mathfrak{q})$ *is weakly CR nondegenerate;*
(2) $\bigcap_{h \geq 0} \bar{\mathfrak{q}}^{(h)} = \mathfrak{q} \cap \bar{\mathfrak{q}}$;
(3) $\bigcap_{h \geq 0} \tilde{\mathfrak{q}}^{(h)} = \mathfrak{q}$.

Sequences (4.1) and (4.2) can be used to measure weak nondegeneracy. Let the *lenght* of a descending chain of vector spaces

$$V_0 \supseteq V_1 \supseteq \cdots \supseteq V_h \supseteq V_{h+1} \supseteq \cdots$$

be the smallest integer ν such that $V_h = V_\nu$ for all $h \geq \nu$. By Lemma 4.2 we have the statement:

Proposition 4.4. *The sequences* (4.1) *and* (4.2) *have the same lenght ν and all their terms with indices smaller than ν are different.*

Definition 4.5. We say that $(\mathfrak{g}_0, \mathfrak{q})$ is $(\nu - 1)$-nondegenerate if the descending chains (4.1) and (4.2) have finite lenght ν.

Remark 4.6. Strict nondegeneracy is 0-nondegeneracy, while weak nondegeneracy is ν-nondegeneracy for some $\nu < +\infty$.

Example 4.7. For *CR* algebras, contact is a weaker notion than weak nondegeneracy. A score of examples can be obtained by considering real orbits in complex flag manifolds (see [1, 3]) whose *CR* algebras are fundamental, but not weakly nondegenerate. We give here a simple example, consisting of the minimal orbit M_0 of $\mathbf{SU}(1, 5)$ in the complex flag manifold M consisting of triples (ℓ_2, ℓ_3, ℓ_4) of complex 2, 3, 4-planes in $\mathbb{C}^6$ with $\ell_2 \subset \ell_3 \subset \ell_4$. A point of M_0 is a flag in the $\mathbf{SU}(1, 5)$-orthogonal of an isotropic line. Let us give the explicit matrix representation. We

define $\mathfrak{g}_0 \simeq \mathfrak{su}(1,5)$ and $\mathfrak{q}, \mathfrak{q}' \subset \mathfrak{sl}_6(\mathbb{C})$ by

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} \lambda & \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 & i\sigma \\ z_1 & it_1 & -\bar{w}_1 & -\bar{w}_2 & -\bar{w}_3 & -\bar{\zeta}_1 \\ z_2 & w_1 & it_2 & -\bar{w}_4 & -\bar{w}_5 & -\bar{\zeta}_2 \\ z_3 & w_2 & w_4 & it_3 & -\bar{w}_6 & -\bar{\zeta}_3 \\ z_4 & w_3 & w_5 & w_6 & it_4 & -\bar{\zeta}_4 \\ is & -\bar{z}_1 & -\bar{z}_2 & -\bar{z}_3 & -\bar{z}_4 & -\bar{\lambda} \end{pmatrix} \middle| \begin{array}{l} \lambda, z_i, \zeta_i, w_i \in \mathbb{C},\ s, \sigma, t_i \in \mathbb{R}, \\[2mm] 2\mathrm{Im}(\lambda) + \sum t_i = 0 \end{array} \right\},$$

$$\mathfrak{q} = \left\{ \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{pmatrix} \right\}, \quad \mathfrak{q}' = \left\{ \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{pmatrix} \right\},$$

$$\mathfrak{q} + \bar{\mathfrak{q}} = \mathfrak{q}' + \bar{\mathfrak{q}}' = \left\{ \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & 0 & 0 & * & * \end{pmatrix} \right\}.$$

Then $(\mathfrak{g}_0, \mathfrak{q})$ is not weakly nondegenerate, because $\mathfrak{q} \subsetneqq \mathfrak{q}' \subset \mathfrak{q} + \bar{\mathfrak{q}}$, while $(\mathfrak{g}_0, \check{\mathfrak{q}}_0, \tilde{\mathfrak{q}}_0)$ is a contact triple which is nondegenerate because $\mathfrak{su}(1,5)$ is simple and therefore does not contain nontrivial ideals.

5. Homogeneous contact structures

Let $\mathbf{G}_0$ be a Lie group, acting transitively on a smooth manifold M_0. Fix a point p_0 of M_0. The injective quotient of

$$\pi : \mathbf{G}_0 \ni x \to x \cdot p_0 \in M_0$$

yields the idenfication $M_0 \approx \mathbf{G}_0/\mathbf{H}_0$ of M_0 with the quotient of $\mathbf{G}_0$ by the stabilizer $\mathbf{H}_0$ of p_0. A $\mathbf{G}_0$-equivariant contact structure on M_0 is the datum of a constant rank distribution $\mathscr{L}^*$ on M_0, which is invariant for the left translations by elements of $\mathbf{G}_0$:

$$x \cdot \mathscr{L}^* = \mathscr{L}^*, \quad \forall x \in \mathbf{G}_0.$$

The pullback $\mathscr{L} = \pi^*(\mathscr{L}^*)$ is a left-invariant distribution on $\mathbf{G}_0$ generated by a subspace $\mathcal{L}_0$ of left-invariant vector fields on $\mathbf{G}_0$, containing the Lie algebra $\mathfrak{h}_0$ of $\mathbf{H}_0$. Moreover, the vector subspace

$$L_{p_0} = \{\Theta_{p_0} \mid \Theta \in \mathscr{L}^*\} \subseteq T_{p_0} M_0$$

must be invariant for the differential at p_0 of the translations by elements of $\mathbf{H}_0$. This yields

$$\mathrm{Ad}(x)(\mathcal{L}_0) = \mathcal{L}_0, \quad \forall x \in \mathbf{H}_0,$$

which also implies that $[\mathfrak{h}_0, \mathcal{L}_0] \subseteq \mathcal{L}_0$ for the Lie algebra $\mathfrak{h}_0$ of $\mathbf{H}_0$.

Vice versa, il $\mathcal{L}_0$ is an $\mathrm{Ad}(\mathbf{H}_0)$-invariant linear subspace of the Lie algebra $\mathfrak{g}_0$ of $\mathbf{G}_0$, then the push-forward on M_0 of the distribution on $\mathbf{G}_0$ generated by the left-invariant vector fields corresponding to the elements of $\mathcal{L}_0$ is a smooth distribution $\mathscr{L}^* = \mathcal{L}_0^*$ on M_0, which is invariant by the $\mathbf{G}_0$-translations on M_0.

Assume now that we do not know a priori that M_0 is a homogeneous space, but we are given a constant rank distribution $\mathscr{L}^*$ on M_0 and a Lie algebra $\mathfrak{g}_0^*$ of smooth vector fields on M_0 which leave $\mathscr{L}^*$ invariant: this means that $[\mathfrak{g}_0^*, \mathscr{L}^*] \subseteq \mathscr{L}^*$.

We say that $\mathfrak{g}_0^*$ is *transitive* at p_0 if

$$\{X^*_{p_0} \in T_{p_0} M_0 \mid X^* \in \mathfrak{g}_0^*\} = T_{p_0} M_0.$$

Let $\mathfrak{g}_0 = \mathfrak{g}_0^{*\mathrm{opp}}$, where the superscript "*opp*" means that, if we denote by X the element of $\mathfrak{g}_0$ corresponding to the vector field X^* of $\mathfrak{g}_0^*$, then

$$[X, Y]^* = -[X^*, Y^*], \quad \forall X, Y \in \mathfrak{g}_0.$$

With $L_{p_0} = \{\Theta_{p_0} \mid \Theta \in \mathscr{L}^*\}$, let us set

$$\mathfrak{h}_0 = \{X \in \mathfrak{g}_0 \mid X^*_{p_0} = 0\}, \quad \mathcal{L}_0 = \{X \in \mathfrak{g}_0 \mid X^*_{p_0} \in L_{p_0}\}.$$

Proposition 5.1. *If $\mathfrak{g}_0^*$ is transitive, then $(\mathfrak{g}_0, \mathfrak{h}_0, \mathcal{L}_0)$ is a transitive contact triple.*

Proof. The quotient $\mathfrak{g}_0/\mathfrak{h}_0$ maps injectively into $T_{p_0} M_0$ and therefore is finite dimensional. Let $X \in \mathcal{L}_0$ and $Y \in \mathfrak{h}_0$. Then we can find a vector field Θ, vanishing at p_0, such that $X^* + \Theta \in \mathcal{L}_0^*$. Then

$$[Y^*, X^* + \Theta] = [X, Y]^* + [Y^*, \Theta] \in \mathcal{L}_0^*.$$

Since $[Y^*, \Theta]$ vanishes at p_0, this means that $[X, Y]^*_{p_0} \in L_{p_0}$, showing that $[Y, X] \in \mathcal{L}_0$. $\qquad\square$

6. $\mathbb{Z}$-graded Lie algebras and a theorem by Tanaka

We will use the following criterion ([24, Section 11]):

Proposition 6.1 (N. Tanaka). *Let*

$$\mathfrak{G} = \bigoplus_{h \geq -\mu} \mathfrak{G}_h \tag{6.1}$$

be a $\mathbb{Z}$-graded real Lie algebra, with $\dim_{\mathbb{R}}(\mathfrak{G}_h) < +\infty$ for all $h \in \mathbb{Z}$, having finitely many summands with negative index. Assume that $\mathfrak{G}$ is transitive: this means that

$$\{\eta \in \mathfrak{G}_h \mid [\eta, \mathfrak{G}_{-1}] = \{0\}\} = \{0\}, \quad \text{if } h \geq 0.$$

Then a necessary and sufficient condition for $\mathfrak{G}$ to be finite dimensional is that

$$\mathfrak{G}' = \{\eta \in \mathfrak{G} \mid [\eta, \mathfrak{G}_h] = \{0\} \text{ for } h \leq -2\} \tag{6.2}$$

be finite dimensional.

Let us comment on this criterion. In the following, we assume that $\mathfrak{G}$ is transitive.

Clearly, $\mathfrak{G}'$ is a $\mathbb{Z}$-graded Lie subalgebra of $\mathfrak{G}$ and

$$[\mathfrak{G}'_h, \mathfrak{G}_{-1}] \subseteq \mathfrak{G}'_{h-1}, \quad \forall h \in \mathbb{Z}. \tag{6.3}$$

Given real vector spaces V, W, let $\mathcal{M}^h(V, W)$ denote the space of W-valued h-multilinear forms on V and $\mathcal{S}ymm^h(V, W)$ the subspace consisting of those which are symmetric.

We define a map $\eta \rightarrow \eta_h$ of $\mathfrak{G}$ into $\mathcal{M}^h(\mathfrak{G}_{-1}, \mathfrak{G})$ by associating to each $\eta \in \mathfrak{G}$ the multilinear form

$$\eta_h(\xi_1, \ldots, \xi_h) = [\eta, \xi_1, \ldots, \xi_h], \quad \text{for } \xi_1, \ldots, \xi_h \in \mathfrak{G}_{-1}.$$

We also consider the alternate $\mathfrak{G}_{-2}$-valued bilinear form on $\mathfrak{G}_{-1}$:

$$\omega(\xi_1, \xi_2) = [\xi_1, \xi_2], \quad \text{for } \xi_1, \xi_2 \in \mathfrak{G}_{-1} \tag{6.4}$$

and set

$$\mathfrak{sp}(\omega) = \{T \in \mathfrak{gl}_{\mathbb{R}}(\mathfrak{G}_{-1}) \mid \omega(T(\xi_1), \xi_2) + \omega(\xi_1, T(\xi_2)) = 0\}. \tag{6.5}$$

Lemma 6.2. *For each $h \geq 0$, the maps*

$$\begin{cases} \mathfrak{G}_0 \ni \eta \longrightarrow \eta_1 \in \mathfrak{gl}_{\mathbb{R}}(\mathfrak{G}_{-1}) & \text{for } h = 0 \\ \mathfrak{G}_h \ni \eta \longrightarrow \eta_h \in \mathcal{M}^h(\mathfrak{G}_{-1}, \mathfrak{G}_0) & \text{for } h > 0 \end{cases} \tag{6.6}$$

are injective.

$$\begin{cases} \mathfrak{G}'_0 \subseteq \{\eta \in \mathfrak{G}_0 \mid \eta_1 \in \mathfrak{sp}(\omega)\} & \text{for } h = 0 \\ \mathfrak{G}'_h \subseteq \{\eta \in \mathfrak{G}_h \mid \eta_k \in \mathcal{S}ymm^k(\mathfrak{G}_{-1}, \mathfrak{G}_{h-k})\} & \text{for } h, k > 0. \end{cases} \tag{6.7}$$

Proof. The fact that the maps in (6.6) are injective is a straightforward consequence of transitivity.

If $\eta \in \mathfrak{G}_0$ and $[\eta, \mathfrak{G}_{-2}] = 0$, then

$$0 = [\eta, [\xi_1, \xi_2]] = [[\eta, \xi_1], \xi_2] + [\xi_1, [\eta, \xi_2]] = \omega(\eta_1(\xi_1), \eta_2) + \omega(\xi_1, \eta_1(\xi_2))$$

shows that $\eta_1 \in \mathfrak{sp}(\omega)$. In the same way, if $\eta \in \mathfrak{G}$ and $[\eta, \mathfrak{G}_{-2}] = 0$, we obtain that

$$\left[\eta, \xi_1, \xi_2\right] = \left[[\eta, \xi_1], \xi_2\right] = \left[[\eta, \xi_2], \xi_1\right] = \left[\eta, \xi_2, \xi_1\right], \ \forall \eta \in \mathfrak{G}', \ \forall \xi_1, \xi_2 \in \mathfrak{G}_{-1}.$$

This gives at once that $[\eta, \xi_1, \xi_2, \xi_3] = [\eta, \xi_2, \xi_1, \xi_3]$ for $\eta \in \mathfrak{G}'$ and $\xi_1, \xi_2, \xi_3 \in \mathfrak{G}_{-1}$, while

$$\begin{aligned}
0 &= \left[\eta, [\xi_1, \xi_2, \xi_3]\right] = \left[\eta, [[\xi_1, \xi_2], \xi_3]\right] = \left[[\eta, [\xi_1, \xi_2]], \xi_3\right] + \left[[\xi_1, \xi_2], [\eta, \xi_3]\right] \\
&= \left[[\xi_1, [\eta, \xi_3]], \xi_2\right] + \left[\xi_1, [\xi_2, [\eta, \xi_3]]\right] = -\left[\eta, \xi_3, \xi_1, \xi_2\right] + \left[\eta, \xi_3, \xi_2, \xi_1\right]
\end{aligned}$$

shows that also $[\eta, \xi_1, \xi_2, \xi_3] = [\eta, \xi_1, \xi_3, \xi_2]$ for $\eta \in \mathfrak{G}'$ and $\xi_1, \xi_2, \xi_3 \in \mathfrak{G}_{-1}$. This yields symmetry on the triples. Arguing recursively on k, we obtain, for all $k > 3$,

$$\begin{aligned}
0 &= \left[[\eta, \xi_1, \ldots, \xi_{k-2}], [\xi_{k-1}, \xi_k]\right] \\
&= \left[\eta, \xi_1, \ldots, \xi_{k-1}, \xi_k\right] - \left[\eta, \xi_1, \ldots, \xi_k, \xi_{k-1}\right].
\end{aligned}$$

This shows that $[\eta, \xi_1, \ldots, \xi_{k-1}, \xi_k] = [\eta, \xi_1, \ldots, \xi_k, \xi_{k-1}]$. Thus η_k stays invariant under the transposition $(k-1, k)$. By the recursive assumption, it is also invariant under the transpositions $(j-1, j)$ for $2 \leq j < k$ and thus is invariant under the full permutation group of $\{1, \ldots, k\}$. $\square$

Example 6.3. Let V be a real vector space of finite dimension n, viewed as a degree (-1)-homogeneous Abelian real Lie algebra. Then its maximal Levi-Tanaka extension is isomorphic to the $\mathbb{Z}$-graded Lie algebra $\mathfrak{P}$ of vector fields with polynomial coefficients in $\mathbb{R}^n$, the grading being defined by

$$\mathfrak{P}_h = \left\{ \sum_{i=1}^{n} p_i(x) \frac{\partial}{\partial x_i} \ \middle| \ p_i \in \mathbb{R}_{h+1}[x_1, \ldots, x_n] \right\}, \quad h \geq -1. \qquad (6.8)$$

Here $\mathbb{R}_{h+1}[x_1, \ldots, x_n]$ denotes the vector space of homogeneous polynomials of degree $(h+1)$ in the $x_1, \ldots, x_n$ variables.

If $n = 2m$ and V has a complex structure, then the Levi-Tanaka extension of $V \oplus \mathfrak{gl}_{\mathbb{C}}(V)$ is isomorphic to the $\mathbb{Z}$-graded complex Lie algebra $\mathfrak{P}^{\mathbb{C}}$, of homolorphic complex vector fields with holomorphic polynomial coefficients, with the gradation defined by

$$\mathfrak{P}_h^{\mathbb{C}} = \left\{ \sum_{i=1}^{m} p_i(z) \frac{\partial}{\partial z_i} \ \middle| \ p_i \in \mathbb{C}_{h+1}[z_1, \ldots, z_n] \right\}, \quad h \geq -1. \qquad (6.9)$$

Here $\mathbb{C}_{h+1}[z_1, \ldots, z_n]$ denotes the vector space of homogeneous holomorphic polynomials of degree $(h+1)$ in the $z_1, \ldots, z_n$ variables.

Proof. The fact that $\mathfrak{P}$ is a maximal transitive extension of $\mathfrak{P}_{-1} \simeq V$ is a consequence of the fact that $[\mathfrak{P}_h, \mathfrak{P}_{-1}] = \mathfrak{P}_{h-1}$ for $h \geq 0$.

Analogously, when V has a complex structure, $\mathfrak{P}^{\mathbb{C}}$ is a maximal transitive extensions of $V \oplus \mathfrak{gl}_{\mathbb{C}}(V) \simeq \mathfrak{P}_{-1}^{\mathbb{C}} \oplus \mathfrak{P}_0^{\mathbb{C}}$ because $[\mathfrak{P}_h^{\mathbb{C}}, \mathfrak{P}_{-1}^{\mathbb{C}}] = \mathfrak{P}_{h-1}^{\mathbb{C}}$ for all $h \geq 0$. $\square$

7. A finitness criterion for *CR* algebras

We recall that the contact triple associated to a fundamental *CR* algebra $(\mathfrak{g}_0, \mathfrak{q})$ is

$$\left(\mathfrak{g}_0, \check{\mathfrak{q}}_0, \tilde{\mathfrak{q}}_0\right) = \left(\mathfrak{g}_0, \mathfrak{q} \cap \bar{\mathfrak{q}} \cap \mathfrak{g}_0, (\mathfrak{q} + \bar{\mathfrak{q}}) \cap \mathfrak{g}_0\right).$$

We consider the canonical filtration (3.1) of the contact pair $(\mathfrak{g}_0, \tilde{\mathfrak{q}}_0)$ and the corresponding $\mathbb{Z}$-graded Lie algebra

$$\mathfrak{G} = \bigoplus_{h \in \mathbb{Z}} \mathfrak{G}_h, \quad \text{with} \quad \mathfrak{G}_h = \mathcal{G}_h / \mathcal{G}_{h+1}. \tag{7.1}$$

Denote by $\pi_h : \mathcal{G}_h \to \mathfrak{G}_h = \mathcal{G}_h / \mathcal{G}_{h+1}$ the projections onto the quotients.

Lemma 7.1 (Partial complex structure). *There is a unique complex structure J on $\mathfrak{G}_{-1}$, defined by*

$$\mathrm{J}(\pi_{-1}(X)) = \pi_{-1}(Y) \quad \text{if and only if} \quad X + iY \in \mathfrak{q}. \tag{7.2}$$

The operator $\mathrm{J} \in \mathfrak{gl}_{\mathbb{R}}(\mathfrak{G}_{-1})$ *satisfies*

$$[\mathrm{J}(\xi_1), \mathrm{J}(\xi_2)] = [\xi_1, \xi_2], \quad [\mathrm{J}(\xi_1), \xi_2] + [\xi_1, \mathrm{J}(\xi_2)] = 0, \quad \forall \xi_1, \xi_2 \in \mathfrak{G}_{-1}. \tag{7.3}$$

Proof. To show that J is well defined, we need to verify that, if $X, Y \in \mathfrak{g}_0$, $X \in \mathcal{G}_0$ and $X + iY \in \mathfrak{q}$, then $Y \in \mathcal{G}_0$. If $X' \in \mathcal{G}_{-1}$, then we can find $Y' \in \mathcal{G}_{-1}$ such that $X' + iY' \in \mathfrak{q}$. Then

$$[X + iY, X' + iY'] = [X, X'] - [Y, Y'] + i([X, Y'] + [Y, X']) \in \mathfrak{q}. \tag{7.4}$$

Since by assumption both $[X, X']$ and $[X, Y']$ belong to $\mathcal{G}_{-1}$, then also $[Y, X']$ and $[Y, Y']$ belong to $\mathcal{G}_{-1}$. This shows that $[Y, \mathcal{G}_{-1}] \subset \mathcal{G}_{-1}$, and then $Y \in \mathcal{G}_0$. Formula (7.4) holds in general for $X + iY, X' + iY' \in \mathfrak{q}$, yielding (7.3). $\square$

Lemma 7.2. *The $\mathfrak{G}_{-2}$-valued form (6.4) is nondegenerate*, i.e.

$$\xi \in \mathfrak{G}_{-1}, \quad \omega(\xi, \xi') = 0, \; \forall \xi' \in \mathfrak{G}_{-1} \Longleftrightarrow \xi = 0.$$

Lemma 7.3. *Let $(\mathfrak{g}_0, \mathfrak{q})$ be a CR algebra and (3.1) the associated $\mathbb{Z}$-filtration. assume that there is a nonnegative integer k such that $\mathcal{G}_k \subset \mathfrak{h}_0 = \mathfrak{q} \cap \bar{\mathfrak{q}} \cap \mathfrak{g}_0$. Then $\mathfrak{G}'_{2k+1} = \{0\}$.*

Proof. By the assumption, any Y of $\mathcal{G}_k$ belongs to $\mathfrak{q}$ and therefore

$$[Y, Z_1, \ldots, Z_{k+2}] \in \mathfrak{q}, \quad \forall Z_1, \ldots, Z_{k+2} \in \mathfrak{q}.$$

In the complexification of $\mathfrak{G}$, this yelds the equation

$$\left[\theta, \xi_1 + i\mathrm{J}(\xi_1), \ldots, \xi_{k+2} + i\mathrm{J}(\xi_{k+2})\right] = 0, \quad \forall \theta \in \mathfrak{G}_k, \quad \forall \xi_1, \ldots, \xi_{k+2} \in \mathfrak{G}_{-1},$$

as $\pi_{-2}(\mathfrak{q}) = \{0\}$, because $\mathfrak{q}$ is contained in the complexification of $\mathcal{G}_{-1}$.

Let now $\eta \in \mathfrak{G}_{2k+1}$. Fix $\xi \in \mathfrak{G}_{-1}$ and consider, for $0 \leq h \leq 2k+3$, the $(2k+4)$ vectors of $\mathfrak{G}_{-2}$:

$$v_h = [\eta, \xi_1, \ldots, \xi_{2k+3}], \text{ with } \xi_i = J(\xi) \text{ for } i \leq h \text{ and } \xi_i = \xi \text{ for } h < i \leq 2k+3.$$

For each choice of $\xi_1, \ldots, \xi_{k+1}$ in $\mathfrak{G}_{-1}$, the multi-commutator $[\eta, \xi_1, \ldots, \xi_{k+1}]$ belongs to $\mathfrak{G}_k$. Thus, setting, for each integer h with $0 \leq h \leq k+1$, $\xi_i^h = J(\xi)$ if $1 \leq i \leq h$ and $\xi_i^h = \xi$ if $h < i \leq k+1$, the real and imaginary parts of

$$\left[\eta, \xi_0^h, \ldots, \xi_k^h, Z_1, \ldots, Z_{k+2}\right] = 0, \quad \text{with } Z_1 = \cdots = Z_{k+2} = \xi + iJ(\xi),$$

yield $(2k+4)$ linear combinations of $v_0, \ldots, v_{2k+3}$ which sum to zero. These can be written in the form

$$(v_0, v_1, \ldots, v_{2k+3}) \cdot M_k = 0, \tag{7.5}$$

where M_k is a real $(2k+4) \times (2k+4)$ matrix whose columns are the coefficients of the real and imaginary parts of the polynomials $t^h(t-i)^{k+2}$, for $0 \leq t \leq (k+1)$.

It is easy to check that these polynomials form a basis of the $(2k+4)$-dimensional $\mathbb{C}$-vector space $\mathbb{C}_{2k+3}[t]$ of polynomials of degree less or equal to $(2k+3)$.

In fact their $\mathbb{C}$-linear span contains the $(2k+4)$ polynomials $(t-i)^{k+h+2}$ and $(t+i)^{k+h+2}$, for $0 \leq h \leq k+1$. These are linearly independent and hence form a basis of $\mathbb{C}_{2k+3}[t]$. Indeed, let a_h, b_h be complex coefficients for which

$$f(t) = \sum_{h=0}^{k+1} \left(a_h(t-i)^{k+h+2} + b_h(t+i)^{k+h+2}\right) = 0.$$

Let $r \leq k+1$ and assume that we already know that $a_h = 0$ and $b_h = 0$ if $h > r$, this being obviously the case when $r = (k+1)$. Then

$$0 = \frac{i}{2 \cdot (2k+r)!} \frac{d^{2k+r-1} f(-i)}{dt^{2k+r-1}} = a_r, \quad 0 = \frac{-i}{2 \cdot (2k+r)!} \frac{d^{2k+r-1} f(i)}{dt^{2k+r-1}} = b_r$$

shows that also $a_r = 0$ and $b_r = 0$. By recurrence, this proves that all coefficients a_h, b_h must be zero and thus the claimed linear independence of the polynomials $(t-i)^{k+h+2}$, $(t+i)^{k+h+2}$.

Hence M_k is nondegenerate and (7.5) tells us that all vectors $v_0, \ldots, v_{2k+3}$ are zero. In particular $v_0 = 0$ and therefore we proved that

$$[\eta, \underbrace{\xi, \ldots \ldots, \xi}_{(2k+3) \text{ times}}] = 0, \quad \forall \xi \in \mathfrak{G}_{-1}.$$

Since, by Lemma 6.2, the multilinear $\mathfrak{G}_{-2}$-valued form η_{2k+3} is symmetric in its arguments, it follows by polarization (see, *e.g.*, [25, page 5]) that

$$[\eta, \xi_1, \ldots, \xi_{2k+3}] = 0 \text{ for all } \xi_1, \ldots, \xi_{2k+3} \in \mathfrak{G}_{-1}.$$

Since ω is nondegenerate, this yields

$$[\eta, \xi_1, \ldots, \xi_{2k+2}] = 0 \text{ for all } \xi_1, \ldots, \xi_{2k+2} \in \mathfrak{G}_{-1}$$

and hence, by transitivity, $\eta = 0$. $\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Thus we obtain, using also [14, Theorem 10.2]:

Theorem 7.4. *A CR algebra* $(\mathfrak{g}_0, \mathfrak{q})$ *for which the associated contact triple* $(\mathfrak{g}_0, \breve{\mathfrak{q}}_0, \tilde{\mathfrak{q}}_0) = (\mathfrak{g}_0, \mathfrak{q} \cap \bar{\mathfrak{q}} \cap \mathfrak{g}_0, (\mathfrak{q} + \bar{\mathfrak{q}}) \cap \mathfrak{g}_0)$ *is nondegenerate is finite dimensional.*

We will prove in Section 9 that, under the assumptions of Theorem 7.4, $(\mathfrak{g}_0, \mathfrak{q})$ has a *maximal extension* and that this is finite dimensional. To this aim we will generalise, in Section 8, the construction of Section 5, by a procedure similar to that of [7,11].

8. Transitive pairs and generalised contact distributions

8.1. Vector fields with formal power series coefficients

Let V be a finite dimensional vector space. The space $\mathcal{F}$ of formal power series associated to V is the infinite direct sum

$$\mathcal{F} = \sum_{h \geq 0} \mathit{Symm}^h(V) = \left\{ \sum_{h=0}^{\infty} \alpha_h \,\middle|\, \alpha_h \in \mathit{Symm}^h(V) \; \forall h \right\}, \tag{8.1}$$

where $\mathit{Symm}^h(V) = \mathit{Symm}^h(V, \mathbb{R})$ are the real-valued homogeneous multilinear symmetric forms of degree h (*cf.* Section 6). The coefficient α_0 is the *value at* 0 of $\sum_{h=0}^{\infty} \alpha_h$. With the standard operations, $\mathcal{F}$ is a local ring, whose maximal ideal $\mathcal{F}_0$ consists of formal power series vanishing at 0.

Each vector v of V defines a derivation D_v on $\mathcal{F}$, whose action on each summand $\mathit{Symm}^h(V)$ is described by

$$\begin{cases} D_v \alpha = 0 \;\text{ if } \alpha \in \mathit{Symm}^0(V) \simeq \mathbb{R}, \\ D_v \alpha = h \cdot v \,\lrcorner\, \alpha \in \mathit{Symm}^{h-1}(V), \;\text{ if } \alpha \in \mathit{Symm}^h(V), \text{ for } h > 0, \\ i.e. \; (D_v \alpha)(v_1, \ldots, v_{h-1}) = h \cdot \alpha(v_1, \ldots, v_{h-1}, v), \; \forall v_1, \ldots, v_{h-1} \in V. \end{cases} \tag{8.2}$$

The set $Der(\mathcal{F})$ of derivations of $\mathcal{F}$ is the left $\mathcal{F}$-module $\mathcal{F} \otimes_{\mathbb{R}} V$ generated by V. Thus any derivation X^* is a formal series

$$X^* = \sum_{h=0}^{\infty} X_h^*, \quad \text{with} \quad X_h^* \in Symm^h(V, V) = Symm^h(V) \otimes_{\mathbb{R}} V. \qquad (8.3)$$

Denote by $Der_0(\mathcal{F})$ the Lie subalgebra $\mathcal{F}_0 \otimes V$ of derivations vanishing at 0.

8.2. The case of Lie groups

Let $\mathbf{G}_0$ be a real Lie group. We recall that the left and right invariant vector fields on $\mathbf{G}_0$ coincide with the infinitesimal generators L_X and R_X of the one-parameter groups

$$\mathbb{R} \times \mathbf{G}_0 \ni (t, x) \to x \cdot \exp(tX) \in \mathbf{G}_0, \quad \mathbb{R} \times \mathbf{G}_0 \ni (t, x) \to \exp(tX) \cdot x \in \mathbf{G}_0,$$

of diffeomorphisms of $\mathbf{G}_0$, respectively. We have the commutation rules

$$\begin{cases} [L_X(v), L_Y(v)] = L_{[X,Y]}(v) \\ [R_X(v), R_Y(v)] = -R_{[X,Y]}(v) \qquad \forall X, Y \in \mathfrak{g}_0. \\ [L_X(v), R_Y(v)] = 0 \end{cases} \qquad (8.4)$$

The exponential map is a diffeomorphism of an open neighbourhood of 0 in its Lie algebra $\mathfrak{g}_0$ onto an open neighborhood of the identity, defining a local chart. In order to determine the Taylor series expansions of L_X, R_X in in these coordinates, it is convenient to consider the identities

$$\exp(tX) \cdot \exp(v) = \exp\left(v + t \cdot R_X(v) + 0(t^2)\right),$$

$$\exp(v) \cdot \exp(tX) = \exp\left(v + t \cdot L_X(v) + 0(t^2)\right),$$

for $v, X \in \mathfrak{g}_0$. Since the maps $X \to L_X$ and $X \to R_X$ are linear, we obtain

$$\exp(tX) \cdot \exp(v) = (\exp(-v) \cdot \exp(-tX))^{-1} = \left(\exp(-v - t \cdot L_X(-v) + 0(t^2))\right)^{-1}$$

$$= \exp\left(v + t \cdot L_X(-v) + 0(t^2)\right)$$

showing that

$$R_X(v) = L_X(-v).$$

We can obtain the Taylor series expansions of L_X and R_X in the $\mathfrak{g}_0$-coordinates from the Baker-Campbell-Hausdorff formula (see, *e.g.*, [8]):

$$\begin{cases} L_X(v) = \sum\limits_{h=0}^{\infty} b_h (\mathrm{ad}(v))^h(X) \\ R_X(v) = \sum\limits_{h=0}^{\infty} (-1)^h b_h (\mathrm{ad}(v))^h(X), \end{cases} \tag{8.5}$$

where the coefficients b_h are defined by

$$\sum_{h=0}^{\infty} b_h t^h = \frac{t}{1 - \exp(-t)} = 1 + \frac{t}{2} + \frac{t^2}{12} + \frac{t^3}{48} + \cdots$$

8.3. Homogeneous spaces

Let $M = \mathbf{G}_0/\mathbf{H}_0$ be a homogeneous space, with base point $\mathrm{p}_0 = [\mathbf{H}_0]$. Fix a linear complement V of $\mathfrak{h}_0$ in $\mathfrak{g}_0$. The map

$$V \times \mathbf{H}_0 \ni (v, x) \longrightarrow \exp(v) \cdot x \in \mathbf{G}_0 \tag{8.6}$$

restricts to a diffeomorphism of the product of a neighborhood of $\{0\} \times \mathbf{H}_0$ onto an open neighbourhood of the identity in $\mathbf{G}_0$. We use the projection on M of $\exp(v)$ to define coordinates near p_0. In fact, if $\varpi : \mathbf{G}_0 \to M$ is the natural projection, with $\varpi(x) = x \cdot \mathrm{p}_0$, then $v \to \varpi(\exp(v))$ is a diffeomorphism of an open neighbourhood of 0 in V onto an open neighbourhood of p_0 in M. Let $\mathcal{F}$ be as in Section 8.1.

In analogy with the definitions of L_X and R_X in Section 8.2, we may introduce a couple of vector fields L_X^* and R_X^* on M, as infinitesimal generators of the one-parameter groups of diffeomorphisms of M, locally defined by

$$\begin{cases} \exp(v) \cdot \exp(tX) = \exp\left(v + t \cdot L_X^*(v) + 0(t^2)\right) \cdot \exp\left(t \cdot H(v) + 0(t^2)\right) \\ \exp(tX) \cdot \exp(v) = \exp\left(v + t \cdot R_X^*(v) + 0(t^2)\right) \cdot \exp\left(t \cdot H'(v) + 0(t^2)\right) \\ \text{with } L_X^*, R_X^* \in \mathcal{F} \otimes V, \quad H, H' \in \mathcal{F} \otimes \mathfrak{h}_0. \end{cases} \tag{8.7}$$

Let us find their formal power series expansions. Set

$$\begin{cases} R_X^*(v) = \sum\limits_{h=0}^{\infty} \mathfrak{x}_h(v), \quad H'(v) = \sum\limits_{h=0}^{\infty} \hbar'_h(v) \\ \text{with } \mathfrak{x}_h \in \mathit{Symm}^h(V, V), \quad \hbar'_h \in \mathit{Symm}^h(V, \mathfrak{h}_0). \end{cases} \tag{8.8}$$

Using (8.5), we obtain

$$\sum_{h=0}^{\infty}(-1)^h b_h(\mathrm{ad}(v))^h(X) = \sum_{h=0}^{\infty}\mathfrak{x}_h(v) + \sum_{h=0}^{\infty}\sum_{r+s=h} b_r(\mathrm{ad}(v))^r(\hslash_s'(v)),$$

yielding by recurrence

$$\begin{cases} X = \mathfrak{x}_0 + \hslash_0' \\ \mathfrak{x}_{h+1}(v) + \hslash_{h+1}'(v) = (-1)^{h+1}b_{h+1}(\mathrm{ad}(v))^{h+1}(X) \\ \qquad\qquad\qquad - \sum_{r=0}^{h} b_{r+1}(\mathrm{ad}(v))^{r+1}(\hslash_{h-r}'(v)) \end{cases} \quad (h \geq 0). \qquad (8.9)$$

Let $\pi : \mathfrak{g}_0 \to V$ be the projection along $\hslash_0$. Equations (8.9) can be used to obtain explicit formulae for $\mathfrak{x}_h$ and $\hslash_h'$:

$$\begin{cases} \mathfrak{x}_0 = \pi(X) \\ \hslash_0' = X - \pi(X) \\ \mathfrak{x}_{h+1}(v) = (-1)^{h+1}b_{h+1}\pi((\mathrm{ad}(v))^{h+1}(X)) \\ \qquad\qquad - \sum_{r=0}^{h} b_{r+1}\pi((\mathrm{ad}(v))^{r+1}(\hslash_{h-r}'(v))) \\ \hslash_{h+1}'(v) = (-1)^{h+1}b_{h+1}(\mathrm{ad}(v))^{h+1}(X) \\ \qquad\qquad - \sum_{r=0}^{h} b_{r+1}(\mathrm{ad}(v))^{r+1}\big(\hslash_{h-r}'(v)\big) - \mathfrak{x}_{h+1}(v). \end{cases} \qquad (8.10)$$

Since the projection $\mathbf{G}_0 \ni x \to x \cdot p_0 \in M$ relates R_X^* with the right invariant vector field on $\mathbf{G}_0$ corresponding to X, we obtain:

Lemma 8.1. *We have*

$$[X^*(v), Y^*(v)] = -[X, Y]^*(v), \quad \forall X, Y \in \mathfrak{g}_0. \qquad (8.11)$$

Analogously, let us set

$$\begin{cases} L_X^*(v) = \sum_{h=0}^{\infty}\mathfrak{y}_h(v), \quad H(v) = \sum_{h=0}^{\infty}\hslash_h(v) \\ \text{with} \quad \mathfrak{y}_h \in \mathit{Symm}^h(V, V), \quad \hslash_h \in \mathit{Symm}^h(V, \hslash_0). \end{cases} \qquad (8.12)$$

From the equation

$$\sum_{h=0}^{\infty} b_h(\mathrm{ad}(v))^h(X) = \sum_{h=0}^{\infty}\mathfrak{y}_h(v) + \sum_{h=0}^{\infty}\sum_{r+s=h} b_r(\mathrm{ad}(v))^r(\hslash_s(v)),$$

we obtain

$$\begin{cases} X = \mathfrak{y}_0 + \hbar_0 \\ \mathfrak{y}_{h+1}(v) + \hbar_{h+1}(v) = b_{h+1}(\mathrm{ad}(v))^{h+1}(X) \\ \qquad\qquad\qquad\qquad - \sum_{r=0}^{h} b_{r+1}(\mathrm{ad}(v))^{r+1}(\hbar_{h-r}(v)), \quad (h \geq 0). \end{cases} \tag{8.13}$$

Equations (8.13) can be used to obtain recursive formulae for $\mathfrak{y}_h$ and $\hbar_h$:

$$\begin{cases} \mathfrak{y}_0 = \pi(X) \\ \hbar_0 = X - \pi(X) \\ \mathfrak{y}_{h+1}(v) = b_{h+1}\pi((\mathrm{ad}(v))^{h+1}(X)) - \sum_{r=0}^{h} b_{r+1}\pi((\mathrm{ad}(v))^{r+1}(\hbar_{h-r}(v))) \\ \hbar_{h+1}(v) = b_{h+1}(\mathrm{ad}(v))^{h+1}(X) - \sum_{r=0}^{h} b_{r+1}(\mathrm{ad}(v))^{r+1}(\hbar_{h-r}(v)) - \mathfrak{y}_{h+1}(v). \end{cases} \tag{8.14}$$

Note that $L_Y^* = 0$ when $Y \in \mathfrak{y}_0$.

To compute the commutator of R_X^* and L_Y^* for a pair $X, Y \in \mathfrak{g}_0$, we use the infinitesimal description of their flows, which can be obtained from (8.7). Set Φ_X for the flow of R_X^* and Ψ_Y for the flow of L_Y^*. Then

$$\exp(v + t^2[R_X^*, L_Y^*](v) + 0(t^3)) = \exp(\Phi_X(-t) \circ \Psi_Y(-t) \circ \Phi_X(t) \circ \Psi_Y(t))(v)).$$

By lifting the action to $\mathbf{G}_0$, we have

$$\begin{cases} \exp(\Phi_X(t)(v)) = \exp(tX)\exp(v)\exp\left(-tH'(v) + 0(t^2)\right) \\ \exp(\Psi_Y(t)(v)) = \exp(v)\exp(tY)\exp\left(-tH(v) + 0(t^2)\right). \end{cases}$$

Then we obtain the composition

$$\begin{aligned} &\exp(-tX)\exp(tX)\exp(v)\exp(tY)\exp\left(-tH(v)+0(t^2)\right)\exp\left(-tH'(v)+0(t^2)\right) \\ &\quad \cdot \exp(-tY)\exp(tH(v))\exp\left(tH'(v)\right) \\ =\, &\exp(v)\exp(tY)\exp(-tH(v))\exp\left(-tH'(v)\right)\exp(-tY)\exp(tH(v))\exp\left(tH'(v)\right) \\ &\quad + 0(t^3) \\ =\, &\exp(v)\exp\left(-t^2[Y, (H(v) + H'(v))]\right)\exp\left(t^2[H(v), H'(v)]\right) + 0(t^3). \end{aligned}$$

This yields:

Lemma 8.2. *For $X, Y \in \mathfrak{g}_0$, we have*

$$\left[R_X^*, L_Y^*\right](v) = L_{[H+H',Y]}^*(v), \tag{8.15}$$

where $H(v)$ and $H'(v)$ are described by (8.14) and (8.10), and the right-hand side is a composition of formal power series.

This lemma tells us that the *infinitesimal translations* of L_w^*, for a $w \in V$, can be expressed as formal power series whose coefficients are L_Y^*'s for Y in the $\mathfrak{h}_0$-module generated by w. In a similar way we also obtain:

Lemma 8.3. *For $Y_1, Y_2 \in \mathfrak{g}_0$, we have*

$$[L_{Y_1}^*, L_{Y_2}^*] = L_{[Y_1,Y_2]+[H_1(v)-H_2(v),Y_1-Y_2]}^*, \tag{8.16}$$

where $H_i \in \mathcal{F} \otimes \mathfrak{h}_0$ is defined by the first line of (8.7), with $X = Y_i$, for $i = 1, 2$.

8.4. General transitive pairs

Let us fix a transitive pair. We recall from Definition 2.1 that it is a couple $(\mathfrak{g}_0, \mathfrak{h}_0)$ consisting of a linearly compact topological Lie algebra $\mathfrak{g}_0$ and of a finite codimensional closed subalgebra $\mathfrak{h}_0$ that does not contain any nontrivial ideal of $\mathfrak{g}_0$.

Let us fix a finite dimensional complement V of $\mathfrak{h}_0$ in $\mathfrak{g}_0$. We define $R_X^*(v)$, $L_X^*(v)$, $H(v)$, $H'(v)$ by (8.8), (8.9), (8.12), (8.13), after noticing that to write these formulae it is not needed that $\mathfrak{g}_0$ be finite dimensional, because the homogeneous summands in the V-coordinates of their Taylor series only involve finite powers of ad(v), finite linear combinations, and the projection $\pi : \mathfrak{g}_0 \to V$ along $\mathfrak{h}_0$. Set

$$\mathfrak{g}_0^* = \{R_X^* \mid X \in \mathfrak{g}_0\}. \tag{8.17}$$

Theorem 8.4. *If $(\mathfrak{g}_0, \mathfrak{h}_0)$ is a transitive pair and V a linear complement of $\mathfrak{h}_0$ in $\mathfrak{g}_0$, then the map*

$$\mathfrak{g}_0 \ni X \longrightarrow R_X^* \in \mathcal{D}er(\mathcal{F}) \tag{8.18}$$

is injective and defines an anti-isomorphism of Lie algebras between $\mathfrak{g}_0$ and $\mathfrak{g}_0^*$, with

$$\mathfrak{h}_0 = \left\{X \in \mathfrak{g}_0 \mid R_X^* \in \mathcal{D}er_0(\mathcal{F})\right\}. \tag{8.19}$$

The correspondence

$$\mathcal{L}_0 \longrightarrow \mathcal{L}_0^* = \mathcal{F} \otimes \left\langle L_w^* \mid w \in V \cap \mathcal{L}_0 \right\rangle \tag{8.20}$$

is a bijection between the set of $\mathfrak{h}_0$-submodules of $\mathfrak{g}_0$ containing $\mathfrak{h}_0$ and $\mathfrak{g}_0^*$-invariant left $\mathcal{F}$-modules of $\mathcal{D}er(\mathcal{F})$. Its inverse is given by

$$\mathcal{L}_0^* \longrightarrow \mathcal{L}_0 = \{Y \in \mathfrak{g}_0 \mid \pi(Y) = \Theta(0) \ \text{for some} \ \Theta \in \mathcal{L}_0^*\}. \tag{8.21}$$

Proof. Let $X \in \mathfrak{g}_0$ and assume that $R_X^* = 0$. We use (8.10). From $\mathfrak{r}_0 = 0$ se obtain that $X \in \mathfrak{h}_0$. Since $b_1 = \frac{1}{2} \neq 0$, the condition that $\mathfrak{r}_1 = 0$ means that $[v, X] \in \mathfrak{h}_0$ for all $v \in V$, and hence $[Y, X] \in \mathfrak{h}_0$ for all $Y \in \mathfrak{g}_0$. In general, we find that $\mathfrak{h}_h(v)$, for each $h \geq 0$, is a multiple of $(\mathrm{ad}(v)^h(X)$ and, arguing by recurrence we obtain that $(\mathrm{ad}(Y)^h(X) \in \mathfrak{h}_0$ for all $h \geq 0$ and $Y \in \mathfrak{g}_0$. This yields that actually $\mathrm{ad}(Y_1) \circ \cdots \circ \mathrm{ad}(Y_h)(X) \in \mathfrak{h}_0$ for all h and $Y_1, \ldots, Y_h \in \mathfrak{g}_0$. To show this fact, we argue again by recurrence on h, as the cases of $h = 0, 1$ are already settled. For $h > 1$, we note that, for $t_1, \ldots, t_h \in \mathbb{R}$,

$$t_1 \cdots t_h \cdot \mathrm{ad}(Y_1) \circ \cdots \circ \mathrm{ad}(Y_h)(X)$$
$$= \frac{1}{h!} \big[\mathrm{ad}(t_1 Y_1 + \cdots + t_h Y_h) \big]^h (X) + \sum_i 0(t_i^2)$$
$$+ \text{ terms of the form } \mathrm{ad}\left(Y_1'\right) \circ \cdots \mathrm{ad}\left(Y_r'\right)(X) \text{ with } r < h.$$

By the recursive assumption, the coefficient of the monomial $t_1 \cdots t_h$ in the right-hand side is an element of $\mathfrak{h}_0$ and hence $\mathrm{ad}(Y_1) \circ \cdots \circ \mathrm{ad}(Y_h)(X) \in \mathfrak{h}_0$. We proved that the kernel of the map $X \to R_X^*$ is an ideal of $\mathfrak{g}_0$ contained in $\mathfrak{h}_0$. Therefore it is $\{0\}$ if $(\mathfrak{g}_0, \mathfrak{h}_0)$ is a transitive pair.

The conclusion of Lemma 8.1 only depends on the formal definition of R_X^* in (8.8) and (8.9) and therefore is still valid, yielding (8.18).

Also the validity of Lemma 8.2 depends only on the formal definitions of R_X^* and L_Y^* and therefore shows that, when $\mathcal{L}_0$ contains $\mathfrak{h}_0$ and $[\mathfrak{h}_0, \mathcal{L}_0] \subseteq \mathcal{L}_0$, then the left $\mathcal{F}$-module $\mathcal{L}_0^*$ generated by $\{L_w^* \mid w \in V \cap \mathcal{L}_0\}$ satisfies $[\mathfrak{g}_0^*, \mathcal{L}_0^*] \subseteq \mathcal{L}_0^*$. Vice versa, if $\mathcal{L}_0^*$ is a left $\mathcal{F}$ submodule of $\mathcal{D}er(\mathcal{F})$ with $[\mathfrak{g}_0^*, \mathcal{L}_0^*] \subseteq \mathcal{L}_0^*$, then the set $\mathcal{L}_0$ of $Y \in \mathfrak{g}_0$ such that $\pi(Y)$ is the value in 0 of a vector field in $\mathcal{L}_0^*$ is a subspace of $\mathfrak{g}_0$ containing $\mathfrak{h}_0$ and satisfying $[\mathfrak{h}_0, \mathcal{L}_0] \subseteq \mathfrak{h}_0$. Indeed $[X, w]$ is the value at 0 of $[R_X^*, L_w^*]$. This yields the correspondence (8.20), completing the proof of the theorem. $\qquad \square$

We already noted that the map $Y \to L_Y^*$ is linear. In particular, it can be extended by $\mathbb{C}$-linearity to the case where Y belongs to the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$. Then the second part of the statement of Theorem 8.4 extends to the case of *complex vector distributions*. We denote by $\mathfrak{h} \subseteq \mathfrak{g}$ the complexification of $\mathfrak{h}_0$.

Theorem 8.5. *The correspondence*

$$\mathcal{L} \longrightarrow \mathcal{L}^* = \mathcal{F} \otimes \langle L_w^* \mid w \in V \cap \mathcal{L} \rangle \tag{8.22}$$

is a bijection between the set of $\mathfrak{h}$-submodules of $\mathfrak{g}$ containing $\mathfrak{h}$ and $\mathfrak{g}_0^*$-invariant left $\mathbb{C} \otimes \mathcal{F}$-modules of $\mathbb{C} \otimes \mathcal{Der}(\mathcal{F})$. Its inverse is given by

$$\mathcal{L}^* \longrightarrow \mathcal{L} = \left\{ Y \in \mathfrak{g} \mid \pi(Y) = \Theta(0) \; for \; some \;\; \Theta \in \mathcal{L}^* \right\}. \qquad (8.23)$$

9. Extensions

Definition 9.1. We say that a contact triple $(\mathfrak{g}_0', \mathfrak{h}_0', \mathcal{L}_0')$ *extends* the contact triple $(\mathfrak{g}_0, \mathfrak{h}_0, \mathcal{L}_0)$ if there is an injective homomorphism of real Lie algebras $\phi : \mathfrak{g}_0 \to \mathfrak{g}_0'$ such that $\phi(\mathcal{L}_0) \subseteq \mathcal{L}_0'$, $\phi(\mathfrak{h}_0) \subseteq \mathfrak{h}_0'$ and the quotient maps $(\mathfrak{g}_0/\mathcal{L}_0) \to (\mathfrak{g}_0'/\mathcal{L}_0')$ and $(\mathfrak{g}_0/\mathfrak{h}_0) \to (\mathfrak{g}_0'/\mathfrak{h}_0')$ induced by ϕ are linear isomorphisms.

We say that a *CR* algebra $(\mathfrak{g}_0', \mathfrak{q}')$ *extends* the *CR* algebra $(\mathfrak{g}_0, \mathfrak{q})$ if there is an injective Lie algebras homomorphism $\phi : \mathfrak{g}_0 \to \mathfrak{g}_0'$, whose complexification we still denote by ϕ, such that $\phi(\mathfrak{q}) \subseteq \mathfrak{q}'$ and the induced map on the quotients $\mathfrak{g}_0/(\mathfrak{q} \cap \bar{\mathfrak{q}} \cap \mathfrak{g}_0) \to \mathfrak{g}_0'/(\mathfrak{q}' \cap \bar{\mathfrak{q}}' \cap \mathfrak{g}_0')$ and $(\mathfrak{g}/\mathfrak{q}) \to (\mathfrak{g}'/\mathfrak{q}')$ are linear isomorphisms. To deal with extensions, it is convenient to introduce a common Lie algebra in which we can embed both a given Lie algebra and its extension.

Proposition 9.2. *Let $(\mathfrak{g}_0, \mathfrak{h}_0, \mathcal{L}_0)$ be a contact triple. Then there is a maximal contact triple $(\mathfrak{g}_0', \mathfrak{h}_0', \mathcal{L}_0')$ extending $(\mathfrak{g}_0, \mathfrak{h}_0, \mathcal{L}_0)$, which is unique modulo isomorphisms.*

Let $(\mathfrak{g}_0, \mathfrak{q})$ be a CR algebra. Then there is a maximal CR algebra $(\mathfrak{g}_0', \mathfrak{q}')$ extending $(\mathfrak{g}_0, \mathfrak{q})$, which is unique modulo isomorphisms.

Proof. The statement follows from Theorems 8.4 and 8.5. If $\mathcal{L}_0^*$ is the $\mathcal{F}$-module corresponding to $\mathcal{L}_0$, we define $\mathfrak{g}_0'^*$ as the Lie algebra of formal vector fields stabilising $\mathcal{L}_0^*$ and $\mathfrak{g}_0'$ equals to its opposite Lie algebra.

Likewise, in the case of a *CR* algebra, we take $\mathfrak{g}_0'^*$ to be the stabiliser of $\tilde{\mathfrak{q}}^*$ in $\mathcal{Der}(\mathcal{F})$ and define $\mathfrak{g}_0'$ to be its opposite Lie algebra. $\square$

The finiteness result of Section 7 applies to give informations about the global and local *CR* automorphisms on homogeneous and locally homogeneous *CR* manifolds.

The analytic Lie subgroup of a Lie group $\mathbf{G}_0$ generated by a Lie subalgebra $\mathfrak{h}_0$ of its Lie algebra $\mathfrak{g}_0$ may fail to be closed. In this case the pair $(\mathfrak{g}_0, \mathfrak{h}_0)$ is associated to a *locally $\mathbf{G}_0$-homogeneous* manifold, *i.e.* a smooth open manifold M_0 having the property that the elements of a small open neighborhood of the origin of $\mathbf{G}_0$ act as a transitive group of local diffeomorphisms on an open neighborhood of a base point p_0 of M_0, and $\mathfrak{h}_0$ is the Lie algebra of the stabilizer of p_0 for this action (see, *e.g.*, [17, 22]). We can give in an obvious way a notion of *locally $\mathbf{G}_0$-homogeneous CR manifold*, that we employ in the formulation of the following result.

Theorem 9.3. *Every contact nondegenerate CR algebra $(\mathfrak{g}_0, \mathfrak{q})$ admits an essentially unique maximal extension $(\mathfrak{g}_0', \mathfrak{q}')$, which is finite dimensional and is therefore a CR algebra of a locally homogeneous CR manifold whose CR automorphisms form a Lie group of transformations.*

Theorem 9.4. *Let $\mathbf{G}_0$ be a Lie group and M_0 a locally $\mathbf{G}_0$-homogeneous CR manifold, with associated CR algebra $(\mathfrak{g}_0, \mathfrak{q})$. If $(\mathfrak{g}_0, \mathfrak{q})$ is fundamental and contact non-degenerate, then the local CR automorphisms of M_0 generate a finite dimensional Lie group.*

References

[1] A. ALTOMANI, C. MEDORI and M. NACINOVICH, *The CR structure of minimal orbits in complex flag manifolds*, J. Lie Theory **16** (2006), 483–530.

[2] A. ALTOMANI, C. MEDORI and M. NACINOVICH, *On homogeneous and symmetric CR manifolds*, Boll. Unione Mat. Ital. (9) **3** (2010), 221–265.

[3] A. ALTOMANI, C. MEDORI and M. NACINOVICH, *Orbits of real forms in complex flag manifolds*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **9** (2010), 69–109.

[4] A. ALTOMANI, C. MEDORI and M. NACINOVICH, *Reductive compact homogeneous CR manifolds*, Transform. Groups **18** (2013), 289–328.

[5] M. S. BAOUENDI, P. EBENFELT and L. P. ROTHSCHILD, "Real Submanifolds in Complex Space and their Mappings", Princeton Mathematical Series, Vol. 47, Princeton University Press, Princeton, NJ, 1999.

[6] J. F. CONN, "Nonabelian Minimal Closed Ideals of Transitive Lie Algebras", Mathematical Notes, Vol. 25, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1981.

[7] J. F. CONN, *On the structure of real transitive Lie algebras*, Trans. Amer. Math. Soc. **286** (1984), 1–71.

[8] E. B. DYNKIN, *Calculation of the coefficients in the Campbell-Hausdorff formula*, Doklady Akad. Nauk SSSR (N.S.) **57** (1947), 323–326.

[9] G. FELS and W. KAUP, *Classification of Levi degenerate homogeneous CR-manifolds in dimension 5*, Acta Math. **201** (2008), 1–82.

[10] V. W. GUILLEMIN, *A Jordan-Hölder decomposition for a certain class of infinite dimensional Lie algebras*, J. Differential Geom. **2** (1968), 313–345.

[11] V. W. GUILLEMIN and S. STERNBERG, *An algebraic model of transitive differential geometry*, Bull. Amer. Math. Soc. **70** (1964), 16–47.

[12] A. ISAEV and D.I ZAITSEV, *Reduction of five-dimensional uniformly Levi degenerate CR structures to absolute parallelisms*, J. Geom. Anal. **23** (2013), 1571–1605.

[13] C. MEDORI and M. NACINOVICH, *Levi-Tanaka algebras and homogeneous CR manifolds*, Compositio Math. **109** (1997), 195–250.

[14] C. MEDORI and M. NACINOVICH, *Algebras of infinitesimal CR automorphisms*, J. Algebra **287** (2005), 234–274.

[15] C. MEDORI and A. SPIRO, *The equivalence problem for five-dimensional Levi degenerate CR manifolds*, Int. Math. Res. Not. IMRN (2014), 5602–5647.

[16] C. MEDORI and A. SPIRO, *Structure equations of Levi degenerate CR hypersurfaces of uniform type*, Rend. Semin. Mat. Univ. Politec. Torino (2015), 127–150.

[17] G. D. MOSTOW, *The extensibility of local Lie groups of transformations and groups on surfaces*, Ann. of Math. (2) **52** (1950), 606–636.

[18] S. POCCHIOLA, *Explicit absolute parallelism for 2-nondegenerate real hypersurfaces $M^5 \subset \mathbb{C}^3$ of constant Levi rank 1*, arXiv:1312.6400, 2013.

[19] C. PORTER, *The local equivalence problem for 7-dimensional, 2-nondegenerate CR manifolds whose cubic form is of conformal unitary type*, arXiv:1511.04019, 2015.

[20] C. PORTER and I. ZELENKO, *Absolute parallelism for 2-nondegenerate CR structures via bigheaded Tanaka prolongation*, arXiv:1704.03999, 2017.

[21] A. Santi, *Homogeneous models for Levi degenerate CR manifolds*, Kyoto J. Math. **60** (2020), 291–334.

[22] A. Spiro, *A remark on locally homogeneous Riemannian spaces*, Results Math. **24** (1993), 318–325.

[23] N. Tanaka, *On generalized graded Lie algebras and geometric structures. I*, J. Math. Soc. Japan **19** (1967), 215–254.

[24] N. Tanaka, *On differential systems, graded Lie algebras and pseudogroups*, J. Math. Kyoto Univ. **10** (1970), 1–82.

[25] H. Weyl, "The Classical Groups. Their Invariants and Representations", Princeton University Press, Princeton, N.J., 1939.

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