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MEAN FIELD GAMES MODELS OF SEGREGATION

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This paper introduces and analyzes some models in the framework of Mean Field Games describing interactions between two populations motivated by the studies on urban settlements and residential choice by Thomas Schelling. For static games, a large population limit is proved. For differential games with noise, the existence of solutions is established for the systems of partial differential equations of Mean Field Game theory, in the stationary and in the evolutive case. Numerical methods are proposed, with several simulations. In the examples and in the numerical results, particular emphasis is put on the phenomenon of segregation between the populations.

Keywords: Mean Field Games, multi-populations models, large populations limit, systems of parabolic equations, finite difference methods, segregation.

AMS Subject Classification: 91A13, 49N70, 35K55

1. Introduction

The theory of Mean Field Games (MFG, in short) is a branch of Dynamic Games which aims at modeling and analyzing complex decision processes involving a large number of indistinguishable rational agents who have individually a very small influence on the overall system and are, on the other hand, influenced by the distribution of the other agents. It originated about ten years ago in the independent work of J. M. Lasry and P.L. Lions, Ref. 41, and of M.Y. Huang, P. E. Caines and R. Malhamé Refs. 37, 36. In the case of independent noises affecting the agents, the main

equations describing MFG are a Hamilton-Jacobi-Bellman parabolic equation for the value function of the representative agent coupled with a Kolmogorov-Fokker-Planck equation for the density of the population, the former backward in time with a terminal condition and the latter forward in time with an initial condition. Recently the theory and applications of MFG have been growing very fast: we refer to P.-L. Lions' courses on the site of the Collège de France <http://www.college-de-france.fr/site/en-pierre-louis-lions/>, the lecture notes Refs. 33 and 14, and the books Refs. 12, 28, and 32. A major recent breakthrough by Cardaliaguet, Delarue, Lasry, and Lions is the solution of a PDE in the space of probability measures, called master equation, which describes MFGs with a common noise affecting all players, and allows to prove general convergence results of N -person differential games to a MFG as $N \rightarrow \infty$ in a suitable sense, see Ref. 15.

The goal of this paper is to propose some models in the framework of Mean Field Games to describe some kinds of interactions between two different populations, each formed by a large number of indistinguishable agents. Such phenomena arise, for instance, in urban settlements, ecosystems, pedestrian dynamics, see, e.g., Refs. 22, 10, and the references therein. We will focus in particular on models of residential choice possibly leading to segregated neighborhoods. We are inspired by the pioneering work of the Nobel Prize in Economics Thomas Schelling, Refs. 44, 45, and some of its developments until recently, see, e.g., Refs. 47, 13, 9, 48, 26, the survey 25, and the references therein. However, different from the sociologic and economic literature where the models are usually discrete in space and time, we propose games continuous in space and either static, for which we derive rigorously the large population limit, or in continuous time, with the dynamics of each player described by a controlled system affected by noise. In the differential game, the preferences of the players are described by a cost functional integrated in time that each player seeks to minimise. We consider finite horizon problems as well as games with long-time average cost (also called ergodic cost).

Our analytic results are on the existence of solutions to the system of the four PDEs associated to the two-population MFG, with Neumann boundary conditions modelling the boundedness of the city where the agents live. The PDEs are elliptic in the case of ergodic cost, with an additive eigenvalue in each of the two H-J-B equations; the case of several populations was treated by the second author and Feleqi with periodic boundary conditions (i.e., the state space of the agents is a torus, Refs. 7, 23), and by the third author with Neumann boundary conditions, Ref. 20. For finite horizon costs, the PDEs are parabolic (two backward and two forward in time) and existence is known for a single population and periodic boundary conditions; we extend it to two populations and Neumann conditions. Uniqueness of solutions holds for a single population under a restrictive monotonicity condition (Ref. 41) and is not expected to hold for several populations. In fact, we provide examples of non uniqueness by showing that the same game can have segregated solutions as well as non-segregated ones, such as uniform distributions of both populations.

One of the most interesting issues about these models is the qualitative behavior of solutions, in particular whether two initially mixed population tend to segregate, i.e., to concentrate in different parts of the city. Schelling's most striking discovery was that very moderate preferences for same-population neighbors at the individual level can lead to complete residential segregation at the macro level. For example, if every agent requires at least half of her neighbors to belong to the same population, and moves only if the percentage is below this threshold, the final outcome, after a sequence of moves, is almost always complete segregation. Nowadays several softwares freely available on the internet allow such simulations and show that segregation eventually occurs, with random initial conditions, even with much milder thresholds, i.e., lower than $1/2$, see, e.g., NetLogo (<http://ccl.northwestern.edu/netlogo/>). Thus Schelling's conclusion was that the "macrobehavior" in a society may not reflect the "micromotives" of its individual members (Ref. 45). His early experiments are considered today among the first prototypes of artificial societies, see, e.g., Ref. 43.

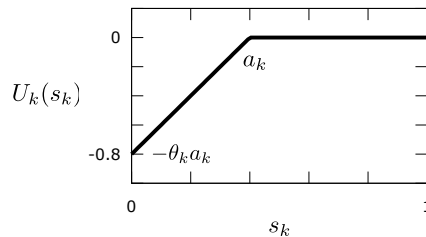
We study the qualitative behavior of solutions by numerical methods. We use the techniques introduced in MFG with a single population and periodic boundary conditions by the first author, Capuzzo Dolcetta, and Camilli in Refs. 4, 2, and 3. We present finite difference schemes for the stationary PDEs associated to ergodic costs as well as for the evolutive backward-forward system of the finite horizon problem. For both cases we show that segregation occurs with low preference thresholds, so Schelling's principle is valid also in our MFG models. We also compare the results for different thresholds, showing that a higher threshold pushes a population to concentrate in a smaller space, and we also observe the instability arising if both populations are rather xenophobic, leading to oscillations in time. Finally we present a 2-d example of pedestrian dynamics with two populations.

More references to the literature on MFG will be given throughout the paper.

The paper is organised as follows. In Section 2, we propose several forms of cost functionals that reflect the preferences described by Schelling, with variants and generalizations. In Section 3, we prove a large population limit for the static game, following the method of Lions and Cardaliaguet, Ref. 14, and give some simple examples of Mean Field equilibria. In Section 4, we first introduce a dynamics driven by a stochastic control system, the long-time average cost, and the stationary MFG PDEs associated to them, followed by an example of coexistence of segregated and non-segregated solutions. Then we describe the finite horizon problem, the evolutive MFG PDEs for it, and prove an existence theorem. Section 5 illustrates the numerical methods for the MFG PDEs. The final Section 6 contains several simulations for the stationary and evolutive cases, in 1 and 2 dimensions.

2. Static games in continuous space inspired by T. Schelling

In this section, we propose a class of static (one-shot) games with two populations of players whose positions are taken in a bounded set $\bar{\Omega} \subset \mathbb{R}^d$. Within each population,

Fig. 1. The utility function U_k ($\theta_k = 2$, $a_k = 0.4$).


all players have the same cost functional to minimize. We choose such functionals in a way that reproduces the main features of the classical models of segregated neighborhoods by Schelling, Ref. 44 and 45, and of some of their subsequent developments. We fix a neighborhood $\mathcal{U}(x)$ for each point $x \in \bar{\Omega}$ and consider the amount of each population living in such neighborhood, $N_1(x), N_2(x)$. In the simplest models, the utility U_k (= minus the cost) of an individual of the k -th species living at the position x depends only on the quantity

$$s_k := \frac{N_k(x)}{N_1(x) + N_2(x)} \quad (2.1)$$

and has the shape shown in Figure 1, that is,

$$U_k(s_k) := \begin{cases} \theta_k(s_k - a_k) & \text{if } s_k < a_k, \\ 0 & \text{else,} \end{cases} \quad (2.2)$$

where $\theta_k > 0$ and $0 \leq a_k \leq 1$. Here s_k is the percentage of population k living in $\mathcal{U}(x)$ and a_k is a threshold of happiness: if s_k is below it the player of the k -th species at the position x has a negative utility, i.e., a positive cost.

In the Schelling's model and in the differential games of Section 4, the agent then moves and looks for a location with a higher value of s_k , possibly $s_k > a_k$. In the static games of this section, we look for equilibrium distributions of the players that are Nash equilibria for the game of minimizing the individual costs. In most of the recent literature the parameter a_k is taken to be $1/2$ for both populations, but in Schelling's original examples it is often below $1/2$, therefore modeling populations that are not xenophobic and that just do not want that their own group be too small in their neighborhood.

The shape of the utility function (2.2) is "peaked at a_k ", as one of those considered in Ref. 9 and a limit case of those in Ref. 47, 48; for the slope θ_k very large it approximates the stair-like utility of Schelling, and for $a_k = 1$ it is the linear utility of Ref. 13; see Ref. 25 for a survey. References 47, 48, and 9 consider also utilities decreasing on the right of a_k : although we do not consider these cases in the numerical simulations, they satisfy the same boundedness conditions as our models and therefore fit into our analysis of Sections 3, 4, and 5.

We will consider also more general cost functionals that depend on $N_1(x)$ and $N_2(x)$ separately, not only via s_k , and definitions of $N_1(x), N_2(x)$ as measures of the number of individuals weighted by the distance from x . Our assumptions will be general enough to include examples in fields different from residential segregation, such as crowd motion and pedestrian dynamics, see Ref. 22 for a general presentation and Ref. 38 and 39 for Mean-Field Games models with two populations.

2.1. A basic game with two populations of N players.

We consider a one-shot game with $2N$ players divided in two populations. The vector (x_1, \dots, x_N) represents the positions of the players of the first population and (y_1, \dots, y_N) those of the players of the second one, where $x_i, y_i \in \bar{\Omega}$ and $\Omega \subset \mathbb{R}^d$ is an open and bounded set. We adopt the conventions and notations of Mean-Field Games, see Ref. 41, and associate to each player a *cost* (instead of a utility) that the player seeks to *minimise* (instead of maximise), it is denoted with $F_i^{1,N}$ for the i -th player of the first population and with $F_i^{2,N}$ for the i -th player of the second population. The first kind of cost functionals we propose are

$$F_i^{1,N}(x_1, \dots, x_N, y_1, \dots, y_N) = \theta_1 \left(\frac{\#\{x_j \in \mathcal{U}(x_i) : j \neq i\}}{\#\{x_j \in \mathcal{U}(x_i) : j \neq i\} + \#\{y_j \in \mathcal{U}(x_i)\} + \eta_1(N-1)} - a_1 \right)^-, \quad (2.3)$$

where $\theta_1 > 0, 0 \leq a_1 \leq 1, \eta_1 \geq 0$, $\#X$ denotes the cardinality of the (finite) set X , $\mathcal{U}(x)$ is some neighborhood of x (for example $B_r(x) \cap \bar{\Omega}$, where $B_r(x)$ is the ball centered at x of radius r , or $S_r(x) \cap \bar{\Omega}$, where $S_r(x)$ is the square centered at x of side length r), and $(t)^-$ denotes the negative part of t , i.e., $(t)^- = -t$ if $t < 0$ and $(t)^- = 0$ if $t \geq 0$. As before, $a_1 \in [0, 1]$ is the ‘‘threshold of happiness’’ of any player of the first population: his cost is null if the ratio of the individuals of his own kind in the neighborhood is above this threshold, whereas the cost is positive with slope θ_1 below the threshold. Note that, for $\eta_1 = 0$ and U_1, s_1 defined by (2.2), (2.1),

$$F_i^{1,N} := -U_1(s_1), \quad N_1(x) = \#\{x_j \in \mathcal{U}(x_i) : j \neq i\}, \quad N_2(x) = \#\{y_j \in \mathcal{U}(x_i)\}.$$

In the following, however, we will assume $\eta_1 > 0$ (and small) in order to avoid the indeterminacy of the ratio s_1 (2.1) as $N_1(x) + N_2(x) \rightarrow 0$. This assumption makes the cost continuous, and it has the following interpretation: suppose that a player is surrounded just by individuals of his own kind, i.e. $\#\{y_j \in \mathcal{U}(x_i)\} = 0$, then the cost he pays is null as long as

$$N_1(x) = \#\{x_j \in \mathcal{U}(x_i) : j \neq i\} \geq \frac{a_1 \eta_1}{1 - a_1} (N - 1).$$

But if $N_1(x)$ becomes too small he pays a positive cost (tending to $\theta_1 a_1$ as $N_1(x) \rightarrow 0$). This means that it is uncomfortable to live in an almost desert neighborhood.

We introduce the notation

$$G(r, s; a, t) := \left(\frac{r}{r + s + t} - a \right)^-, \quad (2.4)$$

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and observe that $G : [0, +\infty) \times [0, +\infty) \times [0, 1] \times (0, 1) \rightarrow [0, +\infty)$ is a continuous and bounded function of r, s for each a, t fixed. We rewrite

$$\begin{aligned} F_i^{1,N}(x_1, \dots, x_N, y_1, \dots, y_N) \\ = \theta_1 G(\#\{x_j \in \mathcal{U}(x_i) : j \neq i\}, \#\{y_j \in \mathcal{U}(x_i)\}; a_1, \eta_1(N-1)), \end{aligned}$$

The cost for each player of the second population is

$$\begin{aligned} F_i^{2,N}(x_1, \dots, x_N, y_1, \dots, y_N) \\ = \theta_2 G(\#\{y_j \in \mathcal{U}(y_i) : j \neq i\}, \#\{x_j \in \mathcal{U}(y_i)\}; a_2, \eta_2(N-1)), \end{aligned}$$

where $a_2 \in [0, 1]$ represents the threshold of happiness of this population and $\theta_2, \eta_2 > 0$. It has the same form as $F_i^{1,N}$, but the three parameters a_2, θ_2, η_2 can be different from a_1, θ_1, η_1 .

We note that the costs depend on the position of the players only via the empirical measures of the two populations. As usual in the theory of Mean-Field Games they can be generated by maps over probability measures as follows

$$F_i^{1,N}(x_1, \dots, x_N, y_1, \dots, y_N) = V^{1,N} \left[\frac{1}{N-1} \sum_{i \neq j} \delta_{x_j}, \frac{1}{N} \sum \delta_{y_j} \right] (x_i), \quad (2.5)$$

where $V^{1,N} : \mathcal{P}(\bar{\Omega}) \times \mathcal{P}(\bar{\Omega}) \rightarrow C(\bar{\Omega})$ is defined by

$$V^{1,N}[m_1, m_2](x) := \theta_1 G \left((N-1) \int_{\mathcal{U}(x)} m_1, N \int_{\mathcal{U}(x)} m_2; a_1, \eta_1(N-1) \right), \quad (2.6)$$

where $\mathcal{P}(\bar{\Omega})$ denotes the set of all probability measures over $\bar{\Omega}$. In the same way,

$$\begin{aligned} F_i^{2,N}(x_1, \dots, x_N, y_1, \dots, y_N) &= V^{2,N} \left[\frac{1}{N} \sum \delta_{x_j}, \frac{1}{N-1} \sum_{i \neq j} \delta_{y_j} \right] (y_i) \\ &= \theta_2 G \left((N-1) \int_{\mathcal{U}(y_i)} \frac{1}{N-1} \sum_{i \neq j} \delta_{y_j}, N \int_{\mathcal{U}(y_i)} \frac{1}{N} \sum \delta_{x_j}; a_2, \eta_2(N-1) \right). \end{aligned} \quad (2.7)$$

In the rest of the paper, we will assume

$$\theta_1 = \theta_2 = 1.$$

This is done merely for simplifying the notations, all the results and proofs of the paper remain valid for any positive values of θ_k .

2.2. Overcrowding and family effects

In the discrete model of Schelling, there is a structural impossibility of overcrowding: every player occupies a position in a chessboard, and every slot can host at most one player. In our continuous model, there is no constraint on the local density and the individuals may even concentrate at a single point of the domain. In order to

avoid this unrealistic phenomenon, we shall introduce an *overcrowding* term in the costs $F_i^{k,N}$:

$$\begin{aligned}\hat{F}_i^{1,N}(x_1, \dots, y_N) &= F_i^{1,N} + C_1[(\#\{x_j \in \mathcal{U}(x_i)\} + \#\{y_j \in \mathcal{U}(x_i)\})/(2N) - b_1]^+, \\ \hat{F}_i^{2,N}(x_1, \dots, y_N) &= F_i^{2,N} + C_2[(\#\{x_j \in \mathcal{U}(y_i)\} + \#\{y_j \in \mathcal{U}(y_i)\})/(2N) - b_2]^+, \end{aligned}$$

for every $i = 1, \dots, N$, so every player starts paying a positive cost when the total number of players in his neighborhood overcomes the threshold $b_k 2N$; thus $b_k \geq 0$ represents the maximum percentage of the whole population that is tolerated at no cost. Here C_k are positive constants, possibly large: when the concentration of players is too high in some regions, the discomfort might be due to overcrowding and not necessarily to an unsatisfactory ratio between the total number of individuals of the two populations (the $F_i^{k,N}$ term).

The maps over probability measures that generate these costs are

$$\begin{aligned}\hat{V}^1[m_1, m_2](x) &:= V^1[m_1, m_2](x) + C_1 \left[\int_{\mathcal{U}(x)} \frac{m_1 + m_2}{2} - b_1 \right]^+, \\ \hat{V}^2[m_1, m_2](x) &:= V^2[m_1, m_2](x) + C_2 \left[\int_{\mathcal{U}(x)} \frac{m_1 + m_2}{2} - b_2 \right]^+ \end{aligned}$$

for the two populations.

Next we take into account that an individual may be influenced also by the opinions of other individuals living around him. A first attempt to model this is adding to the cost of each player the costs paid by the players of his own kind and very close to him, e.g., by his family, leading to

$$\begin{aligned}\bar{F}_i^{1,N}(x_1, \dots, x_N, y_1, \dots, y_N) &= \frac{1}{N} \sum_{l: x_l \in \mathcal{V}(x_i)} F_l^{1,N}(x_1, \dots, x_N, y_1, \dots, y_N), \\ \bar{F}_i^{2,N}(x_1, \dots, x_N, y_1, \dots, y_N) &= \frac{1}{N} \sum_{l: y_l \in \mathcal{V}(y_i)} F_l^{2,N}(x_1, \dots, x_N, y_1, \dots, y_N), \end{aligned}$$

where $\mathcal{V}(x)$ is a neighborhood of x in Ω . This can be refined by assuming that the opinion of other neighbors is weighted by a function that depends upon the distance from the individual

$$\bar{F}_i^{k,N}(x_1, \dots, x_N, y_1, \dots, y_N) = \frac{1}{N} \sum_{l=1}^N F_l^{k,N}(x_1, \dots, x_N, y_1, \dots, y_N) W(x_i, x_l), \quad (2.8)$$

$k = 1, 2$, where $W : \bar{\Omega} \times \bar{\Omega} \rightarrow \mathbb{R}$ is nonnegative and such that $W(x_i, \cdot)$ has support in $\mathcal{V}(x_i)$. Hence, combining (2.5) and (2.7) with (2.8), we arrive at

$$\bar{F}_i^{k,N}(x_1, \dots, x_N, y_1, \dots, y_N) = \int_{\bar{\Omega}} W(x_i, z) V^{k,N}(z) \frac{1}{N} \sum_{l=1}^N \delta_{x_l}(dz), \quad (2.9)$$

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where

$$V^{1,N}(z) := V^{1,N} \left[\frac{1}{N-1} \sum_{i \neq j} \delta_{x_j}, \frac{1}{N} \sum \delta_{y_j} \right] (z),$$

$$V^{2,N}(z) := V^{2,N} \left[\frac{1}{N} \sum \delta_{x_j}, \frac{1}{N-1} \sum_{i \neq j} \delta_{y_j} \right] (z).$$

2.3. More regular cost functionals

The cost functionals proposed so far involve the amount of individuals in a neighborhood of x that can be written as

$$\int_{\mathcal{U}(x)} dm_k(y) = \int_{\Omega} \chi_{\mathcal{U}(x)}(y) dm_k(y),$$

where m_k is the empirical measure of the k -th population and $\chi_{\mathcal{U}(x)}(\cdot)$ is the indicator function of the set $\mathcal{U}(x)$, i.e., $\chi_{\mathcal{U}(x)}(y) = 1$ if $y \in \mathcal{U}(x)$ and $\chi_{\mathcal{U}(x)}(y) = 0$ otherwise. It is useful to consider regularized versions of such integrals where $\chi_{\mathcal{U}(x)}(x, y)$ is approximated by a nonnegative smooth kernel $K(\cdot, \cdot)$ such that $K(x, y) = 1$ if $y \in \mathcal{U}(x)$ and $K(x, y) = 0$ for y out of a small neighborhood of $\mathcal{U}(x)$. The cost functionals of Section 2.1 are modified to

$$V^{1,N} [m_1, m_2] (x) := G \left((N-1) \int_{\Omega} K(x, y) dm_1(y), N \int_{\Omega} K(x, y) dm_2(y); a_1, \eta_1(N-1) \right), \quad (2.10)$$

$$V^{2,N} [m_1, m_2] (x) := G \left((N-1) \int_{\Omega} K(x, y) dm_2(y), N \int_{\Omega} K(x, y) dm_1(y); a_2, \eta_2(N-1) \right). \quad (2.11)$$

As we will see in the next section, these new functionals are continuous on $\mathcal{P}(\overline{\Omega})$ endowed with a suitable notion of distance between measures.

In the present continuous-space setting, they are also more realistic, because individuals near the boundary of $\mathcal{U}(x)$ still count in the cost but with small weights. More generally, K can be a suitable decreasing function of the distance between x and y .

3. Static Mean-Field Games with two populations

In this section, we derive a pair of equations in $\mathcal{P}(\overline{\Omega})$ that describe the one-shot Mean-Field Game with two populations of players. They are obtained by taking the limit as $N \rightarrow \infty$ of Nash equilibria in the game with $N + N$ players. They are the natural extension to two populations of the equation proposed by Lions for a single population in his lectures at the College de France, see Ref. 14.

In the sequel, we consider $\mathcal{P}(\bar{\Omega})$ as a metric space with the Kantorovich-Rubinstein distance^a between two measures μ, ν that we denote with $\mathbf{d}(\mu, \nu)$, whose topology corresponds to the weak* convergence of measures (see, e.g., Ref. 14).

3.1. The large populations limit

Let $F_1^{1,N}, \dots, F_N^{1,N}, F_1^{2,N}, \dots, F_N^{2,N} : \bar{\Omega}^{2N} \rightarrow \mathbb{R}$ be the cost functions of a game with two populations of N players each. Suppose that there exist continuous $V^1, V^2 : \mathcal{P}(\bar{\Omega}) \times \mathcal{P}(\bar{\Omega}) \rightarrow C(\bar{\Omega})$ such that, for all N and $i = 1, \dots, N$,

$$F_i^{1,N}(x_1, \dots, x_N, y_1, \dots, y_N) = V^1 \left[\frac{1}{N-1} \sum_{i \neq j} \delta_{x_j}, \frac{1}{N} \sum \delta_{y_j} \right] (x_i) + o(1) \quad (3.1)$$

$$F_i^{2,N}(x_1, \dots, x_N, y_1, \dots, y_N) = V^2 \left[\frac{1}{N} \sum \delta_{x_j}, \frac{1}{N-1} \sum_{i \neq j} \delta_{y_j} \right] (y_i) + o(1), \quad (3.2)$$

where $o(1) \rightarrow 0$ as $N \rightarrow \infty$ uniformly with respect to x_i, y_j .

For $(\bar{x}_1^N, \dots, \bar{x}_N^N, \bar{y}_1^N, \dots, \bar{y}_N^N) \in \bar{\Omega}^{2N}$, denote the empirical measures with

$$\bar{m}_1^N := \frac{1}{N} \sum_{j=1}^N \delta_{\bar{x}_j^N}, \quad \bar{m}_2^N := \frac{1}{N} \sum_{j=1}^N \delta_{\bar{y}_j^N}.$$

The next result is the large population limit of Nash equilibria.

Proposition 1. *Assume (3.1), (3.2), and that, for all N , $(\bar{x}_1^N, \dots, \bar{x}_N^N, \bar{y}_1^N, \dots, \bar{y}_N^N)$ is a Nash equilibrium for the game with cost functions $F_1^{1,N}, \dots, F_N^{1,N}, F_1^{2,N}, \dots, F_N^{2,N}$. Then, up to subsequences, the sequences of measures $(\bar{m}_1^N), (\bar{m}_2^N)$ converge, respectively, to $\bar{m}_1, \bar{m}_2 \in \mathcal{P}(\bar{\Omega})$ such that*

$$\int_{\bar{\Omega}} V^k[\bar{m}_1, \bar{m}_2](x) d\bar{m}_k(x) = \inf_{\mu \in \mathcal{P}(\bar{\Omega})} \int_{\bar{\Omega}} V^k[\bar{m}_1, \bar{m}_2](x) d\mu(x), \quad k = 1, 2. \quad (3.3)$$

Proof. By compactness, $\bar{m}_k^N \rightarrow m_k$ as $N \rightarrow \infty$ (up to subsequences); we need to prove that \bar{m}_k satisfy (3.3). Let $\epsilon > 0$, for all $N \geq \bar{N} = \bar{N}(\epsilon)$ we have that for all $z \in \bar{\Omega}$, $i = 1, \dots, N$,

$$V^1 \left[\frac{1}{N-1} \sum_{i \neq j} \delta_{\bar{x}_j^N}, \frac{1}{N} \sum \delta_{\bar{y}_j^N} \right] (\bar{x}_i^N) \leq V^1 \left[\frac{1}{N-1} \sum_{i \neq j} \delta_{\bar{x}_j^N}, \frac{1}{N} \sum \delta_{\bar{y}_j^N} \right] (z) + \epsilon$$

by definition of Nash equilibrium and (3.1), so the measure $\delta_{\bar{x}_i^N}$ satisfies for all

^aWe recall that $\mathbf{d}(\mu, \nu) = \sup \{ \int_{\bar{\Omega}} \phi(x) (\mu - \nu)(dx) \mid \phi : \bar{\Omega} \rightarrow \mathbb{R} \text{ is 1-Lipschitz continuous} \}$.

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$\mu \in \mathcal{P}(\bar{\Omega})$

$$\int_{\bar{\Omega}} V^1 \left[\frac{1}{N-1} \sum_{j \neq i} \delta_{\bar{x}_j^N}, \frac{1}{N} \sum_j \delta_{\bar{y}_j^N} \right] (x) d\delta_{\bar{x}_i^N}(x) \leq \int_{\bar{\Omega}} V^1 \left[\frac{1}{N-1} \sum_{j \neq i} \delta_{\bar{x}_j^N}, \frac{1}{N} \sum_j \delta_{\bar{y}_j^N} \right] (x) d\mu(x) + \epsilon.$$

Since $\mathbf{d} \left(\frac{1}{N-1} \sum_{j \neq i} \delta_{\bar{x}_j^N}, \bar{m}_1^N \right) \rightarrow 0$, by continuity of V^1

$$\left| V^1 \left[\frac{1}{N-1} \sum_{j \neq i} \delta_{\bar{x}_j^N}, \frac{1}{N} \sum_j \delta_{\bar{y}_j^N} \right] (x) - V^1[\bar{m}_1^N, \bar{m}_2^N](x) \right| \leq \epsilon$$

for all $x \in \bar{\Omega}$ and $N \geq \bar{N}$, so

$$\int_{\bar{\Omega}} V^1[\bar{m}_1^N, \bar{m}_2^N](x) d\delta_{\bar{x}_i^N}(x) \leq \int_{\bar{\Omega}} V^1[\bar{m}_1^N, \bar{m}_2^N](x) d\mu(x) + 3\epsilon.$$

Then we take the sum for $i = 1, \dots, N$ and the \inf_{μ} , divide by N and get

$$\int_{\bar{\Omega}} V^1[\bar{m}_1^N, \bar{m}_2^N](x) d\bar{m}_1^N(x) \leq \inf_{\mu \in \mathcal{P}(\bar{\Omega})} \int_{\bar{\Omega}} V^1[\bar{m}_1^N, \bar{m}_2^N](x) d\mu(x) + 3\epsilon.$$

Using again that continuity of V^1 , by passing to the limit as $N \rightarrow \infty$ and then $\epsilon \rightarrow 0$ we obtain (3.3) for $k = 1$. The argument for $k = 2$ is analogous, by using (3.2) instead of (3.1). \square

Remark 2. The two equations (3.3) define a *Mean-Field equilibrium* (\bar{m}_1, \bar{m}_2) for any game with two populations associated to the functionals V^1, V^2 . They are easily seen to be equivalent to the equations

$$\forall x \in \text{supp } \bar{m}_k \quad V^k[\bar{m}_1, \bar{m}_2](x) = \min_{z \in \bar{\Omega}} V^k[\bar{m}_1, \bar{m}_2](z), \quad k = 1, 2, \quad (3.4)$$

see Ref. 14, Section 2.2, for the case of a single population.

Remark 3. The assumption of existence of a Nash equilibrium for the $N + N$ game in the previous theorem may look restrictive because Nash equilibria may not exist without further assumptions. However, the classical Nash Theorem guarantees that Nash equilibria exist if we allow players to use *mixed strategies*, i.e., to minimise over elements of $\mathcal{P}(\bar{\Omega})$. Moreover, all players of the same population use the same cost function, so one can consider Nash equilibria in mixed strategies that are symmetric within each population, as in Section 8 of Ref. 14. Then one can derive the equations (3.3) and (3.4) via the large population limit by assuming $(x, m_1, m_2) \mapsto V^k[m_1, m_2](x)$ both Lipschitz continuous, but not the existence of a Nash equilibrium in pure strategies, following Section 2.3 of Ref. 14.

3.2. Examples

Here we show that the models of Section 2 satisfy the assumptions of Proposition 1 or Remark 3 as soon as the amount of players in a neighborhood is regularized as in Section 2.3. This is based on the next simple result.

Lemma 4. *If $K : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is Lipschitz continuous, then the map $\bar{\Omega} \times \mathcal{P}(\bar{\Omega}) \rightarrow \mathbb{R}^d$, $(x, m) \mapsto \int_{\bar{\Omega}} K(x, y) dm(y)$ is Lipschitz continuous.*

Proof. The Lipschitz continuity in x is immediate. For the Lipschitz continuity in m we observe that, if L is a Lipschitz constant for $K(x, \cdot)$, then $y \mapsto K(x, y)/L$ has Lipschitz constant 1, so by the very definition of Kantorovich-Rubinstein distance

$$\left| \int_{\bar{\Omega}} K(x, y) d(m(y) - \mu(y)) \right| = L \left| \int_{\bar{\Omega}} \frac{K(x, y)}{L} d(m(y) - \mu(y)) \right| \leq L \mathbf{d}(m, \mu).$$

□

Example 5 (The basic game). We consider the game with $N + N$ players and cost functions

$$F_i^{1,N}(x_1, \dots, x_N, y_1, \dots, y_N) = V^{1,N} \left[\frac{1}{N-1} \sum_{i \neq j} \delta_{x_j}, \frac{1}{N} \sum \delta_{y_j} \right] (x_i),$$

$$F_i^{2,N}(x_1, \dots, x_N, y_1, \dots, y_N) = V^{2,N} \left[\frac{1}{N} \sum_j \delta_{x_j}, \frac{1}{N-1} \sum_{j \neq i} \delta_{y_j} \right] (y_i),$$

where $V^{k,N}$ are the regularized functionals (2.10) and (2.11) with $K \geq 0$ and Lipschitz, and G is defined by (2.4). Since $G(\gamma r, \gamma s; a, t) = G(r, s; a, \gamma^{-1}t)$ for all $\gamma \neq 0$,

$$V^{k,N}[m_1, m_2](x) = G \left(\int_{\bar{\Omega}} K(x, y) dm_k(y), \frac{N}{N-1} \int_{\bar{\Omega}} K(x, y) dm_{-k}(y); a_k, \eta_k \right).$$

Moreover, for $\eta_i > 0$, G is Lipschitz continuous in the first two entries, so we can pass to the limit as $N \rightarrow \infty$ and get (3.1) and (3.2) with

$$V^k[m_1, m_2](x) := G \left(\int_{\bar{\Omega}} K(x, y) dm_k(y), \int_{\bar{\Omega}} K(x, y) dm_{-k}(y); a_k, \eta_k \right), \quad (3.5)$$

where $m_{-1} = m_2$ and $m_{-2} = m_1$. Furthermore, $(x, m_1, m_2) \mapsto V^k[m_1, m_2](x)$ are Lipschitz continuous by Lemma 4. Then Proposition 1 applies to this example if there are Nash equilibria in pure strategies for the $N + N$ game, and in general Remark 3 applies.

Example 6 (Games with family effects). Here we take the cost functionals with “family effects” of Section 2.2 and we regularize them as in Section 2.3, i.e., $V^{k,N}(x)$ are the regularized functionals (2.10) and (2.11) as in the preceding example and we consider

$$\bar{F}_i^{k,N}(x_1, \dots, x_N, y_1, \dots, y_N) = \frac{1}{N} \sum_{l=1}^N V^{k,N}(x_l) W(x_i, x_l), \quad (3.6)$$

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where $W : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is Lipschitz continuous. In this case, (3.1) and (3.2) are satisfied by

$$\bar{V}^k[m_1, m_2](x) := \int_{\bar{\Omega}} W(x, z) V^k[m_1, m_2](z) dm_k(z) \quad (3.7)$$

and $(x, m_1, m_2) \mapsto \bar{V}^k[m_1, m_2](x)$ are Lipschitz continuous as in the previous example.

Note that the functionals V^k and \bar{V}^k have a remarkably different behavior in areas where both populations are rare. In fact, assume that at some point \bar{x} both $\int_{\bar{\Omega}} K(\bar{x}, y) dm_k(y) = 0$ and, e.g., $\int_{\bar{\Omega}} W(\bar{x}, z) dm_1(z) = 0$. Then

$$V^1[m_1, m_2](\bar{x}) = a_1 = \max G, \quad \bar{V}^1[m_1, m_2](\bar{x}) = 0 = \min G.$$

3.3. Some explicit Mean-Field equilibria

In this section, we give two simple examples of pairs $(\bar{m}_1, \bar{m}_2) \in \mathcal{P}(\bar{\Omega}) \times \mathcal{P}(\bar{\Omega})$ that satisfy the Mean-Field equations (3.4) (or, equivalently, (3.3)) for the basic game of Example 5.

Example 7 (Uniform distributions). In addition to the assumptions of Example 5, suppose that

$$\int_{\bar{\Omega}} K(x, y) dy = c \quad \text{does not depend on } x. \quad (3.8)$$

This says that the kernel K gives the same total weight to the neighborhood $\mathcal{U}(x) := \text{supp}K(x, \cdot)$ of x , for all $x \in \bar{\Omega}$. Consider the uniform distributions

$$\bar{m}_1(x) = \bar{m}_2(x) = 1/|\Omega| \quad \forall x \in \bar{\Omega},$$

where $|\Omega|$ denotes the measure of Ω . Observe that, by (3.8), $V^k[\bar{m}_1, \bar{m}_2](x)$ is constant. Then the pair (\bar{m}_1, \bar{m}_2) solves (3.4) and therefore it is a Mean-Field equilibrium. Note that this occurs for all values of the parameters a_k, η_k , and that the “value of the game” $V^k[\bar{m}_1, \bar{m}_2](x)$ is not necessarily 0 (e.g., for $a_k \geq 1/2, \eta_k > 0$).

Example 8 (Fully segregated solutions). In addition to the assumptions of Example 5, we suppose now that, for some $r > 0$,

$$\text{supp}K(x, \cdot) \subseteq \{z : |z - x| \leq r\} \quad (3.9)$$

and $a_1, a_2 < 1$. We consider two sets $\Omega_1, \Omega_2 \subseteq \Omega$ such that

$$\text{dist}(\bar{\Omega}_1, \bar{\Omega}_2) \geq r, \quad \int_{\bar{\Omega}_k} K(x, y) dy \geq c_k > 0 \quad \forall x \in \bar{\Omega}_k, \quad k = 1, 2.$$

The second condition means that $\bar{\Omega}_k$ has enough weight near x for all $x \in \bar{\Omega}_k$. We consider the distributions

$$\bar{m}_1(x) = \begin{cases} 1/|\Omega_1| & \text{if } x \in \Omega_1, \\ 0 & \text{else,} \end{cases} \quad \bar{m}_2(x) = \begin{cases} 1/|\Omega_2| & \text{if } x \in \Omega_2, \\ 0 & \text{else.} \end{cases} \quad (3.10)$$

In order to check (3.4), we first pick $x \in \text{supp } \bar{m}_1 = \Omega_1$. By (3.9) and the first property of Ω_k we have

$$\frac{\int_{\bar{\Omega}} K(x, y) d\bar{m}_1(y)}{\int_{\bar{\Omega}} K(x, y) d\bar{m}_1(y) + \int_{\bar{\Omega}} K(x, y) d\bar{m}_2(y) + \eta_1} = 1 / \left(1 + \frac{\eta_1 |\Omega_1|}{\int_{\bar{\Omega}_1} K(x, y) dy} \right),$$

and the right-hand side is above or equal to the threshold a_1 if and only if

$$\eta_1 |\Omega_1| \leq \int_{\bar{\Omega}_1} K(x, y) dy \left(\frac{1}{a_1} - 1 \right),$$

which is true for all $x \in \Omega_1$ if

$$\eta_1 |\Omega_1| \frac{a_1}{1 - a_1} \leq c_1.$$

Then for such values of the parameters $V^1[\bar{m}_1, \bar{m}_2](x) = 0$, so the first equation (3.4) is satisfied. Similarly, if $\eta_2 |\Omega_2| a_2 / (1 - a_2) \leq c_2$, for $x \in \text{supp } \bar{m}_2 = \Omega_2$ we have $V^2[\bar{m}_1, \bar{m}_2](x) = 0$ and also the second equation (3.4) is verified. Therefore we have a large set of parameters for which any segregated solution of the form (3.10) is a Mean-Field equilibrium.

3.4. Models with myopic players

In connection with the differential Mean-Field games of the next sections, it is interesting to consider models where the cost functionals $V^k[m_1, m_2](x)$ depend only on $(m_1(x), m_2(x))$. This makes sense only if the measures m_k have a density, and it is a limit case that does not meet the regularity conditions of Section 3.1. We derive such local versions of the cost functionals by letting the size of the neighborhoods $\mathcal{U}(x)$ tend to 0. This corresponds to individuals who compute their cost functional by looking only at a very short distance, that we call *myopic players*.

Suppose that the kernel K in Section 2.3 takes the form

$$K(x, y) = \rho^{-d} \varphi \left(\frac{x - y}{\rho} \right)$$

where φ is a mollifier (i.e., a smooth nonnegative function $\mathbb{R}^d \rightarrow \mathbb{R}$ with support the unit ball centered at 0 and $\int_{\mathbb{R}^d} \varphi(z) dz = 1$). If $m \in L^1(\Omega)$, $\lim_{\rho \rightarrow 0} \int K(x, y) dm(y) = m(x)$ for a.e. x .

Consider first the functionals V^k associated to the basic game (in the large population limit) defined by (3.5) in Example 5. Then

$$\begin{aligned} \lim_{\rho \rightarrow 0} V^k[m_1, m_2](x) &= G(m_k(x), m_{-k}(x); a_k, \eta_k) \\ &= \left(\frac{m_k(x)}{m_k(x) + m_{-k}(x) + \eta_k} - a_k \right)^- =: V_\ell^k[m_1, m_2](x). \end{aligned}$$

Next we consider the game with family effects of Example 6 and assume the kernel W in (3.7) is also of the form

$$W(x, y) = r^{-d} \psi \left(\frac{x - y}{r} \right)$$

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where ψ is a mollifier. In the functionals \bar{V}^k defined by (3.7), we let first $r \rightarrow 0$ and get

$$\lim_{r \rightarrow 0} \bar{V}^k[m_1, m_2](x) = m_k(x) V^k[m_1, m_2](x).$$

This a partially local model that can be interesting in some cases, but we do not study it further in this paper. Finally, we let $\rho \rightarrow 0$ and obtain the local version of \bar{V}^k :

$$\lim_{\rho \rightarrow 0} \lim_{r \rightarrow 0} \bar{V}^k[m_1, m_2](x) = m_k(x) V_\ell^k[m_1, m_2](x) =: \bar{V}_\ell^k[m_1, m_2](x).$$

4. Mean-field differential game models of segregation

4.1. Long-time average cost functionals

In the last section, we designed some one-shot mean field games inspired by the original ideas of the population model by T. Schelling. We obtained the averaged costs V^k, \bar{V}^k by taking the limits as $N \rightarrow \infty$ of Nash equilibria of one-shot games with $2N$ players, and then the local limits V_ℓ^k, \bar{V}_ℓ^k by shrinking the neighborhoods to points. We shall now investigate dynamic mean field games with the same cost functionals in a differential context. We consider the state of a representative agent of the k -th population governed by the controlled stochastic differential equation with reflection

$$dX_s^k = \alpha_s^k ds + \sqrt{2\nu} dB_s^k - n(X_s^k) dl_s^k, \quad (4.1)$$

where B_s^k is a standard d -dimensional Brownian motion defined on some probability space, α_s^k is a control process adapted to B_s^k , $n(x)$ is the outward normal to the open set Ω at the point $x \in \partial\Omega$, and the local time $l_s^k = \int_0^s \chi_{\partial\Omega}(X_s^k) dl_s^k$ is a non-decreasing process adapted to B_s^k . The term $n(X_s^k) dl_s^k$ in the stochastic differential equation prevents the state variable X_s^k to escape from $\bar{\Omega}$ by reflecting it when it reaches the boundary.

The goal of a player of the k -th population is minimizing the long-time average cost, also called *ergodic cost*,

$$J^k(X_0^k, \alpha^1, \alpha^2, m_1, m_2) = \liminf_{T \rightarrow +\infty} \frac{1}{T} \mathbb{E} \left[\int_0^T L(X_s^k, \alpha_s^k) + V^k[m_1, m_2](X_s^k) ds \right], \quad (4.2)$$

where m^k are the distributions of the two populations and L is a Lagrangian function (smooth and convex in its second entry) which represents the cost paid by the player for using the control α_s^k at the position X_s^k .

The equilibrium distributions m_k satisfy, together with $\lambda_k \in \mathbb{R}$ and the functions u_k , the stationary MFG system of two Hamilton-Jacobi-Bellman and two

Kolmogorov-Fokker-Planck equations

$$\begin{cases} -\nu\Delta u_k + H(x, Du_k) + \lambda_k = V^k[m_1, m_2](x) & \text{in } \Omega, k = 1, 2 \\ -\nu\Delta m_k - \operatorname{div}(D_p H(x, Du_k)m_k) = 0, \\ \partial_n u_k = 0, \quad \nu\partial_n m_k + m_k D_p H^k(x, Du_k) \cdot n = 0, \text{ on } \partial\Omega, \end{cases} \quad (4.3)$$

where the Hamiltonian H is the Legendre transform of L with respect to the 2nd entry, λ_k is the (constant) value of the representative agent of the k -th population, and the solutions u_k of the H-J-B equations provide the optimal strategies in feedback form $-D_p H(\cdot, Du_k(\cdot))$. Here the costs V^k might be replaced by \bar{V}^k or by the local versions V_ℓ^k and \bar{V}_ℓ^k defined in the previous section. The connection between systems like (4.3) and stochastic differential games with N players having the same dynamics and individual costs, as $N \rightarrow \infty$, was discovered by Lasry and Lions Ref. 41 in the periodic setting for a single population, and extended to several populations and more general data in Ref. 23 and to Linear-Quadratic problems in Ref. 8, see also Ref. 37 for related results by different methods.

Existence for (4.3) can be proved by means of fixed-point arguments when the cost functionals are bounded.

Theorem 9. *Let Ω be a convex domain. Suppose that $H(x, p) = R|p|^\gamma - H_0(x)$, where $R > 0, \gamma > 1, H_0 \in C^2(\bar{\Omega})$ and $\partial_n H_0 \geq 0$ on $\partial\Omega$. Then, there exists at least one solution $(u_k, \lambda_k, m_k) \in C^{1,\delta}(\bar{\Omega}) \times \mathbb{R} \times W^{1,p}(\Omega)$ to (4.3) with costs either V^k , or \bar{V}^k , or V_ℓ^k , $k = 1, 2$.*

Proof. See Ref. 20, Theorem 6. □

The case of local costs \bar{V}_ℓ^k in dimension $d > 1$ does not fit into the existence theorem because \bar{V}_ℓ^k is unbounded and a-priori estimates on solutions might fail in general. For space dimension $d = 1$ see Ref. 19, Proposition 4.6. We do not expect uniqueness of the solution to the system (4.3).

For non-local V^k, \bar{V}^k solutions can be proved to be classical and existence holds under weaker assumptions (see Theorem 4 in Ref. 20), provided the negative part $(\cdot)^-$ in G is replaced by some smooth regularization. We are interested in qualitative properties of m_1, m_2 , but no methods in this direction are known so far for solutions of PDE systems like (4.3). For such a reason, a numerical analysis will be carried out in Section 6.

4.1.1. The deterministic case in one space dimension

In order to convince ourselves that segregation phenomena might occur also in our differential MFG models, we briefly analyze the deterministic case $\nu = 0$ in space dimension $d = 1$. Suppose that the state space is a closed interval $\bar{\Omega} = [a, b] \subset \mathbb{R}$ and that there is no Brownian motion perturbing the dynamics of the average players

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($\nu = 0$). Suppose also that $H(x, p) = |p|^2/2$. Then, (4.3) simplifies to

$$\begin{cases} \frac{(u'_k)^2}{2} + \lambda_k = V^k[m_1, m_2](x) & \text{in } \Omega, k = 1, 2 \\ (u'_k m_k)' = 0, \\ u'_k = 0, \quad u'_k m_k = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.4)$$

where the Neumann boundary conditions must be interpreted in the viscosity sense, as it is natural when taking the limit as $\nu \rightarrow 0$.

It is possible to construct explicit solutions for this system. For simplicity, we will consider the non-smoothened costs

$$V^k[m_1, m_2](x) = G\left(\int_{\mathcal{U}(x)} m_k, \int_{\mathcal{U}(x)} m_{-k}; a_k, \eta_k\right),$$

where G is defined in (2.4) and $m_{-1} = m_2, m_{-2} = m_1$.

Example 10 (Uniform distributions).

$$m_k = \frac{1}{b-a}, \quad u_k = 0, \quad \lambda_k = V^k[m_1, m_2], \quad k = 1, 2$$

provides a solution: the two populations are distributed uniformly and the cost functions are everywhere zero if the two thresholds a_k are not large (say, below .5 if η is negligible).

Example 11 (Segregated solutions). A family of fully segregated solutions may be written down explicitly. Suppose that $\mathcal{U}(x) = (x-r, x+r) \cap [a, b]$ with $r > 0$ small, and let $a = x_0 < x_1 < x_2 < x_3 < x_4 < x_5 = b$ such that $x_{k+1} - x_k > r$ for $k = 0, \dots, 4$. Set

$$m_1(x) = \frac{1}{x_2 - x_1} \chi_{[x_1, x_2]}(x), \quad m_2(x) = \frac{1}{x_4 - x_3} \chi_{[x_3, x_4]}(x) \quad \forall x \in [a, b].$$

Then, $\int_{\mathcal{U}(x)} m_1$ and $\int_{\mathcal{U}(x)} m_2$ are continuous functions which have support in $(x_1 - r, x_2 + r)$ and $(x_3 - r, x_4 + r)$, respectively. $V^1[m_1, m_2](\cdot)$ is also continuous, and vanishes in $[x_1, x_2]$ (if $a_1 < 1$ and η_1 is small enough); indeed, $\int_{\mathcal{U}(x)} m_2 = 0$, so $\int_{\mathcal{U}(x)} m_1 / \int_{\mathcal{U}(x)} (m_1 + m_2) = 1$. The same is for V^2 , so we define

$$\lambda_k = 0, \quad u_k(x) = \int_a^x (2V^k[m_1, m_2](\sigma))^{1/2} d\sigma, \quad \forall x \in [a, b], k = 1, 2.$$

It is easy to see that the functions (u_1, u_2) verify the two HJB equations of (4.4). Moreover, they satisfy the Neumann boundary conditions $u'_k(a) = u'_k(b) = 0$ in the viscosity sense^b (but not in classical sense, as $(u'_k)^2 = 2V^k \neq 0$ on the boundary of $[a, b]$); indeed, suppose that ϕ is a test function such that $u_1 - \phi$ has a local

^bA function $u \in C([a, b])$ satisfies the homogeneous Neumann boundary conditions in the viscosity sense in a if, for all test functions $\phi \in C^2$ such that $u - \phi$ has a local maximum at a , then $\min\{(\phi'(a))^2 - 2V^1[m_1, m_2](a), \phi'(a)\} \leq 0$, and for all $\phi \in C^2$ such that $u - \phi$ has a local minimum at a , then $\max\{(\phi'(a))^2 - 2V^1[m_1, m_2](a), \phi'(a)\} \geq 0$.

maximum at $x = b$. If we set $s = (2V^1[m_1, m_2](b))^{1/2}$ it follows that $\phi'(b) \leq s$. If $\phi'(b) \geq -s$ then $(\phi'(b))^2 \leq s^2$, so

$$\min\{(\phi'(b))^2 - 2V^1[m_1, m_2](b), \phi'(b)\} \leq 0.$$

Similarly, if $u_1 - \phi$ has a local minimum at $x = b$,

$$\max\{(\phi'(b))^2 - 2V^1[m_1, m_2](b), \phi'(b)\} \geq 0,$$

and in the same way it also holds that $u'_1(a) = u'_2(a) = u'_2(b) = 0$ in the viscosity sense.

It remains to check that m_k are (weak) solutions of the two Kolmogorov equations. To do so, we notice that m_1 is zero outside $[x_1, x_2]$; in $[x_1, x_2]$, however, $V^1[m_1, m_2](x) = 0$, hence $u'_1(x) = 0$. Similarly, $m_2(x)$ or $u'_2(x)$ vanishes, so $(u'_k m_k)' = 0$.

4.2. Finite horizon problems

When the the cost paid by a single player has the form (4.2), which captures the effect of the m_k long-time average, the mean field system of partial differential equations (4.3) which characterizes Nash equilibria is stationary, i.e. no time dependence appears. Suppose, on the other hand, that a time horizon $T > 0$ is fixed, and the cost paid by the average player of the k -th population is of the form

$$J^k(X_0^k, t, \alpha^1, \alpha^2, m_1, m_2) = \mathbb{E} \left[\int_t^T L(x, \alpha_s^k) + V^k[m_1, m_2](X_s^k) ds + G_T^k[m(T)](X_T^k) \right], \quad (4.5)$$

where t is the initial time and $G_T^k[m(T)]$ represents the cost paid at the final time T . Then, the time variable t enters the Mean Field Game system, which becomes

$$\begin{cases} -\partial_t u_k - \nu \Delta u_k + H^k(x, Du_k) = V^k[m](x), & \text{in } \Omega \times (0, T), \\ \partial_t m_k - \nu \Delta m_k - \operatorname{div}(D_p H^k(x, Du_k) m_k) = 0 & \text{in } \Omega \times (0, T), \\ \partial_n u_k = 0, \nu \partial_n m_k + m_k D_p H^k(x, Du_k) \cdot n = 0 & \text{on } \partial\Omega \times (0, T), \\ u_k(x, T) = G_T^k[m(T)](x), m_k(x, 0) = m_{k,0}(x) & \text{in } \Omega \end{cases} \quad (4.6)$$

We observe that (4.6) has a backward-forward structure: the Hamilton-Jacobi-Bellman equation for the value functions V^k is backward in time, being the representative agent able to foresee the outcome of his actions, while his own distribution m_k evolves forward in time. The final cost G_T^k and the initial distributions $m_{k,0}$ are prescribed as final/initial boundary data.

For one population with periodic boundary conditions, the rigorous derivation of such a system from Nash equilibria of $2N$ -persons games in the limits as $N \rightarrow \infty$ was proved very recently in the fundamental paper by Cardaliaguet, Delarue, Lasry, and Lions Ref. 15 on the so-called Master Equation of MFG. For related results by probabilistic methods, see Ref. 24 and the references therein. The fact that from a solution of (4.6) one can synthesize ϵ -Nash equilibria for the $2N$ -persons game, if N

is large enough, is due to Huang, Caines and Malhamé Ref. 37 (for one population) and to Nourian and Caines for problems with major and minor agents, Ref. 42.

We also point out that the system (4.3) captures in some circumstances the behavior of (4.6) as $T \rightarrow \infty$. In particular, for a single population, if the cost V is monotone increasing with respect to m , then solutions of (4.6) converge to solutions of (4.3) (see Ref. 16). It is not clear whether a similar phenomenon can be rigorously proved in our multi-population systems, since monotonicity fails, but we show in Section 6 that it is likely to occur by providing some numerical evidences.

Existence of classical solutions for non-stationary Mean Field Games systems like (4.6) can be stated under rather general assumptions. In Ref. 14 a detailed proof is provided for the single-population case with periodic boundary conditions. Next we state a precise existence result for our system (4.6) and outline its proof, whose main modifications are due to the presence of Neumann boundary conditions. Nevertheless, the general lines of the argument are the same: the fixed point structure of the system is exploited and the regularizing assumptions on V^k, G_T^k assure that suitable a-priori estimates hold.

We recall that the space of probability measures $\mathcal{P}(\bar{\Omega})$ can be endowed with the Kantorovitch-Rubinstein distance, which metrizes the weak* topology on $\mathcal{P}(\bar{\Omega})$. The assumptions on $V^k, G_T^k, m_{i,0}$ we require are

- (1) V^k, G_T^k are continuous in $\bar{\Omega} \times \mathcal{P}(\bar{\Omega})^2$.
- (2) $V^k[m], G_T^k[m]$ are bounded respectively in $C^{1,\beta}(\bar{\Omega}), C^{2,\beta}(\bar{\Omega})$ for some $\beta > 1$, uniformly with respect to $m \in \mathcal{P}(\bar{\Omega})^2$.
- (3) $H^k \in C^1(\bar{\Omega} \times \mathbb{R}^d)$ and it satisfies for some $C_0 > 0$ the growth condition

$$D_p H^k(x, p) \cdot p \geq -C_0(1 + |p|^2).$$

- (4) $m_{i,0} \in C^{2,\beta}(\bar{\Omega})$.
- (5) The following compatibility conditions are satisfied:

$$\begin{aligned} \partial_n G_T^k[m(T)](x) &= 0, \quad \forall m \in \mathcal{P}(\bar{\Omega})^2, x \in \partial\Omega, \\ \partial_n m_{i,0}(x) + m_{i,0} D_p H^k(x, Du_k(x)) \cdot n &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

The assumptions (1) and (2) are satisfied by the non-local costs V^k, \bar{V}^k defined by (3.5) and (3.7) in Section 3.2 if the negative part function $(\cdot)^-$ in G is replaced by a smooth approximation ^c.

Theorem 12. *Under the assumptions listed above there exists at least one classical solution to (4.6).*

Proof. Step 1. We start by an estimate on the Fokker-Planck equation. Suppose that b is a given vector field, continuous in time and Hölder continuous in space (on

^cFor example, $\varphi_\epsilon(t) = \frac{1}{2}(\sqrt{t^2 + \epsilon^2} - t)$, $\epsilon > 0$ small, or $\Psi_{-, \epsilon}(\cdot)$ as in (6.1).

$\bar{\Omega}$), and $m \in L^1(\Omega \times (0, T))$ solves in the weak sense

$$\begin{cases} \partial_t m - \nu \Delta m + \operatorname{div}(b m) = 0 & \text{in } \Omega \times (0, T), \\ \nu \partial_n m(x) - m b \cdot n = 0 & \text{on } \partial\Omega \times (0, T), \\ m(x, 0) = m_0(x) & \text{in } \Omega. \end{cases} \quad (4.7)$$

Then, $m(t)$ is the law of the following stochastic differential equation with reflection

$$\begin{aligned} X_t &= X_0 + \int_0^t b(X_s, s) ds + \sqrt{2\nu} B_t - \int_0^t n(X_s) dl_s \quad X_t \in \bar{\Omega} \\ l_t &= \int_0^t \chi_{\partial\Omega}(X_s) dl_s \\ l(0) &= 0 \quad l \text{ is nondecreasing,} \end{aligned} \quad (4.8)$$

where B_t is a standard Brownian motion over some probability space, X_t, l_t (the so-called local time) are continuous processes adapted to B_t and the law of X_0 is m_0 . This can be verified by exploiting the results of Ref. 46, where it is proved that for all $\varphi \in C^2(\bar{\Omega})$ such that $\partial_n \varphi = 0$ on $\partial\Omega$,

$$M_t := \varphi(X_t) - \int_0^t [\nu \Delta \varphi(X_t) + b(X_t, t) \cdot D\varphi(X_t)] dt \quad (4.9)$$

is a martingale with respect to B_t . As a consequence, taking expectations in (4.9) shows that the law of X_t is the (unique) solution of (4.7).

This kind of stochastic interpretation of (4.7) allows us to derive the following estimate:

$$\begin{aligned} d(m(t), m(s)) &= \sup \left\{ \int_{\Omega} \phi(x) (m(x, t) - m(x, s)) dx : \phi \text{ is 1-Lipschitz continuous} \right\} \\ &\leq \sup \{ \mathbb{E}^x |\phi(X_t) - \phi(X_s)| : \phi \text{ is 1-Lipschitz continuous} \} \leq \mathbb{E}^x |X_t - X_s| \\ &\leq \mathbb{E}^x \left[\int_s^t |b(X_\tau, \tau) d\tau| + \sqrt{2\nu} |B_t - B_s| \right], \end{aligned}$$

for all $s, t \in [0, T]$, where the last inequality follows from Ref. 6. We can then conclude that

$$d(m(t), m(s)) \leq c_0 (1 + \|b\|_{\infty}) |t - s|^{\frac{1}{2}} \quad (4.10)$$

for some c_0 which does not depend on t, s .

Step 2. We set up now the existence argument, which is based on a fixed-point method. Let \mathcal{C} be the set of maps $\mu \in C^0([0, T], \mathcal{P}(\bar{\Omega}))$ such that

$$\sup_{s \neq t} \frac{d(\mu(s), \mu(t))}{|t - s|^{1/2}} \leq C_1, \quad (4.11)$$

for a constant C_1 large enough that will be chosen subsequently. The set \mathcal{C} is convex and compact. To any $(\mu_1, \mu_2) \in \mathcal{C}^2$ we associate the (unique) classical solution (u_1, u_2) of

$$-\partial_t u_k - \nu \Delta u_k + H^k(x, Du_k) = V^k[\mu_1, \mu_2](x), \quad (4.12)$$

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satisfying the Neumann boundary conditions $\partial_n u_k = 0$ on $\partial\Omega$, and then define $m = (m_1, m_2) = \Psi(\mu)$ as the solutions of the two Fokker-Planck equations

$$\partial_t m_k - \nu \Delta m_k - \operatorname{div}(D_p H^k(x, Du_k) m_k) = 0. \quad (4.13)$$

A fixed point of Ψ is clearly a solution of (4.6). Such a mapping is indeed well-defined: existence for the HJB equation (4.12) is guaranteed by Theorem 7.4, p. 491 of Ref. 40 and the well-posedness of (4.13) is stated in Theorem 5.3, p. 320 of Ref. 40. These results incorporate also the Schauder a-priori estimates, that together with (4.10) make Ψ continuous and a mapping from \mathcal{C}^2 into itself, provided that the constant C_1 in (4.11) is large enough. The existence of a fixed point for Ψ follows from the application of the Schauder fixed point theorem. \square

While existence of smooth solutions of (4.6) with costs $V^k, \overline{V^k}$ can be established through standard methods, the local versions $V_\ell^k, \overline{V}_\ell^k$ are not regularizing, so the ideas of Theorem 12 cannot be applied directly; in this case, existence of solutions is a much more delicate issue.

A well-established workaround is to smoothen the costs by convolution with kernels, and pass to the limit in a sequence of approximating solutions (which are obtained by arguing as in Theorem 12); this procedure requires a-priori bounds, that strongly depend on the behavior of the Hamiltonian at infinity, the cost, and the space dimension d . It is not the purpose of this paper to present theoretical results on existence of smooth solutions in full generality. We believe that, under suitable assumptions, solutions can be obtained without substantial difficulties by extending known results for one-population MFG on the torus to the case of two populations with Neumann boundary conditions. Next we briefly explain how.

Suppose that $H^k(x, p)$ behaves like $c|p|^\gamma$ as $p \rightarrow \infty$ ($c > 0$, and $\gamma > 1$). In our setting, the couplings $V_\ell^k, \overline{V}_\ell^k$ are non-negative, and a-priori bounds on $\int |Du_k|^\gamma m_k dxdt$ and $\int V_\ell^k m_k dxdt$ (quantities that are somehow related to the energy of the system) can be easily proved. To carry out the approximation procedure, it is crucial to have a-priori bounds on $\|m_k\|_{L^\infty(\Omega)}$.

Example 13. In the purely quadratic case, namely, $H^k(x, p) = |p|^2/2$, the Hopf-Cole transformation can be used to transform (4.6) into a system of two couples of semilinear equations of the form

$$\begin{cases} -\partial_t \phi_k - \nu \Delta \phi_k + \frac{1}{2\nu} V_\ell^k(\phi_1 \psi_1, \phi_2 \psi_2) \phi_k = 0, \\ \partial_t \psi_k - \nu \Delta \psi_k + \frac{1}{2\nu} V_\ell^k(\phi_1 \psi_1, \phi_2 \psi_2) \psi_k = 0, \end{cases}$$

where $\phi_k = e^{-u_k/2\nu}$ and $\psi_k = m_k e^{u_k/2\nu}$, with the corresponding initial-final data and Neumann boundary conditions. Bounds on $\|m_k\|_{L^\infty(\Omega)} = \|\phi_k \psi_k\|_{L^\infty(\Omega)}$ can be derived by arguing as in Ref. 16, where a Moser iteration method is implemented.

Example 14. If $1 < \gamma < 1 + 1/(d + 1)$, so that H grows almost linearly, it is known that existence of smooth solutions can be established, see the discussion in Ref. 30. In particular, the basic estimate for $\int |Du_k|^\gamma m_k$ implies that the drifts

$D_p H^k$ entering the Fokker-Planck equations belong to $L^p(m_k)$, where $p > d + 2$. It is known that this kind of Lebesgue regularity on the drifts is strong enough to guarantee Hölder bounds for m_k .

Example 15. For other values of γ , we observe that V_ℓ^k are uniformly bounded. Therefore, at least in the subquadratic case (namely, when $\gamma \leq 2$), one might exploit the classical Lipschitz bounds for viscous HJ equations and Hölder estimates for the Fokker-Planck to achieve a-priori regularity for m_k , see Ref. 40.

The setting with the costs \bar{V}_ℓ^k is more delicate, as \bar{V}_ℓ^k is a-priori unbounded in L^∞ . Here, one might reason as in Ref. 30, or Ref. 31 in the superquadratic case (see also Ref. 32), and finely combine regularity of the HJB equation and the Fokker-Planck equation to prove existence of solutions of (4.6), at least if the space dimension is sufficiently small ($d = 1, 2$). We leave these extensions to future work.

5. Numerical methods

Numerical methods for approximating mean field game systems are an important research issue since they are crucial for applications. The finite difference methods described below are reminiscent of the method first introduced and analyzed in Ref. 4 for mean field games with a single population, which, to the best of our knowledge, remains the more robust and flexible technique. The numerical scheme basically relies on monotone approximations of the Hamiltonian and on a suitable weak formulation of the Kolmogorov equation. It has several important features:

- existence and possibly uniqueness for the discretized problems can be obtained by similar arguments as those used in the continuous case
- it is robust when $\nu \rightarrow 0$ (the deterministic limit of the models)
- it can be used for finite and infinite horizon problems
- bounds on the solutions, which are uniform in the grid step, can be proved under reasonable assumptions on the data.

A first result on the convergence to classical solutions was contained in Ref. 4. The method was used for planning problems (the terminal condition is a Dirichlet like condition for m) in Ref. 2. Ref. 3 contains a further analysis of convergence to classical solutions and very general results on the convergence to weak solutions are supplied in Ref. 5. In Ref. 1, similar computational techniques are applied to MFG models in macro-economics.

Discrete time, finite state space mean field games were discussed in Ref. 27. We also refer to Ref. 34, 35 for a specific constructive approach when the Hamiltonian is quadratic. Semi-Lagrangian approximations were investigated in Ref. 17, 18. Finally, augmented Lagrangian methods for the solution of the system of equations arising from the discrete version of a variational mean field game was proposed in Ref. 11.

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5.1. Stationary PDEs

To approximate (4.3), we will implement the strategy proposed in Ref. 4, that consists of taking the long-time limit of the *forward-forward* MFG system

$$\begin{cases} \partial_t u_k - \nu \Delta u_k + H^k(x, Du_k) = V^k[m_1, m_2](x) & (0, T) \times \Omega \\ \partial_t m_k - \nu \Delta m_k - \operatorname{div}(D_p H^k(x, Du_k) m_k) = 0, \\ \partial_n u_k = 0, \quad \nu \partial_n m_k + m_k D_p H^k(x, Du_k) \cdot n = 0, & (0, T) \times \partial\Omega \\ u_k(t=0) = u_{k,0}, \quad m_k(t=0) = m_{k,0}, \quad k = 1, 2. \end{cases} \quad (5.1)$$

This method is reminiscent of long-time approximations for the cell problem in homogenization theory: we expect that there exists some $\lambda_k \in \mathbb{R}$ such that $u_k(\cdot, T) - \lambda_k T$ and $m_k(\cdot, T)$ converge as $T \rightarrow \infty$, respectively, to some $\bar{u}_k(\cdot), \bar{m}_k(\cdot)$ solving (4.3). Although this has not been proven rigorously in general in the MFG setting, Guéant studies some single-population examples where the coupling $V(m)$ is not increasing with respect to the distribution m (so there is no uniqueness of solutions, as in our framework) and justifies the approach (see Ref. 33). Very recently, a proof of the long-time convergence for a class of forward-forward one dimensional MFG has been proved in Ref. 29. We are going to present numerical experiments, even if no rigorous proof of any convergence is available at this stage in our multi-population setting.

We mention that if the Hamiltonians H^k are quadratic, it is possible to simplify (4.3) through the Hopf-Cole change of variables and reduce the number of unknowns (see Ref. 34).

We will develop a finite-difference scheme for (5.1) in space dimension $d = 2$ as in Ref. 4, assuming for simplicity that the Hamiltonians are of the form

$$H^k(x, p) = W^k(x) + \frac{1}{\gamma_k} |p|^{\gamma_k}, \quad \gamma_k > 1, \quad W^k \in C^2(\Omega). \quad (5.2)$$

In space dimension $d \neq 2$, analogous schemes can be set up. Consider a square domain $\Omega = (0, 1)^2$, and a uniform grid with mesh step h , assuming that $1/h$ is an integer N_h ; denote by $x_{i,j}$ a generic point of the grid. Let Δt be a positive time step and $t_n = n\Delta t$. The values of u_k and m_k at $x_{i,j}$, t_n will be approximated by $U_{i,j}^{k,n}$ and $M_{i,j}^{k,n}$ respectively, $k = 1, 2$, $i, j = 1, \dots, N_h$ and $n \geq 0$.

We introduce the usual finite difference operators

$$(D_1^+ U)_{i,j} = \frac{U_{i+1,j} - U_{i,j}}{h}, \quad (D_2^+ U)_{i,j} = \frac{U_{i,j+1} - U_{i,j}}{h},$$

and the numerical Hamiltonians $g^k : \Omega \times \mathbb{R}^4 \rightarrow \mathbb{R}$ of Godunov type defined by

$$g^k(x, q_1, q_2, q_3, q_4) = W^k(x) + \frac{1}{\gamma_k} [|(q_1)^-|^2 + |(q_3)^-|^2 + |(q_2)^+|^2 + |(q_4)^+|^2]^{\gamma_k/2}.$$

Denoting by

$$[D_h U]_{i,j} = ((D_1^+ U)_{i,j}, (D_1^+ U)_{i-1,j}, (D_2^+ U)_{i,j}, (D_2^+ U)_{i,j-1}),$$

the finite difference approximation of the Hamiltonian function H^k will be $g^k(x, [D_h U^k]_{i,j})$.

We choose the classical five-points discrete version of the Laplacian

$$(\Delta_h U)_{i,j} = -\frac{1}{h^2}(4U_{i,j} - U_{i+1,j} - U_{i-1,j} - U_{i,j+1} - U_{i,j-1}).$$

The non-local couplings $V^k[m_1, m_2]$, $\bar{V}^k[m_1, m_2]$ involve terms of the form $\int_{\Omega} K(x, y)m_k(y)dy$; we approximate them via

$$h^2 \sum_{r,s} K(x_{i,j}, x_{r,s})M_{r,s}^{k,n}.$$

On the other hand, local couplings V_{ℓ}^k and \bar{V}_{ℓ}^k will be simply evaluated at $x_{i,j}$, i.e. $(V_{\ell}^k[M^{1,n}, M^{2,n}])_{i,j} = V_{\ell}^k(M_{i,j}^{1,n}, M_{i,j}^{2,n})$.

In order to approximate the Kolmogorov equations in (5.1), we consider their weak formulation. Given any test function ϕ , the divergence term involved can be rewritten as

$$-\int_{\Omega} \operatorname{div}(m_k D_p H^k(x, Du_k))\phi = \int_{\Omega} m D_p H^k(x, Du_k) \cdot D\phi,$$

which is going to be approximated by (boundary terms disappear by Neumann conditions)

$$h^2 \sum_{i,j} M_{i,j}^{k,n} D_q g^k(x, [D_h U^{k,n}]_{i,j}) \cdot [D_h \Phi]_{i,j},$$

where Φ is the finite difference version of ϕ . By introducing the compact notation

$$\mathcal{B}_{i,j}^k(U, M) = \frac{1}{h} \begin{pmatrix} M_{i,j} \partial_{q_1} g^k(x, [D_h U]_{i,j}) - M_{i-1,j} \partial_{q_1} g^k(x, [D_h U]_{i-1,j}) \\ + M_{i+1,j} \partial_{q_2} g^k(x, [D_h U]_{i+1,j}) - M_{i,j} \partial_{q_2} g^k(x, [D_h U]_{i,j}) \\ + M_{i,j} \partial_{q_3} g^k(x, [D_h U]_{i,j}) - M_{i,j-1} \partial_{q_3} g^k(x, [D_h U]_{i,j-1}) \\ + M_{i,j+1} \partial_{q_4} g^k(x, [D_h U]_{i,j+1}) - M_{i,j} \partial_{q_4} g^k(x, [D_h U]_{i,j}) \end{pmatrix},$$

we can finally write the discrete version of (5.1)

$$\begin{cases} \frac{U_{i,j}^{k,n+1} - U_{i,j}^{k,n}}{\Delta t} - \nu(\Delta_h U^{k,n+1})_{i,j} + g^k(x, [D_h U^{k,n+1}]_{i,j}) = (V^k[M^{1,n+1}, M^{2,n+1}])_{i,j}, \\ \frac{M_{i,j}^{k,n+1} - M_{i,j}^{k,n}}{\Delta t} - \nu(\Delta_h M^{k,n+1})_{i,j} - \mathcal{B}_{i,j}^k(U^{k,n+1}, M^{k,n+1}) = 0, \quad k = 1, 2. \end{cases} \quad (5.3)$$

The system above has to be satisfied for internal points of the grid, i.e. $2 \leq i, j \leq N_h - 1$. The finite difference version of the homogeneous Neumann boundary conditions for U is, for all n, k ,

$$\begin{aligned} U_{1,j}^{k,n} &= U_{2,j}^{k,n}, & U_{N_h-1,j}^{k,n} &= U_{N_h,j}^{k,n}, & \forall j &= 2, \dots, N_h - 1 \\ U_{i,1}^{k,n} &= U_{i,2}^{k,n}, & U_{i,N_h-1}^{k,n} &= U_{i,N_h}^{k,n}, & \forall i &= 2, \dots, N_h - 1 \\ U_{1,1}^{k,n} &= U_{2,2}^{k,n}, & U_{N_h,1}^{k,n} &= U_{N_h-1,2}^{k,n}, \\ U_{1,N_h}^{k,n} &= U_{2,N_h-1}^{k,n}, & U_{N_h,N_h}^{k,n} &= U_{N_h-1,N_h-1}^{k,n}. \end{aligned}$$

and on Hamiltonians given by (5.2). The time-step Δt is assumed to be of the form T/N , for a positive integer N . Using the same notations as in § 5.1, the approximate version of (4.6) reads: for any $0 \leq n < N$, $1 < i, j < N_h$,

$$\begin{cases} \frac{U_{i,j}^{k,n+1} - U_{i,j}^{k,n}}{\Delta t} + \nu(\Delta_h U^{k,n})_{i,j} - g^k(x, [D_h U^{k,n}]_{i,j}) & = -(V^k[M^{1,n}, M^{2,n}])_{i,j}, \\ \frac{M_{i,j}^{k,n+1} - M_{i,j}^{k,n}}{\Delta t} - \nu(\Delta_h M^{k,n+1})_{i,j} - \mathcal{B}_{i,j}^k(U^{k,n}, M^{k,n+1}) & = 0, \quad k = 1, 2, \end{cases} \quad (5.5)$$

with the initial and terminal conditions: for $1 \leq i, j \leq N_h$,

$$M_{i,j}^{k,0} = m_{k,0}(x_{i,j}), \quad U_{i,j}^{k,N} = u_{k,T}(x_{i,j}). \quad (5.6)$$

It can be supplemented with discrete Neumann conditions as in § 5.1 or with periodicity conditions. Note that (5.5) consists of a semi-implicit scheme for the (forward) Kolmogorov equation (i.e. implicit with respect to m and explicit with respect to u) coupled with a semi-implicit scheme for the (backward) Hamilton-Jacobi equation (i.e. implicit with respect to u and explicit with respect to m). When dealing with one population only, it was shown in Ref. 4 that the discrete scheme preserves the structure of the continuous problem, which makes it possible to prove existence, and uniqueness/stability under additional assumptions. In the multi-population case also, existence of solutions of the discrete system can be obtained by using a Brouwer fixed point method. Then, assuming that $h^2 \sum_{i,j} M_{i,j}^{k,0} = 1$ for $k = 1, 2$, mass conservation, i.e. $h^2 \sum_{i,j} M_{i,j}^{k,n} = 1$ for any n , $k = 1, 2$, is a consequence of the definition of \mathcal{B}^k . Using the monotonicity of g , we also obtain the nonnegativity of $M^{k,n}$ for any n , $k = 1, 2$, see Ref. 4.

We briefly describe the iterative method used in order to solve (5.5)-(5.6). Since the latter system couples forward and backward (nonlinear) equations, it cannot be solved by merely marching in time. Assuming that the discrete Hamiltonians are C^2 and the coupling functions are C^1 allows us to use a Newton-Raphson method for the whole system of nonlinear equations (which can be huge if $d \geq 2$).

More precisely, we see (5.5)-(5.6) as a fixed point problem. We first define the mapping Ξ which maps the pair of grid functions $(Y_{i,j}^{1,n}, Y_{i,j}^{2,n})_{i,j,n}$ to the pair of grid function $((V^1[M^{1,n}, M^{2,n}])_{i,j}, (V^2[M^{1,n}, M^{2,n}])_{i,j})_{i,j,n}$, where n takes its values in $\{1 \dots, N\}$ and i, j take their values in $\{1 \dots, N_h\}$, and $(M_{i,j}^{1,n}, M_{i,j}^{2,n})$ is found by solving the following system of discrete Bellman and Kolmogorov equations: for any $0 \leq n < N$, $1 < i, j < N_h$,

$$\begin{cases} \frac{U_{i,j}^{k,n+1} - U_{i,j}^{k,n}}{\Delta t} + \nu(\Delta_h U^{k,n})_{i,j} - g^k(x, [D_h U^{k,n}]_{i,j}) & = -Y_{i,j}^{k,n+1}, \\ \frac{M_{i,j}^{k,n+1} - M_{i,j}^{k,n}}{\Delta t} - \nu(\Delta_h M^{k,n+1})_{i,j} - \mathcal{B}_{i,j}^k(U^{k,n}, M^{k,n+1}) & = 0, \end{cases} \quad (5.7)$$

supplemented with (5.6) and discrete Neumann conditions. Finding a fixed point of Ξ is equivalent to solving (5.5)-(5.6).

Note that in (5.7) the discrete Bellman equations do not involve $M^{k,n+1}$. Therefore, one can first solve the Bellman equations for $U^{k,n}$ $0 \leq n \leq N$, $k = 1, 2$ by marching backward in time (i.e. performing a backward loop with respect to the index n). For every time index n , the two systems of nonlinear equations for $U^{k,n}$, $k = 1, 2$ are themselves solved by means of a nested Newton-Raphson method. Once an approximate solution of the Bellman equations has been found, one can solve the (linear) Kolmogorov equations for $M^{k,n}$ $0 \leq n \leq N$, $k = 1, 2$, by marching forward in time (i.e. performing a forward loop with respect to the index n). The solutions of (5.7)-(5.6) are such that $M^{k,n}$ are nonnegative and $h^2 \sum_{i,j} M_{i,j}^{k,n} = 1$ for any n , $k = 1, 2$.

The fixed point equation $\Xi \left(\left(Y_{i,j}^{1,n}, Y_{i,j}^{2,n} \right)_{i,j,n} \right) = \left(Y_{i,j}^{1,n}, Y_{i,j}^{2,n} \right)_{i,j,n}$ is solved numerically by using a Newton-Raphson method. This requires the differentiation of both the Bellman and Kolmogorov equations in (5.7).

A good choice of an initial guess is important, as always for Newton methods. To address this matter, we first observe that the above mentioned iterative method generally quickly converges to a solution when the value of ν is large. This leads us to use a continuation method in the variable ν : we start solving (5.5)-(5.6) with a rather high value of the parameter ν (of the order of 1), then gradually decrease ν down to the desired value, the solution found for a value of ν being used as an initial guess for the iterative solution with the next and smaller value of ν .

6. Numerical simulations

6.1. Stationary PDEs

In this section, we will show some results obtained by implementing the long-time procedure presented in Section 5.1. Here, we choose $d = 1$, $\Omega = (0, 1)$ and Hamiltonians of the form (5.2), with $W \equiv 0$. The mesh step is $h = 1/200$; at each time step n we define the approximate ergodic constant $\lambda_k^n = h(\sum_i U_i^{k,n})/t_n$ and the relative errors $err_m^n = \max_{k=1,2} \|M^{k,n} - M^{k,n-1}\|_\infty / \Delta t$, $err_\lambda^n = \max_{k=1,2} |\lambda_k^n - \lambda_k^{n-1}|$. As mentioned before, we expect that as t_n grows, λ_k^n converges to some constant value; we stop the simulation when the two relative errors become smaller than a fixed threshold, and denote by u_h^k, m_h^k the approximate solutions $U^{k,n}, M^{k,n}$ respectively at the last time iteration.

The initial data are set to be (unless otherwise specified)

$$U^{k,0} \equiv 0, \quad M_i^{1,0} = \chi_{[0,0.5]}(x_i), \quad M_i^{2,0} = \chi_{[0.5,1]}(x_i),$$

while the time step is $\Delta t = 0.02$ as long as the relative error is large, namely when $err_m > 1$ (this happens during the first time iterations), and it is linearly increased to $\Delta t = 2$ as soon as the relative error err_m reaches 0.001. In our simulations, stability in the long-time regime always occurs; in Figure 3 (right) it is shown a typical behavior of the relative errors as the number of time iterations increases.

We will show various tests with different values of H, ν , and different choices of

the cost functionals (see Table 1). Note that if ν is large (say, greater than 0.1), the constant solution only is achieved in the long-time regime, namely $M^{k,n} \rightarrow 1$ as n increases; in this situation the mixing effect of the Brownian noise prevails on the individual preference of players. A richer structure of approximate solutions shows up as ν approaches zero.

Table 1. The data in the tests.

Test	γ	ν	a_1	a_2	Couplings
1	2	0.05, 0.0005	0.3	0.4	V_ℓ
2	2	0.05	0.4	0.8	\bar{V}_ℓ, V_ℓ
3	2	0.001	0.8, 0.3	0.8, 0.3	V
4	$8, \frac{4}{3}$	0.005	0.3	0.3	V_ℓ

Test 1. Here, we obtain two monotone configurations, and observe that segregation between the two populations appears; moreover, it becomes more evident as the viscosity ν goes to zero, see Figure 2. In other words, we find two disjoint intervals Ω_k , $k = 1, 2$ such that $m_h^k > 0$ on Ω_k and $m_h^{3-k} \rightarrow 0$ as $\nu \rightarrow 0$ on Ω_k . Note that segregation occurs even if the two “happiness” thresholds a_k are small: the cost paid by a player can be zero even if the distribution of his own population is less than half of the distribution of both the populations. The optimal feedback control $-D_h u_h^k$ vanishes on Ω_k in the small viscosity regime, because in this region the cost V_ℓ^k is identically zero; $-D_h u_h^k$ acts substantially only on the complement of Ω_k , forcing m_h^k to be close to zero.

Note that if ν is very small, the free boundary between Ω_1 and Ω_2 becomes a point, which varies upon the choice of a_k (see also the other tests); in general, if $a_1 > a_2$ this boundary shifts closer to $x = 0$ if $0 \in \Omega_1$, or to $x = 1$ if $1 \in \Omega_1$: the more xenophobic population concentrates more, while the other one is distributed over a bigger subset of the domain.

The asymptotic behavior of $\int m^1 m^2 dx$ with respect to ν appears to be power-like, that is $\int m^1 m^2 dx \approx c\nu^4$ for some positive c , depending on the “branch” of solutions. For this test, numerical values can be found in Table 2.

We finally mention that if one changes the initial distributions $M^{1,0}, M^{2,0}$, then the approximate solution m_h^k may vary; in the one dimensional case monotone configurations are likely to occur, but it is possible to obtain solutions with more than one stationary point (see Figure 3) by a suitable choice of $M^{k,0}$ (see also Remark 16).

Test 2. In this test, we show how the “family effect” affects the behavior of the two populations, and compare the approximate solutions of (4.3) with local couplings \bar{V}_ℓ and V_ℓ . In general, the presence of the family effect discourages segregation, and the two distributions appear to be a bit more “mixed” in this case, see Figure 4. Nevertheless, full segregation still occurs as ν approaches zero. Note

Fig. 2. m_h (left), u_h (right) at different values of ν : $\nu = 0.05$ is marked with circles, $\nu = 0.0005$ is marked with triangles; solid/red lines are used for (u_1, m_1) , while dashed/blue lines are used for (u_2, m_2) .

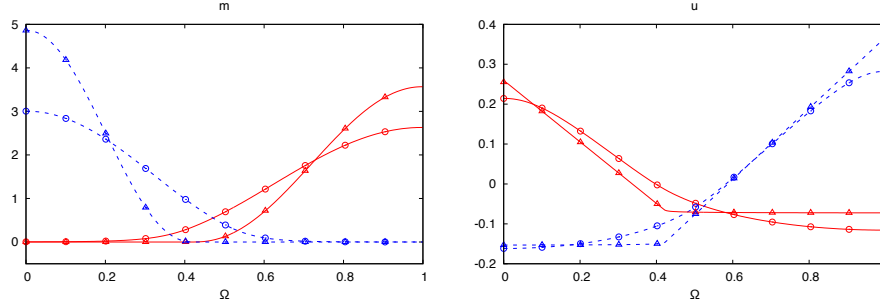


Fig. 3. Another configuration, $\nu = 0.001$, with relative errors.

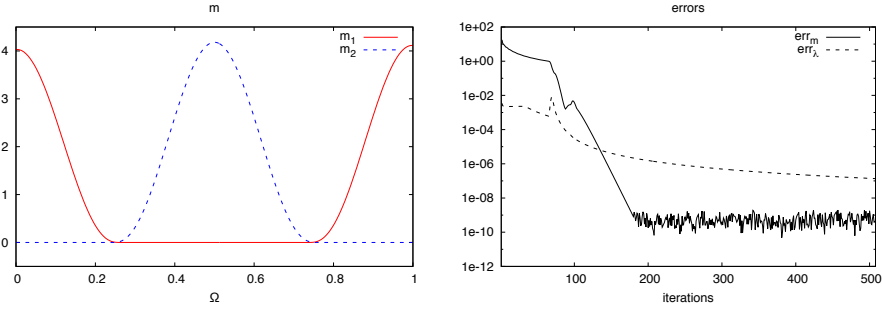


Table 2. The value of $\int m^1 m^2 dx$ versus ν

ν	$h \sum_i m_{h,i}^1 m_{h,i}^2$
0.05	0.09100195573
0.01	0.00017126474
0.005	0.00000811663
0.0005	0.00000000068

that V_ℓ^k is positive where m_h^k is close to zero, while \bar{V}_ℓ is proportional to m_h^k : what happens is that \bar{V}_ℓ is different from zero only in a (very) small region around the free boundary between m_1 , m_2 , that still is sufficient to trigger segregation if the viscosity is small.

Test 3. In the previous tests, we used the local versions of the costs, namely we considered myopic players. Here, we show the results obtained considering the

Fig. 4. The family effect. \bar{V}_ℓ , left. V_ℓ , right.

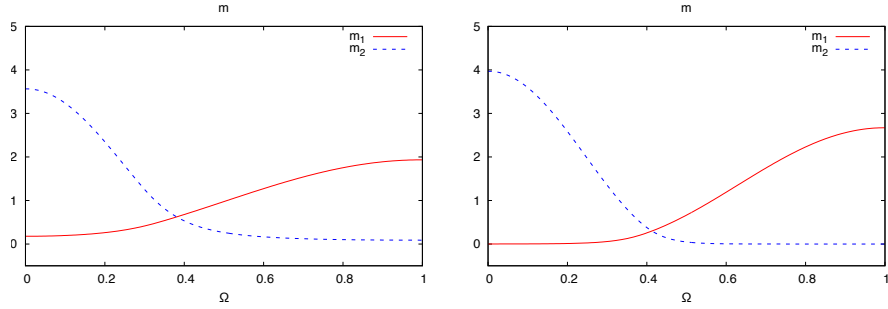
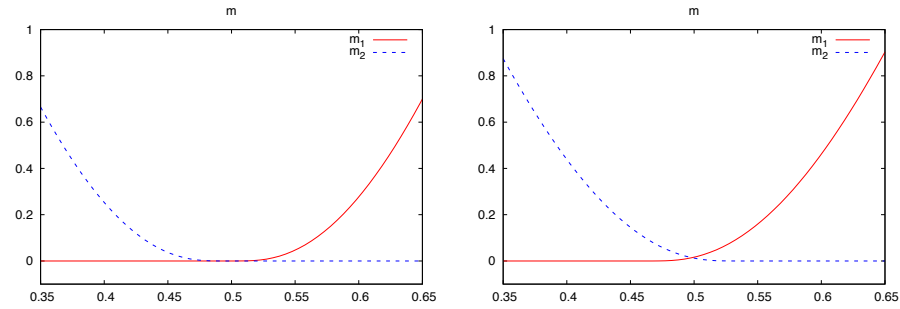


Fig. 5. The non-local case: m_1 and m_2 in the subinterval $[0.35, 0.65]$. $a_1 = a_2 = 0.8$, left. $a_1 = a_2 = 0.3$, right.



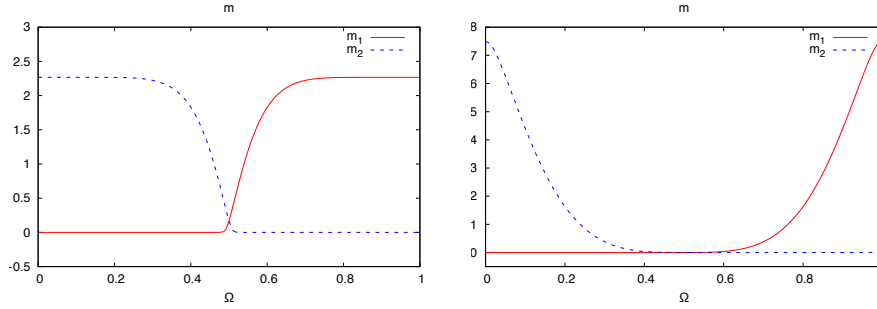
non-local versions of the cost functionals as in (3.5), with kernel

$$K(x, y) = \frac{1}{|[x - \delta, x + \delta] \cap \Omega|} \chi_{[x - \delta, x + \delta] \cap \Omega}(y), \quad \delta \in (0, 1).$$

In Figure 5 the solutions m_h are plotted; here, $\delta = 0.2$. If $a_1 = a_2 = 0.8$, players prefer regions of Ω with prevalent presence of their own population. In this case, one may observe that the set where both the distributions vanish as $\nu \rightarrow 0$ is an interval with non-empty interior; this is a consequence of the fact that the cost at position x paid by a player depends on an entire neighbourhood of x . Nevertheless, if the happiness thresholds are sufficiently low (say, less than 0.5, as in Figure 5 (right)), the free boundary becomes a point, as in the local case V_ℓ .

Test 4. In this test, we choose different parameters for the Hamiltonians. The value of γ affects the shape of the distributions on their support, as shown in Figure 6. Still, different values of γ produce segregation to the same extent.

Remark 16. Numerical simulations suggest the presence of a wide variety of solutions of (4.3) even in space dimension $d = 1$. In Ref. 21, a similar system MFG is considered, where $\gamma = 2$ and $V^k(m_1, m_2)$ are just increasing functions of m_{3-k} ;

Fig. 6. The non-quadratic case. $\gamma = 8$, left. $\gamma = 4/3$, right.

with respect to our models, segregation is even more encouraged, as players aim at avoiding the other population in any case. In their framework, some numerical phenomena arising here have been proven rigorously: existence of branches of solutions having one or more critical points, and segregation as $\nu \rightarrow 0$, namely

$$\int_{\Omega} m_1 m_2 \rightarrow 0.$$

Moreover, in the vanishing viscosity limit, uniform bounds on m are shown, indicating that concentration of the distribution is not likely to happen (therefore, anti-overcrowding terms in the costs as in Section 2.2 might be unnecessary), and segregated configurations can be characterized by optimal partition problems. We believe that such features of (4.3) can be proven also for our Schelling models.

6.2. Evolutive PDEs

Let us discuss the numerical simulations of some finite horizon problems.

6.2.1. A one-dimensional case

Here, we choose $d = 1$, $\Omega = (-0.5, 0.5)$ and the horizon $T = 4$. The parameter ν will take the two values 0.12 and 0.045. The value functions and the densities satisfy Neumann conditions at the two endpoints. The Hamiltonian is $H(x, p) = |p|^2$. The terminal cost is 0 and the coupling terms are of the form $V^1[m_1, m_2](x) = V_{\epsilon}(m_1(x), m_2(x))$ and $V^2[m_1, m_2](x) = V_{\epsilon}(m_2(x), m_1(x))$, with

$$V_{\epsilon}(m, n) = \Psi_{-, \epsilon} \left(\frac{m}{m+n+\epsilon} - 0.7 \right) + \Psi_{+, \epsilon}(m+n-8)$$

where

$$\Psi_{-, \epsilon}(y) = \begin{cases} -y + \frac{\epsilon}{2}(e^{\frac{y}{\epsilon}} - 1) & \text{if } y \leq 0 \\ \frac{\epsilon}{2}(e^{-\frac{y}{\epsilon}} - 1) & \text{if } y \geq 0 \end{cases} \quad \text{and} \quad \Psi_{+, \epsilon}(y) = \begin{cases} \frac{\epsilon}{2}(e^{\frac{y}{\epsilon}} - 1) & \text{if } y \leq 0 \\ y + \frac{\epsilon}{2}(e^{-\frac{y}{\epsilon}} - 1) & \text{if } y \geq 0, \end{cases} \quad (6.1)$$

and $\epsilon = 10^{-5}$. The function V_ϵ is a regularized version of

$$V(m, n) = \left(\frac{m}{m+n} - 0.7 \right)^- + (m+n-8)^+.$$

In this case, the two populations are symmetric to each other. The first part of the coupling term stands for xenophobia: an agent located at x pays a cost if at x , the proportion of agents of its own type is less than 70%. The second part models the aversion to overcrowded locations: an agent located at x pays a cost if the density of agents of both types at x is greater than 4. The initial densities are $m_{1,0}(x) = 3/4 + 1/2\chi_{[-1/2, -1/4] \cup [0, 1/4]}(x)$ and $m_{2,0}(x) = 3/4 + 1/2\chi_{[-1/4, 0] \cup [1/4, 1/2]}(x)$. Since the initial distributions are symmetric to each other and the population have symmetric characteristics, the distributions should remain symmetric for all times.

The spatial grid step is $h = 1/50$ and the time step is $\Delta t = 1/100$.

For $\nu = 0.12$, the evolution of the distributions of agents is displayed on Figure 7, which contains nine snapshots corresponding to different dates between 0 and T . We easily see that the distributions of the two types of agents remain symmetric to each other. The distributions seem to keep oscillating between two configurations in which the populations are segregated and grouped in opposite sides of the domains. A possible explanation of this behavior may be as follows: in that rather particular situation when the two populations are symmetric to each other and strongly xenophobic, a rather high level of noise makes it difficult to reach a global steady equilibrium. We expect that there exists another solution which comes close to a steady equilibrium for times not too close to 0 and T (see the next case with $\nu = 0.045$), but this solution has not been selected by our numerical method.

For $\nu = 0.045$, the evolution of the distributions is displayed on Figure 8. Here again, the two distributions of agents remain symmetric to each other, but this time, we see that the populations are very close to a steady equilibrium when t is not too close to 0 and T . The latter equilibrium is a configuration in which the two populations occupy disjoint subdomains.

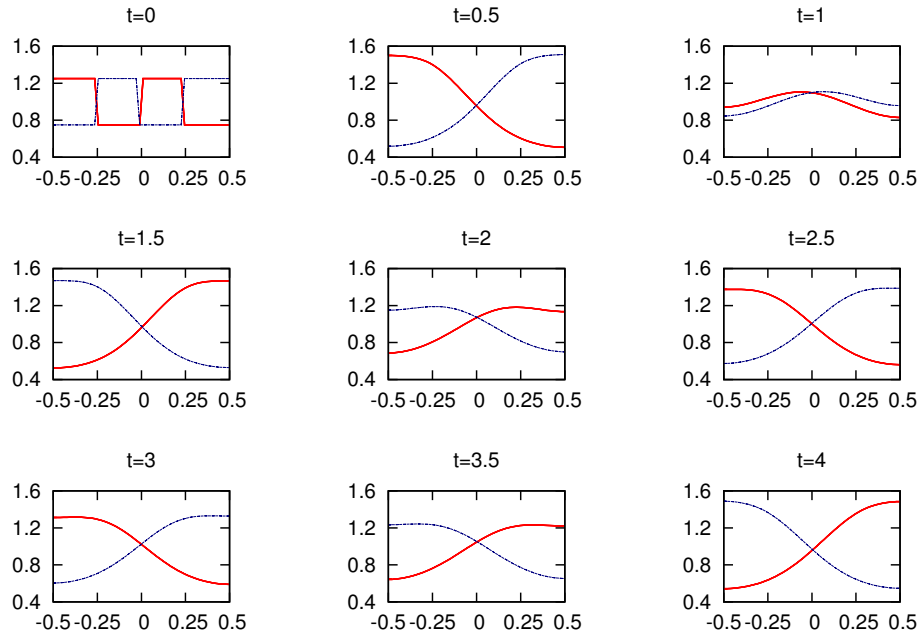
6.2.2. Two bidimensional cases

Case a) Here the domain Ω is obtained by removing a crossed-shaped set from the unit square $(-0.5, 0.5)^2$. We consider two types agents bound to stay in $\bar{\Omega}$, both with “threshold of happiness” a_i below 1/2. More precisely, the model is as follows: the Hamiltonians are $H^1(x, p) = H^2(x, p) = |p|^2$. We take $\nu = 0.038$; the value functions and the densities satisfy Neumann conditions at $\partial\Omega$.

The terminal cost is 0 and the coupling terms are of the form

$$\begin{aligned} V_\epsilon^1[m_1, m_2](x) &= 2\Psi_{-, \epsilon} \left(\frac{m_1(x)}{m_1(x) + m_2(x) + \epsilon} - 0.5 \right) + \Psi_{+, \epsilon}(m_1(x) + m_2(x) - 8), \\ V_\epsilon^2[m_1, m_2](x) &= \Psi_{-, \epsilon} \left(\frac{m_1(x)}{m_1(x) + m_2(x) + \epsilon} - 0.4 \right) + \Psi_{+, \epsilon}(m_1(x) + m_2(x) - 8), \end{aligned}$$

Fig. 7. Evolution of m_h for $\nu = 0.12$: solid/red (respectively dashed/blue) lines are used for m_1 , (respectively m_2).



where $\Psi_{-, \epsilon}$ and $\Psi_{+, \epsilon}$ are defined in § 6.2.1 and $\epsilon = 10^{-5}$. These coupling terms are regularized versions of

$$V^1[m_1, m_2](x) = 2 \left(\frac{m_1(x)}{m_1(x) + m_2(x) + \epsilon} - 0.5 \right)^- + (m_1(x) + m_2(x) - 8)^+,$$

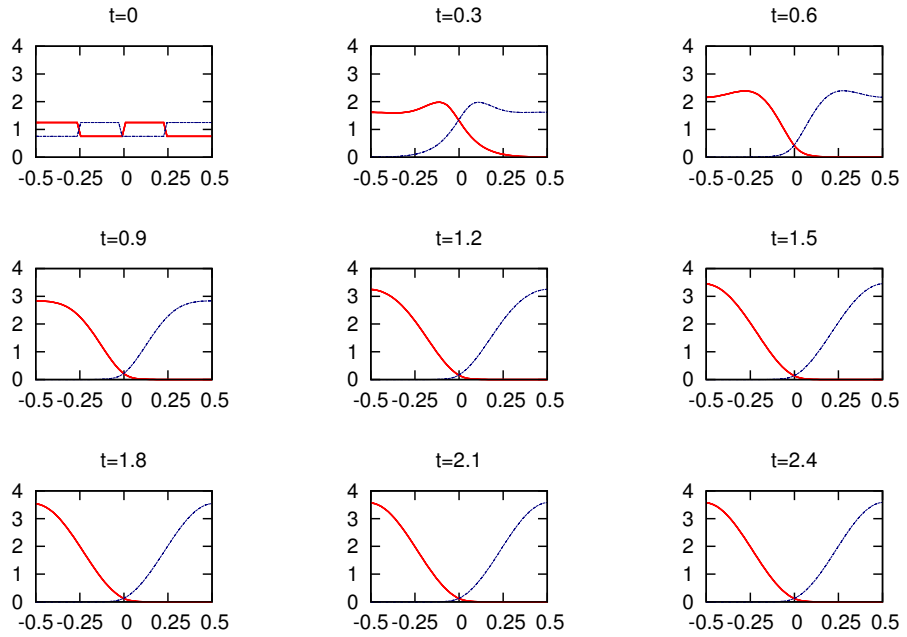
$$V^2[m_1, m_2](x) = \left(\frac{m_2(x)}{m_1(x) + m_2(x) + \epsilon} - 0.4 \right)^- + (m_1(x) + m_2(x) - 8)^+.$$

Note that the first population is less tolerant than the second one.

The agents of the first (respectively second) type are initially uniformly distributed in the top half part (right half part) of the domain, with a density of 2. Therefore, in the top-right corner of the domain, the two populations are initially mixed and the less tolerant agents are in an uncomfortable state. Moreover, the cost for staying in that part of the domain is higher for the first population of agents (by the factor 2 multiplying the term $(\dots)^-$).

In the simulation, the spatial grid step is $1/64$ and the time step is $1/100$. The evolution of the distributions is displayed on Figure 9: we see that the first population leaves the top-right corner and moves toward to the top-left corner of

Fig. 8. Evolution of m_h for $\nu = 0.045$: solid/red (respectively dashed/blue) lines are used for m_1 , (respectively m_2).



the domain. The second population, which is more tolerant, remains in the top-right corner, evolves in a slower manner, and tends to occupy a larger part of the domain than the first one.

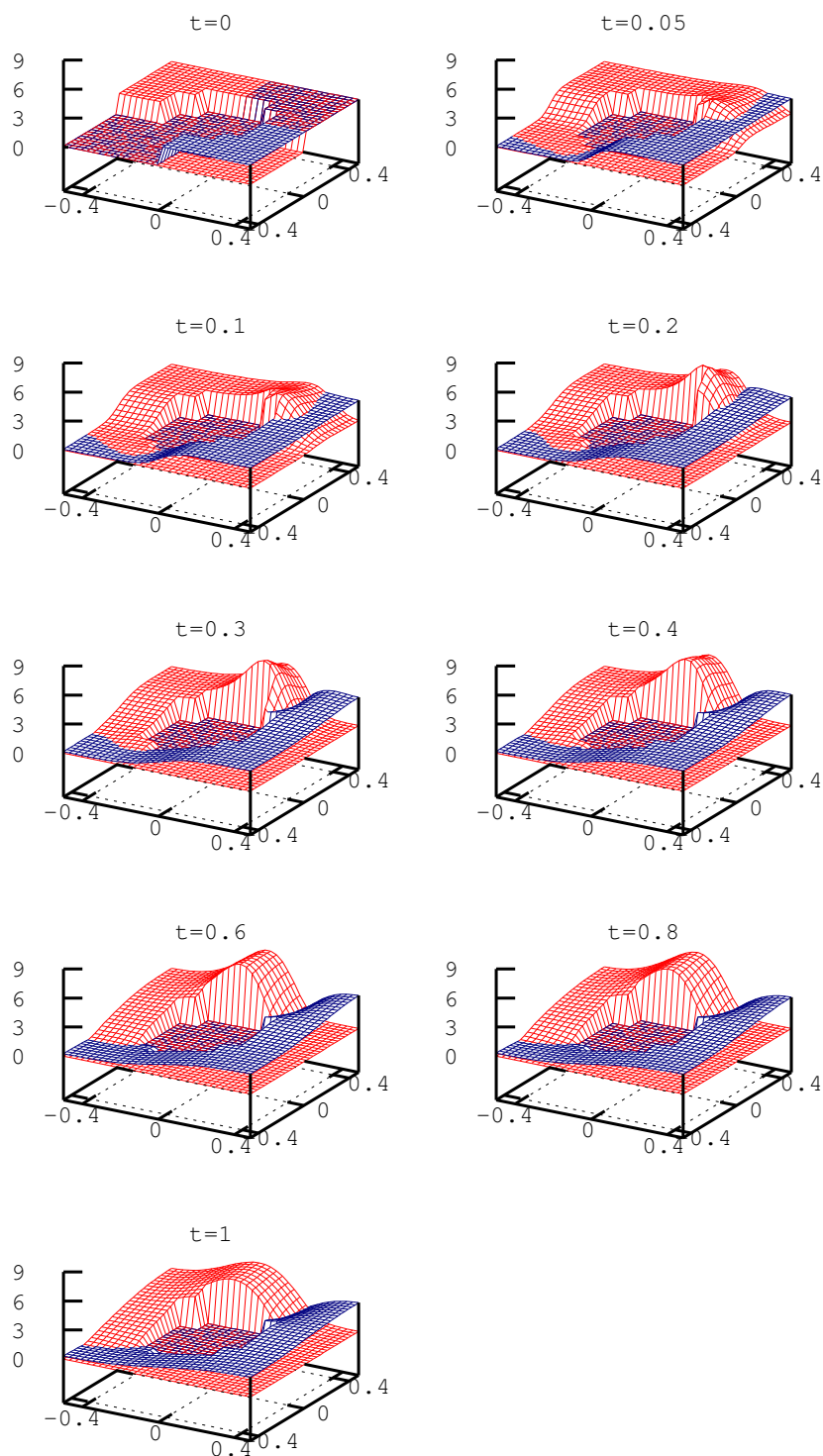
Note the Schelling's phenomenon: segregation occurs even if both thresholds $a_1 = 0.5$, $a_2 = 0.4$ are not xenophobic.

Case b) Here, we consider a case when the two types of agents move in order to reach two different targets: the strategy of the agents consists of reaching the targets while avoiding the agents of the other population. Hence, the dynamics of the agents is not only motivated by xenophobia.

The domain is the unit square $\Omega = (0, 1)^2$ and the horizon is $T = 1$.

The agents of the first (respectively second) type are initially distributed in the top-left (respectively bottom-left) corner of the domain, but are attracted toward the bottom-right (respectively top-right) corner to avoid the running costs. Therefore, the strategy of the agents will be obtained as a trade-off between two opposite tendencies: on the one hand, the agents would like to quickly reach the opposite corner, taking paths which cross each other, but on the other hand the two popu-

Fig. 9. Evolution of m_h for $\nu = 0.038$: red (respectively blue) colors are used for m_1 , (respectively m_2).



lations try to avoid each other.

More precisely, the model is as follows: the Hamiltonians are

$$\begin{aligned} H^1(x, p) &= |p|^2 - 1.4\chi_{[0,0.7] \times [0.2,1]}(x), \\ H^2(x, p) &= |p|^2 - 1.4\chi_{[0,0.7] \times [0,0.8]}(x), \end{aligned}$$

which means in particular that the first (respectively second) type of agents is attracted to the rectangle $[0.7, 1] \times [0, 0.2]$ (respectively $[0.7, 1] \times [0.8, 1]$). We take $\nu = 0.03$; the value functions and the densities satisfy Neumann conditions at $\partial\Omega$. The terminal cost is 0. The coupling terms are

$$\begin{aligned} V^1[m_1, m_2](x) &= 2 \left(\frac{m_1(x)}{m_1(x) + m_2(x) + \epsilon} - 0.8 \right)^- + (m_1(x) + m_2(x) - 8)^+, \\ V^2[m_1, m_2](x) &= \left(\frac{m_2(x)}{m_1(x) + m_2(x) + \epsilon} - 0.6 \right)^- + (m_1(x) + m_2(x) - 8)^+. \end{aligned}$$

The first population is more xenophobic than the second one. The initial distributions of the agents are given by

$$\begin{aligned} m_{1,0}(x) &= 4\chi_{(0,0.2) \times (0.6,1)} + 0.02, \\ m_{2,0}(x) &= 4\chi_{(0,0.2) \times (0,0.4)} + 0.02. \end{aligned}$$

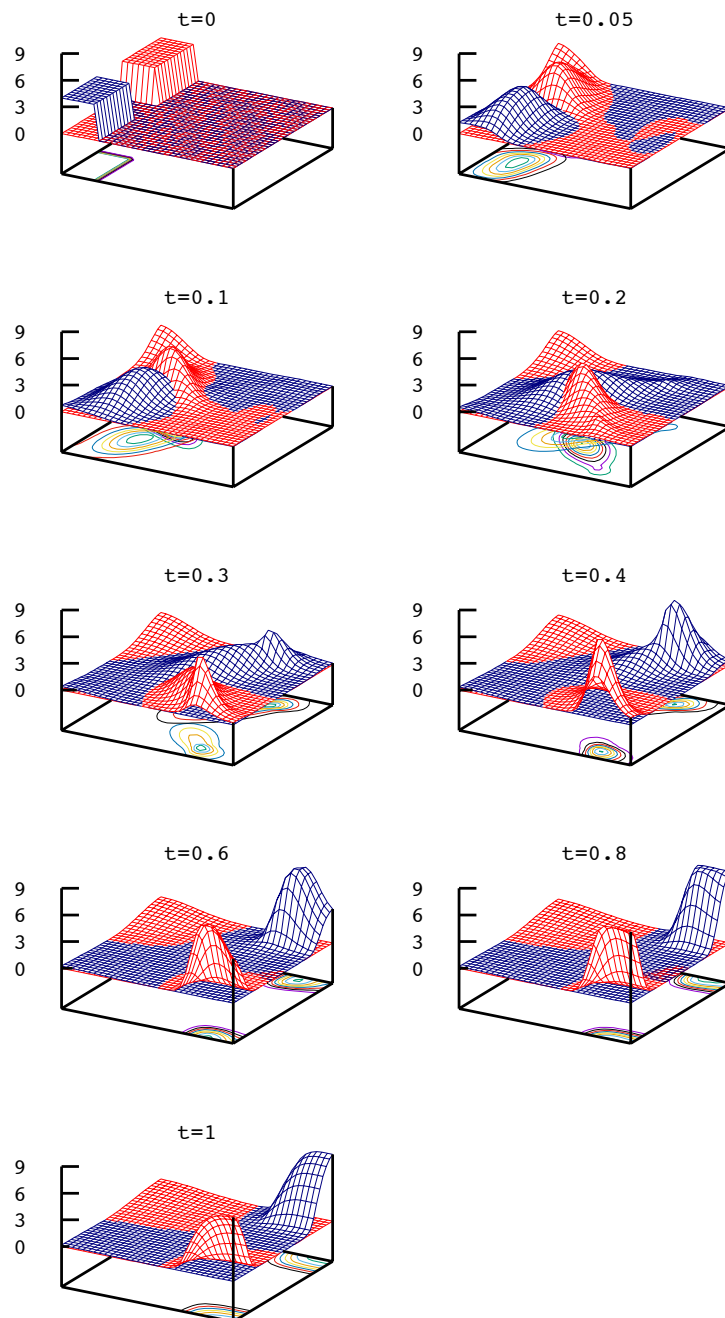
In the simulation, the spatial grid step is $1/64$ and the time step is $1/100$.

The evolution of the distributions is displayed on Figure 10: we see that in the beginning (before $t = 0.2$), a significant part of the first population (the more xenophobic agents) quickly moves to the opposite corner: even if those agents pay an important cost for quickly moving to the opposite corner, this cost is compensated by their quickly reaching a location where there are no agents of type 2. By contrast, for $t \leq 0.2$ the second population is more uniformly distributed. At time $t = 0.2$, the first population is split into two groups: the first group has almost reached the desired corner, whereas the second group has not moved. Next, for $0.2 \leq t \leq 0.6$, this latter group of agents of the first type still does not move, while the whole second population moves to its favorite corner, occupying the center of the domain. Indeed, since the density of the agents of the second type in the middle of the domain has become too important, the agents of the first type prefer waiting rather than meeting them. At $t = 0.6$, most of the second population has reached the desired corner, and the first population can finish crossing the domain.

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Fig. 10. Evolution of m_h for $\nu = 0.03$: red (respectively blue) colors are used for m_1 , (respectively m_2).



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