Risk estimation for short-term financial data through pooling of stable fits

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Abstract We suggest a new, parsimonious, method to fit financial data with a stable distribution. As a result of a stable fitting *via* Maximum Likelihood Estimation (MLE), we find that some assets have similar values as stability indices, independently of the time interval considered. This fact can be exploited to pool the assets in groups and to choose a parameter α as an *ex ante* stability index, valid for every asset in the pool-sector. With this fixed parameter, MLE is used again to obtain the other stable parameters. We discuss an innovative risk measure, based on the Expected Shortfall, which exploits the above procedure. We show that it gives a good estimation of risk even when only short time series are available. Finally we introduce the notion of *Risk Class*, which allows us to classify assets according to their risk exposition and to compare different methods for the computation of the Expected Shortfall.

Keywords Stable distribution · heavy tails · stability index · sector pool · expected shortfall · risk class.

JEL classification C13, C51, C53, G17

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1 Introduction

For a long time, normal distributions have been used to model the log-variation of asset prices. Yet, several empirical studies, such as those by Rachev and Mittnik (2000), Carr *et al.* (2002), Stoyanov *et al.* (2006), Nolan (2014), and Bianchi (2015), showed that the distributions of asset log-returns are *leptokurtic*, *i.e.,* their tails are heavier than those of a Gaussian distribution, thus implying a higher probability of extreme phenomena. Such tails should not be underestimated, because in the last years, and especially after the crisis in 2008, extreme events proved to be crucial for financial models. In 2010, the Basel Committee introduced the stress tests in order to evaluate the performance of financial institutions under extreme events, namely, Value at Risk and Expected Shortfall must be computed on a period of 250 days of financial stress chosen over a period of 1-4 years (they are called respectively stressed Value at Risk and stressed Expected Shortfall). Basel III, which will be fully operational in 2027, prescribes the stressed Expected Shortfall to be the only measure for market risk (BCBS 2013).

As a consequence, alternative distributions have been considered, such as mixtures of normal distributions, hyperbolic, gamma, Student's t and stable distributions (we recall, among the others, Rydberg (1999), Aas and Haff (2006), Marinelli *et al.* (2007) and Dokov *et al.* (2008)). In the Sixties, Mandelbrot (1963) and Fama (1963) proposed to use *stable* distributions of which normal distribution are a special case and which turn out to be very efficient in modeling the case of extreme events, such as market breakdowns or natural catastrophes, the so called *black swans* in the Gaussian universe[1]. The use of stable distributions in a financial context has been widely investigated (see, among others, Rachev and Mittnik (2000), Bradley and Taqqu (2003), Nolan (2003), Rachev *et al.* (2003), Stoyanov *et al.* (2006), Xu *et al.* (2011), Misiorek and Weron (2012) and Gabriel and Lau (2014). As pointed out by Rachev *et al.* (2011, p.7): "*If changes in a stock price, interest rate, or any other financial variable are driven by many independently occurring small shocks, then the only appropriate distributional model for these changes is a stable model* (*normal or non-normal stable*)." The present work aims to contribute to this area, by confirming that stable distributions can be effectively used for risk evaluation and by proposing a specific innovative method for individuating the most appropriate stable distribution, at least when only a short time series is available.

We compare several models for the distribution of the daily logarithmic returns of the assets of NYSE Composite and of NASDAQ Composite, on three time windows of different widths, respectively, 20-years, 10-years and 2-years. As a first, preliminary result, we find that, on all of the three different time windows, the stable distribution fits data better than the Gaussian one. Yet, for a given asset, the Maximum Likelihood Estimation (MLE) stable fit

[1] Nowadays, the concept of a *black swan* fully belongs to the usual language of the "financial culture", also because of the success of the controversial bestseller of Taleb (2010).

on the three time windows generates different parameters, and, in particular, three different stability indices. The longest time series are of course the most reliable, but there is a large number of stocks for which only short time series are available and the estimated stability index does not contain enough information for a reliable evaluation of risk. The main aim of this paper is to propose and implement a method, based on an appropriate choice[2] of the stability index, which allows us to use data from the "shorter" series to obtain an accurate evaluation of risk if compared to risk estimated on longer time series. In order not to make the presentation too heavy, we report here mainly the results related to the 20-years distribution.

The analysis of all the stability indices generated by the stable fit on the 20-years time series shows that assets coming from the same industrial sector have similar αs. To properly identify sectors, we choose the Industry Classification Benchmark (ICB), which is a system of classification launched by Dow Jones and FTSE in 2005. It is a globally recognized standard which categorizes companies and securities, by using a system of ten industries. We then group together assets within the same Industry group and propose to choose the median of the αs in each group as the "representative" stability index for all the assets with the same classification. It will be shown that such an α seems to yield more reliable fits for short time series than the α obtained from the series themselves. Furthermore, such an "α-pool" turns out to be very effective in adjusting or completing the valuation whenever time series are incomplete.

As a next step, we discuss a risk measure, RedES$^{\mathrm{TM}}$ (R* for short) based on our choice of the stability index: the measure R* of a stock is its Expected Shortfall, computed with the stable distribution, with the α-pool parameter[3]. We also introduce the concept of *Risk Class* in order to synthetically compare methods for the computation of the Expected Shortfall. We then show that R* (and the corresponding Risk Class) provide a reliable estimation of risk, comparable with the evaluation based on 20-years data, and therefore that it proves to be a helpful tool in classifying assets according to their risk exposition, also when only a short data set is available.

The paper is organized as follows. In Section 2, we briefly present some preliminaries about Pareto stable distributions. In Section 3, the data set and the fitting process are described. In Section 4, the distribution of the estimated stability indices is analysed, showing how data can be pooled into sectors according to this parameter and the α-pool parameter is defined. In Section 5, the risk measure R* and the notion of Risk Class are introduced. A discussion on a representative set of assets is the subject of Section 6. Section 7 concludes and suggests further developments of research.

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[2] A very preliminary step of this research can be found in Donati and Corazza (2014).

[3] RedES$^{\mathrm{TM}}$ has been developed and introduced in 2004 by Redexe and has, since then, been used for risk management by several Italian financial institutions. The patent is currently pending.

2 Stable distributions: definition and basic properties

Hereafter, we recall some basic notions about Pareto stable distributions: in particular, we concentrate on the properties and parametrization which are mainly used in the paper. For further details, we refer the reader to Feller (1966), Samorodnitsky and Taqqu (1994), Shiryaev (1996), Nolan (2003) and Rachev *et al.* (2011).

Definition 1 A real-valued and non degenerate random variable X is said to be (*Pareto*) *stable* (or to have a *stable distribution*) if, for every $n \geq 2$, there exist n independent random variables $X_1,\ X_2, \ldots, X_n$ distributed as X and constants $a_n > 0$, $b_n \in \mathbb{R}$ such that $X_1 + X_2 + \cdots + X_n$ is distributed as $a_n X + b_n$.

This condition can also be expressed in terms of the characteristic function φ. The random variable (r.v.) X is said to be stable if

$$[\varphi(u)]^n = [\varphi(a_n u)] \exp(\mathrm{i} b_n u)$$

Moreover, it can be proved (see, for instance, Feller (1966), Theorem VI.I.1.) that $a_n = n^{1/\alpha}$, with $0 < \alpha \leq 2$.

We adopt the parametrization of Samorodnitsky and Taqqu (1994) and Rachev *et al.* (2011), according to which a random variable X is stable if and only if its characteristic function has the form

$$\varphi(u) = \begin{cases} \exp\left[\mathrm{i}\mu u - c^\alpha |u|^\alpha \left(1 - \mathrm{i}\beta (\operatorname{sgn} u) \tan \frac{\pi\alpha}{2}\right)\right] & \alpha \neq 1 \\ \exp\left[\mathrm{i}\mu u - c|u|\left(1 + \mathrm{i}\beta \, \frac{2}{\pi} \, (\operatorname{sgn} u) \log |u|\right)\right] & \alpha = 1 \end{cases},$$

where $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$, $\mu \in \mathbb{R}$, $c > 0$. We briefly recall the meaning of the four parameters:

- $\alpha \in (0, 2]$ is the so-called *stability index* (it is also called *characteristic exponent*). It is an indicator of the asymptotic behaviour of the distribution. In particular, α determines the weight of the tails and the peakedness of the distribution, *i.e.,* the kurtosis of the random variable. Small values of α indicate heavy tails and a high peak, *i.e.,* leptokurtosis.
- $\beta \in [-1, 1]$ is a *skewness* parameter. It measures the asymmetry of the distribution: $\beta = 0$ is equivalent to symmetry with respect to μ; $\beta > 0$ (< 0) is equivalent to skewness to the right (to the left).
- $\mu \in \mathbb{R}$ is a *shift* parameter. It only affects the location of the distribution.
- $c > 0$ is a *scale* parameter. It measures the width of the distribution.

We will shortly write $X \sim S_\alpha(\beta, \mu, c)$. The probability density function can be obtained *via* the Fourier inversion Theorem: if $\int_{\mathbb{R}} |\varphi(u)| \mathrm{d}t < +\infty$, then X has density:

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\mathrm{i}ux} \varphi(u) \mathrm{du}$$

However, a closed-form expression for the density function can be given only in three cases, namely for the Gaussian ($\alpha = 2$, independently of β), the Cauchy ($\alpha = 1$, $\beta = 0$) and the Lévy ($\alpha = 1/2$, $\beta = \pm 1$) distributions.

Among the numerous properties of a stable r.v. X, let us mention:

(i) the *invariance of the distributional form under linear transformations* (as a consequence of the stability property), namely:
  - if X_1, X_2 are independent r.v.s with $X_i \sim S_\alpha(\beta_i, \mu_i, c_i)$, $i = 1, 2$, then $X_1 + X_2 \sim S_\alpha(\beta, \mu, c)$, with

$$c = (c_1^\alpha + c_2^\alpha)^{1/\alpha}, \quad \beta = \frac{\beta_1 c_1^\alpha + \beta_2 c_2^\alpha}{c_1^\alpha + c_2^\alpha}, \quad \mu = \mu_1 + \mu_2;$$

  - if $X \sim S_\alpha(\beta, 0, c)$, then $-X \sim S_\alpha(-\beta, 0, c)$
  - if $X \sim S_\alpha(\beta, \mu, c)$, then $X + a \sim S_\alpha(\beta, \mu + a, c)$
  - if $X \sim S_\alpha(\beta, \mu, c)$ and $a \neq 0$, then aX is still stable, with stability index α and the other parameters depending on α.

(ii) the *power-tail decay*: the tail of the density of a (non-normal and with $\beta \neq \pm 1$) stable random variable decays like a power function (Pareto law of index α), *i.e.,* slower than an exponential function as it occurs for normal distributions:

$$\Pr(|X| > x) \propto k x^{-\alpha}, \quad x \to \infty$$

for some appropriate constant k. Hence, the tails of the density function behave as $|x|^{-\alpha-1}$. Moreover, if $\beta = 1$ (respectively $\beta = -1$), the negative tail (respectively the positive tail) decreases faster than the Pareto law of index α.

(iii) *moments*: if $\alpha < 2$, then $\mathbb{E}[|X|^r] < \infty$ if and only if $0 < r < \alpha$. Hence, the expected value of X is finite if and only if $\alpha > 1$ and, in this case, $\mathbb{E}[X] = \mu$. When $\alpha < 2$, variance, skewness and kurtosis are always infinite[4].

(iv) the *generalized Central Limit Theorem*: Gnedenko and Kolmogorov extended the Central Limit theorem, obtaining the following famous necessary and sufficient condition:
Theorem. *A non degenerate random variable X is stable if and only if there exists a sequence of i.i.d. random variables $\{X_n\}$ and constants $a_n > 0$ and b_n such that:*

$$a_n (X_1 + X_2 + \cdots + X_n) - b_n \to X,$$

where the limit is in distribution.

(v) the *infinitely divisible* property: a stable random variable is infinitely divisible, namely for each integer $n \geq 1$, there exist n i.i.d. random variables $X_1, X_2, \ldots, X_n$ such that X is distributed as $X_1 + X_2 + \cdots + X_n$.

[4] We remark that c does not represent the standard deviation, since for $\alpha < 2$ it is infinite. For $\alpha = 2$, namely when X is normally distributed, the standard deviation σ is related to c by the equation $\sigma = \sqrt{2}c$.

3 Data and preliminary remarks

We used Datastream[5] to get the time series of the prices of all 5,502 assets in the NYSE Composite (2,872 assets) and NASDAQ Composite (2,630 assets) indices. The first observation in the set dates back to January 1, 1973, and the last observation date for all assets is March 18, 2019. For a comprehensive view, for each stock, we collect the following data: economic sector, market value and date of the first quotation of the asset on the market. In order to have a sufficiently rich set of data to use for comparing fits of the same assets over time intervals of different lengths, we restricted the sample and identified the following three time windows:

- 20 years: March 18, 1997 to March 18, 2017 (1,764 assets)
- 10 years: March 18, 2007 to March 18, 2017 (3,010 assets)
- 2 years: March 18, 2015 to March 18, 2017 (4,262 assets).

For the 1,764 assets with the 20-years time series, we also collected the data from March 19, 2017 to March 18, 2019 to test our model on "out-of-sample" data.
We use the time series of the daily closing prices, adjusted for possible dividends and stock splits[6], to construct the daily log-returns: $r_t = \log(P_t/P_{t-1})$ where P_t is the daily closing price of an asset.

On such data, we compute the frequency, the empirical mean, the standard deviation, the skewness and the kurtosis. We observe that on each time window and for every asset, the empirical mean is close to 0. The standard deviation is smaller than 0.07, and in general smaller on the 2-years time series. There is a greater variability in the values for the skewness, but, apart from a few cases, it takes values between -1 and 1. Lastly, as expected, kurtosis is significantly greater than 3, suggesting a leptokurtic distribution. Although it is an established practice to employ a normal distribution to fit the distribution of log-returns, several empirical studies have shown that α stable distributions better capture the features of empirical data (as the high kurtosis). We recall the work by Hoechstoetter *et al.* (2005) on German stock data and by Xu *et al.* (2011) on the Chinese market and refer to these papers for references to the analysis of different markets. In analogy with them, for every asset, we fit the time series of the log-returns on each of the mentioned three windows with two distributions: (i) the normal distribution with density function $f(x \mid \mu, \sigma)$ and parameter vector $\underline{\vartheta} = (\mu, \sigma)$; ($ii$) the stable (non-normal) distribution with density function $f(x \mid \alpha, \beta, c, \mu)$ and parameter vector $\underline{\vartheta} = (\alpha, \beta, c, \mu)$.

[5] Datastream is one of the most important international financial time series database, due to Thomson Reuters (Datastream 2019).

[6] In our time series, we observe the presence of null returns for some assets. Most likely, these values correspond to days where, for some reasons, there was no trading on some particular asset, though there have been movements in the market (namely, if there had been some trading on that asset, than its price would have changed). In our opinion, these zeroes do not reflect adequately the real behaviour of the price of the asset, therefore we decided to eliminate them introducing a *filter on zero* (null filter), obtaining time filtered series.

The parameters are estimated *via* MLE method. The estimation of the stable parameters is based on an algorithm developed by Nolan (1997, 1999), which exploits the Zolotarev's (M) parameterization of the characteristic function.

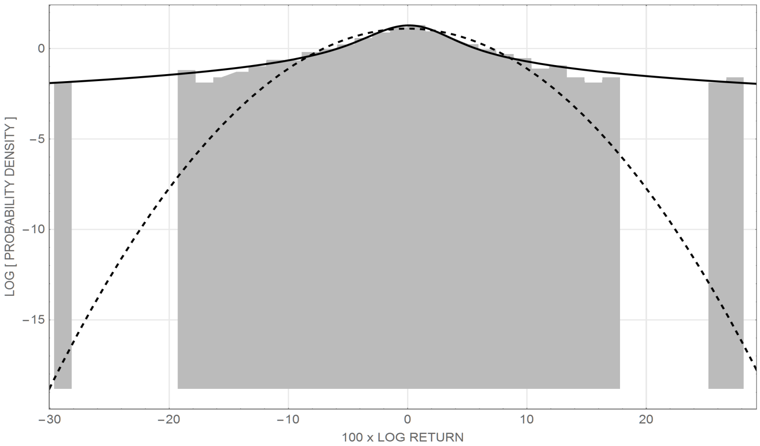

Fig. 1 Normal (dashed line) and α stable (solid line) fits for log-returns of Golden Ocean Group asset (20-years data)

As an example, we show in Fig. 1 the normal and the α stable (non-normal) distribution on the 20-years time series of Golden Ocean Group[7] (GOGL). The grey area represents the relative frequencies of the daily log-returns. For the purpose of drawing the empirical data, the horizontal axis has been divided into 50 return intervals (note that the MLE fitting procedure does not depend on the discretisation).

It is evident that the higher flexibility of the α stable distributions allows for better capturing the extreme values (tails): a family of distributions with four parameters certainly gives better results than a subfamily with only two parameters. Anyway, the question is whether the complication of using four, instead of two, parameters is justified or not by more satisfying results. In our opinion, the increase in the number of parameters is widely repaid, mainly in terms of the better accuracy in describing the tails of the observed distributions, which often represents the most critical point in the usual descriptions, especially when evaluating risk. Moreover, the method proposed in the next section considerably reduces this complication, making the estimation of the parameters faster and more precise, even when a small set of data is available.

[7] Golden Ocean Group Limited is a leading international dry bulk shipping company, founded in 1996, based in Bermuda and listed on NASDAQ and on the Oslo Stock Exchange.

4 Pooling by ICB group

The results of our estimations show us that it is difficult to find a reliable value for the stability index of the stable distribution, because different time windows generate different αs. This may be an issue when estimating risk, because α characterizes the shape, and hence the tails, of the distribution and it is, therefore, one of the most relevant parameters to the purpose of measuring risk (the other one is c, which measures the width of the distribution).

Indeed, when few observations are available, tails tend to be underestimated, both by the empirical and theoretical distributions, if no extreme events occurred (which is likely, for short or medium time windows). On the other hand, tails may be overestimated if extreme values are observed in such a short or medium period, thus leading to a too prudent evaluation of risk. It seems therefore reasonable that the values estimated on the 20-years time series are the most significant[8], and our analysis confirms this assumption. However, for many assets only short data sets may be available, for instance when they have been quoted only for a few years. Therefore, we wonder whether we could find some sort of *ex ante* α for shorter data sets, to be updated from time to time (say, every 5 or 10 years).

To investigate this issue, we analyse the distribution of the stability indices on the different time windows: we report the relative frequencies of the estimated αs in Fig. 2. The stability index takes values mainly in the interval $[1.50, 1.90]$, consistently with the results found by Hoechstoetter *et al.* (2005). On the 2-years time window, we see that α assumes mainly values between

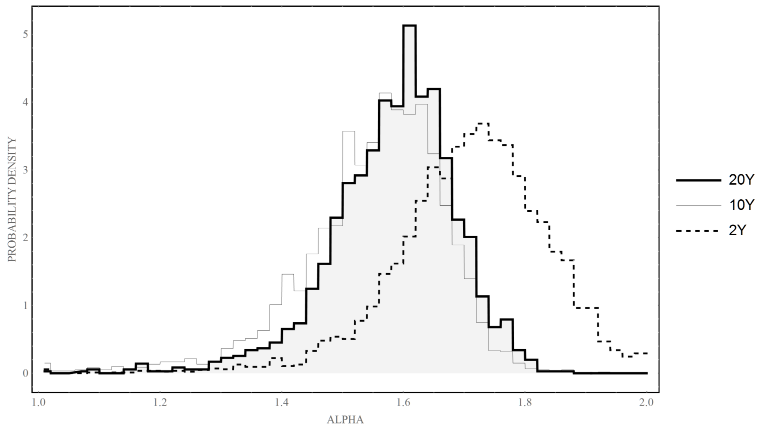

Fig. 2 Relative frequencies of α obtained from stable fitting of the 2-years, 10-years and 20-years series

1.60 and 1.90, that is, the distributions of the log-returns show thin tails,

[8] The analysis in Hoechstoetter *et al.* (2005) and Xu *et al.* (2011) both consider a period of about 15 years.

quite close to those given by normal distributions (for which $\alpha = 2$). As we observed above, in a short time interval, extreme values are rare and if some extreme event occurs, its effect is magnified, producing a very low level of α, which explains why the distribution of α is quite wide: there are some assets for which the 2-years α is smaller than the 20-years one (see the discussion in Section 6). In the 10-years case, where some extreme events are observed, the values of α lie mainly in the interval $[1.45, 1.75]$. Finally, in the 20-years time windows, the distribution of the stability index is more concentrated around its central values (the median is 1.594, the mean 1.584), which are slightly higher than the corresponding values over the 10-years series (the median is 1.564, the mean 1.551). The longest time series incorporate a number of shocks and rare events[9] (*e.g.*, for a firm, relevant increases or decreases of the sales, bankruptcy, mergers, acquisitions, *etc.*; in a more general sense, the consequences of a global phenomenon, as in 2008) sufficient to obtain a reliable value of α for the corresponding asset. At the same time in a 20-years period, the effect of the shocks is scaled down. So the value of the stability index in the last case turns out to be a sort of intermediate value between the cases of 2-years and 10-years series.

Fig. 2 shows that there are several assets with estimated stability indices really close one to another. This suggests the possibility to pool together assets with approximately the same parameter in order to have a larger number of data for its estimation. Since assets coming from the same sector have similar αs, we investigate for a possible link between the value of the stability index of an asset and the sector of the issuing company. In order to identify possible sectors, we follow the ICB (Industry Classification Benchmark), which divides the market into ten groups depending on the nature of the core business of the underlying activities: Basic Materials, Consumer Goods, Consumer Services, Financials, Health Care, Industrials, Oil & Gas, Technology, Telecommunications, and Utilities. We then analyse group by group the different values of α obtained from the stable fits of the previous section. The result from such fittings on the 20-years data is represented in Fig. 3. Bar heights (horizontal lengths) are proportional to the number of assets with α in the given interval; bar widths (vertical segmentation) are automatically determined by the function *DistributionChart* of *Mathematica 11.*

The value of α is represented as a function of the sector. The width of the horizontal bars indicates the numerosity of the assets which share the same α, whereas the values of αs are hosted on the vertical axis. Fig. 3 confirms that, apart from some outliers, the αs of all the assets in the same sectors are close. Therefore we pool together the assets in the same Industrial Group and assign to each ICB group a common value α_p as stability parameter. We choose as α_p the median[10] of all the αs in the group, because it seems to provide a

[9] For the meaning of *rare event*, see also the interesting discussion on the notion of *extremal event* in the Reader Guidelines of the book by Embrechts *et al.* (1996).

[10] An alternative choice for the stability parameter could be the mean of the group. We performed all the computations, assuming the mean as stability parameter: the final results,

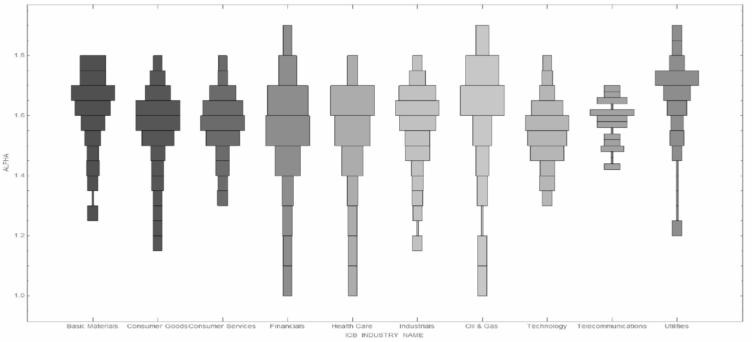

Fig. 3 Frequencies of α depending on sectors, 20-years data

well-balanced parameter (half of the assets in the pool have a lower α, while the other half a higher one).

As mentioned above, our idea is to consider this value as a fixed *ex ante* value for the stable fits on a 2-years data set. In this way, only the three remaining parameters need to be estimated. Furthermore, α_p could provide a better prediction for the fatness of the tails for short time series, since it incorporates information on a longer period. Such a value may remain valid for a few years, before being updated, considerably reducing the computational time. We call *α-pool fitting* this process and *α-pool distribution* the distribution obtained in this way.

In order to verify the reliability of this choice, we test this value on out-of-sample data: namely, we compute the stable fit on the 2-years data set 2017–2019, using the α_p computed on the 20 years 1997–2017. As we will show in the next example, α-pool fitting will turn out to be more adequate than classical stable fitting.

Group	**Samples**	**Mean**	**Median α_p**
Basic Materials	91	1.640	1.652
Consumer Goods	174	1.580	1.589
Consumer Services	176	1.575	1.583
Financials	491	1.570	1.579
Health Care	161	1.567	1.590
Industrials	336	1.585	1.601
Oil & Gas	97	1.643	1.663
Technology	149	1.542	1.548
Telecommunications	21	1.585	1.594
Utilities	68	1.676	1.701

Table 1 ICB main groups and α-statistics on the 20-years data set

in terms of estimation of risk, are very similar. Moreover, in most cases, the median provides a better fit in terms of distributions.

4.1 The α-pool fitting: an example

We now apply the procedure described above to the Golden Ocean Group asset, whose 20-years return frequencies and the above fits are reported in Fig. 1.

In order to test our method, we consider the out-of-sample time series from March 19, 2017 to March 18, 2019. The 2-years stable fit on the log-returns of such a series is represented in Fig. 4 (dashed line): the estimated value for the stability index is 1.776. We then perform the α-pool fitting, setting the stability parameter to be the group parameter α_p, which equals 1.601 because the Golden Ocean Group asset belongs to the Industrials group. The three remaining parameters obtained from the fitting procedure yield the distribution represented by the solid line in Fig. 4. We observe that the α-pool fit presents fatter tails and performs better than the stable fit, since it better captures extreme events.

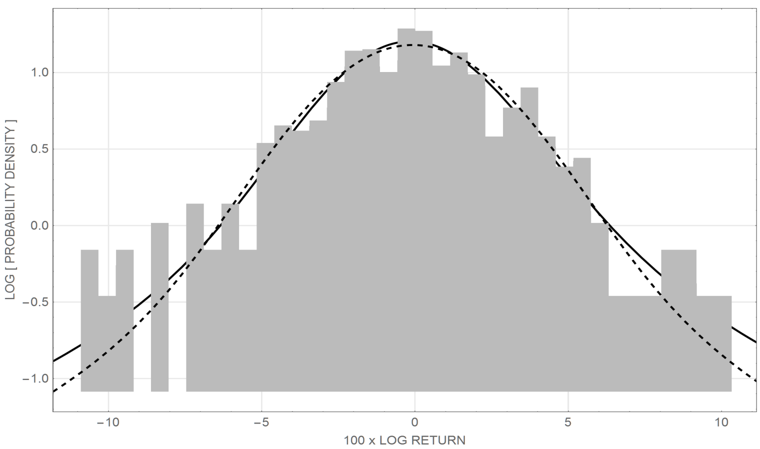

Fig. 4 α stable (dashed line) and α-pool fits (solid line) of Gold Ocean Group asset (2-years data)

The result of the two fittings and of the normal fitting are collected in Table 2, and compared with the stable fit on the 20-years time series from March 18, 1997 to March 18, 2017.

	α	β	μ	c	σ
Normal	(2)	(0)	-0.00006	0.022	0.031
2y Stable	1.776	-0.002	-0.00073	0.019	–
α-pool Stable	1.601	-0.075	-0.00124	0.017	–
20y Stable	1.499	-0.079	-0.00085	0.015	–

Table 2 Parameters estimated on the Golden Ocean Group asset on 2-years data, compared with the parameters estimated on 20-years data

We conclude with a qualitative remark, as a sort of further confirmation of the effectiveness of our method. In Fig. 5, we report the stable fit (dotted line) on the 20-years time series, the stable fit (dashed line) on the 2-years time series and the α-pool fit (solid line), on the empirical distribution on 20 years for the Golden Ocean Group asset. We observe how, differently from the

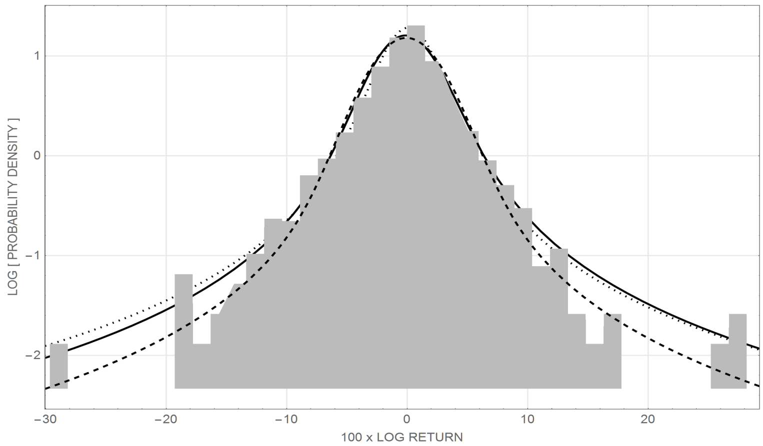

Fig. 5 α stable (dashed line) and α-pool (solid line) fits of Golden Ocean Group asset (2-years data), plotted against α stable (dotted line) and empirical fits (20-years data)

2-years α stable distribution, the α-pool fit well complies with the empirical distribution on 20 years, though three of its parameters come from the 2-years sample, which confirms the prevailing role of the stability parameter with respect to the other parameters. Of course, on the left side of the graphic, the stable fit on the 20-years fits better, since it pops out of 20-years data. However, even if three of the four parameters have been estimated on only two years of data, the α-pool fit turns out to provide a quite accurate representation of the data of the past twenty years, and so to be a good estimator, which can be used when 20-years data are not available.

Summarizing, α-pool gives essentially the following advantages: it gives a more reliable estimation of α for the stocks in the group, reduces the number of the parameters which have to be estimated and, as we will see in the next section, allows us to exploit 2-years data to give a reliable estimation of future potential losses.

5 RedES$^{\mathrm{TM}}$ and Risk Classes

In this section we exploit the results of the previous section in order to suggest a different way to measure risk, as an alternative to classical methods as, for instance, Value at Risk (VaR for short) and Expected Shortfall (ES for short).

Our aim is to provide a method which allows to obtain a reliable value with a reduced data set.

A benchmark widely adopted by financial institutions is the *Value at Risk*[11] which quantifies the potential loss in value of a risky asset or portfolio over a defined period, estimating the loss level which will not be exceeded within a given confidence level. It is however well-known that VaR presents some limitations. For instance, it is not sub-additive and any loss beyond the VaR level is neglected. As the Basel Committee observes, "A number of weaknesses have been identified with using VaR for determining regulatory capital requirements, including its inability to capture 'tail risk' " (BCBS 2013, p. 7). For this reason, a new, more reliable, measure has been introduced in Artzner *et al.* (1999): the *Expected* Shortfall (or Conditional Value at Risk or *Expected Tail Loss*)[12] which estimates the expected loss, when it exceeds VaR, taking into consideration the shape (distribution) of the tails. ES has better theoretical properties and is also more robust from a statistical point of view[13]. Furthermore, generally ES valuations do not present larger statistical errors than VaR estimations (Acerbi, 2004). The Expected Shortfall is usually computed from the empirical distribution of data or from a theoretical distribution, considered as a good fit for data. A normal distribution is often chosen, because of the availability of closed formulas for computation. Stable distributions have been employed as theoretical distributions for the calculation of both VaR and ES (see, among others, Khindanova *et al.* (2001), Aussenegg and Miazhynskaia (2006), Harmantzis *et al.* (2006), Stoyanov *et al.* (2006) and Misiorek and Weron (2012)). However, if data are available only on a short term period, the obtained value for the Expected Shortfall is not reliable since the tails may be underestimated, both by the empirical and theoretical distributions, if no extreme events occurred, or overestimated if extreme values were observed in such a short period. The analysis of the previous sections showed that the α-pool fitting produces a good fit, better than the classical stable fit, when a small data set is available. Therefore we propose the following risk measure:

Definition 2 We call *RedES*™ (in short R*) of a given asset the daily Expected Shortfall at confidence level q, computed on the log-returns of the α-pool distribution.

[11] For a portfolio with a random payoff X and distribution function F_X the *Value at Risk* at confidence level $q \in (0,1)$ is given by

$$\mathrm{VaR}_q(X) = -\inf\{x \in \mathbb{R} : F_X(x) \geq q\}.$$

which coincides with $F_X^{-1}(1)$ when F_X is strictly increasing.

[12] The Expected Shortfall at confidence level q of a portfolio with payoff X is defined as

$$\mathrm{ES}_q(X) = \frac{1}{q}\int_0^q \mathrm{VaR}_r(X)\mathrm{d}r$$

[13] A good discussion about ES can be found in Fabozzi and Tunaru (2006). For a comparison between VaR and ES, see, among others, Yamai and Yoshiba (2002).

In addition, in order to easily compare risk magnitudes, we define the concept of Risk Classes, through a one-to-one correspondence with the risk measures.

Definition 3 Given an asset with some risk level $x \leq 0$, we call *Risk Class* of the asset the number

$$\mathrm{rc}(x) = \log_2(1 - 100x).$$

For completeness, we assign class 0 to risk-free assets. If the risk level x of an asset is the Expected Shortfall or R* at confidence level q, we call the Risk Class ES or R* *Risk Class*.

A Risk Class is a risk measure too, playing the role of a sort of an instantaneous return rate[14]. It does not contain different information with respect to ES or R*, but allows to make homogeneous different risk indicators. Risk Classes provide the set of assets with a preorder: every step up (or down) in a Risk Class approximately corresponds to doubling (or halving) risk. Indeed, simple algebra shows that $\mathrm{rc}(x) = \mathrm{rc}(y) + 1$ if and only if $x = 2y - 0.01$.

To assess the effectiveness of R*, we compute the Expected Shortfall on the log-returns of the 2-years out-of-sample data set (2017–2019), using respectively the normal fit, the 2-years stable fit and the α-pool fit (the latter based on the 20-years in-sample data). For those assets that have a long enough time series, we compare the above ESs with the Expected Shortfall on the stable fit of the 20-years data, intended as a sort of benchmark. According to the most recent Basel Committee recommendations (BCBS 2013), we set $q = 97.5\%$.

We compare the Risk Classes generated by the methods listed above with the benchmark ES Risk Class obtained by the α stable fitting on the 20-years time series[15]. For every asset, we compute the difference between each of the 2-years Risk Classes (normal fit, stable fit, α-pool fit) and the stable 20-years ES Risk Class. This comparison is illustrated in Fig. 6, where we depict the frequency functions of such differences.

We underline that it is not always true, as one could expect, that a lower stability index (hence heavier tails) denotes a riskier asset. All other parameters are involved in the evaluation of the Expected Shortfall. For instance, a higher c, which is usually observed when passing from a 2-years time series to the 20-years one, means that the distribution is wider. This also explains why the stable ES Risk Class computed on the 20-years data set is generally higher.

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[14] Think of the analogy between the classical expression $\delta = \ln(1 + i)$, which connects the force of interest with the annual compound interest rate i, and the expression $\mathrm{rc(ES)} = \log_2(1 + |\mathrm{ES}|)$; ES < 0.

[15] Briefly, when computing the R* measure for a 2-years time series from 2017 to 2019, we use the α-pool parameter calculated on 20-years time series from 1997 to 2017, which does not "overlap" the short series. The first submitted version of this paper, based on data collected in September 2011, used a different approach: the α-pool parameter was calculated on 1991-2011 time series and then used to calculate R* of 2009-2011 time series, entirely contained in the long one. It is noteworthy that both approaches yield remarkably similar results, and we thank the anonymous referee that suggested such a check for leading us to such a double confirmation of our method.

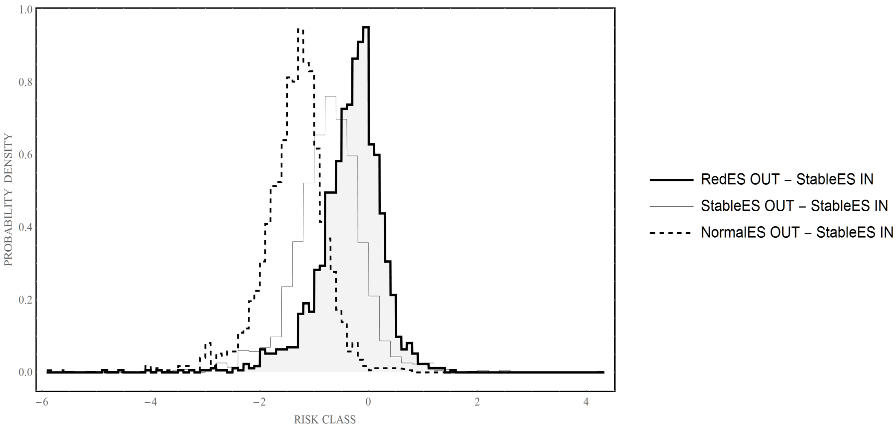

Fig. 6 Frequencies of Risk Class Differences

In most cases, the R* Risk Class is really close to the 20-years stable ES Class. Indeed, the distribution of the differences between R* and 20-years stable Risk Classes (solid, thick line in Fig. 6) is approximately concentrated around 0. Though there are some outliers, for which the difference is smaller than -1 (greater than 1 in absolute value), on average the R* Risk Class seems to be a good approximation of the 20-years stable Risk Class. The 2-years stable distribution presents higher differences (solid, thin line). There is still a considerable number of assets for which Risk Class is close to the 20-years stable Risk Class, but there is also a large number of assets for which the calculated risk is too small. On average, the 2-years stable distribution underestimates the Risk Class of 0.5, hence its performance is worse than R*. As we could expect, the normal distribution (dashed line) behaves really badly, underestimating Risk Classes of approximately 1.5. We can therefore conclude that R* risk measure gives a reliable prediction of risk and works better than other risk measures computed on the same, short, data set.

After testing R* on 1,764 assets, for which the 20-years time series are available, we find that R* Risk Class is, in general, the best approximation of the 20-years stable Risk Class. This leads us to use R*, rather than usual stable fits, to measure the risk of all the assets for which only 2-years time series are available. Of course, there is nothing against using R*, or the R* Risk Classes, also for assets with longer available time series, or in different and/or wider contexts.

5.1 An example

In this section we test our method on the time series of the log-returns of the Golden Ocean Group asset. All results are collected in Table 3.

	Expected Shortfall	**Risk Class**
Empirical, OUT	−0.0842	3.236
Empirical, 20y	−0.0969	3.418
Normal, OUT	−0.0710	3.018
Normal, 20y	−0.0734	3.060
Stable, OUT	−0.1095	3.578
Stable, 20y	−0.1945	4.354
α-Pool	−0.1574	4.066

Table 3 Expected Shortfalls and Risk Classes for Gold Ocean Group asset

As a first step, we compute the Expected Shortfall on the empirical distribution of log-returns, both for the out-of-sample 2-years data and for the 20-years, in-sample, data. The empirical ES Risk Class is smaller for the shorter time series, because in 20 years more rare events occurred (for instance, the asset had its worst performance in December 2015).

We then evaluate the normal fits of log-returns on both time series. The Risk Classes obtained are lower than the corresponding empirical ones. This confirms that normal distributions tend to underestimate the tails and are unfit to measure risk.

With the α stable fit, we obtain more coherent values with the empirical data on both the time windows. In particular, stable ES Risk Classes are greater than empirical ES Risk Classes, namely the estimation of risk is more conservative and, in our opinion, more realistic.

Finally, we compute R* Risk Class and find an intermediate value between the 2-years stable Risk Class and the 20-years stable Risk Class:

$$\underbrace{3.578}_{\text{stable ES Risk Class (2y)}} < \underbrace{4.066}_{\text{R}^* \text{ Risk Class (2y)}} < \underbrace{4.354}_{\text{stable ES Risk Class (20y)}} .$$

Therefore, once again, R* seems to give a good estimate of the overall 20-years risk, even if it is computed on a 2-years windows only.

6 Data analysis

The analysis described in the previous section for the Golden Ocean Group asset has been carried out on all the 1,764 assets with 20-years data. In most cases we observe a behaviour similar to the Golden Ocean Group, but there are a few cases where things seem to go differently. Here we present the results on a set of assets which in our opinion is quite representative since it covers all sectors and try to depict a, as various as possible, scenario. Note that the different behaviors do not depend on the particular sector, but are transversal.

In Table 4 we collect the assets we consider in this last analysis, reporting their sector, their stability indices respectively estimated on the 2-years and on the 20-years data set, and the α-pool stability index α_p. In Table 5 we calculate the Risk Classes computed according to the several methods we mentioned.

Ticker	Company	Sector	α-2Y	α-20Y	α_p
ABX	Barrick Gold Corporation		1.708	1.764	
CLF	Cleveland Cliffs	Basic Material	1.829	1.557	1.652
OLN	Olin		1.841	1.655	
JJSF	J&J SnackFood Corporation		1.769	1.598	
KO	CocaCola Company	Consumer Goods	1.747	1.587	1.589
MAT	Mattel Inc.		1.536	1.642	
ANF	Abercrombie and Fitch		1.612	1.629	
CMCSA	ComCast Corporation		1.733	1.562	
DIS	Walt Disney Company	Consumer Services	1.730	1.623	1.583
FL	Foot Locker Inc.		1.651	1.581	
MCD	McDonald's Corporation		1.607	1.653	
JPM	JP Morgan Chase & Co.		1.727	1.482	
MUFG	Mitsubishi UFJ Fin. Group	Financials	1.871	1.688	1.579
PICO	Pico Holdings		1.761	1.606	
IDXX	Idexx Laboratories		1.739	1.454	
JNJ	Johnson & Johnson	Healthcare	1.649	1.593	1.590
PFE	Pfizer Inc.		1.635	1.662	
BA	Boeing		1.570	1.682	
TXT	Textron	Industrials	1.707	1.570	1.601
PDCE	PDC Energy		1.880	1.666	
XOM	Exxon Mobile Corporation	Oil & Gas	1.702	1.737	1.663
MSFT	Microsoft Corporation		1.534	1.588	
SYMC	Symantec	Technology	1.643	1.482	1.548
USAT	USA Technologies		1.705	1.682	
T	AT&T		1.654	1.569	
TDS	Telephone and Data Sys	Telecommunications	1.637	1.604	1.594
MGEE	MGE Energy Inc.		1.834	1.715	
SWX	Southwest Gas Corporation	Utilities	1.794	1.732	1.701
WTR	Aqua America Inc.		1.726	1.688	

Table 4 Sectors and αs

Ticker	Empirical 20y	Normal 20y	Stable 20y	Empirical 2y	Normal 2y	Stable 2y	R*
ABX	3.127	2.913	3.407	2.768	2.466	3.167	3.338
CLF	3.743	3.360	4.376	3.180	3.127	3.427	3.981
OLN	3.072	2.759	3.555	2.620	2.506	2.843	3.354
JJSF	2.884	2.643	3.552	2.298	2.052	2.523	3.000
KO	2.371	2.125	3.032	1.976	1.624	2.142	2.583
MAT	2.887	2.624	3.288	3.259	3.045	4.136	3.939
ANF	3.512	3.143	3.927	3.468	3.247	4.075	4.208
CMCSA	2.924	2.657	3.663	2.567	2.229	2.870	3.328
DIS	2.769	2.491	3.318	2.190	1.901	2.512	2.917
FL	3.126	2.901	3.815	3.393	3.054	3.494	3.740
MCD	2.427	2.201	2.960	2.009	1.820	2.709	2.788
JPM	3.086	2.795	4.020	2.173	1.927	2.470	2.856
MUFG	2.977	2.795	3.444	2.195	2.054	2.416	3.259
PICO	3.077	2.827	3.718	2.612	2.403	2.752	3.245
IDXX	3.165	2.897	4.065	2.515	2.285	3.185	3.658
JNJ	2.218	2.005	2.855	2.321	1.854	2.583	2.757
PFE	2.583	2.334	3.041	2.118	1.803	2.636	2.767
BA	2.779	2.490	3.247	2.568	2.273	3.285	3.186
TXT	3.215	2.842	3.649	2.429	2.103	2.930	3.284
PDCE	3.486	3.198	3.995	3.105	2.912	3.281	3.957
XOM	2.447	2.225	2.816	2.141	1.853	2.603	2.714
MSFT	2.799	2.511	3.392	2.373	2.117	3.343	3.294
SYMC	3.353	3.002	4.236	3.150	2.830	3.320	3.651
USAT	4.220	4.104	4.623	4.277	3.824	4.059	4.642
T	2.589	2.328	3.360	2.526	2.067	2.861	3.040
TDS	2.899	2.574	3.468	2.837	2.491	3.200	3.349
MGEE	2.547	2.339	3.002	2.162	1.993	2.419	2.820
SWX	2.437	2.209	2.781	2.281	2.031	2.447	2.687
WTR	2.512	2.310	3.018	2.234	1.821	2.680	2.759

Table 5 Risk classes

As a general fact, the data confirm that, in each time window, normal Risk Classes are always the smallest and that empirical Risk Classes tend to be smaller than the corresponding stable ones. Moreover, in many cases, the empirical Risk Class computed on the 20-years data set is smaller than the Risk Class on the 2-years stable distribution, confirming that stable distributions, even on short data set, give a more conservative valuation of risk[16].

Let us mention again that, owing to the role of the remaining parameters, not always a lower stability index denotes a riskier asset. This of course affects the determination of the Risk Classes. In many cases, the 20-years stable Risk Class is higher than the 2-years one, even when the 20-years α is higher than the 2-years α, because of the wider distribution, highlighted by a higher value of c. In these cases, the role of α does not appear particularly relevant. Conversely, in most cases, given α_p, the estimation of the remaining parameters gives values for β, μ and c very close to those of the 2-years stable distribution. Therefore, when comparing the 2-years stable Risk Class and the R* Risk Class on a certain asset, the role of α is predominant. The main difference between the stable and the α-pool distributions is in the tails, which depend on α and a higher stability index corresponds to a smaller Risk Class. In some cases, it seems that taking a smaller α on a 2-year distribution *compensates* the small c due to the short number of data, providing a more conservative evaluation of risk.

In a greater detail, we observe that CLF, KO, CMCSA, FL, JPM, IDXX, TXT, SYMC, T and WTR, which belong to different sectors, behave like Golden Ocean Group. Both the value of α_p and the R* Risk Class are intermediate between the 2-years stable and the 20-years stable fits. Thus, even when the difference between the 20-years stable and R* Risk Classes may appear big (as for SYMC), R* gives however the best estimation among the Risk Classes computed on 2 years, namely the difference between R* Risk Class and the 20-years stable Risk Class is minimal. Moreover, for some of them, as CLF or TXT , R* gives also the best approximation for the empirical Risk Class on 20 years, which in these cases is underestimated by the 2-years stable Risk Class.

For a different set of assets, like OLN, JJSF, DIS, MUFG, PICO, JNJ, PDCE, TDS, MGEE and SWX, the value of α_p is smaller than the other stability indices, and in particular much smaller than the 2-years α. Nevertheless, the resulting Risk Classes are between the 2-years and the 20-years ones. We also observe that some of these assets have a quite high 2-year stability index (greater than 1.8) which implies an underestimation of risk: the 2-year stable Risk Classes are even smaller than the 20-years empirical ones. In such cases R* allows to obtain a more balanced valuation, adjusting the underestimation.

For some assets, like ABX, MCD, PFE and XOM, the 2-years α is smaller than the 20-years one, but the other parameters are such that the 20-years Risk Class is higher than the 2-years one. The parameter α_p is smaller than

[16] In Harmantzis *et al.* (2006), the performance of different models (empirical, Gaussian, generalized Pareto and stable distributions) is empirically tested in evaluating Value at Risk and Expected Shortfall and similar results are obtained.

both of them, as well as in the previous case, but the R* Risk Class is between the 2-years and the 20-years Risk Class, which once more provides a good evidence that R* Risk Class is a well balanced measure.

The asset MAT is anomalous: not only the 2-years α is smaller than the 20-years α, but also the 2-years Risk Class is higher than the 20-years Risk Class. This reflects the same order that we observe in the empirical Risk Classes and suggests that some extreme event occurred during the 2-years window, heavily affecting the empirical and the stable distribution. Indeed, from 2016, Mattel, Inc. suffered heavy losses because of a decrease in the net sales (also as a result of the liquidation of Toys'R'us in 2018) [17]. The presence of an intermediate value for α_p causes the R* Class to better approximate the 20-years Risk Class.

For few cases, like ANF and USAT, the value of α_p is so small with respect to the 2-years and 20-years parameters that the corresponding Risk Class is higher. While for USAT, R* Risk Class is very close to the 20-years stable Risk Class, for ANF, the 2-years stable Risk Class is between the stable 20-years Risk Class (which is the lower) and the R* Risk Class. This suggests that the combination of a low α_p and the occurrence of some extreme event[18] in the last two years, pushes our estimator to be more conservative (which is not necessary a bad thing).

There is a set of assets where things appear to go worse as, for instance, MSFT and BA. For this class of assets the 2-years stable α is smaller than α_p and the R* Risk Class is the smallest, so it underestimates risk. We observe however that for both the assets, the distance between the R* Risk Class and the 2-years stable Risk Class is quite small, meaning that the indications provided by 2-years Risk Classes and R* classes are substantially equivalent. Moreover, R* classes are not too far from the 20-years ones.

Lastly, for some assets, especially in the Financial sector (see, for instance, JPM), the R* Risk Class is much smaller than the 20-years stable one and also smaller than the 20-years empirical Risk Class. Most likely, the effects of the crisis of 2008 is very well present in the data series, hence reflected in the 20-years stable distribution, but affects in a minor way the value of R*. However, among the risk measures based on 2-years data, R* is the one with the better performance.

Obviously, for the assets with a reduced time series of data, the above analysis is not possible. But, in our opinion, the previous results allows us to think of R* as a good candidate for a reliable estimate of risk, exactly because obtained by the examination of a wider "population" (in terms of assets and of data). Nevertheless, without going into details, also for this kind of assets, we could observe that, as a general fact, normal Risk Classes are smaller than the other ones as well, empirical Risk Classes are smaller than the corresponding stable ones and the value of stability index is prevailing in

[17] See Mattel. Annual Reports (2019).

[18] In May 2017, the company was in talks to sell itself. In July, negotiations failed and on 10 July the stock fell 21 percent (Abercrombie & Fitch. Annual Reports 2019; Bray 2017).

determining the shape of the stable distribution. As a consequence, smaller values of the stability indices provide higher Risk Classes and *vice versa*. In most of cases $\alpha_p < \alpha$: α-pool process adjusts stable valuation from 2-years data taking into account longer time series coming from other assets in the pool. So, again α-pool valuation is more conservative than the stable one. In some cases, $\alpha_p > \alpha$, but the values are not too far from each other.

In conclusion, R^* is not to be intended as a “magic recipe” which always works as the best one, but it seems to be reliable and able, most of the times, to adjust and balance over/underestimations of stable fit.

6.1 A simple exercise of backtesting

It is well-known that ES valuation methods are difficult to backtest. Backtesting process results easier for VaR than for ES and this is one of the reasons for which, in the past, the Basel Committee has been reluctant in adopting ES as measure to valuate market risks[19]. There is a vast literature about it and the debate is still open (see, among others, Righi and Ceretta (2013), Acerbi and Székely (2014) and Brie *et al.* (2018)).

Thus, in this paper, we limit ourselves to propose a simple ‘backtesting exercise’, postponing a more specific analysis for a next study. We compare the values of R^* Risk Classes computed on past data with the empirical Risk Classes computed on the last 20 years and see how many times the R^* Risk Classes are smaller than the empirical ones. For the computation of R^*, we exploit the data[20] from September 15, 1991 to September 15, 2011. For the comparison, we take as benchmark the empirical Risk Classes on the 20-years data from 1997 to 2017. These twenty years include 2008, when for many assets the largest losses were registered, so we consider this value quite representative (a sort of stressed value of empirical ES), though the period partially overlaps the period of our estimation. The analysis has been performed on 1,289 assets. The R^* Risk Classes underestimate the risk in 129 cases (10% of the assets), but for the 69 % of these assets the difference between R^* Risk Classes and the empirical ones is lower than 0.1. Only 11.63 % of the assets for which the risk is underestimated (15 over 129, that is 1.16 % of the assets) present a value of R^* Risk Class which is more than 0.5 different from the empirical Risk Class and only for two of them the difference is greater than 0.9 and, in any case, smaller than 1.

As observed in the previous section, there are some assets, like JPM, for which the R^* Risk Class computed on 2017–2019 does not completely reflect the losses of the 20 years 1997–2017. For these assets, the R^* Risk class computed on the 2009-2011 data is sufficiently conservative to be greater than the

[19] Currently, Basel III asks to use ES to estimate capital requirements for market risk and VaR for backtesting (BCBS, 2013, Appendix B).

[20] The data had been collected for a previous version of this paper. In that case, the sector stability index was computed on the 20 years 1991-2011, while R^* was calculated on the 2 years 2009-2011.

empirical Risk Class computed on 20 years[21]. Furthermore, we observe that there is a large set of assets, as ANF, for which in 2011 only a 2-years data set was available. We observe that for these assets, the stability index for the computation of R* was based on the 20-years time series of the assets belonging to the same sector as ANF. The obtained value for R* proves to be a good estimation for future risk[22].

7 Conclusions and further developments

In the last years, the need to properly consider extreme events into financial models repeatedly emerged, due to the fact that rare events are not so negligible as assumed before 2008[23]. In 2010, the Basel Committee introduced the stress tests (even if referred to VaR) in order to evaluate the performance of financial institutions at the presence of the extreme events observed in the last years. Basel III, which will be fully operational in 2027, considers, for market risk, only stressed Expected Shortfall (BCBS 2013). So, in the last years, the necessity to have a tool capable to take into the due consideration extreme events became very pressing, and stable distributions began to play an important role in financial scenarios. In this paper, we supported the use of stable distributions for modeling financial assets and suggested a method to employ them for measuring risk in an efficient way, even when only short time series are available for the estimation of parameters. We fitted stable distributions on a large class of stocks from the US market on different time windows and observed that assets in the same Industry classification have a similar stability index. This suggested to consider a whole Industry group as a "pool" of assets with a common α parameter, thus leading to the identification of ten values for α, each to be used as the *ex ante* value for all the assets in its group. Hence, we derived the (stable) α-pool distribution with α computed from 20-years data and the remaining three parameters from 2-years data. We defined a new risk measure, the Expected Shortfall on the α-pool distribution, and compared it with the Expected Shortfall on the 20-years stable distribution, which we chose as a benchmark, to find that our measure gives, on short time series, a good prediction of future risk.

Thus, by our scheme, a cross-sectional information is used to reduce an estimation error that would otherwise need a longer time series and therefore less recent and relevant information for the current market situation.

This paper represents a first step in this direction and several improvements can be done. First of all, our analysis could be extended to different time windows: we thought of a decade-long investigation, but it could be reasonable

[21] The R* Risk class for JPM estimated in the period 2009-2011 is 3.917. The empirical Risk Class for the period 1997–2017 is 3.086.

[22] The R* Risk class for ANF estimated in the period 2009-2011 is 3.589. The empirical Risk Class for the period 1997–2017 is 3.512.

[23] For instance, between September and October 2008, the Standard & Poor's 500 index decreased of 25.9% and, on August 2015, the Shanghai Composite Index has fallen by 8.49% just on one day.

to work with a different time scaling. It may be useful to investigate whether a finer partitioning of the market (for instance, pooling assets by ICB subsectors) yielded better performances.

Moreover, for the calibration of the parameters, we used the algorithm developed by Nolan (1997, 1999). When α is greater than 1, this method works well and is very precise, but it is quite slow. As a development of the research, we intend to try out different methods, as the one proposed by Hassannejad Bibalan *et al.* (2017). Besides, about the choice of α for the α-pool distribution, it could be interesting to test a different α obtained by combining the assets of a pool as a first step and then fitting a single α *per* pool, while leaving the remaining parameters free for each individual asset.

Also, the impact of the capitalization of each asset in the market on the estimation of the parameter and the extension of the analysis to multidimensional distributions, in order to measure the risk of portfolios, are issues which deserves attention. A further issue is that stable distributions present some critical issues (see, for instance, Varga (1998)): the variance does not exist and generally densities have not a closed form. Stable distributions do not capture the phenomenon of volatility clustering. Empirical data show that the right tails of the distribution of log-returns must be heavier than normal tails, but thinner than stable tails. Modifications of these distributions have been introduced, as smoothly truncated and tempered stable distributions (see, for instance, Boyarchenko and Levendorskiî (2000), Rachev *et al.* (2005), Soltani *et al.* (2010) and Grabchak (2016)). We aim to extend our method to these distributions.

Empirical studies usually ignore the lack of independence of the returns. A quite ambitious project for the future is to find a way to extend our procedure to stable processes which are conditionally heteroscedastic (see, for instance, Panorksa *et al.* (1995), Hauksson and Rachev (2001) and Bonato (2012)).

A final consideration is worth to be mentioned. The transition from the Gaussian world to a stable one and, consequently, to new indicators, like RedES$^{\mathrm{TM}}$, could have an impact similar to the transition from constant to stochastic volatility. The original successful model of Black and Scholes was based on constant volatility, but its application made its inadequacy transparent. Over time, stochastic volatility models were developed and nowadays they are commonly accepted. In our opinion, for Pareto stable distributions, the same innovative perspective change is at hand, opening the doors to richer and more reliable models.

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