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STABILITY WITH RESPECT TO ACTIONS OF REAL REDUCTIVE LIE GROUPS

LEONARDO BILIOTTI AND MICHELA ZEDDA

Abstract. We give a systematic treatment of the stability theory for action of a real reductive Lie group G on a topological space. More precisely, we introduce an abstract setting for actions of non-compact real reductive Lie groups on topological spaces that admit functions similar to the Kempf-Ness function. The point of this construction is that one can characterize stability, semi-stability and polystability of a point by numerical criteria, that is in terms of a function called maximal weight. We apply this setting to the actions of a real non-compact reductive Lie group G on a real compact submanifold M of a Kähler manifold Z and to the action of G on measures of M.

Contents

1. Introduction

Stability theory in Kähler geometry has been intensively studied by many authors and from several points of view, see e.g. [16, 18, 19, 21, 22, 31, 32, 37, 42, 43]. This paper is inspired by the works of I. Mundet i Riera [39] and A. Teleman [44] where a systematical presentation of the stability theory in the non-algebraic Kählerian geometry of complex reductive Lie groups is given, and by the recent paper [8] where the first author jointly with A. Ghigi develops a geometrical

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invariant theory on topological spaces, without assuming the existence of a symplectic structure. In particular, they apply the main results to the action of $U^{\mathbb{C}}$ on measures on a compact Kähler manifold Z, where U is a compact connected Lie group acting in Hamiltonian fashion on Z. This was also motivated by an application to upper bounds for the first eigenvalue of the Laplacian on functions [2, 3, 4, 11, 29].

In this paper we identify an abstract setting to develop the geometrical invariant theory for actions of real reductive Lie groups. More precisely, given a Hausdorff topological space $\mathscr{M}$ with a continuous action of a non-compact real reductive Lie group $G = K\exp(\mathfrak{p})$ and a set of functions formally similar to the classical Kempf-Ness function we define an analogue of the gradient map $\mathfrak{F} : \mathscr{M} \longrightarrow \mathfrak{p}$ and the usual concepts of stability.

The gradient map has been intensively studied in [23, 24, 25, 27] and many other papers. The main idea is to investigate a class of actions of real reductive Lie groups on complex spaces and on real submanifolds using momentum map techniques. This means that we consider a Kähler manifold (Z, ω) acted on by a complex reductive Lie group $U^{\mathbb{C}}$ of holomorphic maps. The Kähler form ω is U-invariant, where U is a compact form of $U^{\mathbb{C}}$, and there exists a momentum map $\mu : Z \longrightarrow \mathfrak{u}^*$. We recall that a momentum map μ is U-equivariant and for any $\xi \in \mathfrak{u}$, the gradient of the function $\mu^{\xi}(x) = \mu(x)(\xi)$ is given by $J(\xi_Z)$, where $\xi_Z(p) = \frac{d}{dt}|_{t=0}\exp(t\xi)p$ is the vector field corresponding to $\xi \in \mathfrak{u}$ and J is the complex structure of Z (see [20, 35] for more details about momentum map). Since U is compact we may identify $\mathfrak{u} \cong \mathfrak{u}^*$ by means of an $\mathrm{Ad}(U)$-invariant scalar product on $\mathfrak{u}$. Hence we may think the momentum map as a $\mathfrak{u}$-valued map, i.e., $\mu : Z \longrightarrow \mathfrak{u}$.

Let $G \subset U^{\mathbb{C}}$ be compatible (see Definition 2 below). Then G is closed and the Cartan decomposition $U^{\mathbb{C}} = U\exp(i\mathfrak{u})$ induces a Cartan decomposition $G = K\exp(\mathfrak{p})$, where $K = G \cap U$ and $\mathfrak{p} = \mathfrak{g} \cap i\mathfrak{u}$. Identifying $i\mathfrak{u} \cong \mathfrak{u}$ the inclusion $\mathfrak{p} \hookrightarrow i\mathfrak{u}$ induces a K-equivariant map $\mu_{\mathfrak{p}} : Z \longrightarrow \mathfrak{p}$. Finally if M is a G-invariant real submanifold of Z, we may restrict $\mu_{\mathfrak{p}}$ to M and so considering $\mu_{\mathfrak{p}} : M \longrightarrow \mathfrak{p}$. The map $\mu_{\mathfrak{p}} : M \longrightarrow \mathfrak{p}$ is called *gradient map*. In Section 7 we extend the construction given in [38] for the gradient map, defining a Kempf-Ness function of (M, G, K).

The G-action on M induces in a natural way a continuous action on measures of M, that we denote by $\mathcal{P}(M)$, with respect to the weak-$*$ topology. In Section 8 we prove there exists a Kempf-Ness function for $(\mathcal{P}(M), G, K)$ and the map

$$\mathfrak{F}(\nu) = \int_M \mu_{\mathfrak{p}}(x)\mathrm{d}\nu(x),$$

is the analogue of the gradient map in this setting. These are our basic examples and the main motivations to develop a geometrical invariant theory for actions of real reductive Lie groups.

Stability and semi-stability are checked using the position of the G-orbit with respect to the vanishing locus of the gradient map. The main point of our construction is that one can characterize stability, semi-stability and polystability of a point by numerical criteria, that is in terms of a function called maximal weight, which is defined on the Tits boundary of the symmetric space of non-compact type G/K. Roughly speaking we extend criteria for stability, semi-stability and polystability due to Teleman [44], Mundet I Riera [38, 39], Kapovich, Leeb and Milson [30], Biliotti and Ghigi [8] and probably many others, for a large class of actions of complex reductive Lie groups, to actions of non-compact real reductive Lie groups. Our criterion for polystability is weaker than those proved by Mundet i Riera [39] and by the first author and Ghigi in [8] for complex reductive Lie gropus. However if $G = K^{\mathbb{C}} = K\exp(\mathbf{i}\mathfrak{k})$ is complex reductive then condition $(P3)$ in Section 3, i.e., $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0}\Psi(x, \exp(tv)) = 0$ if and only if $\exp(\mathbb{R}v) \subset G_x$, does not imply $\exp(\mathbb{C}v) \subset G_x$ as required by $(P3)$ in [8, p. 6]. This condition is crucial in Mundet's proof [39] and in the proof given in [8] for polystability. Indeed, thanks to the K-equivariance of $\mathfrak{F}$, if $\exp(\mathbb{C}v) \subset G_x$, then $\mathfrak{F}(x) \in \mathfrak{k}^v = \{u \in \mathfrak{k} : [u, v] = 0\}$ and thus a sort of a reduction principle applied.

In the abstract setting introduced in this paper, the above condition is equivalent to the following: if $\exp(\mathbb{R}v) \subset G_x$ then $\mathfrak{F}(x) \in \mathfrak{p}^v = \{u \in \mathfrak{p} : [u, v] = 0\}$. This does not hold for a general gradient map $\mathfrak{F}$ since it is only K-equivariant. On the other hand this condition holds for the gradient map [25] and the gradient map defined by the Kempf-Ness function with respect to the G action on measure (Proposition 45). The authors believe that the polystability criterion due to Mundet [39] holds under the above condition. We leave this problem for future investigation.

What is satisfactory of Theorem 33 is that the reductivity of the stabilizer is obtained as a consequence of conditions involving only the maximal weight and the set on which the maximal weight is zero. We also prove a version of the Hilbert-Munford criterion and the arguments in [8, Corollary 3.10] apply verbatim to the present context and imply that the set of stable points is open. Finally we completely characterize stable, semi-stable and polystable measures on real projective spaces.

The paper is organized as follows.

In Section 2 we review basic facts on real reductive Lie groups and Tits boundary of a Hadamard manifold.

In Section 3 we define the abstract setting and the general gradient map with respect to a Kempf-Ness function of $(\mathscr{M}, G, K)$.

In Section 4 we define the maximal weight on the Tits boundary of $X = G/K$. Since the Kempf-Ness function is K-invariant, for any $x \in \mathscr{M}$ the Kempf-Ness function descends to a function $\psi_x : X \longrightarrow \mathbb{R}$ which is geodesically convex. If ψ_x is Lipchitz then Lemma 9 defines what is called maximal weight on the Tits boundary of X. We also point out that the maximal weight is G-equivariant.

In Section 5 we define stable, semi-stable and polystable points giving a numerical criterion for an element $x \in \mathscr{M}$ to be stable (Theorem 23). We give a version of the Hilbert-Munford criterion (Corollary 28) and we prove the openness of the set of stable points (Corollary 29).

In Section 6 we give numerical criteria for semi-stability (Theorem 35) and polystability (Theorem 33) and a Hilbert-Munford criterion for semi-stable points (Corollary 36).

In Section 7 we discuss the basic example, i.e., the classical gradient map, and in Section 8 we apply our setting on the G action on measures. Using the Morse-Bott theory of the gradient map on M we compute rather explicitly the maximal weight. Moreover, under a condition on the gradient map which holds for any real flag manifold, if $0 \in E(\mu_{\mathfrak{p}})$, where $E(\mu_{\mathfrak{p}})$ is the convex hull of the image of the gradient map $\mu_{\mathfrak{p}}$, then any smooth measures is semi-stable (Proposition 55). The condition $0 \in E(\mu_{\mathfrak{p}})$ is always satisfied up to shifting the gradient map with respect to some $\mathrm{Ad}(K)$-fixed point of $\mathfrak{p}$. We also prove that the set of semi-stable points is dense. If 0 lies in the interior of $E(\mu_{\mathfrak{p}})$ then any smooth measures is stable and the set of stable points is open and dense. This condition is always satisfied if M is an adjoint orbit of K in $\mathfrak{p}$ and K acts irreducibly on $\mathfrak{p}$. In a recent paper ([9, Appendix A]) the authors point out that the condition 0 lies in the interior of $E(\mu_{\mathfrak{p}})$ is not restrictive. Indeed, such condition is always satisfied up to replace G with a compatible group $G' = K' \exp(\mathfrak{p}')$ such that $\mu_{\mathfrak{p}'}(M) = \mu_{\mathfrak{p}}(M)$ and up to shift $\mu_{\mathfrak{p}'}$.

In Section 9 we completely describe stable, semi-stable and polystable measures on real projective spaces.

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2. Tits boundary of G/K

Let G be a non-compact real reductive Lie group and denote by $\mathfrak{g}$ its Lie algebra. Recall that such G has a finite number of connected components and its algebra splits as $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z}(\mathfrak{g})$, where $[\mathfrak{g}, \mathfrak{g}]$ is semisimple and $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$. Further, maximal compact subgroups of G always exist and meet every connected components, and any two of them are conjugate under an element of the identity component G^o of G. Assume that there exists a Cartan involution $\theta : G \longrightarrow G$ with fixed points set K and let us denote also by $\theta : \mathfrak{g} \longrightarrow \mathfrak{g}$ its differential. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and the map $f : K \times \mathfrak{p} \to G$, $f(g, v) = g \exp v$ is a diffeomorphism. This means that $G = K \exp(\mathfrak{p})$ and G/K is simply connected. Since $\theta_{|\mathfrak{k}} = \mathrm{Id}$ and $\theta_{|\mathfrak{p}} = -\mathrm{Id}$, we have $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. Therefore if $\mathfrak{a} \subset \mathfrak{p}$ is a Lie subalgebra, then it must be abelian. Moreover, two maximal abelian subalgebras contained in $\mathfrak{p}$ are conjugate with respect to the identity component K^o. We refer the reader to [10, 28, 33] for more details on real reductive Lie groups. Set

$$X := G/K.$$

Observe that G acts on X from the left by:

$$L_g : X \to X, \qquad L_g(hK) := ghK, \qquad g \in G.$$

To simplify the notation, we will often write gx instead of $L_g(x)$. The choice of an $\mathrm{Ad}(K)$-invariant scalar product on $\mathfrak{p}$ induces a G-invariant Riemannian metric on X. It is well known that X endowed with this metric is a symmetric space of non-compact type and thus a Hadamard manifold [14, 28]. The Riemannian exponential map arises by the exponential map of Lie groups.

Hence a geodesic on X is given by $g\exp(tv)K$, where $g \in G$ and $v \in \mathfrak{p}$. In the sequel we denote by $\gamma^v(t) = \exp(tv)K$.

Since X is a Hadamard manifold there is a natural notion of boundary at infinity $\partial_\infty X$ which can be described using geodesics.

Two unit speed geodesic rays $\gamma, \gamma' : (0, +\infty) \to X$ are equivalent, denoted by $\gamma \sim \gamma'$, if $\sup_{t\in(0,+\infty)} d(\gamma(t), \gamma'(t)) < +\infty$. The *Tits boundary* of X, denoted by $\partial_\infty X$, is the set of equivalence classes of unit speed geodesic ray in X.

Set $o := K \in X$. Mapping v to the tangent vector $\dot\gamma^v(0)$ yields an isomorphism $\mathfrak{p} \cong T_o X$. Any geodesic ray in X is equivalent to a unique ray starting from o, so the map:

$$\mathrm{e} : S(\mathfrak{p}) \to \partial_\infty X, \ \mathrm{e}(v) := [\gamma^v], \tag{1}$$

where $S(\mathfrak{p})$ is the unit sphere in $\mathfrak{p}$, is a bijection. The *sphere topology* is the topology on $\partial_\infty X$ such that e is a homeomorphism. (For more details on the Tits boundary see for example [10, §I.2] and [14].)

Since G acts by isometries on X, if γ is a unit speed geodesic in X, then for each $g \in G$ also $g\gamma$ is. Further, since $\gamma \sim \gamma'$ implies $g\gamma \sim g\gamma'$, we get a G-action on the Tits boundary $\partial_\infty X$ by:

$$g \cdot [\gamma] = [g\gamma],$$

which also induces by (1) a G-action on $S(\mathfrak{p})$ given by:

$$g \cdot v = \mathrm{e}^{-1}(g \cdot \mathrm{e}(v)).$$

This action is continuous with respect to the sphere topology on $\partial_\infty X$ (see [10] p. 41), but it is not smooth.

Definition 2. *Let $H \subset G$ be a closed subgroup. Set $L := H \cap K$ and $\tilde{\mathfrak{p}} := \mathfrak{h} \cap \mathfrak{p}$. Following* [25, 26]*, we say that H is* compatible *if $H = L\exp(\tilde{\mathfrak{p}})$.*

If H is a compatible subgroup of G, then it follows that it is a real reductive subgroup of G, the Cartan involution of G induces a Cartan involution of H, L is a maximal compact subgroup of H and finally $\mathfrak{h} = \mathfrak{l} \oplus \tilde{\mathfrak{p}}$. Note that H has finitely many connected components. Moreover, there are totally geodesic inclusions $X' := H/L \hookrightarrow X$ and $\partial_\infty X' \subset \partial_\infty X$.

3. Kempf-Ness functions

Let $\mathscr{M}$ be a Hausdorff topological space and let G be a non-compact real reductive group which acts continuously on $\mathscr{M}$. Observe that with these assumptions we can write $G = K\exp(\mathfrak{p})$, where K is a maximal compact subgroup of G. Starting with these data we consider a function $\Psi : \mathscr{M} \times G \to \mathbb{R}$, subject to five conditions. The first four are the following ones:

($P1$) For any $x \in \mathscr{M}$ the function $\Psi(x, \cdot)$ is smooth on G.

($P2$) The function $\Psi(x, \cdot)$ is left–invariant with respect to K, i.e.: $\Psi(x, kg) = \Psi(x, g)$.

($P3$) For any $x \in \mathscr{M}$, and any $v \in \mathfrak{p}$ and $t \in \mathbb{R}$:

$$\frac{\mathrm{d}^2}{\mathrm{dt}^2}\Psi(x, \exp(tv)) \geq 0.$$

Moreover:

$$\left.\frac{\mathrm{d}^2}{\mathrm{dt}^2}\right|_{t=0}\Psi(x,\exp(tv))=0$$

if and only if $\exp(\mathbb{R}v)\subset G_x$.

$(P4)$ For any $x\in\mathscr{M}$, and any $g,h\in G$:

$$\Psi(x,g)+\Psi(gx,h)=\Psi(x,hg).$$

This equation is called the *cocycle condition.*

As in the previous section, let $X=G/K$. If Ψ is a function satisfying $(P1)$–$(P4)$, then by $(P2)$ the function $g\mapsto\Psi(x,g^{-1})$ descends to a function on X:

$$\psi_x:X\to\mathbb{R},\quad \psi_x(gK):=\Psi(x,g^{-1}),\tag{3}$$

and the cocycle condition $(P4)$ can be rewritten in terms of ψ_x as:

$$\psi_x(ghK)=\psi_x(gK)+\psi_{g^{-1}x}(hK),\tag{$P4'$}$$

which is also equivalent to the following identity between two functions and a constant:

$$L_g^*\psi_x=\psi_{g^{-1}x}+\psi_x(gK),\tag{4}$$

where L_g denotes the action of G on X (see previous section).

In order to state our fifth condition, let $\langle\cdot,\cdot\rangle:\mathfrak{p}^*\times\mathfrak{p}\to\mathbb{R}$ be the duality pairing. For $x\in\mathscr{M}$ define $\mathfrak{F}(x)\in\mathfrak{p}^*$ by requiring that:

$$\mathfrak{F}^v(x)=\langle\mathfrak{F}(x),v\rangle=-(d\psi_x)_o(\dot\gamma^v(0))=\left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0}\psi_x(\exp(-tv)K)=\left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0}\Psi(x,\exp(tv)).$$

The following is the fifth and last condition imposed on the function Ψ:

$(P5)$ The map $\mathfrak{F}:\mathscr{M}\to\mathfrak{p}^*$ is continuous.

We call $\mathfrak{F}$ the *gradient map* of $(\mathscr{M},G,K,\Psi)$. As immediate consequence of the definition of $\mathfrak{F}$ we have the following result.

Proposition 5. *The map $\mathfrak{F}:\mathscr{M}\to\mathfrak{p}^*$ is K-equivariant.*

Proof. It is an easy application of the cocycle condition and the left-invariance with respect to K of $\Psi(x,\cdot)$. Indeed,

$$\begin{aligned}\langle\mathfrak{F}(kx),v\rangle&=\left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0}\Psi(x,\exp(tv)k)=\left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0}\Psi(x,k^{-1}\exp(tv)k)\\&=\left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0}\Psi\left(x,\exp(t\mathrm{Ad}(k^{-1})(v))\right)=\mathrm{Ad}^*(k)(\mathfrak{F}(x))(v).\end{aligned}$$

□

The following definition summarizes the above discussion.

Definition 6. *Let G be a non-compact real reductive Lie group, K a maximal compact subgroup of G and $\mathscr{M}$ a topological space with a continuous G–action. A* Kempf-Ness function *for $(\mathscr{M}, G, K)$ is a function*

$$\Psi : \mathscr{M} \times G \to \mathbb{R},$$

that satisfies conditions $(P1)$–$(P5)$.

Remark 7. *Taking $g = h = e$ in the cocycle condition $(P4)$ we have $\Psi(x, e) = 0$. Hence $\Psi(x, k) = 0$ for every $k \in K$, since $\Psi(x, \cdot)$ is K-invariant on the second factor. Moreover, for any $x \in \mathscr{M}$ and for any $g, h \in G_x$ we have:*

$$\Psi(x, hg) = \Psi(x, g) + \Psi(x, h), \tag{8}$$

which implies that $\Psi(x, \cdot) : G_x \longrightarrow \mathbb{R}$ is a homomorphism.

4. Maximal weights

Let $X = G/K$ and let $u : X \to \mathbb{R}$ be a smooth function. We say that u is *geodesically convex* on X if $u(\gamma(t))$ is a convex function for any geodesic $\gamma(t)$ in X. The following lemma is proven in greater generality by Kapovich, Leeb and Millson in [30, §3.1] (see also [8, §2.3]).

Lemma 9. *Let $u : X \to \mathbb{R}$ be a smooth geodesically convex function on X. Assume that u is globally Lipschitz continuous. Then the function $u_\infty : \partial_\infty X \to \mathbb{R}$ given by:*

$$u_\infty([\gamma]) := \lim_{t \to +\infty} (u \circ \gamma)'(t), \tag{10}$$

is well–defined. Moreover u is an exhaustion if and only if $u_\infty > 0$ on $\partial_\infty X$.

Recall that a continuous function $f : X \to \mathbb{R}$ is an exhaustion if for any $c \in \mathbb{R}$ the set $f^{-1}((-\infty, c])$ is compact, condition which is equivalent for f to be bounded below and proper.

As in [8], the following result holds.

Lemma 11. *The function ψ_x is geodesically convex on X. More precisely, if $v \in \mathfrak{p}$ and $\alpha(t) = g\exp(tv)K$ is a geodesic in X, then $\psi_x \circ \alpha$ is either strictly convex or affine. The latter case occurs if and only if $g\exp(\mathbb{R}v)g^{-1} \subset G_x$. In the case $g = e$, the function $\psi_x \circ \alpha$ is linear if $\exp(\mathbb{R}v) \subset G_x$ and strictly convex otherwise.*

Due to Lemma 11, in order to apply Lemma 9 to ψ_x, we need only to add this last assumption:

$(P6)$ For any $x \in \mathscr{M}$, the function $\psi_x : X \to \mathbb{R}$ is globally Lipschitz on X.

When property $(P6)$ holds, for any $x \in \mathscr{M}$ the function $\lambda_x := (\psi_x)_\infty$ given by:

$$\lambda_x : \partial_\infty X \to \mathbb{R} \qquad \lambda_x([\gamma]) := \lim_{t \to +\infty} \frac{\mathrm{d}}{\mathrm{dt}} \psi_x(\gamma(t)), \tag{12}$$

is well-defined and finite. We call λ_x *maximal weight.* Moreover for any $x \in \mathscr{M}$, any $g \in G$ and any $p \in \partial_\infty X$ we have (see [8, Lemma 2.28] for a proof):

$$\lambda_{g^{-1}x}(p) = \lambda_x(g \cdot p). \tag{13}$$

The following function:

$$\lambda : \mathscr{M} \times \partial_\infty X \longrightarrow \mathbb{R}, \quad \lambda(x,p) := \lambda_x(p), \tag{14}$$

is also well-defined and finite. Since we set the sphere topology on $\partial_\infty X$, i.e., the topology on $\partial_\infty X$ such that $e : S(\mathfrak{p}) \to \partial_\infty X$ is an homeomorphism (see Section 2), by [8, Lemma 4.9], λ is lower semicontinuous and for $v \in S(\mathfrak{p})$ it follows:

$$\lambda_x(\mathrm{e}(v)) = \lim_{t\to+\infty} \frac{\mathrm{d}}{\mathrm{dt}} \psi_x(\exp(tv)K) = \lim_{t\to+\infty} \frac{\mathrm{d}}{\mathrm{dt}} \Psi(x, \exp(-tv)). \tag{15}$$

5. Stability

Let $(\mathscr{M}, G, K)$ be as above and let Ψ be a Kempf-Ness function. In particular, according to Definition 6 we assume that Ψ satisfies conditions $(P1)$–$(P5)$.

Definition 16. *Let $x \in \mathscr{M}$. Then:*

a) *x is* polystable *if $Gx \cap \mathfrak{F}^{-1}(0) \neq \emptyset$.*
b) *x is* stable *if it is polystable and $\mathfrak{g}_x$ is conjugate to a subalgebra of $\mathfrak{k}$.*
c) *x is* semi–stable *if $\overline{Gx} \cap \mathfrak{F}^{-1}(0) \neq \emptyset$.*
d) *x is* unstable *if it is not semi–stable.*

Remark 17. *The four conditions above are G-invariant in the sense that if a point x satisfies one of them, then every point in the orbit of x satisfy the same condition. This follows directly from the definition for polystability, semi–stability and unstability, while for stability it is enough to recall that $\mathfrak{g}_{gx} = \mathrm{Ad}(g)(\mathfrak{g}_x)$.*

The following result establishes a relation between the Kempf-Ness function and polystable points.

Proposition 18. *Let $x \in \mathscr{M}$. The following conditions are equivalent:*

a) *$g \in G$ is a critical point of $\Psi(x, \cdot)$;*
b) *$\mathfrak{F}(gx) = 0$;*
c) *$g^{-1}K$ is a critical point of ψ_x.*

Proof. Let $v \in \mathfrak{p}$. Using the cocycle condition $(P4)$, one gets:

$$\Psi(x, \exp(tv)g) = \Psi(x, g) + \Psi(gx, \exp(tv)).$$

Therefore,

$$\left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \Psi(x, \exp(tv)g) = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \Psi(gx, \exp(tv)) = \langle \mathfrak{F}(gx), v \rangle. \tag{19}$$

Since for any $k \in K$, $\Psi(x, kg) = \Psi(x, g)$, then $\mathfrak{F}(gx) = 0$ if and only if g is a critical point of $\Psi(x, \cdot)$ if and only if $g^{-1}K$ is a critical point of ψ_x. $\square$

Proposition 20. *If $\mathfrak{F}(x) = 0$, then G_x is compatible.*

Proof. Let $g \in G_x$. Then $g = k\exp(v)$ for some $k \in K$ and $v \in \mathfrak{p}$. By Proposition 5, we have $\mathfrak{F}(\exp(v)x) = 0$. Let $f(t) := \mathfrak{F}^v(\exp(tv)x) = \langle \mathfrak{F}(\exp(tv)x), v\rangle$. Then $f(0) = f(1) = 0$ and

$$\frac{\mathrm{d}}{\mathrm{dt}} f(t) = \frac{\mathrm{d}}{\mathrm{dt}} \mathfrak{F}^v(\exp(tv)x) = \frac{\mathrm{d}^2}{\mathrm{dt}^2} \Psi(x, \exp(tv)) \geq 0.$$

Therefore $\frac{\mathrm{d}^2}{\mathrm{dt}^2}\Psi(x, \exp(tv)) = 0$ for $0 \leq t \leq 1$. It follows from $(P3)$ that $\exp(tv)x = x$ for any $t \in \mathbb{R}$ and thus G_x is compatible. □

Next we give a numerical criteria for an element $x \in \mathscr{M}$ to be stable. We begin with the following lemma.

Lemma 21. *If $\mathfrak{a} \subset \mathfrak{g}$ is a subalgebra which is conjugate to a subalgebra of $\mathfrak{k}$, then $\mathfrak{a} \cap \mathfrak{p} = \{0\}$.*

Proof. It is enough to show that $\mathrm{Ad}(g)(\mathfrak{k}) \cap \mathfrak{p} = \{0\}$ for any $g \in G$. Let $X \in \mathrm{Ad}(g)(\mathfrak{k}) \cap \mathfrak{p}$. By the Cartan decomposition $G = K\exp(\mathfrak{p})$, it follows $\Gamma = \exp(\mathbb{R}X)$ is a closed abelian subgroup of G isomorphic to $\mathbb{R}$. On the other hand $X = \mathrm{Ad}(g)(Y)$ for some $Y \in \mathfrak{k}$ which implies $\Gamma = \mathrm{Ad}(g)(\exp(\mathbb{R}Y))$ is a torus. Hence $X = 0$. □

Consider the function:

$$\Lambda : \mathscr{M} \times \mathfrak{p} \to [-\infty, +\infty],$$

$$\Lambda(x, \xi) := \lim_{t\to+\infty} \frac{\mathrm{d}}{\mathrm{dt}} \Psi(x, \exp(t\xi)) = \lim_{t\to+\infty} \frac{\mathrm{d}}{\mathrm{dt}} \psi_x(-t\xi K).$$

The following Lemma is proven in [44, Lemma 2.10].

Lemma 22. *Let V be a subspace of $\mathfrak{p}$. For a point $x \in \mathscr{M}$ the following conditions are equivalent:*

a) *The map $\Psi(x, \exp(\xi))$ is linearly proper on V, i.e. there exist positive constants C_1 and C_2 such that:*

$$||\xi||^2 \leq C_1\Psi(x, \exp(\xi)) + C_2, \quad \forall\, \xi \in V.$$

b) *$\Lambda(x, \xi) > 0$ for any $\xi \in V - \{0\}$.*

Theorem 23. *Let $x \in \mathscr{M}$. Then x is stable if and only if $\Lambda(x, \xi) > 0$ for any $\xi \in \mathfrak{p} - \{0\}$.*

Proof. Let first $x \in \mathscr{M}$ be stable. Then $\mathfrak{F}(gx) = 0$ for some $g \in G$ and by Proposition 18, g is a critical point of $\Psi(x, \cdot)$. Set $y = gx$. We start by proving $\Lambda(y, \xi) > 0$ for any $\xi \in \mathfrak{p} - \{0\}$. By $(P3)$ the function $f(t) = \Psi(y, \exp(t\xi))$ is a convex function. Hence:

$$\Lambda(y, \xi) \geq f'(0) = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \Psi(y, \exp(t\xi)) = \langle \mathfrak{F}(y), \xi\rangle = 0.$$

Assume $\Lambda(y, \xi) = 0$. By assumption f is a convex function satisfying $\lim_{t\to+\infty} f'(t) = 0$ and $f'(0) = 0$. Hence $f'(t) = 0$ for $t \geq 0$ and so $\frac{\mathrm{d}^2}{\mathrm{dt}^2}\Psi(x, \exp(t\xi)) = 0$ for any $t \geq 0$. By $(P3)$ it follows that $\exp(\mathbb{R}\xi) \subset G_y$, so $\xi \in \mathfrak{g}_y \cap \mathfrak{p}$. Since x is stable, $\mathfrak{g}_y = \mathrm{Ad}(g)(\mathfrak{g}_x)$ is conjugate to a subalgebra of $\mathfrak{k}$, thus Lemma 21 implies that $\xi = 0$. By Lemma 22 the function $\Psi(y, \cdot)$ is linearly proper on $\mathfrak{p}$. By the cocycle condition we have

$$\Psi(x, \exp(\xi)) = \Psi(g^{-1}y, \exp(\xi)) = \Psi(y, \exp(\xi)g^{-1}) - \Psi(y, g^{-1}).$$

Write $\exp(\xi)g^{-1} = k(\xi)\exp(\sigma(\xi))$, for $k(\xi) \in K$, $\sigma(\xi) \in \mathfrak{p}$. Then $\Psi(x, \exp(\xi)) = \Psi(y, \exp(\sigma(\xi))) - \Psi(y, g^{-1})$. Using the same arguments in [38, Prop. 3.7 and Lemma 3.8] (see also [44, p. 193]), we get an estimate of the form

$$||\xi||^2 \leq A_1 ||\sigma(\xi)||^2 + A_2,$$

where A_1 and A_2 are positive constants. Therefore the linearly properness of $\Psi(y, \cdot)$ on $\mathfrak{p}$ implies the linearly properness of $\Psi(x, \cdot)$ on $\mathfrak{p}$. Hence, by Lemma 22, $\Lambda(x, \xi) > 0$ for any $\xi \in \mathfrak{p} - \{0\}$.

Assume now that $\Lambda(x, \xi) > 0$ for any $\xi \in \mathfrak{p} - \{0\}$. Then $\Lambda(x, \cdot)$ restricted on the unit sphere $S(\mathfrak{p})$ of $\mathfrak{p}$ has a minimum $C > 0$.

Let $\xi \in S(\mathfrak{p})$ and let $f(t) = \Psi(x, \exp(t\xi))$. The function f is a convex function and $\lim_{t \to +\infty} f'(t) \geq C$, respectively $\lim_{t \to -\infty} f'(t) \leq -C$. Hence f has a global minimum and $\lim_{t \to +\infty} f(t) = +\infty$. Thus, for any $M > 0$, there exists $t(\xi) > 0$ such that $f(t) = \Psi(x, \exp(t\xi)) > M$ for any $t \geq t(\xi)$.

We claim that there exists $\gamma_o > 0$ such that $\Psi(x, \exp(\xi)) > \frac{M}{2}$ for $\xi \in \mathfrak{p}$ with $||\xi|| \geq \gamma_o$. Indeed, otherwise there exist sequences $\xi_n \in S(\mathfrak{p})$ and $t_n \in \mathbb{R}$ with $t_n \mapsto +\infty$ such that $\Psi(x, \exp(t_n \xi_n)) \leq \frac{M}{2}$. We may assume $\xi_n \mapsto \xi_o$. Since $\Psi(x, \exp(t\xi_o)) \geq M$ for $t > t(\xi_o)$ and keeping in mind that the function

$$\mathbb{R} \times S(\mathfrak{p}) \longrightarrow \mathbb{R}, \qquad (t, \xi) \to \Psi(x, \exp(t\xi)),$$

is continuous, there exists a neighborhood U of ξ_o in $S(\mathfrak{p})$ and a neighborhood $(t(\xi_o) - \epsilon, t(\xi_o) + \epsilon)$ of $t(\xi_o)$ in $\mathbb{R}$, such that $\Psi(x, \exp(t\xi)) > \frac{M}{2}$ for any $t \in (t(\xi_o) - \epsilon, t(\xi_o) + \epsilon)$ and for any $\xi \in U$. Now, there exists $\tilde{n} \in \mathbb{N}$ such that $\xi_n \in U$ and $t_n > t(\xi_o)$ for $n \geq \tilde{n}$. Since the function $t \mapsto \Psi(x, \exp(t\xi))$ increases, it means $\Psi(x, \exp(t_n \xi_n)) > \frac{M}{2}$ for $n \geq \tilde{n}$ which is a contradiction. Now, keeping in mind that $\psi_x \circ \exp(\xi) = \Psi(x, \exp(-\xi))$, we have proved that the function $\psi_x \circ \exp$ has a minimum and so a critical point. Since $\exp : \mathfrak{p} \longrightarrow G/K$ is a diffeomorphism, it follows that ψ_x has a critical point. By Proposition 18 the point x is polystable. Let $g \in G$ such that $\mathfrak{F}(gx) = 0$. Set $y = gx$. Since

$$\Lambda(y, \xi) \geq \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \Psi(y, \exp(t\xi)) = \langle \mathfrak{F}(y), \xi \rangle = 0,$$

by the same arguments used before, we have $\Lambda(y, \xi) > 0$ for any $\xi \in \mathfrak{p} - \{0\}$. To conclude the proof we prove $\mathfrak{g}_y \cap \mathfrak{p} = \{0\}$.

Let $\xi \in \mathfrak{g}_y \cap \mathfrak{p}$. By Remark 7 the function $t \mapsto \Psi(y, \exp(t\xi))$ is linear. Since both $\Lambda(y, \xi)$ and $\Lambda(y, -\xi)$ are positive it follows

$$\lim_{t \mapsto +\infty} \frac{\mathrm{d}}{\mathrm{dt}} \Psi(y, \exp(t\xi)) = a \geq 0, \ \lim_{t \mapsto +\infty} \frac{\mathrm{d}}{\mathrm{dt}} \Psi(y, \exp(-t\xi)) = -a \geq 0.$$

This implies $a = 0$, $\Lambda(y, \xi) = 0$ and so $\xi = 0$. By Proposition 20, $\mathfrak{g}_y$ is a compatible subalgebra of $\mathfrak{g}$ with $\mathfrak{g}_y \cap \mathfrak{p} = \{0\}$. Hence $\mathrm{Ad}(g)(\mathfrak{g}_x) = \mathfrak{g}_y \subset \mathfrak{k}$ proving x is stable. □

Remark 24. *It is an immediate consequence of Lemma 22 and the definitions that the condition $\Lambda(x, \xi) > 0$ for any $\xi \in \mathfrak{p} - \{0\}$ is equivalent to ψ_x being an exhaustion.*

Corollary 25. *If $x \in \mathscr{M}$ is stable, then G_x is compact.*

Proof. Let $g \in G$ be such that $\mathfrak{F}(gx) = 0$ and set $y = gx$. By Proposition 20 the stabilizer of y, i.e. G_y, is compatible and so has only finitely many connected components. Moreover G_y^0 is compact since $\mathfrak{g}_y \subset \mathfrak{k}$. It follows that G_y and $G_x = g^{-1}G_y g$ are both compact. □

If $\mathscr{M}'$ is a G-invariant subspace of $\mathscr{M}$, the restriction of Ψ to $G \times \mathscr{M}'$ is a Kempf-Ness function for $(\mathscr{M}', G, K)$. The functions Λ and $\mathfrak{F}$ for $(\mathscr{M}', G, K)$ are simply the restrictions of those for $\mathscr{M}$. If $G' \subset G$ is a compatible subgroup of G, i.e., $G' = K' \exp(\mathfrak{p}')$, then $K' \subset K$, $\mathfrak{p}' \subset \mathfrak{p}$ and $X' := G'/K' \hookrightarrow X$ is a totally geodesic inclusion. If Ψ is a Kempf-Ness function for $(G, K, \mathscr{M})$, then $\Psi^{K'} := \Psi|_{\mathscr{M} \times G'}$ is a Kempf-Ness function for $(G', K', \mathscr{M})$. The related functions are

$$\mathfrak{F}^{K'} : \mathscr{M} \to \mathfrak{p}'^*, \qquad \mathfrak{F}^{K'}(x) := \mathfrak{F}(x)|_{\mathfrak{p}'}, \tag{26}$$

$$\psi_x^{K'} := \psi_x|_{X'}, \qquad \Lambda^{K'} = \Lambda|_{\mathscr{M} \times \mathfrak{p}'}. \tag{27}$$

A subalgebra contained in $\mathfrak{p}$ must be abelian since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. The following Corollary is analogous to the stability part in the Hilbert-Mumford criterion.

Corollary 28. *A point $x \in \mathscr{M}$ is G-stable if and only if it is A-stable for any abelian group $A = \exp(\mathfrak{a})$, where $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$.*

Proof. By Theorem 23 it is enough to prove that we have $\Lambda(x, \xi) > 0$ for any $\xi \in \mathfrak{p} - \{0\}$ if and only if for any abelian group $A = \exp(\mathfrak{a})$, where $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$ we have $\Lambda^A(x, \xi) > 0$ for any $\xi \in \mathfrak{a} - \{0\}$. The necessary condition is trivial, being $\Lambda^A(x, \xi)$ the restriction of $\Lambda(x, \xi)$ to $\mathfrak{a}$. For the sufficient, observe that for any $\xi \in S(\mathfrak{p})$ we can set $\mathfrak{a} = \mathbb{R}\xi$ and conclusion follows since with this choice we have $\Lambda(x, \xi) = \Lambda^A(x, \xi)$. □

We conclude this section with the following interesting result.

Corollary 29. *The function $\Lambda : \mathscr{M} \times S(\mathfrak{p}) \longrightarrow \mathbb{R}$ is lower semincontinuos and the set of stable points is open in $\mathscr{M}$.*

Proof. The proof of [8, Lemma 3.9] works also for Λ proving it is lower semicontinuos. The openness of the stable points can be proved as in [8, Corollary 3.10]. □

6. Polystability and semi-stability

The aim of this section is to characterize polystability and semi-stability of $x \in \mathscr{M}$ in terms of the maximal weight λ_x. Throughout this section we assume that the Kempf-Ness function of $(\mathscr{M}, G, K)$ satisfies not only $(P1)$–$(P5)$ but also $(P6)$. Further, for semi-stability we also assume that $\mathscr{M}$ is compact. This will be enough for the case of measures on a compact manifold.

Let us denote by $\mathscr{M}^{ps}$ the set of polystable points, i.e. according to Definition 16:

$$\mathscr{M}^{ps} = \{x \in \mathscr{M} : Gx \cap \mathfrak{F}^{-1}(0) \neq \emptyset\}.$$

It follows by an easy argument that if $x \in \mathscr{M}$ is polystable then $Gx \cap \mathfrak{F}^{-1}(0)$ contains exactly one K-orbit. Indeed, let $y \in Gx$ be such that $\mathfrak{F}(y) = 0$. We shall prove that $Ky = Gy \cap \mathfrak{F}^{-1}(0)$. Assume that $gy \in \mathfrak{F}^{-1}(0)$. Set $g = k \exp(v)$. By the K-equivariance of $\mathfrak{F}$ it follows $\mathfrak{F}(\exp(v)y) =$

0. As in the proof of Proposition 20, we get $\mathbb{R}v \in \mathfrak{g}_y$ and so $Gy \cap \mathfrak{F}^{-1}(0) = Ky$. Hence we have proven the following result.

Proposition 30. *The inclusion $\mathfrak{F}^{-1}(0) \hookrightarrow \mathscr{M}^{ps}$ induces a bijection*

$$\mathfrak{F}^{-1}(0)/K \longrightarrow \mathscr{M}^{ps}/G.$$

Hence the set of polystable orbits, endowed with the quotient topology, is Hausdorff.

In this section we give a numerical criteria for an element $x \in \mathscr{M}$ to be a polystable point. Let $x \in \mathscr{M}$. We define $Z(x) = \{p \in \partial_\infty X : \lambda_x(p) = 0\}$. We start with the following result.

Proposition 31. *Let $x \in \mathfrak{F}^{-1}(0)$. Then $\lambda_x \geq 0$, $\mathfrak{g}_x = \mathfrak{k}_x \oplus \mathfrak{q} \subset \mathfrak{k} \oplus \mathfrak{p}$ is compatible and $Z(x) = \mathrm{e}(S(\mathfrak{q})) = \partial_\infty G_x/K_x$.*

Proof. By Proposition 20 the stabilizer G_x is compatible. Hence $\mathfrak{g}_x = \mathfrak{k}_x \oplus \mathfrak{q}$ with $\mathfrak{k}_x \subset \mathfrak{k}$ and $\mathfrak{q} \subset \mathfrak{p}$. Further, observe that for $\xi \in \mathfrak{p}$, since $\Psi(x, \exp(t\xi))$ is a convex function, we get:

$$\Lambda(x, \xi) \geq \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \Psi(y, \exp(t\xi)) = \langle \mathfrak{F}(y), \xi \rangle = 0.$$

To conclude, we shall prove that $v \in S(\mathfrak{q})$ if and only if $\lambda_x(\mathrm{e}(-v)) = 0$. Let first $v \in S(\mathfrak{q})$. By Remark 7 the function:

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \qquad t \mapsto \Psi(x, \exp(tv)),$$

is linear. Since $\lambda_x \geq 0$, we have $\lim_{t\to+\infty} f'(t) = a \geq 0$ and $\lim_{t\to+\infty} f'(-t) = -a \geq 0$. Thus, $f(t) = \Psi(x, \exp(tv)) = 0$ and condition $(P3)$ implies $\lambda_x(\mathrm{e}(-v)) = 0$.

Vice-versa, assume $\lambda_x(\mathrm{e}(-v)) = 0$ and consider again the function $f(t) = \Psi(x, \exp(tv))$. Observe that f is convex and by assumptions $\lim_{t\to+\infty} f'(t) = 0$ and $f'(0) = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \Psi(x, \exp(tv)) = \langle \mathfrak{F}(x), v \rangle = 0$. Hence $f'(t) = 0$ for $t \geq 0$. Therefore $f''(t) = \frac{\mathrm{d}^2}{\mathrm{dt}^2}|_{t=0} \Psi(x, \exp(tv)) = 0$. By property $(P3)$ we get $\mathbb{R}v \in \mathfrak{q}$ concluding the proof. $\square$

Note that the inclusion $G_x/K_x \hookrightarrow X$ is totally geodesic. We claim that the converse of Proposition 31 holds as well. We start with the following Lemma.

Lemma 32. *Let $x \in \mathscr{M}$. Assume $\lambda_x \geq 0$ and $Z(x) = \partial_\infty X'$, where X' is a totally geodesic submanifold of X. Then, there exists $g \in G$ such that setting $y = gx$ we have $Z(y) = \partial_\infty G'/K'$, where G' is compatible, $G' \cap K = K'$ and $G' \subset G_y$.*

Proof. Assume first $o = [K] \in X'$. We shall prove that the statement holds for $g = e$. Since X' is a totally geodesic submanifold of X there exists a subspace $\mathfrak{q} \subset \mathfrak{p}$, called *Lie triple system* of $\mathfrak{p}$, such that $X' = \exp(\mathfrak{q})$ and $[[\mathfrak{q}, \mathfrak{q}], \mathfrak{q}] \subset \mathfrak{q}$ (see e.g. [28]). We claim $\mathfrak{q} \subset \mathfrak{g}_x$. Indeed, let $v \in S(\mathfrak{q})$. Since $\lambda_x(e(-v)) = \lambda_x(e(v)) = 0$, the convex function $f(t) = \Psi(x, \exp(tv))$ satisfies $\lim_{t\to\pm\infty} f'(t) = 0$. Hence f' is constant and so

$$f''(0) = \left.\frac{d^2}{dt}\right|_{t=0} \Psi(x, \exp(tv)) = 0.$$

By properties $(P3)$ we have $v \in \mathfrak{g}_x$. Let $\mathfrak{g}' = [\mathfrak{q}, \mathfrak{q}] \oplus \mathfrak{q}$. Observe that $\mathfrak{g}'$ is a subalgebra of $\mathfrak{g}$ due to the fact that $\mathfrak{q}$ is a Lie triple system of $\mathfrak{p}$ (see e.g. [28]). Let G' denote the connected subgroup of G with lie algebra $\mathfrak{g}'$. Hence $G' = (G' \cap K)\exp(\mathfrak{q})$ and $G' \subset G_x$. Therefore $G' = \overline{G''} = (\overline{G''} \cap K)\exp(\mathfrak{q})$ is compatible, $G' \subset G_x$ and if we denote by $K' = G' \cap K$ we have $\partial_\infty X' = \partial_\infty G'/K'$.

In general, for any $g \in G$ we can consider the totally geodesic submanifold defined by $X'' = gX'$. Since by (13) it follows $Z(gx) = g(Z(x))$, we have:

$$Z(gx) = g\partial_\infty X' = \partial_\infty X'',$$

and we are done.

□

Theorem 33. *An element $x \in \mathscr{M}$ is a polystable point if and only if $\lambda_x \geq 0$ and $Z(x) = \partial_\infty X'$ for some totally geodesic submanifold $X' \subset X = G/K$.*

Proof. One direction is proved in Proposition 31. Assume $\lambda_x \geq 0$ and $Z(x) = \partial_\infty X'$ for some totally geodesic submanifold $X' \subset X = G/K$. By the above lemma and property (13), we may assume $Z(x) = \partial_\infty G'/K'$ where $G' = K'\exp(\mathfrak{q}) \subset G_x$, $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{q}$ with $\mathfrak{k}' \subset \mathfrak{k}$ and $\mathfrak{q} \subset \mathfrak{p}$, $G' = K'\exp(\mathfrak{q})$ and $Z(x) = \mathrm{e}(S(\mathfrak{q}))$. Write $\mathfrak{p} = \mathfrak{q} \oplus \mathfrak{q}^\perp$. By a Mostow decomposition, see [24, Th. 9.3 p. 211], any $g \in G$ can be written as $g = k\exp(\theta)h$, where $k \in K$, $h \in G'$ and $\theta \in \mathfrak{q}^\perp$. Therefore by the K-invariants and the cocycle condition of Ψ, keeping in mind that $G' \subset G_x$, we get:

$$\Psi(x, g) = \Psi(x, k\exp(\theta)h) = \Psi(x, \exp(\theta)) + \Psi(x, h).$$

We claim that $\Psi(x, h) = 0$. Indeed, $h = k\exp(v)$ with $k \in K'$ and $v \in \mathfrak{q}$. Hence $\Psi(x, h) = \Psi(x, \exp(v))$. As in the above lemma, we consider the function $f(t) = \Psi(x, \exp(tv))$ which is linear due to Remark 7. Since $\lambda_x(e(\pm v)) = 0$, we have $\lim_{t \mapsto \pm\infty} f'(t) = 0$, which implies $f \equiv 0$ and thus $\Psi(x, h) = 0$. Hence $\Psi(x, g) = \Psi(x, \exp(\theta))$. Since $\Lambda(x, \cdot) > 0$ on $\mathfrak{q}^\perp - \{0\}$, by Lemma 22 there exist positive constants C_1 and C_2 such that

$$||\theta||^2 \leq C_1\Psi(x, \exp(\theta)) + C_2.$$

This means $\Psi(x, \cdot)_{|_{\mathfrak{q}^\perp}}$ is an exhaustion and so it has a minimum. Since $\Psi(x, g) = \Psi(x, \exp(\theta))$ with $\theta \in \mathfrak{q}^\perp$, this means that $\Psi(x, \cdot)$ has a minimum and thus a critical point. By Proposition 18 the point x is polystable. □

Corollary 34. *Let $x \in \mathscr{M}$ be a polystable point. Let $g \in G$ be such that $\mathfrak{F}(gx) = 0$. Then there exists an abelian subalgebra $\mathfrak{a} \subset \mathfrak{p} \cap \mathfrak{g}_{gx}$ such that:*

(i) gx is $G^{\mathfrak{a}}$ polystable, where $G^{\mathfrak{a}}$ is the centralizer of $\mathfrak{a}$ in G;
(ii) gx is G'_{ss} stable, where G'_{ss} denotes the semisimple part of $G^{\mathfrak{a}}$.

Proof. Let $x \in \mathscr{M}$ be a polystable point and let $g \in G$ be such that $\mathfrak{F}(gx) = 0$. Set $y = gx$. By Proposition 31, G_y is compatible, and thus $\mathfrak{g}_y = \mathfrak{k}_y \oplus \mathfrak{p}_y \subset \mathfrak{k} \oplus \mathfrak{p}$, and $Z(y) = \mathrm{e}(S(\mathfrak{p}_y))$. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}_y$. The centralizer of $\mathfrak{a}$ in G, $G^{\mathfrak{a}} = \{g \in G : \mathrm{Ag}(g)(\xi) = \xi$

for all $\xi \in \mathfrak{a}\}$ is a compatible subgroup of G (see [33]) and by (26) it follows $\mathfrak{F}'(y) = 0$ and so y is polystable with respect to $G^{\mathfrak{a}}$. Let G'_{ss} be the semisimple part of $G^{\mathfrak{a}}$. By 26 it follows that y is G'_{ss} polystable and so $(\mathfrak{g}'_{ss})_y$ is compatible. We claim $(\mathfrak{g}'_{ss})_y \cap \mathfrak{p} = \{0\}$. Indeed, if $v \in (\mathfrak{g}'_{ss})_y \cap \mathfrak{p}$, then $v \in \mathfrak{p}_y$ and $[v, \mathfrak{a}] = 0$. Since $v \notin \mathfrak{a}$ and $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{p}_y$ we get a contradiction. Since $(\mathfrak{g}'_{ss})_y$ is compatible it follows $(\mathfrak{g}'_{ss})_y \subset \mathfrak{k}$ and so $(G'_{ss})_y$ is compact. Therefore y is G'_{ss} stable concluding the proof. $\square$

The following theorem, in analogy with [8, Th. 4.17], gives a numerical criteria for semi-stable points in terms of maximal weights. The proof is the same of the proof of [44, Theorem 4.3] and thus it follows by [30, Lemma 3.4] due to Kapovich, Leeb and Millson.

Theorem 35. *If $\mathscr{M}$ is compact, then a point $x \in \mathscr{M}$ is semi-stable if and only if $\lambda_x \geq 0$.*

The following result is a Hilbert-Mumford criterion for semi-stability. The proof is completely similar to that of Corollary 28.

Corollary 36. *A point $x \in \mathscr{M}$ is G semi-stable if and only if it is A semi-stable for any abelian group $A = \exp(\mathfrak{a})$, where $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$.*

We conclude this section with the following corollaries.

Corollary 37. *Let $x \in \mathscr{M}$ be a semi-stable point. Then either x is stable or $\overline{Gx} \cap \mathfrak{F}^{-1}(0) \subset \mathscr{M}^{ps}$.*

Proof. Let $x \in \mathscr{M}$ be a semi-stable point which is not stable. Setting $\mathscr{M}' = \overline{Gx}$, the restriction of Ψ to $G \times \mathscr{M}'$ is a Kempf-Ness function for $(\mathscr{M}', G, K)$ and the functions Λ and $\mathfrak{F}$ for $(\mathscr{M}', G, K)$ are simply the restrictions of those for $\mathscr{M}$. By Corollary 29 the set of stable points of $\mathscr{M}'$ is open. By definition the set of stable points is G-invariant. Hence if a point $z \in \mathscr{M}'$ were stable, then x would also be stable contradicting our assumption. $\square$

Corollary 38. *If $x \in \mathscr{M}$ is semi-stable then so is any $y \in \overline{Gx}$.*

Proof. Let $g_\alpha \in G$ be a net such that $g_\alpha x \to y$ and let $v \in \partial_\infty X$. By the G-equivariance of the maximal weight (14) and the semicontinuity of λ, we get:

$$\lambda_y(v) = \lambda_{g_\alpha^{-1} y}(g_\alpha v) \geq \liminf_\alpha \lambda_{g_\alpha^{-1} x}(g_\alpha v) \geq 0,$$

concluding the proof. $\square$

7. The integral of the gradient map

Let U be a compact connected Lie group and denote by $\mathfrak{u}$ its Lie algebra and by $U^{\mathbb{C}}$ its complexification. Let (Z, ω) be a Kähler manifold on which $U^{\mathbb{C}}$ acts holomorphically. Assume that U acts in a Hamiltonian fashion with momentum map $\mu : Z \longrightarrow \mathfrak{u}^*$. Consider a closed connected subgroup G of $U^{\mathbb{C}}$ compatible with respect to the Cartan decomposition of $U^{\mathbb{C}}$, i.e. $G = K \exp(\mathfrak{p})$, for $K = U \cap G$ and $\mathfrak{p} = \mathfrak{g} \cap i\mathfrak{u}$ (see [25, 26]). The inclusion $i\mathfrak{p} \hookrightarrow \mathfrak{u}$ induces by restriction a K-equivariant map $\mu_{i\mathfrak{p}} : Z \longrightarrow (i\mathfrak{p})^*$. There is a $\mathrm{Ad}\,(U^{\mathbb{C}})$–invariant and non-degenerate bilinear form $B : \mathfrak{u}^{\mathbb{C}} \times \mathfrak{u}^{\mathbb{C}} \longrightarrow \mathbb{R}$ which is positive definite on $i\mathfrak{u}$, negative definite on $\mathfrak{u}$

and such that $B(\mathfrak{u}, i\mathfrak{u}) = 0$ (see [5, p. 585]). Therefore B is $\mathrm{Ad}\,(U^{\mathbb{C}})$–invariant, non-degenerate and its restriction to $\mathfrak{g}$ satisfies the following conditions: B is $\mathrm{Ad}(G)$–invariant, $B(\mathfrak{k}, \mathfrak{p}) = 0$, B restricted to $\mathfrak{k}$ is negative definite and B restricted to $\mathfrak{p}$ is positive definite. Using $\langle \cdot, \cdot \rangle$, we identify $\mathfrak{u} \cong \mathfrak{u}^*$. For $z \in Z$, let $\mu_{\mathfrak{p}}(z) \in \mathfrak{p}$ denote $-i$ times the component of $\mu(z)$ in the direction of $i\mathfrak{p}$. In other words we require that $\langle \mu_{\mathfrak{p}}(z), \beta \rangle = -\langle \mu(z), i\beta \rangle$, for any $\beta \in \mathfrak{p}$. Then, we view $\mu_{\mathfrak{p}}$ as a map:

$$\mu_{\mathfrak{p}} : Z \to \mathfrak{p},$$

which is called the *G-gradient map* or *restricted momentum map* associated to μ. We also set:

$$\mu_{\mathfrak{p}}^{\beta} := \langle \mu_{\mathfrak{p}}, \beta \rangle = \mu_{\mathfrak{p}}^{-i\beta}.$$

By definition, it follows that $\mathrm{grad}\mu_{\mathfrak{p}}^{\beta} = \beta_Z$, where $\beta_Z(x) = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \exp(t\beta)x$.

Throughout this section we fix a G-invariant subset $M \subset Z$ and we consider the gradient map $\mu_{\mathfrak{p}} : M \longrightarrow \mathfrak{p}$ restricted on M. Further, we denote by $\beta_M = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \exp(t\beta)x$. Observe that if M is a manifold, then β_M is the gradient of $\mu_{\mathfrak{p}}^{\beta}$ restricted to M with respect to the induced Riemannian structure on M.

As Mundet pointed out in [40], the existence of the Kempf-Ness function for an action of a complex reductive group on a Kähler manifold given in [38] also holds for the setting introduced in [24, 25, 27].

Theorem 39. *There exists a Kempf-Ness function for (M, G, K) satisfying the conditions $(P1) - (P5)$. Furthermore, if M is a G-invariant compact submanifold of Z, then $(P6)$ holds as well.*

Proof. Fix $x \in M$. Let $\pi_{\mathfrak{p}} : \mathfrak{g} \longrightarrow \mathfrak{p}$ be the linear projection induced by the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and identify T_eG with $\mathfrak{g}$ in the usual way. For $g \in G$ and $v \in T_gG$, one has $dR_{g^{-1}}(v) \in \mathfrak{g}$. Thus, we can define a 1-form σ on G by setting:

$$\sigma_g(v) := \langle \mu_{\mathfrak{p}}(gx), \pi_{\mathfrak{p}}(dR_{g^{-1}}(v)) \rangle.$$

Observe that $\sigma_g \in T_gG^*$ and $\sigma \in \Lambda^1(G)$. When we need to stress the dependence on x we will write σ^x. We claim that σ is closed. In order to prove it, fix $g \in G$, $v, w \in T_gG$ and let $\xi, \eta \in \mathfrak{g}$ be such that $dR_g(\xi) = v$ and $dR_g(\eta) = w$. Further, let also $X, Y \in \mathfrak{X}(G)$ be the fundamental vector fields corresponding to ξ and η under the action of *left* multiplication. In other words X is the right-invariant vector field such that $X(e) = v$, i.e. for $h \in G$,

$$X(h) := dR_h(v) = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \exp(tv)h.$$

For a left action the map that sends a vector in $\mathfrak{g}$ to its fundamental vector field is an anti-isomorphism of Lie algebras. Thus $[X, Y]$ is the fundamental vector field corresponding to

$-[\xi,\eta]$. Hence:

$$[X,Y](g):=dR_g(-[\xi,\eta]),$$
$$\sigma([X,Y])(g)=\langle\mu_{\mathfrak{p}}(gx),\pi_{\mathfrak{p}}([\xi,\eta])\rangle.$$

We can assume by linearity that $\xi,\eta\in\mathfrak{k}\cup\mathfrak{p}$.

It is immediate from the definition that $\sigma(X)=\sigma(Y)=\sigma([X,Y])\equiv 0$ if $\xi,\eta\in\mathfrak{k}$. Thus recalling that:

$$(d\sigma)_g(v,w)=X(g)\sigma(Y)-Y(g)\sigma(X)-\sigma([X,Y])(g),$$

for $\xi,\eta\in\mathfrak{k}$ the claim is proven.

Assume now that $\xi\in\mathfrak{k}$ and $\eta\in\mathfrak{p}$. Then $\sigma(X)\equiv 0$ and for $h\in G$,

$$\sigma(Y)(h)=\langle\mu_{\mathfrak{p}}(hx),\eta\rangle=\mu_{\mathfrak{p}}^{\eta}(hx).$$

By the K-equivariance of the gradient map we have:

$$\begin{aligned}(X\sigma(Y))(g)&=\left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0}\sigma(Y)(\exp(t\xi)g)=\left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0}\mu_{\mathfrak{p}}^{\eta}(\exp(t\xi)gx)\\&=\left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0}\langle\mathrm{Ad}(\exp(t\xi)(\mu_{\mathfrak{p}}(gx)),\eta\rangle=\langle[\xi,\mu_{\mathfrak{p}}(gx)],\eta\rangle.\end{aligned}$$

Thus:

$$\begin{aligned}d\sigma(v,w)&=\langle[\xi,\mu_{\mathfrak{p}}(gx)],\eta\rangle-\langle\mu_{\mathfrak{p}}(gx),\pi_{\mathfrak{p}}([\eta,\xi])\rangle\\&=\langle[\xi,\mu_{\mathfrak{p}}(gx)],\eta\rangle-\langle\mu_{\mathfrak{p}}(gx),[\eta,\xi]\rangle\\&=\langle[\xi,\mu_{\mathfrak{p}}(gx)],\eta\rangle-\langle[\xi,\mu_{\mathfrak{p}}(gx)],\eta\rangle\\&=0.\end{aligned}$$

Finally, we consider the last possibility, $\xi,\eta\in\mathfrak{p}$. In this case $[\xi,\eta]\in\mathfrak{k}$ and thus $\sigma([X,Y])\equiv 0$. On the other hand:

$$(X\sigma(Y))(g)=\left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0}\sigma(Y)(\exp(t\xi)\cdot g)=(d\mu_{\mathfrak{p}}^{\eta})_{(gx)}(\xi_M)=\langle\eta_M,\xi_M\rangle,$$

which is symmetric in ξ and η, implying $d\sigma(v,w)=0$ also in this case.

This shows that σ is closed. Let $\gamma\in\Omega(G,e,e)$. Then there exists $\gamma'\in\Omega(K,e,e)$ such that $\gamma\sim i\circ\gamma'$, where $i:K\hookrightarrow G$, and thus:

$$\int_\gamma\sigma=\int_{\gamma'}i^*\sigma.$$

Since $i^*\sigma=0$, it follows that σ is exact. Let $\Psi_x\in C^\infty(G)$ be the unique function such that $\Psi_x(e)=0$ and $d\Psi_x=\sigma^x$. Since ${\sigma^x}_{|TK}\equiv 0$, then $\Psi_x(h)=0$ for any $h\in K$. Moreover, for any $\eta\in\mathfrak{p}$, we have:

$$(d\Psi_x)_{(e)}(\eta)=\mu_{\mathfrak{p}}^{\eta}(x).$$

Thus, the function:

$$\Psi:M\times G\to\mathbb{R}\qquad\Psi(x,g):=\Psi_x(g),$$

satisfies conditions ($P1$) and ($P5$). In order to prove ($P3$), compute:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\Psi_x(\exp(t\eta)) &= (\sigma^x)_{\exp(t\eta)}\left(\frac{\mathrm{d}}{\mathrm{dt}}\exp(t\eta)\right)\\ &= (\sigma^x)_{\exp(t\eta)}(dR_{\exp(t\eta)}(\eta))\\ &= \langle\mu_{\mathfrak{p}}(\exp(t\eta)x),\eta\rangle\\ &= \mu_{\mathfrak{p}}^{\eta}(\exp(t\eta)x).\end{aligned}$$

Therefore,

$$\frac{\mathrm{d}^2}{dt^2}\Psi_x(\exp(t\eta)) = d\mu^\eta(\eta_M)(\exp(t\eta)x) = ||\eta_M||^2(\exp(t\eta)x),$$

and thus ($P3$) follows.

In order to prove ($P4$), let $g \in G$ and $x \in M$. We claim that $R_g^*\sigma^x = \sigma^{gx}$. Indeed if $v \in T_hG$ and $w = dR_{h^{-1}}(v)$, then:

$$\sigma_h^{gx}(v) = \langle\mu(hgx), \pi_{\mathfrak{p}}(w)\rangle,$$

$$(R_g^*\sigma^x)_h(v) = (\sigma^x)_{hg}(dR_g(v))\rangle = \langle\mu(hgx), \pi_{\mathfrak{p}}(dR_{(hg)^{-1}}dR_g(v))\rangle = \langle\mu(hgx), \pi_{\mathfrak{p}}(w)\rangle.$$

Thus the claim is proven. Therefore $\Psi_{gx} - R_g^*\Psi_x = c$ is a constant. Evaluating at $h = e$ we get:

$$c = 0 - \Psi_x(g)$$

and thus:

$$\Psi_{gx}(h) + \Psi_x(g) = \Psi_x(hg),$$

as desired. Property ($P2$) follows by the cocycle condition together with the fact that for any $x \in M$, $\Psi_x(h) = 0$ for all $h \in K$.

Finally, if M is a compact G-invariant submanifold of Z, then the norm square of the gradient map restricted to M is bounded. Hence ψ_x is Lipschitz since its differential is bounded and thus ($P6$) holds. □

As direct consequence of Corollary 29 we get the following result.

Theorem 40. *Let $M \subset Z$ be a G-invariant subset of Z. Then the set of stable points for the gradient map $\mu_{\mathfrak{p}}\colon M \longrightarrow \mathfrak{p}$ restricted to M is open. Moreover, if $G = A = \exp(\mathfrak{a})$, where $\mathfrak{a} \subset \mathfrak{p}$ is an abelian subalgebra, and $\mu_{\mathfrak{a}} : M \longrightarrow \mathfrak{a}$ is the gradient map of A, then for any $\beta \in \mathfrak{a}$, the set $\{p \in M :\ Ap \cap \mu_{\mathfrak{a}}^{-1}(\beta) \neq \emptyset \text{ and } \mathfrak{a}_p = \{0\}\}$ is open.*

When M is a compact G-invariant submanifold of Z, Theorems 33 and 35 also hold for the gradient map $\mu_{\mathfrak{p}}\colon M \longrightarrow \mathfrak{p}$ restricted on M. More precisely we have:

Theorem 41. *Let $M \subset Z$ be a compact G-invariant submanifold of Z and let $\mu_{\mathfrak{p}} : M \longrightarrow \mathfrak{p}$ be the gradient map restricted to M. Then $x \in M$ is semi-stable if and only if $\lambda_x \geq 0$. Furthermore, a point $x \in M$ is polystable if and only if $\lambda_x \geq 0$ and $Z(x) = \partial_\infty X'$ for some totally geodesic submanifold $X' \subset X = G/K$.*

8. Measures

Let M be a compact Hausdorff space. Denote by $\mathscr{M}(M)$ the vector space of finite signed Borel measures on M. Observe that they are automatically Radon [15, Thm. 7.8, p. 217]. Denote by $C(M)$ the space of real continuous functions on M which is a Banach space with the sup–norm. By the Riesz Representation Theorem (see e.g. [15, p.223]) $\mathscr{M}(M)$ is the topological dual of $C(M)$. We endow $\mathscr{M}(M)$ with the weak-$*$ topology as dual of $C(M)$ that it is usually called the weak topology on measures. We use the symbol $\nu_\alpha \rightharpoonup \nu$ to denote the weak convergence of the net $\{\nu_\alpha\}$ to the measure ν. Finally, we denote by $\mathscr{P}(M) \subset \mathscr{M}(M)$ the set of Borel probability measures on M. It is well-known that $\mathscr{P}(M)$ is a compact convex subset of $\mathscr{M}(M)$. Indeed the cone of positive measures is closed and $\mathscr{P}(M)$ is the intersection of this cone with the closed affine hyperplane $\{\nu \in \mathscr{M}(M) : \nu(M) = 1\}$. Therefore $\mathscr{P}(M)$ is closed and it is contained in the closed unit ball in $\mathscr{M}(M)$, which is compact in the weak topology by the Banach-Alaoglu Theorem [13, p. 425]. Since $C(M)$ is separable, the weak topology on $\mathscr{P}(M)$ is metrizable (see [13, p. 426]).

If $f : M \to N$ is a measurable map between measurable spaces and ν is a measure on M, the *image measure* $f_*\nu$ is the measure on Y such that $f_*\nu(A) := \nu(f^{-1}(A))$. Observe that it satisfies the *change of variables formula*

$$\int_N u(y)d(f_*\nu)(y) = \int_M u(f(x))d\nu(x). \tag{42}$$

If G acts on M, then we have an action on the probability measures on M as follows:

$$G \times \mathscr{P}(M) \to \mathscr{P}(M), \quad (g, \nu) \mapsto g_*\nu. \tag{43}$$

Let U be a compact connected Lie group and $U^{\mathbb{C}}$ its complexification. As in section 7 we assume that $G = K\exp(\mathfrak{p})$ is a compatible subgroup of $U^{\mathbb{C}}$ and M is a G-stable compact subset of a Kähler manifold (Z, ω). One can prove in a totally similar way as in the proof of [8, Lemma 5.5 p. 18] that the action (43) is continuous with respect to the weak topology.

Lemma 44. *Let X be a vector field on Z whose flow $\{\varphi_t\}$ preserves M. If $\nu \in \mathscr{M}(M)$ and X vanishes ν–almost everywhere, then $\varphi_{t*}\nu = \nu$ for any t. Hence, if $v \in \mathfrak{g}$ and $v_M(x) = 0$ for every x outside a set of ν–measure zero, then $\exp(\mathbb{R}v) \subset G_\nu$.*

Proof. Set $N := \{p \in M : X(p) \neq 0\}$. Then $\nu(N) = 0$ and for any $t \in \mathbb{R}$ and any $x \notin N$, $\varphi_t(x) = x$. In particular both N and $M - N$ are φ_t-invariant. If $A \subset M$ is measurable, then

$$\varphi_{-t}(A) = \varphi_{-t}((A - N) \sqcup (N \cap A)) = (A - N) \sqcup \varphi_{-t}(N \cap A).$$

Since $\varphi_{-t}(N \cap A) \subset N$, $\varphi_{t*}\nu(A) = \nu(\varphi_{-t}(A)) = \nu(A - N) = \nu(A)$. $\square$

Proposition 45. *Let M, G, K and $\mu_\mathfrak{p}$ be as in § 7 and let Ψ^M be the Kempf-Ness function of (M, G, K). The function:*

$$\Psi^{\mathscr{P}} : \mathscr{P}(M) \times G \to \mathbb{R}, \quad \Psi^{\mathscr{P}}(\nu, g) := \int_M \Psi^M(x, g)d\nu(x), \tag{46}$$

is a Kempf-Ness function for $(\mathscr{P}(M), G, K)$ satisfying conditions $(P1)-(P5)$. If in addition M is compact then Ψ also satisfies condition $(P6)$. Moreover, if we denote by $X = G/K$, then

$$\psi_\nu^{\mathscr{P}} : X \to \mathbb{R}, \quad \psi_\nu^{\mathscr{P}}(gK) := \Psi^{\mathscr{P}}(\nu, g^{-1}) = \int_M \psi_x^M(gK)d\nu(x), \tag{47}$$

and if $\mathfrak{F}$ denotes the gradient map, then:

$$\mathfrak{F} : \mathscr{P}(M) \to \mathfrak{p}, \quad \mathfrak{F}(\nu) := \int_M \mu_{\mathfrak{p}}(x)d\nu(x). \tag{48}$$

Finally, if $\exp(\mathbb{R}\beta) \subset G_\nu$*, for some $\beta \in \mathfrak{p}$, then $\mathfrak{F}(\nu) \in \mathfrak{p}^\beta$.*

For a sake of completeness we sketch the proof which is totally similar to that of Proposition 5.12 in [8].

Proof. Since Ψ^M is left-invariant with respect to K, the same holds for $\Psi^{\mathscr{P}}$.

Fix $\nu \in \mathscr{P}(M)$. By differentiation under the integral sign $\Psi^{\mathscr{P}}(\nu, \cdot)$ is a smooth function on G and for $v \in \mathfrak{p}$ we have:

$$\frac{\mathrm{d}^2}{\mathrm{dt}^2}\Psi^{\mathscr{P}}(\nu, \exp(tv)) = \int_M \left(\frac{\mathrm{d}^2}{\mathrm{dt}^2}\Psi^M(x, \exp(tv))\right) d\nu(x) \geq 0,$$

since the integrand is non-negative by $(P3)$. If $\frac{\mathrm{d}^2}{\mathrm{dt}^2}\Big|_{t=0} \Psi^{\mathscr{P}}(\nu, \exp(tv)) = 0$, then:

$$\frac{\mathrm{d}^2}{\mathrm{dt}^2}\Big|_{t=0} \Psi^M(\nu, \exp(tv)) = 0 \qquad \nu\text{-almost everywhere.}$$

Again by $(P3)$ this implies that $v_M = 0$ ν-almost everywhere. By Lemma 44 it follows that $\exp(\mathbb{R}v) \subset G_\nu$. We have proven that $\Psi^{\mathscr{P}}$ satisfies $(P1)-(P3)$. The cocycle condition for $\Psi^{\mathscr{P}}$ follows immediately from the cocycle condition for Ψ^M. Fix $\nu \in \mathscr{P}(M)$. It is immediate to verify that the function $\psi^{\mathscr{P}}$ associated to $\Psi^{\mathscr{P}}$ as in (3) is the one given by (47). Therefore it is clearly continuous on $\mathscr{P}(M)$. Finally, it is easy to check that $\Psi_\nu^{\mathscr{P}}$ is Lipschitz whenever M is a compact manifold.

Let $\beta \in \mathfrak{p}$. Since $X^\beta = \{y \in M;\, \beta_M(x) = 0\}$, is the set of fixed points $\{y \in M : \exp(t\beta)y = y, \text{ for all } t \in \mathbb{R}\}$, then X^β is G^β-stable and $\mu_{\mathfrak{p}}(X^\beta) \subset \mathfrak{p}^\beta$ (see [25]). If $\exp(\mathbb{R}\beta) \in G_\nu$, using the same argument of the proof of Proposition 52, then ν is supported on X^β and so $\mathfrak{F}(\nu) \in \mathfrak{p}^\beta$. $\square$

From now on we assume that M is a compact G-invariant submanifold of Z. We shall compute the maximal weight using the geometry of the gradient map. We begin recalling the following slice theorem proved in [24, 25].

Theorem 49 (Linearization Theorem)**.** *Let M, G, K and $\mu_{\mathfrak{p}}$ be as in § 7. If x is a fixed point of G, then there exist an open subset $S \subset T_xM$, stable under the isotropy representation of G, an open G-stable neighborhood Ω of x in M and a G-equivariant diffeomorphism $h : S \to \Omega$. One can further require that $h(0) = x$ and $dh_0 = \mathrm{id}_{T_xM}$.*

Fix $v \in \mathfrak{p}$. The *gradient flow* of a function $f \in C^\infty(M)$ is usually defined as the flow of the vector field $-\operatorname{grad} f$. Let $\{\varphi_t\}$ denote the gradient flow of μ^v. Since $\operatorname{grad}\mu^v = \beta_M$, we have $\varphi_t(x) = \exp(tv)x$. Then the function $\mu_\mathfrak{p}^v$ is a Morse-Bott function [24, 25, 26]. If we denote by $c_1 > \cdots > c_r$ the critical values of μ^v, then the corresponding level sets of $\mu_\mathfrak{p}^v$, $C_i := (\mu^v)^{-1}(c_i)$ are submanifolds which are the components of $\mathrm{Crit}(\mu^v)$. By Theorem 49 it follows that for any $x \in M$ the limit:

$$\alpha(x) := \lim_{t\to-\infty} \varphi_t(x) = \lim_{t\to+\infty} \exp(tv)x,$$

exists. Let us denote by W_i the *unstable manifold* of the critical component C_i for the gradient flow of μ^v:

$$W_i := \{x \in M : \alpha(x) \in C_i\}. \tag{50}$$

Then:

$$M = \bigsqcup_{i=1}^r W_i, \tag{51}$$

and for any i the map:

$$\alpha|_{W_i} : W_i \to C_i,$$

is a smooth fibration with fibres diffeomorphic to $\mathbb{R}^{l_i}$ where l_i is the index (of negativity) of the critical submanifold C_i.

Proposition 52. *Let ν be a polystable measure which is not stable. Hence there exist an abelian subalgebra $\mathfrak{a} \subset \mathfrak{g}_\nu$ such that ν is supported on $M^\mathfrak{a} = \{x \in M : \xi_M(x) = 0$ for any $\xi \in \mathfrak{a}\}$.*

Proof. By Proposition 31, Lemma 32 and Theorem 33, $\mathfrak{g}_\nu = \mathrm{Ad}(g)(\mathfrak{k}' \oplus \mathfrak{q})$, i.e., it is conjugate to a compatible subalgebra of $\mathfrak{g}$ and $\partial_\infty G_\nu/K_\nu = Z(\nu) = g(e(S(\mathfrak{q})))$.

Let $\mathfrak{a}' \subset \mathfrak{q}$ be a maximal abelian subalgebra of $\mathfrak{q}$. Then $\mathfrak{a} = \mathrm{Ad}(g)(\mathfrak{a}')$ is an abelian subalgebra of $\mathfrak{g}_\nu$ and $S(\mathfrak{a}) \subset Z(\nu)$. Let $u \in \mathfrak{a}$. Then $\exp(tu) \in G_\nu$ and thus:

$$\lim_{n\mapsto-\infty} \exp(nu)\nu = \nu.$$

Let $A \subset M$ be a measurable subset. Then $\nu(A) = \lim_{n\mapsto-\infty} \nu(\exp(nu)(A)) = \nu(\alpha(A))$, where α is the gradient flow of $\mu_\mathfrak{p}^u$. Hence ν is supported on the critical submanifolds of $\mu_\mathfrak{p}^u$ for any $u \in \mathfrak{a}$. Hence ν is supported on $M^\mathfrak{a}$. □

Now, we explicitly compute the maximal weights.

Theorem 53. *With the notation above we have*

$$\lambda_\nu(\mathrm{e}(-v)) = \sum_{i=1}^r c_i \nu(W_i).$$

We give a sketch of the proof, which follows essentially that of [8, Th. 5.23].

Proof. By definition of λ_ν and by differentiating under the integral sign we get

$$\begin{aligned}\lambda_\nu(\mathrm{e}(-v)) &= \lim_{t\to+\infty} \frac{\mathrm{d}}{\mathrm{dt}} \int_M \Psi^M(x,\exp(tv))d\nu(x)\\ &= \lim_{t\to+\infty} \int_M \left(\frac{\mathrm{d}}{\mathrm{dt}}\Psi^M(x,\exp(tv))\right) d\nu(x).\end{aligned}$$

Applying the dominated convergence theorem, since $\left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=t_o} \Psi^M(x,\exp(tv)) = \mu_\mathfrak{p}^v(\exp(t_0 v)x)$ and $\mu_\mathfrak{p}^v$ is bounded, we get

$$\begin{aligned}\lambda_\nu(\mathrm{e}(-v)) &= \lim_{t\to+\infty} \int_M \mu_\mathfrak{p}^v(\exp(tv)x)d\nu(x)\\ &= \int_M \mu^v(\alpha(x))d\nu(x) = \sum_{i=1}^r \int_{W_i} \mu_\mathfrak{p}^v(\alpha(x))d\nu(x).\end{aligned}$$

Since for $x \in W_i$, $\alpha(x) \in C_i$ and so $\mu_\mathfrak{p}^v(\alpha(x)) = c_i$, we finally obtain:

$$\lambda_\nu(\mathrm{e}(-v)) = \sum_{i=1}^r c_i \nu(W_i).$$

□

Let $E(\mu_\mathfrak{p})$ denote the convex hull of $\mu_\mathfrak{p}(M) \subset \mathfrak{p}$, i.e. a K-invariant convex body in $\mathfrak{p}$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a abelian subalgebra and let $\pi : \mathfrak{p} \longrightarrow \mathfrak{a}$ be the orthogonal projection onto $\mathfrak{a}$. Then $\mu_\mathfrak{a} = \pi \circ \mu_\mathfrak{p}$ is the gradient map associated to $A = \exp(\mathfrak{a})$. Denote by $P = \mu_\mathfrak{a}(M)$. It is well-known that P is a finite union of polytopes [27] and the convex bodies $E(\mu_\mathfrak{p})$ and the convex hull of P are strongly related [7]. Although, the convexity of $\mu_\mathfrak{a}(M) = P$ is not known. This holds if $G = U^\mathbb{C}$ and M is a complex connected submanifold by the Atiyah-Guillemin-Sternberg convexity theorem [1, 17] or when M is an irreducible semi-algebraic subset of a Hodge manifold Z [6, 27, 34].

In the sequel we always assume that for any $v \in \mathfrak{p}$, a local maxima of $\mu_\mathfrak{p}^v$ is a global maxima. This condition holds for any real flag manifold [5]. In our assumption, the Morse-Bott decomposition of M with respect to $\mu_\mathfrak{p}^v$, i.e., $M = \bigsqcup_{i=0}^r W_i$, has a unique open and dense unstable manifold W_r^u and the others unstable manifolds are proper submanifolds. Therefore, if ν is a smooth measure of M then W_r^u has full measure and so $\lambda_\nu(\mathrm{e}(-v)) = c_r = \max_{x\in M}\mu_\mathfrak{p}^v$. Summing up we have proved the following result.

Corollary 54. *If ν is a smooth measure on M, then for any $v \in \mathfrak{p}$:*

$$\lambda_\nu(\mathrm{e}(-v)) = \max_{x\in M}\mu_\mathfrak{p}^v.$$

Since ν is a probability measures, it follows that $\mathfrak{F}(\nu) \in E(\mu_\mathfrak{p})$. Indeed, $\mathfrak{F}(\nu)$ is the barycenter of the gradient map $\mu_\mathfrak{p}$ with respect to ν and so it lies in $E(\mu_\mathfrak{p})$. If $0 \notin E(\mu_\mathfrak{p})$, then there exists $v \in E(\mu_\mathfrak{p})$ realizing the minimum distance of $E(\mu_\mathfrak{p})$ to the origin. Moreover v is a K fixed point due to the fact that $E(\mu_\mathfrak{p})$ is K-invariant. Hence up to shifting the gradient map we may assume that $0 \in E(\mu_\mathfrak{p})$. Under this assumption we get the following result.

Proposition 55. *If $0 \in E(\mu_{\mathfrak{p}})$ then any smooth measure on M is semi-stable.*

Proof. Let $v \in \mathfrak{p}$. By the above corollary, we have $\lambda_\nu(\mathrm{e}(-v)) = \max_{x\in M}\mu_{\mathfrak{p}}^v$. Since $0 \in E(\mu_{\mathfrak{p}})$, it follows that $\lambda_\nu(\mathrm{e}(-v)) = \max_{x\in M}\mu_{\mathfrak{p}}^v \geq 0$. By Theorem 35 ν is semi-stable. $\square$

Corollary 56. *If $0 \in E(\mu_{\mathfrak{p}})$, then the set $\mathscr{P}_{ss}(M) := \{\nu \in \mathscr{P}(M) : \nu$ is semi-stable$\}$ is dense in $\mathscr{P}(M)$. Moreover, if 0 lies in the interior of $E(\mu_{\mathfrak{p}})$ then the set $\mathscr{P}_s(M) := \{\nu \in \mathscr{P}(M) :$* ν is stable$\}$ *is open and dense.*

Proof. By the above Proposition any smooth measure is semi-stable. Since smooth measures are dense, then the set of semi-stable measures is dense. If 0 belongs to the interior of the $E(\mu_{\mathfrak{p}})$, then for any $v \in \mathfrak{p}$ the function $\mu_{\mathfrak{p}}^v$ change sign and so it has a strictly positive maxima. By Corollary 54 $\lambda_\nu(\mathrm{e}(-v)) > 0$ and by Theorem 23 we get that it is stable. Since by Corollary 29 the set of the stable points is also open, it means $\mathscr{P}_s(M)$ is open and dense. $\square$

9. Measures on real projective spaces

In the recent paper [8] the authors completely describe stable, semi-stable and polystable measures on complex projective spaces (see also [12, 36]). Here we consider the real projective space:

$$\mathbb{P}^n(\mathbb{R}) = \frac{\mathbb{R}^{n+1} - \{0\}}{\sim} = \frac{\mathbb{S}^n}{\{\pm \mathrm{Id}_{n+1}\}},$$

where we denote by Id_{n+1} the identity matrix of order $n+1$. Consider on $\mathbb{P}^n(\mathbb{R})$ the action of $\mathrm{SL}(n+1,\mathbb{R})$ and recall that its Lie algebra $\mathfrak{sl}(n+1)$ decomposes as $\mathfrak{sl}(n+1) = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{so}(n+1) \oplus \mathrm{sym}_0(n+1)$. A gradient map for this action is given by:

$$\mu_{\mathfrak{p}} \colon \mathbb{P}^n(\mathbb{R}) \to \mathfrak{p}, \quad \mu_{\mathfrak{p}}([x]) = \frac{1}{2}\left[\frac{xx^T}{|x|^2} - \frac{1}{n+1}\mathrm{Id}_{n+1}\right].$$

Observe that $\mathrm{sym}_0(n+1)$ admits the maximal abelian subalgebra $\mathfrak{a}$ of traceless diagonal matrices, which we identify with $\mathbb{R}^n \subset \mathbb{R}^{n+1}$. Given an element $\xi \in \mathrm{sym}_0(n+1)$, let $\lambda_1 > \cdots > \lambda_k$ be its eigenvalues and denote by $V_1, \ldots, V_k$ the corresponding eigenspaces. In view of the orthogonal decompositions $\mathbb{R}^{n+1} = V_1 \oplus \cdots \oplus V_k$ we can write $x \in \mathbb{R}^{n+1}$ as $x = x_1 + \cdots + x_k$ with $x_j \in V_j$, $j = 1, \ldots, k$. With this notation we have:

$$\mu_{\mathfrak{a}}^{\xi}([x]) = \frac{1}{2}\frac{\lambda_1|x_1|^2 + \cdots + \lambda_k|x_k|^2}{|x_1|^2 + \cdots + |x_k|^2},$$

where $\langle\cdot,\cdot\rangle$ is the dual pairing. Consider the projection $\pi : \mathbb{R}^{n+1} - \{0\} \to \mathbb{P}^n(\mathbb{R})$. Since $(d\pi)_x\left(\xi_{\mathbb{R}^{n+1}-\{0\}}(x)\right) = \xi_{\mathbb{P}^n(\mathbb{R})}$ and $\xi_{\mathbb{R}^{n+1}-\{0\}}(x) = \lambda_1 x_1 + \cdots + \lambda_k x_k$, one has $\xi_{\mathbb{P}^n(\mathbb{R})} \equiv 0$ iff $\xi_{\mathbb{R}^{n+1}-\{0\}}(x)$ is parallel to x, i.e. iff $x = x_j$ for some $j = 1, \ldots, k$. Thus, critical points of $\mu_{\mathfrak{p}}^{\xi}$ are given by $\mathrm{Crit}(\mu_{\mathfrak{p}}^{\xi}) = \mathbb{P}(V_1) \cup \cdots \cup \mathbb{P}(V_k)$ and critical values are $c_j = \frac{1}{2}\lambda_j$, $j = 1, \ldots, k$.

In order to describe:

$$W_j^{\xi} = \{[x] \in \mathbb{P}^n(\mathbb{R}) : \alpha([x]) \in C_j\},$$

for $j = 1, \ldots, n+1$, where by definition:

$$\alpha([x]) = \lim_{t\to+\infty} \exp(t\xi)x,$$

observe that:

$$\exp(t\xi)x = [\exp(t\lambda_1)x_1 + \cdots + \exp(t\lambda_k)x_k],$$

which implies:

$$\alpha([x]) = \lim_{t\to+\infty}[\exp(t\lambda_1)x_1 + \cdots + \exp(t\lambda_k)x_k] = \begin{cases} [x_1] \text{ if } \ x_1 \neq 0; \\ [x_2] \text{ if } \ x_1 = 0, x_2 \neq 0; \\ \ \vdots \\ [x_k] \text{ otherwise.} \end{cases}$$

Thus, since $[x] \in W_j^\xi$ iff $\alpha([x]) \in \mathbb{P}(V_j)$ we have:

$$W_1^\xi = \mathbb{P}^n(\mathbb{R}) - \mathbb{P}(V_2 \oplus \cdots \oplus V_k),$$

$$W_2^\xi = \mathbb{P}(V_2 \oplus \cdots \oplus V_k) - \mathbb{P}(V_3 \oplus \cdots \oplus V_k),$$

$$\vdots$$

$$W_{k-1}^\xi = \mathbb{P}(V_{k-1} \oplus V_k) - \mathbb{P}(V_k).$$

$$W_k^\xi = \mathbb{P}(V_k).$$

By Theorem 53 it follows:

$$\begin{aligned} \lambda_\nu(e(-\xi)) =& \frac{1}{2}\left(\sum_{j=1}^r \lambda_j \nu(W_j^\xi)\right) \\ =& \frac{1}{2}\left(\lambda_1 - (\lambda_1 - \lambda_2)\nu(\mathbb{P}(V_2 \oplus \cdots \oplus V_k)) - \cdots - (\lambda_{k-1} - \lambda_k)\nu(\mathbb{P}(V_k))\right). \end{aligned} \tag{57}$$

In the following two examples we develop in details the cases $n = 1$ and $n = 2$.

Example 58. Let $n = 1$. We have $\xi = (\lambda_1, -\lambda_1)$ and $\mathbb{R}^2 = V_1 \oplus V_2$. Denote $p_i = \mathbb{P}(V_i)$ for $i = 1, 2$. Then, $\mathrm{Crit}(\mu^\xi) = \{p_1, p_2\}$. If we denote $x = x_1 + x_2$ as before, we have:

$$\alpha(X) = \lim_{t\to+\infty}[\exp(t\lambda_1)x_1 + \exp(t\lambda_2)x_2] = \begin{cases} p_1 \text{ if } \ x_1 \neq 0; \\ p_2 \text{ if } \ x_1 = 0, \end{cases}$$

which implies:

$$W_1^\xi = \mathbb{P}^1(\mathbb{R}) - p_2, \quad W_2^\xi = p_2.$$

It follows that:

$$\lambda_\nu(e(-\xi)) = \frac{\lambda_1}{2}(1 - 2\nu(p_2)).$$

Thus ν is stable iff for any $p \in \mathbb{P}^1(\mathbb{R})$:

$$\nu(p) < \frac{1}{2},$$

semi-stable iff for any $p \in \mathbb{P}^1(\mathbb{R})$:

$$\nu(p) \leq \frac{1}{2},$$

polystable but not stable iff ν is only supported by two points, i.e.:

$$\nu = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_2.$$

Indeed, If ν is polystable, by Corollary 34, there exists $\xi \in \mathfrak{p}$ such that $\exp(t\xi) \in \mathrm{SL}(2,\mathbb{R})_\nu$, ν is supported by two points p_1 and p_2 and by:

$$0 = \lambda_\nu(e(-\xi)) = \frac{\lambda_1}{2}(1 - 2\nu(p_2))$$

it follows $\nu = \frac{1}{2}\delta_{p_1} + \frac{1}{2}\delta_{p_2}$. Vice-versa, if $\nu = \frac{1}{2}\delta_{p_1} + \frac{1}{2}\delta_{p_2}$ with $p_1 \neq p_2$, then there exists $g \in \mathrm{SL}(2,\mathbb{R})$ such that $gp_1 = [1:0]$ and $gp_2 = [0:1]$. It is easy to check that

$$\mathfrak{F}(g\nu) = \frac{1}{2}(\mu_{\mathfrak{p}}([1:0]) - \mu_{\mathfrak{p}}([0:1]) = 0,$$

proving ν is polystable.

Example 59. Let $n = 2$. We have three cases:

(*a*) $\xi = (\lambda_1, \lambda_2, \lambda_3)$, with $\lambda_3 = -\lambda_1 - \lambda_2$, $\mathbb{R}^3 = V_1 \oplus V_2 \oplus V_3$, $\dim(V_j) = 1$;
(*b*) $\xi = (\lambda_1, -\frac{1}{2}\lambda_1, -\frac{1}{2}\lambda_1)$ and $\mathbb{R}^3 = V_1 \oplus V_2$, where $\dim(V_1) = 1$, $\dim(V_2) = 2$;
(*c*) $\xi = (\lambda_1, \lambda_1, -2\lambda_1)$ and $\mathbb{R}^3 = V_1 \oplus V_2$, where $\dim(V_1) = 2$, $\dim(V_2) = 1$.

Let us deal first with the case (*a*). Denote $p_i = \mathbb{P}(V_i) \subset \mathbb{P}^2(\mathbb{R})$ for $i = 1,2,3$ and let $\xi = (\lambda_1, \lambda_2, \lambda_3)$. Then $\mathrm{Crit}(\mu_{\mathfrak{p}}^{\xi}) = \{p_1, p_2, p_3\}$ and:

$$\alpha(x) = \lim_{t\to+\infty}[\exp(t\lambda_1)x_1 + \exp(t\lambda_2)x_2 + \exp(t\lambda_3)x_3] = \begin{cases} p_1 \text{ if } \ x_1 \neq 0; \\ p_2 \text{ if } \ x_1 = 0,\, y_2 \neq 0; \\ p_3 \text{ if } \ [x] = p_3, \end{cases}$$

and

$$W_1^{\xi} = \mathbb{P}^2(\mathbb{R}) - \mathbb{P}(V_2 \oplus V_3), \quad W_2^{\xi} = \mathbb{P}(V_2 \oplus V_3) - p_3, \quad W_3^{\xi} = p_3.$$

It follows that:

$$\begin{aligned} \lambda_\nu(e(-\xi)) =& \frac{\lambda_1}{2} - \frac{\lambda_1 - \lambda_2}{2}\nu(\mathbb{P}(V_2 \oplus V_3)) - \frac{2\lambda_2 + \lambda_1}{2}\nu(p_3) \\ =& \frac{\lambda_1}{2}\left(1 - \left(1 - \frac{\lambda_2}{\lambda_1}\right)\nu(\mathbb{P}(V_2 \oplus V_3)) - \left(2\frac{\lambda_2}{\lambda_1} + 1\right)\nu(p_3)\right). \end{aligned}$$

Observe that from $\lambda_1 > \lambda_2 > -\lambda_1 - \lambda_2$ we get $-1/2 < \lambda_2/\lambda_1 < 1$.

For the case (*b*), namely for $\xi = (\lambda_1, -\frac{1}{2}\lambda_1, -\frac{1}{2}\lambda_1)$, we have $\mathrm{Crit}(\mu^{\xi}) = \{p_1\} \cup \mathbb{P}(V_2)$,

$$\alpha(x) = \lim_{t\to+\infty}[\exp(t\lambda_1)x_1 + \exp(t\lambda_2)x_2] = \begin{cases} p_1 \text{ if } \ x_1 \neq 0; \\ [0:y_2] \text{ if } \ x_1 = 0, \end{cases}$$

and

$$W_1^{\xi} = \mathbb{P}^2(\mathbb{R}) - \mathbb{P}(V_2) = p_1, \quad W_2^{\xi} = \mathbb{P}(V_2) = \mathbb{P}^2(\mathbb{R}) - \{p_1\}.$$

It follows that:

$$\lambda_\nu(e(-\xi)) = \lambda_1\left(\frac{1}{4} - \frac{3}{4}\nu(p_1)\right).$$

Finally, when $\xi = (\lambda_1, \lambda_1, -2\lambda_1)$, $\mathrm{Crit}(\mu^\xi) = \mathbb{P}(V_1) \cup \{p_3\}$,

$$\alpha(x) = \lim_{t \to +\infty} [\exp(t\lambda_1)x_1 + \exp(t\lambda_2)x_2] = \begin{cases} [x_1] \text{ if } \ x_1 \neq 0; \\ p_3 \text{ if } \ x_1 = 0, \end{cases}$$

and

$$W_1^\xi = \mathbb{P}^2(\mathbb{R}) - \{p_3\}, \quad W_2^\xi = \mathbb{P}(V_2) = \{p_3\}.$$

It follows that:

$$\lambda_\nu(e(-\xi)) = \frac{\lambda_1}{2}\left(1 - 3\,\nu(p_3)\right).$$

Denote by $\mathrm{Li} \subset \mathbb{R}^3$ a linear subspace of $\mathbb{R}^3$ of dimension 2 and let $p \in \mathbb{P}^2(\mathbb{R})$. Then, ν is stable iff for any choice of Li and p:

$$\nu(\mathbb{P}(\mathrm{Li})) < \frac{2}{3}, \quad \nu(p) < \frac{1}{3},$$

ν is semi-stable iff for any choice of Li and p:

$$\nu(\mathbb{P}(\mathrm{Li})) \leq \frac{2}{3}, \quad \nu(p) \leq \frac{1}{3},$$

and ν is polystable iff either it is stable or it is one of the following:

$$\nu := \frac{2}{3}\delta_{\mathbb{P}(\mathrm{Li})} + \frac{1}{3}\delta_p, \qquad \nu := \frac{1}{3}\delta_1 + \frac{1}{3}\delta_2 + \frac{1}{3}\delta_3,$$

i.e. it is supported by some $\mathbb{P}(\mathrm{Li})$ and by a point p or by three points (see the proof of Prop. 60 below for details).

We conclude with the following proposition which states necessary and sufficients conditions for stability and polystability in general dimension.

Proposition 60. *The measure ν is stable iff for any choice of a linear subspace* $\mathrm{Li} \subset \mathbb{R}^{n+1}$*:*

$$\nu(\mathbb{P}(\mathrm{Li})) < \frac{\dim(\mathrm{Li})}{n+1},$$

ν is semi-stable iff:

$$\nu(\mathbb{P}(\mathrm{Li})) \leq \frac{\dim(\mathrm{Li})}{n+1}.$$

The measure ν is polystable iff there exists a splitting $\mathbb{R}^{n+1} = \mathrm{Li}_1 \oplus \cdots \oplus \mathrm{Li}_\mathrm{r}$ *such that ν is supported on* $\mathbb{P}(\mathrm{Li}_1) \cup \cdots \cup \mathbb{P}(\mathrm{Li}_r)$*. Moreover*

$$\nu := \sum_j^r \frac{\dim(\mathrm{Li}_j)}{n+1}\delta_{\mathbb{P}(\mathrm{Li}_j)},$$

where $\delta_{\mathbb{P}(\mathrm{Li}_j)}$ is a stable measure of $\mathbb{P}(\mathrm{Li}_j)$ *with respect to* $\mathrm{SL}(\mathrm{Li_j})$*.*

Proof. As before, let $\xi \in \mathfrak{a}$, $\lambda_1 > \cdots > \lambda_k$ be its eigenvalues and $V_1, \ldots, V_k$ be the corresponding eigenspaces, with $\sum_{j=1}^k \dim(V_j)\lambda_j = 0$. From (57) we have $\lambda_\nu(e(-\xi)) > 0$ iff:

$$\lambda_1 - (\lambda_1 - \lambda_2)\nu(\mathbb{P}(V_2 \oplus \cdots \oplus V_k)) - \cdots - (\lambda_{k-1} - \lambda_k)\nu(\mathbb{P}(V_k)) > 0. \tag{61}$$

Assume that $\nu(\mathbb{P}(\mathrm{Li})) < \frac{\dim(\mathrm{Li})}{n+1}$ for any linear subspace $\mathrm{Li} \subset \mathbb{R}^{n+1}$. Then, since $\lambda_j - \lambda_{j+1} > 0$:

$$\begin{aligned} &\lambda_1 - (\lambda_1 - \lambda_2)\nu(\mathbb{P}(V_2 \oplus \cdots \oplus V_k)) - \cdots - (\lambda_{k-1} - \lambda_k)\nu(\mathbb{P}(V_k)) > \\ &\quad > \lambda_1 - (\lambda_1 - \lambda_2)\frac{\dim(V_2) + \cdots + \dim(V_k)}{n+1} - \cdots - (\lambda_{k-1} - \lambda_k)\frac{\dim(V_k)}{n+1} = 0, \end{aligned}$$

where the last equality follows by applying $\sum_{j=1}^{k} \dim(V_j)\lambda_j = 0$ several times. Viceversa, let Li be a linear subspace of $\mathbb{R}^{n+1}$ of dimension $0 < r < n+1$ such that $\nu(\mathbb{P}(\mathrm{Li})) \geq \frac{\dim(\mathrm{Li})}{n+1}$. Then, $\mathbb{R}^{n+1} = \mathrm{Li} \oplus \mathrm{Li}^{\perp}$, where we denote by $\mathrm{Li}^{\perp}$ the orthogonal complement of Li, and we can choose ξ is such a way that $\xi = (\lambda_1, \lambda_2)$, $r\lambda_1 + (n+1-r)\lambda_2 = 0$, with corresponding eigenspaces Li and $\mathrm{Li}^{\perp}$. We can assume without loss of generality that $\lambda_1 > 0$. Conclusion follows since by (57) we have:

$$\lambda_\nu(e(-\xi)) = \lambda_1 - (\lambda_1 - \lambda_2)\nu(\mathbb{P}(\mathrm{Li}^{\perp})) \leq \lambda_1 - \lambda_1\frac{n+1-r}{n+1} - \lambda_1\frac{r}{n+1} = 0,$$

where we use that $r\lambda_1 + (n+1-r)\lambda_2 = 0$.

In order to prove the polystability part, assume that ν is polystable. Then there exists $g \in \mathrm{SL}(n+1, \mathbb{R})$ such that $\mathfrak{F}(g\nu) = 0$. Set $\nu' = g\nu$. By Lemma 32 and Proposition 52 there exists an abelian subalgebra $\mathfrak{a} \subset \mathrm{sym}_0(n+1)$ such that ν' is supported on $\mathbb{P}^n(\mathbb{R})^{\mathfrak{a}}$. We can diagonalize simultaneously any element of $\mathfrak{a}$. Hence there exists an orthogonal splitting:

$$\mathbb{R}^{n+1} = V_1 \oplus \cdots \oplus V_r,$$

such that for any $\xi \in \mathfrak{a}$, we have $\xi_{|V_i} = \lambda_j(\xi)\mathrm{Id}_{V_j}$. Therefore $\mathbb{P}^n(\mathbb{R})^{\mathfrak{a}} = \mathbb{P}(V_1) \cup \cdots \cup \mathbb{P}(V_r)$ and so ν' is supported on $\mathbb{P}(V_1) \cup \cdots \cup \mathbb{P}(V_r)$. This means that $\nu' = \sum_{j=1}^{r} \lambda_i \delta_{\mathbb{P}(V_j)}$, where $\delta_{\mathbb{P}(V_j)} \in \mathscr{P}(\mathbb{P}(V_j))$, $\lambda_j \geq 0$ for $j = 1, \ldots, r$ and $\sum_{j=1}^{r} \lambda_j = 1$. Since $\mathrm{SL}(n+1, \mathbb{R})^{\mathfrak{a}} = \mathrm{SL}(V_1 \oplus \cdots \oplus V_r)$ its semisimple part is given by $\mathrm{SL}(V_1) \times \cdots \times \mathrm{SL}(V_r)$. By Corollary 34 ν' is $\mathrm{SL}(V_1) \times \cdots \times \mathrm{SL}(V_r)$ stable and so its stabilizer:

$$(\mathrm{SL}(V_1) \times \cdots \times \mathrm{SL}(V_r))_{\nu'} = \mathrm{SL}(V_1)_{\delta_{\mathbb{P}(V_1)}} \times \cdots \times \mathrm{SL}(V_r)_{\delta_{\mathbb{P}(V_r)}}$$

is compact. In particular $\mathrm{SL}(V_j)_{\delta_{\mathbb{P}(V_j)}}$ is compact. If we decompose $x = x_1 + \cdots + x_r$ by means of the above splitting, we have:

$$\begin{aligned} 0 = \mathfrak{F}(\nu') &= \int_{\mathbb{P}^n(\mathbb{R})} \mu_{\mathfrak{p}}(x) \mathrm{d}\nu'(x) = \int_{\mathbb{P}^n(\mathbb{R})} \frac{xx^T}{||x||^2}\mathrm{d}\nu'(x) - \frac{1}{n+1}\mathrm{Id}_{n+1} \\ &= \sum_{j=1}^{r} \lambda_j \int_{\mathbb{P}(V_j)} \left(\frac{x_j x_j^T}{||x_j||^2} - \frac{1}{\dim V_j}\mathrm{Id}_{V_j}\right) \mathrm{d}\delta_{\mathbb{P}(V_j)}(x_j) + \sum_{j=1}^{r} \lambda_j \frac{1}{\dim V_j} - \frac{1}{n+1}\mathrm{Id}_{n+1} \\ &= \sum_{j=1}^{r} \lambda_j \mathfrak{F}^j(\delta_{\mathbb{P}(V_j)}) + \sum_{j=1}^{r} \left(\frac{\lambda_j}{\dim V_j} - \frac{1}{n+1}\right)\mathrm{Id}_{V_j}. \end{aligned}$$

In the above formula $\mathfrak{F}^j$ denotes the gradient map with respect to the $\mathrm{SL}(V_j)$ action on $\mathscr{P}(\mathbb{P}(V_j))$. Therefore, keeping in mind that $\sum_{j=1}^{r} \lambda_j \mathfrak{F}^j(\delta_{\mathbb{P}(V_j)})$, which lies in $\mathfrak{sym}_{\mathfrak{o}}(n+1)$, and $\sum_{j=1}^{r} \frac{\lambda_j}{\dim V_j}\mathrm{Id}_{V_j} - \frac{1}{n+1}\mathrm{Id}_{V_j}$ are orthogonal in $\mathfrak{gl}(n+1, \mathbb{R})$, we have $\mathfrak{F}^j(\delta_{\mathbb{P}(V_j)}) = 0$, and so by the above discussion $\delta_{\mathbb{P}(V_j)}$ is stable with respect to $\mathrm{SL}(V_j)$, and $\lambda_j = \frac{\dim V_j}{n+1}$. Set $\mathrm{Li_j} = g^{-1}V_j$ for any

$j=1,\ldots,r$. By the above discussion $\nu=\sum_j^r \frac{\dim(\mathrm{Li}_j)}{n+1}\delta_{\mathbb{P}(\mathrm{Li}_j)}$, where $\delta_{\mathbb{P}(\mathrm{Li}_j)}$ is a measure of $\mathbb{P}(\mathrm{Li}_j)$. We claim $\delta_{\mathbb{P}(\mathrm{Li}_j)}$ is a stable measure with respect to $\mathrm{SL(Li_j)}$. Indeed, $\mathrm{SL(Li_j)=g^{-1}SL(V_j)g}$ and it is easy to check that:

$$\mathrm{Ad}(g^{-1})\circ\mu_{\mathfrak{p}}^{\mathrm{SL}(V_j)}=\mu_{\mathfrak{p}}^{\mathrm{SL(Li_j)}}\circ g^{-1}.$$

Similarly $\mathrm{Ad}(g^{-1})\circ\mathfrak{F}=\mathfrak{F}'\circ g^{-1}$ and so $\mathfrak{F}^{-1}(0)=g\cdot\mathfrak{F}'^{-1}(0)$ proving $\delta_{\mathbb{P}(\mathrm{Li}_j)}$ is a stable measure with respect to $\mathrm{SL(Li_j)}$.

Vice-versa, assume $\nu=\sum_j^r \frac{\dim(\mathrm{Li}_j)}{n+1}\delta_{\mathbb{P}(\mathrm{Li}_j)}$ with respect to a splitting:

$$\mathbb{R}^{n+1}=\mathrm{Li_1}\oplus\cdots\oplus\mathrm{Li_r},$$

where $\delta_{\mathbb{P}(\mathrm{Li}_j)}$ is a stable measure of $\mathbb{P}(\mathrm{Li}_j)$ with respect to $\mathrm{SL(Li_j)}$. Let $g\in\mathrm{SL}(n+1,\mathbb{R})$ such that if we denote by $V_j=g\mathrm{Li_j}$ for $j=1,\ldots,r$, then:

$$\mathbb{R}^{n+1}=V_1\oplus\cdots\oplus V_r,$$

is an orthogonal splitting. By the above computation we get $\mathfrak{F}(g\nu)=\sum_{j=1}^r\frac{\dim V_j}{n+1}\mathfrak{F}^j(\delta_{\mathbb{P}(V_j)})$, where $\delta_{\mathbb{P}(V_j)}=g\delta_{\mathbb{P}(\mathrm{Li}_j)}$ for $j=1,\ldots,r$. By the above discussion since $\delta_{\mathbb{P}(\mathrm{Li}_j)}$ is stable with respect to $\mathrm{SL(Li_j)}$, then $g\delta_{\mathbb{P}(V_j)}$ is stable with respect to $\mathrm{SL}(V_j)$. Hence there exists $g_j\in\mathrm{SL(V_j)}$ such that $\mathfrak{F}^j(g_j\delta_{\mathbb{P}(V_j)})=0$. Let $h=g_1\times\cdots\times g_r\in\mathrm{SL}(V_1)\times\cdots\times\mathrm{SL}(V_r)\subset\mathrm{SL}(n+1,\mathbb{R})$. Then

$$\mathfrak{F}(hg\nu)=\sum_{j=1}^r\frac{\dim V_j}{n+1}\mathfrak{F}^j(g_j\delta_{\mathbb{P}(V_j)})=0,$$

concluding the proof. □

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