MEROMORPHIC LIMITS OF AUTOMORPHISMS

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MEROMORPHIC LIMITS OF AUTOMORPHISMS

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Abstract. Let X be a compact complex manifold in the Fujiki class $\mathcal{C}$. We study the compactification of $\mathrm{Aut}^0(X)$ given by its closure in Barlet cycle space. The boundary points give rise to non-dominant meromorphic self-maps of X. Moreover convergence in cycle space yields convergence of the corresponding meromorphic maps. There are analogous compactifications for reductive subgroups acting trivially on $\mathrm{Alb}\, X$. If X is Kähler, these compactifications are projective. Finally we give applications to the action of $\mathrm{Aut}(X)$ on the set of probability measures on X. In particular we obtain an extension of Furstenberg lemma to manifolds in the class $\mathcal{C}$.

Introduction

Let X be a compact complex manifold and assume that $\mathrm{Aut}^0(X)$, the connected component of $\mathrm{Aut}(X)$ containing the identity, is not trivial. It is interesting to consider pointwise limits of sequences $\{g_n\}$ in $\mathrm{Aut}^0(X)$. Even more interesting is the fact that such limits often exist! We first met this phenomenon in the case of a rational homogeneous space $X = G/P$. Fix an ample class on X and a Cartan involution θ on G. Call *self-adjoint* the elements $g \in G$ such that $\theta(g) = g^{-1}$. These elements form a submanifold of G diffeomorphic to the symmetric space G/K, where $K = \mathrm{Fix}(\theta)$. The ample class allows to fix a particular Satake compactification of G/K. One can prove that if a sequence $\{g_n\}$ of self-adjoint elements converges in the Satake compactification, then the maps $g_n : X \to X$ converge almost everywhere on X (with respect to smooth Lebesgue measures). The limit map is a rational self-map of X and one can describe it rather explicitely, see [6, §3.1]. In particular the pointwise limit of the maps g_n exists, it is holomorphic on a Zariski open subset of X and its image is contained in a proper subvariety of X.

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We later discovered that this phenomenon holds in greater generality. Assume that X is a Kähler manifold and that a compact connected subgroup $K \subset \mathrm{Aut}^0(X)$ acts on X in a Hamiltonian way, i.e. with a momentum mapping. If $\xi \in \mathfrak{k}$ and $x \in X$, then the limit

$$\lim_{t \to +\infty} \exp(it\xi) \cdot x \tag{1}$$

always exists and defines a limit map, see e.g. [7, Prop. 5.18]. This map is not continuous on the whole manifold X, but its restriction to a Zariski open subset is continuous and holomorphic [7, §5.20]. If we set $g_n(x) := \exp(it_n\xi) \cdot x$ for a sequence $\{t_n\}$ converging to $+\infty$, then we observe the same phenomenon as above: the pointwise limit of g_n exists and is holomorphic on a Zariski open subset of X. The proof of these facts relies heavily on the Linearization Theorem proved in the papers [24], [25], [26, §14]. As is well-known the flow $\exp(it\xi)$ in (1) is a Morse-Bott flow. It is interesting to notice that using quite different methods one can make sense of the limit for every Morse-Bott flow, see [23, 33].

In the present paper we study this phenomenon, that is the existence of the limit, in full generality:

Question 1. *Let X be a compact complex manifold and let $\{g_n\}$ be a sequence in $\mathrm{Aut}^0(X)$. For which $x \in X$ does the limit*

$$f(x) := \lim_{n \to \infty} g_n \cdot x$$

exist (up to passing to a subsequence)? What is the structure of the set of such points? What can be said about the limit map f?

The basic idea of our approach is simply to replace a biholomorphism of X by its graph. This idea goes back at least to Douady [15] and is of course common in many areas of mathematics. In the study of biholomorphism groups this idea has already been used very successfully by Barlet, Fujiki and Lieberman [17, 34, 3].

The graph of a biholomorphism is an analytic subvariety of $X \times X$. Subvarieties can be considered either as ideal sheaves, i.e. points in the Douady space (the Hilbert scheme in the projective case), or as cycles, i.e. points in the Barlet cycle space (the Chow scheme in the projective case). For our purposes the choice between these two approaches is not fundamental.

The manifolds for which we can answer the question above are those in Fujiki class $\mathcal{C}$: this class contains by definition all the manifolds that are meromorphic images of compact Kähler manifolds (see Definition 14 below). For these manifolds the irreducible components of both Douady and cycle space are compact. Let $B(X)$ (respectively $F(X)$) denote the irreducible component of the diagonal in the cycle space $C_n(X \times X)$, where $n = \dim X$ (resp. in the Douady space of $X \times X$). Thus $B(X)$ (resp. $F(X)$) is an analytic compactification of $\mathrm{Aut}^0(X)$. Some instances of this compactification have already been considered in the literature. For example Brion [11] has studied $B(X)$ in great detail the case where X is a rational homogeneous space. Using the compactness of $B(X)$ we prove the following result, which gives a rather complete answer to Question 1 for X in the class $\mathcal{C}$ (see §2, especially Theorems 20 and 21).

Theorem 2. *Let $X \in \mathcal{C}$ and let $\{g_n\}$ be a divergent sequence in* $\mathrm{Aut}^0(X)$*. Up to passing to a subsequence there are a meromorphic map $f : X \dashrightarrow X$ and a proper analytic subset $A \subset X$ such that*

(1) f is defined outside A;
(2) $g_n \to f$ uniformly on compact subsets of $X - A$;
(3) f is not dominant, i.e. $f(X)$ is contained in a proper subvariety of X.

An example of complex manifold not in the class $\mathcal{C}$ is provided by Hopf manifolds [43]. We are able to show that for such manifolds our result fails, see Remark 8.

In §3 we consider reductive subgroups of $\mathrm{Aut}^0(X)$. We recall several results from Fujiki's fundamental paper [17]. Fujiki used $F(X)$ instead of $B(X)$. We explain that they are equivalent for our purposes. It follows that for every connected complex reductive subgroup $G \subset \mathrm{Aut}^0(X)$ that acts trivially on $\mathrm{Alb}\, X$, the closure $\overline{G} \subset B(X)$ is analytic. (The corresponding statement in $F(X)$ was proved by Fujiki.) This allows to refine (3) in Theorem 2: if the sequence $\{g_n\}$ lies in G, then $f(X)$ is contained in the fixed set of a positive-dimensional subgroup of G.

The compactification of a reductive $G \subset \mathrm{Aut}^0(X)$ obtained in this way is quite interesting in its own. If X is Kähler we are able to prove the following (see Theorem 33).

Theorem 3. *If X is a Kähler manifold and $G \subset \mathrm{Aut}^0(X)$ is a connected complex reductive subgroup, that acts trivially on* $\mathrm{Alb}\, X$*, then the closure of G inside $B(X)$ is a projective variety.*

In §4 we apply Theorem 2 to study the action of $\mathrm{Aut}^0(X)$ on the set of probability measures on X. A famous lemma due to Furstenberg [21], which is used in the proof of Borel density theorem, says (among other things) that a measure on $\mathbb{P}^n$ whose stabilizer in $\mathrm{PGL}(n+1, \mathbb{C})$ is non-compact, is supported on a union of proper linear subspaces. The previous results allow to generalize this to any manifold in $\mathcal{C}$: a measure on X with non-compact stabilizer in $\mathrm{Aut}^0(X)$ is supported on a proper analytic subset (see Theorem 34).

Finally in Theorem 36 we give an application of the results obtained in the paper to the map F_ν, originally introduced by Bourguignon, Li and Yau [10] and studied in [6, 7]. We are able to give a much shorter proof of one of the main results in [7], although in a slightly less general setting.

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1. Notation and preliminaries

We start by recalling the basic definitions on meromorphic maps and some elementary lemmata needed in the paper. See [4, 16, 22, 38] for more details.

Definition 4. *Let X and Y be reduced complex spaces. A map $\tau : X \to Y$ is a* proper modification *if it is proper and there is an analytic subset $T \subset Y$ with empty interior such that*

(1) $\tau^{-1}(T)$ has empty interior and
(2) the restriction of τ to $X - \tau^{-1}(T)$ is a biholomorphism onto $Y - T$.

The center *of τ is the intersection of all the analytic subset $T \subset Y$ satisfying the above condition. The* exceptional set *of τ is the inverse image of the center.*

Definition 5. *Let X and Y be reduced complex spaces. A* meromorphic map *of X in Y is an analytic subset G of $X \times Y$ such that $p := \pi_1|_G : G \to X$ is a proper modification. If $S \subset X$ is the center of p then $f := \pi_2 \circ p^{-1} : X - S \to Y$ is a holomorphic map. We write $f : X \dashrightarrow Y$. The set G is called the* graph *of f and it is denoted by Γ_f. The* image *of f is $\pi_2(G) \subset Y$. The meromorphic map f is* surjective *if $\pi_2(G) = Y$. The center of p is called the* set of indeterminacy *of f, denoted* $\operatorname{indet}(f)$*, and its complement is called the* domain of definition *of f. We say that f is defined at $x \in X$ if x lies in the domain of definition.*

Remark 1. If $\tau : X \to Y$ is a proper modification and Y is irreducible, then also X is irreducible. In fact $Y - T$ is irreducible and so is $X - \tau^{-1}(T)$. Moreover $X - \tau^{-1}(T)$ is dense in X. As a corollary, if $f : X \dashrightarrow Y$ is a meromorphic map with graph G, and X irreducible, then G is irreducible.

Lemma 6. *Let X and Y be reduced and irreducible compact analytic spaces. Let $f : X \dashrightarrow Y$ be a meromorphic map with graph G and set of indeterminacy $S \subset X$. Then G is the closure of the graph of $f : X - S \to Y$.*

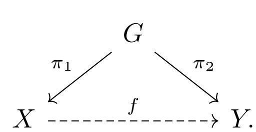
Proof. Since $f : X - S \to Y$ is a holomorphic map, its graph Γ_f is an analytic subset of $(X - S) \times Y$ and it is biholomorphic to $X - S$. By the definition of meromorphic map we have $\Gamma_f = G - (S \times Y)$. Therefore Γ_f is Zariski open in G. By the previous remark G is irreducible, so Γ_f is dense in G for the Hausdorff topology.

Lemma 7. *If X and Y are reduced and irreducible compact analytic spaces and $S \subset X$ is a proper analytic subset, a holomoprhic map $f : X - S \to Y$ is meromorphic if and only if the closure of its graph is an analytic subset of $X \times Y$.*

Proof. We already proved that the condition is necessary. To prove that it is sufficient, assume that $G := \overline{\Gamma_f}$ is analytic in $X \times Y$. Since G is compact the map $p := \pi_1|_G$ is proper. Moreover $\pi_1(G) = X$, since $\pi_1(G)$ is compact and contains $X - S$. Since X is irreducible, also Γ_f and G are irreducible. Finally $p^{-1}(S) = G \cap (S \times X)$ is a proper analytic subset of G, so it is nowhere dense. We have proved that $p : G \to X$ is a proper modification.

Lemma 8. *Let X and Y be reduced and irreducible compact analytic spaces and let $f : X \dashrightarrow Y$ be a meromorphic map. Let $A \subset X$ be a proper analytic subset containing* $\operatorname{indet}(f)$*. If $W \subset X$ is an irreducible analytic subset which is not contained in A, then $f(W - A)$ has analytic closure in Y.*

Proof. Let $G \subset X \times Y$ be the graph of f and let π_1, π_2 be the restrictions of the projections:



Let $\pi_1^{-1}(W) = Z_1 \cup \cdots \cup Z_r$ be the decomposition in irreducible components. Since W is irreducible, we can assume $\pi_1(Z_1) = W$. We claim that $\pi_2(Z_1) = \overline{f(W-A)}$. Indeed since W is irreducible, $W - A$ is also irreducible. Since π_1 is a biholomorphism over $X - A$, also $\pi_1^{-1}(W - A) \cong W - A$ is irreducible. Hence it is contained in a unique irreducible component of $\pi_1^{-1}(W)$, which is necessarily Z_1. This shows that $\pi_1^{-1}(W - A) \subset Z_1 - \pi_1^{-1}(A)$. The opposite inequality being obvious, we get $\pi_1^{-1}(W-A) = Z_1 - \pi_1^{-1}(A)$. Since Z_1 is irreducible, $\pi_1^{-1}(W-A)$ is dense in Z_1. So $f(W - A) = \pi_2\pi_1^{-1}(W - A) = \pi_2(Z_1 - \pi_1^{-1}(A))$ is dense in $\pi_2(Z)$. This means that the closure of $f(W - A)$ is the set $\pi_2(Z_1)$, which is analytic by Remmert Proper Mapping Theorem.

Lemma 9. *Let X and Y be reduced and irreducible compact analytic spaces and let $f : X \to Y$ be a holomorphic map. Let $B \subset Y$ be a proper analytic subset such that for any $y \in Y - B$, the fibre $f^{-1}(y)$ consists of a single point. Then f is a bimeromorphic map.*

Proof. Define $h : Y - B \to X$ by $h(y) := f^{-1}(y)$. Let $G \subset X \times Y$ denote the graph of f, which is an irreducible analytic subset of $X \times Y$. The map $t : X \times Y \to Y \times X$, $t(x, y) := (y, x)$ is a biholomorphism, so also $G' := t(G)$ is analytic and irreducible in $Y \times X$. The set $G' - \pi_1^{-1}(B)$ is Zariski open in G' and it coincides with the graph of h. By Lemma 7 we conclude that h extends to a meromorphic map $Y \dashrightarrow X$. By construction we have $hf = \mathrm{id}_X$ on $X - f^{-1}(B)$ (which is and nonempty and dense in X) and $fh = \mathrm{id}_Y$ on $Y - B$. Therefore h is a meromorphic inverse to f.

We will need the following classical result (see e.g. [41, Cor. 1.20 p. 108] and [13, p. 116]).

Theorem 10. *Let X and Y be compact complex spaces and let $f : X \to Y$ be a proper surjective holomorphic map. Assume that X and Y are reduced and irreducible. Then there are Zariski open subsets $Y^0 \subset Y$ and $X^0 \subset X$ such that $f(X^0) = Y^0$, both X^0 and Y^0 are non-singular and $f|_{X^0} : X^0 \to Y^0$ is a submersion with fibres of dimension equal to* $\dim X - \dim Y$.

We now recall the basic definitions related to Barlet cycle space.

Definition 11. *Let X be a reduced complex space. A n-cycle in X is a locally finite sum $Z = \sum_i n_i Z_i$ where $n_i \in \mathbb{N}$ and Z_i is an irreducible analytic subset of X of dimension n.*

The set of n-cycles in X will be denote by $C_n^{\mathrm{loc}}(X)$. A cycle is *compact* if the subsets Z_i are compact and $n_i \neq 0$ for only finitely many indices. The set of compact n-cycles in X will be denote by $C_n(X)$. It can be provided with the structure of a Banach analytic set. The irreducible components have finite dimension. A family of compact n-dimensional cycles in X parametrized by a topological space S is a map $f : S \to C_n(X)$. We also denote the family by $\{Y_s := f(s)\}_{s\in S}$. The family is called continuous if the corresponding map is continuous. It is called analytic if S is a complex space and the map is holomorphic.

The universal family of compact n-cycles in X is the analytic family corresponding to the identity map of $C_n(X)$ [4, p. 367].

An n-cycle Y on X has a well-defined multiplicity $\operatorname{mult}_x(Y)$ at every point $x \in X$ [4, p. 446].

Let $(Y_s)_{s\in S}$ be an analytic family of n-cycles on X. The *set-theoretic graph* of the family is the analytic subset

$$|G_S| := \{(s,x) \in S \times X : \operatorname{mult}_x(Y_s) \geq 1\}. \tag{2}$$

Let $|G_S| = \cup G_i$ be the decomposition in irreducible components. For each i the function $(s,x) \mapsto \operatorname{mult}_x(Y_s)$ has a generic value n_i on G_i. Then $G := \sum_i n_i G_i$ is the *graph* of the family. It is an $n+q$-cycle on $S \times X$, where S is reduced and has pure dimension q.

This cycle is compact for a continuous family of compact cycles $\{Y_s\}_{s\in S}$ if and only if S is compact.

Theorem 12 ([4, Thm. 3.3.1 p. 448]). *Let $Y_s = \sum_k n_{s,k} Z_{s,k}$ be the decomposition in irreducible components for a for very general $s \in S$. Then $n_{s,k} = n_i$ if $\{s\} \times Z_{s,k} \subset G_i$.*

Theorem 13 ([4, Thm. 3.4.1 p. 449]). *Let S be a normal complex space and let $G \in C^{\mathrm{loc}}_{n+q}(S \times X)$. Assume that the fibres of $\pi : |G| \to S$ have pure dimension n and that π is proper. Then there is a unique analytic family of cycles whose graph is G.*

Definition 14. *A complex manifold X is said to belong to the Fujiki class $\mathcal{C}$ if there is a compact Kähler manifold Y and a surjective meromorphic map $h : Y \dashrightarrow X$. By Hironaka's theorem one can assume that h is holomorphic. Moreover in [44, 5] it is proven that h can be assumed to be bimeromorphic. For more details see [18, §4.3],[43, 44, 5].*

The following result due to Campana and Fujiki is fundamental for the whole paper. See [4, p. 431] for a proof in the Kähler case and [19, 12] for the general case.

Theorem 15. *If X is a reduced complex space in class $\mathcal{C}$, then any irreducible component of $C_n(X)$ is compact.*

2. Limit maps for sequences in $\operatorname{Aut}^0(X)$

Let X be an n-dimensional compact connected complex manifold in the class $\mathcal{C}$. For $f \in \operatorname{Aut}^0(X)$, let $\Gamma_f \subset X \times X$ denote the graph of f. Since X is a connected manifold, the graph is smooth connected submanifold, hence an irreducible analytic subset. In particular $\Gamma_f \in C_n(X \times X)$. This yields a map

$$j : \operatorname{Aut}^0(X) \to C_n(X \times X), \qquad j(f) := \Gamma_f. \tag{3}$$

We denote by $B^0(X)$ the image of j and by $B(X)$ the closure of $B^0(X)$ in $C_n(X \times X)$. We will often identify $f \in \operatorname{Aut}^0(X)$ with $j(f)$ and consider $\operatorname{Aut}^0(X)$ as a subset of $B(X)$. The idea of replacing f by its graph goes back to [15] and has been used in [34] and [17]. Also the following Proposition has been proven in [17, 34].

Proposition 16. *The map j is a holomorphic embedding, $B(X)$ is an irreducible component of $C_n(X \times X)$ and $\partial B(X) := B(X) - B^0(X)$ is an analytic subset of $B(X)$.*

Proof. To prove that j is holomorphic it is enough to prove that the family of cycles $(\Gamma_f)_{f \in \mathrm{Aut}^0(X)}$ is analytic. Indeed $Z := \{(f, x, y) \in \mathrm{Aut}^0(X) \times X \times X : f(x) = y\}$ is a complex submanifold of $\mathrm{Aut}^0(X) \times X \times X$ biholomorphic to $\mathrm{Aut}^0(X) \times X$, hence irreducible. By Theorem 13 it defines an analytic family of compact cycles, which corresponds to the map j. The image of j is contained in a unique irreducible component of $C_n(X \times X)$ that we denote by $B(X)$. The rest is proven in [34, Prop. 2.1].

It follows from Theorem 15 that $B(X)$ is a compact irreducible analytic space. In fact it belongs to class $\mathcal{C}$ [12, Cor. 3]. The inclusion $B(X) \hookrightarrow C_n(X \times X)$ corresponds to a family of n-cycles on $X \times X$ that we denote by $\{Y_b\}_{b \in B(X)}$. In other words $\{Y_b\}_{b \in B(X)}$ is the restriction of the universal family of cycles to $B(X) \subset C_n(X \times X)$. Let $G_{B(X)}$ be the graph of the family $\{Y_b\}_{b \in B(X)}$.

Lemma 17. *For any $b \in B(X)$ and any $x \in X$ the intersection $Y_b \cap (\{x\} \times X)$ is non-empty. It either contains a component of positive dimension or it reduces to a single point. In the latter case this point is a smooth point of Y_b, at which Y_b and $\{x\} \times X$ is intersect transversally.*

Proof. Since $B(X)$ is connected, the homology class of Y_b is constant for $b \in B(X)$. In particular it coincides with the homology class of the diagonal Δ, which is the graph of the identity map of X. Setting for simplicity $F_x := \{x\} \times X$, in the homology ring of $X \times X$ we have for $b \in B(X)$

$$[Y_b] \cdot [F_x] = [\Delta] \cdot [F_x].$$

Since Δ and F_x intersect only at (x, x) and the intersection is transverse, $[\Delta] \cdot [F_x] = 1$ and therefore $[Y_b] \cdot [\{x\} \times X] = 1$. It follows immediately that $Y_b \cap F_x \neq \varnothing$. This intersection is a compact analytic subset of $X \times X$. If there are no components of positive dimension, then $Y_b \cap F_x = \{p_1, \dots, p_k\}$. So Y_b and F_x intersect *properly* and

$$1 = [Y_b] \cdot [F_x] = \sum_{i=1}^{k} I(p_i, Y_b, F_x, X \times X).$$

Since $I(p_i, Y_b, F_x, X \times X) \geq 1$, we conclude that $k = 1$, i.e. $Y_b \cap F_x = \{p_1\}$ and also that $I(p_1, Y_b, F_x, X \times X) = 1$. It follows that both Y_b and F_x are smooth at p_1 and that they are transversal, see [20, p. 137-138].

Given spaces $X_1, X_2, \dots X_n$ we denote by π_i and $\pi_{i,j}$ the natural projections

$$\pi_i : X_1 \times X_2 \times \cdots \times X_n \longrightarrow X_i$$
$$\pi_{ij} : X_1 \times X_2 \times \cdots \times X_n \longrightarrow X_i \times X_j$$

Lemma 18. *Assume that $X \in \mathcal{C}$. Set*

$$\psi := \pi_{12}|_{|G_{B(X)}|} : |G_{B(X)}| \longrightarrow B(X) \times X, \quad \psi(b, x_1, x_2) := (b, x_1).$$

(i) *The map ψ is onto.*
(ii) *The set $\Omega := \{(b,x) \in B(X)\times X : |\psi^{-1}(b,x)| = 1\}$ is Zariski open in $B(X)\times X$. (By $|\psi^{-1}(b,x)|$ we denote the cardinality of the fibre of ψ.)*
(iii) *The restriction $\psi|_{\psi^{-1}(\Omega)} : \psi^{-1}(\Omega) \longrightarrow \Omega$ is a homeomorphism.*
(iv) *If $(b,x) \in \Omega$, then there is an open neighbourhood U of x in X and a holomorphic function $\varphi : U \to X$ such that $Y_b \cap (U\times X)$ coincides with the graph of φ.*
(v) *$B^0(X)\times X \subset \Omega$.*
(vi) *The set-theoretic graph $|G_{B(X)}|$ is irreducible and $G_{B(X)} = |G_{B(X)}|$.*
(vii) *For any $b \in B$ we have $\Omega \cap (\{b\}\times X) \neq \varnothing$.*
(viii) *If $b \in B(X)$ there is one and only one irreducible component Z_b of Y_b such that $\pi_1(Z_b) = X$. This component has multiplicity 1 in Y_b and it is the graph of a meromorphic map $f_b : X \dashrightarrow X$.*

Proof. Recall that $|G_{B(X)}| = \{(b,x_1,x_2) \in B(X)\times X\times X : (x_1,x_2)\in Y_b\}$. So for any $x_1 \in Y_b$

$$\psi^{-1}(b,x_1) = \{b\}\times (Y_b \cap (\{x_1\}\times X)). \tag{4}$$

Thus (i) follows directly from Lemma 17. Next set

$$\Sigma_1(\psi) := \{(b,x_1,x_2) \in |G_{B(X)}| : \dim_{(b,x_1,x_2)} \psi^{-1}(b,x_1) \geq 1\},$$

Since ψ is a proper holomorphic map between reduced complex spaces [4, Thm. II.4.5.3 p. 179] ensures that $\Sigma_1(\psi)$ is an analytic subset of $|G_{B(X)}|$. Since ψ is proper, its image $Z := \psi(\Sigma_1(\psi))$ is also an analytic set by Remmert Proper Mapping theorem. Its complement $\Omega' := B(X)\times X - Z$ is Zariski open and it contains exactly the points of $B(X)\times X$ whose fibre (for ψ) is 0-dimensional. Using (4) and Lemma 17 we conclude that $\Omega' = \Omega$. This proves (ii). Since X, $B(X)$ and $|G_{B(X)}|$ are compact Hausdorff spaces, the map ψ is closed. Hence the same holds for the restriction $\psi|_{\psi^{-1}(\Omega)} : \psi^{-1}(\Omega) \to \Omega$. Since this map is by construction a continuous bijection of $\psi^{-1}(\Omega)$ onto Ω, it is a homeomorphism. This proves (iii). Let $(b,x) \in \Omega$ and assume $Y_b \cap (\{x\}\times X) = \{(x,x')\}$. It follows from Lemma 17 that Y_b is smooth at (x,x') and transverse to $\{x\}\times X$. Hence there is a neighbourhood V of (x,x') in Y_b such $\pi_1|_V$ is a biholomorphism onto a neighbourhood $U \subset X$ of x. Set $\varphi := \pi_2 \circ (\pi_1|_V)^{-1} : U \to X$. Then $V = \Gamma_\varphi$. But $U \subset \Omega$, so $V = Y_b \cap (U\times X)$. This proves (iv). (v) is obvious.

If $b \in B^0(X)$, then Y_b has a unique component of multiplicity 1. Therefore the definition (2) of $G_{B(X)}$ and Theorem 12 imply that $G_{B(X)}$ has a unique component of multiplicity 1, i.e. (vi) holds.

If $b \in B^0(X)$ we have $\{b\}\cap X \subset \Omega$. Assume $b \in \partial B(X)$. By (i) $\pi_1|_{|Y_b|} : Y_b \to X$ is onto. If every fibre had positive dimension, Theorem 10 would imply that $\dim |Y_b| \geq \dim X + 1$, which is absurd. So the fibre over some $x \in X$ has dimension 0. By Lemma 17 $(b,x) \in \Omega$. This proves (vii).

Let $Y_b = \sum_{i=1}^r n_i Z_i$ be the decomposition in irreducible components. Since $\cup_i \pi_1(Z_i) = \pi_1(|Y_b|) = X$, there is at least one index i, such that $\pi_1(Z_i) = X$. Set $T := \{x \in X : (b,x) \notin \Omega\}$. By (ii) T is an analytic subset of X and by (vii) it is a proper subset. If $x \in X - T$, then there is exactly one $y \in X$ such that $(x, y) \in Y_b$. Necessarily $(x, y) \in Z_i$ and $x \notin \pi_1(Z_j)$ for $j \neq i$. This shows that the component Z_i is unique and also that $\pi_1(Z_j) \subsetneq X$ for $j \neq i$. Denote by Z_b the component Z_i. By Theorem 10 applied to $p := \pi_1|_{Z_b} : Z_b \to X$ there are Zariski open subsets $Z^0 \subset Z_b$ and $X^0 \subset X$, such that both Z^0 and X^0 are smooth and $p : Z^0 \to X^0$ is a local biholomorphism. We can assume that $X^0 \subset X - T$. So $p|_{Z^0}$ is injective, hence a biholomorphism. It follows that $p : Z_b \to X$ is a modification with center T, hence $f_b := \pi_2 \circ p^{-1} : X \dashrightarrow X$ is a meromorphic map and the graph of f_b coincides with Z_b by Lemma 6.

Remark 2. In general the map in (iii) is not necessarily a biholomorphism. The point is that a bijective holomorphic is automatically biholomorphic only if the target is weakly normal, see e.g. [4, p. 310-11 and p. 358]. So one can only assert that $\psi|_{\psi^{-1}(\Omega)}$ is a biholomorphism on the weak normalization of $B(X)$. This kind of problem is quite common in the study of cycle spaces. Indeed the weak normalization goes back to [2].

Definition 19. *For $b \in B(X)$ we will denote by Z_b be the unique irreducible component of Y_b such that $\pi_1(Z_b) = X$. We will call Z_b the* meromorphic component *of Y_b. We will denote by f_b the meromorphic map such that $\Gamma_{f_b} = Z_b$.*

We have $b \in B^0(X)$ iff $f_b \in \mathrm{Aut}^0(X)$. We also denote by A_b the set of points $x \in X$ such that $(\{x\} \times X) \cap Y_b$ contains more than one point. This means that

$$\{b\} \times (X - A_b) = \Omega \cap (\{b\} \times X). \tag{5}$$

In other words, if $Y_b = Z_b + \sum_{i=1}^r n_i Z_i$, then

$$A_b := \mathrm{indet}(f_b) \cup \bigcup_{i=1}^r \pi_1(Z_i).$$

The intersection $Y_b \cap ((X - A_b) \times X)$ is the graph of the holomorphic map $f_b|_{X - A_b}$. Let $\mathcal{M}(X)$ denote the set of meromorphic self-maps of X. We have constructed a map

$$\Phi : B(X) \to \mathcal{M}(X), \quad \Phi(b) := f_b. \tag{6}$$

Remark 3. In general the map Φ is not injective: different points $b, b' \in \partial B := B(X) - B^0(X)$ can have the same meromorphic components, i.e. $Z_b = Z_{b'}$. The fibres of the map (6) can be even of positive dimention. We describe such an example for $X = \mathbb{P}^n$ based on the results of Brion [11, p. 621-622]. Set $V = \mathbb{C}^{n+1}$ and $X = \mathbb{P}^n = \mathbb{P}(V)$. Fix a basis $\{v_1, \dots, v_{n+1}\}$ of V. Let $J = \{j_1 < \dots < j_r\}$ be a subset of $\{1, \dots, n\}$. Define

$$\begin{aligned} V_0 &:= \mathrm{span}\,(v_1, \dots, v_{j_1}), \\ V_i &:= \mathrm{span}\,(v_{j_i+1}, \dots, v_{j_{i+1}}), \text{ for } 1 \le i < r, \\ V_r &:= \mathrm{span}\,(v_{j_r+1}, \dots, v_{n+1}), \end{aligned}$$

$$V_{<k} := \bigoplus_{i<k} V_i, \quad V_{>k} := \bigoplus_{i>k} V_i, \quad \text{for } k = 0, \dots, r,$$

$$\tilde{Z}_i = \{(x, y, \ell) \in \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}(V_i) : x \in \mathbb{P}(V_{<i} + \ell), y \in \mathbb{P}(V_{>i} + \ell)\},$$
$$\text{for } i = 0, \dots, r.$$

Denote by $\pi_{12} : \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}(V_i) \longrightarrow \mathbb{P}^n \times \mathbb{P}^n$ the projection. Then the map

$$\rho_i := \pi_{12}|_{\tilde{Z}_i} : \tilde{Z}_i \longrightarrow Z_i := \pi_{12}(\tilde{Z}_i)$$

is a modification. Set

$$\Gamma_J := \sum_{i=0}^{r} Z_i \in C_n(X \times X).$$

We have $\pi_1(Z_i) = X$ iff $i = r$ and $\pi_2(Z_i) = X$ iff $i = 0$. Thus the meromorphic component of Γ_J is Z_r. Since

$$\tilde{Z}_r = \{(x, y, y) \in \mathbb{P}^n \times \mathbb{P}(V_r) \times \mathbb{P}(V_r) : x \in \mathbb{P}(V_{<r} + y)\},$$
$$Z_r = \{(x, y) \in \mathbb{P}^n \times \mathbb{P}(V_r) : x \in \mathbb{P}(V_{<r} + y)\},$$

the meromorphic component Z_r only depends on $V_{<r}$ and V_r and there are infinitely many cycles $b \in B(X)$ sharing some meromorphic component.

Remark 4. The fibres of the map Φ in (6) give an equivalence relation $\sim$ on $B(X)$ and it would be nice to prove that the quotient of $B(X)$ with respect to this equivalence relation has the structure of complex analytic space. This is indeed the case when $X = \mathbb{P}^n$. In fact, as shown above, the meromorphic component of a cycle Γ_J depends only on $V_{<r}$ e V_r. Moreover Γ_J coincides with the graph of the projection onto $\mathbb{P}(V_r)$ with centre $\mathbb{P}(V_r)$. To get the whole of $B(X)$ we let $\mathrm{GL}(n+1, \mathbb{C})$ act on the left and on the right on the various cycles Γ_J. In this way we get the graphs of all the elements of $\mathbb{P}(M_{n+1}(\mathbb{C}))$. Thus in this case $B(X)/\sim = \mathbb{P}(M_{n+1}(\mathbb{C}))$. Unfortunately dealing with the general case seems rather delicate. The fibres of Φ can be of different dimensions, by the previous remark. So [30, Satz 1(b)] shows that in general the relation $\sim$ is not open. Therefore to prove that $B(X)/\sim$ is a complex space one cannot apply directly the main theorem of [30], which says that the quotient of a seminormal complex space by an open analytic relation is a complex space.

Remark 5. In a series of papers Neretin gave a new construction of compactifications of reductive groups and symmetric spaces. In particular he gave a compactification of $\mathbb{P}\mathrm{GL}(n+1, \mathbb{C})$ via so-called *hinges*, see [36, 35]. This compactfication is a semigroup and it coincides with the De Concini-Procesi compactification [14]. By Brion's results [11] it also coincides with $B(X)$ for $X = \mathbb{P}^n$. It would be very interesting to see if also for a general X the space $B(X)$ or some compactification related to it is a semigroup. This would be related to the philosophy put forward at pages 1 and 9-11 of [37]. We hope to come back to these questions in the future.

Consider now the following action of $\mathrm{Aut}^0(X)$ on $X \times X$:

$$g \cdot (x, y) := (x, g \cdot y).$$

This action induces a corresponding action on $C_n(X \times X)$: for $\Gamma \in C_n(X \times X)$ set

$$g \cdot \Gamma := (\mathrm{id}_X \times g)_* \Gamma. \tag{7}$$

This action preserves $B(X)$.

Theorem 20. *For $b \in \partial B(X)$ the stabilizer $\mathrm{Aut}^0(X)_b$ for the action (7) has positive dimension. Moreover $f_b(X) = \pi_2(Z_b) \subset X^{\mathrm{Aut}^0(X)_b}$. In particular $f_b : X \dashrightarrow X$ is non-dominant.*

Proof. The map j of (3) is equivariant with respect to the action of $\mathrm{Aut}^0(X)$ on itself by left multiplication and the action (7) on $C_n(X \times X)$:

$$j(gh) = \Gamma_{gh} = g \cdot \Gamma_h = g \cdot j(h).$$

Thus $B^0(X) = j(\mathrm{Aut}^0(X))$ is an orbit of $\mathrm{Aut}^0(X)$. We know from Proposition 16 that $B(X)$ is irreducible and that $\partial B(X) := B(X) - B^0(X)$ is a proper analytic subset of $B(X)$. Hence any irreducible component of $\partial B(X)$ has dimension strictly less than $\dim B(X)$. Since $\partial B(X)$ is invariant by the action, it follows that for $b \in \partial B(X)$, $\dim \mathrm{Aut}^0(X) \cdot b < \dim B^0(X) = \dim \mathrm{Aut}^0(X)$, so $\dim \mathrm{Aut}^0(X)_b > 0$.

Denote by Y_b the cycle corresponding to b and let Z_b be the meromorphic component. If $g \in \mathrm{Aut}^0(X)$, then clearly $\pi_1(g \cdot \Gamma_b) = \pi_1(\Gamma_b)$. Thus for $g \in \mathrm{Aut}^0(X)_b$, $g \cdot Z_b = Z_b$.

Let $Y := \mathrm{indet}(f_b) \subset X$ be the indeterminacy locus of $f_b : X \dashrightarrow X$. If $x \in X - Y$, then $(\{x\} \times X) \cap Z_b = \{(x, f_b(x))\}$. If $h \in \mathrm{Aut}^0(X)_b$, then $h \cdot (x, f(x)) = (x, hf(x)) \in Z_b$, so $hf(x) = f(x)$. This shows that $f_b(X - Y) \subset X^{\mathrm{Aut}^0(X)_b}$. Since Z_b is the closure of $\{(x, y) \in (X - Y) \times X : y = f_b(x)\}$, we conclude that $\pi_2(Z_b) \subset X^{\mathrm{Aut}^0(X)_b}$.

Finally, since $\mathrm{Aut}^0(X)_b$ has positive dimension, it is not the trivial subgroup, so $X^{\mathrm{Aut}^0(X)_b}$ is a proper analytic subset of X. Therefore the image of f_b is strictly smaller than X.

Remark 6. A refinement of this theorem in the case of a reductive subgroup is given by Theorem 32 below.

Theorem 2 in the Introduction follows from the previous theorem together with the following one.

Theorem 21. *Let X be a compact complex manifold in the class $\mathcal{C}$. Let $\{b_j\}$ be a sequence in $B(X)$ converging to $b \in B(X)$. Then $f_{b_j} \to f_b$ uniformly on compact subsets of $X - A_b$. In particular, if $\{g_j\}$ is a sequence in $\mathrm{Aut}^0(X)$, passing to a subsequence we can find $b \in B(X)$ such that $g_j \to f_b$ uniformly on compact subsets of $X - A_b$.*

Remark 7. In general the set A_b is larger than the indeterminacy set of f_b and the convergence holds only on $X - A_b$. For example if $X = \mathbb{P}^1$ and g_j is the map $g_j(z) = j \cdot z$, then f_b maps every point of $\mathbb{P}^1$ to ∞ and has no indeterminacy point, but convergence does not hold at $0 \in A_b$.

We start the proof with the following elementary observation.

Lemma 22. *Let X and Y be topological spaces and let (Z, d) be a metric space. Let $h : X \times Y \to Z$ be a continuous map. Let $\{x_n\}$ be a sequence in X converging to $\bar{x} \in X$. Set*

$$f_n(y) := h(x_n, y), \quad \bar{f}(y) := h(\bar{x}, y).$$

If Y is compact, $f_n \to \bar{f}$ uniformly on Y.

Proof. Fix $\varepsilon > 0$. Given $y_0 \in Y$, continuity of h yields open neighbourhoods U of $\bar{x}$ in X and V of y_0 in Y, such that $d(h(x,y), h(\bar{x}, y_0)) < \varepsilon/2$ for any $(x,y) \in U \times V$. Since Y is compact we can cover it with a finite number of neighbourhoods like V, that is we can find a list $\{(U_i, V_i, y_i)\}_{i=1}^n$ such that U_i is open in X, V_i is open in Y, $\bar{x} \in U_i$, $y_i \in V_i$, $\cup_i V_i = Y$ and

$$d(h(x,y), h(\bar{x}, y_i)) < \frac{\varepsilon}{2}, \quad \forall (x,y) \in U_i \times V_i. \tag{8}$$

Then $W := \cap_i U_i$ is a neighbourhood of $\bar{x}$, so there is n_0 such that for any $n \geq n_0$, $x_n \in W$. If $y \in Y$, there is i such that $y \in V_i$. Hence for $n \geq n_0$ using twice (8) we get

$$d(h(x_n, y), h(\bar{x}, y)) \leq d(h(x_n, y), h(\bar{x}, y_i)) + d(h(\bar{x}, y), h(\bar{x}, y_i)) < \varepsilon.$$

Proof of Theorem 21. Fix a compact subset $K \subset X - A_b$. By (5) this means that $\{b\} \times K \subset \Omega$, so there is an open subset $V \subset B(X)$ such that $V \times K \subset \Omega$. There is n_0 such that $b_j \in V$ for $n \geq n_0$. Recall from Lemma 18 (iv) that $\psi|_{\psi^{-1}(\Omega)}$ is a homeomorphism. In particular we can invert $f := \psi|_{V \times K}$. Hence we have a well-defined map

$$h := \pi_3 \circ f^{-1} : V \times K \longrightarrow X.$$

By Lemma 21 $h(b_j, \cdot) \to h(b, \cdot)$ uniformly on X (with respect to any metric inducing the topology). But if $n \geq n_0$, $\{b_j\} \times K \subset \Omega$, i.e. $K \subset X - A_{b_j}$. Hence $h(b_j, \cdot) = f_{b_j}$ and $h(b, \cdot) = f_b$. We have proved that $f_{b_j} \to f_b$ uniformly on K.

Remark 8. It is important to notice that Theorem 21 does not hold without the hypothesis $X \in \mathcal{C}$. Consider the following example already studied in [43]. Set $W := \mathbb{C}^2 - \{0\}$ and choose $\alpha \in \mathbb{C}$ with $0 < |\alpha| < 1$. Let α act on W by the rule $\alpha \cdot (x,y) := (\alpha x, \alpha y)$. Then $H_\alpha := W/\langle \alpha \rangle$ is a Hopf surface and $\mathrm{Aut}(H_\alpha) = \mathrm{GL}(2, \mathbb{C})/\langle \alpha \rangle$. Set

$$g := \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix},$$

and consider the sequence $\{g^n\}$ in $\mathrm{Aut}(H_\alpha)$. Set $E_1 = \{[x,y] \in H_\alpha : x = 0\}$ and $E_2 = \{[x,y] \in H_\alpha : y = 0\}$. These are elliptic curves isomorphic to $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}a)$ where $\exp(2\pi i a) = \alpha$. It is easy to check that for $p = [x,y] \notin E_1$ we have $g^n(p) \to \varphi(p) := [x, 0] \in E_2$. While for $p = [0,y] \in E_1$, $g^n(p) = p$. So the limit exists for every $p \in H_\alpha$. On the other hand the map $\varphi : H_\alpha - E_1 \to E_2$ is not meromorphic. In fact call Γ its graph. We claim that $\overline{\Gamma} = \Gamma \cup E_1 \times E_2$. It is clear that $\Gamma \subset H_\alpha \times E_2$ and that Γ is closed in $(H - E_1) \times E_2$. Moreover if $([0,y],[u,0]) \in E_1 \times E_2$, then

$$([\alpha^n u, y], \varphi([\alpha^n u, y])) = ([\alpha^n u, y], [u, 0]) \to ([0,y],[u,0]).$$

So $E_1 \times E_2 \subset \overline{\Gamma}$. This proves that indeed $\overline{\Gamma} = \Gamma \cup E_1 \times E_2$. Now we show that $\overline{\Gamma}$ is not analytic. Call $\pi : W \times W \to H_\alpha \times H_\alpha$ the projection. Fix $p_0 = ([0, y_0], [u_0, 0]) \in$

$E_1 \times E_2$. Let U be a small neighbourhood of $(0, y_0, u_0, 0)$ in $W \times W$ such that $\pi|_U$ is a biholomorphism. Then

$$\pi^{-1}\overline{\Gamma} \cap U = (\{(0, y, u, 0)\} \cap U) \cup \bigcup_{n \in \mathbb{Z}} (\{(x, y, \alpha^n x, 0)\} \cap U),$$

which is not analytic. One can also deduce that $\overline{\Gamma}$ is not analytic from the fact that $E_1 \times E_2 \subset \overline{\Gamma} - \Gamma$ and $\dim E_1 \times E_2 = \dim \Gamma$.

Remark 9. In the literature there are several notions of convergence for meromorphic maps, see for example [28, 29]. It would be interesting to compare the convergence in $B(X)$ with these notions of convergence. We leave this for further inquiry.

3. Compactifications of reductive subgroups

In this section we consider complex reductive subgroups of $\mathrm{Aut}^0(X)$. Since we will only consider *complex* reductive subgroups, we will often refer to them simply as *reductive subgroups* of $\mathrm{Aut}^0(X)$.

Our goal is to construct compactifications of the connected reductive subgroups of $\mathrm{Aut}^0(X)$ that act trivially on $\mathrm{Alb}\, X$. We will take advantage of Fujiki's deep work in [17]. We start by recalling some definitions introduced in that paper.

Let G be a connected complex Lie group. A *meromorphic structure* on G is an analytic compactification G^* (i.e. a compact analytic space G^* containing G as a dense open subset) such that the product map and the inversion extend as meromorphic maps $G^* \times G^* \dashrightarrow G^*$ and $G^* \dashrightarrow G^*$. Two such structures G^* and G^{**} are *equivalent* if id_G extends to a bimeromorphic map $G^* \dashrightarrow G^{**}$. An equivalence class of meromorphic structures is called a *meromorphic group*. We will denote a meromorphic group by G or G^* or (G, G^*).

If G^* is a meromorphic structure on G, a subgroup $H \subset G$ is *meromorphic* if the closure of H in G^* is an analytic subset. If G^{**} is another meromorphic structure which is equivalent to G^*, then H is a meromorphic subgroup with respect to G^* iff it is meromorphic with respect to G^{**}. To prove the last statement one uses Lemma 8. Thus the notion of meromorphic subgroup depends only on the ambient meromorphic group.

If G is a linear algebraic group over $\mathbb{C}$, then it has a canonical meromorphic structure given by taking a faithful representation of $G \to \mathrm{SL}(V)$ and letting G^* be the closure of G inside $\mathbb{P}(\mathrm{End}\, V)$. This structure is well-defined, i.e. does not depend on the choice of the representation [17, Rmk. 2.3]. When G is endowed with this structure we say that it is *meromorphically linear*.

If G is a connected complex Lie group with a meromorphic structure G^* and X is a compact complex space we say that an action $\sigma : G \times X \longrightarrow X$ of G on X is *meromorphic* if σ extends to a meromorphic map $G^* \times X \dashrightarrow X$.

Proposition 23. *Let (G, G^*) be a meromorphic group. Assume that G acts on the compact complex spaces X and Y and that $f : X \dashrightarrow Y$ is a G-equivariant bimeromorphic map. Then the action on X is meromorphic iff the action on Y is meromorphic.*

Proof. Let X^0 and Y^0 be Zariski open subsets such that $f : X^0 \longrightarrow Y^0$ is a biholomorphism. Equivariance is understood in the following sense: if $x \in X^0$ and $g \cdot x \in X^0$, then $f(g \cdot x) = g \cdot f(x)$. Denote by $\sigma : G \times X \to X$ the action on X and by $\tau : G \times Y \to Y$ that on Y. Set $F := \mathrm{id}_{G^*} \times f \times f : G^* \times X^2 \dashrightarrow G^* \times X^2$. Consider the set $\Gamma' := \{(g, x_1, x_2) \in G \times X^2 : x_1, x_2 \in X^0, x_2 = g \cdot x_1\}$. Since $\Gamma' = \Gamma_\sigma \cap (G \times X^0 \times X^0)$, it is a Zariski open subset of Γ_σ. It is clearly non-empty since $(1, x, x) \in \Gamma'$ for any $x \in X^0$. Therefore it is dense in Γ_σ. The same holds for $\Gamma'' = \Gamma_\tau \cap (G \times Y^0 \times Y^0)$: this is a dense Zariski open subset of Γ_τ. The map F is defined on Γ'. The equivariance and the hypothesis on X^0 and Y^0 imply that $F(\Gamma') = \Gamma''$. Denote by W' the closure of Γ_σ in $G^* \times X \times X$. If the action of G on X is meromorphic, W' is an analytic subset of $G^* \times X \times X$ by Lemma 6. Since Γ_σ is closed in $G \times X \times X$, we have $W' \cap (G \times X \times X) = \Gamma_\sigma \cap (G \times X \times X)$. Let A be the complement of $G \times X^0 \times X^0$ in $G^* \times X \times X$. A is an analytic subset and it contains $\mathrm{indet}(F)$. The set W' is irreducible and it is not contained in A. So Lemma 8 implies that $W'' := \overline{F(W' - A)}$ is an analytic subset of $G^* \times Y \times Y$. But by the definition of Γ' we have $\Gamma' = W' - A$. So $W'' = \overline{\Gamma''}$. But we know that Γ'' is dense in Γ_τ, so $\Gamma'' \subset \Gamma_\tau \subset \overline{\Gamma''}$. This finally shows that $\overline{\Gamma}_\tau = W''$ is analytic, i.e. the action on Y is meromorphic.

Assume that X is a compact complex manifold. Let $F(X)$ denote the irreducible component of the Douady space $\mathcal{D}(X \times X)$ containing the diagonal Δ. We let $F(X)_{\mathrm{red}}$ denote the reduction of $F(X)$. We recall some fundamental results of Fujiki.

Theorem 24 (Fujiki). *If $X \in \mathcal{C}$, then $F(X)_{\mathrm{red}}$ is a meromorphic structure on $\mathrm{Aut}^0(X)$, called the* natural *meromorphic structure. Moreover there is an exact sequence of meromorphic groups*

$$0 \to L(X) \to \mathrm{Aut}^0(X) \xrightarrow{\alpha} T(X) \to 0$$

where $L(X)$ is meromorphically linear and $T(X)$ is a torus.

See [17, Prop. 2.2 p. 231] and [17, Thm. 5.5]. If $H \subset \mathrm{Aut}^0(X)$, we say that H is a *meromorphic subgroup with the natural structure* if it is a meromorphic subgroup of $F(X)_{\mathrm{red}}$, i.e. if $\bar{H}$ is an analytic subset of $F(X)_{\mathrm{red}}$.

Let $\mathrm{Alb}\, X$ be the Albanese torus of X. Since $\mathrm{Alb}\, X$ is a compact torus, the group $A(X) := \mathrm{Aut}^0(\mathrm{Alb}\, X)$ is simply the group of translations of $\mathrm{Alb}\, X$. If $x_0 \in X$ is fixed, one defines an Albanese map $\mathrm{alb}_X : X \to \mathrm{Alb}\, X$ with $\mathrm{alb}(x_0) = 0$ and a homomorphism

$$\mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(\mathrm{Alb}\, X), \quad g \mapsto A_g$$

such that $\mathrm{alb}_X \circ g = A_g \circ \mathrm{alb}_X$ for every $g \in \mathrm{Aut}(X)$ [1, p. 101]. The *Jacobi morphism* $\varphi_* : \mathrm{Aut}^0(X) \longrightarrow A(X)$ is defined as the restriction of the morphism $g \mapsto A_g$ to the connected components of the identity.

Proposition 25 ([17, Thm. 5.5 (2) p. 251]). *If $X \in \mathcal{C}$, then $L(X)$ is a finite index subgroup of $\ker \varphi_*$.*

Corollary 26. *If $X \in \mathcal{C}$ and $G \subset \mathrm{Aut}^0(X)$ is a connected subgroup, then G acts trivially on* $\mathrm{Alb}\, X$ *if and only if $G \subset L(X)$.*

Corollary 27. *If X is Kähler and K is a compact connected real Lie group that acts holomorphically on X in Hamiltonian way, then $G := K^{\mathbb{C}}$ is a meromorphic subgroup of* $\mathrm{Aut}^0(X)$.

Proof. The assumption means that there are a Kähler form ω and a momentum mapping $\mu : X \longrightarrow \mathfrak{k}^*$ such that ω is K-invariant, μ is equivariant and $d\langle \mu, v\rangle = i_{\xi_v}\omega$, where $\langle \cdot, \cdot \rangle$ denotes the pairing of $\mathfrak{k}^*$ and $\mathfrak{k}$ and ξ_v is the fundamental vector field corresponding to $v \in \mathfrak{k}$. It is well-known that K acts by biholomorphisms [31, p. 93], that the inclusion $K \subset \mathrm{Aut}^0(X)$ extends to an inclusion $G := K^{\mathbb{C}} \subset \mathrm{Aut}^0(X)$ and that G acts trivially on $\mathrm{Alb}\, X$, [27, Prop. 1].

Theorem 28 (Fujiki). *Let $X \in \mathcal{C}$ and let $G \subset \mathrm{Aut}^0(X)$ be a connected reductive subgroup. Then G is meromorphic (with the natural structure) if and only if it acts trivially on* $\mathrm{Alb}\, X$.

Proof. One implication is proved in [17, Lemma 3.8]. For the other assume that G acts trivially on $\mathrm{Alb}\, X$. By Corollary 26 $G \subset L(X)$. By Theorem 24 $L(X)$ is a meromorphic subgroup of $F(X)_{\mathrm{red}}$ and the meromorphic structure induced from $F(X)_{\mathrm{red}}$ (i.e. the natural structure) is equivalent to the linear one. Since G is reductive, it is an algebraic subgroup of $L(X)$. Hence it is a meromorphic subgroup of $L(X)$ with the natural structure and thus it is itself a meromorphic subgroup of $\mathrm{Aut}^0(X)$ with the natural structure. See [17, Prop. 6.10].

Proposition 29. *If X is a compact complex manifold, then $F(X)_{\mathrm{red}}$ is $\mathrm{Aut}^0(X)$-equivariantly bimeromorphic to $B(X)$.*

Proof. The morphism from Douady space to cycle space restricts to a surjective holomorphic map $f : F(X)_{\mathrm{red}} \longrightarrow B(X)$, see [3, Thm. 8 p. 121]. This map is obviously $\mathrm{Aut}^0(X)$-equivariant. The complex space $\mathrm{Aut}^0(X)$ embeds in both $F(X)_{\mathrm{red}}$ and $B(X)$. If we consider these embeddings as identifications, the map f extends $\mathrm{id}_{\mathrm{Aut}^0(X)}$. In particular f is 1-1 over $\mathrm{Aut}^0(X)$. By Lemma 9 f is bimeromorphic.

Proposition 30. *If G is a meromorphic subgroup with the natural structure, then the closure of G in $B(X)$ is an analytic subset.*

Proof. Consider again the morphism from Douady space to cycle space $f : F(X)_{\mathrm{red}} \longrightarrow B(X)$ as in Proposition 29. Denote by W the closure of G in $F(X)_{\mathrm{red}}$. By assumption W is an analytic subset. By Remmert Proper Mapping Theorem $f(W)$ is an analytic subset of $B(X)$. But it coincides with the closure of G in $B(X)$.

Corollary 31. *If X is Kähler and K is a compact connected real Lie group that acts holomorphically on X in Hamiltonian way, then $G := K^{\mathbb{C}}$ has analytic closure in $B(X)$.*

Proof. By Corollary 27 G is meromorphic.

The next result is a refinement of Theorem 20.

Theorem 32. *Assume that $X \in \mathcal{C}$ and that $G \subset \mathrm{Aut}^0(X)$ is a meromorphic subgroup (in the natural structure). Let $\bar{G}$ denote the closure of G in $B(X)$ and set $\partial G := \bar{G} - G$. Then $\partial G \subset \partial B(X)$. Morever for $b \in \partial G$, the stabilizer G_b for the action (7) has positive dimension and $f_b(X) \subset X^{G_b}$.*

Proof. Let j be the map defined in (3) and consider the action of $\mathrm{Aut}^0(X)$ on $C_n(X \times X)$ defined in (7). As usual we identify elements of $\mathrm{Aut}^0(X)$ with their image through j. So we consider $G \subset \mathrm{Aut}^0(X) = B^0(X) \subset B(X)$. By Proposition 30 $\bar{G}$ is an analytic subset of $B(X)$. In particular G is closed in $B^0(X)$, so $\partial G \subset \partial B(X)$. To prove the second assertion, observe that G is an open orbit of itself in $\bar{G}$. By Proposition 16 $\bar{G}$ is irreducible and ∂G is a proper analytic subset of $\bar{G}$. Hence any irreducible component of ∂G has dimension strictly less than $\dim G$. Since ∂G is invariant by the action, it follows that for $b \in \partial G$, $\dim G \cdot b < \dim G$, so $\dim G_b > 0$. Observing that $G_b \subset \mathrm{Aut}^0(X)_b$ and applying Theorem 20 concludes the proof.

If X is Kähler, we can say something on the geometry of $\bar{G}$. (Compare Theorem 3.)

Theorem 33. *If X is a Kähler manifold and $G \subset \mathrm{Aut}^0(X)$ is a connected reductive subgroup, that acts trivially on* $\mathrm{Alb}\, X$*, then the closure of G inside $B(X)$ is a projective variety.*

Proof. By Corollary 26 $G \subset L(X)$. By Theorem 28 $\bar{G}$ is an analytic subset of $B(X)$. By a result of Varouchas [44] $B(X)$ is a Kähler space, so the same is true of $\bar{G}$. Let $\pi : Z \to \bar{G}$ be a G-resolution of $\bar{G}$ (see e.g. [32, p. 150]). Then π is a projective, hence a Kähler morphism [9, Prop. 4.6 (4)]. Since Z is compact, it follows from [9, Prop. 4.6 (2)] that it is Kähler. Thus Z is a Kähler G-almost homogeneous manifold. We claim that G acts trivially on $\mathrm{Alb}(Z)$. Indeed G acts on $\mathrm{Alb}\, Z$ and being connected it acts by translations. Now up to a finite cover $G = T \rtimes S$ with $T = (\mathbb{C}^*)^r$ and S semisimple and connected. Any morphism $S \to \mathrm{Alb}\, Z$ is trivial, so S acts trivially. Each $\mathbb{C}^*$-factor of T is algebraic in G and hence is a meromorphic subgroup of G. As such $\mathbb{C}^*$ acts meromorphically on X. By [17, Prop. 2.2] it acts meromorphically also on $F(X)$ and on $F(X)_{\mathrm{red}}$. Using Propositions 23 and 29 we conclude that the action of $\mathbb{C}^*$ on Z is meromorphic. Hence every orbit has analytic closure [17, Lemma 2.4 (1)]. Fix $z \in Z$. The closure of $\mathbb{C}^* \cdot z$ contains a closed orbit, i.e. a fixed point. So fixed points exists, hence $\mathbb{C}^*$ acts trivially on $\mathrm{Alb}\, Z$ [42]. By [27, Prop. 2] and [39] we get that $b_1(Z) = 0$ and Z is projective. It follows that $\bar{G}$ is Moishezon, since it is bimeromorphic to the projective manifold Z, see [40, p. 305]. But $\bar{G}$ is also Kähler. Being Moishezon and Kähler $\bar{G}$ is in fact projective by [40, p. 310].

Remark 10. It would be interesting to know if $B(X)$ is projective for any $X \in \mathcal{C}$, without the Kählerness assumption.

4. The action on the set of measures

If X is a compact manifold, denote by $\mathcal{M}(X)$ the vector space of finite signed Borel measures on X endowed with the weak topology. Denote by $\mathcal{P}(X) \subset \mathcal{M}(X)$ the set of Borel probability measures on X.

The following theorem is a generalization of the so-called Furstenberg lemma, which corresponds to the case $X = \mathbb{P}^n$, see [21], [45, IV],[46, Lemma 3.2.1]

Theorem 34. *Let X be a complex manifold in the class $\mathcal{C}$. Let $\mu, \nu \in \mathcal{P}(X)$ and let $\{g_n\}$ be a sequence in $\mathrm{Aut}^0(X)$, such that $g_n \cdot \mu \rightharpoonup \nu$. Then either $\{g_n\}$ has compact closure in $\mathrm{Aut}^0(X)$ or ν is supported on a proper analytic subset of X.*

Proof. If $\{g_n\}$ is divergent in $\mathrm{Aut}^0(X)$, we can extract a subsequence (that we still denote by $\{g_n\}$) converging to some $b \in \partial B(X)$. By Theorems 21 and 20 we have

a) $g_n \to f_b$ uniformly on compact subset of $X - A_b$;
c) $f_b(X - A_b) \subset A' := X^{\mathrm{Aut}^0(X)_b} \subsetneq X$.

Let A_j be the irreducible components of A_b and set $a_j := \dim A_j$. For any fixed j the cycles $g_n \cdot A_j$ belong - for any n - to the same irreducible component of $C_{a_j}(X)$. These components are compact by Theorem 15, so by passing to a subsequence we can assume that $g_n \cdot A_j \to \hat{A}_j$ for any j and for some $\hat{A}_j \in C_{a_j}(X)$. The convergence as cycles implies the analogous convergence as closed subset of the metric space X. [4, Cor. 2.7.13 p. 424]. Hence, writing $\hat{A} := \cup_j \hat{A}_j$, we have

c) $g_n \cdot A \to \hat{A}$ in the Hausdorff topology of closed subsets.

Write $\mu = \mu_1 + \mu_2$ with $\mu_1(X - A) = \mu_2(A) = 0$. Since $\mathcal{P}(X)$ is compact in the weak topology, up to passing to a subsequence we can assume that $g_n \cdot \mu_1 \rightharpoonup \nu_1$ and $g_n \cdot \mu_2 \rightharpoonup \nu_2$. Hence $\nu_1 + \nu_2 = \nu$. We claim that

d) $\mathrm{supp}(\nu_1) \subset \hat{A}$;
e) $\mathrm{supp}(\nu_2) \subset A'$.

To prove (d) fix $u \in C(X)$ such that $\mathrm{supp}(u) \cap \hat{A} = \varnothing$. Then there is $\varepsilon > 0$ such that $\mathrm{supp}(u) \cap (\hat{A})_\varepsilon = \varnothing$. So $\mathrm{supp}(u) \cap (g_n \cdot A) = \varnothing$ for large n. Now

$$\int_X u\, d\nu_1 = \lim_{n\to\infty} \int_X u\, d(g_n \mu_1) = \lim_{n\to\infty} \int_X u(g_n \cdot x)\, d\mu_1(x),$$
$$\text{and} \quad \int_X u(g_n \cdot x)\, d\mu_1(x) = \int_A u(g_n \cdot x)\, d\mu_1(x),$$

since μ_1 is concentrated on A. For large n the last integral vanishes, since u vanishes on $g_n \cdot A$. This proves (d).

To prove (e) fix $u \in C(X)$ with $\mathrm{supp}(u) \cap A' = \varnothing$. As before

$$\int_X u\, d\nu_2 = \lim_{n\to\infty} \int_X u(g_n \cdot x)\, d\mu_2(x).$$

By (a) we have $u(g_n \cdot x) \to u(f_b(x))$ pointwise on $X - A$, hence μ_2-a.e. Since $u \in L^\infty$ we can apply Lebesgue Dominated Convergence Theorem to get

$$\int_X u\, d\nu_2 = \int_X u(f_b(x))\, d\mu_2(x).$$

But $f_b(X - A) \subset A'$ by (b). Since $u \equiv 0$ on A', we conclude that $\int_X u\, d\nu_2 = 0$. So (e) also is proven. (d) and (e) together clearly imply that $\mathrm{supp}(\nu) \subset \hat{A} \cup A'$, so the theorem is proved.

The following was already known in the special case $X = \mathbb{P}^n$, see [46, Cor. 3.2.2, p. 39].

Corollary 35. *If $\nu \in \mathcal{P}(X)$, then*

i) either ν is not supported on a proper analytic subset, in which case $\mathrm{Aut}^0(X)_\nu$ is compact;
ii) or there is a proper irreducible analytic subset Y of X such that
   a) $\nu(Y) > 0$,
   b) the orbit $O := \mathrm{Aut}^0(X)_\nu \cdot Y = \{g \cdot Y | g \in \mathrm{Aut}^0(X)_\nu\}$ is finite; in particular a finite subgroup of $\mathrm{Aut}^0(X)_\nu$ leaves Y invariant.

Proof. If ν is not supported on a proper analytic subset, the previous theorem implies that $\mathrm{Aut}^0(X)_\nu$ is compact. Thus (i) is clear. If ν is supported on a proper analytic subset, then there are proper irreducible analytic subsets with $\nu(Y) > 0$. Take Y to be one of minimal dimension. If $g_1 \cdot Y$ and $g_2 \cdot Y$ are distinct elements of O, then $\nu(g_1 \cdot Y \cap g_2 \cdot Y) = 0$. Otherwise some irreducible component Z of this intersection has positive measure and $\dim Z < \dim Y$. Moreover since $g_i \in \mathrm{Aut}^0(X)_\nu$ we have $\nu(g_1 \cdot Y) = \nu(g_2 \cdot Y)$. Since $\nu(X) = 1$ the orbit must be finite. The rest is clear.

Remark 11. We remark that in fact one might expect a better result: linear subspaces of $\mathbb{P}^n$ can be characterized as fixed sets of subgroups of $\mathrm{PGL}(n+1, \mathbb{C})$. So one might ask if the support of a measure with non-compact stabilizer is in fact contained in the fixed set of a proper subgroup of $\mathrm{Aut}^0(X)$. We leave this point for further inquiry.

Another application concernes the construction of Hersch and Bourguignon-Li-Yau that we now recall briefly, see [7, §§5-6] for more details. Let X be a compact Kähler manifold and let K be a compact connected real Lie group acting almost effectively on X with momentum mapping

$$\mu : X \to \mathfrak{k}^*.$$

If $v \in \mathfrak{k}$, set $\mu^v := \langle \mu, v \rangle$. Then μ is K-equivariant and $d\mu^v = i_{v_X}\omega$. The action of K extends to a holomorphic action of the complexification $G := K^{\mathbb{C}}$. Define $\mathfrak{F} : \mathcal{P}(X) \to \mathfrak{k}^*$ by the formula

$$\mathfrak{F}(\nu) := \int_X \mu(x) d\nu(x).$$

As explained in [7] this map is a momentum mapping for the action of K on $\mathcal{P}(X)$, in an appropriate sense.

Let $E(\mu)$ denote the convex hull of $\mu(X) \subset \mathfrak{k}^*$ and let $\Omega(\mu)$ denote the interior of $E(\mu)$ as a subset of $\mathfrak{k}^*$. Finally set

$$F_\nu : G \longrightarrow \mathfrak{k}^*, \quad F_\nu(a) := \mathfrak{F}(a \cdot \nu).$$

The following should be compared to Theorem 6.14 in [7].

Theorem 36. *Fix $\nu \in \mathcal{P}(X)$ and assume that $\nu(A) = 0$ for any proper analytic subset A of X. Then $F_\nu(G) = \Omega(\mu)$ and $F_\nu : G \to \Omega(\mu)$ is a fibration with compact connected fibres.*

Proof. By Corollary 35 (i) the stabilizer $\mathrm{Aut}^0(X)_\nu$ is compact, so also G_ν is compact. Therefore Theorem 6.4 in [7] implies that the map F_ν is a smooth submersion onto its image, which is an open subset of $\Omega(\mu)$. To conclude it is enough to check that F_ν is proper as a map $G \to \Omega(\mu)$ (see [7, p. 1140] for details). Let $\{g_n\}$ be a diverging sequence in G. Since $E(\mu)$ is compact, we can assume that $F_\nu(g_n) \to \xi \in E(\mu)$. We have to prove that $\xi \in \partial E(\mu)$. If $\bar{G}$ denotes the closure of G in $B(X)$ (which is compact), we can also assume $g_n \to b$ for some $b \in \partial G$. Let ν_0 be a fixed smooth probability measure, i.e. a measure given by a smooth volume form which vanishes nowhere. By Theorem 6.14 of [7] (see also Definition 5.27 in that paper) the map $F_{\nu_0} : G \longrightarrow \Omega(\mu)$, $F_{\nu_0}(a) := \mathfrak{F}(a \cdot \nu_0)$ is proper. Therefore up to passing to a subsequence we can assume that $F_{\nu_0}(g_n)$ converges to some point $\theta \in \partial E(\mu)$. And by Theorem 0.3 in [8], the convex body $E(\mu)$ has the property that all its faces $\overline{\Omega(\mu)} = E(\mu)$ are exposed (see [8, p. 426] for the definitions). Therefore there exists a $v \in \mathfrak{k}$, such that $v \neq 0$ and $\langle \theta, v \rangle = \max_{E(\mu)} \langle \cdot, v \rangle$. On the other hand Theorem 21 we have pointwise convergence $g_n \to f_b(x)$ on $X - A_b$. Since $\nu_0(A_b) = 0$ and $\mu^v := \langle \cdot, v \rangle$ is bounded, the dominated convergence theorem yields

$$\langle F_{\nu_0}(g_n), v \rangle = \int_X \mu^v(g_n \cdot x) d\nu_0(x) \to \int_X \mu^v(f_b(x)) d\nu_0(x).$$

On the other hand $\langle F_{\nu_0}(g_n), v \rangle \to \langle \theta, v \rangle$. Thus

$$\int_X \mu^v(f_b(x)) d\nu_0(x) = \langle \theta, v \rangle = \max_{E(\mu)} \langle \cdot, v \rangle = \max_X \mu^v.$$

This shows that the equality $\mu^v \circ f_b = \max_X \mu^v$ holds ν_0-almost everywhere on $X - A_b$. Since this function is continuous, the equality holds in fact everywhere on $X - A_b$. But since $\nu(A_b) = 0$ by assumption, we can redo this computation with ν instead of ν_0:

$$\langle F_\nu(g_n), v \rangle q = \int_{X - A_b} \mu^v(g_n \cdot x) d\nu(x) \to \int_{X - A_b} \mu^v(f_b(x)) = \max_X \mu^v.$$

Summing up we get $\langle \xi, v \rangle = \max_\mu^v = \max_{E(\mu)} \langle \cdot, v \rangle$. Therefore ξ (just as θ) lies in the face $F_v(E(\mu))$. In particular $\xi \in \partial E(\mu)$.

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