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Optimal Transmit Filters for Constrained Complexity Channel Shortening Detectors

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***Abstract*—We consider intersymbol interference channels with reduced-complexity, mutual information optimized, channel-shortening detectors. For a given channel and receiver complexity, we optimize the transmit filter to use. The cost function we consider is the (Shannon) achievable information rate of the entire transceiver system. By functional analysis, we can establish a general form of the optimal transmit filter, which can then be optimized by standard numerical methods. As a side result, we also obtain an insight of the behaviour of the standard waterfilling algorithm for intersymbol interference channels.**

I. Introduction

The intersymbol interference (ISI) channel has played a central role in communication theory for several decades. It has been heavily researched, and today most of its fundamental properties are known. The capacity of the ISI channel was for example derived by Hirt back in 1988 in [1], and it was shown that Gaussian inputs in combination with the classical waterfilling algorithm achieves capacity. In practice, Gaussian channel inputs are not very common and discrete QAM-type inputs are typically preferred. In this case the ultimate communication limit was found in the early 2000s through a series of papers [2]–[6]. Further results on capacity properties of ISI channels include Kavcic's elegant method [7] to achieve the capacity of the ISI channel with discrete inputs through a generalized version of the Arimoto-Blahut algorithm, and also Soriaga et al.'s evaluation of the low-rate Shannon limit of ISI channels [8].

However, all of the above mentioned papers study ISI channels under the assumption that the receiver can perform optimal maximum-likelihood (ML) or maximum-a-posteriori (MAP) detection. Let $L_{\mathrm{H}}+1$ denote the number of taps in the channel impulse response. Forney showed in 1972 [9] that the complexity of ML/MAP-detection is exponential in $L_{\mathrm{H}}+1$ and, in many practical scenarios L_{H} is far too long for practical implementation of optimal ML/MAP detection. This observation spurred significant research efforts to reduce the computational complexity. One promising approach was channel shortening pioneered by Falconer and Magee in 1973 [10] and further investigated by several researchers; see e.g. [11]–[20]. Traditionally, channel shortening detectors were optimized from a minimum mean-square-error (MMSE) perspective. However, minimizing the MSE does not directly correspond to achieving the highest information rate (in the Shannon sense) that can be supported by a shortening detector. Recently, the achievable rate of channel-shortening detectors was optimized in [21] by utilizing the framework of mismatched mutual information [22], [23]. The result of [21] is a closed-form expression of the achievable information rate of an ISI channel with Gaussian inputs and an optimized channel-shortening detector that considers the channel memory to be $L<L_{\mathrm{H}}$ taps long, where L is a user-defined parameter.

In this paper we shall extend [21] into a closed-loop setting. Namely, *we will solve for the optimal transmit filter to use for a given ISI channel and a given receiver complexity* L. Hence, we essentially redo Hirt's derivations, but this time with the practical constraint of a given receiver complexity.

Our results are not as conclusive as in the unconstrained receiver complexity case. With functional analysis, we can prove that the optimal transmit spectrum is $(L+1)$-dimensional in the sense that it is described by $L+1$ real-valued scalar values. The transmit filter optimization thereby becomes a problem of finite dimensionality, and a numerical optimization provides the optimal spectrum. Note that, in practice, L is limited to rather small values and $L=1$ is an appealing choice from a complexity perspective. This essentially leads to very effective numerical optimizations.

The rest of the paper is organized as follows. In Section II we lay down the system model and formulates the problem that we intend to solve. In Section III we derive a general form of the optimal transmit spectrum. Numerical examples and properties of the numerical optimization is given in Section V. Finally, Section VI concludes the paper.

II. Preliminaries

In this section we give the system model, lay down the fundamentals of channel shortening receivers and their optimization, and formulates the problem that will be solved.

A. System Model

We consider linearly-modulated transmissions over channels affected by intersymbol interference (ISI) and additive white Gaussian noise (AWGN). Under the assumptions of ideal synchronization and finite ISI, the received signal can be described by means of the following discrete-time model

$$y_k=\sum_{\ell=0}^{L_{\mathrm{H}}} a_{k-\ell}h_\ell+w_k, \tag{1}$$

where $\boldsymbol{a}=\{a_k\}$ are the transmitted symbols, $\boldsymbol{h}=\{h_\ell\}_{\ell=0}^{L_{\mathrm{H}}}$ are the ISI coefficients, and $\boldsymbol{w}=\{w_k\}$ are independent and

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identically distributed complex Gaussian random variables, with mean zero and variance N_0 — note that bold letters are used for vectors. The system is studied under the assumption of ideal channel estimation at the receiver side, that is, perfect knowledge of the ISI coefficients and the noise variance. The symbol vector $\boldsymbol{a}$ is a precoded version of the information symbols $\boldsymbol{u} = \{u_k\}$,

$$\boldsymbol{a} = \boldsymbol{u} \star \boldsymbol{p},$$

where "$\star$" denotes convolution and $\boldsymbol{p}$ is a transmit filter subject to the power constraint $\sum_k |p_k|^2 = 1$. Taken together, the received signal can be expressed as

$$\boldsymbol{y} = \boldsymbol{v} \star \boldsymbol{u} + \boldsymbol{w}, \tag{2}$$

where $\boldsymbol{v} = \boldsymbol{h} \star \boldsymbol{p}$. It is convenient to assembly the presentation on matrix notation, so that (2) becomes

$$\boldsymbol{y} = \boldsymbol{V}\boldsymbol{u} + \boldsymbol{w}, \tag{3}$$

where $\boldsymbol{V}$ is a convolutional matrix formed from the vector $\boldsymbol{v}$, and $\boldsymbol{y}$, $\boldsymbol{u}$ and $\boldsymbol{w}$ are now column vectors of appropriate sizes. Assume that the combined channel-precoder response $\boldsymbol{v}$ has $K+1$ non-zero taps. The complexity of ML (implemented through the Viterbi algorithm) and MAP (implemented through the BCJR algorithm) is $\mathcal{O}(U^K)$, where U is the cardinality of the employed alphabet. Falconer and Magee's idea was to reduce this complexity by a linear filtering

$$\boldsymbol{r} = \boldsymbol{y} \star \boldsymbol{q} = (\boldsymbol{v} \star \boldsymbol{q}) \star \boldsymbol{u} + (\boldsymbol{w} \star \boldsymbol{q}). \tag{4}$$

Then, a Viterbi/BCJR algorithm follows that assumes a *target response* $\boldsymbol{t}$ of $L+1$ taps. Presumably, the target response $\boldsymbol{t}$ roughly equals the $L+1$ strongest taps of $(\boldsymbol{v} \star \boldsymbol{q})$, but there must not be an exact match if it turns out that it is not optimal to do so. In matrix notation, this procedure can be viewed as if the receiver decodes on the basis of a mismatched conditional probability distribution (pdf)[1]

$$\tilde{p}(\boldsymbol{y}|\boldsymbol{u}) \propto \exp\left(-\frac{\|\boldsymbol{Q}\boldsymbol{y} - \boldsymbol{T}\boldsymbol{u}\|^2}{N_0}\right) \tag{5}$$

instead of the actual conditional pdf

$$p(\boldsymbol{y}|\boldsymbol{u}) \propto \exp\left(-\frac{\|\boldsymbol{y} - \boldsymbol{V}\boldsymbol{u}\|^2}{N_0}\right). \tag{6}$$

Two questions now emerge: (1) For a given target response $\boldsymbol{t}$, how should the linear filter $\boldsymbol{q}$ be selected? And (2) how should the target response $\boldsymbol{t}$ be selected? These two questions kept researchers busy for several decades, see [11]–[20]. However, in all of those papers, the optimizations of $\boldsymbol{t}$ and $\boldsymbol{q}$ was done with an MMSE cost function, which does not directly correspond to the achievable information rate of the overall system[2].

The optimization for achievable information rate was completely solved in [21] under the assumption of Gaussian input symbols and by using a slightly more general model for channel shortening. This generalization is now described. By expansion of the exponent in (5) we get

$$\begin{aligned}\tilde{p}(\boldsymbol{y}|\boldsymbol{u}) &\propto \exp\left(-\frac{\|\boldsymbol{Q}\boldsymbol{y} - \boldsymbol{T}\boldsymbol{u}\|^2}{N_0}\right)\\ &\propto \exp\left(\frac{2\mathcal{R}\{\boldsymbol{u}^\dagger\boldsymbol{T}^\dagger\boldsymbol{Q}\boldsymbol{y}\} - \boldsymbol{u}^\dagger\boldsymbol{T}^\dagger\boldsymbol{T}\boldsymbol{u}}{N_0}\right),\end{aligned} \tag{7}$$

where all terms independent of $\boldsymbol{u}$ have been left out. A ML algorithm based on (7) was proposed by Ungerboeck in 1974 [24] and an algorithm for MAP detection in 2005 by Colavolpe and Barbieri [25]. In [21], a reduced complexity channel shortening detector is obtained by substituting in (7) $\boldsymbol{T}^\dagger\boldsymbol{Q}$ with $(\boldsymbol{H}^{\mathrm{r}})^\dagger$ and $\boldsymbol{T}^\dagger\boldsymbol{T}$ with $\boldsymbol{G}^{\mathrm{r}}$. In addition, the noise density N_0 is also absorbed into $\boldsymbol{H}^{\mathrm{r}}$ and $\boldsymbol{G}^{\mathrm{r}}$. This results in a mismatched conditional pdf of the form

$$\tilde{p}(\boldsymbol{y}|\boldsymbol{u}) = \exp\left(2\mathcal{R}\{\boldsymbol{u}^\dagger(\boldsymbol{H}^{\mathrm{r}})^\dagger\boldsymbol{y}\} - \boldsymbol{u}^\dagger\boldsymbol{G}^{\mathrm{r}}\boldsymbol{u}\right). \tag{8}$$

While the front-end $\boldsymbol{H}^{\mathrm{r}}$ is unconstrained, the matrix $\boldsymbol{G}^{\mathrm{r}}$ must satisfy

$$G^{\mathrm{r}}_{\ell k} = 0, \quad |\ell - k| > L \tag{9}$$

in order to satisfy the reduced-complexity constraint. The matrix $\boldsymbol{T}^\dagger\boldsymbol{T}$ in (7) must be positive semi-definite, while no such constraint applies to the matrix $\boldsymbol{G}^{\mathrm{r}}$. Hence, a more general model than (5) for channel shortening is obtained. The achievable information rate of a general mismatched receiver is derived in [22], [23] and equals

$$I_{\mathrm{AIR}} = -\mathbb{E}_{\boldsymbol{y}}\left[\log_2\left(\tilde{p}(\boldsymbol{y})\right)\right] + \mathbb{E}_{\boldsymbol{y},\boldsymbol{u}}\left[\log_2\left(\tilde{p}(\boldsymbol{y}|\boldsymbol{u})\right)\right], \tag{10}$$

where $\mathbb{E}_{\boldsymbol{y}}$ denotes the expectation operator with respect to the random variable $\boldsymbol{y}$ and

$$\tilde{p}(\boldsymbol{y}) \triangleq \sum_{\boldsymbol{u}} \tilde{p}(\boldsymbol{y}|\boldsymbol{u})p_{\boldsymbol{u}}(\boldsymbol{u}). \tag{11}$$

The rate I_{LB} is directly impacted by the choices of $\boldsymbol{G}^{\mathrm{r}}$ and $\boldsymbol{H}^{\mathrm{r}}$. The optimization

$$I_{\mathrm{OPT}} = \max_{\boldsymbol{G}^{\mathrm{r}},\boldsymbol{H}^{\mathrm{r}}} I_{\mathrm{AIR}},$$

has been solved in [21] and results in closed-form expressions for $\boldsymbol{G}^{\mathrm{r}}$, $\boldsymbol{H}^{\mathrm{r}}$ and I_{OPT}. We are only interested in I_{OPT} in this paper, and it equals

$$I_{\mathrm{OPT}} = -\log_2(c)$$

with

$$c = b_0 - \mathbf{b}\mathbf{B}^{-1}\mathbf{b}^{\mathrm{T}}, \tag{12}$$

where

$$\boldsymbol{b} = [b_1, b_2, \ldots, b_L],$$

$$\boldsymbol{B} = \mathrm{Toeplitz}([b_0, b_2, \ldots, b_{L-1}]),$$

and

$$\begin{aligned}b_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{N_0}{|V(\omega)|^2 + N_0}\cos(k\omega)d\omega\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{N_0}{|H(\omega)|^2|P(\omega)|^2 + N_0}\cos(k\omega)d\omega.\end{aligned} \tag{13}$$

[1]By $\boldsymbol{T}$ and $\boldsymbol{Q}$ we mean the convolutional matrices formed from the vectors $\boldsymbol{t}$ and $\boldsymbol{q}$, respectively.

[2]With "overall system", we mean the chain: prefilter-channel-reduced complexity receiver.

The reader should observe that from (12) and onwards, we have restricted the analysis to real-valued channels $\boldsymbol{h}$ and transmit filters $\boldsymbol{p}$. This implies that all Fourier transforms are symmetric around the origin.

B. Problem Formulation

The problem we aim at solving is to maximize I_{OPT} over the transmit filter $P(\omega)$, i.e., the Fourier transform of $\boldsymbol{p}$. Thus, we have the following optimization problem at hand

$$\begin{array}{c}\min_{P(\omega)} c[P(\omega)] \\ \text{such that} \\ \int_{-\pi}^{\pi} |P(\omega)|^2 d\omega = 2\pi.\end{array} \tag{14}$$

In (14) we have explicitly written out the dependency of c on $P(\omega)$, but not on N_0 and $H(\omega)$, since these are not subject to optimization.

III. General Form of the Optimal Transmit Filter

The optimization problem (14) is an instance of calculus of variations. We have not been able to solve it in closed form, but we can reduce the optimization problem into an $L+1$ dimensional problem, which can then efficiently be solved by standard numerical methods. The main result of the paper is

Theorem 1: The optimal transmit filter for the channel $H(\omega)$ with a memory L channel shortening detector satisfies

$$|P(\omega)|^2 = \max\left(0, \frac{N_0}{\sqrt{|H(\omega)|^2}}\sqrt{\sum_{\ell=0}^{L} A_\ell \cos(\ell\omega)} - \frac{N_0}{|H(\omega)|^2}\right).$$

where $\{A_\ell\}$ are real-valued scalar constants.

Proof: We first note that $P(\omega)$ only enters the optimization through its square magnitude, and we therefore make the variable substitution $S_p(\omega) = |P(\omega)|^2$ and optimize over $S_p(\omega)$ instead. From Cramer's rule, we get that

$$\mathbf{B}^{-1} = \frac{1}{\det(\mathbf{B})}[C_{ij}],$$

where C_{ij} is the cofactor of entry (i,j) in $\mathbf{B}$. This implies that we can express $\mathbf{b}\mathbf{B}^{-1}\mathbf{b}^{\text{T}}$ as

$$\mathbf{b}\mathbf{B}^{-1}\mathbf{b}^{\text{T}} = \frac{\sum_{m=1}^{M} \alpha_m b_0^{\phi_{m,0}} b_1^{\phi_{m,1}} \cdots b_L^{\phi_{m,L}}}{\sum_{n=1}^{N} \beta_n b_0^{\psi_{n,0}} b_1^{\psi_{n,1}} \cdots b_{L-1}^{\psi_{n,L-1}}}, \tag{15}$$

where M and N are finite constants that depend on L, $\alpha_m, \beta_m \in \{\pm 1\}$, and both $\phi_{m,\ell}$ and $\psi_{n,\ell}$ are non-negative integers which satisfy

$$\sum_{\ell=0}^{L} \phi_{m,\ell} = L+1 \quad \text{and} \quad \sum_{\ell=0}^{L-1} \psi_{n,\ell} = L\,.$$

We next introduce the variable substitution

$$y(\omega) = \frac{N_0}{|H(\omega)|^2 S_p(\omega) + N_0}, \; S_p(\omega) = \frac{N_0}{|H(\omega)|^2}\left[\frac{1}{y(\omega)} - 1\right].$$

The constraint $\int S_p(\omega) d\omega = 2\pi$ translates into

$$e[y(\omega)] = \int_{-\pi}^{\pi} \frac{1}{y(\omega)|H(\omega)|^2} d\omega = \int_{-\pi}^{\pi} \frac{1}{|H(\omega)|^2} d\omega + \frac{2\pi}{N_0}.$$

Furthermore, we have

$$b_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} y(\omega)\cos(k\omega) d\omega.$$

The constrained Euler-Lagrange equation becomes

$$\frac{\delta c}{\delta y} = \lambda \frac{\delta e}{\delta y} = -\frac{\lambda}{|H(\omega)|^2 y^2(\omega)}.$$

An application of the quotient rule to (12) gives

$$\frac{\delta c}{\delta y} = 1 - \frac{\sum_m \alpha_m \frac{\delta\left[b_0^{\phi_{m,0}} \cdots b_L^{\phi_{m,L}}\right]}{\delta y}\left[\sum_n \beta_n b_0^{\psi_{n,0}} \cdots b_{L-1}^{\psi_{n,L-1}}\right]}{\left[\sum_n \beta_n b_0^{\psi_{n,0}} \cdots b_{L-1}^{\psi_{n,L-1}}\right]^2} + \frac{\left[\sum_m \alpha_m b_0^{\phi_{m,0}} \cdots b_L^{\phi_{m,L}}\right]\sum_n \beta_n \frac{\delta\left[b_0^{\psi_{n,0}} \cdots b_{L-1}^{\psi_{n,L-1}}\right]}{\delta y}}{\left[\sum_n \beta_n b_0^{\psi_{n,0}} \cdots b_{L-1}^{\psi_{n,L-1}}\right]^2}. \tag{16}$$

By application of the chain rule we have

$$\frac{\delta\left[b_0^{\phi_{m,0}} b_1^{\phi_{m,1}} \cdots b_L^{\phi_{m,L}}\right]}{\delta y} = \sum_{\ell=0}^{L} \frac{\delta b_\ell^{\phi_{m,\ell}}}{\delta y} \prod_{k \neq \ell} b_k^{\phi_{m,k}} \tag{17}$$

and

$$\frac{\delta\left[b_0^{\psi_{n,0}} b_1^{\psi_{n,1}} \cdots b_{L-1}^{\psi_{n,L}}\right]}{\delta y} = \sum_{\ell=0}^{L-1} \frac{\delta b_\ell^{\psi_{n,\ell}}}{\delta y} \prod_{k \neq \ell} b_k^{\psi_{n,k}}. \tag{18}$$

The functional derivative $\delta b_k^s / \delta y$ equals

$$\begin{aligned}\frac{\delta b_k^s}{\delta y} &= \frac{\delta\left[\int_{-\pi}^{\pi} y(\omega)\cos(k\omega) d\omega\right]^s}{\delta y} \\ &= s\left[\int_{-\pi}^{\pi} y(\omega)\cos(k\omega) d\omega\right]^{s-1}\cos(k\omega) \\ &= s b_k^{s-1}\cos(k\omega).\end{aligned} \tag{19}$$

We now note that b_k, raised to any power, is a *constant* that depends explicitly on y. Therefore, if we plug (17)-(19) into the functional derivative (16) we obtain an expression of the form

$$\frac{\delta c}{\delta y} = 1 - \frac{\sum_{\ell=0}^{L} A_\ell[y]\cos(\ell\omega)}{C[y]},$$

where the constants $A_\ell[y]$ and $C[y]$ depend explicitly on y, e.g.,

$$C[y] = \left[\sum_{n=1}^{N} \beta_n b_0^{\psi_{n,0}} b_1^{\psi_{n,1}} \cdots b_{L-1}^{\psi_{n,L-1}}\right]^2.$$

By manipulation of the Euler-Lagrange equation and by introducing a new set of constants $\{B_\ell[y]\}$, we obtain

$$y(\omega) = \frac{1}{\sqrt{|H(\omega)|^2[\sum_{\ell=0}^{L} B_\ell[y]\cos(\ell\omega)]}}.$$

This translates into a general form of the optimal $S_p(\omega)$ which reads

$$S_p^{\mathrm{opt}}(\omega) = \frac{N_0}{\sqrt{|H(\omega)|^2}}\sqrt{\sum_{\ell=0}^{L} A_\ell \cos(\ell\omega)} - \frac{N_0}{|H(\omega)|^2}.$$

We have now found a general form for any stationary point. Unfortunately, for a given $H(\omega)$, this stationary point may lie outside of the domain of the optimization. The optimal spectrum $S_p(\omega)$ must therefore lie on the boundary of the optimization domain, which in this case implies that $S_p(\omega) = 0$ for $\omega \in \mathcal{I} \subset [-\pi, \pi]$. However, outside $\mathcal{I}$, the general form must apply, so that we can express all feasible $S_p^{\mathrm{opt}}(\omega)$ as

$$S_p^{\mathrm{opt}}(\omega) = \max\left(0, \frac{N_0}{\sqrt{|H(\omega)|^2}}\sqrt{\sum_{\ell=0}^{L} A_\ell \cos(\ell\omega)} - \frac{N_0}{|H(\omega)|^2}\right).$$

□

IV. Interlude: Full Complexity Detectors

Theorem 1 gives a general form of the optimal transmit filter to use for a memory L channel shortening detector. By definition, it becomes the classical waterfilling filter when $L = K$. Hence, it also provides an insight to the behaviour of the transmit filter for the classical waterfilling algorithm. We remind the reader that $L_{\mathrm{H}} + 1$ denotes the duration of the channel impulse response and $K+1$ denotes the duration of the combined transmit filter and channel response. We summarize our finding in the following

Theorem 2: Let $P(\omega)$ be the transmit filter found through the waterfilling algorithm. Then,

$$K \geq L_{\mathrm{H}}.$$

Whereas the statement is trivial when the transmit filter and the channel have a finite impulse response (FIR), the theorem proves that this fact holds also when they have an infinite impulse response (IIR). Thus, for a FIR channel response, the waterfilling solution cannot contain any pole that cancels a zero, while, for IIR channels, the waterfilling solution cannot contain any zero that cancels a pole. Thus, the overall channel cannot be with memory shorter than the original one.

Proof: The waterfilling algorithm will produce a transmit filter that satisfies [1]

$$|P(\omega)|^2 = \max\left(0, \theta - \frac{N_0}{|H(\omega)|^2}\right), \tag{20}$$

for some power constant θ. In view of Theorem 1, $|P(\omega)|^2$ in (20) must also satisfy

$$|P(\omega)|^2 = \max\left(0, \frac{N_0}{\sqrt{|H(\omega)|^2}}\sqrt{\sum_{\ell=0}^{K} A_\ell \cos(\ell\omega)} - \frac{N_0}{|H(\omega)|^2}\right). \tag{21}$$

Equating (20) and (21) yields

$$\theta - \frac{N_0}{|H(\omega)|^2} = \frac{N_0}{\sqrt{|H(\omega)|^2}}\sqrt{\sum_{\ell=0}^{K} A_\ell \cos(\ell\omega)} - \frac{N_0}{|H(\omega)|^2}. \tag{22}$$

From (22), it can be seen that we must have

$$\sum_{\ell=0}^{K} A_\ell \cos(\ell\omega) = \gamma |H(\omega)|^2,$$

for some constant γ. However,

$$|H(\omega)|^2 = \left|\sum_{\ell=0}^{L_{\mathrm{H}}} h_\ell \exp(-\ell\omega)\right|^2 = g_0 + 2\sum_{\ell=1}^{L_{\mathrm{H}}} g_\ell \cos(\ell\omega),$$

where

$$g_\ell = \sum_k h_k h_{k-\ell}.$$

Clearly, to satisfy

$$\sum_{\ell=0}^{K} A_\ell \cos(\ell\omega) = \gamma\left[g_0 + 2\sum_{\ell=1}^{L_{\mathrm{H}}} g_\ell \cos(\ell\omega)\right],$$

K must at least equal L_{H}. □

Theorem 2 reveals the interesting fact that the waterfilling algorithm trades a rate gain for detection complexity. By using the optimal transmit filter, a capacity gain is achieved, but the associated decoding complexity (of a full complexity detector) must inherently increase. Thus, with waterfilling, it is not possible to achieve both a rate gain and a decoding complexity reduction at the same time.

V. Numerical Optimization and Examples

Theorem 1 provides a general form of the optimal transmit filter for channel shortening detection of ISI channels. What remains to be optimized is the $L + 1$ real-valued constants $\{A_\ell\}$. A closed form optimization seems out of reach since the constraint

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} S_p(\omega) d\omega = 1 \tag{23}$$

has no simple analytical form in $\{A_\ell\}$. In fact, the integral

$$\int \sqrt{1 + A\cos(x)}dx$$

is an instance of the incomplete elliptic integral of the second kind, for which no closed form is known to date.

We have applied a straightforward numerical optimization of the variables $\{A_\ell\}$ under the constraints (23) and

$$\sum_{\ell=0}^{L} A_\ell \cos(\omega\ell) \geq 0.$$

With a standard workstation and any randomly generated channel impulse response, the optimization is stable, converges to the same solution no matter the starting position as long as the signal-to-noise-ratio (SNR) is not very high or very low, and is altogether a matter of fractions of a second.

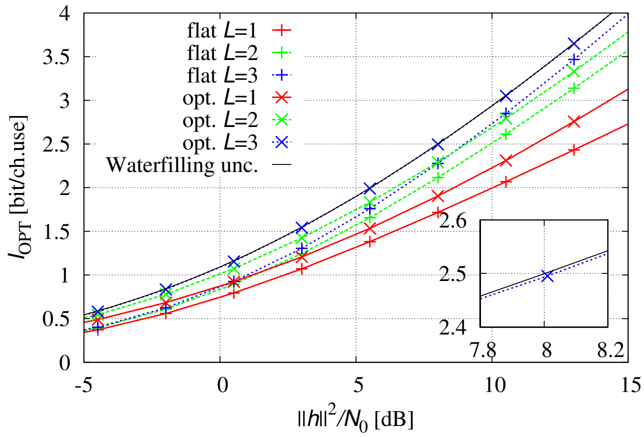

Fig. 1. Achievable information rates for Gaussian symbols on the EPR4 channel, for different values of the considered memory L at the receiver.

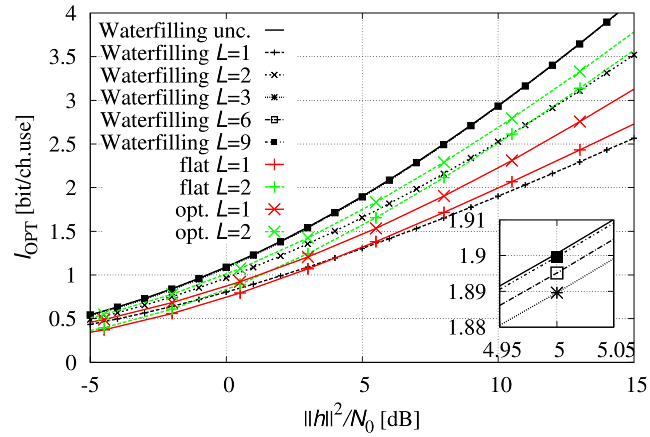

Fig. 2. Achievable information rates for Gaussian symbols with the waterfilling-solution power spectrum, for different values of the considered memory L at the receiver.

Next we turn to several illuminating examples. We consider the EPR4 ISI channel $\boldsymbol{h} = [0.5, 0.5, -0.5, -0.5]$ which has memory $L_H = 3$. Figure 1 shows the achievable information rates I_{OPT} for Gaussian symbols when the transmit filter is optimized for different values of the memory L used by the receiver. For comparison, the figure also gives I_{OPT} for a flat transmit power spectrum (i.e., no transmit filter at all) and the channel capacity (i.e. using the spectrum carried out by means of the waterfilling algorithm and unconstrained complexity of the receiver). It can be seen that using an optimized transmit filter for each L, significant gains are achieved w.r.t the flat power spectrum at all SNRs. The flat spectrum reaches its maximum information rate when $L = L_H$ but suffers a loss to the channel capacity. Differently, we can see that the optimized transmit filter when $L = L_H$ achieves an achievable rate which is close to the channel capacity. However, there is not an exact match. This loss is due to the fact that L_H must be lower than the combined channel-precoder memory K as stated by Theorem 2.

This behaviour is clearly illuminated by Figure 2, which plots the information rate when the transmit filter is found through the waterfilling algorithm and the receiver complexity is constrained with values of the memory L. It can be seen that when the memory L is increasing more and more, even above L_H, the information rate becomes closer and closer to the channel capacity. Moreover, it is important to notice that if, naïvely, a transmit filter found through the waterfilling algorithm was used when the receiver complexity is constrained, a loss w.r.t. the optimized case occurs and it may even be better to not have any transmit filter at all for high SNR values.

Although the results of this paper were so far presented only for Gaussian symbols, we next point out that when the optimized transmit filter and detector for Gaussian inputs are used for low-cardinality discrete alphabets, the ensuing I_{AIR} is still excellent[3]. Figure 3 shows the achievable information rate for a binary phase shift keying (BPSK) modulation. It can be noticed that the behavior among the curves for BPSK reflects the behavior for Gaussian symbols.

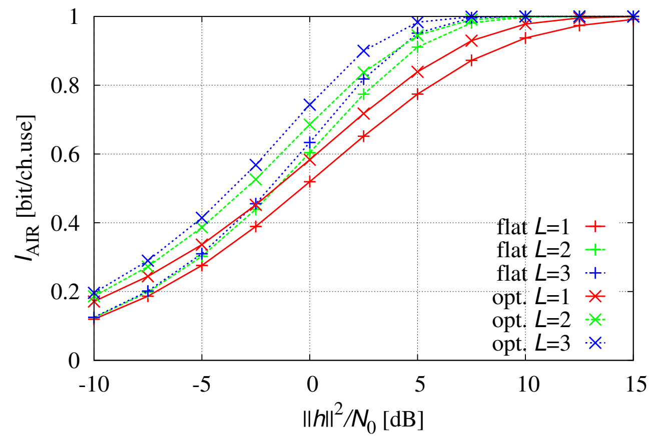

Fig. 3. Achievable information rates for BPSK modulation for different values of the considered memory L at receiver.

The AIRs can be approached in practice with proper modulation and coding formats. Fig. 4 shows the bit error rate (BER) of a BPSK-based system using the DVB-S2 low-density parity-check code with rate 1/2. In all cases, 50 internal iterations were carried out within the LDPC decoder, while 10 global iterations were carried out. It can be noticed that the performance are in accordance with the AIR results.

All simulations that we have presented were also carried out for other channels (e.g., Proakis B and C). Due to lack of space, we have not presented any results for these channels,

[3]We remind the reader that I_{OPT} refers to an optimized detector while I_{AIR} refers to the achievable rate for a not optimized detector. Since the filters have been optimized for Gaussian channel inputs, but we use here low-cardinality constellations, the filters could be further optimized and for these reason we use the notation I_{AIR}.

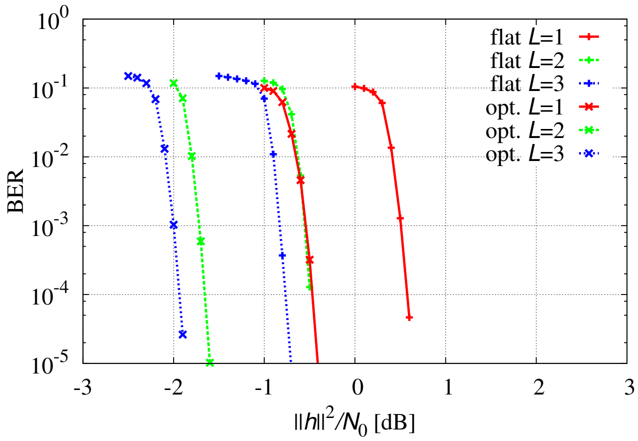

Fig. 4. Bit error rate for BPSK modulation for different values of the considered memory L at receiver.

but we remark that our general conclusions hold also for them.

VI. Conclusion

In this paper we have studied ISI channels with channel shortening detection. The channel shortening detector that we used is optimized from a mutual information perspective and allows for the highest possible data rate. We then optimized the transmit filter for a given receiver complexity and ISI channel. This is an optimization problem of infinite dimensionality, but we managed to reduce it through functional analysis into an optimization problem of a dimension that equals the memory of the receiver plus one. A standard numerical optimization procedure then follows. Since the memory of the receiver L is in practice typically set to a small value, such as $L = 1$, the numerical optimization is feasible to carry out.

As a side result, we also show that the classical waterfilling algorithm for ISI channels can never result in a shorter channel response at the receiver than the length of the channel response itself.

From our numerical experiments, we have found that it is crucial to take the receiver complexity into account when designing the transmit filter, since if the transmit filter found through the waterfilling algorithm is used, then a loss can occur compared with a flat transmit filter.

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