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Kähler Geometry of Scalar Flat Metrics on Line Bundles Over Polarized Kähler–Einstein Manifolds

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Abstract

In view of a better understanding of the geometry of scalar flat Kähler metrics, this paper studies two families of scalar flat Kähler metrics constructed by Hwang and Singer (Trans Am Math Soc 354(6):2285–2325, 2002) on $\mathbb{C}^{n+1}$ and on $\mathcal{O}(-k)$. For the metrics in both the families, we prove the existence of an asymptotic expansion for their ϵ-functions and we show that they can be approximated by a sequence of projectively induced Kähler metrics. Further, we show that the metrics on $\mathbb{C}^{n+1}$ are not projectively induced, and that the Burns–Simanca metric is characterized among the scalar flat metrics on $\mathcal{O}(-k)$ to be the only projectively induced one as well as the only one whose second coefficient in the asymptotic expansion of the ϵ-function vanishes.

Keywords Scalar flat Kähler manifolds · Kähler immersions · TYCZ expansion

Mathematics Subject Classification 32H02 · 53C07 · 53C42

1 Introduction and Statement of the Main Result

An important open problem in Kähler geometry consists in characterizing projectively induced metrics in view of the properties of their curvatures. A Kähler metric g on a complex manifold M is said to be projectively induced if there exists a *local* Kähler immersion into the complex projective space $\mathbb{C}\mathrm{P}^N$, that is if for any $p \in M$ there

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exists an open set $U \subset M$, $p \in U$, and a holomorphic function $f : U \rightarrow \mathbb{C}\mathrm{P}^N$, such that $f^* g_{FS} = g$. Here, we denote by g_{FS} the Fubini–Study metric, i.e., if $[Z_0 : \cdots : Z_N]$ are homogeneous coordinates on $\mathbb{C}\mathrm{P}^N$ and $(z_1, \ldots, z_N)$ are affine coordinates on $U_0 = \{Z_0 \neq 0\}$, g_{FS} is described on U_0 by the Kähler potential $\log(1 + |z_1|^2 + \cdots + |z_N|^2)$. Observe that we allow N to be infinite, where $\mathbb{C}\mathrm{P}^\infty$ is the quotient of $l^2(\mathbb{C}) \setminus \{0\}$ by the usual equivalent relation.

Many examples of projectively induced metrics can be constructed by taking the pull-back of the Fubini–Study metric on holomorphic submanifolds of $\mathbb{C}\mathrm{P}^N$, although it is more difficult to find projectively induced metrics with prescribed curvature. For example, Hulin in [9] proved that the scalar curvature of a compact Kähler–Einstein manifold Kähler immersed into $\mathbb{C}\mathrm{P}^N$ is forced to be positive. Observe that if a compact manifold admits a Kähler immersion in $\mathbb{C}\mathrm{P}^\infty$ then it is also a Kähler submanifold of $\mathbb{C}\mathrm{P}^N$ for some finite N, as the immersion is given by a basis of the space of global holomorphic sections of a suitable holomorphic line bundle, that when the manifold is compact is always finite dimensional. Although this holds true only for *global* Kähler immersions, in fact the flat torus is an example of compact manifold that is *locally* projectively induced in $\mathbb{C}\mathrm{P}^\infty$ but does not admit any Kähler immersion in $\mathbb{C}\mathrm{P}^N$ for finite N, as follows by Calabi's rigidity Theorem in [4, Th. 9] (see also [16] for an overview of Calabi's work). Recently in [1], Arezzo, Li, and Loi proved that there are not Ricci–flat submanifolds of $\mathbb{C}\mathrm{P}^N$ with $N < \infty$. It is still an open question if there exists a Ricci–flat (nonflat) Kähler submanifold of $\mathbb{C}\mathrm{P}^\infty$. It is important to emphasize that when the ambient space is taken to be infinite dimensional the situation could be much different, for example in [15] Kähler-Einstein submanifolds of $\mathbb{C}\mathrm{P}^\infty$ with negative scalar curvature are given. In [13], Loi, Salis, and Zuddas conjectured that the flat metric is the only example of projectively induced Ricci–flat metric and they validate the conjecture when the metric is radial and the immersion is *stable* (see also [17, 18, 22] for other results in the same context).

A very little is known for constant scalar curvature Kähler metrics. In the finite dimensional context, it is conjectured by Loi, Salis, Zuddas in [14] that the only projectively induced constant scalar curvature Kähler metrics lie on flag manifolds (actually their conjecture includes also extremal Kähler metrics). The Burns–Simanca metric on the blow–up of $\mathbb{C}^2$ at one point is an example of scalar flat (nonflat) complete projectively induced Kähler metric, as shown by Cannas Aghedu and Loi in [6]. The Burns–Simanca metric actually satisfies a stronger assumption than to be projectively induced, namely it admits a regular quantization.

A geometric quantization (L, h) of a n-dimensional Kähler manifold (M, ω) consists of an hermitian holomorphic line bundle L over M such that the first Chern class of L is represented by ω and its curvature $\mathrm{Ric}(h) := -i\partial\bar{\partial}\log h$ satisfies $\mathrm{Ric}(h) = \omega$. Let $\mathcal{H}$ be the space of global holomorphic sections of L and denote by $\langle\cdot,\cdot\rangle_h$ the scalar product:

$$\langle s, s\rangle_h := \int_M h(s(x), s(x))\frac{\omega^n}{n!}.$$

When $\mathcal{H} \neq \{0\}$ (condition that is always satisfied when M is compact), we can take an orthonormal basis $\{s_j\}_{j=0,\ldots,d}$ of $\mathcal{H}$, and define a function on M by

$$\epsilon_g(x) := \sum_{j=0}^{d} h(s_j(x), s_j(x)). \tag{1.1}$$

In literature, this ϵ-function was first introduced under the name of *η-function* by Rawnsley in [21], later renamed as *θ-function* in [3] followed by the *distortion function* of Kempf [12] and Ji [11], for the special case of Abelian varieties and of Zhang [24] for complex projective varieties. The geometric quantization (L, h) of (M, ω) is said to be *regular* if $\epsilon_{\alpha g}$ is constant for all large enough $\alpha \in \mathbb{Z}^+$ (when M is noncompact, α is not necessarily an integer). Observe that in this case one considers the geometric quantizations given by (L^α, h_α) such that $\mathrm{Ric}(h_\alpha) = \alpha\omega$. We also say that the metric is regular. Regular metrics enjoy the properties of being projectively induced Kähler metrics of constant scalar curvature. More precisely, for large enough α, one can construct a holomorphic map $F_\alpha : M \to \mathbb{C}\mathrm{P}^{d_\alpha}$, ($d_\alpha \leq +\infty$), called the *coherent states map*, by

$$F_\alpha : M \to \mathbb{C}\mathrm{P}^{d_\alpha}; \quad x \mapsto [s_0(x) : \cdots : s_{d_\alpha}(x)].$$

which satisfies (see e.g., [2]):

$$F_\alpha^*(\omega_{FS}) = \alpha\omega + \frac{i}{2}\partial\bar{\partial}\log\epsilon_{\alpha g}. \tag{1.2}$$

In particular, one has that when $\epsilon_{\alpha g}$ is constant, F_α is a holomorphic and isometric immersion.

Further, in view of Zelditch work [23], when M is compact, the function $\epsilon_{\alpha g}$ admits an asymptotic expansion (the so-called *Tian-Yau-Catlin-Zelditch expansion*):

$$\epsilon_{\alpha g}(x) \sim \sum_{j=0}^{\infty} a_j(x)\alpha^{n-j},$$

where $a_0(x) \equiv 1$ and the $a_j(x)$, $j = 1, 2, \ldots$ are smooth functions on M depending on the curvature and on its covariant derivatives at x of g. For this asymptotic expansion it is meant that, for every integers l, r and every compact $K \subseteq M$,

$$\left\| \epsilon_{\alpha g}(x) - \sum_{j=0}^{l} a_j(x)\alpha^{n-j} \right\|_{C^r} \leq \frac{C(l, r, K)}{\alpha^{l+1}}, \tag{1.3}$$

for some constant $C(l, r, K) > 0$. In particular, Lu [19] computed the first three coefficients, and the first two reads:

$$\begin{cases} a_1 = \frac{1}{2}\sigma_g \\ a_2 = \frac{1}{3}\Delta\sigma_g + \frac{1}{24}\left(|R_g|^2 - 4|\mathrm{Ric}_g|^2 + 3\sigma_g^2\right), \end{cases} \tag{1.4}$$

where σ_g, Ric_g and R_g denote, respectively, the scalar curvature, the Ricci tensor, and the curvature tensor of g, and the norms are taken with respect to g. When M is noncompact, the existence of such an expansion (known as *Engliš expansion*) is not guaranteed and only partial results are given (see Sect. 6 for details). Further, in [8], Engliš computed the a_j's coefficients obtaining the same results as Lu. All the a_j's coefficients of regular metrics are constant. The Burns-Simanca metric shares with the flat metric the property of presenting all the coefficients a_j's equal to zero [6].

Even if the metric is not regular, the existence of an asymptotic expansion for the ϵ function has important geometric consequences. In particular, it turns out that the coherent states map via (1.2) allows to approximate a Kähler metric g with projectively induced ones (see Lemma 5 in Sect. 6).

In this paper, we study families of scalar flat metrics constructed via Calabi ansatz on the total space of a hermitian line bundle over Kähler–Einstein manifolds by Andrew D. Hwang and Michael A. Singer in [10]. The necessary hypotheses for the existence of scalar flat metrics include the so-called sigma constancy, condition that is automatically satisfied by polarized Kähler manifolds, which is the case we are interested in. In particular, we consider the following families:

(A) the 1-parameter family of nontrivial scalar flat Kähler metrics g_β on $\mathbb{C}^{n+1}$, $\beta < 0$ (described in Sect. 4);
(B) the scalar flat metrics g_k on $\mathcal{O}(-k)$ for integers $k > 0$ (described in Sect. 5).

Observe that the metrics g_k in (B) reduce to the Burns–Simanca metric for $k = 1$, and to the Ricci–flat Eguchi–Hanson metric for $k = 2$.

The first result of this paper is the following:

Theorem 1 *Let g_β be the Kähler metric on $\mathbb{C}^{n+1}$ arising from Hwang–Singer construction. Then, cg_β is not projectively induced for any value of $c > 0$ and $\beta < 0$, but it can be approximated by a sequence of projectively induced metrics.*

Our second result characterizes the Burns–Simanca metric among the Hwang–Singer family g_k on $\mathcal{O}(-k)$. More precisely, we prove the following:

Theorem 2 *Let g_k be the Kähler metric arising from Hwang–Singer construction on $\mathcal{O}(-k)$. Then, g_k is projectively induced if and only if its second coefficient vanishes identically, that is if and only if it is the Burns–Simanca metric on the blow-up of $\mathbb{C}^2$ at one point. Moreover, g_k can be approximated by a sequence of projectively induced metrics.*

The paper is organized as follows. In Sect. 2, we recall what we need about Hwang-Singer construction restricted to polarized Kähler–Einstein manifolds. In Sect. 3, we give an overview of Calabi's criterion, deriving a necessary condition for the Hwang-Singer metrics to be projectively induced. Sections 4 and 5 are devoted, respectively, to the description of Hwang-Singer metrics on $\mathbb{C}^{n+1}$ and $\mathcal{O}(-k)$. Section 6 contains the existence results for the ϵ-function associated to the Hwang-Singer metrics on $\mathbb{C}^{n+1}$ and $\mathcal{O}(-k)$, and for its asymptotic expansion, and the proofs of theorems 1 and 2. Finally, the appendix includes some computations regarding the a_2 coefficient.

2 Momentum Construction

A technique to produce complete Kähler metrics with good curvature properties is known as *Calabi ansatz*, firstly introduced by Calabi in [5] and later adopted by several authors. Hwang and Singer generalized this construction on the total space of an hermitian holomorphic line bundle $\pi : L \to M$ with "*σ-constant curvature*" over a Kähler manifold (M, g_M). In this section, we summarize Hwang–Singer construction, restricting our attention to the case of polarized manifolds, where these hypotheses are automatically satisfied.

Let $\pi : (L, h) \to (M, \omega_M)$ be a polarized hermitian holomorphic line bundle with curvature form $\gamma = -i\partial\overline{\partial}\log h \in \Omega^2(M)$ such that $\gamma = \beta\omega_M$ over a Kähler–Einstein manifold of complex dimension n, that is $\rho_M = \lambda\omega_M$, where ρ_M is the Ricci form associated to g_M. This method, also known as *momentum construction*, gives rise to *bundle-adapted metrics* on L, that is Kähler metrics $g_{\varphi,\beta}$ whose Kähler form arises from the Calabi ansatz

$$\omega_{\varphi,\beta} = \pi^*\omega_M + 2i\partial\overline{\partial} f(t),$$

where t is the logarithm of the norm function defined by h and $f : (-\infty, +\infty) \to [0, +\infty)$ is an increasing and strictly convex function of one real variable which makes $\omega_{\varphi,\beta}$ positive definite.

In a coordinate chart $U \subset M$ over which L is trivial, i.e., $\pi^{-1}(U) \cong U \times \mathbb{C}$, there exists a local coordinate system $\tilde{z} = (z, \xi) = (z^1, \dots, z^n, \xi)$ for L where $\xi = \rho e^{i\theta}$ is a fiber coordinate and $z = (z^1, \dots, z^n)$ are pullbacks of coordinates on M, i.e., if $q \in M$ is a point with coordinates z, then every point in the fiber $\pi^{-1}(q)$ can be described by coordinates $\tilde{z}$. In such a chart, there is a smooth positive function $h : U \subset M \to \mathbb{R}$ such that

$$t := \log ||\tilde{z}|| = \frac{1}{2}\log\left(|\xi|^2 h(z)\right).$$

As explained in [10], to simplify the construction of scalar flat Kähler metrics on L, it is advantageous to change coordinates. Setting

$$\tau = f'(t), \quad \varphi(\tau) = f''(t),$$

so that f satisfies the differential equation

$$\begin{cases} f''(t) = \varphi(\tau) \\ f'(0) = \mu_0 > 0, \end{cases}$$

the Kähler metric $\omega_{\varphi,\beta}$ reads as

$$\omega_{\varphi,\beta} = \pi^*\omega_M - \tau\pi^*\gamma + \frac{1}{\varphi}\, d\tau \wedge d^c\tau, \tag{2.1}$$

and along the fiber L_x over $x \in M$ restricts to

$$\omega_{\varphi,\beta}|_{\text{fiber}} = \frac{\varphi(\tau)}{|\xi|^2}\, d\xi \wedge d\overline{\xi}.$$

The explicit expression for the profile function φ for scalar flat polarized metrics is

$$\varphi(\tau) = \frac{2}{(1-\beta\tau)^n}\left(\tau + \frac{\lambda\big((1-\beta\tau)^{n+1} - (1-\beta\tau) + \beta n\tau\big)}{\beta^2(n+1)}\right). \tag{2.2}$$

Observe that the factor $(1-\beta\tau)^n$ arises as the determinant of the endomorphism $\mathrm{Id} - \tau\mathrm{B}$, since $\mathrm{B} := \omega_M^{-1}\gamma = \beta\,\mathrm{Id}$ for $\gamma = \beta\omega_M$.

Remark 1 The function $f' : (-\infty, +\infty) \to (0, +\infty)$ is an increasing and surjective function (see Prop. 1.4. in [10]). In particular,

$$\lim_{t\to-\infty} f'(t) = 0.$$

Remark 2 The derivatives of the function $f(t)$ are expressed recursively in the variable τ as

$$f^{(n)}(t) = \varphi(\tau)(f^{(n-1)}(t))$$

for $n \geq 3$. In particular, we have

$$\begin{aligned} f'''(t) &= \varphi(\tau)\varphi'(\tau), \\ f^{(iv)}(t) &= \varphi(\tau)(\varphi(\tau)\varphi''(\tau) + (\varphi'(\tau))^2). \end{aligned} \tag{2.3}$$

Proposition 1 *Let $c > 0$ be a positive real number. If f is a solution for the* ODE *$y'' = \varphi(y')$ with φ given by (2.2), then $\hat{f} := cf$ is a solution to $y'' = \hat{\varphi}(y')$, where we denote with $\hat{\varphi}$ the profile function with parameters $\hat{\beta} = \frac{\beta}{c}$ and $\hat{\lambda} = \frac{\lambda}{c}$.*

Proof It follows by noticing that

$$\varphi(y') = \frac{2}{(1-\frac{\beta}{c}(cy)')^n}\left(\frac{1}{c}(cy)' + \frac{1}{c^2}\frac{\lambda\big((1-\frac{\beta}{c}(cy)')^{n+1} - (1-\frac{\beta}{c}(cy)') + n\frac{\beta}{c}(cy)'\big)}{\frac{\beta^2}{c^2}(n+1)}\right)$$
$$= \frac{1}{c}\frac{2}{(1-\hat\beta(cy)')^n}\left(y' + \frac{\hat\lambda\big((1-\hat\beta(cy)')^{n+1} - (1-\hat\beta(cy)') + \hat\beta n(cy)'\big)}{\hat\beta^2(n+1)}\right).$$

Thus,

$$c\varphi(y') = \frac{2}{(1-\hat\beta(cy)')^n}\left(y' + \frac{\hat\lambda\big((1-\hat\beta(cy)')^{n+1} - (1-\hat\beta(cy)') + \hat\beta n(cy)'\big)}{\hat\beta^2(n+1)}\right).$$

□

In this setting [10, Theorem B] by Hwang and Singer, reads

Theorem 3 *Let $\pi : (L, h) \to (M, \omega_M)$ be a polarized hermitian holomorphic line bundle over a complete Kähler–Einstein manifold (M, ω_M) such that $\gamma = \beta\omega_M$, with $\beta < 0$. Then, the metric $g_{\varphi,\beta}$ on the total space of L is a complete scalar flat Kähler metric. Moreover, the metric $g_{\varphi,\beta}$ is Ricci–flat if and only if $\rho_M = -\gamma$.*

Remark 3 For $\gamma = 0$, we have local product metrics since they are bundle-adapted metrics on flat-bundles, see ([10], Remark 1.6).

3 Calabi's Criterion Applied to $g_{\varphi,\beta}$

In this section, we recall what we need on Calabi's criterion for projectively induced metrics and compute the diastasis function for our metrics.

Let $(\mathbb{C}\mathrm{P}^N, g_{FS})$ be the complex projective space of dimension $N \leq \infty$ endowed with the Fubini-Study metric. Let $[Z_0 : \cdots : Z_N]$ be homogeneous coordinates and $(z_1, \ldots, z_N)$ the respective affine coordinates on the coordinate charts $U_j = \{Z_j \neq 0\}$ defined by $z_k := \frac{Z_k}{Z_j}$. A Kähler potential for g_{FS} on U_0 is

$$\phi_{FS}(z) = \log\left(1 + \sum_{j=1}^N |z_j|^2\right).$$

In [4], Calabi gives a criterion for a real analytic Kähler manifold (M, g) to admit a holomorphic and isometric (from now on *Kähler*) immersion into a complex space form in terms of the *diastasis function* associated to the metric g. Here, we consider only the case when the ambient space is the complex projective space $\mathbb{C}\mathrm{P}^N$, that is when the metric is projectively induced, which is the one we deal with. Observe that it is not restrictive to assume the manifold to be real analytic, since a metric induced by the pull-back through a holomorphic map of the real analytic Fubini-Study metric is forced itself to be real analytic. The diastasis function can be viewed as a particular

Kähler potential defined as follows. Fix a coordinates system $(z_1, \ldots, z_n)$ on a chart $U \subset M$ and let $\phi : U \to \mathbb{R}$ be a Kähler potential for g on U. By duplicating the variables z and $\overline{z}$, the Kähler potential ϕ on U can be complex analytically extended to a function $\tilde{\phi} : W \to \mathbb{R}$ on a neighborhood W of the diagonal in $U \times \overline{U}$. The diastasis function is defined by

$$D(z, w) := \tilde{\phi}(z, \overline{z}) + \tilde{\phi}(w, \overline{w}) - \tilde{\phi}(z, \overline{w}) - \tilde{\phi}(w, \overline{z}). \tag{3.1}$$

Denote by $p \in U$ the point of coordinates w_0. Observe that fixing $w = w_0$, the diastasis $D_p(z) := D(z, w_0)$ is a Kähler potential for g on U.

The Calabi's criterion is the following:

Theorem 4 (Calabi's criterion [4]) *Let (M, g) be a Kähler manifold. An open neighborhood of a point $p \in M$ admits a Kähler immersion into $\mathbb{C}\mathrm{P}^N$ if and only if the $\infty \times \infty$ hermitian matrix of coefficients (b_{jk}) defined by*

$$e^{D_p(z)} - 1 = \sum_{j,k=0}^{\infty} b_{jk}(z-p)^{m_j}(\overline{z}-\overline{p})^{m_k} \tag{3.2}$$

is positive semidefinite of rank at most N.

Let now (M, ω_M) be a Kähler-Einstein manifold with Einstein constant λ, that is $\rho_M = \lambda\omega_M$. As described in Sect. 2, the momentum construction gives a 1-parameter family of scalar flat Kähler metrics $\omega_{\varphi,\beta}$ on the polarized line bundle (L, h) described by the Kähler potential:

$$\Psi(z, \xi) = \Phi(z) + 4f\left(\frac{1}{2}\log[|\xi|^2 h(z)]\right), \tag{3.3}$$

where we can take as Φ the diastasis function for ω_M, centered at $z = 0$. We now describe the diastasis function for the metrics $g_{\varphi,\beta}$ and give a necessary condition for these metrics to be projectively induced, which follows directly by applying the Calabi's criterion to them.

By (3.1), the diastasis function associated to $\omega_{\varphi,\beta}$, centered at $p = (0, s)$ with $s \in \mathbb{R}^+$ is

$$\begin{aligned} D(z,\xi)|_p = \; & \Phi(z) + 4f\left(\frac{1}{2}\log\left(|\xi|^2 h(z)\right)\right) + 4f\left(\frac{1}{2}\log s^2\right) \\ & -4f\left(\frac{1}{2}\log(\xi s)\right) - 4f\left(\frac{1}{2}\log(\overline{\xi} s)\right), \end{aligned} \tag{3.4}$$

where we set $h(0) = 1$.

In particular, for the fiber metric, we have

$$D_p(\xi)|_{\text{fiber}} = 4f\left(\frac{1}{2}\log\left(|\xi|^2\right)\right) + 4f\left(\frac{1}{2}\log s^2\right) - 4f\left(\frac{1}{2}\log(\xi s)\right) - 4f\left(\frac{1}{2}\log(\overline{\xi} s)\right). \tag{3.5}$$

Proposition 2 *Let $c > 0$ be a positive real number. Then, the metric $c\,\omega_{\varphi,\beta} = \omega_{\hat{\varphi},\hat{\beta}}$, where $\hat{\varphi}$ is the profile function defined by (2.2) with parameters $\hat{\beta} := \beta/c$ and $\hat{\lambda} := \lambda/c$.*

Proof Observe that

$$c\omega_{\varphi,\beta} = c\omega_M + 2i\partial\overline{\partial}cf(t),$$

and $c\omega_M$ is a Kähler–Einstein metric with Einstein constant $\frac{\lambda}{c}$. Conclusion follows since by Proposition 1 $\hat{f} = cf$ is a solution to $y'' = \hat{\varphi}(y')$. □

Remark 4 We note that if g is a scalar flat projectively induced Kähler metric, then its (scalar flat) multiples cg may not be so. If the base manifold is Kähler–Einstein with Einstein constant λ, we find a close connection between the parameters c and λ (as in the previous proposition). Namely, it turns out that varying the parameter c over the positive real line corresponds to construct the Hwang–Singer metrics on the same line bundle over a rescaled Kähler-Einstein manifold with Einstein constant $\frac{\lambda}{c}$. Thus, it is equivalent to study the metric $c\omega_\varphi$ as c varies and the metric ω_φ as λ varies in the base manifold.

Lemma 1 *In the notation above, a necessary condition for the metric $\omega_{\varphi,\beta}$ to be projectively induced is that*

$$n\,(\lambda + 2\beta) \geq -4. \tag{3.6}$$

Proof By Calabi's Criterion Theorem 4, since $\frac{\partial^4(e^{D_p}-1)}{\partial\xi^2\overline{\partial}\xi^2}|_p$ is an element on the diagonal of the matrix (b_{jk}) in (3.2), a necessary condition for the metric $\omega_{\varphi,\beta}$ to be projectively induced is that

$$\begin{aligned}\frac{\partial^4(e^{D_p}-1)}{\partial\xi^2\overline{\partial}\xi^2}|_p &= \frac{1}{s^4}\left(\frac{1}{4}f^{(4)}\left(\frac{\log s^2}{2}\right) - f^{(3)}\left(\frac{\log s^2}{2}\right) + 2f''\left(\frac{\log s^2}{2}\right)^2\right.\\ &\left.\quad + f''\left(\frac{\log s^2}{2}\right)\right) \geq 0.\end{aligned} \tag{3.7}$$

By (2.3) and since $\varphi(\tau) > 0$ for every $\tau \in \mathbb{R}^+$, (3.7) is equivalent to

$$4 + 8\varphi(\mu_0) - 4\varphi'(\mu_0) + \varphi'(\mu_0)^2 + \varphi(\mu_0)\varphi''(\mu_0) \geq 0,$$

where $\mu_0 := f'\left(\frac{\log s^2}{2}\right)$, i.e.,

$$(2-\varphi'(\mu_0))^2+\varphi(\mu_0)\left(8+\varphi''(\mu_0)\right)\geq 0,$$

that is

$$\varphi''(\mu_0)\geq -8-\frac{(2-\varphi'(\mu_0))^2}{\varphi(\mu_0)}. \tag{3.8}$$

By the definition of φ (2.2),

$$\varphi'(\mu_0)=\frac{2\left((n+1)\beta+\lambda\left(1-(1-\beta\mu_0)^n\right)+\left((\beta^2+1)(n^2-1)+\lambda((1-\beta\mu_0)^n)\right)\mu_0\right)}{(n+1)\beta(1-\beta\mu_0)^{n+1}},$$
$$\varphi''(\mu_0)=\frac{2n}{(1-\beta\mu_0)^{n+2}}\left(\lambda+2\beta+(n-1)\beta(\beta+\lambda)\mu_0\right).$$

Since we can choose $s>0$ arbitrarily small, then (3.8) must hold for $\mu_0\to 0$ (see Remark 1). It is not hard to see that as $\mu_0\to 0$,

$$\frac{(2-\varphi'(\mu_0))^2}{\varphi(\mu_0)}\to 0,$$

and

$$\varphi''(\mu_0)\to 2n\left(\lambda+2\beta\right).$$

Thus, (3.8) implies

$$n\left(\lambda+2\beta\right)\geq -4,$$

as wished. □

Remark 5 Considering the j-th derivatives $\frac{\partial^{2j}(e^{D_p}-1)}{\partial\xi^j\partial\bar{\xi}^j}|_p$, we get sharper necessary conditions for the metric $g_{\varphi,\beta}$ to be projectively induced. However, such conditions will always depend on the choice of β and λ.

Remark 6 When $\beta=-\lambda$, the metric $g_{\varphi,\beta}$ is Ricci-flat. In this case, condition (3.6) gives that $g_{\varphi,\beta}$ is not projectively induced for any $\lambda>\frac{4}{n}$. This estimate can be improved to $\lambda\geq 1$ also for $n=2$, 3, and 4, by computing the 4-th derivative $\frac{\partial^8(e^{D_p}-1)}{\partial\xi^4\partial\bar{\xi}^4}|_p$, evaluated at $\mu_0=\frac{1}{100\lambda}$. Further, observe that when $n=1$, $g_{\varphi,\beta}$ is the Eguchi–Hanson metric on $\mathbb{C}\mathrm{P}^1$, which has been proven to be not projectively induced in [18]. As before, observe that such condition can be improved considering higher derivatives but will always depend on the choice of λ, as in the above remark.

4 Hwang–Singer Metrics on $\mathbb{C}^{n+1}$

Let $(M, \omega_M) = (\mathbb{C}^n, \omega_0)$, where ω_0 is the canonical flat metric, i.e., $\omega_0 = \frac{i}{2}\partial\bar\partial||z||^2$. The momentum construction in this case gives a 1-parameter family of scalar flat Kähler metrics $\omega_{\varphi,\beta}$ on $\mathbb{C}^{n+1}$ described by the Kähler potential (see 3.3):

$$\Phi(z,\xi) := ||z||^2 + 4f\left(\frac{1}{2}\log\left[|\xi|^2 e^{-\frac{\beta}{2}||z||^2}\right]\right), \tag{4.1}$$

obtained setting $h(z) = e^{-\frac{\beta}{2}||z||^2}$ in (3.3), for $\beta < 0$, so that $\gamma = -i\partial\bar\partial\log h(z) = \beta\frac{i}{2}\partial\bar\partial||z||^2 = \beta\omega_M$ and by (3.4) the diastasis function for $\omega_{\varphi,\beta}$ centered at $(z,\xi) = (0,s)$ reads

$$D_{(s,0)}(z,\xi) = ||z||^2 + 4f\left(\frac{1}{2}\log\left[|\xi|^2 e^{-\frac{\beta}{2}||z||^2}\right]\right) + 4f\left(\frac{1}{2}\log s^2\right) - 4f\left(\frac{1}{2}\log[\xi s]\right) -4f\left(\frac{1}{2}\log[\bar\xi s]\right). \tag{4.2}$$

The profile function obtained setting $\lambda = 0$ in (2.2) is given by

$$\varphi(\tau) = \frac{2\tau}{(1-\beta\tau)^n}, \tag{4.3}$$

for $\tau \in [0, +\infty)$.

In order to prove the first part of Theorem 1, i.e., that $(\mathbb{C}^{n+1}, c\omega_{\varphi,\beta})$ is not projectively induced for any c and β, let us first show how to drop the dependence on the parameters β and c.

Lemma 2 *Up to an affine change of coordinates on $\mathbb{C}^n$, the metric $c\omega_{\varphi,\beta}$ on $\mathbb{C}^{n+1}$ is equivalent to $\omega_{\varphi,-1}$.*

Proof Let us first deal with β. The metric $\omega_{\varphi,-1}$ is obtained by a momentum construction on $(\mathbb{C}^n, \omega_0)$ with profile $\varphi(\tau) = \frac{2\tau}{(1+\tau)^n}$. Perform a change of coordinates on $\mathbb{C}^n$ by setting $z' = \frac{1}{\sqrt{-\beta}}z$. Then,

$$\omega_0 = \frac{i}{2}\partial\bar\partial||z||^2 = -\beta\frac{i}{2}\partial\bar\partial||z'||^2,$$

Observe that while in the z coordinates $\gamma = -\omega_0$, in the coordinates z', $\gamma = \beta\omega_0$. The determinant of the endomorphism $\mathrm{Id} - \tau\mathrm{B}$ (see Sect. 2 after formula (2.2)), that in the z coordinates was $(1+\tau)^n$, now in z' reads $(1-\beta\tau)^n$. Thus, the change of coordinates transforms the metric $\omega_{\varphi,-1}$ on $\mathbb{C}^{n+1}$ in the metric $\omega_{\varphi,\beta}$.

Let us now prove that the multiplication of $\omega_{\varphi,-1}$ by $c>0$ is equivalent to consider $\omega_{\varphi,-\frac{1}{c}}$. By (4.1), a Kähler potential for $c\omega_{\varphi,-1}$ is given by

$$c\Phi(z,\xi)=c||z||^2+4cf\left(\frac{1}{2}\log\left[|\xi|^2e^{\frac{1}{2}||z||^2}\right]\right).$$

Performing a change of coordinates $z'=\sqrt{c}z$, we get

$$c\Phi(z',\xi)=||z'||^2+4cf\left(\frac{1}{2}\log\left[|\xi|^2e^{\frac{1}{2c}||z'||^2}\right]\right).$$

Conclusion follows observing that by Proposition 1, if f satisfies the ODE given by $\varphi(\tau)=\frac{2\tau}{(1+\tau)^n}$, then cf satisfies the ODE given by $\varphi(\tau)=\frac{2\tau}{(1+\frac{1}{c}\tau)^n}$. □

5 Hwang–Singer Metrics on Line Bundles Over $\mathbb{C}\mathrm{P}^1$

Let L be a holomorphic line bundle over $\mathbb{C}\mathrm{P}^1$ endowed with the Fubini-Study metric normalized so that $\lambda=1$, that is, in affine coordinates $z=\frac{Z_1}{\frac{1}{2}Z_0}$ on $U_0=\{Z_0\neq 0\}$

$$\omega_{\mathrm{FS}}=\frac{i}{2}\partial\bar{\partial}\,4\log\left(1+\frac{1}{4}|z|^2\right).$$

Since L is a holomorphic line bundle over $\mathbb{C}\mathrm{P}^1$, then L is of the form $\mathcal{O}(-k)$, $k\in\mathbb{Z}$.

The natural hermitian metric on the line bundle $\mathcal{O}(-1)$ on $\mathbb{C}\mathrm{P}^1$ is given by restricting the hermitian metric of $\mathbb{C}^2$ to each fiber $l=L_x\subset\mathbb{C}^2$. So if $\{U_0,U_1\}$ is a cover of $\mathbb{C}\mathrm{P}^1$ with

$$U_\alpha=\{Z_\alpha\neq 0\}\quad,\quad \alpha=0,1,$$

then

$$h_\alpha=\frac{|Z_0|^2+|Z_1|^2}{|Z_\alpha|^2}\quad,\quad \alpha=0,1.$$

So, on each open set U_α, if we take z as local coordinate, we have

$$h(z)=1+\frac{1}{4}|z|^2.$$

For $k>0$, the line bundles $\mathcal{O}(-k):=\mathcal{O}(k)^*=\mathcal{O}(1)^*\otimes\cdots\otimes\mathcal{O}(1)^*$ inherit natural hermitian structures given by

$$h_k(z)=\left(1+\frac{1}{4}|z|^2\right)^k.$$

The curvature form is then

$$\gamma = -i\partial\overline{\partial}\log\left(1+\frac{1}{4}|z|^2\right)^k = -\frac{k}{2}\omega_{\rm FS}$$

and the line bundle is polarized, with $\lambda = 1$ and $\beta = -\frac{k}{2}$, with k a positive integer.

So the profile 2.2 reads

$$\varphi_k(\tau) = \frac{2\tau+\tau^2}{1+\frac{k}{2}\tau},$$

and the momentum construction gives a 1-parameter family $\omega_k := \omega_{\varphi,-\frac{k}{2}}$ of scalar flat Kähler metrics on the polarized line bundle $\mathcal{O}(-k)$ described by the potentials

$$\Psi(z,\xi) = 4\log\left(1+\frac{1}{4}|z|^2\right) + 4f\left(\frac{1}{2}\log\left[|\xi|^2\left(1+\frac{1}{4}|z|^2\right)^k\right]\right). \qquad (5.1)$$

Remark 7 For $k=0$, the metric ω_k reduces to the local product metric on $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}$, see Remark 3.

In the proof of Theorem 2, we need the following lemma.

Lemma 3 *The metric ω_k on $\mathbb{C}\mathrm{P}^1$ is not projectively induced for any $k \geq 3$.*

Proof Let $p \in \mathcal{O}(-k)$ be the point of coordinates $(s,0)$ and let D_p be the diastasis function for the metric ω_k as in (3.5). The fourth derivative of $e^{D_p}-1$ evaluated at p is given by

$$\begin{aligned}&\frac{\partial^8(e^{D_p}-1)}{\partial\xi^4\overline{\partial}\xi^4}|_p\\ &= \frac{1}{s^8}\Big(+24f''^4+216f''^3+f'''(3f^{(5)}-45f^{(4)}-66)+(18f^{(4)}-216f'''+242)f''^2\\ &\quad+\frac{1}{64}(f^{(8)}-24f^{(7)}+232f^{(6)}-1152f^{(5)}+136(f^{(4)})^2+3088f^{(4)})\\ &\quad+(f^{(6)}-18f^{(5)}+125f^{(4)}+36f'''^2-396f'''+36)f''+114f'''^2\Big)\left(\frac{\log s^2}{2}\right),\end{aligned}$$

that written in terms of $\varphi(\mu_0)$ with $\mu_0 = f'\left(\frac{\log s^2}{2}\right)$, up to the multiplication by the positive constant $\frac{1}{s^8}$, reads $\frac{1}{64}\varphi(\mu_0)A(\varphi(\mu_0))$, with (to simplify the notation we drop the dependence from μ_0 in $\varphi(\mu_0)$ and its derivatives):

$$\begin{aligned} A(\varphi(\mu_0)) &= \varphi^{(6)}\varphi^5 + ((\varphi')^3 - 12(\varphi')^2 + 44\varphi' - 48)^2 + \varphi(\varphi') - 2)(-8(193\varphi'' + 968) \\ &+ (\varphi')^3(57\varphi'' + 392) - 2(\varphi'))^2(255\varphi'' + 1624) + 4\varphi'(383\varphi'' + 2200)) + 2\varphi^2(16(-36\varphi^{(3)} \\ &+ 29(\varphi'')^2 + 250\varphi'' + 432) + 61\varphi^{(3)}(\varphi')^3 + 2(\varphi')^2(96(9 - 2\varphi^{(3)} + 45(\varphi'')^2 + 436\varphi'')+ \\ &- 4\varphi'(-203\varphi^{(3)} + 102(\varphi'')^2 + 936\varphi'' + 1728)) + 2\varphi^3(17(\varphi'')^3 + 196(\varphi'')^2 \\ &+ 2\varphi^{(4)}(19(\varphi')^2 - 66\varphi' + 58) + 64\varphi^{(3)}(5\varphi' - 9) + 12(\varphi^{(3)}(8\varphi' - 15) + 48)\varphi'' + 768) \\ &+ \varphi^4(15(\varphi^{(3)})^2 + 8\varphi^{(5)}(2\varphi' - 3) + \varphi^{(4)}(26\varphi'' + 64)), \end{aligned}$$

since $\varphi(\mu_0)$ is positive, the sign of $\frac{\partial^8(e^{D_p}-1)}{\partial\xi^4\partial\bar{\xi}^4}|_p$ is the same as that of $A(\varphi(\mu_0))$. From (2.2), we get

$$\varphi_k^{(j)}(\tau) = (-1)^{j+1}\frac{8j!(k-1)k^{j-2}}{(2+k\tau)^{j+1}},$$

that substituted into the expression of $A(\varphi(\mu_0))$ gives

$$\frac{\mu_0^3}{2^7 3(2+k\mu_0)^{12}}P_k(\mu_0),$$

where $P_k(\mu_0)$ is the polynomial in μ_0:

$$P_k(\mu_0) = 105 - 113k + 48k^2 - 8k^3 + \sum_{s=1}^{12} q_s(k)\mu_0^s,$$

for given $q_s(k)$ that are not relevant for our analysis. Since μ_0 can be chosen small enough in $[0, +\infty)$ taking $s \to 0$, the sign of $A(\varphi(\mu_0))$ is the same as the sign of $P_k(\mu_0)$ for positive values of μ_0. Conclusion follows by noticing that

$$\lim_{\mu_0\to 0} P_k(\mu_0) = 105 - 113k + 48k^2 - 8k^3,$$

and the right hand side is negative for any $k \geq 3$. □

6 Asymptotic Expansion of $\omega_{\varphi,\beta}$ and Proofs of Theorems 1 and 2

Throughout this section, let us write $(X, \omega_{\varphi,\beta})$ for either $X = \mathbb{C}^{n+1}$ or $X = \mathcal{O}(-k)$. Let $\hat{L}$ be a holomorphic line bundle over X and let $h_{\hat{L}}$ be an hermitian metric on $\hat{L}$ such that $\mathrm{Ric}(\hat{h}) = \omega_{\varphi,\beta}$. Notice that such an $(\hat{L}, \hat{h})$ exists if and only if $\omega_{\varphi,\beta}$ is integral. In our case, this occurs since the base metric ω_M is an integral form and

$$[\omega_{\varphi,\beta}] = [\pi^*\omega_M + 2i\partial\overline{\partial} f(t)] = [\pi^*\omega_M],$$

where $\pi : L \to M$ is the projection given in Sect. 2 (here $M = \mathbb{C}^n$ or $\mathbb{C}\mathrm{P}^1$). Consider the tensor power $(\hat{L}^\alpha, \hat{h}_\alpha)$ and let $\mathcal{H}_\alpha$ be the space of global holomorphic sections of $\hat{L}^\alpha$. In order to define the ϵ-function for $(X, \omega_{\varphi,\beta})$, we first need to show that $\mathcal{H}_\alpha \neq \{0\}$.

Lemma 4 *In the notation above,* $1 \in \mathcal{H}_\alpha$ *for either* $X = (\mathbb{C}^{n+1}, \omega_{\varphi,\beta})$ *or* $X = (\mathcal{O}(-k), \omega_k)$*).*

Proof Observe that by formula (2.21) in [10], we have

$$\frac{\omega_{\varphi,\beta}^{n+1}}{(n+1)!} = \varphi Q \det(g_M) \frac{1}{|\xi|^2} \left(\frac{i}{2}\right)^{n+1} d\xi \wedge d\bar{\xi} \prod_{j=1}^{n} dz_j \wedge d\bar{z}_j.$$

where Q is the determinant of the endomorphism $\mathrm{Id} - \tau \mathrm{B}$, as after equation (2.2) of the profile.

Let us deal first with the case $X = \mathbb{C}^{n+1}$. In this case, $\mathcal{H}_\alpha$ is the weighted Hilbert space of global holomorphic functions over $\mathbb{C}^{n+1}$ that are L^2 limited in norm, namely:

$$\mathcal{H}_\alpha = \left\{ u \in \mathrm{Hol}(\mathbb{C}^{n+1}) | \int_{\mathbb{C}^{n+1}} |u|^2 e^{-\alpha\Phi} \frac{\omega_{\varphi,\beta}^{n+1}}{(n+1)!} < +\infty \right\},$$

where Φ is given by (4.1).

Due to Lemma 2, we can set $\beta = -1$. In order to prove that $1 \in \mathcal{H}_\alpha$, it is enough to check the convergence of the integral:

$$\begin{aligned} &\int_{\mathbb{C}^{n+1}} e^{-\alpha(||z||^2 + 4f(t))} \frac{2f'(t)}{|\xi|^2} \left(\frac{i}{2}\right)^{n+1} d\xi \wedge d\bar{\xi} \prod_{j=1}^{n} dz_j \wedge d\bar{z}_j \\ &= \pi^{n+1} \int_0^\infty \cdots \int_0^\infty \int_0^\infty e^{-\alpha(\sum_j r_j + 4f(\hat{t}))} \frac{2f'(\hat{t})}{r_0} dr_0 \prod_{j=1}^{n} dr_j, \end{aligned} \tag{6.1}$$

where we set polar coordinates $\xi := \rho_0 e^{i\theta_0}$, $z_j := \rho_j e^{i\theta_j}$ and $r_j := \rho_j^2$, $j = 0, \ldots, n$, and we denote $\hat{t} = \frac{1}{2}(\log r_0 + \frac{1}{2}\sum_j r_j)$. The function under the integral is positive and smooth, since $\frac{\varphi(t)}{|\xi|^2} \to g_{0\overline{0}}$ as $|\xi|^2 \to 0$, so its integral converges inside any closed ball of ray $R > 0$ centered at the origin. Thus, it is enough to check that the integral outside the ball is finite. Using that the function under the integral is positive and that

$$e^{-\alpha(\sum_j r_j + 4f(\hat{t}))} \frac{2f'(\hat{t})}{r_0} \leq e^{-\alpha(4f(\hat{t}))} \frac{2f'(\hat{t})}{r_0} = -\frac{1}{\alpha} \frac{d}{dr_0} e^{-4\alpha f(\hat{t})},$$

we have,

$$\begin{aligned}
&\int_R^\infty \cdots \int_R^\infty \int_R^\infty e^{-\alpha\left(\sum_j r_j + 4f(\hat{t})\right)} \frac{2f'(\hat{t})}{r_0} dr_0 \prod_{j=1}^n dr_j \\
&\leq -\frac{1}{\alpha} \int_R^\infty \cdots \int_R^\infty \int_R^\infty \frac{d}{dr_0} e^{-4\alpha f(\hat{t})} dr_0 \prod_{j=1}^n dr_j \qquad (6.2)\\
&= \frac{1}{\alpha} \int_R^\infty \cdots \int_R^\infty e^{-4\alpha f\left(\frac{1}{2}\log R + \frac{1}{4}\sum_j r_j\right)} \prod_{j=1}^n dr_j.
\end{aligned}$$

The last integral in (6.2) converges at least for $\alpha > n/4$ and for a large enough R, since

$$e^{-4\alpha f(\frac{1}{2}\log R + \frac{1}{4}\sum_j r_j))} \leq \frac{4^n}{r_1^{4\alpha/n} \cdots r_n^{4\alpha/n}}.$$

More precisely, since f is an increasing function and $\frac{\partial^2}{\partial r_j^2} f = \frac{1}{4} f'' > 0$, there exists $R \in \mathbb{R}$ such that for $r_j > R$, $j = 1, \ldots, n$,

$$f\left(\frac{1}{2}\log R + \frac{1}{4}\sum_j r_j\right) \geq f\left(\frac{1}{4} r_j\right) \geq \log\left(\frac{1}{4} r_j\right),$$

thus

$$f\left(\frac{1}{2}\log R + \frac{1}{4}\sum_j r_j\right) \geq \frac{1}{n}\sum_{j=1}^n \log\left(\frac{1}{4} r_j\right).$$

Let us now deal with $X = \mathcal{O}(-k)$ over $\mathbb{C}\mathrm{P}^1$. In this case, it is enough to check the convergence of the following integral over the chart $\mathcal{U}_0 \times \mathbb{C} \simeq \mathbb{C}^2$:

$$\begin{aligned}
&\int_{\mathbb{C}^2} \frac{e^{-4\alpha f(t)}}{\left(1 + \frac{1}{4}|z|^2\right)^{4\alpha+2}} \frac{2f'(t) + f'(t)^2}{|\xi|^2} \left(\frac{i}{2}\right)^2 d\xi \wedge d\bar{\xi} \wedge dz \wedge d\bar{z} \\
&= \pi^2 \int_0^\infty \int_0^\infty \frac{e^{-4\alpha f(\hat{t})}}{\left(1 + \frac{1}{4} r_1\right)^{4\alpha+2}} \frac{2f'(\hat{t}) + f'(\hat{t})^2}{r_0} dr_0 dr_1,
\end{aligned}$$

where we set polar coordinates $\xi = \rho_0 e^{i\theta_0}$, $z_1 = \rho_1 e^{i\theta_1}$, $r_j := \rho_j^2$, $j = 0, 1$, and set $\hat{t} := \frac{1}{2}\log r_0 + \frac{k}{2}\log(1 + \frac{1}{4} r_1)$. As before, since the function we are integrating is smooth on any closed ball of ray $R > 0$ (since $\frac{\varphi(t)}{|\xi|^2} \to g_{0\bar{0}}$ as $|\xi|^2 \to 0$), we reduce to check that the integral converges outside such ball. First observe that

$$I_1 := \int_R^\infty -e^{-4\alpha f(\hat{t}(r_0))} \frac{f'(\hat{t}(r_0))}{r_0} dr_0 = \frac{1}{2\alpha} \int_R^\infty \frac{d}{dr_0} e^{-4\alpha f} = \frac{1}{2\alpha}\left[e^{-4\alpha f}\right]_R^\infty < \infty$$

and

$$I_2 := \int_R^\infty \int_R^\infty \frac{e^{-4\alpha f(\hat{t})}}{\left(1+\frac{1}{4}r_1\right)^2} \frac{2f'(\hat{t})}{r_0} dr_0 dr_1 = -\frac{1}{\alpha} \int_R^\infty \int_R^\infty \frac{1}{\left(1+\frac{1}{4}r_1\right)^2} \frac{d}{dr_0} e^{-4\alpha f} dr_0 dr_1$$
$$= -\frac{1}{\alpha} \int_R^\infty \frac{1}{\left(1+\frac{1}{4}r_1\right)^2} \left[e^{-4\alpha f}\right]_R^\infty dr_1 = \frac{1}{\alpha} \int_R^\infty \frac{e^{-4\alpha f(\hat{t}(r_1))}}{\left(1+\frac{1}{4}r_1\right)^2} dr_1 < \infty.$$

So, since $\alpha > 0$, we have

$$\begin{aligned} &\int_R^\infty \int_R^\infty \frac{e^{-4\alpha f(\hat{t})}}{\left(1+\frac{1}{4}r_1\right)^{4\alpha+2}} \frac{2f'(\hat{t}) + f'(\hat{t})^2}{r_0} dr_0 dr_1 \\ &= \int_R^\infty \int_R^\infty \frac{e^{-4\alpha f(\hat{t})}}{\left(1+\frac{1}{4}r_1\right)^{4\alpha+2}} \frac{2f'(\hat{t})}{r_0} dr_0 dr_1 \\ &+ \int_R^\infty \int_R^\infty \frac{e^{-4\alpha f(\hat{t})}}{\left(1+\frac{1}{4}r_1\right)^{4\alpha+2}} \frac{f'(\hat{t})^2}{r_0} dr_0 dr_1 \\ &\le I_2 + \int_R^\infty \int_R^\infty \frac{e^{-4\alpha f(\hat{t})}}{\left(1+\frac{1}{4}r_1\right)^2} \frac{f'(\hat{t})^2}{r_0} dr_0 dr_1. \end{aligned}$$

It remains to check that

$$I := \int_R^\infty \int_R^\infty \frac{e^{-4\alpha f(\hat{t})}}{\left(1+\frac{1}{4}r_1\right)^2} \frac{f'(\hat{t})^2}{r_0} dr_0 dr_1$$

converges. Integrating by parts, since

$$\frac{e^{-4\alpha f(\hat{t})}}{\left(1+\frac{1}{4}r_1\right)^2} \frac{f'(\hat{t})^2}{r_0} = -\frac{2}{k\alpha} \frac{d}{dr_1} e^{-4\alpha f(\hat{t})} \frac{f'(\hat{t})}{r_0\left(1+\frac{1}{4}r_1\right)}.$$

we get

$$I = -\frac{2}{k\alpha} \int_R^\infty \int_R^\infty \frac{d}{dr_1} e^{-4\alpha f(\hat{t})} \frac{f'(\hat{t})}{r_0\left(1+\frac{1}{4}r_1\right)} dr_0 dr_1$$
$$= -\frac{2}{k\alpha} \left\{ \int_R^\infty \left[e^{-4\alpha f} \frac{f'}{r_0\left(1+\frac{1}{4}r_1\right)} \Big|_R^\infty - \frac{1}{r_0} \int_R^\infty e^{-4\alpha f} \frac{f''\frac{k}{8} - \frac{f'}{4}}{\left(1+\frac{1}{4}r_1\right)^2} dr_1 \right] dr_0 \right\}$$
$$= -\frac{2}{k\alpha} \left\{ \frac{I_1}{\left(1+\frac{1}{4}R\right)} - \frac{k}{8} \int_R^\infty \int_R^\infty \frac{e^{-4\alpha f} f''}{r_0\left(1+\frac{1}{4}r_1\right)^2} dr_1 dr_0 + \frac{1}{8} \int_R^\infty \int_R^\infty \frac{2f' e^{-4\alpha f}}{r_0\left(1+\frac{1}{4}r_1\right)^2} dr_1 dr_0 \right\}$$

$$= -\frac{2}{k\alpha}\left\{\frac{I_1}{(1+\frac{1}{4}R)} - \frac{k}{8}\int_R^\infty\int_R^\infty \frac{e^{-4\alpha f}(2f'+(f')^2)}{r_0\left(1+\frac{1}{4}r_1\right)^2\left(1+\frac{k}{2}f'\right)}dr_1dr_0 + \frac{I_2}{8}\right\}$$
$$\leq -\frac{2}{k\alpha}\left\{\frac{I_1}{(1+\frac{1}{4}R)} - \frac{k}{8}\int_R^\infty\int_R^\infty \frac{e^{-4\alpha f}(2f'+(f')^2)}{r_0\left(1+\frac{1}{4}r_1\right)^2}dr_1dr_0 + \frac{I_2}{8}\right\}$$
$$= -\frac{2}{k\alpha}\left\{\frac{I_1}{(1+\frac{1}{4}R)} - \frac{k}{8}I_2 - \frac{k}{8}I + \frac{I_2}{8}\right\} \tag{6.3}$$

where in the second equality we used that

$$\lim_{r_1\to+\infty}\frac{f'e^{-4\alpha f}}{\left(1+\frac{1}{4}r_1\right)} = 0$$

as follows by applying de l'Hopital and using that $f'' = \frac{2f'+f'^2}{1+\frac{k}{2}f'}$. Further the inequality follows by $(1+\frac{k}{2}f') > 1$, since f' is a positive function. From (6.3), we obtain

$$\left(1-\frac{1}{4\alpha}\right)I \leq C$$

for a suitable constant $C \in \mathbb{R}$. In particular, I converges at least for $\alpha > \frac{1}{4}$. □

Unlike the compact case, for a noncompact manifold, it is not guaranteed in general the existence of the Engliš expansion of the function ϵ_{kg} and only partial results in this direction are known (see e.g., [7] for the case of strongly pseudoconvex bounded domains in $\mathbb{C}^n$ with real analytic boundary). In [20, Theorem 6.1.1], X. Ma and G. Marinescu state sufficient conditions for the expansion to exist in a very general context. A version of their theorem adapted to our setting reads as follows (cf. [17, Theorem 7]):

Theorem 5 *Let (X, g, ω) be a complete Kähler manifold and let $(\hat{L}, \hat{h})$ be an hermitian line bundle on X. Then, $\epsilon_{\alpha g}$ admits an asymptotic expansion in α with coefficients given by 1.4 provided there exist constants $l > 0$ and $c > 0$ such that*

$$iR^{\hat{L}} > l\,\omega, \quad iR^{\det} > -c\,\omega, \quad |\partial\omega|_g < c, \tag{6.4}$$

where $R^{\det}$ denotes the curvature of the connection on $\det(T^{1,0}X)$ induced by g and $R^{\hat{L}}$ the curvature of the connection on $\hat{L}$ induced by the hermitian metric $\hat{h}$.

In the following theorem, we prove that conditions (6.4) hold for Hwang–Singer metrics based on a Kähler–Einstein polarized manifold. Recall that from [10] Section 2, the Ricci form ρ_φ of $\omega_{\varphi,\beta}$ is given by

$$\rho_\varphi = \pi^*\rho_M + \frac{1}{2Q}(\varphi Q)'(\tau)\pi^*\gamma - \frac{1}{2\varphi}\left[\frac{1}{Q}(\varphi Q)'\right]' d\tau\wedge d^c\tau,$$

thus, when ω_M is polarized, we have

$$\begin{aligned}\rho_\varphi &= \pi^*(\lambda\omega_M) + \frac{1}{2Q}(\varphi Q)'(\tau)\pi^*(\beta\omega_M) - \frac{1}{2\varphi}\left[\frac{1}{Q}(\varphi Q)'\right]' d\tau \wedge d^c\tau \\ &= \left(\lambda + \frac{\beta}{2Q}(\varphi Q)'\right)\pi^*\omega_M - \frac{1}{2\varphi}\left[\frac{1}{Q}(\varphi Q)'\right]' d\tau \wedge d^c\tau.\end{aligned} \tag{6.5}$$

Theorem 6 *Let $\omega_{\varphi,\beta}$ be the Hwang–Singer metric on a polarized line bundle over a Kähler–Einstein manifold with integral Kähler form. Then, if $\mathcal{H}_\alpha \neq \{0\}$, the Engliš expansion of the function $\epsilon_{\alpha g_{\varphi,\beta}}$ exists and the coefficients a_j are given by 1.4.*

Proof Let us check that conditions (6.4) hold for $\omega_{\varphi,\beta}$. The first condition is satisfied for $l \in (0, 1)$ since

$$iR^{\hat{L}} = -i\partial\bar{\partial}\log\hat{h} = -i\partial\bar{\partial}\log e^{-\frac{1}{2}\Psi} = \omega_{\varphi,\beta},$$

while the third condition is satisfied for every positive $c > 0$, since $\partial\omega_{\varphi,\beta} = 0$, being the metric Kähler. Let us now deal with the second condition. We want to show that there exists a positive $c > 0$ such that the form given by

$$iR^{\det} + c\omega_{\varphi,\beta} = \rho_\varphi + c\omega_{\varphi,\beta}$$

is positive. Thus, using 6.5 and 2.1, it is sufficient to show that there exists $c > 0$ such that

$$\left(\lambda + \frac{\beta}{2Q}(\varphi Q)' + c(1-\tau\beta)\right)\omega_M + \left(-\frac{1}{2\varphi}\left[\frac{1}{Q}(\varphi Q)'\right]' + \frac{c}{\varphi}\right) d\tau \wedge d^c\tau > 0.$$

Being $\lambda \geq 0$ and $\varphi > 0$, we show that there exists $c > 0$ such that

$$\begin{cases} \frac{\beta}{2Q}(\varphi Q)' + c(1-\tau\beta) > 0 \\ -\frac{1}{2}\left[\frac{1}{Q}(\varphi Q)'\right]' + c \geq 0, \end{cases}$$

namely, we want a positive c that satisfies

$$\begin{cases} c > -\frac{\beta}{2Q}(\varphi Q)'\frac{1}{1-\tau\beta} \\ c \geq \frac{1}{2}\left(\frac{1}{Q}(\varphi Q)'\right)'. \end{cases}$$

Since $\frac{1}{1-\tau\beta} \leq 1$, we reduce to prove that

$$\frac{(\varphi Q)'}{Q}, \quad \left(\frac{(\varphi Q)'}{Q}\right)'$$

are limited functions, proving the existence of such a c. Using the expression of the profile function 2.2 and since $Q(\tau) = (1-\beta\tau)^n$, we get

$$\begin{aligned}\frac{(\varphi Q)'}{Q} &= \frac{2\lambda}{\beta(1-\beta\tau)^n} - \frac{2\lambda}{\beta} + \frac{2}{(1-\beta\tau)^n}\\ &< -\frac{2\lambda}{\beta} + \frac{2}{(1-\beta\tau)^n}\\ &< -\frac{2\lambda}{\beta} + 2,\end{aligned}$$

and

$$\left(\frac{(\varphi Q)'}{Q}\right)' = \frac{2n(\beta+\lambda)}{(1-\beta\tau)^{n+1}} \leq 2n(\beta+\lambda),$$

concluding the proof. □

From the existence of an asymptotic expansion of the ϵ-function, it follows that the metric can be approximated by a sequence of projectively induced ones in the following way (cf. [17, Corollary 9]).

Lemma 5 *Let (M, g) be a polarized Kähler manifold such that the $1 \in \mathcal{H}$, where $\mathcal{H}$ is the weighted Hilbert space of holomorphic functions on M limited in norm. Then, the ϵ-function associated to g exists and, if it admits an asymptotic expansion whose coefficients are given by (1.4), then g can be approximated by a sequence of projectively induced Kähler metrics.*

Proof Denote by ω the Kähler form associated to g. Let $F_\alpha : M \to \mathbb{C}\mathrm{P}^{d_\alpha}$ be the coherent states map, i.e., $F_\alpha(x) = [\sigma_0(x) : \cdots : \sigma_j(x) : \dots]$, where $\{\sigma_j\}_{j=0,1,\dots}$ is an orthonormal basis of $\mathcal{H}$ such that $\sigma_0 \equiv 1$. Since $\mathcal{H} \neq \{0\}$, we can define the ϵ-function for g by (1.1), and we have

$$F_\alpha^* \omega_{FS} = \alpha\omega + \frac{i}{2}\partial\bar{\partial}\log\epsilon_{\alpha g}.$$

By (1.3), since $a_0 = 1$, we have that $\lim_{\alpha\to\infty} \frac{1}{\alpha} F_\alpha^*\, g_{FS} = g$. □

Remark 8 Observe that the assumption $1 \in \mathcal{H}$ is needed to define the coherent states map. When M is a compact polarized Kähler manifold, the existence of F_α is guaranteed by Kodaira's Theorem. In the noncompact case, one can always define the map F_α for example when g is regular, i.e., when $\epsilon_{\alpha g}$ is constant. In this case, (1.1) implies that for each $x \in M$ there exists a nonvanishing $\sigma_j(x)$.

We are now in the position of proving Theorem 1.

Proof of Theorem 1 By Lemma 2, we can reduce ourselves to prove that $\omega_{\varphi,\beta}$ is not projectively induced for a given value of β. By Lemma 1, a necessary condition for the metric $\omega_{\varphi,\beta}$ on $\mathbb{C}^{n+1}$ to be projectively induced is that $\beta n \geq -2$. Thus, it is enough to set $\beta < -\frac{2}{n}$.

The second part follows by Lemma 4, Theorem 6, and Lemma 5. □

Let us now complete the proof of Theorem 2.

Proof of Theorem 2 By Lemma 3, ω_k is not projectively induced for any $k \geq 3$. For $k = 2$, ω_2 is the Eguchi-Hanson metric on the canonical line bundle $\mathcal{O}(-2)$, that is not projectively induced as shown by Loi, Zedda, Zuddas in [18]. For $k = 1$, ω_1 is the Burns-Simanca metric that is projectively induced as shown by Cannas Aghedu and Loi in [6]. The second part follows by Lemma 4, Theorem 6, and Lemma 5. Finally, a direct computation (see Appendix A below) gives

$$a_2 = -\frac{2(k-1)\left(k^2\tau - 2k\tau - 2\right)}{(k\tau + 2)^6},$$

that is identically zero if and only if $k = 1$, concluding the proof. □

Remark 9 In [6], Cannas Aghedu and Loi showed that the Simanca metric g_1 is projectively induced, and this implies that any of its integer multiples kg_1 also are. We note here that these are the only possible multiples that can be Kähler immersed in $\mathbb{C}\mathrm{P}^\infty$. In fact, by momentum construction, the Simanca metric on $\mathcal{O}(-1)$ arises as a metric on a line bundle over $\mathbb{C}\mathrm{P}^1$. In particular, $\mathbb{C}\mathrm{P}^1$ is a Kähler submanifold of $\mathcal{O}(-1)$ (obtained setting the fiber coordinate $\xi = 0$) and the Fubini–Study form is not integral when multiplied by a noninteger factor.

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Appendix A: Computations of a_2

We compute here the a_2 coefficients for the metrics $\omega_{\varphi,\beta}$ in the case where the base manifold M is the complex projective line $\mathbb{C}\mathrm{P}^1$, completing the proofs of Theorem 2.

From (5.1), the metric g_k reads

$$g_k = \begin{pmatrix} \frac{k^2|z|^2 f''(t) + 8kf'(t) + 16}{(|z|^2+4)^2} & \frac{kzf''(t)}{\xi(|z|^2+4)} \\ \frac{k\bar{z}f''(t)}{\bar{\xi}(|z|^2+4)} & \frac{f''(t)}{|\xi|^2} \end{pmatrix}.$$

It follows that

$$\det(g_k) = \frac{\left(1 + \frac{k}{2}\tau\right)\varphi(\tau)}{|\xi|^2\left(1 + \frac{1}{4}|z|^2\right)},$$

and

$$g_k^{-1} = \begin{pmatrix} \frac{\left(|z|^2+4\right)^2}{8(kf'(t)+2)} & -\frac{k\bar{\xi}z\left(|z|^2+4\right)}{8(kf'(t)+2)} \\ -\frac{k\bar{z}\xi\left(|z|^2+4\right)}{8(kf'(t)+2)} & \frac{|\xi|^2\left(k^2|z|^2f''(t)+8kf'(t)+16\right)}{8f''(t)(kf'(t)+2)} \end{pmatrix}.$$

The norms of the Riemann and Ricci tensors are

$$|R|^2 = \frac{1}{16}\left(\frac{f^{(4)}(t)^2}{f''(t)^4} + \frac{f^{(3)}(t)^4}{f''(t)^6} - \frac{8(k^3f^{(3)}(t) - 2kf'(t) - 4)}{(kf'(t)+2)^3} + \frac{8k^4f''(t)^2}{(kf'(t)+2)^4} \right.$$
$$\left. - \frac{16k^2f''(t)}{(kf'(t)+2)^3} - \frac{2f^{(3)}(t)^2f^{(4)}(t)}{f''(t)^5} + \frac{4k^2f^{(3)}(t)^2}{f''(t)^2(kf'(t)+2)^2}\right),$$

and

$$|\mathrm{Ric}|^2 = \frac{1}{16}\left(\frac{16}{(kf'(t)+2)^2} + \frac{f^{(3)}(t)^4}{f''(t)^6} + \frac{2k^4f''(t)^2}{(kf'(t)+2)^4} - \frac{8k^2f''(t)}{(kf'(t)+2)^3} - \frac{2f^{(3)}(t)^2f^{(4)}(t)}{f''(t)^5}\right.$$
$$+ \frac{4k^2f^{(3)}(t)^2}{f''(t)^2(kf'(t)+2)^2} + \frac{2kf^{(3)}(t)f^{(4)}(t)}{f''(t)^3(kf'(t)+2)} - \frac{2k(kf^{(4)}(t) + 4f^{(3)}(t))}{f''(t)(kf'(t)+2)^2} + \frac{f^{(4)}(t)^2}{f''(t)^4} +$$
$$\left. - \frac{2kf^{(3)}(t)^3}{(kf'(t)+2)f''(t)^4}\right).$$

By (2.3) with $\varphi(\tau) = \frac{2\tau+\tau^2}{1+\frac{k}{2}\tau}$, the a_2 coefficient for the metrics ω_k on $\mathcal{O}(-k)$ is given by

$$a_2 = -\frac{2(k-1)\left(k^2\tau - 2k\tau - 2\right)}{(k\tau+2)^6}. \quad \text{(A.1)}$$

Remark 10 A similar computation for the Hwang–Singer metric on $\mathbb{C}^{n+1}$ gives:

$$a_2(0,1) = \frac{\beta^2}{4(1-\beta\tau)^{2(n+2)}}\left(\beta^2n^4\tau^2 + n(\beta^2 2^n\tau^2 + 2\beta(2^n+4)\tau + 2^n - 4) + 2^n(1-\beta^2\tau^2)\right.$$
$$\left. + \beta n^3\tau(\beta(2^n-2)\tau + 4) + \beta n^2\tau(\beta(2^n-3)\tau + 2\left(2^n+2\right))\right).$$

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