Precautionary retirement and precautionary saving

Original
Precautionary retirement and precautionary saving / Magnani, M.. - In: JOURNAL OF ECONOMICS. - ISSN 0931-8658. - 129:1(2019), pp. 49-77. [10.1007/s00712-019-00668-6]

Availability:
This version is available at: 11381/2862310 since: 2021-11-15T10:57:02Z

Publisher:
Springer

Published
DOI:10.1007/s00712-019-00668-6

Precautionary Retirement and Precautionary Saving

Marco Magnani *

Abstract

The paper analyzes the lifetime utility maximization problem of an agent who chooses her saving and timing of retirement in the presence of labor income risk in a simple setting where a pure redistributive pension scheme is in place. In this context, a precautionary motive for retirement, which pushes old workers to replace an uncertain labor income with certain pension payments, and to retire early is identified.

The conditions for precautionary retirement and saving to arise are then characterized and interpreted in two settings.

In the first setting, utility only depends on income, and a sufficiently low level of absolute prudence is necessary for precautionary retirement. A sufficiently high level is necessary however for precautionary saving, which can coexist with precautionary retirement only for intermediate values of absolute prudence.

In the second setting, agent utility also depends on leisure, and three conditions allow the precautionary motive for retirement and saving to jointly operate: prudence, an index of absolute prudence sufficiently low and cross-prudence in leisure.

JEL Classification: D81, E20.

Keywords: precautionary retirement; precautionary saving; retirement decision; labor income risk.

*Università degli Studi di Parma, Dipartimento di Scienze Economiche e Aziendali, via J.F. Kennedy 6, 43125, Parma, Italy; tel: +39 0521 032 530; fax: +39 0521 032 402; e-mail: marco.magnani@unipr.it

1 Introduction

Increasing life expectancies and decreasing fertility rates are threatening the financial sustainability of the pension system in most developed countries (OECD, 2015). Encouraging continued work is often thought to be the main way of alleviating the current financial distress of pension systems. In fact, after a period of intense reform, many OECD countries have increased the official pension age above the age of 65 years and today it can be said that "67 has indeed become the new 65" (OECD, 2015).

The effects of these reforms however, are called into question by an important fact observed in the U.S.. Unlike the past, when a single spike in retirements at the full retirement age was observed (Gruber and Wise, 1999), a second spike in retirements now occurs at age 62, which, under normal conditions, is the first age for early retirement (Gustman and Steinmeier, 2005). The behavior of U.S. workers is puzzling because postponing retirement beyond age 62 is a good deal for most people, given that future Social Security benefits are increased at a better than the actuarially fair rate (Gustman and Steinmeier, 2015, Shoven and Slavov 2012a,b).

This fact has a potentially big impact on retirement age reform and requires investigation. The present analysis addresses this issue, starting from the preliminary observation that retirement is a form of insurance against labor income variability which allows workers who are eligible for retirement to replace an uncertain labor income with a certain stream of pension payments. This means that, similarly to what happens for precautionary saving, retirement can be used to reduce the disutility caused to a risk-averse agent by an uncertain level of labor income. Hence a precautionary motive for retirement may arise and be responsible for the observed retirement peak at age 62, because a risk averse worker may be pushed to plan an earlier exit from the labor market in the face of higher uncertainty in labor income.[1]

We study the precautionary motive for retirement in a simple model where an agent chooses the level of saving and the timing of retirement in the presence of labor income risk. We assume that a pure redistributive pension system is in place where payments are certain and largely independent of individual contributions.[2] This type of system which primarily aims at granting its members a safe minimum benefit of a size known in advance, is a Beveridgean system which partially resembles

[1]On the precautionary motive for saving see, for instance, Leland (1968), Sandmo (1970), Drèze and Modigliani (1972) and Kimball (1990).

[2]Note that because they depend on wage earnings, contributions are affected by labor income risk and are in fact uncertain.

the U.S. Social Security system.[3]

Following a long tradition in decision theory which starts from Pratt (1964), we focus on the case of small risks. Our analysis characterizes and interprets the conditions for the precautionary motive for retirement and for saving to operate in two different settings.

In the first setting, agent utility depends only on income, and a sufficiently low level of the index of absolute prudence is necessary for precautionary retirement. On the contrary, for the precautionary motive for saving to operate a sufficiently high index of absolute prudence is required, implying that for intermediate levels of the index, precautionary saving and precautionary retirement can coexist.

In the second setting, agent utility also depends on leisure. Under the assumption of Edgeworth-Pareto complementarity between income and leisure, three conditions are sufficient for the precautionary motive for retirement and saving to jointly operate: prudence, a sufficiently low index of absolute prudence and cross-prudence in leisure.

Our results enrich the description of the relationship between retirement and labor supply[4] and suggest that a higher level of labor income risk, deriving from the effects of demographic change, may play a role in the increase in retirement at age 62. Papadopoulos et al. (2017) in fact point out that this risk has faced baby-boomers who are now approaching retirement age with diminished labor market prospects and decreased employment stability. This explanation is novel to the literature and contributes to the debate over the causes of the observed spike at age 62 where two alternative explanations are found.

One explanation points at the deterioration in the macroeconomic scenario after the Great Recession, which caused workers suffering a negative earning shock to be pushed into early retirement due to reduced labor market opportunities (Card et al., 2014, Coile and Levine, 2011a,b). A second explanation considers a microeconomic aspect related to worker preferences i.e. the heterogeneity in time preferences (Gustman and Steinmeier, 2005). The presence in the population of individuals with high time preferences causes the spike in the retirement age at the first age for early retirement. The present analysis is mostly related to this last strand of the literature which focuses on the role of agent preferences, and in this context, introduces a new element, i.e

[3]For a definition of a Beveridgean system see for instance Cremer and Pestieau (2003). The U.S. Social Security system is not a pure redistributive system, but a spurious redistributive system, as pension payments partially depend on contributions. But as acknowledged by the classifications by Conde-Ruiz and Profeta (2007) and Krieger and Traub (2013), it is similar in many ways to a Beveridgean systems.

[4]For a complete survey on this issue, see Blundell et al. (2016).

preferences toward risk.

A further contribution of the analysis is the characterization and interpretation of the conditions for precautionary retirement to crowd out precautionary saving. Analyzing the crowding out between retirement and saving is especially relevant in the light of the results obtained by Engen and Gruber (2001), and more recently by Chen et al. (2015). These authors find that unemployment benefits, which, like retirement, are a form of insurance against labor income risk, reduce the scope for precautionary saving. Our results show that in the case of precautionary retirement, this is not necessarily the case. In the setting where utility depends solely on income specific features of worker preferences, including prudence and temperance, are in fact crucial for the crowding out to occur, while in the setting where utility depends on income and leisure, no crowding out occurs.

The paper proceeds as follows. Section 2 presents the basic model where agent utility depends solely on income. Section 3 defines the conditions for the precautionary motive for retirement and for savings to operate, in the presence of labor income risk. Section 4 characterizes the mutual relationship between precautionary retirement and saving. Section 5 introduces agent preferences over leisure, and studies the sufficient conditions for the precautionary motives for saving and for retirement to operate jointly. Section 6 discusses the results, and Section 7 concludes.

2 The basic model

Consider a two-period framework where each agent lives for two periods.

In the first period, the agent works for the whole period and faces no uncertainty. Her labor income, net of the payroll tax, is y and is allocated between consumption and saving, denoted by $s \in \Re^{+}$. Saving is invested in the financial market, and in the second period, yields a return sR where $R \geq 0$ is the gross real rate of return.

In the second period, the agent may choose to retire at any date between γ and the end of the period, where $0 < \gamma < 1$ is the earliest date for retirement. The timing of retirement is described by the variable $\theta \in [0, 1]$ which defines the fraction $(1-\gamma)\,\theta$ of the period that a worker spends as a retiree. It follows that the agent works until date $\gamma + (1-\theta)(1-\gamma)$ and earns $(\gamma + (1-\theta)(1-\gamma))\,y$.

In this setting, if $\theta = 0$, there is no retirement. This however should not be interpreted as a situation where the agent keeps working until the end of her life, but as an approximation for the case where at some mandatory retirement age she retires.[5]

[5]It is possible to show that the inclusion in the model of a period of mandatory

In retirement, each agent receives a certain pension payment, Π, from a pure redistributive pension system, which is independent of her individual contributions. In other words, we consider a defined benefit scheme where the payroll tax is exogenously set[6], and pension payments amount to

$$\Pi = \alpha(\theta)\, y.$$

The function $\alpha(\theta) : [0,1] \to (0,1)$, such that $\alpha'(\theta) < 0$, defines the replacement rate.[7]

In this framework, second period income amounts to

$$\begin{aligned} & sR + (\gamma + (1-\theta)(1-\gamma))\, y + \theta(1-\gamma)\,\alpha(\theta)\, y \\ = & sR + (1 - \theta(1-\gamma)(1-\alpha(\theta)))\, y. \end{aligned}$$

The agent has time-separable preferences described by the utility function

$$V(y_1, y_2) = u(y_1) + v(y_2)$$

where y_t denotes income in period t $(t = 1, 2)$, and u and v are Von Neumann-Morgenstern utility functions defining agent utility respectively in the first period and in the second period. For the sake of simplicity, we assume from now on that the intertemporal discount rate is embedded in the utility function v. Denote by u_1, u_{11}, u_{111}, u_{1111}(v_1, v_{11} v_{111} and v_{1111}) the derivatives of u and v respectively, from the first to the fourth. Functions u and v are assumed to be strictly increasing and strictly concave ($u_1 > 0$, $v_1 > 0$, $u_{11} < 0$, $v_{11} < 0$), and four times continuously differentiable.

3 The precautionary motive for saving and retirement in the presence of labor income risk

In order to characterize the conditions for the precautionary motive for retirement and for saving to operate, in this section we compare the optimal retirement and saving choices in the absence of uncertainty, respectively θ^* and s^*, to the retirement and saving choices in the presence of a labor income risk, respectively θ^{**} and s^{**}. In this framework,

retirement, corresponding to a fixed fraction of the second period, does not qualitatively change the results of the analysis.

[6]These features of the pension system are discussed in detail in Section 6.

[7]The assumption $\alpha(\theta) < 1$ implies that pension payments do not exceed net labor income. As shown by OECD (2017) in fact, the gross replacement rates for mandatory pensions calculated in the year 2017, in no OECD country exceeded 100%.

precautionary retirement occurs if $\theta^{**} > \theta^{*}$, and precautionary saving occurs if $s^{**} > s^{*}$.

In the absence of uncertainty the agent solves the following lifetime-utility maximization problem:

$$\max_{\{\theta,s\}} V_y(\theta, s) = u(y - s) + v(sR + (1 - \theta(1 - \gamma)(1 - \alpha(\theta)))y). \quad (1)$$

In this problem, the timing of retirement and the level of saving are chosen at the same time because they are part of the same decision process. Retirement in fact sets the moment in time when a worker stops earning a labor income and must finance living expenses through pension earnings and the return on saving. Moreover, retirement defines the period where a worker bears a labor income risk.

These two issues are relevant to saving decisions, because, as noted by Gollier (2001), a large share of household saving is invested in long-term funds where for various reasons, mainly depending on tax incentives, an early disinvestment is specially costly. This is typically the case of money saved for retirement and of complementary private pension plans, implying that the cost of withdrawals impairs the ability of saving to forearm against labor income shocks. As a consequence, retirement and long-term saving must be jointly considered when an agent decides how much labor income risk to bear and how to deal with it.[8]

In the absence of uncertainty, the first order condition with regard to θ defines the following inequality

$$\begin{aligned} \frac{\partial V_y(\theta^*, s^*)}{\partial \theta} &= -v_1(s^*R + (1 - \theta^*(1 - \gamma)(1 - \alpha(\theta^*)))y) \\ & y(1 - \gamma)(1 - \alpha(\theta^*) - \theta^* \cdot \alpha'(\theta^*)) < 0 \end{aligned} \quad (2)$$

which implies that the optimal timing of retirement is $\theta^* = 0$.[9] The agent chooses not to retire earlier than the mandatory retirement age, because there is no disutility of labor and because no insurance against labor income risk is needed. Hence, when second period labor income is uncertain, retirement is entirely caused by a precautionary motive.

The optimal level of saving is defined by the following first-order condition:[10]

$$\frac{\partial V_y(0, s^*)}{\partial s} = -u_1(y - s^*) + Rv_1(s^*R + y) = 0. \quad (3)$$

[8]Further details on the timing of the decision process leading to the joint choice of the optimal level of saving and retirement age are reported in Section 6.

[9]Note that $(1 - \alpha(\theta^*) - \theta^* \cdot \alpha'(\theta^*)) > 0$ holds since $\alpha(\theta^*) < 1$ and $\alpha'(\theta^*) < 0$.

[10]It is easy to see that $\frac{\partial'' V_y(0,s^*)}{\partial'' s} \leq 0$ implying that s^* solves Problem (1).

Introduce now labor income risk, and study the precautionary motives for retirement and saving. In this framework, the uncertain level of labor income in the second period is a random variable $\tilde{y}$ such that $\mathbb{E}[\tilde{y}] = y$. Second period income is thus:

$$\begin{aligned} & sR + (\gamma + (1-\theta)(1-\gamma))\,\tilde{y} + \theta(1-\gamma)\,\alpha(\theta)\,y \\ = & sR + \tilde{y} - \theta(1-\gamma)(\tilde{y} - \alpha(\theta)\,y) \end{aligned}$$

and the level of uncertainty faced by the agent depends on the product between the length of the working period, $1-\theta$, and the random labor income $\tilde{y}$. A multiplicative risk is present and agent exposure to risk is scaled up if θ decreases, and scaled down if θ increases.[11]

In this setting, the agent lifetime-utility maximization problem becomes:[12]

$$\max_{\{\theta,s\}} V_{\tilde{y}}(\theta,s) = u(y-s) + E\left[v\left(sR + \tilde{y} - \theta(1-\gamma)(\tilde{y} - \alpha(\theta)\,y)\right)\right]. \quad (4)$$

The optimal timing of retirement, θ^{**} and the optimal level of saving, s^{**}, are defined by the following first-order conditions:

$$\begin{aligned} & \frac{\partial V_{\tilde{y}}(\theta^{**}, s^{**})}{\partial \theta} \\ = & -E\left[v_1\left(s^{**}R + \tilde{y} - \theta^{**}(1-\gamma)(\tilde{y} - \alpha(\theta^{**})\,y)\right)\tilde{y}\right](1-\gamma) + \\ & +E\left[v_1\left(s^{**}R + \tilde{y} - \theta^{**}(1-\gamma)(\tilde{y} - \alpha(\theta^{**})\,y)\right)\right] \\ & (1-\gamma)\,y\left(\alpha(\theta^{**}) + \theta^{**}\cdot\alpha'(\theta^{**})\right) = 0 \qquad (5) \end{aligned}$$

and

$$\begin{aligned} & \frac{\partial V_{\tilde{y}}(\theta^{**}, s^{**})}{\partial s} = -u_1(y - s^{**}) + \qquad (6) \\ & +R\cdot E\left[v_1\left(s^{**}R + \tilde{y} - \theta^{**}(1-\gamma)(\tilde{y} - \alpha(\theta^{**})\,y)\right)\right] = 0. \end{aligned}$$

Following previous literature on two-choice problems, such as Dionnne and Eeckhoudt (1984) and Brianti et al. (2018), we assume that second-order conditions are satisfied everywhere.[13]

[11] This is the main difference with settings where an additive risk is present and the level of uncertainty does not depend on the decision variable. Our analysis thus only applies to the case of multiplicative risks.

[12] As in the standard precautionary saving setting, we assume that, at the beginning of the first period, the distribution of $\tilde{y}$ is known and no information revelation takes place hereafter.

[13] Second-order conditions require that $\frac{\partial'' V_{\tilde{y}}(\theta^{**}, s^{**})}{\partial''\theta} < 0$ and $\frac{\partial'' V_{\tilde{y}}(\theta^{**}, s^{**})}{\partial'' s} < 0$.

Compare now the setting with uncertainty to the setting without uncertainty, assuming that labor income risk is small.

Before proceeding, note that according to the interpretation by Kimball (1990), an agent is prudent if the third derivative of the utility function is positive, i.e. $v_{111} > 0$. Define then the index of absolute prudence for a generic utility function $U(X)$, as

$$P(X) = -\frac{U_{111}(X)}{U_{11}(X)} \qquad \forall X > 0$$

or, in our problem:[14]

$$P = -\frac{v_{111}\left(sR + \left(1 - \theta\left(1-\gamma\right)\left(1-\alpha\left(\theta\right)\right)\right)y\right)}{v_{11}\left(sR + \left(1 - \theta\left(1-\gamma\right)\left(1-\alpha\left(\theta\right)\right)\right)y\right)}.$$

Focus first on the precautionary motive for retirement, which is largely driven by the level of prudence characterizing agent preferences. A level of absolute prudence below a certain threshold is indeed a necessary condition for precautionary retirement. Furthermore, assuming that the agent is imprudent also makes it possible to characterize the sufficient condition for the precautionary motive to operate.

In order to see this, first define $s(\theta) \in \Re^{+}$ as the saving level such that $\frac{\partial V_{\tilde{y}}(\theta, s(\theta))}{\partial s} = 0$ for given $\theta \in (0, 1]$.

Proposition 1 *In the presence of small labor income risk, the precautionary motive for retirement operates, i.e. $\theta^{**} > 0$, if and only if*

$$\begin{aligned}
&-v_1\left(s(\theta) \cdot R + \left(1-\theta\left(1-\gamma\right)\left(1-\alpha\left(\theta\right)\right)\right)y\right)\\
&y\left(1-\alpha\left(\theta\right)-\theta \cdot \alpha'\left(\theta\right)\right) + \\
&-v_{11}\left(s(\theta) \cdot R + \left(1-\theta\left(1-\gamma\right)\left(1-\alpha\left(\theta\right)\right)\right)y\right)\\
&\left(1-\theta\left(1-\gamma\right)\right) var\left[\tilde{y}\right] + \\
&-\frac{1}{2} \cdot v_{111}\left(s(\theta) \cdot R + \left(1-\theta\left(1-\gamma\right)\left(1-\alpha\left(\theta\right)\right)\right)y\right)\\
&y\left(1-\theta\left(1-\gamma\right)\right)^2 var\left[\tilde{y}\right]\left(1-\alpha\left(\theta\right)-\theta \cdot \alpha'\left(\theta\right)\right) > 0
\end{aligned} \qquad (7)$$

The second condition is satisfied under our assumptions. We further assume that $2\alpha'\left(\theta^{**}\right) + \theta^{**}\alpha''\left(\theta^{**}\right) < 0$ so that the first condition also holds. Lastly it must be the case that $\frac{\partial'' V_{\tilde{y}}(\theta^{**}, s^{**})}{\partial'' s} \cdot \frac{\partial'' V_{\tilde{y}}(\theta^{**}, s^{**})}{\partial'' \theta} - \left(\frac{\partial'' V_{\tilde{y}}(\theta^{**}, s^{**})}{\partial\theta\partial s}\right)^2 > 0$. It is possible to show that this condition holds for some commonly used utility functions. In particular, it is always satisfied in the case of a quadratic utility function, in the standard formulation $v\left(w\right) = w - \beta w^2$, since $v_{11}\left(w\right)$ is constant. Furthermore, the previous condition holds for R sufficiently large, in the case of a standard constant relative risk aversion function, $v\left(w\right) = \frac{w^{1-\beta}}{1-\beta}$ with $\beta \leq 1$, such that $v_{11}\left(w\right)$ approaches 0 as w approaches $+\infty$.

[14]For the sake of simplicity, the argument of the function P(X) is henceforth omitted.

holds for some $\theta \in (0, 1]$.

Proof. See the Appendix. ■

Corollary 1 *In the presence of small labor income risk, a necessary condition for the precautionary motive for retirement to operate, is that* $P < \chi(\theta)$ *holds for some* $\theta \in (0, 1]$

where

$$\chi(\theta) = \frac{2}{y\left(1 - \theta\left(1 - \gamma\right)\right)\left(1 - \alpha\left(\theta\right) - \theta \cdot \alpha'\left(\theta\right)\right)} > 0. \tag{8}$$

Proof. See the Appendix. ■

Corollary 2 *In the presence of small labor income risk, if the decision-maker is imprudent (i.e.* $v_{111} < 0$*) a sufficient condition for the precautionary motive for retirement to operate is*

$$-\frac{v_{11}\left(s(0) \cdot R + y\right)}{v_1\left(s(0) \cdot R + y\right)} > \frac{\left(1 - \alpha\left(0\right)\right) y}{var\left[\tilde{y}\right]}. \tag{9}$$

Proof. See the Appendix. ■

Analyze now the effects of retirement. Bringing retirement forward reduces exposure to labor income risk, and, at the same time, reduces second period income. We label the former effect as the substitution effect of retirement. This effect depends on the substitution between an uncertain labor income and certain pension payments. We further label the latter effect as the income effect of retirement which depends on the fact that pension payments are smaller than the net wage.[15]

In order to have precautionary retirement, a necessary condition is that the substitution effect, which pushes a risk averse agent to anticipate retirement to escape labor income risk, prevails over the income effect. The income effect pushes a prudent agent to postpone retirement and to increase expected income in the second period, where there is uncertainty.[16]

[15]The definitions of substitution effect and of income effect derive from the extension to the present context of the interpretation referred to precautionary saving, first proposed by Sandmo (1970) and then adopted by Rothschild and Stiglitz (1971) and by the literature on (multiplicative) interest rate risk (see for instance Li, 2012; Baiardi et al., 2014; Magnani, 2017).

[16]It is worth noting the partial symmetry of this result with the condition for precautionary saving in the presence of a a mean-preserving increase in the interest rate risk. This condition first, studied by Sandmo (1970) and Rothschild and Stiglitz (1971) and later generalized by Chiu et al. (2015) and Wong (2019) to the case of an Nth degree risk increase and of (M, N)th-order stochastic dominance respectively,

The substitution effect is larger than the income effect if the elasticity of agent risk-aversion with respect to the timing of retirement, i.e. the elasticity of $-v_{11}\left(s(\theta)\cdot R+\left(1-\theta\left(1-\gamma\right)\left(1-\alpha\left(\theta\right)\right)\right)y\right)$ with respect to θ, is greater than the elasticity of the variance of labor income with respect to θ.[17]

Clearly if the agent is imprudent, the necessary condition for precautionary retirement is always satisfied, since the reduction in income due to early retirement causes a reduction in the disutility due to risk-aversion. In this case, the sufficient condition for precautionary retirement requires that the marginal cost of reducing the exposure to labor income risk is lower than the marginal benefit which a risk averse agent obtains by a reduction in uncertainty. In other words, the ratio $-\frac{v_{11}}{v_1}$, i.e. the index of absolute risk-aversion, must be sufficiently high.

Consider now the necessary and sufficient conditions for the precautionary motive for saving to operate. As in the case of precautionary retirement these conditions involve the level of prudence of the agent but unlike that case, the precautionary motive for saving operates when the index of absolute prudence is sufficiently high and exceeds a certain threshold. Moreover if the agent is prudent no other conditions are required to have precautionary saving.

requires absolute prudence to exceed a given threshold. In this context, the income effect of saving, which pushes a prudent agent to raise the level of saving in order to increase her income in the period where she faces uncertainty, needs to prevail over the substitution effect. The substitution effect indeed pushes a risk-averse agent to reduce her level of saving because saving itself is a source of uncertainty, and increases exposure to the interest rate risk.

[17]This interpretation is derived from Magnani (2017) and can be better understood by rewriting the condition $P<\chi(\theta)$ as follows

$$\frac{v_{111}\left(s(\theta)\cdot R+\left(1-\theta\left(1-\gamma\right)\left(1-\alpha\left(\theta\right)\right)\right)y\right)y(1-\gamma)\left(1-\alpha\left(\theta\right)-\theta\cdot\alpha'\left(\theta\right)\right)\frac{\theta}{}}{v_{11}\left(s(\theta)\cdot R+\left(1-\theta\left(1-\gamma\right)\left(1-\alpha\left(\theta\right)\right)\right)y\right)}\geq$$
$$-2\left(1-\theta\left(1-\gamma\right)\right)\left(1-\gamma\right)var[\tilde{y}]\frac{\theta}{\left(1-\theta\left(1-\gamma\right)\right)^{2}var[\tilde{y}]}.$$

The left hand side of this inequality is the marginal disutility, due to risk-aversion, borne by a prudent agent after a 1% increase in θ which reduces income in the period where there is uncertainty. It can be considered as a measure of the sensitivity of the income effect to retirement. The right hand side of the inequality is the marginal reduction in the variance of second period income caused by a 1% increase in θ which reduces the time span when labor income is uncertain. It can be considered as a measure of the sensitivity of the substitution effect to retirement. Hence the precautionary motive for retirement operates if a 1 % increase in θ causes a marginal percentage increase in the disutility due to risk-aversion which is lower than the marginal percentage reduction in the variance of second period income.

Proposition 2 *In the presence of small labor income risk, a necessary condition for the precautionary motive for saving to operate, i.e. for* $s^{**} > s^{*}$, *is* $P_{|\theta=0,s=s^{*}} > \zeta(\theta^{**})$ *where*

$$\zeta(\theta^{**}) = -2 \cdot \frac{(1-\gamma)(1-\alpha(0))\, y \cdot \theta^{**}}{(1-\gamma)^{2}(1-\alpha(0))^{2}(y \cdot \theta^{**})^{2} + var[\tilde{y}]} < 0. \tag{10}$$

Proof. See the Appendix. ■

Proposition 3 *In the presence of small labor income risk, the fact that the agent is prudent (i.e.* $v_{111} > 0$*) is a sufficient condition for the precautionary motive for saving to operate.*

Proof. See the Appendix. ■

Proposition 3 includes the standard condition for precautionary saving in the presence of labor income risk, which has been widely studied since the seminal papers by Leland (1968), Sandmo (1970) and Dréze and Modigliani (1972).

In the present setting too the precautionary saving motive operates if the third derivative of the utility function is positive, which ensures that the marginal utility of income is larger in the presence of uncertainty than in the presence of certainty.[18]

But in the presence of retirement, prudence is only a sufficient condition for precautionary saving; it is not a necessary condition. Under specific conditions, an imprudent agent may increase her level of saving when she faces uncertainty. Note indeed that the condition $P_{|\theta=0,s=s^{*}} > \zeta(\theta^{**})$ requires the index of absolute prudence to be higher than a negative threshold, implying that an agent who is risk-averse ($v_{11} < 0$) but imprudent ($v_{111} \leq 0$) can fulfill this condition. In particular, the level of imprudence shall not be too large compared to the level of risk aversion, i.e. it is required that the absolute prudence index is above the threshold $\zeta(\theta^{**})$.

The reason for this has to do with the fact that saving and retirement interact. Before considering this interaction, note that if an agent is risk-averse but imprudent, her preferences display increasing absolute risk aversion, and a reduction in income in the period where there is uncertainty increases her tolerance toward risk, i.e. decreases the disutility caused by it. As a consequence, when facing a risk, she will reduce the level of saving, with respect to the case with certainty.

[18]This condition can be interpreted in the light of Chiu and Eeckhoudt (2010) as the precautionary effect or apportionment effect which captures agent preference to bear a risk when income is higher, or equivalently, to disaggregate the harm of risk and that of lower income.

Consider then that in the present setting, bringing retirement forward causes simultaneous decreases in the levels of income and risk in the second period. These decreases produce respectively an increase in agent risk tolerance and a reduction of the overall level of uncertainty. Both effects lower the incentive to decrease the level of income in the period where there is uncertainty and push the agent to increase the level of saving. This implies that the precautionary motive for retirement makes it more likely that an imprudent agent will also commit to precautionary saving.

4 The relationship between precautionary retirement and saving.

In the present section we consider the relationship between precautionary retirement and saving.

A first result is that by the analysis presented in Section 3, it is possible to identify the agents for whom the precautionary motive for retirement and for saving operates, based on the level of absolute prudence which characterizes their preferences. In order to do this, define $\overline{\chi}$ as the maximum level attained by $\chi(\theta)$ for $\theta \in (0, 1]$, and $\underline{\zeta}$ as the minimum level attained by $\zeta(\theta)$ for $\theta \in (0, 1]$. Note further that because $\chi(\theta)$ is always positive and $\zeta(\theta)$ is always negative, precautionary retirement and saving can coexist. The necessary conditions reported in Corollary 1 and Proposition 2 in fact are mutually compatible.

Corollary 3 *In the presence of small labor income risk:*

- *If $P < \underline{\zeta}$, the agent optimal choice does not imply precautionary saving but can imply precautionary retirement;*
- *If $\underline{\zeta} \leq P \leq \overline{\chi}$, the agent optimal choice can imply precautionary retirement and precautionary saving;*
- *If $P > \overline{\chi}$, the agent optimal choice does not imply precautionary retirement but can imply precautionary saving.*

Proof. Straightforwardly follows from Proposition 2 and Proposition 1. ■

The level of absolute prudence drives agent choice. An imprudent risk-averse agent with sufficiently low absolute prudence in fact, prefers precautionary retirement over precautionary saving. This is the case because precautionary retirement reduces both the mean and the variance of labor income in the period where uncertainty occurs. On the other hand, if absolute prudence is sufficiently high, the agent only chooses precautionary saving. This is the case because even though saving does not decrease labor income variance, it makes it possible to increase total income when uncertainty occurs. The simultaneous presence of precautionary saving and retirement only characterizes the optimal choice of agent whose preferences are characterized by an absolute prudence index which is not very high.

These results show that the precautionary motives for retirement and for saving do not operate under the same set of conditions and may contribute to the wide distribution of accumulated wealth of workers who decide to retire early which is observed, for instance, in U.S. data by Venti and Wise (1999, 2001).

Since for some agents the precautionary motive for both retirement and saving operates, we extend our analysis to investigate the mutual interdependence between the optimal choices of these two variables. Consider the situation where $\underline{\zeta} \leq P \leq \overline{\chi}$ hold, so that the necessary conditions for precautionary saving and precautionary retirement are satisfied.

Note that in the present setting, the pension system plays a role similar to that of unemployment insurance. Like unemployment insurance, retirement reduces labor income variability. The reduction of uncertainty, however, is not due to an insurance payment which partially offsets the loss associated with unemployment. Rather it occurs because the time span where labor income is uncertain is reduced.

Some authors point out that unemployment insurance crowds out precautionary saving (Engen and Gruber, 2001, and Chen et al., 2015).[19] Consider thus whether this occurs in the present setting where the agent (unlike the case of unemployment insurance) can choose, to some extent, the quantity of insurance against labor income risk.

In the present setting, crowding out depends on risk aversion and prudence, as well as on temperance, i.e. on the sign of the fourth derivative of the utility function. Following Kimball (1993), an agent is temperate if $v_{1111} \leq 0$ and intemperate if $v_{1111} > 0$.

Proposition 4 *In the presence of small labor income risk, precautionary retirement crowds out precautionary saving if and only if*

$$-v_{11}\left(s^{**}R + \left(1-\theta^{**}\left(1-\gamma\right)\left(1-\alpha\left(\theta^{**}\right)\right)\right)y\right)$$
$$y\left(1-\alpha\left(\theta^{**}\right)-\theta^{**}\cdot\alpha'\left(\theta^{**}\right)\right)+$$

[19]A similar result is also obtained when, in the presence of a longevity risk, other kinds of social insurance are introduced. In particular, Sheshinski and Weiss (1981), Abel (1985), Kotlikoff et al. (1987), and Hubbard and Judd (1987) show that, in the presence of functioning capital markets but no annuity market, Social Security can significantly decrease precautionary saving, if lifespan is uncertain.

$$
\begin{aligned}
&-\frac{1}{2}\cdot v_{1111}\left(s^{**}R+\left(1-\theta^{**}\left(1-\gamma\right)\left(1-\alpha\left(\theta^{**}\right)\right)\right)y\right) \\
&\left(1-\theta^{**}\left(1-\gamma\right)\right)^{2}y\left(1-\alpha\left(\theta^{**}\right)-\theta^{**}\cdot\alpha'\left(\theta^{**}\right)\right)var\left[\tilde{y}\right]+ \\
&-v_{111}\left(s^{**}R+\left(1-\theta^{**}\left(1-\gamma\right)\left(1-\alpha\left(\theta^{**}\right)\right)\right)y\right) \\
&\left(1-\theta^{**}\left(1-\gamma\right)\right)var\left[\tilde{y}\right]\leq 0. \qquad (11)
\end{aligned}
$$

Proof. See the Appendix. ■

Corollary 4 *In the presence of small labor income risk, the condition $P_{|\theta=\theta^{**},s=s^{**}} \geq \eta(\theta^{**})$ where*

$$\eta(\theta^{**})=\frac{y\left(1-\alpha\left(\theta^{**}\right)-\theta^{**}\cdot\alpha'\left(\theta^{**}\right)\right)}{\left(1-\theta^{u}\left(1-\gamma\right)\right)var\left[\tilde{y}\right]} \qquad (12)$$

- *is necessary for precautionary retirement to crowd out precautionary saving, if the agent is temperate (i.e. if $v_{1111} \leq 0$);*
- *is sufficient for precautionary retirement to crowd out precautionary saving, if the agent is intemperate (i.e. if $v_{1111} > 0$).*

Proof. See the Appendix. ■

The crowding out of precautionary saving by precautionary retirement occurs more easily when the agent is intemperate. In this case the substitution and income effects of bringing retirement forward point in the same direction: the reduction of the disutility deriving from income uncertainty.

The substitution effect i.e. the reduction in labor income risk, clearly decreases this disutility and makes precautionary saving less important. This effect is particularly strong when the level of prudence is high compared to the level of risk aversion, i.e. when the index of absolute prudence exceeds the threshold $\eta(\theta^{**})$.

But the same reduction in disutility also follows from the income effect of bringing retirement forward, i.e. the reduction in second period income. If $v_{1111} > 0$ holds, risk-aversion is a convex function, and a reduction in income also reduces the marginal disutility caused by the uncertainty generated by labor income risk, i.e. the level of prudence.

The decrease in the levels of uncertainty and prudence thus reduces the benefit deriving from precautionary saving. This makes the crowding out of precautionary saving by precautionary retirement more likely.

If the agent is temperate, however, precautionary retirement has a different impact on agent utility. In addition to the effect of the reduction of labor income risk, the decrease in second period income also causes an increase in the level of prudence, since $v_{1111} \leq 0$. This effect clearly increases the benefits of precautionary saving and makes crowding out less likely to occur.

5 The model with income and leisure

In this section, we analyze the precautionary motive for retirement and for saving in a setting where we introduce disutility of labor and a wage rate risk. In particular, we assume that each agent has time-separable preferences described by the utility function

$$V(y_1, y_2, l_1, l_2) = u(y_1, l_1) + v(y_2, l_2). \tag{13}$$

where y_t and l_t denote respectively income and leisure in period t ($t = 1, 2$) and u and v are Von Neumann-Morgenstern utility functions defining agent utility respectively in the first period and in the second period. The intertemporal discount rate is embedded in the utility function v.

We denote by u_i , u_{ij} , u_{ijk} (v_i , v_{ij} , and v_{ijk}, respectively) the first, second and third partial derivatives of u (v, respectively). Functions u and v are assumed to be strictly increasing and strictly concave with regard to each argument ($u_i > 0$, $v_i > 0$, $u_{ii} < 0$, $v_{ii} < 0$), and three times continuously differentiable.

We further assume that the benefit of consumption and leisure together is greater than the sum of their separate benefits, i.e. $u_{12} > 0$ and $v_{12} > 0$. As noted by Samuelson (1974, p. 1270), when considering a multi-argument utility function, a positive cross-derivative with respect to two arguments indicates Edgeworth-Pareto complementarity between them.

Each agent works for the whole first period and has a unit time endowment. Labor supply is fixed and amounts to $1 - \bar{l}$, implying that first period leisure is $l_1 = \bar{l}$.[20] Labor income, net of the payroll tax, is $y_1 = (1 - \bar{l})w$, where w is the net wage rate.

In the second period, an agent retires at date θ and obtains a level of leisure $l_2 = \bar{l} + \theta(1-\gamma)\left(1-\bar{l}\right)$ and a labor income amounting to $(1 - \theta(1-\gamma))\left(1-\bar{l}\right)w$. Pension payments amount to

$$\Pi = \alpha(\theta)\left(1-\bar{l}\right)w.$$

In order to study precautionary retirement and precautionary saving, we compare optimal retirement and saving choices in the absence of uncertainty, respectively $\breve{\theta}^*$ and $\breve{s}^*$, to retirement and saving choices in the presence of a wage rate risk, respectively $\breve{\theta}^{**}$ and $\breve{s}^{**}$.

Consider initially the circumstance where there is no uncertainty on the wage rate. Note that first period income is $y_1 = \left(1-\bar{l}\right)w - s$, while second period income, y_2 amounts to

$$\begin{aligned} y_2 &= sR + (\gamma + (1-\theta)(1-\gamma))\left(1-\bar{l}\right)w + \theta(1-\gamma)\alpha(\theta)\left(1-\bar{l}\right)w \\ &= sR + (1 - \theta(1-\gamma)(1-\alpha(\theta)))\left(1-\bar{l}\right)w. \end{aligned}$$

[20] This assumption is discussed in Section 6.

The agent lifetime-utility maximization problem is:

$$\max_{\{\theta,s\}} V_w(\theta,s) = u\left(\left(1-\bar{l}\right)w - s, \bar{l}\right) + \tag{14}$$
$$+ v\left(sR + \left(1-\theta\left(1-\gamma\right)\left(1-\alpha\left(\theta\right)\right)\right)\left(1-\bar{l}\right)w, \bar{l} + \theta\left(1-\gamma\right)\left(1-\bar{l}\right)\right).$$

The first order condition with regard to θ is

$$\frac{\partial V_w(\theta,s)}{\partial\theta} = v_1(y_2,l_2)\frac{\partial y_2}{\partial\theta} + v_2(y_2,l_2)\frac{\partial l_2}{\partial\theta} = 0. \tag{15}$$

Substituting in the previous equation $\frac{\partial l_2}{\partial\theta} = \left(1-\bar{l}\right)(1-\gamma)$ and

$$\frac{\partial y_2}{\partial\theta} = -\left(1-\bar{l}\right)w(1-\gamma)\left(1-\alpha(\theta)-\theta\alpha'(\theta)\right) < 0$$

gives

$$-v_1(y_2,l_2)\left(1-\bar{l}\right)w(1-\gamma)\left(1-\alpha(\theta)-\theta\alpha'(\theta)\right) + \tag{16}$$
$$+v_2(y_2,l_2)\left(1-\bar{l}\right)(1-\gamma) = 0.$$

The first order condition with regard to s is

$$\frac{\partial V_w(\theta,s)}{\partial s} = -u_1(y_1,l_1) + Rv_1(y_2,l_2) = 0. \tag{17}$$

As in the previous section we assume that the second order conditions for Problem (14) are satisfied everywhere.[21]

We now introduce a wage rate risk into our framework, and study the precautionary motive for retirement and saving. In this context, the uncertain wage rate in the second period is $\tilde{w}$, where $\tilde{w}$ is a random variable such that $\mathbb{E}[\tilde{w}] = w$. Second period income is thus

$$\tilde{y_2} = sR + \left(\gamma + (1-\theta)(1-\gamma)\right)\left(1-\bar{l}\right)\tilde{w} + \theta(1-\gamma)\alpha(\theta)\left(1-\bar{l}\right)w$$
$$= sR + \left(1-\theta(1-\gamma)\right)\left(1-\bar{l}\right)\tilde{w} - \alpha(\theta)\left(1-\bar{l}\right)w.$$

The agent lifetime-utility maximization problem in the presence of a wage risk becomes:

$$\max_{\{\theta,s\}} V_{\tilde{w}}(\theta,s) = u\left(\left(1-\bar{l}\right)w - s\right), \bar{l}) + \tag{18}$$
$$+ E\left[v\left(R + \left(1-\theta(1-\gamma)\right)\left(1-\bar{l}\right)\tilde{w} - \alpha(\theta)\left(1-\bar{l}\right)w, \bar{l} + \theta(1-\gamma)\left(1-\bar{l}\right)\right)\right].$$

---

[21] In particular, our assumptions ensure that $\frac{\partial'' V_w(\theta,s)}{\partial'' s} < 0$ while it is possible to show that if $2\alpha'(\theta^{**}) + \theta^{**}\alpha''(\theta^{**}) < 0$ also $\frac{\partial'' V_w(\theta,s)}{\partial''\theta} < 0$ holds. We need thus to assume that $\frac{\partial'' V_w(\theta,s)}{\partial''\theta} \cdot \frac{\partial'' V_w(\theta,s)}{\partial'' s} - \left(\frac{\partial'' V_w(\theta,s)}{\partial\theta\partial s}\right)^2 > 0$ holds everywhere.

The first-order condition with regard to θ is

$$\frac{\partial V_{\tilde{w}}(\theta, s)}{\partial \theta} = E\left[v_1(\tilde{y_2}, l_2)\frac{\partial \tilde{y_2}}{\partial \theta}\right] + E\left[v_2(\tilde{y_2}, l_2)\frac{\partial \tilde{l_2}}{\partial \theta}\right] = 0.$$

Substituting in the previous equation $\frac{\partial l_2}{\partial \theta} = \left(1-\bar{l}\right)(1-\gamma)$ and

$$\frac{\partial \tilde{y_2}}{\partial \theta} = -\left(1-\bar{l}\right)\tilde{w}(1-\gamma) + \left(1-\bar{l}\right)w(1-\gamma)\left(\alpha(\theta)+\theta\alpha'(\theta)\right)$$

gives

$$\begin{aligned} & -E\left[v_1(\tilde{y_2}, l_2)\tilde{w}\right]\left(1-\bar{l}\right)(1-\gamma)+ \\ & +E\left[v_1(\tilde{y_2}, l_2)\right]w\left(\alpha(\theta)+\theta\alpha'(\theta)\right)\left(1-\bar{l}\right)(1-\gamma)+ \\ & +E\left[v_2(\tilde{y_2}, l_2)\right]\left(1-\bar{l}\right)(1-\gamma)=0. \end{aligned} \tag{19}$$

The first order condition with regard to s is

$$\frac{\partial V_{\tilde{w}}(\theta, s)}{\partial s} = -u_1(y_1, l_1) + R \cdot E\left[v_1(\tilde{y_2}, l_2)\right] = 0. \tag{20}$$

We assume that the second order conditions for Problem (18) are satisfied everywhere.[22]

Compare now the setting with uncertainty to the setting without uncertainty, and assume that wage rate risk is small. Consider the conditions for the precautionary motives for saving and for retirement to jointly operate.

Proposition 5 *In the presence of small wage rate risk, the precautionary motives for retirement and for saving jointly operate, i.e.* $\breve{\theta}^{**} > \breve{\theta}^{*}$ *and* $\breve{s}^{**} > \breve{s}^{*}$ *hold if the following three conditions are satisfied*

- *the agent is prudent* $v_{111} \geq 0$*;*
- *the following condition holds*

$$-\frac{v_{111}(c_2, l_2)}{v_{11}(c_2, l_2)} \leq \frac{2}{\left(1-\breve{\theta}^{*}(1-\gamma)\right)\left(1-\bar{l}\right)w\left(1-\alpha(\breve{\theta}^{*})-\breve{\theta}^{*}\alpha'(\breve{\theta}^{*})\right)} \tag{21}$$

[22]As shown in Subsection 8.8 of the Appendix, inequalities $\frac{\partial'' V_{\tilde{w}}(\theta,s)}{\partial'' \theta} < 0$ and $\frac{\partial'' V_{\tilde{w}}(\theta,s)}{\partial'' s} < 0$ are verified implying that we need to assume that $\frac{\partial'' V_{\tilde{w}}(\theta,s)}{\partial'' \theta} \cdot \frac{\partial'' V_{\tilde{w}}(\theta,s)}{\partial'' s} - \left(\frac{\partial'' V_{\tilde{w}}(\theta,s)}{\partial \theta \partial s}\right)^2 > 0$ holds everywhere.

- *the agent is cross prudent in leisure* $v_{112} \geq 0$.

On the contrary, if all these conditions are reversed the precautionary motives for retirement and for saving do not operate, and $\breve{\theta}^{**} \leq \breve{\theta}^{*}$ *and* $\breve{s}^{**} \leq \breve{s}^{*}$ *hold.*

Proof. See the Appendix. ■

Analyze the conditions in Proposition 5, starting from $v_{111} \geq 0$. This condition is conceptually and analytically equivalent to the standard sufficient condition for precautionary saving in the presence of labor income risk, in a setting with a univariate von Neumann-Morgenstern utility.

Consider now the remaining conditions and analyze the effects of retirement. Bringing retirement forward has three effects: it reduces exposure to wage rate risk, reduces second period income, and increases leisure.

Focus initially on the first two effects which affect second period income. Extending the interpretation proposed in Section 3, we label the reduction in the exposure to wage rate risk as the substitution effect of retirement, and the reduction in second period income as the income effect of retirement.

Also in this context, as in the case of Corollary 1, in order to have precautionary retirement it must be the case that the substitution effect prevails over the income effect.[23]

The third effect of retirement is to increase leisure in the second period. This effect is appreciated by an agent who is cross-prudent in leisure. The interpretation provided by Chiu and Eeckhoudt (2010) helps to explain this. These authors define cross-prudence in leisure as stochastic complementarity effect because it has to do with the fact that the agent prefers more or less leisure when there is a stochastic deterioration in income. The condition $v_{112} \geq 0$ in particular ensures that the marginal utility of leisure is larger in the presence of uncertainty on income than in the presence of certainty. This means that the agent prefers to bear a risk when leisure is higher (precautionary effect), or equivalently, to disaggregate the harm of risk and that of lower leisure (apportionment effect).

Consider now the circumstance where sufficient conditions for precautionary saving and precautionary retirement are satisfied, and analyze the crowding out between them.

[23] Equivalently, it must be the case that a 1 % increase in θ causes a marginal percentage increase in the disutility due to risk-aversion which is lower than the marginal percentage decrease in the variance of second period income. It is indeed possible to rewrite Inequality (21) in terms of elasticities of agent risk aversion and of variance of second period labor income.

Proposition 6 *In the presence of small wage rate risk, if* $v_{111} \geq 0$*,* $v_{112} \geq 0$ *and Inequality (21) holds, precautionary retirement crowds in precautionary saving.*

Proof. Analogous to the proof of Proposition 4. ∎

Crowding out never occurs; on the contrary we find that precautionary retirement crowds in precautionary saving. This result is consistent with the findings of Flodén (2006), which show that labor supply flexibility raises precautionary saving in the presence of wage rate risk. In the present setting indeed, retirement allows for labor-supply flexibility.

Note that precautionary retirement, by causing a reduction in labor income, strengthen the need for precautionary saving. In the face of a wage rate risk, a risk-averse agent who is also cross-prudent in leisure desires to reduce the level of uncertainty and to increase the quantity of leisure by anticipating retirement. Since this causes second period income to decrease, a prudent agent is also willing to increase her level of income in the period where she faces uncertainty by making precautionary saving.

6 Discussion of the Results

The results presented in the previous sections are obtained under a set of assumptions which need to be discussed in detail.

The first relevant assumption concerns the fact that the decisions on saving and retirement are taken at an early age. This assumption is consistent with the fact that this study focuses on a lifetime utility maximization problem. The focus is thus on long-term saving, which requires that the agent chooses how much to save at an early age. In the light of the above, the decision on retirement age also needs to be taken at the same time. Although it may seem counter-intuitive that a commitment on retirement is taken when the agent is still young, it should be noted that the joint definition of the level of saving and the timing of retirement made at an early age will heavily affect future behavior.

As observed by Blundell et al. (2016) retirement behavior might in fact depend on liquidity constraints. Consider for instance the case of a worker, eligible for early retirement, who faces a period of unemployment. This adverse shock may lead the worker to bring retirement forward compared to what she had previously planned. But, as it is illegal to borrow against Social Security benefits, the possibility of retiring at an earlier age is constrained by accumulated wealth and thus by previous decisions on long-term saving. This implies that the choice on saving and retirement made at a young age contributes to the retirement behavior of a forward looking agent in old age.

The second important assumption concerns the type of pension system, which is a pure redistributive, defined benefit scheme where pension payments are certain. The redistributive nature of the pension system is crucial to the analysis since it allows for a precautionary motive for retirement to arise. In this system in fact there can be substitution between an uncertain income stream and a safe income stream when a worker retires. A precautionary motive for retirement further requires that either the level of inflation in the economy is low, or that an indexation mechanism for pension payments exists. In the absence of both these conditions, in fact be the volatility of pension payments could exceed the volatility of labor income.[24]

In this context, it is irrelevant whether the system is unfunded, or partially or fully funded. The results of the analysis in fact apply to all cases of defined benefit systems, for instance, in the U.S, the partially funded public Social Security System and the fully funded private pension scheme "Dollars times service" plan.

Our results do not apply, in principle, to defined contribution and Bismarckian schemes where the size of pension payments is uncertain. But, if pension payments are less uncertain than labor income, a precautionary motive for retirement may still emerge, and our results become applicable. This specific circumstance is likely to occur in many OECD countries, due to the increasing trend in wage dispersion, well documented by Berlingieri et al. (2017), and to the fact that public pension plans usually include a non-contributory old age pension which ensures positive payments after retirement and reduces income volatility.[25]

The above statement holds even though in the present scenario, population aging appears to be threatening the financial stability of unfunded public pension systems and putting future pension benefits at risk.

There are two main reasons for this. The first reason is that actions have been taken which in many cases have improved the sustainability of pension systems (OECD 2015). The second reason is the existence in many countries, especially in the U.S., of defined benefit private pensions which provide a coverage of the labor income risk similar to that provided by a pure redistributive scheme. These pensions play an important role

[24]A precautionary retirement motive can also arise outside a public pension scheme. A worker in fact could purchase an inflation-protected annuity in the private sector before the legal mandatory retirement age and obtain protection against income risk, as in a pure redistributive pension system.

[25]Note that, in fact, during their working life people can even earn no income if they are unemployed and ineligible for an unemployment benefits.

in the retirement choice (Gustman et al. 2010) and are limiting the impact of the current distress of unfunded public pension schemes on retirement behavior.

The third relevant assumption included in the model concerns the setting where preferences on leisure are introduced. We assume that labor supply is fixed, thus excluding that a precautionary motive *a la* Flodén (2006) operates. When a long time span is considered, as in the decision about retirement, the precautionary motive for labor supply is likely to be weak due to the constraints on variations in labor supply imposed by job availability, and by the fact that labor contracts often include a number of working hours which is largely fixed.

7 Final Remarks

The paper analyzes in a simple model where a pure redistributive pension system is in place, the lifetime utility maximization problem of an agent who chooses the level of saving and the timing of retirement in the presence of labor income risk. In this context, a precautionary motive for retirement, which pushes old workers to replace an uncertain labor income with certain pension payments and to retire early is identified.

The conditions for the precautionary motive for retirement and saving to operate are then characterized and interpreted, as are the conditions for precautionary retirement to crowd out precautionary saving.

This analysis is performed in two different settings.

In the first setting, agent utility depends solely on income, and our results make it possible to identify the different types of agents who choose either only precautionary retirement, or only precautionary saving, or both.

If the agent is imprudent and her absolute prudence index is particularly low, only the precautionary motive for retirement operates. If the level of absolute prudence is sufficiently high, only the precautionary motive for saving operates. Lastly, if the agent's absolute prudence index is negative but sufficiently high, or if she is prudent but the index of absolute prudence is not too high, her optimal choice can entail both precautionary retirement and saving.

In the latter case precautionary retirement can crowd out saving, if the agent is temperate and absolute prudence is below the threshold $\eta(\theta^{**})$. But if the agent is intemperate and absolute prudence is lower than $\eta(\theta^{**})$, crowding out occurs with certainty.

In the second setting, the decision of an agent whose utility also depends on leisure is considered. Under the assumption of Edgeworth-Pareto complementarity between income and leisure, three conditions are sufficient for the precautionary motive for retirement and saving to operate jointly. Two of these conditions are analogous to those obtained in the setting where the agent has no disutility from labor, i.e. prudence and a sufficiently low level of the ratio $-\frac{v_{111}}{v_{11}}$. The last condition, cross-prudence in leisure, is specific to this setting where no crowding out occurs.

These results offer a clear insight on how the precautionary motive for retirement affects saving decisions and the choice to leave the labor market. However, it is important to discuss their empirical relevance and effectiveness in explaining actual worker behaviors in the real world.

Providing a direct estimate of the strength of the precautionary motive for retirement lies beyond the scope of the present paper, and is a fruitful avenue for future research. But our analysis can be complemented with some indirect evidence retrieved from the literature.

The first piece of evidence concerns the interactions between labor income risk and health. Health is an important predictor of retirement, and can affect an individuals productivity and wage (Blundell et al. 2016). This is especially true in the old age when health declines and shocks occur more frequently.

In this context, if a young agent is in bad health and/or expects her current situation to persist or worsen in the future, she might plan an early exit from the labor market, primarily to avoid a higher disutility from working, but also to escape increased labor income risk (French and Jones, 2017). Part of the relationship observed in the data between early retirement and health (Gustman and Steinmeier, 2014) can thus be ascribed to a precautionary motive.[26]

The second piece of evidence is obtained combining the results of different studies. Consider the findings by Noussair et al. (2014) who provide experimental data on peoples behavior in the presence of uncertainty, and show that risk aversion, prudence, and temperance are usually positively correlated. They further show that the level of prudence increases with education.

This suggests that early retirement and a low level of precautionary saving characterize the behavior of people with low levels of education. Such people in fact are more likely to be imprudent and to prefer precautionary retirement to precautionary saving. Moreover, their low level of prudence may also mean that their level of temperance is low, and that they are more subject to the crowding out effect.

These results have two main implications.

[26] In real world situations, the set of instruments available to deal with health shocks is wider than retirement and saving. There are other insurance devices, primarily disability insurance, which impact on behavior and reduce the strength of the precautionary motive for retirement.

The first implication is that workers with a low level of education whose precautionary motive for retirement is stronger, should leave the labor market at an earlier age than those with a higher level of education. As shown by Coile (2015) using U.S. data, this is in fact what occurs in the real world, and different patterns in retirement behavior by education can in fact be seen according to education level. Hence, the hypothesis that the precautionary motive is a significant issue for retirement decisions is not contradicted by the data and remains plausible.

The second implication is that workers with a low level of education should also face liquidity constraints due to insufficient saving, which implies that they should be more likely to retire at the first age for early retirement. This is coherent with the empirical evidence which inspired this study. In the presence of a large share of people in the population with a low level of education, the spike at age 62 observed in the U.S., may well reflect that many such people take precautionary retirement.

The indirect evidence reported above highlights heterogeneity in worker characteristics in terms of health and preferences toward risk. Our results, looked at together with the wide variation existing in individual preferences and attributes, suggest that a fixed retirement age is not beneficial, and that more flexibility could be welfare improving. A recent study (Eurofound 2016) in fact finds that almost two-thirds of EU citizens would prefer to combine a part-time job and partial pension to complete retirement. This is consistent with recent calls by OECD (2017) for flexible retirement, defined as the ability to draw a (full or partial) pension benefit while continuing in paid work, and to choose the time of retirement.

Lastly, note that implementing flexible retirement could help to relieve the current distress of pension systems. The combination of (partial) pension payments and labor income in fact reduces the level of uncertainty faced by workers, because pension benefits ensure against the risk of unemployment. Furthermore, if the worker is employed, pension payments reduce total income variance and act as a buffer against labor income risk.

A flexible retirement regime weakens the precautionary motive for retirement, and thus lengthens the working life of many people. Especially in countries where the population is rapidly aging, this would be advantageous in two ways: by directly improving the financial soundness of pension systems, and/or indirectly contributing to pension system strengthening by ensuring greater economic growth and higher tax revenues.

8 Appendix: Proofs of Propositions and Corollaries

8.1 Proof of Proposition 1

By Equation (5), $\theta^{**} > 0$ holds if and only if

$$\begin{aligned} & E\left[v_1\left(s(\theta)\cdot R+\tilde{y}-\theta\left(1-\gamma\right)\left(\tilde{y}-\alpha\left(\theta\right)y\right)\right)\right] \\ & y\left(\alpha\left(\theta\right)+\theta\cdot\alpha'\left(\theta\right)\right) \\ > & E\left[v_1\left(s(\theta)\cdot R+\tilde{y}-\theta\left(1-\gamma\right)\left(\tilde{y}-\alpha\left(\theta\right)y\right)\right)\tilde{y}\right] \end{aligned} \tag{22}$$

for some $\theta \in (0,1]$. Apply now a Taylor expansion around the point y to the terms in Inequality (22) to obtain

$$\begin{aligned} & E\left[v_1\left(s(\theta)\cdot R+\tilde{y}-\theta\left(1-\gamma\right)\left(\tilde{y}-\alpha\left(\theta\right)y\right)\right)\right] \\ & y\left(\alpha\left(\theta\right)+\theta\cdot\alpha'\left(\theta\right)\right) \\ \tilde{=} & v_1\left(s(\theta)\cdot R+\left(1-\theta\left(1-\gamma\right)\left(1-\alpha\left(\theta\right)\right)\right)y\right) \\ & y\left(\alpha\left(\theta\right)+\theta\cdot\alpha'\left(\theta\right)\right)+ \\ & +\frac{1}{2}\cdot v_{111}\left(s(\theta)\cdot R+\left(1-\theta\left(1-\gamma\right)\left(1-\alpha\left(\theta\right)\right)\right)y\right) \\ & y\left(\alpha\left(\theta\right)+\theta\cdot\alpha'\left(\theta\right)\right)\left(1-\theta\left(1-\gamma\right)\right)^2 var\left[\tilde{y}\right] \end{aligned} \tag{23}$$

and

$$\begin{aligned} & E\left[v_1\left(s(\theta)\cdot R+\tilde{y}-\theta\left(1-\gamma\right)\left(\tilde{y}-\alpha\left(\theta\right)y\right)\right)\tilde{y}\right] \\ \tilde{=} & v_1\left(s(\theta)\cdot R+\left(1-\theta\left(1-\gamma\right)\left(1-\alpha\left(\theta\right)\right)\right)y\right)y\ + \\ & +\frac{1}{2}v_{111}\left(s(\theta)\cdot R+\left(1-\theta\left(1-\gamma\right)\left(1-\alpha\left(\theta\right)\right)\right)y\right) \\ & y\left(1-\theta\left(1-\gamma\right)\right)^2 var\left[\tilde{y}\right]+ \\ & +v_{11}\left(s(\theta)\cdot R+\left(1-\theta\left(1-\gamma\right)\left(1-\alpha\left(\theta\right)\right)\right)y\right) \\ & \left(1-\theta\left(1-\gamma\right)\right)var\left[\tilde{y}\right]. \end{aligned} \tag{24}$$

Substituting Equations (23) and (24) into Inequality (22) gives after some algebra, Inequality (7).

8.2 Proof of Corollary 1

Note that Inequality (7) is verified only if

$$\begin{aligned} & \frac{2}{y\left(1-\theta\left(1-\gamma\right)\right)\left(1-\alpha\left(\theta\right)-\theta\cdot\alpha'\left(\theta\right)\right)} > \\ & -\frac{v_{111}\left(s(\theta)\cdot R+\left(1-\theta\left(1-\gamma\right)\left(1-\alpha\left(\theta\right)\right)\right)y\right)}{v_{11}\left(s(\theta)\cdot R+\left(1-\theta\left(1-\gamma\right)\left(1-\alpha\left(\theta\right)\right)\right)y\right)}. \end{aligned}$$

8.3 Proof of Corollary 2

A sufficient condition for $\theta^{**} > 0$ to hold is that Inequality (7) is verified for $\theta = 0$, i.e.

$$-v_1 (s(0) \cdot R + y) (1-\gamma) y (1-\alpha(0)) - v_{11} (s(0) \cdot R + y) (1-\gamma) var[\tilde{y}] + \\ -\frac{1}{2} \cdot v_{111} (s(0) \cdot R + y) \cdot (1-\gamma) y \cdot var[\tilde{y}] (1-\alpha(0)) > 0.$$

If $v_{111} < 0$ this inequality is verified if

$$-v_1 (s(0) \cdot R + y) y (1-\alpha(0)) - v_{11} (s(0) \cdot R + y) var[\tilde{y}] > 0$$

which is equivalent to Inequality (9).

8.4 Proof of Proposition 2

By Equations (3) and (6), $s^{**} > s^*$, if and only if

$$E[v_1 (s^{**} R + \tilde{y} - \theta^{**} (1-\gamma) (\tilde{y} - \alpha(\theta^{**}) y))] > v_1 (s^* R + y).$$

Since $v_{11} < 0$, if $s^{**} > s^*$, the following inequality must hold

$$E[v_1 (s^* R + \tilde{y} - \theta^{**} (1-\gamma) (\tilde{y} - \alpha(\theta^{**}) y))] \\ > E[v_1 (s^{**} R + \tilde{y} - \theta^{**} (1-\gamma) (\tilde{y} - \alpha(\theta^{**}) y))]$$

which further implies that a necessary condition for the precautionary motive for saving to operate is

$$E[v_1 (s^* R + \tilde{y} - \theta^{**} (1-\gamma) (\tilde{y} - \alpha(\theta^{**}) y))] > v_1 (s^* R + y). \quad (25)$$

Consider a Taylor expansion around the point $s^* R + y$ of the term in the left-hand side of Inequality (25):

$$\begin{aligned} & E[v_1 (s^* R + \tilde{y} - \theta^{**} (1-\gamma) (\tilde{y} - \alpha(\theta^{**}) y))] \\ \tilde{=} \; & v_1 (s^* R + y) - v_{11} (s^* R + y) (1-\gamma) (1-\alpha(0)) y \cdot \theta^{**} + \\ & + \frac{1}{2} v_{111} (s^* R + y) \left((1-\gamma)^2 (1-\alpha(0))^2 (y \cdot \theta^{**})^2 + var[\tilde{y}] \right). \end{aligned} \quad (26)$$

Substitute Equation (26) into Inequality (25) to obtain, using simple algebra

$$-\frac{v_{111} (s^* R + y)}{v_{11} (s^* R + y)} > -2 \cdot \frac{(1-\gamma) (1-\alpha(0)) y \cdot \theta^{**}}{(1-\gamma)^2 (1-\alpha(0))^2 (y \cdot \theta^{**})^2 + var[\tilde{y}]}.$$

8.5 Proof of Proposition 3

If $v_{111} > 0$

$$E\left[v_1\left(s^*R + \tilde{y} - \theta^{**}\left(1-\gamma\right)\left(\tilde{y} - \alpha\left(\theta^{**}\right)y\right)\right)\right] >$$
$$v_1\left(s^*R + 1 - \theta^{**}\left(1-\gamma\right)\left(1-\alpha\left(\theta^{**}\right)\right)y\right)$$

holds. Since $\frac{\partial v_1}{\partial \theta} > 0$ the following inequalities are also verified

$$v_1\left(s^*R + \left(1 - \theta^{**}\left(1-\gamma\right)\left(1-\alpha\left(\theta^{**}\right)\right)\right)y\right) > v_1\left(s^*R + y\right)$$

and

$$E\left[v_1\left(s^*R + \tilde{y} - \theta^{**}\left(1-\gamma\right)\left(\tilde{y} - \alpha\left(\theta^{**}\right)y\right)\right)\right] > v_1\left(s^*R + y\right).$$

This implies that by Equation (3) it is

$$E\left[v_1\left(s^*R + \tilde{y} - \theta^{**}\left(1-\gamma\right)\left(\tilde{y} - \alpha\left(\theta^{**}\right)y\right)\right)\right] > \frac{u_1\left(y - s^*\right)}{R} \tag{27}$$

while by Equation (6), we have that

$$E\left[v_1\left(s^{**}R + \tilde{y} - \theta^{**}\left(1-\gamma\right)\left(\tilde{y} - \alpha\left(\theta^{**}\right)y\right)\right)\right] = \frac{u_1\left(y - s^{**}\right)}{R}. \tag{28}$$

Comparing Equations (27) and (28) makes it possible to exclude that $s^{**} = s^*$. Moreover, by the concavity of u and v, Equation (28) requires $s^{**} > s^*$.

8.6 Proof of Proposition 4

By the Implicit Function Theorem, precautionary retirement crowds out precautionary saving when

$$\frac{\partial s^{**}}{\partial \theta^{**}} = -\frac{\frac{\partial'' V_{\tilde{y}}(\theta^{**}, s^{**})}{\partial\theta^{**}\delta s^{**}}}{\frac{\partial'' V_{\tilde{y}}(\theta^{**}, s^{**})}{\partial'' s^{**}}} \leq 0.$$

Under the assumption of risk-aversion, $\frac{\partial'' V_{\tilde{y}}(\theta^{**}, s^{**})}{\partial'' s^{**}} < 0$ holds, and the previous inequality is verified if and only if $\frac{\delta'' V_{\tilde{y}}(\theta^{**}, s^{**})}{\partial\theta^{**}\delta s^{**}} \leq 0$ holds.

Check now when it is $\frac{\partial'' V_{\tilde{y}}(\theta^{**}, s^{**})}{\partial\theta^{**}\delta s^{**}} \leq 0$, and derive Equation (6) with respect to θ to obtain:

$$\begin{aligned} &E\left[v_{11}\left(s^{**}R + \tilde{y} - \theta^{**}\left(1-\gamma\right)\left(\tilde{y} - \alpha\left(\theta^{**}\right)y\right)\right)\right] \\ &y\left(\alpha\left(\theta^{**}\right) + \theta^{**}\cdot\alpha'\left(\theta^{**}\right)\right) + \\ &-E\left[v_{11}\left(s^{**}R + \tilde{y} - \theta^{**}\left(1-\gamma\right)\left(\tilde{y} - \alpha\left(\theta^{**}\right)y\right)\right)\tilde{y}\right] \leq 0. \end{aligned} \tag{29}$$

Apply now a Taylor expansion around y to both terms in Inequality (29) to obtain

$$\begin{aligned}
& E\left[v_{11}\left(s^{**}R+\tilde{y}-\theta^{**}\left(1-\gamma\right)\left(\tilde{y}-\alpha\left(\theta^{**}\right)y\right)\right)\right] \\
& y\left(\alpha\left(\theta^{**}\right)+\theta^{**}\cdot\alpha'\left(\theta^{**}\right)\right) \\
\tilde{=}\, & v_{11}\left(s^{**}R+\left(1-\theta^{**}\left(1-\gamma\right)\left(1-\alpha\left(\theta^{**}\right)\right)\right)y\right) \\
& y\left(\alpha\left(\theta^{**}\right)+\theta^{**}\cdot\alpha'\left(\theta^{**}\right)\right)+ \\
& +\frac{1}{2}\cdot v_{1111}\left(s^{**}R+\left(1-\theta^{**}\left(1-\gamma\right)\left(1-\alpha\left(\theta^{**}\right)\right)\right)y\right) \\
& \left(1-\theta^{**}\left(1-\gamma\right)\right)^{2}y\left(\alpha\left(\theta^{**}\right)+\theta^{**}\cdot\alpha'\left(\theta^{**}\right)\right)var\left[\tilde{y}\right] \qquad (30)
\end{aligned}$$

and

$$\begin{aligned}
& E\left[v_{11}\left(s^{**}R+\tilde{y}-\theta^{**}\left(1-\gamma\right)\left(\tilde{y}-\alpha\left(\theta^{**}\right)y\right)\right)\tilde{y}\right] \\
\tilde{=}\, & v_{11}\left(s^{**}R+\left(1-\theta^{**}\left(1-\gamma\right)\left(1-\alpha\left(\theta^{**}\right)\right)\right)y\right)y+ \\
& +\frac{1}{2}v_{1111}\left(s^{**}R+\left(1-\theta^{**}\left(1-\gamma\right)\left(1-\alpha\left(\theta^{**}\right)\right)\right)y\right) \\
& \left(1-\theta^{**}\left(1-\gamma\right)\right)^{2}y\cdot var\left[\tilde{y}\right]+ \\
& +v_{111}\left(s^{**}R+\left(1-\theta^{**}\left(1-\gamma\right)\left(1-\alpha\left(\theta^{**}\right)\right)\right)y\right) \\
& \left(1-\theta^{**}\left(1-\gamma\right)\right)var\left[\tilde{y}\right]. \qquad (31)
\end{aligned}$$

Substituting Equations (30) and (31) into Inequality (29) gives

$$\begin{aligned}
& -v_{11}\left(s^{**}R+\left(1-\theta^{**}\left(1-\gamma\right)\left(1-\alpha\left(\theta^{**}\right)\right)\right)y\right) \\
& y\left(1-\alpha\left(\theta^{**}\right)-\theta\cdot\alpha'\left(\theta^{**}\right)\right)+ \\
& -\frac{1}{2}\cdot v_{1111}\left(s^{**}R+\left(1-\theta^{**}\left(1-\gamma\right)\left(1-\alpha\left(\theta^{**}\right)\right)\right)y\right) \\
& \left(1-\theta^{**}\left(1-\gamma\right)\right)^{2}y\left(1-\alpha\left(\theta^{**}\right)-\theta^{**}\cdot\alpha'\left(\theta^{**}\right)\right)var\left[\tilde{y}\right]+ \\
& -v_{111}\left(s^{**}R+\left(1-\theta^{**}\left(1-\gamma\right)\left(1-\alpha\left(\theta^{**}\right)\right)\right)y\right) \\
& \left(1-\theta^{**}\left(1-\gamma\right)\right)var\left[\tilde{y}\right]\leq 0.
\end{aligned}$$

8.7 Proof of Corollary 4

If the agent is temperate and $v_{1111}\leq 0$ holds, a necessary condition for Inequality (11) to hold is

$$\begin{aligned}
& 1+\frac{v_{111}\left(s^{**}R+\left(1-\theta^{**}\left(1-\gamma\right)\left(1-\alpha\left(\theta^{**}\right)\right)\right)y\right)}{v_{11}\left(s^{**}R+\left(1-\theta^{**}\left(1-\gamma\right)\left(1-\alpha\left(\theta^{**}\right)\right)\right)y\right)} \\
& \frac{\left(1-\theta^{**}\left(1-\gamma\right)\right)var\left[\tilde{y}\right]}{y\left(1-\alpha\left(\theta^{**}\right)-\theta^{u}\cdot\alpha'\left(\theta^{**}\right)\right)}\leq 0
\end{aligned}$$

or

$$\frac{y\left(1-\alpha\left(\theta^{**}\right)-\theta^{**}\cdot\alpha'\left(\theta^{**}\right)\right)}{\left(1-\theta^{**}\left(1-\gamma\right)\right)var\left[\tilde{y}\right]}\leq \tag{32}$$
$$-\frac{v_{111}\left(s^{**}R+\left(1-\theta^{**}\left(1-\gamma\right)\left(1-\alpha\left(\theta^{**}\right)\right)\right)y\right)}{v_{11}\left(s^{**}R+\left(1-\theta^{**}\left(1-\gamma\right)\left(1-\alpha\left(\theta^{**}\right)\right)\right)y\right)}.$$

If the agent is intemperate and $v_{1111} > 0$ holds, Inequality (32) is a sufficient condition for Inequality (11) to hold.

8.8 Proof of Proposition 5

In order for a precautionary motive for retirement to operate and $\breve{\theta^{**}} \geq \breve{\theta^*}$ to hold, it must be the case that

$$\frac{\partial V_{\tilde{w}}\left(\breve{\theta^{**}},\breve{s^{**}}\right)}{\partial\theta}\leq\frac{\partial V_{\tilde{w}}\left(\breve{\theta^{*}},\breve{s^{**}}\right)}{\partial\theta}$$

since

$$\frac{\partial''V_{\tilde{w}}\left(\theta,s\right)}{\partial''\theta}=v_{11}\left(\tilde{y}_2,l_2\right)\left(\frac{\partial\tilde{y}_2}{\partial\theta}\right)^2+v_1\left(\tilde{y}_2,l_2\right)\frac{\partial''\tilde{y}_2}{\partial''\theta}+ \tag{33}$$
$$+2v_{12}\left(\tilde{y}_2,l_2\right)\cdot\frac{\partial l_2}{\partial\theta}\cdot\frac{\partial\tilde{y}_2}{\partial\theta}+v_{22}\left(\tilde{y}_2,l_2\right)\left(\frac{\partial l_2}{\partial\theta}\right)^2<0$$

because by the assumption $2\alpha'(\theta)+\theta\alpha''(\theta)<0$ it is[27]

$$\frac{\partial''\tilde{y}_2}{\partial''\theta}=\left(1-\bar{l}\right)w\left(1-\gamma\right)\left(2\alpha'(\theta)+\theta\alpha''(\theta)\right)<0.$$

Note further that

$$\frac{\partial''V_{\tilde{w}}\left(\theta,s\right)}{\partial'\theta\partial's}=v_{11}\left(\tilde{y}_2,l_2\right)\frac{\partial\tilde{y}_2}{\partial\theta}R+v_{12}\left(\tilde{y}_2,l_2\right)\frac{\partial l_2}{\partial\theta}R>0 \tag{34}$$

so that if $\breve{s^{**}} \geq \breve{s^*}$, the following inequality must be satisfied

$$\frac{\partial V_{\tilde{w}}\left(\breve{\theta^{*}},\breve{s^{*}}\right)}{\partial\theta}\leq\frac{\partial V_{\tilde{w}}\left(\breve{\theta^{*}},\breve{s^{**}}\right)}{\partial\theta}.$$

As a consequence, a sufficient condition for $\breve{\theta^{**}} \geq \breve{\theta^*}$ to hold when $\breve{s^{**}} \geq \breve{s^*}$, is

$$\frac{\partial V_{\tilde{w}}\left(\breve{\theta^{*}},\breve{s^{*}}\right)}{\partial\theta}\leq\frac{\partial V_{\tilde{w}}\left(\breve{\theta^{**}},\breve{s^{**}}\right)}{\partial\theta}=0$$

[27] On this assumption see note 13.

or by Equation (19)

$$\frac{\partial V_{\tilde{w}}\left(\breve{\theta}^{*},\breve{s}^{*}\right)}{\partial\theta}\leq\frac{\partial V_{w}\left(\breve{\theta}^{*},\breve{s}^{*};w\right)}{\partial\theta}=0.$$

Using Equations (16) and (19) rewrite the previous inequality as follows

$$\begin{aligned}
&-E\left[v_1\left(\tilde{y_2},l_2\right)\tilde{w}\right]+E\left[v_1\left(\tilde{y_2},l_2\right)\right]w\left(\alpha(\breve{\theta}^{*})+\breve{\theta}^{*}\alpha'(\breve{\theta}^{*})\right)+\\
&+E\left[v_2\left(\tilde{y_2},l_2\right)\right]\left(1-\bar{l}\right)\\
&\leq -v_1\left(y_2,l_2\right)\left(1-\bar{l}\right)w\left(1-\alpha(\breve{\theta}^{*})-\breve{\theta}^{*}\alpha'(\breve{\theta}^{*})\right)+v_2\left(y_2,l_2\right)\left(1-\bar{l}\right).
\end{aligned}\tag{35}$$

Consider now a Taylor expansion around w of the three terms in the left hand side of Inequality (35). The first term is

$$\begin{aligned}
E\left[v_1\left(\tilde{y_2},l_2\right)\tilde{w}\right]\tilde{=}&v_1\left(y_2,l_2\right)w+\frac{1}{2}\left(1-\breve{\theta}^{*}\left(1-\gamma\right)\right)\left(1-\bar{l}\right)\\
\left(v_{111}\left(y_2,l_2\right)\left(1-\breve{\theta}^{*}\left(1-\gamma\right)\right)\left(1-\bar{l}\right)+2v_{11}\left(y_2,l_2\right)\right)&var\left[\tilde{w}\right].
\end{aligned}\tag{36}$$

Note further that it is the case that

$$\begin{aligned}
E\left[v_1\left(\tilde{y_2},l_2\right)\right]\tilde{=}&v_1\left(y_2,l_2\right)+\\
+\frac{1}{2}v_{111}\left(y_2,l_2\right)&\left(1-\breve{\theta}^{*}\left(1-\gamma\right)\right)^2\left(1-\bar{l}\right)^2var\left[\tilde{w}\right]
\end{aligned}\tag{37}$$

and

$$\begin{aligned}
E\left[v_2\left(\tilde{y_2},l_2\right)\right]\tilde{=}&v_2\left(y_2,l_2\right)+\\
+\frac{1}{2}v_{112}\left(y_2,l_2\right)&\left(1-\breve{\theta}^{*}\left(1-\gamma\right)\right)^2\left(1-\bar{l}\right)^2var\left[\tilde{w}\right].
\end{aligned}\tag{38}$$

Substituting Equations (36), (37) and (38) into the left hand side of Inequality (35) gives

$$\begin{aligned}
&-v_1\left(y_2,l_2\right)w-\frac{1}{2}\left(1-\breve{\theta}^{*}\left(1-\gamma\right)\right)\left(1-\bar{l}\right)\\
&\quad\left(v_{111}\left(y_2,l_2\right)\left(1-\breve{\theta}^{*}\left(1-\gamma\right)\right)\left(1-\bar{l}\right)+2v_{11}\left(y_2,l_2\right)\right)var\left[\tilde{w}\right]+\\
&+v_1\left(y_2,l_2\right)w\left(\alpha(\breve{\theta}^{*})+\breve{\theta}^{*}\alpha'(\breve{\theta}^{*})\right)+\\
&+\frac{1}{2}v_{111}\left(y_2,l_2\right)\left(1-\breve{\theta}^{*}\left(1-\gamma\right)\right)^2\left(1-\bar{l}\right)^2var\left[\tilde{w}\right]w\left(\alpha(\breve{\theta}^{*})+\breve{\theta}^{*}\alpha'(\breve{\theta}^{*})\right)+\\
&+v_2\left(y_2,l_2\right)\left(1-\bar{l}\right)+\frac{1}{2}v_{112}\left(y_2,l_2\right)\left(1-\breve{\theta}^{*}\left(1-\gamma\right)\right)^2\left(1-\bar{l}\right)^3var\left[\tilde{w}\right]+\\
&\geq -v_1\left(y_2,l_2\right)w\left(1-\alpha(\breve{\theta}^{*})\left(1-\bar{l}\right)-\breve{\theta}^{*}\alpha'(\breve{\theta}^{*})\right)+v_2\left(y_2,l_2\right)\left(1-\bar{l}\right)
\end{aligned}\tag{39}$$

or simplifying and reordering the terms

$$-\frac{v_{111}\left(y_2,l_2\right)}{v_{11}\left(y_2,l_2\right)}w\left(1-\alpha(\breve{\theta}^*)-\breve{\theta}^*\alpha'(\breve{\theta}^*)\right)+\frac{v_{112}\left(y_2,l_2\right)}{v_{11}\left(y_2,l_2\right)}\left(1-\bar{l}\right)$$
$$\leq\frac{2}{\left(1-\bar{l}\right)\left(1-\breve{\theta}^*\left(1-\gamma\right)\right)}.$$

This inequality is verified if $v_{112}\left(c_2,l_2\right)\geq 0$ and

$$-\frac{v_{111}\left(y_2,l_2\right)}{v_{11}\left(y_2,l_2\right)}\leq\frac{2}{\left(1-\breve{\theta}^*\left(1-\gamma\right)\right)\left(1-\bar{l}\right)w\left(1-\alpha(\breve{\theta}^*)-\breve{\theta}^*\alpha'(\breve{\theta}^*)\right)}.$$

In order for a precautionary motive for saving to operate and $\breve{s^{**}}\geq\breve{s^*}$ to hold, it must be the case that

$$\frac{\partial V_{\tilde{w}}\left(\breve{\theta^{**}},\breve{s^{**}}\right)}{\partial s}\leq\frac{\partial V_{\tilde{w}}\left(\breve{\theta^{**}},\breve{s^*}\right)}{\partial s}$$

since

$$\frac{\partial'' V_{\tilde{w}}\left(\theta,s\right)}{\partial'' s}=u_{11}\left(\tilde{y}_1,l_1\right)+v_{11}\left(\tilde{y}_2,l_2\right)R^2<0. \tag{40}$$

Note further that since $\frac{\partial'' V_{\tilde{w}}(\theta,s)}{\partial'\theta\partial' s}\geq 0$, if $\breve{\theta^{**}}\geq\breve{\theta}^*$, the following inequality must be satisfied

$$\frac{\partial V_{\tilde{w}}\left(\breve{\theta}^*,\breve{s^*}\right)}{\partial s}\leq\frac{\partial V_{\tilde{w}}\left(\breve{\theta^{**}},\breve{s^*}\right)}{\partial s}.$$

As a consequence a sufficient condition for $\breve{s^{**}}\geq\breve{s^*}$ to hold when $\breve{\theta^{**}}\geq\breve{\theta}^*$, is

$$\frac{\partial V_{\tilde{w}}\left(\breve{\theta}^*,\breve{s^*}\right)}{\partial s}\leq\frac{\partial V_{\tilde{w}}\left(\breve{\theta^{**}},\breve{s^{**}}\right)}{\partial s}=0$$

or by Equation (20)

$$\frac{\partial V_{\tilde{w}}\left(\breve{\theta}^*,\breve{s^*}\right)}{\partial s}\leq\frac{\partial V_{w}\left(\breve{\theta}^*,\breve{s^*}\right)}{\partial s}. \tag{41}$$

Consider now a Taylor expansion of Equation (41) around w by substituting Equation (37) in its left hand side. Substitute further Equation (17) in the right hand side to obtain

$$\frac{1}{2}v_{111}\left(y_2,l_2\right)\left(1-\breve{\theta}^*\left(1-\gamma\right)\right)^2 var\left[\tilde{w}\right]\geq 0$$

implying that $v_{111}\left(y_2,l_2\right)\geq 0$ is a sufficient condition for the previous inequality to hold.

The sufficient conditions for $\breve{\theta^{**}}\leq\breve{\theta^*}$ and $\breve{s^{**}}\leq\breve{s^*}$ to jointly hold are characterized following the same steps described above.

References

[1] Abel AB (1985) Precautionary saving and accidental bequest. *American Economic Review* 75: 777–791.

[2] Baiardi, D., Magnani, M., Menegatti, M., (2014). Precautionary saving under many risks. *Journal of Economics* 113, 211-228.

[3] Berlingieri G, Blanchenay P, Criscuolo, C (2017) The great divergence (s). OECD Science, Technology and Industry Policy Papers, May 2017.

[4] Blundell, R, French E, Tetlow, G (2016). Retirement incentives and labor supply. In Handbook of the economics of population aging (Vol. 1, pp. 457-566). North-Holland.

[5] Brianti M, Magnani M, Menegatti M (2018) Optimal choice of prevention and cure under uncertainty on disease effect and cure effectiveness. *Research in Economics* 72(2): 327-342.

[6] Card D, Maestas N, Purcell P (2014) Labor market shocks and early Social Security benefit claiming. Working Paper 2014–317, University of Michigan Retirement Research Center.

[7] Chen H, Hsu WY, Weiss MA (2015) The pension option in labor insurance and its effect on household saving and consumption: evidence from Taiwan. *The Journal of Risk and Insurance* 82: 947–975.

[8] Chiu HW, Eeckhoudt L (2010) The effects of stochastic wages and non-labor income on labor supply: update and extensions. *Journal of Economics* 100(1): 69-83.

[9] Chiu HW, Eeckhoudt L, Rey B (2015) On relative and partial risk attitudes: theory and implications. *Economic Theory* 50: 151–167

[10] Coile CC, Levine PB (2011a) The Market Crash and Mass Layoffs: How the Current Economic Crisis May Affect Retirement?. *The B.E. Journal of Economic Analysis & Policy* 11: article 22.

[11] Coile CC, Levine PB (2011b) Recessions, Retirement, and Social Security. *American Economic Review* 101: 23–28.

[12] Coile CC (2015) Economic determinants of worker's retirement decisions. *Journal of Economic Surveys* 29(4): 830-853.
[13] Conde-Ruiz JI, Profeta P (2007) The redistributive design of social security systems. *Economic Journal* 117: 686-712
[14] Cremer H, Pestieau P (2003) Social Insurance Competition between Bismarck and Beveridge. *Journal of Urban Economics* 54(1): 181-196.
[15] Dionne G, Eeckhoudt L (1984) Insurance and Saving: Some Further Results. *Insurance: Mathematics and Economics* 3: 101–110.
[16] Drèze J, Modigliani F (1972) Consumption Decision under Uncertainty. *Journal of Economic Theory* 5: 308–335.
[17] Engen EM, Gruber J (2001) Unemployment insurance and precautionary saving. *Journal of Monetary Economics* 47: 545–579.
[18] Eurofound (2016), Extending Working Lives Through Flexible Retirement Schemes: Partial Retirement, Publications Office of the European Union, Luxembourg.
[19] Flodén M (2006) Labor Supply and Saving Under Uncertainty. *Economic Journal* 116: 721–737.
[20] French E, Bailey JJ (2017) Health, health insurance, and retirement: a survey. *Annual Review of Economics* 9: 383-409.
[21] Gruber J, Wise D (1999) International Comparison of Social Security Systems. University of Chicago Press, Chicago.
[22] Gollier C (2001) The economics of risk and time. MIT Press, Cambridge.
[23] Gustman AL, Steinmeier TL (2005) The social security early entitlement age in a structural model of retirement and wealth. *Journal of Public Economics* 89(2): 441–463.
[24] Gustman, AL, Steinmeier TL, Tabatabai N (2010) Pensions in the Health and Retirement Study. Harvard University Press.
[25] Gustman AL, Steinmeier TL (2014) The role of health in retirement. Working Paper 19902, National Bureau of Economic Research.
[26] Gustman AL, Steinmeier TL (2015) Effects of social security policies on benefit claiming, retirement and saving. *Journal of Public Economics* 129: 51–62.
[27] Hubbard RG, Judd KL (1987) Social security and individual welfare: precautionary saving, liquidity constraints, and the payroll tax. *American Economic Review* 77: 630–646.
[28] Kimball MS (1990) Precautionary Savings in the Small and in the Large. *Econometrica* 58: 53–73.
[29] Kimball MS (1993) Standard Risk Aversion. *Econometrica* 61: 589–611.
[30] Kotlikoff LJ, Shoven J, Spivak A (1987) Annuity markets, savings,

and the capital stock. In: Bodie Z, Shoven J, Wise D (Eds.) Issues in Pension Economics. University of Chicago Press, Chicago.

[31] Krieger T, Traub S (2013) The bismarckian factor: a measure of intra-generational redistribution in international pension systems. CESifo DICE Report 1: 64–66.

[32] Li J (2012) Precautionary saving in the presence of labor income and interest rate risks. *Journal of Economics* 106(3): 251–266.

[33] Leland H (1968) Saving and Uncertainty: the Precautionary Demand for Saving. *The Quarterly Journal of Economics* 82: 465–473.

[34] Magnani M (2017) A new interpretation of the condition for precautionary saving in the presence of an interest-rate risk. *Journal of Economics* 120(1), 79–87.

[35] Noussair CN, Trautmann ST, Van de Kuilen G (2014) Higher order risk attitudes, demographics, and financial decisions. *The Review of Economic Studies* 81: 325–355.

[36] OECD (2015) Pensions at a glance. OECD Publishing.

[37] OECD (2017) Pensions at a glance. OECD Publishing.

[38] Papadopoulos M, Patria M, Triest RK (2017) Population aging, labor demand, and the structure of wages. *The Geneva Papers on Risk and Insurance-Issues and Practice* 42: 453–474.

[39] Pratt JW (1964) Risk aversion in the small and in the large. *Econometrica* 32: 122–136.

[40] Rothschild M, Stiglitz J (1971) Increasing risk II: economic consequences. *Journal of Economic Theory* 3: 66-84.

[41] Samuelson PA (1974) Complementarity: An essay on the 40th anniversary of the Hicks-Allen revolution in demand theory. *Journal of Economic Literature* 12(4): 1255–1289.

[42] Sandmo A (1970) The Effect of Uncertainty on Saving Decisions. *Review of Economic Studies* 37: 353–360.

[43] Sheshinski E, Weiss Y (1981) Uncertainty and optimal social security systems. *The Quarterly Journal of Economics* 96: 189–206.

[44] Shoven JB, Slavov SN (2012a) The decision to delay Social Security benefits: theory and evidence. NBER Working Paper No. 17866.

[45] Shoven JB, Slavov SN (2012b) When does it pay to delay Social Security? The impact of mortality, interest rates, and program rules. NBER Working Paper No. 18210.

[46] Venti SF, Wise DA (1999) Lifetime earnings, saving choices, and wealth at retirement. In: Smith JP, Willis RJ (Eds.) Wealth, Work and Health: Innovations in Measurement in the Social Sciences. University of Michigan Press, Ann Arbor, pp. 87-120.

[47] Venti SF, Wise DA (2001) Choice, chance, and wealth dispersion at retirement. In: Ogura S, Tachibanaki T, Wise D (Eds.) Aging

Issues in the United States and Japan. University of Chicago Press, Chicago.
[48] Wong, K. P., (2019). An interpretation of the condition for precautionary saving: the case of greater higher-order interest rate risk. *Journal of Economics*, 126: 275–286.