invariant convex sets in polar representations

# INVARIANT CONVEX SETS IN POLAR REPRESENTATIONS

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ABSTRACT. We study a compact invariant convex set $E$ in a polar representation of a compact Lie group. Polar rapresentations are given by the adjoint action of $K$ on $\mathfrak{p}$, where $K$ is a maximal compact subgroup of a real semisimple Lie group $G$ with Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. If $\mathfrak{a} \subset \mathfrak{p}$ is a maximal abelian subalgebra, then $P = E \cap \mathfrak{a}$ is a convex set in $\mathfrak{a}$. We prove that up to conjugacy the face structure of $E$ is completely determined by that of $P$ and that a face of $E$ is exposed if and only if the corresponding face of $P$ is exposed. We apply these results to the convex hull of the image of a restricted momentum map.

The boundary of a compact convex set is the union of its faces. Among the faces, the simplest ones are the exposed ones. They are given by the intersection of the convex set with a supporting hyperplane. In [3, 4] we studied the convex hull $\widehat{\mathcal{O}}$ of a $K$-orbit $\mathcal{O}$ in $\mathfrak{p}$, where $\mathfrak{p}$ is given by the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of a reductive Lie algebra $\mathfrak{g}$ and $K$ acts on $\mathfrak{p}$ by the adjoint representation. In this paper we use the results of [4] and show that a substantial part of them holds for any $K$–invariant compact convex set $E$ of $\mathfrak{p}$. More precisely we study the faces of $E$. We show in Proposition 1.2 that for a face $F$ of $E$ there exists a subalgebra $\mathfrak{s} \subset \mathfrak{p}$ such that $F$ is a subset of $\mathfrak{p}^{\mathfrak{s}} = \{x \in \mathfrak{p} : [x, \mathfrak{s}] = 0\}$ and $F$ is invariant with respect to the action of $K^{\mathfrak{s}} = \{h \in K : \mathrm{Ad}(h)(\mathfrak{s}) = \mathfrak{s}\}$, where Ad denotes the adjoint representation.

If we fix a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$, then the set $P = E \cap \mathfrak{a}$ is convex and invariant with respect to the action of the normalizer $\mathcal{N}_K(\mathfrak{a}) = \{h \in K : \mathrm{Ad}(h)(\mathfrak{a}) = \mathfrak{a}\}$ of $\mathfrak{a}$ in $K$. The $\mathcal{N}_K(\mathfrak{a})$–action on $P$ induces an action on the set of faces of $P$. Similarly $K$ acts on the set of faces of $E$. Denote these sets by $\mathscr{F}(P)$ respectively by $\mathscr{F}(E)$. If $\sigma$ is a face of $P$, let $\sigma^\perp$ denote the orthogonal complement in $\mathfrak{a}$ of the affine hull of $\sigma$ (see Section 1). Our main result is

**Theorem 0.1.** *The map $\mathscr{F}(P) \to \mathscr{F}(E)$, $\sigma \mapsto K^{\sigma^\perp} \cdot \sigma$ is well–defined and induces a bijection between $\mathscr{F}(P)/\mathcal{N}_K(\mathfrak{a})$ and $\mathscr{F}(E)/K$.*

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An application of Theorem 0.1 is the following result.

**Theorem 0.2.** *The faces of $E$ are exposed if and only if the faces of $P$ are exposed.*

Interesting $K$–invariant compact subsets of $\mathfrak{p}$ often arise as images of restricted momentum or gradient mappings. More precisely, let $U$ be a compact connected Lie group which acts by biholomorphism and in a Hamiltonian fashion on a compact Kähler manifold $Z$ with momentum map $\mu : Z \longrightarrow \mathfrak{u}$. Let $G \subset U^{\mathbb{C}}$ be a connected Lie subgroup of $U^{\mathbb{C}}$ which is *compatible* with respect to the Cartan decomposition of $U^{\mathbb{C}}$. This means that $G$ is a closed subgroup of $U^{\mathbb{C}}$ such that $G = K \exp(\mathfrak{p})$, where $K = U \cap G$ and $\mathfrak{p} = \mathfrak{g} \cap i\mathfrak{u}$ [13, 15]. Let $X \subset Z$ be a $G$-invariant compact subset of $Z$. We have the restricted momentum map or the gradient map $\mu_{\mathfrak{p}} : X \longrightarrow \mathfrak{p}$ in the sense of [13] (see also Section 3) and we denote by $E = \widehat{\mu_{\mathfrak{p}}(X)}$ the convex hull of the $K$-invariant set $\mu_{\mathfrak{p}}(X)$. If $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{p}$ and $\pi$ is the orthogonal projection onto $\mathfrak{a}$, then $\mu_{\mathfrak{a}} = \pi \circ \mu_{\mathfrak{p}} : X \longrightarrow \mathfrak{a}$ is the gradient map with respect to $A = \exp(\mathfrak{a})$. Since $P = E \cap \mathfrak{a} = \widehat{\mu_{\mathfrak{a}}(X)}$ is a convex polytope (Proposition 3.1), we deduce the following.

**Theorem 0.3.** *All faces of $\widehat{\mu_{\mathfrak{p}}(X)}$ are exposed.*

A reformulation of Theorem 3.1 is that the faces of $E$ correspond to maxima of components of the gradient map. This observation will be used to realize a close connection between the faces of $E$ and parabolic subgroups of $G$. More precisely, for any face $F \subset E$ let $X_F := \mu_{\mathfrak{p}}^{-1}(F)$ and let $Q^F = \{g \in G : g \cdot X_F = X_F\}$. Then $X_F$ is the set of maximum points of an appropriately chosen component of the gradient map and $Q^F$ is a parabolic subgroup of $G$.

If $X$ is a $G$-stable compact submanifold of $Z$, then for any face $F$, one can construct an open neighbourhood $X_F^-$ of $X_F$ in $X$, which is an analogue of an open Bruhat cell. Moreover there is a smooth deformation retraction of $X_F^-$ onto $X_F$. See Theorem 3.1 for more details.

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## 1. Group theoretical description of the faces

We start by recalling the basic definitions and results regarding convex bodies. For more details see e.g. [18]. Let $V$ be a real vector space with scalar product $\langle \cdot, \cdot \rangle$. A *convex body* $E \subset V$ is a convex compact subset of $V$. Let $\mathrm{Aff}(E)$ denote the affine span of $E$. The interior of $E$ in $\mathrm{Aff}(E)$ is called the *relative interior* of $E$ and is denoted by $\mathrm{relint}\, E$. By definition a *face* of $E$ is a convex subset $F \subset E$ such that $x, y \in E$ and $\mathrm{relint}[x, y] \cap F \neq \emptyset$

implies $[x,y] \subset F$. A face distinct from $E$ and $\emptyset$ is called a *proper face*. The *extreme points* of $E$ are the points $x \in E$ such that $\{x\}$ is a face. We will denote by $\operatorname{ext} E$ the set of the extreme points of $E$. The set $\operatorname{ext} E$ completely determines the convex body $E$ since the convex hull of $\operatorname{ext} E$ coincides with $E$ and it is the smallest subset of $E$ with this property. If $F$ is a face of $E$, we denote by $\operatorname{Dir}(F)$ the vector subspace of $V$ defined by $\operatorname{Aff}(F)$, i.e. $\operatorname{Aff}(F) = p + \operatorname{Dir}(F)$. We call $\operatorname{Dir}(F)$ the *direction* of $F$. Every vector $\beta \in V$ defines an *exposed face* $F = F_\beta(E) = \{x \in E : \langle x, \beta\rangle = \max_{y\in E}\langle y, \beta\rangle\}$ with $\operatorname{Dir}(F_\beta(E)) \subset \{\beta\}^\perp$. In general not all faces of a convex set are exposed, see Fig. 1 for an example. For any exposed face $F$ the set

$$C_F = \{\beta \in V : F = F_\beta(E)\}, \tag{1}$$

is a convex cone. The faces of $E$ are closed. If $F_1$ and $F_2$ are faces of $E$ and they are distinct, then $\operatorname{relint} F_1 \cap \operatorname{relint} F_2 = \emptyset$. Moreover the convex body $E$ is the disjoint union of the relative interiors of its faces (see [18, p. 62]).

We are interested in invariant convex bodies in polar representations. A theorem of Dadok [6] asserts that we can restrict ourselves to the following setting.

Let $\mathfrak{g}$ be a semisimple Lie algebra with a Cartan involution $\theta$ and let $B$ be the Killing form of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, is the eigenspace decomposition of $\mathfrak{g}$ in 1 and $-1$ eigenspaces of $\theta$ and they are orthogonal under $B$. Moreover, $B$ restricted to $\mathfrak{k}$, respectively $\mathfrak{p}$, is negative definite, respectively positive definite. In the sequel we denote $\langle \cdot,\cdot\rangle = B_{|\mathfrak{p}\times\mathfrak{p}}$ which is a $K$-invariant scalar product. Out object of study will be a $K$-stable convex body $E \subset \mathfrak{p}$. For for any $A, B \subset \mathfrak{p}$ we set

$$\begin{aligned} A^B &:= \{\eta \in A : [\eta, \xi] = 0, \text{for all } \xi \in B\} \\ G^B &:= \{g \in G : \operatorname{Ad} g(\xi) = \xi, \text{for all } \xi \in B\}, \\ K^B &:= K \cap G^B. \end{aligned}$$

where Ad denotes the adjoint representation. In the sequel we denote by $k \cdot x = \operatorname{Ad}(k)(x)$ the action of $K$ on $\mathfrak{p}$ by linear isometries.

Faces of $K$–invariant convex bodies in $\mathfrak{p}$ are closely connected to orbits of subgroups of $K$ which are given as centralizers. More precisely for any nonzero $\beta$ in $\mathfrak{p}$ we have the Cartan decomposition $\mathfrak{g}^\beta = \mathfrak{k}^\beta \oplus \mathfrak{p}^\beta$ of the Lie algebra of the centralizer $G^\beta$ of $\beta$ in $G$.

**Proposition 1.1.** *Let $F = F_\beta(E)$ be an exposed face of $E$. Then*

a) *$F \subset \mathfrak{p}^\beta$ and $F$ is $K^\beta$–stable;*
b) *$\operatorname{Dir}(F) \subset \beta^\perp$, where $\perp$ is in $\mathfrak{p}$.*

*Proof.* If $x \in F_\beta(E)$, then $\widehat{K \cdot x} \subset E$ since $E$ is $K$-invariant. Moreover, we have

$$\max_{y\in E}\langle y, \beta\rangle = \max_{y\in\widehat{K\cdot x}}\langle y, \beta\rangle = \langle x, \beta\rangle.$$

Corollary 3.1 in [4] implies $F_\beta(\widehat{K\cdot x}) \subset \mathfrak{p}^\beta$. Therefore $x \in \mathfrak{p}^\beta$. This proves a). Part b) follows since $F$ is contained in an affine hyperplane orthogonal to $\beta$. □

For an arbitrary face of $E$ we have the following.

**Proposition 1.2.** *Let $F \subset E$ be a face. Then there exists an abelian subalgebra $\mathfrak{s} \subset \mathfrak{p}$ such that*

a) *$F \subset \mathfrak{p}^{\mathfrak{s}}$ and $F$ is $K^{\mathfrak{s}}$–stable;*
b) *$\mathrm{Dir}(F) \subset \mathfrak{s}^\perp$;*

*Proof.* We may fix a maximal chain of faces $F = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k = E$ (see [3, Lemma 2]). If $k = 0$, then $F = E$ and $\mathfrak{s} = \{0\}$. Assume the theorem is true for a face contained in a maximal chain of length $k$. Then the claim is true for $F_1$ and consequently there exists $\mathfrak{s}_1 \subset \mathfrak{p}$ such that $F_1 \subset \mathfrak{p}^{\mathfrak{s}_1}$, $F_1$ is $K^{\mathfrak{s}_1}$-stable and $\mathrm{Dir}(F_1) \subset \mathfrak{s}_1^\perp$. $F$ is an exposed face of $F_1$. Let $\beta' \in \mathfrak{p}^{\mathfrak{s}_1}$ such that $F = F_{\beta'}(F_1)$ and set $\mathfrak{s} := \mathbb{R}\beta' \oplus \mathfrak{s}_1$. Then $F \subset \mathfrak{p}^{\mathfrak{s}}$, $F$ is $(K^{\mathfrak{s}_1})^{\beta'} = K^{\mathfrak{s}}$–stable and $\mathrm{Dir}(F) \subset \mathfrak{s}^\perp$. □

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra of $\mathfrak{p}$ and let $\pi : \mathfrak{p} \longrightarrow \mathfrak{a}$ be the orthogonal projection onto $\mathfrak{a}$. Then $P = E \cap \mathfrak{a}$ is a convex subset of $\mathfrak{a}$ which is $\mathcal{N}_K(\mathfrak{a})$–stable. The proof of the following Lemma is given in [7].

**Lemma 1.1.** *(i) If $E \subset \mathfrak{p}$ is a $K$–invariant convex subset, then $E \cap \mathfrak{a} = \pi(E)$ and $K \cdot \pi(E) = E$. (ii) If $C \subset \mathfrak{a}$ is a $\mathcal{N}_K(\mathfrak{a})$-invariant convex subset, then $K \cdot C$ is convex and $\pi(K \cdot C) = C$.*

**Lemma 1.2.** *Let $U$ be a compact Lie group and let $\mathfrak{g} \subset \mathfrak{u}^{\mathbb{C}}$ be a semisimple $\theta$-invariant subalgebra. Then any Lie subgroup with finitely many connected components and with Lie algebra $\mathfrak{g}$ is closed and compatible.*

*Proof.* We fix an embedding $U \hookrightarrow \mathrm{U}(n)$ such that the Cartan involution $X \mapsto (X^{-1})^*$ of $\mathrm{GL}(n,\mathbb{C})$ restricts to $\theta$. Then $G$ is closed in $\mathrm{GL}(n,\mathbb{C})$ (see [16, p. 440] for a proof) and hence also in $U^{\mathbb{C}}$. Since $\mathfrak{g}$ is $\theta$-invariant, also $G$ is, and $\theta$ restricts to the Cartan involution of $G$. This shows that $G$ is compatible. □

If $G \subset U^{\mathbb{C}}$ is compatible with Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, then $\mathfrak{g}$ is real reductive and there is a nondegenerate $K$–invariant bilinear form $B : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$ which is positive definite on $\mathfrak{p}$, negative definite on $\mathfrak{k}$ and such that $B(\mathfrak{k}, \mathfrak{p}) = 0$. Indeed, fix a $U$-invariant inner product $\langle\, ,\, \rangle$ on $\mathfrak{u}$. Let $\langle\, ,\, \rangle$ denote also the inner product on $i\mathfrak{u}$ such that multiplication by $i$ be an isometry of $\mathfrak{u}$ onto $i\mathfrak{u}$. Define $B$ on $\mathfrak{u}^{\mathbb{C}}$ imposing $B(\mathfrak{u}, i\mathfrak{u}) = 0$, $B = -\langle\, ,\, \rangle$ on $\mathfrak{u}$ and $B = \langle\, ,\, \rangle$ on $i\mathfrak{u}$. Therefore $B$ is $\mathrm{Ad}\, U^{\mathbb{C}}$–invariant and non-degenerate and its restriction to $\mathfrak{g}$ satisfies the above conditions.

Let $\mathfrak{q}$ be a $K$–invariant subspace of $\mathfrak{p}$. Then $[\mathfrak{q}, \mathfrak{q}]$ is a $K$–invariant linear subspace of $\mathfrak{k}$ and therefore an ideal of $\mathfrak{k}$. Since $K$ is compact, we have the following $K$–invariant splitting $\mathfrak{k} = [\mathfrak{q}, \mathfrak{q}] \oplus \mathfrak{k}'$. In particular $\mathfrak{k}'$ is an ideal of $\mathfrak{k}$ commuting with $[\mathfrak{q}, \mathfrak{q}]$. Let $\mathfrak{p} = \mathfrak{q} \oplus \mathfrak{q}'$ be a $K$–invariant splitting of $\mathfrak{p}$. Since

$$B([\mathfrak{q}, \mathfrak{q}'], \mathfrak{k}) = B(\mathfrak{q}, [\mathfrak{k}, \mathfrak{q}']) \subset B(\mathfrak{q}, \mathfrak{q}') = 0,$$

this shows that $[\mathfrak{q}, \mathfrak{q}'] = 0$ and so $[\mathfrak{q}', [\mathfrak{q}, \mathfrak{q}]] = [\mathfrak{q}, [\mathfrak{q}, \mathfrak{q}']] = 0$. Moreover $\mathfrak{p} = \mathfrak{q} \oplus \mathfrak{q}'$ implies that $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}] \oplus \mathfrak{q}$ and $\mathfrak{h}' = \mathfrak{k}' \oplus \mathfrak{q}'$ are compatible $K$–invariant commuting ideal of $\mathfrak{g}$.

If a $K$–invariant linear subspace $\mathfrak{q} \subset \mathfrak{p}$ is fixed, one gets decomposition of $\mathfrak{g}$, and so of $G$. This is decomposition is the content of the next Proposition. We will need it in the case where $F \subset \mathfrak{p}$ is a $K$–invariant convex body and $\mathfrak{q}$ is such that $\mathrm{Aff}(F) = x_0 + \mathfrak{q}$.

**Proposition 1.3.** *Let $G \subset U^{\mathbb{C}}$ be a compatible subgroup with Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and let $\mathfrak{q} \subset \mathfrak{p}$ be a linear $K$–invariant subspace. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}'$ where $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}] \oplus \mathfrak{q}$ and $\mathfrak{h}' = \mathfrak{h}^{\perp_B}$. Then the following hold.*

a) *$\mathfrak{h}$ and $\mathfrak{h}'$ are compatible $K$–invariant commuting ideal of $\mathfrak{g}$;*
b) *Let $K_1$ be the connected Lie subgroup of $G$ with Lie algebra $\mathfrak{k} \cap [\mathfrak{h}, \mathfrak{h}]$. Then $K_1 \exp(\mathfrak{q})$ is a connected compatible subgroup of $G$ and any two maximal subalgebras of $\mathfrak{q}$ are congiugate by an element of $K_1$.*
c) *Let $K_2$ be the connected Lie subgroup of $G$ with Lie algebra $\mathfrak{k} \cap [\mathfrak{h}', \mathfrak{h}']$. Then any two maximal subalgebras of $\mathfrak{q}'$ are congiugate by an element of $K_2$.*

*Proof.* We have proved (a) in the above discussion. Let $\mathfrak{b} := [\mathfrak{h}, \mathfrak{h}]$. Then $\mathfrak{h} = \mathfrak{z}(\mathfrak{h}) \oplus \mathfrak{b}$ and $\mathfrak{b}$ is semisimple. Denote by $B$ the connected subgroup of $U^{\mathbb{C}}$ with Lie algebra $\mathfrak{b}$. By Lemma 1.2 $B$ is a closed subgroup of $U^{\mathbb{C}}$. Set $\mathfrak{z}_{\mathfrak{p}} := \mathfrak{z}(\mathfrak{h}) \cap \mathfrak{p}$ and $\mathfrak{d} := \mathfrak{b} \oplus \mathfrak{a}$. Then $\mathfrak{d}$ is a reductive Lie algebra and $\exp \mathfrak{a}$ is a compatible abelian subgroup commuting with $B$. Thus $D := B \cdot \exp \mathfrak{a}$ is a connected closed subgroup with Lie algebra $\mathfrak{d}$. Moreover $D \cap U = B \cap U$ and $\exp(\mathfrak{b} \cap \mathfrak{p}) \cdot \exp \mathfrak{a} = \exp(\mathfrak{b} \cap \mathfrak{p} \oplus \mathfrak{a}) = \exp(\mathfrak{d} \cap \mathfrak{p})$. This shows that $D$ is compatible. Since $D \cap U$ coincides with $K_1$ and $D$ is connected the last statement in (b) follows from standard properties of compatible subgroups (see e.g. Prop. 7.29 in [16]; note that a connected compatible subgroup is a reductive group in the sense of [16, p. 446]). This proves (b). For (c) the same argument applies more directly. It is enough to observe that the connected Lie subgroup $H'' \subset G$ with Lie algebra $[\mathfrak{h}', \mathfrak{h}']$ is semisimple, compatible and connected and that $K_2 = H'' \cap U$. □

**Remark 1.1.** *The compatible subgroup $G$ in the previous Proposition is not assumed to be connected. Nevertheless the constructions in (b) and (c) depend only on $G^0$. Thus considering $G^0$ in place of $G$ makes no difference.*

**Lemma 1.3.** *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a reductive Lie algebra and $\mathfrak{g}_i$ ideals. If $\mathfrak{a} \subset \mathfrak{p}$ is a maximal subalgebra, then $\mathfrak{a}_i := \mathfrak{a} \cap \mathfrak{p}_i$ is a maximal subalgebra of $\mathfrak{p}_i$ and $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$.*

If $\sigma$ is a face of $P$, let $\sigma^{\perp}$ denote the orthogonal (inside $\mathfrak{a}$) to the direction of the affine hull of $\sigma$.

**Lemma 1.4.** *Let $F$ be a face and let $\mathfrak{s}$ be as in Proposition 1.2. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra containing $\mathfrak{s}$. Set $\sigma := \pi(F)$. Then $\sigma$ is a face of $P$, $\sigma = F \cap \mathfrak{a}$ and $F = K^{\sigma^\perp} \cdot \sigma$. Moreover $F$ is a proper face if and only if $F \cap \mathfrak{a}$ is.*

*Proof.* By Proposition 1.2 $F \subset \mathfrak{p}^{\mathfrak{s}}$ is a $K^{\mathfrak{s}}$–stable convex set. By Lemma 1.1 we get $\sigma = \pi(F) = F \cap \mathfrak{a}$ and this is a face $P$ by [3, Lemma 11]. Since $\mathrm{Dir}(F)$ is contained in the orthogonal complement of $\mathfrak{s}$, and $\mathrm{Dir}(\sigma) \subset \mathrm{Dir}(F)$, we have $\mathrm{Dir}(\sigma) \subset \mathfrak{a} \cap \mathfrak{s}^\perp$. Then $\sigma^\perp \subset \mathfrak{s}$. Hence $K^{\sigma^\perp} \cdot \sigma \subset K^{\mathfrak{s}} \cdot \sigma \subset F$. We prove the reverse inclusion. If $y \in F$, then $F \cap \widehat{K \cdot y}$ is a face of $\widehat{K \cdot y}$. Set $\tilde{\sigma} = \pi(F \cap \widehat{K \cdot y})$. We have $\tilde{\sigma} \subset \sigma$ and by Proposition 3.6 in [4] we also have that $F \cap \widehat{K \cdot y} = K^{\tilde{\sigma}^\perp} \cdot \tilde{\sigma}$. On the other hand, $\sigma^\perp \subset \tilde{\sigma}^\perp$, so $K^{\tilde{\sigma}^\perp} \subset K^{\sigma^\perp}$ and

$$F \cap \widehat{K \cdot y} = K^{\tilde{\sigma}^\perp} \cdot \tilde{\sigma} \subset K^{\sigma^\perp} \cdot \sigma.$$

This implies $F = K^{\sigma^\perp} \cdot \sigma$. Note that $F$ is proper if $\sigma$ is. It remains to prove that $\sigma$ is proper, when $F$ is proper.

Let $\mathrm{Aff}(E) = x_o + \mathfrak{q}_E$. Note that $\mathfrak{q}_E = \{x - y : x, y \in \mathrm{Aff}(E)\}$ implies that $\mathfrak{q}_E$ is $K$–invariant. Since $K$ acts on $\mathfrak{p}$ by isometries, we may assume that $x_o$ is orthogonal to $\mathfrak{q}$. Note that $x_o$ is uniquely defined by this condition. It follows that $x_o$ is a $K$ fixed point and $E = x_0 + E_1$, where $E_1$ is a $K$–invariant convex body of $\mathfrak{q}_E$. Proposition 1.3 applied to $\mathfrak{q}_E$ yields $K_1, K_2$ such that $G_1 = K_1 \exp(\mathfrak{q}_E)$ is a connected compatible semisimple real Lie group, $K = K_1 \cdot K_2$ and for any $x \in E$ we have

$$K \cdot x = K \cdot (x_o + x_1) = x_o + K \cdot x_1 = x_o + K_1 \cdot x_1 = K_1 \cdot x.$$

since $\mathfrak{q}_E$ is fixed pointwise by $K_2$. By Lemma 1.3, $\mathfrak{a} = \mathfrak{a}_E \oplus \mathfrak{a}'_E$, where $\mathfrak{a}_E$ is a maximal abelian subalgebra of $\mathfrak{q}_E$ and $\mathfrak{a}'_E$ is a maximal abelian subalgebra of $\mathfrak{q}'_E$. Since $\pi(E) = \pi(x_o) + \pi(E_1)$ and $\mathrm{Dir}(E_1) = \mathfrak{q}_E$, it follows that the direction of $\pi(E)$ is $\mathfrak{a}_E$. If $\sigma = \pi(F) = \pi(E) = E \cap \mathfrak{a}$, then $\sigma^\perp = \mathfrak{a}'_E$ and so $K_1 \subset K^{\mathfrak{a}'_E}$. It follows that

$$F = K^{\mathfrak{a}'_E} \cdot (E \cap \mathfrak{a}) = K_1 \cdot (E \cap \mathfrak{a}) = K \cdot (E \cap \mathfrak{a}) = E.$$

where the last equality follows by Lemma 1.1. Hence, if $F$ is proper, then $\sigma = \pi(F) \subsetneq \pi(E) = E \cap \mathfrak{a}$. □

**Proposition 1.4.** *Let $F$ be a proper face and let $\mathfrak{s}$ as in Proposition 1.2. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra containing $\mathfrak{s}$. Then $F$ is exposed if and only if $F \cap \mathfrak{a}$ is.*

*Proof.* Assume that there exists $\beta \in \mathfrak{p}$ such that $F = F_\beta(E)$. Since $F \cap \mathfrak{a} = \sigma$ is a proper face of $P$, the point $\beta$ is not orthogonal to $\mathfrak{a}$. We have $\beta = \beta_1 \oplus \beta_2$, with $\beta_1 \in \mathfrak{a}$ different from zero and $\beta_2$ orthogonal to $\mathfrak{a}$. Therefore $F_\beta(E) \cap \mathfrak{a} = F_{\beta_1}(E) \cap \mathfrak{a} = F_{\beta_1}(P) = \sigma$. Now, assume that there exists $\beta \in \mathfrak{a}$ such that $\sigma = F_\beta(P)$. Let $F' := F_\beta(E)$. By Proposition 1.1 $F' \subset \mathfrak{p}^\beta$. Moreover $\mathfrak{a} \subset \mathfrak{p}^\beta$ since $\beta \in \mathfrak{a}$. By Lemma 1.4 the intersection of a face with

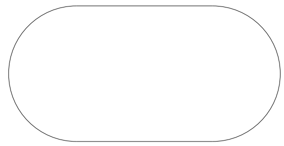

Figure 1.

$\mathfrak{a}$ determines the face. Since $F' \cap \mathfrak{a} = F_\beta(P) = \sigma = F \cap \mathfrak{a}$ we conclude that $F = F'$. Thus $F$ is exposed. □

**Remark 1.2.** *Given a Weyl–invariant convex body $P \subset \mathfrak{a}$, set $E := K \cdot P$. By Lemma 1.1 $E$ is a $K$-invariant convex body in $\mathfrak{p}$ and $P = E \cap \mathfrak{a}$. Thus a general $P$ can be realized as $E \cap \mathfrak{a}$. A general Weyl–invariant convex body $P$ can have non–exposed faces. For example take $G = U^{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ and $K = \mathrm{SU}(2) \times \mathrm{SU}(2)$. Then $\mathfrak{a} = \mathbb{R}^2$ and the Weyl group is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ where the generators are given by the reflections on the axes. The picture in Fig. 1 is a Weyl–invariant $P$ with exactly 4 non–exposed faces. By the Proposition also the corresponding body $E \subset i\mathfrak{su}(2) \oplus i\mathfrak{su}(2)$ has non–exposed faces.*

## 2. Proof of the main results

Let $\mathfrak{a} \subset \mathfrak{p}$ and define the following map

$$\Upsilon : \mathscr{F}(P) \longrightarrow \mathscr{F}(E), \quad \sigma \mapsto K^{\sigma^\perp} \cdot \sigma$$

Since $\sigma$ is $\mathcal{N}_{K^{\sigma^\perp}}(\mathfrak{a})$-invariant, it follows from Lemma 1.1 that $\Upsilon(\sigma)$ is a face of $E$.

**Theorem 0.1.** *The map $\Upsilon$ induces a bijection between $\mathscr{F}(P)/\mathcal{N}_K(a)$ and $\mathscr{F}(E)/K$.*

*Proof.* Set $\mathcal{N} := \mathcal{N}_K(\mathfrak{a})$. We first show that $\Upsilon$ is $\mathcal{N}$-equivariant. Let $w \in \mathcal{N}$. Then $\sigma' = w\sigma$ implies $K^{\sigma'^\perp} = wK^{\sigma^\perp}w^{-1}$ and therefore $\Upsilon(\sigma') = w\Upsilon(\sigma)$. This means that the map

$$\tilde{\Upsilon} : \mathscr{F}(P)/\mathcal{N} \longrightarrow \mathscr{F}(E)/K, \quad [\sigma] \mapsto K^{\sigma^\perp} \cdot \sigma$$

is well-defined. Next, we prove that $\tilde{\Upsilon}$ is injective. Assume for some $g \in K$ $g \cdot F_1 = F_2$ where $F_1 = \Upsilon(\sigma_1)$ and $F_2 = \Upsilon(\sigma_2)$. Since $F_2 = K^{\sigma_2^\perp} \cdot \sigma_2$, the face $F_2$ is a $K^{\sigma_2^\perp}$–invariant convex body. Moreover $\sigma_2 \subset \mathfrak{a} \subset \mathfrak{p}^{\sigma_2^\perp}$ and $\mathfrak{p}^{\sigma_2^\perp}$ is $K^{\sigma_2^\perp}$–invariant. Therefore $F_2$ is contained in $\mathfrak{p}^{\sigma_2^\perp}$. It follows that $\mathrm{Aff}(F_2) = x_o + \mathfrak{q}_{F_2}$, where $\mathfrak{q}_{F_2}$ is a $K^{\sigma_2^\perp}$ invariant subspace of $\mathfrak{p}^{\sigma_2^\perp}$, $x_o$ is a fixed $K^{\sigma_2^\perp}$ point and it is orthogonal orthogonal to $\mathfrak{q}_{F_2}$. We apply Proposition 1.3 to the group $G^{\sigma_2^\perp}$ and $\mathfrak{q}_{F_2}$. Thus $\mathfrak{h}_{F_2} = [\mathfrak{q}_{F_2}, \mathfrak{q}_{F_2}] \oplus \mathfrak{q}_{F_2}$ and its orthogonal complement in $\mathfrak{g}^{\sigma_2^\perp}$, that we denote by $\mathfrak{h}'_{F_2}$, are commuting ideal. The Proposition 1.3 also yields subgroups $K_1, K_2 \subset K^{\sigma_2 \perp}$ such that any two maximal subalgebras in $\mathfrak{q}_{F_2}$, respectively $\mathfrak{q}'_{F_2}$, are interchanged by

$K_1$, respectively $K_2$. Since $\sigma_2 \subset \mathfrak{a}$, also $\mathrm{Dir}(\sigma_2) \subset \mathfrak{a}$ and we may decompose $\mathfrak{a} = \mathrm{Dir}(\sigma_2) \oplus \sigma_2^\perp$. But $\mathrm{Dir}(\sigma_2)$ is contained also in $\mathfrak{q}_{F_2}$ since $\sigma_2 \subset F_2$. So $\sigma_2^\perp \subset \mathfrak{q}_{F_2}^\perp \cap \mathfrak{p} = \mathfrak{q}'_{F_2}$. By dimension $\mathrm{Dir}(\sigma_2)$ is a maximal subalgebra in $\mathfrak{q}_{F_2}$ and $\sigma_2^\perp$ is a maximal subalgebra in $\mathfrak{q}'_{F_2}$. On other hand from $g \cdot F_1 = F_2$ it follows that $g \cdot \mathrm{Dir}(\sigma_1) \subset \mathfrak{q}_{F_2}$ and $g \cdot \sigma_1^\perp \subset \mathfrak{q}_{F_2}$, and they are also maximal subalgebras in these spaces. By the Proposition 1.3 (b) and (c) there exist $k_1 \in K_1, k_2 \in K_2$ such that

$$(k_1 g) \cdot \mathrm{Dir}(\sigma_1) = \mathrm{Dir}(\sigma_2)$$
$$(k_2 g) \cdot \sigma_1^\perp = \sigma_2^\perp .$$

Since $x_0$ is fixed by the larger group $K^{\sigma_2^\perp}$ it follows that $k_1 g \sigma_1 = \sigma_2$. Moreover $k_1 k_2 = k_2 k_1$ since $[\mathfrak{h}_{F_2}, \mathfrak{h}'_{F_2}] = 0$. For the same reason $\mathfrak{q}'_{F_2}$ is fixed pointwise by $K_1$ and $\mathfrak{q}_{F_2}$ is fixed pointwise by $K_2$. Set $k = k_1 k_2$ and $w = kg$. Then $k \in K^{\sigma_2^\perp}$ and $w \in K$. We get

$$w \cdot \mathrm{Dir}(\sigma_1) = \mathrm{Dir}(\sigma_2)$$
$$w \cdot \sigma_1^\perp = \sigma_2^\perp .$$

Thus $w \cdot \mathfrak{a} = \mathfrak{a}$, i.e. $w \in \mathscr{N}$. Since $k \in K^{\sigma_2^\perp}$, $w \cdot F_1 = (kg) \cdot F_1 = k \cdot F_2 = F_2$. Since $\sigma_1 = (x_0 + \mathrm{Dir}(\sigma_1)) \cap F_1$ and similarly for $F_2$, we conclude that $w\sigma_1 = \sigma_2$. Finally we prove that $\tilde{\Theta}$ is surjective. Let $F \subset \widehat{\mathcal{O}}$ be a face. Then $F \subset \mathfrak{p}^{\mathfrak{s}}$ for some abelian subalgebra $\mathfrak{s} \in \mathfrak{p}$. Then there exists $k \in K$ such that $k \cdot \mathfrak{a} \subset \mathfrak{p}^{\mathfrak{s}}$. Therefore $k^{-1} \cdot F \subset \mathfrak{p}^{(k^{-1} \cdot \mathfrak{s})}$ and $\mathfrak{a} \subset \mathfrak{p}^{(k^{-1} \cdot \mathfrak{s})}$. By Proposition 1.4, $k \cdot F = K^{\sigma^\perp} \cdot \sigma$ where $\sigma = (k \cdot F) \cap \mathfrak{a}$ and so $\tilde{\Upsilon}$ is surjective. ☐

As an application of the above theorem and Proposition 1.4, we get the following result.

**Theorem 0.2.** *The faces of $E$ are exposed if and only if the faces of $P$ are exposed.*

*Proof.* By the above Theorem, the map $\sigma \mapsto K^{\sigma^\perp} \cdot \sigma$ induces a bijection between $\mathscr{F}(P)/\mathscr{N}$ and $\mathscr{F}(E)/K$. Hence, keeping in mind that if $F_1 = kF_2$, then $F_1$ is exposed if and only if $F_2$, the result follows from Proposition 1.4. ☐

**Remark 2.1.** *We have proven Theorems 0.1 and 0.2 under the assumption that $G$ is a connected real semisimple Lie group. From this it follows that both theorems hold true for any connected compatible subgroup of $U^{\mathbb{C}}$, since such a subgroup is real reductive in the sense of* [16, p. 446] *and thus it is the product of a semisimple connected subgroup and an abelian subgroup, see e.g.* [16, p. 453].

## 3. Convex hull of the gradient map image

Let $U$ be a compact connected Lie group and $U^{\mathbb{C}}$ its complexification. Let $(Z, \omega)$ be a Kähler manifold on which $U^{\mathbb{C}}$ acts holomorphically. Assume that $U$ acts in a Hamiltonian fashion with momentum map $\mu : Z \longrightarrow \mathfrak{u}^*$. Let $G \subset U^{\mathbb{C}}$ be a closed connected subgroup of $U^{\mathbb{C}}$ which is compatible with respect to the Cartan decomposition of $U^{\mathbb{C}}$. This means that $G$ is a closed subgroup of $U^{\mathbb{C}}$ such that $G = K \exp(\mathfrak{p})$, where $K = U \cap G$ and $\mathfrak{p} = \mathfrak{g} \cap i\mathfrak{u}$ [13, 15]. The inclusion $i\mathfrak{p} \hookrightarrow \mathfrak{u}$ induces by restriction a $K$-equivariant map $\mu_{i\mathfrak{p}} : Z \longrightarrow (i\mathfrak{p})^*$. Using a fixed $U$-invariant scalar product $\langle\, ,\, \rangle$ on $\mathfrak{u}$, we identify $\mathfrak{u} \cong \mathfrak{u}^*$. We also denote by $\langle\, ,\, \rangle$ the scalar product on $i\mathfrak{u}$ such that multiplication by $i$ be an isometry of $\mathfrak{u}$ onto $i\mathfrak{u}$. For $z \in Z$ let $\mu_{\mathfrak{p}}(z) \in \mathfrak{p}$ denote $-i$ times the component of $\mu(z)$ in the direction of $i\mathfrak{p}$. In other words we require that $\langle \mu_{\mathfrak{p}}(z), \beta \rangle = -\langle \mu(z), i\beta \rangle$, for any $\beta \in \mathfrak{p}$. Then we view $\mu_{i\mathfrak{p}}$ as a map

$$\mu_{\mathfrak{p}} : Z \to \mathfrak{p},$$

which is called the *$G$-gradient map* or *restricted momentum map* associated to $\mu$. For the rest of the paper we fix a $G$-stable *compact subset* $X \subset Z$ and we consider the gradient map $\mu_{\mathfrak{p}} : X \longrightarrow \mathfrak{p}$ restricted on $X$. We also set

$$\mu_{\mathfrak{p}}^{\beta} := \langle \mu_{\mathfrak{p}}, \beta \rangle = \mu^{-i\beta}.$$

We will now study the convex hull of $\mu_{\mathfrak{p}}(X)$, that we denote by $E$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra of $\mathfrak{p}$ and let $\pi : \mathfrak{p} \longrightarrow \mathfrak{a}$ be the orthogonal projection onto $\mathfrak{a}$. Then $\pi \circ \mu_{\mathfrak{p}} =: \mu_{\mathfrak{a}}$ is the gradient map associated to $A = \exp(\mathfrak{a})$. Let $Z^A$ be the set of fixed points of $A$. We note that $\mu_{\mathfrak{a}}$ is locally constant on $Z^A$ since $\operatorname{Ker} \mathrm{d}\mu_{\mathfrak{a}}(x) = (\mathfrak{a} \cdot x)^{\perp}$ (see [15]). Let $\mathfrak{b}$ a subspace of $\mathfrak{a}$ and let $X^{\mathfrak{b}} = \{p \in X : \xi_X(p) = 0 \text{ for all } \xi \in \mathfrak{b}\}$, where $\xi_X$ is the vector field induced by the $A$ action on $X$. Then the map $\mu_{\mathfrak{b}} : X^{\mathfrak{b}} \longrightarrow \mathfrak{b}$, that is the composition of $\mu_{\mathfrak{p}}$ with the orthogonal projection onto $\mathfrak{b}$, is locally constant ([11], Section 3). Since $X^{\mathfrak{b}}$ is compact, $\mu_{\mathfrak{b}}(X^{\mathfrak{b}})$ is a finite set. In [11] it also shown that for any $y \in X^{(\mathfrak{b})} := \{p \in X : \mathfrak{a}_p = \mathfrak{b}\}$, where $\mathfrak{a}_p := \{\xi \in \mathfrak{a} : \xi_X(p) = 0\}$, we have that $\mu_{\mathfrak{a}}(A \cdot y) \subset \mu_{\mathfrak{a}}(y) + \mathfrak{b}^{\perp}$ is an open subset of the affine space $\mu_{\mathfrak{a}}(y) + \mathfrak{b}^{\perp}$ (the orthogonal complements are taken in $\mathfrak{a}$). Moreover $\mu_{\mathfrak{a}}(A \cdot y)$ is a convex subset of $\mu_{\mathfrak{a}}(y) + \mathfrak{b}^{\perp}$ (see [10]) and therefore $\mu_{\mathfrak{a}}(\overline{A \cdot y}) = \overline{\mu_{\mathfrak{a}}(A \cdot y)}$ is a convex body.

Let $P := \widehat{\mu_{\mathfrak{a}}(X)}$. If $\beta \in \mu_{\mathfrak{a}}(X)$ is an extremal point of $P$, and $y \in \mu_{\mathfrak{a}}^{-1}(\beta)$, then $\mu_{\mathfrak{a}}(A \cdot y)$ is an open neighborhood of $\mu_{\mathfrak{a}}(y)$ in $\mu_{\mathfrak{a}}(y) + \mathfrak{a}_y^{\perp}$ and it is contained in $\mu_{\mathfrak{a}}(X) \subset P$. Since $\mu_{\mathfrak{a}}(y)$ is an extremal point, it follows that $\mathfrak{a}_y^{\perp} = \{0\}$ and so $y$ is a fixed point of $A$. Since $X$ is compact, the set $X^A$ has finitely many connected components. Therefore $P$ has finitely many extremal points, i.e. it is a polytope. We have shown the following.

**Proposition 3.1.** *Let $X \subset Z$ be a $G$-invariant compact subset of $Z$. Then the image $\mu_{\mathfrak{a}}(X^A)$ is a finite set $\{c_1, \ldots, c_p\}$ and $P = \widehat{\mu_{\mathfrak{a}}(X)}$ is the convex hull of $c_1, \ldots, c_p$.*

As a corollary we get the following result.

**Theorem 0.3.** *Let $X \subset Z$ be a $G$-invariant compact subset of $Z$. Then every face of $E = \widehat{\mu_{\mathfrak{p}}(X)}$ is exposed.*

*Proof.* Since

$$\pi(E) = \pi(\widehat{\mu_{\mathfrak{p}}(X)}) = \widehat{\mu_{\mathfrak{a}}(X)},$$

by Lemma 1.1 (i) we conclude that $E \cap \mathfrak{a} = \pi(E) = P$ and by Proposition 3.1, Remark 2.1 and Theorem 0.2 we get that every face of $E$ is exposed. □

We call $P$ the *momentum polytope*. If $G = U^{\mathbb{C}}$ and $X$ is a complex connected submanifold of $Z$, then $P = \mu_{\mathfrak{a}}(X)$ by the Atiyah-Guillemin-Sternberg convexity theorem [1, 8]. The same holds for $X$ an irreducible semi-algebraic subset of a Hodge manifold $Z$ [17, 11, 5].

Since any proper face $F$ of $E$ is exposed, the set $C_F$ defined in (1) is a non-empty convex cone in $\mathfrak{p}$. Set

$$K^F := \{g \in K : g \cdot F = F\}.$$

By Proposition 5 in [3] we have $C_F^{K^F} := \{\beta \in C_F : \, K^F \cdot \beta = \beta\} \neq \emptyset$. This means that for a proper face $F$ one can find a $K^F$–invariant vector $\beta$ such that $F_\beta(E) = F$. For $\beta \in \mathfrak{p}$, denote by $X^\beta$ the set of points of $X$ that are fixed by $\exp(\mathbb{R}\beta)$. If $\beta \in C_F$, let

$$X^\beta_{\max} := \{x \in X : \mu_{\mathfrak{p}}^\beta(x) = \max_X \mu_{\mathfrak{p}}^\beta\}.$$

Since the function $\mu_{\mathfrak{p}}^\beta$ is $K^\beta$–invariant the set $X^\beta_{\max}$ is $K^\beta$–invariant. Moreover $X^\beta_{\max}$ is a union of finitely many connected components of $X^\beta$ and $X^\beta$ is $G^\beta$-stable. Every connected component of $G^\beta$ meets $K^\beta$. This implies that $G^\beta$ leaves $X^\beta_{\max}$ invariant. Next we show that $X^\beta_{\max}$ does not depend on the choice of $\beta$ in $C_F$.

**Lemma 3.1.** *If $\beta \in C_F$, then $X^\beta_{\max} = \mu_{\mathfrak{p}}^{-1}(F)$. Moreover $F$ is the convex hull of $\mu_{\mathfrak{p}}(X^\beta_{\max})$.*

*Proof.* Fix $x \in X$. Then $\mu_{\mathfrak{p}}(x) \in F$ if and only if $\langle \mu_{\mathfrak{p}}(x), \beta\rangle = \max_{v\in E}\langle v, \beta\rangle$. Moreover $\max_{v\in E}\langle v, \beta\rangle = \max_{v\in\mu_{\mathfrak{p}}(X)}\langle v, \beta\rangle = \max_X \mu_{\mathfrak{p}}^\beta$. So $x \in \mu_{\mathfrak{p}}^{-1}(F)$ if and only if $x$ is a maximum of $\mu_{\mathfrak{p}}^\beta(x)$ restricted to $X$. This shows that $X^\beta_F = \mu_{\mathfrak{p}}^{-1}(F)$. The inclusion $\mu_{\mathfrak{p}}(X^\beta_F) \subset F$ follows from the definition and therefore $\widehat{\mu_{\mathfrak{p}}(X^\beta_F)} \subset F$. By [3, Lemma 3] $\operatorname{ext} F = \operatorname{ext} E \cap F$, so $\operatorname{ext} F \subset \mu_{\mathfrak{p}}(X) \cap F = \mu_{\mathfrak{p}}(X^\beta_F)$. It follows that $F = \widehat{\mu_{\mathfrak{p}}(X^\beta_F)}$. □

Motivated by the above Lemma we set $X_F := X^\beta_{\max}$ where $\beta$ is any vector in $C_F$. We also set

$$Q^F = \{g \in G : \, g \cdot X_F = X_F\}.$$

$Q^F$ is a closed Lie subgroup of $G$.

Given $\beta \in \mathfrak{p}$ define the following subgroups:

$$G^{\beta+} = \{g \in G : \lim_{t \mapsto -\infty} \exp(t\beta) g \exp(-t\beta) \text{ exists}\},$$
$$G^{\beta-} = \{g \in G : \lim_{t \mapsto +\infty} \exp(-t\beta) g \exp(t\beta) \text{ exists}\},$$
$$R^{\beta+} = \{g \in G : \lim_{t \mapsto -\infty} \exp(t\beta) g \exp(-t\beta) = e\},$$
$$R^{\beta-} = \{g \in G : \lim_{t \mapsto +\infty} \exp(-t\beta) g \exp(t\beta) = e\}.$$

$G^{\beta+}$ (respectively $G^{\beta-}$) is a parabolic subgroup, $R^{\beta+}$ (respectively $R^{\beta-}$) is its unipotent radical and $G^\beta$ is a Levi factor. Therefore $G^{\beta+} = G^\beta \rtimes R^{\beta+}$ (respectively $G^{\beta-} = G^\beta \rtimes R^{\beta-}$).

**Lemma 3.2.** *$Q^F \cap K = K^F$.*

*Proof.* If $g \in Q^F \cap K$, then $g \cdot X_F = X_F$. Since $\mu_\mathfrak{p}$ is a $K$-invariant map, $g \cdot \mu_\mathfrak{p}(X_F) = \mu_\mathfrak{p}(X_F)$. Taking the convex hull of both sides and using Lemma 3.1 we get that $g \cdot F = F$, thus $g \in K^F$. Conversely, if $g \in K^F$, the equivariance of $\mu_\mathfrak{p}$ yields $X_F = \mu_\mathfrak{p}^{-1}(F) = \mu_\mathfrak{p}^{-1}(g \cdot F) = gX_F$, thus $g \in Q^F$. □

We are now ready to prove the connection between the set of the faces of $E$ and parabolic subgroups of $G$.

**Proposition 3.2.** *$Q^F$ is a parabolic subgroup of $G$. Moreover $Q^F = G^{\beta+}$ for every $\beta \in C_F^{K^F}$.*

*Proof.* Observe that by definition $Q^F$ is a closed subgroup of $G$. Let $\beta \in C_F^{K^F}$. Then $F = F_\beta(E)$ and, by definition of $K^F$, we get $K^F = K^\beta$. The set $X_F = \{x \in X : \mu_\mathfrak{p}^\beta(x) = \max_X \mu_\mathfrak{p}^\beta\}$ is $G^\beta$-stable. Fix $p \in X_F$ and consider the orbit $G \cdot p$, which is a smooth submanifold contained in $X$. By Proposition 2.5 in [13] (see also Proposition 2.1 in [4]) we get that $\xi_X(x) = 0$ for any $\xi \in \mathfrak{r}^{\beta+}$ and for any $x \in X_F$. Therefore $G^{\beta+} \cdot p \subset X_F$. Hence $G^{\beta+} \subset Q^F$ and the Lie algebra $\mathfrak{q}^F$ of $Q^F$ is parabolic. On the other hand by Lemma 3.2, we have $\mathfrak{q}^F \cap \mathfrak{k} = \mathfrak{g}^{\beta+} \cap \mathfrak{k} = \mathfrak{k}^\beta$ and so by Lemma 3.7 [4] we conclude that $\mathfrak{q}^F = \mathfrak{g}^{\beta+}$. Since $Q^F \subset N_G(\mathfrak{g}^{\beta+}) = G^{\beta+}$ we get $Q^F = G^{\beta+}$. □

**Remark 3.1.** *If $\beta' \in C_F^{K^F}$, then $Q_F = G^{\beta'+} = G^{\beta+}$. By Lemma 2.8 in* [4], *we have $[\beta, \beta'] = 0$, $G^\beta = G^{\beta'}$ and $R^{\beta+} = R^{\beta'+}$.*

Let $Q^{F-} = \Theta(Q^F)$, where $\Theta : G \longrightarrow G$ denotes the Cartan involution. The subgroup $Q^{F-}$ is parabolic and depends only on $F$. The subgroup $L^F := Q^F \cap Q^{F-}$ is a Levi factor of both $Q^F$ and $Q^{F-}$. Let $\beta \in C_F^{K^F}$. Then $Q^F = G^{\beta+}$, $L^F = G^\beta$ and we have the projection

$$\pi^{\beta+} : G^{\beta+} \longrightarrow G^\beta, \qquad \pi^{\beta+}(g) = \lim_{t \mapsto +\infty} \exp(t\beta) h \exp(-t\beta),$$

respectively

$$\pi^{\beta+} : G^{\beta-} \longrightarrow G^{\beta}, \qquad \pi^{\beta-}(g) = \lim_{t \mapsto -\infty} \exp(t\beta) h \exp(-t\beta).$$

**Lemma 3.3.** *If $\beta \in C_F^{K^F}$, then the projections $\pi^{\beta+}$ and $\pi^{\beta-}$ depend only on $F$.*

*Proof.* Let $g \in G^{\beta+}$. We know that $g = hr$, where $h \in G^{\beta}$ and $r \in R^{\beta+}$. Then

$$\pi^{\beta+}(g) = \lim_{t \mapsto +\infty} \exp(t\beta) g \exp(-t\beta) = h \lim_{t \mapsto +\infty} \exp(t\beta) r \exp(-t\beta) = h.$$

Since $G^{\beta} = G^{\beta'}$ and $R^{\beta+} = R^{\beta'+}$ the decomposition $g = hr$ is the same for both groups and $\pi^{\beta+}(g) = \pi^{\beta'+}(g)$. The same argument works for $\pi^{\beta-}$. □

Now assume that $X$ is a $G$-stable *compact submanifold* of $Z$.

For $\beta \in C_F^{K_F}$ set $X_F^{\beta-} := \{p \in X : \lim_{t \mapsto +\infty} \exp(t\beta) \cdot p \in X_F\}$. Then the map

$$p^{\beta-} : X_F^{\beta-} \longrightarrow X_F, \qquad p^{\beta-}(x) = \lim_{t \mapsto +\infty} \exp(t\beta) \cdot x \tag{2}$$

is well-defined, $G^{\beta}$-equivariant, surjective and its fibers are $R^{\beta-}$-stable.

More generally one can consider $p^{\beta-}$ as a map from $X^{\beta-} = \{y \in X : \lim_{t \mapsto +\infty} \exp(t\beta) \cdot x$ exists $\}$ to $X^{\beta}$. In general however this map is not even continuous [14, Example 4.2]. To ensure continuity and smoothness it is enough that the topological Hilbert quotient $X^{\beta-}//G^{\beta}$ exists. Using the notation of [14] and choosing $r = \max_X \mu_{\mathfrak{p}}^{\beta}$, we have $X_F = X_{\max}^{\beta} = X_r^{\beta}$ and $X_r^{\beta-} = X_F^{\beta-}$. Thus Prop. 4.4 of [14] applies and yields that $X_F^{\beta-}$ is an open $G^{\beta-}$–stable subset of $X$ and that (2) is smooth deformation retraction onto $X_F$. Using $\pi^{\beta-}$ one defines an action of $Q^{F-} = G^{\beta-}$ on $X_F$ by setting $g \cdot x = \pi^{\beta-}(q) \cdot x$. This just depends on $F$. With respect to this action the map $p^{\beta-}$ becomes $Q^{F-}$–equivariant.

**Lemma 3.4.** *The set $X_F^{\beta-}$ and the map $p^{\beta-}$ do not depend on the choice of $\beta \in C_F^{K^F}$.*

*Proof.* Set $\Gamma = \exp(\mathbb{R}\beta)$. If $p \in X_F$ by the Slice Theorem [13, Thm. 3.1] there are open neighborhoods $S_p \subset T_pX$ and $\Omega_p \subset X$ and a $\Gamma$-equivariant diffeomorphism $\Psi_p : S_p \longrightarrow \Omega_p$, such that $0 \in S_p$, $p \in \Omega_p$, $\Psi_p(0) = p$. Since $p$ is a maximum of $\mu_{\mathfrak{p}}^{\beta}$ restricted to $X$, the following orthogonal splitting $T_pX = V_0 \oplus V_-$ with respect to the Hessian of $\mu_{\mathfrak{p}}^{\beta}$ holds. Here $V_0$ denotes the kernel of the Hessian of $\mu_{\mathfrak{p}}^{\beta}$ and $V_-$ denotes the sum of eigenspaces of the Hessian of $\mu_{\mathfrak{p}}^{\beta}$ corresponding to negative eigenvalues. We also point out that $V_0 = T_pX_F$ and $S_p = \{x_0 + x_- : x_0 \in S_p \cap V_0, x_- \in V_-\}$, see [15]. It follows that $\Omega_p \subset X_F^{\beta-}$. Set $\Omega := \bigcup_{p \in X_F} \Omega_p$. By what we just proved, $\Omega \subset X_F^{\beta-}$. On the other hand $\Omega$ is an open $\Gamma$-invariant neighbourhood of $X_F$, so $X_F^{\beta-} \subset \Omega$. So $X_F^{\beta-} = \Omega$. If $\beta'$ is another vector of $C_F^{K^F}$, set

$B = \exp(\mathbb{R}\beta \oplus \mathbb{R}\beta')$. This is a compatible abelian subgroup and $X_F \subset X^B$. So we may choose the open subsets $\Omega_p$ above to be $B$-stable. Therefore we get $X^{\beta'-} = \Omega$ as well. This proves that $X_F^{\beta-} = X_F^{\beta'-}$.

Next we show that $p^{\beta-} = p^{\beta'-}$. First observe that $p^{\beta-}(y) = p^{\beta'-}(y)$ if $y \in \Omega$. Indeed if $y \in \Omega_p$ we can study the limit using the diffeomorphism $\Psi_p : S_p \to \Omega_p$. The decomposition $T_pX = V_0 \oplus V_-$ is the same for $\beta$ and $\beta'$ since they commute and attain their maxima on $X_F$. Therefore if $x = \Psi_p^{-1}(y) = x_0 + x_-$, then

$$p^{\beta-}(y) = \Psi_p(x_0) = p^{\beta'-}(y). \tag{3}$$

If $p \in X_F^{\beta-}$ and $q = \lim_{t\mapsto+\infty} \exp(t\beta)\cdot p \in X_F$, there is $t_1 \in \mathbb{R}$, such that $\exp(t\beta)\cdot p \in \Omega$. Therefore

$$\begin{aligned}
\lim_{t\mapsto+\infty} \exp(t\beta')\cdot p &= \lim_{t\mapsto+\infty} \exp(t\beta')(\exp(t_1\beta')\cdot p)\\
&= \lim_{t\mapsto+\infty} \exp(t\beta)(\exp(t_1\beta')\cdot p) \text{ (by 3)}\\
&= \exp(t_1\beta')(\lim_{t\mapsto+\infty} \exp(t\beta)\cdot p)\\
&= \lim_{t\mapsto+\infty} \exp(t\beta)\cdot p.
\end{aligned}$$

□

By the above Lemma if $F$ is a face and $\beta \in C_F^{K^F}$, we can set $X_F^- := X_F^{\beta-}$ and $p^{F-} := p^{\beta-} : X_F^- \longrightarrow X_F$.

**Theorem 3.1.** *For any face $F \subset E$, the set $X_F$ is closed and $L^F$-stable, $X_F^-$ is an open $Q^{F-}$–stable neighborhood of $X_F$ in $X$ and the map $p^{F-}$ is a smooth $Q^{F-}$–equivariant deformation retraction of $X_F^-$ onto $X_F$.*

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