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Abstract

We provide an existence result for stationary fractional mean field game systems, with fractional exponent greater than 1/2. In the case in which the coupling is a nonlocal regularizing potential, we obtain existence of solutions under general assumptions on the Hamiltonian. In the case of local coupling, we restrict to the subcritical regime, that is the case in which the diffusion part of the operator dominates the Hamiltonian term. We consider both the case of local bounded coupling and of local unbounded coupling with power-type growth. In this second regime, we impose some conditions on the growth of the coupling and on the growth of the Hamiltonian with respect to the gradient term.

Nous prouvons un résultat d'existence pour des systèmes de jeux à champ moyen stationnaires et fractionnaires, avec un exposant fractionnaire supérieur à 1/2. Dans le cas où le couplage est un potentiel de régularisation non local, nous obtenons l'existence de solutions sous des hypothèses générales sur l'hamiltonien. Dans le cas où le couplage est local, nous considérons le régime sous-critique, c'est le cas dans lequel la partie de diffusion de l'opérateur domine le terme hamiltonien. Nous considérons à la fois le cas du couplage limité local et du couplage local non borné avec la croissance du type puissance. Dans ce second régime, nous imposons certaines conditions sur la croissance du couplage et sur la croissance de l'hamiltonien par rapport au terme gradient.

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1. Introduction

Mean Field Games (briefly MFG) is a very recent mathematical theory modelling the macroscopic behaviour of a large population of indistinguishable agents who wish to minimize a cost depending on the distribution of all the agents in a noisy environment. It was proposed independently in 2006 by Lasry and Lions ([22]) and Huang, Caines and Malhamé ([21]), and it has a number of potential applications, from economics and finance (growth theory, environmental policy, formation of volatility in financial markets), to engineering and models of social systems, such as crowd motion and traffic control. For the development of the theory and several applications, we refer to the monographs [8], [20], and to the references therein.

Up to now, the noisy environment in which the average game takes place has been usually modeled by standard diffusion. Our aim is to consider a more general framework for the disturbances, and in particular we take into account processes driven by pure jump Lévy processes. This generalization is interesting for applications to financial models, where jump processes are widely used to model sudden crisis and crashes on the markets (see e.g. the monograph [18] for a detailed description of motivations for the use of processes with jumps in financial models and examples of applications of Lévy processes in risk management).

More precisely, even if still heuristically, MFG are noncooperative differential games, with a continuum of players, each of whom controls his own trajectory in the state space, which in our case is the N-dimensional torus. The trajectory of each player is affected by a fractional Brownian motion: it is defined by a stochastic differential equation

$$dX_t = v_t dt + dZ_t,$$

where v_t is the control and Z_t is a N-dimensional, $2s$-stable pure jumps Lévy process, with associated Lévy measure (which describes the distribution of jumps of the process) given by $\nu(dx) = \frac{1}{|x|^{N+2s}}dx$ (see [1]). Each player wants to minimize the long time average cost

$$\liminf_{T\to+\infty} \mathbf{E}\left[\frac{1}{T}\int_0^T L(v_s) + f(X_s, m(X_s))ds\right],$$

where $m(x)$ denotes the density distribution of the population at point x, $L(\cdot)$ is a superlinear convex function and f is a cost function taking into account the position of each player and the density of the whole population. We look for a stable configuration, that is a Nash equilibrium: a configuration where, keeping into account the choices of the others, no player would spontaneously decide to change his own choice. In an equilibrium regime, the corresponding density of the average player is stable as time goes to $+\infty$, and coincides with the population density m.

From a PDE point of view, this equilibrium configuration is characterized by a system of a fractional Hamilton-Jacobi equation with Hamiltonian H given by the Legendre transform of L, coupled with a fractional stationary Fokker-Planck equation describing the long-time distribution of all agents, moving according to the control which minimizes the long time average cost (see [22], [20]).

We recall that MFG with jumps have been very recently considered in the literature by using a completely different approach based on probabilistic techniques in [6], where the theory of non-linear Markovian propagators is used, and in [7], where the players control the intensity of jumps.

In this paper we start with the analysis of stationary fractional mean field game systems, in the periodic setting, with fractional exponent greater than $\frac{1}{2}$. We restrict to this regime since the fractional Laplacian operator with drift presents different properties depending on the fact that the fractional exponent is greater or lower than $\frac{1}{2}$. In the case $s > \frac{1}{2}$, the diffusion component dominates the drift term, and so, the drift term can be treated as a lower-order term. Moreover the kernel of the linear operator defined by the fractional Laplacian with drift can be estimated in terms of the fractional heat kernel (see [9]). We provide in this paper an accurate analysis of steady state solutions to the fractional Fokker-Planck equations in the periodic setting, with bounded drift and fractional exponent s greater than $\frac{1}{2}$, see Section 2. On the other hand, we discuss in Section 2.1 some examples in the case of fractional Laplacian operator with fractional exponent s lower than $\frac{1}{2}$ and bounded drift, which suggest that the study of fractional MFG in the range $s < \frac{1}{2}$ presents structural differences with respect to the range $s > \frac{1}{2}$.

We consider the following ergodic fractional MFG on the N-dimensional torus $Q := \mathbb{R}^N/\mathbb{Z}^N$. The goal is to find a constant $\lambda \in \mathbb{R}$ for which there exists a couple (u, m) solving

$$\begin{cases} (-\Delta)^s u + H(\nabla u) + \lambda = f(x, m), \\ (-\Delta)^s m - \operatorname{div}(m \nabla H(\nabla u)) = 0, \\ \int_Q m \, dx = 1. \end{cases} \tag{1.1}$$

Here we consider the fractional Laplacian $(-\Delta)^s = (-\Delta_Q)^s$ defined on the torus Q with fractional parameter $s \in \left(\frac{1}{2}, 1\right)$. This operator can be defined directly by the multiple Fourier series

$$(-\Delta_Q)^s u(x) := \sum_{k \in \mathbb{Z}^N} |k|^{2s} c_k(u) e^{ik \cdot x}$$

where c_k are the Fourier coefficients of $u : Q \to \mathbb{R}$ (see [27]). We identify functions defined on Q with their periodic extensions to $\mathbb{R}^N$, and it is possible to show that for such functions u, the periodic distribution $(-\Delta_Q)^s u(x)$ coincides with the distributional fractional Laplacian on $\mathbb{R}^N$ of u (see [28, Theorem A]). In particular we denote with $C^{k+\alpha}(Q)$, for $k \geq 0$ and $\alpha \in [0, 1]$, the restriction to Q of Q-periodic functions which are in $C^{k+\alpha}(\mathbb{R}^N)$.

We shall assume that $H : \mathbb{R}^N \to \mathbb{R}$ is locally Lipschitz continuous and strictly convex, and that there exist some $C_H > 0$, $K > 0$ and $\gamma > 1$ such that, for all $\mathscr{p} \in \mathbb{R}^N$,

$$\begin{aligned} &C_H|\mathscr{p}|^\gamma - C_H^{-1} \le H(\mathscr{p}) \le C_H^{-1}(|\mathscr{p}|^\gamma + 1), \\ &\nabla H(\mathscr{p}) \cdot \mathscr{p} - H(\mathscr{p}) \ge C_H|\mathscr{p}|^\gamma - K \quad \text{ and } \quad |\nabla H(\mathscr{p})| \le C_H|\mathscr{p}|^{\gamma-1}. \end{aligned} \tag{1.2}$$

We mention that for general applications it could be interesting to consider Hamiltonians $H(x, \mathscr{p})$ with explicit x dependance. We expect that our analysis can be carried over to this case whenever (1.2) holds uniformly in x and $H(\cdot, \mathscr{p})$ is smooth enough.

As for the function f, we consider both the case of local and the case of nonlocal coupling. We will give more precise assumptions[1] in what follows about this.

Moreover, following [23, 29], for $p > 1$ and $\sigma \ge 0$, we define the Bessel potential space $H_p^\sigma(Q)$ as

$$H_p^\sigma(Q) := \left\{ u \in L^p(Q) : \ (I-\Delta)^{\frac{\sigma}{2}} u \in L^p(Q) \right\} \qquad \text{with } \ \|u\|_{H_p^\sigma(Q)} := \|(I-\Delta)^{\frac{\sigma}{2}} u\|_{L^p(Q)} . \tag{1.3}$$

In this setting, we say that a classical solution to the system (1.1) is a triple $(u, \lambda, m) \in C^{2s+\theta}(Q) \times \mathbb{R} \times H_p^{2s-1}(Q)$, for all $\theta < 2s - 1$ and for all $p > 1$.

Our main result is the following, and it is proved in Theorems 4.1, 5.1 and 6.3.

Theorem 1.1. *Let $s \in \left(\frac{1}{2}, 1\right)$. Then (1.1) admits a classical solution in the following cases.*

1. $\gamma > 1$ *and for some $\alpha < 2s - 1$, f maps continuously the set of probability measures with density in $C^\alpha(Q)$ into a bounded subset of $W^{1,\infty}(Q)$.*
2. $1 < \gamma \le 2s$ *and $f : Q \times [0, +\infty) \to \mathbb{R}$ is continuous and bounded.*
3. $1 < \gamma < \frac{N}{N-2s+1}$ *for $N > 1$, $1 < \gamma \le 2s$ for $N = 1$, and $f : Q \times [0, +\infty) \to \mathbb{R}$ is locally Lipschtiz continuous and satisfies*

$$-Cm^{q-1} - K \le f(x, m) \le Cm^{q-1} + K, \tag{1.4}$$

for some C, $K > 0$ and

$$1 < q < 1 + \frac{(2s-1)}{N} \frac{\gamma}{\gamma - 1}. \tag{1.5}$$

Now, we discuss in more details the results in Theorem 1.1.

[1]With a slight abuse of notation, we write $f[m]$ when we intend the action of the function f to a function m and $f(\cdot, m)$ when we intend the map $x \mapsto f(x, m(x))$. The two cases are structurally different, since $f[m]$ takes into account a "nonlocal setting", in which, for instance, $f[m]$ can be the convolution of f with a kernel (in particular, $f[m](x)$ does not depend only on x and on $m(x)$, but rather on x and on all the values that m may attain). A more precise setting is discussed in Section 4.

In the case (1), that is in the case in which the coupling f is a smoothing potential, we obtain existence of solutions to the MFG system by taking advantadge of a classical approach given in [22], based on the Schauder Fixed Point Theorem. To get the existence result in this case, we use some estimates on the solutions to stationary Fokker-Planck equations obtained in Section 2 and a-priori gradient estimates on solutions of fractional coercive Hamilton-Jacobi equations, inspired by the Bernstein method in [4].

As for the case of the local coupling, we use a different approach. First of all, in order to get a-priori gradient estimates on solutions of fractional coercive Hamilton-Jacobi equations we cannot use anymore the Bernstein method, since the function $x \mapsto f(x, m(x))$ is not in general Lipschtiz continuous. So, we use the so-called Ishii-Lions method (see [2]) to obtain gradient estimates on solutions of fractional coercive Hamilton-Jacobi equations. This method requires, in particular, that $\gamma \leq 2s$, where γ is the growth of the Hamiltonian given in (1.2) and s is the fractional exponent of the Laplacian. The gradient estimates in this case depend only on the L^∞ norm of the solutions and of f.

In case (2), this result permits to conclude the proof of Theorem 1.1, by first regularizing the potential and then passing to the limit.

In case (3), in which the local coupling term is unbounded, we use the variational approach, which goes back to the seminal work [22] (see also [14, 17, 24]): the MFG system is obtained (at least formally) as the optimality condition of an appropriate optimal control problem on the fractional Fokker-Planck equation.

First of all, the function $f(x, \cdot)$ can be unbounded both from below and from above, so in general the energy associated to the MFG system is not even bounded. The condition on the growth of f with respect to m, given in (1.4) and (1.5), is necessary to get boundedness of the energy associated to the system and then to obtain existence of minimizers by direct methods.

Note that our assumption allows us to treat both the case in which the coupling is an increasing function of m, that is a congestion game, in which players aim to avoid regions where the population has a high density, and the opposite case in which the coupling is a decreasing function in m, modelling a game in which every player is attracted by regions where the density of population is high.

Finally we point out that the condition on the growth of the Hamiltonian in (3) of Theorem 1.1, that is

$$1 < \gamma < \frac{N}{N - 2s + 1},$$

is just a technical condition, that can be eliminated once a-priori gradient estimates on the solutions of fractional coercive Hamilton-Jacobi equations depending only on the L^∞ norm of the potential term f and not on the L^∞ norm of the solutions u are available. In the case of the classical Laplacian such a result has been obtained by an improved Bernstein method, based also on Ishii-Lions type arguments, in [13]. We believe that such an approach can be adapted to the fractional case, and this will be the topic of future research.

The paper is organized as follows. In Section 2 we provide some results on a-priori estimates, existence and uniqueness of solutions to stationary fractional Fokker-Planck equations in the periodic setting. These results should be classical, and well known, nevertheless due to the lack of a precise references in the literature, we provide also a sketch of the proofs. In Section 3, we recall the existing results about a-priori gradient bounds for solutions to fractional Hamilton-Jacobi equations with coercive Hamiltonians and on the solvability of ergodic problems in this setting. Section 4 is devoted to the analysis of MFG systems in the case of regularizing nonlocal coupling. Section 5 contains the existence result for MFG systems with local bounded coupling. In Section 6, we consider fractional MFG systems with local unbounded coupling. Finally, Section 7 contains the improvement of regularity of solutions of the MFG system in the case in which the coefficients are more regular, and the uniqueness result for increasing coupling terms.

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2. Steady state solutions to fractional Fokker-Planck equations

We provide here some results on existence, uniqueness and regularity of steady state solutions to fractional Fokker-Planck equations in the periodic setting.

First of all we recall some simple result about Bessel potential spaces. We recall that (see [19]) the norm $\|\cdot\|_{H^\sigma_p(Q)}$ defined in (1.3) is equivalent to the norm

$$\|u\| = \|u\|_{L^p(Q)} + \|(-\Delta)^{\frac{\sigma}{2}} u\|_{L^p(Q)}\,.$$

Observe that the space $H^\sigma_2(Q)$ coincides with $W^{\sigma,2}(Q)$. Moreover we have the following embedding results.

Lemma 2.1. *For every $\sigma \geq 0$, $p > 1$ and $\varepsilon > 0$, we get*

$$H^{\sigma+\varepsilon}_p(Q) \subseteq W^{\sigma,p}(Q) \subseteq H^{\sigma-\varepsilon}_p(Q),$$

with continuous embeddings. Moreover

$$W^{m,p}(Q) \simeq H^m_p(Q) \qquad \text{if } m \in \mathbb{N}. \tag{2.1}$$

In particular there exists a constant $C = C_p > 0$ such that

$$\|\nabla\phi\|_{L^p(Q)} \leq C\|(-\Delta)^{\frac{1}{2}}\phi\|_{L^p(Q)}. \tag{2.2}$$

Proof. The proof of this result is given in [23, Theorem 3.2], for $Q = \mathbb{R}^N$. Then in [23, Section 4], the result is extended to $Q = \Omega$ with Ω bounded open set with regular boundary, since it is proved (see [23, Proposition 4.1]) that $H_p^\sigma(\Omega)$ coincides with the set of restrictions to Ω of functions in $H_p^\sigma(\mathbb{R}^N)$.

The same argument (even simpler) permits to show also the result for the periodic case. $\square$

Lemma 2.2. *Let $w \in H_p^\sigma(Q;\mathbb{R}^N)$, with $\sigma \geq 0$. Then there exists a unique solution $m \in H_p^{2s-1+\sigma}(Q)$ to the problem*

$$(-\Delta)^s m = \operatorname{div}(w), \qquad \textit{with} \int_Q m\,dx = 1. \tag{2.3}$$

Moreover there exists $C > 0$, depending on p, such that

$$\|m\|_{H_p^{2s-1+\sigma}(Q)} \leq C\|w\|_{H_p^\sigma(Q)}. \tag{2.4}$$

Proof. We first show that the following auxiliary problem

$$-\Delta u = \operatorname{div}(w), \qquad \text{with} \int_Q u\,dx = 0, \tag{2.5}$$

admits a unique solution $u \in H_p^{1+\sigma}(Q)$.

Assume first that w is smooth, let u be the unique smooth solution to (2.5), and let $v \in C^\infty(Q)$ be a test function. Multiplying (2.5) by $(-\Delta)^{\frac{\sigma}{2}} v$ and integrating by parts, we get

$$\begin{aligned}\int_Q u\,(-\Delta)^{1+\frac{\sigma}{2}} v\,dx &= -\int_Q (-\Delta)^{\frac{\sigma}{2}} w \cdot \nabla v\,dx \\ &\leq \|w\|_{H_p^\sigma(Q)}\,\|\nabla v\|_{L^{p'}(Q)} \leq C\|w\|_{H_p^\sigma(Q)}\,\|(-\Delta)^{\frac{1}{2}} v\|_{L^{p'}(Q)},\end{aligned}$$

where the last inequality follows from (2.1) with $m = 1$. Here above $p' = \frac{p}{p-1}$ is the conjugate exponent of p.

As a consequence, by taking $\psi := (-\Delta)^{\frac{1}{2}} v$, which is an arbitrary test function with zero average, we get

$$\int_Q u\,(-\Delta)^{\frac{1+\sigma}{2}} \psi\,dx \leq C\|w\|_{H_p^\sigma(Q)}\,\|\psi\|_{L^{p'}(Q)}, \qquad \forall \psi \in C^\infty(Q) \text{ with } \int_Q \psi = 0,$$

which implies that

$$\|u\|_{H_p^{1+\sigma}(Q)} \leq C\|w\|_{H_p^\sigma(Q)}. \tag{2.6}$$

The result in the general case then follows by approximating w with smooth vector fields.

Letting now $m := 1 + (-\Delta)^{1-s} u \in H_p^{2s-1+\sigma}(Q)$, so that $(-\Delta)^s m = -\Delta u$, we have that

$$\int_Q m\,dx = 1$$

and m is the (unique) solution to (2.3). Finally, recalling the definition of m, we get that

$$\|m\|_{H_p^{2s-1+\sigma}(Q)} = \|(I-\Delta)^{s-\frac{1}{2}+\frac{\sigma}{2}} m\|_{L^p(Q)}$$
$$\le \|(I-\Delta)^{\frac{1-\sigma}{2}} u\|_{L^p(Q)} + \|(I-\Delta)^{s-\frac{1}{2}+\frac{\sigma}{2}} u\|_{L^p(Q)} = \|u\|_{H_p^{\sigma+1}(Q)} + \|u\|_{H_p^{\sigma-1+2s}(Q)}. \tag{2.7}$$

Notice now that $2s-1 \in (0,1)$, therefore $\sigma - 1 + 2s \le \sigma + 1$, Hence, (2.7), together with (2.6), gives (2.4), thanks to Lemma 2.1. □

Lemma 2.3. *Let $r > 1$ and $m \in L^1(Q)$ be such that $\int_Q m = 1$ and*

$$\int_Q m(-\Delta)^s \phi \le C\|\nabla \phi\|_{L^{r'}(Q)}, \qquad \forall \phi \in C^1(Q), \tag{2.8}$$

with $r' = \frac{r}{r-1}$, for some $C > 0$. Then $(-\Delta)^{s-\frac{1}{2}} m \in L^r(Q)$ and

$$\|(-\Delta)^{s-\frac{1}{2}} m\|_{L^r(Q)} \le C. \tag{2.9}$$

Proof. Let $m_\varepsilon := m \star \chi_\varepsilon$, where χ_ε is a standard mollifier. Then (2.8) reads

$$\int_Q m_\varepsilon(-\Delta)^s \phi \le C\|\nabla \phi\|_{L^{r'}(Q)}, \qquad \forall \phi \in C^1(Q).$$

Therefore, integrating by parts and recalling (2.2), we obtain

$$\int_Q (-\Delta)^{s-\frac{1}{2}} m_\varepsilon (-\Delta)^{\frac{1}{2}} \phi\, dx = \int_Q m_\varepsilon (-\Delta)^s \phi\, dx \le C\|\nabla \phi\|_{L^{r'}(Q)} \le C\|(-\Delta)^{\frac{1}{2}} \phi\|_{L^{r'}(Q)},$$

from which we obtain the desired inequality (2.9) for m_ε and then for m, letting $\varepsilon \to 0$. □

Finally we consider steady state solutions to the periodic fractional Fokker-Planck equation.

Proposition 2.4. *Let $b \in L^\infty(Q;\mathbb{R}^N)$. Then, there exists a unique solution $m \in H_p^{2s-1}(Q)$, for all $p > 1$, to the problem*

$$(-\Delta)^s m + \operatorname{div}(bm) = 0, \tag{2.10}$$

with $\int_Q m\, dx = 1$, and

$$\|m\|_{H_p^{2s-1}(Q)} \le C,$$

where $C > 0$ depends only on N, p and $\|b\|_{L^\infty(Q;\mathbb{R}^N)}$. In particular, we have that $m \in C^\theta(Q)$, for every $\theta \in (0, 2s-1)$.

Furthermore, we get that there exists a constant $C = C(s, N, b) > 0$ such that

$$0 < C \le m(x) \le C^{-1}, \quad \textit{for any } x \in Q.$$

Proof. Assume b to be smooth, the general case will follow by an approximation argument.

Step 1: Existence and uniqueness of a solution. The existence result follows by the Fredholm alternative.

More precisely, for Λ large enough, by Lax-Milgram Theorem, the equation

$$(-\Delta)^s v - b \cdot \nabla v + \Lambda v = \psi$$

has a unique solution $u \in H_2^s(Q)$, for any fixed $\psi \in L^2(Q)$. Therefore, the mapping $\mathscr{G}_\Lambda$, defined by $v = \mathscr{G}_\Lambda \psi$, is a compact mapping of $L^2(Q)$ into itself.

Now, equation (2.10) may be rewritten as

$$(I - \Lambda \mathscr{G}_\Lambda^*)m = 0. \tag{2.11}$$

By the Fredholm alternative, the number of linearly independent solutions of (2.11) is the same as that of the adjoint problem, that is

$$(I - \Lambda \mathscr{G}_\Lambda)v = 0,$$

that corresponds to

$$(-\Delta)^s v - b \cdot \nabla v = 0. \tag{2.12}$$

Any $v \in H_2^s(Q)$ solving (2.12) is in $C^{2s}(Q)$ (due to [25, Lemma 2.2]), and then it must be constant by the Strong Maximum Principle (see [15]). We conclude that there exists m solving (2.10) (in the distributional sense), and such $m \in L^2(Q)$ is unique up to a multiplicative constant.

Step 2: Positivity. Fix a nonnegative periodic Borel initial datum z_0, and consider the following Cauchy problem,

$$\begin{cases} \partial_t z + (-\Delta)^s z - b \cdot \nabla z = 0 & \text{in } \mathbb{R}^N \times (0, \infty), \\ z(\cdot, 0) = z_0(\cdot). \end{cases} \tag{2.13}$$

We recall estimates of heat kernel of fractional Laplacian perturbed by gradient operators obtained in [9, Theorem 2]: for every $t_0 > 0$, there exists a constant $C > 0$, depending on t_0, s, b and N, such that

$$Cp(t,x,y) \leq p'(t,x,y) \leq C^{-1} p(t,x,y), \quad \text{for any } x, y \in \mathbb{R}^N \text{ and } t \in (0, t_0), \tag{2.14}$$

where $p(t,x,y)$ is the fractional heat kernel and $p'(t,x,y)$ is the kernel associated to the operator $\partial_t + (-\Delta)^s - b \cdot \nabla$.

Now we fix $x_0 \in Q$ and we take a mollifying sequence

$$z_{0,n} \rightharpoonup \delta_{x_0} \tag{2.15}$$

in the sense of measure. Let z_n be the solution to (2.13) with initial datum $z_{0,n}$.

So, the solution z_n of (2.13) satisfies

$$z_n(x, 1) \geq \tilde{C} \int_Q z_{0,n}(x) dx = \tilde{C},$$

where $\tilde{C} > 0$ is a constant depending on b, s and N. In particular, by the comparison principle,

$$z_n(x,t) \geq \tilde{C} \quad \text{for any } t \geq 1.$$

By [3, Theorem 2], there exist a constant c_n and a Q periodic function $\bar{z}_n$ such that $z_n(\cdot,t) - c_n t - \bar{z}_n(\cdot)$ converges uniformly to zero as $t \to +\infty$, where the couple $(c_n, \bar{z}_n)$ solves the stationary problem

$$(-\Delta)^s \bar{z}_n - b \cdot \nabla \bar{z}_n = c_n \quad \text{in } Q. \tag{2.16}$$

Note that $(c_n, \bar{z}_n)$ solving (2.16) must satisfy $c_n = 0$, so that $\bar{z}_n$ is identically constant on Q; hence $z_n(\cdot,t) \to \bar{z}_n \geq \tilde{C}$ uniformly on Q as $t \to +\infty$.

By multiplying the equation in (2.13) by m, the equation in (2.10) by z_n, and integrating by parts on Q, we obtain that, for all $t > 1$,

$$\int_Q \partial_t z_n(x,t) m(x) dx = 0,$$

so

$$\int_Q z_{0,n}(x) m(x) dx = \int_Q z_n(x,t) m(x) dx \to \bar{z}_n \geq \tilde{C} > 0 \tag{2.17}$$

as $t \to +\infty$, since $\int_Q m\, dx = 1$. Now we send $n \to +\infty$ in (2.17) and we get, recalling (2.15),

$$m(x_0) \geq \tilde{C}.$$

Since this is true for every $x_0 \in Q$, we get that there exists a constant $C = C(s, N, b) > 0$ such that $0 < C \leq m(x)$.

Step 3: Boundedness and regularity. The same argument as in Step 2, using the bound from above in (2.14) (instead of the bound from below), gives that there exists a constant $C' = C'(s, N, b) > 0$ such that $m(x) \leq C'$.

Since $bm \in L^\infty(Q)$, by Lemma 2.2 we have that $m \in H_p^{2s-1}(Q)$, for all $p > 1$. In particular, we have that $m \in C^\theta(Q)$, for every $\theta \in (0, 2s-1)$. $\square$

2.1. The case $s < \frac{1}{2}$

Note that in the case $s < \frac{1}{2}$ the solutions to

$$(-\Delta)^s m = \text{div} w,$$

for $w \in H_p^\sigma$, have to be intended in some weak sense. In particular, if $\sigma < 1 - 2s$, the solution m is a distribution.

Moreover the associated kernel of the operator $(-\Delta)^s + b(x) \cdot \nabla$ is not bounded from below by the fractional heat kernel, and it does not produce strictly positive solutions. These phenomena will be discussed in details in the following remarks. This suggests that the study of fractional Mean Field Games in the range $s < \frac{1}{2}$ presents structural differences with respect to the range $s > \frac{1}{2}$.

Remark 2.5. Concerning the optimality of the regularity results in Proposition 2.4, we point out that the solution m may vanish at a point and is not better than C^{2s} for $s \in (0, 1/2)$. To see a one-dimensional example in $\mathbb{R}$, we take $n = 1$, $s \in (0, 1/2)$ and

$$b(x) := -\frac{1}{m(x)} \int_0^x (-\Delta)^s m(y)\, dy,$$

with $m(x) := |x|^\theta$, with $\theta \in (2s, 1)$.

Using the substitution $z = y/x$, we see that

$$\int_{\mathbb{R}} \frac{|x+y|^\theta + |x-y|^\theta - 2|x|^\theta}{|y|^{1+2s}}\, dy = |x|^{\theta-2s} \int_{\mathbb{R}} \frac{|1+z|^\theta + |1-z|^\theta - 2|z|^\theta}{|z|^{1+2s}}\, dz \tag{2.18}$$

and so $(-\Delta)^s m(x) = -c|x|^{\theta-2s}$, for some $c > 0$.

This setting gives that, for small $|x|$,

$$|b(x)| \le \frac{C}{|x|^\theta} \int_0^{|x|} |y|^{\theta-2s}\, dy \le C|x|^{1-2s},$$

up to renaming $C > 0$. This gives that b is locally bounded (and Hölder continuous with exponent $1 - 2s$). Moreover, we have that

$$\operatorname{div}(bm) = (bm)' = -\left(\int_0^x (-\Delta)^s m(y)\, dy\right)' = -(-\Delta)^s m,$$

hence the equation is satisfied.

Remark 2.6. The example of Remark 2.5 can also be used to show that the positivity results of [9] do not hold in general for $s \in (0, 1/2)$. For instance, we take $n = 1$, $s \in (0, 1/2)$, $v(x) := |x|^\theta$, with $\theta \in (2s, 1)$.

From (2.18), we know that $(-\Delta)^s v(x) = -c|x|^{\theta-2s}$, for some $c > 0$. So we define $b(x) := -\frac{c}{\theta}\, |x|^{-2s}x$ and we notice that b is locally bounded (and Hölder continuous with exponent $1 - 2s$) and

$$(-\Delta)^s v - b \cdot \nabla v = -c|x|^{\theta-2s} + \left(\frac{c}{\theta}\, |x|^{-2s}x\right) \cdot (\theta|x|^{\theta-2}x) = 0,$$

hence (2.12) is satisfied.

Since $v \ge 0$ but $v(0) = 0$, this example shows that the strong maximum principle is violated in this case. Note that v solves a.e. the equation, but it is not a viscosity (sub)solution of the equation at $x = 0$. Indeed, by the strong maximum principle proved in Lemma 4.4 in [4], the unique viscosity solutions to $(-\Delta)^s v - b \cdot \nabla v$ are constants.

Moreover, v is also a (stationary) solution of the heat flow associated to (2.12), corresponding to an initial datum which is nonnegative and that does not become strictly positive as time flows (this lack of positivity gain in time can be seen as a counterpart when $s \in (0, 1/2)$ to the positivity of the heat kernel established in [9]).

3. Fractional Hamilton-Jacobi equations with coercive Hamiltonian

We collect some results on the Lipschitz continuity of viscosity solutions to Hamilton-Jacobi equations and on the solution to the ergodic problem. There are different kinds of results, depending on the fact that the Hamiltonian term is dominant or not with respect to the fractional Laplacian term. Actually, the growth $\gamma \leq 2s$ allows the use of the so called Ishii-Lions method, in particular getting a-priori estimates on the gradient of the solution which depend only on the L^∞ norm of the potential term, whereas in the case $\gamma > 2s$ the Bernstein method is used, obtaining a-priori estimates on the gradient of the solution which depend only on the Lipschitz norm of the potential term.

We consider the following Hamilton-Jacobi equation

$$(-\Delta)^s u + H(\nabla u) + \lambda = f(x), \qquad x \in \mathbb{R}^N. \tag{3.1}$$

We assume that $f \in C(\mathbb{R}^N)$, and that f is $\mathbb{Z}^N$-periodic.

Theorem 3.1. *Let $s > \frac{1}{2}$ and $\gamma \leq 2s$.*
Then the following statements hold.

1. *If u is a continuous periodic solution to (3.1), then there exists a constant $K > 0$, depending on $\|u\|_{L^\infty(Q)}$, $\|f\|_{L^\infty(Q)}$ and $|\lambda|$, such that*
$$\|\nabla u\|_{L^\infty(Q)} \leq K.$$
Moreover, there exists a constant $C > 0$, depending only on the period of u and $\|f\|_{L^\infty(Q)}$, such that $\|u\|_{L^\infty(Q)} \leq C$.
2. *There exists a unique constant $\lambda \in \mathbb{R}$ such that (3.1) has a periodic solution $u \in W^{1,\infty}(\mathbb{R}^N)$ and*
$$\lambda = \sup\{c \in \mathbb{R} \;\; s.t. \;\; \exists u \in W^{1,\infty}(\mathbb{R}^N) \; s.t. \; (-\Delta^s)u + H(\nabla u) + c \leq f(x)\}. \tag{3.2}$$
Moreover, u is the unique Lipschitz viscosity solution to (3.1) up to addition of constants.

Finally, if $f \in C^\theta(\mathbb{R}^N)$, for some $\theta \in (0,1]$, then $u \in C^{2s+\alpha}(Q) \cap H^{2s}_p(Q)$, for every $\alpha < \theta$ and every $p > 1$.

Proof. The a-priori estimates on the gradient is proved in [2, Theorem 2]. The a-priori estimate on the L^∞ norm of u can be obtained as in [2, Proposition1] or [4, Lemma 4.2].

The existence of λ and of a unique (up to constants) viscosity solution to (3.1) is given in [3, Theorem 1]. Formula (3.2) can be proved by a standard argument, using the Strong Maximum Principle, which holds for operators as $(-\Delta)^s + b\cdot\nabla$, with $s > \frac{1}{2}$ and b continuous (see [15]).

Finally, if f is Hölder continuous, applying [12], we get that $u \in C^{1+\alpha}(Q)$ for any $\alpha < 2s - 1$, and finally, by the bootstrap argument in [5, Theorem 6], we obtain the desired regularity.

Also, since $(-\Delta)^s u \in L^\infty(Q)$, then by [19, Theorem 2.1] it follows that $u \in H^{2s}_p(Q)$ for every $p > 1$. $\square$

Theorem 3.2. *Let $\gamma > 1$ and assume that $f \in W^{1,\infty}(\mathbb{R}^N)$.*

Then the following statements hold.

1. *If u is a continuous solution to* (3.1)*, then there exists a constant $K > 0$, depending on $\|f\|_{L^\infty(Q)}$, $\|\nabla f\|_{L^\infty(Q;\mathbb{R}^N)}$ and $|\lambda|$, such that*
$$\|\nabla u\|_{L^\infty(Q)} \le K.$$
2. *There exists a unique constant $\lambda \in \mathbb{R}$ such that* (3.1) *has a periodic solution $u \in W^{1,\infty}(\mathbb{R}^N)$ and*
$$\lambda = \sup\{c \in \mathbb{R} \ \ s.t. \ \ \exists u \in W^{1,\infty}(\mathbb{R}^N) \ s.t. \ (-\Delta^s)u + H(\nabla u) + c \le f(x)\}.$$
Moreover u is the unique Lipschitz viscosity solution to (3.1) *up to addition of constants.*

Finally, $u \in C^{2s+\alpha}(Q) \cap H^{2s}_p(Q)$, for every $\alpha < 1$ and every $p > 1$.

Proof. The a-priori estimates on the gradient is proved in [4, Theorem 3.1]. The existence of λ and of a unique (up to constants) viscosity solution to (3.1) is given in [4, Proposition 4.1].

As for the rest we proceed analogously as in Theorem 3.1. □

Remark 3.3 (Case $s \le \frac{1}{2}$)**.** We note that Theorem 3.2 holds for every $s \in (0,1)$. As for Theorem 3.1, it can be proved that, if $\gamma < 2s < 1$, solutions to (3.1) are actually Hölder continuous, with Hölder exponent striclty less than $\frac{2s-\gamma}{1-\gamma}$ (see Remark 1 in [3]). In this case, the uniqueness of the ergodic constant λ remains true, but it is not clear anymore that the solutions of the ergodic problem are unique up to an additive constant.

4. Regularizing coupling

The aim of this section is to prove existence of solutions for (1.1) in the case (1) of Theorem 1.1. For this, we consider the system

$$\begin{cases} (-\Delta)^s u + H(\nabla u) + \lambda = f[m](x), \\ (-\Delta)^s m - \operatorname{div}(m \nabla H(\nabla u)) = 0, \\ \int_Q m\, dx = 1, \end{cases} \tag{4.1}$$

where $f : C^\alpha(Q) \to W^{1,\infty}(Q)$, with $\alpha < 2s - 1$, is a regularizing functional. Let

$$X := \left\{ m \in C^\alpha(Q) : m \ge 0, \int_Q m\, dx = 1 \right\}. \tag{4.2}$$

We suppose that

$$f \text{ maps continuously } X \text{ into a bounded set of } W^{1,\infty}(Q). \tag{4.3}$$

A typical example of f satisfying (4.3) is $f[m](x) := g(x, K \star m(x))$, where $K : \mathbb{R}^N \to \mathbb{R}$ is a Lipschitz kernel and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Lipschitz function, which is $\mathbb{Z}^N$-periodic in x.

Theorem 4.1. *Assume that* (4.3) *holds. Then, there exists a classical solution* (u, λ, m) *to the mean field game system* (4.1)*.*

Proof. The statement follows by the Schauder Fixed Point Theorem in X (we will follow the lines of [16, Section 3]). More precisely, we construct a compact map $T : X \to X$, with $T(m) = \mu$, as follows.

For any $m \in X$, we consider the problem

$$(-\Delta)^s u + H(\nabla u) + \lambda = f[m](x). \tag{4.4}$$

By Theorem 3.2, since $f[m]$ is a Lipschitz function, we get that there exists a unique solution $(u, \lambda) \in W^{1,\infty}(Q) \times \mathbb{R}$. This implies, in particular, that $(-\Delta^s)u \in L^\infty(Q)$, so $u \in H^{2s}_p(Q)$ for all $p > 1$, thanks to [19]. Hence, by a bootstrap argument we get that $u \in C^{2s+\theta}(Q)$, for all $\theta < 1$.

Now, we observe that $\|\nabla H(\nabla u)\|_{L^\infty(Q)} \leq C$, with a constant $C > 0$ independent of m, in virtue of Theorem 3.2.

Let μ be the solution to

$$\begin{cases} (-\Delta)^s \mu - \operatorname{div}(\mu \nabla H(\nabla u)) = 0, \\ \int_Q \mu \, dx = 1. \end{cases} \tag{4.5}$$

By Lemma 2.4, there exists a unique μ solution to (4.5), and $\mu \in H^{2s-1}_p(Q)$ for all $p > 1$, with

$$\|\mu\|_{H^{2s-1}_p(Q)} \leq C\|\nabla H(\nabla u)\|_{L^\infty(Q)}.$$

Now, by Sobolev embedding, $\|\mu\|_{C^\beta(Q)}$ is bounded, for some $\alpha < \beta < 2s - 1$. So, $T : m \mapsto \mu$ is a compact mapping of X into itself.

Therefore, we only need to show that T is also continuous to conclude the existence of a fixed point, that in turn provides the existence of a solution to (4.1). This follows by stability of the equation in (4.4). Indeed, for a given $f[m]$, the couple solving the first equation in (4.4) is unique, if we impose for example $u(0) = 0$ (see Theorem 3.2). $\square$

5. Local bounded coupling

Here we prove Theorem 1.1 under the assumptions of case (2). For this, we now specify the setting in which we work.

We assume that $f : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ is a continuous function, $\mathbb{Z}^N$-periodic in x, that is $f(x + z, m) = f(x, m)$ for all $z \in \mathbb{Z}^N$, all $x \in \mathbb{R}^N$ and all $m \in [0, +\infty)$. Moreover, we assume that there exists $K > 0$ such that

$$|f(x, m)| \leq K \quad \forall m \geq 0. \tag{5.1}$$

We also suppose that

$$1 < \gamma \leq 2s. \tag{5.2}$$

In this framework, we get the following existence result, based on a regularization argument and on the existence result given in Theorem 4.1.

Theorem 5.1. *Under assumptions* (5.1) *and* (5.2)*, there exists a classical solution* (u, λ, m) *to the mean field game system* (1.1).

Proof. We consider the following regularization of the system (1.1):

$$\begin{cases} (-\Delta)^s u + H(\nabla u) + \lambda = f_\varepsilon[m](x), \\ (-\Delta)^s m - \operatorname{div}(m \nabla H(\nabla u)) = 0, \\ \int_Q m\, dx = 1, \end{cases} \tag{5.3}$$

where

$$f_\varepsilon[m](x) = f(x, m \star \chi_\varepsilon) \star \chi_\varepsilon(x) = \int_Q \chi_\varepsilon(x-y) f\left(y, \int_Q m(z)\chi_\varepsilon(y-z)dz\right) dy$$

and χ_ε, for $\varepsilon > 0$, is a sequence of standard mollifiers.

Note that f_ε satisfies assumption (4.3) (see e.g. [16, Example 5]), and therefore, for every $\varepsilon > 0$, there exists a classical solution $(u_\varepsilon, \lambda_\varepsilon, m_\varepsilon)$ to (5.3), thanks to Theorem 4.1.

Now, let $\overline{x}_\varepsilon$ and $\underline{x}_\varepsilon$ be such that $u_\varepsilon(\overline{x}_\varepsilon) = \max u_\varepsilon$ and $u_\varepsilon(\underline{x}_\varepsilon) = \min u_\varepsilon$. Evaluating the Hamilton-Jacobi equations at these points, we get

$$f_\varepsilon[m_\varepsilon](\underline{x}_\varepsilon) - C_H^{-1} \le \lambda_\varepsilon \le f_\varepsilon[m_\varepsilon](\overline{x}_\varepsilon) + C_H^{-1},$$

where C_H is the constant given in (1.2). This and (5.1) imply that $|\lambda_\varepsilon| \le \tilde{C}$, for some $\tilde{C} > 0$. So, up to passing to a subsequence, we can assume that $\lambda_\varepsilon \to \lambda$, as $\varepsilon \to 0$.

Again by assumption (5.1), using Theorem 3.1, we get that there exists $C > 0$, independent of ε, such that

$$\|\nabla u_\varepsilon\|_{L^\infty(Q)} \le C. \tag{5.4}$$

Hence, since u_ε solves (5.3), we get that $(-\Delta)^s u_\varepsilon$ is uniformly bounded in $L^\infty(Q)$, and so $u_\varepsilon \in H_p^{2s}(Q)$, for every $p > 1$, and $\|u_\varepsilon\|_{H_p^{2s}(Q)} \le C$, with $C > 0$ independent of ε.

Therefore, by Sobolev embedding, we obtain also that the sequence u_ε is equibounded in $C^{1+\alpha}(Q)$, for some $\alpha \in (0,1)$.

Moreover, the estimate in (5.4) and (1.2) imply that

$$\|\nabla H(\nabla u_\varepsilon)\|_{L^\infty(Q)} \le C,$$

for some constant $C > 0$. Then, we are in the position to apply Proposition 2.4, and conclude that, for all $p > 1$ and $\alpha \in (0, 2s-1)$, there exist constant C_1, $C_2 > 0$, depending on K and on p and α, respectively, such that

$$\|m_\varepsilon\|_{H_p^{2s-1}(Q)} \le C \qquad \text{and} \qquad \|m_\varepsilon\|_{C^\alpha(Q)} \le C'. \tag{5.5}$$

This implies that, up to subsequences, $m_\varepsilon \to m$ in $H_p^{2s-1}(Q)$, as $\varepsilon \to 0$, for all $p > 1$ (and also uniformly in $C^\alpha(Q)$ for every $\alpha < 2s - 1$, thanks to Sobolev embeddings). Therefore $f_\varepsilon[m_\varepsilon](x)$ is equibounded in $C^\alpha(Q)$, for every $\alpha < 2s-1$. Since u_ε solves (5.3), this, in turn, gives that u_ε are equibounded in $C^{2s+\alpha}(Q)$, for some $\alpha \in (0,1)$. Therefore, by Ascoli-Arzelà Theorem, we can extract a converging subsequence $u_\varepsilon \to u$ in $C^{2s}(Q)$, as $\varepsilon \to 0$.

Note that the convergences obtained are sufficiently strong to pass to the limit in the equations, and so we conclude that (u, λ, m) is a classical solution to (1.1). This completes the proof of Theorem 5.1. □

6. Local unbounded coupling

As in Section 5, we consider the case in which the coupling f is local, that is, we suppose that $f : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ is locally Lipschitz continuous in both variables, and $\mathbb{Z}^N$-periodic in x, namely $f(x+z, m) = f(x, m)$ for all $z \in \mathbb{Z}^N$, all $x \in \mathbb{R}^N$ and all $m \in [0, +\infty)$.

Differently from Section 5, here f can be unbounded, both from above and from below. Nevertheless, we have to restrict the condition on the growth γ of the Hamiltonian H. In particular, we assume that there exist $C > 0$ and $K > 0$ such that

$$- Cm^{q-1} - K \le f(x,m) \le Cm^{q-1} + K, \qquad \text{with } 1 < q < 1 + \frac{(2s-1)}{N}\frac{\gamma}{\gamma - 1},$$

$$\text{and } \begin{cases} 1 < \gamma < \frac{N}{N-2s+1} & \text{if } N > 1, \\ 1 < \gamma \le 2s & \text{if } N = 1. \end{cases} \tag{6.1}$$

Note that if $N > 1$ then $\frac{N}{N-2s+1} < 2s$, in virtue of (6.1).

We also remark that the bound on γ in the case $N > 1$ is just a technical assumption, that could be removed if the a priori bounds on the gradients of solutions to fractional Hamilton-Jacobi equations with coercive Hamiltonian stated in Theorem 3.1 could be shown to be independent on the L^∞ norm of the solutions.

To provide existence of a solution to (1.1) in this setting, we follow the variational approach, see [11, 14, 17, 22]. More precisely, we associate to the mean field game system an energy whose minimizers will be used to construct solutions to (1.1).

6.1. The energy associated to the system

We denote by $\tilde{L}$ the Legendre transform of H, i.e.

$$\tilde{L}(\mathscr{q}) := \sup_{\mathscr{p} \in \mathbb{R}^N} [\mathscr{p} \cdot \mathscr{q} - H(\mathscr{p})], \qquad \text{for any } \mathscr{q} \in \mathbb{R}^N.$$

Observe that, by (1.2), there exists $C_L > 0$ such that

$$C_L|\mathscr{q}|^{\gamma'} - C_L^{-1} \le L(\mathscr{q}) \le C_L^{-1}(|\mathscr{q}|^{\gamma'} + 1), \quad \text{for any } v \in \mathbb{R}^N, \tag{6.2}$$

where $\gamma' = \frac{\gamma}{\gamma-1}$ is the conjugate exponent of γ. Note that, by assumption (6.1),

$$\gamma' > \frac{N}{2s-1}. \tag{6.3}$$

We let

$$\begin{aligned}\mathscr{K} := \Big\{ &(m,w) \in L^1(Q) \cap H^{2s-1}_{\gamma'}(Q) \times L^{\gamma'}(Q) \ \text{ s.t.} \\ &\int_Q m(-\Delta)^s \varphi \, dx = \int_Q w \cdot \nabla \varphi \, dx \quad \forall \varphi \in C^\infty(Q), \\ &\int_Q m \, dx = 1, \quad m \geq 0 \text{ a.e.} \Big\}.\end{aligned} \tag{6.4}$$

We associate to the mean field game (1.1) the following energy

$$\mathscr{E}(m,w) := \begin{cases} \displaystyle\int_Q mL\left(-\frac{w}{m}\right) + F(x,m)\, dx & \text{if } (m,w) \in \mathscr{K}, \\ +\infty & \text{otherwise,} \end{cases} \tag{6.5}$$

where

$$\begin{aligned} L\left(-\frac{w}{m}\right) &:= \begin{cases} \tilde{L}\left(-\frac{w}{m}\right) & \text{if } m > 0, \\ 0 & \text{if } m = 0, w = 0, \\ +\infty & \text{otherwise} \end{cases} \\ \text{and} \qquad F(x,m) &:= \begin{cases} \displaystyle\int_0^m f(x,n)\, dn & \text{if } m \geq 0, \\ +\infty & \text{if } m < 0. \end{cases} \end{aligned} \tag{6.6}$$

Note that, since $\tilde{L}$ is the Legendre transform of H, we have that, for all $m \geq 0$,

$$mH(p) = \sup_w \left[-p \cdot w - mL\left(-\frac{w}{m}\right) \right]. \tag{6.7}$$

Moreover, recalling (6.2), we get that

$$C_L \frac{|w|^{\gamma'}}{m^{\gamma'-1}} - C_L^{-1} m \leq L\left(-\frac{w}{m}\right) \leq C_L^{-1} \frac{|w|^{\gamma'}}{m^{\gamma'-1}} + C_L^{-1} m. \tag{6.8}$$

Now, we provide a-priori estimates for couples $(m,w) \in \mathscr{K}$ with finite energy.

Proposition 6.1. *Assume that $(m,w) \in \mathscr{K}$ such that there exists $K > 0$ with*

$$E := \int_Q \frac{|w|^{\gamma'}}{m^{\gamma'-1}} dx \leq K.$$

Then, there exist $\delta > 0$ and $C > 0$ such that

$$\|m\|^{q(1+\delta)}_{L^q(Q)} \leq C \int_Q \frac{|w|^{\gamma'}}{m^{\gamma'-1}} dx \leq CK \tag{6.9}$$

where q is as in (6.1).

Moreover, for every $\alpha \in \left(0, 2s-1-\frac{N}{\gamma'}\right)$, there exists a constant $C > 0$, depending on α, such that

$$\|m\|_{C^\alpha(Q)} \leq C \int_Q \frac{|w|^{\gamma'}}{m^{\gamma'-1}} dx \leq CK. \tag{6.10}$$

Proof. Note that since $m \in H^{2s-1}_{\gamma'}(Q)$ and γ' satisfies (6.3), then $m \in L^p(Q)$, for every $p \in (1, +\infty]$. Moreover, by Sobolev embeddings, we have that $m \in C^\alpha(Q)$, for every $\alpha \in \left(0, 2s-1-\frac{N}{\gamma'}\right)$.

Now, let $p > 1$ and define r_p as follows

$$\frac{1}{r_p} = \frac{1}{\gamma'} + \left(1 - \frac{1}{\gamma'}\right)\frac{1}{p}. \tag{6.11}$$

In this way, we see that $r_p < \min\{p, \gamma'\}$.

By (6.4), we have that

$$\begin{aligned}\int_Q m(-\Delta)^s \phi\, dx &= \int_Q w \cdot \nabla\phi\, dx \leq \int_Q \left(\frac{|w|^{\gamma'}}{m^{\gamma'-1}}\right)^{\frac{1}{\gamma'}} m^{\frac{1}{\gamma}} |\nabla\phi|\, dx \\ &\leq \left(\int_Q \frac{|w|^{\gamma'}}{m^{\gamma'-1}}\, dx\right)^{\frac{1}{\gamma}} \|m\|^{\frac{1}{\gamma}}_{L^p(Q)} \|\nabla\phi\|_{L^{r'_p}(Q)} \leq E^{\frac{1}{\gamma'}} \|m\|^{\frac{1}{\gamma}}_{L^p(Q)} \|\nabla\phi\|_{L^{r'_p}(Q)},\end{aligned}$$

for any $\phi \in C^\infty(Q)$. Here above we used the notation $r'_p = \frac{r_p}{r_p-1}$.

Therefore, by Lemma 2.3 we get that

$$\|(-\Delta)^{s-\frac{1}{2}} m\|_{L^{r_p}(Q)} \leq C E^{\frac{1}{\gamma'}} \|m\|^{\frac{1}{\gamma}}_{L^p(Q)}, \tag{6.12}$$

for some $C > 0$. Moreover, by interpolation, we get that

$$\|m\|_{L^{r_p}(Q)} \leq \|m\|^{\frac{1}{\gamma}}_{L^p(Q)} \|m\|^{\frac{1}{\gamma'}}_{L^1(Q)} = \|m\|^{\frac{1}{\gamma}}_{L^p(Q)}. \tag{6.13}$$

From (6.12) and (6.13), we conclude that

$$\|m\|_{H^{2s-1}_{r_p}(Q)} \leq C(E^{\frac{1}{\gamma'}} + 1)\|m\|^{\frac{1}{\gamma}}_{L^p(Q)}. \tag{6.14}$$

Now, we prove (6.9). For this, let $r = r_q$, that is, in (6.11) we choose $p = q$, where q is as in (6.1). Let $r^\star$ be such that

$$\frac{1}{r^\star} = \frac{1}{r} - \frac{2s-1}{N} \qquad \text{if } r < \frac{N}{2s-1},$$

$$\text{and} \qquad r^\star = +\infty \qquad \text{if } r \geq \frac{N}{2s-1}.$$

Notice that by (6.1), it is easy to see that

$$q < r^\star. \tag{6.15}$$

Therefore, by Sobolev embedding, there exists $C > 0$ such that

$$\|m\|_{H_r^{2s-1}(Q)} \geq C\|m\|_{L^q(Q)},$$

and so, substituting in (6.14) we get that

$$\|m\|_{L^q(Q)} \leq C(E+1) \qquad \text{and} \qquad \|m\|_{H_r^{2s-1}(Q)} \leq C(E+1).$$

Note that, in virtue of (6.15), by interpolation and using (6.14), we get

$$\|m\|_{L^q(Q)} \leq \|m\|_{L^1(Q)}^{1-\theta}\|m\|_{L^{r^\star}(Q)}^{\theta} \leq \|m\|_{L^1(Q)}^{1-\theta}\|m\|_{H_r^{2s-1}(Q)}^{\theta} \leq C(1+E^{\frac{\theta}{\gamma'}})\|m\|_{L^q(Q)}^{\frac{\theta}{\gamma}},$$

where θ is such that

$$\frac{1}{q} = 1-\theta+\frac{\theta}{r^\star}.$$

It is easy to check that

$$\frac{1}{\theta} = 1-\frac{1}{\gamma'}+\frac{2s-1}{N}\frac{q}{q-1}. \tag{6.16}$$

We then obtain that

$$\|m\|_{L^q(Q)}^{\left(1-\frac{\theta}{\gamma}\right)\frac{\gamma'}{\theta}} \leq C(1+E). \tag{6.17}$$

Using (6.16) we check that

$$\left(1-\frac{\theta}{\gamma}\right)\frac{\gamma'}{\theta} = \gamma'\frac{2s-1}{N}\frac{q}{q-1} = (1+\delta)q,$$

where

$$\delta = \frac{1}{q-1}\left(\frac{\gamma'(2s-1)+N}{N}-q\right) > 0$$

thanks to (6.1). This and (6.17) imply (6.9), as desired.

We prove now (6.10). In virtue of (6.3), we can choose p in (6.11) sufficiently large such that

$$\frac{N}{2s-1} < r_p < \gamma'.$$

So, from (6.14), using Sobolev embeddings and reasoning as above, we obtain that

$$\|m\|_{L^p(Q)} \leq C(E+1) \qquad \text{and} \qquad \|m\|_{H_{r_p}^{2s-1}(Q)} \leq C(E+1). \tag{6.18}$$

From the second estimate in (6.18) and Sobolev embeddings, we obtain (6.10). This completes the proof of Proposition 6.1. □

Using the previous estimates in Proposition 6.1, we deduce the existence of a minimizer of the energy in the class $\mathscr{K}$ introduced in (6.4).

Theorem 6.2. *There exists* $(m,w) \in \mathscr{K}$ *such that*

$$\mathscr{E}(m,w) = \min_{(m,w)\in\mathscr{K}} \mathscr{E}\,.$$

Proof. First of all observe that, by Proposition 6.1 and (6.8), there exists $C > 0$ such that, for every $(m,w) \in \mathscr{K}$,

$$\mathscr{E}(m,w) \geq C\|m\|_{L^q(Q)}^{(1+\delta)q} - C + \int_Q F(x,m)dx\,.$$

From this, recalling assumption (6.1) and the definition of F in (6.6), we conclude that there exists a constant K, depending on q, such that

$$\mathscr{E}(m,w) \geq C\|m\|_{L^q(Q)}^{(1+\delta)q} - C'\|m\|_{L^q(Q)}^q - C' \geq K.$$

Let $e := \inf_{(m,w)\in\mathscr{K}} \mathscr{E}(m,w)$. We fix a minimizing sequence (m_n,w_n). Therefore $\mathscr{E}(m_n,w_n) \leq e+1$, for every n sufficiently large. Note that by our definition of L this implies that $w_n = 0$ where $m_n = 0$.

Therefore, again by assumption (6.1), (6.8) and Proposition 6.1, we get

$$\begin{aligned}
\int_Q \frac{|w_n|^{\gamma'}}{m_n^{\gamma'-1}}\,dx &\leq C_L^{-1}\left(e+1-\int F(x,m_n)dx\right)\\
&\leq C_L^{-1}\left(e+1-C+C\|m_n\|_{L^q(Q)}^q\right)\\
&\leq C_L^{-1}\left(e+1+C'+K\left(\int_Q \frac{|w_n|^{\gamma'}}{m_n^{\gamma'-1}}\,dx+1\right)^{\frac{1}{1+\delta}}\right).
\end{aligned}$$

This implies in particular that $\left(\int_Q \frac{|w_n|^{\gamma'}}{m_n^{\gamma'-1}}\,dx\right)$ is equibounded in n.

By (6.10), this implies that $\|m_n\|_{C^\alpha(Q)} \leq C$, for some $\alpha \in (0,1)$. Therefore, up to a subsequence,

$$m_n \to m \quad \text{uniformly in } Q, \text{ as } n \to +\infty.$$

Therefore, we have that $m_n \to m$ in $L^1(Q)$ and $\int_Q m dx = 1$.

In particular, we see that $0 \leq m_n \leq C$, for every n, and then

$$\int_Q |w_n|^{\gamma'}dx \leq C^{\gamma'-1}\int_Q \frac{|w_n|^{\gamma'}}{m_n^{\gamma'-1}}\,dx.$$

This implies that w_n is equibounded in $L^{\gamma'}(Q)$ and so, up to a subsequence,

$$w_n \to w \quad \text{weakly in } L^{\gamma'}(Q), \text{ as } n \to +\infty.$$

In particular m_n is equibounded in $H_{\gamma'}^{2s-1}$ by Lemma 2.2.

Note that the convergences are strong enough to assure that $(m,w) \in \mathscr{K}$. Then, the desired result follows from the lower semicontinuity of the kinetic part of the functional and by the strong convergence in $L^q(Q)$ of m_n. □

6.2. Existence of solutions to the mean field game system

In order to construct a solution to the mean field game (1.1), we associate to the energy in (6.5) a dual problem, using standard arguments in convex analysis. For additional details see also [24]. First of all, following [11], we pass to a convex problem.

Given a minimizer $(\bar{m}, \bar{w})$ as obtained in Theorem 6.2, we introduce the following functional

$$J(m,w) := \int_Q mL\left(-\frac{w}{m}\right) + f(x,\bar{m})m\,dx. \tag{6.19}$$

We claim that for $(m,w) \in \mathscr{K}$ we have that

$$\int_Q mL\left(-\frac{w}{m}\right)dx - \int_Q \bar{m}L\left(-\frac{\bar{w}}{\bar{m}}\right) \geq -\int_Q f(x,\bar{m})(m-\bar{m})dx.$$

This can be proved as in [11, Proposition 3.1], using the convexity of L and the regularity of F. The idea is to consider, for every $\lambda \in (0,1)$, $m_\lambda := \lambda m + (1-\lambda)\bar{m}$ and the same definition for w_λ, and to observe that by minimality

$$\int_Q m_\lambda L\left(-\frac{w_\lambda}{m_\lambda}\right)dx - \int_Q \bar{m}L\left(-\frac{\bar{w}}{\bar{m}}\right) \geq -\int_Q F(x,\bar{m}_\lambda) - F(x,\bar{m})dx\,.$$

Then, using the convexity to estimate the left hand side and the regularity of F on the right hand side, and finally sending $\lambda \to 0$, we get that

$$\min_{(m,w)\in\mathscr{K}} J(m,w) = J(\bar{m},\bar{w}).$$

Now we complete the proof of Theorem 1.1, by showing the last point (3).

Theorem 6.3. *Let $(\bar{m},\bar{w})$ be a minimizer of J as given by Theorem 6.2.*

Then, $\bar{m} \in H^{2s-1}_p(Q)$, for all $p > 1$, and there exist $\lambda \in \mathbb{R}$ and $u \in C^{2s+\alpha}(Q)$, such that $(u,\lambda,\bar{m})$ is a classical solution to the mean field game (1.1). Finally $\bar{w} = -\bar{m}\nabla H(\nabla u)$.

Proof. The functional in (6.19) is convex, so we can write the dual problem as follows, following standard arguments in convex analysis. Recall that $\bar{m} \in C^\alpha(Q)$, for any $\alpha \in \left(0, 2s-1-\frac{N}{\gamma'}\right)$, thanks to Proposition 6.1.

Now, we consider the following functional

$$\mathscr{A}(m,w,u,c) := \int_Q \left[mL\left(-\frac{w}{m}\right) + f(x,\bar{m})m - m(-\Delta)^s u + \nabla u \cdot w - cm\right]dx + c.$$

It is easy to observe that

$$J(\bar{m},\bar{w}) = \inf_{\{(m,w)\in H^{2s-1}_{\gamma'}(Q)\times L^{\gamma'}(Q),\ \int_Q m=1,\ m\geq 0\}} \sup_{(u,c)\in C^{2s}(Q)\times\mathbb{R}} \mathscr{A}(m,w,u,c), \tag{6.20}$$

so the infimum is actually a minimum.

Note that $\mathscr{A}(\cdot,\cdot,u,c)$ is convex and weakly lower semicontinuous, and $\mathscr{A}(m,w,\cdot,\cdot)$ is linear (so in particular concave). Hence we can use the min-max Theorem, see in particular [10, Thm 2.3.7], to interchange minimum and supremum, that is

$$\begin{aligned} &\min_{\{(m,w)\in H^{2s-1}_{\gamma'}\times L^{\gamma'},\ \int m=1,\ m\geq 0\}} \sup_{(u,c)\in C^{2s}\times\mathbb{R}} \mathscr{A}(m,w,u,c) \\ &= \sup_{(u,c)\in C^{2s}\times\mathbb{R}} \min_{\{(m,w)\in H^{2s-1}_{\gamma'}\times L^{\gamma'},\ \int m=1,\ m\geq 0\}} \mathscr{A}(m,w,u,c). \end{aligned} \tag{6.21}$$

Finally, thanks to the Rockafellar's Interchange Theorem [26] between infimum and integral (based on measurable selection arguments and the lower semicontinuity of the functional) we get, using the fact that H is the Legendre transform of L, that

$$\begin{aligned} &\min_{\{(m,w)\in H^{2s-1}_{\gamma'}\times L^{\gamma'},\ \int m=1,\ m\geq 0\}} \mathscr{A}(m,w,u,c) \\ &= \int_Q \min_{m\geq 0,w} \left[mL\left(-\frac{w}{m}\right) + f(x,\bar{m})m - m(-\Delta)^s u + \nabla u\cdot w - cm\right] dx + c \\ &= \int_Q \min_{m\geq 0} m\left[-H(\nabla u) - (-\Delta)^s u + f(x,\bar{m}) - c\right] dx + c. \end{aligned}$$

Note that

$$\min_{m\geq 0} m\left[-(-\Delta)^s u - H(\nabla u) + f(x,\bar{m}) - c\right] = \begin{cases} 0, & \text{if } -(-\Delta)^s u - H(\nabla u) + f(x,\bar{m}) - c \geq 0, \\ -\infty, & \text{if } -(-\Delta)^s u - H(\nabla u) + f(x,\bar{m}) - c < 0. \end{cases}$$

Therefore, from (6.20), (6.21) and (6.22) we get that

$$\begin{aligned} J(\bar{m},\bar{w}) &= \sup_{(u,c)\in C^{2s}\times\mathbb{R}} \int_Q \min_{m\geq 0} m\left[-H(\nabla u) - (-\Delta)^s u + f(x,\bar{m}) - c\right] dx + c \\ &= \sup\left\{c\in\mathbb{R}\ |\exists u\in C^{2s},\ \text{s.t. } (-\Delta)^s u + H(\nabla u) + c \leq f(x,\bar{m})\right\}. \end{aligned} \tag{6.22}$$

Due to Theorem 3.1, such supremum is actually a maximum: there exist $\lambda\in\mathbb{R}$ and a periodic function $u\in C^{2s+\alpha}(Q)\cap H^{2s}_p(Q)$, for every $\alpha < 2s-1-\frac{N}{\gamma'}$ and every $p>1$, which is unique up to additive constants and solves

$$(-\Delta)^s u + H(\nabla u) + \lambda = f(x,\bar{m}). \tag{6.23}$$

So, equality (6.20) reads

$$\lambda = J(\bar{m},\bar{w}) = \int_Q \bar{m}\left[L\left(-\frac{\bar{w}}{\bar{m}}\right) + f(x,\bar{m})\right] dx.$$

Therefore, recalling that $\int_Q \bar{m} = 1$ and using both (6.23) and (2.3) with test function u, we obtain that

$$\begin{aligned}
0 &= \int_Q \bar{m}\left[L\left(-\frac{\bar{w}}{\bar{m}}\right) + f(x,\bar{m}) - \lambda\right] dx \\
&= \int_Q \bar{m}\left[L\left(-\frac{\bar{w}}{\bar{m}}\right) + (-\Delta)^s u + H(\nabla u)\right] dx \\
&= \int_Q \bar{m}\left[L\left(-\frac{\bar{w}}{\bar{m}}\right) + \nabla u \cdot \frac{\bar{w}}{\bar{m}} + H(\nabla u)\right] dx.
\end{aligned}$$

Using the fact that H is the Legendre transform of L and (6.7), we thus conclude that

$$\frac{\bar{w}}{\bar{m}} = -\nabla H(\nabla u),$$

where $\bar{m} \neq 0$. Moreover, by the definition of L, we get that $\bar{w} = 0$ where $\bar{m} = 0$.

In particular, recalling (2.3), we find that $\bar{m}$ is a solution of

$$(-\Delta)^s m - \operatorname{div}(m\nabla H(\nabla u)) = 0, \qquad \text{with} \qquad \int_Q m = 1.$$

Since $\bar{m}\nabla H(\nabla u) \in L^\infty(Q)$, by Lemmata 2.2 and 2.3, we get that $\bar{m} \in H_p^{2s-1}(Q)$ for every $p > 1$. This implies that $(u, \bar{m})$ is a solution to (1.1). The proof of Theorem 6.3 is thus complete. □

7. Further properties: improved regularity and uniqueness

If we assume some more regularity on f and H, we can obtain more regular solutions.

Proposition 7.1. *Assume that $H \in C^{1+k}(\mathbb{R}^N)$, for some $k \geq 1$, and, in the local case (under assumptions (5.1) or (6.1)), that $f \in C^k(\mathbb{R}^N \times \mathbb{R})$ or, in the nonlocal case (under assumption (4.3)), that f maps continuously X (as defined in (4.2)) in a bounded subset of $C^k(Q)$.*

Then, the system in (1.1) admits a classical solution $(u, \lambda, m) \in C^k(Q) \times \mathbb{R} \times C^{k-1}(Q)$.

Proof. By Theorems 4.1, 5.1 and 6.3, we have a solution $(u, \lambda, m) \in (C^{2s+\alpha}(Q) \cap H_p^{2s}(Q)) \times \mathbb{R} \times H_p^{2s-1}(Q)$.

Using the regularity of m and $\nabla H(\nabla u)$ and the fact that both are in L^∞, we get that $m\nabla H(\nabla u)) \in H_p^{2s-1}(Q)$ for all $p > 1$. This implies by Lemma 2.2 that $m \in H_p^{4s-2}(Q)$ for every $p > 1$.

By the regularity of f, if $k \geq 4s - 2$, also $f(m) \in H_p^{4s-2}(Q)$. Therefore, using the Hamilton-Jacobi equation, we get that

$$(-\Delta)^s u \in H_p^{2s-1}(Q)$$

for every $p > 1$, which gives that $u \in H_p^{4s-1}(Q)$.

Reasoning as above we obtain that $m\nabla H(\nabla u)) \in H_p^{4s-2}(Q)$ for all $p > 1$, and then by Lemma 2.2 we conclude that $m \in H_p^{6s-3}(Q)$ for every $p > 1$.

We iterate the argument up to arriving to $m \in C^{M(2s-1)}(Q)$, where $M := \left[\frac{k}{2s-1}\right]$ (that is, M is the integer part of $\frac{k}{2s-1}$). In particular $m \in C^{k-1}(Q)$. So we conclude that $u \in C^{2s+N(2s-1)}(Q)$. □

It is well known that, under a monotonicity condition on the function f, we have uniqueness of solutions to mean field games system.

Theorem 7.2. *Assume that in the local case (under assumptions* (5.1) *or* (6.1)*) the map $m \mapsto f(x,m)$ is increasing for all $x \in Q$ or in the nonlocal case (under assumption* (4.3)*)*

$$\int_Q (f[m_1](x) - f[m_2](x))(m_1 - m_2)dx > 0, \qquad \textit{for any } m_1, m_2 \in X.$$

Then, the system in (1.1) *admits a unique classical solution (u, λ, m), where u is defined up to addition of constants.*

Proof. The argument is standard, see [22], and the adaptation to the fractional case is straightforward. □

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