Curvature-dependent energies: a geometric and analytical approach

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Abstract. *We consider the total curvature of graphs of curves in high codimension Euclidean space. We introduce the corresponding relaxed energy functional and prove an explicit representation formula. In the case of continuous Cartesian curves, i.e. of graphs c_u of continuous functions u on an interval, we show that the relaxed energy is finite if and only if the curve c_u has bounded variation and finite total curvature. In this case, moreover, the total curvature does not depend on the Cantor part of the derivative of u. We treat the wider class of graphs of one-dimensional BV-functions, and we prove that the relaxed energy is given by the sum of length and total curvature of the new curve obtained by closing with vertical segments the holes in c_u generated by jumps of u.*

In the mathematical literature, functionals depending on second order derivatives have recently been applied e.g. in image restoration processes, in order to overcome some drawbacks typical of approaches based on first order functionals, as the total variation. One instance is the approach by Chan-Marquina-Mulet [6] who proposed to consider regularizing terms given by second order functionals of the type

$$\int_\Omega |\nabla u|\,dx + \int_\Omega \psi(|\nabla u|)\,h(\Delta u)\,dx$$

for scalar-valued functions u defined in two-dimensional domains, where the function ψ satisfies suitable conditions at infinity in order to allow jumps.

The downscaled one-dimensional version of the above functional is given by

$$\int_a^b |\dot u|\,dt + \int_a^b \psi(|\dot u|)\,|\ddot u|^p\,dt\,, \qquad p \ge 1\,, \qquad u : [a,b] \to \mathbb{R} \tag{0.1}$$

and it has been thoroughly studied in [8], where Dal Maso-Fonseca-Leoni-Morini proved an explicit formula for the relaxed energy, under suitable assumptions on the function ψ.

The prototypical example is the *curvature energy functional*, given by choosing

$$\psi_p(t) := \frac{1}{(1+t^2)^{(3p-1)/2}}\,.$$

In this case, in fact, the above functional takes the form

$$\mathcal{E}_p(u) := \int_a^b |\dot u|\,dt + \int_a^b \sqrt{1+\dot u(t)^2}\cdot k_u(t)^p\,dt\,, \qquad k_u(t) := \frac{|\ddot u(t)|}{(1+\dot u(t)^2)^{3/2}}\,.$$

Therefore, in the smooth case, considering the *Cartesian curve* $c_u(t) := (t, u(t))$, since $k_u(t)$ is the curvature at the point $c_u(t)$, and replacing the first term with the integral of $\sqrt{1+\dot u^2}$, by the area formula one obtains an intrinsic formulation on the graph curve c_u as

$$\mathcal{E}_p(u) = \mathcal{L}(c_u) + \int_{c_u} k_u^p\,d\mathcal{H}^1\,,$$

where $\mathcal{L}$ is the length.

In this paper, restricting to the linear case $p = 1$, and leaving the case $p > 1$ to a further research, we deal with the higher codimension analogous of the above curvature functional, proving a complete explicit formula for its relaxed energy.

More precisely, for C^2-functions $u : I \to \mathbb{R}^N$, where $I = [a, b]$, we set

$$\mathcal{E}(u) = \mathcal{L}(c_u) + \int_{c_u} k_u \, d\mathcal{H}^1 \,,$$

where k_u is the curvature of the Cartesian curve $c_u(t) := (t, u(t))$, see (1.6), and we define

$$\overline{\mathcal{E}}(u) := \inf\big\{\liminf_{h\to\infty} \mathcal{E}(u_h) \mid \{u_h\} \subset C^2(I, \mathbb{R}^N)\,,\ u_h \to u \text{ in } L^1\big\} \tag{0.2}$$

for any summable function $u \in L^1(I, \mathbb{R}^N)$.

Any function u with finite relaxed energy (0.2) has bounded variation, with distributional derivative decomposed as usual by $Du = \dot{u}\,\mathcal{L}^1 + D^C u + D^J u$. A crucial role is played here by the *Gauss map* τ_u that is defined a.e. in I by means of the *approximate gradient* $\dot{u}$, namely

$$\tau_u = \frac{\dot{c}_u}{|\dot{c}_u|}\,, \qquad \dot{c}_u = (1, \dot{u}^1, \dots, \dot{u}^N)\,. \tag{0.3}$$

For our purposes, we recall that the *total curvature* $\mathrm{TC}(c)$ of a curve c has been defined by Milnor [19] as the supremum of the total curvature (i.e. the sum of the turning angles) of the polygons inscribed in c. A curve with finite total curvature is rectifiable, and hence it admits a Lipschitz parameterization. Therefore, it is well defined the *tantrix* (or tangent indicatrix), that assigns to a.e. point the oriented unit tangent vector $\mathfrak{t}_c$. Moreover, see Proposition 1.3, the total curvature agrees with the *essential total variation* of the tantrix.

For smooth Cartesian curves c_u the tantrix $\mathfrak{t}_{c_u}$ agrees with the Gauss map τ_u, whence the total curvature $\mathrm{TC}(c_u)$ is equal to the total variation of τ_u. Therefore, for C^2-functions u one has

$$\mathrm{TC}(c_u) = \int_{c_u} k_u \, d\mathcal{H}^1 = \int_I |\dot{\tau}_u| \, dt\,.$$

As we shall describe below, see (0.4), in the relaxation process the role of the tantrix is played by the Gauss map (0.3).

In the case of codimension $N = 1$, in [8] it is proved that the function $t \mapsto \arctan \dot{u}(t)$ has bounded variation. We will recover also this result, and the explicit formula for the relaxed energy (0.2) obtained in [8] is reproduced at the beginning of Sec. 4 below.

The authors of [8] show the existence of functions with finite relaxed energy (0.2) but with a non-trivial Cantor component of the derivative. However, they also prove a concentration property for the Cantor part $D^C u$, namely, its positive and negative parts are concentrated on the set of points where the approximate gradient is equal to $\pm\infty$, see Definition 4.1. Roughly speaking, for continuous functions such a geometric property implies that the Cartesian curve c_u does not have "angles" in correspondence of points in the Cantor set of u. From our analytic/geometric viewpoint this justifies why the Cantor part of the distributional derivative $D^C u$ does not appear in the curvature part of the relaxed energy. We remark that the total variation of $\arctan \dot{u}$ agrees with the total variation of the Gauss map τ_u.

The techniques used in [8] are deeply analytical, as the authors deal with more general functionals of the type (0.1), and hence do not make use of the geometric properties that are specific of the curvature.

Our approach to the study of the relaxed energy (0.2) is much more of geometric flavor, instead, and differently from [8] it allows us to work in any codimension $N \geq 1$. For this reason we shall use some features of the theory of currents with its powerful theoretical results.

Beside the interest by itself, our study is motivated by the fact that functionals depending on the curvature of vector valued curves play a crucial role e.g. in the study of vibration of strings and go back to the work by Euler.

This approach of bridging the singularities of a deformation by means of currents may be compared to the work by Marcellini [18] in the context of finding semicontinuous elastic energies in presence of cavitation, see also [20]. Another similar approach, in the case of parametric curves, may be found in [7].

The mathematical literature concerning curvature functionals is very wide, and our list of references is far from being exhaustive. We recall here that a different approach based on the theory of varifolds

has been considered in [16, 17]. Moreover, further properties concerning curvature functionals of curves have been obtained in [4, 5, 7].

Main results. In this paper, we shall prove in any codimension that for any function $u \in BV(I, \mathbb{R}^N)$ with finite relaxed energy (0.2), the *Gauss map* $\tau_u : I \to \mathbb{S}^N$, defined in (0.3), is a function with *bounded variation*, Theorem 4.7.

Moreover, see Theorem 9.2, we shall prove that any continuous function $u \in BV(I, \mathbb{R}^N)$ has finite relaxed energy (0.2) if and only if the Cartesian curve c_u has *finite total curvature* $\mathrm{TC}(c_u)$, and in this case we shall prove the equality

$$\mathrm{TC}(c_u) = |D\tau_u|(I)\,. \tag{0.4}$$

This second result says that the total variation of the tantrix of the graph of a continuous function u with finite relaxed energy does not read jumps in presence of the Cantor part of the derivative $D^C u$. In some sense, the above fact is the higher codimension analogous of the previously mentioned concentration property from [8] of the Cantor component $D^C u$, in the case $N = 1$.

These geometric facts are at the basis of the proof of the lower bound, Theorem 8.1, and upper bound, Theorem 8.5, yielding to an explicit formula for the relaxed energy (0.2), that we now describe.

For continuous functions such that $\overline{\mathcal{E}}(u) < \infty$, recalling (0.4), we shall obtain in Sec. 9 that the relaxed energy is the sum of the length and of the total curvature of the Cartesian curve c_u, together with the representation of the total curvature:

$$\overline{\mathcal{E}}(u) = \mathcal{L}(c_u) + \mathrm{TC}(c_u)\,, \qquad \mathrm{TC}(c_u) = |D\tau_u|(I)\,. \tag{0.5}$$

In the general case, i.e. when u is a BV-function with a non-trivial Jump set J_u, we shall prove that

$$\overline{\mathcal{E}}(u) = |Dc_u|(I) + |D\tau_u|(I \setminus J_u) + \mathbf{M}(S_u^{Jc})\,. \tag{0.6}$$

The second term in (0.6) involves the total variation of the Gauss map τ_u outside the Jump set J_u. The so called *Jump-corner* term $\mathbf{M}(S_u^{Jc})$ is an energy contribution that "lives" in the Jump set of u, see (5.9) and (5.13). Roughly speaking, it is given by the sum at each discontinuity point $t \in J_u$ of the two turning angles that appear in the graph of the function u when connecting the endpoints of a discontinuity in the graph with a straight line segment. Notice that in codimension $N = 1$, we have $|D\tau_u|(I \setminus J_u) = |D \arctan \dot{u}|(I \setminus J_u)$, whereas the Jump-corner term $\mathbf{M}(S_u^{Jc})$ takes the same value as the corresponding term obtained in [8] for the energy density ψ_p with $p = 1$, compare Corollary 7.9.

At the end, we read formula (0.6) in a simple way by introducing the continuous curve $\widetilde{c}_u$ obtained by closing the graph of u as above. We will prove our main result:

$$\overline{\mathcal{E}}(u) = \mathcal{L}(\widetilde{c}_u) + \mathrm{TC}(\widetilde{c}_u) \tag{0.7}$$

that reduces to (0.5) in case of continuous functions.

Plan of the paper. In Sec. 1, we recall some features concerning the total curvature of curves, and we restrict to the subclass of Cartesian curves, introducing our model energy.

In Sec. 2, we collect some basic facts concerning functions of bounded variations, compare [2]. We shall make use of techniques based on Geometric Measure Theory, and in particular on the theory of *Cartesian currents* by Giaquinta-Modica-Souček [13], that we shall briefly recall.

Our approach to the relaxation problem also relies on some ideas from the theory of *Gauss graphs* of curves, that is the graph of the couple $(c, \mathfrak{t}_c)$, compare [3] and [9], but re-written in the context of Cartesian curves. In Sec. 3, we shall then focus on the class of currents GG_u in $U \times \mathbb{S}^N$, where $U := \mathring{I} \times \mathbb{R}^N$, that are carried by the Gauss graphs of C^2-functions u, which are parameterized by the map $\Phi_u(t) := (c_u(t), \tau_u(t))$.

In Sec. 4, we shall introduce the class of 1-currents in $U \times \mathbb{S}^N$ that arise as *weak limits* (in the sense of currents) *of sequences* $\{GG_{u_h}\}$ *of Gauss graphs of smooth Cartesian curves with equibounded masses.* This class will be denoted by $\mathrm{Gcart}(U \times \mathbb{S}^N)$, compare (4.4). We shall then analyze some properties concerning functions with finite relaxed energy (0.2), proving in particular that $\tau_u \in BV$ in the already cited Theorem 4.7.

In Sec. 5, we shall extend the notion of current GG_u carried by the Gauss graph to any function u with finite relaxed energy (0.2). This current is naturally defined by three terms, $GG_u = GG_u^a + GG_u^C + GG_u^J$, the absolute continuous, Cantor, and Jump component, respectively. Our definition makes sense because we already know that both the function c_u and the Gauss map τ_u have bounded variation. In general the "Gauss graph" GG_u has fractures, or holes. We shall then see that there is a natural (and optimal) way to find a "vertical" current S_u such that $\Sigma_u := GG_u + S_u$ is an integer multiplicity (say i.m.) rectifiable current in $U \times \mathbb{S}^N$ without boundary. The current S_u is made by two terms. The first one is the *corner component* S_u^c, that is used to "fill the holes" in the Gauss graph given by the corners of the curve c_u, i.e., the points where u is continuous but the Gauss map τ_u is discontinuous. The second one is the already mentioned Jump-corner component S_u^{Jc}, that is used to fill the holes in the Gauss graph given by the discontinuity points of the function u.

In Sec. 6, we shall prove some structure properties concerning the class $\mathrm{Gcart}(U \times \mathbb{S}^N)$, Theorem 6.1. We shall then see that a current Σ in Gcart preserves the geometry of Gauss graphs, Theorem 6.3. More precisely, when the first component x of the tangent vector to Σ at a point $z = (x, y) \in U \times \mathbb{S}^N$ is non zero, then it has to be parallel to (and pointing the same way as) the second component y.

In Sec. 7, we shall extend the energy to general currents, by introducing a lower semicontinuous functional $\Sigma \mapsto \mathcal{E}^0(\Sigma)$ that agrees with the energy functional $\mathcal{E}(u)$ when restricted to the Gauss graphs GG_u of smooth functions, see Proposition 7.2. In order to prove the explicit formula (0.6), we will also write more explicitly the action of our energy functional in the case when $\Sigma = \Sigma_u$, yielding in general to the right-hand side of (0.6). In fact, our strategy consists in proving the equality

$$\overline{\mathcal{E}}(u) = \mathcal{E}^0(\Sigma_u)$$

for every function u with finite relaxed energy.

In Sec. 8, we shall in fact prove the energy lower and upper bounds ("$\geq$" and "$\leq$" in the equality above), Theorems 8.1 and 8.5. To prove Theorem 8.1 we will show that among all currents in $\mathrm{Gcart}(U \times \mathbb{S}^N)$ with underlying function given by u, the optimal one in terms of energy is our generalized Gauss graph Σ_u. To this purpose, we shall make use of an average formula of the total curvature proved by Sullivan [21, Prop. 4.1] that goes back to Fáry [11], see Proposition 1.4.

Instead, to prove the upper bound "$\leq$", we show that a density property holds for the Gauss graph Σ_u, taking advantage of the geometric equality (0.4) for continuous functions.

Finally, in Sec. 9, we prove the representations (0.5), (0.6) and (0.7), see Theorems 9.2 and 9.5.

1 Notation and preliminary results

In this section we recall some features concerning the total curvature of curves, and we restrict to the subclass of Cartesian curves, introducing our model energy.

Curvature of curves. Let $n \geq 2$ and $I := [a, b] \subset \mathbb{R}$ denote a non-trivial closed interval. A function $c : [a, b] \to \mathbb{R}^n$ of class C^2 is said to be a *regular curve* in $\mathbb{R}^n$ if the first derivative vector $\dot{c}(t)$ is non-zero everywhere on I.

The unit tangent vector and the curvature are respectively given by

$$\mathbf{t}_c(t) = \frac{\dot{c}(t)}{|\dot{c}(t)|}, \qquad \mathbf{k}_c(t) := \frac{\mathrm{area}[\dot{c}(t), \ddot{c}(t)]}{|\dot{c}(t)|^3} \tag{1.1}$$

where we have denoted by $\mathrm{area}[\dot{c}, \ddot{c}]$ the area of the (possibly degenerate) parallelogram in $\mathbb{R}^n$ spanned by $\dot{c}, \ddot{c}$. Therefore, we have $\mathrm{area}[\dot{c}, \ddot{c}] = |\det[\dot{c}, \ddot{c}]|$ for $n = 2$, and $\mathrm{area}[\dot{c}, \ddot{c}] = |\dot{c} \times \ddot{c}|$ for $n = 3$.

Referring to [15] for the notion of intermediate curvatures and of torsion (the last curvature), we recall that the (first) curvature does not depend on the parameterization of $c(t)$.

Curvature functionals. By the above, one may consider curvature functionals depending on the curvature as follows.

Let $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to [0, +\infty]$ be a sufficiently smooth integrand that is positively homogeneous of

degree one in the second entry, i.e.,

$$f(c, \lambda\dot{c}, k) = \lambda\, f(c, \dot{c}, k) \qquad \forall\, (c, \dot{c}, k) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}\,, \quad \forall\, \lambda > 0\,.$$

To each regular curve $c : [a, b] \to \mathbb{R}^n$ we may associate the curvature functional

$$\mathcal{F}(c) := \int_a^b f(c(t), \dot{c}(t), \mathbf{k}_c(t))\, dt\,.$$

It is readily checked that the definition does not depend on the parameterization. Moreover, the above functional does not depend on the orientation if in addition we assume that

$$f(c, -\dot{c}, k) = f(c, \dot{c}, k) \qquad \forall\, (c, \dot{c}, k) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}\,.$$

Example 1.1 Consider for simplicity the curvature functional

$$\mathcal{F}(c) := \int_{-1}^{1} |\dot{c}(t)|\, (\lambda_1 + \lambda_2\, g(\mathbf{k}_c(t)))\, dt\,, \qquad \lambda_1,\, \lambda_2 \in \mathbb{R}^+ \tag{1.2}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a non-negative Lipschitz function such that $g(0) = 0$ and the limit

$$g^\infty := \lim_{t \to +\infty} \frac{g(t)}{t} \tag{1.3}$$

exists and is finite. The *recession* g^∞ of g comes into the play in case of occurrence of angles.

In fact, let $0 < \alpha < \pi/4$ and $c_\alpha : [-1, 1] \to \mathbb{R}^2$ be the piecewise affine curve with a turning angle of width 2α at the origin, given by

$$c_\alpha(t) := \begin{cases} (t, 0) & \text{if } t < 0 \\ \dfrac{(t, m_\alpha \cdot t)}{\sqrt{1 + m_\alpha^2}} & \text{if } t \geq 0 \end{cases} \qquad m_\alpha := \tan(2\alpha)\,.$$

This function is not of class C^2, so the functional $\mathcal{F}$ is not defined on c. But we now see that, in a relaxed sense,

$$\mathcal{F}(c_\alpha) = \lambda_1 \cdot \mathcal{L}(c_\alpha) + \lambda_2 \cdot 2\alpha \cdot g^\infty\,.$$

In fact, outside the origin the curvature of c_α is zero, and we can approximate c_α near the origin by small regular arcs, as e.g. $c_{\alpha,\varepsilon} : [-\pi/2, -\pi/2 + 2\alpha] \to \mathbb{R}^2$ defined by $c_{\alpha,\varepsilon}(t) := \varepsilon\,(-\sin\alpha + \cos\alpha\cos t, \cos\alpha + \cos\alpha\sin t)$. We thus have $|\dot{c}_{\alpha,\varepsilon}(t)| = \varepsilon\,\cos\alpha$ and $\det\,[\dot{c}_{\alpha,\varepsilon}(t)|\ddot{c}_{\alpha,\varepsilon}(t)] = \varepsilon^2\cos^2\alpha$, so that

$$\int_{-\pi/2}^{-\pi/2+2\alpha} |\dot{c}_{\alpha,\varepsilon}(t)|\, g\Big(\frac{|\det[\dot{c}_{\alpha,\varepsilon}(t)|\ddot{c}_{\alpha,\varepsilon}(t)]|}{|\dot{c}_{\alpha,\varepsilon}(t)|^3}\Big)\, dt = 2\alpha\,\varepsilon\cos\alpha\, g((\varepsilon\cos\alpha)^{-1})$$

and hence $\mathcal{F}(c_{\alpha,\varepsilon}) \to \lambda_1 \cdot \mathcal{L}(c_\alpha) + \lambda_2 \cdot 2\alpha \cdot g^\infty$ as $\varepsilon \to 0^+$.

In Figure 1 we have divided the graph curve $(c(t), \mathbf{k}_c(t))$ by drawing on the left side the image of $c(t)$ and on the right the graph $(t, \mathbf{k}_c(t))$ of the unit tangent vector. This example shows that the relaxed formula of the curvature functional must in general contain an angle term reminding of the regular version.

Total curvature. The *total curvature* $\mathrm{TC}(c)$ of a curve c in $\mathbb{R}^n$ has been defined by Milnor [19] as the supremum of the total curvature of the polygons P inscribed in c. For a polygon P, the total curvature is the sum of the *turning angles* between consecutive segments. As for the length, we thus have

$$\begin{aligned} \mathcal{L}(c) &:= \sup\{\mathcal{L}(P) \mid P \text{ inscribed in } c\}\,, \\ \mathrm{TC}(c) &:= \sup\{\mathrm{TC}(P) \mid P \text{ inscribed in } c\}\,. \end{aligned} \tag{1.4}$$

Proposition 1.2 *If $\{P_h\}$ is a sequence of polygons inscribed in c such that* $\mathrm{mesh}(P_h) \to 0$, *then* $\mathcal{L}(P_h) \to \mathcal{L}(P)$ *and* $\mathrm{TC}(P_h) \to \mathrm{TC}(c)$.

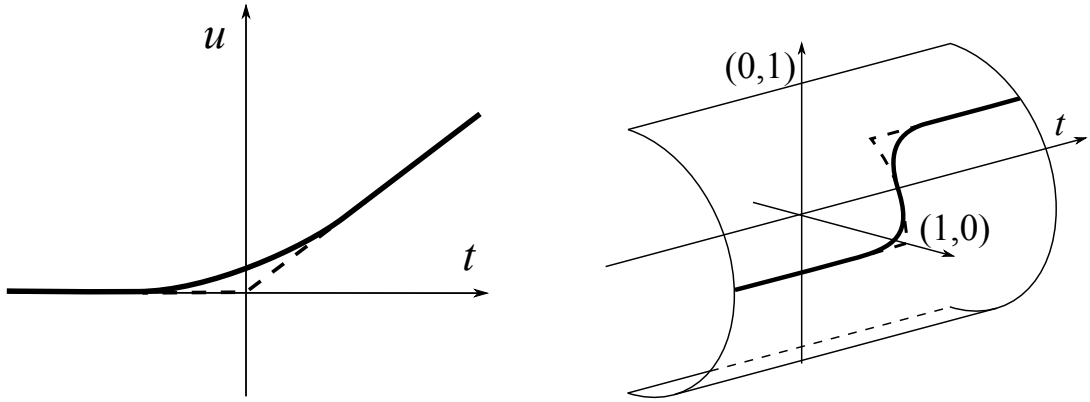

Figure 1: The curve c_α (dashed) and the smooth approximation of $c_{\alpha,\varepsilon}$.
On the right: the graph $(t, \mathbf{k}_c(t))$ of the corresponding unit tangent vector.

A curve c with finite total curvature, say $c \in FTC(\mathbb{R}^n)$, is rectifiable, $\mathcal{L}(c) < \infty$, and hence it admits a Lipschitz parameterization $c : [a,b] \to \mathbb{R}^n$. Since Lipschitz functions are differentiable a.e., it is well defined the *tantrix* (or tangent indicatrix), that assigns to a.e. point the oriented unit tangent vector $\mathfrak{t}_c \in \mathbb{S}^{n-1}$. Moreover, see Sullivan [21], one has

Proposition 1.3 *The total curvature of c agrees with the* essential total variation *of the tantrix, whence $c \in FTC(\mathbb{R}^n)$ if and only if the tantrix $\mathfrak{t}_c \in BV([a,b], \mathbb{S}^{n-1})$.*

Total curvature and projections. The total curvature of a curve does not give an uniform bound on the total curvature of its orthogonal projections. Taking e.g. the curve in $\mathbb{R}^3$

$$c : \quad z = x^2 , \quad y = 0 , \quad |x| \le 1/2$$

one has $\mathrm{TC}(c) = \pi/2$. However, the total curvature of its projection on the plane $z = 0$ is zero, the projected curve being a straight segment, whereas the total curvature of its projection on the plane $x = 0$ is equal to π, the projected curve this time being a segment bent backwards. This remains equal to π even if we restrict to $|x| \le \varepsilon$, in which case $\mathrm{TC}(c)$ goes to zero with ε.

Notwithstanding, following Sullivan [21], the total curvature of a curve is the average of the total curvatures of all its projections on k-planes, for each k. More precisely, for $1 \le k < n$ integer, we denote by $G_k\mathbb{R}^n$ the Grassmannian of k-planes in $\mathbb{R}^n$. It is a compact group, and it can be equipped with a unique rotationally invariant probability measure μ_k. For $p \in G_k\mathbb{R}^n$, we denote by π_p the orthogonal projection of $\mathbb{R}^n$ onto p. The following result is proved in [21, Prop. 4.1], and it goes back to Fáry [11].

Proposition 1.4 *Given a curve c in $\mathbb{R}^n$, and some fixed integer $1 \le k < n$, then*

$$\mathrm{TC}(c) = \int_{G_k\mathbb{R}^n} \mathrm{TC}(\pi_p(c))\, d\mu_k(p) .$$

Cartesian curves. In this paper we shall focus on the action of curvature functionals as in (1.2) on the subclass of *Cartesian curves* in $\mathbb{R}^n$. More precisely, we let $N = n-1 \ge 1$, and consider a C^2-function $u : I \to \mathbb{R}^N$ where $I = [a,b]$, so that the corresponding Cartesian curve is $c_u : I \to \mathbb{R}^{N+1}$ defined by $c_u(t) := (t, u(t))$, where $u = (u^1, \dots, u^N)$ in components. Any smooth Cartesian curve is automatically regular, as $\dot{c}_u(t) = (1, \dot{u}(t))$ for each t. For simplicity of notation *we shall correspondingly denote by τ_u and k_u the tantrix $\mathfrak{t}_{c_u}$ and curvature $\mathbf{k}_{c_u}$ of a Cartesian curve*, respectively. We thus have

$$\tau_u = \frac{\dot{c}_u}{|\dot{c}_u|} , \qquad \dot{c}_u = (1, \dot{u}^1, \dots, \dot{u}^N) , \qquad |\dot{c}_u| = \sqrt{1 + |\dot{u}|^2} .$$

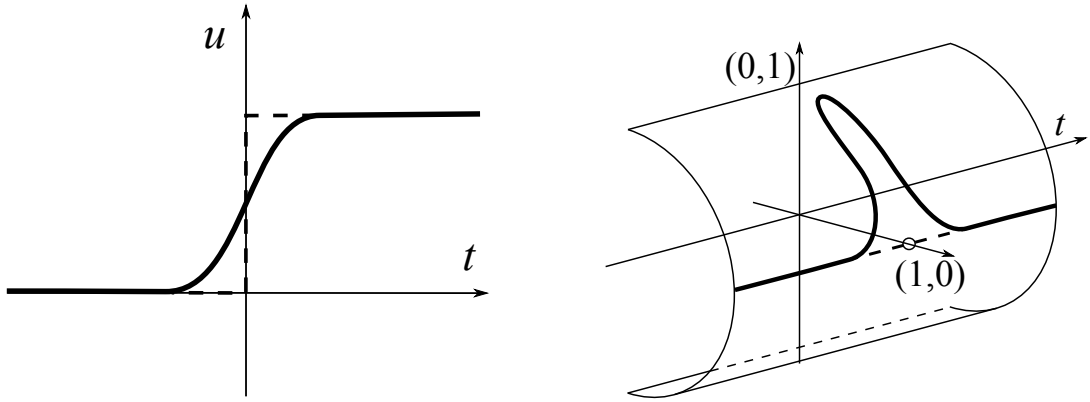

Figure 2: The curve c (dashed) and the smooth approximation of c_{u_h}.
On the right: the corresponding curves in the (t,τ)-space.

In codimension $N=1$, i.e. for $c_u(t)=(t,u(t)):I\to\mathbb{R}^2$, the curvature of c_u at the point $c_u(t)$ is

$$k_u(t)=\frac{|\ddot u(t)|}{\big(1+\dot u(t)^2\big)^{3/2}} \tag{1.5}$$

so that $|\dot c_u(t)|\,k_u(t)=|\dot v(t)|$, where $v(t):=\arctan\dot u(t)$. In higher codimension $N\ge 2$, denoting by $v_1\bullet v_2$ the scalar product of vectors, we obtain:

$$\mathrm{area}[\dot c_u,\ddot c_u]^2=|\dot c_u\wedge\ddot c_u|^2=|\ddot u|^2(1+|\dot u|^2)-(\dot u\bullet\ddot u)^2\,,$$

whence for $c_u(t)=(t,u(t)):I\to\mathbb{R}^{N+1}$ the curvature at the point $c_u(t)$ is

$$k_u(t)=\frac{\big(|\ddot u|^2(1+|\dot u|^2)-(\dot u\bullet\ddot u)^2\big)^{1/2}}{(1+|\dot u|^2)^{3/2}}\,. \tag{1.6}$$

Example 1.5 Let $c_{u_h}:[-1,1]\to\mathbb{R}^2$ be the piecewise affine Cartesian curve $c_{u_h}(t)=(t,u_h(t))$, where

$$u_h(t):=\begin{cases}0 & \text{if } t<-\pi/h\\ ht+\pi & \text{if } -\pi/h\le t\le\pi/h\\ 2\pi & \text{if } t>\pi/h\end{cases}\qquad h\in\mathbb{N}^+\quad\text{large}\,, \tag{1.7}$$

so that c_{u_h} has two corners with two turning angles both of width $\arctan h$ at the points $(-\pi/h,0)$ and $(\pi/h,2\pi)$. With the same notation as in Example 1.1, and in the same relaxed sense since $u_h\notin C^2$,

$$\mathcal{F}(c_{u_h}):=\int_{-1}^{1}|\dot c_{u_h}(t)|\,(\lambda_1+\lambda_2\,g(k_{u_h}(t)))\,dt=\lambda_1\cdot\mathcal{L}(c_{u_h})+\lambda_2\cdot 2\arctan h\cdot g^\infty\,,$$

so that $\mathcal{F}(c_{u_h})\to\lambda_1\cdot(2+2\pi)+\lambda_2\cdot 2\cdot\dfrac{\pi}{2}\cdot g^\infty$ as $h\to\infty$. Although the functions u_h converge to a jump function u, the graphs c_{u_h} converge to a curve c which closes the jump of u with a vertical segment of length 2π, see Figure 2, and we got in a further relaxed sense

$$\mathcal{F}(c)=\lambda_1\cdot\mathcal{L}(c)+\lambda_2\cdot g^\infty\cdot\mathrm{TC}(c)\,.$$

We formalize what we learned from the example. Using the notation from Sec. 2 below, we denote by $G_{u_h}:=c_{u_h\#}[\![-1,1]\!]$ the 1-current in $(-1,1)\times\mathbb{R}$ carried by the graph of u_h. It is easy to check that G_{u_h} weakly converges to the *Cartesian current* $T:=c_\#[\![0,2(1+\pi)]\!]$ given by the integration of 1-forms in $\mathcal{D}^1((-1,1)\times\mathbb{R})$ over the (oriented) limit curve $c:[0,2(1+\pi)]\to\mathbb{R}^2$

$$c(s):=\begin{cases}(s-1,0) & \text{if } 0\le s\le 1\\ (0,s-1) & \text{if } 1<s<1+2\pi\\ (s-2\pi-1,2\pi) & \text{if } 1+2\pi\le s\le 2(1+\pi)\,.\end{cases} \tag{1.8}$$

The above computation suggests to define a suitable functional $\mathcal{F}^0$ in the corresponding class of 1-dimensional Cartesian currents so that $\mathcal{F}^0(G_u) = \mathcal{F}(c_u)$ for currents G_u carried by the graph of smooth functions, and in our example

$$\mathcal{F}^0(T) = \lambda_1 \cdot \mathbf{M}(T) + \lambda_2 \cdot 2 \cdot \frac{\pi}{2} \cdot g^\infty \,, \qquad \mathbf{M}(T) = \mathcal{L}(c) = 2(1+\pi)\,. \tag{1.9}$$

This will be shown in Examples 6.2 and 7.7 below.

The energy functional. In the sequel we shall consider the curvature functional (1.2) where $g(k) := k$ and for simplicity $\lambda_1 = \lambda_2 = 1$. Therefore, the recession function (1.3) is $g^\infty = 1$. For C^2-functions $u : I \to \mathbb{R}^N$ we thus let

$$\mathcal{E}(u) := \mathcal{F}(c_u) = \int_I |\dot{c}_u(t)|\,(1 + k_u(t))\,dt\,. \tag{1.10}$$

Therefore, in codimension $N = 1$ our functional becomes

$$\mathcal{E}(u) = \int_I \sqrt{1+\dot{u}^2}\,dt + \int_I \frac{|\ddot{u}(t)|}{1+\dot{u}(t)^2}\,dt\,.$$

In higher codimension $N \geq 2$ we obtain:

$$\mathcal{E}(u) = \int_I \sqrt{1+|\dot{u}|^2}\,dt + \int_I \frac{\big(|\ddot{u}|^2(1+|\dot{u}|^2) - (\dot{u} \bullet \ddot{u})^2\big)^{1/2}}{1+|\dot{u}|^2}\,dt\,.$$

The relaxed energy. As we shall see below as a byproduct of Proposition 7.2, the following lower semicontinuity property holds:

Proposition 1.6 *If a sequence $\{u_h\} \subset C^2(I,\mathbb{R}^N)$ converges in L^1 to some function $u \in C^2(I,\mathbb{R}^N)$, then $\mathcal{E}(u) \leq \liminf_h \mathcal{E}(u_h)$.*

The functional $u \mapsto \mathcal{E}(u)$ is well defined on Sobolev functions $u \in W^{2,1}(I,\mathbb{R}^N)$, where this time $\dot{u}$ and $\ddot{u}$ denote the approximate first and second derivatives. We wish to extend it to the wider class of functions in $L^1(I,\mathbb{R}^N)$ in order to obtain a lower semicontinuous functional $u \mapsto \overline{\mathcal{E}}(u)$ that agrees with $\mathcal{E}(u)$ for C^2 and $W^{2,1}$ functions.

For this reason, we are interested in studying the *relaxed energy* w.r.t. the L^1-convergence:

$$\overline{\mathcal{E}}(u) := \inf\Big\{\liminf_{h\to\infty} \mathcal{E}(u_h) \mid \{u_h\} \subset C^2(I,\mathbb{R}^N)\,,\ u_h \to u \text{ in } L^1(I,\mathbb{R}^N)\Big\}\,. \tag{1.11}$$

2 1d Cartesian currents and BV-functions

If $u : I \to \mathbb{R}^N$ has finite relaxed energy (1.11), then there exists a sequence $\{u_h\} \subset C^2(I,\mathbb{R}^N)$ of smooth functions such that $u_h \to u$ in $L^1(I,\mathbb{R}^N)$ and $\sup_h \|\dot{u}_h\|_{L^1(I,\mathbb{R}^N)} < \infty$. Therefore, u is a function of *bounded variation*. Moreover, possibly passing to a subsequence the graphs of the functions u_h weakly converge (in the sense of currents) to a *Cartesian current* $T \in \mathrm{cart}(\mathring{I} \times \mathbb{R}^N)$. For this reason, in this section we collect some notation and basic properties, and we refer to [2] and [13] for further results concerning functions of bounded variation and Geometric Measure Theory, respectively.

BV-functions. For any $f : I \to \mathbb{R}^N$ and $t \in I$, we set

$$f(t_+) := \lim_{s\to t^+} f(s)\,, \qquad f(t_-) := \lim_{s\to t^-} f(s)$$

whenever the above one-sided limits exist. It will sometimes be typographically convenient to denote by $f_\pm$ the functions $t \mapsto f(t_\pm)$.

A function $u : I \to \mathbb{R}^N$ is in $BV(I, \mathbb{R}^N)$ if all its components u^j are functions in $L^1(I)$ whose distributional derivative Du^j is a finite measure. If $u \in BV(I, \mathbb{R}^N)$, the functions $u_\pm$ are defined everywhere and each of them may be used as a *precise representative* of u. For every $t \in I$ we have

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{t-\varepsilon}^{t} |u(s) - u_-(t)|\, ds = 0\,, \quad \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{t}^{t+\varepsilon} |u(s) - u_+(t)|\, ds = 0\,.$$

The *Jump set* of u is the at most countable set J_u of discontinuity points of u. If $t \in J_u$, we denote the Jump of u at t by $[u(t)] := u(t_+) - u(t_-)$. The distributional derivative $Du = (Du^1, \dots, Du^N)$ is a vector-valued measure that splits into three mutually singular terms, the "gradient" part, which is absolutely continuous with respect to the Lebesgue measure, the "jump" part, which is atomic, and the "Cantor-type" part. More precisely, one has $Du = D^a u + D^s u$, where $D^a u = \dot{u}\, \mathcal{L}^1$, the vector $\dot{u} = (\dot{u}^1, \dots, \dot{u}^N)$ being the *approximate gradient* of u. The singular part splits as $D^s u = D^J u + D^C u$, where

$$D^J u = [u]\, \mathcal{H}^0 \llcorner J_u\,, \quad D^C u = D^s u \llcorner (I \setminus J_u)\,.$$

Also, a sequence $\{u_h\}$ is said to converge to u *weakly in the BV-sense*, $u_h \rightharpoonup u$, if $u_h \to u$ strongly in L^1 and $Du_h \rightharpoonup Du$ weakly in the sense of (vector-valued) measures.

Remark 2.1 If $u \in BV(I, \mathbb{R}^N)$, then also $c_u \in BV(I, \mathbb{R}^{N+1})$, and one has

$$c_u(t_\pm) = (t, u(t_\pm))\,, \quad \dot{c}_u = (1, \dot{u})\,, \quad D^C c_u = (0, D^C u)\,, \quad D^J c_u = (0, D^J u)\,.$$

As a consequence, one infers that

$$|Dc_u|(I) = \int_I |\dot{c}_u|\, dt + |D^C u|(I) + |D^J u|(I)$$

where, we recall,

$$\int_I |\dot{c}_u|\, dt = \int_I \sqrt{1 + |\dot{u}|^2}\, dt\,, \quad |D^J u|(I) = \sum_{t \in J_u} |[u(t)]| = \sum_{t \in J_u} |[c_u(t)]|\,.$$

Graph currents. The sequence $\{u_h\}$ in Example 1.5 converges in L^1 (and weakly in the BV-sense) to a jump function, whose graph has a "hole". However, the graphs c_{u_h} converge to the curve (1.8) which "fills the hole" in the limit graph. This fact is described by the weak convergence as currents.

Following [13], in fact, there is a canonical way to associate to any function $u \in BV(I, \mathbb{R}^N)$ an integer multiplicity (say i.m.) rectifiable 1-current T_u that fills the holes of the "graph" of u in an optimal way. The current T_u is decomposed into the *absolutely continuous*, *Cantor*, and *Jump* parts

$$T_u := T_u^a + T_u^C + T_u^J\,. \tag{2.1}$$

The component T_u^a agrees with the integration on the "rectifiable graph" of u, the component T_u^C depends on the Cantor part of the derivative, and finally the component T_u^J lives on the jump set of u and is given by the integration along the union of oriented line segments connecting the points $c_u(t_\pm)$.

We are forced to give some details. Denoting by U the open set

$$U := \mathring{I} \times \mathbb{R}^N\,, \qquad \text{so that} \quad \overline{U} := I \times \mathbb{R}^N$$

any form $\omega \in \mathcal{D}^1(U)$ splits as $\omega = \omega^{(0)} + \omega^{(1)}$ according to the number of vertical differentials, i.e.

$$\omega^{(0)} = \phi(t, z)\, dt \qquad \text{and} \qquad \omega^{(1)} = \sum_{j=1}^{N} \phi^j(t, z)\, dz^j \tag{2.2}$$

for some $\phi, \phi^j \in C_c^\infty(U)$. According to Remark 2.1, we define

$$\langle T_u^a, \omega^{(0)} \rangle := \int_I \phi(c_u(t))\, dt\,, \quad \langle T_u^C, \omega^{(0)} \rangle := 0\,, \quad \langle T_u^J, \omega^{(0)} \rangle := 0\,. \tag{2.3}$$

Moreover, taking e.g. $c_{u+}(t) = c_u(t_+)$ as a precise representative of c_u, we set

$$\begin{aligned} \langle T_u^a, \omega^{(1)} \rangle &:= \sum_{j=1}^N \int_I \phi^j(c_u(t))\, \dot{u}^j(t)\, dt \\ \langle T_u^C, \omega^{(1)} \rangle &:= \sum_{j=1}^N \langle D^C u^j, \phi^j \circ c_{u+} \rangle \\ \langle T_u^J, \omega^{(1)} \rangle &:= \sum_{j=1}^N \sum_{t \in J_u} \int_{\gamma_t} \phi^j(t,z)\, dz^j \,. \end{aligned} \tag{2.4}$$

In the third formula, for each $t \in J_u$ we have denoted by $\gamma_t = \gamma_t(u)$ the oriented line segment in $\{t\} \times \mathbb{R}^N$ with end points $c_u(t_\pm)$, so that

$$\partial [\![\gamma_t]\!] = \delta_{c_u(t_+)} - \delta_{c_u(t_-)} \quad \text{and} \quad \mathcal{H}^1(\gamma_t) = |c_u(t_+) - c_u(t_-)| = |[u(t)]| \,,$$

where δ_P denotes the *unit Dirac mass* at the point P. We also remark that the precise representative in the second formula is only formal, since $c_{u+} = c_{u-} = c_u$ on the support of $D^C c_u$.

It turns out that T_u is an integer multiplicity (say i.m.) rectifiable 1-current in $\mathcal{R}_1(U)$, satisfying the *null-boundary condition* $\partial T_u = 0$, i.e.

$$\langle \partial T_u, f \rangle := \langle T_u, df \rangle = 0 \qquad \forall\, f \in C_c^\infty(U) \,. \tag{2.5}$$

In fact, according to Remark 2.1 we have

$$\langle T_u^a + T_u^C, df \rangle = \int_I \nabla f(c_u) \bullet \dot{c}_u\, dt + \int_I \nabla f(c_{u+}) \bullet dD^C c_u$$

whereas

$$\langle T_u^J, df \rangle = \sum_{t \in J_u} \int_{\gamma_t} df = \sum_{t \in J_u} \big(f(c_u(t_+)) - f(c_u(t_-)) \big) \,.$$

Therefore, using that $\int_I D(f \circ c_u) = 0$, as the function $f \circ c_u \in BV(I)$ has support contained in $\mathring{I}$, and recalling that by the chain rule formula

$$D(f \circ c_u) = \nabla f(c_u) \bullet \dot{c}_u\, \mathcal{L}^1 + \nabla f(c_{u+}) \bullet D^C c_u + \big(f(c_{u+}) - f(c_{u-}) \big)\, \mathcal{H}^0 \llcorner J_u$$

one obtains the null-boundary condition (2.5).

Finally, the *mass* of T_u is finite and splits as $\mathbf{M}(T_u) = \mathbf{M}(T_u^a) + \mathbf{M}(T_u^C) + \mathbf{M}(T_u^J)$, where

$$\mathbf{M}(T_u^a) = \int_I |\dot{c}_u|\, dt \,, \quad \mathbf{M}(T_u^C) = |D^C u|(I) \,, \quad \mathbf{M}(T_u^J) = |D^J u|(I) = \sum_{t \in J_u} |[u(t)]| \,,$$

so that clearly $|Du|(I) \le \mathbf{M}(T_u) < \infty$ and actually $\mathbf{M}(T_u) = |Dc_u|(I)$.

Remark 2.2 By the area formula one obtains that $T_u^a = G_u$, where the graph-current G_u is given by integration of 1-forms ω over the *rectifiable graph* $\mathcal{G}_u$.

More precisely, following [13] we denote $\mathcal{G}_u := \{(t, u(t)) \mid t \in R_u\}$, where R_u is the set of Lebesgue points of both u and $\dot{u}$ and $u(t)$ is the Lebesgue value of u at t. The set $R_u \subset I$ has full $\mathcal{L}^1$-measure, $|R_u| = |I|$. Also, $\mathcal{G}_u$ is a countably 1-rectifiable set, oriented a.e. by $\tau_u(t) := \dot{c}_u(t)/|\dot{c}_u(t)|$, which in the smooth case is the tantrix of the curve c_u. We thus have

$$\langle G_u, \omega \rangle := \int_{\mathcal{G}_u} \langle \omega, \tau_u \rangle\, d\mathcal{H}^1 \,, \qquad \omega \in \mathcal{D}^1(U)$$

and hence by the area formula one obtains that $G_u = c_{u\#} [\![I]\!]$, i.e., $\langle G_u, \omega \rangle = \int_I c_u^\# \omega$ for every $\omega \in \mathcal{D}^1(U)$, where the pull-back is computed in terms of the approximate gradient $\dot{u}$, so that according to the notation from (2.2) one has

$$c_u^\# \omega^{(0)} = \phi(c_u(t))\, dt \qquad \text{and} \qquad c_u^\# \omega^{(1)} = \sum_{j=1}^N \phi^j(c_u(t))\, \dot{u}^j\, dt \,.$$

In particular, the mass of G_u agrees with the *length of the rectifiable graph* of u, i.e. $\mathbf{M}(G_u) = \mathcal{H}^1(\mathcal{G}_u)$.

If $u \in BV(I, \mathbb{R}^N)$ is continuous, one has $T_u^J = 0$. Moreover, if u is a Sobolev function in $W^{1,1}(I, \mathbb{R}^N)$ one has $T_u^C = T_u^J = 0$ and hence $T_u = G_u$. Finally, in codimension $N = 1$, the current T_u agrees with the boundary of the current carried by the subgraph of u, compare [13] and Remark 2.5 below.

Using a convolution argument, one also obtains for every $u \in BV(I, \mathbb{R}^N)$:

Proposition 2.3 *There exists a sequence of smooth functions $\{u_h\} \subset C^\infty(I, \mathbb{R}^N)$ such that $u_h \rightharpoonup u$ weakly in the BV-sense, $G_{u_h} \rightharpoonup T_u$ weakly in $\mathcal{D}_1(U)$,[1] and $\mathbf{M}(G_{u_h}) \to \mathbf{M}(T_u)$ as $h \to \infty$.*

Cartesian currents. We now consider the class of 1-currents T in $\mathcal{D}_1(U)$ that are weak limit points of sequences $\{G_{u_h}\}$ of graphs of smooth functions $u_h : I \to \mathbb{R}^N$ with equibounded $W^{1,1}$-norms.

Any such current T belongs to the class $\mathrm{cart}(\mathring{I} \times \mathbb{R}^N)$, see [13, 14]. By Federer-Fleming's closure theorem, the current T is an i.m. rectifiable current in $\mathcal{R}_1(U)$, with finite mass, $\mathbf{M}(T) < \infty$, and it satisfies the null boundary condition $\partial T = 0$, compare (2.5). Moreover, denoting by $u = u_T \in BV(I, \mathbb{R}^N)$ the weak BV-limit of the corresponding subsequence $\{u_h\}$, similarly to (2.1) the current T can be decomposed into four terms:

$$T = T_u^a + T_u^C + T^J + T^s \,.$$

The first two terms have already been defined and only depend on the underlying function $u = u_T$, whereas both the *Jump* and *singular* components T^J and T^s are "vertical", i.e., according to (2.2) and (2.3), one has

$$\langle T, \omega^{(0)} \rangle = \langle T_u^a, \omega^{(0)} \rangle = \langle G_u, \omega^{(0)} \rangle = \int_I \phi(c_u(t))\, dt \,.$$

The extra singular term T^s is necessary, as the ensuing well-known Example 2.4 shows.

The Jump component T^J is defined as in the third line of (2.4), where this time $\gamma_t = \gamma_t(T)$ is an oriented rectifiable arc in $\{t\} \times \mathbb{R}^N$ with length $\mathcal{L}(\gamma_t(T))$ and end points given by the one sided limits $c_u(t_\pm)$, so that we again have $\partial [\![\gamma_t(T)]\!] = \delta_{c_u(t_+)} - \delta_{c_u(t_-)}$.

Arguing as for (2.5), one then obtains the null-boundary condition

$$\partial (T_u^a + T_u^C + T^J) = 0 \,.$$

As a consequence, the fourth term T^s is a (vertical) i.m. rectifiable current such that $\partial T^s = 0$. Moreover, the above structure property implies a *decomposition in mass*, i.e.

$$\mathbf{M}(T) = \mathbf{M}(T_u^a) + \mathbf{M}(T_u^C) + \mathbf{M}(T^J) + \mathbf{M}(T^s)$$

where, we recall,

$$\mathbf{M}(T_u^a) = \int_I |\dot{c}_u|\, dt \,, \quad \mathbf{M}(T_u^C) = |D^C u|(I) \,, \quad \mathbf{M}(T^J) = \sum_{t \in J_u} \mathbf{M}([\![\gamma_t(T)]\!]) \,.$$

In particular one again has $|Dc_u|(I) \le \mathbf{M}(T) < \infty$. Moreover, in the above decomposition we may and do assume that for every $t \in J_u$ the current $[\![\gamma_t(T)]\!]$ is a-cyclical, or indecomposable. Therefore, a density property similar to Proposition 2.3 holds true for each current $T \in \mathrm{cart}(\mathring{I} \times \mathbb{R}^N)$ such that $T^s = 0$.

Example 2.4 Let $I = [-\pi, 3\pi]$ and consider the sequence of functions from I into $\mathbb{S}^1 \subset \mathbb{R}^2$

$$u_h(t) := \begin{cases} (\cos ht, \sin ht) & \text{if } t \in [0, 2\pi/h] \\ (1, 0) & \text{elsewhere} \end{cases}$$

so that we have

$$\int_I |\dot{u}_h(t)|\, dt = 2\pi \,, \qquad \mathcal{L}(c_{u_h}) = \int_I |\dot{c}_{u_h}(t)|\, dt = 2\pi \left(2 - \frac{1}{h} + \frac{\sqrt{1+h^2}}{h} \right) .$$

[1] The weak convergence $T_h \rightharpoonup T$ as currents in $\mathcal{D}_1(U)$ is defined by duality as

$$\langle T_h, \omega \rangle \to \langle T, \omega \rangle \qquad \forall\, \omega \in \mathcal{D}^1(U) \,.$$

Moreover $u_h \rightharpoonup u_\infty$ weakly in the BV-sense, where $u_\infty(t) \equiv (1,0)$, but the degree

$$\deg\, u_h = 1 \quad \forall\, h\,, \qquad \deg\, u_\infty = 0$$

and

$$\begin{aligned} \int_I |\dot u_\infty(t)|\, dt = 0 \quad &< \quad 2\pi = \lim_{h\to\infty} \int_I |\dot u_h(t)|\, dt\,, \\ \mathcal{L}(c_{u_\infty}) = 4\pi \quad &< \quad 6\pi = \lim_{h\to\infty} \mathcal{L}(c_{u_h})\,, \end{aligned}$$

whence the weak BV convergence fails to preserve the geometry and to read the energy concentration.

On the other hand, the graphs G_{u_h} weakly converge to the Cartesian current $T = G_{u_\infty} + T^s$, where $G_{u_\infty} = [\![-\pi, 3\pi]\!] \times \delta_{(1,0)}$ and the singular term $T^s = \delta_0 \times [\![\mathbb{S}^1]\!]$ is a vertical 1-cycle. The total variation and degree can be defined on T in such a way that

$$\text{total variation } T = 2\pi\,, \qquad \deg\, T = 1\,, \qquad \mathbf{M}(T) = 6\pi\,.$$

Therefore, one recovers concentration and loss of geometry from the limit of graphs.

Remark 2.5 For any $u \in BV(I, \mathbb{R}^N)$, the current T_u belongs to the class $\mathrm{cart}(\mathring{I} \times \mathbb{R}^N)$. Moreover, the mass of T_u is lower than (or equal to) the mass of any Cartesian current T with underlying function $u_T = u$. Finally, in codimension $N = 1$ one has $\mathrm{cart}(\mathring{I} \times \mathbb{R}) = \{T_u \mid u \in BV(I, \mathbb{R})\}$, and T_u actually agrees (up to the sign) with the boundary of the 2-current naturally associated to the subgraph of u in $\mathring{I} \times \mathbb{R}$.

3 Gauss graphs of smooth Cartesian curves

Our approach to the relaxation problem makes use of some features from the theory of *Gauss graphs* re-written in the context of Cartesian curves. Some ideas are therefore taken from [3], see also [9].

We first recall that for a smooth rectifiable 1-1 curve with support $\mathcal{C} \subset \mathbb{R}^{N+1}_x$, the Gauss graph can be viewed as the graph in $\mathbb{R}^{N+1}_x \times \mathbb{S}^N$ of the unit tangent vector $\mathfrak{t}_{\mathcal{C}}(x) \in \mathbb{S}^N \subset \mathbb{R}^{N+1}_y$ at $x \in \mathcal{C}$, i.e.,

$$\mathcal{M}_{\mathcal{C}} := \{(x, \mathfrak{t}_{\mathcal{C}}(x)) \mid x \in \mathcal{C}\}\,,$$

and an i.m. rectifiable current is naturally associated to $\mathcal{M}_{\mathcal{C}}$. In the sequel we shall then denote by $(e_0, e_1, \dots, e_N)$ and $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N)$ the canonical basis in $\mathbb{R}^{N+1}_x$ and $\mathbb{R}^{N+1}_y$, respectively.

Assume that c_u is a smooth Cartesian curve defined as the graph of a C^2-function $u : I \to \mathbb{R}^N$, so that $\mathcal{C} = \{(t, u(t)) \mid t \in I\}$ and $\mathfrak{t}_{\mathcal{C}}(x) = \tau_u(t)$ if $x = c_u(t)$. We thus introduce the map $\Phi_u : I \to \overline{U} \times \mathbb{S}^N$

$$\Phi_u(t) := (c_u(t), \tau_u(t))\,, \qquad t \in I$$

where

$$c_u(t) := (t, u(t)) = t\, e_0 + \sum_{j=1}^N u^j(t)\, e_j\,, \qquad \tau_u(t) := \frac{1}{\sqrt{1 + |\dot u(t)|^2}} \Big(\varepsilon_0 + \sum_{j=1}^N \dot u^j(t)\, \varepsilon_j \Big)\,. \tag{3.1}$$

Therefore, the Gauss graph associated to c_u is identified by

$$\mathcal{GG}_u := \{\Phi_u(t) \mid t \in I\}\,, \qquad \mathcal{GG}_u \subset \overline{U} \times \mathbb{S}^N.$$

Moreover, the set $\mathcal{GG}_u$ is the support of the curve Φ_u, it is 1-rectifiable and naturally oriented by the unit vector

$$\xi_u(t) := \frac{\dot\Phi_u(t)}{|\dot\Phi_u(t)|}\,. \tag{3.2}$$

Remark 3.1 Take a sequence $\{u_h\}$ of smooth functions, and assume that their graphs c_{u_h} converge weakly as currents to a rectifiable (not necessarily Cartesian) curve c; for each h the tantrix τ_{u_h} has positive first component $\tau^0_{u_h}$, so also for the limit curve c the tantrix has non-negative first component, i.e. it takes values into the half-sphere

$$\mathbb{S}^N_+ := \{y = (y_0, y_1, \dots, y_N) \in \mathbb{R}^{N+1}_y \,:\, |y| = 1\,,\ y_0 \ge 0\}\,.$$

Curvature. Denoting as before by $\bullet$ the scalar product in $\mathbb{R}^N$, we compute for each $t \in I$

$$\dot\Phi_u(t) = e_0 + \sum_{j=1}^N \dot u^j\, e_j + \frac{-(\dot u \bullet \ddot u)}{(1+|\dot u|^2)^{3/2}}\,\varepsilon_0 + \sum_{j=1}^N \frac{\ddot u^j(1+|\dot u|^2) - \dot u^j\,(\dot u \bullet \ddot u)}{(1+|\dot u|^2)^{3/2}}\,\varepsilon_j\,. \tag{3.3}$$

If $N=1$ one has $\dot\Phi_u(t) = e_0 + \dot u\, e_1 - \widetilde{k}_u\, \dot u\, \varepsilon_0 + \widetilde{k}_u\, \varepsilon_1$ where $\widetilde{k}_u$ is the signed curvature of c_u

$$\widetilde{k}_u(t) := \frac{\ddot u(t)}{(1+\dot u(t)^2)^{3/2}}\,, \tag{3.4}$$

so that $|\widetilde{k}_u| = k_u$, compare (1.5), and hence

$$|\dot\Phi_u(t)| = |\dot c_u(t)|\,\sqrt{1+k_u(t)^2}\,, \qquad |\dot c_u(t)| = \sqrt{1+|\dot u(t)|^2}\,, \qquad |\dot c_u| k_u = |\dot\tau_u|\,. \tag{3.5}$$

The above formulas (3.5) continue to hold in higher codimension $N \geq 2$, where this time the curvature k_u of the Cartesian curve c_u is given by (1.6).

Notice that $2^{-1/2}(1+k_u) \leq \sqrt{1+k_u^2} \leq (1+k_u)$. This gives that

$$2^{-1/2}|\dot c_u|\,(1+k_u) \leq |\dot\Phi_u| \leq |\dot c_u|\,(1+k_u)\,. \tag{3.6}$$

In particular $\dot\Phi_u$ is summable in I if and only if both $|\dot u|$ and $|\dot c_u|\,k_u$ are summable. Recalling (1.10), we thus have for every $u \in C^2(\mathring{I}, \mathbb{R}^N)$

$$|\dot\Phi_u| \in L^1(I) \qquad \Longleftrightarrow \qquad \mathcal{E}(u) < \infty\,.$$

Example 3.2 The length of the Gauss graph is some sort of an average between the length and the total curvature. Let e.g. $N=2$ and $u_h : [0, 2\pi/h] \to \mathbb{R}^2$ be given by $u_h(t) = R(\cos(ht), \sin(ht))$, so that the curve c_{u_h} parameterizes one turn of the helix of radius $R>0$ and step $2\pi/h$. The tantrix τ_{u_h} describes a circle in $\mathbb{S}^2_+$ of radius $R(h) = Rh/\sqrt{1+R^2h^2}$ that converges to one as $h \to \infty$. Moreover, the limit curve c_R is a circle of radius R and total curvature 2π. In fact, we have $\dot u_h \bullet \ddot u_h = 0$ and

$$|\dot\Phi_{u_h}(t)| = \sqrt{1+R^2h^2}\,\sqrt{1+\frac{R^2h^4}{(1+R^2h^2)^2}} = \sqrt{\frac{1+2R^2h^2+(R^2+R^4)h^4}{1+R^2h^2}}$$

for every $t \in [0, 2\pi/h]$, so that the limit

$$\lim_{h\to\infty} \int_0^{2\pi/h} |\dot\Phi_{u_h}(t)|\,dt = 2\pi\sqrt{1+R^2}$$

is equal to the length of the Gauss graph of the curve c_R. Since moreover $|\dot c_{u_h}|\,k_{u_h} = Rh^2/\sqrt{1+R^2h^2}$, then the limit of the total curvature functional gives

$$\lim_{h\to\infty} \mathrm{TC}(c_{u_h}) = \lim_{h\to\infty} \int_0^{2\pi/h} |\dot c_{u_h}|\,k_{u_h}\,dt = 2\pi$$

that is the total curvature of the limit curve c_R.

Cartesian Gauss graphs. If $u : \mathring{I} \to \mathbb{R}^N$ is a C^2-function, recalling that $U = \mathring{I} \times \mathbb{R}^N$, we may associate to the Gauss graph of the Cartesian curve c_u a one-dimensional current $GG_u \in \mathcal{D}_1(U \times \mathbb{S}^N)$ defined by integrating 1-forms on the set $\mathcal{GG}_u$, which is naturally oriented by the unit vector ξ_u defined in (3.2). Then we have

$$\langle GG_u, \omega\rangle := \int_{\mathcal{GG}_u} \omega = \int_{\mathcal{GG}_u} \langle \omega, \xi_u\rangle\, d\mathcal{H}^1\,, \qquad \omega \in \mathcal{D}^1(U \times \mathbb{S}^N)$$

and by the Remark 3.1 it turns out that $\operatorname{spt} GG_u \subset \overline{U} \times \mathbb{S}^N_+$.

Moreover, if $|\dot\Phi_u| \in L^1(I)$, by means of the area formula we compute

$$\langle GG_u, \omega\rangle = \int_{\mathcal{G}\mathcal{G}_u} \langle \omega, \dot\Phi_u\rangle\, |\dot\Phi_u|^{-1}\, d\mathcal{H}^1 = \int_I \langle \omega(\Phi_u(t)), \dot\Phi_u(t)\rangle\, dt\,.$$

Therefore, it turns out that $GG_u = [\![\, \mathcal{G}\mathcal{G}_u, 1, \xi_u \,]\!]$ is an i.m. rectifiable current in $\mathcal{R}_1(U \times \mathbb{S}^N)$ with null interior boundary, $\partial GG_u = 0$, and finite mass:

$$\mathbf{M}(GG_u) = \mathcal{H}^1(\mathcal{G}\mathcal{G}_u) = \int_I |\dot\Phi_u(t)|\, dt < \infty\,, \qquad |\dot\Phi_u| = |\dot c_u|\, \sqrt{1+k_u^2}\,.$$

Remark 3.3 By (3.6), recalling the definition (1.10), for smooth functions $u : \mathring{I} \to \mathbb{R}^N$ we have

$$2^{-1/2}\mathcal{E}(u) \le \mathbf{M}(GG_u) \le \mathcal{E}(u)$$

and hence u *has finite energy if and only if the corresponding current* GG_u *has finite mass.*

4 Functions with finite relaxed energy

In this section we focus on the class of functions with finite relaxed energy (1.11). We shall thus denote:

$$\mathcal{E}(I, \mathbb{R}^N) := \{u \in L^1(I, \mathbb{R}^N) \mid \overline{\mathcal{E}}(u) < \infty\}\,. \tag{4.1}$$

We first recall from [8] how the relaxation problem has been solved in the codimension one case. We then outline the main properties of functions u in $\mathcal{E}(I, \mathbb{R}^N)$. We shall in particular prove that the Gauss map τ_u is of bounded variation, Theorem 4.7.

The case $N = 1$. In codimension $N = 1$, the relaxed functional (1.11) has been studied in [8], where the authors introduce the following notation:

Definition 4.1 *The class* $X(I)$ *is given by the real valued functions* u *in* $BV(I) = BV(I, \mathbb{R})$ *satisfying the following properties:*

(*a*) *the function* $t \mapsto \arctan(\dot u(t))$ *belongs to* $BV(I)$ *;*

(*b*) *the positive and negative parts* $(D^C u)^\pm$ *of the Cantor-type component are respectively concentrated on the sets*

$$Z^\pm[\dot u] := \Big\{ t \in I \,:\, \lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} \dot u(s)\, ds = \pm\infty \Big\}\,.$$

Remark 4.2 The class $X(I)$ trivially contains the Sobolev space $W^{2,1}(I)$. Therefore, a function $u \in SBV(I)$ with a finite Jump set, $\mathcal{H}^0(J_u) < \infty$, belongs to $X(I)$ if it is a pure Jump-function, i.e. $\dot u = 0$ a.e., or more generally if $\arctan(\dot u) \in BV(I)$. However, in [8] it is shown the existence of functions u in $X(I)$ with non-trivial Cantor component, $D^C u \neq 0$.

In [8] it is proved that $\mathcal{E}(I, \mathbb{R}) \subset X(I)$. Moreover, compare Corollary 7.9 below, the explicit representation of the relaxed functional is given for $u \in X(I)$ by

$$\overline{\mathcal{E}}(u) = |Dc_u|(I) + \mathcal{G}(u)\,, \tag{4.2}$$

where the second term is given by

$$\mathcal{G}(u) := |D\arctan(\dot u)|(I \setminus J_u) + \sum_{t\in J_u} \Phi(\nu_u(t), \dot u(t_-), \dot u(t_+)) \tag{4.3}$$

(here and a few lines below we keep the original notation Φ from [8], which clearly has nothing to do with our Φ and Φ_u). For Sobolev functions $u \in W^{2,1}(I, \mathbb{R})$, the functional $\mathcal{G}(u)$ agrees with the total curvature functional

$$\int_{c_u} \mathbf{k}_{c_u}\, d\mathcal{H}^1 = \int_I \frac{|\ddot u(t)|}{1+\dot u(t)^2}\, dt\,, \qquad c_u(t) = (t, u(t))\,.$$

In general, the second addendum in the definition of $\mathcal{G}(u)$ depends on the sign $\nu_u(t)$ of the jump $[u(t)] := u(t_+) - u(t_-)$ and on the left and right limits of $\dot{u}$ at the Jump point of u

$$\dot{u}(t_-) := \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} \dot{u}(s)\,ds\,, \qquad \dot{u}(t_+) := \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_{t}^{t+\varepsilon} \dot{u}(s)\,ds\,.$$

Such limits always exist in $\overline{\mathbb{R}}$ at all points $t \in I$ provided that $u \in X(I)$. In fact, compare [8], we have $\dot{u} = \tan v$, where v is a good representative of the BV-function $\arctan \dot{u}$. Finally, the general definition of $\Phi(\nu_u(t), \dot{u}(t_-), \dot{u}(t_+))$ from [8], for the case of the curvature functional as in our context, agrees with the sum of the two turning angles

$$\arccos\Big(\frac{(1, \dot{u}(t_\pm)) \bullet (0, \nu_u(t))}{|(1, \dot{u}(t_\pm))|}\Big)\,, \quad t \in J_u$$

provided that $\dot{u}(t_\pm) \in \mathbb{R}$, and with the obvious extensions in the case $|\dot{u}(t_\pm)| = \infty$, yielding to the corresponding terms 0 or π according to the sign of the product $\dot{u}(t_\pm)\,\nu_u(t)$.

Remark 4.3 If $u \in X(I)$, so that $v := \arctan \dot{u} \in BV(I)$, setting $\tau_u := (1, \dot{u})/\sqrt{1+\dot{u}^2}$, then $\tau_u = (\cos v, \sin v)$, whence $\tau_u \in BV(I, \mathbb{R}^2)$, and by the chain-rule formula one has $|D\tau_u|(A) = |Dv|(A)$ for each Borel set $A \subset I$.

Weak limit currents. Let now $N \geq 1$ and $u \in \mathcal{E}(I, \mathbb{R}^N)$. By Remark 3.3, the class $\mathcal{E}(I, \mathbb{R}^N)$ is characterized by the L^1-functions u for which we can find a smooth sequence $\{u_h\} \subset C^2(I, \mathbb{R}^N)$ such that $u_h \to u$ in L^1 and $\sup_h \mathbf{M}(GG_{u_h}) < \infty$. Since $u_h \rightharpoonup u$ weakly in BV, we deduce that $u \in BV$, and hence $c_u \in BV$.

Recalling that $U := \mathring{I} \times \mathbb{R}^N$, the object of our analysis is the subclass of 1-currents in $\mathcal{D}_1(U \times \mathbb{S}^N)$ that are *weak limits* (in the sense of currents) *of sequences* $\{GG_{u_h}\}$ *of Gauss graphs of smooth Cartesian curves with equibounded masses.* We thus introduce the class $\mathrm{Gcart}(U \times \mathbb{S}^N)$, defined by

$$\begin{aligned}\mathrm{Gcart}(U \times \mathbb{S}^N) := \{\Sigma \in \mathcal{D}_1(U \times \mathbb{S}^N) \mid \exists \{u_h\} \subset C^2(I, \mathbb{R}^N) \text{ such that} \\ GG_{u_h} \rightharpoonup \Sigma \text{ in } \mathcal{D}_1(U \times \mathbb{S}^N)\,,\ \sup_h \mathbf{M}(GG_{u_h}) < \infty\}\,.\end{aligned} \tag{4.4}$$

The structure properties of the class $\mathrm{Gcart}(U \times \mathbb{S}^N)$ we shall analyze here and in more detail in Sec. 6, are inherited (by weak convergence with equibounded masses) from the corresponding properties of the approximating currents GG_{u_h}. To this purpose, we shall denote by Π_x and Π_y the canonical projections of $\mathbb{R}^{N+1}_x \times \mathbb{R}^{N+1}_y$ onto the first and second factor.

Proposition 4.4 *Let* $\Sigma \in \mathrm{Gcart}(U \times \mathbb{S}^N)$ *and let* $\{u_h\} \subset C^2(I, \mathbb{R}^N)$ *be such that* $\sup_h \mathbf{M}(GG_{u_h}) < \infty$ *and* $GG_{u_h} \rightharpoonup \Sigma$ *weakly in* $\mathcal{D}_1(U \times \mathbb{S}^N)$. *Then we have:*

i) The current Σ *is* i.m. rectifiable *in* $\mathcal{R}_1(U \times \mathbb{S}^N)$, *with finite* mass

$$\mathbf{M}(\Sigma) \leq \liminf_{h\to\infty} \mathbf{M}(GG_{u_h}) < \infty$$

and it satisfies the null-boundary condition $\partial\Sigma = 0$.

ii) The sequence $\{u_h\}$ *weakly converges in the* BV*-sense to some function* $u \in BV(I, \mathbb{R}^N)$.

iii) The projection $T = T(\Sigma) := \Pi_{x\#}\Sigma$ *is a Cartesian current in* $\mathrm{cart}(\mathring{I} \times \mathbb{R}^N)$, *see Sec.* 2, *with underlying function* $u_T = u$.

Proof: Property i) follows from the classical Federer-Fleming's closure theorem, observing that the null-boundary condition $\partial GG_{u_h} = 0$ is preserved by the weak convergence. Moreover, we have $\Pi_{x\#}GG_{u_h} = G_{u_h}$, where $G_{u_h} \in \mathcal{R}_1(\mathring{I} \times \mathbb{R}^N)$ is the graph current associated to the Cartesian curve $c_{u_h}(t) = (t, u_h(t))$, and the following mass bound holds:

$$\mathbf{M}(G_{u_h}) = \int_I |\dot{c}_{u_h}|\,dt \leq \int_I |\dot{c}_{u_h}|\,\sqrt{1+k_{u_h}^2}\,dt = \mathbf{M}(GG_{u_h}) \tag{4.5}$$

where, we recall, $|\dot{c}_{u_h}| := \sqrt{1+|\dot{u}_h|^2}$. By the closure property of the class $\mathrm{cart}(\mathring{I}\times\mathbb{R}^N)$ it turns out that G_{u_h}, which we already know to converge to the current $\Pi_{x\#}\Sigma$, weakly converges in $\mathcal{D}_1(U)$ to some Cartesian current $T\in\mathrm{cart}(\mathring{I}\times\mathbb{R}^N)$, and $u_h \rightharpoonup u_T$ weakly in the BV-sense. Properties ii) and iii) readily follow. □

Good parameterizations. Motivated by the construction from Example 1.5, we now choose a suitable parameterization of currents in the class $\mathrm{Gcart}(U\times\mathbb{S}^N)$.

Proposition 4.5 *Let $\Sigma\in\mathrm{Gcart}(U\times\mathbb{S}^N)$ and $T(\Sigma):=\Pi_{x\#}\Sigma\in\mathrm{cart}(\mathring{I}\times\mathbb{R}^N)$. Let $\{u_h\}\subset C^2(I,\mathbb{R}^N)$ such that $GG_{u_h}\rightharpoonup\Sigma$ weakly in $\mathcal{D}_1(U\times\mathbb{S}^N)$ and $\sup_h\mathbf{M}(GG_{u_h})<\infty$. Then there exist a number $L\geq|I|$, and a Lipschitz function $c:I_L\to\overline{U}$, where $I_L:=[0,L]$, satisfying the following properties:*

i) the first component c^0 of c is a non-decreasing and surjective function $c^0:I_L\to I$;

ii) the image current $c_\#[\![I_L]\!]$ agrees with the Cartesian current $T(\Sigma)=\Pi_{x\#}\Sigma$;

iii) the gradient $s\mapsto\dot{c}(s)$ is a function with bounded variation in $BV(I_L,\mathbb{R}^{N+1})$, with $|\dot{c}|=1$ a.e. in I_L;

iv) setting $\Phi(s):=(c(s),\dot{c}(s))$, then

$$|D\Phi|(I_L)\leq\liminf_{h\to\infty}\int_I|\dot{\Phi}_{u_h}(t)|\,dt\quad\text{and}\quad|Dc|(I_L)+|D\dot{c}|(I_L)\leq\liminf_{h\to\infty}\mathcal{E}(u_h)\,.$$

Proof: Denoting $L_h:=\int_I|\dot{c}_{u_h}(t)|\,dt$, since $L_h=\mathbf{M}(G_{u_h})$ by (4.5) we have $|I|\leq\inf_h L_h\leq\sup_h L_h\leq K<\infty$, and possibly passing to a subsequence we may assume that $L_h\to L\in[|I|,K]$.

Recalling that $c_{u_h}(t)=(t,u_h(t))$ and that $I=[a,b]$, for every h denote by $\psi_h:I\to I_L$ the transition function

$$\psi_h(t):=\frac{L}{L_h}\int_a^t|\dot{c}_{u_h}(\lambda)|\,d\lambda$$

so that $\dot{\psi}_h(t)=(L/L_h)|\dot{c}_{u_h}(t)|\geq(L/L_h)$ for every $t\in I$. The inverse function

$$\varphi_h:I_L\to I \tag{4.6}$$

is a smooth diffeomorphism such that $0<\dot{\varphi}_h(s)\leq L_h/L$ for every $s\in I_L$. The corresponding smooth function $c_h(s):=c_{u_h}(\varphi_h(s)):I_L\to\overline{U}$ has constant velocity $|\dot{c}_h(s)|\equiv L_h/L$, with a strictly increasing and bijective first component $c_h^0=\varphi_h:I_L\to I$.

Define $\Phi_h(s):=\Phi_{u_h}(\varphi_h(s))=(c_h(s),\tau_{u_h}(\varphi_h(s)))$, so that $\Phi_h:I_L\to\overline{U}\times\mathbb{S}^N_+$. By the change of variable $t=\varphi_h(s)$ one infers that $\Phi_{h\#}[\![I_L]\!]=GG_{u_h}$ and $c_{h\#}[\![I_L]\!]=G_{u_h}$. By Ascoli's theorem, possibly passing to a subsequence we infer that the sequence $\{c_h\}$ uniformly converges in I_L to a Lipschitz function $c\in\mathrm{Lip}(I_L,\overline{U})$. In particular, the transition functions φ_h uniformly converge to the first component c^0 of c, and property i) holds. Using that $\sup_h\|\dot{\varphi}_h\|_\infty<\infty$, we also deduce that $\varphi_h\rightharpoonup c^0$ weakly-* in $W^{1,\infty}$. Since moreover $G_{u_h}\rightharpoonup T(\Sigma)$, and by the uniform convergence also $c_{h\#}[\![I_L]\!]\rightharpoonup c_\#[\![I_L]\!]$, we obtain property ii).

The function c being Lipschitz-continuous, by Rademacher's theorem it is differentiable a.e. in I_L. Its distributional derivative is given by $Dc=\dot{c}\,dt$, hence $\dot{c}\in L^\infty(I_L,\mathbb{R}^{N+1})$. Also, by the uniform convergence of the Lipschitz functions c_h to c and since $L_h/L\to1$, it turns out that $\|\dot{c}\|_\infty\leq1$.

Now, we have

$$\tau_{u_h}(\varphi_h(s))=\frac{\dot{c}_h(s)}{|\dot{c}_h(s)|}=\frac{L}{L_h}\,\dot{c}_h(s) \tag{4.7}$$

so that

$$\frac{L}{L_h}\,\ddot{c}_h(s)=D(\tau_{u_h}\circ\varphi_h)(s)=\dot{\tau}_{u_h}(\varphi_h(s))\,\dot{\varphi}_h(s)$$

for each $s\in I_L$, whence by (3.5)

$$\frac{L}{L_h}\int_{I_L}|\ddot{c}_h(s)|\,ds=\int_{I_L}|D(\tau_{u_h}\circ\varphi_h)(s)|\,ds=\int_I|\dot{\tau}_{u_h}(t)|\,dt\leq\mathcal{E}(u_h)$$

for every h. Since $\sup_h \mathcal{E}(u_h) < \infty$, see Remark 3.3, we infer that a subsequence of $\{\dot{c}_h\}$ weakly converges in the BV-sense to some function $v \in BV(I_L, \mathbb{R}^{N+1})$.

We claim that $v = \dot{c}$ a.e. in I_L, which entails that the whole sequence $\dot{c}_h$ converges to $\dot{c}$,

$$\dot{c}_h \rightharpoonup \dot{c} \quad \text{weakly in } BV(I_L, \mathbb{R}^{N+1})\,. \tag{4.8}$$

In fact, still denoting by c_h the subsequence, setting $V(t) := c(0) + \int_0^t v(s)\,ds$, and recalling that $c_h(t) = c_h(0) + \int_0^t \dot{c}_h(s)\,ds$, by the pointwise convergence $c_h(0) \to c(0)$ and the weak BV convergence $\dot{c}_h \rightharpoonup v$, which implies strong L^1-convergence, we have $c_h \to V$ in L^∞, hence $\dot{c}_h \rightharpoonup \dot{V} = v$. But we already know that $c_h \to c$ in L^∞, thus $v = \dot{c}$.

The weak BV-convergence of $\dot{c}_h$ to $\dot{c}$ implies the strong convergence in L^1, thus

$$L = \lim_{h\to\infty} L_h = \lim_{h\to\infty} \int_{I_L} |\dot{c}_h(s)|\,ds = \int_{I_L} |\dot{c}(s)|\,ds\,.$$

Using that $\|\dot{c}\|_\infty \le 1$, this yields that $|\dot{c}| = 1$ a.e. in I_L, and hence property iii) holds true.

Finally, by (4.7) the sequence $\{\tau_{u_h} \circ \varphi_h\}$ weakly converges in the BV-sense to $\dot{c}$, too. By the change of variables $t = \varphi_h(s)$ we check

$$\int_{I_L} |\dot{\Phi}_h(s)|\,ds = \int_I |\dot{\Phi}_{u_h}(t)|\,dt\,,$$
$$\int_{I_L} \big(|\dot{c}_h(s)| + |D(\tau_{u_h} \circ \varphi_h)(s)|\big)\,ds = \int_I \big(|\dot{c}_{u_h}(t)| + |\dot{\tau}_{u_h}(t)|\big)\,dt\,,$$

whence property iv) follows from lower semicontinuity, on account of (3.5). □

Remark 4.6 In order to prove the geometric property from Theorem 6.3 and the energy lower bound, see Proposition 8.2 below, we now modify the above argument to recover the weak limit current $\Sigma \in \mathrm{Gcart}(U \times \mathbb{S}^N)$. In fact, in Proposition 4.5 we are not claiming that the image current $(c, \dot{c})_\#[\![I_L]\!]$ agrees with Σ. For $N = 1$, this will be shown in Example 6.2 below, working on Example 1.5.

For this purpose, we have to consider a transition function depending on the whole derivative $|\dot{\Phi}_{u_h}|$. More precisely, denoting $\widetilde{L}_h := \int_I |\dot{\Phi}_{u_h}(t)|\,dt$, so that $\widetilde{L}_h = \mathcal{E}(u_h) \le \widetilde{K}$ for some real constant $\widetilde{K}$, possibly passing to a subsequence $\widetilde{L}_h \to \widetilde{L} \in [|I|, \widetilde{K}]$. Denoting by $\widetilde{\psi}_h : I \to I_{\widetilde{L}}$ the transition function

$$\widetilde{\psi}_h(t) := \frac{\widetilde{L}}{\widetilde{L}_h} \int_a^t |\dot{\Phi}_{u_h}(\lambda)|\,d\lambda\,,$$

the inverse function $\widetilde{\varphi}_h : I_{\widetilde{L}} \to I$ is again a smooth diffeomorphism. Consider the corresponding smooth function $\Psi_h(s) := \Phi_{u_h}(\widetilde{\varphi}_h(s)) = (c_{u_h}(\widetilde{\varphi}_h(s)), \tau_{u_h}(\widetilde{\varphi}_h(s)))$, so that $\Psi_h : I_{\widetilde{L}} \to \overline{U} \times \mathbb{S}^N$. The function Ψ_h has again constant velocity equal to $\widetilde{L}_h/\widetilde{L}$, and by the change of variable $t = \widetilde{\varphi}_h(s)$ one recovers that $\Psi_{h\#}[\![I_{\widetilde{L}}]\!] = GG_{u_h}$. This time, by Ascoli's theorem, possibly passing to a subsequence we have that $\{\Psi_h\}$ uniformly converges in $I_{\widetilde{L}}$ to a Lipschitz function $\Psi \in \mathrm{Lip}(I_{\widetilde{L}}, \overline{U} \times \mathbb{S}^N)$, so that $\Psi_{h\#}[\![I_{\widetilde{L}}]\!] \rightharpoonup \Psi_\#[\![I_{\widetilde{L}}]\!]$. Since we know that $GG_{u_h} \rightharpoonup \Sigma$, we obtain that the image current $\Psi_\#[\![I_{\widetilde{L}}]\!]$ agrees with Σ.

BV-property of the Gauss map. Let now $u \in \mathcal{E}(I, \mathbb{R}^N)$ and let $\Phi_u(t) := (c_u(t), \tau_u(t))$ be defined a.e. as in the smooth case, see (3.1), but in terms of the approximate gradient $\dot{u}$ of the BV-function u. We already know that $c_u \in BV(I, \overline{U})$. On account of the previous parameterization, we now prove that also *the Gauss map* $\tau_u : I \to \mathbb{S}^N_+$ *is a function with bounded variation*, compare Remark 4.3.

Theorem 4.7 *Let $N \ge 1$ and $u \in \mathcal{E}(I, \mathbb{R}^N)$. Let $\{u_h\} \subset C^2(I, \mathbb{R}^N)$ such that $u_h \to u$ in L^1 and $\sup_h \mathcal{E}(u_h) < \infty$. Then we have:*

i) the function $t \mapsto \Phi_u(t)$ belongs to $BV(I, \overline{U} \times \mathbb{S}^N)$;

ii) possibly passing to a subsequence $\{\Phi_{u_h}\}$ converges weakly in the BV-sense to the function $\Phi_u(t)$;

iii) by lower semicontinuity, $|D\Phi_u|(I) \le \liminf_h \int_I |\dot\Phi_{u_h}(t)|\,dt$.

Proof: Property iii) is a consequence of ii). Since $\sup_h \mathbf{M}(GG_{u_h}) < \infty$, see Remark 3.3, possibly passing to a subsequence we infer by (4.4) that $GG_{u_h} \rightharpoonup \Sigma$ weakly in $\mathcal{D}_1(U \times \mathbb{S}^N)$ to some current $\Sigma \in \mathrm{Gcart}(U \times \mathbb{S}^N)$, and setting $T = T(\Sigma) := \Pi_{x\#}\Sigma \in \mathrm{cart}(\mathring{I} \times \mathbb{R}^N)$, by Proposition 4.4 we have $u_T = u$.

We consider the corresponding Lipschitz function $c : I_L \to U$ given by Proposition 4.5. The image current $c_\#[\![I_L]\!]$ is oriented at $\mathcal{H}^1$-a.e. point $c(s)$ in its support by the unit tangent vector $\dot c(s)$. Denote

$$\widetilde{I}_L := \{s \in I_L \mid \dot c^0(s) > 0\}\,. \tag{4.9}$$

The first component $c^0(s)$ being non-decreasing, by changing variable $t = c^0(s)$ and using that $|\dot c| = 1$ a.e. we obtain

$$|\widetilde{I}_L| \ge \int_{\widetilde{I}_L} \dot c^0(s)\,ds = |I|\,.$$

Therefore, the set $\widetilde{I}_L$ has positive measure, and it identifies $\mathcal{H}^1$-essentially the set of points in the support of the image current $c_\#[\![I_L]\!]$ where the unit tangent vector is "non-vertical". In fact, for $\mathcal{L}^1$-a.e. $s \in \widetilde{I}_L$ the vector space generated by $\dot c(s)$ has projection on the first coordinate of rank one.

In a similar way, recalling the notation collected in Sec. 2, we observe that the set of points in the support of the Cartesian current $T(\Sigma)$ where the unit tangent vector is "non-vertical" is identified $\mathcal{H}^1$-a.e. by the rectifiable graph $\mathcal{G}_u$, see Remark 2.2. Moreover, at $\mathcal{H}^1$-a.e. such points $c_u(t) \in \mathcal{G}_u$ the orientation is provided by the unit vector $\tau_u(t) = \dot c_u(t)/|\dot c_u(t)|$.

Since we know that $c_\#[\![I_L]\!] = T(\Sigma)$, by the previous facts we deduce that (up to $\mathcal{H}^1$-null sets) the restriction of the first component $t = c^0(s)$ to the set $\widetilde{I}_L$ establishes a 1-1 correspondence with the set R_u of the Lebesgue points of both u and $\dot u$. In particular we have

$$\dot c(s) = \tau_u(c^0(s)) \quad \text{for } \mathcal{L}^1\text{-a.e. } s \in \widetilde{I}_L \tag{4.10}$$

and by the change of variable $t = c^0(s)$ we deduce that for every bounded and smooth function $g \in C_b^\infty(I, \mathbb{R}^{N+1})$

$$\int_I \tau_u(t) \bullet g(t)\,dt = \int_{R_u} \tau_u(t) \bullet g(t)\,dt = \int_{\widetilde{I}_L} \dot c(s) \bullet g(c^0(s))\,\dot c^0(s)\,ds\,. \tag{4.11}$$

We are now ready to prove that (up to a subsequence) the sequence $\{\tau_{u_h}\}$ converges weakly in the BV-sense to τ_u. In fact, since $\int_I |\dot\tau_{u_h}(t)|\,dt \le \mathcal{E}(u_h)$, we may assume that $\tau_{u_h} \rightharpoonup w$ weakly to some $w \in BV(I, \mathbb{R}^{N+1})$. We first show that for every bounded and smooth function $\phi \in C_b^\infty(I, \mathbb{R}^{N+1})$ such that $\phi(a) = \phi(b) = 0$

$$\int_I w(t) \bullet \dot\phi(t)\,dt = \int_I \tau_u(t) \bullet \dot\phi(t)\,dt\,. \tag{4.12}$$

In fact, by (4.11) we have

$$\int_I \tau_u(t) \bullet \dot\phi(t)\,dt = \int_{\widetilde{I}_L} \dot c(s) \bullet \dot\phi(c^0(s))\,\dot c^0(s)\,ds\,.$$

Recalling (4.6) and setting $\phi_h(s) := \phi \circ \varphi_h(s)$, we also infer that $\phi_h \to \phi \circ c^0$ uniformly in I_L, and $\dot\phi_h(s) = \dot\phi(\varphi_h(s))\,\dot\varphi_h(s)$. Moreover, we know that $\tau_{u_h}(\varphi_h(s)) = \dot c_h(s)/|\dot c_h(s)|$, with $|\dot c_h(s)| \equiv L_h/L \to 1$. By changing variable $t = \varphi_h(s)$, and by the weak convergence of $\dot c_h$ to $\dot c$, see (4.8), we compute

$$\begin{aligned}\int_I w(t) \bullet \dot\phi(t)\,dt &= \lim_{h\to\infty} \int_I \tau_{u_h}(t) \bullet \dot\phi(t)\,dt = \lim_{h\to\infty} \frac{L}{L_h} \int_{I_L} \dot c_h(s) \bullet \dot\phi_h(s)\,ds \\ &= -\lim_{h\to\infty} \langle D\dot c_h, \phi_h\rangle = -\langle D\dot c, \phi \circ c^0\rangle\end{aligned}$$

where we used that $\phi_h(0) = \phi_h(L) = 0$, as $\varphi_h(0) = a$, $\varphi_h(L) = b$, and $\phi(a) = \phi(b) = 0$. Since moreover

$$-\langle D\dot c, \phi \circ c^0\rangle = \int_{I_L} \dot c(s) \bullet D(\phi \circ c^0)(s)\,ds = \int_{\widetilde{I}_L} \dot c(s) \bullet \dot\phi(c^0(s))\,\dot c^0(s)\,ds$$

as $\phi \circ c^0(0) = \phi \circ c^0(L) = 0$, formula (4.12) holds true.

Now, for every bounded and smooth function $g \in C_b^\infty(I, \mathbb{R}^{N+1})$ with zero average integral

$$\overline{g} := \fint_I g(t)\, dt = 0\,,$$

writing (4.12) for the primitive $\phi \in C_b^\infty(I, \mathbb{R}^{N+1})$ of g such that $\phi(a) = \phi(b) = 0$, we get

$$\int_I w(t) \bullet g(t)\, dt = \int_I \tau_u(t) \bullet g(t)\, dt\,.$$

On the other hand, for each constant $\overline{g} = (\overline{g}^0, \dots, \overline{g}^N) \in \mathbb{R}^{N+1}$ we obtain, similarly as above,

$$\begin{aligned} \int_I w(t) \bullet \overline{g}\, dt &= \lim_{h\to\infty} \int_I \tau_{u_h}(t) \bullet \overline{g}\, dt = \lim_{h\to\infty} \frac{L}{L_h} \int_{I_L} \dot{c}_h(s) \bullet \overline{g}\, \dot{\varphi}_h(s)\, ds \\ &= \sum_{j=0}^N \overline{g}^j \lim_{h\to\infty} \int_{I_L} \dot{c}_h^j(s)\, \dot{\varphi}_h(s)\, ds \end{aligned} \tag{4.13}$$

where by the weak BV-convergence $\dot{c}_h \rightharpoonup c$ and the weak-* convergence $\varphi_h \rightharpoonup c^0$ in $W^{1,\infty}$ we get

$$\lim_{h\to\infty} \int_{I_L} \dot{c}_h^j(s)\, \dot{\varphi}_h(s)\, ds = \lim_{h\to\infty} \int_{I_L} \dot{c}^j(s)\, \dot{\varphi}_h(s)\, ds = \int_{I_L} \dot{c}^j(s)\, \dot{c}^0(s)\, ds \tag{4.14}$$

for each j. By (4.13) and (4.14) we deduce

$$\int_I w(t) \bullet \overline{g}\, dt = \sum_{j=0}^N \overline{g}^j \int_{I_L} \dot{c}^j(s)\, \dot{c}^0(s)\, ds = \int_{\widetilde{I}_L} \dot{c}(s) \bullet \overline{g}\, \dot{c}^0(s)\, ds$$

so that applying (4.11) with $g(t) \equiv \overline{g}$ we obtain:

$$\int_I w(t) \bullet \overline{g}\, dt = \int_I \tau_u(t) \bullet \overline{g}\, dt \qquad \forall\, \overline{g} \in \mathbb{R}^{N+1}.$$

Therefore, decomposing any bounded and smooth function $g \in C_b^\infty(I, \mathbb{R}^{N+1})$ as $g(t) = \big(g(t) - \overline{g}\big) + \overline{g}$, by linearity we deduce that

$$\int_I w(t) \bullet g(t)\, dt = \int_I \tau_u(t) \bullet g(t)\, dt \qquad \forall\, g \in C_b^\infty(I, \mathbb{R}^{N+1})\,.$$

This yields that $w = \tau_u$, whence τ_u is a function with bounded variation and the proof is complete. □

Remark 4.8 Formula (4.10) yields that $\dot{c}^0(s) = |\dot{c}_u(c^0(s))|^{-1}$ for $\mathcal{L}^1$-a.e. $s \in \widetilde{I}_L$. By changing variable $t = c^0(s)$, similarly to (4.11) we also obtain for each Borel set $A \subset I$

$$\int_A \dot{c}_u(t)\, dt = \int_A \tau_u(t)\, |\dot{c}_u(t)|\, dt = \int_{\widetilde{A}} \tau_u(c^0(s))\, |\dot{c}_u(c^0(s))|\, \dot{c}^0(s)\, ds = \int_{\widetilde{A}} \dot{c}(s)\, ds$$

where $\widetilde{A} = \widetilde{I}_L \cap (c^0)^{-1}(A)$, so that $\widetilde{A} = \widetilde{I}_L$ if $A = I$.

Convergence of the approximate gradient. As a consequence, we have:

Corollary 4.9 *Let $N \geq 1$ and $u \in \mathcal{E}(I, \mathbb{R}^N)$. If $\{u_h\} \subset C^2(I, \mathbb{R}^N)$ is such that $u_h \to u$ in L^1 and $\sup_h \mathcal{E}(u_h) < \infty$, then possibly passing to a subsequence $\dot{u}_h \to \dot{u}$ a.e. in I.*

Proof: In fact, by Theorem 4.7 we know that up to a subsequence $\tau_{u_h} \rightharpoonup \tau_u$ weakly in the BV-sense, hence $\tau_{u_h} \to \tau_u$ a.e. in I. Denoting by Ω the set of points in I such that $|\dot{u}(t)| < \infty$ and $\tau_{u_h}(t) \to \tau_u(t)$, we have that $|I \setminus \Omega| = 0$ and

$$\dot{u}^j(t) = \frac{\tau_u^j(t)}{\tau_u^0(t)} = \lim_{h\to\infty} \frac{\tau_{u_h}^j(t)}{\tau_{u_h}^0(t)} = \lim_{h\to\infty} \dot{u}_h^j(t)$$

for each $j = 1, \dots, N$ and every $t \in \Omega$, as required. □

Remark 4.10 In general, we cannot conclude that $\dot{u}_h \to \dot{u}$ a.e. for a smooth sequence $\{u_h\} \subset C^2(I, \mathbb{R}^N)$ weakly converging in the BV-sense to a smooth function u, if the bound $\sup_h \int_I |\dot{\tau}_{u_h}(t)|\, dt < \infty$ on the total curvature of the Cartesian curves c_{u_h} is not satisfied.

Taking e.g. $N = 1$, $I = [0, 2\pi]$, and $u_h(t) := \sin(ht)/h$, the sequence $\{u_h\}$ converges both weakly in the BV-sense and uniformly to the null function $u \equiv 0$, but we have $\int_0^{2\pi} |\dot{\tau}_{u_h}(t)|\, dt = h \cdot \pi$, and it is false that $\dot{u}_h(t) = \cos(ht) \to 0$ for a.e. $t \in [0, 2\pi]$.

5 Closing the Gauss graph of Cartesian curves

In this section we extend the notation from Sec. 3 to the wider class of functions u in $\mathcal{E}(I, \mathbb{R}^N)$, see (4.1), i.e. with finite relaxed energy (1.11). Our definition relies on the fact that the Gauss map $\tau_u : I \to \mathbb{S}^N$ is a function of bounded variation, Theorem 4.7.

We shall define a current GG_u carried by the "Gauss graph" of u that has three components

$$GG_u = GG_u^a + GG_u^C + GG_u^J$$

the absolute continuous, Cantor, and Jump ones, respectively. It turns out that $GG_u^J = 0$ if u has a continuous representative, and that also $GG_u^C = 0$ if $u \in W^{1,1}(I, \mathbb{R}^N)$. The component GG_u^a is well-defined in terms of the approximate gradient of the BV-function Φ_u.

We shall then see that there is a natural way to find a "vertical" current $S_u \in \mathcal{D}_1(U \times \mathbb{S}^N)$ such that the current

$$\Sigma_u := GG_u + S_u \tag{5.1}$$

is i.m. rectifiable in $\mathcal{R}_1(U \times \mathbb{S}^N)$ and has no interior boundary. Moreover, in the case $N = 1$ it turns out that the mass of Σ_u essentially agrees with the relaxed energy $\overline{\mathcal{E}}(u)$ as it is computed in [8], see Corollary 7.9.

Gauss graph of Cartesian curves. In the sequel we shall denote by $(dx^0, dx^1, \dots, dx^N)$ and $(dy^0, dy^1, \dots, dy^N)$ the canonical bases of 1-forms dual to the bases $(e_0, e_1, \dots, e_N)$ and $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N)$ in $\mathbb{R}_x^{N+1}$ and $\mathbb{R}_y^{N+1}$, respectively.

Let $u \in \mathcal{E}(I, \mathbb{R}^N)$. As a consequence of Theorem 4.7, it turns out that the *approximate gradient function* $t \mapsto \dot{\Phi}_u(t)$ is well-defined a.e. in I as in (3.3), where this time $\ddot{u}$ denotes the approximate gradient of $\dot{u}$, and $\dot{\Phi}_u \in L^1(I, \overline{U} \times \mathbb{R}_y^{N+1})$. Moreover, taking good representatives of each component, it turns out that the right and left limits $\Phi_u(t_\pm)$ exist at each point $t \in \mathring{I}$, with $\Phi_u(t_\pm) = (t, u(t_\pm), \tau_u(t_\pm))$, where $\tau_u = (\tau_u^0, \tau_u^1, \dots, \tau_u^N)$.

The absolute continuous component. As in the smooth case, since $\dot{\Phi}_u$ is summable, see Theorem 4.7, we define the current $GG_u^a \in \mathcal{D}_1(U \times \mathbb{S}^N)$ by setting for each $\omega \in \mathcal{D}^1(U \times \mathbb{S}^N)$

$$\langle GG_u^a, \omega \rangle := \int_I \langle \omega(\Phi_u(t)), \dot{\Phi}_u(t) \rangle\, dt\,. \tag{5.2}$$

To our purposes, we compute $\langle \partial GG_u^a, f \rangle$ for any $f \in C_c^\infty(U \times \mathbb{S}^N)$. By the definition of boundary current we obtain:

$$\langle \partial GG_u^a, f \rangle := \langle GG_u^a, df \rangle = \int_I \langle df(\Phi_u(t)), \dot{\Phi}_u(t)\, dt \rangle = \int_I \nabla f(\Phi_u) \bullet \dot{\Phi}_u\, dt\,.$$

Moreover, the composition function $f \circ \Phi_u$ belongs to $BV(I)$, and since $f \in C_c^\infty(U \times \mathbb{S}^N)$ by the definition of distributional derivative we deduce that

$$\int_I D(f \circ \Phi_u) = 0\,.$$

Also, choosing $\Phi_{u+}(t) = \Phi_u(t_+)$ as a precise representative, by the chain-rule formula we get

$$D(f \circ \Phi_u) = \nabla f(\Phi_u) \bullet \dot{\Phi}_u\, dt + \nabla f(\Phi_{u+}) \bullet D^C \Phi_u + \big(f(\Phi_{u+}) - f(\Phi_{u-})\big)\, \mathcal{H}^0 \llcorner J_{\Phi_u}\,.$$

Therefore, we obtain that for each $f \in C_c^\infty(U \times \mathbb{S}^N)$

$$\langle \partial GG_u^a, f\rangle = -\int_I \nabla f(\Phi_{u+}) \bullet dD^C\Phi_u - \sum_{t \in J_{\Phi_u}} \big(f(\Phi_u(t_+)) - f(\Phi_u(t_-))\big)\,. \tag{5.3}$$

The Cantor component. We have $D^C\Phi_u \bullet e_0 = 0$, $D^C\Phi_u \bullet e_j = D^C u^j$, and

$$D^C\Phi_u \bullet \varepsilon_0 = D^C\tau_u^0\,, \qquad D^C\Phi_u \bullet \varepsilon_j = D^C\tau_u^j$$

for $j = 1, \dots, N$. We define the Cantor component $GG_u^C \in \mathcal{D}_1(U \times \mathbb{S}^N)$ extending by linearity the action on basic forms. For any $g \in C_c^\infty(U \times \mathbb{S}^N)$ we set:

i) $\langle GG_u^C, g(x,y)\, dx^0\rangle := 0$

ii) $\langle GG_u^C, g(x,y)\, dx^j\rangle := \displaystyle\int_I g(\Phi_{u+})\, dD^C u^j\,,\ j = 1, \dots, N$

iii) $\langle GG_u^C, g(x,y)\, dy^0\rangle := \displaystyle\int_I g(\Phi_{u+})\, dD^C \tau_u^0$

iv) $\langle GG_u^C, g(x,y)\, dy^j\rangle := \displaystyle\int_I g(\Phi_{u+})\, dD^C \tau_u^j\,,\ j = 1, \dots, N\,.$

Therefore, for each $f \in C_c^\infty(U \times \mathbb{S}^N)$ we clearly obtain

$$\langle \partial GG_u^C, f\rangle := \langle GG_u^C, df\rangle = \int_I \nabla f(\Phi_{u+}) \bullet dD^C\Phi_u\,. \tag{5.4}$$

The Jump component. In Sec. 2, for each Jump point $t \in J_u$ we denoted by $\gamma_t(u)$ the oriented line segment in $U = \mathring{I} \times \mathbb{R}^N$ with initial point $c_u(t_-)$ and final point $c_u(t_+)$. Since $\gamma_t(u)$ is oriented by the unit vector

$$\Big(0, \frac{[u(t)]}{|[u(t)]|}\Big) \in \mathbb{S}_+^N\,, \qquad [u(t)] := (u_+(t) - u_-(t)) \in \mathbb{R}^N \setminus \{0\}\,,$$

we correspondingly denote by $\widetilde{\gamma}_t(u)$ the oriented rectifiable arc in $U \times \mathbb{S}_+^N$

$$\widetilde{\gamma}_t(u) := \Big(\gamma_t(u), \Big(0, \frac{[u(t)]}{|[u(t)]|}\Big)\Big)$$

and we set

$$\langle GG_u^J, \omega\rangle := \sum_{t \in J_u} \int_{\widetilde{\gamma}_t(u)} \omega\,, \qquad \omega \in \mathcal{D}^1(U \times \mathbb{S}^N)\,. \tag{5.5}$$

In particular, for each $f \in C_c^\infty(U \times \mathbb{S}^N)$ we obtain

$$\langle \partial GG_u^J, f\rangle := \langle GG_u^J, df\rangle = \sum_{t \in J_u} \int_{\widetilde{\gamma}_t(u)} df = \sum_{t \in J_u} \big(f(P_+(t)) - f(P_-(t))\big)\,, \tag{5.6}$$

where we have set

$$P_\pm(t) := \Big(t, u(t_\pm), 0, \frac{[u(t)]}{|[u(t)]|}\Big)\,, \qquad t \in J_u\,. \tag{5.7}$$

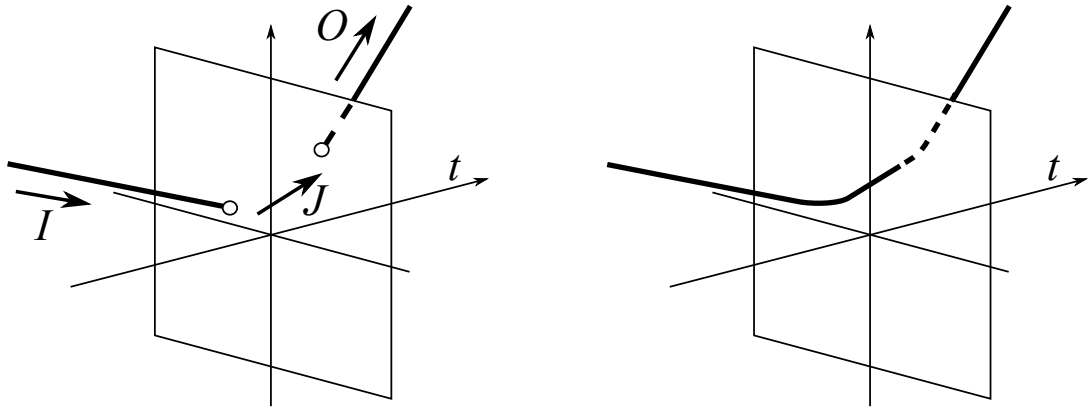

Figure 3: The codimension-two "curve" on the left has a jump-corner point at $t = 0$, with incoming, jump, and outgoing directions given by I, J, and O, respectively.
On the right: a codimension-two smooth approximating Cartesian curve.

In conclusion, by the formulas (5.3), (5.4), and (5.6), we deduce that for any $f \in C_c^\infty(U \times \mathbb{S}^N)$

$$\begin{aligned} \langle \partial GG_u, f\rangle &= \langle GG_u^a, df\rangle + \langle GG_u^C, df\rangle + \langle GG_u^J, df\rangle \\ &= \sum_{t\in J_u} \big(f(P_+(t)) - f(P_-(t))\big) - \sum_{t\in J_{\Phi_u}} \big(f(\Phi_u(t_+)) - f(\Phi_u(t_-))\big)\,. \end{aligned} \tag{5.8}$$

Closing the Gauss graph. By our definition it is readily checked that the x-projection of the current GG_u agrees with the Cartesian current T_u, compare Sec. 2, i.e. $\Pi_{x\#}GG_u = T_u$. However, even if we always have $\partial T_u = 0$, the current GG_u has in general a non-zero boundary with possibly infinite mass, given by the formula (5.8).

We thus define a "vertical" current S_u in such a way that if $\Sigma_u := GG_u + S_u$ as in (5.1), then Σ_u has no boundary in $U \times \mathbb{S}^N$, and again $\Pi_{x\#}\Sigma_u = T_u$. As a consequence, we will obtain that Σ_u is i.m. rectifiable in $\mathcal{R}_1(U \times \mathbb{S}^n)$. The current S_u lives upon the Jump points t in $J_{\Phi_u} = J_u \cup J_{\dot u}$. It is given by two terms:

$$S_u = S_u^{Jc} + S_u^c$$

a "Jump-corner" component S_u^{Jc} that is concentrated upon the discontinuity set J_u, and a "corner" component S_u^c that is concentrated upon the discontinuity points of the approximate gradient $\dot u$ where u is continuous, the so called "corner" points in $J_{\dot u} \setminus J_u$. Roughly speaking, the first component takes into account of the turning angles that appear when the "graph" of u meets a vertical part of the Cartesian current T_u, possibly giving rise to two corners at the points $(t, u_\pm(t))$, where one side of each corner is "vertical", since it follows the jump. The second component deals with the turning angles where u is continuous but $\dot u$ has a jump.

In Figures 3 and 4 we illustrate an example in codimension $N = 2$ with occurrence of a jump-corner term. This is due to a jump point of both the graph function $c_u(t) = (t, u(t))$ and of its derivative. A crucial role is played by the incoming, jump, and outgoing directions, denoted by I, J, and O. The jump direction is determined by the last $N+1$ components in the formula (5.7). The incoming and outgoing directions are determined by the last $N+1$ components of $\Phi_u(t_-)$ and $\Phi_u(t_+)$, respectively, i.e. by the left and right limits $\tau_u(t_\pm)$ of the Gauss map.

A similar example with a corner term is readily obtained by gluing together the two line segments of the graph $c_u(t)$. In this case, only the incoming and outgoing directions $\tau_u(t_\pm)$ come into play.

The Jump-corner component. For each point $t \in J_u$, we denote by $\Gamma_t^\pm(u)$ an oriented geodesic arc in $\{c_u(t_\pm)\} \times \mathbb{S}_+^N$ with initial point $P_\pm(t)$, see (5.7), and final point $\Phi_u(t_\pm)$, and we set

$$\langle S_u^{Jc}, \omega\rangle := \sum_{t\in J_u} \Big(\int_{\Gamma_t^+(u)} \omega - \int_{\Gamma_t^-(u)} \omega\Big)\,, \qquad \omega \in \mathcal{D}^1(U \times \mathbb{S}^N)\,. \tag{5.9}$$

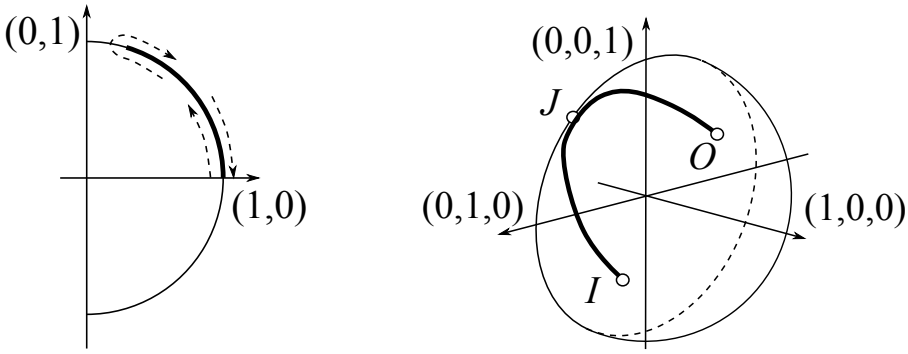

Figure 4: On the left: we revise the codimension-one curve in Example 1.5, drawing the image of the tantrix of the smooth approximation of c_{u_h}. It corresponds to the τ-projection of the curve on the right-hand side of Figure 2.
On the right: the image on the 2-sphere of the tantrix of the codimension-two smooth approximating Cartesian curve from Figure 3.

Therefore, we clearly have $\Pi_{x\#}S_u^{Jc} = 0$, and for each $f \in C_c^\infty(U \times \mathbb{S}^N)$ we compute

$$\langle \partial S_u^{Jc}, f\rangle := \langle S_u^{Jc}, df\rangle = \sum_{t\in J_u} \big(f(\Phi_u(t_+)) - f(P_+(t)) - f(\Phi_u(t_-)) + f(P_-(t))\big)\,. \tag{5.10}$$

The corner component. Instead, for each point $t \in J_{\dot u} \setminus J_u$, we denote by $\Gamma_t(u)$ an oriented geodesic arc in $\{c_u(t)\} \times \mathbb{S}^N_+$ with initial point $\Phi_u(t_-)$ and final point $\Phi_u(t_+)$, and we set

$$\langle S_u^c, \omega\rangle := \sum_{t\in J_{\dot u}\setminus J_u} \int_{\Gamma_t(u)} \omega\,, \qquad \omega \in \mathcal{D}^1(U \times \mathbb{S}^N)\,.$$

We again have $\Pi_{x\#}S_u^c = 0$, whereas this time for each $f \in C_c^\infty(U \times \mathbb{S}^N)$ we get

$$\langle \partial S_u^c, f\rangle := \langle S_u^c, df\rangle = \sum_{t\in J_{\dot u}\setminus J_u} \big(f(\Phi_u(t_+)) - f(\Phi_u(t_-))\big)\,. \tag{5.11}$$

Remark 5.1 The current $S_u := S_u^{Jc} + S_u^c$ is "vertical" in the sense that for any $g \in C_c^\infty(U \times \mathbb{S}^N)$

$$\langle S_u, g(x,y)\,dx^j\rangle = 0 \qquad \forall\, j = 0,\dots,N\,.$$

In fact, all the arcs $\Gamma_t^\pm(u)$ and $\Gamma_t(u)$ have tangent vector that is everywhere perpendicular to the horizontal directions $e_0, e_1, \dots, e_N$.

Properties. The current $\Sigma_u \in \mathcal{D}_1(U \times \mathbb{S}^N)$ is supported in $\overline{U} \times \mathbb{S}^N_+$, and it satisfies $\Pi_{x\#}\Sigma_u = T_u$. Moreover the null-boundary condition $\partial\Sigma_u = 0$ holds. In fact, by (5.8), (5.10), and (5.11), we check

$$\langle \partial\Sigma_u, f\rangle = \langle \partial GG_u, f\rangle + \langle \partial S_u^{Jc}, f\rangle + \langle \partial S_u^c, f\rangle = 0 \qquad \forall\, f \in C_c^\infty(U \times \mathbb{S}^N)\,.$$

Mass decomposition. It is readily seen that the mass of Σ_u decomposes as

$$\mathbf{M}(\Sigma_u) = \mathbf{M}(GG_u^a) + \mathbf{M}(GG_u^C) + \mathbf{M}(GG_u^J) + \mathbf{M}(S_u^{Jc}) + \mathbf{M}(S_u^c)\,. \tag{5.12}$$

In fact, all the involved arcs $\widetilde{\gamma}_t(u)$, $\Gamma_t^\pm(u)$, and $\Gamma_t(u)$ are $\mathcal{H}^1$-essentially disjoint, as they possibly meet only at the end points, whereas the absolute continuous and Cantor parts $D^a\Phi_u$ and $D^C\Phi_u$ are null on the at most countable set J_{Φ_u}. More explicitly, we have

$$\mathbf{M}(GG_u^a) = \int_I |\dot\Phi_u(t)|\,dt\,, \qquad |\dot\Phi_u| = |\dot c_u|\,\sqrt{1+k_u^2}$$

where, we recall,

$$|\dot{c}_u| = \sqrt{1+|\dot{u}|^2}\,, \qquad k_u^2 = \frac{|\ddot{u}|^2(1+|\dot{u}|^2)-(\dot{u}\bullet\ddot{u})^2}{(1+|\dot{u}|^2)^3}\,.$$

As to the Cantor component, we have $\mathbf{M}(GG_u^C) = |D^C\Phi_u|(I)$, and hence

$$2^{-1/2}\big(|D^C u|(I)+|D^C\tau_u|(I)\big) \le \mathbf{M}(GG_u^C) \le |D^C u|(I)+|D^C\tau_u|(I)\,.$$

Finally, for the mass of the other three components we compute

$$\begin{aligned}
\mathbf{M}(GG_u^J) &= \sum\nolimits_{t\in J_u}\mathcal{H}^1(\widetilde{\gamma}_t(u)) = \sum\nolimits_{t\in J_u}|[c_u(t)]| = |D^J u|(I)\\
\mathbf{M}(S_u^{J_c}) &= \sum\nolimits_{t\in J_u}\big(\mathcal{H}^1(\Gamma_t^+(u))+\mathcal{H}^1(\Gamma_t^-(u))\big)\\
\mathbf{M}(S_u^c) &= \sum\nolimits_{t\in J_{\dot{u}}\setminus J_u}\mathcal{H}^1(\Gamma_t(u))\,.
\end{aligned} \tag{5.13}$$

Rectifiability. Finally, in Corollary 8.4 we shall prove that Σ_u *is an i.m. rectifiable current in* $\mathcal{R}_1(U\times\mathbb{S}^N)$, actually an integral 1-cycle in $U\times\mathbb{S}^N$. This fact will be used only in the proof of the density theorem 8.5 below.

6 Gauss graphs of Cartesian currents

In Proposition 4.4 we outlined some basic facts concerning currents in the class $\mathrm{Gcart}(U\times\mathbb{S}^N)$ defined in (4.4). In this section we prove further structure properties, Theorem 6.1, and give an explicit example. We also show that currents in $\mathrm{Gcart}(U\times\mathbb{S}^N)$ preserve the geometry of Gauss graphs of Cartesian curves, Theorem 6.3.

A first structure theorem. We extend Proposition 4.4.

Theorem 6.1 *Let* $\Sigma\in\mathrm{Gcart}(U\times\mathbb{S}^N)$ *and let* $\{u_h\}\subset C^2(I,\mathbb{R}^N)$ *be such that* $\sup_h\mathbf{M}(GG_{u_h})<\infty$ *and* $GG_{u_h}\rightharpoonup\Sigma$ *weakly in* $\mathcal{D}_1(U\times\mathbb{S}^N)$. *Then we have:*

i) The sequence $\{u_h\}$ *converges weakly in the* BV*-sense to some function* $u\in\mathcal{E}(I,\mathbb{R}^N)$, *i.e. with finite relaxed energy* (1.11).

ii) The function $t\mapsto\Phi_u(t)$ *of Theorem* 4.7 *belongs to* $BV(I,\overline{U}\times\mathbb{S}^N)$ *and it is equal to the weak* BV*-limit of the sequence* $\{\Phi_{u_h}\}$.

iii) The current Σ *decomposes as*

$$\Sigma = GG_u^a + GG_u^C + \widetilde{\Sigma} \tag{6.1}$$

where GG_u^a *and* GG_u^C *are the absolute continuous and Cantor component of the current* GG_u *defined w.r.t. the limit function* u *as in Sec.* 5.

iv) If T_u *is the current defined in* (2.1)–(2.4), *we have*

$$\Pi_{x\#}GG_u^a = T_u^a\,, \quad \Pi_{x\#}GG_u^C = T_u^C\,.$$

v) The component $\widetilde{\Sigma}$ *has support contained in* $\overline{U}\times\mathbb{S}^N_+$, *and it satisfies the* verticality condition

$$\langle\widetilde{\Sigma}, g(x,y)\,dx^0\rangle = 0 \qquad \forall\, g\in C_c^\infty(U\times\mathbb{S}^N)$$

and the boundary condition

$$\partial\widetilde{\Sigma} = \sum_{t\in J_{\Phi_u}}\big(\delta_{\Phi_u(t_+)}-\delta_{(\Phi_u(t_-))}\big) \quad \text{on } C_c^\infty(U\times\mathbb{S}^N)\,.$$

vi) The following decomposition in mass *holds:*

$$\widetilde{\Sigma} = \widehat{\Sigma} + \sum_{t\in J_{\Phi_u}} \Gamma_{t,\Sigma}\,, \qquad \mathbf{M}(\widetilde{\Sigma}) = \mathbf{M}(\widehat{\Sigma}) + \sum_{t\in J_{\Phi_u}} \mathbf{M}(\Gamma_{t,\Sigma})\,,$$

where the current $\widehat{\Sigma} \in \mathcal{R}_1(U\times\mathbb{S}^N)$ *satisfies the null-boundary condition* $\partial\widehat{\Sigma}=0$*, and* $\Gamma_{t,\Sigma}$ *is for each* $t\in J_{\Phi_u}$ *an* a-cyclic *i.m. rectifiable current in* $\mathcal{R}_1(U\times\mathbb{S}^N)$*, supported in* $\{t\}\times\mathbb{R}^N\times\mathbb{S}^N_+$*, and with boundary*

$$\partial\Gamma_{t,\Sigma} = \delta_{\Phi_u(t_+)} - \delta_{\Phi_u(t_-)}\,. \tag{6.2}$$

Proof: Properties i) and ii) follow from Proposition 4.4 and Theorem 4.7, respectively, while property iii) is an immediate consequence of the general structure properties from [13] concerning the class of Cartesian currents in $\mathrm{cart}(\mathring{I}\times\mathbb{R}^D)$, where $D:=2N+1$. Property iv) is a consequence of iii) and of Proposition 4.4. Regarding property v), the support condition follows from the fact that $\mathrm{spt}\,GG_{u_h}\subset\overline{U}\times\mathbb{S}^N_+$ for any h, the verticality condition from the convergence

$$\lim_{h\to\infty}\langle GG_{u_h}, g(x,y)\,dx^0\rangle = \lim_{h\to\infty}\int_I g(\Phi_{u_h}(t))\,dt = \int_I g(\Phi_u(t))\,dt\,,$$

a consequence of iv), recalling from Sec. 5 that

$$\langle GG^a_u, g(x,y)\,dx^0\rangle = \int_I g(\Phi_u(t))\,dt\,, \qquad \langle GG^C_u, g(x,y)\,dx^0\rangle = 0\,,$$

whereas the boundary condition follows from (5.3) and (5.4), using that $\partial\Sigma=0$. Finally, property vi) follows from v) on account of the classical decomposition theorem, see [12, 4.2.25]. □

Example 6.2 We explicitly compute all the current Σ in a simple case. Referring to Example 1.5, we let Σ denote the weak limit of the sequence of Gauss graphs $\{GG_{v_h}\}$ corresponding to a "smoothing" $v_h:[-1,1]\to\mathbb{R}$ of the sequence $\{u_h\}$ from (1.7) at the corner points $(-\pi/h,0)$ and $(\pi/h,2\pi)$. The smoothing can be performed as in Example 1.1. By a diagonal argument, we may choose the smooth sequence $\{v_h\}$ in an optimal way, so that the total curvature of the Cartesian curve c_{v_h} is equal to $2\arctan h$ for each h, and v_h agrees with u_h outside two small intervals centered at the points $\pm\pi/h$, in such a way that $\|v_h-u_h\|_\infty\to 0$ as $h\to\infty$, see Figure 2.

Then $L=2(1+\pi)$ and the parameterization $s\mapsto c(s)$ from Proposition 4.5 is equal to the one defined in (1.8). We thus have

$$\dot c(s) := \begin{cases} (1,0) & \text{if } 0\le s<1\\ (0,1) & \text{if } 1<s<1+2\pi\\ (1,0) & \text{if } 1+2\pi<s<2(1+\pi)\end{cases}$$

and hence in this case the image current $(c,\dot c)_\#[\![\,I_L\,]\!]$ agrees with the current GG_u from Sec. 5, where $u:[-1,1]\to\mathbb{R}$ is the weak limit BV-function

$$u(t) := \begin{cases} 0 & \text{if } t<0\\ 2\pi & \text{if } t>0\,.\end{cases} \tag{6.3}$$

More precisely, we have $GG_u = GG^a_u + GG^C_u + GG^J_u$, where the absolute continuous component is given by (5.2) with $I=[-1,1]$ and

$$\Phi_u(t) = \begin{cases} (t,0,1,0) & \text{if } t<0\\ (t,2\pi,1,0) & \text{if } t>0\end{cases}$$

i.e. $GG^a_u = \Phi_{u\#}[\![\,I\,]\!]$; the Cantor component is zero, $GG^C_u=0$, and the Jump component GG^J_u, according to (5.5), agrees with the integration on the oriented line segment in $\{0\}\times\mathbb{R}\times\mathbb{S}^1$ with initial point $(0,0,0,1)$ and final point $(0,2\pi,0,1)$. In particular, since $J_{\Phi_u}=J_u=\{0\}$, in accordance with (5.8) and (5.7) we have

$$\partial(c,\dot c)_\#[\![\,I_L\,]\!] = \partial GG_u = -\big(\delta_{\Phi_u(0_+)}-\delta_{\Phi_u(0_-)}\big) + \big(\delta_{P_+(0)}-\delta_{P_-(0)}\big)$$

on $C_c^\infty(U\times\mathbb{S}^1)$, where

$$\Phi_u(0_+)=(0,2\pi,1,0)\,,\quad \Phi_u(0_-)=(0,0,1,0)\,,$$
$$P_+(0)=(0,2\pi,0,1)\,,\quad P_-(0)=(0,0,0,1)\,.$$

Moreover, by our optimal choice of the sequence $\{v_h\}$, it turns out that the weak limit current Σ of the sequence $\{GG_{v_h}\}$ agrees with the current Σ_u from Sec. 5. This means that $\Sigma=\Sigma_u=GG_u+S_u^{Jc}+S_u^c$, where the corner component $S_u^c=0$, since there are no points in $J_{\dot u}\setminus J_u$, and the Jump-corner component S_u^{Jc}, according to (5.9), is given by

$$S_u^{Jc}=[\![\,\Gamma_0^+(u)\,]\!]-[\![\,\Gamma_0^-(u)\,]\!]\,, \tag{6.4}$$

where $\Gamma_0^\pm(u)$ is the oriented geodesic arc in $\{c_u(0_\pm)\}\times\mathbb{S}^1_+$ with initial point $P_\pm(0)$ and final point $\Phi_u(0_\pm)$. We thus have

$$\mathbf{M}(\Sigma_u)=\mathbf{M}(GG_u^a)+\mathbf{M}(GG_u^J)+\mathbf{M}(S_u^{J_c})\,,$$

where

$$\mathbf{M}(GG_u^a)=2\,,\quad \mathbf{M}(GG_u^J)=|Du|(I)=2\pi\,,\quad \mathbf{M}(S_u^{J_c})=2\cdot\frac{\pi}{2}\,. \tag{6.5}$$

Structure of Gauss graphs. We now see that a current Σ in the class $\mathrm{Gcart}(U\times\mathbb{S}^N)$ preserves the geometry of Gauss graphs: indeed, for a regular Gauss graph (c_u,τ_u) the second component τ_u is a normalization of the tangent vector to the first component. We prove that when the first component of the tangent vector to Σ at a point $z=(x,y)\in U\times\mathbb{S}^N_+$ is non zero, then it has to be parallel (and with the same verse) to the second component y of the point z, see (6.6) below.

For this purpose, we let $\Sigma\in\mathrm{Gcart}(U\times\mathbb{S}^N)$. In Remark 4.6 we have shown the existence of a Lipschitz function $\Psi\in\mathrm{Lip}(I_{\widetilde L},\overline{U}\times\mathbb{S}^N)$ such that the image current $\Psi_\#[\![\,I_{\widetilde L}\,]\!]$ agrees with Σ. Recall that Π_x and Π_y denote the orthogonal projections of $\overline{U}\times\mathbb{S}^N$ onto the x and y coordinates, respectively.

Theorem 6.3 *For a.e. $s\in J_{\widetilde L}$ such that $|\Pi_x(\dot\Psi(s))|\neq 0$, we have*

$$\frac{\Pi_x(\dot\Psi(s))}{|\Pi_x(\dot\Psi(s))|}=\Pi_y(\Psi(s))\in\mathbb{S}^N_+\,. \tag{6.6}$$

Proof: Choose a sequence $\{u_h\}\subset C^2(I,\mathbb{R}^N)$ satisfying $\sup_h\mathbf{M}(GG_{u_h})<\infty$ and $GG_{u_h}\rightharpoonup\Sigma$ weakly in $\mathcal{D}_1(U\times\mathbb{S}^N)$. In Remark 4.6 we correspondingly defined the smooth functions $\Psi_h(s):=\Phi_{u_h}(\widetilde\varphi_h(s)):I_{\widetilde L}\to\overline{U}\times\mathbb{S}^N$ with constant velocity $\widetilde L_h/\widetilde L$ such that $\Psi_{h\#}[\![\,I_{\widetilde L}\,]\!]=GG_{u_h}$ for each h. Since GG_{u_h} is the current carried by the Gauss graph of a smooth function, writing for simplicity $t(s)=\widetilde\varphi_h(s)$ we have

$$\frac{\widetilde L}{\widetilde L_h}\,\dot\Psi_h(s)=\frac{\dot\Phi_{u_h}(t(s))}{|\dot\Phi_{u_h}(t(s))|}\,,\quad \dot\Phi_{u_h}(t)=(\dot c_{u_h}(t),\dot\tau_{u_h}(t))$$

for each $s\in I_{\widetilde L}$. Recalling that $\dot\tau_{u_h}=\dot c_{u_h}/|\dot c_{u_h}|$, this implies that for every $s\in I_{\widetilde L}$

$$\frac{\Pi_x(\dot\Psi_h(s))}{|\Pi_x(\dot\Psi_h(s))|}=\Pi_y(\Psi_h(s))\in\mathbb{S}^N_+\,. \tag{6.7}$$

Setting

$$a_h(s):=\Pi_x(\dot\Psi_h(s))\,,\qquad b_h(s):=\Pi_y(\Psi_h(s))\,,$$

formula (6.7) yields that the two vectors $a_h(s)$ and $b_h(s)$ are always parallel and pointing the same way.

This geometric property is a.e. preserved when passing to the limit as $h\to\infty$. More precisely, setting $a(s):=\Pi_x(\dot\Psi(s))$ and $b(s):=\Pi_y(\Psi(s))$ we prove that the two vectors $a(s)$ and $b(s)$ are parallel and pointing the same way, for a.e. $s\in I_{\widetilde L}$. This clearly implies the validity of (6.6).

Now, given two vectors $a,b\in\mathbb{R}^{N+1}$, with $a\neq 0$ and $|b|=1$, they are parallel and pointing the same way if and only if $a/|a|=b$, that is equivalent to $a\bullet b=|a|$, since

$$|(a/|a|)-b|^2=1+1-2\frac{a\bullet b}{|a|}\,.$$

Possibly passing to a subsequence, by Remark 4.6 we know that $\{\Psi_h\}$ uniformly converges in $I_{\widetilde{L}}$ to the above mentioned Lipschitz function $\Psi \in \mathrm{Lip}(I_{\widetilde{L}}, \overline{U} \times \mathbb{S}^N)$, whence $b_h(s) \to b(s) \in \mathbb{S}^N_+$ uniformly in $I_{\widetilde{L}}$. Since moreover $\dot{\Psi}_h \rightharpoonup \dot{\Psi}$ weakly-* in L^∞, we deduce that $a_h(s) \rightharpoonup a(s)$ weakly-* in L^∞, whence

$$\lim_{h\to\infty} \int_{I_{\widetilde{L}}} a_h(s) \bullet b_h(s)\, ds = \int_{I_{\widetilde{L}}} a(s) \bullet b(s)\, ds\,.$$

Since moreover $a_h(s) \rightharpoonup a(s)$ weakly in L^1, the lower semicontinuity

$$\int_{I_{\widetilde{L}}} |a(s)|\, ds \le \liminf_{h\to\infty} \int_{I_{\widetilde{L}}} |a_h(s)|\, ds$$

holds, and hence

$$\int_{I_{\widetilde{L}}} \big(|a(s)| - a(s) \bullet b(s)\big)\, ds \le \liminf_{h\to\infty} \int_{I_{\widetilde{L}}} \big(|a_h(s)| - a_h(s) \bullet b_h(s)\big)\, ds\,.$$

By (6.7) we have seen that $|b_h| = 1$, $|a_h| \neq 0$, and $a_h/|a_h| = b_h$ for all s and h. Therefore, we obtain

$$\int_{I_{\widetilde{L}}} \big(|a(s)| - a(s) \bullet b(s)\big)\, ds \le 0\,.$$

The integrand being non-negative by the Schwartz inequality $a \bullet b \le |a||b| = |a|$, we deduce that $|a(s)| - a(s) \bullet b(s) = 0$ for a.e. $s \in I_{\widetilde{L}}$, whence the two vectors $a(s)$ and $b(s)$ are parallel and pointing the same way, as required. □

7 The energy functional on currents

In this section we define a lower semicontinuous functional $\Sigma \mapsto \mathcal{E}^0(\Sigma)$ in the class $\mathcal{R}_1(U \times \mathbb{S}^N)$ that agrees with the energy functional $\mathcal{E}(u)$ when restricted to the Gauss graph GG_u of a smooth function, see Proposition 7.2. Since we aim at showing that $\overline{\mathcal{E}}(u) = \mathcal{E}^0(\Sigma_u)$, we shall finally write more explicitly the action of the energy functional in the case of Gauss graphs $\Sigma = \Sigma_u$ of BV-functions u such that $\overline{\mathcal{E}}(u) < \infty$.

The energy on currents. In order to define the energy functional on the class $\mathrm{Gcart}(U \times \mathbb{S}^N)$, we remark that these currents are of the type $S = [\![\, \mathcal{M}, \theta, \xi \,]\!]$, i.e.

$$\langle S, \omega \rangle = \int_{\mathcal{M}} \langle \omega, \xi \rangle\, \theta\, d\mathcal{H}^1 \qquad \forall\, \omega \in \mathcal{D}^1(U \times \mathbb{S}^N)\,,$$

where $\mathcal{M}$ is a countably 1-rectifiable set, ξ is the orienting unit vector and θ is the integer-valued non-negative multiplicity function, so that $\mathbf{M}(S) = \int_{\mathcal{M}} \theta\, d\mathcal{H}^1$. The unit vector ξ in $\mathbb{R}^{N+1}_x \times \mathbb{R}^{N+1}_y$ orienting $\mathcal{M}$ at $\mathcal{H}^1 \llcorner \mathcal{M}$-a.e. point can be decomposed as $\xi = (\xi^{(x)}, \xi^{(y)})$, where $\xi^{(x)} := \Pi_x(\xi)$ and $\xi^{(y)} := \Pi_y(\xi)$.

Definition 7.1 *For any current $S = [\![\, \mathcal{M}, \theta, \xi \,]\!]$ we let*

$$\mathcal{E}^0(S) := \int_{\mathcal{M}} \theta \left(|\xi^{(x)}| + |\xi^{(y)}|\right) d\mathcal{H}^1\,.$$

Proposition 7.2 *The following properties hold:*

i) (smooth maps) *If $S = GG_u$ for some smooth function $u \in C^2(I, \mathbb{R}^N)$, then $\mathcal{E}^0(GG_u) = \mathcal{E}(u)$.*

ii) (lower semicontinuity) *Let $\{u_h\} \subset C^2(I, \mathbb{R}^N)$ be such that $GG_{u_h} \rightharpoonup \Sigma$ weakly in $\mathcal{D}_1(U \times \mathbb{S}^N)$ to some $\Sigma \in \mathrm{Gcart}(U \times \mathbb{S}^N)$. Then $\mathcal{E}^0(\Sigma) \le \liminf_h \mathcal{E}(u_h)$.*

iii) (Energy decomposition) *If* $\Sigma \in \mathrm{Gcart}(U \times \mathbb{S}^N)$ *decomposes as in* (6.1)*, then*

$$\mathcal{E}^0(\Sigma) = \int_I |\dot{c}_u|(1+k_u)\,dt + |D^C u|(I) + |D^C \tau_u|(I) + \mathcal{E}^0(\widetilde{\Sigma})\,.$$

Proof: If $S = GG_u$ for some smooth u, then $\mathcal{M} = \mathcal{GG}_u$, $\theta \equiv 1$, and $\xi = \xi_u$ is given by (3.2), so that

$$\mathcal{E}^0(GG_u) = \int_{\mathcal{GG}_u} \big(|\xi_u^{(x)}| + |\xi_u^{(y)}|\big)\,d\mathcal{H}^1\,, \qquad |\xi_u^{(x)}| = \frac{|\dot{c}_u|}{|\dot{\Phi}_u|}\,, \qquad |\xi_u^{(y)}| = \frac{|\dot{c}_u|\,k_u}{|\dot{\Phi}_u|}\,.$$

By the area formula, from the definition (1.10) we thus obtain property i), as

$$\mathcal{E}(u) = \int_I |\dot{c}_u|(1+k_u)\,dt = \int_I |\dot{\Phi}_u|\,\big(|\xi_u^{(x)}| + |\xi_u^{(y)}|\big)\,dt = \int_{\mathcal{GG}_u} \big(|\xi_u^{(x)}| + |\xi_u^{(y)}|\big)\,d\mathcal{H}^1\,. \tag{7.1}$$

The lower semicontinuity property ii) follows from the fact that the functional $S \mapsto \mathcal{E}^0(S)$ is a parametric integrand, see [13, Vol. II, Sec. 3.3.1].

More precisely, denote by $\mathcal{T}(U \times \mathbb{S}^N)$ the class of i.m. rectifiable currents $\Sigma \in \mathcal{R}_1(U \times \mathbb{S}^N)$ such that $\Pi_{x\#}\Sigma \in \mathrm{cart}(\mathring{I} \times \mathbb{R}^N)$. Notice that $\mathrm{Gcart}(U \times \mathbb{S}^N) \subset \mathcal{T}(U \times \mathbb{S}^N)$ and $\Sigma_u \in \mathcal{T}(U \times \mathbb{S}^N)$ for every $u \in BV(I, \mathbb{R}^N)$ such that $\overline{\mathcal{E}}(u) < \infty$. Now, following [13], one obtains that the energy functional $\Sigma \mapsto \mathcal{E}^0(\Sigma)$ agrees on the class $\mathcal{T}(U \times \mathbb{S}^N)$ with the *parametric convex lower semicontinuous extension* of the functional $u \mapsto \mathcal{E}(u)$. This implies the lower semicontinuity property ii).

To show property iii), we first see that from the decomposition (6.1) we get

$$\mathcal{E}^0(\Sigma) = \mathcal{E}^0(GG_u^a) + \mathcal{E}^0(GG_u^C) + \mathcal{E}^0(\widetilde{\Sigma})\,,$$

where the three terms are correspondingly computed as in Definition 7.1. Now, from the definition of GG_u^a, arguing as in the smooth case, see (7.1), for the absolute continuous component we get

$$\mathcal{E}^0(GG_u^a) = \int_I |\dot{c}_u|(1+k_u)\,dt\,.$$

As to the Cantor component we similarly obtain $\mathcal{E}^0(GG_u^C) = |D^C u|(I) + |D^C \tau_u|(I)$. □

The energy on Gauss graphs. Let now $\Sigma = \Sigma_u$ be the Gauss graph of a BV-function u such that $\overline{\mathcal{E}}(u) < \infty$, see Sec. 5. Notice that the current Σ_u actually decomposes as (6.1), where the third term $\widetilde{\Sigma} = \widetilde{\Sigma}_u$ is given by the sum of the Jump, Jump corner and corner components. Then Σ_u is of the type $[\![\,\mathcal{M}, 1, \xi\,]\!]$, so that the energy $\mathcal{E}^0(\Sigma_u)$ is well defined. We thus have

$$\mathcal{E}^0(\Sigma_u) = \int_I |\dot{c}_u|(1+k_u)\,dt + |D^C u|(I) + |D^C \tau_u|(I) + \mathcal{E}^0(\widetilde{\Sigma}_u)\,, \quad \widetilde{\Sigma}_u = GG_u^J + S_u^{Jc} + S_u^c.$$

Moreover, we recall from Sec. 5 that

$$\widetilde{\Sigma}_u = \sum_{t \in J_u} \big(-[\![\,\Gamma_t^-(u)\,]\!] + [\![\,\widetilde{\gamma}_t(u)\,]\!] + [\![\,\Gamma_t^+(u)\,]\!]\big) + \sum_{t \in J_{\dot{u}} \setminus J_u} [\![\,\Gamma_t(u)\,]\!]$$

where the oriented arcs satisfy:

i) $\widetilde{\gamma}_t(u)$ is the line segment in $\{t\} \times \mathbb{R}^N \times \mathbb{S}^N_+$ connecting the points $P_\pm(t)$ defined by (5.7), so that $\partial[\![\,\widetilde{\gamma}_t(u)\,]\!] = \delta_{P_+(t)} - \delta_{P_-(t)}$

ii) $\Gamma_t^\pm(u)$ is a geodesic arc in $\{t\} \times \mathbb{R}^N \times \mathbb{S}^N_+$ with initial point $P_\pm(t)$ and final point $\Phi_u(t_\pm)$, so that $\partial[\![\,\Gamma_t^\pm(u)\,]\!] = \delta_{\Phi_u(t_\pm)} - \delta_{P_\pm(t)}$

iii) for any $t \in J_u$ we thus have $\partial\big(-[\![\,\Gamma_t^-(u)\,]\!] + [\![\,\widetilde{\gamma}_t(u)\,]\!] + [\![\,\Gamma_t^+(u)\,]\!]\big) = \delta_{\Phi_u(t_+)} - \delta_{\Phi_u(t_-)}$

iv) for any $t \in J_{\dot{u}} \setminus J_u$, instead, $\Gamma_t(u)$ is a geodesic arc in $\{c_u(t)\} \times \mathbb{S}^N_+$ with initial point $\Phi_u(t_-)$ and final point $\Phi_u(t_+)$, so that again $\partial[\![\,\Gamma_t(u)\,]\!] = \delta_{\Phi_u(t_+)} - \delta_{\Phi_u(t_-)}$.

We now see how the energy on smooth curves splits into the sum of the length of the projections.

Remark 7.3 Let $\gamma : [-M, M] \to U \times \mathbb{S}^N$ denote a simple Lipschitz curve, that decomposes as $\gamma = (\Pi_x(\gamma), \Pi_y(\gamma))$. Setting $\Gamma := \gamma_\# [\![-M, M]\!]$, we have $\Gamma = [\![\mathcal{M}, 1, \xi_\Gamma]\!]$ where $\mathcal{M} = \gamma([-M, M])$ and $\xi_\Gamma(x, y) = \dot\gamma(s)/|\dot\gamma(s)|$ if $(x, y) = \gamma(s)$, for a.e. $s \in [-M, M]$. By the area formula, we then compute

$$\begin{aligned} \mathcal{E}^0(\Gamma) &= \int_{\gamma([-M,M])} \big(|\xi_\Gamma^{(x)}| + |\xi_\Gamma^{(y)}|\big)\, d\mathcal{H}^1 \\ &= \int_{-M}^{M} |\Pi_x(\dot\gamma(s))|\, ds + \int_{-M}^{M} |\Pi_y(\dot\gamma(s))|\, ds = \mathcal{L}(\Pi_x(\gamma)) + \mathcal{L}(\Pi_y(\gamma))\,. \end{aligned}$$

In our case, we thus readily obtain by this remark:

$$\mathcal{E}^0(\widetilde\Sigma_u) = \sum_{t \in J_u} \big(\mathcal{H}^1(\Gamma_t^+(u)) + \mathcal{H}^1(\widetilde\gamma_t(u)) + \mathcal{H}^1(\Gamma_t^-(u))\big) + \sum_{t \in J_{\dot u} \setminus J_u} \mathcal{H}^1(\Gamma_t(u))$$

and hence, on account of (5.13),

$$\mathcal{E}^0(\widetilde\Sigma_u) = \mathbf{M}(GG_u^J) + \mathbf{M}(S_u^c) + \mathbf{M}(S_u^{Jc})\,.$$

We have thus proved:

Corollary 7.4 *For every* $u \in \mathcal{E}(I, \mathbb{R}^N)$ *the current* Σ_u *defined in* (5.1) *satisfies*

$$\mathcal{E}^0(\Sigma_u) = \int_I |\dot c_u|(1 + k_u)\, dt + |D^C u|(I) + |D^C \tau_u|(I) + \mathbf{M}(GG_u^J) + \mathbf{M}(S_u^c) + \mathbf{M}(S_u^{Jc})\,. \tag{7.2}$$

Remark 7.5 From the mass estimates after (5.12), see also (3.6), we deduce that

$$2^{-1/2}\, \mathcal{E}^0(\Sigma_u) \le \mathbf{M}(\Sigma_u) \le \mathcal{E}^0(\Sigma_u)$$

and hence:

Corollary 7.6 *For every* $u \in \mathcal{E}(I, \mathbb{R}^N)$ *we have*

$$\mathbf{M}(\Sigma_u) < \infty \iff \mathcal{E}^0(\Sigma_u) < \infty\,.$$

Example 7.7 Returning to Example 6.2, that refers to Example 1.5, we recall that $\Sigma_u = GG_u + S_u^{Jc}$, where $u : [-1, 1] \to \mathbb{R}$ is the piecewise constant function in (6.3), and the Jump-corner component S_u^{Jc} is defined by (6.4), i.e. it is the sum of two oriented arcs in $\{c_u(0_\pm)\} \times \mathbb{S}^1_+$ both of length $\pi/2$. By using (6.5) and (7.2), we thus obtain

$$\mathcal{E}^0(\Sigma_u) = 2(1 + \pi) + 2 \cdot \frac{\pi}{2}\,,$$

so that the expected formula from (1.9) holds, as with our choices $\lambda_1 = \lambda_2 = 1$ and $g^\infty = 1$.

For our purposes, we now write an equivalent formula for the energy (7.2):

Proposition 7.8 *If* $u \in \mathcal{E}(I, \mathbb{R}^N)$, *we have*

$$\mathcal{E}^0(\Sigma_u) = |Dc_u|(I) + |D\tau_u|(I \setminus J_u) + \mathbf{M}(S_u^{Jc})\,,$$

where the jump-corner *term* $\mathbf{M}(S_u^{Jc})$ *is given by formula* (5.13).

Proof: In fact, recalling that $\mathbf{M}(GG_u^J) = |D^J u|(I)$ and that the graph map $c_u(t) = (t, u(t))$ satisfies

$$|D^a c_u|(I) = \int_I |\dot c_u(t)|\, dt\,, \qquad |D^C c_u|(I) = |D^C u|(I)\,, \qquad |D^J c_u|(I) = |D^J u|(I)\,,$$

where $|\dot c_u| = \sqrt{1 + |\dot u|^2}$, we have

$$\int_I |\dot c_u|\, dt + |D^C u|(I) + \mathbf{M}(GG_u^J) = |Dc_u|(I)\,.$$

Since moreover the term $\mathbf{M}(S_u^c)$ is given by the formula (5.13), we check that

$$|D^a\tau_u|(I) = \int_I |\dot{c}_u(t)|\, k_u\, dt\,, \qquad |D^J\tau_u|(I \setminus J_u) = \mathbf{M}(S_u^c)\,.$$

Using that $|D^C\tau_u|(I) = |D^C\tau_u|(I \setminus J_u)$, we get

$$\int_I |\dot{c}_u|\, k_u\, dt + |D^C\tau_u|(I) + \mathbf{M}(S_u^c) = |D\tau_u|(I \setminus J_u)\,.$$

The claim follows from (7.2). □

The case of codimension one. Assume now that $N = 1$. In this case the curvature $k_u := |\widetilde{k}_u|$ is defined a.e. in I by (3.4), but in terms of the first and second approximate gradient of u, whence

$$|\dot{c}_u(t)|\, k_u(t) = |\dot{v}(t)|\,, \quad |D\tau_u|(I) = |Dv|(I)\,, \tag{7.3}$$

where $v := \arctan \dot{u} \in BV(I)$. Recalling from [8] the formulas (4.2) and (4.3) for the relaxed energy, we readily obtain:

Corollary 7.9 *Let $N = 1$ and $u \in L^1(I, \mathbb{R})$ be such that $\overline{\mathcal{E}}(u) < \infty$. Then*

$$\overline{\mathcal{E}}(u) = \mathcal{E}^0(\Sigma_u)$$

where the energy $\mathcal{E}^0(\Sigma_u)$ is given by Proposition 7.8.

Proof: In fact, by (7.3) we have

$$|D \arctan \dot{u}|(I \setminus J_u) = |D\tau_u|(I \setminus J_u)\,.$$

Moreover, comparing formula (5.13) for the mass of the jump-corner component S_u^{Jc} with the explicit computation for the last addendum in (4.3), in the case of the curvature functional, we readily check that

$$\mathbf{M}(S_u^{Jc}) = \sum_{t \in J_u} \Phi(\nu_u(t), \dot{u}(t_-), \dot{u}(t_+))\,.$$

The claim follows from the formulas (4.2) and (4.3), on account of Proposition 7.8. □

8 Energy bounds

In the case of codimension one, in Corollary 7.9 we deduced that for every $u \in \mathcal{E}(I, \mathbb{R})$

$$\overline{\mathcal{E}}(u) = \mathcal{E}^0(\Sigma_u)\,.$$

We will show, see Corollary 9.1, that the above formula holds true in higher codimension $N \geq 1$, too. More precisely, in the first part of this section we shall prove the lower bound "$\geq$", Theorem 8.1. In the second part we shall prove the upper bound "$\leq$" by means of the *density theorem* 8.5.

The energy lower bound. For any $\Sigma \in \mathrm{Gcart}(U \times \mathbb{S}^N)$, we shall denote by u_Σ the function $u \in BV(I, \mathbb{R}^N)$ for which decomposition (6.1) holds. Correspondingly, we define

$$\mathrm{Gcart}_u := \{\Sigma \in \mathrm{Gcart}(U \times \mathbb{S}^N) \mid u_\Sigma = u\}\,, \qquad u \in BV(I, \mathbb{R}^N)\,. \tag{8.1}$$

By the definition of the class $\mathrm{Gcart}(U \times \mathbb{S}^N)$, according to (4.1) we clearly have

$$u \in \mathcal{E}(I, \mathbb{R}^N) \Longrightarrow \mathrm{Gcart}_u \neq \emptyset\,.$$

On the other hand, since the weak convergence $GG_{u_h} \rightharpoonup \Sigma$ implies the weak convergence $u_h \rightharpoonup u = u_\Sigma$ in the BV-sense, we conversely deduce:

$$\forall\, u \in BV(I, \mathbb{R}^N), \quad \mathrm{Gcart}_u \neq \emptyset \Longrightarrow u \in \mathcal{E}(I, \mathbb{R}^N)\,.$$

Theorem 8.1 (Energy lower bound). *For every* $u \in \mathcal{E}(I, \mathbb{R}^N)$ *we have* $\overline{\mathcal{E}}(u) \geq \mathcal{E}^0(\Sigma_u)$, *where* $\mathcal{E}^0(\Sigma_u)$ *is given by Corollary* 7.4.

Proof: Choose a sequence $\{u_h\} \subset C^2(I, \mathbb{R}^N)$ such that $u_h \to u$ in $L^1(I, \mathbb{R}^N)$. We have to show that

$$\mathcal{E}^0(\Sigma_u) \leq \liminf_{h\to\infty} \mathcal{E}(u_h)\,.$$

Without loss of generality, we assume that the above lower limit is a finite limit, and that GG_{u_h} weakly converges to a current $\Sigma \in \mathrm{Gcart}(U \times \mathbb{S}^N)$. Since by the L^1-convergence $u_\Sigma = u$, then $\Sigma \in \mathrm{Gcart}_u$. By lower semicontinuity we have $\liminf_h \mathcal{E}(u_h) \geq \mathcal{E}^0(\Sigma)$, and hence we readily obtain that

$$\overline{\mathcal{E}}(u) \geq \inf\{\mathcal{E}^0(\Sigma) \mid \Sigma \in \mathrm{Gcart}_u\}\,.$$

Therefore, the claim follows if we show that $\mathcal{E}^0(\Sigma) \geq \mathcal{E}^0(\Sigma_u)$ for every $\Sigma \in \mathrm{Gcart}_u$.

Now, the decomposition formula (6.1) holds true for any $\Sigma \in \mathrm{Gcart}_u$. Therefore, on account of Proposition 7.2 and formula (7.2) the inequality $\mathcal{E}^0(\Sigma) \geq \mathcal{E}^0(\Sigma_u)$ holds true if we show that

$$\mathcal{E}^0(\widetilde{\Sigma}) \geq \mathbf{M}(GG_u^J) + \mathbf{M}(S_u^c) + \mathbf{M}(S_u^{Jc})\,. \tag{8.2}$$

To this purpose, we make use of the mass decomposition given by property vi) from the structure theorem 6.1. In particular, from Definition 7.1 we readily obtain the energy estimate:

$$\mathcal{E}^0(\widetilde{\Sigma}) \geq \sum_{t\in J_{\Phi_u}} \mathcal{E}^0(\Gamma_{t,\Sigma})\,.$$

As a consequence, by (5.13) we deduce that the lower bound (8.2) holds true provided that the two following properties are verified:

(a) if $t \in J_{\dot{u}} \setminus J_u$, then $\mathcal{E}^0(\Gamma_{t,\Sigma}) \geq \mathcal{H}^1(\Gamma_t(u))$;

(b) if $t \in J_u$, then $\mathcal{E}^0(\Gamma_{t,\Sigma}) \geq \mathcal{H}^1(\Gamma_t^+(u)) + \mathcal{H}^1(\widetilde{\gamma}_t(u)) + \mathcal{H}^1(\Gamma_t^-(u))$.

Property (a) is readily checked, by the minimality of the geodesic arc $\Gamma_t(u)$. In fact, if $t \in J_{\dot{u}} \setminus J_u$ we have $\Phi_{u\pm}(t) = (t, u(t), \tau_{u\pm}(t))$, and the i.m. rectifiable 1-current $\Gamma_{t,\Sigma}$ is supported in $\{t\} \times \mathbb{R}^N \times \mathbb{S}^N_+$ and has boundary given by (6.2). Therefore, the mass of $\Gamma_{t,\Sigma}$ is bounded from below by the length of a geodesic arc in $\mathbb{S}^N_+$ connecting the points $\tau_{u\pm}(t)$, i.e. by $\mathcal{H}^1(\Gamma_t(u))$.

In order to prove property (b), we fix a Jump point $t \in J_u$ and recall that $\Gamma_{t,\Sigma}$ is an a-cyclic (or indecomposable) i.m. rectifiable current in $\mathcal{R}_1(U \times \mathbb{S}^N)$ with boundary given by (6.2).

By Federer's structure theorem [12, 4.2.25], we find a Lipschitz and injective function $\gamma_t : [-M, M] \to U \times \mathbb{S}^N$ such that $|\dot{\gamma}_t| = 1$ a.e., $2M = \mathbf{M}(\Gamma_{t,\Sigma})$, and $\gamma_{t\#}[\![-M, M]\!] = \Gamma_{t,\Sigma}$.

Therefore, as in Remark 7.3 we obtain

$$\mathcal{E}^0(\Gamma_{t,\Sigma}) = \mathcal{L}(\Pi_x(\gamma_t)) + \mathcal{L}(\Pi_y(\gamma_t))\,.$$

Moreover, using the boundary condition (6.2) we deduce that

$$\delta_{\gamma_t(M)} - \delta_{\gamma_t(-M)} = \partial\gamma_{t\#}[\![-M, M]\!] = \delta_{\Phi_u(t_+)} - \delta_{\Phi_u(t_-)}$$

and hence that $\gamma_t(\pm M) = \Phi_u(t_\pm) = (c_u(t_\pm), \tau_u(t_\pm))$. Since then $\Pi_x \circ \gamma_t(\pm M) = c_u(t_\pm)$, we estimate

$$\mathcal{L}(\Pi_x(\gamma_t)) \geq |c_u(t_+) - c_u(t_-)| = |[u(t)]|$$

where $|[u(t)]| = \mathcal{H}^1(\widetilde{\gamma}_t(u))$, whence

$$\mathcal{L}(\Pi_x(\gamma_t)) \geq \mathcal{H}^1(\widetilde{\gamma}_t(u))\,.$$

Therefore, property (b) holds if we show that

$$\mathcal{L}(\Pi_y(\gamma_t)) \geq \mathcal{H}^1(\Gamma_t^+(u)) + \mathcal{H}^1(\Gamma_t^-(u))\,. \tag{8.3}$$

For this purpose, we first exploit the geometric property from Theorem 6.3 to prove the following

Proposition 8.2 *There exists an oriented rectifiable curve c_t in U with initial point $c_u(t_-)$, final point $c_u(t_+)$, initial velocity $\tau_u(t_-)$, and final velocity $\tau_u(t_-)$ such that*

$$\mathcal{L}(\Pi_y(\gamma_t)) \geq \mathrm{TC}(c_t)\,.$$

Proof: Recalling that $\Psi_\#[\![I_{\widetilde{L}}]\!] = \Sigma$ and $\gamma_{t\#}[\![-M, M]\!] = \Gamma_{t,\Sigma}$, property (6.6) clearly implies the analogous one concerning the orienting vector $\dot\gamma_t$, namely that for a.e. $\lambda \in [-M, M]$ such that $|\Pi_x(\dot\gamma_t(\lambda))| \neq 0$

$$\frac{\Pi_x(\dot\gamma_t(\lambda))}{|\Pi_x(\dot\gamma_t(\lambda))|} = \Pi_y(\gamma_t(\lambda)) \in \mathbb{S}^N_+\,.$$

Since moreover $\Pi_y \circ \gamma_t(\pm M) = \tau_u(t_\pm)$ and $\Pi_x \circ \gamma_t(\pm M) = c_u(t_\pm)$, and the function γ_t is Lipschitz-continuous, we obtain the existence of the required curve c_t. □

By means of the average estimate from Proposition 1.4, we now prove the following inequality:

Proposition 8.3 *Let c_t be the curve given by Proposition 8.2. We have*

$$\mathrm{TC}(c_t) \geq \mathcal{H}^1(\Gamma^+_t(u)) + \mathcal{H}^1(\Gamma^-_t(u))\,.$$

In fact, this property implies (8.3), by Proposition 8.2, and concludes the proof of Theorem 8.1. □

Proof of Proposition 8.3: Consider a polygonal curve $\mathcal{P}_t$ in $\mathbb{R}^{N+1}$ given by three consecutive line segments, the first one oriented by $\tau_u(t_-)$, the second one by $(0, [u(t)])$, where $[u(t)] := u(t_+) - u(t_-)$, and the third one by $\tau_u(t_+)$. The total curvature $\mathrm{TC}(\mathcal{P}_t)$ is equal to the sum of the two corresponding turning angles, that are equal to the length of the arcs $\Gamma^\pm_t(u)$, i.e. to $\mathcal{H}^1(\Gamma^\pm_t(u))$. We thus have

$$\mathrm{TC}(\mathcal{P}_t) = \mathcal{H}^1(\Gamma^+_t(u)) + \mathcal{H}^1(\Gamma^-_t(u))\,.$$

We apply the average formula from Proposition 1.4, with $n = N+1$ and $k = 2$. More precisely, if μ_2 is the Haar measure associated to the Grassmannian $G_2\mathbb{R}^{N+1}$ of 2-planes in $\mathbb{R}^{N+1}$, we have

$$\mathrm{TC}(\mathcal{P}_t) = \int_{G_2\mathbb{R}^{N+1}} \mathrm{TC}(\pi_p(\mathcal{P}_t))\, d\mu_2(p)\,.$$

In a similar way, for the curve c_t we have

$$\mathrm{TC}(c_t) = \int_{G_2\mathbb{R}^{N+1}} \mathrm{TC}(\pi_p \circ c_t)\, d\mu_2(p)\,.$$

It then suffices to show that for μ_2-a.e. $p \in G_2\mathbb{R}^{N+1}$ one has

$$\mathrm{TC}(\pi_p \circ c_t) \geq \mathrm{TC}(\pi_p(\mathcal{P}_t))\,. \tag{8.4}$$

In fact, assuming that (8.4) holds true, by monotonicity of the integrals w.r.t. the measure μ_2 we get

$$\mathrm{TC}(c_t) \geq \mathrm{TC}(\mathcal{P}_t) = \mathcal{H}^1(\Gamma^+_t(u)) + \mathcal{H}^1(\Gamma^-_t(u))\,.$$

Now, the projection curve $\pi_p \circ c_t$ has end points $\pi_p(c_u(t_\pm))$, initial velocity $\pi_p(\tau_u(t_-))$, and final velocity $\pi_p(\tau_u(t_+))$. Moreover, for μ_2-a.e. projection π_p one has $\pi_p(c_u(t_-)) \neq \pi_p(c_u(t_+))$. The passage to planes is due to the following: for any such *planar* curve $\pi_p \circ c_t$ in $p \simeq \mathbb{R}^2$, the total curvature $\mathrm{TC}(\pi_p \circ c_t)$ cannot be lower than the sum of the two turning angles between the two couples of vectors $\pi_p(\tau_u(t_-))\,,\ \pi_p(0, [u(t)])$, and $\pi_p(\tau_u(t_+))\,,\ \pi_p(0, [u(t)])$. Since the sum of such two turning angles is equal to the total curvature $\mathrm{TC}(\pi_p(\mathcal{P}_t))$ of the p-projection of the polygonal curve $\mathcal{P}_t$, inequality (8.4) follows, as required. □

Corollary 8.4 *If $u \in \mathcal{E}(I, \mathbb{R}^N)$, then $\mathbf{M}(\Sigma_u) < \infty$. In particular Σ_u is an i.m. rectifiable current in $\mathcal{R}_1(U \times \mathbb{S}^N)$.*

Proof: From Theorem 8.1 and Corollary 7.6 we deduce that the current Σ_u has finite mass. Thus Σ_u is a normal current in $\mathcal{D}_1(U \times \mathbb{S}^1)$ that is concentrated on a 1-rectifiable set. Then by the rectifiable slices theorem, compare [10, Thm. 3.2], we obtain that Σ_u *is an i.m. rectifiable current in* $\mathcal{R}_1(U \times \mathbb{S}^N)$, actually an integral 1-cycle in $U \times \mathbb{S}^N$. □

The energy upper bound. We now prove a density property for the i.m. rectifiable currents Σ_u associated to the Gauss graph of functions u with finite relaxed energy.

Theorem 8.5 (Energy upper bound). *For every $u \in \mathcal{E}(I, \mathbb{R}^N)$, there exists a sequence of smooth functions $\{u_h\} \subset C^2(I, \mathbb{R}^N)$ such that $u_h \to u$ strongly in L^1, $GG_{u_h} \rightharpoonup \Sigma_u$ weakly in $\mathcal{D}_1(U \times \mathbb{S}^N)$ and $\mathcal{E}(u_h) \to \mathcal{E}^0(\Sigma_u)$ as $h \to \infty$, where $\mathcal{E}^0(\Sigma_u)$ is given by Corollary 7.4.*

The one-dimensional case. Before giving the proof of the density theorem 8.5, we recall how the corresponding approximation result is proved in the cited paper [8] for the case $N = 1$. The proof from [8] is divided in three steps. In the first step the authors assume that $\dot{u}$ is an L^∞-function, so that $D^C u = 0$, by the membership of u to the class $X(I)$ of Definition 4.1, and that the Jump set J_u is finite. In the second step they only assume that J_u is finite, and in the third one the above restriction is removed.

In Proposition 8.6 below, we shall make use of the higher codimension analogous of the following density argument, that goes back to the first step of the proof of [8, Thm. 2.5].

Assume $u \in \mathcal{E}(I, \mathbb{R})$ is in $W^{1,1}$ with $\dot{u} \in L^\infty$. Let $\{v_h\} \subset C^2(I)$ be such that $v_h \to \arctan \dot{u}$ strongly in the BV-sense, with $\sup_h \|v_h\|_\infty < \|\arctan \dot{u}\|_\infty < \pi/2$. Define $w_h(t) := \tan(v_h(t))$ and $u_h(t) := u(a) + \int_a^t w_h(s)\, ds$. One has $w_h \to \dot{u}$ strongly in L^1, hence $u_h(t) \to u(a) + \int_a^t \dot{u}(s)\, ds = u(t)$ a.e. and actually in $L^1(I)$, by dominated convergence. Moreover, one obtains that $\int_I |\dot{c}_{u_h}|\, dt \to \int_I |\dot{c}_{u_h}|\, dt$ as $h \to \infty$. Also, $\dot{u}_h = \tan v_h$, $\ddot{u}_h = (1 + \tan^2 w_h)\, \dot{v}_h$, and hence $\ddot{u}_h/(1 + \dot{u}_h^2) = \dot{v}_h$, that yields

$$\lim_{h\to\infty} \int_I |\dot{c}_{u_h}|\, k_{u_h}\, dt = \lim_{h\to\infty} \int_I |\dot{v}_h|\, dt = |D(\arctan \dot{u})|(I)$$

and we recall that $|D(\arctan \dot{u})|(I) = |D\tau_u|(I)$ in codimension $N = 1$.

Proof of Theorem 8.5: The proof is divided into four steps. We let $u \in \mathcal{E}(I, \mathbb{R}^N)$, so that $\overline{\mathcal{E}}(u) < \infty$. In the first step we assume in addition that u is a Sobolev function in $W^{1,1}(I, \mathbb{R}^N)$ with $\dot{u} \in L^\infty(I, \mathbb{R}^N)$. In the second one we only assume that $u \in W^{1,1}(I, \mathbb{R}^N)$, in the third one that u is continuous, and in the last one we deal with the more general case.

Step 1: We prove the following

Proposition 8.6 *Let $u \in \mathcal{E}(I, \mathbb{R}^N)$ be a Sobolev function in $W^{1,1}(I, \mathbb{R}^N)$ with $\dot{u} \in L^\infty(I, \mathbb{R}^N)$. There exists a smooth sequence $\{u_h\} \subset C^\infty(I, \mathbb{R}^N)$ such that $u_h \to u$ in $W^{1,1}$ and*

$$\lim_{h\to\infty} \int_I |\dot{c}_{u_h}|\, dt = \int_I |\dot{c}_u|\, dt\,, \qquad \lim_{h\to\infty} \int_I |\dot{c}_{u_h}|\, k_{u_h}\, dt = |D\tau_u|(I)\,.$$

Since $\mathbf{M}(S_u^{J_c}) = 0$ if $J_u = \emptyset$, by Proposition 7.8 we thus obtain the validity of Theorem 8.5 for the subclass of "smooth" functions u satisfying the hypotheses of Proposition 8.6.

Proof of Proposition 8.6: Recall that $\Phi_u = (c_u, \tau_u)$ is a BV-function, where $c_u(t) = (t, u(t))$ and

$$|\dot{c}_u| = \sqrt{1 + |\dot{u}|^2}\,, \qquad \tau_u^0 := \frac{1}{|\dot{c}_u|}\,, \qquad \tau_u^j := \frac{\dot{u}^j}{|\dot{c}_u|}\,, \qquad j = 1, \dots, N\,.$$

By means of a convolution argument, we may find a sequence $v_h = (v_h^1, \dots, v_h^N)$ with $v_h^j \in C^\infty(I)$ that converges strongly in the BV-sense to $(\tau_u^1, \dots \tau_u^N)$, i.e. $v_h^j \to \tau_u^j$ in $L^1(I)$ and $\int_I |\dot{v}_h^j|\, dt \to |D\tau_u^j|(I)$ as $h \to \infty$. Since moreover $\|\dot{u}\|_\infty < \infty$, the vector $(\tau_u^1, \dots, \tau_u^N)$ belongs to $B^N(0, 1 - 2\varepsilon)$ for some $\varepsilon > 0$, thus $\|v_h\|_\infty \le 1 - \varepsilon$ for large h and defining $v_h^0 := \sqrt{1 - |v_h|^2}$ we similarly deduce that

the sequence $\{v_h^0\} \subset C^\infty(I)$ converges strongly in the BV-sense to τ_u^0, i.e. $v_h^0 \to \tau_u^0$ in $L^1(I)$ and $\int_I |\dot v_h^0|\,dt \to |D\tau_u^0|(I)$. Moreover, we compute

$$\dot v_h^0 = -\frac{v_h \bullet \dot v_h}{\sqrt{1-|v_h|^2}}\,.$$

Setting then for $j = 1, \dots, N$

$$w_h^j(t) := \frac{v_h^j(t)}{\sqrt{1-|v_h(t)|^2}}\,, \qquad u_h^j(t) := u^j(a) + \int_a^t w_h^j(s)\,ds\,, \qquad t \in I$$

we now check the following convergences as $h \to \infty$:

i) $w_h^j \to \dot u^j$ strongly in L^1, for each j;

ii) $u_h^j(t) \to u^j(a) + \displaystyle\int_a^t \dot u^j(s)\,ds = u^j(t)$ a.e. and strongly in $L^1(I)$;

iii) $\displaystyle\int_I \sqrt{1+|\dot u_h|^2}\,dt \to \int_I \sqrt{1+|\dot u|^2}\,dt$, hence $u_h \to u$ in $W^{1,1}(I,\mathbb{R}^N)$;

iv) $\displaystyle\int_I \sqrt{|\dot v_h|^2 + (\dot v_h^0)^2}\,dt \to |D\tau_u|(I)$.

In fact, properties i) and ii) hold true by a.e. convergence, using Lebesgue theorem, and the convergence of the integral in property iii) is similarly obtained, so that the $W^{1,1}$-convergence follows from an observation in [1, Thm. 2.2], as a consequence of a continuity theorem by Reshetnyak. Finally, property iv) holds true as the sequence $(v_h^0, v_h) : I \to \mathbb{S}^N$ converges to τ_u strongly in the BV-sense.

We now claim that

$$|\dot c_{u_h}|\,k_{u_h} = \left(|\dot v_h|^2 + \frac{(v_h \bullet \dot v_h)^2}{1-|v_h|^2}\right)^{1/2} = \sqrt{|\dot v_h|^2 + (\dot v_h^0)^2}\,. \tag{8.5}$$

This claim concludes the proof, by property iv). In order to prove (8.5), we compute

$$\dot u_h^j = w_h^j = \frac{v_h^j}{\sqrt{1-|v_h|^2}}\,, \quad \ddot u_h^j = \frac{1}{(1-|v_h|^2)^{3/2}}\left\{(1-|v_h|^2)\,\dot v_h^j + (v_h \bullet \dot v_h)\,v_h^j\right\}.$$

Using that $(1+|\dot u_h|^2) = (1-|v_h|^2)^{-1}$, we thus have

$$|\ddot u_h|^2\,(1+|\dot u_h|^2) = \frac{1}{(1-|v_h|^2)^4}\left\{(1-|v_h|^2)^2\,|\dot v_h|^2 + (2-|v_h|^2)\,(v_h \bullet \dot v_h)^2\right\}$$

whereas

$$\dot u_h \bullet \ddot u_h = \frac{v_h \bullet \dot v_h}{(1-|v_h|^2)^2}$$

so that we get

$$|\ddot u_h|^2\,(1+|\dot u_h|^2) - (\dot u_h \bullet \ddot u_h)^2 = \frac{1}{(1-|v_h|^2)^3}\left\{(1-|v_h|^2)\,|\dot v_h|^2 + (v_h \bullet \dot v_h)^2\right\}.$$

Therefore, recalling formula (1.6), we obtain

$$k_{u_h} := \frac{\left(|\ddot u_h|^2(1+|\dot u_h|^2) - (\dot u_h \bullet \ddot u_h)^2\right)^{1/2}}{(1+|\dot u_h|)^{3/2}} = \left\{(1-|v_h|^2)\,|\dot v_h|^2 + (v_h \bullet \dot v_h)^2\right\}^{1/2}$$

and finally (8.5), using that $|\dot c_{u_h}| = \sqrt{1+|\dot u_h|^2}$. □

Step 2: We prove the following

Proposition 8.7 *Let $u \in \mathcal{E}(I,\mathbb{R}^N) \cap W^{1,1}$. There exists a sequence $\{u_h\} \subset W^{1,\infty}(I,\mathbb{R}^N)$ such that*

$$\Sigma_{u_h} \rightharpoonup \Sigma_u \quad \textit{weakly in } \mathcal{D}_1(I \times \mathbb{S}^N) \quad \textit{and} \quad \mathcal{E}^0(\Sigma_{u_h}) \to \mathcal{E}^0(\Sigma_u) \tag{8.6}$$

as $h \to \infty$.

Now, weak convergence together with convergence in energy clearly yield convergence of $\{u_h\}$ to u strongly in the BV-sense, whence in L^1. Therefore, by Proposition 7.8 and a diagonal argument we deduce that Proposition 8.7 implies the validity of Theorem 8.5 for the subclass of functions $u \in W^{1,1}(I,\mathbb{R}^N)$.

Proof of Proposition 8.7: We shall use a truncation argument for τ_u. To this purpose, since $\tau_u \in BV(I,\mathbb{R}^{N+1})$, we recall that the left and right limits $\tau_u(t_\pm)$ are everywhere well-defined, and that both $\tau_{u\pm}(t) := \tau_u(t_\pm) \in \mathbb{S}^N_+$ are good representatives that agree outside an at most countable set. If $\tau^0_u(t_\pm) \in]0,1]$, then $\arctan|\dot u(s)| \to \theta^\pm_u(t) \in [0,\pi/2[$ as $s \to t^\pm$, and the limit of $\dot u(s)$ as $s \to t^\pm$ is finite, too. Otherwise, if $\tau^0_u(t_\pm) = 0$, then $\arctan|\dot u(s)| \to \pi/2$ as $s \to t^\pm$, and

$$\lim_{s\to t^\pm} \frac{|\dot u(s)|}{\sqrt{1+|\dot u(s)|^2}} = 1\,.$$

This time we again have the existence of the limit

$$\frac{\dot u}{|\dot u|}(t_\pm) := \lim_{s\to t^\pm} \frac{\dot u(s)}{|\dot u(s)|} = \lim_{s\to t^\pm} \frac{\sqrt{1+|\dot u(s)|^2}}{|\dot u(s)|} \cdot (\tau^1_u,\dots,\tau^N_u)(s) = \lim_{s\to t^\pm} (\tau^1_u,\dots,\tau^N_u)(s).$$

Therefore, we can write for every $t \in I$

$$\tau_u(t_\pm) = \Big(\cos\theta^\pm_u(t), \sin\theta^\pm_u(t)\,\frac{\dot u}{|\dot u|}(t_\pm)\Big)\,, \qquad \theta^\pm_u(t) := \lim_{s\to t^\pm} \arctan|\dot u(s)| \in [0,\pi/2]\,.$$

Truncation. On account of the previous remark, we choose a positive and increasing sequence of angles $\{\theta_h\} \nearrow \pi/2$ and we truncate the BV-function τ_u by setting $\mathcal{L}^1$-a.e.

$$\tau_h(t) := \begin{cases} \tau_u(t) & \text{if} \quad \theta_u(t) \in [0,\theta_h] \\ \Big(\cos\theta_h, \sin\theta_h\,\dfrac{\dot u(t)}{|\dot u(t)|}\Big) & \text{if} \quad \theta_u(t) \in [\theta_h,\pi/2] \end{cases}$$

where we have set

$$\theta_u(t) := \theta^+_u(t) = \theta^-_u(t)\,, \qquad \frac{\dot u(t)}{|\dot u(t)|} := \frac{\dot u}{|\dot u|}(t_+) = \frac{\dot u}{|\dot u|}(t_-)\,.$$

Notice that if $\theta_u(t) \in [0,\theta_h]$, then

$$\tau_u(t) = \Big(\cos\theta_u(t), \sin\theta_u(t)\,\frac{\dot u(t)}{|\dot u(t)|}\Big)\,, \qquad \tan\theta_u(t) = |\dot u(t)|\,.$$

By dominated convergence, we obtain that $\tau_h \to \tau_u$ strongly in L^1.

Approximating sequence. Define $w_h = (w^1_h,\dots,w^N_h) : I \to \mathbb{R}^N$ by

$$w^j_h(t) := \frac{\tau^j_h(t)}{\tau^0_h(t)}\,, \qquad j = 1,\dots,N$$

so that $w_h(t) = \dot u(t)$ if $\theta_u(t) \in [0,\theta_h]$, and $w_h(t) = \tan\theta_h\,\dfrac{\dot u(t)}{|\dot u(t)|}$ if $\theta_u(t) \in [\theta_h,\pi/2]$. By dominated convergence we have

$$\lim_{h\to\infty} \int_I \sqrt{1+|w_h|^2}\,dt = \int_I \sqrt{1+|\dot u|^2}\,dt\,,$$

whence $w_h \to \dot{u}$ strongly in L^1. Setting then $u_h : I \to \mathbb{R}^N$ by

$$u_h(t) := u(a) + \int_a^t w_h(s)\, ds\,, \qquad t \in I = [a,b]$$

we clearly have that $u_h \to u$ strongly in $W^{1,1}$, with $\{u_h\} \subset W^{1,\infty}(I,\mathbb{R}^N)$. Since $J_{u_h} = J_u = \emptyset$ and $S^{Jc}_{u_h} = S^{Jc}_u = 0$, on account of Proposition 7.8 we readily infer that the weak convergence with the energy (8.6) holds true if we show that

$$\lim_{h\to\infty} |D\tau_{u_h}|(I) = |D\tau_u|(I)\,. \tag{8.7}$$

Now, the above convergence of the total variation of the distributional derivatives follows from the fact that actually $\tau_{u_h} = \Lambda_h \circ \tau_u$ for every h, where $\Lambda_h : \mathbb{S}^N_+ \to \mathbb{S}^N_+$ is the smooth retraction function with Lipschitz constant $\operatorname{Lip}\Lambda_h = 1$ defined by

$$\Lambda_h(y^0,\widetilde{y}) := \begin{cases} (y^0,\widetilde{y}) & \text{if} \quad y^0 \geq \cos\theta_h \\ \Big(\cos\theta_h, \sin\theta_h \dfrac{\widetilde{y}}{|\widetilde{y}|}\Big) & \text{if} \quad y^0 \leq \cos\theta_h \end{cases} \qquad y = (y^0,\widetilde{y}) \in \mathbb{R}\times\mathbb{R}^N$$

which concludes the proof of Proposition 8.7. □

Step 3: We now assume that u is BV and continuous. In the proof of Step 3 we make use of a result which will be presented in the next section, but whose proof only uses Proposition 8.7 above. As before, the Jump component $D^J u = 0$, but this time in general the Cantor component $D^C u$ is non-trivial. By Step 2 and a diagonal argument, it clearly suffices to show the existence of a sequence $\{u_h\} \subset W^{1,1}(I,\mathbb{R}^N)$ such that

$$\lim_{h\to\infty} \int_I \sqrt{1+|\dot{u}_h|^2}\, dt = \int_I \sqrt{1+|\dot{u}|^2}\, dt + |D^C u|(I)\,, \tag{8.8}$$

$\{\Sigma_{u_h}\}$ converges to Σ_u weakly in $\mathcal{D}_1(U\times\mathbb{S}^N)$, and also (8.7) holds true. By the construction which will be performed in Theorem 9.2, the above property follows from (9.4), on account of (9.2).

Remark 8.8 The convergence (8.7) in Step 3 extends to the higher codimension case the property already observed in [8] when $N = 1$, namely that the occurrence of a Cantor part of Du does not give a contribution to the relaxed energy.

We point out that the argument used at page 2369 of [8] is correct if one assumes in addition that u is continuous. Otherwise, with the notation from [8], one cannot conclude in general that at the end points of the interval I^+_{kj} the property "$(u')^a_\wedge \leq k$" holds, if such end points belong to the Jump set of u. The proof of [8, Thm. 2.5] may be modified by assuming at this point that u is continuous, and by treating at the following step the general case when $J_u \neq \emptyset$.

Step 4: We finally remove the additional assumptions on $u \in \mathcal{E}(I,\mathbb{R}^N)$ so that in general $D^J u \neq 0$. By Step 3 and a diagonal argument, it suffices to find a sequence of continuous functions $\{u_h\} \subset BV(I,\mathbb{R}^N)$ such that

$$\lim_{h\to\infty} \Big(\int_I \sqrt{1+|\dot{u}_h|^2}\, dt + |D^C u_h|(I) \Big) = \int_I \sqrt{1+|\dot{u}|^2}\, dt + |D^C u|(I) + |D^J u|(I) \tag{8.9}$$

and $\{\Sigma_{u_h}\}$ converges to Σ_u weakly in $\mathcal{D}_1(U\times\mathbb{S}^N)$. Moreover, again by Proposition 7.8, the convergence in energy holds true if we prove in addition that

$$\lim_{h\to\infty} |D\tau_{u_h}|(I) = |D\tau_u|(I\setminus J_u) + \mathbf{M}(S^{Jc}_u)\,. \tag{8.10}$$

We first observe that it suffices to consider the case when the Jump set J_u is finite. In fact, if J_u is countable, say $J_u = \{t_1, t_2, \ldots\}$, setting $J^h_u = \{t_1,\ldots,t_h\}$ one defines

$$\widetilde{u}_h(t) := u_+(a) + \int_a^t \dot{u}(s)\, ds + D^C u(]a,t[) + \sum_{s<t\,,\, s\in J^h_u} [u](s)\,. \tag{8.11}$$

It is then readily checked that $\widetilde{u}_h \to u$ strongly in L^1, $\Sigma_{\widetilde{u}_h} \rightharpoonup \Sigma_u$ weakly in $\mathcal{D}_1(U\times\mathbb{S}^N)$, and $\mathcal{E}^0(\Sigma_{\widetilde{u}_h}) \to \mathcal{E}^0(\Sigma_u)$ as $h\to\infty$. Therefore, since the Jump set $J_{\widetilde{u}_h} = J_u^h$ is finite for each h, a diagonal argument will conclude the proof in the general case.

Assuming then that J_u is finite, we denote $J_u = \{t_i\}_{i=1}^m$, where $a := \inf I < t_1 < t_2 < \cdots < t_m < \sup I =: b$, and we let $t_0 = a$, $t_{m+1} = b$. We then choose a decreasing sequence $\delta_h \searrow 0$ such that

$$\dot u \text{ is continuous at } t_i \pm \delta_h \text{ and } \lim_{h\to\infty} \dot u(t_i \pm \delta_h) = \dot u_\pm(t_i) \tag{8.12}$$

for $i = 1,\dots,m$. Notice that for h large enough we have $t_i - t_{i-1} > 3\delta_h$ for all i.

We now define $u_h : I \to \mathbb{R}^N$ by the formula:

$$u_h(t) := \begin{cases} u_+(a) + Du(]a,t[) & \text{in } [a, t_1-\delta_h] \\ u_+(t_{i-1}+\delta_h) + Du(]t_{i-1}+\delta_h, t[) & \text{in } [t_{i-1}+\delta_h, t_i-\delta_h]\,, \quad i\ge 2 \\ u_+(t_m+\delta_h) + Du(]t_m+\delta_h, t[) & \text{in } [t_m+\delta_h, b] \\ \lambda_h^{(i)}(t) & \text{in } [t_i-\delta_h, t_i+\delta_h]\,, \quad i\ge 1\,. \end{cases}$$

In the last line of the previous definition, for each $i = 1,\dots,m$ we have chosen the affine function $\lambda_h^{(i)} : [t_i-\delta_h, t_i+\delta_h] \to \mathbb{R}^N$ such that $\lambda_h^{(i)}(t_i\pm\delta_h) = u(t_i\pm\delta_h)$.

The sequence of continuous functions $\{u_h\} \subset BV(I,\mathbb{R}^N)$ converges to u strongly in L^1, and the convergence of the total variation (8.9) of c_{u_h} is readily checked. The convergence $\Sigma_{u_h} \rightharpoonup \Sigma_u$ weakly as currents holds true once we show that (8.10) is satisfied. To this purpose, we observe that clearly

$$\lim_{h\to\infty} |D\tau_{u_h}|(I\setminus I_h) = |D\tau_u|(I\setminus J_u)$$

where we have set $I_h := \bigcup_{i=1}^m [t_i-\delta_h, t_i+\delta_h]$, so that (8.10) holds true if one has

$$\lim_{h\to\infty} |D\tau_{u_h}|(I_h) = \mathbf{M}(S_u^{Jc})$$

where the jump-corner term $\mathbf{M}(S_u^{Jc})$ is given by formula (5.13). Therefore, it suffices to show that

$$\lim_{h\to\infty} |D\tau_{u_h}|([t_i-\delta_h, t_i+\delta_h]) = \mathcal{H}^1(\Gamma_{t_i}^+(u)) + \mathcal{H}^1(\Gamma_{t_i}^-(u))$$

for $i = 1,\dots,m$, where, we recall, $\Gamma_{t_i}^\pm(u)$ denotes an oriented geodesic arc in $\{c_u(t_{i\pm})\}\times\mathbb{S}_+^N$ with initial point $P_\pm(t_i)$, see (5.7), and final point $\Phi_u(t_{i\pm}) = (t_i, u(t_{i\pm}), \tau_u(t_{i\pm}))$.

Now, by the definition of u_h on the interval $[t_i-\delta_h, t_i+\delta_h]$, it turns out that the total variation $|D\tau_{u_h}|([t_i-\delta_h, t_i+\delta_h])$ is equal to the sum of the two turning angles between the couples of vectors

$$\frac{(1,\dot u(t_i\pm\delta_h))}{|(1,\dot u(t_i\pm\delta_h)|}\,, \qquad \frac{(1, v_{i,h}^\pm)}{|(1,v_{i,h}^\pm)|}$$

where we have set

$$v_{i,h}^\pm := \frac{u(t_i+\delta_h) - u(t_i-\delta_h)}{2\delta_h}\,.$$

Since $\delta_h \searrow 0$ we have

$$\lim_{h\to\infty} \frac{(1,v_{i,h}^\pm)}{|(1,v_{i,h}^\pm)|} = \Big(0, \frac{[u(t_i)]}{|[u(t_i)]|}\Big)\,,$$

whereas by (8.12) we deduce that

$$\lim_{h\to\infty} \frac{(1,\dot u(t_i\pm\delta_h))}{|(1,\dot u(t_i\pm\delta_h)|} = \tau_u(t_{i\pm})\,.$$

This yields that the total variation $|D\tau_{u_h}|([t_i-\delta_h, t_i+\delta_h])$ converges to the sum of the two turning angles between the two couples of vectors

$$\tau_u(t_{i\pm})\,, \qquad \Big(0, \frac{[u(t_i)]}{|[u(t_i)]|}\Big)$$

that clearly agrees with the sum of the length of the two geodesic arcs $\Gamma_{t_i}^\pm(u)$, as required. □

9 Main results

This final section contains the main results of this paper. For greater clarity we shall postpone the proof of the two main theorems below to the second part of the section.

Relaxed energy. Let $N \geq 1$ and $u \in \mathcal{E}(I, \mathbb{R}^N)$, i.e. $u : I \to \mathbb{R}^N$ is an L^1-function with finite relaxed energy $\overline{\mathcal{E}}(u)$, see (1.11) and (4.1). Then $u \in BV(I, \mathbb{R}^N)$ and in Theorem 4.7 we showed that the Gauss map $\tau_u : I \to \mathbb{S}^N$ has bounded variation. Moreover, recalling the notation from (4.4) and (5.1), by Theorem 8.5 we already know that $\Sigma_u \in \mathrm{Gcart}(U \times \mathbb{S}^N)$, and actually that $\Sigma_u \in \mathrm{Gcart}_u$, see (8.1).

By Theorems 8.1 and 8.5 we thus deduce that the relaxed energy of u is equal to the least energy among all the currents in $\mathrm{Gcart}(U \times \mathbb{S}^N)$ with underlying function $u_\Sigma = u$, and that the energy minimum is attained by the optimal current Σ_u :

Corollary 9.1 *For every function u with finite relaxed energy we have*

$$\overline{\mathcal{E}}(u) = \min\{\mathcal{E}^0(\Sigma) \mid \Sigma \in \mathrm{Gcart}_u\} = \mathcal{E}^0(\Sigma_u)\,.$$

By using the explicit formula for $\mathcal{E}^0(\Sigma_u)$ from Proposition 7.8, we readily obtain formula (0.6) from the introduction:

$$\forall\, u \in \mathcal{E}(I, \mathbb{R}^N)\,, \qquad \overline{\mathcal{E}}(u) = |Dc_u|(I) + |D\tau_u|(I \setminus J_u) + \mathbf{M}(S_u^{Jc}) \tag{9.1}$$

where the jump-corner term $\mathbf{M}(S_u^{Jc})$ is given by (5.13).

Continuous functions. In case of continuous functions, we shall prove the following theorem, where we heavily exploit the geometric structure of the energy.

Theorem 9.2 *Let $u \in L^1(I, \mathbb{R}^N)$ be a continuous function. Then u has finite relaxed energy if and only if the Cartesian curve c_u has finite length and total curvature, i.e.*

$$u \in \mathcal{E}(I, \mathbb{R}^N) \iff \mathcal{L}(c_u) + \mathrm{TC}(c_u) < \infty\,.$$

In this case, moreover, the total variation $|D\tau_u|(I)$ agrees with the total curvature $\mathrm{TC}(c_u)$ of the Cartesian curve c_u, i.e.

$$|D\tau_u|(I) = \mathrm{TC}(c_u)\,. \tag{9.2}$$

Roughly speaking, the above result says that the total variation of the tantrix of a continuous function $u \in \mathcal{E}(I, \mathbb{R}^N)$ does not read jumps in presence of the Cantor part of the derivative $D^C u$. Using the polar decomposition $D^C u = g_C\, |D^C u|$, where $g_C : I \to \mathbb{S}^{N-1}$ is a Borel function, this should imply that at $|D^C u|$-a.e. point $t \in I$ one has $(0, g_C(t)) = \tau_u(t) \in \{0\} \times \mathbb{S}^{N-1}$. In the case of codimension $N = 1$, it is easy to check that this last property is actually equivalent to property (b) from Definition 4.1. This will be subject of further work.

Remark 9.3 The proof of Theorem 9.2 is postponed, and it only makes use of the results from Sec. 4 and of Proposition 8.7. Therefore, we were entitled to use it in Step 3 of the proof of Theorem 8.5.

As a consequence of Theorem 9.2, we readily obtain formula (0.5) from the introduction:

Corollary 9.4 *For every continuous function $u \in \mathcal{E}(I, \mathbb{R}^N)$ we have $\overline{\mathcal{E}}(u) = \mathcal{L}(c_u) + \mathrm{TC}(c_u)$.*

Proof: In fact, clearly $|Dc_u|(I) = \mathcal{L}(c_u)$. Since moreover $J_u = \emptyset$, by (5.13) we infer that $\mathbf{M}(S_u^{Jc}) = 0$. Also, by (9.2) we get $|D\tau_u|(I \setminus J_u) = |D\tau_u|(I) = \mathrm{TC}(c_u)$. The claim follows from the general formula (9.1). □

Functions with jumps. Let $u \in BV(I, \mathbb{R}^N)$ be possibly with discontinuities. We denote by $\widetilde{c}_u$ the oriented curve obtained by connecting the jumps in the graph of u with oriented line segments from $c_u(t_-)$ to $c_u(t_+)$ at each point $t \in J_u$, so that its length is $\mathcal{L}(\widetilde{c}_u) = |Dc_u|(I)$. Similarly to Theorem 9.2, but this time using the density theorem 8.5, we shall then prove:

Theorem 9.5 *A function $u \in L^1(I, \mathbb{R}^N)$ has finite relaxed energy if and only if the curve $\widetilde{c}_u$ has finite length and total curvature, i.e.*

$$u \in \mathcal{E}(I, \mathbb{R}^N) \iff \mathcal{L}(\widetilde{c}_u) + \mathrm{TC}(\widetilde{c}_u) < \infty\,.$$

In this case, moreover, we have

$$|D\tau_u|(I \setminus J_u) + \mathbf{M}(S_u^{J_c}) = \mathrm{TC}(\widetilde{c}_u)\,. \tag{9.3}$$

As a consequence of Theorem 9.5, we readily obtain the formula (0.7) from the introduction:

Corollary 9.6 *For every function $u \in \mathcal{E}(I, \mathbb{R}^N)$ we have $\overline{\mathcal{E}}(u) = \mathcal{L}(\widetilde{c}_u) + \mathrm{TC}(\widetilde{c}_u)$.*

Proof: In fact, this time we clearly have $|D\widetilde{c}_u|(I) = \mathcal{L}(\widetilde{c}_u)$. The claim follows from the general formula (9.1), on account of (9.3). □

Proofs. It remains to prove Theorems 9.2 and 9.5.

Proof of Theorem 9.2: If $u \in \mathcal{E}(I, \mathbb{R}^N) \cap C^0$, we already know that $u \in BV(I, \mathbb{R}^N)$ and $\tau_u \in BV(I, \mathbb{S}^N)$, whereas $D^J u = 0$. Recalling Proposition 4.5 we find a Lipschitz function $c : I_L \to \overline{U}$ such that $|\dot{c}(s)| = 1$ a.e. and the image current $c_\#[\![I_L]\!]$ agrees with the Cartesian current $T(\Sigma) = \Pi_{x\#}\Sigma$. This yields that at $\mathcal{H}^1$-a.e. point x in the support of the oriented Cartesian curve c_u the unit tangent vector is equal to $\dot{c}(s)$ for some point $s \in I$ such that $c(s) = x$. As a consequence, we may recover the total variation of the tantrix of c_u by means of the total variation of the derivative function $\dot{c}$, see Proposition 1.3, obtaining that

$$\mathrm{TC}(c_u) \le |D\dot{c}|(I_L) < \infty\,.$$

Since we already know that c_u has bounded variation, this yields that for continuous functions

$$u \in \mathcal{E}(I, \mathbb{R}^N) \Longrightarrow \mathcal{L}(c_u) + \mathrm{TC}(c_u) < \infty\,.$$

Assuming now that $\mathcal{L}(c_u) + \mathrm{TC}(c_u) < \infty$, and recalling the definition (1.4) of length and total curvature, we choose for each $h \in \mathbb{N}^+$ a partition of the interval I with mesh of order at most $1/h$, say $\{t_i^h\}_{i=1}^{m_h}$, with $a = t_0^h$, $b = t_{m_h}^h$, and $0 < t_i^h - t_{i-1}^h < 1/h$ for every i. By the uniform continuity of c_u, using the points $c_u(t_i^h)$ we find for each h a polygon P_h inscribed in the curve c_u such that $\mathrm{mesh}(P_h) \to 0$. Therefore, by Proposition 1.2 we have $\mathcal{L}(P_h) \to \mathcal{L}(c_u)$ and $\mathrm{TC}(P_h) \to \mathrm{TC}(c_u)$.

Correspondingly, we define the continuous function $u_h : I \to \mathbb{R}^N$ such that $c_{u_h}(t_i^h) = c_u(t_i^h)$ for all i, and u_h is affine in each interval $[t_{i-1}^h, t_i^h]$ of the partition. It is readily checked that u_h is a Sobolev function in $W^{1,1}(I, \mathbb{R}^N)$, with

$$\mathcal{L}(P_h) = \int_I \sqrt{1 + |\dot{u}_h|^2}\, dt\,, \qquad \mathrm{TC}(P_h) = |D\tau_{u_h}|(I)\,.$$

In fact, the total curvature of the Cartesian curve c_{u_h}, i.e. the sum of the turning angles at the edges of P_h, is equal to the total variation of the distributional derivative $D\tau_{u_h}$, as u_h is piecewise affine.

Since the length of c_u is given by

$$\mathcal{L}(c_u) = \int_I \sqrt{1 + |\dot{u}|^2}\, dt + |D^C u|(I)\,,$$

this yields that

$$\lim_{h\to\infty} \int_I \sqrt{1 + |\dot{u}_h|^2}\, dt = \int_I \sqrt{1 + |\dot{u}|^2}\, dt + |D^C u|(I)\,, \quad \lim_{h\to\infty} |D\tau_{u_h}|(I) = \mathrm{TC}(c_u) \tag{9.4}$$

whereas clearly $u_h \to u$ strongly in $L^1(I, \mathbb{R}^N)$, by the Poincaré inequality.

Now, by Proposition 8.7 we deduce for each Sobolev function $v_\infty \in W^{1,1}(I, \mathbb{R}^N) \cap \mathcal{E}(I, \mathbb{R}^N)$ the existence of a sequence of smooth functions $\{v_h\} \subset C^2(I, \mathbb{R}^N)$ such that $v_h \to v_\infty$ strongly in $W^{1,1}(I, \mathbb{R}^N)$

and $\mathcal{E}^0(\Sigma_{v_h}) \to \mathcal{E}^0(\Sigma_{v_\infty})$ as $h \to \infty$. Since $\mathcal{E}^0(\Sigma_{v_h}) = \int_I \sqrt{1+|\dot{v}_h|^2}\,dt + |D\tau_{v_h}|(I)$, the convergence $\mathcal{E}^0(\Sigma_{v_h}) \to \mathcal{E}^0(\Sigma_{v_\infty})$ together with $v_h \to v_\infty$ in $W^{1,1}$ imply that $\mathcal{L}(c_{v_h}) \to \mathcal{L}(c_{v_\infty})$ and $|D\tau_{v_h}|(I) \to |D\tau_{v_\infty}|(I)$.

Applying this density property to each u_h we find by a diagonal argument the existence of a sequence of smooth functions $\{w_h\} \subset C^2(I,\mathbb{R}^N)$ such that $w_h \to u$ strongly in the BV-sense, $\sup_h \mathcal{E}(w_h) < \infty$, and moreover

$$\mathcal{L}(c_{w_h}) \to \mathcal{L}(c_u)\,, \qquad |D\tau_{w_h}|(I) \to \mathrm{TC}(c_u)\,.$$

We have just shown the reverse implication: for each $u \in L^1(I,\mathbb{R}^N)$ continuous

$$\mathcal{L}(c_u) + \mathrm{TC}(c_u) < \infty \Longrightarrow u \in \mathcal{E}(I,\mathbb{R}^N)\,.$$

We now apply Proposition 4.5 to the function u and w.r.t. the sequence $\{w_h\}$. Since $L_h := \mathcal{L}(c_{w_h}) \to \mathcal{L}(c_u)$, and with our notation $L_h \to L$, we correspondingly find that the Lipschitz function $c : I_L \to \overline{U}$ satisfies $L = \mathcal{L}(c_u)$. Using that $|\dot{c}| = 1$ a.e., this implies that the support of the curve $c(I_L)$ agrees with the support of the image current $c_\#[\![I_L]\!]$, that is equal to the Cartesian current $T(\Sigma) = \Pi_{x\#}(\Sigma)$. Again by the convergence $\mathcal{L}(c_{w_h}) \to \mathcal{L}(c_u)$, we deduce that $T(\Sigma) = T_u := G_u + G_u^C$.

We have thus obtained the equalities

$$c_\#[\![I_L]\!] = T_u = G_u + G_u^C\,, \qquad \mathbf{M}(c_\#[\![I_L]\!]) = \mathcal{L}(c_u)\,.$$

This yields that *the gradient map* $s \mapsto \dot{c}(s)$ *agrees* $\mathcal{H}^1$*-essentially with the tantrix of the Cartesian curve* c_u, whence $\mathrm{TC}(c_u) = |D\dot{c}|(I_L)$, by Proposition 1.3.

We finally show that $|D\dot{c}|(I_L) = |D\tau_u|(I)$, which yields (9.2) and concludes the proof. For this purpose, recall that in the proof of Theorem 4.7 we have proved the formula

$$\int_I \tau_u(t) \bullet \dot{\phi}(t)\,dt = -\langle D\dot{c}, \phi \circ c^0\rangle \qquad \forall\, \phi \in C^1_c(I,\mathbb{R}^{N+1})\,. \tag{9.5}$$

Since moreover the function $c : I_L \to \overline{U}$ is the arc-length parameterization of the continuous Cartesian curve c_u, recalling (4.9) we deduce that $|I_L \setminus \widetilde{I}_L| = 0$, otherwise $J_u \neq \emptyset$, a contradiction. Therefore we have $\dot{c}^0(s) > 0$ a.e. on I_L, whence the function $c^0 : I_L \to I$ is strictly increasing, hence bijective. The function c^0 being Lipschitz, it turns out that its inverse ${c^0}^{-1} : I \to I_L$ is continuous. Therefore,

$$\phi \in C^0_c(I,\mathbb{R}^{N+1}) \Longrightarrow \widetilde{\phi} := \phi \circ c^0 \in C^0_c(I_L,\mathbb{R}^{N+1})$$

and conversely

$$\widetilde{\phi} \in C^0_c(I_L,\mathbb{R}^{N+1}) \Longrightarrow \phi := \widetilde{\phi} \circ (c^0)^{-1} \in C^0_c(I,\mathbb{R}^{N+1})\,.$$

This yields that

$$\begin{aligned} |D\dot{c}|(I_L) &:= \sup\{\langle D\dot{c}, \widetilde{\phi}\rangle \mid \widetilde{\phi} \in C^0_c(I_L,\mathbb{R}^{N+1})\,,\ \|\widetilde{\phi}\|_\infty \le 1\} \\ &= \sup\{\langle D\dot{c}, \phi \circ c^0\rangle \mid \phi \in C^0_c(I,\mathbb{R}^{N+1})\,,\ \|\phi\|_\infty \le 1\} \\ &= \sup\{\langle D\dot{c}, \phi \circ c^0\rangle \mid \phi \in C^1_c(I,\mathbb{R}^{N+1})\,,\ \|\phi\|_\infty \le 1\} \end{aligned}$$

and hence by (9.5) we obtain:

$$|D\dot{c}|(I_L) = \sup\{\langle \tau_u, \dot{\phi}\rangle \mid \phi \in C^1_c(I,\mathbb{R}^{N+1})\,,\ \|\phi\|_\infty \le 1\} = |D\tau_u|(I)\,,$$

as required. □

Proof of Theorem 9.5: Assume that $u \in \mathcal{E}(I,\mathbb{R}^N)$. By Theorem 8.5, we may and do apply Proposition 4.5 to the current $\Sigma_u \in \mathrm{Gcart}(U \times \mathbb{S}^N)$ and w.r.t. the strongly converging sequence $\{u_h\} \subset C^2(I,\mathbb{R}^N)$, so that $\mathcal{E}(u_h) \to \mathcal{E}^0(\Sigma_u)$. The Lipschitz function $c : I_L \to \overline{U}$ satisfies $|\dot{c}(s)| = 1$ a.e. and the image current $c_\#[\![I_L]\!]$ agrees with the Cartesian current $\Pi_{x\#}\Sigma_u = T_u$, see (2.1), so that $\mathbf{M}(T_u) = |Dc_u|(I) = \mathcal{L}(\widetilde{c}_u)$. By lower semicontinuity of the energy functional, we deduce that

$L = \mathbf{M}(T_u)$. Therefore, the function $c : I_L \to \overline{U}$ is the arc-length parameterization of the curve $\widetilde{c}_u$, and hence $|D\dot{c}|(I_L) = \mathrm{TC}(\widetilde{c}_u)$, again by Proposition 1.3. Whence:

$$u \in \mathcal{E}(I, \mathbb{R}^N) \Longrightarrow \mathcal{L}(\widetilde{c}_u) + \mathrm{TC}(\widetilde{c}_u) < \infty\,.$$

Assume now that $\mathcal{L}(\widetilde{c}_u) + \mathrm{TC}(\widetilde{c}_u) < \infty$. Similarly as before, we choose for each $h \in \mathbb{N}^+$ a partition of the set $I \setminus J_u$ with mesh of order at most $1/h$, say $\{t_i^h\}_{i=1}^{m_h}$, with $a = t_0^h$, $b = t_{m_h}^h$, and $0 < t_i^h - t_{i-1}^h < 1/h$ for every i. Moreover, we denote by J_h the finite set of Jump points $t \in J_u$ such that $|[u(t)]| > 1/h$. For each $t \in J_h$, we divide the line segment with end points $c_u(t_\pm)$ into a finite number of vertical segments of length lower than $1/h$. By using both the finite set in $\overline{U}$ given by the points $c_u(t_h)$, and the end points of the vertical segments this way obtained, we clearly define a polygon P_h inscribed in the curve $\widetilde{c}_u$ such that $\mathrm{mesh}(P_h) \to 0$, so that $\mathcal{L}(P_h) \to \mathcal{L}(\widetilde{c}_u)$ and $\mathrm{TC}(P_h) \to \mathrm{TC}(\widetilde{c}_u)$, by Proposition 1.2.

Correspondingly, we define for each h the function $u_h : I \to \mathbb{R}^N$ that satisfies $u_h(t_i^h) = u(t_i^h)$ for all i, $u_h(t_\pm) = u(t_\pm)$ for every $t \in J_h$, and u_h is affine inside each segment connecting two consecutive points of the set $J_h \cup \{t_i^h\}_{i=0}^{m_h}$. It turns out that u_h is a function in $BV(I, \mathbb{R}^N)$ with $D^C u_h = 0$, $J_{u_h} = J_h$, and actually $u_h \in \mathcal{E}(I, \mathbb{R}^N)$. Moreover, the related current Σ_{u_h} satisfies

$$\mathcal{E}^0(\Sigma_{u_h}) = |Dc_{u_h}|(I) + |D\tau_{u_h}|(I \setminus J_h) + \mathbf{M}(S_{u_h}^{J_c})$$

and by the definition of u_h one has:

$$\mathcal{L}(P_h) = |Dc_{u_h}|(I)\,, \qquad \mathrm{TC}(P_h) = |D\tau_{u_h}|(I \setminus J_h) + \mathbf{M}(S_{u_h}^{J_c})\,.$$

We also check that the sequence $\{u_h\}$ converges to u strongly in the BV-sense, and $\Sigma_{u_h} \rightharpoonup \Sigma_u$ weakly in $\mathcal{D}_1(U \times \mathbb{S}^N)$. Therefore, by applying to each u_h the density theorem 8.5, and by a diagonal argument, we find the existence of a sequence of smooth functions $\{w_h\} \subset C^2(I, \mathbb{R}^N)$ such that $w_h \to u$ strongly in the BV-sense, $\Sigma_{w_h} \rightharpoonup \Sigma_u$ weakly in $\mathcal{D}_1(U \times \mathbb{S}^N)$, and moreover

$$\mathcal{L}(c_{w_h}) \to \mathcal{L}(\widetilde{c}_u)\,, \qquad |D\tau_{w_h}|(I) \to \mathrm{TC}(\widetilde{c}_u)\,.$$

This proves the reverse implication

$$\mathcal{L}(\widetilde{c}_u) + \mathrm{TC}(\widetilde{c}_u) < \infty \Longrightarrow u \in \mathcal{E}(I, \mathbb{R}^N)\,.$$

We now prove formula (9.3), and we first consider the case when J_u is a finite set. Going back to the arc-length parameterization c previously defined, since $|D\dot{c}|(I_L) = \mathrm{TC}(\widetilde{c}_u)$, it suffices to prove that

$$|D\tau_u|(I \setminus J_u) + \mathbf{M}(S_u^{J_c}) = |D\dot{c}|(I_L)\,. \tag{9.6}$$

Denoting $J_u = \{t_i\}_{i=1}^m$, where $a = \inf I < t_1 < t_2 < \cdots < t_m < \sup I = b$, we find for each i a closed interval I_L^i contained in I_L such that $c(I_L^i)$ parameterizes the straight line segment with end points $c_u(t_{i\pm})$. Recalling the formula (9.5), and setting $\widehat{I}_L := I_L \setminus \bigcup_{i=1}^m I_L^i$, arguing as in the proof of Theorem 9.2 we deduce that $\widetilde{I}_L \subset \widehat{I}_L$, see (4.9), and

$$|D\dot{c}|(\widehat{I}_L) = |D\tau_u|(I \setminus J_u)\,.$$

Moreover, for each i we also check:

$$|D\dot{c}|(I_L^i) = \mathcal{H}^1(\Gamma_{t_i}^+(u)) + \mathcal{H}^1(\Gamma_{t_i}^-(u))$$

so that by using (5.13) we obtain the formula (9.6).

In the general case when J_u is countable, say $J_u = \{t_1, t_2, \ldots\}$, setting $J_u^h = \{t_1, \ldots, t_h\}$ we define as in (8.11)

$$u_h(t) := u_+(a) + \int_a^t \dot{u}(s)\,ds + D^C u(]a,t[) + \sum_{s<t\,,\,s\in J_u^h} [u](s)\,.$$

It is then readily checked that $\{u_h\} \subset \mathcal{E}(I, \mathbb{R}^N)$, whereas $|D\tau_{u_h}|(I \setminus J_{u_h}) \to |D\tau_u|(I \setminus J_u)$, $\mathbf{M}(S_{u_h}^{J_c}) \to \mathbf{M}(S_u^{J_c})$, and also $\mathrm{TC}(\widetilde{c}_{u_h}) \to \mathrm{TC}(\widetilde{c}_u)$ as $h \to \infty$. Since we have already proved formula (9.3) for each u_h, passing to the limit we obtain (9.3) for u, as required. □

References

[1] E. Acerbi and G. Dal Maso. New lower semicontinuity results for polyconvex integrals. *Calc. Var.* **2** (1994), 329–372

[2] L. Ambrosio, N. Fusco and D. Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford Math. Monographs, Oxford, 2000.

[3] G. Anzellotti, R. Serapioni and I. Tamanini. Curvatures, Functionals, Currents. *Indiana Univ. Math. J.* **39** (1990), 617–669.

[4] G. Bellettini, G. Dal Maso and M. Paolini. Semicontinuity and relaxation properties of a curvature depending functional in 2D. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **20** (1993), 247–297.

[5] G. Bellettini and M. Paolini. Variational properties of an image segmentation functional depending on contours curvature. *Adv. Math. Sci. Appl.* **5** (1995), 681–715.

[6] T. Chan, A. Marquina and P. Mulet. High-order total variation-based image restoration. *Siam J. Sci. Comput.* **22** (2000), 503–516.

[7] A. Coscia. On curvature sensitive image segmentation. *Nonlinear Anal.* **39** (2000), 711-730.

[8] G. Dal Maso, I. Fonseca, G. Leoni and M. Morini. A higher order model for image restoration: the one dimensional case. *Siam J. Math. Appl.* **40** No. 6 (2009), 2351–2391.

[9] S. Delladio. Special generalized Gauss graphs and their application to minimization of functionals involving curvatures. *J. reine angew. Math.* **486** (1997), 17–43.

[10] C. De Lellis. Some fine properties of currents and applications to distributional Jacobians, *Proc. Royal Soc. Edinburgh Sect. A* **132** (2002), 815–842.

[11] I. Fáry. Sur la courbure totale d'une courbe gauche faisant un nœud. *Bull. Soc. Math. France* **77** (1949), 128–138.

[12] H. Federer. *Geometric measure theory.* Grundlehren math. Wissen. 153, Springer, New York, 1969.

[13] M. Giaquinta, G. Modica and J. Souček. *Cartesian currents in the calculus of variations,* Voll I, II. Ergebnisse Math. Grenzgebiete (III Ser), 37, 38, Springer, Berlin, 1998.

[14] M. Giaquinta and D. Mucci. *Maps into manifolds and currents: area and* $W^{1,2}$*-,* $W^{1/2}$*-, BV-energies.* CRM Series, Edizioni della Normale, Pisa, 2006.

[15] E. Gutkin. Curvatures, volumes and norms of derivatives for curves in Riemannian manifololds. *J. Geom. Phys.* **61** (2001), 2147–2161.

[16] J. E. Hutchinson. Second fondamental form for varifolds and the existence of surface minimizing curvature. *Indiana Univ. Math. J.* **35** (1986), 45-71.

[17] C. Mantegazza. Curvature varifolds with boundary. *J. Differential Geom.* **43** (1996), 807–843.

[18] P. Marcellini. The stored-energy for some discontinuous deformations in nonlinear elasticity. In: Partial differential equations and the calculus of variations, Vol. II, 767–786. (Progr. Nonlinear Differential Equations Appl., 2, Birkhuser Boston, Boston, MA, 1989).

[19] J. W. Milnor. On the total curvature of knots. *Ann. of Math.* **52** (1950), 248–257.

[20] S. Müller, S. J. Spector. An existence theory for nonlinear elasticity that allows for cavitation. *Arch. Rational Mech. Anal.* **131** (1995), 1–66.

[21] J. M. Sullivan. Curves of finite total curvature. *Discrete Differential Geometry.* (Bobenko, Schröder, Sullivan, and Ziegler, eds.), Oberwolfach Seminars, vol. 38, Birkäuser, 2008.