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NONLINEAR STABILITY RESULTS FOR THE MODIFIED MULLINS-SEKERKA AND THE SURFACE DIFFUSION FLOW

E. Acerbi, N. Fusco, V. Julin, M. Morini

Abstract

It is shown that any three-dimensional periodic configuration that is strictly stable for the area functional is exponentially stable for the surface diffusion flow and for the Mullins-Sekerka or Hele-Shaw flow. The same result holds for three-dimensional periodic configurations that are strictly stable with respect to the sharp-interface Ohta-Kawaski energy. In this case, they are exponentially stable for the so-called modified Mullins-Sekerka flow.

Contents

1. Introduction

In this paper we establish new global-in-time existence and long-time behavior results in three-space dimensions for two physically relevant geometric motions; namely, the modified Mullins-Sekerka and the surface diffusion flows. Let Ω be a bounded open set of $\mathbb{R}^N$. We start by recalling that a smooth flow of sets $(E_t)_t \subset\subset \Omega$, defined on some (maximal) time interval $(0, T^*)$, is a solution of the modified (or nonlocal) Mullins-Sekerka flow if the evolution is governed by the following law

$$\begin{cases} V_t = [\partial_{\nu_t} w_t] & \text{on } \partial E_t, \\ \Delta w_t = 0 & \text{in } \Omega \setminus \partial E_t, \\ w_t = H_t + 4\gamma v_t & \text{on } \partial E_t, \\ -\Delta v_t = u_{E_t} - \fint_\Omega u_{E_t}\,, & \text{in } \Omega, \end{cases} \tag{1.1}$$

Key words and phrases. Mullins-Sekerka flow, Hele-Shaw flow, Ohta-Kawaski energy, gradient flows, asymptotic stability, global-in-time existence, large-time behavior, stable periodic structures.

where both w_t and v_t are subject to homogeneous Neumann boundary conditions on $\partial\Omega$ or to *periodic boundary conditions* in the case $\Omega = \mathbb{T}^N$, with $\mathbb{T}^N$ denoting the N-dimensional flat torus. Here and in the following V_t stands for the outer normal velocity of the moving boundary ∂E_t, H_t denotes the mean curvature of ∂E_t, $\gamma \geq 0$ is a fixed parameter, $u_{E_t} := 2\chi_{E_t} - 1$ and $[\partial_{\nu_t} w_t]$ is a short notation for the jump of the normal derivative of w_t at ∂E_t; more precisely, $[\partial_{\nu_t} w_t] := \partial_{\nu_t} w_t^+ - \partial_{\nu_t} w_t^-$, with w_t^+ and w_t^- denoting the restrictions of w_t to $\Omega \setminus E_t$ and E_t, respectively. In the case $\gamma = 0$ the *potential* v_t becomes irrelevant and we recover the classical Mullins-Sekerka flow (see [**35**]), which is also sometimes referred to as the *two-phase Hele-Shaw flow with surface tension* (see [**16**]). Such models arise as singular limits of the Cahn-Hilliard equation in the case $\gamma = 0$, as formally derived in [**38**] and then rigorously proved in [**2**], and of a modified (nonlocal) version of the Cahn-Hilliard equation in the case $\gamma > 0$. Such a modified equation has been proposed in [**37**] to describe phase separation in diblock copolymer melts and its convergence to (1.1) has been established in [**29**]. Under Neumann boundary conditions if $\gamma = 0$ and $(E_t)_t \subset\subset \Omega$ Alexandrov's Theorem implies that the only possible equilibria for (1.1) are union of balls. On the contrast, in the periodic case or when $\gamma > 0$ the sets of equilibria has a much richer structure as we will see below.

The second geometric flow we are dealing with is the motion of sets *by surface diffusion*; in this case the evolution of E_t is governed by the law

$$V_t = \Delta_\tau H_t \qquad \text{on } \partial E_t, \tag{1.2}$$

where Δ_τ denotes the surface Laplacian or Laplace-Beltrami operator on ∂E_t. Such a law has been proposed in the physical literature to describe the evolution of interfaces between solid phases driven by surface diffusion of atoms under the action of a chemical potential (see for instance [**21**] and the references therein).

The two flows share several features: they are both *volume preserving* and may be regarded as suitable gradient flows of the (nonlocal) area functional (also known as sharp-interface Ohta-Kawasaki energy):

$$J(E) := P_\Omega(E) + \gamma \int_\Omega \int_\Omega G(x,y) u_E(x) u_E(y)\, dxdy\,, \tag{1.3}$$

where P_Ω is the standard perimeter (or area) functional in Ω, while G stands for the Green's function in Ω and $u_E := 2\chi_E - 1$. More precisely, (1.1) can be seen as the gradient flow of (1.3) with respect to a suitable $H^{-\frac{1}{2}}$-Riemannian structure (see for instance [**29**]) formally defined on the space of shapes, while (1.2) is the gradient flow of the area functional, that is of (1.3) with $\gamma = 0$, with respect to a H^{-1}-type Riemannian structure (see [**7**]). In contrast with the more standard mean curvature flow, one cannot expect a comparison principle to hold for (1.1) and (1.2). This makes it very difficult to apply weak methods such as those based on the notion of viscosity solution.

Since in fact singularities (such as pinching) may form in finite time (see for instance [**5**, **33**]), as far as smooth flows are concerned one can only expect in general local-in-time existence and uniqueness: see [**8**] and [**16**, **39**] for the Hele-Shaw model in the two-dimensional and the n-dimensional case, respectively, [**15**] for the modified Mullins-Sekerka flow, and [**12**] and [**14**] for the motion by surface diffusion in two and higher dimensions, respectively. For a very weak (distributional) notion of global-in-time solution to the Mullins-Sekerka flow in three dimensions, obtained via a minimizing movements approach, we refer to [**45**]. Finally,

we remark that, again in contrast with the motion by mean curvature, both (1.1) and (1.2) do not preserve convexity (see [**25**, **13**]).

The nonlocal area functional (1.3) is the sharp-interface limit of the so-called ε-diffuse Ohta-Kawasaki energy, which was proposed in [**37**] to model the behavior of a class of two-phase materials called *diblock coplymers*. From the mathematical point of view, the main new feature is the presence of a nonlocal Green's function term, which acts as a long-range repulsive interaction of Coulombic type. While the perimeter term favors the formation of large connected regions of pure phases with minimal interface area, the double integral term prefers scattered configurations with several tiny connected components that try to separate from each other as much as possible, due to the repulsive nature of their interaction. The two competing trends often lead to the formation of stable nontrivial patterns, with a rather complex structure. We refer to [**34**] and the references therein for a review on the Ohta-Kawasaki energy and some related mathematical results.

We now describe the results of our paper. As already mentioned, we are interested in finding a class of initial data for which we can prove the existence of a global-in-time solution and study its long-time behavior. We focus on the periodic setting in three-dimensions; that is, we take $\Omega = \mathbb{T}^3$ in (1.1) and (1.2) and we assume spatial one-periodicity both on the evolving sets and the functions involved. In other words, finding a solution in $\mathbb{T}^3$ is equivalent to finding a solution in the whole space $\mathbb{R}^3$, which is one-periodic in space. All the results and arguments that we present clearly hold also for $N = 2$. However, for the sake of presentation we decided to stick to the physically relevant case $N = 3$.

Because of the gradient flow structure of the two flows, it is very natural to expect that if the initial set is sufficiently close to a stable critical point (or a local minimizer) F of the energy functional J, then the flow exists for all times and asymptotically converges to F.

The proper notion of criticality and stability can be defined in terms of the first and second variation of the energy by a standard procedure that we recall in the following: We say that a smooth subset $F \subset \mathbb{T}^3$ is *critical* for (1.3) if for any (admissible) smooth one-parameter family of volume preserving diffeomorphisms $(\Phi_t)_t$ with $\Phi_0 = Id$ we have that $\frac{d}{dt}J(\Phi_t(F))\big|_{t=0} = 0$. It turns out (see for instance [**9**]) that a smooth set F is critical if and only if

$$H_{\partial F} + 4\gamma v_F = \text{constant} \qquad \text{on } \partial F, \tag{1.4}$$

where $H_{\partial F}$ is the mean curvature of ∂F and $v_F(\cdot) := \int_{\mathbb{T}^3} G(\cdot, y)(2\chi_F(y) - 1)\, dy$ is the potential associated with F (see also (1.1) where v_t stands for v_{F_t}). When $\gamma = 0$ one recovers the classical constant mean curvature condition. Next, given a critical set F we may compute its *second variation*: By the results of [**9**] (see also [**1**, **27**, **36**]), we associate with it a quadratic form $\partial^2 J(F)$ defined over all functions $\varphi \in \widetilde{H}(\partial F) := \{\varphi \in H^1(\partial F) : \int_{\partial F} \varphi\, d\mathcal{H}^2 = 0\}$. This quadratic form is related to the second variation of J by the following equality

$$\frac{d^2}{dt^2} J(\Phi_t(F))\bigg|_{t=0} = \partial^2 J(F)[X \cdot \nu]\,, \tag{1.5}$$

where $X \cdot \nu$ is the (outer) normal component of the velocity field X of $(\Phi_t)_t$ on ∂F. The expression of $\partial^2 J(F)$ can be computed explicitly, see (2.9). Note that the condition $\int_{\partial F} \varphi\, d\mathcal{H}^2 = 0$ is related to the fact that we allow only volume preserving variations.

The notion of stability amounts to requiring that $\partial^2 J$ is positive definite in a suitable sense. However, we have to take into account that J is translation invariant, so that in particular $J(F) = J(F + t\eta)$ for all $\eta \in \mathbb{R}^3$ and $t \in \mathbb{R}$. By differentiating twice this identity with respect to t, one obtains $\partial^2 J(F)[\eta \cdot \nu] = 0$, thus showing that there is always a finite dimensional

subspace of *infinitesimal translations*

$$T(\partial F) := \{\varphi \in \widetilde{H}(\partial F) \,:\, \varphi = \eta \cdot \nu,\ \eta \in \mathbb{R}^3\} \tag{1.6}$$

where the second variation vanishes. In view of these observations, we say that the critical set F is *strictly stable* if

$$\partial^2 J(F)[\varphi] > 0 \qquad \text{for all } \varphi \in T^{\perp}(\partial F) \setminus \{0\}. \tag{1.7}$$

In [**1**, Theorem 1.1] (see also [**27**] for the case of Neumann boundary conditions) it is shown that strictly stable critical sets are in fact *isolated local minimizers* of the functional J with respect to small L^1-perturbations. The main purpose of this paper is to show that the latter (static) stability property extends to the evolutionary case. In Theorems 3.4 and 4.3 we show that any strictly stable critical set is *asymptotically stable* for both (1.1) and (1.2). More precisely, we have:

Main Result. *Let $F \subset \mathbb{T}^3$ be a smooth set satisfying* (1.4) *and* (1.7) *(with $\gamma = 0$ in the case of the surface diffusion flow). If E_0 is sufficiently close to F, then both the periodic modified Mullins-Sekerka flow and the periodic surface diffusion flow starting from E_0 are defined for all times and converge to a translate of F exponentially fast.*

For the proper notion of closeness to F and of exponential convergence we refer to the precise statements of the aforementioned theorems.

Let us now comment on the class of initial data to which our main result can be applied. In the three-dimensional case and for the area functional ($\gamma = 0$) the stable periodic sets are classified (see for instance [**46**]): they are *lamellae* or *balls* or *cylinders* or *triply periodic structures* such as *gyroids*. It is rather easy to see that the first three configurations are in fact strictly stable (with respect to volume preserving variations), while the strict stability of triply periodic sets has been established in some cases (see for instance in [**22**, **23**, **47**]). Due to our results, in all these cases these structures are exponentially stable for the periodic versions of (1.1) and (1.2).

As for the case $\gamma > 0$ a complete classification of the stable periodic structures is still missing. However, it has been shown that lamellar configurations are strictly stable if the number of interfaces is larger than a minimum value $k(\gamma)$, where $k(\gamma) \to +\infty$ as $\gamma \to \infty$ (see [**1**, **9**]). Moreover, again by the results of [**1**] one can show that if F is any periodic set that is strictly stable for the area functional, then for all $\gamma > 0$ sufficiently small it is possible to find sets F_γ that are strictly stable for (1.3) (with the corresponding γ) in such a way that $F_\gamma \to F$ smoothly as $\gamma \to 0^+$. If instead we fix the value of γ and F is as before, then we may find sets E that are stable for the the functional J and closely resemble a rescaled version of F. More precisely, the following has been shown in [**11**]: Let $F \subset \mathbb{T}^3$ be strictly stable for the area functional, and for any $k \in \mathbb{N}$ denote by F_k the $1/k$-periodic set $\frac{F}{k}$. Then, for every $\varepsilon > 0$ there exists $\bar{k} = \bar{k}(\gamma, \varepsilon) \in \mathbb{N}$ such that for all $k \geq \bar{k}$ we may find a set E, which is ε-close to F_k in a C^1-sense and strictly stable for J with respect to $1/k$-periodic variations. Moreover, the set E can be constructed in such a way that its mean curvature is uniformly close to a constant. Our main result clearly applies to all such sets, yielding that they are exponentially stable for the $1/k$-periodic version of the modified Mullins-Sekerka flow.

A few comments about previous related results are in order: most of them treat the exponential stability of N-dimensional spheres both for the Hele-Shaw ([**8**, **17**, **39**]) and the surface diffusion flow ([**14**, **48**]), with few exceptions in the case of the surface diffusion flow,

like the infinite cylinders considered in [**30**, **31**] and the two-dimensional triple junctions configurations studied in [**19**] (under Neumann conditions). It seems also that no asymptotic stability results for the modified Mullins-Sekerka flow were known before. Moreover, all the previous works deal with specific examples, but to the best of our knowledge no general *linear versus nonlinear stability principle* has been established for (1.1) and (1.2) prior to our main result.

Most of the aforementioned papers use semigroup techniques combined with an ad hoc center manifold analysis in order to deal with the translation invariance, see also [**40**]. Our approach instead is completely different, more variational in nature, and based on the derivation of suitable energy identities. In this respect, our method is closer in spirit to that of [**8**] and [**48**], where energy identities are the key tool to establish the desired exponential stability.

Although many technical details in the proofs of our main Theorems 3.4 and 4.3 are different, the underlying general argument and strategy is the same. We overview it for the convenience of the reader. The starting crucial observation is that the following energy identity holds along the flow $(E_t)_{t\in(0,T^*)}$ (see Lemmas 3.5 and 4.4): Setting $\mathcal{E}(E_t) := -\frac{d}{dt}J(E_t)$, we have

$$-\frac{d^2}{dt^2}J(E_t) = \frac{d}{dt}\mathcal{E}(E_t) = -2\partial^2 J(E_t)[V_t] + R(E_t)\,, \tag{1.8}$$

where $\partial^2 J$ is the second variation quadratic form introduced in (1.5), V_t is the normal velocity of the moving boundary and $R(E_t)$ is a remainder whose explicit expression depends on whether $(E_t)_t$ solves (1.1) or (1.2). Next we implement a stopping time argument; namely, we consider the maximal time $\bar{t}$ such that

$$\mathrm{dist}_{C^1}(E_t, F) < \varepsilon_0 \quad \text{and} \quad \mathcal{E}(E_t) < 2\delta_0 \qquad \text{for all } t \in (0, \bar{t}), \tag{1.9}$$

where $\mathrm{dist}_{C^1}(E_t, F)$ stands for a suitable C^1-distance of E_t from the stable critical set F and ε_0, δ_0 are (small) positive constants to be chosen. Clearly, by choosing the initial set E_0 so close to F that

$$\mathrm{dist}_{C^1}(E_0, F) < \varepsilon_0 \quad \text{and} \quad \mathcal{E}(E_0) \le \delta_0 \tag{1.10}$$

we can ensure that $\bar{t} > 0$. The purpose is to show that $\bar{t}$ coincides with the maximal time of existence T^*. The argument now proceeds by contradiction, assuming that $\bar{t} < T^*$ and that $\mathcal{E}(E_{\bar{t}}) = 2\delta_0$ or $\mathrm{dist}_{C^1}(E_{\bar{t}}, F) = \varepsilon_0$. Assume first that

$$\mathcal{E}(E_{\bar{t}}) = 2\delta_0\,. \tag{1.11}$$

At this point, the idea is to exploit the strict stability assumption on F, and the closeness of E_t to F (ensured by (1.9), with δ_0 smaller if needed) to show that the quadratic form $\partial^2 J(E_t)$ remains positive definite outside the space of infinitesimal translations $T(\partial E_t)$ (see (1.6)). This observation, together with a delicate estimate showing that V_t remains bounded away from $T(\partial E_t)$, allows one to conclude that

$$\partial^2 J(E_t)[V_t] \ge \sigma \|V_t\|^2_{H^1(\partial E_t)} \tag{1.12}$$

in $(0, \bar{t})$ for a suitable constant $\sigma > 0$. Next, one has to control the remainder $R(E_t)$ in (1.8); more precisely, one shows that

$$|R(E_t)| \le \varepsilon \|V_t\|^2_{H^1(\partial E_t)}\,, \tag{1.13}$$

where the constant ε can be made arbitrarily small, provided that ε_0 and δ_0 are chosen properly (small) in (1.10). The above inequality relies on delicate boundary estimates for

harmonic extensions in the case of the Mullins-Sekerka flow (see Proposition 3.6) and on the geometric interpolation inequality established in Lemma 4.7 in the case of the surface diffusion flow. From the technical point of view, this is where the dimension restriction $N \leq 3$ plays a role in our argument. Finally, one has to show that

$$\mathcal{E}(E_t) \leq C\|V_t\|^2_{H^1(\partial E_t)}\,, \tag{1.14}$$

with the constant $C > 0$ depending only on the C^1-bounds on ∂E_t provided by (1.9). Collecting (1.8) and (1.12)–(1.14) yields the existence of $c_0 > 0$ such that

$$\frac{d}{dt}\mathcal{E}(E_t) \leq -c_0\,\mathcal{E}(E_t)\,,$$

so that, by integration,

$$\mathcal{E}(E_t) \leq \mathcal{E}(E_0)\mathrm{e}^{-c_0 t} \leq \delta_0 \mathrm{e}^{-c_0 t} \tag{1.15}$$

for $t \in [0, \bar{t}]$. The above inequality contradicts (1.11). Now it is not too difficult to see (using the explicit expression of $\mathcal{E}(E_t)$) that under the C^1-bound of (1.9) the decay of $\mathcal{E}(E_t)$ obtained in (1.15) forces E_t to remain close to F in a C^1-sense, so that assuming $\mathrm{dist}_{C^1}(E_{\bar{t}}, F) = \varepsilon_0$ also leads to a contradiction. Thus, the stopping time $\bar{t}$ coincides with the maximal time and both (1.9) and (1.15) hold for the whole lifespan of the solution. A little refinement of the estimates above allows one also to control the Hölder-norm of the curvatures of ∂E_t, so that we may use the local-in-time existence theorems available for the two flows, together with a standard continuation argument, to infer that the solution exists for all times.

Once global-in-time existence has been established, one proceeds in the following way: A compactness argument, based on (1.9) and (1.15), yields the existence of a sequence $t_n \to \infty$ and of a set F', critical for J, such that $E_{t_n} \to F'$ (in a suitable sense). Since necessarily F' is close to F and of course $|F| = |F'|$, we may use the results from [**1**] (see also Proposition 2.7) to conclude that F' is a translate of F. The exponential convergence of the flow to F' then follows from (1.15) via suitable elliptic estimates.

We conclude the introduction by remarking that although the presentation is restricted to the periodic case, our methods would equally work in the Neumann case, under the additional assumption that the evolving interfaces do not touch $\partial\Omega$ or equivalently that $F \subset\subset \Omega$, see Theorem 3.8. It would certainly be interesting to extend our result to the general Neumann setting and to arbitrary space dimensions. This will probably require the use of some of the techniques developed in [**32**], see also [**4**], and will be the subject of future investigations. We finally mention that our methods would apply also to the *volume-preserving mean curvature flow* (see [**24**]). However, for the sake of presentation we decided to treat only the more difficult flows (1.1) and (1.2).

The plan of the paper is the following: In Section 2 we introduce the precise definition of the energy functional (1.3), recall the formulas of the first and the second variation and other related results that are useful for our analysis. In Section 3 we prove our main nonlinear stability result for the modified Mullins-Sekerka flow, while the corresponding result in the case of the surface diffusion flow is treated in Section 4. Finally, in Section 5 we gather the proofs of several auxiliary and technical results used along the way.

2. The nonlocal perimeter and its first and second variations

As already explained in the introduction the geometric evolutions considered in this paper may be regarded as suitable gradient flows of (a non-local variant of) the perimeter functional.

In this section we introduce such a non-local energy and recall the first and second variation formulas, that were derived in [**9**] (see also [**1**, **27**, **36**]).

To this end, we start by recalling that the (unit) flat torus $\mathbb{T}^3$ is the quotient of $\mathbb{R}^3$ with respect to the equivalence relation $x \sim y \iff x - y \in \mathbb{Z}^3$. The functional spaces $W^{k,p}(\mathbb{T}^3)$, $k \in \mathbb{N}$, $p \geq 1$, can be identified with the subspace of $W^{k,p}_{loc}(\mathbb{R}^3)$ of functions that are one-periodic with respect to all coordinate directions. Similarly, $C^{k,\alpha}(\mathbb{T}^3)$, $\alpha \in (0,1)$ may be identified with the space of one-periodic functions in $C^{k,\alpha}(\mathbb{R}^3)$.

A set $E \subset \mathbb{T}^3$ will be called of class $W^{k,p}$, C^k, or smooth if its one-periodic extension to $\mathbb{R}^3$ is of class $C^{k,\alpha}$, $W^{k,p}$, or smooth. In the following we will (often) identify E with such a periodic extension. Finally, by saying that $E_n \to E$ in $W^{k,p}$ (or $C^{k,\alpha}$) we mean that there exists a sequence (Ψ_n) of smooth diffeomorphisms from $\mathbb{T}^3$ to $\mathbb{T}^3$ such that $\Psi_n \to Id$ in $W^{k,p}$ (or $C^{k.\alpha}$) and $E_n = \Psi_n(E)$ for all n sufficiently large. When E is sufficiently smooth this is equivalent to saying that for every $\varepsilon > 0$, there exists $\bar{n}$ such that

$$|E \Delta E_n| \leq \varepsilon \quad \text{and} \quad \partial E_n = \{x + \psi_n(x)\nu_E(x) : x \in \partial E\}\,, \qquad \text{with } \|\psi_n\|_{W^{k,p}(\partial E)} \leq \varepsilon \ (\text{or } \|\psi_n\|_{C^{k,\alpha}(\partial E)} \leq \varepsilon)$$

for all $n \geq \bar{n}$. Here and in the following we have used the notation ν_E to denote the outer unit normal to E.

Given a smooth set $E \subset \mathbb{T}^3$, we say that a tubular neighborhood of ∂E is *regular*, if both the *signed distance* function d_E from the set E and the orthogonal projection onto ∂E are smooth functions in U. Recall that

$$d_E(x) := \begin{cases} \mathrm{dist}(x, \partial E) \text{ if } x \notin E, \\ -\mathrm{dist}(x, \partial E) \text{ if } x \in E. \end{cases} \tag{2.1}$$

In this periodic setting, the (relative) perimeter of a set $E \subset \mathbb{T}^3$ is defined as

$$P_{\mathbb{T}^3}(E) := \sup\left\{ \int_E \operatorname{div} \varphi \, dz : \ \varphi \in C^1(\mathbb{T}^3; \mathbb{R}^3)\,, \|\varphi\|_\infty \leq 1 \right\}.$$

Let $\gamma \geq 0$ be fixed and for every $E \subset \mathbb{T}^3$ set

$$J(E) := P_{\mathbb{T}^3}(E) + \gamma \int_{\mathbb{T}^3} |Dv_E|^2 \, dx\,, \tag{2.2}$$

where v_E is the periodic solution of

$$\begin{cases} -\Delta v_E = u_E - m, \\ \displaystyle\int_{\mathbb{T}^3} v_E \, dx = 0. \end{cases} \tag{2.3}$$

Here $u_E = \chi_E - \chi_{\mathbb{T}^3 \setminus E}$ and $m = 2|E| - 1$. It is useful to recall that v_E can be represented as

$$v_E(x) := \int_{\mathbb{T}^3} G_{\mathbb{T}^3}(x,y) u_E(y) \, dy\,, \tag{2.4}$$

where $G_{\mathbb{T}^3}$ is the Laplacian's Green function in the torus; that is, for $x \in \mathbb{T}^3$, $G_{\mathbb{T}^3}(x, \cdot)$ is the unique solution of

$$\begin{cases} -\Delta_y G_{\mathbb{T}^3}(x, \cdot) = \delta_x - 1 \quad \text{in } \mathbb{T}^3, \\ \int_{\mathbb{T}^3} G_{\mathbb{T}^3}(x,y) \, dy = 0\,. \end{cases}$$

We stress that the relevant particular case $\gamma = 0$ (corresponding to the standard perimeter) is always included in all the discussion below.

Throughout the paper we will make repeated use of the following notation: For any one-parameter family of functions $(g_t)_t \in (0,T)$ the symbol $\dot{g}_t$ will denote the partial derivative with respect to s of the map $s \mapsto g_{t+s}$ evaluated at $s = 0$; that is,

$$\dot{g}_t := \frac{\partial}{\partial s} g_{t+s}\Big|_{s=0}.$$

Definition 2.1. Let $E \subset \mathbb{T}^N$ be a smooth set.

(i) We say that a one-parameter family $(\Phi_t)_{t\in I}$ of diffeomorphisms from $\mathbb{T}^3$ to $\mathbb{T}^3$, with I a real interval containing 0 and $\Phi_0 = Id$, is *admissible* if the map $(x,t) \mapsto \Phi_t(x)$ belongs to $C^\infty(\mathbb{T}^3 \times I; \mathbb{T}^3)$ and

$$|\Phi_t(E)| = |E| \qquad \text{for all } t \in I.$$

(ii) Denote by X_t the velocity field at time t, that is,

$$X_t := \dot{\Phi}_t \circ \Phi_t^{-1}$$

and set for simplicity $X := X_0$. If the family $(\Phi_t)_{t\in I}$ is admissible and X_t is independent of t, i.e., $X_t = X$, then we say that $(\Phi_t)_{t\in I}$ is an *admissible flow.*

We recall that given a vector X, its tangential part on some smooth $(N-1)$-manifold $\mathcal{M}$ is defined as $X_\tau := X - (X\cdot\nu)\nu$, with ν being a unit normal vector to $\mathcal{M}$. In particular, we will denote by D_τ the tangential gradient operator given by $D_\tau\varphi := (D\varphi)_\tau$. Finally $\operatorname{div}_\tau X$ will stand for the *tangential divergence* of X on $\mathcal{M}$ defined as $\operatorname{div}_\tau X := \operatorname{div} X - \partial_\nu X \cdot \nu$.

Theorem 2.2 ([**1**, **9**])**.** *Let E, $(\Phi_t)_{t\in I}$, X_t be as in Definition* 2.1-*(ii), and set*

$$\dot{v}_E := \frac{\partial}{\partial t} v_{\Phi_t(E)}\Big|_{t=0},$$

and $v_{\Phi_t(E)}$ is the potential defined in (2.4)*, with E replaced by $\Phi_t(E)$. Then,*

$$\dot{v}_E = 2\int_{\partial E} G_{\mathbb{T}^3}(\cdot,y)X(y)\cdot\nu_E(y)\,d\mathcal{H}^2 \tag{2.5}$$

and

$$\frac{d}{dt}J(\Phi_t(E))\Big|_{t=0} = \int_{\partial E}(H_{\partial E} + 4\gamma v_E)X\cdot\nu_E\,d\mathcal{H}^2, \tag{2.6}$$

where ν_E denotes the outer unit normal to ∂E, $H_{\partial E}$ stands for the sum of its principal curvatures, and we wrote X instead of X_0. If in addition $(\Phi_t)_{t\in I}$ is an admissible flow according to Definition 2.1-*(ii), then*

$$\begin{aligned}\frac{d^2}{dt^2}J(\Phi_t(E))\Big|_{t=0} &= \int_{\partial E}\Big(|D_\tau(X\cdot\nu_E)|^2 - |B_{\partial E}|^2(X\cdot\nu_E)^2\Big)\,d\mathcal{H}^2\\ &\quad + 8\gamma\int_{\partial E}\int_{\partial E} G_{\mathbb{T}^3}(x,y)(X\cdot\nu_E)(x)(X\cdot\nu_E)(y)\,d\mathcal{H}^2(x)\,d\mathcal{H}^2(y)\\ &\quad + 4\gamma\int_{\partial E}\partial_{\nu_E}v_E(X\cdot\nu_E)^2\,d\mathcal{H}^2 + R,\end{aligned} \tag{2.7}$$

where the remainder R is defined as

$$R := -\int_{\partial E} (4\gamma v_E + H_{\partial E}) \operatorname{div}_\tau\big(X_\tau (X\cdot\nu_E)\big)\, d\mathcal{H}^2 + \int_{\partial E} (4\gamma v_E + H_{\partial E})(\operatorname{div} X)(X\cdot\nu_E)\, d\mathcal{H}^2 \,. \tag{2.8}$$

In the above formulas $B_{\partial E}$ denotes the second fundamental form of ∂E so that the square $|B_{\partial E}|^2$ of its Euclidean norm coincides with the sum of the squares of the principal curvatures.

Recall now that if Φ_t is admissible, then $|\Phi_t(E)| = |E|$ for all $t \in [0,1]$ and thus

$$0 = \frac{d}{dt}|\Phi_t(E)|\Big|_{t=0} = \int_E \frac{d}{dt} J\Phi\Big|_{t=0} = \int_E \operatorname{div} X\, dx = \int_{\partial E} X\cdot\nu_E\, d\mathcal{H}^2 \,,$$

that is, the normal component $X\cdot\nu_E$ has zero average on ∂E. Then (2.6) together with a simple extension argument (see [**1**, Corollary 3.4]) implies that

$$\frac{d}{dt} J(\Phi_t(E))\Big|_{t=0} = 0 \qquad \text{for all admissible } \Phi_t$$

if and only if

$$\int_{\partial E} (H_{\partial E} + 4\gamma v_E)\varphi\, d\mathcal{H}^2 = 0 \quad \text{for all } \varphi \in C^\infty(\partial E) \text{ s.t. } \int_{\partial E} \varphi\, d\mathcal{H}^2 = 0.$$

This motivates the following definition.

Definition 2.3 (Critical sets)**.** A smooth subset $F \subset \mathbb{T}^3$ is said to be *critical* for the functional J if there exists a constant $\lambda \in \mathbb{R}$ such that

$$H_{\partial F} + 4\gamma v_F = \lambda \qquad \text{on } \partial F.$$

It is now easy to see that for critical sets the remainder (2.8) vanishes so that the second variation depends (quadratically) only on $X\cdot\nu_F$. Denoting

$$\widetilde{H}(\partial F) := \Big\{\varphi \in H^1(\partial F) : \int_{\partial F} \varphi\, d\mathcal{H}^2 = 0\Big\},$$

we are led to consider the quadratic form $\partial^2 J(F) : \widetilde{H}(\partial F) \to \mathbb{R}$ defined as

$$\begin{aligned} \partial^2 J(F)[\varphi] := &\int_{\partial F} |D_\tau \varphi|^2\, d\mathcal{H}^2 - \int_{\partial F} |B_{\partial F}|^2 \varphi^2\, d\mathcal{H}^2 \\ &+ 8\gamma \int_{\partial F}\int_{\partial F} G_{\mathbb{T}^3}(x,y)\varphi(x)\varphi(y)\, d\mathcal{H}^2(x) d\mathcal{H}^2(y) \\ &+ 4\gamma \int_{\partial F} \partial_{\nu_F} v_F\, \varphi^2\, d\mathcal{H}^2 \,, \end{aligned} \tag{2.9}$$

so that if F is critical, then

$$\frac{d^2}{dt^2} J(\Phi_t(F))\Big|_{t=0} = \partial^2 J(F)[X\cdot\nu_F],$$

thanks to (2.7). In order to give the proper notion of stability we have to take into account that the functional J is invariant under translations of sets. Thus, if one considers the (admissible) flow $\Phi(t,x) = x + t\,\eta$, $\eta \in \mathbb{R}^3$, then $\Phi_t(F) = F + t\eta$ and $J(\Phi_t(F)) = J(F)$ for all t. Therefore,

$$0 = \frac{d^2}{dt^2} J(\Phi_t(F))\Big|_{t=0} = \partial^2 J(F)[\eta\cdot\nu_F] \qquad \text{for all } \eta \in \mathbb{R}^3.$$

We conclude that the quadratic form $\partial^2 J(F)$ always vanishes on the finite dimensional subspace $T(\partial F) \subset \widetilde{H}(\partial F)$ defined as

$$T(\partial F) := \left\{ \eta \cdot \nu_F \,:\, \eta \in \mathbb{R}^3 \right\}.$$

The above observation motivates the following definition.

Definition 2.4. Let $F \subset \mathbb{T}^3$ be a smooth critical set, according to Definition 2.3. We say that F is *strictly stable* if

$$\partial^2 J(F)[\varphi] > 0 \qquad \text{for all } \varphi \in T^{\perp}(\partial F) \setminus \{0\},$$

where $T^{\perp}(\partial F)$ stands for the space orthogonal to $T(\partial F)$ with respect to the $L^2(\partial F)$ scalar product.

Let F be a smooth critical set. Observe that we may choose an orthogonal base $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ of $\mathbb{R}^3$ such that the functions $\tilde{e}_i \cdot \nu_F$, $i = 1, 2, 3$, are orthogonal in $L^2(\partial F)$ (see [**1**, Section 3]). Then we set

$$\Pi_F := \operatorname{span}\{\tilde{e}_i \,:\, i \in I_F\}, \tag{2.10}$$

where

$$I_F := \{i \,:\, \tilde{e}_i \cdot \nu_F \ \text{ is not identically zero}\}. \tag{2.11}$$

Remark 2.5. Setting for $\varphi \in \widetilde{H}(\partial E)$

$$v_\varphi(x) := \int_{\partial E} G_{\mathbb{T}^3}(x, y)\varphi(y)\, d\mathcal{H}^2(y)$$

and $\mu_\varphi := \varphi\, \mathcal{H}^2 \llcorner \partial E$, it follows from the properties of the Green's function (see [**28**, Chapter 18]) that v_φ satisfies $-\Delta v_\varphi = \mu_\varphi$ in $\mathbb{T}^3$ or, equivalently,

$$\int_{\mathbb{T}^3} Dv_\varphi \cdot D\psi dx = \int_{\partial E} \varphi\, \psi\, d\mathcal{H}^2 \qquad \text{for all } \psi \in H^1(\mathbb{T}^3). \tag{2.12}$$

Therefore,

$$\int_{\partial E}\int_{\partial E} G_{\mathbb{T}^3}(x, y)\varphi(x)\varphi(y)\, d\mathcal{H}^2(x)\, d\mathcal{H}^2(y) = \int_{\partial E} \varphi\, v_\varphi\, d\mathcal{H}^2 = \int_{\mathbb{T}^3} |Dv_\varphi|^2\, dx\,,$$

where the last equality follows from (2.12).

We conclude this section by stating two facts that will be used throughout.

The first lemma states that when a set is sufficiently close to a strictly stable critical point then the quadratic form associated with the second variation remains positive. More precisely, we have:

Lemma 2.6. *Fix $p > 2$ and let F be a smooth strictly stable critical set in the sense of Definition 2.4. Then, for every $\varepsilon \in (0, 1]$ there exist $\sigma_\varepsilon > 0$ and $\delta_1 > 0$ such that*

$$\partial^2 J(E)[\varphi] \geq \sigma_\varepsilon \|\varphi\|^2_{H^1(\partial E)} \tag{2.13}$$

for all $\varphi \in \widetilde{H}(\partial E)$ satisfying

$$\min_{\eta \in \Pi_F} \|\varphi - \eta \cdot \nu_E\|_{L^2(\partial E)} \geq \varepsilon \|\varphi\|_{L^2(\partial E)},$$

provided that $E \subset \mathbb{T}^3$ is δ_1-close to F in a $W^{2,p}$-sense, that is

$$\partial E = \{x + \psi(x)\nu_F(x) \,:\, x \in \partial F \text{ for some smooth } \psi \text{ with } \|\psi\|_{W^{2,p}(\partial F)} \leq \delta_1\}.$$

The proof of the above lemma is given in Section 5.

The final result of this section states the crucial observation that in the vicinity of a given strictly stable critical set there are no other critical sets.

Proposition 2.7. *Let p and F be as in Lemma 2.6. Then there exists $\delta_2 > 0$ such that if $F' \subset \mathbb{T}^3$ is a smooth critical set in the sense of Definition 2.3, $|F'| = |F|$, $|F\Delta F'| \le \delta_2$ and*

$$\partial F' = \{x + \psi(x)\nu_F(x) : x \in \partial F \text{ for some smooth } \psi \text{ with } \|\psi\|_{W^{2,p}(\partial F)} \le \delta_2\},$$

then $F' = F + \sigma$ for some $\sigma \in \mathbb{R}^3$.

Proof. This fact is essentially proven in [**1**, Proof of Theorem 3.9]. There, it is shown that for every $p > 2$ there exists $\delta_2 > 0$ with the following property: if $F' \subset \mathbb{T}^3$ is a smooth set with $|F'| = |F|$, $|F\Delta F'| \le \delta_2$ and

$$\partial F' = \{x + \psi(x)\nu_F(x) : x \in \partial F \text{ for some smooth } \psi \text{ with } \|\psi\|_{W^{2,p}(\partial F)} \le \delta_2\},$$

then we may find a small vector $\sigma \in \mathbb{T}^3$ and an admissible flow Φ_t such that $\Phi_0(F) = F$, $\Phi_1(F) = F' + \sigma$ and

$$\frac{d^2}{dt^2} J(\Phi_t(F))\Big|_{t=s} \ge c|E\Delta(F' + \sigma)|^2$$

for all $s \in [0, 1]$, where c is a positive constant independent of F'. Assume that F' is a smooth critical set which is not translate of F. Then $\frac{d}{dt} J(\Phi_t(F))\Big|_{t=0} = 0$ and from the above formula we have that $\frac{d}{dt} J(\Phi_t(F))\Big|_{t=1} > 0$. Therefore $F' + \sigma$ and, in turn F', is not critical. q.e.d.

3. Nonlinear stability for the modified Mullins-Sekerka flow

In this section we consider the modified Mullins-Sekerka flow. In order to speak about classical solutions, we need to define first the notion of a smooth flow.

Definition 3.1 (Smooth flows of sets)**.** We say that a one-parameter family of sets $(E_t)_{t\in(0,T)}$ is a *smooth flow* on the interval $(0, T)$ if there exists a smooth *reference set* $F \subset \mathbb{T}^3$ and a map $\Psi \in C^\infty(\mathbb{T}^3 \times (0, T); \mathbb{T}^3)$ such that $\Psi_t := \Psi(\cdot, t)$ is a smooth diffeomorphism from $\mathbb{T}^3$ into $\mathbb{T}^3$ and $E_t = \Psi_t(F)$ for all $t \in [0, T)$.

We will make use of the following notation: Given a (smooth) set $E \subset \mathbb{T}^3$, we denote by w_E the unique solution in $H^1(\mathbb{T}^3)$ to the following problem

$$\begin{cases} \Delta w_E = 0 & \text{in } \mathbb{T}^3 \setminus \partial E \\ w_E = H_{\partial E} + 4\gamma v_E & \text{on } \partial E, \end{cases} \tag{3.1}$$

where v_E is the potential introduced in (2.3). Moreover, we denote by w_E^+ and w_E^- the restrictions $w_E|_{\mathbb{T}^3\setminus E}$ and $w_E|_E$, respectively. Finally, denoting as usual by ν_E the outer unit normal to E, we set

$$[\partial_{\nu_E} w_E] := \partial_{\nu_E} w_E^+ - \partial_{\nu_E} w_E^- = -(\partial_{\nu_{E^c}} w_E^+ + \partial_{\nu_E} w_E^-)\,.$$

In the following, given $\alpha \in (0, 1)$ and $k, m \in \mathbb{N}$ we denote

$$h^{k,\alpha}(\mathbb{R}^m) := \{f \in C^{k,\alpha}(\mathbb{R}^m) : \exists\{f_n\} \subset C^\infty(\mathbb{R}^m) \text{ s.t. } f_n \to f \text{ locally in } C^{k,\alpha}(\mathbb{R}^m)\}\,.$$

The space $h^{k,\alpha}(M)$, when $M \subset \mathbb{R}^m$ is a smooth manifold, can be then defined by means of local charts. In turn, we will say that a set $F \subset \mathbb{T}^3$ is of class $h^{k,\alpha}$, $\alpha \in (0, 1)$, if for each

point $x \in \partial F$ there exists a a neighborhood V of x, a function $f \in h^{k,\alpha}(\mathbb{R}^2)$, and a suitable coordinate system such that $F \cap V = \{(x', x_N) \in V : x_N \leq f(x')\}$.

Definition 3.2 (Modified Mullins-Sekerka flows)**.** Let $E_0 \subset \mathbb{T}^3$ be of class $h^{2,\alpha}$ for some $\alpha \in (0,1)$. We say that the one-parameter family $(E_t)_{t\in(0,T)}$ is a *classical solution* to the modified Mullins-Sekerka flow on the interval $(0,T)$ with initial datum E_0 if it is a smooth flow in the sense of Definition 3.1, $E_t \to E_0$ in $C^{2,\alpha}$ as $t \to 0^+$, and the following evolution law holds:

$$V_t = [\partial_{\nu_t} w_t] \quad \text{on } \partial E_t \text{ for all } t \in (0,T)\,, \tag{3.2}$$

where V_t stands for the outer normal velocity of the moving boundary ∂E_t. Here we used the simplified notation $\partial_{\nu_t} w_t$ in place of $\partial_{\nu_{E_t}} w_{E_t}$.

As explained in the introduction the modified Mullins-Sekerka flow is volume preserving. This can be easily checked by the following computation (using also the notation introduced in Definition 3.2):

$$\frac{d}{dt}|E_t| = \int_{\partial E_t} V_t \, d\mathcal{H}^2 = \int_{\partial E_t} [\partial_{\nu_t} w_t] \, d\mathcal{H}^2 = 0\,,$$

where the last equality follows from the Divergence Theorem and the fact that w_t is harmonic in $\mathbb{T}^3 \setminus \partial E_t$.

We use the following notation: Given a smooth set $F \subset \mathbb{T}^3$ and a regular tubular neighborhood U of ∂F, we denote by $\mathfrak{C}^1_M(F,U)$, $M > 0$, the class of all smooth sets $E \subset F \cup U$ such that

$$\partial E = \{x + \psi_E(x)\nu_F(x) : x \in \partial F\}\,, \tag{3.3}$$

for some $\psi_E \in C^\infty(\partial F)$, with $\|\psi_E\|_{C^1(\partial F)} \leq M$. For $\alpha \in (0,1)$ and $k \in \mathbb{N}$ we also let $\mathfrak{h}^{k,\alpha}_M(F,U)$ be the collection of sets $E \in \mathfrak{C}^1_M(F,U)$ such that $\|\psi_E\|_{h^{k,\alpha}(\partial F)} \leq M$. We are now ready to state a local-in-time existence and uniqueness result proved in [**15**]. [1]

Theorem 3.3 (Local-in-time existence and uniqueness, [**15**])**.** *Let $F_0 \subset \mathbb{T}^3$ be a smooth set and U a regular tubular neighborhood of ∂F_0. Then, for every $M > 0$ and $\alpha \in (0,1)$ there exists $T > 0$ with the following property: For every $E_0 \in \mathfrak{h}^{2,\alpha}_M(F_0,U)$ there exists a unique classical solution to the modified Mullins-Sekerka flow in $(0,T)$ with initial datum E_0.*

Our purpose is to show that for special initial data the flow exists for all time and then to study its long-time behavior.

The main result is the following.

Theorem 3.4 (Main result)**.** *Let $F \subset \mathbb{T}^3$ be a strictly stable critical set according to Definition 2.4 and let U be a regular tubular neighborhood of ∂F. Then, for every $M > 0$ and $\alpha \in (0,1)$ there exists $\delta_0 > 0$ with the following property: Let $E_0 \in \mathfrak{h}^{2,\alpha}_M(F,U)$ be such that*

$$|E_0| = |F|\,, \qquad |E_0 \Delta F| \leq \delta_0\,, \qquad \textit{and} \qquad \int_{\mathbb{T}^3} |Dw_{E_0}|^2 \, dx \leq \delta_0\,.$$

Then, the unique classical solution $(E_t)_t$ to the Mullins-Sekerka flow with initial datum E_0 is defined for all $t > 0$. Moreover, $E_t \to F + \sigma$ in $W^{5/2,2}$ exponentially fast as $t \to +\infty$, for

[1] In fact [**15**] deals with the evolution in the whole space $\mathbb{R}^N$, but it is clear that the same arguments go through in the periodic case.

some $\sigma \in \mathbb{R}^3$. More precisely, there exist η, $c_F > 0$ such that for all $t > 0$, writing

$$\partial E_t = \{x + \psi_{\sigma,t}(x)\nu_{F+\sigma}(x) : x \in \partial F + \sigma\},$$

we have

$$\|\psi_{\sigma,t}\|_{W^{5/2,2}(\partial F+\sigma)} \le \eta \mathrm{e}^{-c_F t}.$$

Both $|\sigma|$ and η vanish as $\delta_0 \to 0^+$.

Note that the $H^1(\mathbb{T}^3)$ norm of w_E is equivalent to the $H^{1/2}(\partial E)$ norm of $H_{\partial E} + 4\gamma v_E$ which in turn controls the $W^{5/2,2}(\partial F)$ norm of ψ_E. This explains the $W^{5/2,2}$ convergence in the above theorem.

The proof of the result is postponed until the end of this section. It will be achieved through several auxiliary results, that we state in the following and whose proofs can be found in the final section.

Lemma 3.5 (Energy identities)**.** *Let $(E_t)_{t\in(0,T)}$ be a smooth flow satisfying* (3.2)*. The following energy idienties hold:*

$$\frac{d}{dt}J(E_t) = -\int_{\mathbb{T}^3} |Dw_t|^2\,dx\,, \tag{3.4}$$

and

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{T}^3}|Dw_t|^2\,dx\right) = -\partial^2 J(E_t)\Big[[\partial_{\nu_t} w_t]\Big] + \frac{1}{2}\int_{\partial E_t}(\partial_{\nu_t} w_t^+ + \partial_{\nu_t} w_t^-)[\partial_{\nu_t} w_t]^2\,d\mathcal{H}^2\,, \tag{3.5}$$

where $\partial^2 J(E_t)$ is the quadratic form defined in (2.9) *(with E_t in place of E) and, as usual, the subscript t stands for E_t.*

The proof of the lemma is given in the final section. Note that if E_t is not critical then $\frac{d^2}{dt^2}J(E_t)$ is not equal to the second variation of $J(E_t)$ evaluated at $[\partial_{\nu_t} w_t]$. However, quite surprisingly the formulas above show that the leading order term of $\frac{d^2}{dt^2}J(E_t)$ is indeed twice the quadratic form $\partial^2 J(E_t)$ at $[\partial_{\nu_t} w_t]$. The same holds for the surface diffusion flow, see (4.3). The next proposition provides crucial boundary estimates for harmonic functions. Some of them are perhaps well-known to the experts. However, for the convenience of the reader we provide a self-contained proof in the final section.

Proposition 3.6 (Boundary estimates for harmonic functions)**.** *Let $E \subset \mathbb{T}^3$ be of class $C^{1,\alpha}$, $f \in C^\alpha(\partial E)$ (with zero average on ∂E) and let $u \in H^1(\mathbb{T}^3)$ be the solution of*

$$-\Delta u = f\mathcal{H}^2 \llcorner \partial E$$

with zero average in $\mathbb{T}^3$. Denote $u^- = u\big|_E$ and $u^+ = u\big|_{\mathbb{T}^3\setminus E}$ and assume that u^- and u^+ are of class C^1 up to the boundary ∂E. Then, for every $1 < p < \infty$ there exists a constant C, which depends only on the $C^{1,\alpha}$ bounds on ∂E and on p, such that:

(i)
$$\|u\|_{L^p(\partial E)} \le C\|f\|_{L^p(\partial E)}\,;$$

(ii)
$$\|\partial_{\nu_E} u^+\|_{L^2(\partial E)} + \|\partial_{\nu_E} u^-\|_{L^2(\partial E)} \le C\|u\|_{H^1(\partial E)}\,;$$

(iii)
$$\|\partial_{\nu_E} u^+\|_{L^p(\partial E)} + \|\partial_{\nu_E} u^-\|_{L^p(\partial E)} \le C\|f\|_{L^p(\partial E)}\,.$$

(iv)
$$\|u\|_{C^{0,\beta}(\partial E)} \le C\|f\|_{L^p(\partial E)}$$
for all $p > 2$, $\beta \in (0, \frac{p-2}{p})$, *with* C *depending also on* β.

(v) *Moreover, if* $f \in H^1(\partial E)$, *then for every* $2 \le p < +\infty$ *there exists a constant* C, *which depends only on the* $C^{1,\alpha}$ *bounds on* ∂E *and on* p, *such that*
$$\|f\|_{L^p(\partial E)} \le C\|f\|_{H^1(\partial E)}^{\frac{p-1}{p}} \|u\|_{L^2(\partial E)}^{\frac{1}{p}}.$$

We will need also the following:

Lemma 3.7 (Compactness of sets)**.** *Let* $F \subset \mathbb{T}^3$ *be a smooth set and denote by* U *a fixed regular tubular neighborhood of* ∂F. *Let* $\{E_n\}_n \subset \mathfrak{C}^1_M(F,U)$ *be a sequence of sets such that*
$$\sup_n \int_{\mathbb{T}^3} |Dw_{E_n}|^2\, dx < +\infty\,.$$
Then there exists $F' \in \mathfrak{C}^1_M(F,U)$ *of class* $W^{\frac{5}{2},2}$ *such that, up to a (non relabeled) subsequence,* $E_n \to F'$ *in* $W^{2,p}$ *for all* $1 \le p < 4$. *Moreover, if*
$$\int_{\mathbb{T}^3} |Dw_{E_n}|^2\, dx \to 0\,,$$
then F' *is critical in the sense of Definition* 2.3 *and the convergence holds in* $W^{\frac{5}{2},2}$.

We give now the proof of Theorem 3.4.

***Proof of Theorem* 3.4.** Throughout the proof C will denote a constant depending only on the $C^{1,\alpha}$-bounds on the boundary of the set. The value of C may change from line to line. We start by the trivial observation that if $\{E_n\}_n \subset \mathfrak{h}^{2,\alpha}_M(F,U)$ and $|E_n \Delta F| \to 0$, then $E_n \to F$ in $C^{2,\beta}$ for all $\beta \in (0,\alpha)$. For any set $E \in \mathfrak{C}^1_M(F,U)$ consider
$$D(E) := \int_{E\Delta F} \mathrm{dist}\,(x, \partial F)\, dx = \int_E d_F\, dx - \int_F d_F\, dx, \tag{3.6}$$
where d_F is the signed distance function defined in (2.1). Using coarea formula the reader may check that
$$|E\Delta F| \le C\|\psi_E\|_{L^1(\partial F)} \le C\|\psi_E\|_{L^2(\partial F)} \le C\sqrt{D(E)} \tag{3.7}$$
for a constant C depending only on F. Thus, for every $\varepsilon_0 > 0$ sufficiently small, there exists $\delta_0 \in (0,1)$ so small that for any set $E \in \mathfrak{C}^1_M(F,U)$ the following implications hold true:
$$E \in \mathfrak{h}^{2,\alpha}_M(F,U) \text{ and } D(E) \le \delta_0 \Longrightarrow \|\psi_E\|_{C^1(\partial F)} \le \frac{\varepsilon_0}{2} \tag{3.8}$$
and
$$\|\psi_E\|_{C^1(\partial F)} \le \varepsilon_0 \text{ and } \int_{\mathbb{T}^3} |Dw_E|^2\, dx \le 1 \Longrightarrow \|\psi_E\|_{W^{2,3}(\partial F)} \le \omega(\varepsilon_0) \le 1\,, \tag{3.9}$$
where ω is a positive non-decreasing function such that $\omega(\varepsilon_0) \to 0$ as $\varepsilon_0 \to 0^+$. The last implication is true thanks to Lemma 3.7. In the following ε_0, δ_0 will denote two constants in $(0,1)$ satisfying (3.8) and (3.9). The final choice of ε_0, δ_0 will be made throughout the proof. Choose an initial set $E_0 \in \mathfrak{h}^{2,\alpha}_M(F,U)$ such that
$$D(E_0) \le \delta_0 \qquad \text{and} \qquad \int_{\mathbb{T}^3} |Dw_{E_0}|^2\, dx \le \delta_0\,. \tag{3.10}$$

Let $(E_t)_{t\in(0,T(E_0))}$ be the unique classical solution to the modified Mullins-Sekerka flow provided by Theorem 3.3. Here $T(E)\in(0,+\infty]$ stands for the maximal time of existence of the classical solution starting from E. By the same theorem, there exists $T_0>0$ such that

$$T(E)\geq T_0 \qquad \text{for all } E\in\mathfrak{h}^{2,\alpha}(F,U). \tag{3.11}$$

We now split the rest of the proof into several steps.

Step 1. *(Stopping-time)* Let $\bar t\leq T(E_0)$ be the maximal time such that

$$\|\psi_t\|_{C^1(\partial F)}<\varepsilon_0 \quad\text{and}\quad \int_{\mathbb{T}^3}|Dw_t|^2\,dx<2\delta_0. \quad \text{for all } t\in(0,\bar t), \tag{3.12}$$

Here and in the following the subscript t stands for the subscript E_t. Note that such a maximal time is well defined in view of (3.8) and (3.10). We claim that by taking ε_0 and δ_0 smaller if needed, we have $\bar t=T(E_0)$. This claim will be proved in Step 3 below.

Step 2. *(Estimate of the translational component of the flow)* We claim that there exists small $\varepsilon>0$ such that

$$\min_{\eta\in\Pi_F}\big\|\,[\partial_{\nu_t}w_t]-\eta\cdot\nu_t\big\|_{L^2(\partial E_t)}\geq\varepsilon\|[\partial_{\nu_t}w_t]\|_{L^2(\partial E_t)} \qquad \text{for all } t\in(0,\bar t)\,, \tag{3.13}$$

where Π_F is defined in (2.10). To this aim, let $\eta_t\in\Pi_F$ be such that

$$[\partial_{\nu_t}w_t]=\eta_t\cdot\nu_t+g, \tag{3.14}$$

where g is orthogonal to the subspace of $L^2(\partial E_t)$ spanned by $\tilde e_i\cdot\nu_t$ with $i\in I_F$ (see (2.11)). We argue by contradiction assuming $\|g\|_{L^2(\partial E_t)}<\varepsilon\|[\partial_{\nu_t}w_t]\|_{L^2(\partial E_t)}$, for some $\varepsilon>0$ that will be chosen below. First of all, by (2.6) and the translation invariance of the energy we have

$$0=\frac{d}{ds}J(E_t+s\eta_t)\big|_{s=0}=\int_{\partial E_t}(H_t+4\gamma v_t)\eta_t\cdot\nu_t\,d\mathcal{H}^2=\int_{\partial E_t}w_t(\eta_t\cdot\nu_t)\,d\mathcal{H}^2\,.$$

Thus, multiplying (3.14) by $w_t-\hat w_t$, with $\hat w_t:=\fint_{\mathbb{T}^3}w_t\,dx$, and integrating over ∂E_t, we get

$$\begin{aligned}\int_{\mathbb{T}^3}|Dw_t|^2\,dx&=-\int_{\partial E_t}w_t[\partial_{\nu_t}w_t]\,d\mathcal{H}^2=-\int_{\partial E_t}(w_t-\hat w_t)[\partial_{\nu_t}w_t]\,d\mathcal{H}^2\\&=-\int_{\partial E_t}(w_t-\hat w_t)g\,d\mathcal{H}^2\\&\leq\varepsilon\|w_t-\hat w_t\|_{L^2(\partial E_t)}\|[\partial_{\nu_t}w_t]\|_{L^2(\partial E_t)}.\end{aligned} \tag{3.15}$$

Note that in the second and the third equality above we have used the fact that $[\partial_{\nu_t}w_t]$ and ν_t, respectively, have zero average on ∂E_t. Let us denote the (periodic) harmonic extension of $\eta_t\cdot\nu_t$ to $\mathbb{T}^3$ by f. Since

$$\int_{\partial F}|\tilde e_i\cdot\nu_F|^2\,d\mathcal{H}^2>0 \qquad \text{for } i\in I_F$$

from (3.12) it follows that if ε_0 is small enough then $||\tilde e_i\cdot\nu_t||_{L^2(\partial E_t)}\geq c_0>0$ for all $i\in I_F$. Hence $|\eta_t|\leq C\|[\partial_{\nu_t}w_t]\|_{L^2(\partial E_t)}$. By (3.9) we have

$$\|Df\|_{L^2(\mathbb{T}^3)}\leq C\|\eta_t\cdot\nu_t\|_{H^{1/2}(\partial E_t)}\leq C|\eta_t|\|\nu_t\|_{W^{1,3}(\partial E_t)}\leq C\|[\partial_{\nu_t}w_t]\|_{L^2(\partial E_t)}\,. \tag{3.16}$$

Note now that

$$\Delta w_t=[\partial_\nu w_t]\mathcal{H}^2\llcorner\partial E \qquad \text{in } \mathbb{T}^3. \tag{3.17}$$

We may then apply Proposition 3.6-(i) to obtain

$$\|w_t - \hat{w}_t\|_{L^2(\partial E_t)} \le C\|[\partial_{\nu_t} w_t]\|_{L^2(\partial E_t)}. \tag{3.18}$$

Thus, combining (3.14) with (3.15)–(3.18), we infer

$$\begin{aligned}\|\eta_t \cdot \nu_t\|^2_{L^2(\partial E_t)} &= \int_{\partial E_t} [\partial_{\nu_t} w_t](\eta_t \cdot \nu_t)\, d\mathcal{H}^2 = -\int_{\mathbb{T}^3} Df \cdot Dw_t\, dx \\ &\le \left(\int_{\mathbb{T}^3} |Df|^2\, dx\right)^{1/2} \left(\int_{\mathbb{T}^3} |Dw_t|^2\, dx\right)^{1/2} \\ &\le C\varepsilon^{1/2} \|[\partial_{\nu_t} w_t]\|^2_{L^2(\partial E_t)}\,.\end{aligned}$$

If ε is chosen so small that $C\varepsilon^{\frac{1}{2}} + \varepsilon^2 < 1$ in the last inequality, then we reach a contradiction to (3.14) and the fact that $\|g\|_{L^2(\partial E_t)} < \varepsilon\|[\partial_{\nu_t} w_t]\|_{L^2(\partial E_t)}$. This shows that for this choice of ε condition (3.13) holds. Recall now that by Lemma 2.6 and Proposition 2.7, there exist σ_ε and $\delta_1 > 0$ with the following properties: for any set $E \in \mathfrak{C}^1_M(F, U)$

$$\begin{aligned}\|\psi_E\|_{W^{2,3}(\partial F)} \le \delta_1 \Longrightarrow \partial^2 J(E)[\varphi] \ge \sigma_\varepsilon \|\varphi\|^2_{H^1(\partial E)} &\text{ for all } \varphi \in \widetilde{H}(\partial E) \\ \text{s.t. } \min_{\eta \in \Pi_F} \|\varphi - \eta \cdot \nu_E\|_{L^2(\partial E)} \ge \varepsilon \|\varphi\|_{L^2(\partial E)}&\end{aligned} \tag{3.19}$$

and

$$F' \text{ critical, } |F| = |F'| \quad \text{and} \quad \|\psi_{F'}\|_{W^{2,3}(\partial F)} \le \delta_1 \Longrightarrow F' = F + \sigma \tag{3.20}$$

for a suitable $\sigma \in \mathbb{R}^3$. By taking ε_0 (and δ_0) smaller, if needed, we may ensure that

$$\omega(\varepsilon_0) \le \delta_1\,, \tag{3.21}$$

where ω is the modulus of continuity introduced in (3.9).

Step 3. *(The stopping time $\bar{t}$ equals the maximal time $T(E_0)$)* Here we show that, by taking δ_0 smaller if needed, we have $\bar{t} = T(E_0)$. To this aim, assume by contradiction that $\bar{t} < T(E_0)$. Then,

$$\|\psi_{\bar{t}}\|_{C^1(\partial F)} = \varepsilon_0 \quad \text{or} \quad \int_{\mathbb{T}^3} |Dw_{\bar{t}}|^2\, dx = 2\delta_0$$

We further split into two sub-steps, according to the two alternatives above.

Step 3-(a). Assume that

$$\int_{\mathbb{T}^3} |Dw_{\bar{t}}|^2\, dx = 2\delta_0 \tag{3.22}$$

Recall that (3.13) holds. Thus, by (3.9), (3.12), (3.19), and (3.21) we have

$$\partial^2 J(E_t)\left[[\partial_{\nu_t} w_t]\right] \ge \sigma_\varepsilon \|[\partial_{\nu_t} w_t]\|^2_{H^1(\partial E)} \text{ for all } t \in (0, \bar{t}).$$

In turn, by Lemma 3.5 we may estimate

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{T}^3} |Dw_t|^2\, dx\right) \le -\sigma_\varepsilon \|[\partial_{\nu_t} w_t]\|^2_{H^1(\partial E)} + \frac{1}{2}\int_{\partial E_t} (\partial_{\nu_t} w_t^+ + \partial_{\nu_t} w_t^-)[\partial_{\nu_t} w_t]^2\, d\mathcal{H}^2$$

for every $t \le \bar{t}$. By Proposition 3.6-(iii) and (3.17), we may estimate the last term by

$$\begin{aligned}\int_{\partial E_t} (\partial_{\nu_t} w_t^+ + \partial_{\nu_t} w_t^-)[\partial_{\nu_t} w_t]^2 \, d\mathcal{H}^2 &\le C \int_{\partial E_t} (|\partial_{\nu_t} w_t^+|^3 + |\partial_{\nu_t} w_t^-|^3) \, d\mathcal{H}^2 \\ &\le C \int_{\partial E_t} |[\partial_{\nu_t} w_t]|^3 \, d\mathcal{H}^2 .\end{aligned}$$

Now, Proposition 3.6-(v) implies

$$\|[\partial_{\nu_t} w_t]\|_{L^3(\partial E_t)} \le C \|[\partial_{\nu_t} w_t]\|_{H^1(\partial E_t)}^{2/3} \|w_t - \hat{w}_t\|_{L^2(\partial E_t)}^{1/3} .$$

Therefore, combining the last three estimates, we get

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{T}^3} |Dw_t|^2 \, dx\right) &\le -\sigma_\varepsilon \|[\partial_{\nu_t} w_t]\|_{H^1(\partial E_t)}^2 + C \|w_t - \hat{w}_t\|_{L^2(\partial E_t)} \|[\partial_{\nu_t} w_t]\|_{H^1(\partial E_t)}^2 \\ &\le -\frac{\sigma_\varepsilon}{2} \|[\partial_{\nu_t} w_t]\|_{H^1(\partial E_t)}^2 \end{aligned} \tag{3.23}$$

for every $t \le \bar{t}$, where the last inequality holds provided that δ_0 is small enough since by (3.12) and by trace theorem

$$\|w_t - \hat{w}_t\|_{L^2(\partial E_t)}^2 \le C \int_{\mathbb{T}^3} |Dw_t|^2 \, dx \le C\delta_0 .$$

We use (3.18) to conclude

$$\begin{aligned}\int_{\mathbb{T}^3} |Dw_t|^2 \, dx &= -\int_{\partial E_t} w_t [\partial_{\nu_t} w_t] \, d\mathcal{H}^2 = -\int_{\partial E_t} (w_t - \hat{w}_t)[\partial_{\nu_t} w_t] \, d\mathcal{H}^2 \\ &\le \|w_t - \hat{w}_t\|_{L^2(\partial E_t)} \|[\partial_{\nu_t} w_t]\|_{L^2(\partial E_t)} \\ &\le C \|[\partial_{\nu_t} w_t]\|_{L^2(\partial E_t)}^2 .\end{aligned}$$

Combining the above inequality with (3.23), we finally obtain

$$\frac{d}{dt} \int_{\mathbb{T}^3} |Dw_t|^2 \, dx \le -c_0 \int_{\mathbb{T}^3} |Dw_t|^2 \, dx$$

for every $t \le \bar{t}$ and for a suitable $c_0 > 0$. Integrating the differential inequality and recalling (3.10), we get

$$\int_{\mathbb{T}^3} |Dw_t|^2 \, dx \le \mathrm{e}^{-c_0 t} \int_{\mathbb{T}^3} |Dw_{E_0}|^2 \, dx \le \delta_0 \mathrm{e}^{-c_0 t} , \tag{3.24}$$

which for $t = \bar{t}$ gives a contradiction to (3.22).

Step 3-(b). Assume that

$$\|\psi_{\bar{t}}\|_{C^1(\partial F)} = \varepsilon_0 . \tag{3.25}$$

Recalling (3.6) and denoting by X_t the velocity field of the flow (see Definition 2.1), we may compute

$$\begin{aligned}\frac{d}{dt} D(E_t) &= \frac{d}{dt} \int_{E_t} d_F \, dx = \int_{E_t} \operatorname{div}(d_F X_t) \, dx \\ &= \int_{\partial E_t} d_F (X_t \cdot \nu_t) \, d\mathcal{H}^2 = \int_{\partial E_t} d_F [\partial_{\nu_t} w_t] \, d\mathcal{H}^2 \\ &= -\int_{\mathbb{T}^3} Dh \cdot Dw_t \, dx ,\end{aligned}$$

where h denotes the harmonic extension of d_F to $\mathbb{T}^3 \setminus \partial E_t$. Note that

$$\|Dh\|_{L^2(\mathbb{T}^3)} \le C\|d_F\|_{C^1(\partial E_t)} \le C\,.$$

Thus, also by (3.24), we have

$$\frac{d}{dt}D(E_t) \le C\|Dw_t\|_{L^2(\mathbb{T}^3)} \le C\sqrt{\delta_0}\mathrm{e}^{-\frac{c_0}{2}t}$$

for all $t \le \bar t$. By integrating over $(0,\bar t)$ and recalling (3.7) we get

$$\|\psi_{\bar t}\|_{L^2(\partial F)} \le C\sqrt{D(E_{\bar t})} \le C\sqrt{D(E_0) + C\sqrt{\delta_0}} \le C\sqrt[4]{\delta_0}\,, \tag{3.26}$$

provided that δ_0 is small enough. Since by (3.12) and (3.9) we also have uniform $W^{2,3}$-bounds on $\psi_{\bar t}$, by standard interpolation we infer from (3.26) that $\|\psi_{\bar t}\|_{C^1(\partial F)} \le C\delta_0^\theta$ for a suitable $\theta \in (0,1)$. Thus if δ_0 is small enough we reach a contradiction to (3.25).

The combination of Step 3-(a) (see also (3.24)) and Step 3-(b) yields $\bar t = T(E_0)$ and

$$\|\psi_t\|_{C^1(\partial F)} < \varepsilon_0 \text{ and } \int_{\mathbb{T}^3} |Dw_t|^2\,dx \le \mathrm{e}^{-c_0 t}\int_{\mathbb{T}^3}|Dw_{E_0}|^2\,dx \quad \text{for all } t \in (0, T(E_0)). \tag{3.27}$$

Step 4. *(Global-in-time existence)* Here we show that, by taking δ_0 smaller if needed, we have $T(E_0) = +\infty$, that is the classical solution exists for all times. To this aim, recall that by (3.23) and by the fact that $\bar t = T(E_0)$ we have

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{T}^3}|Dw_t|^2\,dx\right) + \frac{\sigma_\varepsilon}{2}\|[\partial_{\nu_t} w_t]\|^2_{H^1(\partial E_t)} \le 0$$

for all $t \in (0, T(E_0))$. Assume now by contradiction $T(E_0) < +\infty$. Integrating over $\big(T(E_0) - \frac{T_0}{2}, T(E_0) - \frac{T_0}{4}\big)$, where T_0 is as in (3.11), we obtain

$$\begin{aligned}\sigma_\varepsilon \int_{T(E_0)-\frac{T_0}{2}}^{T(E_0)-\frac{T_0}{4}} \|[\partial_{\nu_t} w_t]\|^2_{H^1(\partial E_t)}\,dt &\le \int_{\mathbb{T}^3}|Dw_{T(E_0)-\frac{T_0}{2}}|^2\,dx - \int_{\mathbb{T}^3}|Dw_{T(E_0)-\frac{T_0}{4}}|^2\,dx\\ &\le \delta_0\,,\end{aligned}$$

where the last inequality follows from (3.27) and (3.10). Thus, by the mean value theorem there exists $\hat t \in \big(T(E_0) - \frac{T_0}{2}, T(E_0) - \frac{T_0}{4}\big)$ such that $\|[\partial_{\nu_{\hat t}} w_{\hat t}]\|^2_{H^1(\partial E_t)} \le \frac{8\delta_0}{T_0\sigma_\varepsilon}$. Note that for any measurable set $E \subset \mathbb{T}^3$ we have $\|v_E\|_{C^1(\mathbb{T}^3)} \le L$ for some absolute constant L and that w_F is constant. Thus, since $H^1(\partial E_{\hat t})$ embeds into $L^p(\partial E_{\hat t})$ for all $p > 1$, by Proposition 3.6 we in turn infer that

$$\begin{aligned}&[H_{\hat t}(\cdot + \psi_{\hat t}(\cdot)\nu_F(\cdot) - H_F]^2_{C^{0,\alpha}(\partial F)}\\ &\quad\le C[w_{\hat t}(\cdot + \psi_{\hat t}(\cdot)\nu_F(\cdot) - w_F]^2_{C^{0,\alpha}(\partial F)} + C[v_{\hat t}(\cdot + \psi_{\hat t}(\cdot)\nu_F(\cdot) - v_{\hat t}]^2_{C^{0,\alpha}(\partial F)} + C[v_{\hat t} - v_F]^2_{C^{0,\alpha}(\partial F)}\\ &\quad\le C[w_{\hat t}]^2_{C^{0,\alpha}(\partial E_{\hat t})} + CL^2\|\psi_{\hat t}\|^2_{C^1(\partial F)} + C\|u_{\hat t} - u_F\|^2_{L^2(\mathbb{T}^3)}\\ &\quad\le C\frac{\delta_0}{T_0\sigma_\varepsilon} + CL^2\|\psi_{\hat t}\|^2_{C^1(\partial F)} + C|E_{\hat t}\Delta F|^2,\end{aligned}$$

where $[\cdot]_{C^{0,\alpha}(\partial E_{\hat t})}$ stands for the α-Hölder seminorm on $\partial E_{\hat t}$. Thus, if we choose δ_0 sufficiently small, the above inequality together with (3.12), (3.7) and (3.27) ensures that $E_{\hat t} \in \mathfrak{h}^{2,\alpha}_M(F,U)$. In turn, by (3.11) the time span of existence of the classical solution starting from $E_{\hat t}$ is at least T_0, which means that $(E_t)_t$ can be continued beyond $T(E_0)$. This is clearly a contradiction.

Step 5. *(Convergence, up to subsequences, to a translate of F)* Let $t_n \to +\infty$. Then by (3.27) the sets E_{t_n} satisfy the hypotheses of Lemma 3.7. Thus, up to a (not relabeled) subsequence we have that there exists a critical set $F' \in \mathfrak{C}^1_M(F,U)$ such that $E_{t_n} \to F'$ in $W^{\frac{5}{2},2}$. Due to (3.9) and (3.21) we also have $\|\psi_{F'}\|_{W^{2,3}(\partial F)} \le \delta_1$. But then (3.20) implies that $F' = F + \sigma$ for a suitable (small) $\sigma \in \mathbb{R}^3$.

Step 6. *(Exponential convergence of the full sequence)* Consider now

$$D_\sigma(E) := \int_{E\Delta(F+\sigma)} \mathrm{dist}\,(x, \partial F + \sigma)\, dx\,.$$

The very same calculations performed in Step 3-(b) show that

$$\frac{d}{dt} D_\sigma(E_t) \le C\|Dw_t\|_{L^2(\mathbb{T}^3)} \le C\sqrt{\delta_0}\mathrm{e}^{-\frac{c_0}{2}t} \tag{3.28}$$

for all $t > 0$. From this inequality it is easy to deduce that $\lim_{t\to+\infty} D_\sigma(E_t)$ exists. Thus, by the previous step $D_\sigma(E_t) \to 0$ as $t \to +\infty$. In turn, integrating (3.28) and writing $\partial E_t = \{x + \psi_{\sigma,t}(x)\nu_{F+\sigma}(x) : x \in \partial F + \sigma\}$ we get

$$\|\psi_{\sigma,t}\|^2_{L^2(\partial F+\sigma)} \le C D_\sigma(E_t) \le \int_t^{+\infty} C\sqrt{\delta_0}\mathrm{e}^{-\frac{c_0}{2}s}\, ds \le C\sqrt{\delta_0}\mathrm{e}^{-\frac{c_0}{2}t}\,. \tag{3.29}$$

Since by the previous steps $\|\psi_{\sigma,t}\|_{W^{2,3}(\partial F+\sigma)}$ is bounded, we infer from (3.29) and standard interpolation estimates that also $\|\psi_{\sigma,t}\|_{C^{1,\beta}(\partial F+\sigma)}$ decays exponentially for $\beta \in (0, \frac{1}{3})$. For all $\beta \in (0,1)$, setting $p = \frac{2}{1-\beta}$, we have by (3.29) and by (3.7)

$$\begin{aligned} \|v_t - v_{F+\sigma}\|_{C^{1,\beta}(\mathbb{T}^3)} &\le C\|v_t - v_{F+\sigma}\|_{W^{2,p}(\mathbb{T}^3)} \le C\|u_t - u_{F+\sigma}\|_{L^p(\mathbb{T}^3)} \\ &\le C|E_t\Delta(F+\sigma)|^{\frac{1}{p}} \le C\|\psi_{\sigma,t}\|^{\frac{1}{p}}_{L^2(\partial F+\sigma)} \\ &\le C\delta_0^{\frac{1}{4p}}\mathrm{e}^{-\frac{c_0}{4p}t}, \end{aligned} \tag{3.30}$$

where we recall that u_t stands for u_{E_t}, see (2.3). Denote the average of w_t on ∂E_t by $\bar{w}_t$. Since by (3.27) we have that

$$\|w_t\big(\,\cdot + \psi_{\sigma,t}(\cdot)\nu_{F+\sigma}(\cdot)\big) - \bar{w}_t\|_{H^{\frac{1}{2}}(\partial F+\sigma)} \le C\|w_t - \bar{w}_t\|_{H^{\frac{1}{2}}(\partial E_t)} \le C\|Dw_t\|_{L^2(\mathbb{T}^3)} \le C\sqrt{\delta_0}\mathrm{e}^{-\frac{c_0}{2}t}\,,$$

it follows (taking into account also (3.30)) that

$$\begin{aligned} \Big\| \big[H_t\big(\,\cdot + \psi_{\sigma,t}(\cdot)\nu_{F+\sigma}(\cdot)\big) - \overline{H}_t\big] \\ - [H_{\partial F+\sigma} - \overline{H}_{\partial F+\sigma}]\Big\|_{H^{\frac{1}{2}}(\partial F+\sigma)} \to 0 \quad \text{exponentially fast,} \end{aligned} \tag{3.31}$$

where $\overline{H}_t$ and $\overline{H}_{\partial F+\sigma}$ stand for the average of H_t on ∂E_t and of $H_{\partial F+\sigma}$ on $\partial F+\sigma$, respectively. Let d_σ be the signed distance function from $F+\sigma$ and let Ψ_t denote a diffeomorphism such that $\Psi_t(F+\sigma) = E_t$. Clearly we can find such a diffeomorphism with the additional property that $\Psi_t(x) = x + \psi_{\sigma,t}(x)\nu_{F+\sigma}(x)$ on $\partial F + \sigma$ and $\|\Psi_t - Id\|_{C^1(\mathbb{T}^3)} \le C\|\psi_{\sigma,t}\|_{C^1(\partial F+\sigma)}$. Then, denoting the tangential divergence on ∂E_t by div_{τ_t} and the tangential Jacobian of Ψ_t by $J_\tau\Psi_t$,

we have

$$\begin{aligned}
&\left|\int_{\partial E_t} H_t \nabla d_\sigma \cdot \nu_t \, d\mathcal{H}^2 - \int_{\partial F+\sigma} H_{\partial F+\sigma} \, d\mathcal{H}^2\right| \\
&\qquad = \left|\int_{\partial E_t} \mathrm{div}_{\tau_t} \nabla d_\sigma \, d\mathcal{H}^2 - \int_{\partial F+\sigma} \mathrm{div}_\tau \nabla d_\sigma \, d\mathcal{H}^2\right| \\
&\qquad \le \left|\int_{\partial F+\sigma} \big(\mathrm{div}_{\tau_t} \nabla d_\sigma \circ \Psi_t J_\tau \Psi_t - \mathrm{div}_\tau \nabla d_\sigma\big) \, d\mathcal{H}^2\right| \\
&\qquad \le C \|\psi_{\sigma,t}\|_{C^1(\partial F+\sigma)} \,,
\end{aligned} \tag{3.32}$$

where the constant C also depends on the C^2-bounds on ∂F. Moreover,

$$\begin{aligned}
\left|\int_{\partial E_t} (H_t \nabla d_\sigma \cdot \nu_t - H_t) \, d\mathcal{H}^2\right| &= \left|\int_{\partial E_t} H_t (\nabla d_\sigma - \nu_t) \cdot \nu_t \, d\mathcal{H}^2\right| \\
&\le \|H_t\|_{L^1(\partial E_t)} \|\nabla d_\sigma - \nu_t\|_{L^\infty(\partial E_t)} \le C \|\psi_{\sigma,t}\|_{C^1(\partial F+\sigma)} \,,
\end{aligned} \tag{3.33}$$

where we have also used the uniform bounds on H_t established in the previous steps. Combining (3.32) and (3.33), we get that $\overline{H}_t - \overline{H}_{\partial F+\sigma}$ decays exponentially and in turn, thanks to (3.31)

$$\big\|H_t\big(\,\cdot + \psi_{\sigma,t}(\cdot)\nu_{F+\sigma}(\cdot)\big) - H_{\partial F+\sigma}\big\|_{H^{\frac12}(\partial F+\sigma)} \to 0 \quad \text{exponentially fast.}$$

The conclusion follows arguing as in the end of the proof of Lemma 3.7. q.e.d.

Theorem 3.4 can be readily extended to the Neumann case, at least when the stable critical set F is well contained in Ω. Recall in this case the energy (2.2) must be replaced with

$$J_N(E) := P_\Omega(E) + \gamma \int_\Omega |\nabla v_E|^2 \, dx,$$

where $P_\Omega(E)$ denotes the perimeter of E inside Ω and the function v_E is the solution of

$$\begin{cases} -\Delta v_E = u_E - m & \text{in } \Omega \\ \displaystyle\int_\Omega v_E \, dx = 0 \,, \quad \frac{\partial v_E}{\partial \nu} = 0 \,, & \text{on } \partial\Omega \,. \end{cases}$$

Here $u_E = 2\chi_E - 1$ and $m = \fint_\Omega u_E \, dx$. As in (2.4) we have

$$v_E(x) = \int_\Omega G(x,y) u_E(y) \, dy \,,$$

where G is the solution of

$$\begin{cases} -\Delta_y G(x,y) = \delta_x - \frac{1}{|\Omega|} & \text{in } \Omega \\ \displaystyle\int_\Omega G(x,y) \, dy = 0 \,, \quad \nabla_y G(x,y) \cdot \nu(y) = 0 \,, & \text{if } y \in \partial\Omega \,. \end{cases}$$

As in the periodic case, we say that a smooth subset $F \subset\subset \Omega$ is a *critical set* for the functional J_N if there exists a constant $\lambda \in \mathbb{R}$ such that

$$H_{\partial F}(x) + 4\gamma v_F(x) = \lambda \qquad \text{for all } x \in \partial F.$$

The quadratic form associated with the second variation $\partial^2 J_N(E)$ is also defined as in (2.9). If $F \subset\subset \Omega$ is a smooth local minimizer of J_N under volume constraint, then it is also critical and $\partial^2 J_N(E)[\varphi] \geq 0$ for all $\varphi \in \widetilde{H}(\partial F)$.

Note that, unlike in the periodic case, the functional J_N is not translation invariant. Therefore we say that a smooth critical set F is *strictly stable* if

$$\partial^2 J_N(E)[\varphi] > 0 \qquad \text{for all } \varphi \in \widetilde{H}(\partial E) \setminus \{0\}.$$

With these definitions in hand we can state the following counterpart of Theorem 3.4.

Theorem 3.8. *Let Ω be an open set in $\mathbb{R}^3$ and let $F \subset\subset \Omega$ be a smooth strictly stable critical set and U a regular tubular neighborhood of ∂F. Then, for every $M > 0$ and $\alpha \in (0,1)$ there exists $\delta_0 > 0$ with the following property: Let $E_0 \in \mathfrak{h}^{2,\alpha}_M(F,U)$ be such that*

$$|E_0| = |F|\,, \qquad |E_0 \Delta F| \leq \delta_0\,, \qquad \textit{and} \qquad \int_\Omega |Dw_{E_0}|^2\,dx \leq \delta_0\,.$$

Then, the unique classical solution $(E_t)_t$ to the Mullins-Sekerka flow (1.1) with initial datum E_0 is defined for all $t > 0$. Moreover, $E_t \to F$ in $W^{5/2,2}$ exponentially fast as $t \to +\infty$.

The proof of this result is similar to the one of Theorem 3.4. Actually it is simpler since we do not need the argument used in Step 2, where we controlled the translational component of the flow. Note that in the statement of Lemma 2.6 now (2.13) holds for all $\varphi \in \widetilde{H}(\partial E)$. Finally, observe that under the assumptions of Proposition 2.7 we may conclude that $F' = F$, i.e., that there are no other critical sets close to F.

The assumption that F does not touch the boundary may seem restrictive. However we remark that in two and three dimensions there are examples of strictly stable critical sets which consist of either a single or multiple almost spherical sets well contained in Ω. The precise conditions on the parameters m, γ and $|\Omega|$ under which these strictly stable sets exist are given in [**42**, **43**, **44**]. Other examples of local minimizers well contained in Ω are given in [**10**]. An example of a local minimizer touching the boundary is provided in [**41**].

4. Nonlinear stability for the surface diffusion flow

Throughout the section we assume $\gamma = 0$ in (2.2), so that we will be dealing only with the standard local perimeter. We will show how to adapt the strategy devised in the previous section to the case of the surface diffusion equation. For the definition of sets of class $h^{2,\alpha}$ we refer to the previous section.

Definition 4.1 (Surface diffusion flows)**.** Let $E_0 \subset \mathbb{T}^3$ be of class $h^{2,\alpha}$ for some $\alpha \in (0,1)$. We say that the one-parameter family $(E_t)_{t\in(0,T)}$ is a *classical solution* to the *surface diffusion equation* on the interval $(0,T)$ with initial datum E_0 if it is a smooth flow in the sense of Definition 3.1, $E_t \to E_0$ in $C^{2,\alpha}$ as $t \to 0^+$, and the following evolution law holds:

$$V_t = \Delta_\tau H_t \quad \text{on } \partial E_t \text{ for all } t \in (0,T)\,, \tag{4.1}$$

where, as usual, V_t stands for the outer normal velocity of the moving boundary ∂E_t, H_t stands for $H_{\partial E_t}$ and Δ_τ is the Laplace-Beltrami operator on ∂E_t.

It is well-known that the surface diffusion flow is volume preserving. This can be straightforwardly checked by the following computation:

$$\frac{d}{dt}|E_t| = \int_{\partial E_t} V_t\,d\mathcal{H}^2 = \int_{\partial E_t} \Delta_\tau H_t\,d\mathcal{H}^2 = 0\,.$$

The following local-in-time existence and uniqueness result has been established in [**14**][2] . We make use of the notation introduced in the previous section.

Theorem 4.2 (Local-in-time existence and uniqueness, [**14**])**.** *Let $F_0 \subset \mathbb{T}^3$ be a smooth set and U a regular tubular neighborhood of ∂F_0. Then, for every $M > 0$ and $\alpha \in (0,1)$ there exists $T > 0$ with the following property: For every $E_0 \in \mathfrak{h}^{2,\alpha}_M(F_0, U)$ there exists a unique classical solution to the surface diffusion flow in $(0,T)$ with initial datum E_0.*

As before we are interested in the asymptotic stability of strictly stable configurations. The main result of the section is the following.

Theorem 4.3 (Main result)**.** *Let $F \subset \mathbb{T}^3$ be a strictly stable critical set according to Definition* 2.4 *and let U be a regular tubular neighborhood of ∂F. Then, for every $M > 0$ and $\alpha \in (0,1)$ there exists $\delta_0 > 0$ with the following property: Let $E_0 \in \mathfrak{h}^{2,\alpha}_M(F, U)$ be of class $W^{3,2}$ such that*

$$|E_0| = |F|\,, \qquad |E_0 \Delta F| \le \delta_0\,, \qquad \text{and} \qquad \int_{\partial E_0} |D_\tau H_{\partial E_0}|^2 \, d\mathcal{H}^2 \le \delta_0\,.$$

Then, the unique classical solution $(E_t)_t$ to the surface diffusion flow with initial datum E_0 is defined for all $t > 0$. Moreover, $E_t \to F + \sigma$ in $W^{3,2}$ as $t \to +\infty$, for some $\sigma \in \mathbb{R}^3$. The convergence is exponentially fast; more precisely, there exist η, $c_F > 0$ such that for all $t > 0$, writing

$$\partial E_t = \{x + \psi_{\sigma,t}(x)\nu_{F+\sigma}(x) : x \in \partial F + \sigma\}\,,$$

we have

$$\|\psi_{\sigma,t}\|_{W^{3,2}(\partial F+\sigma)} \le \eta \mathrm{e}^{-c_F t}\,.$$

Both $|\sigma|$ and η vanish as $\delta_0 \to 0^+$.

Note that the $H^1(\partial E)$ norm of $H_{\partial E}$ is equivalent to the $W^{3,2}(\partial F)$ norm of ψ_E. This explains the $W^{3,2}$ convergence in the above theorem.

As before, the proof of the theorem, which is close in spirit to the proof of Theorem 3.4, is postponed until the end of the section. We first collect some auxiliary results, whose proofs are given in Section 5.

Lemma 4.4 (Energy identities)**.** *Let $(E_t)_{t\in(0,T)}$ be a smooth flow satisfying* (4.1)*. The following energy idienties hold:*

$$\frac{d}{dt} J(E_t) = -\int_{\partial E_t} |D_\tau H_t|^2 \, dx\,, \tag{4.2}$$

and

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}\int_{\partial E_t} |D_\tau H_t|^2 \, dx\right) = &- \partial^2 J(E_t)\,[\Delta_\tau H_t] - \int_{\partial E_t} B_t\,[D_\tau H_t]\,\Delta_\tau H_t \, d\mathcal{H}^2 \\ &+ \frac{1}{2}\int_{\partial E_t} H_t |D_\tau H_t|^2 \Delta_\tau H_t \, d\mathcal{H}^2\,, \end{aligned} \tag{4.3}$$

where $\partial^2 J(E_t)$ is the quadratic form defined in (2.9) *(with E_t in place of E and with $\gamma = 0$) and, as usual, the subscript t stands for E_t. Note also that we have used the notation $B_t[\cdot]$ to denote the second fundamental quadratic form on ∂E_t, which we recall is defined as $B_t[\tau] := (D_\tau \nu_t \tau) \cdot \tau$ for all $\tau \in \mathbb{R}^3$.*

[2]In fact [**14**] deals with the evolution in the whole space $\mathbb{R}^N$, but it is clear that the same arguments go through in the periodic case.

Lemma 4.5 (Interpolation on boundaries)**.** *Let $F \subset \mathbb{T}^3$ be a smooth set, U a regular tubular neighborhood of ∂F, and $M > 0$, $p \in (2, +\infty)$ fixed constants. Then, there exists $C > 0$ with the following property: for every $E \in \mathfrak{C}^1_M(F, U)$ and $f \in H^1(\partial E)$ it holds*

$$\|f\|_{L^p(\partial E)} \leq C \left(\|D_\tau f\|^\theta_{L^2(\partial E)} \|f\|^{1-\theta}_{L^2(\partial E)} + \|f\|_{L^2(\partial E)} \right),$$

with $\theta := 1 - \frac{2}{p}$. Moreover, the following Poincaré inequality holds

$$\|f - \bar{f}\|_{L^p(\partial E)} \leq C \|D_\tau f\|_{L^2(\partial E)},$$

where $\bar{f}$ denotes the piecewise constant function defined as $\fint_{\Gamma^i} f \, d\mathcal{H}^2$ on each connected component Γ^i of ∂E.

The proof of the above lemma can be found in [**3**, Theorem 3.70].

For the next lemma we introduce the following notation: for every sufficiently regular f defined on ∂E we set

$$\delta_i f := D_\tau f \cdot e_i \qquad \text{and} \qquad D^2_\tau f := (\delta_i \delta_j f)_{i,j}, \tag{4.4}$$

where e_i is the i-th element of the canonical basis of $\mathbb{R}^3$.

Lemma 4.6 (H^2-estimates on boundaries)**.** *Let F, U, and M be as in Lemma 4.5. Then there exists a constant $C > 0$ such that if $E \in \mathfrak{C}^1_M(F, U)$ and $f \in H^1(\partial E)$, with $\Delta_\tau f \in L^2(\partial E)$, then $f \in H^2(\partial E)$ and*

$$\|D^2_\tau f\|_{L^2(\partial E)} \leq C \|\Delta_\tau f\|_{L^2(\partial E)} (1 + \|H_{\partial E}\|^2_{L^4(\partial E)}).$$

The following lemma provides the crucial "geometric interpolation" that will be needed in the proof of the main theorem.

Lemma 4.7 (Geometric interpolation)**.** *Let F, U, and M be as in Lemma 4.5. There exists a constant $C > 0$ such that if $E \in \mathfrak{C}^1_M(F, U)$ the following estimates holds:*

$$\int_{\partial E} |B_{\partial E}| |D_\tau H_{\partial E}|^2 |\Delta_\tau H_{\partial E}| \, d\mathcal{H}^2 \\ \leq C \|D_\tau(\Delta_\tau H_{\partial E})\|^2_{L^2(\partial E)} \|D_\tau H_{\partial E}\|_{L^2(\partial E)} \left(1 + \|H_{\partial E}\|^3_{L^6(\partial E)}\right).$$

The next lemma highlights an interesting property of the mean curvature. Note that since ∂E can be disconnected (as in the case of lamellae) one can not expect Poincaré inequality to hold on ∂E. However, if E is sufficiently close to a stable critical set then the Poincaré inequality holds for $H_{\partial E}$.

Lemma 4.8 (Geometric Poincaré Inequality)**.** *Fix $p > 2$, let $F \subset \mathbb{T}^3$ be a strictly stable critical set according to Definition 2.4 and let δ_1 be the constant provided by Lemma 2.6, with $\varepsilon = 1$ (and $\gamma = 0$). Then, there exists $C > 0$ such that*

$$\int_{\partial E} |H_{\partial E} - \overline{H}_{\partial E}|^2 \, d\mathcal{H}^2 \leq C \int_{\partial E} |D_\tau H_{\partial E}|^2 \, d\mathcal{H}^2, \tag{4.5}$$

provided that

$$\partial E = \{x + \psi(x)\nu_F(x) : x \in \partial F \text{ for some smooth } \psi \text{ with } \|\psi\|_{W^{2,p}(\partial F)} \leq \delta_1\}.$$

Here $\overline{H}_{\partial E}$ stands for the average $\fint_{\partial E} H_{\partial E} \, d\mathcal{H}^2$.

Finally, we have:

Lemma 4.9 (Compactness of sets)**.** *Let F, U, and M be as in Lemma* 4.5*. Let $\{E_n\}_n \subset \mathfrak{C}^1_M(F,U)$ be a sequence of sets such that*

$$\sup_n \int_{\partial E_n} |D_\tau H_{\partial E_n}|^2 \, dx < +\infty \, .$$

Then there exists $F' \in \mathfrak{C}^1_M(F,U)$ of class $W^{3,2}$ such that, up to a (non relabeled) subsequence, $E_n \to F'$ in $W^{2,p}$ for all $p \in [1,+\infty)$. Moreover, if (4.5) *holds for every set E_n (with C independent of n) and*

$$\int_{\partial E_n} |D_\tau H_{\partial E_n}|^2 \, dx \to 0 \, ,$$

then F' is critical in the sense of Definition 2.3 *and the convergence holds in $W^{3,2}$.*

The proof of this lemma is similar to the proof of Lemma 3.7 given in Subsection 5.1 and thus we omit it.

***Proof of Theorem* 4.3.** The proof of the theorem is very close in spirit to the proof of Theorem 3.4. In the following, C will denote a constant depending only on the C^1-bounds on the boundary of the set. The value of C may change from line to line. For every $\varepsilon_0 > 0$ sufficiently small, there exists $\delta_0 \in (0,1)$ so small that for any set $E \in \mathfrak{C}^1_M(F,U)$ the following implications hold true:

$$E \in \mathfrak{h}^{2,\alpha}_M(F,U) \text{ and } D(E) \le \delta_0 \Longrightarrow \|\psi_E\|_{C^1(\partial F)} \le \frac{\varepsilon_0}{2} \, , \tag{4.6}$$

where $D(E)$ is defined in (3.6), and

$$\|\psi_E\|_{C^1(\partial F)} \le \varepsilon_0 \text{ and } \int_{\partial E} |D_\tau H_{\partial E}|^2 \, d\mathcal{H}^2 \le 1 \Longrightarrow \|\psi_E\|_{W^{2,6}(\partial F)} \le \omega(\varepsilon_0) \le 1 \, , \tag{4.7}$$

where ω is a positive non-decreasing function such that $\omega(\varepsilon_0) \to 0$ as $\varepsilon_0 \to 0^+$. Note that the last implication is true thanks to Lemma 4.9.

Note also that by Lemma 4.8, there exists $C > 0$ such that if ε_0 is small enough, then

$$\|\psi_E\|_{W^{2,6}(\partial F)} \le \omega(\varepsilon_0) \Longrightarrow \int_{\partial E} |H_{\partial E} - \overline{H}_{\partial E}|^2 \, d\mathcal{H}^2 \le C \int_{\partial E} |D_\tau H_{\partial E}|^2 \, d\mathcal{H}^2 \, , \tag{4.8}$$

where $\overline{H}_{\partial E}$ is the average of $H_{\partial E}$ over ∂E. Fix ε_0, $\delta_0 \in (0,1)$ satisfying (4.6), (4.7) and (4.8), and choose an initial set $E_0 \in \mathfrak{h}^{2,\alpha}_M(F,U)$ such that

$$D(E_0) \le \delta_0 \qquad \text{and} \qquad \int_{\partial E_0} |D_\tau H_{\partial E_0}|^2 \, d\mathcal{H}^2 \le \delta_0 \, . \tag{4.9}$$

Let $(E_t)_{t \in (0,T(E_0))}$ be the unique classical solution to the surface diffusion flow provided by Theorem 4.2, with $T(E_0)$ denoting the maximal time of existence. By the same theorem, there exists $T_0 > 0$ such that (3.11) holds. We now split the rest of the proof into several steps as in the proof of Theorem 3.4.

Step 1. *(Stopping-time)* Let $\bar{t} \le T(E_0)$ be the maximal time such that

$$\|\psi_t\|_{C^1(\partial F)} < \varepsilon_0 \quad \text{and} \quad \int_{\partial E_t} |D_\tau H_t|^2 \, d\mathcal{H}^2 < 2\delta_0. \quad \text{for all } t \in (0,\bar{t}), \tag{4.10}$$

As before, we claim that by taking ε_0 and δ_0 smaller if needed, we have $\bar{t} = T(E_0)$. This claim will be proved in Step 3 below.

Step 2. *(Estimate of the translational component of the flow)* We claim that there exists $\varepsilon > 0$ such that

$$\min_{\eta\in\Pi_F} \left\|\Delta_\tau H_t - \eta\cdot\nu_t\right\|_{L^2(\partial E_t)} \geq \varepsilon\|\Delta_\tau H_t\|_{L^2(\partial E_t)} \qquad \text{for all } t\in(0,\bar t)\,, \tag{4.11}$$

where Π_F is defined in (2.10). To this aim, let $\eta_t\in\Pi_F$ be such that

$$\Delta_\tau H_t = \eta_t\cdot\nu_t + g, \tag{4.12}$$

where g is orthogonal to the subspace of $L^2(\partial E_t)$ spanned by $\tilde e_i\cdot\nu_t$ with $i\in I_F$ (see (2.11)). As in Step 2 of the proof of Theorem 3.4 we will show that if ε is small enough, then assuming $\|g\|_{L^2(\partial E_t)} < \varepsilon\|\Delta_\tau H_t\|_{L^2(\partial E_t)}$ leads to a contradiction. Recall that $\Delta_\tau H_t$ has zero average. Therefore, setting $\overline{H}_t := \fint_{\partial E_t} H_t\,d\mathcal{H}^2$, and recalling also (4.7) and (4.8), we get

$$\begin{aligned}
\|H_t-\overline{H}_t\|^2_{L^2(\partial E_t)} &\leq C\int_{\partial E_t}|D_\tau H_t|^2\,d\mathcal{H}^2\\
&= -C\int_{\partial E_t}\Delta_\tau H_t H_t\,d\mathcal{H}^2 = -C\int_{\partial E_t}\Delta_\tau H_t(H_t-\overline{H}_t)\,d\mathcal{H}^2\\
&\leq C\|H_t-\overline{H}_t\|_{L^2(\partial E_t)}\|\Delta_\tau H_t\|_{L^2(\partial E_t)}\,.
\end{aligned} \tag{4.13}$$

Recall now that $\int_{\partial E_t} H_t\nu_t\,d\mathcal{H}^2 = \int_{\partial E_t}\nu_t\,d\mathcal{H}^2 = 0$. Thus, multiplying (4.12) by $H_t-\overline{H}_t$, integrating over ∂E_t, and using (4.13), we get

$$\begin{aligned}
\left|\int_{\partial E_t}(H_t-\overline{H}_t)\Delta_\tau H_t\,d\mathcal{H}^2\right| &= \left|\int_{\partial E_t}(H_t-\overline{H}_t)g\,d\mathcal{H}^2\right|\\
&< \varepsilon\|H_t-\overline{H}_t\|_{L^2(\partial E_t)}\|\Delta_\tau H_t\|_{L^2(\partial E_t)}\\
&\leq C\varepsilon\|\Delta_\tau H_t\|^2_{L^2(\partial E_t)}\,.
\end{aligned}$$

Arguing as in Step 2 of the proof of Theorem 3.4 we have that, if ε_0 is small enough there exists a constant C such that $|\eta_t|\leq C\|\Delta_\tau H_t\|_{L^2(\partial E_t)}$. Hence

$$\begin{aligned}
\|\eta_t\cdot\nu_t\|^2_{L^2(\partial E_t)} &= \int_{\partial E_t}\Delta_\tau H_t(\eta_t\cdot\nu_t)\,d\mathcal{H}^2 = -\int_{\partial E_t}D_\tau H_t\cdot D_\tau(\eta_t\cdot\nu_t)\,d\mathcal{H}^2\\
&\leq |\eta_t|\|D_\tau\nu_t\|_{L^2(\partial E_t)}\|D_\tau H_t\|_{L^2(\partial E_t)}\\
&\leq C\|D_\tau\nu_t\|_{L^2(\partial E_t)}\|\Delta_\tau H_t\|_{L^2(\partial E_t)}\left(-\int_{\partial E_t}(H_t-\overline{H}_t)\Delta_\tau H_t\,d\mathcal{H}^2\right)^{1/2}\\
&\leq C\|D_\tau\nu_t\|_{L^2(\partial E_t)}\varepsilon^{1/2}\|\Delta_\tau H_t\|^2_{L^2(\partial E_t)} \leq C\varepsilon^{1/2}\|\Delta_\tau H_t\|^2_{L^2(\partial E_t)}\,,
\end{aligned}$$

where in the last inequality the constant C depends also on the curvature bounds provided by (4.7). If ε is chosen so small that $C\varepsilon^{\frac12}+\varepsilon^2<1$ in the last inequality, then we reach a contradiction to (4.12) and the fact that $\|g\|_{L^2(\partial E_t)} < \varepsilon\|\Delta_\tau H_t\|_{L^2(\partial E_t)}$.

As in Step 2 of the proof of Theorem 3.4, by taking ε_0 (and δ_0) smaller if needed, we may ensure that (3.21) holds, with ω the modulus of continuity introduced in (4.7) and δ_1 satisfying (3.19) and (3.20), with $W^{2,3}(\partial F)$ replaced by $W^{2,6}(\partial F)$.

Step 3. *(The stopping time $\bar t$ equals the maximal time $T(E_0)$)* Here we assume by contradiction that $\bar t < T(E_0)$ and thus

$$\|\psi_{\bar t}\|_{C^1(\partial F)} = \varepsilon_0 \quad\text{or}\quad \int_{\partial E_{\bar t}}|D_\tau H_{\bar t}|^2\,d\mathcal{H}^2 = 2\delta_0\,.$$

We further split into two sub-steps, according to the two alternatives above.

Step 3-(a). Assume that

$$\int_{\partial E_{\bar t}} |D_\tau H_{\bar t}|^2 \, d\mathcal{H}^2 = 2\delta_0 \,. \tag{4.14}$$

Recall that (4.11) holds. Thus, by (4.7), (4.10), (3.19) (with $W^{2,3}(\partial F)$ replaced by $W^{2,6}(\partial F)$), and (3.21) we have

$$\partial^2 J(E_t)\,[\Delta_\tau H_t] \geq \sigma_\varepsilon \|\Delta_\tau H_t\|^2_{H^1(\partial E)} \text{ for all } t \in (0, \bar t).$$

Note also that (4.13), together with the Poincaré inequality (4.5), yields

$$\|D_\tau H_t\|_{L^2(\partial E_t)} \leq C \|\Delta_\tau H_t\|_{L^2(\partial E_t)} \,. \tag{4.15}$$

Now, we may use Lemma 4.4 to estimate

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\int_{\partial E_t} |D_\tau H_t|^2 \, d\mathcal{H}^2\right) \quad &\leq \quad -\sigma_\varepsilon \|\Delta_\tau H_t\|^2_{H^1(\partial E_t)} + 2\int_{\partial E_t} |B_t||D_\tau H_t|^2 |\Delta_\tau H_t| \, d\mathcal{H}^2 \\
&\overset{\text{Lemma 4.7}}{\leq} \quad -\sigma_\varepsilon \|\Delta_\tau H_t\|^2_{H^1(\partial E_t)} \\
&\qquad + C\|D_\tau(\Delta_\tau H_t)\|^2_{L^2(\partial E_t)} \|D_\tau H_t\|_{L^2(\partial E_t)} \left(1 + \|H_t\|^3_{L^6(\partial E_t)}\right) \\
&\overset{(4.10)}{\leq} \quad -\sigma_\varepsilon \|\Delta_\tau H_t\|^2_{H^1(\partial E_t)} \\
&\qquad + C\sqrt{\delta_0} \|D_\tau(\Delta_\tau H_t)\|^2_{L^2(\partial E_t)} \left(1 + \|H_t\|^3_{L^6(\partial E_t)}\right) \\
&\overset{(4.7)}{\leq} \quad -\sigma_\varepsilon \|\Delta_\tau H_t\|^2_{H^1(\partial E_t)} + C\sqrt{\delta_0} \|D_\tau(\Delta_\tau H_t)\|^2_{L^2(\partial E_t)}
\end{aligned}$$

for every $t \leq \bar t$. Thus, if we choose δ_0 small enough we have

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\partial E_t} |D_\tau H_t|^2 \, d\mathcal{H}^2\right) \leq -\frac{\sigma_\varepsilon}{2} \|\Delta_\tau H_t\|^2_{H^1(\partial E_t)} \leq -c_0 \|D_\tau H_t\|^2_{L^2(\partial E_t)} \,,$$

where the last inequality follows from (4.15).

Integrating the differential inequality and recalling (4.9), we obtain

$$\int_{\partial E_t} |D_\tau H_t|^2 \, d\mathcal{H}^2 \leq \mathrm{e}^{-c_0 t} \int_{\partial E_0} |D_\tau H_{E_0}|^2 \, d\mathcal{H}^2 \leq \delta_0 \mathrm{e}^{-c_0 t} \tag{4.16}$$

which gives a contradiction to (4.14) for $t = \bar t$.

Step 3-(b). Assume now that

$$\|\psi_{\bar t}\|_{C^1(\partial F)} = \varepsilon_0 \,. \tag{4.17}$$

Then, arguing as in Step 3-(b) of the proof of Theorem 3.4, we can compute

$$\begin{aligned}
\frac{d}{dt} D(E_t) &= \int_{E_t} \operatorname{div}(d_F X_t) \, dx = \int_{\partial E_t} d_F \, \Delta_\tau H_t \, d\mathcal{H}^2 \\
&= -\int_{\partial E_t} D_\tau d_F \cdot D_\tau H_t \, d\mathcal{H}^2 \leq C \|D_\tau H_t\|_{L^2(\partial E_t)} \leq C\sqrt{\delta_0} \mathrm{e}^{-\frac{c_0}{2} t} \,,
\end{aligned}$$

where the last inequality clearly follows from (4.16). We may now argue exactly as in the end of Step 3-(b) of the proof of Theorem 3.4 and reach a contradiction to (4.17) if δ_0 is small enough.

Thus $\bar{t} = T(E_0)$, and as a byproduct of (4.16) and of Step 3-(b) we also have

$$\|\psi_t\|_{C^1(\partial F)} < \varepsilon_0 \quad \text{and} \quad \int_{\partial E_t} |D_\tau H_t|^2 \, d\mathcal{H}^2 \le \mathrm{e}^{-c_0 t} \int_{\partial E_0} |D_\tau H_{E_0}|^2 \, d\mathcal{H}^2 \quad \text{for all } t \in (0, T(E_0)). \tag{4.18}$$

Step 4. *(Global-in-time existence)* Here we assume by contradiction $T(E_0) < +\infty$. Then, we may argue exactly as in Step 4 of the proof of Theorem 3.4 to find $\hat{t} \in \big(T(E_0) - \frac{T_0}{2}, T(E_0) - \frac{T_0}{4}\big)$ such that $\|\Delta_\tau H_{\hat{t}}\|^2_{H^1(\partial E_{\hat{t}})} \le \frac{8\delta_0}{T_0 \sigma_\varepsilon}$. Thus, also by Lemma 4.6

$$\|D^2_\tau H_{\hat{t}}\|^2_{L^2(\partial E_{\hat{t}})} \le C \|\Delta_\tau H_{\hat{t}}\|^2_{L^2(\partial E_{\hat{t}})} \Big(1 + \|H_{\hat{t}}\|^4_{L^4(\partial E_{\hat{t}})}\Big) \le C\delta_0 \,,$$

where in the last inequality we also used the curvature bounds provided by (4.7). In turn, for p large enough

$$[H_{\hat{t}}]^2_{C^{0,\alpha}(\partial E_{\hat{t}})} \le C \|D_\tau H_{\hat{t}}\|^2_{L^p(\partial E_{\hat{t}})} \le C \|D_\tau H_{\hat{t}}\|^2_{H^1(\partial E_{\hat{t}})} \le C\delta_0 \,,$$

where in the last equality we used also (4.18).

Thus, if we choose δ_0 sufficiently small, then $E_{\hat{t}} \in \mathfrak{h}^{2,\alpha}_M(F, U)$ and, by (3.11) the time span of existence of the classical solution starting from $E_{\hat{t}}$ is at least T_0. This implies that $(E_t)_t$ can be continued beyond $T(E_0)$, leading to a contradiction.

We can now proceed exactly as in Steps 5 and 6 of the proof of Theorem 3.4, using Lemma 4.9 instead of Lemma 3.7, to get the desired conclusion. We leave the details to the reader. q.e.d.

5. Proofs of technical lemmas

In this final section we collect the proofs of the several technical lemmas stated in the previous sections.

5.1. The modified Mullins-Sekerka flow: proof of technical lemmas.

Proof of Lemma 2.6. **Step 1.** First we claim that the strict stability of F (Definition 2.4) implies

$$\partial^2 J(F)[\varphi] > 0 \qquad \text{for all } \varphi \in \widetilde{H}(\partial F) \setminus T(\partial F). \tag{5.1}$$

To this aim we observe that from (2.4) we get

$$Dv_F(x) = 2 \int_F D_x G_{\mathbb{T}^3}(x, y) \, dy = -2 \int_F D_y G_{\mathbb{T}^3}(x, y) \, dy = -2 \int_{\partial F} G_{\mathbb{T}^3}(x, y) \nu(y) \, d\mathcal{H}^2(y).$$

Setting $\nu_i = e_i \cdot \nu_F$ we have by [**20**, Lemma 10.7]

$$-\Delta_\tau \nu_i - |B_{\partial F}|^2 \nu_i = -\delta_i H_{\partial F}$$

where δ_i is defined as in (4.4). Since F is critical it satisfies $H_{\partial F} + 4\gamma v_F = const.$ and by the above identities, we have

$$-\Delta_\tau \nu_i - |B_{\partial F}|^2 \nu_i = -4\gamma \partial_\nu v_F \nu_i - 8\gamma \int_{\partial F} G_{\mathbb{T}^3}(x, y) \nu_i(y) \, d\mathcal{H}^2(y).$$

This can be written as $L(\nu_i) = 0$, where $L : H^1(\partial F) \to H^{-1}(\partial F)$ is self-adjoint, linear operator defined as

$$L(\varphi) := -\Delta_\tau \varphi - |B_{\partial F}|^2 \varphi + 4\gamma \partial_\nu v_F \varphi + 8\gamma \int_{\partial F} G_{\mathbb{T}^3}(x,y)\varphi(y)\, d\mathcal{H}^2(y).$$

Let now $\varphi \in \widetilde{H}(\partial F) \setminus T(\partial F)$. We may write $\varphi = \psi + \eta \cdot \nu_F$ for some $\eta \in \mathbb{R}^3$, where $\psi \in T^\perp(\partial F) \setminus \{0\}$. Since L is self-adjoint, we then conclude

$$\begin{aligned}
&\partial^2 J(F)[\varphi] = \langle L(\varphi), \varphi\rangle_{H^{-1}\times H^1} \\
&= \langle L(\psi), \psi\rangle_{H^{-1}\times H^1} + 2\langle L(\eta\cdot\nu_F), \psi\rangle_{H^{-1}\times H^1} + \langle L(\eta\cdot\nu_F), \eta\cdot\nu_F\rangle_{H^{-1}\times H^1} = \partial^2 J(F)[\psi] > 0,
\end{aligned}$$

where the last inequality follows from the strict stability assumption on F.

Having proved (5.1) we show next that for every $\varepsilon \in (0,1]$ it holds

$$\begin{aligned}
(5.2)\quad m_\varepsilon := \inf\Big\{ &\partial^2 J(F)[\varphi] : \varphi \in \widetilde{H}(\partial F)\,, \|\varphi\|_{H^1(\partial F)} = 1 \\
&\text{and } \min_{\eta\in\Pi_F} \|\varphi - \eta\cdot\nu_F\|_{L^2(\partial F)} \geq \varepsilon \|\varphi\|_{L^2(\partial F)} \Big\} > 0\,.
\end{aligned}$$

Indeed, let φ_h be a minimizing sequence for the infimum in (5.2) and assume that $\varphi_h \rightharpoonup \varphi_0 \in \widetilde{H}(\partial F)$ weakly in $H^1(\partial F)$. Let us first assume that $\varphi_0 \neq 0$. Since

$$\min_{\eta\in\Pi_F} \|\varphi_0 - \eta\cdot\nu_F\|_{L^2(\partial F)} \geq \varepsilon \|\varphi_0\|_{L^2(\partial F)},$$

we conclude $\varphi_0 \in \widetilde{H}(\partial F) \setminus T(\partial F)$. Thus,

$$m_\varepsilon = \lim_h \partial^2 J(F)[\varphi_h] \geq \partial^2 J(F)[\varphi_0] > 0\,,$$

where the last inequality follows from (5.1). If $\varphi_0 = 0$, then

$$m_\varepsilon = \lim_h \partial^2 J(F)[\varphi_h] = \lim_h \int_{\partial F} |D_\tau \varphi_h|^2\, d\mathcal{H}^2 = 1\,.$$

Step 2. In order to conclude the proof of the lemma it is enough to show the existence of $\delta > 0$ such that if $\partial E = \{x + \psi(x)\nu_F(x) : x \in \partial F\}$ with $\|\psi\|_{W^{2,p}(\partial F)} \leq \delta$, then

$$\begin{aligned}
(5.3)\quad \inf\Big\{ &\partial^2 J(E)[\varphi] : \varphi \in \widetilde{H}(\partial E)\,, \|\varphi\|_{H^1(\partial E)} = 1 \\
&\text{and } \min_{\eta\in\Pi_F} \|\varphi - \eta\cdot\nu_E\|_{L^2(\partial E)} \geq \varepsilon \|\varphi\|_{L^2(\partial E)} \Big\} \geq \sigma_\varepsilon := \frac{1}{2}\min\{m_{\varepsilon/2}, 1\}\,,
\end{aligned}$$

where $m_{\varepsilon/2}$ is defined in (5.2), with $\varepsilon/2$ in place of ε. Assume by contradiction that there exists a sequence E_h, with $\partial E_h = \{x + \psi_h(x)\nu_F(x) : x \in \partial F\}$ and $\|\psi_h\|_{W^{2,p}(\partial F)} \to 0$, and a sequence $\varphi_h \in \widetilde{H}(\partial E_h)$, with $\|\varphi_h\|_{H^1(\partial E_h)} = 1$ and $\min_{\eta\in\Pi_F} \|\varphi_h - \eta\cdot\nu_{E_h}\|_{L^2(\partial E_h)} \geq \varepsilon \|\varphi_h\|_{L^2(\partial E_h)}$, such that

$$(5.4)\qquad \partial^2 J(E_h)[\varphi_h] < \sigma_\varepsilon\,.$$

Assume first that $\lim_h \|\varphi_h\|_{L^2(\partial E_h)} = 0$ and observe that by Sobolev embedding $\|\varphi_h\|_{L^q(\partial E_h)} \to 0$ for every $q > 1$. Thus, since ψ_h are uniformly bounded in $W^{2,p}$ for $p > 2$ we obtain

$$\lim_h \partial^2 J(E_h)[\varphi_h] = 1,$$

which is a contradiction to (5.4).

Thus we may assume that

$$\lim_h \|\varphi_h\|_{L^2(\partial E_h)} > 0. \tag{5.5}$$

The idea now is to read φ_h as a function on ∂F. For $x \in \partial F$ set

$$\tilde\varphi_h(x) := \varphi_h\big(x + \psi_h(x)\nu_F(x)\big) - \fint_{\partial F} \varphi_h(y + \psi_h(y)\nu_F(y))\, d\mathcal{H}^2(y)\,.$$

As $\psi_h \to 0$ in $W^{2,p}(\partial F)$, we have in particular that

$$\tilde\varphi_h \in \widetilde{H}(\partial F)\,, \qquad \|\tilde\varphi_h\|_{H^1(\partial F)} \to 1\,, \qquad \text{and} \qquad \frac{\|\tilde\varphi_h\|_{L^2(\partial F)}}{\|\varphi_h\|_{L^2(\partial E_h)}} \to 1\,. \tag{5.6}$$

Note also that $\nu_{E_h}(\cdot + \psi_h(\cdot)\nu_F(\cdot)) \to \nu_F$ in $W^{1,p}(\partial F)$ and thus in $C^{0,\alpha}(\partial F)$ for a suitable $\alpha \in (0,1)$ depending on p. Using also this, and taking into account the third limit in (5.6) and (5.5), one can easily show that

$$\liminf_h \frac{\min_{\eta\in\Pi_F} \|\tilde\varphi_h - \eta\cdot\nu_F\|_{L^2(\partial F)}}{\|\tilde\varphi_h\|_{L^2(\partial F)}} \ge \liminf_h \frac{\min_{\eta\in\Pi_F} \|\varphi_h - \eta\cdot\nu_{E_h}\|_{L^2(\partial E_h)}}{\|\varphi_h\|_{L^2(\partial E_h)}} \ge \varepsilon\,.$$

Thus, for h large enough we have

$$\|\tilde\varphi_h\|_{H^1(\partial F)} \ge \frac{3}{4} \qquad \text{and} \qquad \min_{\eta\in\Pi_F} \|\tilde\varphi_h - \eta\cdot\nu_F\|_{L^2(\partial F)} \ge \frac{\varepsilon}{2}\|\tilde\varphi_h\|_{L^2(\partial F)}\,.$$

In turn, by Step 1 we infer

$$\partial^2 J(F)[\tilde\varphi_h] \ge \frac{9}{16} m_{\varepsilon/2}\,. \tag{5.7}$$

Moreover, the $W^{2,p}$ convergence of E_h to F and standard elliptic estimates for the problem (2.3) imply

$$B_{\partial E_h}\big(\cdot + \psi_h(\cdot)\nu_F(\cdot)\big) \to B_{\partial F} \text{ in } L^p(\partial F), \qquad v_{E_h} \to v_F \text{ in } C^{1,\beta}(\mathbb{T}^3) \text{ for all } \beta < 1. \tag{5.8}$$

We now check that

$$\begin{aligned} \int_{\partial E_h}\int_{\partial E_h} G_{\mathbb{T}^3}(x,y)\varphi_h(x)\varphi_h(y)\, d\mathcal{H}^2(x)d\mathcal{H}^2(y) \\ - \int_{\partial F}\int_{\partial F} G_{\mathbb{T}^3}(x,y)\tilde\varphi_h(x)\tilde\varphi_h(y)\, d\mathcal{H}^2(x)d\mathcal{H}^2(y) \to 0 \end{aligned} \tag{5.9}$$

as $h \to \infty$. Indeed, thanks to Remark 2.5 this is equivalent to

$$\int_\Omega \big(|Dz_h|^2 - |D\tilde z_h|^2\big)\, dz \to 0\,, \tag{5.10}$$

where

$$-\Delta z_h = \mu_h := \varphi_h \mathcal{H}^2 \llcorner \partial E_h\,, \qquad -\Delta \tilde z_h = \tilde\mu_h := \tilde\varphi_h \mathcal{H}^2 \llcorner \partial F\,,$$

under periodicity condition. In turn, (5.10) is clearly implied by

$$\mu_h - \tilde\mu_h \to 0 \quad \text{in } H^{-1}(\mathbb{T}^3),$$

which can be easily checked (see [**1**, Proof of Theorem 3.9] for the details).

Finally, we observe that since $p > 2$, the Sobolev Embedding theorem and the $W^{2,p}$-convergence of ∂E_h to ∂F imply

$$\int_{\partial E_h} |B_{\partial E_h}|^2 \varphi_h^2\, d\mathcal{H}^2 - \int_{\partial F} |B_{\partial F}|^2 \tilde\varphi_h^2\, d\mathcal{H}^2 \to 0\,. \tag{5.11}$$

Combining (5.8), (5.9), and (5.11) we conclude that all terms of $\partial^2 J(E_h)[\varphi_h]$ are asympotically close to the corresponding terms of $\partial^2 J(E)[\tilde{\varphi}_h]$ and thus

$$\partial^2 J(E_h)[\varphi_h] - \partial^2 J(F)[\tilde{\varphi}_h] \to 0\,.$$

Recalling (5.4), we have a contradiction to (5.7). This establishes (5.3) and concludes the proof of the lemma. q.e.d.

Proof of Lemma 3.5. In the following Ψ and Ψ_t are as in Definition 3.1 and the subscript t stands for the subscript E_t. We denote by X_t the associated velocity field, that is, $X_t := \dot{\Psi}_t \circ \Psi_t^{-1}$. In particular, by (3.2) we have that

$$X_t \cdot \nu_t = [\partial_{\nu_t} w_t] \qquad \text{on } \partial E_t. \tag{5.12}$$

Fix $t \in (0,T)$, set $\Phi_s := \Psi_{t+s} \circ \Psi_t^{-1}$, and note that $(\Phi)_{s\in(-t,T-t)}$ is an admissible one-parameter family of diffeomorphisms according to Definition 2.1. Then we may apply Theorem 2.2 to get

$$\begin{aligned}
\frac{d}{dt} J(E_t) &= \frac{d}{ds} J(\Phi_s(E_t))\Big|_{s=0} = \int_{\partial E_t} (H_t + 4\gamma v_t) X_t \cdot \nu_t \, d\mathcal{H}^2 \\
&\overset{(3.1)}{=} \int_{\partial E_t} w_t \, X_t \cdot \nu_t \, d\mathcal{H}^2 \overset{(5.12)}{=} \int_{\partial E_t} w_t [\partial_{\nu_t} w_t] \, d\mathcal{H}^2 \\
&= -\int_{\mathbb{T}^3} |Dw_t|^2 \, dx\,,
\end{aligned}$$

where the last equality follows from integration by parts and the fact that w_t is harmonic in $\mathbb{T}^3 \setminus \partial E_t$. This establishes (3.4). In order to get (3.5), we need to introduce some auxiliary functions: For each $t \in (0,T)$, we let d_t denote the signed distance function from E_t, which, we recall, is smooth in a suitable tubular neighborhood of ∂E_t. We then set $\nu_t := Dd_t$, $H_t := \Delta d_t = \operatorname{div} \nu_t$, and $B_t := D^2 d_t = D\nu_t$. Note that ν_t, H_t, and B_t represent smooth extensions of the outer unit normal field, the mean curvature and the second fundamental form, respectively, to a neighborhood of ∂E_t. We start by recalling the following identity (see [**6**, Lemma 3.8]):

$$\partial_{\nu_t} H_t = DH_t \cdot \nu_t = -|B_t|^2 \qquad \text{on } \partial E_t \tag{5.13}$$

and

$$\dot{\nu}_t := \frac{\partial}{\partial s} \nu_{t+s}\Big|_{s=0} = -D_\tau (X_t \cdot \nu_t) = -D_\tau\big([\partial_{\nu_t} w_t]\big) \qquad \text{on } \partial E_t, \tag{5.14}$$

where the last equality follows again by (5.12). Moreover, by differentiating with respect to s the identity $D\nu_{t+s}[\nu_{t+s}] = 0$, we get $D\dot{\nu}_t[\nu_t] + D\nu_t[\dot{\nu}_t] = 0$. Multiplying the latter equality by ν_t and recalling that $D\nu_t$ is symmetric we get $D\dot{\nu}_t[\nu_t] \cdot \nu_t = -D\nu_t[\nu_t] \cdot \dot{\nu}_t = 0$. In turn, this implies that

$$\operatorname{div}_\tau \dot{\nu}_t = \operatorname{div} \dot{\nu}_t \qquad \text{on } \partial E_t. \tag{5.15}$$

Also,

$$
\begin{aligned}
\frac{\partial}{\partial s}(H_{t+s}\circ\Phi_s)\Big|_{s=0} &= \dot H_t + DH_t\cdot X_t = \\
&\overset{(5.15)}{=} \operatorname{div}_\tau \dot\nu_t + \partial_\nu H_t(X_t\cdot\nu_t) + D_\tau H_t\cdot X_t \\
&\overset{(5.13)}{=} \operatorname{div}_\tau \dot\nu_t - |B_t|^2[\partial_{\nu_t} w_t] + D_\tau H_t\cdot X_t \\
&\overset{(5.14)}{=} -\Delta_\tau[\partial_{\nu_t} w_t] - |B_t|^2[\partial_{\nu_t} w_t] + D_\tau H_t\cdot X_t
\end{aligned}
\tag{5.16}
$$

We can now compute

$$
\begin{aligned}
&\frac{d}{ds}\left(\frac{1}{2}\int_{E_{t+s}} |Dw_{t+s}|^2\,dx\right)\Big|_{s=0} = \frac{d}{ds}\left(\frac{1}{2}\int_{E_t} |(Dw_{t+s})\circ\Phi_s|^2\,J\Phi_s\,dx\right)\Big|_{s=0} \\
&= \frac{1}{2}\int_{E_t} |Dw_t|^2 \operatorname{div} X_t\,dx + \int_{E_t} Dw_t\cdot\left(D^2 w_t[X_t] + D\dot w_t\right)\,dx \\
&= \frac{1}{2}\int_{E_t} \operatorname{div}(|Dw_t|^2 X_t)\,dx + \int_{E_t} Dw_t\cdot D\dot w_t dx \\
&= \frac{1}{2}\int_{\partial E_t} |Dw_t^-|^2 X_t\cdot\nu_t\,d\mathcal{H}^2 + \int_{\partial E_t} \dot w_t^-\,\partial_{\nu_t} w_t^-\,d\mathcal{H}^2 \\
&= \frac{1}{2}\int_{\partial E_t} |Dw_t^-|^2[\partial_{\nu_t} w_t]\,d\mathcal{H}^2 + \int_{\partial E_t} \dot w_t^-\,\partial_{\nu_t} w_t^-\,d\mathcal{H}^2\,.
\end{aligned}
\tag{5.17}
$$

In order to write $\dot w_t^-$ explicitly we use

$$
w_{t+s}^- = H_{t+s} + 4\gamma\, v_{t+s} \qquad \text{on } \partial E_{t+s}\,,
$$

which in turn is equivalent to

$$
w_{t+s}^-\circ\Phi_s = H_{t+s}\circ\Phi_s + 4\gamma\, v_{t+s}\circ\Phi_s \qquad \text{on } \partial E_t.
$$

By differentiating the above identity with respect to s at $s=0$, we get

$$
\dot w_t^- + Dw_t^-\cdot X_t = \dot H_t + DH_t\cdot X_t + 4\gamma\dot v_t + 4\gamma\, Dv_t\cdot X_t \qquad \text{on } \partial E_t.
$$

We now use (5.16) (and of course (5.12)) to get

$$
\begin{aligned}
\dot w_t^- =& -(\partial_{\nu_t} w_t^-)[\partial_{\nu_t} w_t] - \Delta_\tau[\partial_{\nu_t} w_t] - |B_t|^2[\partial_{\nu_t} w_t] \\
&+ 4\gamma\,\dot v_t + 4\gamma\,\partial_{\nu_t} v_t[\partial_{\nu_t} w_t] + D_\tau(H_t + 4\gamma\, v_t - w_t)\cdot X_t \\
=& -(\partial_{\nu_t} w_t^-)[\partial_{\nu_t} w_t] - \Delta_\tau[\partial_{\nu_t} w_t] - |B_t|^2[\partial_{\nu_t} w_t] \\
&+ 4\gamma\,\dot v_t + 4\gamma\,\partial_{\nu_t} v_t[\partial_{\nu_t} w_t] \qquad \text{on } \partial E_t\,,
\end{aligned}
\tag{5.18}
$$

where in the last equality we have used the fact that $w_t = H_t + 4\gamma\, v_t$ on ∂E_t. Therefore from (2.5), (5.17) and (5.18) we get

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\int_{E_t}|Dw_t|^2\,dx\right) =& -\int_{\partial E_t}\partial_{\nu_t}w_t^-\,\Delta_\tau[\partial_{\nu_t}w_t]\,d\mathcal{H}^2 - \int_{\partial E_t}|B_t|^2\,\partial_{\nu_t}w_t^-\,[\partial_{\nu_t}w_t]\,d\mathcal{H}^2\\ &+ 8\gamma\int_{\partial E_t}\int_{\partial E_t}G_{\mathbb{T}^3}(x,y)\,\partial_{\nu_t}w_t^-(x)\,[\partial_{\nu_t}w_t(y)]\,d\mathcal{H}^2(y)d\mathcal{H}^2(x)\\ &+ 4\gamma\int_{\partial E_t}\partial_{\nu_t}v_t\,\partial_{\nu_t}w_t^-\,[\partial_{\nu_t}w_t]\,d\mathcal{H}^2\\ &+ \frac{1}{2}\int_{\partial E_t}|Dw_t^-|^2[\partial_{\nu_t}w_t]\,d\mathcal{H}^2 - \int_{\partial E_t}(\partial_{\nu_t}w_t^-)^2[\partial_{\nu_t}w_t]\,d\mathcal{H}^2.\end{aligned} \tag{5.19}$$

The analogous calculations in $\mathbb{T}^3\setminus E_t$ yield

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{T}^3\setminus E_t}|Dw_t|^2\,dx\right) =& \int_{\partial E_t}\partial_{\nu_t}w_t^+\,\Delta_\tau[\partial_{\nu_t}w_t]\,d\mathcal{H}^2 + \int_{\partial E_t}|B_t|^2\partial_{\nu_t}w_t^+\,[\partial_{\nu_t}w_t]\,d\mathcal{H}^2\\ &- 8\gamma\int_{\partial E_t}\int_{\partial E_t}G_{\mathbb{T}^3}(x,y)\,\partial_{\nu_t}w_t^+(x)\,[\partial_{\nu_t}w_t(y)]\,d\mathcal{H}^2(y)d\mathcal{H}^2(x)\\ &- 4\gamma\int_{\partial E_t}\partial_{\nu_t}v_t\,\partial_{\nu_t}w_t^+\,[\partial_{\nu_t}w_t]\,d\mathcal{H}^2\\ &- \frac{1}{2}\int_{\partial E_t}|Dw_t^+|^2[\partial_{\nu_t}w_t]\,d\mathcal{H}^2 + \int_{\partial E_t}(\partial_{\nu_t}w_t^+)^2[\partial_{\nu_t}w_t]\,d\mathcal{H}^2.\end{aligned} \tag{5.20}$$

Combining (5.19) and (5.20), integrating by parts, and recalling (2.9) we get

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{T}^3}|Dw_t|^2\,dx\right) =& -\partial^2 J(E_t)\left[[\partial_{\nu_t}w_t]\right] + \int_{\partial E_t}\left((\partial_{\nu_t}w_t^+)^2 - (\partial_{\nu_t}w_t^-)^2\right)[\partial_{\nu_t}w_t]\,d\mathcal{H}^2\\ &- \frac{1}{2}\int_{\partial E_t}(|Dw_t^+|^2 - |Dw_t^-|^2)[\partial_{\nu_t}w_t]\,d\mathcal{H}^2.\end{aligned}$$

The result follows from the identity

$$|Dw_t^+|^2 - |Dw_t^-|^2 = (\partial_{\nu_t}w_t^+)^2 - (\partial_{\nu_t}w_t^-)^2 = (\partial_{\nu_t}w_t^+ + \partial_{\nu_t}w_t^-)[\partial_{\nu_t}w_t].$$

q.e.d.

We now prove Proposition 3.6.

Proof of Proposition 3.6. To simplify the notation, throughout the proof we write ν instead of ν_E.

Proof of (i): Observe that we may write u as

$$u(x) = \int_{\partial E} G_{\mathbb{T}^3}(x,y)f(y)\,d\mathcal{H}^2(y).$$

Note that $G_{\mathbb{T}^3}(x,y) = h(x-y) + r(x-y)$ where h is one-periodic, smooth away from 0 and $h(t) = \frac{1}{4\pi|t|}$ in a neighborhood of 0, while r is smooth and one-periodic. The conclusion then follows since for $v(x) := \int_{\partial E}\frac{f(y)}{|x-y|}\,d\mathcal{H}^2(y)$ it holds

$$\|v\|_{L^p(\partial E)} \le C\|f\|_{L^p(\partial E)}.$$

Proof of (ii): Here we adapt the proof of [**26**] to the periodic setting. First observe that since u is harmonic in $E \subset \mathbb{T}^3$ we have

$$\operatorname{div}\big(2(Du\cdot x)Du - |Du|^2 x + uDu\big) = 0. \tag{5.21}$$

Moreover, by the $C^{1,\alpha}$-regularity of ∂E there exist $r>0$, C_0 and N, depending on the $C^{1,\alpha}$ bounds on ∂E, such that we may cover ∂E with at most N balls $B_r(x_k)$ such that, up to a translation,

$$\frac{1}{C_0} \le x\cdot\nu(x) \le C_0 \qquad \text{for } x \in \partial E \cap B_{2r}(x_k). \tag{5.22}$$

Therefore if $0 \le \varphi_k \le 1$ is a smooth function with compact support in $B_{2r}(x_k)$ such that $\varphi_k \equiv 1$ in $B_r(x_k)$ and $|D\varphi_k| \le C/r$, by integrating

$$\operatorname{div}\big(\varphi_k\big(2(Du\cdot x)Du - |Du|^2 x + uDu\big)\big)$$

over E and using (5.21) we easily get

$$\begin{aligned}\int_{\partial E} &\varphi_k|\partial_\nu u^-|^2(x\cdot\nu) - \varphi_k|D_\tau u|^2(x\cdot\nu)\,d\mathcal{H}^2\\ &= -\int_{\partial E}\varphi_k u\partial_\nu u^-\,d\mathcal{H}^2 - 2\int_{\partial E}\varphi_k(D_\tau u\cdot x)\partial_\nu u^-\,d\mathcal{H}^2\\ &\quad + \int_E D\varphi_k\cdot\big(2(Du\cdot x)Du - |Du|^2x + uDu\big)\,dx.\end{aligned}$$

This implies using the Poincaré inequality on the torus (recall that u has zero average) and (5.22)

$$\begin{aligned}\int_{\partial E\cap B_r(x_k)} |\partial_\nu u^-|^2\,d\mathcal{H}^2 &\le C\int_{\partial E}(u^2+|D_\tau u|^2)\,d\mathcal{H}^2 + C\int_{\mathbb{T}^3}(u^2+|Du|^2)\,dx\\ &\le C\int_{\partial E}(u^2+|D_\tau u|^2)\,d\mathcal{H}^2 + C\int_{\mathbb{T}^3}|Du|^2\,dx.\end{aligned}$$

Adding up all the estimates and repeating the argument for $\mathbb{T}^3\setminus E$ we get

$$\int_{\partial E}(|\partial_\nu u^-|^2 + |\partial_\nu u^+|^2)\,d\mathcal{H}^2 \le C\int_{\partial E}(u^2+|D_\tau u|^2)\,d\mathcal{H}^2 + C\int_{\mathbb{T}^3}|Du|^2\,dx.$$

The result follows by observing that

$$\int_{\mathbb{T}^3}|Du|^2\,dx = \int_{\partial E} u(\partial_\nu u^- - \partial_\nu u^+)\,d\mathcal{H}^2.$$

Proof of (iii): The result would follow from the boundary estimates on C^1-domains established in [**18**]. However, it turns out that in the case of $C^{1,\alpha}$-domains the argument can be greatly simplified, as shown in the following.

Let us define

$$Kf(x) := \int_{\partial E} D_x G_{\mathbb{T}^3}(x,y)\cdot\nu(x)f(y)\,d\mathcal{H}^2(y)\,.$$

We first show that the above integral is defined for every $x\in\partial E$ and that

$$\|Kf\|_{L^p(\partial E)} \le C\|f\|_{L^p(\partial E)}. \tag{5.23}$$

By the decomposition recalled at the beginning of the proof we have $D_xG_{\mathbb{T}^3}(x,y) = D_xh(x-y) + D_xr(x-y)$, where $D_xh(x-y) = -\frac{1}{4\pi}\frac{x-y}{|x-y|^3}$ in a neighborhood of the origin and $D_xr(x-y)$

is smooth. Thus, by a standard partition of unity argument we may localize the estimate and reduce to show that if $\varphi \in C^{1,\alpha}(\mathbb{R}^2)$ and $U \subset \mathbb{R}^2$ is a bounded domain, setting $\Gamma := \{(x', \varphi(x')) : x' \in U\}$ and

$$Tf(x) := \int_\Gamma \frac{(x-y)\cdot\nu(x)}{|x-y|^3} f(y)\, d\mathcal{H}^2(y) \quad x \in \Gamma,$$

where ν is the upper normal to Γ, then $Tf(x)$ is well defined at every $x \in \Gamma$ and

$$\|Tf\|_{L^p(\Gamma)} \le C\|f\|_{L^p(\Gamma)}.$$

To show this we observe that we may write

$$Tf(x) := \int_U \frac{\varphi(x') - \varphi(y') - D\varphi(x')\cdot(x'-y')}{(|x'-y'|^2 + (\varphi(x') - \varphi(y'))^2)^{3/2}} f(y', \varphi(y'))\, dy'.$$

Therefore

$$\begin{aligned} |Tf(x)| &\le C \int_U \frac{|x'-y'|^{1+\alpha}}{(|x'-y'|^2 + (\varphi(x') - \varphi(y'))^2)^{3/2}} |f(y', \varphi(y'))|\, dy' \\ &\le C \int_U \frac{|f(y', \varphi(y'))|}{|x'-y'|^{2-\alpha}}\, dy'. \end{aligned}$$

Thus the estimate (5.23) follows from a standard convolution estimate.

For $x \in E$ we have

$$Du(x) = \int_{\partial E} D_x G_{\mathbb{T}^3}(x,y) f(y)\, d\mathcal{H}^2(y).$$

Therefore for $x \in \partial E$ it holds

$$Du(x - t\nu(x))\cdot\nu(x) = \int_{\partial E} D_x G_{\mathbb{T}^3}(x - t\nu(x), y)\cdot\nu(x) f(y)\, d\mathcal{H}^2(y).$$

We claim that

$$\lim_{t\to 0+} Du(x - t\nu(x))\cdot\nu(x) = Kf(x) + \frac{1}{2} f(x) \tag{5.24}$$

for every $x \in \partial E$. Then the lemma follows from (5.23) and (5.24).

To show (5.24) we first recall that for $z \in E$ and for $x \in \partial E$ it holds

$$\begin{aligned} &\int_{\partial E} D_x G_{\mathbb{T}^3}(z,y)\cdot\nu(y)\, d\mathcal{H}^2(y) = 1 - |E| \quad \text{and} \\ &\int_{\partial E} D_x G_{\mathbb{T}^3}(x,y)\cdot\nu(y)\, d\mathcal{H}^2(y) = \frac{1}{2} - |E|. \end{aligned} \tag{5.25}$$

Therefore, we may write

(5.26)

$$\begin{aligned} Du(x - t\nu(x))\cdot\nu(x) = &\int_{\partial E} D_x G_{\mathbb{T}^3}(x - t\nu(x), y)\cdot\nu(x)(f(y) - f(x))\, d\mathcal{H}^2(y) \\ &+ f(x)\int_{\partial E} D_x G_{\mathbb{T}^3}(x - t\nu(x), y)\cdot(\nu(x) - \nu(y))\, d\mathcal{H}^2(y) + f(x)(1 - |E|). \end{aligned}$$

Let us now prove that

$$\lim_{t\to 0}\int_{\partial E} D_x G_{\mathbb{T}^3}(x-t\nu(x),y)\cdot\nu(x)(f(y)-f(x))\,d\mathcal{H}^2(y) \\ = \int_{\partial E} D_x G_{\mathbb{T}^3}(x,y)\cdot\nu(x)(f(y)-f(x))\,d\mathcal{H}^2(y).$$

To establish this, first observe that since ∂E is C^1 then for $|t|$ sufficiently small we have

$$|x-y-t\nu(x)| \geq \frac{1}{2}|x-y| \qquad \text{for all } y\in\partial E\,. \tag{5.27}$$

Then, in view of the decomposition of D_xG recalled before, it is enough show that

$$\lim_{t\to 0}\int_{\partial E}\frac{(x-y-t\nu(x))\cdot\nu(x)}{|x-y-t\nu(x)|^3}(f(y)-f(x))\,d\mathcal{H}^2(y) \\ = \int_{\partial E}\frac{(x-y)\cdot\nu(x)}{|x-y|^3}(f(y)-f(x))\,d\mathcal{H}^2(y)\,,$$

which follows from the Dominated Convergence Theorem, after observing that due to the α-Hölder continuity of f and to (5.27), the absolute value of both integrands can be estimated from above by $C/|x-y|^{2-\alpha}$ for some constant $C>0$.

Hence (5.24) follows by letting $t\to 0$ in (5.26) and recalling (5.25).

Proof of (iv): Fix $p>2$ and $\beta\in(0,\frac{p-2}{p})$. As before, due to the properties of the Green's function it is sufficient to establish the statement for the function

$$v(x) := \int_{\partial E}\frac{f(y)}{|x-y|}\,d\mathcal{H}^2(y)\,.$$

For x_1, $x_2\in\partial E$ we have

$$|v(x_1)-v(x_2)| \leq \int_{\partial E}|f(y)|\frac{\big||x_1-y|-|x_2-y|\big|}{|x_1-y|\,|x_2-y|}\,d\mathcal{H}^2(y)\,.$$

In turn, by an elementary inequality, we have

$$\frac{\big||x_1-y|-|x_2-y|\big|}{|x_1-y|\,|x_2-y|} \leq C(\beta)\frac{\big||x_1-y|^{1-\beta}+|x_2-y|^{1-\beta}\big|}{|x_1-y|\,|x_2-y|}|x_1-x_2|^\beta\,.$$

Thus, by Hölder inequality we have

$$|v(x_1)-v(x_2)| \leq C(\beta)\int_{\partial E}|f(y)|\frac{\big||x_1-y|^{1-\beta}+|x_2-y|^{1-\beta}\big|}{|x_1-y|\,|x_2-y|}\,d\mathcal{H}^2(y)\;|x_1-x_2|^\beta \\ \leq C'(\beta)\|f\|_{L^p}|x_1-x_2|^\beta\,,$$

where we set

$$C'(\beta) := 2C(\beta)\bigg(\sup_{z_1,\,z_2\in\partial E}\int_{\partial E}\frac{1}{|z_1-y|^{\beta p'}\,|z_2-y|^{p'}}\,d\mathcal{H}^2(y)\bigg)^{\frac{1}{p'}}\,.$$

Proof of (v): We start by observing that

$$\|f\|_{L^2(\partial E)} \leq C\|f\|^{\frac{1}{2}}_{H^1(\partial E)}\|f\|^{\frac{1}{2}}_{H^{-1}(\partial E)},$$

where C is a constant depending only on the $C^{1,\alpha}$ bounds on ∂E. If $p > 2$ we have also, see Lemma 4.5,

$$\|f\|_{L^p(\partial E)} \le C\|f\|_{H^1(\partial E)}^{\frac{p-2}{p}}\|f\|_{L^2(\partial E)}^{\frac{2}{p}}.$$

Therefore, by combining the two previous inequalities we get that for $p \ge 2$

$$\|f\|_{L^p(\partial E)} \le C\|f\|_{H^1(\partial E)}^{\frac{p-1}{p}}\|f\|_{H^{-1}(\partial E)}^{\frac{1}{p}}.$$

Hence the claim follows once we show

$$\|f\|_{H^{-1}(\partial E)} \le C\|u\|_{L^2(\partial E)}.$$

Let us fix $\varphi \in H^1(\partial E)$ and with abuse of notation denote its harmonic extension to $\mathbb{T}^3$ by φ. Then by integrating by parts twice and by (ii) we get

$$\begin{aligned}\int_{\partial E} \varphi f \, d\mathcal{H}^2 = -\int_{\partial E} u[\partial_\nu \varphi] \, d\mathcal{H}^2 &\le \|u\|_{L^2(\partial E)}\|[\partial_\nu\varphi]\|_{L^2(\partial E)} \\ &\le \|u\|_{L^2(\partial E)}\left(\|\partial_\nu\varphi^+\|_{L^2(\partial E)} + \|\partial_\nu\varphi^-\|_{L^2(\partial E)}\right) \\ &\le C\|u\|_{L^2(\partial E)}\|\varphi\|_{H^1(\partial E)}.\end{aligned}$$

Therefore

$$\|f\|_{H^{-1}(\partial E)} = \sup_{\|\varphi\|_{H^1(\partial E)}\le 1} \int_{\partial E} \varphi f \, d\mathcal{H}^2 \le C\|u\|_{L^2(\partial E)}.$$

q.e.d.

We now prove Lemma 3.7. Before that we recall that for $E \subset \mathbb{T}^3$ the $H^{\frac{1}{2}}(\partial E)$ *Gagliardo seminorm* of a function $f \in L^2(\partial E)$ is defined by setting

$$[f]^2_{\frac{1}{2},\partial E} := \int_{\partial E} d\mathcal{H}^2(x) \int_{\partial E} \frac{|f(x)-f(y)|^2}{|x-y|^3} \, d\mathcal{H}^2(y)\,.$$

Starting from this definition and using a standard partition of unity argument in order to straighten the boundary of E locally, the reader may reconstruct the proof of the following technical lemma.

Lemma 5.1. *Let $E \subset \mathbb{T}^3$ be an open set of class $C^{1,\alpha}$ for some $\alpha \in (0,1)$. For every $\gamma \in [0, \frac{1}{2})$, there exists a constant C depending only on γ and on the $C^{1,\alpha}$ bounds on ∂E such that if $f \in H^{\frac{1}{2}}(\partial E)$ and $g \in W^{1,4}(\partial E)$ then*

$$[fg]_{\frac{1}{2}} \le \left([f]_{\frac{1}{2}}\|g\|_{L^\infty} + \|f\|_{L^{\frac{4}{1+\gamma}}}\|g\|_{L^\infty}^{\gamma}\|D_\tau g\|_{L^4}^{1-\gamma}\right).$$

Next lemma is probably well known to the expert, but we give its proof for reader's convenience

Lemma 5.2. *l Let F, U be as in Lemma 3.7. Let E be a set in $\mathfrak{h}^{1,\alpha}_M(F,U)$, for some $\alpha > 0$. If $H_{\partial E} \in H^{\frac{1}{2}}(\partial E)$, then E is of class $W^{\frac{5}{2},2}$ and*

$$\|\psi_E\|_{W^{\frac{5}{2},2}(\partial F)} \le C(M)\left(1 + \|H_{\partial E}\|^2_{H^{\frac{1}{2}}(\partial E)}\right),$$

where ψ_E is defined as in (3.3).

Proof. We assume without loss of generality that ψ_E is smooth. To simplify the notation we will drop the subscript from ψ_E and $H_{\partial E}$. Fix $\varepsilon > 0$. By straightening locally the boundary of F, we may reduce to the case where the function ψ is defined in a disk $B' \subset \mathbb{R}^2$ and $\|\psi\|_{C^1(B')} \le \varepsilon$. Fix a cut-off function φ with compact support in B'. Then

$$\Delta(\varphi\psi) - \frac{D^2(\varphi\psi)D\psi D\psi}{1+|D\psi|^2} = \varphi H\sqrt{1+|D\psi|^2} + R(x,\psi,D\psi)\,, \tag{5.28}$$

where the remainder term R is a smooth Lipschitz function. Then, using Lemma 5.1 with $\gamma = 0$ and recalling that $\|\psi\|_{C^1} \le \varepsilon$, we estimate

$$[\Delta(\varphi\psi)]_{\frac12} \le C(M)\big(\varepsilon^2[D^2(\varphi\psi)]_{\frac12} + [H]_{\frac12}(1+\|D\psi\|_{L^\infty}) + \|H\|_{L^4}(1+\|\psi\|_{W^{2,4}}) + 1 + \|\psi\|^2_{W^{2,4}}\big).$$

Observe that by Calderón-Zygmund estimates $\|\psi\|_{W^{2,4}(B')} \le C(M)(1+\|H\|_{L^4(\partial E)})$. Moreover, a simple integration by part argument shows that if u is a smooth function with compact support in $\mathbb{R}^2$ then

$$[\Delta u]_{\frac12,\mathbb{R}^2} = [D^2u]_{\frac12,\mathbb{R}^2}\,.$$

Thus, choosing ε sufficiently small, we may conclude that

$$[D^2(\varphi\psi)]_{\frac12} \le C(M)\big(1+[H]_{\frac12,\partial E} + \|H\|^2_{L^4(\partial E)}\big) \le C(M)\big(1+\|H\|^2_{H^{\frac12}(\partial E)}\big)\,.$$

From this estimate the conclusion follows. q.e.d.

Proof of Lemma 3.7. **Step 1.** Throughout the proof we write w_n, H_n, and v_n instead of w_{E_n}, $H_{\partial E_n}$, and v_{E_n}, respectively. Moreover we denote by $\hat{w}_n$ the average of w_n in $\mathbb{T}^3$ and we set $\tilde{w}_n = \fint_{\partial E_n} w_n\, d\mathcal{H}^2$ and $\tilde{H}_n = \fint_{\partial E_n} H_n\, d\mathcal{H}^2$. First, recall that

$$w_n = H_n + 4\gamma v_n \quad \text{on } \partial E_n \qquad \text{and} \qquad \sup_n \|v_n\|_{C^{1,\alpha}(\mathbb{T}^3)} < +\infty\,. \tag{5.29}$$

The last bound follows from standard elliptic estimates. Moreover, from the trace inequality

$$\|w_n - \tilde{w}_n\|^2_{H^{\frac12}(\partial E_n)} \le \|w_n - \hat{w}_n\|^2_{H^{\frac12}(\partial E_n)} \le C\int_{\mathbb{T}^3} |Dw_n|^2\, dx \tag{5.30}$$

with C depending only on the C^1-bounds on ∂E_n. We claim that

$$\sup_n \|H_n\|_{H^{\frac12}(\partial E_n)} < \infty. \tag{5.31}$$

To see this note that by the uniform C^1-bounds on ∂E_n, we may find a fixed cylinder of the form $C := B' \times (-L,L)$, with $B' \subset \mathbb{R}^2$ a ball centered at the origin, and functions f_n, with

$$\sup_n \|f_n\|_{C^1(\overline{B}')} < +\infty\,, \tag{5.32}$$

such that $\partial E_n \cap C = \{(x',x_n) \in B' \times (-L,L) : x_n = f_n(x')\}$ with respect to a suitable coordinate frame (depending on n). Thus we have

$$\begin{aligned}\int_{B'} (H_n - \tilde{H}_n)\, dx' + \tilde{H}_n|B'| &= \int_{B'} \operatorname{div}\left(\frac{\nabla_{x'} f_n}{\sqrt{1+|\nabla_{x'} f_n|^2}}\right) dx' \\ &= \int_{\partial B'} \frac{\nabla_{x'} f_n}{\sqrt{1+|\nabla_{x'} f_n|^2}} \cdot \frac{x'}{|x'|}\, d\mathcal{H}^1\,.\end{aligned}$$

Hence, recalling (5.32) and the fact that $\|H_n - \tilde{H}_n\|_{H^{\frac12}(\partial E_n)}$ is bounded thanks to (5.29) and (5.30), we get that $\tilde{H}_n$ are bounded. Therefore the claim (5.31) follows.

By applying the Sobolev embedding theorem on each connected component of ∂F we have that $\|H_n\|_{L^4(E_n)}$ is bounded. This fact, together with the uniform C^1 bounds on ∂E_n implies that if we write

$$\partial E_n := \{x + \psi_n(x) : x \in \partial F\},$$

then $\sup_n \|\psi_n\|_{W^{2,4}(\partial F)} < +\infty$. This follows by standard elliptic estimates, see [**1**, Lemma 7.2 and Remark 7.3]. Thus, up to a (not relabeled) subsequence, there exists a set $F' \in \mathfrak{C}^1_M(F,U)$ such that

$$\psi_n \to \psi_{F'} \text{ in } C^{1,\alpha}(\partial F) \quad \text{and} \quad v_n \to v_{F'} \text{ in } C^{1,\beta}(\mathbb{T}^3) \quad \text{for all } \alpha \in (0,\tfrac12) \text{ and } \beta \in (0,1).$$

From (5.31) and Lemma 5.2 we have that the functions ψ_n are bounded in $W^{\frac52,2}(\partial F)$. Hence the first part of the statement follows.

Step 2. For the second part we first observe that if

$$\int_{\mathbb{T}^3} |Dw_n|^2\,dx \to 0$$

then the above arguments yield the existence of $\lambda \in \mathbb{R}$ and a (not relabelled) subsequence such that $w_n\big(\,\cdot + \psi_n(\cdot)\nu_F(\cdot)\big) \to \lambda$ in $H^{\frac12}(\partial F)$. In turn,

$$H_n\big(\,\cdot + \psi_n(\cdot)\nu_F(\cdot)\big) \to \lambda - 4\gamma v_{F'}\big(\,\cdot + \psi_{F'}(\cdot)\nu_F(\cdot)\big) = H_{\partial F'}\big(\,\cdot + \psi_{F'}(\cdot)\nu_F(\cdot)\big) \qquad \text{in } H^{\frac12}(\partial F).$$

To conclude the proof we need to show that ψ_n converge to $\psi := \psi_{F'}$ in $W^{\frac52,2}(\partial F)$. To this aim, fix $\varepsilon > 0$. By straightening locally the boundary of F, we may always reduce to the case where the functions ψ_n are defined on a disk $B' \subset \mathbb{R}^2$, are bounded in $W^{\frac52,2}(B')$, converge in $W^{2,p}(B')$ for all $p \in [1,4)$ to $\psi \in W^{\frac52,2}(B')$ and $\|D\psi\|_{L^\infty(B')} \le \varepsilon$. We fix a cut-off function φ with compact support in B' and we write

$$\begin{aligned}\frac{\Delta(\varphi\psi_n)}{\sqrt{1+|D\psi_n|^2}} - \frac{\Delta(\varphi\psi)}{\sqrt{1+|D\psi|^2}} &= (D^2(\varphi\psi_n) - D^2(\varphi\psi))\frac{D\psi D\psi}{(1+|D\psi|^2)^{\frac32}}\\ &\quad + D^2(\varphi\psi_n)\left(\frac{D\psi_n D\psi_n}{(1+|D\psi_n|^2)^{\frac32}} - \frac{D\psi D\psi}{(1+|D\psi|^2)^{\frac32}}\right)\\ &\quad + \varphi(H_n - H) + R(x,\psi_n,D\psi_n) - R(x,\psi,D\psi),\end{aligned}$$

where the remainder term is R is similar to the one in (5.28). Then, using Lemma 5.1 with $\gamma \in (0,\frac12)$, an argument similar to the one of the proof of Lemma 5.2 shows that

$$\begin{aligned}\left[\frac{\Delta(\varphi\psi_n)}{\sqrt{1+|D\psi_n|^2}} - \frac{\Delta(\varphi\psi)}{\sqrt{1+|D\psi|^2}}\right]_{\frac12} &\le C(M)\big(\varepsilon^2[D^2(\varphi\psi_n) - D^2(\varphi\psi)]_{\frac12}\\ &\quad + \|D^2(\varphi\psi_n) - D^2(\varphi\psi)\|_{L^{\frac{4}{1+\gamma}}}\|D\psi\|^\gamma_{L^\infty}\|D^2\psi\|^{1-\gamma}_{L^4} + [D^2(\varphi\psi_n)]_{\frac12}\|D\psi_n - D\psi\|_{L^\infty}\\ &\quad + \|D^2(\varphi\psi_n)\|_{L^{\frac{4}{1+\gamma}}}\|D\psi_n - D\psi\|^\gamma_{L^\infty}(\|D^2\psi_n\|_{L^4} + \|D^2\psi\|_{L^4})^{1-\gamma}\\ &\quad + \|H_n - H\|_{H^{\frac12}} + \|\psi_n - \psi\|_{W^{2,2}}\big).\end{aligned}$$

Using Lemma 5.1 again to estimate $[\Delta(\varphi\psi_n) - \Delta(\varphi\psi)]_{\frac12}$ with the seminorm on the left hand side of the previous inequality and arguing as in the proof of Lemma 5.2 we finally get

$$[D^2(\varphi\psi_n) - D^2(\varphi\psi)]_{\frac12} \le C(M)\big(\|\psi_n - \psi\|_{W^{2,\frac{4}{1+\gamma}}} + \|D\psi_n - D\psi\|^\gamma_{L^\infty} + \|H_n - H\|_{H^{\frac12}}\big),$$

from which the conclusion follows. q.e.d.

5.2. The surface diffusion flow: proof of technical lemmas. We start by providing the computations leading to the crucial energy identities of Lemma 4.4.

Proof of Lemma 4.4. Let Ψ, Ψ_t, X_t be as in the proof of Lemma 3.5, and note that by (4.1) we have

$$X_t \cdot \nu_t = \Delta_\tau H_t \qquad \text{on } \partial E_t. \tag{5.33}$$

Fix $t \in (0,T)$, and as in Lemma 3.5 set $\Phi_s := \Psi_{t+s} \circ \Psi_t^{-1}$, so that $(\Phi)_{s\in(-t,T-t)}$ is an admissible one-parameter family of diffeomorphisms according to Definition 2.1. Then, by Theorem 2.2 we get

$$\begin{aligned}\frac{d}{dt}J(E_t) &= \frac{d}{ds}J(\Phi_s(E_t))\Big|_{s=0}\\ &= \int_{\partial E_t} H_t X_t \cdot \nu_t \, d\mathcal{H}^2 = \int_{\partial E_t} H_t \Delta_\tau H_t \, d\mathcal{H}^2 = -\int_{\partial E_t} |D_\tau H_t|^2 \, d\mathcal{H}^2 .\end{aligned}$$

This establishes (4.2). Let us fix a time $t > 0$. To continue we observe that, by redefining the velocity field if needed (in a time interval centered at t), we may assume that X_t has only a normal component on ∂E_t; that is,

$$X_t = (X_t \cdot \nu_t)\nu_t \qquad \text{on } \partial E_t. \tag{5.34}$$

Recall that all the geometric quantities can be extended in a neighborhood of ∂E_t by means of the gradient of the signed distance function from E_t (see the proof of Lemma 3.5). Now, arguing as in (5.14), we have

$$\dot\nu_t = -D_\tau(X_t \cdot \nu_t) = -D_\tau \Delta_\tau H_t \qquad \text{on } \partial E_t, \tag{5.35}$$

where the last equality follows again by (5.33). In turn, using also (5.34) and (5.14)

$$\frac{\partial}{\partial s}(DH_{t+s} \circ \Phi_s)\Big|_{s=0} = D\,\mathrm{div}_\tau(\dot\nu_t) + D^2 H_t[X_t] = -D(\Delta_\tau(\Delta_\tau H_t)) + (\Delta_\tau H_t) D^2 H_t \nu_t \tag{5.36}$$

on ∂E_t. Denoting by $D_{\tau_{t+s}}$ the tangential differential on ∂E_{t+s} and by $J_\tau \Phi_s$ the tangential Jacobian of Φ_s, we have

$$\begin{aligned}&\frac{d}{ds}\left(\frac{1}{2}\int_{\partial E_{t+s}} |D_\tau H_{t+s}|^2 \, d\mathcal{H}^2\right)\Big|_{s=0} = \frac{d}{ds}\left(\frac{1}{2}\int_{\partial E_t} |D_{\tau_{t+s}} H_{t+s}|^2 \circ \Phi_s J_\tau \Phi_s \, d\mathcal{H}^2\right)\Big|_{s=0}\\ &= \frac{1}{2}\int_{\partial E_t} |D_\tau H_t|^2 \,\mathrm{div}_\tau(\Delta_\tau H_t \, \nu_t) \, d\mathcal{H}^2 + \int_{\partial E_t} D_\tau H_t \cdot \frac{\partial}{\partial s}\left(D_{\tau_{t+s}} H_{t+s} \circ \Phi_s\right)\Big|_{s=0} d\mathcal{H}^2.\end{aligned} \tag{5.37}$$

We write the last term as

$$D_{\tau_{t+s}} H_{t+s} \circ \Phi_s = [I - \nu_{t+s} \circ \Phi_s \otimes \nu_{t+s} \circ \Phi_s] \, DH_{t+s} \circ \Phi_s$$

and get by (5.34), (5.13), (5.35) and (5.36)

$$\begin{aligned}\frac{\partial}{\partial s}\left(D_{\tau_{t+s}} H_{t+s} \circ \Phi_s\right)\Big|_{s=0} &= (-\dot\nu_t \otimes \nu_t - \nu_t \otimes \dot\nu_t) DH_t + [I - \nu_t \otimes \nu_t] \frac{\partial}{\partial t}(DH_t \circ \Phi_t)\\ &= -|B_t|^2 D_\tau \Delta_\tau H_t - DH_t \cdot \dot\nu_t \, \nu_t - D_\tau \Delta_\tau \Delta_\tau H_t + \Delta_\tau H_t \, [I - \nu_t \otimes \nu_t] \, D^2 H_t \nu_t .\end{aligned} \tag{5.38}$$

In order to calculate $D^2 H_t \nu_t$ we differentiate the equation (5.13) and get

$$-D|B_t|^2 = D(DH_t \cdot \nu_t) = D^2 H_t \nu_t + D\nu_t DH_t.$$

Therefore, since $B_t = D\nu_t$ and $B_t\nu_t = 0$ we get

$$D^2 H_t \nu_t = -D|B_t|^2 - BD_\tau H_t.$$

Plugging the last identity in (5.38) and using again (5.35), we may continue from (5.37) to obtain

$$\begin{aligned}\frac{d}{ds}\left(\frac{1}{2}\int_{\partial E_{t+s}} |D_\tau H_{t+s}|^2\, d\mathcal{H}^2\right)\Big|_{s=0} &= \frac{1}{2}\int_{\partial E_t} H_t |D_\tau H_t|^2 \Delta_\tau H_t\, d\mathcal{H}^2 \\ &- \int_{\partial E_t} |B_t|^2\, D_\tau H_t \cdot D_\tau \Delta_\tau H_t\, d\mathcal{H}^2 - \int_{\partial E_t} D_\tau H_t \cdot D_\tau \Delta_\tau \Delta_\tau H_t\, d\mathcal{H}^2 \\ &- \int_{\partial E_t} (\Delta_\tau H_t) D_\tau |B_t|^2 \cdot D_\tau H_t\, d\mathcal{H}^2 - \int_{\partial E_t} B[D_\tau H_t]\Delta_\tau H_t\, d\mathcal{H}^2.\end{aligned} \tag{5.39}$$

Integrating the third term on the right-hand side by parts twice, we get

$$-\int_{\partial E_t} D_\tau H_t \cdot D_\tau \Delta_\tau \Delta_\tau H_t\, d\mathcal{H}^2 = -\int_{\partial E_t} |D_\tau \Delta_\tau H_t|^2\, d\mathcal{H}^2\,.$$

Integrating the second last term on the right-hand side by parts once, we have

$$\begin{aligned}&-\int_{\partial E_t} (\Delta_\tau H_t) D_\tau |B_t|^2 \cdot D_\tau H_t\, d\mathcal{H}^2 \\ &= \int_{\partial E_t} |B_t|^2 D_\tau H_t \cdot D_\tau \Delta_\tau H_t\, d\mathcal{H}^2 + \int_{\partial E_t} |B_t|^2 |\Delta_\tau H_t|^2\, d\mathcal{H}^2.\end{aligned}$$

Plugging the last two identities into (5.39) and recalling (2.9) (with $\gamma = 0$), the identity (4.3) follows. q.e.d.

Proof of Lemma 4.6. In the following proof, in order to simplify the notation we drop the dependence on ∂E from all the geometric objects and the L^p spaces involved. Let us first show

$$\int_{\partial E} |D_\tau^2 f|^2\, d\mathcal{H}^2 \le C\int_{\partial E} |\Delta_\tau f|^2\, d\mathcal{H}^2 + C\int_{\partial E} |B|^2 |D_\tau f|^2\, d\mathcal{H}^2. \tag{5.40}$$

Indeed, recalling the following formula (see [**20**, Eq. (10.16)])

$$\delta_i\delta_j = \delta_j\delta_i + (\nu_i\delta_j\nu_k - \nu_j\delta_i\nu_k)\delta_k \tag{5.41}$$

and integrating by parts we get

$$\begin{aligned}&\int_{\partial E} |D_\tau^2 f|^2\, d\mathcal{H}^2 = \int_{\partial E} (\delta_i\delta_j f)\,(\delta_i\delta_j f)\, d\mathcal{H}^2 \\ &= \int_{\partial E} (\delta_i\delta_j f)\,(\delta_j\delta_i f)\, d\mathcal{H}^2 + \int_{\partial E} (\delta_i\delta_j f)(\nu_i\delta_j\nu_k - \nu_j\delta_i\nu_k)\delta_k f\, d\mathcal{H}^2 \\ &= -\int_{\partial E} \delta_j f\,(\delta_i\delta_j\delta_i f)\, d\mathcal{H}^2 + \int_{\partial E} H\nu_i\delta_j f\,(\delta_j\delta_i f)\, d\mathcal{H}^2 + \int_{\partial E} (\delta_i\delta_j f)(\nu_i\delta_j\nu_k - \nu_j\delta_i\nu_k)\delta_k f\, d\mathcal{H}^2 \\ &\le -\int_{\partial E} \delta_j f\,(\delta_i\delta_j\delta_i f)\, d\mathcal{H}^2 + C\int_{\partial E} |B|\,|D_\tau f|\,|D_\tau^2 f|\, d\mathcal{H}^2.\end{aligned}$$

Using (5.41) and integrating by parts again, we obtain

$$\int_{\partial E} |D_\tau^2 f|^2\, d\mathcal{H}^2 \le \int_{\partial E} (\delta_i\delta_i f)\,(\delta_j\delta_j f)\, d\mathcal{H}^2\, d\mathcal{H}^2 + C\int_{\partial E} |B|\,|D_\tau f|\,|D_\tau^2 f|\, d\mathcal{H}^2.$$

The inequality (5.40) follows since $\Delta_\tau f = \delta_i\delta_i f$.

We estimate the last term in (5.40) by Lemma 4.5:

$$\int_{\partial E} |B|^2 |D_\tau f|^2 \, d\mathcal{H}^2 \le \|B\|_{L^4}^2 \|D_\tau f\|_{L^4}^2$$
$$\le C\|B\|_{L^4}^2 \left(\|D_\tau^2 f\|_{L^2} \|D_\tau f\|_{L^2} + \|D_\tau f\|_{L^2}^2 \right) .$$

Plugging in (5.40) and by an application of Young's inequality, we get

$$\begin{aligned} \|D_\tau^2 f\|_{L^2}^2 &\le C \left(\|\Delta_\tau f\|_{L^2}^2 + \|D_\tau f\|_{L^2}^2 (\|B\|_{L^4}^2 + \|B\|_{L^4}^4) \right) \\ &\le C \left(\|\Delta_\tau f\|_{L^2}^2 + \|D_\tau f\|_{L^2}^2 (1 + \|B\|_{L^4}^4) \right) . \end{aligned} \tag{5.42}$$

Now, note that (with the same notation introduced in Lemma 4.5)

$$\begin{aligned} \|D_\tau f\|_{L^2}^2 &= -\int_{\partial E} f \Delta_\tau f \, d\mathcal{H}^2 = -\int_{\partial E} (f - \bar{f}) \Delta_\tau f \, d\mathcal{H}^2 \\ &\le \|f - \bar{f}\|_{L^2} \|\Delta_\tau f\|_{L^2} \le C \|D_\tau f\|_{L^2} \|\Delta_\tau f\|_{L^2} . \end{aligned} \tag{5.43}$$

Note that in the second equality above we have used the fact that $\Delta_\tau f$ has zero average on each connected component of ∂E. Thus, from (5.42) we deduce

$$\|D_\tau^2 f\|_{L^2}^2 \le C \|\Delta_\tau f\|_{L^2}^2 (1 + \|B\|_{L^4}^4).$$

By a standard application of Calderon-Zygmund estimate we have

$$\|B\|_{L^4} \le C(1 + \|H\|_{L^4}),$$

with C depending only the C^1-bounds on ∂E, and the conclusion follows. q.e.d.

We now show the geometric interpolation used in the proof of Theorem 4.3.

Proof of Lemma 4.7. Also here to simplify the notation we drop the dependence on ∂E both from the geometric objects and the L^p spaces. First by Hölder's inequality

$$\int_{\partial E} |B| |D_\tau H|^2 |\Delta_\tau H| \, d\mathcal{H}^2 \le \|\Delta_\tau H\|_{L^3} \left(\int_{\partial E} |B|^{\frac{3}{2}} |D_\tau H|^3 \, d\mathcal{H}^2 \right)^{2/3} .$$

By the Poincaré Inequality stated in Lemma 4.5 we get

$$\|\Delta_\tau H\|_{L^3} \le C \|D_\tau (\Delta_\tau H)\|_{L^2}.$$

In turn, Hölder's inequality implies

$$\left(\int_{\partial E} |B|^{\frac{3}{2}} |D_\tau H|^3 \, d\mathcal{H}^2 \right)^{2/3} \le \left(\int_{\partial E} |D_\tau H|^4 \, d\mathcal{H}^2 \right)^{1/2} \left(\int_{\partial E} |B|^6 \, d\mathcal{H}^2 \right)^{1/6} .$$

Lemma 4.5 yields

$$\left(\int_{\partial E} |D_\tau H|^4 \, d\mathcal{H}^2 \right)^{1/2} \le C \left(\|D_\tau^2 H\|_{L^2} \|D_\tau H\|_{L^2} + \|D_\tau H\|_{L^2}^2 \right) .$$

Combining all the inequalities above, we get

$$\int_{\partial E} |B| |D_\tau H|^2 |\Delta_\tau H| \, d\mathcal{H}^2 \le C \|D_\tau (\Delta_\tau H)\|_{L^2} \, \|B\|_{L^6} \, \|D_\tau H\|_{L^2} (\|D_\tau^2 H\|_{L^2} + \|D_\tau H\|_{L^2}).$$

By Lemma 4.6 and (5.43) (with $D_\tau H$ in place of $D_\tau f$), the right-hand side of the above inequality can be estimated from above by

$$C \|D_\tau (\Delta_\tau H)\|_{L^2} \, \|B\|_{L^6} \, \|\Delta_\tau H\|_{L^2} \, \|D_\tau H\|_{L^2} \, (1 + \|H\|_{L^4}^2).$$

The conclusion follows from the Poincaré Inequality

$$\|\Delta_\tau H\|_{L^2} \le C\|D_\tau(\Delta_\tau H)\|_{L^2}.$$

and the Calderon-Zygmund estimate

$$\|B\|_{L^6} \le C(1 + \|H\|_{L^6})\,.$$

q.e.d.

We conclude with the proof of the geometric Poincaré Inequality stated in Lemma 4.8.

Proof of Lemma 4.8. Since $\int_{\partial E}(H_{\partial E} - \overline{H}_{\partial E})\nu_E\, d\mathcal{H}^2 = 0$, we may apply Lemma 2.6, with $\varepsilon = 1$ and $\varphi := H_{\partial E} - \overline{H}_{\partial E}$, and recall (2.9) (with $\gamma = 0$) to obtain

$$\sigma \int_{\partial E} |H_{\partial E} - \overline{H}_{\partial E}|^2\, d\mathcal{H}^2 \le \int_{\partial E} |D_\tau H_{\partial E}|^2\, d\mathcal{H}^2 - \int_{\partial E} |B_{\partial E}|^2 |H_{\partial E} - \overline{H}_{\partial E}|^2\, d\mathcal{H}^2 \le \int_{\partial E} |D_\tau H_{\partial E}|^2\, d\mathcal{H}^2\,.$$

The conclusion follows. q.e.d.

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