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Original
Short intervals asymptotic formulae for binary problems with primes and powers, II: density 1 / Languasco, Alessandro; Zaccagnini, Alessandro. - In: MONATSHEFTE FÜR MATHEMATIK. - ISSN 0026-9255. - 181:2(2016), pp. 419-435. [10.1007/s00605-015-0871-z]

Availability:
This version is available at: 11381/2819798 since: 2019-01-03T09:53:53Z

Publisher:
Springer-Verlag Wien

Published
DOI:10.1007/s00605-015-0871-z

SHORT INTERVALS ASYMPTOTIC FORMULAE FOR BINARY PROBLEMS WITH PRIMES AND POWERS, II: DENSITY 1

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Abstract. We prove that suitable asymptotic formulae in short intervals hold for the problems of representing an integer as a sum of a prime square and a square, or a prime square. Such results are obtained both assuming the Riemann Hypothesis and in the unconditional case.

1. Introduction

In this second paper devoted to study asymptotic formulae in short intervals for additive problems with primes and squares, we focus our attention on density-one problems, *i.e.*, on representing integers as sum of two squares. We considered the case of the sum of a prime and a square in our paper [5].

We will consider two separate cases depending on the number of prime squares involved in the summations. Let $\varepsilon > 0$, N be a sufficiently large integer and let further H be an integer such that $N^\varepsilon < H = o(N)$ as $N \to \infty$. Taking $n \in [N, N+H]$, the key quantities are

$$r''_{2,2}(n) = \sum_{p_1^2+p_2^2=n} \log p_1 \log p_2 \quad \text{and} \quad r'_{2,2}(n) = \sum_{p^2+m^2=n} \log p.$$

Since it is well known that the expected behaviour of such functions is erratic, to work in a more regular situation we will study their average asymptotics over a suitable short interval.

We have the following results which extend and improve the ones cited in the Introduction of the paper by Daniel [1]. We write $f = \infty(g)$ for $g = o(f)$.

Theorem 1. *Assume the Riemann Hypothesis (RH) holds. Then*

$$\sum_{n=N+1}^{N+H} r''_{2,2}(n) = \frac{\pi}{4} H + \mathcal{O}\Big(\frac{H^2}{N} + H^{1/2} N^{1/4} (\log N)^{3/2}\Big)$$

as $N \to \infty$ uniformly for $\infty(N^{1/2}(\log N)^3) \le H \le o(N)$.

Theorem 2. *Let $\varepsilon > 0$. Then there exists a constant $C = C(\varepsilon) > 0$ such that*

$$\sum_{n=N+1}^{N+H} r''_{2,2}(n) = \frac{\pi}{4} H + \mathcal{O}\Big(H \exp\Big(-C\Big(\frac{\log N}{\log\log N}\Big)^{1/3}\Big)\Big)$$

as $N \to \infty$ uniformly for $N^{7/12+\varepsilon} \le H \le N^{1-\varepsilon}$.

We remark that Plaksin [7] (see Lemma 11 there) proves the case $H = N$ of Theorem 2 with a stronger error term of the form $N \exp(-C(\log N)^{1/2})$. Following its proof it is clear that it can be further improved to $N \exp(-C(\log N)^{3/5}(\log\log N)^{-1/5})$. The comparative weakness of our error term is due to the use of the zero-density estimates for the Riemann zeta-function (we need

2010 *Mathematics Subject Classification.* Primary 11P32; Secondary 11P55, 11P05.
Key words and phrases. Waring-Goldbach problem, Hardy-Littlewood method.

them to be able to get a short interval result). A direct trial following the lines of Lemma 11 of Plaksin [7] leads to weaker uniformity ranges: $H \gg N^{3/4}L^A$, for some $A > 0$, assuming RH and $H \gg N^{7/24+1/2+\varepsilon}$ unconditionally. Here $L = \log N$.

Concerning the sum of a prime square and a square, we have

Theorem 3. *Assume the Riemann Hypothesis holds. Then*

$$\sum_{n=N+1}^{N+H} r'_{2,2}(n) = \frac{\pi}{4}H + \mathcal{O}\Big(\frac{H^2}{N} + \frac{H\log\log N}{(\log N)^{1/2}}\Big)$$

as $N \to \infty$ *uniformly for* $\infty(N^{1/2}(\log N)^2) \le H \le o(N)$.

Theorem 4. *Let* $\varepsilon > 0$. *Then there exists a constant* $C = C(\varepsilon) > 0$ *such that*

$$\sum_{n=N+1}^{N+H} r'_{2,2}(n) = \frac{\pi}{4}H + \mathcal{O}\Big(H\exp\Big(-C\Big(\frac{\log N}{\log\log N}\Big)^{1/3}\Big)\Big)$$

as $N \to \infty$ *uniformly for* $N^{7/12+\varepsilon} \le H \le N^{1-\varepsilon}$.

An argument similar to the proof Lemma 11 of Plaksin [7] proves the case $H = N$ of Theorem 4 with a stronger error term of the form $N\exp(-C(\log N)^{3/5}(\log\log N)^{-1/5})$. As in the previous case, the comparative weakness of our error term is due to the use of the zero-density estimates for the Riemann zeta-function. A direct trial following the lines of Lemma 11 of Plaksin [7] leads to weaker uniformity ranges: $H \gg N^{3/4}L^A$, for some $A > 0$, assuming RH and $H \gg N^{7/24+1/2+\varepsilon}$ unconditionally.

Concerning the problem about the sum of two squares, i.e. the asymptotic formula for

$$r_{2,2}(n) = \sum_{m_1^2+m_2^2=n} 1,$$

our method leads to a weaker result than the one that follows from the well-known formula $\sum_{n=1}^{N} r_{2,2}(n) = \frac{\pi}{4}N - N^{1/2} + \mathcal{O}(N^{\alpha})$, with $\alpha \in (1/4, 1/3)$.

In the proofs we will use the original Hardy-Littlewood circle method setting. This depends on the fact in the standard finite sums method the approximation needed to detect the main term contribution leads to an error term which is under control essentially only for $H > N^{2/3+\varepsilon}$, see also Remark 1 at the bottom of the proof of Theorem 2.

Acknowledgements. This research was partially supported by the grant PRIN2010-11 *Arithmetic Algebraic Geometry and Number Theory*. We wish to thank the referee for his/her remarks.

2. Definitions and Lemmas

Let $L = \log N$ and $\ell \ge 1$ be an integer. We define

$$\widetilde{S}_\ell(\alpha) = \sum_{n=1}^{\infty} \Lambda(n)e^{-n^\ell/N}e(n^\ell\alpha), \quad z = 1/N - 2\pi i\alpha \quad \text{and} \quad U(\alpha, H) = \sum_{1\le m\le H} e(m\alpha), \tag{1}$$

where $e(\alpha) = e^{2\pi i\alpha}$. From now on, we denote

$$\widetilde{E}_\ell(\alpha) := \widetilde{S}_\ell(\alpha) - \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}}. \tag{2}$$

We will also need the following unconditional version of Lemma 3 of [4]; the proof is essentially the same used there and so we skip part of the argument. We just repeat the definition of the main quantities involved and write how to use the zero-density estimates to conclude the proof. In fact all of the following lemmas will be used just for $\ell = 1, 2$ but we take this occasion to describe the general case.

Lemma 1. *Let ε be an arbitrarily small positive constant, $\ell \geq 1$ be an integer and N be a sufficiently large integer. Then there exists a positive constant $c_1 = c_1(\varepsilon)$, which does not depend on ℓ, such that*

$$\int_{-\xi}^{\xi} |\widetilde{E}_\ell(\alpha)|^2 \,\mathrm{d}\alpha \ll_\ell N^{2/\ell-1} \exp\Big(-c_1\Big(\frac{L}{\log L}\Big)^{1/3}\Big)$$

uniformly for $0 \leq \xi < N^{-1+5/(6\ell)-\varepsilon}$.

Proof. Since $z^{-\rho/\ell} = |z|^{-\rho/\ell} \exp\big(-i(\rho/\ell)\arctan 2\pi N\alpha\big)$, by Stirling's formula we have that

$$\frac{1}{\ell}\sum_{\rho} z^{-\rho/\ell}\Gamma\Big(\frac{\rho}{\ell}\Big) \ll_\ell \sum_{\rho} |z|^{-\beta/\ell}|\gamma|^{\beta/\ell-1/2} \exp\Big(\frac{\gamma}{\ell}\arctan 2\pi N\alpha - \frac{\pi}{2\ell}|\gamma|\Big).$$

Recalling the Vinogradov-Korobov zero-free region, *i.e.*, there are no zeros $\beta+i\gamma$ of the Riemann zeta function having

$$\beta \geq 1 - \frac{c'}{(\log(|\gamma|+2))^{2/3}(\log\log(|\gamma|+2))^{1/3}} = 1-\delta(\gamma), \tag{3}$$

say, where $c' > 0$ is an absolute constant, for $|\alpha| \leq 1/N$ or $\gamma\alpha < 0$ we get

$$\begin{aligned}\sum_{\rho} z^{-\rho/\ell}\Gamma(\rho/\ell) &\ll N^{1/\ell}\sum_{\rho} N^{-\delta(\gamma)/\ell}|\gamma|^{1/\ell-1/2}\exp\Big(-C\frac{|\gamma|}{\ell}\Big)\\ &\ll_\ell N^{1/\ell}\sum_{\rho} N^{-\delta(\gamma)/\ell}\exp\Big(-C_1\frac{|\gamma|}{\ell}\Big) \ll_\ell N^{(1-\varepsilon)/\ell},\end{aligned}$$

where $C, C_1 > 0$ are absolute positive constants and $\varepsilon \in (0,1)$ is suitably small. Hence, by the explicit formula for $\widetilde{S}_\ell$ which is Lemma 2 of [4], we have

$$I(N,\xi,\ell) := \int_{-\xi}^{\xi} \Big|\widetilde{S}_\ell(\alpha) - \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}}\Big|^2 \mathrm{d}\alpha \ll_\ell N^{2(1-\varepsilon)/\ell}\xi \tag{4}$$

if $0 \leq \xi \leq 1/N$, and

$$I(N,\xi,\ell) \ll_\ell \int_{1/N}^{\xi} \Big|\sum_{\rho:\,\gamma>0} z^{-\rho/\ell}\Gamma\Big(\frac{\rho}{\ell}\Big)\Big|^2 \mathrm{d}\alpha + \int_{-\xi}^{-1/N} \Big|\sum_{\rho:\,\gamma<0} z^{-\rho/\ell}\Gamma\Big(\frac{\rho}{\ell}\Big)\Big|^2 \mathrm{d}\alpha + N^{2/\ell-1-2\varepsilon/\ell} \tag{5}$$

if $\xi > 1/N$. We will treat only the first integral on the right hand side of (5), the second being completely similar. Clearly

$$\int_{1/N}^{\xi} \Big|\sum_{\rho:\,\gamma>0} z^{-\rho/\ell}\Gamma\Big(\frac{\rho}{\ell}\Big)\Big|^2 \mathrm{d}\alpha = \sum_{k=1}^{K}\int_{\eta}^{2\eta} \Big|\sum_{\rho:\,\gamma>0} z^{-\rho/\ell}\Gamma\Big(\frac{\rho}{\ell}\Big)\Big|^2 \mathrm{d}\alpha + \mathcal{O}(1) \tag{6}$$

where $\eta = \eta_k = \xi/2^k$, $1/N \leq \eta \leq \xi/2$ and K is a suitable integer satisfying $K = \mathcal{O}(L)$. Writing $\arctan 2\pi N\alpha = \pi/2 - \arctan(1/2\pi N\alpha)$ and using the Saffari-Vaughan technique we have

$$\int_{\eta}^{2\eta} \Big|\sum_{\rho:\,\gamma>0} z^{-\rho/\ell}\Gamma\Big(\frac{\rho}{\ell}\Big)\Big|^2 \mathrm{d}\alpha \leq \int_1^2 \Big(\int_{\delta\eta/2}^{2\delta\eta} \Big|\sum_{\rho:\,\gamma>0} z^{-\rho/\ell}\Gamma\Big(\frac{\rho}{\ell}\Big)\Big|^2 \mathrm{d}\alpha\Big)\mathrm{d}\delta$$

$$= \sum_{\rho_1 \colon \gamma_1>0} \sum_{\rho_2 \colon \gamma_2>0} \Gamma\Big(\frac{\rho_1}{\ell}\Big)\overline{\Gamma\Big(\frac{\rho_2}{\ell}\Big)}\, e^{\frac{\pi}{2\ell}(\gamma_1+\gamma_2-i(\beta_1-\beta_2))} \cdot J, \qquad (7)$$

say, where

$$J = J(N,\eta,\ell,\beta_1,\beta_2,\gamma_1,\gamma_2) = \int_1^2 \Big(\int_{\delta\eta/2}^{2\delta\eta} f_1(\alpha) f_2(\alpha)\,\mathrm{d}\alpha\Big)\mathrm{d}\delta,$$

$$f_1(\alpha) = |z|^{-w}, \quad f_2(\alpha) = \exp\Big(-\frac{\gamma_1+\gamma_2-i(\beta_1-\beta_2)}{\ell}\arctan\frac{1}{2\pi N\alpha}\Big),$$

and $w = w(\ell,\beta_1,\beta_2,\gamma_1,\gamma_2) = (\beta_1+\beta_2)/\ell + (i/\ell)(\gamma_1-\gamma_2)$. Arguing exactly as in the proof of Lemma 3 in [4], see pages 6-7 there, we get

$$J \ll_\ell \eta^{1-(\beta_1+\beta_2)/\ell} \frac{1+\ (\frac{1+\gamma_1+\gamma_2}{N\eta})^2}{1+|\gamma_1-\gamma_2|^2} \exp\Big(-c\Big(\frac{\gamma_1+\gamma_2}{N\eta}\Big)\Big),$$

hence from (7) and Stirling's formula we have

$$\int_\eta^{2\eta} \Big|\sum_{\rho\colon \gamma>0} z^{-\rho/\ell}\Gamma\Big(\frac{\rho}{\ell}\Big)\Big|^2 \mathrm{d}\alpha \ll_\ell \sum_{\rho_1 \colon \gamma_1>0} \sum_{\rho_2 \colon \gamma_2>0} \eta^{1-(\beta_1+\beta_2)/\ell} \gamma_1^{\beta_1/\ell-1/2}\gamma_2^{\beta_2/\ell-1/2}$$

$$\times \frac{1+(\frac{1+\gamma_1+\gamma_2}{N\eta})^2}{1+|\gamma_1-\gamma_2|^2}\exp\Big(-c\Big(\frac{\gamma_1+\gamma_2}{N\eta}\Big)\Big). \qquad (8)$$

Sorting real and imaginary parts it is clear that

$$\gamma_1^{\beta_1/\ell-1/2}\gamma_2^{\beta_2/\ell-1/2}\Big\{1+\Big(\frac{1+\gamma_1+\gamma_2}{N\eta}\Big)^2\Big\}\exp\Big(-c\Big(\frac{\gamma_1+\gamma_2}{N\eta}\Big)\Big) \ll_\ell \gamma_1^{2\beta_1/\ell-1}\exp\Big(-\frac{c}{2}\frac{\gamma_1}{N\eta}\Big),$$

hence the r.h.s. of (8) becomes

$$\ll_\ell \sum_{\rho_1 \colon \gamma_1>0} \eta^{1-2\beta_1/\ell}\gamma_1^{2\beta_1/\ell-1}\exp\Big(-\frac{c}{2}\frac{\gamma_1}{N\eta}\Big) \sum_{\rho_2\colon \gamma_2>0;\beta_2\le\beta_1} \frac{1}{1+|\gamma_1-\gamma_2|^2}$$

$$\ll_\ell \sum_{\rho_1 \colon \gamma_1>0} \Big(\frac{\gamma_1}{\eta}\Big)^{2\beta_1/\ell-1}\exp\Big(-\frac{c}{4}\frac{\gamma_1}{N\eta}\Big) \qquad (9)$$

since the number of zeros $\rho_2 = \beta_2 + i\gamma_2$ with $n \le |\gamma_1-\gamma_2| \le n+1$ is $\mathcal{O}(\log(n+\gamma_1))$.

Now we use (3) and the Ingham-Huxley zero-density estimate, *i.e.*, for $1/2 \le \sigma \le 1$ we have that $N(\sigma,t) \ll t^{(12/5)(1-\sigma)}(\log t)^B$. Hence, uniformly for $1/N < \eta < N^{-1+5/(6\ell)-\varepsilon}$, by (6.17) of Saffari and Vaughan [8] we get that (9) is

$$\sum_{\rho_1\colon \gamma_1>0}\Big(\frac{\gamma_1}{\eta}\Big)^{2\beta_1/\ell-1}\exp\Big(-\frac{c}{4}\frac{\gamma_1}{N\eta}\Big) \ll \sum_{\substack{\beta_1\ge 1/2\\ 0<\gamma_1\le N^4}}\Big(\frac{\gamma_1}{\eta}\Big)^{2\beta_1/\ell-1}\exp\Big(-\frac{c}{4}\frac{\gamma_1}{N\eta}\Big)$$

$$\ll_\ell \max_{1/2\le\sigma\le 1-\delta(N^4)} \int_0^{N^4} t^{(12/5)(1-\sigma)}(\log t)^B \Big[\Big(\frac{t}{\eta}\Big)^{2\sigma/\ell-1}\exp\Big(-\frac{c}{8}\frac{t}{N\eta}\Big)\Big]'\,\mathrm{d}t$$

$$\ll_\ell \max_{1/2\le\sigma\le 1-\delta(N^4)} \int_0^{\infty} (Nu)^{2\sigma/\ell-1}(N\eta u)^{(12/5)(1-\sigma)}\exp\Big(-\frac{c}{8}u\Big)\,\mathrm{d}u$$

$$\ll_\ell \max_{1/2\le\sigma\le 1-\delta(N^4)} \Big((N\eta)^{(12/5)(1-\sigma)}N^{2\sigma/\ell-1}\Big) \ll_\ell N^{2/\ell-1}\exp\Big(-c_1\Big(\frac{L}{\log L}\Big)^{1/3}\Big), \qquad (10)$$

where $c_1 = c_1(\varepsilon)$ is a positive constant which does not depend on ℓ. From (4)-(6) and (8)-(10) we get

$$\int_{-\xi}^{\xi} \Big| \sum_{\rho:\ \gamma>0} z^{-\rho/\ell}\Gamma\Big(\frac{\rho}{\ell}\Big)\Big|^2 \mathrm{d}\alpha \ll_\ell N^{2/\ell-1} \exp\Big(-c_1\Big(\frac{L}{\log L}\Big)^{1/3}\Big) \tag{11}$$

uniformly for $1/N < \xi < N^{-1+5/(6\ell)-\varepsilon}$. Lemma 1 follows from (4)-(5) and (11). □

We need also the following analogue of Lemma 1 of [5]. Let

$$\omega_\ell(\alpha) = \sum_{m=1}^{\infty} e^{-m^\ell/N} e(m^\ell\alpha) = \sum_{m=1}^{\infty} e^{-m^\ell z}. \tag{12}$$

We explicitly remark that for $\ell = 1$ the proof of Lemma 2 gives just trivial results; in this case a non-trivial estimate, which, in any case, is not useful in this context, can be obtained following the line of Corollary 3 of [3].

Lemma 2. *Let $\ell \geq 2$ be an integer and $0 < \xi \leq 1/2$. Then*

$$\int_{-\xi}^{\xi} |\omega_\ell(\alpha)|^2 \,\mathrm{d}\alpha \ll_\ell \xi N^{1/\ell} + \begin{cases} L & \text{if } \ell = 2 \\ 1 & \text{if } \ell > 2 \end{cases}$$

and

$$\int_{-\xi}^{\xi} |\widetilde{S}_\ell(\alpha)|^2 \,\mathrm{d}\alpha \ll_\ell \xi N^{1/\ell} L + \begin{cases} L^2 & \text{if } \ell = 2 \\ 1 & \text{if } \ell > 2. \end{cases}$$

Proof. By symmetry we can integrate over $[0, \xi]$. We use Corollary 2 of Montgomery and Vaughan [6] (see also the remark after their statement) with $T = \xi$, $a_r = \exp(-r^\ell/N)$ and $\lambda_r = 2\pi r^\ell$ thus getting

$$\int_0^{\xi} |\omega_\ell(\alpha)|^2 \,\mathrm{d}\alpha = \sum_{r\geq 1} e^{-2r^\ell/N}\big(\xi + \mathcal{O}(\delta_r^{-1})\big) \ll_\ell \xi N^{1/\ell} + \sum_{r\geq 1} r^{1-\ell} e^{-2r^\ell/N}$$

since $\delta_r = \lambda_r - \lambda_{r-1} \gg_\ell r^{\ell-1}$. The last term is $\ll_\ell 1$ if $\ell > 2$ and $\ll L$ otherwise. This proves the first part of Lemma 2. Arguing analogously with $a_r = \Lambda(r)\exp(-r^\ell/N)$, by the Prime Number Theorem we get

$$\int_0^{\xi} |\widetilde{S}_\ell(\alpha)|^2 \,\mathrm{d}\alpha = \sum_{r\geq 1} \Lambda(r)^2 e^{-2r^\ell/N}\big(\xi + \mathcal{O}(\delta_r^{-1})\big) \ll_\ell \xi N^{1/\ell} L + \sum_{r\geq 1} \Lambda(r)^2 r^{1-\ell} e^{-2r^\ell/N}.$$

The last term is $\ll_\ell 1$ if $\ell > 2$ and $\ll L^2$ otherwise. The second part of Lemma 2 follows. □

Let now

$$\widetilde{T}_\ell(\alpha) = \sum_{p=2}^{\infty} \log p\, e^{-p^\ell/N} e(p^\ell\alpha). \tag{13}$$

We also have

Lemma 3. *Let $\ell \geq 1$ be an integer. Then $|\widetilde{S}_\ell(\alpha) - \widetilde{T}_\ell(\alpha)| \ll_\ell N^{1/(2\ell)}$.*

Proof. Clearly we have

$$|\widetilde{S}_\ell(\alpha) - \widetilde{T}_\ell(\alpha)| \leq \sum_{k\geq 2}\sum_{p\geq 2} \log p\, e^{-p^{k\ell}/N} \ll_\ell N^{1/(2\ell)}$$

where in the last inequality we used the Prime Number Theorem. □

Letting $\omega(\alpha) = \omega_2(\alpha)$ and

$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{-n^2/N} e(n^2\alpha) = \sum_{n=-\infty}^{\infty} e^{-n^2 z} = 1 + 2\omega(\alpha),$$

the functional equation of the θ-function (see, *e.g.*, Proposition VI.4.3, page 340, of Freitag and Busam [2]) gives that $\theta(z) = (\pi/z)^{1/2}\theta(\pi^2/z)$. Hence we have

$$\omega(\alpha) = \frac{1}{2}\left(\frac{\pi}{z}\right)^{1/2} - \frac{1}{2} + \left(\frac{\pi}{z}\right)^{1/2} \sum_{\ell=1}^{+\infty} e^{-\ell^2\pi^2/z}. \tag{14}$$

Lemma 4. *Let N be a large integer, $z = 1/N - 2\pi i\alpha$, $\alpha \in [1/2, 1/2]$ and $Y = \Re(1/z) > 0$. We have*

$$\Big|\sum_{\ell=1}^{+\infty} e^{-\ell^2\pi^2/z}\Big| \ll \begin{cases} e^{-\pi^2 Y} & \text{for } Y \geq 1 \\ Y^{-1/2} & \text{for } 0 < Y \leq 1. \end{cases}$$

Proof. It is clear that

$$\Big|\sum_{\ell=1}^{+\infty} e^{-\ell^2\pi^2/z}\Big| \leq \sum_{\ell=1}^{+\infty} e^{-\ell^2\pi^2 Y} \leq \sum_{\ell=1}^{+\infty} e^{-\ell\pi^2 Y} = \frac{e^{-\pi^2 Y}}{1 - e^{-\pi^2 Y}} \ll e^{-\pi^2 Y}$$

for $Y \geq 1$. Moreover, for $Y > 0$, we also have

$$\sum_{\ell=1}^{+\infty} e^{-\ell^2\pi^2 Y} \leq 1 + \int_1^{+\infty} e^{-t^2\pi^2 Y}\mathrm{d}t \ll 1 + Y^{-1/2}$$

and the lemma is proved. □

Since

$$Y = \Re(1/z) = \frac{N}{1 + 4\pi^2\alpha^2 N^2} \geq \frac{1}{5\pi^2}\begin{cases} N & \text{if } |\alpha| \leq 1/N \\ (\alpha^2 N)^{-1} & \text{if } |\alpha| > 1/N, \end{cases}$$

from Lemma 4 we get

$$\Big|\sum_{\ell=1}^{+\infty} e^{-\ell^2\pi^2/z}\Big| \ll \begin{cases} \exp(-\pi^2 N) & \text{if } |\alpha| \leq 1/N \\ \exp(-\pi^2/(\alpha^2 N)) & \text{if } 1/N < |\alpha| = o\big(N^{-1/2}\big) \\ 1 + N^{1/2}|\alpha| & \text{otherwise.} \end{cases} \tag{15}$$

We also recall that

$$|U(\alpha, H)| \leq \min\big(H; |\alpha|^{-1}\big), \tag{16}$$

$$|z|^{-1} \ll \min\big(N, |\alpha|^{-1}\big) \tag{17}$$

and we finally define

$$B = B(N, c) = \exp\Big(c\Big(\frac{L}{\log L}\Big)^{1/3}\Big), \tag{18}$$

where $c = c(\varepsilon) > 0$ will be chosen later.

3. Proof of Theorem 1

Recalling (1) and (13), it is an easy matter to see that

$$\begin{aligned}\sum_{n=N+1}^{N+H} e^{-n/N} r''_{2,2}(n) &= \int_{-1/2}^{1/2} \widetilde{T}_2(\alpha)^2 U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha \\ &= \int_{-1/2}^{1/2} (\widetilde{T}_2(\alpha)^2 - \widetilde{S}_2(\alpha)^2) U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha \\ &\quad + \int_{-1/2}^{1/2} \frac{\pi}{4z} U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha + \int_{-1/2}^{1/2} \Bigl(\widetilde{S}_2(\alpha)^2 - \frac{\pi}{4z}\Bigr) U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha \\ &= I_0 + I_1 + I_2, \end{aligned} \tag{19}$$

say. Using the identity $f^2 - g^2 = 2f(f-g) - (f-g)^2$ and the Cauchy-Schwarz inequality we have

$$\begin{aligned} I_0 &\ll \int_{-1/2}^{1/2} |\widetilde{S}_2(\alpha)||\widetilde{S}_2(\alpha) - \widetilde{T}_2(\alpha)||U(\alpha,H)|\,\mathrm{d}\alpha + \int_{-1/2}^{1/2} |\widetilde{S}_2(\alpha) - \widetilde{T}_2(\alpha)|^2 |U(\alpha,H)|\,\mathrm{d}\alpha \\ &\ll N^{1/4} \Bigl(\int_{-1/2}^{1/2} |\widetilde{S}_2(\alpha)|^2 |U(\alpha,H)|\mathrm{d}\alpha\Bigr)^{1/2} \Bigl(\int_{-1/2}^{1/2} |U(\alpha,H)|\,\mathrm{d}\alpha\Bigr)^{1/2} + N^{1/2} \int_{-1/2}^{1/2} |U(\alpha,H)|\,\mathrm{d}\alpha, \end{aligned}$$

by Lemma 3. By Lemma 2, (16) and a partial integration argument we obtain

$$\begin{aligned} \int_{-1/2}^{1/2} |\widetilde{S}_2(\alpha)|^2 |U(\alpha,H)|\mathrm{d}\alpha &\ll H \int_{-1/H}^{1/H} |\widetilde{S}_2(\alpha)|^2 |U(\alpha,H)|\mathrm{d}\alpha + \int_{1/H}^{1/2} |\widetilde{S}_2(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha} \\ &\ll H\Bigl(\frac{N^{1/2}L}{H} + L^2\Bigr) + N^{1/2}L + \int_{1/H}^{1/2} (N^{1/2}\xi L + L^2) \frac{\mathrm{d}\xi}{\xi^2} \\ &\ll N^{1/2}L^2 + HL^2. \end{aligned}$$

Hence

$$I_0 \ll N^{1/4}(N^{1/2}L^2 + HL^2)^{1/2} L^{1/2} + N^{1/2}L \ll N^{1/2}L^{3/2} + H^{1/2}N^{1/4}L^{3/2}. \tag{20}$$

Now we evaluate I_1. Using Lemma 4 of [4] we immediately get

$$I_1 = \frac{\pi}{4} \sum_{n=N+1}^{N+H} e^{-n/N} + \mathcal{O}\Bigl(\frac{H}{N}\Bigr) = \frac{\pi H}{4e} + \mathcal{O}\Bigl(\frac{H^2}{N}\Bigr). \tag{21}$$

Now we estimate I_2. Again using the identity $f^2 - g^2 = 2f(f-g) - (f-g)^2$, by (2) we obtain

$$I_2 \ll \int_{-1/2}^{1/2} |\widetilde{E}_2(\alpha)| \frac{|U(\alpha,H)|}{|z|^{1/2}}\,\mathrm{d}\alpha + \int_{-1/2}^{1/2} |\widetilde{E}_2(\alpha)|^2 |U(\alpha,H)|\,\mathrm{d}\alpha = J_1 + J_2, \tag{22}$$

say. Using (16)-(17), Lemma 3 of [4] and a partial integration argument we have

$$J_2 \ll H \int_{-1/H}^{1/H} |\widetilde{E}_2(\alpha)|^2\,\mathrm{d}\alpha + \int_{1/H}^{1/2} |\widetilde{E}_2(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha} \ll N^{1/2}L^2 + N^{1/2}L^2\Bigl(1 + \int_{1/H}^{1/2} \frac{\mathrm{d}\xi}{\xi}\Bigr) \ll N^{1/2}L^3. \tag{23}$$

Using the Cauchy-Schwarz inequality and arguing as for J_2 we get

$$J_1 \ll HN^{1/2}\Bigl(\int_{-1/N}^{1/N} \mathrm{d}\alpha\Bigr)^{1/2} \Bigl(\int_{-1/N}^{1/N} |\widetilde{E}_2(\alpha)|^2\,\mathrm{d}\alpha\Bigr)^{1/2} + H\Bigl(\int_{1/N}^{1/H} \frac{\mathrm{d}\alpha}{\alpha^{1/2}}\Bigr)^{1/2} \Bigl(\int_{1/N}^{1/H} |\widetilde{E}_2(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha^{1/2}}\Bigr)^{1/2}$$

$$+\Big(\int_{1/H}^{1/2}\frac{d\alpha}{\alpha^{3/2}}\Big)^{1/2}\Big(\int_{1/H}^{1/2}|\widetilde{E}_2(\alpha)|^2\frac{d\alpha}{\alpha^{3/2}}\Big)^{1/2}$$
$$\ll HN^{-1/4}L + H^{3/4}N^{1/4}L\Big(\frac{1}{H}+\int_{1/N}^{1/H}\frac{d\xi}{\xi^{1/2}}\Big)^{1/2} + H^{1/4}N^{1/4}L\Big(H^{1/2}+\int_{1/H}^{1/2}\frac{d\xi}{\xi^{3/2}}\Big)^{1/2}$$
$$\ll H^{1/2}N^{1/4}L. \qquad (24)$$

Combining (22)-(24) we finally obtain

$$I_2 \ll H^{1/2}N^{1/4}L + N^{1/2}L^3. \qquad (25)$$

Now using (19)-(21) and (25) we have

$$\sum_{n=N+1}^{N+H} e^{-n/N} r''_{2,2}(n) = \frac{\pi H}{4e} + \mathcal{O}\Big(\frac{H^2}{N} + N^{1/2}L^3 + H^{1/2}N^{1/4}L^{3/2}\Big) \qquad (26)$$

which is an asymptotic formula for $\infty(N^{1/2}L^3) \le H \le o(N)$. From $e^{-n/N} = e^{-1} + \mathcal{O}(H/N)$ for $n \in [N+1, N+H]$, we get

$$\sum_{n=N+1}^{N+H} r''_{2,2}(n) = \frac{\pi H}{4} + \mathcal{O}\Big(\frac{H^2}{N} + N^{1/2}L^3 + H^{1/2}N^{1/4}L^{3/2}\Big) + \mathcal{O}\Big(\frac{H}{N}\sum_{n=N+1}^{N+H} r''_{2,2}(n)\Big). \qquad (27)$$

Using $e^{n/N} \le e^2$ and (26) for H in the previously mentioned range, it is easy to see that the last error term is $\ll H^2N^{-1}$. Combining (27) and the last remark, Theorem 1 hence follows for $\infty(N^{1/2}L^3) \le H \le o(N)$. □

4. Proof of Theorem 2

Recalling (1) and (13), it is an easy matter to see that

$$\sum_{n=N+1}^{N+H} e^{-n/N} r''_{2,2}(n) = \int_{-1/2}^{1/2} \widetilde{T}_2(\alpha)^2 U(-\alpha,H)e(-N\alpha)\,d\alpha$$
$$= \int_{-1/2}^{1/2} (\widetilde{T}_2(\alpha)^2 - \widetilde{S}_2(\alpha)^2)U(-\alpha,H)e(-N\alpha)\,d\alpha + \int_{-B/H}^{B/H}\frac{\pi}{4z}U(-\alpha,H)e(-N\alpha)\,d\alpha$$
$$+ \int_{-B/H}^{B/H}\Big(\widetilde{S}_2(\alpha)^2 - \frac{\pi}{4z}\Big)U(-\alpha,H)e(-N\alpha)\,d\alpha + \int_{[-1/2,-B/H]\cup[B/H,1/2]}\widetilde{S}_2(\alpha)^2U(-\alpha,H)e(-N\alpha)\,d\alpha$$
$$= I_0 + I_1 + I_2 + I_3, \qquad (28)$$

say, where B is defined in (18). I_0 can be estimated as in (20) and gives

$$I_0 \ll N^{1/2}L^{3/2} + H^{1/2}N^{1/4}L^{3/2}. \qquad (29)$$

Now we evaluate I_1. Using Lemma 4 of [4] and (16) we immediately get

$$I_1 = \frac{\pi}{4}\sum_{n=N+1}^{N+H} e^{-n/N} + \mathcal{O}\Big(\frac{H}{N}\Big) + \mathcal{O}\Big(\int_{B/H}^{1/2}\frac{d\alpha}{\alpha^2}\Big) = \frac{\pi H}{4e} + \mathcal{O}\Big(\frac{H^2}{N} + \frac{H}{B}\Big). \qquad (30)$$

Now we estimate I_2. Using the identity $f^2 - g^2 = 2f(f-g) - (f-g)^2$, by (2) and (16) we obtain

$$I_2 \ll H\Big(\int_{-B/H}^{B/H}|\widetilde{E}_2(\alpha)|\frac{d\alpha}{|z|^{1/2}} + \int_{-B/H}^{B/H}|\widetilde{E}_2(\alpha)|^2\,d\alpha\Big) = H(J_1 + J_2), \qquad (31)$$

say. Using Lemma 1 with $\ell = 2$ we have

$$J_2 \ll \exp\Big(-c_1\Big(\frac{L}{\log L}\Big)^{1/3}\Big) \tag{32}$$

provided that $\infty(1/N) < B/H < N^{-7/12-\varepsilon/2}$, i.e. $N^{7/12+\varepsilon} \le H \le o(N)$ suffices.

Using the Cauchy-Schwarz inequality and arguing as for J_2 we get

$$J_1 \ll \Big(\int_{-B/H}^{B/H} \frac{\mathrm{d}\alpha}{|z|}\Big)^{1/2} \Big(\int_{-B/H}^{B/H} |\widetilde{E}_2(\alpha)|^2 \,\mathrm{d}\alpha\Big)^{1/2} \ll \exp\Big(-\frac{c_1}{4}\Big(\frac{L}{\log L}\Big)^{1/3}\Big), \tag{33}$$

provided that $\infty(1/N) < B/H < N^{-7/12-\varepsilon/2}$, i.e. $N^{7/12+\varepsilon} \le H \le o(N)$.

Combining (31)-(33), for $N^{7/12+\varepsilon} \le H \le o(N)$ we finally obtain

$$I_2 \ll H \exp\Big(-\frac{c_1}{4}\Big(\frac{L}{\log L}\Big)^{1/3}\Big). \tag{34}$$

Now we estimate I_3. By (16), Lemma 2 and a partial integration argument we get

$$I_3 \ll \int_{B/H}^{1/2} |\widetilde{S}_2(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha} \ll N^{1/2}L + \frac{HL^2}{B} + L\int_{B/H}^{1/2} (\xi N^{1/2} + L)\frac{\mathrm{d}\xi}{\xi^2} \ll \Big(N^{1/2} + \frac{H}{B}\Big)L^2. \tag{35}$$

Now using (28)-(30) and (34)-(35), and choosing $0 < c < c_1/4$ in (18), we have that there exists a constant $C = C(\varepsilon) > 0$ such that

$$\sum_{n=N+1}^{N+H} e^{-n/N} r''_{2,2}(n) = \frac{\pi H}{4e} + \mathcal{O}\Big(H \exp\Big(-C\Big(\frac{L}{\log L}\Big)^{1/3}\Big) + \frac{H^2}{N}\Big)$$

uniformly for $N^{7/12+\varepsilon} \le H \le o(N)$. Theorem 2 hence follows for $N^{7/12+\varepsilon} \le H \le N^{1-\varepsilon}$ since the exponential weight $e^{-n/N}$ can be removed as we did at the bottom of the proof of Theorem 1. □

Remark 1. *Using the finite-sum approach we need to define* $T_2(\alpha) = \sum_{1\le m^2 \le N} e(m^2\alpha)$ *and* $f_2(\alpha) = (1/2)\sum_{1\le m\le N} m^{-1/2} e(m\alpha)$. *Theorem 4.1 of Vaughan [9] gives* $|T_2(\alpha) - f_2(\alpha)| \ll (1 + |\alpha|N)^{1/2}$. *The main term comes from the integral of* $f_2(\alpha)^2 U(-\alpha, H)$ *but we also need to evaluate the quantity*

$$\Big|\int_{-B/H}^{B/H} (T_2(\alpha)^2 - f_2(\alpha)^2) U(-\alpha, H) e(-N\alpha)\,\mathrm{d}\alpha\Big| \ll \frac{NB^{1/2}}{H^{1/2}}.$$

Since the expected order of magnitude of the main term is H, *the previous estimate is under control if and only if* $H \ge N^{2/3}B^{1/3}$ *which is weaker than the result we obtain. Similar remarks apply for the other problems studied in the remaining sections.*

5. Proof of Theorem 3

Letting $1 < A = A(N) < H/2$ to be chosen later, by (1) and (12)-(14) it is an easy matter to see that

$$\sum_{n=N+1}^{N+H} e^{-n/N} r'_{2,2}(n) = \int_{-1/2}^{1/2} \widetilde{T}_2(\alpha)\omega(\alpha) U(-\alpha, H) e(-N\alpha)\,\mathrm{d}\alpha$$
$$= \int_{-1/2}^{1/2} (\widetilde{T}_2(\alpha) - \widetilde{S}_2(\alpha))\omega(\alpha) U(-\alpha, H) e(-N\alpha)\,\mathrm{d}\alpha$$

$$
\begin{aligned}
&+\int_{-A/H}^{A/H}\Big(\frac{\pi}{4z}-\frac{\pi^{1/2}}{4z^{1/2}}\Big)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha+\int_{-A/H}^{A/H}\widetilde{E}_2(\alpha)\omega(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha\\
&+\int_{-A/H}^{A/H}\frac{\pi}{2z}\Big(\sum_{\ell=1}^{+\infty}e^{-\ell^2\pi^2/z}\Big)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha+\int\limits_{[-1/2,-A/H]\cup[A/H,1/2]}\widetilde{S}_2(\alpha)\omega(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha\\
&=I_0+I_1+I_2+I_3+I_4,
\end{aligned}
\tag{36}
$$

say. Using Lemma 3 and the Cauchy-Schwarz inequality we have

$$
I_0 \ll N^{1/4}\Big(\int_{-1/2}^{1/2}|\omega(\alpha)|^2|U(\alpha,H)|\,\mathrm{d}\alpha\Big)^{1/2}\Big(\int_{-1/2}^{1/2}|U(\alpha,H)|\,\mathrm{d}\alpha\Big)^{1/2}.
$$

By Lemma 2, (16) and a partial integration argument we obtain

$$
I_0 \ll N^{1/4}(N^{1/2}L+HL)^{1/2}L^{1/2} \ll N^{1/2}L+H^{1/2}N^{1/4}L. \tag{37}
$$

Now we evaluate I_1. Using Lemma 4 of [4] and (16) we immediately get

$$
I_1=\sum_{n=N+1}^{N+H}\Big(\frac{\pi}{4}-\frac{1}{4n^{1/2}}\Big)e^{-n/N}+\mathcal{O}\Big(\frac{H}{N}\Big)+\mathcal{O}\Big(\int_{A/H}^{1/2}\frac{\mathrm{d}\alpha}{\alpha^2}\Big)=\frac{\pi H}{4e}+\mathcal{O}\Big(\frac{H}{N^{1/2}}+\frac{H^2}{N}+\frac{H}{A}\Big). \tag{38}
$$

To have that $\pi H/(4e)$ dominates in I_0+I_1 we need that $A\to\infty$, $H=o(N)$ and $H=\infty(N^{1/2}L^2)$.

Now we estimate I_3. Assuming $H=\infty(N^{1/2}A)$, by (15)-(17), we have

$$
\begin{aligned}
I_3 &\ll \frac{HN}{e^{\pi^2N}}\int_{-1/N}^{1/N}\mathrm{d}\alpha+\frac{H}{e^{\pi^2H^2/N}}\int_{1/N}^{1/H}\frac{\mathrm{d}\alpha}{\alpha}+\int_{1/H}^{A/H}\frac{\mathrm{d}\alpha}{\alpha^2e^{\pi^2/(N\alpha^2)}}\\
&\ll \frac{H}{e^{\pi^2N}}+\frac{HL}{e^{\pi^2H^2/N}}+\frac{H}{e^{\pi^2H^2/(NA^2)}}
\end{aligned}
\tag{39}
$$

which is $o(H)$ provided that $H=\infty(N^{1/2}\log L)$ and $H=\infty(N^{1/2}A)$.

Now we estimate I_2. Recalling $H=\infty(N^{1/2}A)$, for every $|\alpha|\le A/H$ we have, by (14)-(15), that $|\omega(\alpha)|\ll|z|^{-1/2}$. Hence

$$
I_2 \ll \int_{-A/H}^{A/H}|\widetilde{E}_2(\alpha)|\frac{|U(\alpha,H)|}{|z|^{1/2}}\,\mathrm{d}\alpha.
$$

Using (17) and the Cauchy-Schwarz inequality and Lemma 3 of [4] we get

$$
\begin{aligned}
I_2 &\ll HN^{1/2}\Big(\int_{-1/N}^{1/N}\mathrm{d}\alpha\Big)^{1/2}\Big(\int_{-1/N}^{1/N}|\widetilde{E}_2(\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2}+H\Big(\int_{1/N}^{1/H}\frac{\mathrm{d}\alpha}{\alpha^{1/2}}\Big)^{1/2}\Big(\int_{1/N}^{1/H}|\widetilde{E}_2(\alpha)|^2\frac{\mathrm{d}\alpha}{\alpha^{1/2}}\Big)^{1/2}\\
&\qquad+\Big(\int_{1/H}^{A/H}\frac{\mathrm{d}\alpha}{\alpha^{3/2}}\Big)^{1/2}\Big(\int_{1/H}^{A/H}|\widetilde{E}_2(\alpha)|^2\frac{\mathrm{d}\alpha}{\alpha^{3/2}}\Big)^{1/2}\\
&\ll HN^{-1/4}L+H^{3/4}N^{1/4}L\Big(\frac{1}{H}+\int_{1/N}^{1/H}\frac{\mathrm{d}\xi}{\xi^{1/2}}\Big)^{1/2}+H^{1/4}N^{1/4}L\Big(H^{1/2}+\int_{1/H}^{A/H}\frac{\mathrm{d}\xi}{\xi^{3/2}}\Big)^{1/2}\\
&\ll H^{1/2}N^{1/4}L.
\end{aligned}
\tag{40}
$$

Remark that $I_2=o(H)$ provided that $H=\infty(N^{1/2}L^2)$.

Now we estimate I_4. By (16), Lemma 2 and a partial integration argument we get

$$
I_4 \ll \int_{A/H}^{1/2}|\widetilde{S}_2(\alpha)\omega(\alpha)|\frac{\mathrm{d}\alpha}{\alpha} \ll \Big(\int_{A/H}^{1/2}|\widetilde{S}_2(\alpha)|^2\frac{\mathrm{d}\alpha}{\alpha}\Big)^{1/2}\Big(\int_{A/H}^{1/2}|\omega(\alpha)|^2\frac{\mathrm{d}\alpha}{\alpha}\Big)^{1/2}
$$

$$
\begin{aligned}
&\ll \Big(N^{1/2}L + \frac{HL^2}{A} + L\int_{A/H}^{1/2}(\xi N^{1/2}+L)\frac{\mathrm{d}\xi}{\xi^2}\Big)^{1/2}\Big(N^{1/2} + \frac{HL}{A} + \int_{A/H}^{1/2}(\xi N^{1/2}+L)\frac{\mathrm{d}\xi}{\xi^2}\Big)^{1/2}\\
&\ll L^{3/2}\Big(N^{1/2} + \frac{H}{A}\Big) \qquad (41)
\end{aligned}
$$

which is $o(H)$ provided that $A = \infty(L^{3/2})$ and $H = \infty(N^{1/2}L^{3/2})$.

Combining the conditions on H and A we can choose $A = L^2/(\log L)$ and $H = \infty(N^{1/2}L^2)$. Hence using (36)-(41) we can write

$$
\sum_{n=N+1}^{N+H} e^{-n/N} r'_{2,2}(n) = \frac{\pi H}{4e} + \mathcal{O}\Big(\frac{H^2}{N} + \frac{H\log L}{L^{1/2}} + N^{1/2}L^{3/2} + H^{1/2}N^{1/4}L\Big).
$$

Theorem 3 follows for $\infty(N^{1/2}L^2) \le H \le o(N)$ since the exponential weight $e^{-n/N}$ can be removed as we did at the bottom of the proof of Theorem 1. □

6. Proof of Theorem 4

By (1) and (12)-(14), it is an easy matter to see that

$$
\begin{aligned}
\sum_{n=N+1}^{N+H} e^{-n/N} r'_{2,2}(n) &= \int_{-1/2}^{1/2} \widetilde{T}_2(\alpha)\omega(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha\\
&= \int_{-1/2}^{1/2} (\widetilde{T}_2(\alpha) - \widetilde{S}_2(\alpha))\omega(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha\\
&\quad + \int_{-B/H}^{B/H}\Big(\frac{\pi}{4z} - \frac{\pi^{1/2}}{4z^{1/2}}\Big)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha + \int_{-B/H}^{B/H}\widetilde{E}_2(\alpha)\omega(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha\\
&\quad + \int_{-B/H}^{B/H}\frac{\pi}{2z}\Big(\sum_{\ell=1}^{+\infty} e^{-\ell^2\pi^2/z}\Big)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha + \int\limits_{[-1/2,-B/H]\cup[B/H,1/2]} \widetilde{S}_2(\alpha)\omega(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha\\
&= I_0 + I_1 + I_2 + I_3 + I_4, \qquad (42)
\end{aligned}
$$

say, where B is defined in (18). I_0 can be estimated as in (37) and gives

$$
I_0 \ll N^{1/2}L + H^{1/2}N^{1/4}L. \qquad (43)
$$

I_1 can be evaluated as in (38) and we get

$$
I_1 = \frac{\pi H}{4e} + \mathcal{O}\Big(\frac{H^2}{N} + \frac{H}{B}\Big). \qquad (44)
$$

Now we estimate I_2. Using (16) and the Cauchy-Schwarz inequality we obtain

$$
I_2 \ll H\Big(\int_{-B/H}^{B/H} |\widetilde{E}_2(\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2}\Big(\int_{-B/H}^{B/H} |\omega(\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2} = H(J_1J_2)^{1/2}, \qquad (45)
$$

say. Using Lemma 1 we can write

$$
J_1 \ll \exp\Big(-c_1\Big(\frac{L}{\log L}\Big)^{1/3}\Big) \qquad (46)
$$

provided that $\infty(1/N) < B/H < N^{-7/12-\varepsilon/2}$, i.e. $N^{7/12+\varepsilon} \le H \le o(N)$ suffices.

Using Lemma 2 with $\ell = 2$ we have

$$
J_2 \ll \frac{N^{1/2}B}{H} + L \ll L. \qquad (47)
$$

Combining (45)-(47) for $N^{7/12+\varepsilon} \le H \le o(N)$ we finally obtain

$$I_2 \ll H \exp\Big(-\frac{c_1}{4}\Big(\frac{L}{\log L}\Big)^{1/3}\Big). \tag{48}$$

Now we estimate I_3. By (15)-(17), we have

$$I_3 \ll \frac{HN}{e^{\pi^2 N}} \int_{-1/N}^{1/N} \mathrm{d}\alpha + \frac{H}{e^{\pi^2 H^2/(NB^2)}} \int_{1/N}^{B/H} \frac{\mathrm{d}\alpha}{\alpha} \ll H \exp\Big(-\frac{c_1}{4}\Big(\frac{L}{\log L}\Big)^{1/3}\Big), \tag{49}$$

since $N^{7/12+\varepsilon} \le H \le o(N)$.

I_4 can be estimated as in (41) and gives

$$I_4 \ll L^{3/2}\Big(N^{1/2} + \frac{H}{B}\Big). \tag{50}$$

Now using (42)-(44) and (48)-(50), and choosing $0 < c < c_1$ in (18), we have that there exists a constant $C = C(\varepsilon) > 0$ such that

$$\sum_{n=N+1}^{N+H} e^{-n/N} r'_{2,2}(n) = \frac{\pi H}{4e} + \mathcal{O}\Big(H \exp\Big(-C\Big(\frac{L}{\log L}\Big)^{1/3}\Big) + \frac{H^2}{N}\Big)$$

uniformly for for $N^{7/12+\varepsilon} \le H \le o(N)$. Theorem 4 hence follows for $N^{7/12+\varepsilon} \le H \le N^{1-\varepsilon}$ since the exponential weight $e^{-n/N}$ can be removed as we did at the bottom of the proof of Theorem 1. □

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