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Dualities and Flows:

Sine-Dilaton Gravity, DSSYK and $T\overline{T}$-Deformed Chiral Yang–Mills

Coordinatore:

Chiar.ma Prof.ssa Raffaella Burioni

Tutore:

Chiar.mo Prof. Luca Griguolo

Dottorando:

Dott. Leonardo Bossi

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Abstract.

This dissertation explores the interplay between two-dimensional Quantum Gravity, Gauge Theories, and $T\overline{T}$-deformation, with the aim of clarifying the emergence of dualities and flows between theories in low-dimensional systems.
The first part focuses on the double-scaled SYK model (DSSYK) and its proposed gravitational dual, Sine-Dilaton gravity. Building on the parallel duality between JT-gravity and SYK, we review how the two models are classically connected through the particle-on-$SU_q(1,1)$ formulation and the Poisson Sigma Model framework. A notable tension arises between the linear entropy expected from a candidate bulk dual and the non-monotonic entropy characteristic of DSSYK. It is clear that, to resolve the mismatch within Sine-Dilaton gravity or its dual formulations, additional prescriptions should emerge. In particular, a suitable gauge fixing of the $SU_q(1,1)$ particle leads to a tractable theory, referred to as q-Liouville, in which a positivity constraint, motivated by the holographic dictionary, enforces the correct entropy behavior. Our main result establishes that the equivalence with DSSYK persists up to one-loop order, both at the level of the partition function and of the two-point correlators.
In the second part, after summarizing the theoretical background, we turn to a comprehensive study of the $T\overline{T}$ deformation of two-dimensional chiral $U(N)$ Yang-Mills theory on the torus. By combining the exact solvability of $T\overline{T}$-deformed theories with the rich mathematical structure of two-dimensional Yang-Mills theory, we develop a systematic approach circumventing the lack of geometric moduli necessary for direct application of the Cardy flow equation. Our central innovation is a sequential deformation procedure that leverages the Okuyama-Sakai formulation of chiral Yang-Mills as a local bosonic theory with controlled anti-holomorphic dependence. We demonstrate that this construction yields a well-defined deformation preserving fundamental dualities and structural properties. The $T\overline{T}$ flow manifests as a square-root differential operator acting on the modular parameter, whose action we analyze through perturbative techniques. We derive explicit results for the deformed partition function, establish a (first-order) deformed version of the holomorphic anomaly equation, and prove the persistence of boson-fermion duality under deformation.

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Advisor: Prof. Luca Griguolo

Coordinator: Prof. Raffaella Burioni

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Contents

1 Introduction

This chapter is meant to introduce the context of this dissertation, which is devoted to Sine-Dilaton gravity, Double-Scaled SYK and $T\overline{T}$-deformation of chiral U(N) Yang-Mills on torus topology.

Quantum Gravity. In the previous century, General Relativity has survived numerous experimental tests, moreover recent observations have provided strong evidence for the existence of Black Holes (binary systems, gravitational waves and supermassive galactic-centers [1–3]). On the other hand, Quantum Field Theory successfully describes three of the fundamental forces, but does not allow for a consistent description of gravity, which is non-renormalizable[4]. Cosmological evidence for early-universe inflation points to a regime where quantum gravitational effects were relevant. Indeed, the Cosmic Microwave Background spectrum suggests a quantum unification of forces, with the Universe likely emerging from an initial quantum gravitational state [5, 6]. Finally, Hawking's work revealed a major tension between General Relativity and Quantum Field Theory, showing that black holes evaporate into thermal radiation. This process seems to violate unitarity, and is referred as the black hole information paradox [7].
This conceptual clash evokes a typical scenario in Kuhn's analysis of Scientific Revolutions [8]: when two successful paradigms are in contradiction at their interface, sudden advances can emerge, much like in many examples of unification in physics. Hence, we believe that the quest of quantum gravity is essential for understanding nature's extreme environments.
However, quantization of gravity requires background-independent frameworks with dynamical spacetime and faces non-renormalizability or other profound issues [9]. String theory seems to offers a UV-complete description of quantum gravity, however the analysis of its vast landscape of vacua seems to prevent unique predictions for our Universe [10].

Two-dimensional Quantum Gravity. While a complete framework for a higher-dimensional quantum theory of gravity remains elusive, substantial progress has been achieved through the study of simplified, lower-dimensional models [11–20] Gravitational theories in two spacetime dimensions provide an exceptionally powerful theoretical laboratory in this context [11, 12, 21–24]. Although they lack local propagating gravitational degrees of freedom, they retain a rich and non-trivial geometric structure of non-local degrees of freedom. Sometimes it is even possible to perform a precise and non-perturbative analysis of central phenomena. These low-dimensional toy–models serve as essential conceptual and technical playgrounds. Insights could shed light on the structure or expected properties of a higher dimensional quantum theory of gravity. It is interesting to focus on dilaton gravity [20]. Indeed, two dimensional pure Einstein gravity is topological, since the Einstein-Hilbert action reduces to the Euler characteristic of the corresponding Riemann surface. The simplest extension is dilaton gravity, which includes a dilaton field coupled to curvature and already exhibits nontrivial phenomena. Among these theories, the simplest is JT gravity (after Jackiw and Teitelboim [11, 12]), in which the dilaton is linearly coupled. The model has seen a revival in the last decade due to its solvability and holographic properties. In the last years, a deformation of JT, referred as *sine-dilaton gravity*, has gained some attention and is one of the subjects of this dissertation. Since lot of the work in this new context is inspired by the Literature on JT gravity, it is useful to mention the main results in that field.

JT Gravity. JT gravity arises as the effective theory controlling the near-horizon geometry and low-energy dynamics of higher-dimensional near-extremal black holes. For instance, in this regime, the geometry of a Reissner-Nordström black hole becomes $\text{AdS}_2 \times S^2$ and JT gravity emerges as the theory describing gravitational fluctuations in AdS_2 after integrating out the sphere degrees of freedom [25–27]. The dilaton encodes the size of the transverse sphere and as a consequence, its entropy.
A key feature of Euclidean AdS-JT gravity is its exact solvability at the quantum level. On a disk topology, the bulk dynamics reduces to a boundary theory whose action is described by the Schwarzian derivative. The corresponding Schwarzian theory is a quantum mechanics governing the reparametrizations of the thermal circle and respecting the symmetry group of low-energy JT gravity [28–32]. The Schwarzian is one-loop exact [33], indeed JT-gravity can be expressed as a supersymmetric BF gauge theory and localized [30, 34]; the model can even be solved in a two dimensional conformal-bootstrap approach [35].
The interest in JT gravity was sparked by the breakthrough realization that the Sachdev-Ye-Kitaev (SYK) model, in its low-energy limit, is also governed by the same Schwarzian action [36]. SYK is a quantum system of randomly coupled Majorana fermions in $(0+1)$-dimensions firstly proposed in [24, 37]. The similarity in the IR triggered a proposal [28,

36] for a lower-dimensional duality establishing one of the most controlled examples of holographic correspondence [38–40]. In particular, in its collective-field formulation (referred as G-Σ), the SYK model becomes exactly solvable as a sum over melonic diagrams.

The solvability of SYK further led to the conjecture that SYK is related to a matrix model description [36] (a precursor being the $c = 1$ string theory, which is equivalent to a double-scaled matrix model [41]). In their foundational work, Saad, Shenker, and Stanford [42] identified the specific double-scaled matrix model dual to JT gravity. This enabled the use of powerful matrix model techniques, such as the Eynard-Orantin topological recursion [43], to study the theory non-perturbatively and extract the spectral-density of the model. On the gravitational side, it was recognized [44] that applying this recursion is equivalent to performing a sum over all topologies with an AdS boundary, via Mirzakhani's recursion relation for hyperbolic surfaces [45]. This monumental result provided a non-perturbative, background-independent definition of JT gravity as a topological expansion on Riemann surface with AdS_2 Boundaries. Furthermore, it shed new light on the nature of holographic dualities, leading to a possible refined paradigm that certain theories of quantum gravity are dual not to a single CFT, but to an ensemble of such theories [42, 46], providing a first concrete example of a conjecture by Witten [47].

Besides holography, another deep result lies in the connection to quantum-chaos, which was conjectured to be maximal in BH physics [48–50]. Indeed the Schwarzian action was proven to be maximally chaotic, exhibiting a Lyapunov exponent which saturates the Maldacena-Shenker-Stanford bound [28, 29, 31, 36, 51], defined in [52, 53]. Moreover, the random matrix universality class of the dual matrix model (in particular the presence of a spectral form factor with a characteristic "ramp" and "plateau") confirms that JT gravity exhibits typical features of spectral correlations of chaotic systems [42, 54]. This concrete example illustrates how black holes in quantum gravity realize maximal chaos.

Finally, old proposals for a possible resolution to the black hole information paradox are tested in JT gravity by incorporating fundamental wormhole geometries via the matrix model formulation. Wormholes induce deviations from Hawking's thermal spectrum, restoring unitarity through the characteristic ramp and plateau in the spectral form factor [46, 54]. The problem of unitarity was further studied by means of entanglement islands in [55–57].

While the extensive Literature on JT gravity precludes a comprehensive treatment here, we conclude by mentioning a fascinating implication of JT. Recent works [58, 59] conjecture that near-extremal black holes may exhibit transparency due to quantum gravitational effects in the throat, which is governed by JT gravity. This effect renders the BH invisible to a scattering process involving low-energy electromagnetic and gravitational waves.

Sine-Dilaton Gravity. One of the two main topics of this dissertation is *Sine-Dilaton gravity*. As we have shown, JT gravity is a wonderful arena to test significant conjectures in the field of Quantum Gravity. However, our observable universe is dominated by a positive cosmological constant, making de-Sitter space the relevant cosmological background [5, 60]. Consequently, the developments in AdS-QG are not directly applicable to realistic settings such as astrophysical black holes or the early universe.

To better understand quantum gravity in de-Sitter space, the double-scaling limit of the SYK model (DSSYK) proves useful [28, 61, 62]. Indeed, the theory is well-defined at all energy scales as a genuinely UV-complete quantum mechanical system [63, 64]. DSSYK interpolates between a de-Sitter-like regime at high energies and an anti-de-Sitter-like regime at low energies. In the UV, the model exhibits key features such as a bounded Hilbert space and a maximum entropy state [65–72]. At the same time, DSSYK reduces to the Schwarzian effective action in its low-energy limit, thereby reproducing the infrared dynamics of JT gravity [61]. These properties suggest a holographic perspective, indicating the existence of a UV-complete theory of gravity describing DSSYK from the bulk.

Although alternative proposals for the holographic dual of DSSYK exist (such as 3d de-Sitter models [70, 73, 74] or the Complex Liouville string [75–78]), this work will focus on the Sine-Dilaton gravity formulation [79–87]. Indeed it is natural to postulate that an extension of JT gravity itself could admit a UV completion on the disk; the candidate should reduce to JT in the infrared while remaining well-defined at high energies. Sine-Dilaton gravity, with its periodic potential, emerges as the minimal candidate.

The classical dynamics of the model encodes a cosmological horizon in addition to the black hole horizon present in JT gravity, making it a candidate for de-Sitter quantum gravity. This condition proves to be an ideal framework for exploring quantum gravity in cosmological settings [85], the emergence of spacetime discretization from a UV-complete microscopic description [80] and the interplay between black hole and cosmological horizons. A technical advantage is that the entire chain of dualities of the JT-SYK correspondence admits a natural extension to the double-scaled regime, allowing for a controlled construction. Specifically, while JT gravity possesses a topological formulation as a BF theory with boundary defects, its sine-dilaton counterpart is described by a Poisson sigma model [82, 88]. The classical particle on the $\mathrm{SL}(2,\mathbb{R})$ group manifold describing JT gravity, is promoted to a particle on the $SU_q(1,1)$ quantum group [79]. Similarly, the two-field Liouville formulation of the Schwarzian theory can be deformed into q-Liouville theory, and the four-field formulation admits an analogous deformation, referred as q-Schwarzian [79]. Finally, the $G\Sigma$ collective field theory of SYK can be precisely mapped into its double-scaled counterpart [61, 70]. This consistent deformation of the entire chain is a first positive indication that the approach merits serious consideration.

However, Sine-Dilaton gravity has not yet been fully resolved as a gravitational path integral theory. So far, methodologies based on the Wheeler-DeWitt equation and canonical quantization à la Harlow-Jafferis have been applied [80, 82]. The principal difficulty comes from the need for additional prescriptions in the path integral in order to reproduce the bounded spectrum of DSSYK. Indeed a naive gravitational theory yields an entropy obeying the Hawking area law, which grows unbounded. This was expected, in fact, even in classically dual theories such as q-Liouville, extra conditions must be imposed at the path-integral level in order to recover the right entropy.

A crucial step in verifying this conjecture has been the explicit one-loop computation of the partition function and correlators in q-Liouville theory [81], as detailed in Chapter 3 of this dissertation. On the side of DSSYK there is a well-defined notion of semiclassical limit of the theory, even at one-loop or at non-perturbative level, as shown in [89–91]. The fact that the computation matches DSSYK observables at one-loop provides further evidence that a deeper understanding of the gravitational path integral is required. This may involve refining the path-integral measure, incorporating intrinsic defect configurations, or imposing constraints that naturally implement UV-completeness.

2D Yang-Mills. The second main topic of this dissertation is $T\overline{T}$-deformation of Chiral Yang-Mills on the torus. Two-dimensional Yang–Mills theory provides a simplified setting to explore interesting features of gauge theories. It is an exactly solvable model [92–95], whose partition function admits both a group-theoretic and a dual instanton representation.

Since our primary focus is quantum gravity, we emphasize the deep interplay between gravity and gauge theories, most notably through the reformulation of gravitational dynamics within the first order Palatini formalism. The common element in this descriptions is the Poisson Sigma Model, which is actually a gauge theory. This model provides a unified formalism for a broad class of solvable theories in two dimensions, including dilaton gravities, Liouville gravity, and WZW [96–100]. The solvability of these theories arises due the topological nature of the PSM. As a result, when the theory is defined on a manifold with boundary, it localizes as an effective dynamical theory on the boundary itself.

Even two-dimensional Yang-Mills theory on Riemann surfaces can be described within the framework of the PSM [98, 100]. In this case, the fundamental result of Cattaneo and Felder demonstrate how the theory's partition function can be exactly reproduced from the quantization of the corresponding PSM.

Focusing on $U(N)$ Yang-Mills on the two dimensional torus, the connection to gravita-

tional theories further extends. Indeed, the partition function is identified as counting BPS microstates of an ensemble of four-dimensional supersymmetric black holes [101]. Furthermore, via the Ooguri-Strominger-Vafa (OSV) conjecture [102], the same object factorizes as a chiral product of topological string partition functions.
The connection to topological strings was made concrete in the large-N limit. Indeed Gross and Taylor [103] have proven that the $1/N$ expansion of the partition function can be organized as a sum over maps from a two-dimensional worldsheet to the torus target space. Thus providing compelling evidence for a string-theoretic interpretation. However the factorization holds up to perturbative $1/N$ corrections, and the string interpretation applies separately to each chiral component. Various techniques can be applied to study the model [104–107], and the partition function can be elegantly expressed in terms of quasi-modular Eisenstein series.
Although a precise formulation of the string theory describing the chiral sector remains unknown (see [108–110] for recent proposals), there are other strong evidences. A hallmark of the topological nature is the appearance of a holomorphic anomaly equation, analogous to the one discovered in Kodaira–Spencer theory [111, 112], as emphasized by Dijkgraaf and Okuyama [113, 114]. The analysis is based on a dual QFT formulation of the theory, in terms of a local chiral boson or, after fermionization, as a system of interacting fermions [107, 113, 115, 116].
In particular, in [114] it is shown that treating the boson theory as local is equivalent to omit the contributions of winding modes. The resulting partition function is naturally enhanced by an anti-holomorphic dependence, which originates from the boson propagator. The analysis further clarifies that the equivalence between the fermionic and bosonic descriptions holds only in the anti-holomorphic limit. In this enhanced theory, the holomorphic anomaly relates the components of the perturbative genus expansion and enables a recursive computation of higher-order terms.
On the other hand, in the fermionic formulation starting from the recursion relation proposed in [107], it is possible to extract non-perturbative $1/N$ contributions in the large-N regime [117] by means of resurgence analysis. This analysis can be extended to the full theory and closely parallels the developments in the context of topological strings [118–120], thus reinforcing the original interpretation put forward by Gross and Taylor.

$T\overline{T}$-Deformation Chiral Yang-Mills can be interpreted as a chiral deformation of the compact boson on the torus, which is a 2d CFT [104, 113, 121]. In recent years, the study of deformations of CFTs has emerged as a central theme, driven by $T\overline{T}$ and its generalizations. The interest in this irrelevant deformation arises from the discovery that it provides a powerful connection between quantum field theories, gravity, and holographic dualities.

In the modern formulation, the $T\overline{T}$-deformation consists in coupling a general two-dimensional QFT to the $T\overline{T}$ operator. The latter was defined by Zamolodchikov as an OPE (the determinant of the stress-energy tensor) in the pinching limit and up to derivative corrections[122]. The author has also shown the factorization properties of the operator, leading to explicit evaluation on a cylinder.
A crucial advancement came from an iterative redefinition of the deformation, using the stress tensor of the deformed theory itself. This idea leads to a closed flow equation for the spectrum [123, 124], known as Burger's equation. A simple application revealed that the deformation of a free boson reproduces the Nambu-Goto action in static gauge, mapping a local QFT to a non-local, string-like model. Notably, the flow is non-Wilsonian, preserving degrees of freedom while altering locality. In a geometric approach, it was shown that also the partition function inherits a diffusion equation, allowing to extend the formalism to non Lorentz-invariant theories [125, 126].
As we mentioned, there are also strong connection to gravity. Indeed, the deformation is classically equivalent to a coupling between the QFT and a topological sector (without boundary terms) describing flat JT gravity [127, 128] with some path integral restriction on an unstable saddle. The intuition coming from the Nambu-Goto action lead to the equivalence of deformed CFTs to non-critical string theories [129, 130]. Holographically, the deformation corresponds to placing the dual AdS$_3$ theory at a finite radial cutoff [131].
Regarding deformations on torus topology, the main object of study were CFTs [132, 133]. The deformed partition function remains modular invariant up to a rescaling in the deformation parameter and the flow is led to a phase transition between Cardy and Hagedorn density. The phenomenon should interpreted as a change of background effect rather than a change in fundamental degrees of freedom [132, 134]. There are also some results on the deformation finite-N full Yang-Mills theory on the torus [135, 136].

$T\overline{T}$-deformation of Chiral Yang-Mills. The aim of our analysis, reported in chapter 5, is to extract the perturbative expansion of the $T\overline{T}$ deformation of the chiral $U(N)$ Yang–Mills theory on the torus. A natural question we want to address concerns the effect of a double deformation, obtained by first applying the chiral flow to the compact boson and subsequently $T\overline{T}$.
We prove that the construction is well-defined due to the anti-holomorphic dependence [114] of the chiral boson, which can be non-trivially deformed by applying the Cardy flow equation [125].
On the other hand, in doing so, we test the effects of $T\overline{T}$ deformation on the chiral topological string. In principle these theories should be insensitive to $T\bar{T}$ due to their topological nature. However, due to the controlled dependence on the anti-holomorphic moduli, the deformation is actually non-trivial. In this sense, we show that there is a

window where $T\bar{T}$ can become meaningful even in this context.
Regarding the deformation of chiral YM, our proposal is that it can be obtained by first introducing the Okuyama-Sakai anti-holomorphic dependence, then applying the $T\overline{T}$ flow, and finally removing the anti-holomorphic dependence by a limiting procedure. Within this framework, the partition function resums consistently, yielding an explicit prescription for computing the deformed coefficients at arbitrary order.
From our analysis we find that, formally, the $T\overline{T}$ flow is equivalent, to the action on the undeformed theory of a square-root type differential operator, a typical object appearing in this context.
From a broader perspective, since the chiral boson admits a natural string-theoretic interpretation, it is worth asking to what extent the stringy features persist. We find that (at first order) the deformed partition function obeys a recursion relation, which is naturally interpreted as the analogue of the Holomorphic Anomaly Equation. Finally, the duality between the bosonic and fermionic formulations (interpreted by Dijkgraaf as a manifestation of mirror symmetry on the torus [113]), seems to consistently extend, thereby guaranteeing that the two descriptions remain related and physically meaningful.

Document Structure: We conclude this Section by outlining the content of the Dissertation. In Chapter 2, we treat known results on Sine-Dilaton gravity, DSYYK and related dual theories. Chapter 3 contains the one-loop analysis of q-Liouville theory. Chapter 4 is meant as a review of the main concepts of $T\overline{T}$-deformation and 2d Yang-Mills. Chapter 5 deals with the $T\overline{T}$-deformation of chiral Yang-Mills. Finally, a set of Appendices (A, B, C) follows the Bibliography.

Part I

Double-Scaled-SYK
And
Sine-Dilaton Gravity

A Bulk description of Double-Scaled-SYK

2

As previously mentioned, the SYK model is a theory of N Majorana fermions with random all-to-all p-body interactions, defined by a disordered Hamiltonian. To date, no closed-form solution of the SYK model exists outside the IR regime, where analytic control is available, with the notable exception of the double-scaling limit, which admits a solvable formulation. The study of this limit may provide valuable insights toward a general solution of the SYK model. Nonetheless, it is already of significant interest in its own right, due to its rich holographic implications.
In this chapter, we present a review of the key results that lead to a bulk description of the double-scaled SYK model, based on its gravitational dual. While the material is not original, the results are reorganized here in a coherent and contiguous way, aiming to serve as a structured entry point for readers approaching this topic for the first time. This chapter emerged from the author's own need to build a well-organized understanding of the subject and is conceived to give a broader context to the following chapter of the dissertation, in which non-trivial tests of the duality are performed at one–loop.

2.1 Double-Scaled SYK

2.1.1 Introduction

This section serves as an introduction to the model. As will become clear, the distinctive feature of DSSYK lies in its remarkable solvability, while still being a UV-complete theory and having non-monotonic bounded entropy. The theory has been extensively studied in the literature, mainly due to its holographic implications.
As a first step, it is useful to provide a historical overview, focusing on the most relevant developments. The first appearance was in [61], in which the double scaling limit is proposed as a tool to evaluate the Schwarzian partition function, with the aim of verifying

the Maldacena–Stanford results for JT gravity [28]. Actually the model is there solved in the triple scaling limit (which consists in sending both the double scaling parameter and the energy to zero) confirming the expected results. Moreover, in the same paper it is recognized that the model can be reformulated as a collective field theory (analogous to the G-Σ formulation of SYK). This is obtained by introducing Hubbard–Stratonovich fields and integrating out the Majorana fermions. The new fields are bilocal, but they can be interpreted as fields on a two-dimensional manifold, thus leading to a Liouville action on Lorentzian space. The equations of motion for these fields are the analogue of the Schwinger–Dyson equations in SYK.
In the same work, it is observed that DSSYK is closely related to quantum spin glass models, which had already been exactly solved in [137]. This observation paved the way for the studies of Berkooz et al., aimed at the exact solution of the model beyond the small-λ limit. In particular, in [63] it is proposed that the theory can be solved through the introduction of chord diagrams, which at this stage are nothing but a graphical way to organize Wick contractions. One must take into account the fermionic commutation rules, which, in this specific double scaling limit, are tied to Poisson statistics. It can be shown that the partition function of the model is equivalent to summing over all chord diagrams; and it is thus natural to introduce a transfer matrix, which plays the role of iteratively constructing the chord diagrams, while also accounting for their contribution. The partition function of the model can be interpreted as a canonical thermodynamical system with known spectral density. The entropy turns out to be non-monotonic as a function of the energy, signaling a bounded statistical system.
In [64], a natural definition of matter operators is introduced; these are essentially Hamiltonian like operators but with a different scaling dimension. It is then proven that correlation functions (two-point, four-point, and, in principle, higher) can be computed via a proper transfer matrix formalism that accounts for the matter operators.
The central role of the transfer matrix in the solvability of the model leads to the recognition that this object is not merely a formal tool, but instead encodes the Hamiltonian of the model in a Hilbert space [138]. In particular, it is shown that chords numbers are the proper variables in the Hilbert space and that there is a natural notion of inner product. The underlying idea is to identify a bulk formulation of DSSYK, keeping in mind the SYK–JT gravity duality.
The works [89, 90] focus on the semiclassical formulation, inspired by the effective Liouville theory, showing that it reproduces DSSYK observables up to the one-loop level. Although mathematically consistent, this redefinition does not allow for a holographic interpretation at the level of the gravitational path integral. Moreover, it is not transparent the physical origin of some thermodynamical properties of DSSYK, such as the two

notions of temperature [62, 139] or, similarly, the non-monotonicity of the entropy. A similar semiclassical approach is introduced in [140] in order to study the spectral form factor of the model, testing for the existence of a possible matrix model formulation [84]. Since there is strong evidence that the high-temperature regime of DSSYK might be an effective description of de-Sitter spacetime [65–67, 141, 142], it has been natural to look at the model from the perspective of a suitable dimensional reduction of a theory defined on dS_3 [67–70, 73, 74, 79, 86, 88]. As an example, in [74], two copies of the DSSYK model at the same energy are introduced and interpreted as the worldliness of particles in dS_3. Another relevant perspective aims to link DSSYK to the study of non-commutative spaces. In particular, in the triple scaling limit, the theory reproduces the Schwarzian action, which can also be described as a particle moving on the group manifold $SL(2, \mathbb{R})$. It is thus conjectured that DSSYK can be described by an appropriate quantum deformation of the symmetry group, namely $SU_q(1,1)$[1]. As will be seen in more detail in the following sections, this idea leads to several reformulations, culminating in the possible identification of a bulk dual candidate to DSSYK consisting of a dilaton gravity theory with a sine potential [79, 80, 82, 88]. It should be observed that the literature on the model is very extensive, and here we have summarized only the results relevant to our investigation; a broader discussion can be found in [143]. We now move on to treat the properties of the model in more detail.

2.1.2 Observables

The *Hamiltonian* of SYK model is:

$$H = i^{p/2} \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq N} J_{i_1 \cdots i_p} \psi_{i_1} \cdots \psi_{i_p} \,, \qquad \{\psi_i, \psi_j\} = 2\delta_{ij} \qquad i, j = 1, \ldots, N \,. \tag{2.1}$$

As the commutation relation suggests, the fields are Majorana fermions. The model is a quantum mechanics in $0 + 1$ dimensions (only time). The interactions are between a groups of p fermions among N, with random couplings J_I, which for technical convenience are usually drawn from a Gaussian distribution:

$$\langle J_I \rangle_J = 0 \,, \qquad \langle J_I J_K \rangle_J = \binom{N}{p}^{-1} \mathcal{J}^2 \delta_{IK} = \frac{N}{2p^2} \binom{N}{p}^{-1} \mathbb{J}^2 \delta_{IK} \,, \tag{2.2}$$

The symbol $\langle \ldots \rangle_J$ denotes averaging over the disorder realizations. Having in mind the double scaling limit, we fix the normalization $\mathrm{Tr}(\mathbb{I}) = 1$, which prevents divergences in the entropy. Moreover, for simplicity in the notation, it is customary to fix $\mathcal{J} = 1$. The

[1] where $q = \exp(-\frac{\lambda}{2})$

double scaling limit consists in sending both N and p to infinity, while also fixing their ratio to a finite constant λ:

$$N, p \to \infty\,, \qquad \lambda \equiv 2p^2/N. \tag{2.3}$$

The limit emerges naturally when considering the effective Liouville description (having in mind a saddle point analysis), or in the chord diagram approach, as we will see.
As in standard quantum mechanics, the *partition function* can be expressed as a trace over the Hilbert space. In the present context, we are interested in its annealed[2] disorder-averaged form:

$$Z(\beta) = \left\langle \mathrm{Tr}\, e^{-\beta H} \right\rangle_J . \tag{2.4}$$

Another relevant object is the *thermofield double state (TFD)*, which lives in a doubled Hilbert space formed by taking two copies of the original Hilbert space[3], denoted as left and right. Starting from the maximally entangled state:

$$|0\rangle = \sum_{k=1}^{\infty} |k\rangle_L \, |k\rangle_R \ , \tag{2.5}$$

one can define the TFD state at inverse temperature β as:

$$|\mathrm{TFD}\rangle = e^{-\frac{\beta}{4}(H_L+H_R)}\,|0\rangle \ . \tag{2.6}$$

Where $H_{L,R}$ are the Hamiltonians acting on the left and right copies. Then the disorder-averaged partition function can be expressed as an amplitude between these states:

$$Z = \left\langle 0|e^{-\frac{\beta}{2}(H_L+H_R)}|0\right\rangle_J = \left\langle \mathrm{TFD}|\mathrm{TFD}\right\rangle_J . \tag{2.7}$$

This construction allows to reinterpret the thermal observables as overlaps in a doubled space, providing a convenient framework for the analysis of DSSYK.
In order to explore the quantum properties of DSSYK, one also introduces *probe operators* M_Δ, defined analogously, but with a different interaction number[4] $\tilde{p}$ and independent Gaussian couplings $\tilde{J}_I$:

$$M_\Delta = i^{\tilde{p}/2} \sum_{i_1<\cdots<i_{\tilde{p}}} \tilde{J}_{i_1\cdots i_{\tilde{p}}} \psi_{i_1}\cdots\psi_{i_{\tilde{p}}}, \qquad \Delta := \frac{\tilde{p}}{p}. \tag{2.8}$$

[2]Since we are considering a large-N limit, annealed and quenched average are actually equivalent.
[3]A detailed discussion of the Hilbert space will follow in Section (2.1.8).
[4]Here, the term refers to the number of fermions in a given interaction of the DSSYK probe operator. In the chord diagram representation, this number corresponds to the length of the chord connecting different points and it is denoted "fermion length".

These operators do not modify the dynamics of the system, but rather act as probes; indeed, *correlation functions* like:

$$\tilde{G}_{\Delta}(\beta_1, \beta_2) = \left\langle \mathrm{Tr}\left(e^{-\beta_1 H} M_{\Delta} e^{-\beta_2 H} M_{\Delta}\right)\right\rangle_{J,\tilde{J}} \tag{2.9}$$

measure properties of the quantum state and serve as instruments to investigate the theory. They are chosen to have a well-defined scaling dimension Δ and to respect the symmetries of the original theory, analogously to single-trace operators in holography. In the chord diagram representation, intersections of M_{Δ}-chords with Hamiltonian chords will carry a distinct weight, reflecting their combinatorial structure, but the key idea is that these probes allow one to extract information without extending or altering the Hamiltonian itself.

A natural requirement for probe operators in DSSYK is that their scaling dimensions in the infrared-limit match those of operators in the conformal regime of SYK. Indeed, in this regime, the SYK Hamiltonian flows to the Schwarzian fixed point, where fermionic bilinears acquire specific conformal dimensions. Hence, a probe operator in DSSYK is expected to correspond in the IR to a conformal primary of dimension Δ. This condition guarantees a consistent holographic dictionary.

Another relevant object is the *four-point function*, which in the non-trivial crossed-channel configuration, is defined as:

$$G(\tau_1, \tau_3; \tau_2, \tau_4) = \left\langle \mathrm{Tr}\left(e^{-\beta H} M_1(\tau_1) M_2(\tau_2) M_1(\tau_3) M_2(t_4)\right)\right\rangle_{J,\tilde{J},} \tag{2.10}$$

In the chord-diagram representation, each insertion of M_i will act as a counting operator. This structure parallels the role of bi-local operators in the Schwarzian theory [29]. The crossed four-point function in (2.10) serves also as a probe for the out-of-time-order correlator (OTOC). This correlator is a key diagnostic for quantum chaos, capturing the exponential growth of operator commutators at early times. We know that in the IR regime of SYK, the OTOCs exhibit a Lyapunov exponent that saturates the Maldacena–Shenker–Stanford chaos bound [53]. This saturation indicates maximal scrambling [49, 52, 144] in the system. In DSSYK we can probe different energy regimes and interestingly it can be shown that, away from the IR limit, the exponent is no longer saturated.

2.1.3 Chord diagrams

We now focus on the study of DSSYK observables using the chord diagram approach, which is a key technical instrument to show that the model is exactly solvable. The approach emerges quite naturally when considering the partition function and expanding in moments m_k:

$$Z(\beta) = \sum_{k=0}^{\infty} \frac{(-\beta)^k}{k!} m_k \,, \tag{2.11}$$

$$m_k := \left\langle \mathrm{Tr}\left(H^k\right)\right\rangle_J = i^{kp/2} \sum_{I_1,\dots,I_k} \langle J_{I_1} \cdots J_{I_k} \rangle_J \, \mathrm{Tr}\left(\Psi_{I_1} \cdots \Psi_{I_k}\right) \,. \tag{2.12}$$

Where Ψ_I denotes a set of p fermions. The main advantage of defining the model with Gaussian couplings is that we can now apply Wick's theorem to compute the disorder average on the couplings. In particular, the second moment of J_I reads:

$$\langle J_I J_K \rangle \propto \delta_{I,K} \,. \tag{2.13}$$

Which implies that each Wick contraction between two J's identifies their indices. Since each J_I is also related to a product of fermions Ψ_I, after performing the ensemble average, the trace contains pairs of fermions with the same index set, e.g.:

$$\mathrm{Tr}(\Psi_I \cdots \Psi_J \cdots \Psi_I \cdots \Psi_J) \,. \tag{2.14}$$

Clearly the only non-vanishing moments are even and can be written as:

$$m_k = i^{kp/2} \sum_{I_1,\dots,I_k} \binom{N}{p}^{-k/2} \sum_{I_1,\cdots,I_{k/2}} \sum_{\text{Wick}} \mathrm{Tr}\left(\Psi_{I_1} \Psi_{I_2} \cdots \Psi_{I_1} \cdots \Psi_{I_2} \cdots\right) \tag{2.15}$$

The pairs of fermions with identical indices can then be brought together and annihilated as $\Psi_I^2 \propto \mathbf{1}$, while carefully keeping track of the signs arising from their commutation relations.

We can represent the contractions diagrammatically: because the trace is cyclic, each group of p fermions can be depicted as a point on a circle, with contractions drawn as lines connecting them, referred to as chords (see Figure (2.1)).

Clearly, Wick's theorem tells us to sum over all possible interactions, so it is useful to find an efficient way to organize this sum. The diagrammatic approach provides such a simplification, as it graphically leads us to consider all possible chords connecting a fixed set of nodes for a fixed moment m_k. From this perspective, we can build a transfer matrix which naturally constructs all possible chord diagrams.

So far, we have not invoked the double-scaling limit. However, when computing the moments, the main challenge arises from handling the commutations of indices. This is because, when bringing together Ψ_I and Ψ_K, the set that we move inside the diagram may contain some fermion(s) $\psi_{i_1 \dots i_k}$ that also appear in other sets along the way. In par-

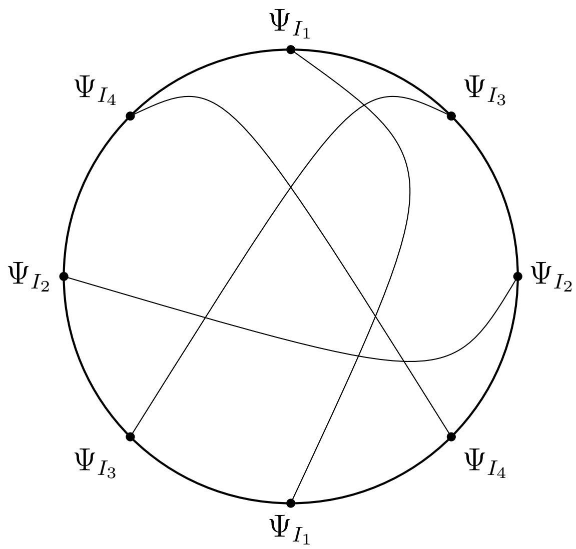


Figure 2.1: An example of chord diagram representing $\mathrm{Tr}\left(\Psi_{I_1}\Psi_{I_4}\Psi_{I_2}\Psi_{I_3}\Psi_{I_1}\Psi_{I_4}\Psi_{I_3}\right)$

ticular, commuting Ψ_I and Ψ_K introduces a factor of $(-1)^m$, where m is the number of common indices. But how many combinations of sets give m intersections?

The claim which solves the problem is that, in the double scaling limit, the number m of common indices is Poisson distributed. Let's consider the problem of choosing two lists I and K, each containing p distinct fermion indices, out of a total of N possible ones. Sets with m overlapping indices can be constructed by first choosing the p indices in I, then selecting m of them that belong to K, and finally completing K with $p-m$ indices taken outside I.

Firstly we count the sample-space dimension. We know that the number of possible combinations for each list is $\binom{N}{p}$. Hence, overall there are $\binom{N}{p}^2$ possible ways of choosing the two lists.

Among these pairs we want to know how many have m common indices. In order to count, we first choose p indices of the list I (there are $\binom{N}{p}$ possibilities). Moreover, among these sets of p indices, we select m indices to be shared with K ($\binom{p}{m}$ possibilities). Finally, we should remember that in K there are remaining $(p-m)$ indices out of the $(N-p)$ indices that are not contained in I ($\binom{N-p}{p-m}$ choices). The probability of having m overlaps is therefore the number of configurations with m intersections, divided by the total number of pairs:

$$P(m) = \frac{\binom{N}{p}\binom{p}{m}\binom{N-p}{p-m}}{\binom{N}{p}^2}. \tag{2.16}$$

By expanding the binomial coefficients and rearranging terms, the dependence on p^2/N becomes manifest:

$$P(m) = \frac{1}{m!}\left(\frac{p!}{(p-m)!}\right)^2 \frac{(N-p)!}{N!}\,\frac{(N-p)!}{(N-2p+m)!} \tag{2.17}$$

$$= \frac{1}{m!}\left(\frac{p^2}{N}\right)^m \prod_{j=0}^{p-1}\frac{N-p-j}{N-j}\prod_{\ell=0}^{m-1}\frac{\left(1-\frac{\ell}{p}\right)^2}{1-\frac{2p-\ell}{N}} \tag{2.18}$$

At this point we take the double scaling limit (here m is finite); the first product is approximated as:

$$\left(1-\frac{p}{N}\right)^p \xrightarrow[N\to\infty]{} e^{-p^2/N} \tag{2.19}$$

and the second product is of order 1. The probability therefore reduces to:

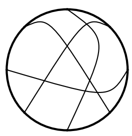
$$P(m) = \frac{1}{m!}\,(\lambda/2)^m e^{-\lambda/2}. \tag{2.20}$$

This is exactly a Poisson distribution with mean $\lambda/2$ for the number m of overlapping indices.

Hence, by summing over the possible number of common indices m, with the corresponding probability $P(m)$ and commutation relation $(-1)^m$, we are able to compute the result of commuting two sets of Majorana fermions:

$$[\Psi_I, \Psi_K] = e^{-\lambda/2}\sum_{m=0}^{\infty}\frac{(\lambda/2)^m}{m!}(-1)^m = \exp\{-\lambda\} \equiv q \tag{2.21}$$

Now, if we want to compute the factor associated to a particular chord diagram, we should start to disentangle the chords, in order to perform the contractions. In doing so, we must commute a number of sets which is precisely identical to the number of intersection of chords in the diagram. As an illustration:



$$= q^6\,. \tag{2.22}$$

As a result, we exactly know, in the double scaling limit, the factor associated to an arbitrary diagram. Finally, we can write the moments of the partition function as:

$$m_k = \sum_{\substack{\text{chord diagrams}\\ \text{with } k \text{ nodes}}} q^{\text{n. intersections}} \tag{2.23}$$

The next step is to determine how to count all possible chord diagrams for large k. A convenient way to do this is by considering the transfer matrix approach, which associates a mathematical structure to the operation of creating a diagram.

2.1.4 Transfer matrix

There are two ways to introduce the transfer matrix of DSSYK, depending on whether we want to focus on a boundary description, as first presented in [63], or on a bulk description, as discussed in [138]. In either case, the idea is to construct the diagrams by sequentially adding or removing chords. We will begin with the boundary description, which is, in some sense, simpler, although its holographic interpretation is less transparent.

We can picture the construction of a chord diagram by moving along the circle, starting from a point with no nodes. At this stage, the chord state is defined as $|0\rangle$. By convention, we move clockwise[5]. As we proceed, we encounter a total of k nodes. At the first node, we are forced to open a chord, which takes us to the state $|1\rangle$. At the next node, we have a choice: we can either close the chord (returning to $|0\rangle$), or open a new one (moving to the state $|2\rangle$).

Once the number of open chords exceeds two, the situation becomes richer: we can close a chord in two distinct ways, depending on which open chord is chosen. Each such choice determines whether we create 0 or 1 chord intersections and as a consequence a different chord diagram and a different weight. Indeed each intersection is weighted by a factor q.

To keep track of all possible outcomes within a single framework (i.e. including all chord diagrams at once), we can notice that the action of the annihilation ladder operator on the state $|2\rangle$ is:

$$a\,|2\rangle = (1+q)\,|1\rangle\,. \tag{2.24}$$

And in general, for a state with $|n\rangle$ open chords, there can be $1, \dots, n-1$ intersections. Hence we write:

$$a\,|n\rangle = (1+q+\cdots+q^{n-1})\,|n-1\rangle = \frac{1-q^n}{1-q}\,|n-1\rangle\,. \tag{2.25}$$

On the other hand, for the creation operator, in this convention there is only one possible way of increasing the number of chords, and no intersections need to be counted:

$$a^\dagger\,|n\rangle = |n+1\rangle\,. \tag{2.26}$$

[5] Moving in the opposite direction would correspond to defining the conjugate transfer matrix.

Finally we notice that at the final node, the transition $|1\rangle \to |0\rangle$ is necessarily enforced. Each step, whether corresponding to creation or annihilation, is therefore encoded in the operator:

$$T = a^\dagger + a \,, \tag{2.27}$$

which is referred as the transfer matrix. Clearly, k successive iterations of T are required in order to generate all possible diagrams with k nodes.

We can take seriously this construction and interpret the transfer matrix as a Hamiltonian acting on a Hilbert space (which lives on a circle). In order to do that, we should also define an inner product:

$$\langle m|n\rangle = \delta_{mn}[n]_q! \,, \tag{2.28}$$

The alternative construction of the Hilbert space, introduced by Lin [138], will be discussed later in the context of the bulk Hilbert space of DSSYK (2.1.8).

Using the properties of the transfer matrix, the moments and the partition function can be formally written as[6]:

$$m_k = \langle 0|T^k|0\rangle \,, \tag{2.29}$$

$$Z(\beta) = \langle 0|\sum_{k=0}^{\infty}\frac{(-\beta)^k}{k!}T^k|0\rangle \ = \langle 0|e^{-\beta T}|0\rangle \,. \tag{2.30}$$

Now we are in position of solving the model, since we recognize that ladder operators obey the algebra of q-deformed Arik-Coon oscillators:

$$[a,a^\dagger]_q \equiv aa^\dagger - qa^\dagger a = 1\,, \qquad [n,a] = -a, \qquad [n,a^\dagger] = a^\dagger \,. \tag{2.31}$$

However it is better to first diagonalize the transfer matrix by a similarity transformation, as shown in [63], in order to make it Hermitian. The Hermitian transfer matrix $\hat{T}$ solves an eigenvalue problem:

$$\hat{T}\psi = E\psi \tag{2.32}$$

where ψ is a fixed energy state. It is also possible to show that the spectrum is continuous and bounded, hence we can parameterize the energy as:

$$E(\theta) = \frac{2\cos(\theta)}{\sqrt{\lambda(1-q)}}, \qquad \theta \in [0,\pi] \tag{2.33}$$

[6]Interestingly, this result can be interpreted as a duality between the actual DSSYK model and the amplitude of a thermal system whose Hamiltonian is the transfer matrix.

We can then rewrite (2.32) as:

$$\hat{T}\left|\theta\right\rangle = \frac{2\cos(\theta)}{\sqrt{1-q}}\left|\theta\right\rangle \tag{2.34}$$

We then project onto the chord basis and use the definition in terms of ladder operators:

$$\begin{aligned}
\frac{2\cos(\theta)}{\sqrt{1-q}}\psi_n(\theta) &= \left\langle n\right|\hat{T}\left|\theta\right\rangle \\
&= \left\langle n\right|\hat{T}\left|n'\right\rangle\left\langle n'|\theta\right\rangle \\
&= \left\langle n\right|(\hat{a}+\hat{a}^\dagger)\left|n'\right\rangle\left\langle n'|\theta\right\rangle \\
&= \left(\frac{1-q^{n'}}{1-q}\delta_{n,n'-1}+\delta_{n,n'+1}\right)\left\langle n'|\theta\right\rangle \\
&= \frac{1-q^{n+1}}{1-q}\psi_{n+1}(\theta)+\psi_{n-1}(\theta)
\end{aligned}$$

In the last line we recognize the recursion relation of the q-harmonic oscillator, whose solution can be written in terms of a q-Hermite polynomial H_n:

$$\psi_n(\theta) = \sqrt{\rho(\theta)}\frac{(1-q)^{n/2}}{(q;q)_n}H_n\left(\cos(\theta)|q\right)\ , \tag{2.35}$$

where $(x;q)_n$ denotes the q-Pochhammer symbol, and we introduce a function:

$$\rho(\theta) = \frac{(q;q)_\infty(e^{2i\theta};q)_\infty(e^{-2i\theta};q)_\infty}{2\pi} \tag{2.36}$$

which is interpreted as the density of states of DSSYK. Indeed, having the spectrum, we can expand the thermal amplitude in the basis of energies:

$$Z(\beta) = \int_0^\pi d\theta\int_0^\pi d\theta'\,\langle 0|\theta\rangle\langle\theta|e^{-\beta T}|\theta'\rangle\langle\theta'|0\rangle = \int_0^\pi\frac{d\theta}{2\pi}\rho(\theta)e^{-\frac{2\beta\cos(\theta)}{\sqrt{\lambda(1-q)}}}\ . \tag{2.37}$$

This is the *exact solution* for the partition function of DSSYK. Notice that this result holds at arbitrary temperature β.

2.1.5 Collective field formalism in the double-scaling limit

In the case of DSSYK there exists a reformulation that is analogous to the $G-\Sigma$ formalism of SYK, first introduced in [61] and later revisited in [62, 90, 145]. The peculiarity of this construction is that one can derive an effective Liouville path integral, whose convergence properties are subtle and will be discussed below. Formally, the result is obtained by introducing a suitable rescaling of the G and Σ bilocal fields as defined in

SYK, a step justified by the goal of extracting an action that naturally admits a saddle point approximation. From this perspective, the saddle is at the origin of the solvability of the model. Indeed the semiclassical path-integral can be interpreted as the generating functional of the chord diagrams.

We now review the derivation in more detail, starting from the path integral formulation of SYK in Euclidean time and after the Gaussian disorder averaging:

$$\langle Z\rangle_J = \int D\psi\ \exp\left\{-\int d\tau\left[\frac{1}{4}\sum_i \psi_i\partial_\tau\psi_i\right] + \frac{\mathbb{J}^2}{4p^2N^{p-1}}\int d\tau_1 d\tau_2\left(\sum_i \psi_i(\tau_1)\psi_i(\tau_2)\right)^p\right\}. \tag{2.38}$$

Clearly, as shown in (2.13), the effect of averaging is to couple the fermions. We note that the last integral encodes the structure of all possible chord diagrams. We can rewrite it in terms of a bilocal field G, introduced together with a Lagrange multiplier Σ, enforcing the (on-shell) equality:

$$G(\tau_1,\tau_2) = \frac{1}{N}\sum_i \psi_i(\tau_1)\psi_i(\tau_2) \tag{2.39}$$

Since G is a bilocal field defined as the product of two fermions, it is antisymmetric under the exchange of times $\tau_1 \leftrightarrow \tau_2$. The same property holds for Σ. With these definitions, the partition function takes the form:

$$\begin{aligned}\langle Z\rangle_J = \int DGD\Sigma D\psi\, \exp\left\{-\int d\tau\left[\frac{1}{4}\sum_i \psi_i\partial_\tau\psi_i\right] + N\frac{\mathbb{J}^2}{4p^2}\int d\tau_1 d\tau_2\, G(\tau_1,\tau_2)^p\right\}\cdot \\ \cdot\exp\left\{-\frac{N}{2}\int d\tau_1 d\tau_2\,\Sigma(\tau_1,\tau_2)\left(G(\tau_1,\tau_2) - \frac{1}{N}\sum_i\psi_i(\tau_1)\psi_i(\tau_2)\right)\right\}.\end{aligned} \tag{2.40}$$

We can now integrate out the fermions which appear quadratically, and obtain the G–Σ formulation of SYK:

$$\langle Z\rangle_J = \int DGD\Sigma\, e^{-I[G,\Sigma]} \tag{2.41}$$

$$\begin{aligned} I = &-\frac{N}{2}\log\det\Big(\delta(\tau_{12})\partial_{\tau_2} - 2\Sigma(\tau_1,\tau_2)\Big) + \\ &+\frac{N}{2}\int d\tau_1 d\tau_2\left(\Sigma(\tau_1,\tau_2)G(\tau_1,\tau_2) - \frac{\mathbb{J}^2}{2p^2}G(\tau_1,\tau_2)^p\right)\end{aligned} \tag{2.42}$$

In SYK, this result is particularly useful in several ways: it provides a large-N effective action where the dynamics is governed by saddle points; it makes the conformal low-energy regime and the emergence of the Schwarzian description manifest; it allows for a systematic study of $1/N$ corrections through fluctuations around the saddle, thereby clarifying the JT gravity dual interpretation.

However, if we look at (2.42) in the regime in which both N and p are large, it is not clear how to organize a semiclassical analysis, since there are two competing large parameters. With the goal of taking a double scaling limit, we introduce the following ansatz, which is specifically designed in order to make the dependence on λ explicit:

$$\Sigma(\tau_1, \tau_2) = \frac{\sigma(\tau_1, \tau_2)}{p}, \qquad G(\tau_1, \tau_2) = \text{sign}(\tau_{12})\left(1 + \frac{g(\tau_1, \tau_2)}{p}\right), \tag{2.43}$$

with σ antisymmetric and g symmetric under $\tau_1 \leftrightarrow \tau_2$. Now, if we denote $K_0(\tau_1, \tau_2) = \delta(\tau_{12})\,\partial_{\tau_2}$, the determinant term reads:

$$-\tfrac{N}{2}\log\det\left(K_0 - \tfrac{2}{p}\sigma\right) \tag{2.44}$$

Since p is large, we can expand the determinant, by applying the standard properties:

$$\log\det A = \text{Tr}\log A \tag{2.45}$$

$$\text{Tr}\log(1 - X) = -\sum_{s=1} \tfrac{1}{s}\text{Tr}(X^s) \tag{2.46}$$

we obtain:

$$-\frac{N}{2}\log\det\left(K_0 - \frac{2}{p}\sigma\right) = -\frac{N}{2}\log\det K_0 + \frac{N}{2}\sum_{s=1}\frac{1}{s}\,\text{Tr}\left(\left(\frac{2}{p}K_0^{-1}\sigma\right)^s\right). \tag{2.47}$$

The inverse kernel of K_0 is the free fermion propagator in $0+1$ dimensions (i.e. an irrelevant constant), as a result, expanding in p the determinant contributes to the action as:

$$\frac{N}{p}\,\text{Tr}\left(K_0^{-1}\sigma\right) + \frac{N}{p^2}\,\text{Tr}\left(K_0^{-1}\sigma\, K_0^{-1}\sigma\right) + O\left(\frac{N}{p^3}\right) = \tag{2.48}$$

$$= \frac{N}{2p}\int d\tau_1 d\tau_2\,\text{sign}(\tau_{12})\,\sigma_{21} + \frac{N}{4p^2}\int d\tau_1 d\tau_2 d\tau_3 d\tau_4\,\text{sign}(\tau_{12})\,\sigma_{23}\,\text{sign}(\tau_{34})\,\sigma_{41} \tag{2.49}$$

where $\sigma_{ij} \equiv \sigma(\tau_i, \tau_j)$, and we used $K_0^{-1}(\tau_1, \tau_2) = \frac{1}{2}\text{sign}(\tau_{12})$. Since λ is finite and p is large, we can omit the third-order corrections.
The mixed term $\frac{N}{2}\int \Sigma G$, after the substitution (2.43), becomes:

$$\frac{N}{2p}\int d\tau_1 d\tau_2\,\text{sign}(\tau_{12})\,\sigma_{12} + \frac{N}{2p^2}\int d\tau_1 d\tau_2\,\text{sign}(\tau_{12})\,\sigma\, g_{12} \tag{2.50}$$

Due to the antisymmetry of σ, the linear contribution at order $1/p$ cancels exactly between (2.49) and (2.50). The first nontrivial terms are therefore of order $N/2p^2 \equiv \lambda$. Finally, for even p, the interaction term from G^p produces the Liouville potential because the large-p limit gives:

$$G^p = \left[\text{sign}(\tau_{12})\left(1+\frac{g}{p}\right)\right]^p = e^g\left(1+O(1/p)\right). \tag{2.51}$$

Factoring out $\frac{N}{4p^2}$, and writing the result in a symmetric notation yields:

$$\begin{aligned} I =& \frac{N}{4p^2}\left[\int d\tau_1 d\tau_2 d\tau_3 d\tau_4 \,\text{sign}(\tau_{12})\,\sigma(\tau_2,\tau_3)\,\text{sign}(\tau_{34})\,\sigma(\tau_4,\tau_1)\right. \\ &\left. + 2\int d\tau_1 d\tau_2 \left(\text{sign}(\tau_{12})\,\sigma(\tau_1,\tau_2)\,g(\tau_1,\tau_2) - \mathbb{J}^2\, e^{g(\tau_1,\tau_2)} + O(1/p)\right)\right]. \end{aligned} \tag{2.52}$$

We can perform the integral on σ which appears quadratically:

$$I = \frac{1}{8\lambda}\int d\tau_1 d\tau_2 \left[\partial_{\tau_1} g(\tau_1,\tau_2)\partial_{\tau_2} g(\tau_1,\tau_2) - 4\mathbb{J}^2 e^{g(\tau_1,\tau_2)}\right]. \tag{2.53}$$

Instead of interpreting g as a bilocal field, we can upgrade it as a field in a two dimensional kinematic space of coordinates τ_1, τ_2. Actually, g depends only on $\tau_1 - \tau_2$, so it is convenient to introduce light-cone coordinates:

$$\tau = \tau_1 - \tau_2, \quad ; \quad \nu = \tau_1 + \tau_2. \tag{2.54}$$

As a result we are led to the following Liouville-type action:

$$I = \frac{N}{16p^2}\int d\tau d\nu \left[-\frac{1}{2}(\partial_\tau g)^2 + \frac{1}{2}(\partial_\nu g)^2 - 2\mathbb{J}^2 e^{g(\tau,\nu)}\right]. \tag{2.55}$$

We notice that the resulting Liouville-like action contains a potential term of the form $\mathbb{J}^2 e^{g(\tau_1,\tau_2)}$. Such a term does not provide a natural convergence prescription for the integral. As a consequence, for real values of the field g the action is unbounded from below, and the functional integral does not possess a natural convergent contour of integration. Therefore the resulting action is only formal. Nevertheless, in the double scaling limit the dominant contribution comes from the saddle point of the action, and the effective theory is well-defined as an expansion around such a saddle. From this point of view, the Liouville action can be understood as an effective description encoding the generating function of chord diagrams rather than a globally convergent path integral. Clearly, if our goal is to identify a genuine gravitational bulk description of DSSYK, this cannot be regarded as the final answer.

2.1.6 Evaluating the two-point functions

In this section, we focus on the two-point function (2.9), extending the chord diagram formalism to include matter operators and presenting the exact result. Recall that the operator M is a probe operator of the same form as H, but with length $\tilde{p}$ and independent Gaussian couplings, hence the ensemble average is taken separately over both sets. As in the computation of the partition function, we expand the exponential as:

$$\mathrm{Tr}\Big(e^{-\beta_1 H} M e^{-\beta_2 H} M\Big) = \sum_{k_1,k_2} \frac{(-\beta_1)^{k_1}}{k_1!} \frac{(-\beta_2)^{k_2}}{k_2!} \, \mathrm{Tr}\Big(H^{k_1} M H^{k_2} M\Big) \,. \tag{2.56}$$

We can think at the trace as a moment m_{k_1,k_2} in the double expansion; whose explicit expression is:

$$m_{k_1,k_2} = \sum_{\substack{I_1,\dots,I_{k_1}\\ I'_1,I'_2\\ J_1,\dots,J_{k_2}}} \left\langle J_{I_1} \cdots J_{I_{k_1}} \tilde{J}_{I'_1} J_{J_1} \cdots J_{J_{k_2}} \tilde{J}_{I'_2} \right\rangle \mathrm{Tr}\left(\Psi_{I_1} \cdots \Psi_{I_{k_1}} \Psi_{I'_1} \Psi_{I'_2} \Psi_{J_1} \cdots \Psi_{J_{k_2}}\right). \tag{2.57}$$

Again, we can apply Wick contraction in order to compute the disorder average, having in mind that now there are two non-interacting ensembles. This process enforces pairwise contractions of the J's and separately of the $\tilde{J}$'s, producing all possible chord diagrams with H chords and a single M chord.
The presence of the probe operators M modifies slightly the picture: the M chord can intersect with the H chords, giving a contribution of weight $\tilde{q} = e^{-2p\tilde{p}/N}$[7]. From a graphical point of view: H chords intersecting among themselves give a factor q, while each crossing of an H chord with the M chord gives a factor $\tilde{q}$.
In order to evaluate the resulting sum over diagrams one extends the transfer matrix formalism. We move along the circle and keep track of the number $\hat{n}$ of open H chords. Each time one crosses the distinguished M chord, all currently open H chords intersect it, so the configuration acquires a factor $\tilde{q}^{\hat{n}}$. We can resum the moments and discover that the transfer matrix still describes the thermal evolution, but in two different time regions separated by the matter operator. As a result the two-point matter correlator takes the compact form:

$$\langle 0|\, e^{-\beta_1 T}\, \tilde{q}^{\hat{n}}\, e^{-\beta_2 T}\, |0\rangle, \tag{2.58}$$

We conclude that the disorder–averaged two–point function is reduced to a transfer matrix computation with a single insertion.

[7]The combinatorics is still related to a Poisson distribution.

In order to give a solution, we can insert a completion in the energy basis:

$$\mathbb{1} = \int_0^\pi d\theta\, \rho(\theta)\, |\theta\rangle\langle\theta|, \tag{2.59}$$

where $\rho(\theta)$ is the density of states and $|\theta\rangle$ are the corresponding eigenstates, normalized such that $\langle\theta|\theta'\rangle = \delta(\theta-\theta')/\rho(\theta)$. The two-point function becomes:

$$\tilde{G}_\Delta(\beta_1,\beta_2) = \int_0^\pi d\theta_1 \int_0^\pi d\theta_2\, \rho(\theta_1)\rho(\theta_2)\, e^{-\beta_1 E(\theta_1)-\beta_2 E(\theta_2)}\, \langle\theta_1|\tilde{q}^{\hat{n}}|\theta_2\rangle. \tag{2.60}$$

In order to evaluate the contribution of the insertion $\tilde{q}^{\hat{n}}$, it is convenient to go back to the combinatorial basis $\{|n\rangle\}_{n\geq 0}$, which diagonalizes the number operator $\hat{n}$:

$$\hat{n}\,|n\rangle = n\,|n\rangle, \qquad \Rightarrow \qquad \tilde{q}^{\hat{n}}\,|n\rangle = \tilde{q}^{\,n}\,|n\rangle.$$

On the other hand, the eigenstates of the transfer matrix, denoted by $|\theta\rangle$, admit an expansion in this basis of the form:

$$|\theta\rangle = \sum_{n=0}^{\infty} \psi_n(\theta)\,|n\rangle,$$

where $\psi_n(\theta)$ are the normalized wavefunctions, explicitly given in terms of continuous q–Hermite polynomials. With this decomposition, the matrix element of interest becomes:

$$\langle\theta_1|\tilde{q}^{\hat{n}}|\theta_2\rangle = \sum_{n=0}^{\infty} \tilde{q}^{\,n}\,\psi_n(\theta_1)\,\psi_n(\theta_2). \tag{2.61}$$

The theory of q–Hermite polynomials provides an exact resummation of this series in terms of q–Pochhammer symbols which leads to the closed expression:

$$\langle\theta_1|\tilde{q}^{\hat{n}}|\theta_2\rangle = \frac{(\tilde{q}^2;q)_\infty}{(\tilde{q}e^{i(\pm\theta_1\pm\theta_2)};q)_\infty}. \tag{2.62}$$

As a result, one obtains:

$$\tilde{G}_\Delta(\beta_1,\beta_2) = \int_0^\pi d\theta_1 \int_0^\pi d\theta_2\, \rho(\theta_1)\rho(\theta_2)\, e^{-\beta_1 E(\theta_1)-\beta_2 E(\theta_2)}\, \frac{(\tilde{q}^2;q)_\infty}{(\tilde{q}e^{i(\pm\theta_1\pm\theta_2)};q)_\infty}, \tag{2.63}$$

Which is the exact full solution for the two point function in DSSYK. It will be relevant for our analysis to extract its semiclassical expansion at small λ.
It is simple to extend the analysis to uncrossed four-point function. While, for crossed four-point amplitudes, the ordering introduces additional combinatorial complexity associated to overlapping matter chords. This requires keeping track of the number of Hamiltonian chords crossing each region defined by the matter insertions and prop-

erly accounting for their weights. However the computation is feasible by extending the transfer matrix formalism and including appropriate weight factors for the intersections. What matters in this discussion is that a systematic framework exists to compute the crossed four-point function, and in general higher-point correlation functions.

2.1.7 Semiclassical limits and one–loop corrections

Partition function. In the DSSYK model, there are two distinct semiclassical limits. The first one corresponds to studying the saddle point arising also in the collective field formulation, namely by taking the limit $\lambda \to 0$ while keeping the temperature βJ fixed [89, 90]. The second possibility is the triple scaling limit [61], where one simultaneously sends $\lambda \to 0$ and explores the low–temperature regime $\beta J \gg 1$.
In this work, we are primarily interested on the former type, as it is the most relevant from the viewpoint of gravitational physics. Our motivation is to establish a framework in which a comparison with a bulk gravitational dual becomes meaningful. Indeed, in low dimensional quantum gravity, it is customary to evaluate the action on classical solutions, and possibly include one-loop corrections. Moreover, we expect this limit to capture more properties, given that we are not fixing the temperature. The aim is to create a benchmark against which a candidate bulk gravitational dual can be tested.
The idea is to perform a saddle point approximation, starting from expression (2.37). Since the density of states $\rho(\theta)$ depends on q, we should firstly expand the q-Pochammer:

$$(x;q)_\infty \approx \exp\left[-\frac{1}{\lambda}\,\mathrm{Li}_2(x)\right] \tag{2.64}$$

As a result:

$$\rho(\theta) \approx (q;q)_\infty \exp\left[-\frac{1}{\lambda}\Big(\mathrm{Li}_2(e^{2i\theta}) + \mathrm{Li}_2(e^{-2i\theta})\Big)\right] \tag{2.65}$$

Here we did not expand the factor $(q;q)_\infty$, since it does not depend on θ and therefore acts only as an overall constant in the partition function, not affecting the saddle-point analysis. We can now write the partition function as:

$$Z \overset{\text{class.}}{=} \int_0^\pi \frac{d\theta}{2\pi}(q;q)_\infty \exp\left[-\frac{F(\theta)}{\lambda}\right] \tag{2.66}$$

$$F(\theta) = \Big(\mathrm{Li}_2(e^{2i\theta}) + \mathrm{Li}_2(e^{-2i\theta}) + 2\beta\mathbb{J}\cos(\theta)\Big) \tag{2.67}$$

The saddle point equation is obtained by extremizing the effective action with respect to θ:

$$\frac{\partial F(\theta)}{\partial\theta} = 0 \tag{2.68}$$

It can be shown that [8]:

$$\frac{d}{d\theta}\left(\mathrm{Li}_2(e^{2i\theta})+\mathrm{Li}_2(e^{-2i\theta})\right)=2i\left[\log\left(1-e^{-2i\theta}\right)-\log\left(1-e^{2i\theta}\right)\right]=2(\pi-2\theta). \quad (2.69)$$

Where in the last equality we have specified the branch ($0<\theta<\pi$). Including also the derivative of the energy term we get the saddle point equation, which relates the on-shell energy θ^* and the inverse temperature β:

$$\beta\mathbb{J}=\frac{\pi v}{\cos\frac{\pi v}{2}}. \quad (2.70)$$

Here we adopt the standard definition of v as in [28]:

$$\pi v := 2\theta^*-\pi. \quad (2.71)$$

Now we are in position to compute the semiclassical approximation of the partition function. Even though $(q;q)_\infty$ does not enter the saddle point computation, it is relevant in the evaluation of the partition function since it contains a constant of order $1/\lambda$, therefore, we start by writing its asymptotic expansion:

$$(q;q)_\infty\approx\sqrt{2\pi}\exp\left[-\frac{\pi^2}{6\lambda}+\frac{1}{2}\log\lambda+\frac{\lambda}{24}\right]. \quad (2.72)$$

In order to evaluate $F(\theta^*)$ we use the standard series expansion of the dilogarithm and apply Fourier series identities to write:

$$\begin{aligned}\mathrm{Li}_2(e^{2i\theta^*})+\mathrm{Li}_2(e^{-2i\theta^*})&=\sum_{k=1}^{\infty}\frac{\cos(2k\theta^*)+\cos(-2k\theta^*)}{k^2} && (2.73)\\ &=\frac{\pi^2}{3}-2\pi\theta^*+2(\theta^*)^2 && (2.74)\\ &=-\frac{\pi^2}{6}+\frac{\pi^2v^2}{2}. && (2.75)\end{aligned}$$

Moreover:

$$2\beta\mathbb{J}\cos\theta^*=-2\pi v\tan\frac{\pi v}{2}. \quad (2.76)$$

[8]To evaluate the derivative of the dilogarithm terms, we use the standard identity: $\frac{d}{dz}\mathrm{Li}_2(z)=-\frac{\log(1-z)}{z}$, which implies: $\frac{d}{d\theta}\mathrm{Li}_2(e^{\pm 2i\theta})=\mp 2i\log\left(1-e^{\pm 2i\theta}\right)$.

Putting everything together, we finally obtain the classical approximation for the partition function:[9]:

$$\begin{aligned}\mathcal{Z} &\approx \exp\left[-\frac{\pi^2}{6\lambda}+\frac{1}{2}\log\lambda+\frac{\lambda}{24}-\frac{1}{\lambda}\left(-2\pi v\tan\frac{\pi v}{2}-\frac{\pi^2}{6}+\frac{\pi^2 v^2}{2}\right)\right] \\ &= \exp\left[\frac{1}{\lambda}\left(-\frac{\pi^2 v^2}{2}+2\pi v\tan\frac{\pi v}{2}\right)\right] \\ &= \exp\left[\frac{1}{\lambda}\left(-2u^2+4u\tan u\right)\right]\end{aligned} \tag{2.77}$$

We now move to the computation of the *one-loop corrections*, mainly following [90]. In order to keep track of each contribution, it is required to properly expand both the measure $\rho(\theta)d\theta$ and the action. By substituting the small-λ asymptotics of the q-Pochhammer symbols (up to $O(\lambda)$ corrections), the density becomes:

$$\rho(\theta)\sim\sqrt{\frac{2\pi}{\lambda}}\exp\Big[-\frac{1}{\lambda}\Big(\mathrm{Li}_2(e^{2i\theta})+\mathrm{Li}_2(e^{-2i\theta})+\frac{\pi^2}{6}\Big)+\tfrac{1}{2}\log\Big((1-e^{2i\theta})(1-e^{-2i\theta})\Big)\Big]. \tag{2.78}$$

The energy appearing in the Boltzmann weight is:

$$\beta E(\theta) \;=\; \frac{2\beta\mathbb{J}\cos\theta}{\lambda} \;+\; \frac{\beta\mathbb{J}\cos\theta}{2} \;+\; O(\lambda), \tag{2.79}$$

where the second term is the subleading $O(\lambda^0)$ correction coming from the expansion of the exact expression for $E(\theta)$. Keeping track of the scaling in λ, we can write the effective action as:

$$\mathcal{Z}(\beta) \;\simeq\; \sqrt{\frac{2\pi}{\lambda}}\int_0^\pi\frac{d\theta}{2\pi}\exp\Big[-\frac{1}{\lambda}F(\theta)-\frac{\pi^2}{6\lambda}+h(\theta)\Big]. \tag{2.80}$$

Where we define:

$$h(\theta) \;:=\; \tfrac{1}{2}\log\Big((1-e^{2i\theta})(1-e^{-2i\theta})\Big)-\tfrac{\beta\mathbb{J}}{2}\cos\theta \tag{2.81}$$

We expect that he explicit factor $\sqrt{2\pi/\lambda}$ in front of the partition function will combine with the Gaussian factor arising from the saddle integral to produce a finite λ-independent multiplicative correction. In order to include fluctuations around the saddle, we give the following ansatz for the improved saddle point solution:

$$\theta=\theta_*+\sqrt{\lambda}\,\varepsilon, \tag{2.82}$$

[9]In the last line we introduce the convention $u := \frac{\pi v}{2}$ of [28].

so that the integration measure becomes $d\theta = \sqrt{\lambda}\, d\varepsilon$. Expanding the action around this improved saddle, we find:

$$\frac{1}{\lambda}\left(F(\theta) - F(\theta_*)\right) = \frac{1}{2\lambda}F''(\theta_*)\,(\theta-\theta_*)^2 + \mathcal{O}((\theta-\theta_*)^3) \tag{2.83}$$

$$= \frac{1}{2}F''(\theta_*)\,\varepsilon^2 + \mathcal{O}(\sqrt{\lambda}). \tag{2.84}$$

And the second derivative evaluates to:

$$F''(\theta_*) = 4\Big(1 + u\tan u\Big). \tag{2.85}$$

As a result, the partition function reduces to a Gaussian integral over ε:

$$Z(\beta) \;\simeq\; \sqrt{\frac{2\pi}{\lambda}}\,\exp\!\left[-\frac{1}{\lambda}F(\theta_*) - \frac{\pi^2}{6\lambda} + h(\theta_*)\right]\int_{-\infty}^{\infty}\frac{\sqrt{\lambda}\,d\varepsilon}{2\pi}\,\exp\!\Big[-2(1+u\tan u)\,\varepsilon^2\Big]\,. \tag{2.86}$$

The Gaussian integral is:

$$\int_{-\infty}^{\infty} d\varepsilon\,\exp\!\Big[-2(1+u\tan u)\,\varepsilon^2\Big] = \sqrt{\frac{\pi}{2(1+u\tan u)}}. \tag{2.87}$$

Moreover:

$$e^{h(\theta_*)} = 2\cos u\, e^{u\tan u} \tag{2.88}$$

Putting everything together, we obtain the one-loop partition function:

$$Z(\beta(u)) = \frac{\cos u}{\sqrt{1+u\tan u}}\,\exp\!\left[\frac{1}{\lambda}\Big(-2u^2 + 4u\tan u\Big) + u\tan u\right]. \tag{2.89}$$

Normalized two-point function. In what follows, we evaluate the *normalized two-point function* in the semiclassical approximation. As before, we start by expanding the exact non-normalized two-point function (2.63) in the limit $\lambda \to 0$, in order to obtain an effective action:

$$\tilde{G}_2 = \frac{2\pi}{\lambda}\int\prod_{k=1,2}\frac{d\theta_k}{2\pi}e^{-\frac{\tilde{F}}{\lambda}+\tilde{h}+\mathcal{O}(\lambda)}, \tag{2.90}$$

where $\tilde{F}$ and $\tilde{h}$ depend on the specific asymptotics and will be specified later. We then want to compute the resulting integral by means of the saddle point method and divide by the partition function.

As it is well known, since the matter chord splits the diagram into two thermal evolutions, the corresponding integral depends on two inverse temperatures, β_1 and β_2, as well as on the associated energies, θ_1 and θ_2. In addition, it carries a dependence on Δ through the parameter $\tilde{q}$. As a result we expect two saddle point equations defining the

values of the leading order energies θ_1^* and θ_2^* as functions of Δ, β_1 and β_2. Actually this set of variables is not the most convenient.
It is useful to settle some notation in order to render the computation more transparent. Since we have in mind to normalize the two-point function it is natural to relate the inverse temperature β_i to the inverse temperature β on the full thermal circles:

$$\beta = \beta_1 + \beta_2 = \frac{2u}{\cos u} \tag{2.91}$$

We chose to define a particular two-point chord diagram by specifying the locations on the thermal circle of the matter operator M, which we denote as τ_1 and τ_2. We can parametrize β_1 as the distance:

$$\beta_1 = \tau_{12} = \tau_1 - \tau_2 \tag{2.92}$$

As a result:

$$\beta_2 = \beta - \tau_{12} \tag{2.93}$$

However, if we formulate the saddle point equations in terms of the variables $\{u, \tau_{12}\}$, the symmetry between the two equations is not manifest. It is far more convenient to regard β_i as arising from a common β with an offset[10] ϕ, namely:

$$\beta_1 = \frac{u-\phi}{\cos u}, \quad \beta_2 = \frac{u+\phi}{\cos u}. \tag{2.94}$$

The saddle point equation is encoded in the term $\tilde{F}$ which is:

$$\tilde{F} = \sum_{k=1,2}\left[2\left(\theta_k - \frac{\pi}{2}\right)^2 - 2\beta_k \cos\theta_k\right] + \mathrm{Li}_2(e^{-2\Delta}) - 4\sum_{n=1}^{\infty}\frac{e^{-\Delta n}}{n^2}\cos(n\theta_1)\cos(n\theta_2), \tag{2.95}$$

We can see that in the limit $\Delta \to 0$ the saddle point equations of $\tilde{F}$ become formally equivalent to the one of F in the evaluation of the partition function. Therefore, it is natural to write the following ansatz[11] for θ_k:

$$\theta_k = \frac{\pi}{2} - u + a_k\Delta + b_k\Delta^2 + \mathcal{O}(\Delta^3). \tag{2.96}$$

[10]Clearly: $\phi := \left(1 - \frac{2\tau_{12}}{\beta}\right) u$.
[11]Where $\theta_k(\Delta = 0) = \theta^*$.

The saddle-point equations can be written as:

$$2\theta_1 - \pi + \beta_1 \sin\theta_1 + \arctan\left(\frac{\sin(\theta_1+\theta_2)}{e^{\Delta} - \cos(\theta_1+\theta_2)}\right) + \arctan\left(\frac{\sin(\theta_1-\theta_2)}{e^{\Delta} - \cos(\theta_1-\theta_2)}\right) = 0$$

$$2\theta_2 - \pi + \beta_2 \sin\theta_2 + \arctan\left(\frac{\sin(\theta_1+\theta_2)}{e^{\Delta} - \cos(\theta_1+\theta_2)}\right) - \arctan\left(\frac{\sin(\theta_1-\theta_2)}{e^{\Delta} - \cos(\theta_1-\theta_2)}\right) = 0 \tag{2.97}$$

Expanding in Δ, at order 0, the saddle point equations both become:

$$\Big(- u + \beta_1 \cos u + \arctan(a_1 - a_2)\Big) = 0 \quad \Rightarrow \quad a_1 - a_2 = \tan(\phi) \tag{2.98}$$

While at order 1:

$$\begin{aligned} 2a_1 + \beta_1 a_1 \sin u\, \frac{1}{2}\Big(- (a_1+a_2)\cos 2u - \tan u\Big(1 - (a_1+a_2)\sin 2u\Big)\Big) = 0. \\ 2a_2 + \beta_2 a_2 \sin u\, \frac{1}{2}\Big(- (a_1+a_2)\cos 2u - \tan u\Big(1 - (a_1+a_2)\sin 2u\Big)\Big) = 0 \end{aligned}$$

Which implies:

$$a_1 + a_2 = \frac{(1+\phi\tan\phi)\tan u}{1+u\tan u} \tag{2.99}$$

We can go further and insert the ansatz at order Δ^2, getting:

$$b_1 - b_2 = \frac{1}{2\cos^2\phi}\left[-(1+u\tan u)\tan\phi + \frac{\phi\tan^2 u(1+\phi\tan\phi)}{1+u\tan u}\right] \tag{2.100}$$

Since we want to evaluate the term $\tilde{F}$ at order one in λ and in the regime $\frac{\Delta}{\lambda}=$ finite, we should expand at order Δ^2/λ. We choose to define:

$$-\tilde{F}_* = f_0 + f_1\Delta + \frac{1}{2} I\Delta^2 + \mathcal{O}\left(\Delta^3\right) \tag{2.101}$$

With the purpose of recognizing the expression of f_1 and I, we can insert θ_k in $\tilde{F}$ and derive with respect to Δ:

$$\begin{aligned} \frac{d\tilde{F}_*}{d\Delta} &= \frac{\partial\tilde{F}_*}{\partial\Delta} + \sum_{k=1,2}\frac{\partial\theta_k}{\partial\Delta}\frac{\partial\tilde{F}_*}{\partial\theta_k} \\ &= \frac{\partial\tilde{F}_*}{\partial\Delta} \\ &= \log\frac{\cosh^2\Delta}{[\cosh\Delta - \cos(\theta_1+\theta_2)]\,[\cosh\Delta - \cos(\theta_1-\theta_2)]} \end{aligned} \tag{2.102}$$

Expanding the previous result at small Δ we recognize:

$$f_1 = \log \frac{\cos^2 u}{\cos^2 \phi}, \qquad I = \frac{f(\phi) f(-\phi)}{1 + u \tan u}, \tag{2.103}$$

where:

$$f(\phi) := (1 + u \tan u) \tan \phi - (1 + \phi \tan \phi) \tan u. \tag{2.104}$$

Looking back at (2.90) and having in mind to normalize with respect to the partition function (hence simplifying f_0), it is clear that f_1 fully encodes the semiclassical contribution, which is:

$$G_2 = \left(\frac{\cos^2 u}{\cos^2 \phi} \right)^{\frac{\Delta}{\lambda}} \tag{2.105}$$

In order to extract the one–loop contribution, beside knowing I, it is necessary to compute the one–loop determinant $(F_{ij} = \frac{\partial^2 F}{\partial \theta_i \partial \theta_j})^{-1/2}$, as done in [89]. It turns out that this contributes as $\Delta \mathcal{A}$ in the action, where:

$$\begin{aligned} \mathcal{A} = \frac{1}{4(1 + u \tan u)} \Bigg[&- \frac{(1 + u \tan u)^2}{\cos^2 \phi} + \\ &+ \frac{(1 + \phi \tan \phi)^2}{\cos^2 u} + \phi^2 (\tan^2 u - \tan^2 \phi) - \frac{1 + \phi \tan \phi}{1 + u \tan u} + 1 \Bigg]. \end{aligned} \tag{2.106}$$

As a consequence the one–loop two-point correlator is:

$$\begin{aligned} G_2 &= \left(\frac{\cos^2 u}{\cos^2 \phi} \right)^{\frac{\Delta}{\lambda}} \left(1 + \frac{\Delta^2}{2\lambda} I + \Delta \mathcal{A} \right) \\ &\approx e^{\frac{\Delta^2}{2\lambda} I} \left[\frac{\cos^2 u}{\cos^2 \phi} (1 + \lambda \mathcal{A}) \right]^{\frac{\Delta}{\lambda}}. \end{aligned} \tag{2.107}$$

Moreover, it is useful to notice that $\mathcal{A}$ and I are related by a differential equation in ϕ:

$$\left(\partial_\phi^2 - \frac{2}{\cos^2 \phi} \right) \mathcal{A} = \frac{1}{\cos^2 \phi} I. \tag{2.108}$$

These results will be relevant in the discussion of the gravitational dualities of the theory.

Non-Monotonic Entropy. In this paragraph we want to discuss the thermodynamical properties of DSSYK, which are a key feature we want to reproduce in the bulk model. As we know, the theory is naturally equipped with a Planck scale $|\log q|$ that controls the semiclassical limit. In order to look at the thermal properties, we can simply evaluate the partition function on the classical saddle, and write it as a canonical system:

$$Z_{\text{DSSYK}}(\beta) \stackrel{\text{class}}{=} \int_0^\pi \mathrm{d}\cos(\theta) \exp\left(\frac{\pi\theta - \theta^2}{|\log q|} + \beta \frac{\cos(\theta)}{2|\log q|}\right) \tag{2.109}$$

It is straightforward to recognize that the energy and the entropy of the model take the following form (parametrized as a function of θ):

$$E(\theta) = \frac{\cos(\theta)}{2|\log q|} \quad , \quad S(\theta) = \frac{\pi\theta - \theta^2}{|\log q|}. \tag{2.110}$$

By definition, we can also extract the temperature (at fixed θ):

$$\beta(\theta) = \frac{dS/d\theta}{dE/d\theta} = \frac{2\pi - 4\theta}{\sin(\theta)} \tag{2.111}$$

The relevant property of (2.110) is that the theory is non-monotonic, signaling a system with bounded energy spectrum. Being DSSYK in the low energy regime equivalent to Schwarzian QM and in turn to JT gravity, we expect the system to have a gravitational interpretation. However such a system would be rather peculiar, not exhibiting the standard linear dependence of the entropy with respect to its energy, as shown in Figure (2.2). There is still much to understand about gravity, when trying to reproduce a UV-complete system from the point of view of the bulk theory.

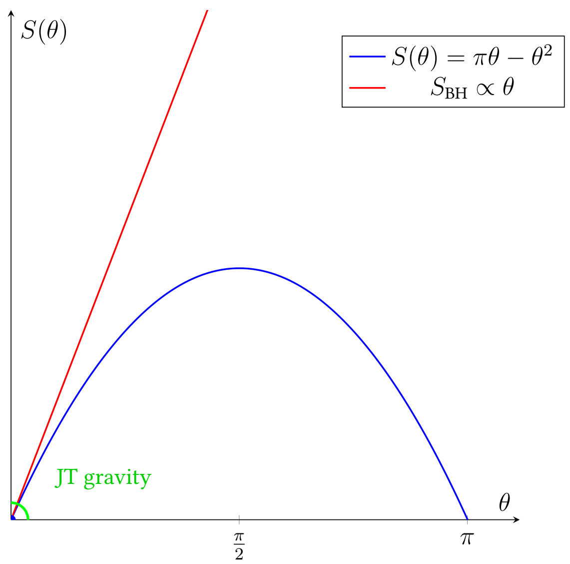


Figure 2.2: Comparison of the entropy of the DSSYK model and the one of a standard Black Hole of energy θ, obeying the Bekenstein-Hawking law.

2.1.8 The Bulk Hilbert Space

It is well known that the AdS/CFT correspondence can be understood either as an identification of partition functions or as an identification of the Hilbert spaces of two dual models. The SYK model is known to admit a holographic correspondence with JT gravity, at least in a certain regime, which is most transparent at the level of the path integral through the appearance of the Schwarzian theory. However, a correspondence has not been established at the level of the Hilbert space. In fact, a Hilbert space formulation of SYK has not been identified, and the $G\Sigma$ formulation does not help in this respect, since it involves bilocal fields which are not straightforward to split open.

The situation becomes more tractable in the double-scaling limit. As we shall see, there exists a web of on-shell dualities connecting DSSYK to sine-dilaton gravity, thereby providing a candidate path-integral formulation of holographic duality. This line of research has also been stimulated by the work of Lin [138], who explicitly aimed at identifying the bulk Hilbert space of DSSYK. The key observation is to interpret chord diagrams not merely as a combinatorial tool, but rather as the natural framework where a notion of Hilbert space emerges. The fundamental idea is that chord diagrams can be regarded as a two-sided wormhole, obtained by gluing together two half-diagrams (each representing the bra and ket of a thermofield double state).

In particular, we shall see that in the theory without matter insertions the Hilbert space is spanned by eigenfunctions of the length ℓ, which becomes discretized. This length is directly related to the chord number, which we interpret here somewhat differently than in the approach of Berkooz, who considered it from the thermal circle perspective. The Hilbert space formulation can be also extended to the matter theory, and the resulting structure resembles a lattice field theory characterized by discretized lengths. This is particularly interesting from the perspective of quantum gravity, since the gravitational dual should exhibits the emergence of a minimal length, which can be interpreted as a Planck scale.

We now want to analyze in detail how the two-sided chord Hilbert space is constructed in the absence of matter insertions. The guiding idea comes from the canonical quantization of JT gravity proposed by Harlow and Jafferis [146], in which the Hilbert space has a phase space spanned by the geodesic length of the two-sided wormhole and its conjugate momentum (see Appendix (A.2)). What emerges from that formulation is the Liouville Hamiltonian, which plays the role of connecting the two sides of the wormhole through Euclidean time evolution.

In the case of DSSYK, we would like to interpret a chord diagram as a two-sided wormhole. The first observation is that such an object can consistently be cut into two halves (the left and right states), along some line connecting two-points of the thermal circle.

We interpret the chord number differently than in Berkooz's approach [63, 64]: here it is defined as the number of chords intersecting the separation line (see Fig. 2.3), avoiding multiple intersections. In this discretized two-dimensional space, the chord number is

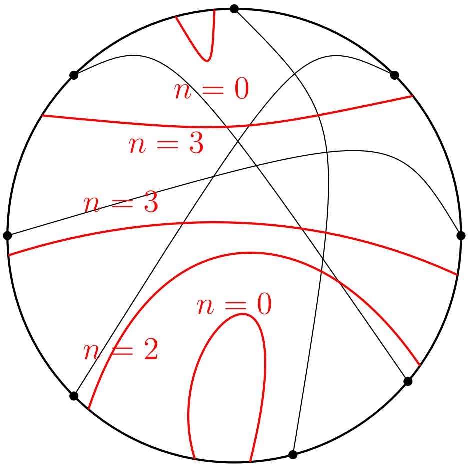


Figure 2.3: Chord diagram with the insertion of slicings. Red lines indicate slicing lines, which intersect each chord exactly once; this allows for well-defined chord number observable.

interpreted as a bulk observable. If we continuously evolve the slicing line and declare at each time the chord number, we obtain a complete description of the chord diagram. There is always a slicing choice such that in the far past and the far future the line intersects no chord. Taking such a slicing, we can define the vacuum states $|0\rangle$, which correspond to slices without any open chords. In this picture, the transfer matrix is the operator that creates or annihilates open chords. More precisely, an open chord can be created in only one way, but it can be annihilated in several ways (since, in general, an open chord can close with different others). One can write down a transfer matrix which is completely analogous to the one of Berkooz, although with a different interpretation and a different normalization convention:

$$T = \tfrac{1}{\sqrt{\lambda}}\left(\alpha^\dagger + \alpha W\right)$$

$$W_n = \mathfrak{q}^0 + \mathfrak{q}^1 + \cdots + \mathfrak{q}^{n-1} = \tfrac{1-\mathfrak{q}^n}{1-\mathfrak{q}}, \tag{2.112}$$

$$\alpha = a\sqrt{\tfrac{1}{n}}, \quad \alpha^\dagger = \sqrt{\tfrac{1}{n}}a^\dagger$$

As before, a and $a^\dagger$ acts on the space of chord diagrams as ladder operators. Once again, we can perform a change of basis in order to diagonalize the transfer matrix, leading to:

$$H = -\frac{1}{\sqrt{\lambda(1-\mathfrak{q})}}\left[e^{i\lambda k}\sqrt{1-e^{-\ell}} + \sqrt{1-e^{-\ell}}e^{-i\lambda k}\right] \tag{2.113}$$

$$\ell = \lambda n. \tag{2.114}$$

Which at this stage is interpreted as a generalization of the Liouville Hamiltonian. The length ℓ is then viewed as the geodesic distance between the two sides of the wormhole. Since it is in direct correspondence with the chord number, it is a discretized variable. The final step in order to provide a well-defined Hilbert space (in addition to specifying the states and the Hamiltonian) is to introduce an inner product. To this end, still at the level of diagrams, one can imagine decomposing a given chord diagram into three regions, separated by two slicing lines as shown in Fig. (2.4). In this way we naturally

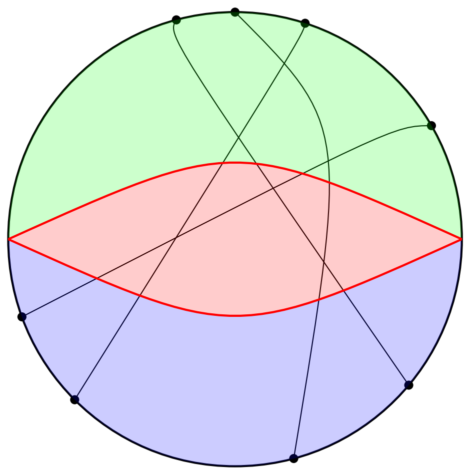


Figure 2.4: Chord diagram with inner product. The upper (green) and lower (blue) regions correspond to the left and right states, while the red shaded middle region represents the intermediate state, playing the role of the inner product. The chords should cross the red line representing the slicing only once.

identify a left state, a right state, and an intermediate state connecting them, which plays the role of the inner product. From a computational perspective, a chord diagram with $k = a + b$ nodes can be represented through the transfer matrix T^k. By factorizing the transfer matrix into two pieces and inserting two completeness relations in between, we arrive at the natural definition of inner product:

$$g^{mn} := \langle m|n\rangle \,. \tag{2.115}$$

Indeed:

$$\begin{aligned}
\langle 0| \, T^k \, |0\rangle &= \langle 0| \, T^{a+b} \, |0\rangle && (2.116)\\
&= \sum_n \langle 0| \, T^a \, |n\rangle \, \langle n| \, T^b \, |0\rangle && (2.117)\\
&= \sum_{n,m} \langle 0| \, T^a \, |n\rangle \, \langle n|m\rangle \, \langle m| \, T^b \, |0\rangle && (2.118)\\
&= T^a_{0,n} \, g^{nm} \, T^b_{m,0}. && (2.119)
\end{aligned}$$

Since every chord entering the middle region must also exit, g^{mn} is necessarily orthogonal, namely:

$$\langle n|m\rangle = S_n \, \delta_{n,m}.$$

Moreover, up to a multiplicative weight factor, we can reduce by one the number of chords inside the middle region, so that:

$$\langle m|n\rangle = W_n \, \langle m-1|n-1\rangle \, . \tag{2.120}$$

Combining the two relations we obtain a recursion relation:

$$S_n = S_{n-1} \, W_n,$$

which can be solved together with the normalization condition $S_0 = 1$. The Hilbert space formalism described above can be extended to DSSYK in the presence of matter insertions. However, such a discussion is not essential for our purposes here. With the above evidence motivating a path-integral formulation of the bulk DSSYK dual, we now proceed to review the main results along this line of investigation.

2.2 Particle on $\mathrm{SU}_q(1,1)$: Representations

Motivation and strategy. This section is mainly based on [79], in which the purpose is to build a bulk description using the information on the observables of DSSYK. The searched theory is dynamical, meaning that it admits a path integral representation. The more direct way is to use the intuition on the amplitudes in Sections (2.1.4) (2.1.6) and to construct a quantum mechanics on a group, which acts as a bridge to the dynamical description. This step is inspired by the parallel duality between JT gravity and the particle on $\mathrm{SL}(2,\mathbb{R})$. As a by-product, the representation theory has led to the formulation in terms of the q-Schwarzian and q-Liouville path integrals. The task of studying the proper gravitational dual is left to following sections.

It was expected that the representation theory of DSSYK is somehow related to quantum groups, simply because amplitudes contain q-Hermite polynomials [147] and in general q-deformed functions. A natural candidate is the quantum group $\mathrm{SU}_q(1,1)$, which reduces to $\mathrm{SL}(2,\mathbb{R})$ for $q \to 1^-$.
Building on this intuition, we want to systematically construct a representation theory for such a group. The strategy is well established in the context of quantum group theory. Firstly, one studies the Gauss decomposition, which allows to obtain a global group description starting form the infinitesimal algebra formulation. Then, extract the Hamiltonian of the system as the q-Casimir operator and write a Schrödinger-like equation, which is solved by proper representation matrices. Standard representation theory allows to write the Haar measure and naturally define an inner product. As typical in DSSYK the Hamiltonian is non-Hermitian, and attention is needed when treating left and right eigenfunctions.

2.2.1 Gauss decomposition and algebra

When studying Lie groups, a key step is to exponentiate the algebra to a group. In the case of quantum groups, the algebra is non-commutative and it is important to rigorously define the parametrization of a group element g in therms of the coordinates on the quantum group. Fortunately the $U_q(\mathfrak{su}(1,1))$ algebra is well-known:

$$[H,E] = E\,, \quad [H,F] = -F\,, \quad [E,F] = \frac{q^{2H} - q^{-2H}}{q - q^{-1}}\,. \tag{2.121}$$

If we introduce the coordinates γ, ϕ, β , a generic element of the quantum group can be written in therms of the generators of the algebra, in the following way [148]:

$$g = e^{\gamma F}_{q^{-2}}\, e^{2\phi H}\, e^{\beta E}_{q^2}\,, \quad 0 < q < 1 \tag{2.122}$$

where we use the definition of q-exponential:

$$e^x_q = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}\,, \quad [n]_q = \frac{1-q^n}{1-q} \tag{2.123}$$

and the coordinates form a non-commutative space, encoded in the algebra:

$$[\gamma, \phi] = -\log q\,\gamma\,, \quad [\beta,\phi] = -\log q\,\beta\,, \quad [\beta,\gamma] = 0. \tag{2.124}$$

It is then trivial to find a Casimir operator, i.e. an operator commuting with all the generators. In the case of $\mathrm{SU}_q(1,1)$, the Casimir is:

$$C = \frac{q^{2H+1} + q^{-2H-1}}{(q-q^{-1})^2} + FE \tag{2.125}$$

The previous can be taken as the operatorial definition of the Hamiltonian of the quantum mechanics on the quantum group manifold $\mathrm{SU}_q(1,1)$.

2.2.2 Differential representation

In order to write down a Schrödinger equation starting from the Hamiltonian, it is required to know a differential representation of the generators[12]. We proceed to derive such objects, which we denote with L_A, for $A \in \{E, F, H\}$. Generally, these representations should act infinitesimally on functions of the group manifold:

$$L_A \cdot f(g) = f(-X_A\, g). \tag{2.126}$$

Being the variables non-commutative, from now on, we choose the specific ordering $\{\gamma, \phi, \beta\}$, and we expand the function on the manifold as:

$$f(\gamma, \phi, \beta) = \sum_{n,m,p} f_{nmp}\, \gamma^n \phi^m \beta^p\,. \tag{2.127}$$

As a result, the action of the representation of F on g, written in Gauss decomposition, is:

$$L_F \cdot g = L_F\, e_{q^{-2}}^{\gamma F}\, e^{2\phi H}\, e_{q^2}^{\beta E} = -F\, e_{q^{-2}}^{\gamma F}\, e^{2\phi H}\, e_{q^2}^{\beta E}\,. \tag{2.128}$$

But knowing the standard definition of q-derivative in the literature, one can prove that its action on a q-exponential is naturally:

$$\left(\frac{d}{d\gamma}\right)_{q^{-2}} e_{q^{-2}}^{\gamma F} = F\, e_{q^{-2}}^{\gamma F}\,, \tag{2.129}$$

hence we recognize that:

$$L_F = -\left(\frac{d}{d\gamma}\right)_{q^{-2}}\,. \tag{2.130}$$

Using the commutation relation between generators (2.121) and the previous expression (2.130), it is possible to prove that the representation L_H does not depend on q and

[12] In the following, the generators are denoted as X_A.

therefore it coincides to the corresponding classical $\mathrm{SL}(2,\mathbb{R})$ representation:

$$L_H = -\frac{1}{2}\frac{d}{d\phi} + \gamma\frac{d}{d\gamma} \tag{2.131}$$

Similarly, using commutators of generators and variables, and introducing rescaling R^x_q and translation operators T^ϕ_a:

$$R^x_q f(x) := f(qx) = \exp\left(\log q\, x\frac{d}{dx}\right) f(x)\,, \tag{2.132}$$

$$T^\phi_a := \exp\left(a\,\frac{d}{d\phi}\right)\,, \tag{2.133}$$

one can find the expression for L_E:

$$L_E = -e^{-2\phi} R^\gamma_{q^2}\left(\frac{d}{d\beta}\right)_{q^2} T^\phi_{-\log q} - \gamma\,\frac{T^\phi_{\log q} - R^\gamma_{q^2} T^\phi_{-\log q}}{q-q^{-1}}\,. \tag{2.134}$$

By construction this representation satisfies (2.121). Now, combining (2.125), (2.130), (2.131) and (2.134), the representation of the Casimir becomes:

$$L_C = \frac{qT^\phi_{\log q} + q^{-1}T^\phi_{-\log q}}{(q-q^{-1})^2} + \left(\frac{d}{d\gamma}\right)_{q^{-2}} e^{-2\phi} R^\gamma_{q^2}\left(\frac{d}{d\beta}\right)_{q^2} T^\phi_{-\log q}\,. \tag{2.135}$$

Actually, due to the fact that the variables on the quantum manifold do not commute, the previous representation of the Hamiltonian, in principle, only holds in the left-regular representation. We should also consider the right-regular representation of the operators, namely:

$$R_A \cdot f(g) = f(g\, X_A)\,. \tag{2.136}$$

However, it is possible to repeat a similar derivation and show that the representation of the Casimir is the same, i.e.: $L_C = R_C$. Moreover, it is possible to show that left and right representations do commute $[L_A, R_B] = 0$.

2.2.3 Irreducible representations

We have chosen a representation of the group SU$q(1,1)$ which has left-right symmetry and whose Hamiltonian coincides with Casimir. As usual, the spectrum is organized in irreducible representations of SU$q(1,1)$.

The proof is the following: we can simultaneously diagonalize the Hamiltonian, one left operator and a right one. We label the corresponding eigenstates as $|\theta,\, \mu_1,\, \mu_2\rangle$, for which, by definition:

$$\mathbf{H}\,|\theta,\ \mu_1\mu_2\rangle = E(\theta)\,|\theta,\ \mu_1\mu_2\rangle\,, \tag{2.137}$$

$$L_A\,|\theta,\ \mu_1\mu_2\rangle = -\mu_1\,|\theta,\ \mu_1\mu_2\rangle\,, \tag{2.138}$$

$$R_B\,|\theta,\ \mu_1\mu_2\rangle = \mu_2\,|\theta,\ \mu_1\mu_2\rangle\,. \tag{2.139}$$

We can look at the representation on $L^2(G)$ of the eigenvalue problem, by defining $f^\theta_{\mu_1\mu_2}(g) := \langle g|\theta\,\mu_1\mu_2\rangle$ and we obtain:

$$\mathbf{H}\cdot f^\theta_{\mu_1\mu_2}(g) = E(\theta)\,f^\theta_{\mu_1\mu_2}(g), \tag{2.140}$$

$$L_A\cdot f^\theta_{\mu_1\mu_2}(g) = -\mu_1\,f^\theta_{\mu_1\mu_2}(g), \tag{2.141}$$

$$R_B\cdot f^\theta_{\mu_1\mu_2}(g) = \mu_2\,f^\theta_{\mu_1\mu_2}(g). \tag{2.142}$$

We claim that the representation matrix elements of SU$q(1,1)$:

$$R_{\theta\,\mu_1\mu_2}(g) := \langle\theta\,\mu_1|\,g\,|\theta\,\mu_2\rangle\ , \tag{2.143}$$

are the solutions of the previous equations, i.e.:

$$f^\theta_{\mu_1\mu_2}(g) = R_{\theta,\,\mu_1\mu_2}(g) \tag{2.144}$$

Indeed, for the left generator:

$$\begin{aligned}
(L_A\cdot f^\theta_{\mu_1\mu_2})(g) &= f^\theta_{\mu_1\mu_2}(-X_A g)\\
&= \langle\theta,\mu_1|\,(-X_A g)\,|\theta,\mu_2\rangle\\
&= -\,\langle\theta,\mu_1|\,X_A\,|\theta,\mu'\rangle\,\langle\theta,\mu'|\,g\,|\theta,\mu_2\rangle\\
&= -\mu_1\,\langle\theta,\mu_1|\theta,\mu'\rangle\,\langle\theta,\mu'|\,g\,|\theta,\mu_2\rangle\\
&= -\mu_1\,\langle\theta,\mu_1|\,g\,|\theta,\mu_2\rangle\\
&= -\mu_1 f^\theta_{\mu_1\mu_2}(g),
\end{aligned}$$

Similarly, for the right one:

$$\begin{aligned}
(R_B\cdot f^\theta_{\mu_1\mu_2})(g) &= f^\theta_{\mu_1\mu_2}(gX_B)\\
&= \langle\theta,\mu_1|\,gX_B\,|\theta,\mu_2\rangle\\
&= \langle\theta,\mu_1|\,g\,|\theta,\mu'\rangle\,\langle\theta,\mu'|\,X_B\,|\theta,\mu_2\rangle\\
&= \mu_2 f^\theta_{\mu_1\mu_2}(g),
\end{aligned}$$

Finally, for the Hamiltonian operator:

$$\begin{aligned}
\mathbf{H} \cdot f^{\theta}_{\mu_1\mu_2}(g) &= \langle \theta, \mu_1 | \, L_C g \, | \theta, \mu_2 \rangle \\
&= \langle \theta, \mu_1 | \, g R_C \, | \theta, \mu_2 \rangle \\
&= \langle \theta, \mu_1 | \, g \, | \theta, \mu_2 \rangle \, E(\theta) \\
&= E(\theta) f^{\theta}_{\mu_1\mu_2}(g).
\end{aligned}$$

The previous result will be useful in the next section, in which boundary conditions are imposed.

2.2.4 Boundary conditions

At this stage, we could interpret the Hamiltonian as a dynamical action, and we would get a 6-fields path integral, which is the q-deformation of the bulk unconstrained action of JT gravity[13]. Recalling the JT-gravity picture, it is known that the (undeformed) unconstrained dynamical system coming from the BF description of JT gravity reduces to the Liouville path integral, once two generators are gauge-fixed [14]. On the other hand, one could also get the more standard Schwarzian path integral, by gauge-fixing only one generator[15].

Here, we have in mind to find the q-deformations of such two theories, by imposing analogue boundary conditions to the particle on a group. At this stage, the boundary conditions that reproduce DSSYK are not yet known, nor is it clear whether they admit a holographic interpretation. The authors propose a possible choice; however, it lacks a derivation from first principles.

Clearly, the boundary conditions should match the undeformed constraints of JT gravity (when sending $q \to 1^-$), which are:

$$L_F = -ip_\gamma , \quad R_E = i \, p_\beta \, . \tag{2.145}$$

Moreover, the authors of [79] are inspired by the representation theory of the modular double $U_q(\mathfrak{sl}(2,\mathbb{R})) \otimes U_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$, derived in the context of $2d$ Liouville gravity [149].

[13] See for instance eq. (2.9) in [79].

[14] In the gravitational picture we are imposing asymptotically AdS boundary condition in a two-sided geometry.

[15] Fixing AdS boundary conditions in a one-sided geometry.

They claim that the generalization of (2.145) is:

$$L_F = i\frac{q^{1/2}}{q-q^{-1}}q^{L_H}\,, \quad R_E = -i\frac{q^{1/2}}{q-q^{-1}}q^{-R_H}\,. \tag{2.146}$$

Although Cartan elements appear in the RHS, the two constraints do commute one with the other and with the Hamiltonian, hence the choice is perfectly legitimate. We can chose a representation (2.137), (2.138), (2.139), in which L_F and R_E are diagonal. As a result, if also the constraint (2.146) holds, $\langle\mu_1|\, q^H$ and $\langle\mu_1|$ are both eigenvectors of F; similarly $q^H\,|\mu_2\rangle$ and $|\mu_2\rangle$ are both eigenvectors of E. As a result, we can consider identical labels $\mu_1 = \mu_2 = \mathrm{i}$ and a particular representation:

$$f^\theta_{\mathrm{ii}}(g) = R_{\theta,\mathrm{ii}}(g) = \langle\theta\,\mathrm{i}|\, g\,|\theta\,\mathrm{i}\rangle \tag{2.147}$$

In this representation, the constraints are:

$$E\,|\mathrm{i}\rangle = -\mathrm{i}\frac{q^{1/2}}{q-q^{-1}}\,q^{-H}\,|\mathrm{i}\rangle\,, \quad \langle\mathrm{i}|\,F = -\mathrm{i}\frac{q^{1/2}}{q-q^{-1}}\,\langle\mathrm{i}|\,q^{-H}\,. \tag{2.148}$$

It is possible to show that (2.148) is invariant under the action of the Cartans $U_L = q^{\alpha H}$ and $U_R = q^{\beta H}$. As a consequence, we can simultaneously act with the gauge transformation:

$$g \to g' = q^{\alpha H}\cdot g\cdot q^{\beta H}, \tag{2.149}$$

which, at the level of the Gauss decomposition acts only on the coordinates γ and β. Gauge invariance allows to fix $\gamma = \beta = 0$ and as a result, the reduced system only depends on the variable ϕ. The gauge fixed matrix element is simply:

$$\langle\phi|\theta\rangle \equiv R_{\theta\,\mathrm{ii}}(e^{2\phi H}) = \langle\theta\,\mathrm{i}|\,e^{2\phi H}\,|\theta\,\mathrm{i}\rangle\,. \tag{2.150}$$

Moreover, we can interpret this gauge fixed space as the physical one, whose Hamiltonian is:

$$(q-q^{-1})^2\mathbf{H} = qT^\phi_{\log q} + (q^{-1} - qe^{-2\phi})T^\phi_{-\log q} \tag{2.151}$$

This system is a quantum deformation of the Liouville formulation of the Schwarzian. Remarkably, the Schrödinger equation reduces to a difference equation:

$$q\,\langle\phi + \log q|\theta\rangle + (q^{-1} - qe^{-2\phi})\,\langle\phi - \log q|\theta\rangle = (q-q^{-1})^2 E(\theta)\,\langle\phi|\theta\rangle\,. \tag{2.152}$$

It is also possible to impose only one constraint, there will be a gauge degree of freedom left and the resulting dynamical action will be the deformation of the Schwarzian quantum mechanics, as we will see.

Degeneracy. A problem arises when trying to naively solve the eigenvector problem (2.152), indeed for each value of $E(\theta)$, if $\langle\phi|\theta\rangle$ is a solution, also $\langle\phi|\theta\rangle\, f_p(\phi)$ is a solution, where $f_p(\phi)$ is a generic function of periodicity $\log(q)$. This seems to induce a degeneracy in the spectrum.
The workaround consists in looking at the Haar measure of SU$q(1,1)$ and noticing that it samples the eigenfunctions in a discrete set of points $\phi = -n\log q$, in such a way that multiplying the eigenfunction by a periodic function amounts to adding a null state. As we will see, the variable $\langle\phi|$ is by all means discretized to the physical Hilbert space coordinate $\langle n|$. This is a fundamental ingredient when trying to match DSSYK amplitudes, indeed, if we parametrize the energy as:

$$E(\theta) = \frac{1}{(q-1/q)^2} 2\cos(\theta) \tag{2.153}$$

and fix the initial condition to:

$$\langle 0|\theta\rangle = 1 \tag{2.154}$$

we get the following difference equation (for discrete and positive n):

$$q\,\langle n-1|\theta\rangle + (q^{-1} - q^{2n+1})\,\langle n+1|\theta\rangle = 2\cos(\theta)\,\langle n|\theta\rangle\ . \tag{2.155}$$

Having in mind the recursion relation for q-Hermite polynomials H_n, it is simple to show that the solution is:

$$\langle n|\theta\rangle = R_{\theta\,\mathrm{ii}}(n) = \frac{q^n}{(q^2;q^2)_n} H_n(\cos(\theta)|q^2) \tag{2.156}$$

We recognize that the previous are exactly the eigenfunctions of the transfer matrix in DSSYK. Moreover, the constrained Hamiltonian of the dynamical system reduces to the transfer matrix (up to non physical normalization factors).

2.2.5 Haar measure and inner products

Now we look in detail on the choice of Haar measure. The authors propose a definition which acts on a generic function of the coordinates in the Gauss decomposition by reducing the integral to a discrete sum of integrals (in this way, the integral is sampled at discrete points). Moreover it implements the roughest sampling which admits a solution to the difference equation. As mentioned, choosing a sampling means avoiding infinitely dense Hilbert spaces. The integration prescription on the Gauss decomposition is:

$$\int (\mathrm{d}g)_q = \int_0^\infty (\mathrm{d}\gamma)_{q^{-2}} \int_0^\infty (\mathrm{d}e^{2\phi})_{q^2} \int_0^\infty (\mathrm{d}\beta)_{q^2} . \tag{2.157}$$

where we introduced the notion of Jackson integral:

$$\int_0^\infty (\mathrm{d}x)_a\, f(x) := (1-a) \sum_{n=-\infty}^{+\infty} a^{-n}\, f(a^{-n}) \,, \tag{2.158}$$

This definition reduces to the one of $\mathrm{SL}(2,\mathbb{R})$ in the undeformed limit. Moreover, it is possible to show that it preserves left and right invariance[16]. Finally, we can naturally define an inner product:

$$\langle \psi_1 | \psi_2 \rangle = \int (\mathrm{d}g)_q\, \langle \psi_1 | g \rangle \langle g | \psi_2 \rangle \ , \tag{2.159}$$

which in the constrained system becomes:

$$\langle \theta_1 | \theta_2 \rangle = \int_0^\infty (de^{2\phi})_{q^2}\, \langle \theta_1 | \phi \rangle \langle \phi | \theta_2 \rangle = \sum_{n=-\infty}^{+\infty} q^{-2n} \langle \theta_1 | n \rangle \langle n | \theta_2 \rangle \ . \tag{2.160}$$

As we can see, the Jacskon integral naturally evaluates the states at discrete points. The previous expression will be useful in the next section, when computing amplitudes.

2.2.6 Amplitudes

Partition function. Since the Hamiltonian is non-Hermitian, we need to keep in mind that left and right eigenvectors are not complex conjugate one with the other:

$$\langle n | \theta \rangle \neq \langle \theta | n \rangle^* \ . \tag{2.161}$$

[16] It is needed to use the rescaling property of the Jackson integral and the fact that this object is the inverse of a q-derivative.

Having the right Schrödinger equation and the notion of inner product, we can derive the left equation and its corresponding eigenfunction. The computation is the following:

$$
\begin{aligned}
\langle\theta_1|\,\mathbf{H}\,|\theta_2\rangle &= \sum_{n=-\infty}^{+\infty} q^{-2n+1}\,\langle\theta_1|n\rangle\,\langle n-1|\theta_2\rangle + \sum_{n=-\infty}^{+\infty}\left(q^{-2n-1}-q\right)\langle\theta_1|n\rangle\,\langle n+1|\theta_2\rangle \\
&= \sum_{n=-\infty}^{+\infty} q^{-2n-1}\,\langle\theta_1|n+1\rangle\,\langle n|\theta_2\rangle + \sum_{n=-\infty}^{+\infty}\left(q^{-2n+1}-q\right)\langle\theta_1|n-1\rangle\,\langle n|\theta_2\rangle \\
&= 2\cos(\theta_1)\sum_{n=-\infty}^{+\infty} q^{-2n}\,\langle\theta_1|n\rangle\,\langle n|\theta_2\rangle
\end{aligned}
\tag{2.162}
$$

Where we introduced a completion and used the action of the Hamiltonian on a right state. As a result, the left Schrödinger equation and its solutions are:

$$
q^{-1}\,\langle\theta_1|n+1\rangle + q\left(1-q^{2n}\right)\langle\theta_1|n-1\rangle = 2\cos(\theta_1)\,\langle\theta_1|n\rangle \tag{2.163}
$$

$$
\langle\theta|n\rangle = q^n H_n\left(\cos(\theta)\mid q^2\right) \tag{2.164}
$$

Which are slightly different from (2.155) and (2.156). Now it is possible to show that states at different energies $E(\theta_1)$ and $E(\theta_2)$ are orthogonal:

$$
\begin{aligned}
\langle\theta_1|\theta_2\rangle &= \sum_{n=0}^{+\infty} q^{-2n}\,\langle\theta_1|n\rangle\,\langle n|\theta_2\rangle \\
&= \sum_{n=0}^{+\infty}\frac{1}{(q^2;q^2)_n}\,H_n\Big(\cos(\theta_1)\mid q^2\Big)\,H_n\Big(\cos(\theta_2)\mid q^2\Big) \\
&= \frac{\delta(\theta_1-\theta_2)}{(e^{\pm 2\mathrm{i}\theta_1};q^2)_\infty}\,.
\end{aligned}
\tag{2.165}
$$

We recognize that the same spectral density of DSSYK appears in the denominator:

$$
\rho(\theta) = (e^{\pm 2\mathrm{i}\theta};q^2)_\infty\,. \tag{2.166}
$$

Moreover the constrained partition function can be computed in a similar fashion, matching the expected result[17]:

$$
\langle 0|\,e^{-\beta H}\,|0\rangle = \int_0^\pi \mathrm{d}\theta\;\rho(\theta)\,e^{-\beta\frac{2\cos(\theta)}{(q-1/q)^2}}\,\langle 0|\theta\rangle\,\langle\theta|0\rangle = \int_0^\pi \mathrm{d}\theta\;\rho(\theta)\,e^{-\beta\frac{2\cos(\theta)}{(q-1/q)^2}}\,. \tag{2.167}
$$

Two-point functions The missing step is to find the counterpart of the two-point functions of DSSYK. If we impose boundary conditions in the quantum mechanics of $SUq(1,1)$ we must require that an operator in the theory also preserve the constraints.

[17] As we can see, the chosen initial condition $\langle 0|\theta\rangle = 1$is required to match the normalization of the partition function in DSSYK.

As a consequence, such an operator is not charged under the constrained currents:

$$-q^{-L_H} L_F\,, \quad q^{R_H} R_E\,. \tag{2.168}$$

And it can be given in a representation that preserves the Casimir eigenvalue and is invariant under the currents:

$$\mathcal{O}_\Delta = \langle \Delta\, 0 |\, g\, | \Delta\, 0 \rangle = R_{\Delta\, 00}(g)\,. \tag{2.169}$$

This operator should satisfy a constrained Casimir equation, which drastically simplifies[18]:

$$(q T^\phi_{\log q} + q^{-1} T^\phi_{-\log q}) \mathcal{O}_\Delta = 2\cosh(2\log q(\Delta - 1/2)) \mathcal{O}_\Delta\,. \tag{2.170}$$

Clearly, here g reduces to the only non-trivial variable ϕ. Moreover, we can solve the equation keeping in mind the discrete sampling $\phi = -n\log q$, and find that the eigenfunction is:

$$f(\phi) = q^{2n\Delta}, \tag{2.171}$$

which coincides with the matter operator in DSSYK.
The result allows to compute the gravitational two-point function, by inserting two completeness relations in the energy states:

$$\begin{aligned} &\langle 0 |\, e^{-\beta_1 H} \mathcal{O}_\Delta e^{-\beta_2 H}\, | 0 \rangle = \\ &= \int_0^\pi \mathrm{d}\theta_1\, \rho(\theta_1)\, e^{-\beta_1 \frac{1}{(q-1/q)^2} 2\cos(\theta_1)} \int_0^\pi \mathrm{d}\theta_2\, \rho(\theta_2)\, e^{-\beta_2 \frac{1}{(q-1/q)^2} 2\cos(\theta_2)} \langle \theta_1 |\, \mathcal{O}_\Delta\, | \theta_2 \rangle \end{aligned} \tag{2.172}$$

Given (2.171), a generic matrix element between energy eigenstates θ_1 and θ_2 results:

$$\begin{aligned} \langle \theta_1 |\, \mathcal{O}_\Delta\, | \theta_2 \rangle &= \sum_{n=0}^{+\infty} q^{-2n} \langle \theta_1 |\, q^{2n\Delta}\, | n \rangle \langle n | \theta_2 \rangle \\ &= \sum_{n=0}^{+\infty} \frac{q^{2n\Delta}}{(q^2;q^2)_n} H_n(\cos(\theta_1)|q^2) H_n(\cos(\theta_2)|q^2) \\ &= \frac{(q^{4\Delta};q^2)_\infty}{(q^{2\Delta \pm 2i\theta_1 \pm 2i\theta_2};q^2)_\infty}\,. \end{aligned} \tag{2.173}$$

Inserting this expression in (2.171), we obtain perfect agreement with the DSSYK computation [63, 64, 138].
As a result, the authors of [79] point out that a theory dual to DSSYK is the (constrained) quantum mechanics of a particle on $SU_q(1,1)$. An important ingredient is a proper no-

[18]We can parametrize the Casimir eigenvalue by a parameter Δ, such that: $\lambda_C = \frac{1}{(q-1/q)^2}\, 2\cosh(2\log q(\Delta - 1/2))\,, \quad \Delta > 0\,.$

tion of Haar measure, which in this setup, implements the discretization of the Hilbert space. Having in mind the JT-gravity picture, this formulation is the analogue of describing JT gravity in terms of a particle on $\mathrm{SL}(2,\mathbb{R})$ manifold. We also know that this formulation allows the Schwarzian (or Liouville, depending on how many edges are considered) path integral to be derived. The authors of [79], therefore continue the search for a gravitational formulation of DSSYK building on the constrained particle-on-a-manifold representation.

2.3 Particle on $\mathrm{SU}_q(1,1)$: Dynamical actions

2.3.1 Towards a commutative geometry

For what follows, it is instructive to show how a canonical dynamical description emerges from the q-group picture. When dealing with quantum mechanics on non-commutative spaces it is useful to introduce a Darboux basis of canonical coordinates by means of a non-canonical variable change relating the two [150]. In this way, one can define a standard dynamical system on a set of commutative coordinates, whose dynamical action is then instrumental to build the path integral description.
Consider a classical 6d phase space with canonical coordinates $\{\gamma, p_\gamma, \varphi, p_\varphi, \beta, p_\beta\}$, equipped with the symplectic form:

$$\omega = d\gamma \wedge dp_\gamma + d\varphi \wedge dp_\varphi + d\beta \wedge dp_\beta. \tag{2.174}$$

Clearly the canonical Poisson brackets derived from (2.174) are given by:

$$\{\gamma, p_\gamma\} = \{\varphi, p_\varphi\} = \{\beta, p_\beta\} = 1, \tag{2.175}$$

with all others vanishing. Upon quantization, these turn into commutation relations:

$$[\gamma, p_\gamma] = [\varphi, p_\varphi] = [\beta, p_\beta] = i\hbar, \tag{2.176}$$

We now introduce a non-canonical coordinate change defined by:

$$\phi := \varphi + i \log q\,(\gamma p_\gamma + \beta p_\beta), \quad p_\phi := p_\varphi, \tag{2.177}$$

while leaving the other variables unchanged. The claim is that the algebra of Poisson brackets (or commutators) is exactly the one of classical (or quantum) $\mathrm{SU}_q(1,1)$. Indeed, the (non-symplectic) form becomes:

$$\omega = d\gamma \wedge dp_\gamma + d\beta \wedge dp_\beta + d\phi \wedge dp_\phi - i \log q \, d(\gamma p_\gamma + \beta p_\beta) \wedge dp_\phi, \tag{2.178}$$

where the additional term implies that ϕ alone is not canonical. In particular, the non-vanishing Poisson brackets involving ϕ are explicitly:

$$\begin{aligned} \{\phi, \gamma\} &= -i \log q \, \gamma, \qquad \{\phi, p_\gamma\} = i \log q \, p_\gamma, \\ \{\phi, \beta\} &= -i \log q \, \beta, \qquad \{\phi, p_\beta\} = i \log q \, p_\beta. \end{aligned} \tag{2.179}$$

Accordingly, quantization leads to non-trivial commutators

$$\begin{aligned} [\phi, \gamma] &= \log q \, \gamma, \qquad [\phi, p_\gamma] = -\log q \, p_\gamma, \\ [\phi, \beta] &= \log q \, \beta, \qquad [\phi, p_\beta] = -\log q \, p_\beta. \end{aligned} \tag{2.180}$$

While φ is a canonical coordinate commuting with γ, β and their conjugate momenta, the new coordinate ϕ is non-canonical and generates a fundamental non-commutative structure, reflecting the quantum deformation induced by the parameter q. Notice that (2.180) contains the quantum $\mathrm{SU}_q(1,1)$ algebra mentioned in (2.124).
As a first step, we want to extract the classical limit of the Casimir operator (i.e. the putative classical Hamiltonian) and then express it in Darboux coordinates. Finally, benefiting from the symplectic structure, we can perform a Legendre transform, obtain the classical Lagrangian and thus the action of the dynamical system. The (left) classical symmetry generators are obtained from the classical limit of (2.130), (2.134) and (2.131). These objects act on the non-commutative phase space as:

$$\begin{aligned} h_L &= \frac{1}{2} p_\phi - \gamma p_\gamma \\ f_L &= -\frac{e^{-2\mathrm{i} \log q \, \gamma p_\gamma} - 1}{2\mathrm{i} \log q \, \gamma} \\ e_L &= e^{-2\phi} \, e^{i \log q \, (-p_\phi + 2\gamma p_\gamma)} \, \frac{e^{2\mathrm{i} \log q \, \beta p_\beta} - 1}{2\mathrm{i} \log q \, \beta} - \gamma \, e^{\mathrm{i} \log q \, p_\phi} \, \frac{e^{2\mathrm{i} \log q \, (\gamma p_\gamma - p_\phi)} - 1}{2\mathrm{i} \log q} \end{aligned} \tag{2.181}$$

and clearly they satisfy the classical version of the $U_q(\mathfrak{su}(1,1))$ algebra. The same story holds for the right generators. Having the generators, we can find the classical Hamiltonian that commutes with all the generators, which is also the classical limit of the q-Casimir:

$$H = -\frac{1}{4} \left[\frac{e^{i \log q \, p_\phi} + e^{-i \log q \, p_\phi}}{(\log q)^2} \quad + e^{-2\phi} \frac{e^{-i \log q \, p_\phi}}{(\log q)^2 \, \beta\gamma} \left(e^{2i \log q \, \beta p_\beta} - 1 \right) \left(e^{2i \log q \, \gamma p_\gamma} - 1 \right) \right] \tag{2.182}$$

If we then translate the Hamiltonian in Darboux coordinates using (2.177), we are allowed to perform the Legendre transform, obtaining the path-integral dynamical description:

$$\mathcal{Z} = \int \mathcal{D}\gamma\,\mathcal{D}\varphi\,\mathcal{D}\beta\,\mathcal{D}p_\gamma\,\mathcal{D}p_\varphi\,\mathcal{D}p_\beta\ \exp\Bigg[\mathrm{i}\int dt\ p_\varphi\varphi' + p_\beta\beta' + p_\gamma\gamma' + \frac{1}{4(\log q)^2}\left(e^{\mathrm{i}\log q\, p_\varphi} + \right.$$

$$\left. + e^{-\mathrm{i}\log q\, p_\varphi}\right) + \frac{1}{4(\log q)^2}e^{-2\varphi}e^{-\mathrm{i}\log q\, p_\varphi}\cdot\frac{1}{\beta\gamma}\left(e^{-2\mathrm{i}\log q\,\beta p_\beta} - 1\right)\left(e^{-2\mathrm{i}\log q\,\gamma p_\gamma} - 1\right)\Bigg] \tag{2.183}$$

2.3.2 q-Liouville theory

In order to obtain the dynamical action of the constrained two-sided geometry which generalizes the Liouville formulation of JT gravity, we impose the classical version of the constraints (2.146), which are:

$$\begin{aligned}\psi_L &= \frac{1}{\gamma}(e^{-3\mathrm{i}\log q\,\gamma p_\gamma} - e^{-\mathrm{i}\log q\,\gamma p_\gamma})\,e^{\mathrm{i}\log q\,p_\varphi/2} = \mathrm{i}\\ \psi_R &= \frac{1}{\beta}(e^{\mathrm{i}\log q\,\beta p_\beta} - e^{-\mathrm{i}\log q\,\beta p_\beta})\,e^{-\mathrm{i}\log q\,p_\varphi/2} = -\mathrm{i}\,.\end{aligned} \tag{2.184}$$

Since these constraints do commute with the classical Hamiltonian, they are of first class, and are conserved under time flow. It is possible to see that there are two gauge modes, which are invariant under the currents ψ_L and ψ_R, but non-physical. The gauge fixing is again $\gamma = \beta = 0$ and the constrained Hamiltonian becomes:

$$-4(\log q)^2\mathbf{H} = (1 - e^{-2\varphi})\,e^{-\mathrm{i}\log q\,p_\varphi} + e^{\mathrm{i}\log q\,p_\varphi}\,. \tag{2.185}$$

The resulting q-Liouville path integral encoding a two sided geometry is:

$$\mathcal{Z} = \int \mathcal{D}\phi\mathcal{D}p_\phi \exp\left(\mathrm{i}\int \mathrm{d}t\left(p_\phi\,\phi' + \frac{1}{4}\frac{e^{-\mathrm{i}\log q\,p_\phi}}{(\log q)^2} + \frac{1}{4}\frac{e^{\mathrm{i}\log q\,p_\phi}}{(\log q)^2} - e^{-2\phi}\,\frac{1}{4}\frac{e^{-\mathrm{i}\log q\,p_\phi}}{(\log q)^2}\right)\right). \tag{2.186}$$

After some manipulation, one can show that the classical Hamiltonian of the system is formally equivalent to the transfer matrix in Section (2.1.4)[19]. In order to match the two objects, one is forced to identify:

$$\phi = |\log q|\mathbf{n}\,. \tag{2.187}$$

[19]There are minor differences due to the fact that the Haar measure has an additional factor q^n, while the path integral measure is flat.

“Fake” q-Liouville thermodynamics. Following [82], we want to study the semi-classical path-integral of q-Liouville theory. It will be clear that it is necessary to impose further conditions on the fields in order to recover the exact partition function of DSSYK. This fact is the avatar of the requirement of choosing the Haar measure in the particle on a group manifold formalism. We start by rescaling the fields in such a way that the semiclassical form of the path integral becomes explicit. We perform the following field redefinitions:

$$k := -\mathrm{i}|\log q| \cdot p_\phi \qquad \Leftrightarrow \qquad p_\phi := \frac{\mathrm{i}}{|\log q|}k \tag{2.188}$$

$$\tau := \mathrm{i}\,|\log q|\,t\,, \qquad \Leftrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t} := \mathrm{i}|\log q|\frac{\mathrm{d}}{\mathrm{d}\tau} \tag{2.189}$$

$$\varphi := \phi \tag{2.190}$$

Under these substitutions, the kinetic term transforms as

$$p_\phi \frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\mathrm{i}}{|\log q|}k \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{1}{|\log q|}k\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\,, \tag{2.191}$$

and the potential terms become:

$$\frac{1}{4}\frac{e^{\mathrm{i}\log q\, p_\phi} + e^{-\mathrm{i}\log q\, p_\phi}}{(\log q)^2} = \frac{1}{2}\frac{\cosh(k)}{(\log q)^2}\,, \tag{2.192}$$

$$-\frac{1}{4}e^{-2\phi}\frac{e^{-\mathrm{i}\log q\, p_\phi}}{(\log q)^2} = -\frac{1}{4}e^{-2\varphi}\frac{e^{-k}}{(\log q)^2}\,. \tag{2.193}$$

Combining and going to Euclidean signature: $\mathrm{d}t = \frac{\mathrm{d}\tau}{\mathrm{i}\,|\log q|}$, we obtain the following path integral:

$$\int \mathcal{D}\varphi\mathcal{D}p\, \exp\left\{\frac{1}{|\log q|}\int_0^\beta \mathrm{d}\tau\left(k\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} + \frac{1}{2}\cosh(k) - \frac{1}{4}e^{k}e^{-2\varphi}\right)\right\}. \tag{2.194}$$

We can perform a saddle-point analysis in the regime $|\log q| \ll 1$, and we naively expect to reproduce the classical contributions. At this stage, the inverse temperature β appears as a formal Euclidean time periodicity. However, to make physical contact with the DSSYK model or with the dual bulk interpretation, one will eventually have to specify its value appropriately. As we will see, two natural choices for β arise: one corresponding to the periodicity of the thermal circle and the other selecting a particular region of the saddle point solution. Each option leads to a different result and will be discussed below. What we can immediately notice is that the path integral is dominated by the stationary points of the exponent. Varying with respect to k and φ the effective action yields the equations of motion:

$$\dot{\varphi} = -\frac{1}{2}\sinh(k) + \frac{1}{4}e^k e^{-2\varphi}, \tag{2.195}$$

$$\dot{k} = \frac{1}{2}e^k e^{-2\varphi}. \tag{2.196}$$

By differentiating equation (2.195) and substituting (2.196), one obtains a second-order equation for φ:

$$\ddot{\varphi} = -\frac{1}{4}e^{-2\varphi}. \tag{2.197}$$

This is a standard Liouville-type equation with negative potential. A possible solution is[20]:

$$e^{-2\varphi} = \frac{\sin^2(\theta)}{\sin^2\left(\frac{\tau\sin(\theta)}{2}\right)} \tag{2.198}$$

where θ is a constant of integration which parametrizes the classical phase space and will later be related to the ADM energy of the dual black hole. Clearly (2.198) exactly determines k:

$$e^{-k} = \frac{\sin(\theta)}{\tan(\sin(\theta)\tau/2)} + \cos(\theta), \tag{2.199}$$

and we are in position to compute the classical saddle-point approximation to the q-Liouville partition function. In order to do that, we should evaluate the Hamiltonian of the system:

$$H = -\frac{1}{2}\cosh(k) + \frac{1}{4}e^k e^{-2\varphi}, \tag{2.200}$$

on the classical solutions obtained above. One finds that the classical Hamiltonian is fixed to a constant energy, as expected:

$$H_{\text{on-shell}} = -\frac{1}{2}\cos(\theta). \tag{2.201}$$

As we will see this energy coincides with the ADM energy of the sine-dilaton black hole geometry studied in (2.5.7). In particular, the parameter θ, introduced as an integration constant in the Liouville solution, is now seen to encode the classical energy of the configuration.

Let us now analyze the periodicity properties of the classical saddle solutions (2.198), (2.199). The first solution is singular whenever the denominator vanishes, i.e., for $\tau = (2\pi n)/\sin(\theta)$; $n \in \mathbb{Z}$. This implies that the natural period of the solution is

$$\beta = \frac{2\pi}{\sin(\theta)}. \tag{2.202}$$

[20] Clearly we fixed one of the constants of integration to zero and denoted as θ the other, a more general solution does exist.

Therefore, it is better for the Euclidean time direction be compactified on a thermal

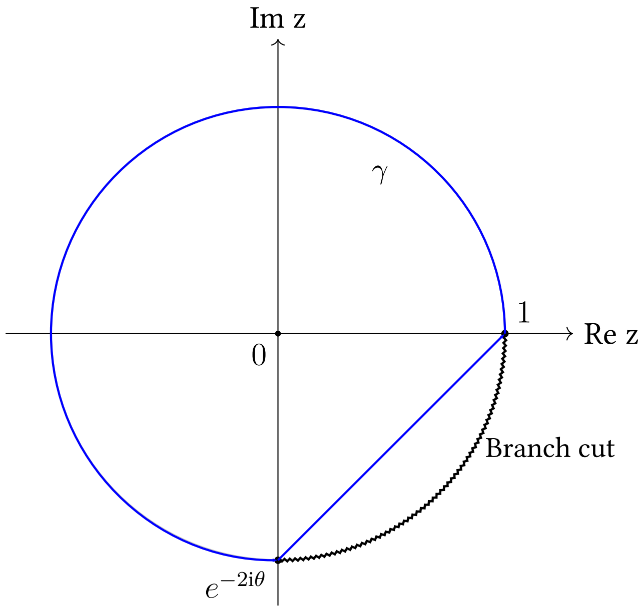


Figure 2.5: Integration contour γ in the complex z-plane. The black line denotes the branch cut of the function $k(z)$, extending along the arc of the unit circle from $z = 1$ to $z = e^{-2i\theta}$. The integration contour γ follows the unit circle in the counterclockwise direction from $z = 1$ to $z = e^{-2i\theta}$, then returns to $z = 1$ along a straight line below the branch cut. This prescription ensures that only the pole at $z = 0$ contributes to the contour integral.

circle of the same periodicity. This quantity will be identified with the Hawking inverse temperature in the dual gravitational theory.

Having determined both the classical periodicity and on-shell energy, we are now left with the task of evaluating the kinetic term. This term actually computes the entropy of the thermodynamical Black Hole solution, as discussed in (2.5.5) and (2.5.6). The on-shell kinetic corresponds to the integral:

$$e^{S_{q\text{-Liouv.}}} = \exp\left(\frac{1}{2|\log q|}\oint k\mathrm{d}\varphi\right). \tag{2.203}$$

It is useful to perform the integral using complex analysis. Indeed, introducing the variable z, we can map the real τ-circle of period β to the unit circle in the complex plane:

$$z = e^{i\,\sin(\theta)\,\tau}. \tag{2.204}$$

In this parametrization, the relevant functions become:

$$\frac{d\varphi}{dz} = -\frac{1}{2z} - \frac{1}{1-z}, \qquad k(z) = -\log\left(-i\sin(\theta)\frac{1+z}{1-z} + \cos(\theta)\right). \tag{2.205}$$

The function $\varphi'(z)$ has two poles in $z = 0$ and in $z = 1$, while $k(z)$ has a branch cut running from $z = 1$ to $z = e^{-2i\theta}$. The entropy, given semiclassically by the integral:

$$S_{q\text{-Liouv.}} = \frac{1}{2|\log q|} \oint_\gamma dz \left(\frac{1}{z} + \frac{2}{1-z}\right) \log\left(-i\sin(\theta)\frac{1+z}{1-z} + \cos(\theta)\right) , \tag{2.206}$$

can then be computed by choosing the contour γ to encircle the pole at $z = 0$, while staying to the left of the branch cut, as shown in Figure (2.5). With this contour, the integral picks only the residue at $z = 0$:

$$\oint_{z=0} \frac{dz}{z} \log\left(-i\sin(\theta)\frac{1+z}{1-z} + \cos(\theta)\right) = 2\pi i \cdot \log\left(e^{-i\theta}\right) = 2\pi\theta\,, \tag{2.207}$$

leading to an entropy linear in θ:

$$S_{q\text{-Liouv.}} = \frac{\pi\theta}{|\log q|}\,. \tag{2.208}$$

Finally, the corresponding semiclassical q-Schwarzian partition function is:

$$Z^{\text{naive}}_{q\text{-Liouv.}}(\beta) \overset{\text{class}}{=} \int dE(\theta)\, \exp\left(\frac{\pi\theta}{|\log q|} + \beta\frac{\cos(\theta)}{2|\log q|}\right) , \tag{2.209}$$

which precisely matches the (naive) semiclassical result obtained in the sine-dilaton gravity framework, as we will see in Section (2.5.4).

"True" q-Liouville Thermodynamics.

So far, we have discussed the *fake* thermodynamics of the q-Liouville model. The reader may have already noticed that the motivation for this adjective lies in the fact that, up to this point, we have not succeeded in reproducing the key observables of DSSYK model, most notably the (semiclassical) partition function (2.77). In particular, while the expression for the energy and inverse temperature are indeed correct, the corresponding entropy is not. Moreover, we have not yet imposed the constraint on the field φ, which is proportional to the number of chords in DSSYK and therefore strictly positive.

These two issues are, in fact, closely connected. In this section, we will present a way to address them simultaneously by modifying the path integral prescription and restricting the space of classical solutions. This will lead to a complete match with DSSYK, including correlation functions. The intuition build in this step will be major hint in the construction of a consistent gravitational dual to DSSYK.

Firstly, let us consider a more general classical solution to the Liouville equation (2.198), which reads:

$$e^{-2\varphi} = \frac{\sin(\theta)^2}{\sin\left(\frac{\sin(\theta)\tau}{2} + c\right)^2} . \tag{2.210}$$

We observe that the integration constant c simply translates the function $\varphi(\tau)$ along the Euclidean time direction; precisely, turning on c amounts to replacing $\tau \to \tau + \frac{2c}{\sin(\theta)}$, thus displacing the profile of φ without altering its shape. This freedom can be used to impose the initial condition $\varphi(0) = 0$. It is straightforward to see that this is achieved by setting $c = \theta$[21]. As a consistency check, we notice that this solution immediately reproduces the classical two-point function (2.105) when evaluating $\exp\{-2\Delta\varphi\}$.
Next, we impose the positivity condition $\varphi(\tau) \geq 0$ for all values of τ. Since the original solution is periodic and becomes negative after a finite interval, we restrict the Euclidean time circle to end precisely when $\varphi(\tau)$ returns to zero. This happens when:

$$\beta = \frac{2\pi - 4\theta}{\sin(\theta)} , \tag{2.211}$$

In other words, we define the period β, such that the full support of the solution lies within the region where $\varphi \geq 0$. In this way, we have fixed the positivity condition.

Clearly, the conjugate momentum $k(\tau)$ must also be modified accordingly to the different expression of $\varphi(\tau)$ and it becomes:

$$e^{-k} = \frac{\sin(\theta)}{\tan(\sin(\theta)\tau/2 + \theta)} + \cos(\theta) \tag{2.212}$$

As a check, one could compute the on-shell Hamiltonian for this new set of solutions. Indeed, it is important to notice that the classical Hamiltonian should remain unchanged (2.201). This follows from the fact that the constrained solution is simply a restriction of the original one, and that the Hamiltonian density is constant along the entire classical trajectory.
The last step is now to compute the entropy of the system. The classical solution (2.210) is no longer periodic with period β_{BH}, but rather with a reduced period (2.211). As before, we choose to go to complex variable $z = e^{i\sin(\theta)\tau}$, but we restrict to the region in which the function is positive definite. This modification implies that the integration contour γ_1 no longer closes into a full circle around the origin (as was the case for the unconstrained, "fake" thermodynamics). Instead, it covers only an arc from $z = 1$ to $z = e^{-4i\theta}$. As a result, we are not allowed to compute the entropy via a standard residue at $z = 0$. However, we can proceed by formally closing the contour into a full circle by adding

[21] An alternative choice exists (namely $c = -\theta$), it would lead to negative values of $\varphi(\tau)$ for $\tau > 0$, immediately violating the physical constraint $\varphi \geq 0$.

the missing segment γ_2 from $z = e^{-4i\theta}$ to $z = 1$. This defines a new closed contour $\gamma_0 = \gamma_1 \cup \gamma_2$ which encircles the origin. We can then write:

$$\int_{\gamma_1} \mathrm{d}z\, g(z) = \underbrace{\int_{\gamma_0} \mathrm{d}z\, g(z)}_{\text{residue at } z=0} - \int_{\gamma_2} \mathrm{d}z\, g(z)\,. \tag{2.213}$$

Clearly, the integral over γ_0 reproduces the same fake entropy as before, while the in-

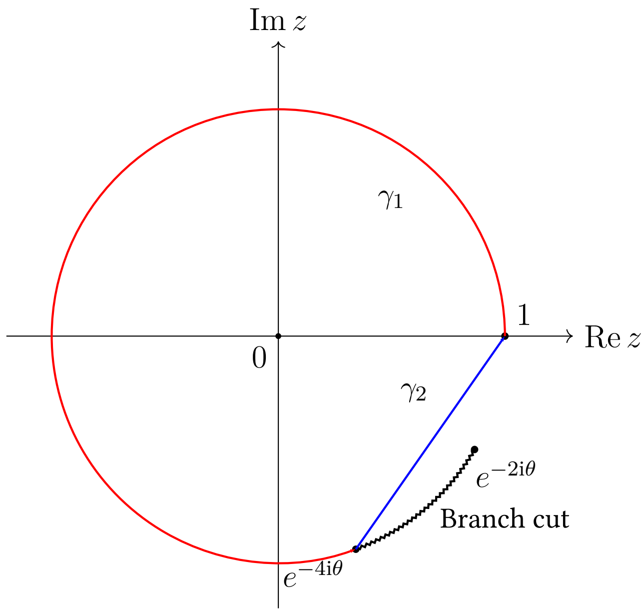


Figure 2.6: Integration contour γ_0 in the complex z-plane. The black line denotes the branch cut of the function $g(z)$, extending along the arc of the unit circle from $z = e^{-4i\theta}$ to $z = e^{-2i\theta}$. The integration contour γ_0 is split into the arc γ_1 and the line γ_2.

tegral over γ_2 must be subtracted to correctly implement the positivity constraint on φ. As a result, the entropy, is:

$$S_{\text{q-sch}} = \frac{1}{|\log q|}\int_{\gamma_1} \mathrm{d}z\, g(z) = \frac{\pi\theta}{|\log q|} - \frac{1}{|\log q|}\int_{\gamma_2} \mathrm{d}z\, g(z)\,, \tag{2.214}$$

Where we are integrating a function $g(z)$:

$$g(z) = \left(\frac{1}{z} + \frac{2e^{2i\theta}}{1 - e^{2i\theta} z}\right) \log\left(\frac{e^{-i\theta} - e^{3i\theta} z}{1 - e^{2i\theta} z}\right), \tag{2.215}$$

which comes from rescaling the argument of $\varphi'(z)k(z)$ in (2.206). Indeed, as discussed earlier, the modified solution (2.210) is obtained by introducing a shift in Euclidean time. This shift changes the functional dependence of both $\varphi(\tau)$ and $k(\tau)$, and correspondingly alters their analytic continuation to the complex plane. Concretely, it manifests as a multiplicative rescaling in the complex variable: $z \to e^{2i\theta} z$. The original branch cut extending from $z = 1$ to $z = e^{-2i\theta}$ is now rotated, and stretches from $z = e^{-4i\theta}$ to $z = e^{-2i\theta}$.

It is possible to evaluate the integral over γ_2, as follows:

$$\int_{\gamma_2} \mathrm{d}z\, g(z) = \pi\theta - \frac{\pi^2}{12} - \frac{1}{2}\mathrm{Li}_2(e^{-2i\theta}) - \frac{1}{2}\mathrm{Li}_2(e^{2i\theta}) - \frac{1}{2}\log\left(-e^{2i\theta}\right)^2 . \tag{2.216}$$

Now, we want to simplify this result, using the identity:

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(z^{-1}) = -\frac{\pi^2}{6} - \frac{1}{2}\log^2(-z)\,. \tag{2.217}$$

One finds:

$$\int_{\gamma_2} \mathrm{d}z\, g(z) = \pi\theta - \frac{1}{4}\log^2(-e^{2i\theta}) \tag{2.218}$$

$$= \pi\theta - (i\pi + 2i\theta)^2 \tag{2.219}$$

$$= \frac{\pi^2}{4} + \theta^2 \tag{2.220}$$

Therefore, the entropy becomes:

$$S_{\mathrm{qsch}} = \frac{\pi\theta - \theta^2 - \pi^2/4}{|\log q|}\,, \tag{2.221}$$

which matches the semiclassical DSSYK entropy, up to an overall constant which was omitted in the DSSYK computation (see Section (2.1.7)). This concludes the evaluation on the classical "true" thermodynamics of the q-Liouville theory.

2.3.3 q-Schwarzian theory

As we know, the Schwarzian formulation of JT gravity can be obtained by imposing a single boundary condition to the particle on a group manifold. This corresponds to fixing one AdS boundary in the gravitational picture. It was mentioned in Section (2.2.4) that imposing only one gauge fixing to the particle on $\mathrm{SU}_q(1,1)$ reduces the system to a one-sided geometry, referred as the q-Schwarzian theory. In particular, one imposes only the left boundary condition $\psi_L = \mathrm{i}$, at the classical level. This corresponds to gauge fix $\gamma = 0$ and the Hamiltonian becomes:

$$\mathbf{H} = -\frac{1}{4}\frac{e^{\mathrm{i}\log q\, p_\phi}}{(\log q)^2} - \frac{1}{4}\frac{e^{-\mathrm{i}\log q\, p_\phi}}{(\log q)^2} + \frac{1}{2\log q}\, e^{-2\phi}\, e^{-\mathrm{i}\log q\, 3p_\phi/2}\, \frac{e^{2\mathrm{i}\log q\, \beta p_\beta} - 1}{2\mathrm{i}\log q\, \beta}\,. \tag{2.222}$$

While the corresponding q-Schwarzian path integral is:

$$\int \mathcal{D}\phi\mathcal{D}\beta\mathcal{D}p_\phi\mathcal{D}p_\beta \exp\left(\mathrm{i}\int \mathrm{d}t\left(p_\varphi\,\varphi' + p_\beta\,\beta' + \frac{1}{4}\frac{e^{\mathrm{i}\log q\,p_\varphi}}{(\log q)^2} + \frac{1}{4}\frac{e^{-\mathrm{i}\log q\,p_\varphi}}{(\log q)^2} + \right.\right.$$

$$\left.\left. + \frac{1}{2\log q}\,e^{-2\phi}\,e^{-\mathrm{i}\log q\,3p_\phi/2}\,\frac{e^{2\mathrm{i}\log q\,\beta p_\beta}-1}{2\mathrm{i}\log q\,\beta}\right)\right). \tag{2.223}$$

In the Schwarzian picture, it is possible to integrate out fields, in order to obtain the well-known Schwarzian derivative in the action; however the same is not possible here. Due to this complication, the q-Schwarzian formalism is not fully studied in the literature. For example, the one-boundary solutions are not known in this setting, and even a semi-classical computation of the partition function is missing to our knowledge. Due to its links to the transfer matrix of DSSYK, it is simpler to study the q-Liouville fully-gauged formulation.

2.4 Poisson Sigma Model

In this section, we introduce the Poisson Sigma model, which serves as a bridge between DSSYK and the gravitational formulation. It can be regarded as the analogue of the BF model in the JT gravity framework. Here, in order to make the flow self-contained, we wish to give an overview of the relevant properties, mainly referring to [100]. We also review [88], which shows the relations with the particle on $\mathrm{SU}_q(1,1)$.

2.4.1 The action

We consider the following action:

$$S_{\text{PSM}} = \int_\Sigma \left(-J_A \wedge \mathrm{d}A^A + \frac{1}{2}\alpha^{AB}(J)A_A \wedge A_B\right) - \int_{\partial\Sigma}\mathbf{H}(J_A) \tag{2.224}$$

The model is defined in two-dimensional Lorentzian spacetime $\Sigma = \mathbb{R}^+ \times \mathbb{R}$ and parametrized by coordinates (u,t). The fields J_A (where $A \in \{0,1,H\}$) are scalar 0-form fields taking values in a 3d target Poisson manifold. While the fields A_A are 1-forms on the worldsheet, interpreted as gauge connections. The function $\alpha^{AB}(J)$ is an antisymmetric bivector on the target space, depending on the scalars J_A, and it defines the Poisson structure of the target space. There is a natural concept of time-like boundary at $u=0$ and a corresponding boundary term in the action. Following [88] and [100], we want to show that this model is topological and localize the 3D target space description to a boundary 6d classical phase space (which can be then quantized). Then show that a specific choice of α_{AB} leads to identify the variables on the boundary to the generators of $U_q(\mathfrak{su}(1,1))$.

2.4.2 A topological theory

In order to follow the program, we start by expanding the wedge products:[22]

$$S_{\text{PSM}} = \int_0^\infty \mathrm{d}u \int \mathrm{d}t \Bigg(-(A_u)_B \dot{J}_B + (A_t)_B \Big(J'_B + \alpha_{BC}(J_A)\,(A_u)_C \Big) \Bigg) - \int \mathrm{d}t\,\mathbf{H}(J_A)\,. \tag{2.225}$$

An immediate observation, useful later, is that, from the previous expression, the Poisson brackets (at equal time t_0) are:

$$\Big\{ (A_u)_B(u_1, t_0), J_C(u_2, t_0) \Big\} = \delta_{BC}\, \delta(u_1 - u_2)\,. \tag{2.226}$$

As we have mentioned, the structure constants α_{AB} do implement the symmetry group. In the case of Lie algebras, α_{AB} are linear in the fields J_A:

$$\alpha_{BC}(J_A) = f_{BC}{}^A J_A, \tag{2.227}$$

This allows for an explicit integration over J_A, reducing the theory to a topological one. The bulk action localizes on flat connections after integrating out J_A.
However, as we will see, in order to recover the algebra of the quantum group, the structure constants $\alpha_{BC}(J_A)$ must be nonlinear in J_A, obstructing the direct integration over J_A in the path integral. As a result, the usual argument for topological invariance by integrating out J_A does not straightforwardly apply.
Luckily, Cattaneo and Felder provided a more general and refined argument to demonstrate the topological nature of the Poisson Sigma Model [100]. Their approach is based on the existence of the following constraint, obtained by interpreting the A_t components of the gauge field as a Lagrange multiplier[23]:

$$J'_B(u, t_0) = -\sum_C \alpha_{BC}(J_A(u, t_0))\,(A_u)_C(u, t_0)\,. \tag{2.228}$$

We notice that (2.228) is a differential equation in the spatial coordinate u at fixed $t = t_0$. Therefore it constrains the possible configurations of the fields $J_A(u)$ and $(A_u)_B(u)$. In this sense, it imposes a restriction on the phase space associated defined on a Cauchy slice at time t_0, effectively reducing the dimension of the degrees of freedom. In partic-

[22] The dot denotes differentiation with respect to t, and the prime with respect to u.
[23] Notice that it is not equivalent to integrate out A_u, due to the structure of the worldsheet Σ, which has a boundary at $u = 0$, inducing a boundary term in the action

ular, the physical phase space is determined by solving (2.228) and modding out by the gauge transformations generated by the constraint.
Basically, the system (2.228) is equivalent to a standard 1d classical mechanics problem and in order to solve it, we can look at its canonical Hamiltonian formulation:

$$\int_0^{+\infty} \mathrm{d}u \left(\sum_A P_A X'_A - \mathbf{G}(J_A(X_A, P_A)) \right). \tag{2.229}$$

where $X_A(u)$ and $P_A(u)$ are canonically conjugate variables and the Hamiltonian is designed to satisfy the (2.228). One should keep in mind that the Poisson brackets in the original model are non-canonical, whereas the auxiliary system we introduced is built with canonical coordinates. The advantage of working in this canonical phase space is that the equations of motion take the familiar simple form:

$$X'_A = \frac{\partial \mathbf{G}}{\partial P_A}, \qquad P'_A = -\frac{\partial \mathbf{G}}{\partial X_A}. \tag{2.230}$$

To relate this auxiliary system back to the original Poisson Sigma model, we recall that the fields $J_A = J_A(X, P)$ satisfy a non-trivial Poisson algebra:

$$\{J_A, J_B\} = \alpha_{AB}(J), \tag{2.231}$$

which implies (by chain rule) the following equations of motion in the J_A-coordinates:

$$J'_B = \{J_B, J_C\} \frac{d\mathbf{G}}{dJ_C}, \qquad X'_B = \frac{dJ_C}{dP_B} \frac{d\mathbf{G}}{dJ_C}. \tag{2.232}$$

Now, comparing the equation of motion (2.232) with the constraint (2.228), we get:

$$-\alpha_{BC}(J)(A_u)_C = \alpha_{BC}(J) \frac{\partial \mathbf{G}}{\partial J_C}. \tag{2.233}$$

Assuming the antisymmetric matrix $\alpha_{BC}(J)$ to be invertible on the relevant subspace, we multiply both sides by the inverse and obtain:

$$(A_u)_C = -\frac{\partial \mathbf{G}}{\partial J_C}. \tag{2.234}$$

The previous equation holds throughout the bulk, along any fixed-time Cauchy slice. However, it plays a crucial role at the boundary $u = 0$, where the initial data are specified. In particular, once the Hamiltonian $\mathbf{G}(J)$ is chosen, and boundary values are fixed:

$$J_A(0) = j_A, \qquad X_A(0) = x_A, \tag{2.235}$$

the above relation fully determines the profile of the gauge field components $(A_u)_C(u)$ throughout the bulk. That is, the functional form of A_u is not arbitrary but is constrained by the structure of $\mathbf{G}$ and the initial data for J and X at the boundary. In this way, we have shown that the physical phase space of this Poisson Sigma model reduces to the classical 6d phase space of the classical Hamiltonian problem. More generally, given a Poisson sigma model with a target space of dimension D, its physical phase space will have dimension $2D$. Now we want to show that a particular choice of the structure constants leads to a boundary formulation.

Firstly we want to show that the variables j_A and x_A satisfy the same Poisson algebra as the corresponding fields, namely (2.231). We consider a fixed Cauchy slice $t = t_0$ and integrate the constraint (2.228) along the spatial slice, obtaining:

$$j_A(t_0) = J_A(\infty, t_0)) + \int_0^\infty \mathrm{d}u_1\, \alpha_{AD}(J_C(u_1, t_0))\, (A_u)_D(u_1, t_0)\,. \tag{2.236}$$

In order to show that the Poisson algebra is inherited, we simply compute the Poisson bracket with $J_B(u_2, t_0)$; the term enters the integral, giving:

$$\Big\{j_A(t_0), J_B(u_2, t_0)\Big\} = \int_0^\infty \mathrm{d}u_1\, \alpha_{AD}(J_C(u_1))\, \Big\{(A_u)_D(u_1), J_B(u_2)\Big\} = \alpha_{AB}(J_C(u_2, t_0))\,. \tag{2.237}$$

Now, if we also restrict J_B to the boundary $u_2 = 0$, the Poisson bracket becomes:

$$\Big\{j_A, j_B\Big\} = \alpha_{AB}(j_C)\,. \tag{2.238}$$

We can also integrate the second equation in (2.232) and use (2.234) to substitute for the derivative of $\mathbf{G}$:

$$x_A = X_A(\infty) + \int_0^\infty \mathrm{d}u_1\, \frac{\mathrm{d}J_D}{\mathrm{d}P_A}\, (A_u)_D(u_1) \tag{2.239}$$

The Poisson bracket at the boundary becomes:

$$\Big\{x_A, j_B\Big\} = \frac{\mathrm{d}j_B}{\mathrm{d}p_A}\,,. \tag{2.240}$$

Hence $p_A = P_A(u = 0)$ must be canonically conjugate to x_A in order to satisfy the relation. Moreover, since the constraint (2.228) is satisfied, the dynamical action reduces only to the boundary term: $\mathbf{H}(J_A(0)) = \mathbf{H}(j_A)$. Therefore, we can write the Poisson Sigma model as an explicitly topological theory defined on the boundary $u = 0$:

$$\exp\left(\mathrm{i}\int \mathrm{d}t \bigg(\sum_A p_A \dot{x}_A - \mathbf{H}(j_A)\bigg)\right) \tag{2.241}$$

2.4.3 Matching the particle on a q-group

In order to recover the q-Schwarzian formulation the correct choice is the set of structure constants (we introduce a parameter b which is related to q as shown in (2.251)):

$$\alpha_{H0} = -J_1, \quad \alpha_{H1} = -J_0, \quad \alpha_{01} = \frac{\sin(2\pi b^2 J_H)}{2\pi b^2} = \frac{1}{2} V_{\text{q-sch.}}(J_H)\,. \tag{2.242}$$

The boundary Poisson algebra is inherited, while the boundary Hamiltonian is:

$$\mathbf{H}(j_A) = -j_1^2 + j_0^2 + \frac{\cos(2\pi b^2 j_h)}{2\pi^2 b^4}\,. \tag{2.243}$$

In order to prove this claim, we notice that the boundary Hamiltonian should be identified with the classical q-deformed Casimir of $\mathrm{SU}_q(1,1)$. In the classical limit the left-invariant generators satisfy the algebra:

$$\{h, e\} = e, \qquad \{h, f\} = -f, \qquad \{e, f\} = \frac{\sin(2\log q\, h)}{\log q}. \tag{2.244}$$

and can be written as functions of the canonical variables $(\varphi, \beta, \gamma, p_\varphi, p_\beta, p_\gamma)$:

$$\begin{aligned} h &= \frac{1}{2} p_\phi - \gamma p_\gamma \\ f &= -\frac{e^{-2\mathrm{i}\log q\, \gamma p_\gamma} - 1}{2\mathrm{i}\log q\, \gamma} \\ e &= e^{-2\phi}\, e^{i\log q\,(-p_\phi + 2\gamma p_\gamma)}\, \frac{e^{2\mathrm{i}\log q\, \beta p_\beta} - 1}{2\mathrm{i}\log q\, \beta} - \gamma\, e^{\mathrm{i}\log q\, p_\phi}\, \frac{e^{2\mathrm{i}\log q\,(\gamma p_\gamma - p_\phi)} - 1}{2\mathrm{i}\log q}, \end{aligned} \tag{2.245}$$

We now define combinations:

$$j_0 := -\frac{1}{2}(e + f), \qquad j_1 := \frac{1}{2}(e - f), \qquad j_h := h. \tag{2.246}$$

Computing the Poisson brackets among the new generators, we obtain:

$$\{j_h, j_0\} = -j_1, \qquad \{j_h, j_1\} = -j_0, \tag{2.247}$$

$$\{j_0, j_1\} = \frac{1}{4}\{f + e, e - f\} = \frac{1}{2} \cdot \frac{\sin(2\log q\, j_h)}{\log q}. \tag{2.248}$$

These relations define the Poisson algebra of the variables j_A, which are inherited by J_A. Therefore, the structure functions of the Poisson sigma model are identified as in (2.242).

Regarding the Hamiltonian, we start by computing the classical Casimir, in the (e, f, h)

basis:

$$\mathbf{H} = ef + \left(\frac{\sin(\log q\, j_h)}{\log q}\right)^2. \tag{2.249}$$

Rewriting ef it in terms of (j_0, j_1) yields $ef = j_0^2 - j_1^2$, hence, the PSM Hamiltonian becomes:

$$\mathbf{H}(J_A) = j_0^2 - j_1^2 + \left(\frac{\sin(\log q\, h)}{\log q}\right)^2. \tag{2.250}$$

Finally, we relate the deformation parameter q to the bulk parameter b[24] by choosing:

$$q = e^{i\pi b^2}, \qquad b \in \mathbb{R}, \tag{2.251}$$

so that $|q| = 1$ and the deformation is consistent with a Lorentzian bulk Poisson sigma model. The corresponding boundary Hamiltonian becomes exactly (2.243). This concludes the systematic derivation.

It is well known that the Poisson Sigma model is related to dilaton gravities. In particular with the present choice of Boundary Hamiltonian and algebra, it is mapped into Sine-Dilaton gravity, which is the next topic.

2.5 Sine-Dilaton gravity

Before starting with the standard definition of Sine-Dilaton gravity [79–82, 84, 86], we want to treat its first order formulation, which is in direct link with the PSM. Next, we move to the main topics, including the discussion of the semiclassical solutions, the notions of temperature and fake temperature, the discretization of spacetime (in its various declinations), defects and canonical quantization.

2.5.1 First-order Formalism

An intuitive reason to adopt a first order formulation is the parallel well-known story of how AdS_3 gravity is related to the WZW model: Indeed, 3d gravity can be translated in the Palatini formalism (introducing zweibein and spin-connection fields) where it takes the explicit form of a $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$ Chern-Simons theory. Moreover, when trying to introduce a boundary term in Chern-Simons (e.g. when considering a cylinder topology), the model is no more strictly topological, since a boundary term must be added in order to keep the variational principle well-defined. Hence the theory becomes dynamical on the border, with action coinciding with that of the WZW model. When imposing

[24]Having in mind that the analytic continuation of the theory to a sinh potential is related to Liouville theory.

Brown-Henneux boundary condition to WZW one can further reduce the model to 2d Liouville gravity. A similar chain of dualities holds in the context of JT gravity, where the theory is shown to have a BF formulation which is then interpreted as a particle on a manifold and reduced to the Schwarzian theory when introducing boundary conditions. This section is mainly based on [88, 97, 98, 151]. The most important aspect is the treatment of the boundary terms in gravity, indeed the q-Schwarzian dynamics emerges from there, as we want to highlight.
The action of the bulk theory can be expressed as:

$$S^{\text{first-order}} = \int_{\Sigma} \Big(-\tilde{\Phi}\mathrm{d}\omega + \frac{\sin(2|\log q|\tilde{\Phi})}{2|\log q|} e^0 \wedge e^1 - \Phi_0(\mathrm{d}e^0 - \omega \wedge e^1) \\ - \Phi_1(\mathrm{d}e^1 - \omega \wedge e^0) \Big) \tag{2.252}$$

where $\tilde{\Phi}, \Phi_0 \Phi_1$ are zero-form scalar fields; in particular, $\tilde{\Phi}$ will be interpreted as the dilaton field and the other two fields act as Lagrangian multipliers enforcing vanishing torsion. Moreover, e^a are zweibein one-forms and ω it the spin-connection.
Since we are interested in a holographic theory reproducing DSSYK, a Gibbons–Hawking–York term must be added. A posteriori, the right choice is:

$$S_{\text{GHY}} = \int_{\partial\Sigma} \tilde{\Phi}\, \omega\, dt \tag{2.253}$$

The inclusion of the GHY term ensures a well-posed variational principle and, in the chosen gauge, its contribution precisely reproduces the ADM energy associated with the boundary time evolution. A counterterm forcing holographic renormalization must also be added in order to recover DSSYK, the proposal by [88] is:

$$S_{\text{count}} = \int_{\partial\Sigma} dt \left(\Phi_0\, e^0_t + \Phi_1\, e^1_t - \mathbf{H}(\Phi_0, \Phi_1, \tilde{\Phi}) \right). \tag{2.254}$$

In addition, a boundary Hamiltonian, function only of the scalar fields, can be introduced:

$$\mathbf{H}(\Phi_0, \Phi_1, \tilde{\Phi}) = -\Phi_0^2 + \Phi_1^2 - \frac{\cos\left(2|\log q|\tilde{\Phi}\right)}{2|\log q|}. \tag{2.255}$$

The key observation in rewriting the Sine-Dilaton gravity action lies in organizing the target space fields and gauge connections into unified multiplets. Specifically, the identification is the following:

$$\tilde{\Phi} = -J_H\,, \quad \Phi_0 = -J_1\,, \quad \Phi_1 = -J_0\,, \quad \omega = A_H\,, \quad e^1 = A_0\,, \quad e^0 = A_1\,. \tag{2.256}$$

This step leads the bulk theory to be reformulated as a Poisson Sigma Model:

$$S^{\text{first order}} = \int_\Sigma \left(J_B\, dA_B + \frac{1}{2}\alpha_{BC}(J) A_B \wedge A_C \right). \tag{2.257}$$

The sine-type potential from the dilaton sector is encoded in α_{BC}, which takes the same form as in (2.242). The advantage of this reformulation is that it captures the topological nature of the theory while making the gauge structure and boundary dynamics manifest. Indeed, one can show that the boundary term of sine-dilaton gravity becomes:

$$S_{\text{bdy.}} = -\int_{\partial\Sigma} (J_B A_B + dt\, \mathbf{H}(J)) \tag{2.258}$$

The final step consists in integrating by parts the derivative term in the bulk, applying Stokes' theorem:

$$\int_{\mathcal{M}} J_B\, dA^B = -\int_\Sigma dJ_B \wedge A^B + \int_{\partial\Sigma} J_B A^B.$$

Indeed the boundary term (2.258) explicitly includes: $\int_{\partial\mathcal{M}} J_B A_B$. This means that we can translate the bulk term in the form $-\int_{\mathcal{M}} dJ_B \wedge A^B$, avoiding double counting the boundary contribution. As a result we can write the action as:

$$S_{\text{PSM}} = \int_0^\infty \mathrm{d}u \int \mathrm{d}t \Big(-(A_u)_B \dot{J}_B + (A_t)_B \Big(J_B' + \{J_B, J_C\}\ (A_u)_C \Big) \Big) - \int \mathrm{d}t\, \mathbf{H}(J_A)\,, \tag{2.259}$$

It is important to notice that the term $\mathbf{H}$ which enforces holographic renormalization in sine-dilaton, reduces to the boundary term of the PSM; moreover, it should be clear from Section (2.4) that it also defines the dynamical action of the particle on the quantum group. This highlights the fact that, in accordance with general expectations from holography, the boundary term assumes a central role in governing the dynamics of the theory.

2.5.2 Second-Order Formalism

The standard second-order formulation of sine-dilaton gravity is given by the following action:

$$S = \left(\frac{1}{2} \int \mathrm{d}x \sqrt{g} \left(\tilde{\Phi} R + \frac{\sin\left(2|\log q|\tilde{\Phi}\right)}{|\log q|} \right) + \int \mathrm{d}\tau \sqrt{h} \left(\tilde{\Phi} K - \mathrm{i}\, \frac{e^{-\mathrm{i}|\log q|\tilde{\Phi}}}{2|\log q|} \right) \right). \tag{2.260}$$

To obtain this second-order formulation starting from (2.252), we perform the following steps:

- First, we integrate out the Lagrange multipliers Φ_0 and Φ_1, which impose the torsionless conditions $\mathrm{d}e^0 = \omega \wedge e^1$ and $\mathrm{d}e^1 = \omega \wedge e^0$. These allow us to express the spin connection ω uniquely in terms of the zweibein.
- Next, we express the curvature two-form $\mathrm{d}\omega$ in terms of the Ricci scalar $R = 2\epsilon^{ab} e^{\mu}_a e^{\nu}_b R_{\mu\nu}$, so that the term $-\tilde{\Phi}\,\mathrm{d}\omega$ becomes $\frac{1}{2}\tilde{\Phi} R\sqrt{g}\,\mathrm{d}^2x$ in the metric formulation.
- Using the identity $e^0 \wedge e^1 = \sqrt{g}\,\mathrm{d}^2x$, we obtain the dilaton potential $V(\tilde{\Phi}) = \frac{\sin\left(2|\log q|\,\tilde{\Phi}\right)}{2|\log q|}$ in the Lagrangian.
- The total derivative from integration by parts of $-\tilde{\Phi}\,\mathrm{d}\omega$ gives a Gibbons- Hawking–York (GHY) term:

$$\int_{\Sigma} \mathrm{d}(\tilde{\Phi}\omega) = \int_{\partial\Sigma} \tilde{\Phi}\omega_\tau\,\mathrm{d}\tau = \int_{\partial\Sigma} \sqrt{h}\,\tilde{\Phi} K\,\mathrm{d}\tau\,,$$

  where K is the extrinsic curvature of the boundary and h the induced metric.
- The counterterm in the second-order formulation,

$$S_{\text{ct}} = -\int dt\,\frac{\cos\left(2|\log q|\,\tilde{\Phi}\right)}{2|\log q|}\,, \tag{2.261}$$

  arises directly from the boundary term in the first-order action when evaluated on-shell by plugging the classical solution $e^0_\tau = \sqrt{F(r)}$ and using the radial gauge $\Phi_1 = 0$, $\Phi = r$.

As a result, after expressing all terms in terms of the metric $g_{\mu\nu}$ and the dilaton $\tilde{\Phi}$, and reducing the boundary term accordingly (including the Gibbons–Hawking–York term and the counterterm), we arrive at the standard second-order action (2.260). Having in mind to perform a semiclassical analysis in order to match the results in DSSYK, it is convenient to rescale the dilaton field as: $\tilde{\Phi} \to \Phi = |\log q|\tilde{\Phi}$. The rescaled action becomes:

$$S = \frac{1}{2|\log q|}\left\{\frac{1}{2}\int_{\Sigma} \mathrm{d}^2x\sqrt{g}\,(\Phi R + 2\sin\Phi) + \int_{\partial\Sigma} \mathrm{d}\tau\sqrt{h}\left(\Phi K - i\,e^{-i\Phi/2}\right)\right\}. \tag{2.262}$$

As we will see, this form is more convenient for a saddle point analysis ($|\log q| \to 0$) and highlights the geometric structure of the theory, with the dilaton Φ coupling directly to curvature and driving the dynamics through its potential.

2.5.3 A Chain of Dualities

Here it is useful to stop and resume the derivation in the previous Sections. We have seen that the starting point of the construction is the description of the boundary dynamics of DSSYK model in terms of a particle propagating on the group manifold $\text{SU}_q(1,1)$. This system naturally exhibits a large reparametrization symmetry and encodes the boundary degrees of freedom of a topological gauge theory.

In order to uplift this 1D boundary theory to a 2D bulk description, one promotes the particle-on-a-group dynamics to a Poisson Sigma Model. The PSM framework is a topological field theory whose target space is a Poisson manifold, in this case, the dual of the Lie algebra $U_q(\mathfrak{su}(1,1))$. The Poisson structure is deformed by a q-parameter, reflecting the q-deformation of the underlying symmetry.

By performing a field redefinition and matching to geometric variables (zweibeins and spin connection), the PSM can be rewritten in terms of a first-order dilaton gravity.

Finally, by integrating out the auxiliary fields Φ_a and solving for the spin connection using the torsionless condition, one obtains the second-order metric formulation of sine-dilaton gravity, which exhibits a nontrivial dilaton potential governed by the q-deformation. We conclude that this second-order action is the one that governs the bulk dynamics.

However, since the mentioned identities are proven only at the classical level and the two systems are really different objects, the connection to DSSYK requires further specification. The treatment of the particle on $\text{SU}_q(1,1)$ gives us some hints on how to obtain perfect matching. To obtain the dynamics of DSSYK, two additional conditions must be imposed:

- Boundary conditions: In analogy with the BF formulation of JT gravity, one must impose a boundary condition that selects a fixed asymptotic geometry. This amounts to reducing the symmetry of the particle-on-a-group down to physical degrees of freedom. In the present case, there are two possible geometrical configurations: the one-sided geometry, leading to the q-Schwarzian theory; The two-sided wormhole, corresponding to the q-Liouville theory. These boundary conditions break gauge invariance and induce a nontrivial dynamics at the boundary.

- Measure prescription: Once the two-sided q-Liouville geometry is selected, an additional prescription must be imposed to reproduce the quantum structure of DSSYK. This takes the form of a path integral measure constraint, analogous to choosing the Haar measure on the quantum group manifold. This ensures the correct weighting of different field configurations.

Both steps are crucial to recover the expected structure. As we shall see, the gravitational framework admits multiple approaches. Nonetheless, in what follows, we begin with a naive semiclassical analysis to illustrate how the unconstrained theory *fails* to reproduce the observables.

2.5.4 Sine-Dilaton Thermodynamics

We want to study the rescaled version of the action (2.262) on disk topology and Euclidean signature. Clearly the saddle point approximation holds in the regime $|\log q| \to 0$. Hence, when considering the bulk part of the action, the classical contribution to the saddle is the classical solution. This model belongs to the general class of 2d dilaton gravities, the solution of which is well known (A.10), as explained in appendix (A.1). Specifying to the sine potential (in radial gauge) we have:

$$\mathrm{d}s^2 = F(r)\mathrm{d}\tau^2 + \frac{1}{F(r)}\mathrm{d}r^2\,, \qquad F(r) = -2\cos(r) + 2\cos(\theta)\,, \qquad \Phi = r\,. \tag{2.263}$$

We implicitly fixed the free parameter to a specific value, θ, hence parameterizing the energy of the model. This solution contains a black hole horizon at $r = \theta$ and a cosmological horizon at $r = 2\pi - \theta$. As usual, by looking at the near-horizon geometry and requiring no conical defects in the metric expressed in polar coordinates, one is led to require periodicity in the τ direction: $\tau \sim \tau + \beta_{\text{BH}}$. In this way, the Hawking temperature of the black hole is fixed:

$$\beta_{\text{BH}} = \frac{2\pi}{\sin(\theta)}\,. \tag{2.264}$$

We now want to discuss the bulk on-shell action. For a chosen value of θ, the classical solution implies:

$$\Phi = r, \qquad R = 2\cos(r), \qquad \sqrt{g} = 1, \tag{2.265}$$

Hence the bulk action becomes:

$$S_{\text{bulk}} = \frac{1}{4|\log q|}\int_0^{\beta}\mathrm{d}\tau\int_{\theta}^{\Phi_{\text{bdy}}}\mathrm{d}r\ \left(2r\cos(r) + 2\sin(r)\right). \tag{2.266}$$

We can compute the integral explicitly, evaluating this expression between $r = \theta$ and $r = \Phi_{\text{bdy}}$, we obtain:

$$S_{\text{bulk}} = \frac{\beta}{4|\log q|} \left[2r\cos(r) + 2\sin(r) - 2\cos(r)\right]_{\theta}^{\Phi_{\text{bdy}}} . \tag{2.267}$$

In order to compute this term we need to choose the boundary conditions and fix the asymptotic behavior of the fields. Following [82, 88], a possible choice for the boundary asymptotics of the geometry and of the dilaton is:

$$\sqrt{h}\, e^{i\Phi_{\text{bdy}}/2} = i\,, \qquad \Phi_{\text{bdy}} = \frac{\pi}{2} + i\infty\,, \tag{2.268}$$

These conditions are chosen so as to ensure a consistent semiclassical limit and to properly tune the on-shell action. In particular, the asymptotic behavior of the fields is such that the counterterm cancels the divergent part of the Gibbons-Hawking-York boundary contribution, yielding a finite boundary action. Furthermore, the location precisely corresponds to the one of the asymptotic AdS_2 screen in the Weyl-rescaled effective geometry perceived by the matter probes. This boundary condition is needed for canonical quantization, as we will see in Section (2.5.7), and suggests a boundary location in the real screen (although not uniquely).

The set of boundary conditions translates at the level of the Poisson Sigma Model, indeed: $f = \sqrt{h}$ and $h = -\Phi_{\text{bdy}}$. As a consequence the constraints (2.184) on the currents in the particle picture are also fixed; implying the q-Liouville description.

Looking back at the action (2.267), we notice that the upper limit $\Phi_{\text{bdy}} = \pi/2 + \mathrm{i}\infty$ introduces a divergence. However, this divergence is precisely canceled by the boundary term in (2.262), which we now evaluate. The induced metric on the boundary is $\sqrt{h} = \sqrt{F(\Phi_{\text{bdy}})}$. Moreover, the extrinsic curvature is given by:

$$K = \frac{1}{2}\frac{F'(\Phi_{\text{bdy}})}{F(\Phi_{\text{bdy}})^{1/2}} . \tag{2.269}$$

Using the explicit expression for the metric function $F(r)$, the combination $\sqrt{h}K$ simplifies to:

$$\sqrt{h}\, K = \sin(\Phi_{\text{bdy}})\,, \tag{2.270}$$

and the full boundary term becomes:

$$S_{\text{bdy}} = \frac{\beta}{2|\log q|} \left(\Phi_{\text{bdy}}\, \sin(\Phi_{\text{bdy}}) - \mathrm{i}\sqrt{F(\Phi_{\text{bdy}})}\, e^{-\mathrm{i}\Phi_{\text{bdy}}/2}\right) , \tag{2.271}$$

where we have used the fact that the integrand has not τ dependence. At the boundary configuration $\Phi_{\text{bdy}} = \pi/2 + \mathrm{i}\infty$, both terms in (2.271) exhibit exponential divergences, but their leading asymptotic contributions cancel precisely against the bulk term. Indeed, the full on-shell action is:

$$S_{\text{on shell}} = \left[\frac{1}{2|\log q|}\left(2\pi\theta + \beta F(\Phi_{\text{bdy}}) - \mathrm{i}\beta\sqrt{F(\Phi_{\text{bdy}})}\, e^{-\mathrm{i}\Phi_{\text{bdy}}/2}\right)\right]. \tag{2.272}$$

When expanding around Φ_{bdy} we have:

$$S_{\text{on shell}} = \frac{\pi\theta}{|\log q|} + \beta\frac{\cos(\theta)}{2|\log q|} \tag{2.273}$$

We can go further and observe that, at the classical level, the path integral localizes on a phase space effectively parametrized by the variable θ. Consequently, the semiclassical approximation of the gravitational partition function reduces to an integral over the corresponding energies $E(\theta)$:[25]

$$Z_{\text{grav}}(\beta) \overset{\text{class}}{=} \int \mathrm{d}E(\theta)\, \exp\left(\frac{\pi\theta}{|\log q|} + \beta\frac{\cos(\theta)}{2|\log q|}\right). \tag{2.274}$$

This expression tells us the classical thermodynamics of the theory, since we can deduce the ADM-energy and the entropy of the black hole:

$$E = -\frac{\cos(\theta)}{2|\log q|}, \qquad S_{\text{BH}} = \frac{\pi\theta}{|\log q|}. \tag{2.275}$$

However, this expression fails to match the semiclassical limit of the DSSYK partition function (2.109). This solution yields a horizon area that grows linearly with θ, which is a hallmark of standard black hole thermodynamics in two-dimensional dilaton gravity models, such as JT gravity. Indeed, in dilaton-gravity, the Bekenstein-Hawking entropy is proportional to the value of the dilaton at the horizon.

This semiclassical behavior is in tension with the known thermodynamic properties of DSSYK model. As we have shown in (2.1.8), the bulk theory is expected to exhibit a notion of discretized length. As a consequence, the entropy in DSSYK is not a smooth, monotonically increasing function of a continuous parameter, but instead exhibits a non-monotonic behavior. This discrepancy implies that, while the sine-dilaton model captures some aspects of the effective low-energy dynamics of DSSYK, it does not reproduce the classical thermodynamics, unless additional conditions are imposed on the gravitational path integral. These elements are not present at the level of the classical gravity

[25] Here the path-integral measure is taken to be classical, but more care is needed in order to properly define the gravitational path-integral.

action, but may emerge only after a proper treatment of the periodic symmetry in the dilaton potential.

2.5.5 Temperature vs "fake" Temperature

In order to deduce what is missing in our bulk gravitational description, it is convenient to compare with the partition function of DSSYK. In that model, the integral is dominated by a saddle point at the classical energy $E_0 = -\cos(\theta)/(2|\log q|)$, determined by the stationarity condition:

$$\frac{d}{dE}\left(S(E)-\beta E\right)\Big|_{E_0} = 0\,. \tag{2.276}$$

This relation identifies the inverse temperature β of the system as the derivative of the entropy with respect to energy. In the specific case of DSSYK:

$$\beta = \frac{dS}{dE} = \frac{2\pi - 4\theta}{\sin\theta}. \tag{2.277}$$

This differs from the naive gravitational inverse temperature and ultimately, it is related to the quadratic θ^2-term in the entropy. If we want to reproduce DSSYK, it is necessary to modify the Euclidean time periodicity in the bulk gravity description. Changing the inverse temperature would clearly also fix the entropy mismatch.

However, from another point of view, the proposal of engineering the system in order to change the value of the temperature might seem puzzling. Indeed, we remember that in DSSYK the two-point correlation functions do actually decay with the scale β_{BH}, related to the "fake" Euclidean periodicity. The semiclassical two-point function of an operator with dimension Δ is [64]:

$$\langle\mathcal{O}(\tau)\mathcal{O}(0)\rangle \propto \frac{\sin(\theta)^{2\Delta}}{\sin\left(\frac{\sin(\theta)}{2}\tau+\theta\right)^{2\Delta}}. \tag{2.278}$$

Expanding for small τ, this correlator decays exponentially, with characteristic inverse time scale:

$$\tau^{-1} \sim \beta_{\mathrm{BH}} = \frac{2\pi}{\sin\theta},$$

which corresponds to the inverse temperature defined by the naive Euclidean time periodicity in gravity.

Therefore, we notice that even in DSSYK, there are two different notions of temperature [62]. The mismatch arises because the energy spectrum controlling the partition function differs from the effective decay-time of correlators. Physically, β_{BH} can be viewed as the geometric (Euclidean) temperature tied to the gravitational background, while β

is the physical temperature felt by the boundary theory. This distinction motivates a change in the Euclidean time, in order to reconcile these two scales.
A possible way to fix the periodicity is to insert a conical defect, as we will see in the next section. Another option arises in the context of canonical quantization of the WdW system, where it is clear that gauging the symmetry of the Hamiltonian leads to the right notion of temperature, while also fixing divergences in the partition function. Indeed, the identification of the right temperature(s) is an avatar of choosing the proper Haar measure in the particle framework or of imposing positivity in the q-Liouville description.

2.5.6 Defects

In order to fix the entropy mismatch, we seek a deformation of the bulk theory which naturally enforces the true temperature condition (2.277) at the level of geometry. Imposing such a constraint directly on the temperature would be nontrivial, as it would break diffeomorphism invariance or modify the variational principle. A more natural approach is to correct the action itself, following the physical intuition developed from the emergence of the true DSSYK temperature.
As we have seen in section 2.5.4, when considering the sine-dilaton action, the Euclidean time periodicity β appears explicitly as a parameter fixing the smoothness of the Euclidean geometry at the horizon. As a first attempt, we consider modifying this smoothness condition by inserting a conical defect at the center of the disk [82, 88]:

$$\mathcal{V}_\gamma = \int \mathrm{d}x \sqrt{g}\, e^{-(2\pi-\gamma)\Phi} \,. \tag{2.279}$$

Hence we are changing the required periodicity of the Euclidean time circle from $\beta_{\text{BH}} = \frac{2\pi}{\sin\theta}$ to a more general $\beta = \frac{\gamma}{\sin\theta}$, where γ is the opening angle of the conical defect. We expect the modified classical partition function to become:

$$\mathcal{Z}^{(\text{defect})}_{\text{sine-dilaton}} = \int \mathrm{d}E(\theta) \exp\left(\frac{\gamma\theta}{2|\log q|} + \beta \frac{\cos(\theta)}{2|\log q|}\right). \tag{2.280}$$

However we should keep in mind that θ and γ are conjugate variables in this formulation and we cannot fix both. The correct procedure is to dynamically fix γ as a solution of the saddle-point equation. In other words, we promote γ to a dynamical field which is integrated over with a Gaussian weight peaked around $\gamma = 2\pi - 4\theta$:

$$Z(\beta) = \int d\gamma \, \exp\left[\frac{(2\pi-\gamma)^2}{16|\log q|}\right] \int dE(\theta)\, e^{-S_\gamma(E)} \,. \tag{2.281}$$

This ansatz reproduces, up to prefactors, the DSSYK partition function and captures the shift between the black hole temperature β_{BH} and the true thermodynamic temperature $\beta = \frac{2\pi - 4\theta}{\sin\theta}$.
This choice matches only semiclassically the partition function of DSSYK. However, it is interesting to notice that the exact quantum formulation of DSSYK leads to a rather similar object, which can be written as:

$$\begin{aligned}
&\text{Tr}\Big(e^{-\beta H_{\text{SYK}}}\Big) = \\
&= \int \mathrm{d}\gamma\, \exp\left(\frac{(2\pi-\gamma)^2}{16|\log q|}\right) \sum_{n=-\infty}^{+\infty} \int \mathrm{d}E(\theta)\, \sinh\left(\frac{\gamma(\theta+2\pi n)}{2|\log q|}\right) \exp\left(\beta \frac{\cos(\theta)}{2|\log q|}\right) \\
&= \int \mathrm{d}E(\theta)\, \rho(E)\, \exp\left(\beta \frac{\cos(\theta)}{2|\log q|}\right)
\end{aligned} \tag{2.282}$$

Where:

$$\rho(E) = \sum_{n=-\infty}^{+\infty} (-1)^n e^{-\frac{1}{|\log q|}\left(\theta+\pi(n-\frac{1}{2})\right)^2} \tag{2.283}$$

A possible interpretation is the following: the main difference is that in sine-dilaton, we have inserted a single defect, whereas in the full DSSYK theory there is a sum over multiple saddles. We can postulate that, being the dilaton potential periodic, the metric possesses singular points for each period, requiring the insertion of multiple defects. Perhaps, to fully reproduce DSSYK, we must sum over a lattice of defects.
Staying with the semiclassical single-defect reformulation of sine-dilaton gravity, we expect that matter correlators still decay with period β_{BH}, as they probe the smooth geometry surrounding the defect, and are thus insensitive to the thermodynamic modification induced by the constraint. This would reproduce the semiclassical behavior of correlation functions in DSSYK, where the decay is governed by the black hole temperature despite the underlying ensemble being weighted with the true temperature β.
From a more technical point of view, we should remember that the correlators are computed in Lorentzian signature, as shown in [82]. We cannot simply perform an analytic continuation; the construction instead follows a standard thermofield double approach: the density matrix is prepared as a path integral over a Euclidean manifold, which can be viewed as composed of two halves, left and right. To preserve the geometric interpretation in the resulting Lorentzian geometry, each half is modified by inserting a half-defect operator. This results in a smooth Euclidean geometry with a total angular opening of γ once the two halves are glued. The Euclidean path integral, with half a defect on each side, prepares a two-sided entangled state. This state is then analytically continued to Lorentzian signature by evolving forward in real time on both sides.
The important point is that, since the total Euclidean geometry has a conical opening

angle γ, the Euclidean time circle must have periodicity such that there is no conical singularity. This ensures that the resulting Lorentzian geometry is smooth at the horizon and corresponds to a black hole with Hawking temperature $\beta_{\rm BH}^{-1}$. Hence, matter fields propagating in the Lorentzian black hole background experience thermal correlators with periodicity $\beta_{\rm BH}$, despite the fact that the total weight of the Euclidean path integral is controlled by the original parameter β.
As a result, thermodynamic quantities are governed by the true temperature β^{-1}, while the correlators, which depend on the Lorentzian geometry, are sensitive to the periodicity of the Euclidean circle and hence decay with $\beta_{\rm BH}$.

2.5.7 Canonical Quantization

The canonical quantization of sine-dilaton gravity provides a natural framework to appreciate how the periodicity of the dilaton potential leads to a discretization of lengths and enforces a positivity constraint. These features align remarkably well with the discretization of states and the length positivity constraint which emerge in DSSYK model, especially in the Bulk Hilbert Space formulation (2.1.8). This section is mainly based on the procedure of canonical quantization of sine-dilaton proposed in [82]. The procedure was inspired by [146], which we review in Appendix A.2. The idea is that the on-shell solution of 2d dilaton gravity can be brought into AdS_2-form by Weyl rescaling. In this way, the canonical quantization procedure becomes quite similar to the original by Harlow-Jafferis, except some details. In the analysis one is led to a 1D canonical system (with some symmetries), which can be quantized.
We consider the rescaled action of sine-dilaton gravity (2.262). Again, we choose the boundary conditions (2.268) and the standard classical solution (2.263). We can identify the energy of the saddle as the classical ADM Hamiltonian of the system, which depends on the position of the horizon θ. Basically, in order to identify the phase space, we can repeat the arguments in Appendix A.2, but impose the new value of the Hamiltonian. One reduces to a 1D quantum system:

$$\begin{aligned} \dot{T} &= 1\,, \\ \dot{\theta} &= 0\,, \\ H &= -\frac{\cos(\theta)}{2|\log q|} \\ \omega &= \mathrm{d}T\wedge \mathrm{d}H\,. \end{aligned} \tag{2.284}$$

Keeping in mind a Wheeler-deWitt approach, it is better to go to a different formulation of the canonical system, which is similar to a "particle on a circle". A similar change of variables was introduced by Harlow and Jafferis when quantizing JT gravity. The main idea is to introduce a notion of Weyl-rescaled length, which captures the action of matter on a particle non minimally coupled to the dilaton:

$$L = \int \mathrm{d}s \, e^{-i\Phi/2} \, . \tag{2.285}$$

The Weyl rescaling maps sine-dilaton gravity into the Rindler wedge of AdS_2, in which the particle is minimally coupled:

$$\mathrm{d}s^2 e^{i\Phi} = -\sin(\theta)^2 \sinh(\rho)^2 \mathrm{d}t^2 + \mathrm{d}\rho^2 \tag{2.286}$$

The usual argument on the absence of conical singularities gives a BH temperature $\beta_{AdS} = 2\pi/\sin(\theta)$; hence we are computing the length of the Einstein-Rosen bridge in AdS_2, which (up to renormalization) is well known:

$$e^{-L} = \frac{\sin(\theta)^2}{\cosh(\sin(\theta)T/2)^2} . \tag{2.287}$$

The same expression for the length is found in the quantization of JT gravity (eq. 2.32 in [146]), however, going back to the sine-dilaton gravity picture, the relation between ADM energy E and position of the horizon θ is different. Precisely, the map is:

$$e^{-L} = \frac{1 - 4|\log q|^2 E^2}{\cosh\left(\left(1 - 4|\log q|^2 E^2\right)^{1/2} T/2\right)^2} \, , \tag{2.288}$$

In order to invert the Hamiltonian in terms of the new variables, it is required to find the conjugate momentum P such that:

$$\omega = \frac{1}{2|\log q|} \mathrm{d}L \wedge \mathrm{d}P \, . \tag{2.289}$$

The right choice is:

$$e^{-\mathrm{i}P} = -\mathrm{i} \sin(\theta) \tanh(\sin(\theta)T/2) + \cos(\theta) \, . \tag{2.290}$$

Remarkably the system (2.288) (2.290) can be explicitly inverted[26], and by substitution in the Hamiltonian (2.284) we get the particle-on-a-circle representation:

$$H = -\frac{\cos(P)}{2|\log q|} + \frac{1}{4|\log q|} e^{iP} e^{-L} \,. \tag{2.291}$$

If one identifies a $L = 2|\log q|n$, this expression is exactly the transfer matrix of DSSYK. In order to quantize, it is required a commutation relation, which is inferred from the symplectic form: $[L, P] = 2\mathrm{i}|\log q|$ and again, matches the quantization condition in DSSYK. However, a really important point is that, in DSSYK, the chord number n is a positive definite quantity, while, so far, in the gravitational picture we have not given a motivation for the condition $L > 0$. Actually a more careful treatment of the divergences in the energy (due to the periodicity of the potential), induces a gauge fixing condition, which implies that the variable L is definite positive. In such a treatment also the discretization of the Weyl-rescaled geodesics length emerges naturally, as shown in [80].

The existence of a bound on the length states, is compatible with the Wheeler-de Witt formulation of the gravitational path integral, in which the evolution between vacuum states is implemented by the gravitational Hamiltonian operator. Having this in mind, we can write the thermal partition function of this two-sided geometry:

$$Z(\beta) = \langle L = 0 | \, e^{-\beta \mathbf{H}_{\text{grav}}} \, | L = 0 \rangle \ . \tag{2.292}$$

Formally, this solution is exactly equivalent to the thermal partition function of DSSYK. As a result, we can interpret DSSYK as the system which encodes the canonical quantization of two sided sine-dilaton gravity.

As a final comment, we notice how, in this case, the discretization of space-time emerges naturally, differently from JT gravity. Indeed, if we admit that the variable L is continuous (with P periodic), we arrive at a paradox: since P is canonically conjugate to L, we cannot physically distinguish between states which do not differ for integer multiples of $|\log(q)|$.

The next Chapter, entirely based on [81], is devoted to the analysis of one-loop properties of the q-Liouville formulation. For reasons of logical coherence, we found it useful to preserve the work as much as possible in its entirety, even though certain sections present slight overlaps with the introductory chapter just concluded.

[26]To our knowledge, the only two examples in which it is possible are sine-dilaton gravity and JT gravity.

Exploring one-loop quantum corrections 3

In this chapter, based on [81], we provide non-trivial checks of the proposed duality between DSSYK and sine-dilaton gravity, studying the path integral at one-loop level. Specifically, we compute the logarithmic correction to the free energy of sine-dilaton gravity and, up to potential ordering ambiguities, we find a match with the corresponding quantity in DSSYK. The computation relies on the description of sine-dilaton gravity in terms of a version of the q-Liouville theory, the quantum deformation of the standard Schwarzian model dual to JT gravity. A crucial aspect of the calculation is selecting the correct Hartle-Hawking vacuum for the gravitational theory, which implies a specific choice of boundary conditions for the one-loop determinant, computed using a generalization of the Gel'fand-Yaglom's theorem. We also evaluate the gravitational one-loop correction to the boundary to boundary propagator of a non-minimally coupled matter field in the bulk theory, showing a perfect agreement with the corresponding quantum correction of matter correlators in DSSYK.

3.1 Context

The Sachdev-Ye-Kitaev (SYK) model [24, 28, 37, 152] is a quantum mechanical system comprising N Majorana fermions interacting through random all-to-all p-local interactions. Historically, it was first introduced[1] in condensed matter physics as a model for strange metals [24, 37, 155]. More recently, it has garnered significant attention within the holographic community [29, 51, 61, 145, 152, 156–159]. The SYK model serves as a toy model for testing quantum gravity, representing one of the rare cases that is both analytically tractable in the infrared (IR) and other regimes while also featuring a maximal chaos exponent.

More specifically, in the IR, the model exhibits a conformal regime with a pattern

[1] Earlier incarnations in nuclear physics appeared in [153, 154]

of symmetries encoded by Schwarzian quantum mechanics, which governs the effective dynamics of gravity on AdS_2 coupled to a scalar field [28, 29, 64, 152, 160]. These insights have been derived by analyzing the SYK model in the large-N limit, where the interaction order p is held fixed. Within this framework, one can formulate Schwinger-Dyson consistency equations for the two-point function [28], which are solvable in the IR using a conformal ansatz [28, 29, 35]. At low energies, an emergent reparametrization symmetry gives rise to Schwarzian theory as the effective low-energy description of the SYK model, establishing a duality with the well-known JT gravity [29, 31, 51, 161].

This correspondence has advanced our understanding of quantum black holes and quantum gravity in AdS_2. However, finding the holographic bulk dual of the complete SYK model remains an open question. To move beyond the low-energy limit, a different large-N scaling has been proposed, offering a more comprehensive solution across all energy scales. In this approach, p is not held fixed as $N \to \infty$; instead a new parameter $|\log q| = \frac{p^2}{N}$ [2] is kept finite, defining the so-called double-scaled SYK (DSSYK) model [61, 63, 64, 143]. The JT regime is recovered in the limit $|\log q| \to 0$, while focusing on low-energy dynamics.

Importantly, all DSSYK amplitudes have been calculated using Hamiltonian methods [63, 64, 143], revealing a remarkable structural similarity to the amplitudes computed for JT gravity [35, 162–165]. This resemblance naturally raises the question of whether DSSYK admits a gravitational dual. Such a dual could provide a UV completion of JT gravity and possibly capture additional features of the full SYK model [28, 79, 82, 88, 138, 166–168]. A plausible direction is to identify a dual gravity model within the general class of two-dimensional dilaton gravity theories.

A remarkable example is sinh-dilaton gravity, also known as Liouville gravity [88, 149, 169–173], which can be viewed not only as a 2D quantum gravity theory but also as a non-critical string theory involving one time-like and one space-like Liouville field. This model has garnered significant attention, particularly due to its string-theoretical construction [172], which extends the so-called minimal string framework [170, 174, 175]. Its amplitudes can be expressed as worldsheet CFT correlators integrated over the moduli space of Riemann surfaces, and a non-perturbative formulation has been proposed using random matrix theory [172, 176, 177]. On the field theory side, the model is exactly solvable. This solvability can be explained in terms of its symmetry structure, which is governed by a quantum group: the modular double of $SL_q(2,\mathbb{R})$ [170].

Another noteworthy example is sine-dilaton gravity [79, 82], distinguished by a periodic dilaton potential. In this case, the symmetry structure is still governed by a quantum

[2] We follow here the convention of [79, 82, 88], where the definition of q differs from that in [63, 64, 138] by $q^2 = q_{\text{there}}$. Hence $\lambda_{\text{there}} = 2|\log q|$.

group, specifically $SU_q(1,1)$. The model can also be formulated in terms of two Liouville fields with complex conjugate central charges [82], and a proposed string-theoretical formulation has recently been introduced [75]. An intriguing feature of sine-dilaton gravity is that the effective local cosmological constant lacks a definite sign, thereby opening the possibility of describing de-Sitter geometries within this framework. This aligns with the expectation that DSSYK is related to de-Sitter physics, or in general to cosmological spacetimes [66, 67, 70, 73, 74, 141].

The dilaton gravities discussed above share fundamental characteristics with their solvable linear counterpart, JT gravity. JT gravity can be expressed in a first-order formulation as a topological BF theory, with all its features encoded in the boundary dynamics. Its description involves a quantum particle on a non-compact group manifold, which is directly linked to the $SL(2,\mathbb{R})$ symmetry of the Schwarzian action. Similarly, sinh-dilaton and sine-dilaton gravities are also topological, as they can be reformulated as Poisson sigma models [79, 82, 97, 98, 100, 151, 161]. While their dynamics remain confined to the boundary, the group manifold is now described by a quantum group the modular double or $SU_q(1,1)$ depending on whether the deformation parameter is real or complex.

The energy in these systems corresponds to the Casimir operator of the relevant quantum group, whose classical limit aligns with the Hamiltonian derived from the quantum Schwarzian [79]. Notably, this Hamiltonian coincides with the DSSYK transfer matrix [64, 138], suggesting a clear duality between the gravitational theory and the statistical model. Furthermore, the quantum group framework provides valuable insights into the boundary conditions required for the dual gravitational description [79].

By starting with a continuum dynamical system describing a particle on $SU_q(1,1)$ and imposing appropriate constraints, one can derive a q-Schwarzian phase space path integral for a single asymptotic boundary, and a q-Liouville phase space path integral for a Cauchy slice with two asymptotic boundaries [79, 88]. The amplitudes of the resulting constrained quantum mechanical system are equivalent to those of DSSYK.

The precise holographic duality between DSSYK and 2d sine-dilaton gravity has been explored more directly in [82]. There, the ADM Hamiltonian of sine-dilaton gravity was reformulated as a q-Liouville model via a canonical transformation, similar to what was done in the JT case [146]. However, this matching is subtle and requires careful consideration of key elements in the gravity model, such as the distinction between fake and real temperatures, the choice of the Hartle-Hawking vacuum, and the imposition of non-trivial geometric constraints [82].

This approach also allows for the study of correlation functions. For example, in the JT case, correlation functions of a scalar field in the bulk can be analyzed via their duality with conformal bilocal operators on the boundary [33, 35, 178]. In the BF formalism, this corresponds to evaluating the vacuum expectation values of a system of anchored Wilson lines. Similarly, for sine-dilaton gravity, one can introduce a suitably non-minimally coupled probe in the bulk [82], which corresponds to dual bilocal operators in the q-Schwarzian system.

The equivalence between the Hamiltonian of sine-dilaton gravity and the DSSYK transfer matrix naturally leads to the identification of the same amplitudes. However, it is important to investigate the path-integral formulation of the theory in detail, as it would offer a deeper understanding of the underlying gravitational dynamics. This approach was originally used in the context of JT gravity, where the path integral was performed over the Schwarzian mode, associated with the spontaneous breaking of conformal symmetry [29].

While the canonical quantization picture of sine-dilaton gravity is well understood [80], this paper provides a computation of both the partition function and the two-point correlators of sine-dilaton gravity, going beyond the leading semiclassical regime within the path integral framework. The motivation is twofold: first, to provide a non-trivial check of the proposed duality between DSSYK and the dilaton gravity model with a sine potential; and second, to gain a better understanding of the path integral formulation of sine-dilaton gravity. As we will see, our results can be interpreted as measuring the first quantum correction to the effective scalar curvature in sine-dilaton gravity and, through this, we will be able to test the reliability of the semiclassical weak-gravity expansion of the gravitational path integral for sine-dilaton gravity.

It is worth noting that the one-loop path integral over the Schwarzian mode in JT gravity already provides a complete answer, thanks to localization [30, 34]. However, this is not the case here. For the correlators, the matching is even more complicated in standard JT gravity [179–182], and going beyond the leading-order behavior is necessary to extract any physical meaning from the exact expressions. In our case, explicit computations could provide insight into more intricate aspects of the gravitational theory, such as the chaotic properties encoded in out-of-time-order correlators or the scattering of shock waves.

From a more formal perspective, one could also explore the contributions of non-trivial saddles (or instantons), if they exist. For instance, it should be possible to explain the structure of the physical spectrum. A similar effect was observed in JT gravity at finite cutoff, where the inclusion of non-perturbative saddles accounted for modifications to the spectral properties [183].

Concretely, we compute here the logarithmic correction to the free energy of sine-dilaton gravity and find that it agrees with the corresponding quantity in DSSYK, differing only by a simple numerical factor, which can be attributed to potential ordering ambiguities. A key technical aspect of the calculation involves the selection of the correct Hartle-Hawking vacuum for the gravitational theory, as this choice determines the boundary conditions for the one-loop determinant, that is evaluated using a non-trivial generalization of Gel'fand-Yaglom's theorem. Furthermore, we compute the one-loop correction to the boundary-to-boundary propagator of a non-minimally coupled matter field in the bulk theory, finding a perfect match with the corresponding quantum correction of matter correlators in DSSYK.

The structure of the paper is as follows: In Section 3.2, we provide the necessary background, reviewing the transfer matrix of double-scaled SYK, the duality between sine-dilaton gravity and DSSYK, and its semiclassical expansion. Section 3.3 is dedicated to the one-loop computation of the partition function, starting from the path integral formulation. We discuss the technical details regarding the choice of boundary conditions and the evaluation of the relevant functional determinant. Section 3.4 extends our analysis to the two-point correlation functions, showing a precise match with the corresponding result from the DSSYK model. Our conclusions and future directions are presented in Section 3.5. Additionally, we include two technical appendices: one for the computation of the Green functions used in the main text, and another for the Poisson sigma model formulation of sine-dilaton gravity.

3.2 Background material

3.2.1 The transfer matrix of double-scaled SYK

In this section, we will settle the notation and recover the material needed for the technical work. We will focus on the transfer matrix of DSSYK. Indeed, the system becomes exactly solvable via an auxiliary quantum mechanics [63, 64, 138]. This overview is not aimed to be self-contained, but rather to introduce the essential elements required to support the discussion of the gravitational dual in the following sections. The general context should be clear from the previous chapters of this work.

We known that the SYK model consists of a system of N Majorana fermions ψ_i ($i = 1, \ldots, N$) that satisfy the anti-commutation relations $\{\psi_i, \psi_j\} = 2\delta_{i,j}$. The dynamics are governed by an all-to-all p-body interaction, described by the Hamiltonian

$$H_{\text{SYK}} = i^{p/2} \sum_{1 \leq i_1 < \cdots < i_p \leq N} J_{i_1 \cdots i_p} \psi_{i_1} \cdots \psi_{i_p} \,, \tag{3.1}$$

where the couplings $J_{i_1 \cdots i_p}$ are typically assumed to be Gaussian random variables.

In the double-scaling limit, N and p are taken to infinity, while keeping the ratio

$$|\log q| = \frac{p^2}{N} \tag{3.2}$$

constant and finite. To explain the transfer-matrix approach to DSSYK, it is useful to focus on the moments

$$m_k = \langle \mathrm{tr} H^k \rangle_J \tag{3.3}$$

of the partition function $\langle \mathrm{tr} e^{-\beta H} \rangle_J$, where $\langle \rangle_J$ represents the averaging over the random couplings.

The various moments can be visually represented using what are known as chord diagrams [63, 64, 138]. To construct the chord diagram corresponding to the k$^{\mathrm{th}}$ moment, we begin by drawing a circle to represent the cyclic trace, and place k nodes on it, each corresponding to an insertion of the Hamiltonian as in (3.3). We then connect pairs of nodes with chords, which arise from the application of Wick's theorem when we perform the averaging over the random couplings. See for instance Figure 1 in [143].

We finally need to sum over all possible Wick contractions, which corresponds to summing over all chord diagrams with k nodes. In this sum, the value of each diagram is weighted by the number of chord intersections. Specifically, as determined in [63, 64], each intersection contributes a factor of q^2 in the double scaled regime, and the total expectation value of the moment is given by

$$m_k = \sum_{\text{chord diagrams with } k \text{ nodes}} q^{2\#}, \tag{3.4}$$

where $\#$ stands for the total number of intersections associated with the diagram.

The sum in (3.4) is generally difficult to compute, but one can handle it by iteratively constructing all chord diagrams using a transfer matrix [63, 64]. Starting from a point on the circle with no open chords, we move clockwise, opening or closing chords at each node. After k steps, all chords must be closed. This method uniquely generates all possible chord diagrams [3]. During the construction, if there are n open chords, opening a new chord adds no intersections and increases the number of open chords to $n + 1$. When closing a chord, the number of intersections depends on its position in the stack, contributing factors of $1, q^2, \ldots, q^{2n-2}$, according to (3.4). Summing over all possible diagrams gives a factor of $\frac{1-q^{2n}}{1-q^2}$ for each chord closure. We can thus define the creation and annihilation operators, $\alpha^\dagger$ and α, as well as the chord number operator $\hat{n}$, through

[3]See for instance Figure 2 in [143]

their action on the chord Hilbert space with basis states $|n\rangle$:

$$\hat{\alpha}\,|n\rangle = |n-1\rangle\,, \qquad \hat{\alpha}^\dagger\,|n\rangle = |n+1\rangle\,, \qquad \hat{n}\,|n\rangle = n\,|n\rangle\,. \tag{3.5}$$

The transfer matrix can now be expressed in terms of these operators as [4]:

$$\sqrt{2|\log q|}\hat{T} = \hat{\alpha}^\dagger + \hat{\alpha}\frac{1-q^{2\hat{n}}}{1-q^2}. \tag{3.6}$$

The moments are computed as the transition matrix elements of the k^{th} power of the transfer matrix between the initial and final states with no chords, i.e. $m_k = \langle 0|\hat{T}^k|0\rangle$. Finally, summing over the moments, the partition function of double-scaled SYK is given by:

$$Z_{\text{DSSYK}}(\beta) = \langle 0|e^{-\beta\hat{T}}|0\rangle. \tag{3.7}$$

The spectrum and eigenstates of the transfer matrix (3.6) are known, enabling us to compute the partition function exactly, as we will shortly demonstrate in 3.2.3.

Matter operators $\mathcal{O}_\Delta$ can also be introduced in DSSYK [64] and are defined as

$$\mathcal{O}_\Delta = \mathrm{i}^{\Delta p/2} \sum_{i_1<\cdots<i_{\Delta p}} M_{i_1\ldots i_{\Delta p}}\psi_{i_1}\ldots\psi_{i_{\Delta p}}\,, \tag{3.8}$$

where $M_{i_1\ldots i_{\Delta p}}$ are (Gaussian) random variables drawn independently from the random couplings of the Hamiltonian and the dimension Δ of the operator is related to the number of interacting fermions in (3.8), with $0<\Delta<1$. The two-point correlation function of (3.8), with the matter operators placed at $\tau_1 = \tau$ and $\tau_2 = \beta-\tau$, is computed by $\langle \mathcal{O}_\Delta(\tau)\mathcal{O}_\Delta(\beta-\tau)\rangle = \mathrm{Tr}\left(e^{-\tau H_{\text{SYK}}}\mathcal{O}_\Delta e^{-(\beta-\tau)H_{\text{SYK}}}\mathcal{O}_\Delta\right)$. In the double scaling limit, this observable can be evaluated once again using the auxiliary quantum mechanical system (3.6) as a transition matrix element:

$$\langle \mathcal{O}_\Delta(\tau)\mathcal{O}_\Delta(\beta-\tau)\rangle = \langle 0|\, e^{-\tau\hat{T}}\, e^{-2\Delta|\log q|\hat{n}}\, e^{-(\beta-\tau)\hat{T}}\,|0\rangle\,. \tag{3.9}$$

We point out that, at this stage, the transfer matrix serves merely as a combinatorial tool to solve the system in the double-scaled regime. However, we will later uncover a physical interpretation of this auxiliary quantum mechanics in terms of a dual bulk gravitational model.

[4] The normalization of $\hat{T}$ in left hand side of (3.6) is standard. See for instance [138].

3.2.2 Sine-dilaton gravity and DSSYK: the duality

As we know, in [79, 82, 88] a new holographic duality was proposed between DSSYK and sine-dilaton gravity. We remind that the theory is described by the following path integral:

$$\int \mathcal{D}g\mathcal{D}\Phi\, \exp\left(\frac{1}{2}\int \mathrm{d}x\sqrt{g}\left(\Phi R + \frac{\sin(2|\log q|\Phi)}{|\log q|}\right) + \int \mathrm{d}\tau\sqrt{h}\left(\Phi K - \mathrm{i}\,\frac{e^{-\mathrm{i}|\log q|\Phi}}{2|\log q|}\right)\right), \tag{3.10}$$

where the boundary terms correspond to the usual GHY term and the appropriate counterterm required for holographic renormalization [82].

We will now briefly show how the canonical quantization of sine-dilaton gravity (3.10) exactly reproduces the auxiliary quantum mechanics (3.6) of DSSYK. In order to do that, we need to classify the classical phase space of the theory we are going to quantize. To find the classical solutions, it is useful to rescale $2|\log q|\Phi \to \Phi$ and minimize the rescaled action

$$\frac{1}{2|\log q|}\left\{\frac{1}{2}\int \mathrm{d}x\sqrt{g}\left(\Phi R + 2\sin(\Phi)\right) + \int \mathrm{d}\tau\sqrt{h}\left(\Phi K - \mathrm{i}\,e^{-\mathrm{i}\Phi/2}\right)\right\}, \tag{3.11}$$

that, because of the overall $1/|\log q|$, is characterized by a reliable semiclassical regime when $|\log q| \ll 1$. [5] The classical solutions for a general dilaton gravity model have been classified [184, 185] in a gauge where the dilaton parametrizes the radial direction $\Phi = r$. For the sine potential, the solution for the metric takes the form

$$\mathrm{d}s^2 = F(r)\mathrm{d}\tau^2 + \frac{1}{F(r)}\mathrm{d}r^2\,, \qquad F(r) = -2\cos(r) + 2\cos(\theta)\,. \tag{3.12}$$

The geometry exhibits a black hole horizon at $r = \theta$ and a cosmological horizon at $r = 2\pi - \theta$, corresponding to surfaces with minimal and maximal areas, respectively. Actually the metric above suggests there is an infinite set of black holes horizons at $r = \theta + 2\pi n$ and an infinite set of cosmological horizons at shifted locations as well. As analyzed in [80, 82], the role of these infinite copies of the original geometry is crucial for recovering the full spectral density of DSSYK, as it is reproduced by an infinite set of saddles on the gravity side. However, for the purposes of this paper, we will focus only on the 'original' patch of the geometry. The theory (3.11) is complemented by the following boundary conditions [82]:

$$\sqrt{F}\,e^{\mathrm{i}\Phi_{\mathrm{bdy}}/2} = \mathrm{i}\,, \quad \Phi_{\mathrm{bdy}} = \frac{\pi}{2} + \mathrm{i}\infty\,. \tag{3.13}$$

[5] Since $|\log q|$ plays the role of G_N in (3.11), the $|\log q|$ expansion of DSSYK corresponds to a semiclassical gravity expansion in the gravity model (3.11).

Notice the asymptotics of the metric (3.12) is consistent with (3.13) as we expand around the boundary location Φ_{bdy}. The first condition in (3.13) is a Brown-Henneaux-type boundary condition, which can be motivated in the context of the *Poisson sigma model* reformulation of sine-dilaton gravity, as we have shown. The boundary location will instead by justified around equation (3.19).

The ADM energy of the gravity theory can then be easily identified as the subleading part of the metric (3.12) which remains finite as we approach the holographic boundary, yielding [6]

$$E_{\text{ADM}} = -\frac{\cos(\theta)}{2|\log q|}\,, \tag{3.14}$$

once we restore the dependence on $|\log q|$. We are now in a position to characterize the classical phase space of the theory. Given the structure of the metric (3.12), the black hole horizon area θ naturally emerges as a phase space variable in sine-dilaton gravity. However, as in JT gravity [146], we expect a two-dimensional phase space for this theory. To determine the second phase space variable, we observe that the Lorentzian form of two classical metrics, both described by (3.12) and sharing the same θ, can still differ due to the amount of Lorentzian time evolution T of two-sided spatial slices in their Kruskal extension. [7]

The classical phase space is thus spanned by (θ, T) and is characterized by the following symplectic measure:

$$\omega = \mathrm{d}T \wedge \mathrm{d}H_{\text{grav}} = \frac{\sin(\theta)}{2|\log q|}\,\mathrm{d}T \wedge \mathrm{d}\theta \tag{3.15}$$

where we identified the gravitational Hamiltonian H_{grav} with the ADM energy (3.14).

Following [146], we will now introduce more convenient variables in the classical phase space. Because of the form (3.32) of the semiclassical DSSK correlator looking the same as a correlator on a AdS$_2$ background, it is natural to look for a length variable that probes the latter geometry. This is given by the Weyl-rescaled length L [82]:

$$L = \int \mathrm{d}s e^{-\mathrm{i}\Phi/2}\,. \tag{3.16}$$

Indeed by setting

$$r = \frac{\pi}{2} + \mathrm{i}\log(\rho + \mathrm{i}\cos(\theta)) \tag{3.17}$$

[6]We have reintroduced the dependence on $|\log q|$, i.e. $F(\Phi_{\text{bdy}}) = -e^{-\mathrm{i}\Phi_{\text{bdy}}} + 4|\log q|E$.

[7]Different values of T will correspond to distinct initial conditions in sine-dilaton gravity.

the Weyl rescaling (3.16) takes the sine-dilaton metric (3.12) to the effective metric

$$\mathrm{d}s^2_{\text{eff}} = e^{-\mathrm{i}\Phi}\mathrm{d}s^2 = \left(\rho^2 - \sin(\theta)^2\right)\mathrm{d}\tau^2 + \frac{\mathrm{d}\rho^2}{\rho^2 - \sin(\theta)^2}\,, \tag{3.18}$$

which corresponds to an AdS_2 background. One can thus compute the length of the two-sided ERB associated to the Weyl-rescaled AdS_2 black hole (3.18), which is known to be given by:

$$e^{-L} = \frac{\sin(\theta)^2}{\cosh\left(\sin(\theta)T/2\right)^2}\,. \tag{3.19}$$

The length (3.19) is calculated by sending the holographic boundary to asymptotic infinity, following the standard procedure in AdS_2 holography. The position of the holographic screen in sine-dilaton gravity can then be inferred by taking $\rho \to +\infty$ in (3.17), which justifies the boundary condition chosen in (3.13). Interestingly, by performing the Weyl rescaling directly at the level of the action, one can rewrite the sine-dilaton gravity in a form where the classical solutions are directly (3.17) and (3.18), which are a real solution for the metric and a complex solution for the dilaton field.

$$S = \frac{1}{|\log q|}\left[\int dx\sqrt{g}\left(R\Phi - ie^{i\Phi}\Phi\nabla^2\Phi + 2\sin\Phi\right) + \int dx\sqrt{h}\left(K\Phi + \frac{i}{2}e^{\frac{i\Phi}{2}}\Phi n^\rho\partial_\rho\Phi - i\right)\right]. \tag{3.20}$$

Now we can find the canonical conjugate to the length L. This can be done by inverting (3.19), plugging it into the symplectic measure (3.15) and requiring the latter to take the canonical form in terms of the new variables L and P. One can readily show that by setting

$$e^{-\mathrm{i}P} = -\mathrm{i}\sin(\theta)\tanh(\sin(\theta)T/2) + \cos(\theta)\,, \tag{3.21}$$

the symplectic form indeed becomes $\omega = \frac{1}{2|\log q|}\mathrm{d}L \wedge \mathrm{d}P$. By inverting the expressions (3.19) and (3.21), one expresses the gravitational Hamiltonian H_{grav} in terms of L and P, resulting in:

$$H_{\text{grav}}\left(L, P\right) = -\frac{\cos(P)}{2|\log q|} + \frac{1}{4|\log q|}e^{iP}e^{-L} \tag{3.22}$$

When promoting the phase space variables L, P to quantum operators $\hat{L}, \hat{P}$ [8] and comparing the gravitational Hamiltonian (3.22) of sine-dilaton gravity with the double-scaled SYK transfer matrix (3.6), we establish the following dictionary:

$$\hat{L} \equiv 2|\log q|\,\hat{n}, \qquad \hat{\alpha} = -e^{\mathrm{i}\hat{P}}, \tag{3.23}$$

[8]The symplectic measure induces the following commutation relation: $\left[\hat{L}, \hat{P}\right] = 2\mathrm{i}|\log q|$, which is indeed consistent with the DSSYK commutation relation $[\hat{n}, \hat{\alpha}] = -\hat{\alpha}$ [138]. This can be easily proven using that $\left[A, e^{xB}\right] = xe^{xB}\left[A, B\right]$.

which shows that the two systems are governed by the same Hamiltonian, and are therefore equivalent [9] .

3.2.3 DSSYK semiclassics and one-loop corrections

In this section, we analyze the semiclassical approximation ($|\log q| \ll 1$) and the first one-loop correction to the DSSYK partition function (3.7) and the matter two-point function (3.9). These quantities will be our primary focus for comparison with the corresponding results in the gravitational model. We will first present the exact expressions for DSSYK and then explore their behavior in the limit $|\log q| \to 0$.

The eigenvalue equation associated with the transfer matrix $\hat{T}$ (3.6) is a difference equation. The spectrum of $\hat{T}$ is continuous and bounded, and can be parameterized by an angle $\theta \in [0, \pi]$. Specifically, we have:

$$\hat{T}\,|\theta\rangle = E(\theta)\,|\theta\rangle\,, \qquad E(\theta) = \frac{2\cos(\theta)}{\sqrt{2|\log q|(1-q^2)}}. \tag{3.24}$$

The eigenvectors $|\theta\rangle$ are given in the chord basis by the continuous q-Hermite polynomials, i.e. $\langle\theta|n\rangle = (1-q^2)^{n/2} H_n\left(\cos(\theta)|q^2\right)$[10]. By expanding (3.7) on the eigenvectors basis, the DSSYK partition function is given by [63, 64]

$$Z_{\text{DSSYK}}(\beta) = \int_0^\pi \mathrm{d}\theta\, (q^2, e^{\pm 2i\theta}; q^2)_\infty \, \exp\left(-\beta \frac{\cos(\theta)}{\sqrt{2|\log q|\,(1-q^2)}}\right), \tag{3.26}$$

where the DSSYK spectral density $\rho(\theta) = (q^2, e^{\pm 2i\theta}; q^2)_\infty$ is determined by ensuring the completeness relation $\int_0^\pi \mathrm{d}\theta \rho(\theta)\,|\theta\rangle\,\langle\theta| = \mathbb{1}$ [11]. The two-point function of DSSYK matter operators $\mathcal{O}_\Delta$ can also be computed exactly, by inserting two completeness relations in

[9] The semiclassical limit of the DSSYK transfer matrix, for $|\log q| \ll 1$, reduces to the classical gravitational Hamiltonian (3.22) in our conventions as $\hat{T} \xrightarrow[|\log q|\to 0]{} \frac{1}{2} H_{\text{grav}}$, which explains why our definition of the inverse temperature is related by $\beta_{\text{here}} = 2\beta_{\text{there}}$ [89]. Actually, as explained in [82], the two hamiltonians agree up to a harmless similarity transformation which preserves the identity.

[10] One can work with with the Hermitian version of (3.6), as for instance is done in [138], or compute separately the left and right eigenvectors

$$\langle\theta|n\rangle = (1-q^2)^{n/2} H_n\left(\cos(\theta)|q^2\right) \qquad \langle n|\theta\rangle = (1-q^2)^{-n/2}\frac{H_n\left(\cos(\theta)|q^2\right)}{(q^2;q^2)_n} \tag{3.25}$$

as in [79]. Here $\left(a;q^2\right)_n = \prod_{k=0}^{n-1}\left(1-aq^{2k}\right)$ denotes the q-Pochhammer symbol.

[11] We can compute the spectral density from the overlap

$$\langle\theta|\theta'\rangle = \sum_{n=0}^{+\infty}\langle\theta|n\rangle\langle n|\theta'\rangle = \sum_{n=0}^{+\infty}\frac{1}{(q^2;q^2)_n} H_n\left(\cos(\theta)|q^2\right) H_n\left(\cos(\theta')|q^2\right) = \frac{\delta\left(\theta-\theta'\right)}{(e^{\pm 2i\theta};q^2)_\infty} \tag{3.27}$$

.

(3.9) and obtaining

$$\langle \mathcal{O}_\Delta(\tau)\mathcal{O}_\Delta(\beta-\tau)\rangle = \frac{1}{Z(\beta)}\times$$
$$\int_0^\pi \mathrm{d}\theta_1\,\rho(\theta_1)\int_0^\pi \mathrm{d}\theta_2\,\rho(\theta_2)\exp\left(-\frac{(\beta-\tau)\cos(\theta_1)}{\sqrt{2|\log q|\,(1-q^2)}}-\frac{\tau\cos(\theta_2)}{\sqrt{2|\log q|\,(1-q^2)}}\right)\langle\theta_1|\,q^{2\Delta\hat{n}}\,|\theta_2\rangle\ , \tag{3.28}$$

where the final matrix element yields [63, 64, 79]:

$$\langle\theta_1|\,q^{2\Delta\hat{n}}\,|\theta_2\rangle = \frac{(q^{4\Delta};q^2)_\infty}{(q^{2\Delta}e^{\pm\mathrm{i}\theta_1\pm\mathrm{i}\theta_2};q^2)_\infty}\ . \tag{3.29}$$

The semiclassical analysis of the DSSYK partition function and two-point function has been covered in [89, 90]. Specifically, the leading order and first one-loop correction can be extracted by performing a saddle point approximation of the exact integrals (3.26) and (3.28) in the limit $|\log q| \ll 1$. Here we will not repeat the derivation but just quote the results. The semiclassical approximation for the DSSYK partition function is:

$$Z_{\mathrm{cl}} = \exp\left[-\frac{\left(\frac{\pi}{2}-\theta\right)^2-(\pi-2\theta)\cot(\theta)}{|\log q|}\right]\ , \tag{3.30}$$

while the one-loop correction is given by [89]

$$Z_{\mathrm{one-loop}} = \frac{\sin(\theta)}{\sqrt{1+(\frac{\pi}{2}-\theta)\cot(\theta)}}e^{(\frac{\pi}{2}-\theta)\cot(\theta)}\ , \tag{3.31}$$

The semiclassical approximation for the DSSYK two-point function is:

$$\langle \mathcal{O}_\Delta(\tau)\mathcal{O}_\Delta(\beta-\tau)\rangle_{\mathrm{cl}} = \frac{\sin(\theta)^{2\Delta}}{\sin(\sin(\theta)\tau/2+\theta)^{2\Delta}}\ . \tag{3.32}$$

Including the one-loop correction, one obtains

$$\langle \mathcal{O}_\Delta(\tau)\mathcal{O}_\Delta(\beta-\tau)\rangle = \langle \mathcal{O}_\Delta(\tau)\mathcal{O}_\Delta(\beta-\tau)\rangle|_{\mathrm{cl}}\left(1+|\log q|\left(\Delta^2\mathcal{I}+\Delta\mathcal{A}\right)+\mathcal{O}(|\log q|^2)\right)\ . \tag{3.33}$$

where $\mathcal{I}$ and $\mathcal{A}$ are non-trivial functions of τ and θ, given by [90]:

$$\begin{aligned}
\mathcal{I} &= -\frac{(\tan\zeta + \tan\zeta(\zeta+u)\tan u + \tan u)(\tan\zeta + \tan u(\tan\zeta(u-\zeta)-1))}{u\tan u + 1} \\
\mathcal{A} &= \frac{1}{2(1+u\tan u)}\left[-\frac{(1+u\tan u)^2}{\cos^2\zeta} + \frac{(1+\zeta\tan\zeta)^2}{\cos^2 u} + \zeta^2(\tan^2 u - \tan^2\zeta)\right. \\
&\qquad \left. - \frac{1+\zeta\tan\zeta}{1+u\tan u} + 1\right]
\end{aligned} \tag{3.34}$$

where we defined $u = \frac{\pi}{2} - \theta$ and $\zeta = \frac{\pi}{2} - \theta - \frac{\tau}{2}\sin\theta$. We will recover the one-loop expressions (3.31) and (3.34) from the gravity side in 3.4.

3.3 Path integral formulation

In the case of JT gravity, the gravitational path integral reduces to a description in terms of Schwarzian quantum mechanics [29, 31, 51, 161]. The path integral over the Schwarzian mode enables the calculation of several important gravitational observables at the disk level. In this section, we provide a path integral formulation of sine-dilaton gravity, which will serve as the framework for the computations presented in sections 3.3.2 and 3.4, the main results of this work.

Analogously to the JT case, one can construct a path integral formulation for the quantum mechanics of sine-dilaton gravity, given by (3.22). Following Feynman prescription, this is given by

$$\mathcal{Z}_{\text{grav}}^{\text{naive}} = \int \mathcal{D}\varphi\mathcal{D}p \quad \exp\left\{\frac{1}{|\log q|}\int_0^\beta \mathrm{d}\tau \left(\mathrm{i}p\frac{\mathrm{d}}{\mathrm{d}\tau}\varphi + \frac{1}{2}\cos(p) - \frac{1}{4}e^{\mathrm{i}p}e^{-2\varphi}\right)\right\}, \tag{3.35}$$

where the action just follows from the Legendre transform of the Hamiltonian (3.22) and, for convenience, we defined $L \equiv 2\varphi$. Based on the derivation presented in 3.2.2, equation (3.35) captures the full gravitational path integral (3.10) for sine-dilaton gravity, just as the Schwarzian path integral computes the JT gravity partition function [30, 35].

Notably, the path integral (3.35) was actually derived from the first time in [79] in the context of quantum groups. There, the same quantum system above is embedded into a more general theory with 6 fields $(\phi, \beta, \gamma, p_\phi, p_\beta, p_\gamma)$, which describes the motion of a particle on $SU_q(1,1)$ [12]. This more general theory reduces to the so-called q-Liouville theory, which exactly corresponds to the path integral (3.35), when two constraints are imposed on the classical phase space of the group manifold [13]. In the gravity context,

[12]The fields (ϕ, β, γ) represent the coordinates over the quantum group manifold.

[13]These constraints allow one to gauge-fix $\beta = \gamma = 0$ and, in doing so, reduce the theory to the form

these constraints on the quantum group implement two asymptotic Brown-Henneaux-type boundary conditions of the form (3.13) [79]. The precise correspondence of these quantities ia a result of the link between the first order formulation of sine-dilaton gravity and the quantum group $SU_q(1,1)$.

Since sine-dilaton gravity can be viewed as a particular deformation of JT gravity [14], the associated path integral (3.35) can be interpreted as the corresponding deformation of the boundary path integral. One can, in fact, rescale $p \to p|\log q|$ in the action (3.35), expand as $|\log q| \ll 1$ and integrate out p, recovering the usual Liouville action for φ, corresponding to the two-sided formulation of JT gravity [33, 79, 146] [15]. Unlike JT though, due to the structure of the action (3.35), it is not possible to simply integrate out the momentum and reduce the system to a single boundary mode. As we shall see, this complicates computations in this theory.

Before moving on, we should actually consider a modification of the gravitational path integral (3.35) if we aim to recover the DSSYK amplitudes. A puzzling aspect of the duality between sine-dilaton gravity and DSSYK is that, despite both systems being described by the same Hamiltonian, as shown in (3.2.2), a naive semiclassical analysis of the sine-dilaton path integral (3.10) does not align with the corresponding analysis in DSSYK. This mismatch arises from the difference between the microscopic temperature of DSSYK:

$$\beta_{\text{DSSYK}} = \frac{2\pi - 4\theta}{\sin(\theta)} \tag{3.36}$$

and the so-called "fake" temperature [62, 68]

$$\beta_{\text{fake}} = \frac{2\pi}{\sin(\theta)},$$

which corresponds, in the gravity framework, to the Hawking temperature associated with the black hole background metric (3.12). This issue has been extensively addressed in [82]. The resolution relies on a different choice of the Hartle-Hawking vacuum for the gravity theory. Naively, one might associate this vacuum with the state $|-\infty\rangle$, because the length L, as computed in (3.19), corresponds to the holographically renormalized geodesic distance in the effective AdS_2 geometry (3.18), which approaches $-\infty$ as the Euclidean separation of the boundary operators approaches $L_{\text{real}} = 0$. Therefore, the naive partition function $\langle -\infty | \, e^{-\beta H_{\text{grav}}} \, |-\infty\rangle$, associated with the path integral (3.35), would reproduce the incorrect result with the fake temperature, which fails to match

(3.35).

[14]The leading term in the expansion of the dilaton potential in the action (3.10) when $|\log q| \ll 1$ yields the linear dilaton potential of JT gravity. To recover JT in the quantum amplitudes, one also has to zoom in the region $\theta \ll 1$.

[15]In this language, the ordinary Schwarzian is the one-boundary description of JT gravity

DSSYK.

On the other hand, the dictionary (3.23) and the DSSYK partition function (3.7) suggest choosing $|L=0\rangle$ as the Hartle-Hawking vacuum state. This choice represents a non-trivial geometric constraint for the gravity theory. The justification for this different choice of the Hartle-Hawking vacuum will be rigorously provided in upcoming work [80], where gauging a global symmetry in the bulk theory (3.10) will be shown to render all negative-length states $|L<0\rangle$ as null states, thus projecting them out of the physical Hilbert space.

While we will not delve into these details here, we state that the correct modification of the naive gravitational path integral, which matches DSSYK at temperature (3.36), is:

$$\mathcal{Z}_{\text{grav}} = \int_{\varphi(0)=\varphi(\beta)=0} \mathcal{D}\varphi\mathcal{D}p \ \exp\left\{\frac{1}{|\log q|}\int_0^\beta \mathrm{d}\tau \left(\mathrm{i}p\frac{\mathrm{d}}{\mathrm{d}\tau}\varphi + \frac{1}{2}\cos(p) - \frac{1}{4}e^{\mathrm{i}p}e^{-2\varphi}\right)\right\} \tag{3.37}$$

where we explicitly impose the boundary condition $\varphi(0)=\varphi(\beta)=0$, which ensures the correct periodicity β for the thermal circle.

In DSSYK, we also introduced the two-point function (3.9) of matter operators (3.8). We can ask ourselves what the dual observable is in the gravity theory (3.10), which turns out to be the boundary-to-boundary propagator of a massive bulk probe that couples to the length L (3.19). The action of this probe is given by [82]:

$$S_{\text{matter}} = \int \mathrm{d}^2x\sqrt{g}\left(g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + m^2 e^{-2\mathrm{i}|\log q|\Phi}\chi^2\right) = \int \mathrm{d}^2x\sqrt{g_{\text{eff}}}\left(g^{\mu\nu}_{\text{eff}}\partial_\mu\chi\partial_\nu\chi + m^2\chi^2\right), \tag{3.38}$$

which describes a scalar field non-minimally coupled to the metric, meaning it also couples to the dilaton, as shown in (3.38). This coupling ensures that the matter field probes the effective metric (3.18), which corresponds to an AdS_2 geometry. According to standard AdS/CFT holography, the dual of the boundary-to-boundary propagator of the matter field χ is represented by the bilocal operator $e^{-2\Delta\varphi}$ [79, 164] [16]. The expectation value of $e^{S_{\text{matter}}}$ in the gravitational path integral (3.10), in the regime where the probe is light and does not back-react on the geometry, is thus computed by inserting $e^{-2\Delta\phi}$ into the path integral (3.37), i.e.,

$$\langle e^{S_{\text{matter}}}\rangle_{\text{grav}} \simeq \int_{\varphi(0)=\varphi(\beta)=0} \mathcal{D}\varphi\mathcal{D}p \ e^{-2\Delta\varphi} \ \exp\left\{-\frac{S_{\text{qLiouville}}(\varphi,p)}{|\log q|}\right\}, \tag{3.39}$$

where the symbol $\simeq$ indicates that we consider only the saddle-point approximation of the matter path integral, as is typical in the AdS/CFT context.

In the following sections, we will test both (3.35) and (3.39) at the semiclassical and

[16] The usual AdS/CFT relation between the mass and the conformal dimension Δ holds.

one-loop levels.

3.3.1 Semiclassical comparison

The semiclassical analysis of the path integral (3.37) was performed in [82], showing a perfect matching with the DSSK semiclassics 3.2.3 [17]. In this section, we briefly recall the basic ingredients of the on-shell action derivation, before focusing on one-loop corrections in the following sections. The equations of motion derived from the q-Liouville action (3.37) are

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\varphi = -\frac{1}{4}e^{-2\varphi}\,, \quad \frac{\mathrm{d}}{\mathrm{d}\tau}e^{-\mathrm{i}p} = -\frac{1}{2}e^{-2\varphi}\,. \tag{3.40}$$

The first equation represents Liouville's equation, whose solutions correspond to AdS_2 conformal factors. This is consistent with interpreting φ as the length of the Einstein-Rosen bridge (ERB) in the effective AdS_2 geometry (3.18). Introducing an integration constant θ, the classical solutions are given by:

$$e^{-2\varphi} = \frac{\sin(\theta)^2}{\sin(\sin(\theta)\tau/2+\theta)^2}\,, \quad e^{-\mathrm{i}p} = \frac{\sin(\theta)}{\tan(\sin(\theta)\tau/2+\theta)} + \cos(\theta)\,. \tag{3.41}$$

Notice the solution for φ is consistent with the boundary conditions $\varphi(0) = \varphi\left(\frac{2\pi-4\theta}{\sin(\theta)}\right) = 0$, as it should. On the solutions (3.41), one can explicitly verify that the Hamiltonian is constant:

$$E_{\mathrm{grav}} = -\frac{\cos(p)}{2|\log q|} + \frac{1}{4|\log q|}e^{\mathrm{i}p}e^{-2\varphi} = -\frac{\cos(\theta)}{2|\log q|}\,. \tag{3.42}$$

This indeed corresponds to the ADM energy (3.14) of the gravitational model.

The non-trivial contribution to the on-shell action is the entropy part. Introducing the complex variable $z = e^{\mathrm{i}\sin(\theta)\tau}$, the semiclassical entropy can be rewritten as [82]:

$$S_{\mathrm{grav}} = \frac{1}{|\log q|}\int_0^\beta \mathrm{d}\tau\,(\mathrm{i}p\varphi') = \frac{1}{2|\log q|}\int_\gamma \mathrm{d}z\,\left(\frac{1}{z} + \frac{2e^{2\mathrm{i}\theta}}{1-e^{2\mathrm{i}\theta}z}\right)\log\left(\frac{e^{-\mathrm{i}\theta}-e^{3\mathrm{i}\theta}z}{1-e^{2\mathrm{i}\theta}z}\right), \tag{3.43}$$

where γ is a contour starting from $z=1$ and ending at $z=e^{-4\mathrm{i}\theta}$, encircling the pole at $z=0$. The integral can be explicitly evaluated and yields [18]

$$\begin{aligned}|\log q|S_{\mathrm{grav}} &= \frac{\pi^2}{12} + \frac{1}{2}\mathrm{Li}_2\left(e^{-2\mathrm{i}\theta}\right) + \frac{1}{2}\mathrm{Li}_2\left(e^{2\mathrm{i}\theta}\right) + \frac{1}{2}\log\left(-e^{2\mathrm{i}\theta}\right)^2 \\ &= \frac{1}{4}\log\left(-e^{2\mathrm{i}\theta}\right)^2 = -\left(\frac{\pi}{2}-\theta\right)^2.\end{aligned} \tag{3.44}$$

[17] A similar analysis was done in [88] in the case of Liouville gravity, where $q = e^{\pi \mathrm{i} b^2}$ and the dual boundary theory is still described by a version of the q-Schwarzian, though with different boundary conditions.

[18] In the last step we made use of the identity $\mathrm{Li}_2(x) + \mathrm{Li}_2(1/x) = -\frac{\pi^2}{6} - \frac{1}{2}\log^2(-x)\,,$

Combining the energy and entropy terms (3.42) and (3.44), we conclude the semiclassical free energy for sine-dilaton gravity is given by:

$$\mathcal{F}_{\text{cl}} = -\log(Z_{\text{cl}}) = -S_{grav} + \beta H_{\text{grav}} = \frac{\left(\frac{\pi}{2}-\theta\right)^2}{|\log q|} - \frac{(\pi-2\theta)\cot(\theta)}{|\log q|} \tag{3.45}$$

This indeed matches with the semi-classical partition function of DSSYK (3.30).

Moreover, the semiclassical approximation of the path integral (3.39) amounts to evaluate the bilocal operator $e^{-2\Delta\varphi}$ on the classical solution (3.41) for φ, which yields

$$\langle e^{-2\Delta\varphi}\rangle_\beta \simeq_{|\log q|^0} e^{-2\Delta\varphi_{\text{cl}}(\tau)} = \left(\frac{\sin(\theta)^2}{\sin\left(\sin(\theta)\tau/2+\theta\right)^2}\right)^\Delta \tag{3.46}$$

which shows perfect agreement with the semiclassical DSSYK matter correlator (3.32).

3.3.2 The one-loop determinant for the partition function

In the previous section we reviewed how the duality works at the semiclassical level. Now, we aim to take a step further and evaluate the first quantum correction to the q-Liouville path integral. To achieve this, we expand around the classical solutions (3.41):

$$\Phi_i = \Phi_i^{(\text{cl})} + \sqrt{|\log q|}\,\delta\Phi_i \tag{3.47}$$

where we defined:

$$\Phi_1 := \varphi \quad ; \quad \Phi_2 := p \quad ; \quad \delta\Phi_1 := \delta\varphi \quad ; \quad \delta\Phi_2 := \delta p \tag{3.48}$$

and we regard $\delta\Phi_i$ as the quantum fluctuations. As a result, the action has the following functional expansion:

$$\frac{\mathcal{S}}{|\log q|} = \frac{\mathcal{S}_{\text{cl}}}{|\log q|} + \frac{1}{\sqrt{|\log q|}}\int d\tau \left.\frac{\delta\mathcal{L}}{\delta\Phi_j}\right|_{\text{cl}} \delta\Phi_j(\tau) + \frac{1}{2}\int\int d\tau d\tau' \left.\frac{\delta^2\mathcal{L}}{\delta\Phi_i\delta\Phi_j}\right|_{\text{cl}} \delta\Phi_i(\tau)\delta\Phi_j(\tau') + \dots \tag{3.49}$$

The linear term vanishes due to the equations of motion, while the second order contribution reads:

$$\mathcal{S}^{(2)} = \int\int \mathrm{d}\tau\mathrm{d}\tau' \left[\frac{\mathrm{i}}{2}\delta p\left(\tau'\right)\ \delta\dot{\varphi}\left(\tau\right)\ - \frac{\mathrm{i}}{2}\ \delta\varphi\left(\tau'\right)\ \delta\dot{p}\left(\tau\right) - \frac{1}{4}\cos\left(p_{\mathrm{cl}}\right)\delta p\left(\tau'\right)\delta p\left(\tau\right) + \right.$$
$$\left. + \frac{1}{8}e^{-2\varphi_{\mathrm{cl}}+\mathrm{i}p_{\mathrm{cl}}}\left(\delta p\left(\tau'\right) + 2\mathrm{i}\delta\varphi\left(\tau'\right)\right)\left(\delta \mathrm{p}\left(\tau\right) + 2\mathrm{i}\delta\varphi\left(\tau\right)\right)\right]\delta(\tau-\tau') \tag{3.50}$$

The previous term is quadratic in the perturbations, as expected, and can be written as:

$$\mathcal{S}^{(2)} = \int\int \mathrm{d}\tau\mathrm{d}\tau'\ \delta\Phi_i(\tau')\ K_{ij}(\tau,\tau')\ \delta\Phi_j(\tau) \tag{3.51}$$

where we introduced $K_{ij}(\tau,\tau') = \delta(\tau-\tau')L_{ij}(\tau)$, with $L_{ij}(\tau)$ being a matrix operator of the form:

$$L_{ij}(\tau) = \begin{pmatrix} -\frac{1}{2}e^{-2\varphi_{\mathrm{cl}}(\tau)+\mathrm{i}p_{\mathrm{cl}}(\tau)} & \frac{\mathrm{i}}{4}e^{-2\varphi_{cl}(\tau)+\mathrm{i}p_{\mathrm{cl}}(\tau)} - \frac{\mathrm{i}}{2}\frac{\mathrm{d}}{\mathrm{d}\tau} \\ \frac{\mathrm{i}}{4}e^{-2\varphi_{\mathrm{cl}}(\tau)+\mathrm{i}p_{\mathrm{cl}}(\tau)} + \frac{\mathrm{i}}{2}\frac{\mathrm{d}}{\mathrm{d}\tau} & -\frac{1}{4}\cos(p_{\mathrm{cl}}(\tau)) + \frac{1}{8}e^{-2\varphi_{\mathrm{cl}}(\tau)+\mathrm{i}p_{\mathrm{cl}}(\tau)} \end{pmatrix}. \tag{3.52}$$

As a result, the path integral over field perturbations is gaussian and the computation of the one-loop partition function is reduced to evaluating the functional determinant of L on the circle, i.e.

$$\mathcal{Z}_{\mathrm{grav}}^{\mathrm{one-loop}} = e^{-\frac{\mathcal{S}\left(\Phi^{(\mathrm{cl})}\right)}{|\log q|}}\int_{\delta\varphi(0)=\delta\varphi(\beta)=0}\mathcal{D}\delta\varphi\ \mathcal{D}\delta p\ e^{-\mathcal{S}^{(2)}} = \frac{e^{-\frac{\mathcal{S}_{\mathrm{cl}}}{|\log q|}}}{\sqrt{\mathrm{Det}\left(L\right)}}. \tag{3.53}$$

where the boundary conditions for the quantum fluctuations $\delta\varphi(0) = \delta\varphi(\beta) = 0$ follow from (3.35). Therefore, we will now focus on the computation of the aforementioned object $\det(L)$. Naively, one approach is to extract the spectrum of the L operator and compute its determinant using zeta-function regularization [186–193]. However, due to the complexity of the operator, this procedure is not straightforward, as it is not simple to solve the spectral problem. Since the problem is one-dimensional, we can take advantage of results along the lines of Gel'fand Yaglom's theorem [194]. The benefit of this approach is that it is not required to compute the set of eigenvalues, indeed the determinant can be written in terms of the solutions of an initial value problem. Strictly speaking, Gel'fand Yaglom's theorem applies to Schrödinger operators on the line, with Dirichlet boundary conditions. However, the result has been extended to matrix elliptic operators of any order, with arbitrary boundary conditions [193, 195–203]. We will rely primarily on Forman's results [195], which we briefly resume below.

Determinant of elliptic differential operators Let's consider an elliptic differential operator, of the form:

$$\mathcal{O} = P_0(\tau)\frac{\mathrm{d}}{\mathrm{d}\tau} + P(\tau) \tag{3.54}$$

where the $P_0(\tau)$ and $P(\tau)$ are complex $n \times n$ matrices on the interval $I = [a, b]$ [19]. Let's admit that we are in a position to calculate the fundamental solution $Y(\tau)$ of the following homogeneous problem:

$$\mathcal{O}\, Y(\tau) = 0 \quad ; \quad Y(a) = \mathbb{1} \tag{3.55}$$

where $Y(\tau)$ is an $N \times N$ matrix constructed by stacking independent solutions (tuples) of (3.54), which satisfy the condition of having no divergence within the interval I. This object provides a basis of solutions of the homogeneous problem, indeed, it is clear that if we define:

$$f(\tau) := Y(\tau) f_a \quad ; \quad f_a \in \mathbb{C}^N \tag{3.56}$$

Then $f(\tau)$ is solution of the problem with arbitrary Cauchy boundary conditions:

$$\mathcal{O} f(\tau) = 0 \quad ; \quad f(a) = f_a \tag{3.57}$$

Moreover, if we want to impose a more general condition on both the extremes, it is sufficient to introduce the matrices M and N, such that:

$$M f(a) + N f(b) = 0 \tag{3.58}$$

As in Gel'fand Yaglom's theorem, the functional determinant, up to regularization, is encoded in the chosen boundary value problem $\{\mathcal{O}, M, N\}$, through the formula [195]:

$$\mathrm{Det}^{(F)}(\mathcal{O}) = \left[\exp\left(\int_a^b \mathrm{Tr}\left[R_{C_{\theta_1\theta_2}}(\tau)\, P(\tau)\, P_0(\tau)^{-1}\right] \mathrm{d}\tau\right) \det(M + N Y_L(b))\right] \tag{3.59}$$

where $R_{C_{\theta_1\theta_2}}$ is a projector defined as follows. If we consider the eigenvalues of the matrix $-iP_0$, they can belong either to the cone $C_{\theta_1\theta_2} = \{z \in \mathbb{C} | \, \theta_1 < z < \theta_2\}$ or its opposite $\overline{C}_{\theta_1\theta_2}$ (for some choice of θ_1 and θ_2). As a result, we can define two sets of eigenvalues $\mathcal{S}_C$ and $\mathcal{S}_{\overline{C}}$, depending in which region they fall. Then $R_{C_{\theta_1\theta_2}}(\tau)$ is the projector onto the subspace generated by the eigenvectors corresponding to the eigenvalues in $\mathcal{S}_C$. The choice of angles θ_1 and θ_2 might seem arbitrary, however, since the number

[19] The operator is elliptic if $\det(P_0(\tau)) \neq 0 \; \forall \, \tau \in I$.

of eigenvalues is finite, there is a limited set of possible projectors R. This arbitrariness of R is a remnant of the possible choices of principal angles with which to carry out analytic continuation in ζ-function regularization. In our analysis it will be clear what is the proper choice of R.

Actually (3.59) is only formal, since it could diverge. In order to make sense of it, the strategy is to introduce a regularization scheme. Let's consider a simpler operator $\hat{\mathcal{O}}$ which shares the same principal symbol P_0 with $\mathcal{O}$ and for which the calculation of the spectrum and ζ-function regularization is easy. Then, the ζ-regularized result is given by [193, 195] [20]:

$$\mathrm{Det}^{(\zeta)}(\mathcal{O}) = \frac{\mathrm{Det}^{(F)}(\mathcal{O})}{\mathrm{Det}^{(F)}(\hat{\mathcal{O}})}\,\mathrm{Det}^{(\zeta)}(\hat{\mathcal{O}}) \tag{3.60}$$

Computation of the one-loop determinant We now use this machinery to analyze the problem of our interest, which is defined on the circle $I = [0, \beta]$. By comparing the form of L in (3.52) with (3.54), we infer that P_0 and $P(\tau)$ are the following 2×2 matrices:

$$P_0 = \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} \qquad\qquad P(\tau) = \begin{pmatrix} h(\tau) & -\frac{i}{2}h(\tau) \\ -\frac{i}{2}h(\tau) & -\frac{\cos(\theta)}{4} \end{pmatrix} \tag{3.61}$$

where

$$h(\tau) := \frac{\sin^2(\theta)}{\cos(\tau\sin(\theta) + 3\theta) - \cos(\theta)} \tag{3.62}$$

is obtained by evaluating (3.52) on the classical solutions (3.41). At this stage, we observe that while $h(0)$ is well-defined, $h(\beta)$ becomes divergent. Consequently, we expect the solutions $Y(\tau)$ to exhibit some divergence as τ approaches β. To address this issue, we introduce a cutoff at the upper limit of the interval, redefining it as $I^{(\epsilon)} = [0, \beta_\epsilon]$. After completing the calculation, we will take the limit $\epsilon \to 0$ and recover the desired result, assuming no pathologies arise during this process.

The appropriate boundary conditions are implemented via the formalism: $M_{ij}\delta\Phi_j(0)+ N_{ij}\,\delta\Phi_j(\beta_\epsilon) = 0$, hence, the correct choice is:

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.63}$$

We will face two main tasks: the calculation of the 2×2 determinant involving the fundamental solution $Y(\beta_\epsilon)$ and the computation of the exponential factor in (3.59).

As a first step, we need to solve the differential system $L_{ij}\,\delta\Phi_j = 0$. Plugging the second

[20] Alternatively, if one focuses on operators on the circle, one can rely on the results of [201], who reports a formula similar to Forman's, but already regularized.

row of the system into the first one, we obtain the following second order differential equation for $\delta\varphi$:

$$h(\tau)\,\delta\varphi(\tau) + \sec(\theta)\left[-\delta\varphi''(\tau) + \delta\varphi(\tau)\,\left(h(\tau)^2 + h'(\tau)\right)\right] = 0 \tag{3.64}$$

whose general solution is:

$$\delta\phi(\tau) = \frac{-c_1\tau\cos(\theta) + c_1\csc(\theta)\sin(\tau\sin(\theta)+3\theta) + c_2}{\cos(\tau\sin(\theta)+3\theta) - \cos(\theta)} \tag{3.65}$$

where c_1 and c_2 are integration constants. Depending on their values, we obtain a solution of the system $\delta\vec{\Phi}^{(c_1,c_2)}(\tau)$. To construct the fundamental matrix $Y(\tau)$, we consider two independent solutions $\delta\vec{\Phi}^{(1,0)}(\tau)$, $\delta\vec{\Phi}^{(0,1)}(\tau)$, by setting respectively: $(c_1, c_2) = \{(1,0);(0,1)\}$, and build up the invertible matrix $H(\tau)$ by stacking the two independent solution vectors as columns:

$$H(\tau) := (\delta\vec{\Phi}^{(1,0)}(\tau)\,,\ \delta\vec{\Phi}^{(0,1)}(\tau)) \tag{3.66}$$

We finally normalize it, by defining $Y(\tau) := H(\tau)H^{-1}(0)$, which clearly satisfies $Y(0) = \mathbb{1}$, as required in (3.55). We are now in position to evaluate the determinant in (3.59):

$$\begin{aligned}&\det(M + NY_L(b)) =\\ &= \frac{\csc(\theta)\sin\left(\frac{b}{2}\sin(\theta)\right)\left[b\cos\left(\frac{b}{2}\sin(\theta)+3\theta\right) - 2\csc(\theta)\sec(\theta)\sin\left(\frac{b}{2}\sin(\theta)\right)\right]}{2[\cos(\theta) - \cos(b\sin(\theta)+3\theta)]}\end{aligned} \tag{3.67}$$

As expected, evaluating the determinant at $b = \beta$ results in a divergence that, however, can be regularized by introducing a small cutoff ϵ. After evaluating the determinant at the regularized endpoint β_ϵ, and retaining only the leading term in the expansion as $\epsilon \to 0$, we obtain:

$$\det(M + NY_L(\beta_\epsilon)) \sim -\frac{2\cot(\theta)((\pi - 2\theta)\cot(\theta) + 2)}{\epsilon}. \tag{3.68}$$

In order for the divergence to disappear it is necessary for the exponential in (3.59), to contribute at ϵ order, this condition guides us to the choice of the correct projector $R_{C_{\theta_1\theta_2}}(\tau)$. The eigenvalues of $-iP_0$ are $\{\frac{i}{2}, -\frac{i}{2}\}$, hence, depending on the choice of θ_1 and θ_2 there are four possibilities: $R \in \{\pi_{i/2}, \pi_{-i/2}, \mathbb{1}, 0\}$. In retrospect, the only significant one is:

$$R(\tau) = \pi_{i/2} = \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix} \tag{3.69}$$

Indeed the integral that appears in (3.59) has the following form:

$$\mathcal{I} := \int_0^{\beta_\epsilon} \left(\frac{\sin^2(\theta)}{\cos(\tau \sin(\theta) + 3\theta) - \cos(\theta)} - \frac{\cos(\theta)}{4} \right) d\tau \tag{3.70}$$

Performing the integral and expanding up to first order in ϵ we arrive at:

$$e^{\mathcal{I}} \sim \frac{\epsilon}{4}\, e^{-\frac{1}{2}(\pi-2\theta)\cot(\theta)} \csc(\theta)\sec(\theta) \tag{3.71}$$

Multiplying the two results (3.71) and (3.68), we note that the divergence disappears and it is safe to send $\epsilon \to 0$. Since the result is already non-divergent, the only factor obtained from the strict application of (3.60) is a numerical constant, which turns out to be -1. Hence we find:

$$\mathrm{Det}^{(\zeta)}(L) = -\mathrm{Det}^{(F)}(L) = \frac{1}{2} e^{-\frac{1}{2}(\pi-2\theta)\cot(\theta)}((\pi-2\theta)\cot(\theta)+2)\csc^2(\theta) \tag{3.72}$$

Including the one-loop correction, the free energy of sine-dilaton gravity is then given by

$$\begin{aligned}\mathcal{F}_{\text{one-loop}} =& \frac{\left(\frac{\pi}{2}-\theta\right)^2}{|\log q|} - \frac{(\pi-2\theta)\cot(\theta)}{|\log q|} + \frac{1}{2}\log\left(1+\left(\frac{\pi}{2}-\theta\right)\cot(\theta)\right) \\ &- \log(\sin(\theta)) - \frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\cot(\theta)\end{aligned} \tag{3.73}$$

We find a perfect match with the one-loop partition function of DSSYK in (3.31), except for an additional factor of $1/2$ for the last term in (3.73). We point out that this last term comes from the first quantum correction to the classical ADM energy (3.14) of sine-dilaton gravity and should therefore be related to different orderings for the gravitational Hamiltonian (3.22). A more refined analysis of the discretization leading to the path integral (3.35) could indeed pinpoint the correct choice of ordering for the Hamiltonian (3.22) leading to this small discrepancy, but we haven't investigated this aspect further in this work.

3.4 One-loop correction to the matter two-point function

The aim of this section is to analyze the one-loop gravitational correction to the boundary to boundary propagator of a scalar matter field (3.8), coupled non-minimally to sine-dilaton gravity. By virtue of (3.39), this is equivalent to extracting the $|\log q|$ correction to the q-Liouville path integral $\langle e^{-2\Delta\phi(\tau)}\rangle_\beta$, in a regime where this insertion does not back react on the geometry. After expanding each field as in (3.47), we obtain:

$$\langle e^{-2\Delta\varphi}\rangle_\beta = \frac{1}{\mathcal{Z}(\beta)} e^{-\frac{\mathcal{S}\left(\Phi^{(\mathrm{cl})}\right)}{|\log q|}} e^{-2\Delta\varphi_{\mathrm{cl}}(\tau)} \times \int_{\delta\varphi(0)=\delta\varphi(\beta)=0} \mathcal{D}\delta\phi\mathcal{D}\delta p \, e^{-2\Delta\sqrt{|\log q|}\delta\varphi(\tau)} \;\; e^{-\mathcal{S}^{(2)}-\sqrt{|\log q|}\mathcal{S}^{(3)}-|\log q|\mathcal{S}^{(4)}+\cdots}\,, \tag{3.74}$$

where we denoted with $S^{(n)}$ the term in the expansion of the action that contains n fields. For future reference, we report below the explicit expression of $\mathcal{S}^{(3)}$:

$$\mathcal{S}^{(3)} = \int \mathrm{d}\tau \bigg[\frac{1}{3} e^{\mathrm{i}p_{\mathrm{cl}}-2\varphi_{\mathrm{cl}}}\delta\varphi(\tau)^3 - \frac{\mathrm{i}}{2} e^{\mathrm{i}p_{\mathrm{cl}}-2\varphi_{\mathrm{cl}}}\delta\varphi(\tau)^2\delta p(\tau) - \frac{1}{4} e^{\mathrm{i}p_{\mathrm{cl}}-2\varphi_{\mathrm{cl}}}\delta\varphi(\tau)\delta p(\tau)^2 + \left(\frac{\mathrm{i}}{24} e^{\mathrm{i}p_{\mathrm{cl}}-2\varphi_{\mathrm{cl}}} + \frac{1}{12}\sin(p_{\mathrm{cl}})\right)\delta p(\tau)^3\bigg] \tag{3.75}$$

As an initial observation regarding (3.74), we note that by neglecting all quantum corrections in $|\log q|$ beyond $|\log q|^0$, the path integral contributes only a factor of the form $\det L^{-\frac{1}{2}}$. Dividing by the partition function at the same order, we recover the semiclassical result (3.46), as expected. More interestingly, we aim to extend the expansion in $|\log q|$ and compute the first correction to the above result, incorporating non-trivial interacting terms from the action. At this order, the normalized contribution from the path integral takes the form:

$$\sim (\det L)^{\frac{1}{2}} \int \mathcal{D}\delta\varphi\mathcal{D}\delta p \, e^{-\mathcal{S}^{(2)}} \left[1 - 2\Delta\delta\varphi\sqrt{|\log q|} + |\log q| \left(2\Delta^2\delta\varphi^2 - 2\Delta\delta\varphi\mathcal{S}^{(3)}\right) + \cdots\right], \tag{3.76}$$

Notably, since the term $\propto |\log q|^{\frac{1}{2}}$ vanishes in the quadratic path integral, we find at order $|\log q|$ the following expression for the correlator:

$$\langle e^{-2\Delta\varphi}\rangle_\beta \simeq e^{-2\Delta\varphi_{\mathrm{cl}}(\tau)} \left[1 + 2|\log q| \left(\Delta^2\langle\delta\varphi(\tau)^2\rangle - \Delta\langle\delta\varphi(\tau)\mathcal{S}^{(3)}\rangle\right) + \mathcal{O}\left(|\log q|^2\right)\right], \tag{3.77}$$

where the expectation values are taken with respect to the quadratic action and are normalized by the one-loop determinant that contributes at order $|\log q|^0$ to the partition function. In order to compute these quantities, it is convenient to analyze the following generating functional obtained by adding a source to the gaussian path integral:

$$Z[\{J_i\};\beta] = \int_{\delta\varphi(0)=\delta\varphi(\beta)=0} \mathcal{D}\delta\varphi\mathcal{D}\delta p\; e^{-\mathcal{S}_{(2)}+\int_0^\beta \mathrm{d}t J_i(t)\delta\Phi_i(t)}, \tag{3.78}$$

where at the exponent we have the following quantity:

$$\frac{1}{2}\int\int \mathrm{d}t\mathrm{d}t'\delta\Phi_i(t)K_{ij}(t,t')\delta\Phi_j(t') + \int\int \mathrm{d}t\mathrm{d}t' J_i(t)\delta(t-t')\delta\Phi_i(t') \tag{3.79}$$

with $K_{ij}(t,t') = L_{ij}(t)\delta(t-t')$ already introduced in (3.52). If we now introduce the Green function $G_{ij}(t,t')$ for the quadratic operator $K_{ij}(t,t')$, defined as:

$$\int_0^\beta \mathrm{d}t'' K_{ij}(t,t'')\, G_{jk}(t'',t') = \delta(t-t')\delta_{ik}, \tag{3.80}$$

and we perform the standard change of variable inside the path integral

$$\delta\tilde{\Phi}_i(t) = \delta\Phi_i(t) + \int \mathrm{d}t' G_{ij}(t,t')J_j(t'), \tag{3.81}$$

we immediately obtain the following result for the generating function:

$$Z[\{J_i\};\beta] = (\det L)^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\int_0^\beta\int_0^\beta \mathrm{d}t\mathrm{d}t' J_i(t)G_{ij}(t,t')J_j(t')\right]. \tag{3.82}$$

Interestingly, we observe that the determinant appearing as a prefactor to the exponential precisely cancels the normalization factor of the expectation values in (3.97). As a result, we can disregard this determinant and treat the expectation values in (3.97) as arising from the functional differentiation with respect to the sources in the generating functional (3.82). Using this method, it is immediate to verify that:

$$\langle\delta\varphi(\tau)^2\rangle = \frac{\delta}{\delta J_\varphi(\tau)}\frac{\delta}{\delta J_\varphi(\tau)} \exp\left[-\frac{1}{2}\int_0^\beta\int_0^\beta \mathrm{d}t\mathrm{d}t' J_i(t)G_{ij}(t,t')J_j(t')\right] = -G_{\varphi\varphi}(\tau,\tau). \tag{3.83}$$

The challenging task, of course, is to compute the Green's function G_{ij} for the matrix operator L in (3.61), subject to the boundary conditions $\delta\varphi(0) = \delta\varphi(\beta) = 0$ and periodic boundary conditions for δp. However, it is possible to construct the Green's function once the solutions to the homogeneous problem are known. As derived in Appendix (B.1), the Green's function can be expressed as:

$$G(t,\tau) = \theta(t-\tau)G^{(a)}(t,\tau) + \theta(\tau-t)G^{(r)}(t,\tau), \tag{3.84}$$

where $G^{(a)}(t,\tau)$ and $G^{(r)}(t,\tau)$ represent the retarded and advanced Green's functions, respectively. These are given by [21]:

$$G^{(a)}(t,\tau) = -H(t)\left[\mathcal{R}^{-1}NH(\beta_\epsilon) - 1\right]H^{-1}(\tau)P_0^{-1}$$
$$G^{(r)}(t,\tau) = -H(t)\mathcal{R}^{-1}NH(\beta_\epsilon)H^{-1}(\tau)P_0^{-1} \tag{3.85}$$

where $H(\tau)$ is the fundamental matrix solution of the homogeneous problem (3.66), $\mathcal{R}$ is defined as:

$$\mathcal{R} = MH(0) + NH(\beta_\epsilon),$$

with M and N being the matrices already introduced in (3.63) to implement the boundary conditions. Using (3.85), the complete form of the Green's function for L is derived in detail in (B.16) and (B.17), but it is rather intricate. For our purposes, we just report here the $G_{\varphi\varphi}(\tau,\tau)$ component [22], which is the one of our interest because of (3.83):

$$G_{\varphi\varphi}(\zeta,\zeta) = \frac{(\tan(\zeta)+\tan(\zeta)(\zeta+u)\tan(u)+\tan(u))(\tan(\zeta)+\tan(u)(\tan(\zeta)(u-\zeta)-1))}{u\tan(u)+1} \tag{3.86}$$

where we have introduced $\zeta = \frac{\pi}{2} - \theta - \frac{\tau}{2}\sin(\theta)$ and $u = \frac{\pi}{2} - \theta$ as in 3.2.3. We can already appreciate the matching between this quantity and $\mathcal{I}$ in (3.34).

The remaining term in (3.97) that we need to compute is $\langle \delta\varphi(\tau)\mathcal{S}_{(3)}\rangle$. To derive an expression for this expectation value, we could, in principle, examine each term in (3.75) and evaluate the corresponding integrals by systematically applying Wick contractions, as prescribed by Wick's theorem. While this method is perfectly valid, it would involve extensive calculations. Instead, we will adopt a more efficient approach that avoids these lengthy computations. The key insight here is that it is advantageous to express $\langle \delta\varphi(\tau)\mathcal{S}_{(3)}\rangle$ using a covariant notation and investigate the more general correlator:

$$\mathcal{A}_l(\tau) = \left\langle \delta\Phi_l(\tau)\int_0^\beta \mathrm{d}t\, T_{ijk}(t)\delta\Phi_i(t)\delta\Phi_j(t)\delta\Phi_k(t)\right\rangle, \tag{3.87}$$

where T_{ijk} are the vertices of the cubic interaction term from the action (3.75). The non-vanishing components of the fully symmetric tensor $T_{ijk}(t)$ can be extracted from (3.75) by substituting the classical solution (3.41), which gives the following structure for the vertices:

$$T_{ppp}(t) = \frac{\mathrm{i}}{2}\cot\left(\theta + \frac{t}{2}\sin\theta\right), \qquad T_{pp\varphi}(t) = \frac{\sin^2(\theta)}{6\cos(3\theta + t\sin\theta) - 6\cos(\theta)},$$
$$T_{p\varphi\varphi}(t) = 2\mathrm{i}\, T_{pp\varphi}(t), \qquad T_{\varphi\varphi\varphi}(t) = 2\mathrm{i}\, T_{p\varphi\varphi}(t). \tag{3.88}$$

[21] Once again, we regulate the divergence of $H(\beta)$, which is exactly canceled by $\mathcal{R}^{-1}$, by introducing the cutoff ϵ and finally sending $\epsilon \to 0$.

[22] In computing the Green's function at coincident points from (3.84), we are assuming $\theta(0) = \frac{1}{2}$.

In the case where $l = \varphi$, the newly defined correlator becomes $\mathcal{A}_\varphi(\tau) = \langle \delta\varphi(\tau)\mathcal{S}^{(3)}\rangle$, which is precisely the quantity we wish to analyze. Similarly, for $l = p$, we have $\mathcal{A}_p(\tau) = \langle \delta p(\tau)\mathcal{S}^{(3)}\rangle$. The benefit of introducing this unified notation, treating both correlators on an equal footing, is that it enables us to derive a differential equation governing the correlator of interest, allowing us to determine its functional form. To proceed we first observe that, by applying Wick's theorem to (3.87), we obtain:

$$\mathcal{A}_l(\tau) = 3\int_0^\beta \mathrm{d}t\; T_{ijk}(t)G_{li}(\tau,t)G_{jk}(t,t), \tag{3.89}$$

where the factor of 3 comes from the multiplicity of the contraction. Let us now apply a useful technique: we act with the quadratic operator $L_{ij}(\tau) = P_0\frac{\mathrm{d}}{\mathrm{d}\tau} + P(\tau)$ (as defined in (3.52)) to both sides of equation (3.89). Recalling the definition of the Green's function from (3.80), this leads to the following system of coupled differential equations:

$$\begin{cases} 2P_{pp}(\tau)\mathcal{A}_p(\tau) + 2P_{p\varphi}(\tau)\mathcal{A}_\varphi(\tau) + \mathrm{i}\dot{\mathcal{A}}_\varphi(\tau) = 3T_{pjk}(\tau)G_{jk}(\tau,\tau) \\ 2P_{\varphi p}(\tau)\mathcal{A}_p(\tau) + 2P_{\varphi\varphi}(\tau)\mathcal{A}_\varphi(\tau) - i\dot{\mathcal{A}}_p(\tau) = 3T_{\varphi jk}(\tau)G_{jk}(\tau,\tau) \end{cases}. \tag{3.90}$$

Solving for $\mathcal{A}_p(t)$ from the first equation and plugging it in the second equation, we end up with:

$$\begin{aligned} -\frac{\ddot{\mathcal{A}}_\varphi(\tau)}{2P_{pp}} + \mathcal{A}_\varphi(\tau)\left(2P_{\varphi\varphi} + i\frac{\dot{P}_{p\varphi}}{P_{pp}} - \frac{2\left(P_{p\varphi}\right)^2}{P_{pp}}\right) &= 3T_{\varphi jk}(\tau)G_{jk}(\tau,\tau) + \\ &+ \left(\frac{3\mathrm{i}}{2P_{pp}}\frac{\mathrm{d}}{\mathrm{d}\tau} - \frac{3P_{p\varphi}}{P_{pp}}\right)T_{pjk}(\tau)G_{jk}(\tau,\tau). \end{aligned} \tag{3.91}$$

We now focus on the left-hand side of the equation and substitute the explicit forms of P_{pp}, $P_{\varphi\varphi}$, and $P_{\varphi p}$, given in (3.61). After some straightforward algebraic manipulations, we arrive at the expression:

$$-\frac{\sin(\theta)^2}{8P_{pp}}\left(\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} - \frac{2}{\cos(\zeta)^2}\right)\mathcal{A}_\varphi(\zeta) = \dots \tag{3.92}$$

where we introduced the same ζ variable as before. The analysis of the right-hand side is more involved and requires nontrivial identities among the Green's functions. In particular, we need an expression for $\frac{d}{d\tau}G(\tau,\tau)$ in terms of the Green's function itself. This derivation is carried out in detail in Appendix B.1, and we present here the final result from (B.25), written in component form [23]:

$$\begin{cases} \frac{d}{d\tau}G_{pp}(\tau,\tau) = -4\mathrm{i}\left(G_{pp}(\tau,\tau)P_{p\varphi}(\tau) + G_{p\varphi}(\tau,\tau)P_{\varphi\varphi}(\tau)\right) \\ \frac{d}{d\tau}G_{p\varphi}(\tau,\tau) = 2\mathrm{i}\left(G_{pp}(\tau,\tau)P_{pp}(\tau) - G_{\varphi\varphi}(\tau,\tau)P_{\varphi\varphi}(\tau)\right) \\ \frac{d}{d\tau}G_{\varphi\varphi}(\tau,\tau) = 4\mathrm{i}\left(G_{p\varphi}(\tau,\tau)P_{pp}(\tau) + G_{\varphi\varphi}(\tau,\tau)P_{p\varphi}(\tau)\right) \end{cases} \tag{3.93}$$

Using these relations, we can express the right-hand side of (3.91) entirely in terms of the Green function components. This yields:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} - \frac{2}{\cos(\zeta)^2}\right)\mathcal{A}_\varphi(\zeta) = -\frac{24P_{pp}}{\sin(\theta)^2}\left(c_{pp}(\tau)G_{pp}(\tau,\tau) + c_{p\varphi}(\tau)G_{p\varphi}(\tau,\tau) + c_{\varphi\varphi}(\tau)G_{\varphi\varphi}(\tau,\tau)\right) \tag{3.94}$$

where the coefficients are:

$$\begin{aligned} c_{\varphi\varphi}(\tau) &= T_{\varphi\varphi\varphi} + \frac{\mathrm{i}}{2P_{pp}}\dot{T}_{p\varphi\varphi} - \frac{P_{\varphi p}T_{p\varphi\varphi}}{P_{pp}} + \frac{2T_{pp\varphi}P_{\varphi\varphi}}{P_{pp}} - \frac{2T_{p\varphi\varphi}P_{p\varphi}}{P_{pp}} = -\frac{\sin(\theta)^2}{24P_{pp}}\frac{1}{\sin\left(\theta+\frac{\tau}{2}\sin\theta\right)^2} \\ c_{p\varphi}(\tau) &= -2\frac{P_{\varphi p}T_{pp\varphi}}{P_{pp}} + \frac{\mathrm{i}}{P_{pp}}\dot{T}_{pp\varphi} + \frac{2T_{ppp}P_{\varphi\varphi}}{P_{pp}} = 0 \\ c_{pp}(\tau) &= T_{\varphi pp} + \frac{\mathrm{i}}{2P_{pp}}\dot{T}_{ppp} - \frac{P_{\varphi p}}{P_{pp}}T_{pp\phi} + \frac{2T_{ppp}P_{p\varphi}}{P_{pp}} - \frac{2T_{pp\varphi}P_{pp}}{P_{pp}} = 0 \end{aligned} \tag{3.95}$$

and are computed using (3.61) and (3.88). Thus, we finally arrive at the following differential equation obeyed by $\mathcal{A}_\varphi$ [24]:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} - \frac{2}{\cos(\zeta)^2}\right)\mathcal{A}_\varphi(\zeta) = \frac{G_{\varphi\varphi}(\tau(\zeta),\tau(\zeta))}{\cos(\zeta)^2}, \tag{3.96}$$

To summarize, from (3.97) the one-loop correction to the correlator takes the form

$$\langle e^{-2\Delta\varphi}\rangle_\beta \simeq e^{-2\Delta\varphi_{\mathrm{cl}}(\tau)}\left[1 - 2|\log q|\left(\Delta^2 G_{\varphi\varphi} + \Delta\mathcal{A}_\varphi\right) + \mathcal{O}\left(|\log q|^2\right)\right], \tag{3.97}$$

By comparing with (3.33), we observe that $G_{\varphi\varphi} = -\mathcal{I}$ and $\mathcal{A}_\varphi = -\mathcal{A}$. Indeed, substituting the expression for $G_{\varphi\varphi}$ from (3.86) and $\mathcal{A}_\varphi = -\mathcal{A}$ from (3.34) into (3.96), we verify that the differential equation is satisfied. Furthermore, since $\mathcal{A}_\varphi(\tau = 0) = \mathcal{A}_\varphi(\tau = \beta) = 0$, due to the boundary conditions imposed on $\delta\varphi$, the solution $\mathcal{A}_\varphi = -\mathcal{A}$ to (3.96) is unique. This completes the proof and demonstrates perfect agreement between the one-loop gravitational correction to the bilocal operator and the one-loop correction in DSSYK.

[23] One can also explicitly verify that the components of the Green's function (3.84), expressed in terms of the retarded and advanced Green's functions (B.16) and (B.17), indeed satisfy the relations in (3.93).

[24] The same differential equation was also found in [90] in the context of the bilocal Liouville action derived from the $G\Sigma$ formulation of SYK.

3.5 Conclusions and Outlooks

In this chapter we have explicitly checked the duality between sine-dilaton gravity and DSSYK model, performing at one-loop order a path integral computation of the free energy and of the two-point function of bilocal operators. We applied the description of sine-dilaton gravity in terms of a deformed version of the standard Schwarzian model dual to JT gravity, the so called q-Liouville theory. Particular attention has been devoted to the boundary condition of the one-loop determinant, crucially related to the correct Hartle-Hawking vacuum for the gravitational theory. Technically we have exploited a generalization of the Gel'fand-Yaglom theorem, adapting the standard formalism to our particular cases. We have found a small discrepancy between our result and previous computations [89, 90] for the partition function, that nevertheless does not affect the the two-point correlator. We attribute the discrepancy to a nontrivial ordering of the operators in the gravitational Hamiltonian that is involved during the quantization process.

An interesting question regards the physical meaning of the one-loop corrections: one expects that this contribution should be related to the so called large p limit of SYK, where p is taken to be large, but independent of N. We already observed that the q-Liouville field φ represents conformal factor for the effective AdS$_2$ metric (3.41). At the leading order, this just describes different finite patches of the geometry as a function of β. In [89], the quantum correction $\delta\varphi$ was computed using the semiclassical saddle-point expansion of the DSSYK two-point function. This allowed for the calculation of the first quantum correction to the scalar curvature in the putative bulk dual geometry via $R = 2e^{2\varphi}\nabla^2\varphi$. From this result, it was observed that, as we move inside the bulk increasing τ, the curvature significantly deviates from the the AdS$_2$ semiclassical saddle, indicating that the semiclassical approximation breaks down and the gravity theory becomes strongly coupled. Our computation of the boundary to boundary propagator of a matter field in sine-dilaton gravity, leading to the same result $\delta\varphi$ for the effective geodesic length measured by the probe [25], represents indeed a direct bulk derivation of the observation of [89]. As a result, we expect sine-dilaton gravity to also become strongly coupled as we move deeper into the bulk, rendering the semiclassical small-$|\log q|$ expansion unreliable in this regime. Nevertheless, the expansion may still be understood as an asymptotic series. By applying techniques such as resurgence theory, it might be possible to identify instantons, which could provide insights into the strongly coupled regime of sine-dilaton gravity. At the level of the partition function, the exis-

[25]The quantum correction $\delta\varphi$ can just be found for instance by taking the derivative with respect to Δ of the result (3.97).

tence of such instantonic saddles has indeed been discussed in [80].

As already remarked in the main text, the partition function of JT gravity can be exactly computed through equivariant localization [30] (see [34] for an approach through supersymmetric localization), meaning that the one-loop approximation is basically exact. The same seems not to be true for the q-Schwarzian, where a full perturbative series in the coupling constant $|\log q|$ is generated. One possibility is that a localization procedure could exploit the full geometrical structure of the Poisson Sigma model or some supersymmetric generalization of it, involving directly the quantum group symmetry. A somehow related question concerns the characterization of the bilocal operator from a bulk point of view: in the gauge theory formulation in JT gravity they appear naturally as anchored Wilson lines [164, 165] therefore one suspects that the same role could be played by a "quantum" holonomy in the Poisson sigma model.

There is also a certain number of direct extensions of the present paper that should be feasible in near future. In JT gravity it is possible to implement twisted boundary conditions in the Schwarzian in order to study the trumpet partition function and bilocal correlator on the trumpet [204]: it would be nice to engineer a similar process here for the q-Schwarzian, capturing the insertion of a defect in geometry of sine-dilaton gravity. On the other hand, the semiclassical expansion around classical solutions already suggests the presence of a geometrical defect to explain the restoration of the real periodicity in the model [82]. A more comprehensive study of the trumpet geometries from the bulk point of view will be presented in [205].

There is a comprehensive study of JT gravity on arbitrary topologies, via matrix-model formulation [42, 54], giving a background independent non-perturbative extension of the theory. Given the relation between Sine-Dilaton gravity and JT-gravity, it would be really interesting to study the theory on arbitrary topologies. From the purely gravitational point of view, the process seems non-trivial (even finding classical solutions on the torus). However, from the matrix model point of view, there has been some interesting progress [84, 176, 206–209]. Nevertheless, from the matrix model perspective, remarkable progress has recently been achieved [84, 206, 208, 209]. Building on matrix model intuition, Ref. [206] conjectured the existence of a discrete q-analog of the Weil–Petersson volumes. These objects were consistently derived in [208], where they were shown to satisfy a discrete q-deformed Mirzakhani recursion relation. It would be really interesting to better understand the geometric meaning of volumes (e.g. in terms of counting maps) and their physical properties.

One would like also to understand better the out-of-time-order correlator in the q–Liouville, starting from shock waves processes in sine-dilaton gravity, as originally done in JT gravity [180]. Recently there have been also interesting proposals to connect

DSSYK with three dimensional physics and de-Sitter gravity [70, 73, 74, 85] and with higher dimensional supersymmetric theories [210]: some non-trivial features of sine-dilaton gravity could be encoded there.

Part II

$T\overline{T}$-deformation of 2d Yang-Mills on torus topology

Background

4

4.1 $T\overline{T}$-deformation

4.1.1 Basic definitions

Hypotheses The concept of the $T\overline{T}$ operator was first introduced in the seminal work of Zamolodchikov [122], and was later shown to generate an integrable flow [123, 124]. As we will see, the operator arises naturally in the framework of two-dimensional CFTs, but its definition extends to generic quantum field theories. To set the stage, let us consider a general two-dimensional QFT or CFT defined on the Euclidean plane $\mathbb{R}^2$. We assume that the theory is:

- 1) Local: so it can be formulated in terms of a Lagrangian built from local operators.
- 2) Invariant under local translations: as a result, the expectation value of any local operator $\mathcal{O}_i(z)$ is independent of the insertion point:

$$\partial_z \langle \mathcal{O}_i(z) \rangle = 0\,. \tag{4.1}$$

- 3) Lorentz invariant: hence the stress-energy tensor is symmetric:

$$T_{\alpha\beta} = T_{\beta\alpha}\,. \tag{4.2}$$

These conditions are sufficient to formulate the formal construction of the $T\overline{T}$ operator and the associated deformation. However, Lorentz invariance is not strictly necessary, as an extension to non-Lorentz-invariant QFTs has been given[126].

The $\hat{\mathcal{C}}_{T\overline{T}}$ operator We review the standard definition, which is based on a conventional CFT perspective and we follow the original paper by Zamolodchikov [122]. We denote the components of the stress energy tensor, in complex Euclidean coordinates, as: T_{zz}, $T_{\bar{z}\bar{z}}$, $T_{z\bar{z}} = T_{\bar{z}z}$. Following the standard conventions of conformal field theory, we can also define the chiral components:

$$T = -2\pi T_{\mathrm{zz}}, \quad \bar{T} = -2\pi T_{\bar{\mathrm{z}}\bar{\mathrm{z}}}, \quad \Theta = 2\pi T_{\mathrm{z}\bar{\mathrm{z}}}$$

Due to translational invariance, the energy-momentum tensor satisfies a continuity equation, which reads:

$$\partial_{\bar{z}} T(z) = \partial_z \Theta(z), \qquad \partial_z \bar{T}(z) = \partial_{\bar{z}} \Theta(z). \tag{4.3}$$

We now consider the following product of operators, denoted by $\hat{\mathcal{C}}_{T\overline{T}}$, which depends on two distinct points z and z':

$$\hat{\mathcal{C}}_{T\overline{T}}(z, z') := T(z)\, \bar{T}(z') - \Theta(z)\, \Theta(z')\,. \tag{4.4}$$

The operator $\hat{\mathcal{C}}_{T\overline{T}}(z, z')$ is a composite operator, and can be understood as the determinant of the energy-momentum tensor in the complex basis, up to a multiplicative factor[1]. However, its most remarkable property is that it possesses a well-behaved operator product expansion (OPE):

$$\hat{\mathcal{C}}_{T\overline{T}}(z, z') = \mathcal{O}_{T\bar{T}}\,(z') + \text{ derivative terms} \tag{4.5}$$

Where clearly $\mathcal{O}_{T\bar{T}}\,(z')$ is a local operator. In order to prove this property, one can begin by computing the first order derivatives of the operator $\hat{\mathcal{C}}_{T\overline{T}}$ and employing the continuity equation (4.3). The resulting derivatives are recast as OPEs, showing that any derivative of $\hat{\mathcal{C}}_{T\overline{T}}$ admits an OPE involving only derivative operators. Hence, the OPE of $\hat{\mathcal{C}}_{T\overline{T}}$ contains operators $\mathcal{O}_i(z)$ that must obey one of the following conditions:

- 1) $\mathcal{O}_i(z)$ is a local operator dependent on z, and it multiplies coefficient $F_i\,(z - z')$ which actually does not depend on z nor z'.

- 2) $\mathcal{O}_i(z)$ is a derivative operator.

One can prove the previous statement by contradiction [211]. We notice that the key property (4.5) directly follows from the continuity equation and as a consequence it can be extended to composite operators built from the energy-momentum tensor and other currents, or more generally from any pair of conserved currents.

[1]A more precise statement, which explicitly accounts for the spatial dependence, is:

$$\hat{\mathcal{C}}(z, z') \propto -\epsilon^{\alpha\beta} \epsilon^{\gamma\delta}\, T_{\alpha\gamma}(z)\, T_{\beta\delta}(z').$$

The $T\overline{T}$ operator We are now in a position to define the $T\overline{T}$ operator, making use of (4.5), which can be schematically expressed as:

$$T\overline{T}(z) = \lim_{z' \to z} \left(\hat{\mathcal{C}}_{T\overline{T}}(z, z') \right) - \text{derivative operators} \tag{4.6}$$

This is the so-called point-splitting definition of $T\overline{T}$. Morally speaking, the $T\overline{T}$ operator can then be understood as the determinant of the stress-energy tensor, obtained by taking the coincident-point limit $z \to z'$ and discarding derivative operators.[2] These derivative terms are irrelevant for our purposes, since in a translationally invariant QFT their expectation values vanish. Indeed, when integrated over space, these contributions cancel exactly, as they correspond to total derivatives. The physical content is thus entirely captured by the $T\overline{T}$ operator, which defines a genuine, non-vanishing observable.
Now we can briefly comment on the historical origin of the name $T\overline{T}$. In the original case of CFTs $\Theta \propto T_{z\bar{z}} = 0$, and the operator $\hat{\mathcal{C}}_{T\overline{T}}$ reduces to $T(z)\,\bar{T}(z')$, so that in CFTs:

$$T\overline{T}(z) = \lim_{z' \to z} \left(T(z)\,\bar{T}(z') \right) - \text{derivative operators.} \tag{4.7}$$

However, in a generic quantum field theory the stress-energy tensor is not traceless, and the more general definition of the $T\overline{T}$ operator is given by (4.6).

The irrelevant deformation In order to introduce the concept of the $T\overline{T}$ deformation, we recall that in Wilson's Renormalization Group framework a theory flows from a high-energy Lagrangian to a low-energy fixed point. The dimension of perturbation (relevant, irrelevant, or marginal), determines the behavior of the flow around the Gaussian fixed point. If we follow the usual RG flow, it is customary to focus on relevant operators, indeed they modify the IR theory and the renormalization procedure is mathematically consistent. On the other hand, irrelevant operators are more problematic. Firstly, we notice that they give physical effects only in the UV regime. If we try to follow the irrelevant deformation at high energies we usually run into a Landau pole (i.e. a divergence). Therefore, in order to study an irrelevant operator at high energies, it is usually necessary to define an infinite amount of counter-terms when we apply the renormalization procedure. This also implies that we should fix an infinite amount of experimental data which are not at our disposal. These problems may be interpreted in a physical intuitive motivation: It is unlikely that we can recover the physical description of a high energy system starting from a system with less degrees of freedom.
An important question now is whether all irrelevant deformations are ill-defined. At

[2] In the case of two-dimensional Yang–Mills theory, as we will show, the energy-momentum tensor is position-independent. Therefore, no divergences arise from derivative operators in the OPE of $\hat{\mathcal{C}}_{T\overline{T}}$.

this stage the concept of $T\overline{T}$-deformation enters in the story as an example of integrable irrelevant deformation[3]. In a certain sense, exploiting integrability, we can define and derive an infinite tower of conserved quantities, which are used to fix the experimental data along the flow. This suggests that along a $T\overline{T}$-flow the number of DOFs of the theory is conserved and the theory is UV complete in a rather particular way. Moreover, it can be shown that the $T\overline{T}$ deformation renders the theory non-local [127, 212, 213]. These properties are incompatible with a standard Wilsonian flow, which assumes that the Lagrangian remains local at each scale.
Now we can give the formal definition of $T\overline{T}$-deformation, at first order around the fixed point:

$$\mathscr{L}^{(\mu+\delta\mu)} - \mathscr{L}^{(\mu)} = -\delta\mu \ T\overline{T}^{(\mu)} \tag{4.8}$$

where μ is the deformation coupling constant. Clearly, we can iteratively apply (4.8) and give a formulation [124, 135, 214] at the level of Lagrangian or Hamiltonian flow:

$$\partial_\mu \mathscr{L}^{(\mu)} = T\overline{T}^{(\mu)} \quad , \quad T^{(\mu)}_{\alpha\beta} = \frac{-2}{\sqrt{g}} \frac{\delta S^{(\mu)}}{\delta g^{\alpha\beta}} \tag{4.9}$$

Where $\mathscr{L}$ is the Lagrangian, $g^{\alpha\beta}$ is the two-dimensional metric and S is the action. For simple theories [214, 215] it is possible to directly apply (4.9) and study the effects of $T\overline{T}$-deformation[4]; however, if we need to focus on more complex theories it is better to give a formulation in terms of the spectral properties, as shown in the next section.

Zamolodchikov's factorization formula A particularly convenient setting to study the $T\overline{T}$ deformation is provided by finite-volume theories. For instance, one can consider an (undeformed) QFT defined on a finite cylinder[5] of radius R. The spectrum of the corresponding $T\overline{T}$-deformed theory can then be analyzed starting from Zamolodchikov's factorization formula [122]:

$$\langle n|T\bar{T}|n\rangle = \langle n|T|n\rangle\langle n|\bar{T}|n\rangle - \langle n|\Theta|n\rangle\langle n|\Theta|n\rangle. \tag{4.10}$$

Where $|n\rangle$ is an eigenvalue of the Hamiltonian of the theory. The above equation provides a convenient formulation of the $T\overline{T}$-deformed theory in terms of a flow equation

[3] The coupling constant is of dimension m^{-2}.

[4] As an example, if we start from a 2d non-interacting free boson and follow the Lagrangian flow driven by $T\overline{T}$, we are surprisingly led to the Nambu-Goto string action. This is a first suggestion that $T\overline{T}$ is somehow linked to a non-local UV completion of IR theories.

[5] In the case of 2d Yang-Mills, this property can be extended to other topologies via the standard cylinder-gluing procedure.

expressed through the observables of the undeformed theory. The proof of (4.10) consists in three steps:

- 1) Prove that the expectation value $\langle \hat{\mathcal{C}}_{T\overline{T}}(z,w)\rangle$ is a constant.
- 2) Factorize the products of operators.
- 3) Compute the pinching limit of the factorized expression.

The Burger's equation Since the result (4.10) is based on Lorentz invariance and conservation of the stress-energy tensor, it holds along all the $T\overline{T}$ flow. Therefore, we can try to translate the formula in the language of a differential equation which relates the deformed solution to the undeformed one. We consider a theory defined on a cylinder of radius R and start from the the first formula in (4.8), which implies that the Hamiltonian varies as:

$$H(\mu+\delta\mu) = H(\mu) + \delta\mu \int_0^R dx\, T\bar{T}(x), \tag{4.11}$$

The variation of an energy eigenvalue with respect to a continuous parameter of the Hamiltonian is governed by the Feynman-Hellmann theorem:

$$\frac{\partial}{\partial\mu} E_n(\mu) = \langle n(\mu)|\, \frac{\partial H(\mu)}{\partial\mu}\, |n(\mu)\rangle. \tag{4.12}$$

Applying this to the present case gives:

$$\frac{\partial}{\partial\mu} E_n(R,\mu) = \int_0^R dx\, \langle n|T\bar{T}(x)|n\rangle. \tag{4.13}$$

By translation invariance along the spatial circle, the expectation value $\langle n|T\bar{T}(x)|n\rangle$ is independent of x. Therefore the integral yields:

$$\frac{\partial}{\partial\mu} E_n(R,\mu) = R\,\langle n|T\bar{T}(0)|n\rangle \tag{4.14}$$

This is the basic flow equation for the energy levels under the $T\bar{T}$ deformation. Combining (4.10) and (4.14), we obtain a preliminary version of the Burgers' equation:

$$\frac{\partial}{\partial\mu} E_n(R,\mu) = R\langle n|T|n\rangle\langle n|\bar{T}|n\rangle - R\langle n|\Theta|n\rangle\langle n|\Theta|n\rangle. \tag{4.15}$$

Finally, in a QFT on a cylinder of radius R the (dimensionless) eigenvalues of energy ($\mathcal{E}_n$) and momentum (P_n) are related to the stress energy tensor in the following way:

$$\frac{\mathcal{E}_n(R,t)}{R} = -\left\langle n\left|T\right|n\right\rangle, \quad \partial_R\mathcal{E}_n(R,t) = -\left\langle n\left|\overline{T}\right|n\right\rangle, \quad P_n = -\mathrm{i}R\left\langle n\left|\Theta\right|n\right\rangle. \tag{4.16}$$

Therefore we obtain the expression of the celebrated Burgers' equation [6]:

$$\partial_\mu \mathcal{E}_n(R,\mu) = \mathcal{E}_n(R,\mu)\partial_R \mathcal{E}_n(R,\mu) + \frac{1}{R}P_n^2(R), \tag{4.17}$$

which is a partial differential equation linear in μ [123]. In classical fluid mechanics it is the simplest equation describing the phenomenon of shock waves, which has nonlinear origins. We notice that the momentum eigenvalues are not modified by the deformation; this is expected, since the spectrum associated with the compact dimension looping around the cylinder is discrete and cannot be continuously deformed by a flow equation. The initial condition is given by the undeformed spectrum at $\mu = 0$.
The existence of a closed flow equation for the energy eigenvalues under the $T\bar{T}$ deformation is highly non-trivial and has important consequences. Unlike generic irrelevant deformations, the spectrum evolves autonomously: the derivative $\partial_\mu E_n$ depends only on the energy and momentum of the same state, leading to a deterministic flow. Moreover, the inviscid Burgers equation suggests that the deformation of the spectrum is mathematically analogous to the nonlinear dynamics of a one-dimensional fluid, explaining the occurrence of singularities at large energies in the perturbative analysis. Finally, the equation suggests that the structure of the flow is universal, independent of the microscopic details of the undeformed theory.

4.1.2 Physical Interpretations

In this section we treat the physical and geometrical meaning of the $T\overline{T}$ deformation, which was firstly suggested in [125, 127, 128].

$T\overline{T}$ and random geometries We start from the random geometry point of view, introduced by Cardy [125]. As we will see, this allows us to interpret the deformation as an infinitesimal random shift of the underlying geometry. Analysing standard geometries (torus, cylinder, disk, and more general simply connected domains), Cardy shows that the partition function evolves under a linear diffusion-type equation, effectively describing a random walk in moduli space. The construction can be generalized to non-Lorentz-invariant theories.
We now proceed to make this formalism explicit, starting from the expression of the $T\overline{T}$

[6]Here we are omitting the fact that, in principle, the deformation includes also contributions coming from derivatives of operators. However, also a more rigorous derivation is possible [216]: it consists in computing the partition function and then extract the spectrum from it.

flow:

$$S^{(\mu+\delta\mu)} = S^{(\mu)} + \frac{\delta\mu}{2} \int_{\mathcal{M}} \epsilon_{ik}\, \epsilon_{jl}\, T^{ij} T^{kl}\, \mathrm{d}^2 x\,. \tag{4.18}$$

Here we consider a two-dimensional QFT defined on arbitrary compact manifolds. Since locally any such manifold admits a flat metric, its analysis can be reduced to the study of simply connected patches. As before, we take the undeformed theory to correspond to $\mu = 0$. We can schematically write the partition function of the model after the infinitesimal transformation as:

$$Z^{(\mu+\delta\mu)}[\mathcal{M}] = \int \mathcal{D}\phi\; e^{-S^{(\mu)}[\phi]}\, \exp\left[-\frac{\delta\mu}{2} \int_{\mathcal{M}} \epsilon_{ik}\epsilon_{jl}\, T^{ij} T^{kl}\, \mathrm{d}^2 x \right]. \tag{4.19}$$

Where ϕ denotes the set of fields of the theory prior to the infinitesimal transformation. Here T depends on the theory defined at $\mu \neq 0$. It is convenient to interpret the transformation as a gaussian integral, via Hubbard-Stratonovich trick:

$$\mathrm{e}^{\frac{\delta\mu}{2} \int_{\mathcal{M}} \epsilon_{ik}\epsilon_{jl} T^{ij} T^{kl}\, \mathrm{d}^2 x} \propto \int \mathcal{D}h\, \mathrm{e}^{-\frac{1}{2\delta\mu} \int_{\mathcal{M}} \epsilon^{ik}\epsilon^{jl} h_{ij} h_{kl}\, \mathrm{d}^2 x + \int_{\mathcal{M}} h_{ij} T^{ij}\, \mathrm{d}^2 x}. \tag{4.20}$$

Now we recall that in a QFT coupled to a background metric g_{ij}, an infinitesimal variation of the metric results in a variation of the action of the form:

$$\delta S = \frac{1}{2} \int_{\mathcal{M}} \sqrt{g}\, T^{ij}\, \delta g_{ij}\, \mathrm{d}^2 x. \tag{4.21}$$

Namely, any infinitesimal deformation of the action with respect to the metric necessarily involves a term proportional to T^{ij}. Comparing this general expression with the linear coupling:

$$\int_{\mathcal{M}} h_{ij} T^{ij}\, \mathrm{d}^2 x, \tag{4.22}$$

which appears after performing the Hubbard–Stratonovich transformation, we can interpret h_{ij} as an infinitesimal fluctuation of the background metric δg_{ij}. We will later specify the nature of such a fluctuation.

We have seen that the Hubbard–Stratonovich transformation also generates a purely quadratic term in h_{ij}:

$$S[h] = -\frac{1}{2\delta\mu} \int_{\mathcal{M}} \epsilon^{ik}\epsilon^{jl}\, h_{ij} h_{kl}\, \mathrm{d}^2 x. \tag{4.23}$$

This term governs the dynamics of the auxiliary metric field in the absence of matter sources. In this sense it plays the role of an effective action for the “gravitational sector”.

It is important to stress, however, that $S[h]$ is not the Einstein–Hilbert action: in two dimensions the Einstein–Hilbert term is purely topological and does not generate local dynamics. Rather, $S[h]$ is a Gaussian weight ensuring that, away from the saddle point, the fluctuations of h_{ij} are suppressed, ($\delta\mu$ is infinitesimal).
We now turn to a more detailed analysis of the metric fluctuations. In two dimensions, the most general deformation of a metric can be written as:

$$h_{ij} = \partial_i \alpha_j + \partial_j \alpha_i + g_{ij}\Phi. \tag{4.24}$$

where we use the fact that h_{ij} is locally conformally flat. The first terms encode an infinitesimal diffeomorphism of the Euclidean metric $x_i \to x'_i = x_i + \alpha_i(x)$; while the second one is an infinitesimal change of scale.
However it is possible to show that $g_{ij}\Phi$ can be absorbed in a redefinition of $\alpha_i(x)$. Indeed one can evaluate the stress-energy tensor T_{ij} at the saddle point, by means of (4.24); impose the conservation of the stress energy tensor and extract the condition $\partial^j\partial_j\Phi = 0$; show that Φ is a sum of holomorphic and anti-holomorphic functions; provide a rescaling of α_z and $\alpha_{\bar{z}}$ which absorbs Φ. In the rescaled setting, the conservation equation translates into:

$$\partial_z \alpha_{\bar{z}} = \partial_{\bar{z}} \alpha_z.$$

As a consequence the on-shell metric deformation becomes:

$$h_{ij} = 2\partial_i \alpha_j \tag{4.25}$$

We are therefore in position to evaluate the variation of the action as:

$$\delta S = \frac{2}{\delta\mu} \int_{\partial\mathcal{M}} (\epsilon^{ik}\epsilon^{jl}\alpha_j \partial_k \alpha_l)\, \mathrm{d}n_i - 2\int_{\partial\mathcal{M}} \alpha_j T^{ij}\, \mathrm{d}n_i \tag{4.26}$$

where we have used the fact that the quadratic term can be expressed as a total derivative, allowing us to apply Stokes' theorem. Since it reduces to a boundary term, it is clear that the infinitesimal transformation is topological. This is a fundamental point, since topological theories are solvable, and this is the primary reason for the solvability of the $T\overline{T}$ deformation. The solvability was already seen in the context of the S-matrix [212].

Cardy's diffusion equation From the practical point of view, the random geometry construction leads to a diffusion equation governing the deformed partition function, which we now specify in the case of torus topology. One can argue that the only contribution to the deformation comes from constant flat metrics. Indeed, the stress-energy tensor T_{ij} is constant on the torus, due to translational invariance. As a result, any position-dependent fluctuations of the auxiliary metric $h_{ij}(x)$ do not change the linear coupling. Consequently, the dominant contribution to the functional integral comes from uniform, position-independent h_{ij}. This reduces the functional integral over $h_{ij}(x)$ to an ordinary integral over the four constant components h_{ij}, greatly simplifying the derivation of the flow equation for the deformed partition function. The part of the path integral involving the infinitesimal $T\overline{T}$-deformation reduces to an ordinary integral:

$$\mathrm{e}^{\delta\mu \int \det(T_{ij})\mathrm{d}^2x} \propto \int \prod_{i,j=1}^{2} \mathrm{d}h_{ij}\mathrm{e}^{-\frac{A}{\delta\mu}(h_{11}h_{22}-h_{12}^2)+Ah_{ij}T^{ij}} \tag{4.27}$$

where A is the area of the torus. Following Cardy [125], we consider a two-dimensional complex torus defined by the parallelogram with vertices 0, L, L', and $L+L'$, where $L = L_1 + iL_2$ and $L' = L'_1 + iL'_2$. In the same paper it is shown that given a Gaussian integral of the form above (4.27), its solution obeys a diffusion-type equation. Which is:

$$\partial_\mu Z = \frac{1}{2A}\epsilon_{ik}\epsilon_{jl}\left(\frac{\partial^2 Z}{\partial g_{ij}\partial g_{kl}}\right). \tag{4.28}$$

Given the discussion in the previous paragraph it is simple to relate the variation of the metric to a variation of the four parameters defining the shape of the torus:

$$\partial_\mu Z = \frac{1}{2}\epsilon_{ik}\epsilon_{jl}(L_i\partial_{L_j} + L'_i\partial_{L'_j})(1/A)(L_k\partial_{L_l} + L'_k\partial_{L'_l})Z. \tag{4.29}$$

As argued by Cardy [125], the diffusion equation for the torus partition function holds even for non-relativistic theories, such as Lifshitz-like or Galilean systems. The only necessary requirements are translational invariance and a consistent definition of $T\overline{T}$ as the determinant of the stress-energy tensor, while no assumption of Lorentz invariance is needed. As a result, this allows us to deform a broader class of theories than those assumed in the seminal works. This observation was a fundamental step when considering $T\overline{T}$ the deformation of Okuyama-Sakai's local Boson formulation of chiral Yang-Mills, which is non-Lorentz-invariant.

Coupling a QFT to 2d topological gravity To provide a complete nonperturbative path integral reformulation of the $T\overline{T}$-deformation, extending Cardy's idea, it has been proposed [127, 128] that the deformation is equivalent to coupling the undeformed theory to a two-dimensional topological gravity sector. This analysis was motivated by the observation that the $T\overline{T}$-deformation acts at the level of the S-matrix through the multiplication by a CDD factor [123, 124, 214]. We think it is relevant to mention this gravitational point of view, which captures the essential features of the deformation and naturally connects to the other central theme of this thesis, namely two-dimensional dilaton gravity.

Explicitly, the proposal by Dubovsky and collaborators begins by considering Jackiw–Teitelboim gravity on flat spacetime, coupled to the undeformed theory:

$$S_{\mathrm{T}\overline{\mathrm{T}}} = S_0\left(g_{\mu\nu}, \psi\right) + \int \sqrt{-g}(\varphi R - \Lambda)\mathrm{d}^2 x \tag{4.30}$$

Here S_0 denotes the undeformed action. Clearly, the Riemann metric $g_{\mu\nu}$ appears only as a determinant in the volume form, but not as a degree of freedom. The undeformed theory S_0 is not coupled directly to the dilaton field, but only to the metric. This type of coupling is rather novel in the literature, as previous works more commonly considered couplings of topological gravity to topological sigma-models [217]. Here the dilaton field is denoted as φ, while the cosmological constant Λ is also related to the deformation parameter by the equation: $\mu = -2/\Lambda$.

The main statement of [128] is that the previous action is actually equivalent to the standard definition of $T\overline{T}$ deformation. As a first step, we can separate the action in a matter sector and a purely gravitational one:

$$S_{\mathrm{TT}} = S_{\text{grav.}} + S_{\mathrm{m}} \tag{4.31}$$

$$S_{\text{grav.}} = \int \sqrt{-g}\phi R\, \mathrm{d}^2 x \quad , \quad S_{\mathrm{m}} = S_0\left(g_{\alpha\beta}, \psi\right) - \Lambda \int \sqrt{-g}\, \mathrm{d}^2 x \tag{4.32}$$

Now, we want to write the action in first order formalism. This framework makes it possible to analyze the symmetries of the model in an explicit way and, from the technical point of view, seems to solve problems such as gauge fixing and compactification of the vacuum configurations. The metric can be written in a convenient form, by introducing the vielbeins $e_{a\alpha}$, a spin connection ω and a Lagrangian multiplier λ^{α}. As usual, we define:

$$g_{\alpha\beta} = e_{a\alpha} e_{b\beta} \eta^{ab}, \tag{4.33}$$

where $\eta^{ab}$is the Minkowski metric. The actions become:

$$S_m = S_0 - \frac{\Lambda}{2} \int \epsilon^{\alpha\beta} \epsilon^{ab}\, e_{a\alpha}\, e_{b\beta}\;\; \mathrm{d}^2 x \tag{4.34}$$

$$S_{\text{grav.}} = \int \epsilon^{\alpha\beta} \left(\lambda^a \left(\partial_\alpha e_{a\beta} - \epsilon_a^b \omega_\alpha e_{b\beta}\right) + \varphi \partial_\alpha \omega_\beta\right) \mathrm{d}^2 x \tag{4.35}$$

where ϵ_a^b are the Levi-Civita symbols. When integrated, the dilaton sets the curvature of the spin connection to zero:

$$\epsilon^{\alpha\beta}\partial_\alpha\omega_\beta = 0. \tag{4.36}$$

The previous equation is clearly satisfied if we reduce ω_α to a longitudinal degree of freedom: $\omega_\alpha = \partial_\alpha\omega$. Moreover, the integration of λ^α sets the following equation:

$$\partial_\alpha e_{a\beta} - \epsilon_a^b\omega_\alpha e_{b\beta} = 0 \tag{4.37}$$

At this stage, we notice that ω acts as a Stueckelberg field, i.e. it generates the rotations in the Lorentz space. We can naturally gauge fix this internal Lorentz symmetry by imposing the condition: $\omega = 0$. Finally, after an integration by parts, the gravitational action becomes:

$$S_{\text{grav}} = -\int \epsilon^{\alpha\beta}\partial_\alpha\lambda^a e_{a\beta}\mathrm{d}^2x \tag{4.38}$$

Clearly, in the previous expression the only surviving symmetry is the Lorentz global symmetry: $\lambda^a \to const. + \lambda^a$. After imposing the equations of motion on the vielbeins it is possible to show that they are related to the derivative of the Lagrangian multipliers (λ^α), therefore, it is convenient to introduce the redefinition of the fields:

$$X^a = \Lambda^{-1}\epsilon_b^a\lambda^b \tag{4.39}$$

The previous step is analogous to the field dependent coordinates transformations appearing in the S-matrix formalism. As a result, we can write the derivatives of X^α as functions of the energy-momentum tensor coming from the variation of the matter action, in the following way:

$$\partial_+X^- = -\frac{T_{++}}{\Lambda}, \quad \partial_-X^+ = -\frac{T_{--}}{\Lambda}, \quad \partial_+X^+ = \partial_-X^- = \frac{T_{+-}}{\Lambda}. \tag{4.40}$$

The total action (adding a total derivative) becomes:

$$S_{T\bar{T}} = \frac{\Lambda}{2}\int \epsilon^{\alpha\beta}\epsilon_{ab}\left(\partial_\alpha X^a - e_\alpha^a\right)\left(\partial_\beta X^b - e_\beta^b\right) + S_0\left(\psi, g_{\alpha\beta}\right) \tag{4.41}$$

Clearly, the partition function consists in integrating all the degrees of freedom, but the undeformed partition function factorizes, since it is not coupled to the gravitational degrees of freedom:

$$Z_{T\overline{T}} = \int \frac{\mathcal{D}e\mathcal{D}X}{V_{\text{diff}}} e^{-\frac{\Lambda}{2}\int \epsilon^{\alpha\beta}\epsilon_{ab}\left(\partial_\alpha X^a - e_\alpha^a\right)\left(\partial_\beta X^b - e_\beta^b\right)} Z_0\left(g_{\alpha\beta}\right) \tag{4.42}$$

The integration of the vielbeins is restricted to the torus topology; while the $\mathcal{D}X$ integration counts the mappings (with winding equal to one) between a torus worldsheet into a torus target space.
The next step is meant to derive explicitly the spectrum on the torus topology. It requires to use the Faddeev-Popov method to express the partition function as an ordinary integral over the variables parametrizing the torus. Then, it is possible to perform a gaussian integration and argue that the resulting partition function should localize around a saddle point.
Finally, the solution of the saddle point equation precisely reproduces the inviscid Burgers equation governing the spectrum. Therefore, it is possible to state that the $T\overline{T}$-deformation is equivalent to coupling a QFT to a two-dimensional gravity sector. This interpretation grounds the deformation in a clear geometric and gravitational framework, explaining solvability. A potential source of confusion might be that in the modern approach to JT gravity, the theory does possess degrees of freedom, which are encoded in the Schwarzian modes [11, 12]. Nevertheless, it is important to emphasize that the topological sector considered here differs in important details: the action lacks a Gibbons–Hawking–York boundary term and is defined on flat space (not AdS_2). As a result, the sector, does not enjoy the parametrization invariance and no additional degrees of freedom are introduced through the deformation.

Other interpretations. There exist, of course, several other perspectives on the $T\overline{T}$-deformation, which we shall not discuss in detail here. We only highlight the holographic viewpoint [131] , as it provides a particularly illuminating connection to gravity. In this interpretation, $T\overline{T}$-deformed CFTs describe AdS gravity at finite radial cutoff. A paradigmatic example in $D = 3$ is the duality between the BTZ black hole and a CFT_2 defined on the holographic boundary with asymptotic Dirichlet conditions. Applying the $T\bar{T}$ deformation to the boundary theory corresponds to moving the holographic screen into the bulk. An analogous behavior emerges in JT gravity, where an effective $T\overline{T}$-deformation applied to the Schwarzian theory describes JT gravity at finite-cutoff [183, 218].

4.2 Two dimensional Yang-Mills

4.2.1 Solving 2d Yang-Mills on Riemann surfaces

In this section, we introduce the main features of two-dimensional Yang–Mills theory on Riemann surfaces, focusing on the case of Large-N torus topology, which will be relevant for our analysis.

The model We consider Yang–Mills theory [95, 219, 220], on an orientable two–dimensional Riemann surface Σ of genus $\mathrm{g} = 1 - \chi/2$. The gauge group is taken to be a compact Lie group G, with associated Lie algebra $\mathfrak{g}$. We introduce a gauge connection A, a $\mathfrak{g}$-valued one-form, with gauge invariant curvature: $F = \mathrm{d}A - i\, A \wedge A$. The action[7] of pure Yang–Mills theory is then given by:

$$S_{\mathrm{ym}} = \frac{1}{2g_{\mathrm{ym}}^2} \int_\Sigma \mathrm{tr}\,(F \wedge \star F) \tag{4.43}$$

The two-form $\mathrm{tr}\,(F \wedge \star F)$ can be written in terms of the Riemann metric g_{ij} on the manifold. The trace acts on the Lie-algebra and g_{ym} is the gauge coupling constant. Since we are in dimensions two, up to an arbitrary normalization constant, we can introduce a volume form η on the Riemann surface and write: $F = \eta f$. This clearly implies:

$$S_{\mathrm{ym}} = \frac{1}{2g_{\mathrm{ym}}^2} \int_\Sigma \eta\ \mathrm{tr}\left(f^2\right) \tag{4.44}$$

the previous expression means that the action only depends on the choice of the measure of integration and on the gauge group G (determining the trace). The only invariant quantity corresponding to a measure for a 2d orientable Riemann surface is the total area:

$$a = \int_\Sigma \eta \tag{4.45}$$

We now observe that the action is invariant not only under gauge transformations, but also under a larger class of symmetries, the so-called "area-preserving diffeomorphisms"[8] [92]. Indeed, consider a rescaling of the volume form $\eta \to t\,\eta$. Consequently, the normalized field strength rescales as $f = F/\eta \to f/t$. If, at the same time, we rescale the coupling constant as $g_{\mathrm{ym}}^2 \to g_{\mathrm{ym}}^2/t$, the Yang–Mills action S_{ym} remains invariant. In particular, the product $a\, g_{\mathrm{ym}}^2$ is left unchanged under these transformations:

$$g_{\mathrm{ym}}^2\, a \ = \ g_{\mathrm{ym}}^2 \int_\Sigma \eta \ \to \ \frac{g_{\mathrm{ym}}^2}{t} \int_\Sigma \eta\, t \ \equiv \ g_{\mathrm{ym}}^2\, a \tag{4.46}$$

Therefore, we conclude that the action is only a function of the effective area $\tilde{a} = g_{\mathrm{ym}}^2\, a$. The previous statement extends to the Yang-Mills partition function, which can be de-

[7] In a rigorous mathematical setting, the action is obtained as the integral over the Riemann surface of the squared L^2-norm of the curvature F.

[8] As we will see below, this terminology is somewhat misleading: the actual invariant quantity is the combination $a\, g_{\mathrm{ym}}^2$.

fined via the usual path integral formulation:

$$Z(\tilde{a}) = \int \mathcal{D}A \;\; e^{-S_{ym}(\tilde{a})} \tag{4.47}$$

This enlarged symmetry group renders the theory almost-topological. As a consequence, the model is both integrable and exactly solvable [95]. From the above discussion, it follows that the theory carries no local propagating degrees of freedom: after gauge fixing, only global modes remain. In particular, there are no transverse gluons in the spectrum and the theory is built out of non-trivial vacua alone. Although this might suggest triviality, the model instead displays a remarkably rich structure.

The Migdal heat kernel expansion Several techniques are available to obtain explicit expressions for the spectrum of the theory. A first solution was provided by Migdal and Rusakov [92, 93], based on a lattice formulation employing the Wilson action [221]. An alternative approach was later developed by Witten [95, 219], which consists in cutting the Riemann surface along a circle and factorizing the path integral into a product of states in a Hilbert space. In this formulation, the Hilbert space is given by the class functions on the gauge group G. This arguments can be extended to study the $T\overline{T}$–deformed partition function, as it exploits the underlying topological structure of the theory. The resulting expression for the partition function, known as the Migdal representation, is:

$$Z(\tilde{a}) = \sum_{R} (\dim R)^{\chi} \;\; e^{-\tilde{a}\, C_2(R)/2} \tag{4.48}$$

and takes the form of a sum over inequivalent irreducible representations R of the gauge group G, weighted by the Euler characteristic χ of the Riemann surface and multiplied by an appropriate Boltzmann factor. Here $C_2(R)$ denotes the eigenvalue of the quadratic Casimir operator in the representation R. We emphasize that the torus topology is a special case: since $\chi = 0$, the sum reduces to a purely Boltzmann-weighted expression.
The formula can be extended to surfaces with boundaries by associating to each boundary a gauge-invariant holonomy and a character [95, 219]. The expression is particularly convenient when generalizing properties that already hold for genus-zero surfaces such as the disk and the cylinder. In close analogy with JT gravity, observables for higher-genus surfaces can be obtained by a gluing procedure: gluing two boundaries corresponds, at the level of the partition function, to integrating the associated characters of the holonomies with respect to the Haar measure of the gauge group G.
We should finally mention that, parallel to the Migdal expansion, there is a "dual" description which organizes the partition function as a sum of instanton contributions, encoding the structure of nontrivial vacua of the theory.

4.2.2 2d Yang-Mills on torus topology

The Gross-Taylor string expansion Since the early days of large-N gauge theories, it has been hoped that a string-theoretical reformulation of QCD could provide an effective framework to study the model even in the strong-coupling, low-energy regime [222]. This idea motivated attempts to express QCD in terms of an emergent string description, having in mind to continue it to strong coupling. A first concrete realization of this program appeared in two-dimensional Yang–Mills theory [223, 224]. However, in case of sphere topology, the presence of a strong/weak phase transition [225] raised important questions about the actual convenience of the string picture.
The idea of a dual string description was studied in several variants, depending on the genus of the Riemann surface, its orientability, and the choice of gauge group. In their seminal paper [223], Gross and Taylor focused on the $U(N)$ theory defined on the torus. They showed that the large-N Migdal partition function (4.48) can be decomposed into two chiral sectors. These sectors are distinguished by the sign of the number of boxes in the Young tableaux associated with a given representation. The key assumption is that, in the large-N limit, only representations with a small number of boxes contribute significantly. This allows one to analyze each chiral sector separately and derive its asymptotic expansion. The analysis leads to a chiral sector whose free energy is:

$$\mathcal{F}(g, \tilde{\alpha}, N) = \sum_{G=0}^{\infty} N^{2-2G} f_G^g(\tilde{\alpha}) \tag{4.49}$$

where:

$$f_G^g(\tilde{\alpha}) := \sum_n \sum_i \omega_{G,g}^{n,i} e^{-\frac{n\tilde{\alpha}}{2}} (\tilde{\alpha})^i \tag{4.50}$$

The proposal by Gross and Taylor is that the free energy $\mathcal{F}$ (in the small area limit) is a perturbative string expansion, counting branched covering maps from a genus G surface to the torus target space ($g = 1$). The maps cover the target space n times and therefore have winding number equal to n. The number of branchpoints is counted by i. In particular, two elements in the sum are considered:

- In (4.50) the contributions with the maximum powers of the area, are given by the Riemann-Hurwitz equation: $2(G-1) = 2n(g-1) + i$ and the corresponding coefficient: $\omega_{G,g}^{n,i}$ is equal to the number of topologically distinct continuous maps of the genus-G surface into the torus, with winding number n and with i branchpoints. Since the number of branchpoints is non negative and $g = 1$, the previous relation converts to $i = 2G - 2$ and imposes that the first contributions to the free energy start at $G = 1$. This also means that the free energy has a first contribution of order N^0 followed by a first correction of order N^{-2}.

- The other terms $\omega^{n,i}_{G,g}$ (satisfying the condition: $2(g-1) > 2n(g-1)+i$) can be interpreted in terms of branchpoints, collapsed handles, and two types of "tubes" connecting sheets of the covering space. These, in some sense, form a set of Feynman rules for this string theory.

This interpretation clarifies why the construction is referred to as a string expansion. However, in the full theory, additional $1/N$ corrections do contribute and it is not clear whether a consistent string-theoretic formulation exists.

Effective descriptions of chiral U(N) Yang-Mills on $\mathbb{T}^2$ When focusing on the partition function of $U(N)$ Yang Mills theory on torus topology, several equivalent formulations do emerge. The theory can be reformulated by employing conformal field theory methods [226]. As a result, the dynamics can be mapped to a system of quasirelativistic free fermions on a circle, or equivalently bosonized into a collective field description, which allows one to compute the partition function explicitly. The starting point is the Migdal exact representation of the partition function (4.48), which in the case of torus topology appears as:

$$Z(A,N) \;=\; \sum_R \exp\Big(-\frac{A}{N}\,C_2(R)\Big)\,, \tag{4.51}$$

Using the correspondence between Young diagrams and fermionic occupation numbers, this sum is reformulated as the partition function of free fermions on a circle. The author considers the second Casimir, which is a quasirelativistic system of the form:

$$H_{U(N)} = NL_0 + N\bar{L}_0 + \oint dz z^2 : \partial\psi^+\partial\psi : + \oint d\bar{z}\bar{z}^2 : \bar{\partial}\bar{\psi}^+\bar{\partial}\bar{\psi} : \tag{4.52}$$

where $L_0 + \bar{L}_0$ is the relativistic Hamiltonian. After bosonization:

$$\phi(z,\bar{z}) := \phi_L(z) + \phi_R(\bar{z}), \qquad e^{i\phi_L(z)} := \psi(z), \qquad e^{-i\phi_L(z)} := \psi^+(z). \tag{4.53}$$

the dynamics is expressed in terms of a compact boson $\phi(z)$ on the torus, coupled to a chiral interaction:

$$H \;=\; H_0 + \frac{1}{N}H_I, \qquad H_I \;\sim\; \oint dz\, z^2 : (\partial\phi(z))^3 : . \tag{4.54}$$

Hence we identify chiral Yang–Mills theory as a chiral deformation of a free CFT. Rudd [105] has shown that the functions $F_{g\to 1}$ are made of Eisenstein quasi-modular forms.

The appearance of these series strongly suggests that the underlying theory admits an interpretation as a topological string theory. Indeed, a similar quasi-modular structure was found in the topological string partition function for certain non-compact Calabi-Yau threefolds, where the free energies obey the holomorphic anomaly equation of Bershadsky–Cecotti–Ooguri–Vafa [111, 112]. In this sense, the two-dimensional Yang-Mills theory on the torus can be viewed as an emergent string theory, with the quasi-modularity encoding the rich symmetry structure characteristic of topological strings.
Since the torus is the simplest one-dimensional Calabi-Yau manifold, according to Dijkgraaf [115], there exist a mirror symmetry description, in which the invariant free energies $F_g(\tau)$ on the original torus can also be interpreted as an invariant of the mirror torus[9]. The main goal was to confirm the general predictions of mirror symmetry in the case of the deformed compact boson, whose action can be conveniently written as:

$$S = \int \left(\partial\varphi\bar{\partial}\varphi + \lambda(\partial\varphi)^3\right) \tag{4.55}$$

An elliptic curve has two moduli: the complex modulus $\tau \in \mathbb{H}$, describing its complex structure, and the Kähler modulus $t \in \mathbb{H}$, describing the area of the curve. Mirror symmetry exchanges these two parameters: remarkably, the author has shown that the chiral boson action (4.55) can be expressed as a path integral of free fermions b, c on the mirror torus:

$$Z = \int [db\,dc]\, e^{-S(b,c)}, \tag{4.56}$$

where the action $S(b, c)$ is defined as:

$$S(b,c) = \int (b\,\bar{\partial}c + \lambda b\,\partial^2 c).$$

From this representation, one finds explicitly that for $g \geq 2$, the functions F_g are quasi-modular forms of weight $6g - 6$. Finally, using the boson/fermion correspondence in two dimensions, the functions F_g can also be computed combinatorially as a sum over trivalent graphs $\mathcal{G}_g$ on the mirror torus:

$$F_g = \sum_{r\in\mathcal{G}_g} \frac{I_r}{|\mathrm{Aut}(r)|}, \tag{4.57}$$

where vertices are labeled by points on the mirror elliptic curve and propagators are given by the Weierstrass $\wp$-function. This construction provides a rigorous realization of mirror symmetry.

[9] This idea is based on the breakthrough work of Bershadsky, Cecotti, Ooguri, and Vafa [111, 112] who performed a similar analysis in the context of Kodaira-Spencer theory.

At first sight the deformation appearing in the compact boson formulation (4.54) looks complicated, because it breaks conformal symmetry and introduces a nontrivial interaction. But the remarkable fact is that, once reformulated in terms of the b, c system, the action (4.2.2) becomes quadratic. Hence, underneath, chiral 2D Yang–Mills is governed by a free model (albeit chirally deformed). This explains its solvability.

From this point of view, the partition function can be interpreted as a generalized character of a chiral algebra, specifically the infinite algebra $W_{1+\infty}$. The cubic term $(\partial\varphi)^3$ is a spin-3 generator of this algebra, and the deformation corresponds to "switching on" only that direction. In this language, the partition function becomes a character of the corresponding pre-Lie algebra with an exponential insertion of the corresponding charge operator.

Inspired by topological strings, in which there exist an anomalous structure intertwining holomorphy and modularity, in a following paper [113], the same author focuses in studying an holomorphic anomaly equation which allows to recursively compute the genus expansion of the partition function. Dijkgraaf shows that also for 2d Yang-Mills one can write down a holomorphic anomaly equation. It takes the form:

$$\partial_{\bar{\tau}} Z = \frac{1}{N^2 \tau_2^2} \partial_\tau^2 Z. \tag{4.58}$$

This expression can be interpreted as the two-dimensional counterpart of the BCOV equations in six dimensions (here $\tau_2 = \operatorname{Im}\tau$ is related to the area on the torus).

In order to make sense of the holomorphic anomaly equation, Dijkgraaf proposes the introduction of an explicit anti-holomorphic dependence by means of the transformation $E_2 \to \hat{E}_2$. However, Okuyama-Sakai [114] have pointed out a limitation in Dijkgraaf's approach to the holomorphic anomaly, as we will explain in the next section.

Okuyama–Sakai anti-holomorphic deformation It is well known that, in order to make sense of the holomorphic anomaly equation, one must restore the anti-holomorphic dependence on $\bar{\tau}$. A common prescription replaces the quasi-modular form $E_2(\tau)$ with its non-holomorphic completion $E_2(\tau, \bar{\tau})$, exploiting the interplay between the holomorphic and modular anomalies. However, this procedure does not explicitly satisfy the anomaly equation (4.58) proposed in [113]. Besides the free energy genus expansion, there is a deep theoretical reason for believing in the existence of an Holomorphic Anomaly Equation. Indeed, by relating the partition function of two-dimensional Yang–Mills theory to that of four-dimensional BPS black holes and invoking the OSV conjecture [102], the dual string theory can be identified as a topological string theory formulated on a double line bundle over an elliptic curve [101]. Prior to Okuyama and Sakai's work [114], no consistent prescription for restoring the $\bar{\tau}$-dependence was known in two-dimensional Yang–Mills theory on a torus, nor was a correct holomorphic anomaly equation derived. The authors successfully addressed this issue, as we will see. The proposal was further tested nonperturbatively using resurgence techniques in [227].

We emphasize that, for the purposes of studying the $T\bar{T}$ deformation, it is essential to identify the correct anti-holomorphic dependence of the theory. We will therefore focus on a detailed analysis of the interplay between the holomorphic anomaly equation (HAE) and the anti-holomorphic dependence.

Okuyama and Sakai's proposal takes into account the conformal anomaly, which naturally restores the anti-holomorphic dependence through the propagator. The authors argue that the holomorphic anomaly arises consistently within the framework of the local bosonic formulation (4.55). By recovering this dependence, one uncovers a holomorphic anomaly equation (HAE) differing from Dijkgraaf's original proposal, yet well justified and likely related to the BCOV equation.

In two-dimensional Yang–Mills theory on the torus, the Eisenstein series E_2 originates from two sources: the propagator and the period integral. Standard treatments restore the holomorphic anomaly by replacing $E_2 \to \hat{E}_2$, but this shift should affect only the propagator, not the period integral. The occurrence of the period integral producing the Eisenstein function is accidental in this specific theory, and the holomorphic anomaly does not act on it.

As discussed in [114], there exists a correspondence between the bosonic and fermionic formulations, though they are not fully identical. In the bosonic theory, one can consider only local degrees of freedom, neglecting the winding modes. This defines a completely local theory, which can be studied via Feynman diagrams. In this setting, the anti-holomorphic dependence arises naturally, providing a trivial dependence that can be removed through a precise limiting procedure. Surprisingly, the purely local theory generalizes the non-local theory, which includes winding modes. Indeed, the propagator introduces the anti-holomorphic dependence on $\bar{\tau}$, but this is canceled once the winding modes are incorporated.

A more detailed analysis of the technical aspects discussed in [114] will be presented in the next chapter; here we summarize the main ideas relevant to our investigation. The deformation is systematically derived by assuming only locality and the specific form of the propagator, built from the Weierstrass function $\wp$ and the modified Eisenstein series $\hat{E}_2$. Evaluating the Feynman integrals, constructed from contractions of compact boson propagators, leads to period integrals of $\wp$, which reproduce the Eisenstein series E_2. In the anti-holomorphic limit, all one-particle-reducible diagrams vanish, yielding the undeformed partition function. The deformed genus-g free energies are polynomials in

$S := (4\pi\ \mathrm{Im}\,\tau)^{-1}$, with coefficients that are quasi-modular forms under $\mathrm{SL}(2,\mathbb{Z})$. Through explicit diagrammatic computations, it is shown that in a local theory the naive replacement $E_2 \to \hat{E}_2$ is inconsistent, since E_2 arises from distinct sources within the Feynman graph expansion. Moreover, because of the explicit dependence on S, the free energies are not modular invariant; this modular anomaly reflects the asymmetric treatment of the torus cycles. Once the anti-holomorphic dependence is consistently restored, the authors propose a holomorphic anomaly equation (HAE) for the partition function of the form

$$\left(2\partial_S - S^{-1}\right)\mathcal{Z} = g_s^2(D+S)D\mathcal{Z}, \tag{4.59}$$

where $D \propto \partial_\tau$. The corresponding equation for the renormalized partition function is

$$2\partial_S\widehat{\mathcal{Z}} = g_s^2\left(D + S - \frac{\widehat{E}_2}{24}\right)\left(D - \frac{\widehat{E}_2}{24}\right)\widehat{\mathcal{Z}}, \tag{4.60}$$

which is formally analogous to the BCOV equation for Calabi–Yau threefolds.

4.3 $T\overline{T}$-deformation of 2d Yang–Mills

In this section we briefly review the main results on the $T\overline{T}$ deformation of two-dimensional Yang–Mills theory [135, 136, 228], with emphasis on the sphere topology and its exact non-perturbative features [229–231]. In theories whose dynamics depend only on the area of the background manifold, Cardy diffusion equation takes a particularly simple form. This analysis is meant to highlight the differences with our main object of interest, namely large-N Yang–Mills theory on the torus.

Flow equation in Area–dependent theories As shown in [135, 136, 228, 230], for a theory whose partition function depends only on the area a of the Riemann surface on which are defined, the $T\overline{T}$-deformed partition function satisfies the simplified diffusion equation:

$$\frac{\partial Z}{\partial \mu} + 2a\frac{\partial^2 Z}{\partial a^2} = 0 \tag{4.61}$$

We can see why this expression makes sense in the particular case of the topological formulation of two dimensional Yang-Mills theory. It is convenient to start from the BF formulation of Yang-Mills theory, which is encoded in the action:

$$S = \int_\Sigma \mathrm{i}\,\mathrm{Tr}(\phi F) + \frac{g_{\mathrm{YM}}^2}{2}\,\eta\ \mathrm{Tr}(\phi^2). \tag{4.62}$$

The field ϕ is a Lie algebra–valued scalar on the two–dimensional surface Σ, while F denotes the curvature two–form of a gauge connection A. The first term, is the topological BF action; the second breaks topological invariance by introducing a quadratic potential for the auxiliary field ϕ, with the density η encoding the background volume form. The field ϕ appears quadratically and can be integrated out exactly, giving back (4.43) [219, 220].
Expression (4.62) serves as a convenient starting point for exploring deformed gauge theories. We simply replace $\mathrm{Tr}(\phi^2)$ by a generalized gauge-invariant structure [219]:

$$\mathrm{Tr}(\phi^2) \to \mathscr{U}(v,\mu) \quad ; \quad v := \mathrm{Tr}(\phi^2) = \lim_{\mu\to 0} \mathscr{U}(v,\mu). \tag{4.63}$$

Given the deformed action:

$$\mathrm{S}(\mu) = \int_\Sigma \mathrm{i}\,\mathrm{Tr}(\phi F) + \frac{g_{\mathrm{YM}}^2}{2}\,\eta\,\mathscr{U}(v,\mu), \tag{4.64}$$

we can compute the stress-energy tensor by applying (4.9) and using the fact that the first term in the integral does not depend on the metric, while the second only depends through the volume form. We find:

$$T^{\kappa\lambda} = \frac{-g_{\mathrm{YM}}^2}{2}\mathscr{U}(v,\mu)g^{\kappa\lambda} \tag{4.65}$$

The $T\overline{T}$ operator results:

$$T\overline{T} = 2\left(\frac{-g_{\mathrm{YM}}^2}{2}\right)^2 \; \mathscr{U}^2(v,\mu) \tag{4.66}$$

Now we can apply the definition of $T\overline{T}$ at the level of the action, getting:

$$\frac{g_{\mathrm{YM}}^2}{2}\int_\Sigma \eta\;\partial_\mu \mathscr{U}(v,\mu) = \int_\Sigma \eta\;\frac{g_{\mathrm{YM}}^4}{2}\;\mathscr{U}^2(v,\mu), \tag{4.67}$$

which leads to a flow equation for the deformed Casimir:

$$\partial_\mu \mathscr{U}(v,\mu) = g_{\mathrm{YM}}^2\;\mathscr{U}^2(v,\mu) \tag{4.68}$$

For reasons of simplicity, lets' focus on the gauge group $U(N)$. Then v coincides with the quadratic Casimir $C_2(R)$ and one can show that the solution to (4.68) is:

$$\mathscr{U}(v,\mu) = \frac{C_2(R)}{1-\mu\,g_{YM}^2\,C_2(R)} \tag{4.69}$$

By heuristically substituting (4.69) into (4.48) in place of the Casimir, one obtains the deformed partition function, which in turn satisfies (4.61).
As shown in [230], in order to make the argument consistent for general gauge theories on arbitrary topologies, one can repeat Witten's quantization procedure [219], in the presence of a deformed Casimir.
Knowing the expression of the deformed Casimir (4.69), in the case of the spherical topology without boundaries, the Migdal expansion becomes:

$$Z(\alpha,\tau) = \sum_R (\dim R)^2 e^{-\frac{\alpha}{2N}\frac{C_2(R)}{1-\tau C_2(R)/N^3}} \tag{4.70}$$

Where it is introduced an effective deformation parameter $\tau = \mu g_{ym}^2 N^3$, which is useful in the study of the large$-N$ limit.

Non-perturbative analysis on the Sphere In this paragraph we briefly mention the non-perturbative results in the case of spherical topology. The motivation to study the partition function at the non-perturbative level is clear if we look at the expression of the Hamiltonian:

$$H = \frac{g_{\mathrm{YM}}^2 C_2(R)/2}{1-\tau C_2(R)/N^3} \tag{4.71}$$

Indeed the spectrum obtained from the naive perturbative expansion in τ has some pathologies. It is possible to prove that for $\tau \neq 0$ there is an infinite number of representations whose energy is arbitrary close to the limit value $-1/2\mu$. In the t'Hooft limit we know that $\mu \to 0$ and the previous term diverges. In particular, we can distinguish a "good sign" and a "bad sign" regime as follows:

- Good sign) We run into a divergence in the Hamiltonian when $C_2(R) \to (N^3/\tau)^{\pm}$. Moreover, the eigenvalues seem to have a physical meaning (positive Hamiltonian) only when $C_2(R) < N^3/\tau$. This is a problem, indeed the Migdal partition function contains a sum over all the irreducible representations.

- Bad sign) This corresponds to the choice $\tau < 0$. A naive divergence appears, but the eigenvalues beyond the divergence are not pathological. It is crucial to determine whether such terms should be included in the physical spectrum.

A natural solution to the pathologies is to study the exact solution to the flow equation, taking into account nonperturbative contributions. The analysis employs resurgence on the deformed instanton representation [230].
The phase diagram of the theory, obtained via characteristic curves derived from the

leading-order free energy at $\tau = 0$, depends on the parameters τ and α. The phase diagram exhibits two phase transitions: the Douglas-Kazakov third-order transition and a second-order transition driven by nonperturbative effects in the deformation parameter, which intersect at a triple point. Three distinct regions emerge, corresponding to different instanton behaviors, showing that the irrelevant deformation significantly modifies the phase structure.

$T\overline{T}$-deformation of chiral Yang-Mills on $\mathbb{T}^2$ 5

5.1 Introduction

The study of irrelevant deformations of two-dimensional quantum field theories has been profoundly advanced by the $T\overline{T}$ operator, defined as the determinant of the stress-energy tensor. In Zamolodchikov's original proposal, the expectation value of $T\overline{T}$ was defined in terms of the stress-energy tensor of the undeformed theory. The key conceptual advancement came with the realization that the $T\overline{T}$ operator should be defined iteratively, using at each step the stress-energy tensor of the infinitesimally deformed theory itself [123, 124]. This leads to a fully closed flow equation for the Lagrangian, enabling exact computations even for non-perturbative effects.

A striking early application was the $T\overline{T}$ deformation of the free boson [124], whose action was shown to be equivalent to the Nambu-Goto action in static gauge. This provided a controlled procedure for obtaining a non-local, string-like model starting from a local quantum field theory. However, it became clear that this flow should not be interpreted in the Wilsonian sense, primarily because the number of degrees of freedom remains preserved along the flow, and a local theory is mapped to a non-local one.

The solvability of $T\overline{T}$-deformed theories manifests through several remarkable structures. The energy spectrum satisfies an inviscid Burgers equation [123], and the partition function on various geometries obeys diffusion-type equations [125, 126]. Cardy's formulation naturally extends the deformation to theories lacking Lorentz invariance and reveals that modular invariance persists when the deformation parameter is appropriately rescaled. This geometric interpretation enables the analysis of high-energy asymptotics, revealing a universal phase transition from Cardy-like behavior in the infrared to Hagedorn-like growth in the ultraviolet [134].

Among the two-dimensional geometries, the torus case has been particularly studied: very explicit results for the $T\overline{T}$-deformed partition function of conformal field theories have been derived [128, 132, 133, 232, 233] and in this context also non-perturbative properties, through resurgence techniques, have been explored [234]. Parallel to these developments, deformed Yang-Mills theories have been thoroughly studied on a two-dimensional sphere, deriving exact results for the partition function and obtaining the full phase diagram in the large-N limit [230, 231]. It is therefore natural to attempt similar investigations for the YM theory on the torus, focusing on the large N limit where a chiral-antichiral factorization appears.

Two-dimensional chiral $U(N)$ Yang-Mills theory on this geometry has been extensively studied due to its profound connections to conformal field theory and topological string theory [107, 113, 115]. The exact solvability of two-dimensional Yang-Mills theory [94, 95, 235] combined with the Gross-Taylor string interpretation [223, 224] provides a rich testing ground for gauge/string duality. In the large-N limit, the partition function factorizes into chiral sectors and admits expansion in terms of maps from worldsheets to the target space.

Regarding the string theory interpretation, there has been a renewed interest over the past few years [108–110, 236]. In particular, [108, 110] revisits a previous formulation by Hořava [237], which allows for a description of non-holomorphic mappings from the worldsheet.

Instead, in [109, 236] a new candidate bosonic string action is presented. The duality arises in the absence of an additional spatial "holographic" dimension, and the analysis aims to clarify, from a theoretical perspective, previous studies of confining theories on the lattice [238]. The model may represent a first example of an unusual class of holographic dualities in which no additional scalar modes appear on the string worldsheet. Interestingly, the duality is extended to symmetric product orbifolds for seed CFTs with $c < 24$ and their $T\overline{T}$ and $J\overline{T}$ deformations; a similar worldsheet action has been studied in the context of $T\overline{T}$ deformations [130]. Concretely, the authors propose a worldsheet action corresponding to a noncritical version of a non-relativistic string, which reproduces the free energy and scattering amplitudes of large-N chiral Yang-Mills theory. Although not directly relevant for our analysis, which is primarily based on the topological string interpretation [101], we consider this work interesting from a parallel holographic perspective and potentially inspiring for attempts to formulate an avatar of $T\overline{T}$ deformations of QFTs within a string-theoretic framework.

A crucial insight from Okuyama, Sakai, Dijkgraaf, and others [113, 114] is that the chiral boson description relevant for the equivalence with Yang-Mills theory is a local one, which naturally introduces anti-holomorphic dependence governed by a holomorphic anomaly equation [111, 112]. This controlled departure from strict holomorphicity provides the essential window for making $T\overline{T}$ deformations meaningful in this context.

This work aims to bridge these two domains by constructing the $T\overline{T}$ deformation of chiral $U(N)$ Yang-Mills theory on the torus. Our main proposal involves a sequential deformation procedure that leverages the local bosonic description. We demonstrate the consistency of this construction and we extract the perturbative expansion of the deformed theory.

Structure. The chapter is organized as follows: Section 5.2 provides comprehensive mathematical background on Cardy diffusion equation of the torus and chiral Yang-Mills theory. Section 5.3 presents our main results on the sequential deformation procedure (deformed partition function in the Local Compact Boson framework and in Chiral Yang-Mills). Section 5.4 presents additional result on the deformed holomorphic anomaly equation (at first order) and establishes the persistence of fundamental dualities. We conclude with discussion and outlook in Section 5.5.

5.2 Basic notions

5.2.1 Cardy flow equation

Perturbative expansion In order to deal with the $T\overline{T}$ deformation of a non-Lorentz invariant theory it is convenient to work at the level of the Cardy flow equation [125, 126]:

$$\partial_t \mathcal{Z}(\{X_i\}, t) = C(\{X_i\}) \left[\mathcal{Z}(\{X_i\}, t)\right] \tag{5.1}$$

This is a PDE which is first order in the deformation parameter, here denoted as t, and second order in $\{X_i\}$ (which is the set of relevant variables appearing in the partition function of the theory). $C(\{X_i\})$ denotes the second order Cardy operator.

Since the Cardy flow equation (5.1) is linear in t, it admits a simple formal perturbative solution for an arbitrary deformed partition function, whose expansion[1] is:

$$\mathcal{Z}(\{X_i\}, t) = \sum_{k=0}^{\infty} t^k \, \mathcal{Z}^{(k)}(\{X_i\}) \tag{5.2}$$

Each coefficient of the expansion obeys the condition:

$$\mathcal{Z}^{(k)}(\{X_i\}) = \frac{C^k \left[\mathcal{Z}^{(0)}(\{X_i\})\right]}{k!} \tag{5.3}$$

where $\mathcal{Z}^{(0)}$ is the undeformed partition function, which serves as a boundary condition for the PDE and C^k denotes k-th order iteration of the Cardy operator.

[1]Here we denote the dimensional deformation parameter as t. In the literature, a dimensionless rescaled parameter is also sometimes considered [116]. The rescaling is performed using the torus parameter ω_1. As will be seen, attention must be paid to the fact that Cardy's operator acts on ω_1.

For a generic theory, it is not straightforward to compute the analytical expression of the $\mathcal{Z}^{(k)}$ terms and to establish whether the formal series (5.2) is well-defined (i.e. convergent/divergent and Borel resummable). Clearly, the previous expression misses possible non-perturbative contributions in t. We could improve the result searching for a refinement of our ansatz which includes a non-perturbative sector or we could apply Borel resummation once the asymptotics of $\mathcal{Z}^{(k)}$ is known. Clearly the details of this analysis depend on the precise form of the Cardy operator.

Cardy operator on Torus topology Now we want to specify the expression of the Cardy operator in a convenient form, which depends on the geometry of T^2, which we view as a complex manifold. Some details and conventions are contained in (C.1). A result by Cardy states that the differential operator C takes the following expression [125, 126]:

$$C := \left(\partial_{L_1}\partial_{L_2'} - \partial_{L_2}\partial_{L_1'}\right) - (1/A)\left[\left(L_2\partial_{L_2} + L_2'\partial_{L_2'}\right) + \left(L_1\partial_{L_1} + L_1'\partial_{L_1'}\right)\right] \tag{5.4}$$

Here L_1, L_1', L_2, L_2' are real variables parametrizing the two dimensional complex torus with area A. These variables are the components of a generic couple of lattice vectors $(\omega_1(L_1, L_2), \omega_2(L_1', L_2'))$ and are related to the torus modular parameter (and its conjugate) in the following way:

$$\tau := \frac{L_1' + iL_2'}{L_1 + iL_2} \quad ; \quad \bar{\tau} := \frac{L_1' - iL_2'}{L_1 - iL_2} s \tag{5.5}$$

In the literature it is also common to define the components of the modular parameter as:

$$\tau := \tau_1 + i\tau_2 \tag{5.6}$$

As mentioned above, the orientation and the scale of the lattice are fixed by imposing $\omega_1 = 1$. However, we notice that the Cardy operator C is acting on a theory defined for the most general lattice. In order to perform the computation it is therefore important to think the partition function as a function of four independent real parameters and only as a final step fix the lattice system to be the standard one, by means of the global scaling transformation.

In particular, as we will see, the boundary condition $Z_g^{(0)}$ only depends on the variables τ and $\bar{\tau}$ (or τ and $S(\tau, \bar{\tau})$), which can be thought as two independent parameters. We should have in mind that the Cardy operator depends on: $\{\partial_{L_1}, \partial_{L_1'}, \partial_{L_2}, \partial_{L_2'}\}$, or, after a convenient variable change: $\{\partial_\tau, \partial_{\bar{\tau}}, \partial_{\omega_1}, \partial_{\bar{\omega}_1}\}$. Notice that the iterated action of ∂_{ω_1} and/or $\partial_{\bar{\omega}_1}$ in C^k on functions of the form $f(\tau, \bar{\tau})$ could be non-trivial if the Cardy operator contains functions of ω_1 and/or $\bar{\omega}_1$. In order to extract the perturbative $T\overline{T}$ deformation of the partition function we precisely want to iterate the Cardy operator, therefore, fixing the lattice to the standard convention before the computation is not allowed.

5.2.2 Okuyama-Sakai Local Chiral Boson formulation

Chiral $U(N)$ Yang-Mills theory on T^2 arises as a factorization of Yang-Mills theory at large-N. The starting point is the exactly solvable finite-N partition function [94, 95, 235, 239]:

$$Z(g_{YM}^2\, A) = \sum_R e^{-g_{YM}^2\, A\; C_2(R)/2} \tag{5.7}$$

where $C_2(R)$ is the quadratic Casimir of the unitary irreducible representation R of the $U(N)$ gauge group, A is the area of the torus and g_{YM}^2 is the Yang-Mills coupling constant. At large-N it is possible to to split the sum (5.7) as a double sum over chiral representations [105, 223, 226]. In the case of the torus ($g = 1$), this implies a factorization of the partition function into a product of two chiral partition functions, up to $\mathcal{O}(1/N)$ interactions which are suppressed in the large-N limit. In this chapter we are interested in computing the $T\overline{T}$ deformation of chiral $U(N)$ Yang-Mills theory at large N. We stress that this purpose is different from calculating the $T\overline{T}$ deformation of full Yang-Mills theory on T^2 at finite-N (which is known [135]) or at large-N (which has yet to be understood). We are only focusing on a chiral sector and, as we will see, the convenient field theoretic description of this model applies to a torus with area different from A, therefore, we expect different results from the ones in [135].

There are many different formulations of chiral $U(N)$ Yang-Mills theory on the torus T^2 ([104–106, 113, 114, 116, 117, 121, 223, 226, 240]). Indeed the theory can be interpreted as a non-critical string theory [223, 240], as a system of non-relativistic free-fermions on a circle [104, 106, 117, 121], as a compact chiral boson field theory on the torus [226] or as a fermion field theory on the mirror manifold [106]. Moreover, there is some ambiguity related to the contributions from (topological) classical winding modes, both in the fermion and boson formulations. On the boson side, if we only consider local quantum contributions, the partition function exhibits an anti-holomorphic deformation; the chiral Yang-Mills partition is recovered only in the limit in which the anti-holomorphic parameter is turn off. It was shown [114, 116] that if we include the contributions to the partition function of (non-local) winding modes, there is no anti-holomorphic dependence and we obtain the genuine Yang-Mills partition function.

For our purposes we will start the analysis from the local chiral boson theory recently studied by Okuyama and Sakai [114]. The reason is that the above is the simplest formu-

lation, to our knowledge, that respects the postulates of $T\overline{T}$ deformation and, above all, for which the Cardy flow equation is guaranteed to work [126]. The Cardy flow equation allows for a straightforward calculation of the partition function at arbitrary order in perturbation theory. The undeformed theory must satisfy the requirement of being a local quantum field theory on the torus T^2 with translational invariance. As shown by Cardy, it is not essential that the theory is Lorentz-invariant. Okuyama and Sakai disregard winding modes in the computation of the partition function and diagrammatically extract a general expression for the partition function, which corresponds to a local theory. Clearly winding modes are non-local phenomena [241]. Furthermore, in [114] it is explicitly stated that the definition of the deformed partition function $\mathcal{Z}(\tau,\bar{\tau})$ takes into account only of local quantum fluctuations, while winding modes are not included.
In principle, one could also choose the dual fermionic field theory as the unformed partition function, taking into account only the local degrees of freedom. However, this object is less studied and we are not aware of the perturbative expansion at generic order. When trying to apply the Cardy flow equation directly to the standard fermion partition function $Z(\tau)$ (including winding modes contributions and forgetting the locality requirement), the deformation is trivially zero, precisely because there is no anti-holomorphic dependence. Another, option could be to $T\overline{T}$-deform the 1D non-relativistic N fermion system. Indeed in the literature the notion of $T\overline{T}$ deformation was extended to non-relativistic systems [242, 243]. It would be interesting to calculate the first contributions in perturbation theory and compare them with our result.

Now let's instead review Okuyama-Sakai formulation in more detail [114], highlighting the most relevant aspects for our purposes. The chiral boson theory which describes large-N chiral Yang Mills has the following partition function:

$$\mathcal{Z}^{(0)} = \int \mathcal{D}\varphi \exp\left[\int_{T^2}\left(\bar{\partial}\varphi\partial\varphi + \frac{g_s}{3!}(\partial\varphi)^3\right)\right] \tag{5.8}$$

and it is defined on the complex torus $T^2 = \mathbb{C}/L(2\pi, 2\pi\tau)$. Here φ is the compactified boson field; $g_s := 1/N$ is the "string" coupling; $\partial := \partial_z$, where z is the coordinate on the torus.
Clearly, the free massless boson (which describes a 2d CFT) is deformed by a cubic derivative interaction and can be interpreted as a chiral deformation of a CFT [113]. Indeed, the $(\partial\varphi)^3$ operator is chiral and carries spin. As a result of the interaction, conformal symmetry is broken and the theory is no more Lorentz/rotation invariant.
This partition function is regarded as genuinely local. As a result, the partition function theory exhibits an anti-holomorphic dependence on $S(\tau,\bar{\tau})$, which generalizes the ordinary chiral $U(N)$ partition function $Z^{(0)}(\tau)$. Remarkably, the non-relativistic free

fermions partition function $Z^{(0)}$ and the local chiral boson partition function $\mathcal{Z}^{(0)}$ are related by the following formal identities [114]:

$$Z^{(0)}(\tau) = \lim_{S\to 0}\left(\mathcal{Z}^{(0)}(\tau,S)\cdot\frac{1}{\sqrt{2\pi S}}\right) \tag{5.9}$$

$$\mathcal{Z}^{(0)}(t,S) = \int_{-\infty}^{\infty} d\phi\, e^{-\frac{\phi^2}{2S}+\frac{g_s\phi^3}{6}} Z^{(0)}\left(\frac{i\alpha}{2\pi}-\frac{i\phi}{2\pi N}\right) \tag{5.10}$$

Where $\alpha = g_{YM}^2 NA$ and the variable S encodes a simple anti-holomorphic dependence:

$$S := -\frac{1}{2\pi i(\tau-\bar{\tau})} = \frac{\pi}{4\pi^2\,\mathrm{Im}(\tau)} \tag{5.11}$$

As usual it is useful to introduce the free energy:

$$\ln\mathcal{Z}^{(0)} = \mathcal{F}^{(0)} = \sum_{g=1}^{\infty} g_s^{2g-2}\mathcal{F}_g^{(0)} \tag{5.12}$$

which is naturally organized as a genus expansion. The order g free energies are evaluated as the sum of all possible connected Feynman diagrams with $2g-2$ vertices:

$$\mathcal{F}_g^{(0)} = \left\langle\left(\frac{1}{3!}\int(\partial\varphi)^3\right)^{2g-2}\right\rangle_{\text{connected}} \tag{5.13}$$

In order to perform Wick contractions and evaluate the expectation value, it is important to introduce the boson propagator on the torus [104, 113]:

$$G\left(z_1,z_2\right) := \left\langle\partial\varphi\left(z_1\right)\partial\varphi\left(z_2\right)\right\rangle = -\wp\left(z_1-z_2\right) - \frac{\hat{E}_2(\tau,\bar{\tau})}{12} \tag{5.14}$$

Where $\wp$ is the Weierstrass function. We also introduced the modular function $\hat{E}_2$ of weight two, consisting of the (quasi-modular) Eisenstein series E_2 and of an anti-holomorphic factor S (which restores modular invariance of the function):

$$\hat{E}_2(\tau,\bar{\tau}) = E_2(\tau) - 12\,S(\tau,\bar{\tau}) \tag{5.15}$$

If we also considered winding modes, the dependence on S in the propagator would disappear. It is important to stress that S is the only source of anti-holomorphic dependence in the partition function. For our purposes, the presence of S is very important, since the Cardy operator gives zero on functions only of τ, as we will see.

The normalization of the path integral fixes the genus-1 free energy to the expression[2]:

$$\mathcal{F}_1(\tau,\bar{\tau}) = \ln\sqrt{2\pi S(\tau,\bar{\tau})} - \ln\eta(\tau) \tag{5.16}$$

which induces a naive divergence when computing the anti-holomorphic limit ($S \to 0$). The problem is solved by applying the regularization prescription (5.9). Higher genus contributions to the free energy are finite polynomial in $\{S, E_2, E_4, E_6\}$, where E_2, E_4 and E_6 are Eisenstein modular functions arising from period integrals of the Weierstrass function [114]. These expressions are a non-trivial generalization of chiral Yang-Mills standard partition function. For example, the first non-trivial corrections to the standard chiral Yang-Mills free energy come from 1PR diagrams, which are discarded if we also take into account of winding modes contributions. In particular, the integrals corresponding to a connected diagram with $2g-2$ ($g > 1$) trivalent vertices obeys the general expression:

$$I_\Gamma = \sum_{k=0}^{3g-3}\left[S^k f_{6g-6-2k}(\tau)\right] \tag{5.17}$$

where f_{2n} is a weight $2n$ quasi-modular function made of E_2, E_4, E_6. As a result the same form is transferred to the free energies, which are weighted sums of diagrams (see (5.13)):

$$\mathcal{F}^{(0)}_{g>1} = \sum_{k=0}^{3\gamma-3}\left[S^k\ \phi_{6g-6-2k}(\tau)\right] \tag{5.18}$$

where ϕ is again a quasi modular form of weight $2n$.
Finally, combining (5.12) (5.16) and (5.18) it is possible to show that the local boson partition function can be formally written as (see also Appendix C):

$$\mathcal{Z}^{(0)}(\tau,S) = \sum_{g=1}^{\infty}\left[N^{2g-2}\,\frac{\sqrt{2\pi}}{\eta(\tau)}\sum_{k=0}^{3g-3}\left(S^{k+1/2}\ \ \psi_{g,k}(\tau)\right)\right] \tag{5.19}$$

Where $\psi_{g,k}$ is some polynomial in E_2, E_4, E_6, i.e a quasi-modular function. We remark that (5.19) is the convenient boundary condition of the Cardy flow equation. The genus-g contribution to the partition function is:

$$\mathcal{Z}^{(0)}_g(\tau,S) = \sum_{k=0}^{3g-3}\left(S^{k+1/2}\ \ \tilde{\psi}_{g,k}(\tau)\right) \tag{5.20}$$

[2]Here $\eta(\tau)$ is the Dedekind eta function.

In the previous expression, we introduce the following compact definition in order to make manifest the factorization of functions of S and τ:

$$\tilde{\psi}_{g,k}(\tau) := \frac{\sqrt{2\pi}\ \ \psi_{g,k}(\tau)}{\eta(\tau)} \tag{5.21}$$

As shown in [114] the limiting procedure (5.9) exactly reproduces the results of standard chiral $U(N)$ Yang-Mills theory. In particular the only surviving terms are the functions $\tilde{\psi}_{g,0}/\sqrt{2\pi}$, which coincide with the fermionic free energies Z_g. In some sense, the local description of the chiral boson theory is a generalization of the standard fermionic theory. Therefore, it is natural to expect that the anti-holomorphic limit of $T\overline{T}$-deformed Okuyama-Sakai formulation should be equivalent to the $T\overline{T}$ deformation of chiral $U(N)$ Yang-Mills. In order to perform the computation we will extended the same limiting procedure (5.9), to the deformed partition function.

5.3 Computing the deformation

5.3.1 Rewriting the Cardy operator

Since the boundary condition is a function of τ and $\bar{\tau}$, while the Cardy operator is a function of the components of the lattice vectors, it is better to perform a variable change, indeed we want the two objects to be expressed with the same set of variables. This is necessary if we want to deal with the subtleties associated with modular invariance and the calculation of the anti-holomorphic limit. We perform the following variable change, thinking τ, $\bar{\tau}$, ω_1 and $\bar{\omega}_1$ as independent variables:

$$\begin{aligned} &\partial_{L_1} = -\frac{\tau}{\omega_1}\,\partial_\tau - \frac{\bar{\tau}}{\bar{\omega}_1}\,\partial_{\bar{\tau}} + \partial_{\omega_1} + \partial_{\bar{\omega}_1}, && \partial_{L_1'} = \frac{1}{\omega_1}\,\partial_\tau + \frac{1}{\bar{\omega}_1}\,\partial_{\bar{\tau}}, \\ &\partial_{L_2} = i\Big(- \frac{\tau}{\omega_1}\,\partial_\tau + \frac{\bar{\tau}}{\bar{\omega}_1}\,\partial_{\bar{\tau}} + \partial_{\omega_1} - \partial_{\bar{\omega}_1}\Big), && \partial_{L_2'} = i\Big(\frac{1}{\omega_1}\,\partial_\tau - \frac{1}{\bar{\omega}_1}\,\partial_{\bar{\tau}}\Big). \end{aligned} \tag{5.22}$$

In these variables, the area of the torus is defined as:

$$A := L_1 L_2' - L_2 L_1' = \mid \omega_1 \mid^2 \frac{(\tau - \bar{\tau})}{2i} = \mid \omega_1 \mid^2 \operatorname{Im}(\tau) \tag{5.23}$$

Putting together these information, the Cardy operator can be written in the following form:

$$C = \frac{-2i}{\tau - \bar{\tau}} \left[\frac{1}{\bar{\omega}_1}\partial_{\omega_1} + \frac{1}{\omega_1}\partial_{\bar{\omega}_1}\right] + 2i \left[\frac{1}{\omega_1}\partial_{\bar{\omega}_1}\partial_\tau - \frac{1}{\bar{\omega}_1}\partial_{\omega_1}\partial_{\bar{\tau}} + \frac{\tau - \bar{\tau}}{\omega_1\bar{\omega}_1}\partial_\tau\partial_{\bar{\tau}}\right] \tag{5.24}$$

More details about the derivation can be found in Appendix C. As a check, it can be verified (in the same Appendix) that the operator expressed in this form returns the correct expression for the $T\overline{T}$ deformation of the free boson on the torus [132, 244]. The mentioned calculation also serves as an evidence that at this stage we should consider the more generic lattice when defining the torus.
For the sake of convenience, it is appropriate to rewrite Cardy operator as a function of S instead of $\bar{\tau}$, using (5.11). The variable change is the following:

$$\begin{cases} \bar{\tau} = \frac{1}{2\pi i S} + \tilde{\bar{\tau}} \\ \tau = \tilde{\tau} \end{cases} \Rightarrow \begin{cases} \partial_\tau = \partial_{\tilde{\tau}} + 2\pi i S^2 \partial_S, \\ \partial_{\bar{\tau}} = -2\pi i S^2 \partial_S \end{cases} \tag{5.25}$$

The previous relation (5.25) is equivalent to (3.1) in [114]; since $\bar{\tau}$ only appears through S, clearly we can interchange the two variables. Moreover, the new variable $\tilde{\tau}$ is independent from S and $\partial_{\tilde{\tau}} S = 0$, by construction. Hence, we are free to commute ∂_S and $\partial_{\tilde{\tau}}$. As a result (omitting the symbol $\tilde{}$), the Cardy operator takes the following evocative form:

$$C = -4\pi \left[\frac{1}{\bar{\omega}_1}\partial_{\omega_1} + \frac{1}{\omega_1}\partial_{\bar{\omega}_1}\right]\left(S + S^2\partial_S\right) + 2i\frac{1}{\omega_1}\partial_{\bar{\omega}_1}\partial_\tau + \frac{2iS}{\omega_1\bar{\omega}_1}\partial_S\partial_\tau - \frac{4\pi S}{\omega_1\bar{\omega}_1}\left(2S\partial_S + S^2\partial_S^2\right) \tag{5.26}$$

It is very important to notice that C is naturally split in two operators $C^{(0)}$ and $C^{(1)}$ with a different action on S. Indeed we can write:

$$C = C^{(0)} + C^{(1)}, \tag{5.27}$$

where:

$$C^{(0)} := \frac{2i}{\omega_1}\left(\partial_{\bar{\omega}_1} + \frac{S}{\bar{\omega}_1}\partial_S\right)\partial_\tau \tag{5.28}$$

$$C^{(1)} := -4\pi S\left[\left(\frac{1}{\bar{\omega}_1}\partial_{\omega_1} + \frac{1}{\omega_1}\partial_{\bar{\omega}_1}\right)(1 + S\partial_S) + \frac{1}{\omega_1\bar{\omega}_1}\left(2S\partial_S + S^2\partial_S^2\right)\right] \tag{5.29}$$

Clearly $C^{(0)}$ leaves the power of S invariant, while $C^{(1)}$ raises the power of 1

$$C^{(0)}\left[S^{k+1/2}\psi(\tau)\right] \propto \frac{2i\ S^{k+1/2}}{\omega_1\bar{\omega}_1}\ \partial_\tau\psi(\tau) \tag{5.30}$$

$$C^{(1)}\left[S^{k+1/2}\psi(\tau)\right] \propto \frac{S^{k+1/2+1}}{\omega_1\bar{\omega}_1}\psi(t) \tag{5.31}$$

This splitting in the Cardy operator allows for a concise computation of the perturbative $T\overline{T}$ deformation in the limit $S \to 0$, at arbitrary order.

5.3.2 $T\overline{T}$ deformation of the Local Chiral Boson

Here we show the first perturbative contributions to the $T\overline{T}$ deformation of Okuyama-Sakai theory as an expansion in t, and $1/N$, which is a simple byproduct of our analysis. The computation of the first order contribution goes as follow. We use the Cardy flow equation:

$$\partial_t \mathcal{Z}(\tau, S) = C\, \mathcal{Z}(\tau, S) \tag{5.32}$$

which induces the perturbative ansatz:

$$\mathcal{Z}(\tau, S, t) = \sum_k \left(t^k \mathcal{Z}^{(k)}(\tau, S)\right) \tag{5.33}$$

and the formal solution is:

$$\mathcal{Z}^{(k)}(\tau, S) = \frac{C^k}{k!} \mathcal{Z}^{(0)}(\tau, S) \tag{5.34}$$

For the first order perturbative contribution the Cardy operator (5.24) gives:

$$\mathcal{Z}^{(1)}(\tau, S) = C^1\, \mathcal{Z}^{(0)}(\tau, \bar{\tau}) = \frac{2i(\tau - \bar{\tau})}{4\pi^2}\, \partial_\tau \partial_{\bar{\tau}} \mathcal{Z}^{(0)}(\tau, \bar{\tau}) \tag{5.35}$$

where we already set $\omega_1 \bar{\omega}_1 = 4\pi^2$
Equivalently, using (5.26):

$$\mathcal{Z}^{(1)}(\tau, S) = \frac{S}{\pi}\left[-\left(2S\partial_S + S^2\partial_S^2\right) + \frac{i}{2\pi}\partial_S \partial_\tau\right] \mathcal{Z}^{(0)}(\tau, S) \tag{5.36}$$

Where again ∂_S and ∂_τ are independent differential variables. Recalling that $\tau = \frac{i\,\alpha}{2\pi}$, we can also write:

$$\mathcal{Z}^{(1)}(\alpha, S) = \frac{S}{\pi}\left[-\left(2S\partial_S + S^2\partial_S^2\right) + \partial_S \partial_\alpha\right] \mathcal{Z}^{(0)}(\alpha, S) \tag{5.37}$$

In the regime $\alpha > 0$, the first order deformed partition function is real. Using (5.20), (C.15), and (5.36) the first term in the $1/N$ expansion is:

$$\mathcal{Z}_1^{(1)}(\tau, S) = \frac{\sqrt{S}}{\sqrt{2\pi}\,\eta(\tau)} \frac{[E_2(\tau) - 36S]}{24} = \frac{\sqrt{S}}{\sqrt{2\pi}\,\eta(\tau)} \frac{\left[\hat{E}_2(\tau, S) - 24S\right]}{24} \tag{5.38}$$

Similarly for the second order:

$$\begin{aligned}\mathcal{Z}_2^{(1)}(\tau,S) =& \frac{\sqrt{S}}{\sqrt{2\pi}\,\eta(\tau)}\left[\frac{-25E_2^4+57E_2^2E_4-30E_4^2-2E_2E_6}{1244160}+\frac{25\,(E_2^2-2E_4)\,S^2}{2304}+\right.\\ &\left.+\frac{(225E_2^3-2322E_2E_4+2232E_6)\,S}{1244160}+\frac{245E_2S^3}{576}-\frac{105S^4}{16}\right]=\\ &=\frac{\sqrt{S}}{\sqrt{2\pi}\,\eta(\tau)}\left[\frac{-30E_4^2-2E_6\hat{E}_2+57E_4\hat{E}_2^2-25\hat{E}_2^4}{1244160}+\right.\\ &\left.+\frac{(2208E_6-954E_4\hat{E}_2-975\hat{E}_2^3)S}{1244160}-\frac{3E_4S^2}{80}+\frac{5\hat{E}_2S^3}{8}\right]\end{aligned} \tag{5.39}$$

It is known that the modular S-transformation acts as [114, 132]:

$$\begin{aligned}&E_{2n}(\tau)\to\tau^{2n}E_{2n}(\tau),\ (n\geq 2),\quad E_2(\tau)\to\tau^2E_2(\tau)+\frac{6\tau}{\pi i},\quad S(\tau,\bar\tau)\to\tau^2S(\tau,\bar\tau)+\frac{\tau}{2\pi i},\\ &\widehat{E}_2(\tau,\bar\tau)\to\tau^2\widehat{E}_2(\tau,\bar\tau),\quad \eta(\tau)\to\sqrt{-i\tau}\,\eta(\tau),\quad t\to t\end{aligned} \tag{5.40}$$

If we apply (5.40) to (5.38) or (5.39) it is clear that the expressions do not satisfy modular invariance, indeed these are polynomials in S which does not transform properly[3]. This was expected and extends at each order in $T\overline{T}$ perturbation theory.

5.3.3 Effective $T\overline{T}$-deformation of chiral Yang-Mills

In this section, we will describe the computation of the $T\overline{T}$ deformation of chiral Yang-Mills theory. Being able to derive the deformation of Okuyama-Sakai theory, we calculate its anti-holomorphic limit[4]. Since the $T\overline{T}$ deformation of chiral Yang-Mills theory must be continuously connected to the corresponding undeformed theory, we deduce that the anti-holomorphic limit must be extracted using the same procedure as in Okuyama-Sakai theory (5.9). Therefore, for the genus-g partition function at order k in

[3]The same reasoning applies to the unformed theory [114].

[4]If one blindly applies the Cardy diffusion equation to the partition function of chiral Yang–Mills, the result is trivial: the deformation produces no effect. Indeed, the differential operator introduced by Cardy acts as a derivative with respect to $\bar\tau$, and therefore vanishes identically on any function that depends on τ but not on $\bar\tau$. We believe the correct physical interpretation of this fact is that chiral Yang–Mills theory is exactly equivalent to the b–c fermion system, or, in its bosonic formulation, to the non-local chiral boson, which has no dependence on $\bar\tau$. Consequently, applying the deformation in this context is formally meaningless, as it amounts to deforming a theory that is intrinsically non-local.

$T\overline{T}$ perturbation theory, we impose:

$$Z_g^{(k)}(\tau) = \lim_{S\to 0}\left(\frac{1}{\sqrt{2\pi S}}\cdot \mathcal{Z}_g^{(k)}(\tau, S)\right) \tag{5.41}$$

Then applying the perturbative ansatz and relation (5.3):

$$\begin{aligned} Z_g^{(k)}(\tau) &= \lim_{S\to 0}\left[\frac{1}{\sqrt{2\pi S}}\cdot\left(\frac{C^k}{k!}\mathcal{Z}^{(0)}(\tau,\bar{\tau})\right)\right] = \\ &= \sum_{m=0}^{3g-3}\left\{\lim_{S\to 0}\left[\frac{1}{\sqrt{2\pi S}}\cdot\left(\frac{C^{k-1}}{k!}\left(C^{(0)}+C^{(1)}\right)\left(S^{\,m+1/2}\ \tilde{\psi}_{g,m}(\tau)\right)\right)\right]\right\} \end{aligned} \tag{5.42}$$

In the previous step we took advantage of the fact that Cardy operator can be decomposed into two operators (5.27) and we introduced expression (5.20). Clearly, in the anti-holomorphic limit, only $C^{(0)}$ gives a non-zero contribution (see (5.28), (5.30)) when acting on the order $m = 0$ of the Okuyama-Sakai partition function (5.20), therefore:

$$Z_g^{(k)}(\tau) = \lim_{S\to 0}\left\{\frac{1}{\sqrt{2\pi S}}\cdot\left[\frac{C^{k-1}}{k!}C^{(0)}\left(S^{1/2}\ \tilde{\psi}_{g,0}(\tau)\right)\right]\right\} \tag{5.43}$$

By iteration of the previous procedure, we can write:

$$Z_g^{(k)}(\tau) = \lim_{S\to 0}\left\{\frac{1}{\sqrt{2\pi S}}\cdot\left[\frac{\left(C^{(0)}\right)^k}{k!}\left(S^{1/2}\ \tilde{\psi}_{g,0}(\tau)\right)\right]\right\} \tag{5.44}$$

Now it is important to introduce the explicit form of $C^{(0)}$, which is factorizable into the product of a piece acting on τ and a piece acting on S[5]. The operator is applied to $S^{1/2}\ \tilde{\psi}_{g,0}(\tau)$, therefore we can continue the computation in the following way:

$$Z_g^{(k)}(\tau) = \lim_{S\to 0}\left\{\frac{1}{\sqrt{2\pi S}}\frac{(2i)^k}{\omega_1^k\, k!}\left[\left(\partial_{\bar{\omega}_1}+\frac{S}{\bar{\omega}_1}\partial_S\right)^k S^{1/2}\right]\left[\partial_\tau^k\tilde{\psi}_{g,0}(\tau)\right]\right\} \tag{5.45}$$

$$= \frac{(2i)^k}{\omega_1^k k!}\ \partial_\tau^k\ \tilde{\psi}_{g,0}(\tau)\ \cdot\ \lim_{S\to 0}\left\{\frac{1}{\sqrt{2\pi S}}\cdot\left[\left(\partial_{\bar{\omega}_1}+\frac{S}{\bar{\omega}_1}\partial_S\right)^k S^{1/2}\right]\right\} \tag{5.46}$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{2i}{\omega_1\bar{\omega}_1}\right)^k \mathcal{N}_k\ \partial_\tau^k\ \tilde{\psi}_{g,0}(\tau) \tag{5.47}$$

Here we introduced the purely numerical factor:

$$\mathcal{N}_k = \frac{\bar{\omega}_1^k}{k!}\lim_{S\to 0}\left\{\frac{1}{\sqrt{S}}\cdot\left[\left(\partial_{\bar{\omega}_1}+\frac{S}{\bar{\omega}_1}\partial_S\right)^k S^{1/2}\right]\right\} \tag{5.48}$$

[5] This is precisely why in (5.26) we rewrote the Cardy operator so that ∂_τ and ∂_S are independent differential operators.

It is simple to extrapolate the general expression of $\mathcal{N}_k$, by direct computation of the first terms:

$$\mathcal{N}_k = \frac{(-1)^{k+1}\,\Gamma(k-1/2)}{2\;\Gamma(1/2)\;\;k!} \tag{5.49}$$

Now that the Cardy operator has been applied to the boundary condition at arbitrary order in perturbation theory, we are free to set the standard reference system on the torus:

$$(\omega_1,\omega_2) = (2\pi, 2\pi\tau) \;\;\Rightarrow\;\; \omega_1\bar{\omega}_1 = 4\pi^2 \tag{5.50}$$

As a result, the deformed chiral Yang-Mills partition function at genus-g is[6]:

$$Z_g(\tau,t) = \sum_{k=0}^{\infty}\left[t^k\;\left(\frac{2i}{4\pi^2}\right)^k \frac{(-1)^{k+1}\,\Gamma(k-1/2)}{2\;\Gamma(k+1)\;\Gamma(1/2)}\,\partial_\tau^k\right]\frac{\tilde{\psi}_{g,0}(\tau)}{\sqrt{2\pi}} \tag{5.51}$$

Moreover, it is simple to resum (5.51), obtaining the formal differential operator:

$$\sum_{k=0}^{\infty}\left[t^k\;\left(\frac{2i}{4\pi^2}\right)^k \frac{(-1)^{k+1}\,\Gamma(k-1/2)}{2\;\Gamma(k+1)\;\Gamma(1/2)}\,\partial_\tau^k\right] = \sqrt{1+\frac{it\partial_\tau}{2\pi^2}} \tag{5.52}$$

We also notice that the operator is acting on a function of τ which is exactly the undeformed chiral Yang-Mills partition function at genus g[7]:

$$Z_g(\tau,t=0) = \frac{\tilde{\psi}_{g,0}(\tau)}{\sqrt{2\pi}} = \frac{\psi_{g,0}(\tau)}{\eta(\tau)} \tag{5.53}$$

Therefore we can state, as a main result, that the action of the $T\overline{T}$ deformation on the chiral $U(N)$ Yang-Mills partition function is:

$$Z_g(\tau,t) = \sqrt{1+\frac{it\,\partial_\tau}{2\pi^2}}\;Z_g(\tau,t=0) \tag{5.54}$$

And recalling that $\tau = (i\alpha)/(2\pi)$ (for $\alpha > 0$), we obtain:

$$Z(\alpha,t) = \sqrt{1+\frac{t\,\partial_\alpha}{\pi}}\;Z(\alpha,t=0) \tag{5.55}$$

Moreover, at first order in perturbation theory, the relation is simply:

$$Z^{(1)}\left(\frac{i\alpha}{2\pi}\right) = \frac{\partial_\alpha}{2\pi}\;Z^{(0)}\left(\frac{i\alpha}{2\pi}\right) \tag{5.56}$$

[6] Notice that we reproduced the dimensionless coupling $\lambda = \frac{t}{\omega_1\bar{\omega}_1} = \frac{t}{4\pi^2}$ as in [132].
[7] This is a simple consequence of relations (5.20), (5.21) and (5.9).

It is simple to compute the perturbative deformation to the genus-1 and genus-2 partition function, which is:

$$
\begin{aligned}
Z_1(\tau, x = t/\pi) &= \frac{1}{\eta(\tau)}\left[1 + x\, a_1 + x^2\, a_2 + x^3\, a_3 + x^4\, a_4 + \mathcal{O}\left(x^5\right)\right],\\
a_1 &:= \frac{E_2(\tau)}{48}, \quad a_2 := \frac{E_2^2(\tau) - 2E_4(\tau)}{4608},\\
a_3 &:= \frac{3E_2^3(\tau) - 18E_2(\tau)E_4(\tau) + 16E_6(\tau)}{221184},\\
a_4 &:= \frac{5\left(15E_2^4(\tau) - 180E_2^2(\tau)E_4(\tau) - 156E_4^2(\tau) + 320E_2(\tau)E_6(\tau)\right)}{42467328}.
\end{aligned}
\tag{5.57}
$$

$$
\begin{aligned}
Z_2(\tau, x = t/\pi) =&\frac{1}{\eta(\tau)}\left[b_0 + xb_1 + x^2 b_2 + \mathcal{O}\left(x^3\right)\right],\\
b_0 :=&(51840)^{-1}\left[5E_2^3(\tau) - 3E_2(\tau)E_4(\tau) - 2E_6(\tau)\right],\\
b_1 :=&(2488320)^{-1}\left[-25E_2^4(\tau) + 57E_2^2(\tau)E_4(\tau) - 30E_4^2(\tau) - 2E_2(\tau)E_6(\tau)\right],\\
b_2 :=&(238878720)^{-1}\left[-175E_2^5(\tau) + 827E_2^3(\tau)E_4(\tau) - 654E_2(\tau)E_4^2(\tau)\right.\\
&\left.-482E_2^2(\tau)E_6(\tau) + 484E_4(\tau)E_6(\tau)\right].
\end{aligned}
\tag{5.58}
$$

Since the free energy is defined as $F(\tau, t) := \log Z(\tau, t)$, it is clear that:

$$
F(\tau, t) = F^{(0)}(\tau) + \sum_{g=1}\sum_{k=1}\left[\left(\frac{t}{\pi}\right)^k\ N^{2-2g}\ F_g^{(k)}(\tau)\right] \tag{5.59}
$$

Where $F^{(0)}$ is the chiral $U(N)$ Yang-Mills free energy and $F_g^{(k)}$ is a polynomial in $\{E_2, E_4, E_6\}$, hence a quasi modular form[8] of weight $6g - 6 + 2k$. We remind that the local boson is not modular invariant, and not even quasi-modular invariant, this is due to the integration prescription, which does not treat the torus cycles symmetrically. However, chiral $U(N)$ Yang-Mills is quasi-modular invariant, and this property is inherited by the deformed theory. Indeed $\partial_\tau \propto D$ as defined in [114], and the application of this operator to quasi-modular functions gives polynomials in $\{E_2, E_4, E_6\}$.

5.4 Supplementary Results

5.4.1 $T\overline{T}$-Holomorphic Anomaly Equation

Remarkably, a holomorphic anomaly equation was found for the local boson, which allows the systematic computation of the perturbative contributions in $1/N$ of the par-

[8] Indeed E_2 is quasi-modular under $SL(2, \mathbb{Z})$.

tition function [111, 114]. This kind of topological recursion relations are common in topological string theories on Calabi-Yau manifolds and the above expression points to the interpretation of chiral Yang-Mills as a topological string [120, 223, 240, 245]. The equation is:

$$[-1+2S\partial_S]\,\mathcal{Z}^{(0)}(\tau,S)=\frac{1}{N^2}\left[SD^2+S^2D\right]\mathcal{Z}^{(0)}(\tau,S) \tag{5.60}$$

Where $D:=(2\pi i)^{-1}\partial_\tau$ also acts on S as $DS=S^2$ (here τ and S are not independent differential variables). It is interesting to explore the existence of an analogous equation for the $T\overline{T}$-deformed theory, pointing for a topological string interpretation also in this case.

From (5.36) we know that $\mathcal{Z}^{(1)}$ can be computed by applying the C^* operator to the unformed partition function, where:

$$C^*:=\frac{2i(\tau-\bar{\tau})}{4\pi^2}\partial_\tau\partial_{\bar{\tau}} \tag{5.61}$$

As a result:

$$\partial_\tau\partial_{\bar{\tau}}\mathcal{Z}_h^{(0)}=-4\pi^3 S\mathcal{Z}_h^{(1)} \tag{5.62}$$

Moreover, by means of a variable change, (5.63) can be also written as:

$$\left(\frac{i}{\pi S^2}\,\partial_{\bar{\tau}}-\frac{1}{S}\right)\mathcal{Z}_{g+1}^{(0)}(\tau,\bar{\tau})=\left(\left(\frac{\partial\tau}{2\pi i}+S\right)\frac{\partial\tau}{2\pi i}\right)\mathcal{Z}_g^{(0)}(\tau,\bar{\tau}) \tag{5.63}$$

In order to obtain a recursion relation for the first order contribution to the deformed partition function $\mathcal{Z}^{(1)}$, we can manipulate (5.63) by adding derivatives of τ and $\bar{\tau}$ on the left such that the expression only contains terms of the form:

$$\partial_\tau^m\partial_{\bar{\tau}}^n\,\mathcal{Z}_h^{(0)}(\tau,\bar{\tau})\propto\partial_\tau^{m-1}\partial_{\bar{\tau}}^{n-1}\,\mathcal{Z}_h^{(1)}(\tau,\bar{\tau})\qquad\text{for } m\geqslant 1, n\geqslant 1 \tag{5.64}$$

However, a problem in this strategy consists in applying the Leibniz rule to terms in (5.63) containing positive powers of S, indeed:

$$\partial_\tau S^p\propto\begin{cases}0 & \text{if } p=0,\\ S^{p+1} & \text{if } p\neq 0\end{cases}\quad;\quad\partial_{\bar{\tau}}S^p\propto\begin{cases}0 & \text{if } p=0,\\ S^{p+1} & \text{if } p\neq 0\end{cases} \tag{5.65}$$

This is undesirable, since we would like the differential operator, at a certain iteration, to act only on the partition function and not on multiplicative coefficients. Therefore, it is important to firstly multiply by S^{-1} and only then apply the proper derivative operator $\partial_\tau^4\,\partial_{\bar{\tau}}^3$. We obtain:

$$\begin{aligned}
&\Big[-192\pi^2\partial_{\bar\tau}^3+912\pi^2\partial_\tau\partial_{\bar\tau}^2-\frac{144i\pi}{S}\partial_\tau\partial_{\bar\tau}^3-672\pi^2\partial_\tau^2\partial_{\bar\tau}+\\
&+\frac{304i\pi}{S}\partial_\tau^2\partial_{\bar\tau}^2+\frac{24}{S^2}\partial_\tau^2\partial_{\bar\tau}^3+72\pi^2\partial_\tau^3-\frac{84i\pi}{S}\partial_\tau^3\partial_{\bar\tau}-\\
&-\frac{19}{S^2}\partial_\tau^3\partial_{\bar\tau}^2+\frac{i}{\pi S^3}\partial_\tau^3\partial_{\bar\tau}^3\Big]\left(\partial_\tau\partial_{\bar\tau}Z_{g+1}^{(0)}(\tau,\bar\tau)\right)=\\
&=\frac{i}{4\pi^2 S}\left[6\pi S\partial_\tau^4\partial_{\bar\tau}^2-6\pi S\partial_\tau^5\partial_{\bar\tau}+i\partial_\tau^5\partial_{\bar\tau}^2\right]\left(\partial_\tau\partial_{\bar\tau}Z_g^{(0)}(\tau,\bar\tau)\right)
\end{aligned}\tag{5.66}$$

Finally, we can substitute (5.62) and after some manipulation we get:

$$\begin{aligned}
&\Big[S^{-3}(2-2S\partial_S+S^2\partial_S^2-2S^3\partial_S^3)D^3+\\
&+S^{-2}(8-8S\partial_S+49S^2\partial_S^2+18S^3\partial_S^3)D^2+\\
&-2S^{-1}(-12+102S\partial_S+107S^2\partial_S^2+18S^3\partial_S^3)D\\
&+2(54+116S\partial_S+55S^2\partial_S^2+6S^3\partial_S^3)\Big]\,\mathcal{Z}^{(1)}(\tau,S)=\\
&=-\frac{1}{N^2}\Big[S^{-2}(-1+S\partial_S+S^2\partial_S^2)D^5+S^{-1}(-6+3S\partial_S+2S^2\partial_S^2)D^4\\
&+4(-3+7S\partial_S+2S^2\partial_S^2)D^3+12S(4+11S\partial_S+2S^2\partial_S^2)D^2\\
&+24S^2(15+15S\partial_S+2S^2\partial_S^2)D+24S^3(30+19S\partial_S+2S^2\partial_S^2)\Big]\,\mathcal{Z}^{(1)}(\tau,S)
\end{aligned}\tag{5.67}$$

Here, as in (5.63), τ and S are not independent differential variables; clearly D and ∂_S do not commute and the order is relevant. As a consistency check, we have verified that by substituting $\mathcal{Z}_1^{(1)}$ and $\mathcal{Z}_2^{(1)}$, the equation returns an identity.

We notice that (5.67) can be interpreted as an Holomorphic Anomaly Equation since it is relating different topological contributions in $\mathcal{Z}^{(1)}$, however its structure is more complex than (5.63). Formally we can also write the rescaled first order partition function as:

$$\mathcal{Z}^{(1)}(\tau,S)=\sqrt{\frac{S}{2\pi\,\eta(\tau)}}\,\hat{\mathcal{Z}}^{(1)}(\tau,S)\tag{5.68}$$

Where:

$$\hat{\mathcal{Z}}^{(1)}(\tau,S):=\sum_{g=1}^{\infty}\left[N^{2-2g}\,\hat{\mathcal{Z}}_g^{(1)}(\tau,S)\right]\tag{5.69}$$

and $\hat{\mathcal{Z}}_g^{(1)}\equiv\hat{\mathcal{Z}}_g^{(1)}(S,\hat{E}_2,E_4,E_6)$ is a polynomial of (quasi-) modular weight $2(3g-2)$. This statement is compatible with the structure of equation (5.67), which can be written as:

$$H^{(0)}\,\hat{\mathcal{Z}}_{g+1}^{(1)}(\tau,S)=H^{(6)}\,\hat{\mathcal{Z}}_g^{(1)}(\tau,S)\tag{5.70}$$

Where $H^{(0)}$ and $H^{(6)}$ are differential operators which respectively leave the modular weight unchanged or increase it by 6[9]. Finally, we notice that the operators ∂_S and D and the propagator S must have the same diagrammatic interpretation as in [114].
We can also explore the existence of an expression similar to (5.67), holding for the chiral Yang–Mills partition function $Z^{(1)}(\tau)$. We conclude that a closed recursion relation does not exist, as expected. Indeed, we can start with the assumption[10]:

$$\mathcal{Z}^{(1)}(\tau,S) = (2\pi S)^{1/2}\, Z^{(1)}(\tau) + (2\pi S)^{3/2}\, f(\tau) + \mathcal{O}(S^{5/2})$$

Here $f(\tau)$ is a series in N^{-2} which can be determined by Taylor–expanding $\mathcal{Z}^{(1)}(\tau,S)$ at first order in S. We can then insert in (5.67), compute the limit $S\to 0$, and get the condition:

$$3\,\partial_\tau^2 Z^{(1)}(\tau) + 2i\,\partial_\tau^3 f(\tau) = -\frac{3i}{8\pi^3 N^2}\,\partial_\tau^5\, Z^{(1)}(\tau)$$

Clearly the term $\partial_\tau^3 f(\tau)$ does not allow to close the recursion relation since it is itself a series in N^{-2}. Again, we interpret this result as a hint that the proper deformed topological string should contain anti-holomorphic dependence. We stress that this result was derived only at first order in the deformation parameter. Further work is needed in order to extend the analysis at arbitrary order, but the analysis is sufficient to conclude that an holomorphic anomaly equation is not allowed for the deformed holomorphic sector.

5.4.2 $T\overline{T}$-"Fermionization" map

It is also possible to extract the deformed version of the fermionization map (5.10) at least at first order in $T\overline{T}$ deformation. We start the derivation by recalling that, at first order, we have found the results (5.37) (5.56):

$$\mathcal{Z}^{(1)}(\alpha,S) = \frac{S}{\pi}\left[-\left(2S\partial_S + S^2\partial_S^2\right) + \partial_S\partial_\alpha\right]\mathcal{Z}^{(0)}(\alpha,S) = \tilde{C}\,\mathcal{Z}^{(0)}(\alpha,S) \tag{5.71}$$

$$Z^{(1)}\left(\frac{i\alpha}{2\pi}\right) = \frac{\partial_\alpha}{2\pi}\,Z^{(0)}\left(\frac{i\alpha}{2\pi}\right) \tag{5.72}$$

Now, if we apply the $\tilde{C}$ operator in (5.71) to the zero-order fermionization map (5.10) we

[9]This can be realised simply by counting the power of operators S, D, ∂_S in (5.67) and knowing that they change the modular weight by $2, 2, -2$, respectively. The exact same pattern is found in (5.63).

[10]From our analysis it is clear that $Z^{(1)}(\tau)$ is the only non-vanishing term in the anti-holomorphic limit ($S\to 0$). On the other hand, by applying (5.67) to (5.38), it is clear that the first-order deformed Okuyama–Sakai partition function $\mathcal{Z}^{(1)}(\tau,S)$ organizes itself as a polynomial in S.

obtain an integral representation of $\mathcal{Z}^{(1)}$ in terms of $Z^{(0)}$:

$$\begin{aligned}\mathcal{Z}^{(1)}\left(\frac{i\alpha}{2\pi},S\right) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\phi\,\Bigg[\frac{\phi^2}{S}\partial_\alpha Z^{(0)}\left(\frac{i\alpha}{2\pi}-\frac{i\phi}{2\pi N}\right) - \\ -\frac{\phi^4}{2S}Z^{(0)}\left(\frac{i\alpha}{2\pi}-\frac{i\phi}{2\pi N}\right)\Bigg]\cdot\exp\left[\frac{-\phi^2}{2S}+\frac{\phi^3}{6N}\right]\end{aligned} \tag{5.73}$$

Then we can compute a further derivative with respect to α (which does not act on S here), in this way we can replace $Z^{(1)}$ instead of $Z^{(0)}$, using (5.72):

$$\begin{aligned}\partial_\alpha\,\mathcal{Z}^{(1)}\left(\frac{i\alpha}{2\pi},S\right) = \int_{-\infty}^{+\infty} d\phi\,\frac{\phi^2}{S}\Bigg[\partial_\alpha Z^{(1)}\left(\frac{i\alpha}{2\pi}-\frac{i\phi}{2\pi N}\right) - \\ -\frac{\phi^2}{2}Z^{(1)}\left(\frac{i\alpha}{2\pi}-\frac{i\phi}{2\pi N}\right)\Bigg]\cdot\exp\left[\frac{-\phi^2}{2S}+\frac{\phi^3}{6N}\right]\end{aligned} \tag{5.74}$$

Finally, we can integrate in α and obtain the first order deformed “fermionization" map, which relates the deformed chiral Yang-Mills partition function to the deformed local boson partition function (at first order in $T\overline{T}$):

$$\begin{aligned}\mathcal{Z}^{(1)}\left(\frac{i\alpha}{2\pi},S\right) = \int_{-\infty}^{+\infty} d\phi\,\Bigg\{\frac{\phi^2}{S}\Bigg[-\frac{\phi^2}{2}\int_0^\alpha d\alpha'\,Z^{(1)}\left(\frac{i}{2\pi}\left(\alpha'-\frac{\phi}{N}\right)\right) + \\ +Z^{(1)}\left(\frac{i}{2\pi}\left(\alpha-\frac{\phi}{N}\right)\right)\Bigg]\cdot\exp\left[-\frac{\phi^2}{2S}+\frac{\phi^3}{6N}\right]\Bigg\}\end{aligned} \tag{5.75}$$

This perturbative result, together with the deformed anti-holomorphic prescription (5.41), is a first step in showing the existence of a deformed mirror-symmetry relation.

5.5 Discussions and Outlooks

An interesting problem is to generalize the study of the deformed Holomorphic Anomaly Equation to arbitrary order in t, thereby further testing the interpretation of the deformed theory as a topological string. Likewise, given the generality of the deformed bosonization map, it would be natural to explore its fermionization counterpart at arbitrary order. A possible framework for parallel progress could be to study the $T\overline{T}$ deformation of fermions on the circle, building on the definition of one-dimensional $T\overline{T}$ introduced by Gross and collaborators [218]. Another promising direction is to gain a deeper understanding of the deformation of the Gross–Taylor string, building on the recently proposed string actions [108–110]. Chiral Yang–Mills theory admits a deep geometric interpretation in terms of winding maps [105]; it would be interesting to understand whether the same extends to the deformed case. Typically, $T\overline{T}$ does not introduce local propagating degrees of freedom, but instead modifies the background of the original theory. As a result, it is expected to alter the combinatorial weights (Hurwitz numbers) in the Gross–Taylor expansion. We believe that the $T\overline{T}$-deformation of a theory described by a topological string yields a dressed topological theory, preserving the classification of worldsheet contributions, while modifying the relative weights and dependence on global metric data. Two possible starting points for such an analysis could be our perturbative result or, alternatively, a study of the deformation of the newly proposed string theories.

It would also be interesting to investigate the effects of the $T\overline{T}$-deformation of the *full* Yang–Mills theory on the torus at large-N. So far, only the deformation of the chiral sector has been studied, or alternatively the deformation of the theory at finite N [135, 136, 228]. A difficulty in the full large-N theory is the presence of $1/N$ contributions due to the interactions between the two chiral sectors; nevertheless, one might be able to restrict to a regime where the $T\overline{T}$ deformation dominates.

Concerning UV-completeness, it remains unclear in which sense the $T\overline{T}$ deformation allows the theory to flow to higher energy regimes. What is established in the community is that $T\overline{T}$ does not implement a standard Callan–Symanzik flow. The reason is that the deformation is applied only at order one and is closed at one loop [128]. As a consequence, as emphasized by Cardy [125] and Komargodski–Richkov [216], the flow is not Hamiltonian, and the resulting theory is not a standard QFT. This aligns with the well-known fact that $T\overline{T}$-deformed CFTs exhibit non-local features.

On the other hand, as pointed out by Datta [132], in the context of CFTs, coupling to the $T\overline{T}$ operator effectively redefines the energy scales of the theory, so that it appears to probe different energies. From a more mature perspective, this effect could simply be the result of defining the theory on a different background. In this sense, the background change (which in certain cases can be reinterpreted as a coupling to a topological gravity sector) can be viewed as providing a form of UV-completeness. Testing these conjectures in the context of chiral Yang-Mills would be interesting.

In this broader context, one finds an analogy with the recent work of Verlinde, Tietto and Blommaert [86], who show in a different setting that coupling SYK to another SYK acting as a clock yields sine-dilaton gravity, which is postulated to be UV complete. In other words, one may think of this as coupling Liouville theory to a copy of itself, where one of the two theories effectively plays the role of an observer. The general lesson that may be drawn from these developments is that a possible key to introduce UV-completeness in quantum gravity could lie in the explicit introduction of an observer, realized in practice by coupling the theory to another system. It would be interesting to test these ideas in the context of $T\overline{T}$-deformed Yang–Mills theory.

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Appendix A

A.1 General Dilaton-Gravities

In [184, 246], a general and concise formulation is derived for a two-dimensional dilaton gravity theory involving at most two derivatives in the action, which admits black hole solutions. Additional details are provided in [185, 247]. A reparametrization of the theory's variables allows for a clearer understanding of the gravitational interpretation of the classical solutions. The authors construct an observable associated with the Killing vectors which remains invariant under reparametrizations and is identified with the energy of the system. Moreover, starting from the Killing vector, one can derive the surface gravity for a generic 2D black hole, and consequently, its entropy. In the paper, the following action is considered:

$$S[h,\phi_0]=\frac{1}{2G}\int d^2x\sqrt{-h}\left(\frac{1}{2}h^{\alpha\beta}\partial_\alpha\phi_0\partial_\beta\phi_0-\frac{1}{l^2}\tilde{V}(\phi_0)+D(\phi_0)R\right) \tag{A.1}$$

Via Weyl rescaling, and reparametrizing the dilaton, it is possible to remove the kinetic term[1], obtaining:

$$S[g,\phi]=\frac{1}{2G}\int_{M^2}d^2x\sqrt{-g}\left(\phi R-\frac{1}{l^2}V(\phi)\right) \tag{A.2}$$

The classical equations arising from the this action, take the simple form:

$$\begin{aligned} R&=\frac{1}{l^2}\frac{dV}{d\phi},\\ \nabla_\mu\nabla_\nu\phi+\frac{1}{2l^2}g_{\mu\nu}V&=0. \end{aligned} \tag{A.3}$$

The general solutions can be found in the conformal gauge, as explained in [247]. The strategy is to perform a Bäcklund transformation: provided that:

$$H\left(g_{\mu\nu},\phi\right)=\nabla_\rho\phi\nabla^\rho\phi\neq 0 \tag{A.4}$$

[1] We require that the theory is well-defined ([185]), the kinetic energy is non-degenerate if $D'(\phi_o\neq 0)$, hence we can redefine: $\phi=D(\phi_0)$

we can transform the fields as:

$$
\begin{aligned}
M = W(\phi) - \nabla_\rho \phi \nabla^\rho \phi, \quad W(\phi) = \int^\phi d\phi' V(\phi') \\
\nabla_\mu \psi = \frac{\nabla_\mu \phi}{\nabla_\rho \phi \nabla^\rho \phi}
\end{aligned}
\tag{A.5}
$$

and EOMs (A.3) become:

$$
\begin{aligned}
\nabla_\mu \nabla^\mu \psi = 0, \\
\nabla_\mu M = 0.
\end{aligned}
\tag{A.6}
$$

Hence M is interpreted as a conserved quantity related to the ADM energy of the system. Moreover, looking at the first equation we notice that ψ is a free field. Remembering that in two dimensions, any metric is locally conformally flat, we can go in the conformal light-cone coordinates and write the metric as:

$$
ds^2 = 4\rho(u,v)dudv. \tag{A.7}
$$

In locally flat metric it is simple to solve D'Alembert equation. The solutions to (A.6) are given by:

$$
\psi = U(u) + V(v), \quad M = M_0. \tag{A.8}
$$

We can then write M and ϕ as functions of ψ and M:

$$
\begin{aligned}
\frac{d\psi}{d\phi} = \frac{1}{W(\phi) - M} \\
\rho = [W(\phi) - M]\partial_u \psi \partial_v \psi
\end{aligned}
\tag{A.9}
$$

Finally, the metric, which is the general solution of 2d dilaton gravities is:

$$
ds^2 = -[W(\phi) - M]dT^2 + [W(\phi) - M]^{-1} d\phi^2 \tag{A.10}
$$

Notice that this on-shell solution only depends on ϕ, namely, a general solution is (0+1)-dimensional. This is due to the fact that any solution has a Killing vector, as shown in [184]:

$$
k^\mu = l\,\eta^{\mu\nu}\phi_{,\nu} \tag{A.11}
$$

where: $\eta^{\mu\nu} = -\eta^{\nu\mu} = \frac{1}{\sqrt{-g}}\epsilon^{\mu\nu}$. It is simple to show that Killing equation is satisfied starting from (A.3). This fact hints that the theory can be reduced to a one-dimensional dynamical system, and it is a first suggestion that 2D dilaton gravity should have a topological description (See Section (2.4)). On the other hand one can be tempted to perform a quantization of this effective 1D model as shown in [247] by Cavaglià. The author states that the quantization can be performed via quantum Birkoff theorem [247]. From a modern, holography-oriented approach Dirac quantization does not take into account boundary terms. As a consequence, in the case of sine-dilaton gravity, we do not expect that this approach could recover the observables of the putative dual system, i.e. DSSYK. In the case of JT gravity or sine-dilaton gravity a convenient

approach is a recent proposal made by Harlow-Jafferis [80, 146], in which the quantization relies on the equivalence of the on-shell metric of sine-dilaton gravity to the one in AdS_2 (via Weyl rescaling).
Now that the classical phase space is known, we can look at the thermodynamical properties of the solutions [184, 185]. We can compute the surface gravity starting from the Killing vector:

$$\kappa^2 = -\frac{1}{2}\nabla^\mu k^\nu \nabla_\mu k_\nu, \tag{A.12}$$

which, evaluated at an horizon ϕ_h such that $W(\phi_h) - M = 0$, gives:

$$\kappa = -\frac{1}{2l} V\left(\phi_h\right) \tag{A.13}$$

By requiring periodicity in Euclidean time, we can compute the Hawking temperature, which is the inverse of the period:

$$T_H = \frac{\kappa}{2\pi} \tag{A.14}$$

A.2 Canonical quantization of JT gravity

The paper [146] focuses on showing that AdS JT gravity is a sensible quantum-gravitational model without a well-defined CFT dual. The idea is to study the factorization properties of the Hilbert space in gravity compared to the factorizable Hilbert space of the putative CFT dual theory (the idea behind the factorization problem is that gauge constraints in the gravitational action, are potentially not compatible with a CFT tensor product decomposition). The result holds for JT gravity which can be interpreted as a low energy description of sine-dilaton gravity. As a by-product of considering the puzzle in the context of JT gravity, the authors develop a technique of canonical quantization which also inspired related works in the context on 2d gravity with periodic dilaton potential [82]. This is the main reason of this focus on canonical quantization. The main idea is to quantize the classical phase space of JT gravity, starting from the classical solutions. The Lorentzian AdS formulation of JT gravity has action:

$$S = \Phi_0\left(\int_M d^2x\sqrt{-g}R + 2\int_{\partial M}\sqrt{|\gamma|}K\right) + \int_M d^2x\sqrt{-g}\Phi(R+2) + 2\int_{\partial M} dt\sqrt{|\gamma|}\Phi(K-1) \tag{A.15}$$

Where Φ_0 is a constant, Φ denotes the dilaton field, K is the trace of the extrinsic curvature on the boundary. As is customary, a holographic renormalization term is included to ensure the finiteness of the action. The variation of the action fixes the curvature in the manifold to $R = -2$, the only non-trivial degree of freedom resides in the dilaton field, satisfying the equation:

$$(\nabla_\mu\nabla_\nu - g_{\mu\nu})\Phi = 0 \tag{A.16}$$

Similarly, the boundary terms:

$$\int_{\partial M} dx\sqrt{|\gamma|}\left[2(K-1)\delta\Phi+\left(r^{\nu}\nabla_{\nu}\Phi-\Phi\right)\gamma^{\alpha\beta}\delta\gamma_{\alpha\beta}\right] \tag{A.17}$$

are chosen to vanish under arbitrary variation in the space of configurations, a possible choice is:

$$\begin{aligned}\gamma_{tt}|_{\partial M} &= r_c^2\\ \Phi|_{\partial M} &= \phi_b r_c\end{aligned} \tag{A.18}$$

Not each diffeomorphisms of the bulk theory keeps (A.18) invariant, but only the subset ξ_μ, which asymptotically satisfies the condition:

$$\gamma_\mu{}^\alpha\gamma_\nu{}^\beta\nabla_{(\alpha}\xi_{\beta)}\Big|_{\partial M}=0 \tag{A.19}$$

we can see that the diffeomorphisms on the boundary are restricted to be time translations.

A.2.1 Classical phase space

Now we can look at the classical EOMs of JT gravity on the topology $\mathbb{R}\times[0,1]$, expecting two asymptotically-AdS boundaries, namely a cylinder. From an holographic point of view we predict the existence of a right and left ADM Hamiltonian on those boundaries. Given the fact that curvature is fixed to be the one in AdS_2, the solution should be a part of that space and the natural way to find solution for this topology is to embed AdS_2 in $1+2$ Minkowski space:

$$ds^2=-dT_1^2-dT_2^2+dX^2. \tag{A.20}$$

with the constraint:

$$T_1^2+T_2^2-X^2=1 \tag{A.21}$$

and identify the boundaries at $X\to\pm\infty$. The solutions are of the form:

$$\Phi=\Phi_h T_1, \tag{A.22}$$

If we want to recover the 2d metric, we simply chose a set of coordinates satisfying (A.21):

$$T_1=\sqrt{1+x^2}\cos\tau \tag{A.23}$$

$$T_2=\sqrt{1+x^2}\sin\tau$$

$$X=x, \tag{A.24}$$

As a result:

$$\begin{aligned} ds^2 &= -(1+x^2)d\tau^2 + \frac{dx^2}{1+x^2} \\ \Phi &= \Phi_h\sqrt{1+x^2}\cos\tau. \end{aligned} \tag{A.25}$$

Clearly we have a two-sided geometry with horizons in $X \to \pm\infty$, as we wanted. Between the two horizons there is a global AdS_2 geometry, which is refereed as the wormhole region in the literature[2]. As in Reissner-Nordstrom solutions, the maximal extension of this space has an infinite set of horizons. However the dynamical problem is well-defined only in the region between two inner horizons Φ_h. The inner horizon actually is not an horizon in this geometry, but it would appear after a small fluctuation is added. Hence the authors decide to focus on that region, having in mind higher dimensional problems.

In order to perform the quantization, it is better to introduce Schwarzschild coordinates:

$$\begin{aligned} T_1 &= r/r_s \\ T_2 &= \sqrt{(r/r_s)^2-1}\,\sinh(r_s t) \\ X &= \sqrt{(r/r_s)^2-1}\,\cosh(r_s t), \end{aligned} \tag{A.26}$$

This is the conventional radial gauge (see Appendix (A.1)) in which:

$$\begin{aligned} ds^2 &= -(r^2-r_s^2)dt^2 + \frac{dr^2}{r^2-r_s^2} \\ \Phi &= \phi_b r. \end{aligned} \tag{A.27}$$

In this set of coordinates the foliation is more transparent, indeed we notice that $r = r_c$ is the place where the boundary conditions (A.18) are imposed, moreover t is identified with the boundary time. Having an AdS boundary, we can identify the corresponding (left or right) CFT boundary stress-energy tensor as:

$$T^{\mu\nu}_{CFT} = 2r_c^3\gamma^{\mu\nu}\left(r^\lambda\nabla_\lambda\Phi - \Phi\right)|_{\partial M}. \tag{A.28}$$

which in this case is simply:

$$H_L = H_R = \frac{\Phi_h^2}{\phi_b} \tag{A.29}$$

and the full canonical Hamiltonian is:

$$H = H_L + H_R = \frac{2\Phi_h^2}{\phi_b} \tag{A.30}$$

[2]In a sense which differs from Saad, Shenker and Stanford [42], indeed there are no higher genus surfaces between the two horizons.

Actually, we are forgetting something, indeed it seems that the Hamiltonian only depends on a single parameter Φ_h. This should not be the case, indeed, a generic Hamiltonian has an even phase space. The other parameter did not appear explicitly in the Hamiltonian, but we must identify it in order to perform canonical quantization. The problem is that, when changing coordinates between global and Schwarzschild coordinates, we performed an illegal gauge transformation. The real parameter space of the Hamiltonian can be either recovered by going in global coordinates, either by performing another illegal gauge transformation of the class which approaches an asymptotic symmetry at infinity. Then, we should mode out legal gauge transformation, and get the correct phase space. As we said before, the only legal gauge transformation are time translations on the left or right asymptotic boundaries. But the time coordinates at the boundary are not independent parameters, indeed being the left and right Hamiltonian identical, the operator $H_L - H_R$ does not induce evolution in the phase space. Instead the full Hamiltonian does, and we define as δ the parameter controlling how much (time) we evolved the solution by $H = H_L + H_R$. Looking at the coordinate changes, it is actually possible to relate the times on the two sides, to δ:

$$\delta = \frac{t_L + t_R}{2}. \tag{A.31}$$

Hence we conclude that the phase space of this Hamiltonian is controlled by δ and Φ_h and its formulation is:

$$\begin{aligned} \dot{\delta} &= 1 \\ \dot{\Phi}_h &= 0 \\ H &= \frac{2\Phi_h^2}{\phi_b}. \end{aligned} \tag{A.32}$$

where $\Phi_h > 0$, $-\infty < \delta < \infty$. We can also write down the symplectic form:

$$\omega = \frac{4\Phi_h}{\phi_b} d\delta \wedge d\Phi_h, \tag{A.33}$$

the relation between H and Phi_h suggests a more natural phase space:

$$\omega = d\delta \wedge dH. \tag{A.34}$$

this means that we can interpret the δ as a time variable canonically conjugate to the Hamiltonian. In particular, δ controls the amount of evolution of the TFD state.

Another set of useful canonical coordinates (even in sine-dilaton gravity) for this space can be identified with a properly renormalized geodesics distance between the two holographic boundaries and its conjugate. In order to avoid infinities when $r_c \to \infty$, we define:

$$L \equiv L_{bare} - 2\log(2\Phi|_{\partial M}). \tag{A.35}$$

Where the bare distance can be computed by looking at a space-like geodesics in global coordinates (A.25) by setting $t_L = t_R$[3] and considering the extrema $x_c = x(r = r_c)$; $-x_c = x(r = -r_c)$:

$$L_{bare} = \int_{-x_c}^{x_c} \frac{dx}{\sqrt{1+x^2}} \tag{A.36}$$

As a result, the variable $L \in \mathbb{R}$ is:

$$L = 2\log\left(\frac{\cosh\left(\Phi_h \delta/\phi_b\right)}{\Phi_h}\right) = 2\log\left(\cosh\left(\sqrt{\frac{E}{2\phi_b}}\delta\right)\right) - \log\frac{\phi_b E}{2} \tag{A.37}$$

It is also possible to find its symplectic conjugate $P \in \mathbb{R}$ in order to have a standard measure $\omega = dL \wedge dP$:

$$P = 2\Phi_h \tanh\left(\Phi_h\delta/\phi_b\right) = \sqrt{2\phi_b E}\tanh\left(\sqrt{\frac{E}{2\phi_b}}\delta\right) \tag{A.38}$$

The two sided Hamiltonian of JT-gravity in this framework becomes the mechanics of a non-relativistic particle moving in an exponential potential:

$$H = \frac{P^2}{2\phi_b} + \frac{2}{\phi_b}e^{-L} \tag{A.39}$$

whose solutions are the ones of a scattering problem with waves coming from $L = \infty$. Being sine–dilaton gravity a completion of JT–gravity, we expect the two sided canonical Hamiltonian of that theory to be a deformation of (A.39).

A.2.2 Quantum Hamiltonian

We could start to quantize the Hamiltonian in the phase space $\{H, \delta\}$. However (since the Hamiltonian is bounded $H > 0$), we should chose the time operator δ to be Hermitian, but not self-adjoint, which is somewhat annoying. It is simpler to go in the formulation $\{L, P\}$, indeed both the operators are unbounded. Hence, we expect P to be self-adjoint and L to have a basis of delta-function renormalized states. Following the quantum particle intuition, we write a Schrödinger equation acting on L_2-normalizable functions of L:

$$-\frac{1}{2\phi_b}\Psi_E''(L) + \frac{2}{\phi_b}e^{-L}\Psi_E(L) = E\Psi_E(L). \tag{A.40}$$

The solution of this quantum scattering problem is written in terms of modified Bessel functions:

$$\Psi_E(L) = \frac{2^{1-2i\sqrt{2E\phi_b}}}{\Gamma(-2i\sqrt{2E\phi_b})}K_{2i\sqrt{2E\phi_b}}\left(4e^{-L/2}\right). \tag{A.41}$$

where the condition $E > 0$ is automatically satisfied. We emphasize that this is the full quantum solution of (two boundary) AdS Lorentzian JT gravity. In the economy of the paper, finding

[3] we can use a gauge transformation implemented by $H_L - H_R$, which does not change the state.

such a solution means that the Hilbert space is not factorizable in a tensor product of two Hilbert spaces on the boundary, being the operator L and P intrinsically defined in the bulk and the energy is a global variable (it cannot be defined locally on the boundary). It can be seen that a similar analysis can be extended to sine-dilaton gravity [80], with a strong implication: if we believe in the quantum mechanical formulation, it should not be possible to describe the theory in terms of two factorized Hilbert spaces on the boundary. Coherently DSSYK cannot be interpreted as a tensor product of CFTs on some boundary.

Appendix B

B.1 Green's functions

In this appendix, we want to show that it is possible to obtain Green's functions of a linear differential system if the fundamental solution of the associated homogeneous problem is known. For clarity, we will firstly consider a differential operator that is diagonal in the derivatives (unlike the one we studied in the paper). As we shall see, however, it is straightforward to extend the methodology to off-diagonal operators. For more, see [248], from which part of this appendix was inspired.

Diagonal operators. Let $A(t)$ be a matrix of dimension $n \times n$ continuous for $t \in (\alpha, \beta)$ and let $a(t)$ be a vector of dimension $n \times 1$, continuous in the same interval. Let M, N be constant matrices of dimension $n \times n$. We will be interested in finding the matrix Green's function of the following differential system (with homogeneous bilocal boundary conditions in $t_1, t_2 \in (\alpha, \beta)$):

$$\begin{cases} \dot{x}(t) = A(t)x(t) + a(t) \\ Mx(t_1) + Nx(t_2) = 0 \end{cases} \tag{B.1}$$

Let's assume we know the fundamental (matrix) solution $K(t)$ of the associated homogeneous problem. By definition, then:

$$\dot{K}(t) = A(t)K(t) \tag{B.2}$$

Moreover, it is useful to introduce the following matrix, which encodes information about the homogeneous problem and the boundary conditions:

$$\mathcal{R}(t_1, t_2) := [MK(t_1) + NK(t_2)] \tag{B.3}$$

If $\det \mathcal{R} \neq 0$ and $\det K \neq 0$, it is always possible to define the following auxiliary bilocal function χ:

$$\chi(t_1, t_2) := -\mathcal{R}^{-1} NK(t_2) \int_{t_1}^{t_2} K^{-1}(\tau)\, a(\tau) d\tau \tag{B.4}$$

A first fundamental statement is that a solution $x(t)$ of the non-homogeneous problem (B.1) can be written in terms of χ and $K(t)$, as follows:

$$x(t) = K(t)\,\chi(t_1, t_2) + K(t)\int_{t_1}^{t} K^{-1}(\tau)a(\tau)d\tau \tag{B.5}$$

We can prove this result simply by applying the definitions, indeed:

$$\begin{aligned} \dot{x}(t) &= \dot{K}(t)\left[\chi + \int_{t_1}^{t} K^{-1}(\tau)a(\tau)d\tau\right] + K(t)K^{-1}(t)a(t) = \\ &= A(t)K(t)\left[\chi + \int_{t_1}^{t} K^{-1}(\tau)a(\tau)d\tau\right] + a(t) = A(t)x(t) + a(t) \end{aligned} \tag{B.6}$$

Furthermore, also the boundary conditions are satisfied[1]:

$$Mx\,(t_1) + Nx\,(t_2) = MK\,(t_1)\,\chi + N\left[K\,(t_2)\,\chi + K\,(t_2)\int_{t_1}^{t_2} K^{-1}(\tau)a(\tau)d\tau\right] = 0 \tag{B.7}$$

Now, for the purpose of extracting Green's functions, we notice that (B.5) can be rewritten as an integral kernel, indeed:

$$\begin{aligned} &x(t) = K(t)\left(\chi + \int_{t_1}^{t} K^{-1}(\tau)a(\tau)d\tau\right) = \\ =&K(t)\left[-\mathcal{R}^{-1}NK\,(t_2)\int_{t_1}^{t_2} K^{-1}(\tau)a(\tau)d\tau + \int_{t_1}^{t_2} K^{-1}(\tau)a(\tau)d\tau\right] = \\ =&-K(t)\mathcal{R}^{-1}NK\,(t_2)\int_{t_1}^{t} K^{-1}(\tau)a(\tau)d\tau - K(t)\mathcal{R}^{-1}NK\,(t_2)\int_{t}^{t_2} K^{-1}(\tau)a(\tau)d\tau + \\ &+ K(t)\int_{t_1}^{t} K^{-1}(\tau)a(\tau)d\tau = \\ =&\int_{t_1}^{t}\left[-K(t)\left(\mathcal{R}^{-1}NK\,(t_2) - 1\right)K^{-1}(\tau)\right]a(\tau)d\tau + \int_{t}^{t_2}\left[-K(t)\mathcal{R}^{-1}NK\,(t_2)\,K^{-1}(\tau)\right]a(\tau)d\tau \end{aligned} \tag{B.8}$$

Introducing the Heaviside step function θ, we can now define:

$$\begin{aligned} \tilde{G}(t,\tau) &:= \theta(t-\tau)\,\tilde{G}^{(a)}(t,\tau) + \theta(\tau - t)\,\tilde{G}^{(r)}(t,\tau) \\ \tilde{G}^{(a)}(t,\tau) &:= -K(t)\left[\mathcal{R}^{-1}NK\,(t_2) - 1\right]K^{-1}(\tau) \quad \text{if} \quad \tau < t \\ \tilde{G}^{(r)}(t,\tau) &:= -K(t)\mathcal{R}^{-1}NK\,(t_2)\,K^{-1}(\tau) \quad \text{if} \quad \tau > t \end{aligned} \tag{B.9}$$

[1] We notice that we can also express (B.4) as:

$$\left[MK\,(t_1) + NK\,(t_2)\right]\chi = -NK\,(t_2)\int_{t_1}^{t_2} K^{-1}(\tau)a(\tau)d\tau$$

and write (B.8) as:

$$x(t) = \int_{t_1}^{t_2} \tilde{G}(t,\tau)a(\tau)d\tau \tag{B.10}$$

The last point is to show that $\tilde{G}(t,\tau)$ is indeed a Green's function associated to the differential operator of our interest $\tilde{L}(t) = \mathbb{1}\, d/dt - A(t)$. By simply applying this operator to (B.10), we get:

$$\begin{aligned}\tilde{L}(t)x(t) &= \tilde{L}(t)\int_{t_1}^{t_2} \tilde{G}(t,\tau)a(\tau)d\tau \\ a(t) &= \int_{t_1}^{t_2} \tilde{L}(t)\tilde{G}(t,\tau)a(\tau)d\tau\end{aligned} \tag{B.11}$$

Hence, the following identity holds:

$$\tilde{L}(t)\tilde{G}(t,\tau) = \delta(t-\tau)\mathbb{1} \tag{B.12}$$

which is the defining property of a (matrix) Green's function.

Off-diagonal operators. As mentioned above, for the purposes of our paper, we want to search for the Green's function $G(t,\tau)$ of a more general differential operator:

$$L(t) = P_0\tilde{L}(t) \tag{B.13}$$

where P_0 is a constant, $n \times n$ and invertible matrix. As a first step, we notice that the fundamental matrix of L (denoted in the main text as $H(\tau)$) is equivalent to $K(\tau)$, the fundamental matrix of $\tilde{L}$. Since the two systems are in some way related, we want to find the link between the corresponding Green's functions. By inversion of (B.13) and using (B.12), we have:

$$\begin{gathered} P_0^{-1}L(t)\tilde{G}(t,\tau) = \delta(t-\tau)\mathbb{1} \\ \Downarrow \\ L(t)\tilde{G}(t,\tau) = \delta(t-\tau)P_0 \\ \Downarrow \\ L(t)\tilde{G}(t,\tau)P_0^{-1} = \delta(t-\tau)\mathbb{1} \end{gathered} \tag{B.14}$$

Hence we recognize the relation:

$$G(t,\tau) = \tilde{G}(t,\tau)P_0^{-1} \tag{B.15}$$

Explicit evaluation of Green's functions. We now turn our attention to the problem of finding the Green's function of the operator $L(t)$ in (3.52) of the main text. As observed in section 3.3.2, the fundamental matrix of the system $H(\tau)$, presents a divergence when evaluated on the value $\tau = \beta$. Therefore, once again, we need to regularize it by introducing a cut-off ϵ

and studying the problem in the interval $[0, \beta_\epsilon]$, where:

$$\beta_\epsilon = \frac{2\pi - \epsilon - 4\theta}{\sin(\theta)}$$

This time the divergences coming from $H(\beta_\epsilon)$ are exactly canceled by the contributions coming from the matrix $\mathcal{R}^{-1}$ and using the formulas (B.9) and (B.15) we can easily obtain the expressions for the Green's functions:

$$\begin{aligned}
G^{(a)}_{\varphi\varphi}(\zeta, \upsilon) &= \frac{(\tan(\zeta)+\tan(\zeta)(\zeta+u)\tan(u)+\tan(u))(\tan(u)((u-\upsilon)\tan(\upsilon)-1)+\tan(\upsilon))}{u\tan(u)+1} \\
G^{(a)}_{\varphi p}(\zeta, \upsilon) &= -\frac{\mathrm{i}\sec(\upsilon)\csc(u+\upsilon)(2(u-\upsilon)\sin(u)+\cos(u+2\upsilon)+3\cos(u))(\tan(\zeta)(\zeta+u)\sin(u)+\tan(\zeta)\cos(u)+\sin(u))}{2(u\sin(u)+\cos(u))} \\
G^{(a)}_{p\varphi}(\zeta, \upsilon) &= -\frac{\mathrm{i}\csc(\zeta+u)(\sin(\zeta)(\tan(\zeta)+\tan(u))+\sec(\zeta)(\zeta+u)\tan(u))(\sin(u)((u-\upsilon)\tan(\upsilon)-1)+\cos(u)\tan(\upsilon))}{u\tan(u)+1} \\
G^{(a)}_{pp}(\zeta, \upsilon) &= -\frac{\sec(\zeta)\cos(u)\sec(\upsilon)\csc(u+\upsilon)(\sin(\zeta)+(\zeta+u)\sin(u)\csc(\zeta+u))(2(u-\upsilon)\sin(u)+\cos(u+2\upsilon)+3\cos(u))}{2(u\sin(u)+\cos(u))}
\end{aligned} \tag{B.16}$$

$$\begin{aligned}
G^{(r)}_{\varphi\varphi}(\zeta, \upsilon) &= \frac{(\tan(\zeta)+\tan(u)(\tan(\zeta)(u-\zeta)-1))((u+\upsilon)\tan(\upsilon)\tan(u)+\tan(u)+\tan(\upsilon))}{u\tan(u)+1} \\
G^{(r)}_{\varphi p}(\zeta, \upsilon) &= -\frac{\mathrm{i}\csc(u+\upsilon)(\sin(u)(\tan(\zeta)(u-\zeta)-1)+\tan(\zeta)\cos(u))(\sin(\upsilon)(\tan(u)+\tan(\upsilon))+(u+\upsilon)\tan(u)\sec(\upsilon))}{u\tan(u)+1} \\
G^{(r)}_{p\varphi}(\zeta, \upsilon) &= -\frac{\mathrm{i}\sec(\zeta)\csc(\zeta+u)((u+\upsilon)\tan(\upsilon)\tan(u)+\tan(u)+\tan(\upsilon))(2(u-\zeta)\sin(u)+\cos(2\zeta+u)+3\cos(u))}{2u\tan(u)+2} \\
G^{(r)}_{pp}(\zeta, \upsilon) &= -\frac{\sec(\zeta)\cos(u)\sec(\upsilon)\csc(\zeta+u)(2(u-\zeta)\sin(u)+\cos(2\zeta+u)+3\cos(u))((u+\upsilon)\sin(u)\csc(u+\upsilon)+\sin(\upsilon))}{2(u\sin(u)+\cos(u))}
\end{aligned} \tag{B.17}$$

In order to write the expressions above we only retained the non-vanishing terms of the ϵ-series expansion; we also introduced $\zeta := \frac{\pi}{2} - \theta - \frac{\tau}{2}\sin(\theta)$ and $u := \frac{\pi}{2} - \theta$ as in 3.2.3. From (B.16) and (B.17) it is then simple to obtain the complete Green's function by means of the definition $G(t,\tau) = \theta(t-\tau)\, G^{(a)}(t,\tau) + \theta(\tau - t)\, G^{(r)}(t,\tau)$.

Derivatives of Green's functions. Here we want to find an expression for $\frac{d}{dt}G(t,t)$ in terms of the Green's function itself. By definition we have that:

$$\frac{d}{d\tau}G(\tau,\tau) = \frac{d}{d\tau}\left(\frac{G^{(a)}(\tau,\tau) + G^{(r)}(\tau,\tau)}{2}\right) \tag{B.18}$$

and for both the advanced and the retarded Green's function the following holds:

$$\frac{d}{d\tau}G^{(i)}(\tau,\tau) = \left(\frac{\partial}{\partial x}G^{(i)}(x,y) + \frac{\partial}{\partial y}G^{(i)}(x,y)\right)\Big|_{x=y=\tau}. \tag{B.19}$$

For the partial derivative with respect to the first argument we know that the Green's function has to satisfy:

$$\frac{\partial}{\partial x}G^{(a)}(x,y) = -P_0^{-1}P(x)G^{(a)}(x,y) + \frac{1}{2}P_0^{-1}\delta(x,y) \tag{B.20}$$

while by also observing that $G^{(a)}(x,y) = G^{(r)}(y,x)^\top$ we find the following relation:

$$\frac{\partial}{\partial y} G^{(a)}(x,y) = \frac{\partial}{\partial y} G^{(r)}\,(y,x)^\top = -G^{(r)}(y,x)^\top P(y)^\top \left(P_0^{-1}\right)^\top + \frac{1}{2}\left(P_0^{-1}\right)^\top \delta(x-y) \quad \text{(B.21)}$$

At the end, putting everything together, and observing that $\left(P(x)^{-1}\right)^\top = P(x)$ and that $P_0^{-1} = -\left(P_0^{-1}\right)^\top$, we end up with the result:

$$\frac{\partial}{\partial x} G^{(a)}(x,y) + \frac{\partial}{\partial y} G^{(a)}(x,y) = -P_0^{-1} P(x) G^{(a)}(x,y) + G^{(a)}(x,y) P(y) P_0^{-1} \quad \text{(B.22)}$$

Therefore, the derivative we are interested in is evaluated as:

$$\frac{d}{d\tau} G^{(a)}(\tau,\tau) = -P_0^{-1} P(\tau) G^{(a)}(\tau,\tau) + G^{(a)}(\tau,\tau) P(\tau) P_0^{-1} \quad \text{(B.23)}$$

Clearly the same is true for the retarded Green's function upon interchanging the indexes $a \leftrightarrow r$. Thus, upon summation, an analogous relation holds for the complete Green's function without indexes:

$$\frac{d}{d\tau} G(\tau,\tau) = -P_0^{-1} P(\tau) G(\tau,\tau) + G(\tau,\tau) P(\tau) P_0^{-1} \quad \text{(B.24)}$$

which in components and reads:

$$\begin{cases} \frac{d}{d\tau} G_{pp}(\tau,\tau) = -4i\left(G_{pp}(\tau,\tau) P_{p\phi}(\tau) + G_{p\phi}(\tau,\tau) P_{\phi\phi}(\tau)\right) \\ \frac{d}{d\tau} G_{p\phi}(\tau,\tau) = 2i\left(G_{pp}(\tau,\tau) P_{pp}(\tau) - G_{\phi\phi}(\tau,\tau) P_{\phi\phi}(\tau)\right) \\ \frac{d}{d\tau} G_{\phi\phi}(\tau,\tau) = 4i\left(G_{p\phi}(\tau,\tau) P_{pp}(\tau) + G_{\phi\phi}(\tau,\tau) P_{p\phi}(\tau)\right) \end{cases} \quad \text{(B.25)}$$

Appendix C

C.1 The torus as a complex manifold

In this brief review we will mainly follow [249]. Let's consider the complex plane $\mathbb{C}$ and a two dimensional lattice $L(\omega_1, \omega_2) \equiv \{\omega_1\, m + \omega_2\, n \mid m, n \in \mathbb{Z}\}$. Where ω_1 and ω_2 are two non-vanishing complex lattice vectors satisfying the condition: $\omega_2/\omega_1 \notin \mathbb{R}$.
We can define the coset space: $T^2 = \mathbb{C}/L(\omega_1, \omega_2)$. For a point $z \in T^2$ the following equivalence class holds: $z \sim z + \omega_1\, m + \omega_2\, n$ for some $m, n \in \mathbb{Z}$. T^2 is a compact complex manifold and is homeomorphic to the real two dimensional torus. Considering the torus as a complex manifold has some advantages.
The notion of complex manifolds M is similar to the one of real manifold: it is a topological space provided with a family of atlases $\{(U_i, \varphi_i)\}$ where $\{U_i\}$ is a family of open sets which covers M and φ_i is a homeomorphism from U_i to an open subset of $\mathbb{C}^k$; moreover at the intersection of two sets $U_i \cap U_j$ the map between φ_i and φ_j is holomorphic.
If the union of two atlases is again an atlas, we say that they define the same complex structure. In the case of T^2 the pair of complex vectors (ω_1, ω_2) defines a complex structure on T^2. Two pairs of complex vectors (ω_1, ω_2) and $(\tilde{\omega}_1, \tilde{\omega}_2)$ define the same complex structure if and only if they are related by the transformation:

$$\begin{pmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 \end{pmatrix} = \lambda\, A \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \tag{C.1}$$

where λ is a complex number implementing a dilatation in the lattice and $A \in PSL(2, \mathbb{Z})$.
The notion of complex structure is needed if we want to study the effects of conformal transformations on the two dimensional torus thought as a Riemann surface [250]. There is a one to one correspondence between the complex structures and the conformal invariant classes on the metric of the Riemann surface. If T^2 and $\tilde{T}^2$ have the same complex structure, they are conformally equivalent.
In the literature it is common to get rid of the dilatation symmetry by setting $\lambda = 2\pi/\omega_1$ and fix

the generators of the lattice to be $(2\pi, 2\pi\,\tau)$, where we have introduced the modular parameter

$$\tau := \frac{\omega_2}{\omega_1} \in \mathbf{H} \tag{C.2}$$

Clearly the invariance under $PSL(2,\mathbb{Z})$ survives and as a result T^2_τ and $\tilde{T}^2_{\tilde{\tau}}$ are conformally equivalent if and only if τ and $\tilde{\tau}$ are related by the transformation:

$$\tilde{\tau} = \frac{a\tau + b}{c\tau + d} \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z}) \tag{C.3}$$

the previous one is called modular transformation. From a physical point of view, for CFTs defined on a torus, we expect the observables to be invariant under modular transformations.

C.2 Trivial Variable changes

We show in detail the manipulations on the Cardy operator. As a first step we trade the components for the respective lattice vectors and their conjugates (thought of as independent variables):

$$\begin{cases} \omega_2 := L_1' + iL_2' \\ \omega_1 := L_1 + iL_2 \\ \bar{\omega_2} := L_1' - iL_2' \\ \bar{\omega_1} := L_1 - iL_2 \end{cases} \longleftrightarrow \begin{cases} L_1 = \frac{w_1 + \bar{w}_1}{2} \\ L_1' = \frac{w_2 + \bar{\omega}_2}{2} \\ L_2 = \frac{w_1 - \bar{w}_1}{2i} \\ L_2' = \frac{w_2 - \bar{w}_2}{2i} \end{cases}$$

Where $L_1, L_2, L_1', L_2' \in \mathbb{R}$. As a consequence, the differential operators take the form:

$$\begin{aligned} \partial_{L_1} &= \partial_{\omega_1} + \partial_{\bar{\omega}_1}, & \partial_{L_2} &= i\left(\partial_{\omega_1} - \partial_{\bar{\omega}_1}\right), \\ \partial_{L_1'} &= \partial_{\omega_2} + \partial_{\bar{\omega}_2}, & \partial_{L_2'} &= i\left(\partial_{\omega_2} - \partial_{\bar{\omega}_2}\right) \end{aligned} \tag{C.4}$$

The second order operators appearing in the Cardy operator are:

$$\partial_{L_1}\partial_{L_2'} = i\left(\partial_{\omega_1}\partial_{\omega_2} - \partial_{\omega_1}\partial_{\bar{\omega}_2} + \partial_{\bar{\omega}_1}\partial_{\omega_2} - \partial_{\bar{\omega}_1}\partial_{\bar{\omega}_2}\right)$$

$$\partial_{L_2}\partial_{L_1'} = i\left(\partial_{\omega_1}\partial_{\omega_2} + \partial_{\omega_1}\partial_{\bar{\omega}_2} - \partial_{\bar{\omega}_1}\partial_{\omega_2} - \partial_{\bar{\omega}_1}\partial_{\bar{\omega}_2}\right)$$

As a consequence, we can evaluate the combination:

$$\begin{aligned} \partial_{L_1}\partial_{L_2'} - \partial_{L_2}\partial_{L_1'} &= i\Big(\partial_{\omega_1}\partial_{\omega_2} - \partial_{\omega_1}\partial_{\bar{\omega}_2} + \partial_{\bar{\omega}_1}\partial_{\omega_2} - \partial_{\bar{\omega}_1}\partial_{\bar{\omega}_2} \\ &\qquad - \partial_{\omega_1}\partial_{\omega_2} - \partial_{\omega_1}\partial_{\bar{\omega}_2} + \partial_{\bar{\omega}_1}\partial_{\omega_2} + \partial_{\bar{\omega}_1}\partial_{\bar{\omega}_2}\Big) \\ &= 2i\left(\partial_{\bar{\omega}_1}\partial_{\omega_2} - \partial_{\omega_1}\partial_{\bar{\omega}_2}\right) \end{aligned} \tag{C.5}$$

Moreover, the linear terms become:

$$L_1\partial_{L_1} = \frac{\omega_1+\bar{\omega}_1}{2}\left(\partial_{\omega_1}+\partial_{\bar{\omega}_1}\right), \qquad L_2\partial_{L_2} = \frac{\omega_1-\bar{\omega}_1}{2i}\, i\left(\partial_{\omega_1}-\partial_{\bar{\omega}_1}\right),$$
$$L_1'\partial_{L_1'} = \frac{\omega_2+\bar{\omega}_2}{2}\left(\partial_{\omega_2}+\partial_{\bar{\omega}_2}\right), \qquad L_2'\partial_{L_2'} = \frac{\omega_2-\bar{\omega}_2}{2i}\, i\left(\partial_{\omega_2}-\partial_{\bar{\omega}_2}\right) \tag{C.6}$$

As a consequence:

$$L_1\partial_{L_1}+L_2\partial_{L_2} = \omega_1\partial_{\omega_1}+\bar{\omega}_1\partial_{\bar{\omega}_1}, \quad L_1'\partial_{L_1'}+L_2'\partial_{L_2'} = \omega_2\partial_{\omega_2}+\bar{\omega}_2\partial_{\bar{\omega}_2} \tag{C.7}$$

The torus area is:

$$\begin{aligned} A &= L_1L_2' - L_2L_1' \\ &= \frac{1}{4i}\Big[(\omega_1+\bar{\omega}_1)(\omega_2-\bar{\omega}_2) - (\omega_1-\bar{\omega}_1)(\omega_2+\bar{\omega}_2)\Big] \\ &= \frac{1}{2i}\left(\bar{\omega}_1\omega_2 - \omega_1\bar{\omega}_2\right) \end{aligned} \tag{C.8}$$

As a final result, the Cardy operator takes the form:

$$C = 2i\left(\partial_{\bar{\omega}_1}\partial_{\omega_2} - \partial_{\omega_1}\partial_{\bar{\omega}_2}\right) - \frac{2i}{\bar{\omega}_1\omega_2-\omega_1\bar{\omega}_2}\cdot\left(\omega_1\partial_{\omega_1}+\bar{\omega}_1\partial_{\bar{\omega}_1}+\omega_2\partial_{\omega_2}+\bar{\omega}_2\partial_{\bar{\omega}_2}\right) \tag{C.9}$$

Now we want to introduce the τ dependence, and we consider the variable change:

$$\begin{cases} \omega_1 = \tilde{\omega}_1 \\ \omega_2 = \tau\tilde{\omega}_1 \end{cases} \quad\longleftrightarrow\quad \begin{cases} \tilde{\omega}_1 = \omega_1 \\ \tau = \frac{\omega_2}{\omega_1} \end{cases}$$

The previous transformation breaks the manifest antisymmetry between ω_1 and ω_2, indeed:

$$\begin{aligned} \frac{\partial}{\partial\omega_1} &= \frac{\partial\tilde{\omega}_1}{\partial\omega_1}\,\partial_{\tilde{\omega}_1} + \frac{\partial\tau}{\partial\omega_1}\,\partial_\tau = \partial_{\tilde{\omega}_1} - \frac{\omega_2}{\omega_1^2}\,\partial_\tau = \partial_{\tilde{\omega}_1} - \frac{\tau}{\tilde{\omega}_1}\,\partial_\tau, \\ \frac{\partial}{\partial\omega_2} &= \frac{\partial\tau}{\partial\omega_2}\,\partial_\tau + \frac{\partial\tilde{\omega}_1}{\partial\omega_2}\,\partial_{\tilde{\omega}_1} = \frac{1}{\tilde{\omega}_1}\,\partial_\tau \end{aligned} \tag{C.10}$$

As a consequence:

$$\omega_1\partial_{\omega_1} = \tilde{\omega}_1\partial_{\tilde{\omega}_1} - \tau\partial_\tau \quad , \quad \omega_2\partial_{\omega_2} = \tau\partial_\tau \quad , \quad \omega_1\partial_{\omega_1}+\omega_2\partial_{\omega_2} = \tilde{\omega}_1\partial_{\tilde{\omega}_1} \tag{C.11}$$

Similarly, for conjugate quantities:

$$\bar{\omega}_1\partial_{\bar{\omega}_1} = \bar{\tilde{\omega}}_1\partial_{\bar{\tilde{\omega}}_1} - \bar{\tau}\partial_{\bar{\tau}}, \quad \bar{\omega}_2\partial_{\bar{\omega}_2} = \bar{\tau}\partial_{\bar{\tau}}, \quad \bar{\omega}_1\partial_{\bar{\omega}_1}+\bar{\omega}_2\partial_{\bar{\omega}_2} = \bar{\tilde{\omega}}_1\partial_{\bar{\tilde{\omega}}_1} \tag{C.12}$$

Therefore, omitting the notation˜, we obtain (5.24):

$$C = \frac{-2i}{\tau-\bar{\tau}}\left[\frac{1}{\bar{\omega}_1}\partial_{\omega_1}+\frac{1}{\omega_1}\partial_{\bar{\omega}_1}\right]+2i\left[\frac{1}{\omega_1}\partial_{\bar{\omega}_1}\partial_\tau-\frac{1}{\bar{\omega}_1}\partial_{\omega_1}\partial_{\bar{\tau}}+\frac{\tau-\bar{\tau}}{\omega_1\bar{\omega}_1}\partial_\tau\partial_{\bar{\tau}}\right]$$

Clearly the above expression is explicitly symmetric under conjugation. The last variable change is:

$$\begin{cases} S=\frac{-1}{2\pi i(\tau-\bar{\tau})} \\ \tilde{\tau}=\tau \end{cases} \longleftrightarrow \begin{cases} \bar{\tau}=\frac{1}{2\pi i S}+\tilde{\tau} \\ \tau=\tilde{\tau} \end{cases}$$

Which implies:

$$\begin{aligned}\partial_\tau &= \frac{\partial\tilde{\tau}}{\partial\tau}\,\partial_{\tilde{\tau}}+\frac{\partial S}{\partial\tau}\,\partial_S=\partial_{\tilde{\tau}}+2\pi i\,S^2\,\partial_S,\\ \partial_{\bar{\tau}} &= \frac{\partial\tilde{\tau}}{\partial\bar{\tau}}\,\partial_{\tilde{\tau}}+\frac{\partial S}{\partial\bar{\tau}}\,\partial_S=-2\pi i\,S^2\,\partial_S\end{aligned} \tag{C.13}$$

As a final result:

$$\begin{aligned}C &= \frac{-2i}{\tau-\bar{\tau}}\left[\frac{1}{\bar{\omega}_1}\partial_{\omega_1}+\frac{1}{\omega_1}\partial_{\bar{\omega}_1}\right]+2i\left[\frac{1}{\omega_1}\partial_{\bar{\omega}_1}\partial_\tau-\frac{1}{\bar{\omega}_1}\partial_{\omega_1}\partial_{\bar{\tau}}+\frac{\tau-\bar{\tau}}{\omega_1\bar{\omega}_1}\partial_\tau\partial_{\bar{\tau}}\right]\\ &= -4\pi S\left[\frac{1}{\bar{\omega}_1}\partial_{\omega_1}+\frac{1}{\omega_1}\partial_{\bar{\omega}_1}\right]+2i\Bigg[\frac{1}{\omega_1}\partial_{\bar{\omega}_1}\partial_\tau+2\pi i\frac{1}{\omega_1}\partial_{\bar{\omega}_1}S^2\partial_S^2+2\pi i\frac{1}{\bar{\omega}_1}\partial_{\omega_1}S^2\partial_S^2\\ &\quad+\frac{1}{S\,2\pi i\,\omega_1\bar{\omega}_1}\left(\partial_\tau+2\pi i\,S^2\partial_S\right)\left(2\pi i\,S^2\partial_S\right)\Bigg]\end{aligned} \tag{C.14}$$

Which is exactly expression (5.26):

$$C=-4\pi\left[\frac{1}{\bar{\omega}_1}\partial_{\omega_1}+\frac{1}{\omega_1}\partial_{\bar{\omega}_1}\right]\left(S+S^2\partial_S\right)+2i\frac{1}{\omega_1}\partial_{\bar{\omega}_1}\partial_\tau+\frac{2iS}{\omega_1\bar{\omega}_1}\partial_S\partial_\tau-\frac{4\pi S}{\omega_1\bar{\omega}_1}\left(2S\partial_S+S^2\partial_S^2\right)$$

C.3 Details: Okuyama-Sakai partition function

By definition of partition function and free energy:

$$\begin{aligned}\mathcal{Z}^{(0)}(\tau,S) &= e^{\mathcal{F}^{(0)}(\tau,S)}=e^{\mathcal{F}_1^{(0)}+\mathcal{F}_2^{(0)}/N^2+\mathcal{F}_3^{(0)}/N^4+\dots}\\ &= e^{\mathcal{F}_1^{(0)}}\left[1+\frac{1}{N^2}\mathcal{F}_2^{(0)}+\frac{1}{N^4}\left(\mathcal{F}_3^{(0)}+2\left(\mathcal{F}_2^{(0)}\right)^2\right)+\dots\right]\end{aligned}$$

We remind that the first order undeformed free energy is:

$$\mathcal{F}_1^{(0)}(\tau,S)=\log\left(\frac{\sqrt{2\pi S}}{\eta(\tau)}\right)$$

Hence:

$$\mathcal{Z}^{(0)}(\tau, S) = \left[\frac{\sqrt{2\pi S}}{\eta(\tau)} + \frac{1}{N^2}\frac{\sqrt{2\pi S}}{\eta(\tau)}\mathcal{F}_2^{(0)}(\tau, S) + \frac{1}{N^4}\frac{\sqrt{2\pi S}}{\eta(\tau)}\left(\left(\mathcal{F}_3^{(0)}(\tau, S)\right) + 2\left(\mathcal{F}_2^{(0)}(\tau, S)\right)^2\right) + \ldots\right]$$

Using (5.20) and (5.21) we recognize the precise form of the first terms appearing in the Okuyama-Sakai partition function:

$$\tilde{\psi}_{1,0}(\tau) \equiv \frac{\sqrt{2\pi}}{\eta(\tau)}, \tag{C.15}$$

$$\tilde{\psi}_{2,0}(\tau) = \frac{\sqrt{2\pi}}{\eta(\tau)}\mathcal{F}_2^{(0)} = \frac{\sqrt{2\pi}}{\eta(\tau)} \cdot \left[\frac{5E_2^3 - 3E_2E_4 - 2E_6}{51840}\right] \tag{C.16}$$

C.4 Details: Deformed Local Compact Boson

As a consistency check of the expression of the Cardy operator, we compute the $T\overline{T}$ deformation of the free compact boson and we compare the result with the literature. We show in detail this simple computation, since it is interesting to study how the ω_1 dependence appears during the perturbative computation of the $T\overline{T}$ deformation. As it is known [251], the partition function of the free boson on torus topology, is:

$$\mathcal{Z}^{(0)}(\tau, \bar{\tau}) = \frac{1}{\sqrt{\tau_2}|\eta(\tau)|^2} \tag{C.17}$$

Where η is the Dedekind Eta function, the torus is described by the lattice vectors $(\omega_1, \omega_2) = (1, \tau)$ and $\tau_2 = \mathrm{Im}(\tau)$.
By direct diagrammatic computation [244] and using the CFT approach [233] it was shown that the first perturbative contributions to the deformed partition function satisfy the relations[1]:

$$\mathcal{Z}^{(1)} = -4\tau_2\partial_\tau\partial_{\bar{\tau}}\mathcal{Z}^{(0)}, \tag{C.18}$$

$$\mathcal{Z}^{(2)} = 16\left(\tau_2^2\partial_\tau^2\partial_{\bar{\tau}}^2 + i\tau_2\left(\partial_\tau^2\partial_{\bar{\tau}} - \partial_{\bar{\tau}}^2\partial_\tau\right)\right)\mathcal{Z}^{(0)} \tag{C.19}$$

It is simple to reproduce this result using (C.17) as the boundary condition of the Cardy equation in order to compute the $T\overline{T}$ perturbative expansion, this procedure does not require the technicalities of renormalization. Clearly we will use the Cardy operator (5.24) written as a function of $(\tau, \bar{\tau}, \omega_1, \bar{\omega}_1)$.
If we consider the first order contribution:

$$\mathcal{Z}^{(1)} = C\mathcal{Z}^{(0)}(\tau, \bar{\tau})$$

[1] Notice that in these papers the convention for the deformation parameter is different and can be recovered by sending $t \to -t$.

since the boundary condition $\mathcal{Z}^{(0)}$ does not depend on $\omega_1, \bar{\omega}_1$, it is clear that the only non-zero contribution is given by the application of the last term in (5.24). Hence:

$$\mathcal{Z}^{(1)}(\tau,\bar{\tau},\omega_1,\bar{\omega}_1) = 2i\frac{\tau-\bar{\tau}}{\omega_1\bar{\omega}_1}\partial_\tau\partial_{\bar{\tau}}Z^{(0)}(\tau,\bar{\tau}) \tag{C.20}$$

Which is equivalent to (C.18) if we set the the values of the lattice vector $\omega_1 = \bar{\omega}_1 = 1$. However, for the next part of the computation, it is important to keep the ω_1 dependence, indeed the Cardy operator now acts non-trivially:

$$\begin{aligned}
\mathcal{Z}^{(2)} &= C\,\mathcal{Z}^{(1)}(\tau,\bar{\tau},\omega_1,\bar{\omega}_1) \\
&= 4\,\partial_\tau\partial_{\bar{\tau}}Z^{(0)}(\tau,\bar{\tau})\cdot\left[\frac{1}{\bar{\omega}_1}\partial_{\omega_1}+\frac{1}{\omega_1}\partial_{\bar{\omega}_1}\right]\left(\frac{1}{\omega_1\bar{\omega}_1}\right) \\
&\quad -4\left[\frac{1}{\omega_1}\partial_{\bar{\omega}_1}\partial_\tau-\frac{1}{\bar{\omega}_1}\partial_{\omega_1}\partial_{\bar{\tau}}+\frac{\tau-\bar{\tau}}{\omega_1\bar{\omega}_1}\partial_\tau\partial_{\bar{\tau}}\right]\left(\frac{\tau-\bar{\tau}}{\omega_1\bar{\omega}_1}\partial_\tau\partial_{\bar{\tau}}Z^{(0)}(\tau,\bar{\tau})\right) = \\
&= 4\,\partial_\tau\partial_{\bar{\tau}}Z^{(0)}(\tau,\bar{\tau})\cdot\left[\frac{1}{\bar{\omega}_1}\partial_{\omega_1}+\frac{1}{\omega_1}\partial_{\bar{\omega}_1}\right]\left(\frac{1}{\omega_1\bar{\omega}_1}\right) \\
&\quad -4\left[\frac{1}{\omega_1}\partial_{\bar{\omega}_1}+\frac{1}{\bar{\omega}_1}\partial_{\omega_1}\right]\left(\frac{1}{\omega_1\bar{\omega}_1}\partial_\tau\partial_{\bar{\tau}}Z^{(0)}(\tau,\bar{\tau})\right) \\
&\quad +4\left(\frac{\tau-\bar{\tau}}{\omega_1^2\bar{\omega}_1^2}\partial_\tau^2\partial_{\bar{\tau}}Z^{(0)}(\tau,\bar{\tau})-\frac{\tau-\bar{\tau}}{\omega_1^2\bar{\omega}_1^2}\partial_\tau\partial_{\bar{\tau}}^2Z^{(0)}(\tau,\bar{\tau})\right) \\
&\quad -4\left[\frac{\tau-\bar{\tau}}{\omega_1\bar{\omega}_1}\partial_\tau\partial_{\bar{\tau}}\right]\left(\frac{\tau-\bar{\tau}}{\omega_1\bar{\omega}_1}\partial_\tau\partial_{\bar{\tau}}Z^{(0)}(\tau,\bar{\tau})\right) \\
&= 8\left(\frac{\tau-\bar{\tau}}{\omega_1^2\bar{\omega}_1^2}\partial_\tau^2\partial_{\bar{\tau}}-\frac{\tau-\bar{\tau}}{\omega_1^2\bar{\omega}_1^2}\partial_\tau\partial_{\bar{\tau}}^2\right)Z^{(0)}(\tau,\bar{\tau}) \\
&\quad -4\left(\frac{\tau-\bar{\tau}}{\omega_1\bar{\omega}_1}\right)^2\partial_\tau^2\partial_{\bar{\tau}}^2Z^{(0)}(\tau,\bar{\tau})
\end{aligned} \tag{C.21}$$

And clearly the last expression is equivalent to (C.19) if we set $\omega_1 = \bar{\omega}_1 = 1$ and $\tau - \bar{\tau} = 2i\tau_2$. We should also notice that the Cardy operator allows for a straightforward calculation which does not requires diagrams, therefore, in this set-up we can reach higher order contributions, as it is shown in [132] with a similar approach[2]. In fact, this method is so powerful that it allows the calculation of the entire formal perturbative series, in the case of chiral $U(N)$ Yang-Mills theory.

[2] In this paper the deformation parameter is rescaled by a factor of 4.