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CONVEXITY THEOREMS FOR THE GRADIENT MAP ON PROBABILITY MEASURES

LEONARDO BILIOTTI AND ALBERTO RAFFERO

Abstract. Given a Kähler manifold (Z, J, ω) and a compact real submanifold $M \subset Z$, we study the properties of the gradient map associated with the action of a noncompact real reductive Lie group G on the space of probability measures on M. In particular, we prove convexity results for such map when G is Abelian and we investigate how to extend them to the non-Abelian case.

1. Introduction

Let (Z, J, ω) be a compact connected Kähler manifold and let U be a compact connected Lie group with Lie algebra $\mathfrak{u}$. Assume that U acts on Z by holomorphic isometries and in a Hamiltonian fashion with momentum mapping $\mu : Z \to \mathfrak{u}^*$. It is well-known that the U-action extends to a holomorphic action of the complexification $\mathrm{U}^{\mathbb{C}}$ of U. Moreover, the latter gives rise to a continuous action of $\mathrm{U}^{\mathbb{C}}$ on the space of Borel probability measures on Z endowed with the weak* topology. We denote such space by $\mathscr{P}(Z)$.

Recently, the first author and Ghigi [5] studied the properties of the $\mathrm{U}^{\mathbb{C}}$-action on $\mathscr{P}(Z)$ using momentum mapping techniques. Although it is still not clear whether any reasonable symplectic structure on $\mathscr{P}(Z)$ may exist (but see [16] for something similar on the Euclidean space), in this setting it is possible to define an analogue of the momentum mapping, namely

$$\mathfrak{F} : \mathscr{P}(Z) \to \mathfrak{u}^*, \quad \mathfrak{F}(\nu) = \int_Z \mu(z) \mathrm{d}\nu(z).$$

$\mathfrak{F}$ is called *gradient map*. Using it, the usual concepts of stability appearing in Kähler geometry [17, 20, 21, 22, 23, 30, 32, 35, 37, 38] can be defined for probability measures, too.

In [5], the authors were interested in determining the conditions for which the $\mathrm{U}^{\mathbb{C}}$-orbit of a given probability measure $\nu \in \mathscr{P}(Z)$ has non-empty intersection with $\mathfrak{F}^{-1}(0)$, whenever 0 belongs to the convex hull of $\mu(Z)$. This problem is motivated by an application to upper bounds for the first eigenvalue of the Laplacian acting on functions (see also [1, 3, 4, 11, 29]). Furthermore, they obtained various stability criteria for measures.

Stability theory for the action of a compatible subgroup G of $\mathrm{U}^{\mathbb{C}}$ was analyzed by the first author and Zedda in [9].

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Recall that a closed subgroup G of $\mathrm{U}^{\mathbb{C}}$ is called *compatible* if the Cartan decomposition $\mathrm{U}^{\mathbb{C}} = \mathrm{U}\exp(i\mathfrak{u})$ induces a Cartan decomposition $\mathrm{G} = \mathrm{K}\exp(\mathfrak{p})$, where $\mathrm{K} := \mathrm{G} \cap \mathrm{U}$ and $\mathfrak{p} := \mathfrak{g} \cap i\mathfrak{u}$ is a K-stable linear subspace of $i\mathfrak{u}$.

Identify $\mathfrak{u}^*$ with $\mathfrak{u}$ by means of an Ad(U)-invariant scalar product on $\mathfrak{u}$. For each $z \in Z$, let $\mu_{\mathfrak{p}}(z)$ denote $-i$ times the component of $\mu(z)$ in the direction of $i\mathfrak{p} \subset \mathfrak{u}$. This defines a K-equivariant map $\mu_{\mathfrak{p}} : Z \longrightarrow \mathfrak{p}$, which is called G-*gradient map* associated with μ [24, 26, 27]. Since $\mathrm{U}^{\mathbb{C}}$ acts holomorphically on Z, the fundamental vector field $\beta_Z \in \mathfrak{X}(Z)$ of any $\beta \in \mathfrak{p}$ is the gradient of the function $\mu_{\mathfrak{p}}^{\beta}(\cdot) := \langle \mu_{\mathfrak{p}}(\cdot), \beta \rangle$ with respect to the Riemannian metric $\omega(\cdot, J\cdot)$, $\langle \cdot, \cdot \rangle$ being an Ad(K)-invariant scalar product on $\mathfrak{p}$.

If M is a compact G-stable real submanifold of Z, we can restrict $\mu_{\mathfrak{p}}$ to M. Moreover, the G-action on M extends in a natural way to a continuous action on $\mathscr{P}(M)$, and the map

$$\mathfrak{F}_{\mathfrak{p}} : \mathscr{P}(M) \to \mathfrak{p}, \quad \mathfrak{F}_{\mathfrak{p}}(\nu) = \int_M \mu_{\mathfrak{p}}(x) \mathrm{d}\nu(x),$$

is the analogue of the G-gradient map in this setting. It is not hard to prove that its image coincides with the convex hull of $\mu_{\mathfrak{p}}(M)$ in $\mathfrak{p}$ (cf. Lemma 3.4).

Fix a probability measure $\nu \in \mathscr{P}(M)$. Having in mind the classical convexity results for the momentum mapping [2, 19, 31] and for the G-gradient map [24, 28], in this paper we are interested in studying the behaviour of $\mathfrak{F}_{\mathfrak{p}}$ on the orbit $\mathrm{G} \cdot \nu$.

Let $\mathfrak{a} \subset \mathfrak{p}$ be an Abelian subalgebra of $\mathfrak{g}$. The Abelian Lie group $\mathrm{A} := \exp(\mathfrak{a})$ is compatible and the corresponding A-gradient map is given by $\mu_{\mathfrak{a}} := \pi_{\mathfrak{a}} \circ \mu_{\mathfrak{p}}$, where $\pi_{\mathfrak{a}}$ is the orthogonal projection onto $\mathfrak{a}$. In Section 4, we prove a result which can be regarded as the analogue of a theorem by Atiyah [2] in our setting (see also [28]):

Theorem. *The image of the map $\mathfrak{F}_{\mathfrak{a}} : \mathrm{A} \cdot \nu \to \mathfrak{a}$ is an open convex subset of an affine subspace of $\mathfrak{a}$ with direction $\mathfrak{a}_{\nu}^{\perp}$. Moreover, $\mathfrak{F}_{\mathfrak{a}}(\overline{\mathrm{A} \cdot \nu})$ is the convex hull of $\mathfrak{F}_{\mathfrak{a}}(\overline{\mathrm{A} \cdot \nu} \cap \mathscr{P}(M)^{\mathrm{A}})$, where $\mathscr{P}(M)^{\mathrm{A}}$ is the set of* A*-fixed measures.*

As an immediate consequence of this theorem, we get that $\mathfrak{F}_{\mathfrak{a}}(\mathrm{A} \cdot \nu)$ is a convex subset of $\mathfrak{a}$ whenever the Lie algebra of the isotropy group A_{ν} is trivial (Corollary 4.4). The image of the map $\mathfrak{F}_{\mathfrak{a}}$ is contained in the convex hull of $\mu_{\mathfrak{a}}(M)$. Hence, when $P := \mu_{\mathfrak{a}}(M)$ is a polytope, it is natural to investigate under which conditions $\mathfrak{F}_{\mathfrak{a}}(A \cdot \nu)$ coincides with $\mathrm{int}(P)$. We point out that the convexity of P is not known for a generic A-invariant closed submanifold M of Z. It holds if $\mathrm{G} = \mathrm{U}^{\mathbb{C}}$ and M is a complex connected submanifold by the Atiyah-Guillemin-Sternberg convexity theorem [2, 19], or, more in general, if Z is a Hodge manifold and M is an irreducible semi-algebraic subset of Z with irreducible real algebraic Zariski closure [7, 24]. In the recent paper [10], the authors gave a short proof of this property when M is an A-invariant compact connected real analytic submanifold of $\mathbb{P}^n(\mathbb{C})$. The key point is that for any $\beta \in \mathfrak{a}$ the Morse-Bott function $\mu_{\mathfrak{p}}^{\beta}$ has a unique local maximum. Under this assumption, in Theorem 4.7 we show that if A_{ν} is trivial and for any $\beta \in \mathfrak{a}$ the unstable manifold corresponding to the unique maximum of $\mu_{\mathfrak{p}}^{\beta}$ has full measure, then $\mathfrak{F}_{\mathfrak{a}}(A \cdot \nu)$ coincides with $\mathrm{int}(P)$. It is worth underlining here that a further result shown in [10] allows to obtain an alternative proof of the convexity properties of the map $\mathfrak{F}_{\mathfrak{a}}$ along the A-orbits. Nevertheless, in our proof the image of $\mathfrak{F}_{\mathfrak{a}}$ along the orbits is better understood. Moreover, it is completely determined for a large class of probability measures in Theorem 4.7.

In Section 5, we focus our attention on the non-Abelian case. Let $\Omega(\mu_{\mathfrak{p}})$ denote the interior of the convex hull of $\mu_{\mathfrak{p}}(M)$ in $\mathfrak{p}$. In Theorem 5.2, we prove that, under a mild regularity assumption on the measure ν, $\mathfrak{F}_{\mathfrak{p}}(\mathrm{G}\cdot\nu) = \Omega(\mu_{\mathfrak{p}})$ and that the map

$$F_\nu : \mathrm{G} \to \Omega(\mu_{\mathfrak{p}}), \quad F_\nu(g) \coloneqq \mathfrak{F}_{\mathfrak{p}}(g\cdot\nu),$$

is a smooth fibration. Notice that the assumptions in Theorem 4.7 are weaker than those of Theorem 5.2. Finally, if ν is a K-invariant smooth measure on M, we show that the map F_ν descends to a map on G/K which is a diffemorphism onto $\Omega(\mu_{\mathfrak{p}})$. (Corollary 5.3). These results may be regarded as a generalization of those obtained in [5] when $\mathrm{G} = \mathrm{U}^{\mathbb{C}}$ and $M = Z$ is a Kähler manifold. However, our proofs are slightly different, since the real case is more involved than the complex one and a new technical result is needed (cf. Appendix A). Moreover, Corollary 5.3 suggests that when M is an adjoint orbit and ν is a K-invariant smooth measure, then a potential compactification of G/K is given by the convex hull of M. This is an analogue of a classical result due to Korányi [34].

The present paper is organized as follows. In Section 2, we review the main properties of compatible groups and of the G-gradient map. In Section 3, we recall some useful results on measures and we introduce the gradient map. The convexity properties of the gradient map in the Abelian and in the non-Abelian case are investigated in Section 4 and in Section 5, respectively. Finally, in Appendix A, we prove a technical result which is of interest in Section 5.

2. Preliminaries

2.1. **Cartan decomposition and compatible subgroups.** Let U be a compact connected Lie group, denote by $\mathfrak{u}$ its Lie algebra and by $\mathrm{U}^{\mathbb{C}}$ its complexification. It is well-known (see for instance [33]) that $\mathrm{U}^{\mathbb{C}}$ is a complex reductive Lie group with Lie algebra $\mathfrak{u}^{\mathbb{C}} = \mathfrak{u} \oplus i\mathfrak{u}$ and that it is diffeomorphic to $\mathrm{U} \times i\mathfrak{u}$ via the real analytic map

$$\mathrm{U} \times i\mathfrak{u} \to \mathrm{U}^{\mathbb{C}}, \quad (u, i\xi) \mapsto u\exp(i\xi).$$

The resulting decomposition $\mathrm{U}^{\mathbb{C}} = \mathrm{U}\exp(i\mathfrak{u})$ is called *Cartan decomposition* of $\mathrm{U}^{\mathbb{C}}$.

A closed connected subgroup $\mathrm{G} \subseteq \mathrm{U}^{\mathbb{C}}$ with Lie algebra $\mathfrak{g}$ is said to be *compatible* with the Cartan decomposition of $\mathrm{U}^{\mathbb{C}}$ if $\mathrm{G} = \mathrm{K}\exp(\mathfrak{p})$, where $\mathrm{K} \coloneqq \mathrm{G} \cap \mathrm{U}$ and $\mathfrak{p} \coloneqq \mathfrak{g} \cap i\mathfrak{u}$ is a K-stable linear subspace of $i\mathfrak{u}$ (cf. [26, 27]). In such a case, K is a maximal compact subgroup of G. The Lie algebra of G splits as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} \coloneqq \mathrm{Lie}(\mathrm{K})$, and the following inclusions hold

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

On the Lie algebra $\mathfrak{u}^{\mathbb{C}} = \mathfrak{u} \oplus i\mathfrak{u}$ there exists a nondegenerate, $\mathrm{Ad}(\mathrm{U}^{\mathbb{C}})$-invariant, symmetric $\mathbb{R}$-bilinear form $B : \mathfrak{u}^{\mathbb{C}} \times \mathfrak{u}^{\mathbb{C}} \to \mathbb{R}$ which is positive definite on $i\mathfrak{u}$, negative definite on $\mathfrak{u}$ and such that the decomposition $\mathfrak{u} \oplus i\mathfrak{u}$ is B-orthogonal (see e.g. [6, p. 585]). In what follows, we let $\langle\cdot,\cdot\rangle \coloneqq B|_{i\mathfrak{u}\times i\mathfrak{u}}$.

Whenever $\mathrm{G} = \mathrm{K}\exp(\mathfrak{p})$ is a compatible subgroup of $\mathrm{U}^{\mathbb{C}}$, the restriction of B to $\mathfrak{g}$ is $\mathrm{Ad}(\mathrm{K})$-invariant, positive definite on $\mathfrak{p}$, negative definite on $\mathfrak{k}$, and fulfils $B(\mathfrak{k}, \mathfrak{p}) = 0$.

2.2. **The G-gradient map.** Let U and $\mathrm{U}^{\mathbb{C}}$ be as in §2.1. Consider a compact Kähler manifold (Z, J, ω), assume that $\mathrm{U}^{\mathbb{C}}$ acts holomorphically on it and that a Hamiltonian action of U on Z is defined. Then, the Kähler form ω is U-invariant and there exists a *momentum mapping* $\mu : Z \to \mathfrak{u}^*$. By definition, μ is U-equivariant and for each $\xi \in \mathfrak{u}$

$$d\mu^\xi = \iota_{\xi_Z}\omega,$$

where $\mu^\xi \in \mathcal{C}^\infty(Z)$ is defined by $\mu^\xi(z) = \mu(z)(\xi)$, for every point $z \in Z$, and $\xi_Z \in \mathfrak{X}(Z)$ is the *fundamental vector field* of ξ induced by the U-action, namely the vector field on Z whose value at $z \in Z$ is

$$\xi_Z(z) = \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi) \cdot z.$$

Since U is compact, we can identify $\mathfrak{u}^*$ with $\mathfrak{u}$ by means of an Ad(U)-invariant scalar product on $\mathfrak{u}$. Consequently, we can regard μ as a $\mathfrak{u}$-valued map.

Let $\mathrm{G} = \mathrm{K}\exp(\mathfrak{p})$ be a compatible subgroup of $\mathrm{U}^{\mathbb{C}}$. The composition of μ with the orthogonal projection of $\mathfrak{u}$ onto $i\mathfrak{p} \subset \mathfrak{u}$ defines a K-equivariant map $\mu_{i\mathfrak{p}} : Z \to i\mathfrak{p}$, which represents the analogue of μ for the G-action. Following [24, 26, 27], in place of $\mu_{i\mathfrak{p}}$ we consider

$$\mu_{\mathfrak{p}} : Z \to \mathfrak{p}, \quad \mu_{\mathfrak{p}}(z) \coloneqq -i\,\mu_{i\mathfrak{p}}(z).$$

As the $\mathrm{U}^{\mathbb{C}}$-action on Z is holomorphic, for every $\beta \in \mathfrak{p}$ the fundamental vector field $\beta_Z \in \mathfrak{X}(Z)$ induced by the G-action is the gradient of the function

$$\mu_{\mathfrak{p}}^\beta : Z \to \mathbb{R}, \quad \mu_{\mathfrak{p}}^\beta(z) \coloneqq \langle \mu_{\mathfrak{p}}(z), \beta\rangle,$$

with respect to the Riemannian metric $\omega(\cdot, J\cdot)$. This motivates the following.

Definition 2.1. $\mu_{\mathfrak{p}}$ is called G-*gradient map* associated with μ.

Let M be a G-stable submanifold of Z. We use the symbol $\mu_{\mathfrak{p}}$ to denote the G-gradient map restricted to M, too. Then, for any $\beta \in \mathfrak{p}$ the fundamental vector field $\beta_M \in \mathfrak{X}(M)$ is the gradient of $\mu_{\mathfrak{p}}^\beta : M \to \mathbb{R}$ with respect to the induced Riemannian metric on M. Moreover, if M is compact, $\mu_{\mathfrak{p}}^\beta$ is a Morse-Bott function (see e.g. [6, Cor. 2.3]). Thus, denoted by $c_1 < \cdots < c_r$ the critical values of $\mu_{\mathfrak{p}}^\beta$, M decomposes as

$$M = \bigsqcup_{j=1}^{r} W_j, \tag{2.1}$$

where for each $j = 1, \ldots, r$, W_j is the unstable manifold of the critical component $(\mu_{\mathfrak{p}}^\beta)^{-1}(c_j)$ for the gradient flow of $\mu_{\mathfrak{p}}^\beta$ (see for instance [25, 26] for more details).

3. Measures

In the first part of this section we recall some known results about measures. The reader may refer for instance to [13, 15] for more details.

Let M be a compact manifold and let $\mathscr{M}(M)$ denote the vector space of finite signed Borel measures on M. By [15, Thm. 7.8], such measures are Radon. Then, by the Riesz Representation Theorem [15, Thm. 7.17], $\mathscr{M}(M)$ is the topological dual of the Banach space

$(\mathcal{C}(M), \|\cdot\|_\infty)$, namely the space of real valued continuous functions on M endowed with the sup-norm. As a consequence, $\mathscr{M}(M)$ is endowed with the weak* topology [15, p. 169].

The set of *Borel probability measures* on M is the compact convex subset $\mathscr{P}(M) \subset \mathscr{M}(M)$ given by the intersection of the cone of positive measures on M and the affine hyperplane $\{\nu \in \mathscr{M}(M) \mid \nu(M) = 1\}$. Observe that the weak* topology on $\mathscr{P}(M)$ is metrizable, since $\mathcal{C}(M)$ is separable [13, p. 426].

Given a measurable map $f : M \to N$ between measurable spaces and a measure ν on M, the *image measure* $f_*\nu$ of ν is the measure on N defined by $f_*\nu(A) := \nu(f^{-1}(A))$ for every measurable set $A \subseteq N$. $f_*\nu$ satisfies the following *change of variables formula*

$$\int_N h(y)d(f_*\nu)(y) = \int_M h(f(x))d\nu(x). \tag{3.1}$$

When a Lie group G acts continuously on a compact manifold M, it is possible to define an action of G on $\mathscr{P}(M)$ as follows:

$$\mathrm{G} \times \mathscr{P}(M) \to \mathscr{P}(M), \quad (g, \nu) \mapsto g_*\nu := (\mathcal{A}_g)_*\nu, \tag{3.2}$$

where for each $g \in \mathrm{G}$

$$\mathcal{A}_g : M \to M, \quad \mathcal{A}_g(x) = g \cdot x,$$

is the homeomorphism induced by the G-action on M. By [5, Lemma 5.5], the action (3.2) is continuous with respect to the weak* topology on $\mathscr{P}(M)$. In what follows, we denote this action by a dot, i.e., $g \cdot \nu := g_*\nu$ whenever $g \in \mathrm{G}$ and $\nu \in \mathscr{P}(M)$.

The next lemma is an immediate consequence of [5, Lemma 5.8].

Lemma 3.1. *Let M be a compact manifold endowed with a smooth action of a Lie group* G*. Consider $\nu \in \mathscr{M}(M)$, $\xi \in \mathfrak{g}$, and suppose that ξ_M vanishes ν-almost everywhere. Then,* $\exp(\mathbb{R}\xi)$ *is contained in the isotropy group* G_ν *of ν.*

Proof. Since ξ_M vanishes ν-almost everywhere, its flow

$$\varphi_t : M \to M, \quad \varphi_t(x) = \exp(t\xi) \cdot x,$$

satisfies $\varphi_{t*}\nu = \nu$ for any $t \in \mathbb{R}$ by [5, Lemma 5.8]. □

Let us focus on the setting $(M, \mathrm{G}, \mathrm{K}, \mu_\mathfrak{p})$ introduced at the end of §2.2. From now on, we assume that the G-stable submanifold $M \subset Z$ is compact. By the above results, the group $\mathrm{G} = \mathrm{K}\exp(\mathfrak{p})$ acts continuously on $\mathscr{P}(M)$. Moreover, albeit a reasonable symplectic structure on $\mathscr{P}(M)$ does not seem to exist, it is possible to define a map which can be regarded as the analogue of the G-gradient map $\mu_\mathfrak{p}$ for the action of G on $\mathscr{P}(M)$.

Definition 3.2. The *gradient map* associated with the action of G on $\mathscr{P}(M)$ is

$$\mathfrak{F} : \mathscr{P}(M) \to \mathfrak{p}, \quad \mathfrak{F}(\nu) = \int_M \mu_\mathfrak{p}(x)d\nu(x).$$

Remark 3.3. By [9, Prop. 45], $\mathfrak{F}$ is precisely the gradient map of a Kempf-Ness function for $(\mathscr{P}(M), \mathrm{G}, \mathrm{K})$. Thus, it is continuous and K-equivariant (cf. [9, Sect. 3]).

Using $\mathfrak{F}$, the usual concepts of stability [17, 20, 21, 22, 23, 30, 32, 35, 37, 38] can be defined for probability measures, too (see also [5, 9]). For instance, a measure $\nu \in \mathscr{P}(M)$ is said to be *stable* if

$$\mathrm{G} \cdot \nu \cap \mathfrak{F}^{-1}(0) \neq \emptyset$$

and $\mathfrak{g}_\nu := \mathrm{Lie}(\mathrm{G}_\nu)$ is conjugate to a subalgebra of $\mathfrak{k}$. In such a case, G_ν is compact [5, Cor. 3.5].

In the light of previous considerations, it is natural to ask whether established results for the G-gradient map [2, 12, 19, 24, 28] can be proved also for the gradient map $\mathfrak{F}$. Here, we focus our attention on convexity properties of $\mathfrak{F}$. We begin with the following observation.

Lemma 3.4. *The image of the gradient map $\mathfrak{F} : \mathscr{P}(M) \to \mathfrak{p}$ coincides with the convex hull $E(\mu_\mathfrak{p})$ of $\mu_\mathfrak{p}(M)$ in $\mathfrak{p}$.*

Proof. Consider $\nu \in \mathscr{P}(M)$. Observe that $\mathfrak{F}(\nu)$ is the barycenter of the measure $\mu_{\mathfrak{p}*}\nu \in \mathscr{P}(\mu_\mathfrak{p}(M))$, since by the change of variables formula (3.1) we have

$$\mathfrak{F}(\nu) = \int_M \mu_\mathfrak{p}(x) d\nu(x) = \int_\mathfrak{p} \beta \, d(\mu_{\mathfrak{p}*}\nu)(\beta).$$

Thus, $\mathfrak{F}(\nu)$ lies in $E(\mu_\mathfrak{p})$. Conversely, for any $\gamma \in E(\mu_\mathfrak{p})$, we can write

$$\gamma = \sum_{j=1}^m \lambda_j \gamma_j,$$

for a suitable m, where $\sum_{j=1}^m \lambda_j = 1$, $\lambda_j \geq 0$ and $\gamma_j \in \mu_\mathfrak{p}(M)$. For each $j = 1, \ldots, m$, let $x_j \in M$ be a point in the preimage of γ_j and let δ_{x_j} denote the Dirac measure supported at x_j. Then, $\gamma = \mathfrak{F}(\widetilde{\nu})$, where

$$\widetilde{\nu} := \sum_{j=1}^m \lambda_j \delta_{x_j}.$$

□

Due to the previous result, in the next sections we shall study the behaviour of $\mathfrak{F}$ on the orbits of the G-action.

4. Convexity properties of $\mathfrak{F}$: Abelian case

Let $\mathfrak{a} \subset \mathfrak{p}$ be a Lie subalgebra of $\mathfrak{g}$. Since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{a}$ is Abelian. The corresponding Abelian Lie group $\mathrm{A} := \exp(\mathfrak{a}) \subset \mathrm{G}$ is compatible with the Cartan decomposition of $\mathrm{U}^\mathbb{C}$ and an A-gradient map $\mu_\mathfrak{a} : M \to \mathfrak{a}$ is given by $\mu_\mathfrak{a} := \pi_\mathfrak{a} \circ \mu_\mathfrak{p}$, where $\pi_\mathfrak{a}$ is the orthogonal projection onto $\mathfrak{a}$. Therefore, the gradient map associated with the A-action on $\mathscr{P}(M)$ is

$$\mathfrak{F}_\mathfrak{a} : \mathscr{P}(M) \to \mathfrak{a}, \quad \mathfrak{F}_\mathfrak{a}(\nu) = \int_M \mu_\mathfrak{a}(x) d\nu(x).$$

Fix a probability measure $\nu \in \mathscr{P}(M)$. We want to study the behaviour of $\mathfrak{F}_\mathfrak{a}$ on the orbit $\mathrm{A} \cdot \nu$. First of all, we show that A_ν is always compatible.

Lemma 4.1. *The isotropy group A_ν of ν is compatible, namely $\mathrm{A}_\nu = \exp(\mathfrak{a}_\nu)$.*

Proof. Let $\alpha \coloneqq \mathfrak{F}_{\mathfrak{a}}(\nu) \in \mathfrak{a}$. Since $\mathfrak{a}$ is Abelian, $\widetilde{\mu}_{\mathfrak{a}} \coloneqq \mu_{\mathfrak{a}} - \alpha$ is still an A-gradient map and the corresponding gradient map $\widetilde{\mathfrak{F}}_{\mathfrak{a}} : \mathscr{P}(M) \to \mathfrak{a}$ satisfies

$$\widetilde{\mathfrak{F}}_{\mathfrak{a}}(\nu) = \int_M \widetilde{\mu}_{\mathfrak{a}}(x) d\nu(x) = \mathfrak{F}_{\mathfrak{a}}(\nu) - \alpha\, \nu(M) = 0.$$

Then, A_ν is compatible by [9, Prop. 20]. □

Consider the decomposition

$$\mathfrak{a} = \mathfrak{a}_\nu \oplus \mathfrak{a}_\nu^\perp,$$

where $\mathfrak{a}_\nu^\perp$ is the orthogonal complement of $\mathfrak{a}_\nu$ in $\mathfrak{a}$ with respect to $B|_{\mathfrak{a}\times\mathfrak{a}}$. We denote by $\pi : \mathfrak{a} \to \mathfrak{a}_\nu^\perp$ the orthogonal projection onto $\mathfrak{a}_\nu^\perp$ and we let $\hat{\mathrm{A}} \coloneqq \exp(\mathfrak{a}_\nu^\perp)$. Since $\exp : \mathfrak{a} \to \mathrm{A}$ is an isomorphism of Abelian Lie groups, we have $\mathrm{A} = \hat{\mathrm{A}}\mathrm{A}_\nu$ and $\mathrm{A} \cdot \nu = \hat{\mathrm{A}} \cdot \nu$.

We are now ready to state the main result of this section.

Theorem 4.2. *The image $\mathfrak{F}_{\mathfrak{a}}(\mathrm{A} \cdot \nu)$ of the orbit $\mathrm{A} \cdot \nu$ is an open convex subset of an affine subspace of $\mathfrak{a}$ with direction $\mathfrak{a}_\nu^\perp$.*

Before proving Theorem 4.2, we show a preliminary lemma.

Lemma 4.3. *The projection of $\mathfrak{F}_{\mathfrak{a}}(\hat{\mathrm{A}} \cdot \nu)$ onto $\mathfrak{a}_\nu^\perp$ is convex.*

Proof. By [9, Thm. 39], there exists a Kempf-Ness function $\Psi : M \times \mathrm{A} \to \mathbb{R}$ for $(M, \mathrm{A}, \{e\})$, where $e \in \mathrm{A}$ is the identity element. Recall that for each point $x \in M$ the function $\Psi(x, \cdot)$ is smooth on A, and that for every $\gamma \in \mathfrak{a}$

$$\frac{d^2}{dt^2}\Psi(x, \exp(t\gamma)) \geq 0, \tag{4.1}$$

and it vanishes identically if and only if $\exp(\mathbb{R}\gamma) \subset \mathrm{A}_x$. Moreover, for every $a, b \in \mathrm{A}$, the following condition is satisfied

$$\Psi(x, ab) = \Psi(x, b) + \Psi(b \cdot x, a). \tag{4.2}$$

Ψ is related to the A-gradient map $\mu_{\mathfrak{a}}$ by

$$\left.\frac{d}{dt}\right|_{t=0} \Psi(x, \exp(t\gamma)) = \langle \mu_{\mathfrak{a}}(x), \gamma \rangle. \tag{4.3}$$

We define a function $f : \mathfrak{a}_\nu^\perp \to \mathbb{R}$ as follows

$$f(\alpha) \coloneqq \int_M \Psi(x, \exp(\alpha)) d\nu(x).$$

We claim that f is strictly convex. By (4.1) and (4.2), for every $\alpha, \beta \in \mathfrak{a}_\nu^\perp$

$$\frac{d^2}{dt^2} f(t\beta + \alpha) = \int_M \frac{d^2}{dt^2}\Psi(\exp(\alpha) \cdot x, \exp(t\beta)) d\nu(x) \geq 0.$$

If it was identically zero, then $\frac{d^2}{dt^2}\Psi(\exp(\alpha) \cdot x, \exp(t\beta))$ would vanish ν-almost everywhere. As a consequence, for every point x outside a set of ν-measure zero we would have $\exp(\mathbb{R}\beta) \subset \mathrm{A}_{\exp(\alpha)\cdot x} = \mathrm{A}_x$, which implies that $\beta_M(x) = 0$. Therefore, $\exp(\mathbb{R}\beta) \subset \mathrm{A}_\nu$ by Lemma 3.1, which is a contradiction. By a standard result in convex analysis (see for instance [18,

p. 122]), the pushforward $df : \mathfrak{a}_\nu^\perp \to (\mathfrak{a}_\nu^\perp)^*$ is a diffeomorphism onto an open convex subset of $(\mathfrak{a}_\nu^\perp)^*$. Now, using (3.1), (4.2), (4.3), for each $\alpha, \beta \in \mathfrak{a}_\nu^\perp$ we have

$$\begin{aligned} df(\alpha)(\beta) &= \frac{d}{dt}\Big|_{t=0} f(t\beta + \alpha) \\ &= \int_M \frac{d}{dt}\Big|_{t=0} \Psi(\exp(\alpha)\cdot x, \exp(t\beta)) d\nu(x) \\ &= \langle \int_M \mu_{\mathfrak{a}}(\exp(\alpha)\cdot x) d\nu(x), \beta\rangle \\ &= \langle \int_M \mu_{\mathfrak{a}}(y) d(\exp(\alpha)\cdot \nu)(y), \beta\rangle \\ &= \langle \mathfrak{F}_{\mathfrak{a}}(\exp(\alpha)\cdot\nu), \beta\rangle \\ &= \langle \pi(\mathfrak{F}_{\mathfrak{a}}(\exp(\alpha)\cdot\nu)), \beta\rangle, \end{aligned}$$

from which the assertion follows. □

Corollary 4.4. *If $\mathfrak{a}_\nu = \{0\}$, then $\mathfrak{F}_{\mathfrak{a}}(\mathrm{A}\cdot\nu)$ is convex in $\mathfrak{a}$ and the map*

$$F_\nu^{\mathrm{A}} : \mathrm{A} \to \mathfrak{a}, \quad F_\nu^{\mathrm{A}}(a) := \mathfrak{F}_{\mathfrak{a}}(a\cdot\nu),$$

is a diffeomorphism onto $\mathfrak{F}_{\mathfrak{a}}(\mathrm{A}\cdot\nu)$.

Proof of Theorem 4.2. Since A_ν is compatible, it follows from the proof of [9, Prop. 52] that ν is supported on

$$M^{\mathfrak{a}_\nu} := \{x \in M \mid \xi_M(x) = 0 \ \forall\ \xi \in \mathfrak{a}_\nu\}.$$

By [25, 26], there exists a decomposition

$$M^{\mathfrak{a}_\nu} = M_1 \sqcup \cdots \sqcup M_n,$$

where each M_j is an A-stable connected submanifold of M. Consequently,

$$\nu = \sum_{j=1}^n \lambda_j \nu_j,$$

where for $j = 1, \ldots, n$, ν_j is a probability measure on M_j, $\lambda_j \geq 0$ and $\sum_{j=1}^n \lambda_j = 1$. By [27], for every $x \in M_j$ the image $\mu_{\mathfrak{a}}(\mathrm{A}\cdot x)$ of $\mathrm{A}\cdot x$ is contained in an affine subspace $\alpha_j + \mathfrak{a}_\nu^\perp$ of $\mathfrak{a}$. Then, since M_j is A-stable, there is a map $\widetilde{\mu}_j : M_j \to \mathfrak{a}_\nu^\perp$ such that $\mu_{\mathfrak{a}}(a\cdot x) = \alpha_j + \widetilde{\mu}_j(a\cdot x)$, for every $a \in \mathrm{A}$. Now, we have

$$\begin{aligned} \mathfrak{F}_{\mathfrak{a}}(a\cdot\nu) &= \int_M \mu_{\mathfrak{a}}(x) d(a\cdot\nu)(x) \\ &= \int_M \mu_{\mathfrak{a}}(a\cdot x) d\nu(x) \\ &= \sum_{j=1}^n \lambda_j \int_{M_j} \mu_{\mathfrak{a}}(a\cdot x) d\nu_j(x) \\ &= \sum_{j=1}^n \lambda_j \alpha_j + \sum_{j=1}^n \lambda_j \int_{M_j} \tilde{\mu}_j(a\cdot x) d\nu_j(x). \end{aligned}$$

Hence, $\mathfrak{F}_{\mathfrak{a}}(\mathrm{A}\cdot\nu) \subseteq \alpha + \mathfrak{a}_\nu^\perp$, where $\alpha := \sum_{j=1}^n \lambda_j \alpha_j$. Using Lemma 4.3, we can conclude that $\mathfrak{F}_{\mathfrak{a}}(\mathrm{A}\cdot\nu)$ is an open convex subset of the affine subspace $\alpha + \mathfrak{a}_\nu^\perp$ of $\mathfrak{a}$. □

From the previous result and the compactness of $\mathscr{P}(M)$, it follows that $\mathfrak{F}_{\mathfrak{a}}(\overline{\mathrm{A}\cdot\nu}) = \overline{\mathfrak{F}_{\mathfrak{a}}(\mathrm{A}\cdot\nu)}$ is a compact convex subset of $\mathfrak{a}$. Moreover, if we denote by

$$\mathscr{P}(M)^{\mathrm{A}} := \{\nu \in \mathscr{P}(M) \mid \mathrm{A}\cdot\nu = \nu\}$$

the set of A-fixed measures, then we have the

Proposition 4.5. *$\mathfrak{F}_{\mathfrak{a}}\left(\overline{\mathrm{A}\cdot\nu}\right)$ is the convex envelope of $\mathfrak{F}_{\mathfrak{a}}(\overline{\mathrm{A}\cdot\nu} \cap \mathscr{P}(M)^{\mathrm{A}})$.*

Proof. By [36, Cor. 1.4.5], it is sufficient to show that every extremal point $\beta \in \mathfrak{F}_{\mathfrak{a}}\left(\overline{\mathrm{A}\cdot\nu}\right)$ is the image of an A-fixed measure. Consider $\widetilde{\nu} \in \overline{\mathrm{A}\cdot\nu}$ such that $\mathfrak{F}_{\mathfrak{a}}(\widetilde{\nu}) = \beta$. By Theorem 4.2, $\mathfrak{F}_{\mathfrak{a}}(\mathrm{A}\cdot\widetilde{\nu})$ is an open convex subset of an affine subspace $\alpha + \mathfrak{a}_{\widetilde{\nu}}^\perp \subset \mathfrak{a}$. Since β is an extremal point, we have necessarily $\mathfrak{a}_{\widetilde{\nu}}^\perp = \{0\}$. Thus, $\widetilde{\nu} \in \mathscr{P}(M)^{\mathrm{A}}$. □

Let $P := \mu_{\mathfrak{a}}(M)$. It was proved in [24, Sect. 5] that P is a finite union of polytopes, while in [8] the authors showed that its convex hull is closely related to $E(\mu_{\mathfrak{p}})$. Moreover, even if P is not necessarily convex, there exist suitable hypothesis guaranteeing that it is a polytope. This happens for instance if for each $\beta \in \mathfrak{a}$ any local maximum of the Morse-Bott function $\mu_{\mathfrak{p}}^\beta$ is a global maximum [10]. Classes of manifolds satisfying this property include real flag manifolds [6], and real analytic submanifolds of the complex projective space [10].

In the sequel, we always assume that for each $\beta \in \mathfrak{a}$ the function $\mu_{\mathfrak{p}}^\beta$ has a unique local maximum. As a consequence, P is a polytope, and the Morse-Bott decomposition (2.1) of M with respect to $\mu_{\mathfrak{p}}^\beta$ has a unique unstable manifold which is open and dense, namely W_r, while the remaining unstable manifolds are submanifolds of positive codimension.

Definition 4.6. Let $\mathscr{W}(M, \mathrm{A})$ denote the set of probability measures on M for which the open unstable manifold W_r has full measure for every $\beta \in \mathfrak{a}$.

A typical example of probability measures belonging to $\mathscr{W}(M, \mathrm{A})$ is given by *smooth* ones, namely those having a smooth positive density in any chart of the manifold with respect to the Lebesgue measure of the chart (cf. [15, Sect. 11.4]).

In a similar way as in [5, Prop. 6.8], we can prove the following

Theorem 4.7. *Let $\nu \in \mathscr{W}(M, \mathrm{A})$ and assume that $\mathrm{A}_\nu = \{e\}$. Then, $\mathfrak{F}_{\mathfrak{a}}(\mathrm{A}\cdot\nu)$ coincides with $\mathrm{int}(P)$.*

Proof. For simplicity of notation, let $\mathcal{O} := \mathfrak{F}_{\mathfrak{a}}(\mathrm{A}\cdot\nu) \subset \mathfrak{a}$. We already know that $\mathcal{O} \subseteq \mathrm{int}(\mathrm{P})$. Suppose by contradiction that $\mathcal{O}$ is strictly contained in $\mathrm{int}(\mathrm{P})$. Then, $\overline{\mathcal{O}} \subset P$, since $\mathcal{O}$ and P are both convex. Consider $\alpha_0 \in P - \overline{\mathcal{O}}$, $\alpha_1 \in \mathcal{O}$ and the line segment $\sigma(t) := (1-t)\,\alpha_0 + t\,\alpha_1$. Let $\overline{t} := \inf\{t \in [0,1] \mid \sigma(t) \in \overline{\mathcal{O}}\}$ and $\overline{\alpha} := \sigma(\overline{t})$. As $\overline{\mathcal{O}}$ is closed, $\overline{\alpha} \in \overline{\mathcal{O}}$ and $\overline{t} \in (0,1)$. We claim that $\overline{\alpha} \in \partial\mathcal{O} \cap \mathrm{int}(P)$. Indeed, it is clear that $\overline{\alpha} \in \partial\mathcal{O}$, while $\overline{\alpha} \in \mathrm{int}(P)$ follows from $\alpha_1 \in \mathcal{O} \subset \mathrm{int}(P)$ and $\overline{t} > 0$. By [36], every boundary point of a compact convex set lies on an exposed face, that is, it admits a support hyperplane. Therefore, there exists $\beta \in \mathfrak{a}$ such that

$$\langle \overline{\alpha}, \beta\rangle = \max_{\alpha \in \overline{\mathcal{O}}} \langle \alpha, \beta\rangle = \sup_{\alpha \in \mathcal{O}} \langle \alpha, \beta\rangle = \sup_{\gamma \in \mathfrak{a}} \langle \mathfrak{F}_{\mathfrak{a}}(\exp(\gamma)\cdot\nu), \beta\rangle.$$

Since $\nu \in \mathscr{W}(M, \mathrm{A})$ and $\mu_{\mathfrak{p}}^{\beta} = \mu_{\mathfrak{a}}^{\beta}$ for every $\beta \in \mathfrak{a}$, it follows from [9, Cor. 54] and from the proof of [9, Thm. 53] that

$$\max_M \mu_{\mathfrak{a}}^{\beta} = \lim_{t \to +\infty} \int_M \mu_{\mathfrak{a}}^{\beta}(\exp(t\beta) \cdot x) d\nu(x) = \lim_{t \to +\infty} \langle \mathfrak{F}_{\mathfrak{a}}(\exp(t\beta) \cdot \nu), \beta \rangle.$$

Consequently,

$$\langle \overline{\alpha}, \beta \rangle = \sup_{\gamma \in \mathfrak{a}} \langle \mathfrak{F}_{\mathfrak{a}}(\exp(\gamma) \cdot \nu), \beta \rangle \geq \max_M \mu_{\mathfrak{a}}^{\beta} = \max_{\rho \in P} \langle \rho, \beta \rangle.$$

That being so, the linear function $\alpha \mapsto \langle \alpha, \beta \rangle$ attains it maximum on P at $\overline{\alpha} \in \mathrm{int}(P)$. Since P is convex, β must be zero, which is a contradiction. □

5. Convexity properties of $\mathfrak{F}$: general case

The goal of this section is to prove a result similar to Theorem 4.7 when the group acting on $\mathscr{P}(M)$ is non-Abelian.

Let $\mathrm{G} = \mathrm{K} \exp(\mathfrak{p})$ be a compatible subgroup of $\mathrm{U}^{\mathbb{C}}$ and fix $\nu \in \mathscr{P}(M)$. To our purpose, it is useful to consider the map [4, 5, 11, 29]

$$F_\nu : \mathrm{G} \to \mathfrak{p}, \quad F_\nu(g) \coloneqq \mathfrak{F}(g \cdot \nu),$$

where $\mathfrak{F} : \mathscr{P}(M) \to \mathfrak{p}$ is the gradient map associated with the action of G on $\mathscr{P}(M)$. In [5, Thm. 6.4], the authors showed that F_ν is a smooth submersion when $\mathrm{G} = \mathrm{U}^{\mathbb{C}}$ and G_ν is compact. This is true for a compatible subgroup of $\mathrm{U}^{\mathbb{C}}$, too.

Proposition 5.1. *If G_ν is compact, then F_ν is a smooth submersion.*

Proof. We have to prove that the pushforward $dF_\nu(g) : T_g\mathrm{G} \to \mathfrak{p}$ of F_ν is surjective for every $g \in \mathrm{G}$. Let us consider the curve $\sigma(t) \coloneqq \exp(t\beta) \cdot g$ in G, where $\beta \in \mathfrak{p}$. Using the change of variables formula (3.1), we can write

$$F_\nu(\sigma(t)) = \int_M \mu_{\mathfrak{p}}(\exp(t\beta) \cdot x) d\widetilde{\nu}(x),$$

where $\widetilde{\nu} \coloneqq g \cdot \nu \in \mathscr{P}(M)$. Suppose that $dF_\nu(g)(\dot{\sigma}(0)) = 0$. Then, denoted by $\|\cdot\|$ the Riemannian norm on M, we have

$$0 = \langle dF_\nu(\dot{\sigma}(0)), \beta \rangle = \int_M \left. \frac{d}{dt} \right|_{t=0} \mu_{\mathfrak{p}}^{\beta}(\exp(t\beta) \cdot x) \, d\widetilde{\nu}(x) = \int_M \|\beta_M\|^2 (x) \, d\widetilde{\nu}(x),$$

since $\mathrm{grad}(\mu_{\mathfrak{p}}^{\beta}) = \beta_M$. Therefore, β_M vanishes $\widetilde{\nu}$-almost everywhere. By Lemma 3.1, $\exp(\mathbb{R}\beta)$ is contained in $\mathrm{G}_{\widetilde{\nu}} = g \, \mathrm{G}_\nu g^{-1}$, which is compact. Thus, $\beta = 0$. We can conclude that $dF_\nu(g)$ is injective on the subspace $dR_g(e)(\mathfrak{p})$ of $T_g\mathrm{G}$, R_g being the right translation on G. By dimension reasons, $dF_\nu(g)$ is surjective. □

As in the previous section, whenever $\mathfrak{a} \subset \mathfrak{p}$ is a maximal Abelian subalgebra of $\mathfrak{g}$ with corresponding Abelian Lie group $\mathrm{A} \coloneqq \exp(\mathfrak{a})$, we assume that the Morse-Bott function $\mu_{\mathfrak{p}}^{\beta}$ has a unique local maximum for every $\beta \in \mathfrak{a}$. In the non-Abelian case, we can exploit the so-called KAK *decomposition* of G (cf. [33, Thm. 7.39]) to show the following.

Theorem 5.2. *Let $\nu \in \mathscr{P}(M)$ be a probability measure which is absolutely continuous with respect to a* K*-invariant smooth probability measure $\nu_0 \in \mathscr{P}(M)$ and assume that* 0 *belongs to the interior $\Omega(\mu_{\mathfrak{p}})$ of $E(\mu_{\mathfrak{p}})$ in $\mathfrak{p}$. Then, $\mathfrak{F}(\mathrm{G}\cdot\nu) = \Omega(\mu_{\mathfrak{p}})$ and $F_\nu : \mathrm{G} \to \Omega(\mu_{\mathfrak{p}})$ is a smooth fibration with compact connected fibres diffeomorphic to* K*.*

Before proving the theorem, we make some remarks on its content. First, we observe that the hypothesis on ν is satisfied by smooth probability measures, which constitute a dense subset of $\mathscr{P}(M)$ (see for instance [13]). Moreover, it guarantees that whenever $\{k_n\}$ is a sequence in K converging to some $k \in \mathrm{K}$, then the sequence $\{k_n \cdot \nu\} \subset \mathscr{P}(M)$ converges to $k \cdot \nu$ in the norm

$$\|\nu\| := \sup\left\{\int_M h d\nu \mid h \in \mathcal{C}(M),\ \sup_M |h| \leq 1\right\},$$

by [5, Lemma 6.11]. Finally, we underline that the assumption $0 \in \Omega(\mu_{\mathfrak{p}})$ is not restrictive, as such condition is always satisfied up to replace G with a compatible group $\mathrm{G}' = \mathrm{K}'\exp(\mathfrak{p}')$ such that $\mu_{\mathfrak{p}'}(M) = \mu_{\mathfrak{p}}(M)$ and up to shift $\mu_{\mathfrak{p}'}$. We will show this assertion in Proposition A.1 of Appendix A, since most of its proof is rather technical.

Proof of Theorem 5.2. First of all, notice that $\nu \in \mathscr{W}(M, \mathrm{A})$ for any $\mathfrak{a} \subset \mathfrak{p}$, since it is absolutely continuous with respect to the smooth probability measure ν_0. As $0 \in \Omega(\mu_{\mathfrak{p}})$, for every $\beta \in \mathfrak{p}$ the function $\mu_{\mathfrak{p}}^{\beta}$ has a strictly positive maximum. This implies that ν is stable (cf. [9, Cor. 56]). Thus, G_ν is compact. Now, by Proposition 5.1, $F_\nu : \mathrm{G} \to \mathfrak{p}$ is a smooth submersion. In particular, its image is an open subset of $\mathfrak{p}$ contained in $E(\mu_{\mathfrak{p}})$. Therefore, $F_\nu(\mathrm{G}) \subseteq \Omega(\mu_{\mathfrak{p}})$ and we can regard F_ν as a map $F_\nu : \mathrm{G} \to \Omega(\mu_{\mathfrak{p}})$. We claim that such map is proper. Let $\{g_n\}$ be a sequence in G such that $\{F_\nu(g_n)\}$ converges to a point of $\Omega(\mu_{\mathfrak{p}})$. We need to show that there exists a convergent subsequence of $\{g_n\}$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal Abelian subalgebra of $\mathfrak{g}$ and set $\mathrm{A} := \exp(\mathfrak{a})$. By the KAK-decomposition of G, every $g_n \in \mathrm{G}$ can be written as $k_n \exp(\alpha_n)\, l_n^{-1}$, where $k_n, l_n \in \mathrm{K}$ and $\alpha_n \in \mathfrak{a}$. Passing to subsequences, we have that $k_n \to k$ and $l_n \to l$, for some $k, l \in \mathrm{K}$. Since F_ν is K-equivariant, it follows that the sequence $\{F_\nu(\exp(\alpha_n)\, l_n^{-1})\}$ is convergent in $\Omega(\mu_{\mathfrak{p}})$. A computation similar to [5, p. 1139] gives

$$\left|F_\nu(\exp(\alpha_n)\, l_n^{-1}) - F_\nu(\exp(\alpha_n)\, l^{-1})\right| \leq \sup_M |\mu_{\mathfrak{p}}| \left\|l_n^{-1}\cdot\nu - l^{-1}\cdot\nu\right\|.$$

Then, by the hypothesis on ν, we get $F_\nu(\exp(\alpha_n)\, l_n^{-1}) - F_\nu(\exp(\alpha_n)\, l^{-1}) \to 0$. Therefore, the sequence $\{F_\nu(\exp(\alpha_n)\, l^{-1})\}$ is convergent in $\Omega(\mu_{\mathfrak{p}})$, too. Consequently, $\{F_\nu(l\exp(\alpha_n)\, l^{-1})\}$ converges to some point of $\Omega(\mu_{\mathfrak{p}})$, being $F_\nu(l\exp(\alpha_n)\, l^{-1}) = \mathrm{Ad}(l)F_\nu(\exp(\alpha_n)\, l^{-1})$. The points $l\exp(\alpha_n)\, l^{-1}$ belong to the Abelian group $\mathrm{A}' := l\mathrm{A}l^{-1}$, which is compatible. The A'-gradient map is $\pi_{\mathfrak{a}'} \circ \mu_{\mathfrak{p}}$, where $\pi_{\mathfrak{a}'} : \mathfrak{p} \to \mathfrak{a}'$ is the orthogonal projection onto the Lie algebra $\mathfrak{a}'$ of A'. Denote by $P := \mu_{\mathfrak{a}'}(M)$ the image of $\mu_{\mathfrak{a}'}$. $P = \pi_{\mathfrak{a}'}(\mu_{\mathfrak{p}}(M))$ is a polytope and $\pi_{\mathfrak{a}'}(\Omega(\mu_{\mathfrak{p}})) \subset \mathrm{int}(P)$. Observe that $0 \in \mathrm{int}(P)$. This implies that ν is stable with respect to A'. Thus, $\mathfrak{a}'_\nu = \{0\}$ by [9, Lemma 21]. Hence, by the results of §4, $\mathfrak{F}_{\mathfrak{a}'}(\mathrm{A}'\cdot\nu) = \mathrm{int}(P)$ and the map $F_\nu^{\mathrm{A}'} : \mathrm{A}' \to \mathfrak{a}'$, $F_\nu^{\mathrm{A}'}(a) = \mathfrak{F}_{\mathfrak{a}'}(a\cdot\nu)$, is a diffeomorphism onto $\mathrm{int}(P)$. Since $\{l\exp(\alpha_n)\, l^{-1}\} \subset \mathrm{A}'$ and $\{\pi_{\mathfrak{a}'}(F_\nu(l\exp(\alpha_n) l^{-1})) = F_\nu^{\mathrm{A}'}(l\exp(\alpha_n)\, l^{-1})\}$ converges to some point of $\mathrm{int}(P)$, the sequences $\{l\exp(\alpha_n) l^{-1}\} \subset \mathrm{A}'$ and $\{\exp(\alpha_n)\} \subset \mathrm{A}$ admit convergent subsequences. The claim is then proved. As a consequence, $F_\nu : \mathrm{G} \to \Omega(\mu_{\mathfrak{p}})$ is a closed

map. Since it is also open, it is surjective. In particular, it is a locally trivial fibration by Ehresmann theorem [14]. As the base $\Omega(\mu_{\mathfrak{p}})$ is contractible, G is diffeomorphic to $\Omega(\mu_{\mathfrak{p}}) \times \mathrm{F}$, where F denotes the fibre. Hence, F is connected. Moreover, $F_\nu^{-1}(0)$ is a K-orbit, since $0 \in \Omega(\mu_{\mathfrak{p}})$ and F_ν is K-equivariant. Therefore, F is diffeomorphic to K. □

Corollary 5.3. *If $\nu \in \mathscr{P}(M)$ is a* K*-invariant smooth probability measure on M and $0 \in \Omega(\mu_{\mathfrak{p}})$, then F_ν descends to a diffeomorphism $\overline{F}_\nu : \mathrm{G/K} \to \Omega(\mu_{\mathfrak{p}})$.*

Proof. Since ν is K-invariant, for every $g \in \mathrm{G}$ and $k \in \mathrm{K}$ we have $F_\nu(gk) = \mathfrak{F}(gk \cdot \nu) = \mathfrak{F}(g \cdot \nu) = F_\nu(g)$. Thus, F_ν descends to a map $\overline{F}_\nu : \mathrm{G/K} \to \Omega(\mu_{\mathfrak{p}})$. By Theorem 5.2, $\overline{F}_\nu$ is a proper map and a local diffeomorphism. Thus, it is a covering map. As $\Omega(\mu_{\mathfrak{p}})$ is contractible, $\overline{F}_\nu : \mathrm{G/K} \to \Omega(\mu_{\mathfrak{p}})$ is a diffeomorphism. □

Remark 5.4. The above corollary may be regarded as an analogue of a classical result by Korányi [34]. Indeed, it suggests that when M is an adjoint orbit and ν is a K-invariant probability measure, then a potential compactification of G/K is given by the convex hull of M.

Appendix A.

Let U be a compact connected Lie group acting in a Hamiltonian fashion on a compact Kähler manifold (Z, J, ω) with momentum mapping $\mu : Z \to \mathfrak{u}$, and assume that the action of $\mathrm{U}^{\mathbb{C}}$ on Z is holomorphic. As mentioned in §5, we are going to show the following result.

Proposition A.1. *Let $\mathrm{G} = \mathrm{K}\exp(\mathfrak{p})$ be a compatible subgroup of $\mathrm{U}^{\mathbb{C}}$. Consider a* G*-stable submanifold M of Z and let $\mu_{\mathfrak{p}} : M \to \mathfrak{p}$ be the* G*-gradient map associated with μ. Then*

i) *there exists a subgroup $\mathrm{G}' = \mathrm{K}'\exp(\mathfrak{p}') \subset \mathrm{G}$ compatible with $\mathrm{U}^{\mathbb{C}}$ such that the interior of $\mu_{\mathfrak{p}'}(M)$ is nonempty in $\mathfrak{p}'$ and $\mu_{\mathfrak{p}'}(M) = \mu_{\mathfrak{p}}(M)$;*
ii) *up to shift $\mu_{\mathfrak{p}'}$, $0 \in \Omega(\mu_{\mathfrak{p}'})$.*

For the sake of clarity, we first prove some lemmata which will be useful in the proof of the above proposition.

Let $\mathfrak{u}' := \mathfrak{k} \oplus i\mathfrak{p}$. It is immediate to check that $\mathfrak{u}'$ is a subalgebra of $\mathfrak{u}$.

Lemma A.2. *Let $\beta_0 \in \mathfrak{p}$ be a* K*-fixed point. Then, $[\beta_0, \mathfrak{g}] = 0$ and $[i\beta_0, \mathfrak{u}'] = 0$.*

Proof. First, observe that $[\beta_0, \mathfrak{k}] = 0$, since β_0 a K-fixed point. Moreover, $[\beta_0, \mathfrak{p}] \subset \mathfrak{k}$ and $B(\mathfrak{k}, [\beta_0, \mathfrak{p}]) = -B([\beta_0, \mathfrak{k}], \mathfrak{p}) = 0$, as B is $\mathrm{Ad}(\mathrm{U}^{\mathbb{C}})$-invariant. Thus, $[\beta_0, \mathfrak{p}] = 0$ and, consequently, $[\beta_0, \mathfrak{g}] = 0$. Finally, from the definition of $\mathfrak{u}'$, it follows that $[i\beta_0, \mathfrak{u}'] = 0$. □

Consider $\mathrm{U}'' := \overline{\exp(\mathfrak{u}')} \subseteq \mathrm{U}$. U'' is a compact subgroup of U and G is a compatible subgroup of $(\mathrm{U}'')^{\mathbb{C}}$, too. Denote by $\mathfrak{u}''$ the Lie algebra of U''. The momentum mapping for the U''-action on (Z, J, ω) is given by $\pi_{\mathfrak{u}''} \circ \mu$, where $\pi_{\mathfrak{u}''} : \mathfrak{u} \to \mathfrak{u}''$ is the projection. Moreover, a result similar to Lemma A.2 also holds for $\mathfrak{u}''$.

Lemma A.3. *Let $\beta_0 \in \mathfrak{p}$ be a* K*-fixed point. Then, $[i\beta_0, \mathfrak{u}''] = 0$.*

Proof. Let $s \in \mathrm{U}''$ and let $\{\xi_n\}$ be a sequence in $\mathfrak{u}'$ such that $\exp(\xi_n) \to s$. By Lemma A.2, we have that $\exp(it\beta_0)\exp(\xi_n) = \exp(\xi_n)\exp(it\beta_0)$, for every $t \in \mathbb{R}$. Therefore, $\exp(it\beta_0)\, s = s\, \exp(it\beta_0)$, that is, $\exp(it\beta_0) \in \mathcal{Z}(\mathrm{U}'')$. □

In the light of the previous observations, up to replace U with U″, we can assume that $\mathrm{G} = \mathrm{K}\exp(\mathfrak{p})$ is a compatible subgroup of $\mathrm{U}^{\mathbb{C}}$ with Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and that for every K-fixed point $\beta_0 \in \mathfrak{p}$ we have $[\beta_0, \mathfrak{g}] = 0$ and $[i\beta_0, \mathfrak{u}] = 0$.

Let us focus on the convex hull $E := E(\mu_{\mathfrak{p}})$ of $\mu_{\mathfrak{p}}(M)$ in $\mathfrak{p}$. E is a K-invariant convex body. Let $\mathrm{Aff}(E)$ denote the affine hull of E. Then, $\mathrm{Aff}(E) = \beta_0 + \mathfrak{p}'$, where $\mathfrak{p}' \subseteq \mathfrak{p}$ is a linear subspace. Pick $\beta_0 \in E$ such that $\|\beta_0\| = \min_E \|\beta\|$. Observe that such β_0 is fixed by the K-action. Therefore, $\mathfrak{p}'$ is K-invariant. Hence, up to shift μ by $-i\beta_0$, we may assume that $E \subseteq \mathfrak{p}'$ and that the interior of E in $\mathfrak{p}'$ is nonempty. Summarizing, we have proved the following

Lemma A.4. *Up to shift the momentum mapping μ, there exists a* K*-invariant subspace $\mathfrak{p}' \subseteq \mathfrak{p}$ such that $E(\mu_{\mathfrak{p}})$ is contained in $\mathfrak{p}'$ and its interior in $\mathfrak{p}'$ is nonempty.*

Proof of Proposition A.1.

i) Consider the subspace $\mathfrak{p}'$ of $\mathfrak{p}$ obtained in Lemma A.4. Since $\mathfrak{p}'$ is K-invariant, $[\mathfrak{p}', \mathfrak{p}']$ is an ideal of $\mathfrak{k}$. Let $\mathfrak{h} := [\mathfrak{p}', \mathfrak{p}'] \oplus \mathfrak{p}'$. The Lie algebra $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$, where $\mathfrak{h}^{\perp}$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to B. By [8, Prop. 1.3], $\mathfrak{h}$ and $\mathfrak{h}^{\perp}$ are compatible K-invariant commuting ideals of $\mathfrak{g}$. Set $\mathrm{K}_1 := \exp([\mathfrak{p}', \mathfrak{p}'])$ and $\mathrm{H} = \mathrm{K}_1 \exp(\mathfrak{p}')$. Then, the group $\mathrm{G}' := \overline{\mathrm{H}} = \overline{\mathrm{K}}_1 \exp(\mathfrak{p}')$ is a compatible subgroup of $\mathrm{U}^{\mathbb{C}}$ and the G'-gradient map $\mu_{\mathfrak{p}'} : M \to \mathfrak{p}'$ associated with μ satisfies $\mu_{\mathfrak{p}'}(M) = \mu_{\mathfrak{p}}(M)$.

ii) Let ν be a K-invariant measure on $\mathfrak{p}'$ such that $\nu(E(\mu_{\mathfrak{p}'})) = 1$. Define $\theta := \int_{E(\mu_{\mathfrak{p}'})} \beta d\nu(\beta)$. θ is a K-fixed point of $E(\mu_{\mathfrak{p}'})$. In particular, $[i\theta, \mathfrak{u}] = 0$. We claim that $\theta \in \Omega(\mu_{\mathfrak{p}'})$. Indeed, otherwise there would exist $\xi \in \mathfrak{p}'$ such that $\langle \theta, \xi \rangle = c$, while $\langle \beta, \xi \rangle < c$ for every $\beta \in \Omega(\mu_{\mathfrak{p}'})$. From this follows that

$$\langle \theta, \xi \rangle = \int_{E(\mu_{\mathfrak{p}'})} \langle \beta, \xi \rangle d\nu(\beta) = \int_{\Omega(\mu_{\mathfrak{p}'})} \langle \beta, \xi \rangle d\nu(\beta) < c,$$

which is a contradiction. Therefore, up to shift μ by $-i\theta$, we have that $0 \in \Omega(\mu_{\mathfrak{p}'})$.

□

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