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Abstract

We develop envelope theorems for optimization problems in which the value function takes values in a general Banach lattice. We consider both the special case of a convex choice set and a concave objective function and the more general case case of an arbitrary choice set and a general objective function. We apply our results to discuss the existence of a well-defined notion of marginal utility of wealth in optimal discrete-time, finite-horizon consumption-portfolio problems with an unrestricted information structure and preferences allowed to display habit formation and state dependency.

Key words: Envelope theorem, Banach lattice, state-dependent utility, value function, Gateaux differential, Fréchet differential.

1 Introduction

Envelope theorems constitute one of the genuine workhorses of economics, and their applications are ubiquitous. Several extensions of the traditional Envelope theorems have emerged over the years, as a response to the necessity of analyzing the behavior of the value function of optimization problems lacking the assumptions for the applicability of the standard Envelope results from graduate textbooks. In concave dynamic programming, a seminal paper is due to Benveniste and Scheinkman [3] who assume the choice set

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to be convex and the objective function to be concave (see also [14], footnote 5, for more references). More recently, Rincón-Zapatero and Santos [18] considered a dynamic framework where the information structure is generated by an exogenous stochastic process, and proved an Envelope theorem that allows the choice set to depend on the parameters of the problem and the optimal solution to lie on the boundary of the feasible set. Cruz-Suárez and Montes-de Oca [8] established Envelope theorems for optimization problems on Euclidean spaces for both the concave and the unrestricted case. The paper which is closer to the spirit of our contribution is the one by Milgrom and Segal [14]. Their Envelope results do not make any assumption on the choice set of the optimization problem nor they require the concavity of the objective functions. In particular, Milgrom and Segal first show that the traditional Envelope formula holds at any differentiability point of the value function, and then they establish conditions for the (left, right or full) differentiability of the value function. Other interesting recent developments can be found in Morand et alii. [15].

The first contribution of this paper is to supply Envelope results for the general case of optimization problems in which the objective function takes its values in a Banach lattice. To start our analysis we extend to this Banach lattice setting the results for concave programming supplied by Benveniste and Scheinkman [3]. Next we maintain the assumption that the objective function takes values in a Banach lattice, and we allow the choice set to be arbitrary and the parameters of the problem to belong to a general Banach space. In this more general case we supply Envelope results that extend to the Banach lattice setting those supplied by Milgrom and Segal [14] for the real-valued case. To develop our general Envelope formulae we replace the standard notion of differentiability for real-valued functions with the more general notion of differentiability in Banach lattices, namely Gateaux and Fréchet differentiability.

The second contribution of this paper comes from applying our general Envelope results for Banach lattices to asset pricing. We consider a security market with a general information filtration. In this framework, we analyze the discrete-time, finite-horizon consumption-portfolio problem for an agent with utility function that can display habit formation and is allowed to be state-dependent. At any time t, the maximum remaining utility (continuation utility) for an agent with wealth W is represented by a value function of the form:

$$H(t, W) = \underset{(c,\theta)\in\mathcal{B}_t(W)}{\text{ess sup}}\ U_t(c, \theta)$$

where $U_t(c, \theta)$ is the expected utility from the consumption plan c and the dynamic portfolio strategy θ conditional on information available at time t, and $B_t(W)$ is the set of consumption plans and portfolio strategies that are budget feasible from time t onwards. It is well-known that the essential supremum of an arbitrary family of random variables is well-defined and it is a random variable itself ([11], Theorem A.18), so the problem is well-posed. If the information structure was restricted to be the one generated by a finite number of random variables, as, for instance, when the model posits the existence of some Markov process of state variables, then the value function would be a real-valued function of these state variables and of wealth. In that case, classical Envelope results would guarantee that, under standard assumptions, the marginal utility of wealth would be well defined and, at the optimum, it would coincide with the marginal utility from consumption. In our general case of a completely unrestricted information structure, however, the value function can no longer be assumed to be real-valued, rather it is a map from the Banach space of current level of wealth to the Banach lattice of integrable random variables. This is where our Envelope theorem for Banach lattices comes into play: we can employ it to show that, under a certain set of assumptions, the marginal utility of wealth is still well defined and, most importantly, at the optimum it still equates the marginal utility of consumption.

The remainder of this paper is as follows. In the next section, we set up the Banach lattice-valued optimization problem and prove our extension of the Envelope theorem to Banach lattices first for a concave objective function and then in the general case. In Section 3 we introduce the optimal consumption-portfolio problem with a general information structure and apply our results from Section 2 to discuss the conditions under which the marginal utility of wealth is well defined and coincides with the marginal utility from consumption. Section 4 concludes, while the Appendix reviews the notions of differentiability in Banach spaces and collects some fundamental results on the relationship between concavity and differentiability in this framework.

2 The general results

Let X be a Banach space and Y a order complete Banach lattice. We refer essentially to Aliprantis and Border [1] and Birkhoff [4] for the main definition and results on Banach lattices. As usual, we adjoin to Y the abstract maximal and minimal elements $\{\pm\infty\}$ and denote by $\bar{Y}$ the enlarged space.

We take an open set U in X as the set of parameters and an arbitrary choice set Θ. Let $F : \Theta \times U \to Y$ be the objective function. For each parameter $u \in U$, we define the value function as:

$$V(u) = \sup_{\theta \in \Theta} F(\theta, u). \tag{2.1}$$

We set $V(u) = -\infty$ if $\Theta = \emptyset$. When Θ is not empty, $V(u)$ belongs to Y if and only if the set $(F(\theta,u))_{\theta\in\Theta}$ is bounded from above. Otherwise, $V(u)$ may be well-defined (and not necessarily $+\infty$), even though $(F(\theta,u))_{\theta\in\Theta}$ is not bounded from above[1] but in this case $V(u)$ does not belong to Y.

We denote with $F'(\theta, u^*; x)$ (resp. $V'(u^*; x)$) the directional derivative of $F(\theta, \cdot\,)$ (resp. V) at u^* in the direction $x \in X$. Since Fréchet differentiability implies Gateaux differentiability, and since the two differentials coincide when they both exist, we use the notation $DF(\theta, u^*)$ (resp. $DV(u^*)$) for the differential of $F(\theta, \cdot\,)$ (resp. V) at u^*, and we specify the type of differential only when it is not apparent from the context.

2.1 A preliminary result

Our first result, which in fact extends Theorem 2 in [14], shows that the Envelope formula holds at any differentiability point of the value function, provided that both the objective and the value functions are differentiable.[2]. For any parameter $u \in U$ we denote with, $\Theta^*(u) = \{\theta \in \Theta : F(\theta, u) = V(u)\}$ the set of optimal choices.

Theorem 2.1 *Let $u^* \in U$ and assume that there exists some $r > 0$ such that $V(x) \in Y$ for every $x \in B(u^*, r)$[3]. Let $\Theta^*(u^*) \neq \emptyset$. Then for all $\theta \in \Theta^*(u^*)$:*

[1] One can for instance take $Y = L^p$ for some $p \geq 1$. If the set $(F(\theta,u))_{\theta\in\Theta}$ is bounded from above in L^q with $q < p$ but not in L^p, then $V(u) \in L^q$ but it does not belong to L^p.

[2] Weaker definitions of differentiability can also be given (see, for instance, Papageorgiou [17]) and bounds derived for the differential of the value function as is done by Morand, Reffet, Tarafdar [15] for the real-valued case, when the value function is not sufficiently smooth.

[3] $B(u^*, r)$ denotes as usual the open ball with center u^* and radius r

1. *if both $F(\theta, \cdot)$ and $V(\cdot)$ admits directional derivative at u^* in some direction $x \in X$, then $F'(\theta, u^*; x) \leq V'(u^*; x)$;*

2. *if both $F(\theta, \cdot)$ and $V(\cdot)$ are Gateaux-differentiable at u^*, then their Gateaux-differentials coincide, i.e. $DF(\theta, u^*) = DV(u^*)$;*

3. *if both $F(\theta, \cdot)$ and $V(\cdot)$ are Fréchet-differentiable at u^*, then their Fréchet-differentials coincide, i.e. $DF(\theta, u^*) = DV(u^*)$.*

Proof.

1. Let $h \in \Re$ and $x \in X$ such that $\|hx\|_X < r$. Then:

$$F(\theta, u^* + hx) - F(\theta, u^*) \leq V(u^* + hx) - V(u^*).$$

In particular, taking h_n in $\Re^+$, which decreases to 0 as $n \to +\infty$, and dividing both sides of the inequalities by h_n, we obtain

$$\frac{F(\theta, u^* + h_n x) - F(\theta, u^*)}{h_n} \leq \frac{V(u^* + h_n x) - V(u^*)}{h_n}. \tag{2.2}$$

If $F(\theta, \cdot)$ admits directional derivative at u^* along x, then according to the definition of directional derivative, we have that $(F(\theta, u^* + h_n x) - F(\theta, u^*))/h_n$ converges in Y-norm to $F'(\theta, u^*; x)$. Then there exists a subsequence which converges in order to the same limit[4].

Analogously, if V has a derivative at u^* along x, then $(V(u^* + h_n x) - V(u^*))/h_n$ will converge in Y-norm, and, up to a subsequence, in order, to $V'(u^*; x)$. Hence

$$F'(\theta, u^*; x) \leq V'(u^*; x). \tag{2.3}$$

2. If the two functions are Gateaux-differentiable, they admit directional derivatives along all directions. In particular, given $x \in X$, they admit directional derivatives along x and $-x$. The previous result implies that

$$DF(\theta, u^*)(x) = F'(\theta, u^*; x) \leq V'(u^*; x) = DV(u^*)(x)$$

[4] In an order complete Banach lattice, the norm convergence is equivalent to *relative uniform star convergence*, which in turn implies order convergence. As a consequence, if a sequence x_n converges in norm to x, then there exist a subsequence which is order convergent to x (see Birkhoff [4], Chapter 15, Theorems 19 and 20).

and

$$DF(\theta, u^*)(-x) = F'(\theta, u^*; -x) \leq V'(u^*; -x) = DV(u^*)(-x).$$

Moreover, since the Gateaux differential is homogeneous, $DF(\theta, u^*)(-x) = -DF(\theta, u^*)(x)$ and $DV(u^*)(-x) = -DV(u^*)(x)$. Therefore, $DF(\theta, u^*)(x) = DV(u^*)(x)$.

3. If $F(\theta, \cdot)$ and V are Fréchet differentiable at u^*, then they are *a fortiori* Gateaux differentiable and the differentials coincide. □

2.2 The concave case

In the framework of dynamic optimization models of economics, Benveniste and Scheinkman [3] determined a set of sufficient conditions for the value function to be differentiable under the assumption that the set of parameters and the set of possible choices are convex, and the objective function is concave with respect to both variables. In this subsection we extend their results to the case where the objective function takes values in a Banach lattice. We make the following assumption:

Assumption 2.1 *The sets Θ and U are convex and the objective function F is concave with respect to both θ and u.*

The next result is an immediate implication of this assumption:

Lemma 2.1 *The value function V is concave.*

Proof. For any $\lambda \in [0, 1]$, $u_1, u_2 \in X$, $\theta_1, \theta_2 \in \Theta$, we have the following inequalities:

$$\begin{aligned} V(\lambda u_1 + (1-\lambda)u_2) &\geq F(\theta_1 + (1-\lambda)\theta_2, \lambda u_1 + (1-\lambda)u_2) \\ &\geq \lambda F(\theta_1, u_1) + (1-\lambda)F(\theta_2, u_2). \end{aligned}$$

As a consequence,

$$V(\lambda u_1 + (1-\lambda)u_2) \geq \lambda \sup_{\theta_1 \in \Theta} F(\theta_1, u_1) + (1-\lambda) \sup_{\theta_2 \in \Theta} F(\theta_2, u_2) = \lambda V(u_1) + (1-\lambda)V(u_2). \qquad \square$$

We show that, as in the real-valued case, concavity is still a sufficient condition for the differentiability of the value function and, as a consequence, an Envelope formula holds at any differentiability point of the objective function given an optimal choice.

Theorem 2.2 *Let $u^* \in U$ and $\Theta^*(u^*) \neq \emptyset$. Then for all $\theta \in \Theta^*(u^*)$:*

1. *if $F(\theta, \cdot)$ is continuous and Gateaux differentiable at u^*, then V is continuous and Gateaux differentiable at u^* as well, and $DV(u^*) = DF(\theta, u^*)$;*

2. *if $F(\theta, \cdot)$ is Fréchet differentiable at u^*, then V is Fréchet differentiable at u^* as well, and $DV(u^*) = DF(\theta, u^*)$.*

Proof.

1. By Lemma 2.1, we know that V is concave. Moreover, $V(u) \geq F(\theta, u)$ for all $u \in U$. Therefore V is continuous at u^* thanks to Proposition A.1. Proposition A.3 implies that $\partial V(u^*)$ is non-empty, where $\partial V(u^*)$ denotes the superdifferential set of V at u^*. Take $L \in \partial V(u^*)$: then, $Lx \geq V(u^*+x) - V(u^*) \geq F(\theta, u^*+x) - F(\theta, u^*)$, namely $L \in \partial F(\theta, u^*)$. This means that $\partial V(u^*) \subset \partial F(\theta, u^*) = \{DF(\theta, u^*)\}$, where the last equality is a consequence of Proposition A.4. Since $\partial V(u^*)$ is non-empty, it must be necessarily $\partial V(u^*) = \{DF(\theta, u^*)\}$, hence by Proposition A.4, V is Gateaux-differentiable at u^* and $DV(u^*) = DF(\theta, u^*)$.

2. If $F(\theta)$ is Fréchet differentiable at u^*, then it is continuous and Gateaux differentiable at u^*. In virtue of the previous theorem, V is continuous and Gateaux-differentiable at u^* and $DV(u^*) = DF(\theta, u^*)$. Moreover, since V is concave, the differential is a superdifferential, hence the following inequalities hold for all $x \in X$:

$$DV(u^*)(x) \geq V(u^* + x) - V(u^*) \geq F(\theta, u^* + x) - F(\theta, u^*)$$

or, equivalently,

$$0 \geq V(u^* + x) - V(u^*) - DV(u^*)(x) \geq F(\theta, u^* + x) - F(\theta, u^*) - DF(\theta, u^*)(x).$$

The inequalities are clearly reversed when taking the absolute values, that is:

$$0 \leq |V(u^* + x) - V(u^*) - DV(u^*)(x)| \leq |F(\theta, u^* + x) - F(\theta, u^*) - DF(\theta, u^*)(x)|$$

Passing to the norms and dividing by $\|x\|_X$ one obtains:

$$0 \leq \frac{\|V(u^* + x) - V(u^*) - DV(u^*)(x)\|_Y}{\|x\|_X} \leq \frac{\|F(\theta, u^* + x) - F(\theta, u^*) - DF(\theta, u^*)\|_Y}{\|x\|_X}.$$

One can then take the limit as $\|x\|_X$ goes to 0: the right-hand term goes to 0 because of the Fréchet differentiability of $F(\theta)$. As a consequence, the middle term goes to 0, which implies that V is Fréchet differentiable. $\square$

2.3 The general case

We deal now with the case in which the objective function is not required to be concave, and the choice set is arbitrary. We first present two alternative sets of conditions on the objective function that guarantee that the value function in (2.1) is continuous. We then discuss conditions under which the value function is either Gateaux or Fréchet differentiable and we state our Envelope results.

2.3.1 Continuity

Given a parameter $u^* \in U$, we present two alternative sets of conditions for the continuity of the value function at u^*. The first set is based on a suitable extension of the standard notion of Lipschitz-continuity to the case in which the function takes values in a Banach lattice. We recall that in Banach lattices $|y| = y \vee (-y)$, where $\vee$ denotes the lattice operation of supremum.

Assumption 2.2 *The objective function $F(\theta, \cdot\,)$ is o-Lipschitz-continuous*[5] *at u^* uniformly in θ, namely, there exists $r > 0$ such that for all $w, v \in B(u^*, r)$, for some $l_{u^*} \in Y$,*

$$\sup_{\theta \in \Theta} |F(\theta, w) - F(\theta, v)| \leq l_{u^*} \|w - v\|_X$$

Proposition 2.1 *Assume that there exists $r > 0$ such that $V(w) \in Y$ for all $w \in B(u^*, r)$. If F satisfies Assumption 2.2, then the value function V is continuous in a neighborhood of u^*.*

Proof. Let $v, w \in B(u^*, r)$. Then,

$$|V(w) - V(v)| \;=\; \left|\sup_{\theta_1} \; F(\theta_1, w) - \sup_{\theta_2} \; F(\theta_2, v)\right| \leq \sup_{\theta} \; |F(\theta, w) - F(\theta, v)|\,.$$

[5] For the notion of o-Lipschitz function see also Papageorgiou [16, 17].

If Assumption 2.2 holds, it follows immediately that

$$\|V(w) - V(v)\|_Y \leq \|l_{u^*}\|_Y \|w - v\|_X$$

hence V is continuous on $B(u^*, r)$. □

An alternative set of sufficient conditions for the continuity of the value function is based on differentiability properties of the objective function. It is an established fact that Lipschitz-continuity implies Gateaux-differentiability outside of a null set (see for instance Lindenstrauss and Preiss [12]). On the other hand, it is well known that Gateaux differentiability in general does not imply continuity. The Gateaux differentiability of the objective function at a given point, therefore, does not even guarantee the continuity of the objective function itself, let alone the continuity of the value function. For this reason, our differentiability-based conditions for the continuity of the value function start from requiring the objective function to be Fréchet differentiable at u^* uniformly in θ.

Assumption 2.3 *The objective function $F(\theta, \cdot)$ is Fréchet differentiable at u^* for every $\theta \in \Theta$. Moreover, we require that*

$$F(\theta, u^* + x) - F(\theta, u^*) = DF(\theta, u^*)(x) + \sigma(\theta, u^*, x) \cdot \|x\|_X$$

where $|\sigma(\theta, u^*, x)| \leq \xi \|x\|_X$, for $\xi \in Y$, for all $\theta \in \Theta$, for $x \in X$ such that $u^* + x \in U$.

Intuitively, Assumption 2.3 requires that each member of the family of functions $(F(\theta, \cdot))_{\theta \in \Theta}$ admits a first-order expansion at u^*, with the error term uniformly bounded in the choice variable θ. In particular, Assumption 2.3 implies that the ratio

$$\frac{\|F(\theta, u^* + x) - F(\theta, u^*) - DF(\theta, u^*)(x)\|_Y}{\|x\|_X}$$

goes to 0 uniformly in θ as $\|x\|_X$ tends to 0, that is the objective function F is Fréchet equi-differentiable at u^*. To control the variation of the value function, the next assumption requires the differential of F to be bounded uniformly in the choice variable θ as well.

Assumption 2.4 *For every $x \in X$ there exists a vector $y_x \in Y$ such that for all $\theta \in \Theta$.*

$$|DF(\theta, u^*)(x)| \leq y_x \|x\|_X$$

Exploiting these two assumptions[6] we can establish the following result on the continuity of the value function.

Proposition 2.2 *Assume that there exists* $r > 0$ *such that* $V(w) \in Y$ *for all* $w \in B(u^*, r)$. *If* F *satisfies Assumptions 2.3 and 2.4, then the value function* V *is continuous at* u^*.

Proof. Let $w \in B(u^*, r)$, and $x = w - u^*$. Then exploiting Assumption 2.3 and Assumption 2.4 we have

$$F(\theta, w) - F(\theta, u^*) = F(\theta, u^* + x) - F(\theta, u^*) = DF(\theta, u^*)(x) + \sigma(\theta, u^*, x) \cdot \|x\|_X .$$

As a consequence

$$\begin{array}{rcl} |V(w) - V(u^*)| & \leq & \sup\limits_\theta \; |F(\theta, w) - F(\theta, u^*)| \\ & \leq & \sup\limits_\theta |F(\theta, w) - F(\theta, u^*) - DF(\theta, u^*)(x)| + \sup\limits_\theta |DF(\theta, u^*)(x)| \\ & \leq & \sup\limits_\theta |\sigma(\theta, u^*, x)| \cdot \|x\|_X + \sup\limits_\theta |DF(\theta, u^*)(x)| \\ & \leq & \xi \, \|x\|_X^2 + y_x \|x\|_X \end{array}$$

We can then take the Y-norms of both sides. Assumption 2.4 implies that $\|DF(\theta, u^*)(x)\|_Y \leq M_x$ for all $\theta \in \Theta$, where $M_x = \|y_x\|_Y \, \|x\|_X$, that is, the family of operators $(DF(\theta, u^*))_{\theta \in \Theta}$ is pointwise bounded. The Banach-Steinhaus theorem implies that there exists a constant Λ such that $\|DF(\theta, u^*)(x)\|_Y \leq \Lambda \, \|x\|_X$ for all $\theta \in \Theta$. As a consequence,

$$\|V(w) - V(u^*)\|_Y \leq \xi \, \|w - u^*\|_X^2 + \Lambda \, \|w - u^*\|_X ,$$

which shows that V is continuous at u^*. $\square$

Remark Our results on the continuity of the value function do not require the existence of an optimal choice for u^*. In other words, while we ask for the supremum defining the value function to be well defined in Y in a neighborhood of u^*, we do not require such supremum to be attained for some optimal choice parameter.

[6] Similar requirements of equi-differentiability with respect to the choice parameter and of boundedness of the derivative of F are made by Milgrom and Segal [14] to obtain the continuity and the differentiability of the value function for the case in which the objective function takes values in $\Re$.

2.3.2 Differentiability

We consider now the differentiability issue. We first present conditions under which an Envelope formula holds in terms of Gateaux differentials. We then move to Fréchet differentiability, and present a set of conditions that guarantee that an Envelope formula holds in terms of Fréchet differentials.

Assumption 2.5 *(i) There exists some $r > 0$, such that $V(w) \in Y$ and the set $\Theta^*(w)$ is non-empty for all $w \in B(u^*, r)$;*

(ii) the objective function $F(\theta, \cdot)$ is Gateaux differentiable at u^* for every $\theta \in \Theta$. Moreover, for every direction $x \in X$ and for every sufficiently small $h \in \Re^+$

$$\sup_{\theta \in \Theta} \left| \frac{F(\theta, u^* + hx) - F(\theta, u^*)}{h} - DF(\theta, u^*)(x) \right| \leq \Sigma(u^*, x, h)$$

for some $\Sigma(u^*, x, h) \in Y$ such that $\lim_{h \to 0} \|\Sigma(u^*, x, h)\|_Y = 0$;

(iii) for every direction x and $h \in \Re^+$ such that $\|hx\|_X < r$, for every $\theta^* \in \Theta^*(u^*)$, the Gateaux differential of F at u^* satisfies:

$$\lim_{h \to 0} \sup_{\theta_{hx} \in \Theta^*(u^* + hx)} \|DF(\theta_{hx}, u^*)(x) - DF(\theta^*, u^*)(x)\|_Y = 0.$$

We are now ready to state our first main result.

Theorem 2.3 *If Assumption 2.5 holds, then the value function V is Gateaux-differentiable at u^* and $DV(u^*) = DF(\theta, u^*)$ for $\theta \in \Theta^*(u^*)$.*

To prove this result we need the following lemma:

Lemma 2.2 *Suppose that Assumption 2.5 holds. Let $\theta \in \Theta^*(u^*)$. Moreover, for a fixed direction $x \in X$ and $h \in \Re^+$ such that $\|hx\|_X < r$, let $\theta_{hx} \in \Theta^*(u^* + hx)$. Then*

$$\lim_{h \to 0} \left\| \frac{F(\theta_{hx}, u^* + hx) - F(\theta_{hx}, u^*)}{h} - DF(\theta, u^*)(x) \right\|_Y = 0$$

Proof of Lemma. We have, thanks to Assumption 2.5 (ii), that:

$$\left| \frac{F(\theta_{hx}, u^* + hx) - F(\theta_{hx}, u^*)}{h} - DF(\theta_{hx}, u^*)(x) \right| \leq \Sigma(u^*, h, x).$$

Hence, the following inequalities hold:

$$\begin{aligned}
&\left\| \frac{F(\theta_{hx}, u^* + hx) - F(\theta_{hx}, u^*)}{h} - DF(\theta, u^*)(x) \right\|_Y \\
\leq\;& \left\| \frac{F(\theta_{hx}, u^* + hx) - F(\theta_{hx}, u^*)}{h} - DF(\theta_{hx}, u^*)(x) \right\|_Y + \|DF(\theta_{hx}, u^*)(x) - DF(\theta, u^*)(x)\|_Y \\
\leq\;& \|\Sigma(u^*, h, x)\|_Y + \|DF(\theta_{hx}, u^*)(x) - DF(\theta, u^*)(x)\|_Y \\
\leq\;& \|\Sigma(u^*, h, x)\|_Y + \sup_{\theta_{hx} \in \Theta^*(u^* + hx)} \|DF(\theta_{hx}, u^*)(x) - DF(\theta, u^*)(x)\|_Y
\end{aligned}$$

Assumption 2.5 ((ii) and (iii)) implies that the right-hand term in the previous inequality goes to 0 as $h \to 0$. Hence the claim follows. $\square$

Proof of Theorem 2.3. Let $\theta_{u^*} \in \Theta^*(u^*)$. Then, $V(u^*) = F(\theta_{u^*}, u^*) \geq F(\theta, u^*)$ for any $\theta \in \Theta$. Now, for a given direction x, let $h \in \Re^+$ be such that $\|hx\|_X < r$ and take θ_{hx} in $\Theta^*(u^* + hx)$, which is not empty by Assumption 2.5 (i). Then

$$V(u^* + hx) = F(\theta_{hx}, u^* + hx) \geq F(\theta, u^* + hx) \qquad \text{for any } \theta \in \Theta.$$

In particular, $V(u^*) \geq F(\theta_{hx}, u^*)$ and $V(u^* + hx) \geq F(\theta_{u^*}, u^* + hx)$. Thus, we can write:

$$F(\theta_{u^*}, u^* + hx) - F(\theta_{u^*}, u^*) \leq V(u^* + hx) - V(u^*) \leq F(\theta_{hx}, u^* + hx) - F(\theta_{hx}, u^*).$$

Dividing by h and subtracting the differential $DF(\theta_{u^*}, u^*)(x)$, we obtain the following inequalities:

$$\begin{aligned}
\frac{F(\theta_{u^*}, u^* + hx) - F(\theta_{u^*}, u^*)}{h} - DF(\theta_{u^*}, u^*)(x) \;&\leq\; \frac{V(u^* + hx) - V(u^*)}{h} - DF(\theta_{u^*}, u^*)(x) \\
&\leq\; \frac{F(\theta_{hx}, u^* + hx) - F(\theta_{hx}, u^*)}{h} - DF(\theta_{u^*}, u^*)(x)
\end{aligned}$$

Take now the limit as $h \to 0^+$. Since the first and the last term converge to 0 in Y-norm, the middle term must converge to 0 as well. This implies that V is differentiable at u^* along the direction x and $V'(u^*; x) = F'(\theta_{u^*}, u^*; x) = DF(\theta_{u^*}, u^*)(x)$. The same argument can be repeated for every $x \in X$, hence V is directionally differentiable along every direction x and $V'(u^*; \cdot) = DF(\theta_{u^*}, u^*)$. Since this is a linear and continuous operator, we can finally say that V is Gateaux-differentiable at u^* and $DV(u^*) = DF(\theta_{u^*}, u^*)$. $\square$

To conclude we address the issue of Fréchet-differentiability. To this end, by suitably readjusting Assumption 2.5 we state:

Assumption 2.6 *(i) There exists some $r > 0$, such that $V(w) \in Y$ and the set $\Theta^*(w)$ is non-empty for all $w \in B(u^*, r)$;*

(ii) assumption 2.3 holds, namely, the objective function $F(\theta, \cdot)$ is Fréchet differentiable at u^* for every $\theta \in \Theta$. Moreover,

$$F(\theta, u^* + x) - F(\theta, u^*) = DF(\theta, u^*)(x) + \sigma(\theta, u^*, x) \cdot \|x\|_X$$

where $|\sigma(\theta, u^*, x)| \leq \xi \|x\|_X$, for $\xi \in Y$, for all $\theta \in \Theta$, for $x \in X$ such that $u^* + x \in U$;

(iii) for $w \in B(u^*, r)$, for every $\theta \in \Theta^*(u^*)$, the Fréchet differential of F at u^* satisfies:

$$\lim_{\|w-u^*\|_X \to 0} \sup_{\theta_w \in \Theta^*(w)} \sup_{\|v\|_X \leq 1} \|DF(\theta_w, u^*)(v) - DF(\theta, u^*)(v)\|_Y = 0.$$

Assumption 2.6 (ii) implies Assumption 2.5 (ii), where, in addition, the error term goes to zero uniformly in x, namely $\lim_{h\to 0} \sup_{\|x\|_X \leq 1} \|\Sigma(u^*, x, h)\|_Y = 0$. An argument similar to that employed in the proof of Theorem 2.3 establishes now the Fréchet-differentiability of the value function. Formally, we have:

Theorem 2.4 *Let Assumption 2.6 hold. Then the value function V is Fréchet-differentiable at u^* and $DV(u^*) = DF(\theta, u^*)$ for $\theta \in \Theta^*(u^*)$.*

Remark. Our results can be strenghtened by requiring Assumptions 2.5, respectively 2.6, to hold for a given selection of optimal choices, instead of imposing them on the whole set of optimal choices. This stronger version would be particularly useful when the optimization problem has enough structure so that an optimal selection on which to test our assumptions can be readily identified.

3 Envelope results for asset pricing models with a general information filtration

In this section we focus on security markets with a general information filtration. In this framework, the first-order conditions for optimality of an agent maximizing a smooth utility can be formulated as the martingale property of prices, after normalizing by a state-price process. Such state-price process can be characterized in terms of the agent's utility gradient (see for instance Duffie [9]). For a wide class of state-dependent utilities, the Envelope theorems of the previous section enable us to extend the link between the differential of the optimal utility of consumption and the differential of the maximum remaining utility of wealth to this case of a general information filtration.

3.1 The model

We consider a frictionless security market in which J assets are traded over the investment horizon $\mathcal{T} = \{0, 1, \ldots, T\}$. We take as given a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t=0}^T)$. As usual, we assume that $\mathcal{F}$ is augmented with all the $P-$null sets, $\mathcal{F}_0$ is the trivial sigma-algebra $\{\varnothing, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. All the processes are adapted to $\mathcal{F}$. We denote by $d_j(t)$ (resp. $S_j(t)$) the cash flow distributed by (resp. the ex-dividend price of) asset j at date t, with $j = 1, ..., J$. Given $p \in [1, +\infty[$, we assume that $S_j(t)$, $d_j(t) \in L^p(\mathcal{F}_t)$ for all t. Without loss of generality, we assume that the assets distribute no cash flow at date 0 and a liquidating one at date T, that is $d_j(0) = S_j(T) = 0$ almost surely.

A *dynamic investment strategy* $\theta = \{\theta(t)\}_{t=0}^{T-1}$ is a J-dimensional process where $\theta_j(t)$ represents the position (in number of units) in assets j taken at date t and liquidated at date $t+1$. We denote by $V_\theta = \{V_\theta(t)\}_{t=0}^T$ the *value process* of θ, given by

$$V_\theta(t) = \begin{cases} \theta(t) \cdot S(t) & t < T \\ \theta(T-1) \cdot d(T) & t = T. \end{cases}$$

The cash flow $x_\theta(t)$ generated by the strategy θ at t is

$$x_\theta(t) = \begin{cases} -V_\theta(0) & t = 0 \\ \theta(t-1) \cdot [S(t) + d(t)] - V_\theta(t) & t = 1, \ldots, T-1 \\ V_\theta(T) & t = T. \end{cases} \tag{3.1}$$

Henceforth, we call the sequence $x_\theta = \{x_\theta(t)\}_{t=0}^T$ the *cash-flow process* of θ. We call *admissible* any dynamic investment strategy θ such that $V_\theta(t)$, $x_\theta(t) \in L^p(\mathcal{F}_t)$ for $t = 0, 1, \ldots, T$. We denote with Θ the set of all *admissible* dynamic investment strategies.

An agent in this market is identified by an *initial endowment* $e_0 \geq 0$ and a *complete and transitive preference relation* on the set $\mathcal{C} = \prod_{t=0}^{T} L^p(\mathcal{F}_t)$ of consumption sequences $c = (c(0), c(1), \ldots, c(T))$, with $c(t) \in L^p(\mathcal{F}_t)$ for all t. In choosing the optimal intertemporal consumption and asset allocation, each agent $(e_0, \succeq)$ in $\mathcal{A}$ faces the budget constraint

$$B(e_0) = \{c \in \mathcal{C} \mid c(0) \leq x_\theta(0) + e_0,\ c(t) \leq x_\theta(t)\ \forall\ t > 0 \quad \textit{for some } \theta \in \Theta\}\,.$$

We consider the class of agents whose preferences have a time-additive von Neumann-Morgenstern representation, such that the period-utilities are allowed to depend on the state ω and on both the individual's past and present consumptions. In this way our model is able to accomodate habit formation models of both the internal and external type (see for instance [7]). For sake of notation, we denote with $c_{t,s}$ the stream of consumption from time t to time s, that is $c_{t,s} = (c(t), \ldots, c(s))$. Analogously, we let $x_{t,s}(\theta) = (x_\theta(t), \ldots, x_\theta(s))$ denote the cash-flow generated by a dynamic trading strategy θ from t to s. We assume that the preference $U(c)$ of an agent over consumption sequences $c \in \mathcal{C}$ takes the form

$$U(c) = \sum_{t=0}^{T} \int_\Omega u_t(c_{0,t}(\omega), \omega) dP(\omega) = \sum_{t=0}^{T} E\left[u_t\left(c_{0,t}\right)\right] \tag{3.2}$$

where, for all $t \leq T$, the period utilities $u_t : \Re^{t+1} \times \Omega \to \Re$ satisfy the following conditions:

(i) for all t, the function $u_t(\gamma, \omega) : \Re^{t+1} \times \Omega \to \Re$ is measurable with respect to the product σ-algebra $\mathcal{B}(\Re^{t+1}) \otimes \mathcal{F}_t$ (where $\mathcal{B}(\Re)$ denotes the Borel σ-algebra);

(ii) for all $c \in B(e_0)$, the integrals $\int_\Omega u_t(c_{0,t}(\omega), \omega) dP(\omega)$ are well defined and they are either finite or they take the value $-\infty$ so that $U(c) < +\infty$ for all $c \in B(e_0)$;

(iii) for every t, the function $u_t(\,\cdot\,, \omega) : \Re^{t+1} \to \Re$ is real-valued and strictly increasing[7] for almost every ω.

An *optimal consumption-portfolio choice* for such an agent is a couple $(c^*, \theta^*) \in \mathcal{C} \times \Theta$ such that $c^*(0) \leq x_{\theta^*}(0) + e_0$, $c^*(t) \leq x_{\theta^*}(t)$ for $t = 1, \ldots, T$ and $U(c^*) \geq U(c)$ for all $c \in \mathcal{C}$ such that $c(0) \leq x_\theta(0) + e_0$, $c(t) \leq x_\theta(t)$ for $t = 1, \ldots, T$ for some $\theta \in \Theta$. We make the following assumption:

Assumption 3.1 *There exists an optimal solution to the consumption-portfolio problem for an agent with preferences as in (3.2) and initial endowment e_0.*

[7] The function $u : \Re^{t+1} \to \Re$ is strictly increasing if $u(c_0, \ldots, c_t) > u(\tilde{c}_0, \cdots, \tilde{c}_t)$ for every pair $(c_s)_{0 \leq s \leq t}, (\tilde{c}_s)_{0 \leq s \leq t}$ such that $c_s \geq \tilde{c}_s$ for all s and $c_{\bar{s}} > \tilde{c}_{\bar{s}}$ for at least one $\bar{s}$.

As a consequence of the strict monotonicity of the period-utilities, the budget constraint is binding at the optimum, namely $c^*(0) = x_{\theta^*}(0) + e_0$ and $c^*(t) = x_{\theta^*}(t)$ for $t = 1, \ldots, T$. To any optimal consumption-portfolio choice (c^*, θ^*) for an agent with preferences as in (3.2) and initial endowment e_0, we associate the optimal intertemporal wealth $W^* = \{W^*(t)\}_{t=0}^T$ generated by θ^*, that is

$$W^*(t) = \begin{cases} e_0 & t = 0 \\ \theta^*(t-1) \cdot [S(t) + d(t)], & t = 1, \ldots, T. \end{cases}$$

Note that $W^*(t) = x_{\theta^*}(t) + V_{\theta^*}(t) = c^*(t) + V_{\theta^*}(t)$ for $t = 1, \ldots, T-1$.

Fix now $t \in \{0, 1, \ldots, T-1\}$, a sequence of past consumptions $c_{0,t-1}$ and a $\mathcal{F}_t$-measurable random variable W. An agent who, at time t, has a level of wealth W, can choose among the infinitely many consumption-portfolio pairs such that $c(t) + V_\theta(t) \le W$ and $c(s) \le x_\theta(s)$ for $s > t$. Every pair determines a "remaining utility" (conditionally to the information available at time t) which is a $\mathcal{F}_t$-measurable random variable. The proper notion of supremum to employ is therefore the one of *essential supremum.* We recall here that for any set Φ of random variables, there exists a random variable φ^*, called the *essential supremum* of Φ and denoted as $\varphi^* = \operatorname*{ess\,sup}_{\varphi\in\Phi} \varphi$, such that: (i) $\varphi^* \ge \varphi$ P-a.s for all $\varphi \in \Phi$; (ii) any other random variable $\tilde{\varphi}$ such that $\tilde{\varphi} \ge \varphi$ for all $\varphi \in \Phi$ satisfies $\tilde{\varphi} \ge \varphi^*$ P-a.s. ([11], Theorem A.18). The maximum remaining utility (or continuation utility) at time t for an agent whose current level of wealth is W, is the random variable defined as follows

$$\begin{aligned} H(t, c_{0,t-1}, W) \equiv \operatorname*{ess\,sup}_{(c,\theta)\in\mathcal{C}\times\Theta} \sum_{s=t}^{T} E_t\left[u_s(c_{0,t-1}, c_{t,s})\right] & \\ s.t. \begin{cases} c(t) + V_\theta(t) \le W & \\ c(s) \le x_\theta(s) & s = t+1, \ldots, T \end{cases} \end{aligned} \tag{3.3}$$

for $t = 0, 1, \ldots, T$, where $E_t[\ \cdot\ ]$ denotes the conditional expectation with respect to $\mathcal{F}_t$. We assume that the integrals $E\left[u_s(c_{0,t-1}, c_{t,s})\right]$ (and hence the conditional expectations in (3.3)) are well defined and, for all consumption levels satisfying the budget constraint at time t, they are either finite or take the value $-\infty$ (in which case we set $E_t\left[u_s(c_{0,t-1}, c_{t,s})\right] = -\infty$). In particular, when $c_{0,t-1} = c^*_{0,t-1} = (c^*_0, \ldots, c^*_{t-1})$ and $W = W^*(t)$, then $H(t, c^*_{0,t-1}, W^*(t))$ represents the maximum remaining utility given the optimal past consumptions and the optimal wealth. Battauz et al. [2] in Proposition 1 prove that, in this framework, the value function H is well-defined and finite at the optimum, and it satisfies the Dynamic Programming Principle, that is

$$\begin{aligned} H(t, c^*_{0,t-1}, W^*(t)) &= \sum_{s=t}^{T} E_t\left[u_s(c^*_{0,s})\right] \\ &= u_t(c^*_{0,t}) + E_t\left[H(t+1, c^*_{0,t}, W^*(t+1))\right]. \end{aligned} \tag{3.4}$$

3.2 The Envelope results

In what follows, we consider the time t as fixed and we let $H(c_{0,t-1}, W) \equiv H(t, c_{0,t-1}, W)$ to streamline the notation. Given the stream of past-optimal consumptions $c^*_{0,t-1}$, a strategy θ and a level of wealth W, we define the map $F(c^*_{0,t-1}, \theta, W)$ as follows:

$$F(c^*_{0,t-1}, \theta, W) = u_t(c^*_{0,t-1}, W - V_\theta(t)) + E_t\left[\sum_{s=t+1}^{T} u_s\left(c^*_{0,t-1}, W - V_\theta(t), x_{t+1,s}(\theta)\right)\right] \tag{3.5}$$

The strict monotonicity of the period-utilities forces the constraints to be binding, which implies that problem (3.3) is equivalent to

$$H(c^*_{0,t-1}, W) \equiv \operatorname*{ess\,sup}_{\theta \in \Theta_t} F(c^*_{0,t-1}, \theta, W). \tag{3.6}$$

where Θ_t is the linear space of strategies that are admissible from time t on, that is the set of sequences $\theta = \{\theta(s)\}_{s=t}^{T-1}$ of J-dimensional, $\mathcal{F}_s-$measurable random variables such that $V_\theta(s)$, $x_\theta(s) \in L^p(\mathcal{F}_s)$ for $s = t, \ldots, T$. From now on we assume $W \in L^p(\mathcal{F}_t)$ and that $F(c^*_{0,t-1}, \theta, W)$ takes values in $L^1(\mathcal{F}_t)$ in a neighborhood of the optimal wealth $W^*(t)$, so that our initial problem takes the form (2.1), with the parameter W lying in the Banach space $L^p(\mathcal{F}_t)$ and the objective function taking values in the Banach lattice $L^1(\mathcal{F}_t)$.

We are interested in the first place to relate the differentiability of F with respect to wealth and the differentiability of the period utilities with respect to consumption. In particular, we observe that, since all future period utilities depend on current consumption, the differentiability (and the differential) of F with respect to current wealth, will depend not only on the differentiability (and differential) of the current period utility with respect to current consumption, but on those of all future period-utilities. To see this more precisely, let

$$\tilde{u}_{s,\theta}(\,\cdot\,) \equiv u_s(c^*_{0,t-1}, \,\cdot\,, x_{t+1,s}(\theta)) \qquad s = t, \ldots, T, \tag{3.7}$$

take any strategy $\theta \in \Theta_t$, any two wealth levels $W_1, W_2 \in L^p(\mathcal{F}_t)$, let $c_i(t) = W_i - V_\theta(t)$ for $i = 1, 2$ and exploit (3.5) to see that

$$\begin{aligned} F(c^*_{0,t-1}, \theta, W_1) - F(c^*_{0,t-1}, \theta, W_2) &= \tilde{u}_{t,\theta}(W_1 - V_\theta(t)) - \tilde{u}_{t,\theta}(W_2 - V_\theta(t)) \\ &\quad + E_t \left\{ \textstyle\sum_{s=t+1}^T \left[\tilde{u}_{s,\theta}(W_1 - V_\theta(t)) - \tilde{u}_{s,\theta}(W_2 - V_\theta(t)) \right] \right\} \\ &= \tilde{u}_{t,\theta}(c_1(t)) - \tilde{u}_{t,\theta}(c_2(t)) \\ &\quad + E_t \left\{ \textstyle\sum_{s=t+1}^T \left[\tilde{u}_{s,\theta}(c_1(t)) - \tilde{u}_{s,\theta}(c_2(t)) \right] \right\} \end{aligned}$$

which shows that, since in this case all future period utilities from time t on depend in general also on time t consumption, the increment of F corresponds now to the sum of the conditional expected increments in all future period utilities. The correct relation is stated in the next proposition:

Proposition 3.1 *Assume that the functions $\tilde{u}_{s,\theta}(\,\cdot\,) = u_s(c^*_{0,t-1}, \,\cdot\,, x_{t+1,s}(\theta))$ are Gateaux (Fréchet) differentiable at some point $W - V_\theta(t)$ for all $s = t, \ldots, T$. Then the function $F(c^*_{0,t-1}, \theta, \,\cdot\,)$ is Gateaux (Fréchet) differentiable at W and, in this case,*

$$DF(c^*_{0,t-1}, \theta, W)(X) = \sum_{s=t}^{T} E_t \left[D\tilde{u}_{s,\theta}(W - V_\theta(t))(X) \right]. \tag{3.8}$$

This proposition is an immediate consequence of the following lemma.

Lemma 3.1 *Let $g : L^p \to L^1$ be Gateaux (resp. Fréchet differentiable) at some point $W \in L^p$. Then the function $G(W) = E_t[g(W)]$ is Gateaux (resp. Fréchet differentiable at W) and $DG(W)(X) = E_t[Dg(W)(X)]$.*

Proof. The arguments for the Gateaux and Fréchet differential case are the same. Therefore, we will prove the result only for the case of Fréchet differentiability. Since g is Fréchet differentiable at W, we have that

$$\lim_{\|X\|_{L^p} \to 0} \frac{E\left[|g(W+X) - g(W) - Dg(W)(X)| \right]}{\|X\|_{L^p}} = 0.$$

On the other hand, exploiting the properties of conditional expectation and, in particular, Jensen's inequality, we obtain that:

$$\begin{aligned} E\left[|G(W+X) - G(W) - E_t\left[Dg(W)(X) \right]| \right] &= E\left[|E_t\left[g(W+X) - g(W) - Dg(W)(X) \right]| \right] \\ &\leq E\left[E_t\left[|g(W+X) - g(W) - Dg(W)(X)| \right] \right] \\ &= E\left[|g(W+X) - g(W) - Dg(W)(X)| \right]. \end{aligned}$$

It follows immediately that

$$\lim_{\|X\|_{L^p}\to 0} \frac{E\left[\left|G(W+X)-G(W)-E_t\left[\mathcal{D}g(W)(X)\right]\right|\right]}{\|X\|_{L^p}} = 0$$

which implies that G is Fréchet differentiable at W and $DG(W) = E_t\left[Dg(W)\right]$. $\square$

We are now ready to show, as a first result, that the Envelope condition holds when the value function and the period utilities are differentiable.

Proposition 3.2 *Let (c^*, θ^*) be an optimal consumption-portfolio pair. Assume that all the time s-period utilities $\tilde{u}_{s,\theta^*}(\cdot) = u_s(c^*_{0,t-1}, \cdot, c^*_{t+1,s})$ are Gateaux (Fréchet)-differentiable at the optimal consumption $c^*(t)$ and the time t value function $H(c^*_{0,t-1}, \cdot)$ is Gateaux (Fréchet)-differentiable at the optimal wealth $W^*(t) = c^*(t) + V_{\theta^*}(t)$. Then*

$$DH(c^*_{0,t-1}, W^*(t))(X) = \sum_{s=t}^{T} E_t\left[D\tilde{u}_{s,\theta^*}(c^*(t))(X)\right]. \tag{3.9}$$

Proof. An immediate consequence of the Dynamic Programming Principle (3.4) is that the optimal strategy θ^* satifies $F(c^*_{0,t-1}, \theta^*, W^*(t)) = H(c^*_{0,t-1}, W^*(t))$. Proposition 3.1 implies that $F(c^*_{0,t-1}, \theta^*, W)$ is differentiable at $W^*(t)$ and

$$DF(c^*_{0,t-1}, \theta^*, W^*(t))(X) = \sum_{s=t}^{T} E_t\left[D\tilde{u}_{s,\theta}(W^*(t) - V^*_\theta(t))(X)\right].$$

Theorem 2.1 yields the claim. $\square$

Remark. In the special case where the period-utilities do not depend on past consumptions, but only on current consumption, that is $u_t(c(0), \ldots, c(t)) \equiv u_t(c(t))$, the differentiability of F with respect to current wealth is equivalent to the differentiability uniquely of the current period-utility u_t with respect to current consumption. In particular, in this case, formula (3.8) becomes $DF(\theta, W) = Du_t(W - V_\theta(t)) = Du_t(c(t))$ and the Envelope formula (3.9) reduces to $DH(W^*(t)) = Du_t(c^*(t))$.

A first set of conditions on the primitives of the optimal consumption-portfolio problem which guarantees that the value function is differentiable consists in requiring the period utilities to be concave. If all the period utilities from t on are concave then $F(c_{0,t-1}, \theta, W)$ is manifestly concave with respect to both θ and W. The following result is then an immediate consequence of Theorem 2.2 in Section 2.

Proposition 3.3 *If the period utilities u_s are concave, and Gateaux (resp. Fréchet)-differentiable at the optimal consumption $c^*(t)$, for all $s = t \dots, T$, then the value function is concave, continuous and Gateaux (resp. Fréchet)-differentiable at the optimal wealth $W^*(t)$ and equality* (3.9) *holds.*

Consider now the general case in which concavity of the period utilities is not required. To apply our general Envelope results to this asset pricing setting, we rephrase the assumptions of Section 2.3 in terms of the period utilities u_s or, equivalently, in terms of the transformed period utilities $\tilde{u}_{s,\theta}$ defined in (3.7).

Remark. To see why imposing conditions on u_s is tantamount to imposing them on $\tilde{u}_{s,\theta}$, let

$$\mathcal{C}^*(W^*(t)) = \left\{ c \in \prod_{s=t}^{T} L^p(\mathcal{F}_s) : \begin{array}{l} c(t) = W^*(t) - V_\theta(t) \\ c(s) = x_\theta(s) \text{ for } s = t \dots, T, \end{array} \qquad \text{for some } \theta \in \Theta_t \right\}$$

be the set of consumption streams admissible from t onward, given the optimal wealth $W^*(t)$, and observe that $\tilde{u}_{s,\theta}$ is in fact the restriction of u_s to $\mathcal{C}^*(W^*(t))$. Let now $c(t) = W^*(t) - V_\theta(t)$, $c(s) = x_\theta(s)$ for $s = t \dots, T$, $\theta \in \Theta_t$ and consider a perturbation $X \in L^p(\mathcal{F}_t)$ of $c(t)$. Then

$$u_s(c^*_{0,t-1}, c(t) + X, c_{t+1,s}) - u_s(c^*_{0,t-1}, c(t), c_{t+1,s}) = \tilde{u}_{s,\theta}(c(t) + X) - \tilde{u}_{s,\theta}(c(t))$$

from which, taking the proper limits, one sees that the period-utility $u_s(c^*_{0,t-1}, \cdot, c_{t+1,s})$ is differentiable at $c(t)$ if and only if $\tilde{u}_{s,\theta}$ is differentiable at $c(t)$, and $Du_s(c(t)) = D\tilde{u}_{s,\theta}(c(t))$. Assumptions 2.3 and 2.4 become then a sort of equidifferentiability and uniform boundedness, respectively, of the transformed period utilities $\tilde{u}_{s,\theta}$ over the set of these admissible consumptions levels produced by deviations from the optimal strategy.

Assumption 3.2 *(i) For every $\theta \in \Theta_t$, the function $\tilde{u}_{s,\theta}$ is Fréchet-differentiable at $c(t) = W^*(t) - V_\theta(t)$ and for every $X \in L^p(\mathcal{F}_t)$ with a sufficiently small norm*

$$\tilde{u}_{s,\theta}(c(t) + X) - \tilde{u}_{s,\theta}(c(t)) = D\tilde{u}_{s,\theta}(c(t))(X) + \sigma_s(c(t), X)$$

where $|\sigma_s(c(t), X)| \leq \xi_s \|X\|_{L^p}$ and $\xi_s \in L^1(\mathcal{F}_t)$.

(ii) For every $X \in L^p(\mathcal{F}_t)$ there exists an integrable random variable Λ_X such that for all $\theta \in \Theta_t, c(t) = W^*(t) - V_\theta(t)$

$$|D\tilde{u}_{s,\theta}(c(t))(X)| < \Lambda_X \, \|X\|_{L^p} \ .$$

If this assumption is satisfied, the continuity of the value function follows from our general Proposition 2.2.

Proposition 3.4 *If for all $s = t \dots, T$ the period utilities u_s satisfy Assumption 3.2, then the value function is finite in a neighborhood of the optimal wealth and continuous at $W^*(t)$.*

Proof. Assumption 3.2 (i) implies the equidifferentiability of $F(c^*_{0,t-1}, \theta, W^*(t))$ (Assumption 2.3). Indeed, let $c(t) = W^*(t) - V_\theta(t) \in \mathcal{C}^*(W^*(t))$. Then, exploiting Assumption 3.2 (i) and Proposition 3.1 we get

$$\begin{aligned} F(c^*_{0,t-1}, \theta, W^*(t) + X) - F(c^*_{0,t-1}, \theta, W^*(t)) &= E_t \left\{ \sum_{s=t}^{T} \left[\tilde{u}_{s,\theta}(c(t) + X) - \tilde{u}_{s,\theta}(c(t)) \right] \right\} \\ &= \sum_{s=t}^{T} E_t \left[D\tilde{u}_{s,\theta}(W^*(t) - V_\theta(t))(X)) + \sigma_s(c(t), X) \right] \\ &= DF(c^*_{0,t-1}, \theta^*, W^*(t))(X) + \sum_{s=t}^{T} E_t \left[\sigma_s(c(t), X) \right]. \end{aligned}$$

Moreover, Jensen's inequality and Assumption 3.2 (i) imply that

$$|E_t \left[\sigma_s(c(t), X) \right]| \leq E_t \left[|\sigma_s(c(t), X)| \right] \leq \xi_s \|X\|_{L^p}$$

hence Assumption 2.3 is fulfilled. The above equality also shows that Assumption 2.3 (ii) is equivalent to Assumption 2.4. Then we can apply Proposition 2.2 to get the claim proved. □

To obtain the differentiability of the value function (and, as a consequence, the Envelope formula) we reformulate Assumptions 2.5 and 2.6 in terms of the Gateaux and the Fréchet differentials, respectively, of the transformed period utilities $\tilde{u}_{s,\theta}$.

Let $\Theta^*(W)$ denote the set of the optimal strategies from time t on, given the wealth level W and the optimal past consumption stream $c^*_{0,t-1}$, that is:

$$\Theta^*(W) = \left\{ \theta \in \Theta_t : F(c^*_{0,t-1}, \theta, W) = H(t, c^*_{0,t-1}, W) \right\}.$$

An immediate consequence of the Dynamic Programming Principle (3.4) is that $\Theta^*(W)$ is non-empty when $W = W^*(t)$, where $W^*(t)$ is the time t wealth generated by the optimal consumption-portfolio strategy (c^*, θ^*).

Assumption 3.3 *(i) There exists $r > 0$ such that for each $W \in B(W^*(t), r)$ the set $\Theta^*(W)$ is not empty.*

(ii) *For every $\theta \in \Theta_t$, the function $\tilde{u}_{s,\theta}$ is Gateaux differentiable at $c(t) = W^*(t) - V_\theta(t)$ and*

$$\left| \frac{\tilde{u}_{s,\theta}(c(t) + hX) - \tilde{u}_{s,\theta}(c(t))}{h} - D\tilde{u}_{s,\theta}(c(t))(X) \right| \leq \Sigma_t(W^*, X, h)$$

for every $X \in L^p(\mathcal{F}_t)$, for very sufficiently small $h \in \Re^+$, for some $\Sigma_t(W^*, X, h) \in L^1(\mathcal{F}_t)$ such that $\lim_{h \to 0} \|\Sigma_t(W^*, X, h)\|_{L^1} = 0$.

(iii) *Let $X \in L^p(\mathcal{F}_t)$ and $h \in \Re^+$ such that $W = W^* + hX \in B(W^*(t), r)$; for every $\theta^* \in \Theta^*(W^*)$,*

$$\lim_{h \to 0} \sup_{\theta_W \in \Theta^*(W)} \left\| D\tilde{u}_{s,\theta}(W^* - V_{\theta_W}(t))(X) - D\tilde{u}_{s,\theta^*}(W^* - V_\theta^*(t))(X) \right\|_{L^1} = 0$$

Assumption 3.4

(i) *There exists $r > 0$ such that for each $W \in B(W^*(t), r)$ the set $\Theta^*(W)$ is not empty.*

(ii) *For every $\theta \in \Theta_t$, the function $\tilde{u}_{s,\theta}$ is Fréchet-differentiable at $c(t) = W^*(t) - V_\theta(t)$ and*

$$\tilde{u}_{s,\theta}(c + X) - \tilde{u}_{s,\theta}(c) = D\tilde{u}_{s,\theta}(c)(X) + \sigma_t(c, X)$$

where $|\sigma_t(c, X)| \leq \xi \|X\|_{L^p}$ and $\xi \in L^1(\mathcal{F}_t)$.

(iii) *Let $W \in B(W^*(t), r)$; for every $\theta^* \in \Theta^*(W^*)$,*

$$\lim_{\|W - W^*\|_{L^p} \to 0} \sup_{\theta_W \in \Theta^*(W)} \sup_{\|X\|_{L^p} \leq 1} \left\| D\tilde{u}_{s,\theta}(W^* - V_{\theta_W}(t))(X) - D\tilde{u}_{s,\theta^*}(W^* - V_\theta^*(t))(X) \right\|_{L^1} = 0$$

Theorems 2.3 and 2.4 imply immediately the following *Envelope theorem for state-dependent utilities and for a general information structure.*

Proposition 3.5 *1. If for all $s = t \dots, T$ the period utilities u_s satisfy Assumption 3.3, then the value function is Gateaux -differentiable at the optimal wealth $W^*(t)$.*

2. If for all $s = t \dots, T$ the period utilities u_s satisfy Assumption 3.4, then the value function is Fréchet-differentiable at the optimal wealth $W^*(t)$.

3. In both cases (3.9) *holds.*

An important application of this general result consists in giving a precise definition of marginal utility of wealth for this general framework, and in showing how the intuitive meaning of Envelope condition as the equality between marginal utility of consumption and wealth extends to this general framework. Given the differential of H, we define a linear and continuous functional $\mathcal{E}_H : L^p(\mathcal{F}_t) \to \Re$ via

$$\mathcal{E}_H(Y) = E\left[DH(c^*_{0,t-1}, W^*(t))(Y)\right]$$

for all $Y \in L^p(\mathcal{F}_t)$.

Definition 3.1 *We call* marginal utility of optimal time t-wealth *the unique random variable*[8] $\pi_t^H \in L^q(F_t)$ *such that* $E\left[\pi_t^H Y\right] = E\left[DH(c^*_{0,t-1}, W^*(t))(Y)\right]$ *for all* $Y \in L^p(\mathcal{F}_t)$

To define the marginal utility of time t-consumption, we first define the marginal time s-utility of optimal time t-consumption:

Definition 3.2 *We call* marginal time s-utility of optimal time t-consumption *the unique random variable* $\pi_{ts}^u \in L^q(\mathcal{F}_t)$ *such that* $E\left[D\tilde{u}_{s,\theta^*}(c^*(t))(Y)\right] = E\left[\pi_{ts}^u Y\right]$ *for all* $Y \in L^p(\mathcal{F}_t)$.

In words, π_{ts}^u measures the impact of a marginal change in the time t optimal consumption on the time $s \geq t$ period utility. The total impact of a marginal change in the time t optimal consumption is then measured by the quantity π_t^u defined implicitly by summing up the impacts on each of the future period utilities, that is:

$$E\left[\pi_t^u Y\right] = \sum_{s=t}^{T} E\left[D\tilde{u}_{s,\theta^*}(c^*(t))(Y)\right] = \sum_{s=t}^{T} E\left[\pi_{ts}^u Y\right] = E\left[\sum_{s=t}^{T} \pi_{ts}^u Y\right]$$

Therefore, exploiting the uniqueness of the Riesz decomposition, we give the following definition:

Definition 3.3 *We call* marginal utility of optimal time t-consumption *the random variable*

$$\pi_t^u = \sum_{s=t}^{T} \pi_{ts}^u.$$

[8] Existence and uniqueness of such a random variable are guaranteed by Riesz representation theorem (see also Duffie and Skiadas [10] for the definition of state-price densities by the Riesz representation property of the utility gradient).

Remark. In the special case in which u_t is state-independent and depend only on current consumption (i.e. $u_t : \Re \to \Re$) and differentiable, then $Du_t(c^*(t))(Y) = u_t'(c^*(t))Y$, which shows how in that special case $\pi_t^u = u_t'(c^*(t))$ coincides with the standard notion of marginal utility.

Since, by the law of iterated expectation, $E\left[E_t\left[D\tilde{u}_{s,\theta^*}(c^*(t))(Y)\right]\right] = E\left[D\tilde{u}_{s,\theta^*}(c^*(t))(Y)\right] = E\left[\pi_{ts}^u Y\right]$, Proposition 3.2 and, once again, the uniqueness of the Riesz decomposition, together imply the classical Envelope condition for marginal utilities.

Corollary 3.1 *Under the assumptions of Proposition 3.2, $\pi_t^H = \pi_t^u$, namely, the marginal utility of optimal time t-wealth equals the marginal utility of optimal time t-consumption.*

The Dynamic Programming Principle allows to relate marginal utilities of today with marginal utilities of tomorrow. In [2], we showed (Proposition 2 and Corollary 1) that when the value function is differentiable at the optimum wealth and there exists a numeraire with bounded returns, the Dynamic Programming Principle holds and the marginal utilities of optimal wealth are state-price densities. We recall that a numeraire with bounded returns[9] is a strictly positive self financing portfolio V_{BN} such that $V_{\theta^{BN}}(t+1)/V_{\theta^{BN}}(t) \in L^\infty$ for all t. We can use that result to state a Euler's equation in terms of the marginal utility of consumptions.

Corollary 3.2 *Under the assumptions of Proposition 3.2, assume that there exists a numeraire with bounded returns. Then $(\pi_t^H)_{0\leq t\leq T}$, or equivalently $(\pi_t^u)_{0\leq t\leq T}$, are state-price densities, namely*

(i) $\pi_t^H \in L^q(\mathcal{F}_t)$ *and* $P(\pi_t^H > 0) = 1$

(ii) $S_j(t) = \dfrac{1}{\pi_t^H} E_t\left[\pi_{t+1}^H \ (S_j(t+1) + (d_j(t+1))\right] = \dfrac{1}{\pi_t^u} E_t\left[\pi_{t+1}^u \ (S_j(t+1) + (d_j(t+1))\right]$
for $j = 1, \ldots, J, t = 0 \ldots, T-1$.

[9] An example is the standard money market account

4 Conclusions

In this paper we have extended a general class of Envelope results, due to Benveniste and Scheinkman [3] and Milgrom and Segal [14], to the case in which the objective function takes values in a general Banach lattice, and not necessarily the real line. Employing the concept of differentiability in Banach spaces, our main results consist in identifying a set of assumptions under which the value function is differentiable, and its differential coincides with the differential of the objective function, seen as a function of the parameters. We then apply our general result to the consumption-portfolio problem of an agent with time additive but possibly state-dependent utility, in a context in which a general information structure is considered. In this setting, at any time t the value function (maximum remaining utility) is in fact a random variable itself, and not just a real-valued function defined on a set of state variables. To investigate if the value function for this problem has a well-defined marginal utility of wealth, defined as the differential of the value seen as a function of wealth levels accumulated up to time t, we recognize that the value function takes values in L^1, the space of integrable random variables, and that L^1 is indeed a Banach lattice. This allows us to bring to full bearing our general results to identify a set of conditions under which the marginal utility of wealth is well defined and coincides with the marginal utility of consumption, when the last one exists.

A Differentiability and concavity in normed vector spaces

In this section we summarize the main definition and results for cone-concave functions on vector spaces and, in particular, on the relation between concavity and differentiability. Let X, Y be normed vector spaces and G a mapping defined on an open domain $U \subset X$, with values in Y.

Definition A.1 *We say that G admits derivative at a point $u \in U$ in a direction $x \in X$ if the limit:*

$$G'(u;x) := \lim_{h \to 0^+} \frac{G(u+hx) - G(u)}{h} \tag{A.1}$$

exists, where the limit is meant in Y-norm.

The function G is said to be ***Gateaux differentiable*** *at u if it is directional differentiable at u in every direction $x \in X$ and the directional derivative $G'(u; \cdot\,) : X \to Y$ is a* *continuous and linear operator. In this case, we denote this operator with $DG(u)$ (namely, $DG(u)(x) = G'(u;x)$) and call it the* ***Gateaux differential*** *of G at u.*

Definition A.2 *We say that G is* ***Fréchet-differentiable*** *at u if there exists a continuous and linear operator $DG(u) : X \to Y$ such that*

$$\lim_{\|x\|_X \to 0} \frac{\|G(u+x) - G(u) - DG(u)(x)\|_Y}{\|x\|_X} = 0 \tag{A.2}$$

The operator $DG(u)$ is called the ***Fréchet differential*** *of G at u.*

The results which follow can be found in [21, 6, 16]. Usually, definition and results are stated for convex function. Since we work under a concavity assumption, we reformulated them in the appropriate form for concave functions with values in the order complete Banach lattice $(Y, C, \geq)$, where C is the positive cone of Y.

Definition A.3 *A function $F : X \to Y$ is C-**concave** (or simply **concave**) if for all $x, y \in X$, $\lambda \in [0,1]$*

$$F(\lambda x + (1-\lambda)y) \geq \lambda F(x) + (1-\lambda)F(y),$$

namely, $F(\lambda x + (1-\lambda)y) - \lambda F(x) + (1-\lambda)F(y) \in C$.

The sets of points at which F is finite is called the *essential domain* of F and denoted by $domF$. The algebraic interior of F is denoted $coreF$.

Proposition A.1 *(Borwein [6], Proposition 2.3) Let $G : X \to \bar{Y}$ be concave. Assume that there exists a function $F : X \to \bar{Y}$ such that $G(x) \geq F(x)$ for all $x \in X$. If F is continuous at some point $x_0 \in X$, then G is continuous at x_0.*

Let now $\mathcal{L}(X,Y)$ denote the set of continous and linear operators between X and Y and let F be a concave function from X to $\bar{Y}$.

Definition A.4 *An operator $L \in \mathcal{L}(X,Y)$ is called a* ***superdifferential*** *for F at x_0 if for all $x \in X$*

$$L(x) \geq F(x_0 + x) - F(x_0).$$

The ***superdifferential set*** *is denoted by $\partial F(x_0)$.*

Proposition A.2 *(Borwein [6], Proposition 3.2 (a) and Proposition 3.7 (a)) If $F : X \to \bar{Y}$ is concave, with $x_0 \in coreF$, then*

$$F^{>}(x_0, x) = \sup_{h>0} \frac{F(x_0 + hx) - F(x_0)}{h}$$

exists and is everywhere finite and superlinear.

Proposition A.3 *(Valadier [21], Proposition 4 and Théorème 6) If $F : X \to \bar{Y}$ is concave and $x_0 \in coreF$ then:*

(i) $L \in \mathcal{L}(X, Y)$ is a superdifferential for F at x_0 if and only if $L(x) \geq F^{>}(x_0, x)$ for all $x \in X$;

(ii) if in addition F is continuous at x_0, then $\partial F(x_0)$ is non-empty, convex and equicontinuous in $\mathcal{L}(X, Y)$ and

$$F^{>}(x_0, x) = \min\{L(x) : L \in \partial F(x_0)\}.$$

Proposition A.4 *(Papageorgiou [16], Theorem 4.6) Let $F : X \to \bar{Y}$ be a concave function. If F is continuous at x_0, then F is Gateaux-differentiable at x_0 if and only if $\partial F(x_0)$ is a singleton.*

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