Lipschitz Bounds and Nonuniform Ellipticity

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Abstract

We consider nonuniformly elliptic variational problems and give optimal conditions guaranteeing the local Lipschitz regularity of solutions in terms of the regularity of the given data. The analysis catches the main model cases in the literature. Integrals with fast, exponential-type growth conditions as well as integrals with unbalanced polynomial growth conditions are covered. Our criteria involve natural limiting function spaces and reproduce, in this very general context, the classical and optimal ones known in the linear case for the Poisson equation. Moreover, we provide new and natural growth a priori estimates whose validity was an open problem. Finally, we find new results also in the classical uniformly elliptic case. Beyond the specific results, the paper proposes a new approach to nonuniform ellipticity that, in a sense, allows us to reduce nonuniform elliptic problems to uniformly elliptic ones via potential theoretic arguments that are for the first time applied in this setting. © 2019 the Authors. *Communications on Pure and Applied Mathematics* is published by the Courant Institute of Mathematical Sciences and Wiley Periodicals, Inc.

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Communications on Pure and Applied Mathematics, Vol. LXXIII, 0944–1034 (2020)

1 Introduction and Results

In this paper we prove regularity theorems and estimates for solutions to nonuniformly elliptic problems. Roughly speaking, these involve elliptic equations whose rate of ellipticity might change depending on the solution itself. The operators we are considering mostly stem from variational problems and, as such, they arise when considering the Euler-Lagrange equation of functionals of the type

$$w \mapsto \mathscr{F}(w;\Omega) := \int_\Omega [F(Dw) - fw]dx, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open subset and $n \geq 2$. In this case the nonuniform ellipticity of the integrand $F(\cdot)$ corresponds to the fact that the ratio between the highest and the lowest eigenvalue of the elliptic tensor $\partial^2 F(z)$ can become unbounded when $|z| \to \infty$; see also (1.13) below. Nonuniform ellipticity is ubiquitous and appears in several different contexts, often motivated by geometric and physical problems [23,37,56,63,64]. It is a classical topic in the field of partial differential equations. In the context of the calculus of variations its study has been carried out systematically in a series of remarkable papers of Marcellini [44–47].

A classical regularity issue for minimizers u of (1.1) is to find sharp criteria on f implying $Du \in L^\infty_{\mathrm{loc}}$. This problem has been widely studied and understood, but only in the uniformly elliptic case. On the contrary, in the nonuniformly elliptic case the issue has remained essentially untouched since the usual uniformly elliptic methods do not seem to apply. The aim of this paper is now twofold:

- To identify minimal conditions on f in order to guarantee that minimizers to (1.1) are locally Lipschitz-continuous, thus filling a remarkable gap in the literature.
- To introduce a new potential theoretic approach allowing us to reduce in a natural way the treatment of nonuniformly elliptic problems to the one of uniformly elliptic ones. This technique yields optimal and new local estimates when $f \equiv 0$ and also when applied to the standard uniformly elliptic case.

Let us summarize the present situation. A number of recent results (see [1,15–17,24,35,36] and references) are concerned with the issue of determining optimal conditions on f ensuring that solutions to uniformly elliptic equations of the type

$$-\operatorname{div}(\tilde{a}(|Du|)Du) = f \tag{1.2}$$

are locally Lipschitz-continuous. Equation (1.2) is considered in these papers under the *uniform ellipticity* assumption

$$\begin{cases} -1 < i_a \leq \dfrac{\tilde{a}'(t)t}{\tilde{a}(t)} \leq s_a < \infty \quad \text{for every } t > 0 \\ \tilde{a}\colon (0,\infty) \to [0,\infty) \text{ is of class } C^1_{\mathrm{loc}}(0,\infty). \end{cases} \tag{1.3}$$

A prominent example in this context is given by the nonhomogeneous p-Laplacian equation, i.e.,

$$-\operatorname{div}(|Du|^{p-2}Du) = f, \quad p > 1. \tag{1.4}$$

This is the Euler-Lagrange equation of the functional (1.1) when taking $F(z) \equiv |z|^p/p$. Solutions to equations of the type in (1.2) are local minimizers of the functional

$$w \mapsto \int_\Omega [A(|Dw|) - fw]dx, \quad A(t) := \int_0^t \tilde{a}(s)s\,ds \ \text{ for } t > 0, \tag{1.5}$$

which is defined on the Orlicz-Sobolev function space $W^{1,A}(\Omega)$, i.e., the space of all functions $w \in W^{1,1}(\Omega)$ such that $A(|Dw|) \in L^1(\Omega)$. Cianchi and Maz′ya [15, 16] consider precisely this setting. Assumptions (1.3) imply that $A(\cdot)$ satisfies the so-called Δ_2-condition; i.e.,

$$A(2t) \leq c(i_a, s_a)A(t) \tag{1.6}$$

holds for every $t > 0$ (see [15, prop. 2.9]). Moreover, fast-growth conditions (such as $A(t) \equiv \exp(t)$) are immediately ruled out here, in the sense that (1.3) also implies that $t^{i_a+2} \lesssim A(t) \lesssim t^{s_a+2}$ holds for t sufficiently large. Apart from the specific assumptions considered, all the above-mentioned contributions assert that a sufficient condition on f to have $Du \in L^\infty_{\text{loc}}$ is that f belongs to the Lorentz space $L(n,1)(\Omega)$. This amounts to requiring that

$$\|f\|_{L(n,1)(\Omega)} := \int_0^\infty \bigl|\{x \in \Omega\colon |f(x)| > \lambda\}\bigr|^{1/n}\,d\lambda < \infty. \tag{1.7}$$

Such a condition turns out to be optimal in the linear case $\tilde{a}(\cdot) \equiv 1$. This is essentially the nonlinear extension of a classical, linear, and sharp result of Stein [59], asserting the local Lipschitz continuity of solutions to the Poisson equation $\triangle u \in L(n,1)$. In this respect, notice that the strict inclusions $L^{n+\varepsilon} \subset L(n,1) \subset L^n$ hold for every $\varepsilon > 0$. Indeed, the original result of Stein states that a function whose gradient belongs to $L(n,1)$ is continuous; the assertion for the Poisson equations then follows recalling that Lorentz spaces are interpolation spaces and using classical Calderón-Zygmund theory. Also notice that solutions to $\triangle u \in L^n$ are in general not locally Lipschitz-continuous. For basic issues of optimality we refer to [13]. A remarkable feature of condition (1.7) is that it is independent of the operator in question [1, 20, 35]. In particular, it is independent of the exponent p when looking at (1.4). These results strongly rely on the uniform ellipticity of the operators; i.e., (1.3) is assumed to be in force. No counterpart of any similar local Lipschitz result is available in the general nonuniformly elliptic case, even when assuming higher integrability on f.

In this paper we prove that condition (1.7) is sufficient for Lipschitz continuity in the nonuniformly elliptic setting too. This actually holds when $n > 2$; we give a similar borderline characterization when $n = 2$. We cover the main relevant examples of nonuniformly elliptic functionals appearing in the literature. For instance, we are able to treat functionals as in (1.1) with fast growth conditions [12,23,40,46] such as

$$w \mapsto \int_\Omega [\exp(\exp(\cdots \exp(|Dw|^p)\cdots)) - fw]\,dx, \quad p \ge 1, \tag{1.8}$$

for which the $\triangle_2$-condition (1.6) fails, as well as functionals with polynomial but yet unbalanced growth conditions, as those satisfying so-called (p,q)-growth conditions, i.e.,

$$|z|^p \lesssim F(z) \lesssim |z|^q + 1, \quad 1 < p < q. \tag{1.9}$$

These have been pioneered by Marcellini [44, 45] (see also [5, 7–10, 21, 39, 40, 53, 63] for some special cases). Typical examples in this case are given by anisotropic variational integrals as

$$\begin{gathered} w \mapsto \int_\Omega \left[(\max\{|Dw| - d, 0\})^p + \sum_{i=1}^n |D_i w|^{q_i} - fw\right] dx \\ 1 < p, q_1, \dots, q_n, \quad d \ge 0, \end{gathered} \tag{1.10}$$

or by perturbations of functionals with standard p-growth conditions as

$$w \mapsto \int_\Omega [|Dw|^p \log(\mathrm{e} + |Dw|) - fw]\,dx, \quad p > 1. \tag{1.11}$$

In this paper we treat both the scalar and the vectorial case, i.e., when minimizers and competitors are scalar-valued and vector-valued maps. We are going to present the results in the following sections, while here we recall the natural notion of (local) minimizer adopted in the present setting.

DEFINITION 1.1. A function $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ is a *local minimizer* of the functional $\mathscr{F}$ in (1.1) with $f \in L^n_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ if, for every open subset $\tilde\Omega \Subset \Omega$, we have $\mathscr{F}(u;\tilde\Omega) < \infty$ and if $\mathscr{F}(u;\tilde\Omega) \le \mathscr{F}(w;\tilde\Omega)$ holds for every competitor $w \in u + W^{1,1}_0(\tilde\Omega;\mathbb{R}^N)$.

In the vectorial case $N > 1$ in (1.1), by fw we still denote the scalar product between f and w. In Definition 1.1 we have started with $f \in L^n_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, so that by Sobolev embedding it follows that $fu \in L^1_{\mathrm{loc}}(\Omega)$, and we conclude with $F(Du) \in L^1_{\mathrm{loc}}(\Omega)$. Notice that, as described above, the L^n-integrability of f is not a restrictive assumption, as we are interested in the local Lipschitz continuity of minimizers.

We also remark that, in this setting, the focal point of regularity is the Lipschitz continuity. Indeed, once minimizers are known to be locally Lipschitz regular, the equation becomes uniformly elliptic at infinity and classical methods apply; see [34], provided suitable assumptions are satisfied. The results of this paper are now presented in the following sections. We first start with special but relevant instances, i.e., functionals with (p,q)-growth conditions and functionals with exponential-type growth, in Sections 1.1 and 1.2, respectively. Starting from Section 1.3 we shall then present our results in full generality. For the notation used in this paper we refer to Section 2 below.

1.1 Nonuniform Ellipticity at Polynomial Rates

We start considering functionals with (p,q)-growth as in (1.9), where the integrand $F\colon \mathbb{R}^n \to \mathbb{R}$ is assumed to be a convex function that is locally C^2-regular in $\mathbb{R}^n \setminus \{0\}$ and satisfies the growth and ellipticity conditions

$$\begin{cases} \nu(|z|^2+\mu^2)^{p/2} \le F(z) \le \Lambda(|z|^2+\mu^2)^{q/2} + \Lambda(|z|^2+\mu^2)^{p/2}, \\ (|z|^2+\mu^2)|\partial^2 F(z)| \le \Lambda(|z|^2+\mu^2)^{q/2} + \Lambda(|z|^2+\mu^2)^{p/2}, \\ \nu(|z|^2+\mu^2)^{(p-2)/2}|\xi|^2 \le \langle \partial^2 F(z)\xi,\xi\rangle, \end{cases} \tag{1.12}$$

for every choice of $z,\xi \in \mathbb{R}^n$ such that $|z| \neq 0$ and for exponents $1 \le p \le q$. Here $0 < \nu \le 1 \le \Lambda$ are fixed ellipticity constants and $\mu \in [0,1]$ serves to distinguish the so-called degenerate case ($\mu = 0$) and nondegenerate case ($\mu > 0$). Assumptions (1.12) allow the ellipticity ratio

$$\mathscr{R}(z) := \frac{\text{highest eigenvalue of } \partial^2 F(z)}{\text{lowest eigenvalue of } \partial^2 F(z)} \lesssim (|z|^2+\mu^2)^{(q-p)/2} + 1 \tag{1.13}$$

to become unbounded for $p < q$ as $|z| \to \infty$, and, therefore, the Euler-Lagrange equation of $\mathscr{F}$, that is, $-\operatorname{div} \partial F(Du) = f$, is nonuniformly elliptic. Under the assumptions in (1.12) it is known that sufficient [44, 45] and necessary [28, 45] conditions for regularity of minimizers are of the type $q/p < 1 + \mathrm{o}(n)$, where $\lim_{n\to\infty} \mathrm{o}(n) = 0$. This controls the rate of nonuniform ellipticity quantified in (1.13). Different expressions for $\mathrm{o}(n)$ have been derived. For instance, in [45] the bound

$$\frac{q}{p} < 1 + \frac{2}{n} \tag{1.14}$$

is shown to guarantee the local Lipschitz continuity of minimizers of $\mathscr{F}$ when $p \ge 2$, $\mu = 1$, and $f \in L^\infty$. In the uniformly elliptic case $p = q$ the assumptions in (1.12) coincide with the classical ones considered by Ladyzhenskaya and Ural′tseva [37], otherwise they are known as (p,q)-growth conditions and have been the object of intensive investigation; see, for instance, [2, 9, 10, 18, 19, 26, 38, 44, 45, 48, 58]. As it is natural, conditions as (1.14) also play a role in our setting, as shown in the following:

THEOREM 1.2 (Scalar (p,q)-estimates). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional $\mathscr{F}$ in* (1.1) *under assumptions* (1.12) *with $1 < p \le q$ and $n > 2$. Assume*

$$\frac{q}{p} < 1 + \min\left\{\frac{2}{n}, \frac{4(p-1)}{p(n-2)}\right\} \quad \text{and} \quad f \in L(n,1)(\Omega). \tag{1.15}$$

Then Du is locally bounded in Ω. Moreover, the local a priori estimate

$$
\begin{aligned}
\|Du\|_{L^\infty(B/2)} &\le c\left(\fint_B F(Du)dx + \|f\|_{L(n,1)(B)}^{\frac{p}{p-1}}\right)^{\frac{1}{p}} \\
&\quad + c\left(\fint_B F(Du)dx + \|f\|_{L(n,1)(B)}^{\frac{p}{p-1}}\right)^{\frac{2}{(n+2)p-nq}} \\
&\quad + c\|f\|_{L(n,1)(B)}^{\frac{4}{4(p-1)-(n-2)(q-p)}}
\end{aligned}
\tag{1.16}
$$

holds for a constant $c \equiv c(n,p,q,\nu,\Lambda)$ whenever $B \Subset \Omega$ is a ball. When $p \ge 2-4/(n+2)$ or when $f \equiv 0$, condition (1.15) *can be replaced by condition* (1.14).

When applied to special situations, (1.15) reproduces several known and classical estimates. In the familiar case of the p-Laplacian equations in (1.4), we then take $p = q$ and $\mu \equiv 0$ so that (1.16) reduces to the following optimal local estimate (see, for instance, [34, 36] and related references):

$$
\|Du\|_{L^\infty(B/2)} \lesssim \left(\fint_B |Du|^p\, dx\right)^{\frac{1}{p}} + \|f\|_{L(n,1)(B)}^{\frac{1}{p-1}}. \tag{1.17}
$$

In turn, when $f \equiv 0$, this is the classical $L^\infty - L^p$ estimate for p-harmonic functions; see [43]. Again for $f \equiv 0$ and $p \le q$ as in (1.14), estimate (1.16) reduces to Marcellini's basic estimate, proved in [45, theorem 3.1] for $\mu = 1$ and $q \ge p \ge 2$ as in (1.14), that is, to

$$
\|Du\|_{L^\infty(B/2)} \lesssim \left(\fint_B F(Du)dx\right)^{\frac{2}{(n+2)p-nq}} + 1. \tag{1.18}
$$

In the vector-valued case $u\colon \Omega \to \mathbb{R}^N$, $N > 1$, similar results hold provided a radial structure is assumed on the integrand. This is precisely the statement of the next

THEOREM 1.3 (Vectorial (p,q)-estimates). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ be a local minimizer of the functional $\mathscr{F}$ in* (1.1) *under assumptions* (1.12) *with $1 < p \le q$ and $n > 2$. Assume*

$$
\frac{q}{p} < 1 + \min\left\{\frac{1}{n}, \frac{2(p-1)}{p(n-2)}\right\} \quad \textit{and} \quad f \in L(n,1)(\Omega;\mathbb{R}^N), \tag{1.19}
$$

and that there exists a $C^1_{\mathrm{loc}}[0,\infty) \cap C^2_{\mathrm{loc}}(0,\infty)$-regular function $\widetilde{F}\colon [0,\infty) \to [0,\infty)$ such that $F(z) = \widetilde{F}(|z|)$ for every $z \in \mathbb{R}^{N\times n}$. Finally, assume that

$$
t \mapsto \frac{\widetilde{F}'(t)}{(t^2+\mu^2)^{\frac{p-2}{2}}\, t} \quad \textit{is nondecreasing.} \tag{1.20}
$$

Then Du is locally bounded in Ω and an estimate similar to (1.16) *holds. Moreover, assuming $p \ge 2$ in* (1.20) *gives that* (1.19) *can be relaxed to* (1.15).

Theorems 1.2 and 1.3 deal with the case $n > 2$ (as in [14, 15, 17, 33] in the uniformly elliptic setting). When $n = 2$ we consider a condition different from (1.7) and involving the Orlicz space $\mathrm{L}^2(\mathrm{Log\,L})^\alpha(\Omega)$, $\alpha \geq 0$. This space is defined via

$$f \in \mathrm{L}^2(\mathrm{Log\,L})^\alpha(\Omega) :\Longleftrightarrow \int_\Omega |f|^2 \log^\alpha(1+|f|)dx < \infty. \tag{1.21}$$

This is an example of Orlicz spaces (see Section 2 below for the relevant definitions). Its borderline character is explained by the strict inclusions $L^{2+\varepsilon} \subset \mathrm{L}^2(\mathrm{Log\,L})^\alpha \subset L^2$, for every $\varepsilon, \alpha > 0$. Moreover, while solutions to $\triangle u \in L^{2+\varepsilon}$ are C^1-regular, they are in general not locally Lipschitz when $\triangle u \in L^2$. A corresponding borderline condition is given by $\triangle u \in \mathrm{L}^2(\mathrm{Log\,L})^\alpha(\Omega)$ for $\alpha > 1$, ensuring that Du is locally bounded. We reproduce almost the same borderline result by requiring $\alpha > 2$.

THEOREM 1.4. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional in* (1.1) *under assumptions* (1.12) *with $1 < p \leq q$ and $n = 2$. Assume*

$$q < 2p \quad \text{and} \quad f \in \mathrm{L}^2(\mathrm{Log\,L})^\alpha(\Omega) \ \text{ for some } \alpha > 2. \tag{1.22}$$

Then Du is locally bounded in Ω.

Correspondingly, in the vectorial case we have the following:

THEOREM 1.5. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ be a local minimizer of the functional in* (1.1) *under assumptions* (1.12) *with $1 < p \leq q$ and $n = 2$. Assume*

$$q < \frac{3p}{2} \quad \text{and} \quad f \in \mathrm{L}^2(\mathrm{Log\,L})^\alpha(\Omega;\mathbb{R}^N) \ \text{ for some } \alpha > 2, \tag{1.23}$$

and that there exists a $C^1_{\mathrm{loc}}[0,\infty) \cap C^2_{\mathrm{loc}}(0,\infty)$-regular function $\widetilde{F}\colon [0,\infty) \to [0,\infty)$ such that $F(z) = \widetilde{F}(|z|)$ for every $z \in \mathbb{R}^{N\times n}$. Finally, assume that (1.20) *is in force. Then Du is locally bounded in Ω. When $p \geq 2$, assumption* (1.23) *can be relaxed to* (1.22).

Remark 1.6. The local Lipschitz regularity results of Theorems 1.2 and 1.4 extend to minimizers of the functional in (1.10) by taking $q \equiv \max\{q_1, \dots, q_n\}$ and considering the bounds in (1.15)–(1.22). In this case it is sufficient to apply Theorem 1.9 below with $T \geq 2d$. Estimates similar to the one in (1.16) hold.

1.2 Nonuniform Ellipticity at Fast Rates

Here we deal with nonpolynomial growth conditions. The simplest example we have in mind is given by the functional

$$W^{1,1}(\Omega;\mathbb{R}^N) \ni w \mapsto \int_\Omega \exp(|Dw|^p)dx, \qquad p \geq 1; \tag{1.24}$$

see, for instance, [23, 40, 46]. More generally, with $\{p_k\}$ being a sequence of real numbers such that $p_0 > 1$, $p_k > 0$ for every $k \in \mathbb{N}$, we inductively define the functions $\mathbf{e}_k\colon [0,\infty) \to \mathbb{R}$ as

$$\begin{cases} \mathbf{e}_{k+1}(t) := \exp[(\mathbf{e}_k(t))^{p_{k+1}}], \\ \mathbf{e}_0(t) := \exp(t^{p_0}). \end{cases} \tag{1.25}$$

For every integer $k \geq 0$, we consider the variational integral

$$W^{1,1}(\Omega;\mathbb{R}^N) \ni w \mapsto \mathscr{E}_k(w) := \int_\Omega [\mathbf{e}_k(|Dw|) - fw]\,dx. \tag{1.26}$$

The functional in (1.8) is a special instance of those displayed in (1.26). Because of the radial structure of the integrand, the following result holds directly in the vectorial case with $u\colon \Omega \to \mathbb{R}^N$ for $N \geq 1$.

THEOREM 1.7 (Exponential estimates). *Let $k \geq 0$ be an integer number and let $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ be a local minimizer of the functional $\mathscr{E}_k$ in* (1.26)*.*

- *If $f \in L(n,1)(\Omega;\mathbb{R}^N)$ and $n > 2$, then Du is locally bounded in Ω.*
- *When $n = 2$ the same conclusion holds provided $f \in \mathrm{L}^2(\mathrm{Log\,L})^\alpha(\Omega;\mathbb{R}^N)$ for some $\alpha > 2$.*
- *Finally, when $f \equiv 0$, the local estimate*

$$\|Du\|_{L^\infty(B/2)} \leq c\,\mathbf{e}_k^{-1}\left(\fint_B \mathbf{e}_k(|Du|)\,dx\right) + c \tag{1.27}$$

holds for a constant $c \equiv c(n,N,k,p_0,\ldots,p_k)$ and for every ball $B \Subset \Omega$.

When $f \equiv 0$, Marcellini [46–48] proved that $Du \in L^\infty_{\mathrm{loc}}$ for the functionals in (1.26); see [23, 40] for special cases. Instead, the natural growth estimate (1.27) is new. This is linked to the novel approach to Lipschitz estimates we give here, which is an alternative to Marcellini's and Lieberman's and that provides sharper estimates than those available in the literature. Moreover, such an approach could be used to give alternative proofs also in the settings of Lieberman [39–41] and Simon [56].

To describe the progress, let us specify the estimate (1.27) to the simplest model case, namely the functional in (1.24). We here get the natural growth estimate

$$\|Du\|^p_{L^\infty(B/2)} \leq c\log\left(\fint_B \exp(|Du|^p)\,dx\right) + c \quad \text{with } c \equiv c(n,N,p) \tag{1.28}$$

for any local minimizer $u \in W^{1,1}(\Omega)$ of the functional (1.24). This estimate parallels the one for p-harmonic functions in (1.17) (when $f \equiv 0$) in that it exhibits the correct growth in the right-hand side. On the other hand, the best Lipschitz estimate for local minimizers of (1.24) available in the literature up to now was in [46–48] and read as

$$\|Du\|_{L^\infty(B/2)} \leq c_\varepsilon\left(\fint_B \exp(|Du|^p)\,dx\right)^{1+\varepsilon} + c_\varepsilon \quad \text{for every } \varepsilon > 0.$$

This bound exhibits a loss of an exponential scale with respect to (1.28). The situation worsens when considering faster growth conditions as in (1.8). The estimates in (1.27)–(1.28) are special occurrences of a general result that will be described in Theorem 1.16 below.

Another interesting class of exponential-type functionals is given by the following anisotropic version of the one in (1.24), this time obviously considered in the scalar case only:

$$(1.29)\quad w \mapsto \mathcal{E}_a(w) := \int_\Omega \left(\exp(A_0|Dw|^p) + \sum_{i=1}^n \exp(A_i|D_i w|^p) - fw \right) dx,$$

where here it is $p > 1$ and $0 < A_0 \le A_1 \le \cdots \le A_n$. When $f \equiv 0$, this functional falls in the realm of those in [47, theorem 2.1]. Marcellini's assumptions, that is, [47, (11)–(14)], prescribe that

$$(1.30)\quad \frac{A_n}{A_0} < 1 + \frac{2}{n}$$

holds, and this implies the local Lipschitz continuity of (scalar) minimizers (compare with (1.15)). The same result actually holds when $f \in L(n,1)(\Omega)$ for the functional (1.29).

THEOREM 1.8. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional $\mathcal{E}_a$ in* (1.29)*. Assume that $f \in L(n,1)(\Omega)$ holds when $n > 2$ together with condition* (1.30)*. Then Du is locally bounded in Ω. When $n = 2$ the same conclusion holds provided $f \in \mathrm{L}^2(\mathrm{Log\,L})^\alpha(\Omega)$ holds for some $\alpha > 2$.*

1.3 General Growth Conditions

In the scalar case $N = 1$, Theorems 1.2–1.8 are relevant special occurrences of more comprehensive results, namely, Theorems 1.9 and 1.11 below. These are devised to cover a large number of model problems, and therefore their formulation involves a generous set of different parameters. Ellipticity is described via two locally bounded and measurable functions $g_1, g_2\colon (0,\infty) \to [0,\infty)$, aimed at controlling the lower and the upper eigenvalues of $\partial^2 F(z)$, respectively, when $|z| \ge T$ and for a fixed number $T > 0$. They are assumed to be continuous on $[T,\infty)$ and such that $g_1(T), g_2(T) > 0$; their behavior on $(0,T)$ will be essentially irrelevant in what follows. As a minimal requirement on $g_1(\cdot), g_2(\cdot)$, we assume that $[T,\infty) \ni t \mapsto g_2(t)/g_1(t)$ and $t \mapsto g_1(t)t$ are almost nondecreasing and nondecreasing, respectively. This means that

$$(1.31)\quad T \le s \le t \implies \frac{g_2(s)}{g_1(s)} \le c_a \frac{g_2(t)}{g_1(t)} \quad\text{and}\quad g_1(s)s \le g_1(t)t,$$

holds for some constant $c_a \ge 1$. Notice that the second condition in (1.31) implies that $g_1(t) > 0$ whenever $t \ge T$; in turn, this and the first condition imply that $g_2(t) > 0$ for $t \ge T$ as well. The ellipticity/convexity properties of the integrand

$z \mapsto F(z)$ can be now described via $g_1(\cdot)$ and $g_2(\cdot)$ as follows:

$$\begin{cases} z \mapsto F(z) \geq 0 & \text{is convex,} \\ z \mapsto F(z) & \text{is } C^2\text{-regular on } \{|z| > T\}, \\ |\partial^2 F(z)| \leq g_2(|z|) & \text{for every } z \in \mathbb{R}^n \text{ with } |z| > T, \\ g_1(|z|)|\xi|^2 \leq \langle \partial^2 F(z)\xi, \xi\rangle & \text{for every } z, \xi \in \mathbb{R}^n \text{ with } |z| > T. \end{cases} \tag{1.32}$$

Finally, the necessary match between ellipticity (1.32) and coercivity properties of $z \mapsto F(z)$ is described via parameters τ and β_0; namely, we assume that

$$\begin{cases} \nu(t^2+\mu^2)^{\tau/2} \leq g_1(t) & \text{for } t \geq T \text{ for some } \tau > -1, \\ \left(\int_T^{|z|} g_1(s)s\,ds\right)^{\beta_0} \leq F(z) & \text{for } |z| \geq T \text{ for some } \frac{1}{2} < \beta_0 \leq 1 \\ & \text{with } \gamma := \beta_0(\tau+2) > 1, \end{cases} \tag{1.33}$$

where $\nu > 0$ and $0 \leq \mu \leq 1$ are fixed constants. We then have the following:

THEOREM 1.9 (General scalar estimates). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional in* (1.1) *under the assumptions* (1.31)–(1.33) *for $n > 2$. Assume $f \in L(n,1)(\Omega)$ and that the inequality*

$$\frac{g_2(t)}{g_1(t)} \leq c_b \min\left\{\left(\int_T^t g_1(s)s\,ds\right)^{\frac{2\beta_0-\sigma}{n}}, \left(\frac{1}{t^{1/\beta_1}}\int_T^t g_1(s)s\,ds\right)^{\frac{4\beta_1}{n-2}}\right\} + c_b' \tag{1.34}$$

holds for every $t \geq T$, for some σ with $0 < \sigma \leq 2\beta_0$, and some fixed positive constants $\beta_1 < 1$ and $c_b \geq 1$. Then Du is locally bounded in Ω. Moreover, for every ball $B \Subset \Omega$, the estimate

$$\begin{aligned} \int_T^{\|Du\|_{L^\infty(B/2)}} g_1(s)s\,ds &\leq c\left(\fint_B F(Du)dx + \|f\|_{L^n(B)}^{\frac{\gamma}{\gamma-1}} + T^\gamma + \mu^\gamma\right)^{\frac{1}{\beta_0}} \\ &\quad + c\left(\fint_B F(Du)dx + \|f\|_{L^n(B)}^{\frac{\gamma}{\gamma-1}} + T^\gamma + \mu^\gamma\right)^{\frac{2}{\sigma}} \\ &\quad + c\|f\|_{L(n,1)(B)}^{\frac{\tau+2}{\tau+1}} + c\|f\|_{L(n,1)(B)}^{\frac{1}{1-\beta_1}} + c(T+\mu)\|f\|_{L(n,1)(B)} \end{aligned} \tag{1.35}$$

holds for a constant c depending only on n, ν, τ, c_a, c_b, σ, β_0, and β_1, but otherwise independent of T. Finally, when $f \equiv 0$, assumption (1.34) *can be replaced by the weaker assumption*

$$\frac{g_2(t)}{g_1(t)} \leq c_b\left(\int_T^t g_1(s)s\,ds\right)^{\frac{2\beta_0-\sigma}{n}} + c_b \quad \forall t \geq T. \tag{1.36}$$

In view of (1.32), the nonuniform ellipticity ratio $\mathscr{R}(z)$ in (1.13) can be estimated as $\mathscr{R}(z) \lesssim g_2(|z|)/g_1(|z|)$ for $|z| > T$, and indeed (1.32)–(1.33) reduce to (1.12) with the choices in (6.1) below; compare with Section 6 below. In this respect, the main assumption (1.34) serves to bound the growth of $\mathscr{R}(z)$ with respect to $|z|$. It reduces to (1.15) in the case of Theorem 1.2. Moreover, when $f \equiv 0$ our assumptions essentially reduce to those considered by Marcellini [47] in the scalar case; see also Section 6.6 below. The local estimate (1.35) might seem somehow involved, but it is actually the most general one in this setting. It provides optimal a priori estimates when applied to relevant model cases, via the tuning of the parameters $\beta_0, \beta_1, \sigma, \tau$, and μ. Indeed, all the estimates in (1.16), (1.17), (1.18), (1.27), and (1.28) can be generated as particular cases of (1.35). The parameter β_0 describes the interplay between ellipticity $(1.32)_4$ and coercivity $(1.33)_2$ via (1.34); in most of the model cases it is $\beta_0 = 1$. The number T bounds the set where the functional loses its ellipticity properties. If Du belongs to $\{|z| > T\}$, then regularity is implied by ellipticity $(1.32)_4$. Otherwise, there is nothing to prove. The shape of (1.35), which trivializes when $\|Du\|_{L^\infty(B/2)} \le T$, reflects this fact. For assumptions (1.33) (with the parameter τ), see the comments in Remark 1.12 below.

Remark 1.10. The structure assumptions (1.32)–(1.33) cover the case of uniform ellipticity (1.3)–(1.5) by taking $g_1(\cdot) \approx g_2(\cdot) \approx \tilde{a}(\cdot)$; see (6.18) below. In particular, in (1.33) they are $\tau = i_a$, $\beta_0 = 1$, and any choice of $T > 0$ is possible. For this we refer to Theorem 1.15 below and its proof in Section 6.5.

As anticipated in Theorem 1.4, the corresponding two-dimensional version of Theorem 1.9 involves the Orlicz space $\mathrm{L}^2(\mathrm{Log\,L})^\alpha$ and it is contained in the following:

THEOREM 1.11 (General two-dimensional scalar estimates). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional in* (1.1) *under the assumptions* (1.31)–(1.33) *for $n = 2$. Assume that $f \in \mathrm{L}^2(\mathrm{Log\,L})^\alpha(\Omega)$ for some $\alpha > 2$ and that the inequality*

$$\frac{g_2(t)}{g_1(t)} \le c_b \left(\int_T^t g_1(s) s \, ds \right)^{\frac{2\beta_0 - \sigma}{2}} + c_b \quad \forall t \ge T \tag{1.37}$$

holds for $c_b \ge 1$, where σ is such that $0 < \sigma \le \beta_0$. Then Du is locally bounded in Ω. Moreover, for every $\theta \in (0, \sigma)$ the estimate

$$\begin{aligned} &\int_T^{\|Du\|_{L^\infty(B/2)}} g_1(s) s \, ds \\ &\le c \left(\fint_B F(Du) dx + \|f\|_{L^2(B)}^{\frac{\gamma}{\gamma-1}} + T^\gamma + \mu^\gamma \right)^{\frac{2}{\sigma - \theta}} \\ &\quad + c \|f\|_{\mathrm{L}^2(\mathrm{Log\,L})^\alpha(B)}^{\frac{\tau+2+\theta}{\tau+1}} + c[(T + \mu)\|f\|_{\mathrm{L}^2(\mathrm{Log\,L})^\alpha(B)}]^{1+\theta} + c \end{aligned} \tag{1.38}$$

holds for every ball $B \Subset \Omega$ with radius $R \leq 1/2$, where $c \equiv c(\nu, \tau, c_a, c_b, \sigma, \beta_0, \theta, \alpha)$ is a constant that is independent of T.

Remark 1.12 ($\triangle_2$ and ∇_2 conditions). A main point in this paper is the treatment of functionals that necessarily satisfy the $\triangle_2$-condition in (1.6), as in (1.8). A condition that is in some sense dual to the one in (1.6) is the so-called ∇_2-condition. This means, by using the notation (1.6), that

$$A(2t) \geq (2+\varepsilon)A(t) \quad \text{for some } \varepsilon > 0 \tag{1.39}$$

holds for every $t > 0$. This implies that the function $A(\cdot)$ has superlinear growth, i.e., $t^{\widetilde{\gamma}} \lesssim A(t)$, where $\widetilde{\gamma} = \log(2+\varepsilon)/\log 2 > 1$. Indeed, condition (1.39) serves to rule out so-called nearly linear growth and therefore functionals of the type

$$w \mapsto \int_\Omega |Dw| \log(1+|Dw|)dx. \tag{1.40}$$

See, for instance, the results in [27, 48] for related regularity results.

The theorems in this paper, while considering the case where (1.6) fails, fall in the realm of (1.39). Assumptions (1.33) automatically rule out functionals as in (1.40). On the other hand, problems in approaching nearly linear growth conditions as in (1.40) appear by looking at the bound (1.15), where $q/p \to 1$ when $p \to 1$. Providing a general theory going beyond the validity of (1.39) is an interesting issue that will be treated in forthcoming work. We refer to [3, 4] and related references for results in this direction.

1.4 The Vectorial Case

In the vectorial case $N > 1$ it is well-known that minimizers of functionals as in (1.1) can develop singularities [51, 60]. This already occurs in the uniformly elliptic setting and when $f \equiv 0$, and is a genuinely vectorial phenomenon. The best one can do in the general case is to look for singular sets dimension estimates; cf. [32]. However, a radial structure $F(z) = \widetilde{F}(|z|)$ as in Theorem 1.3 rules out singularities, as initially noticed by Uhlenbeck [61] (see also Ural′ceva [62]). In this case the Euler-Lagrange equation of the functional (1.1) reads as

$$-\operatorname{div} a(Du) = f \quad \text{with } a(z) = \widetilde{a}(|z|)z, \text{ where } \widetilde{a}(|z|) := \frac{\widetilde{F}'(|z|)}{|z|}, \tag{1.41}$$

for every $z \in \mathbb{R}^{N \times n}$ such that $|z| \neq 0$. This will be essentially the new assumption we consider in the vectorial case that works beside the usual convexity of F. The other ones we consider to prove a vectorial version of Theorem 1.9 are, up to a suitable reformulation, essentially the same, and actually simpler. The ellipticity properties of the mapping $z \mapsto \partial^2 F(z)$ can still be described via the two control functions $g_1(\cdot), g_2(\cdot)$ as in (1.31)–(1.32), while, differently from the scalar case, (1.33) can be replaced by

$$\begin{cases} t \mapsto \dfrac{\widetilde{a}(t)}{(t^2+\mu^2)^{\frac{\gamma-2}{2}}} & \text{is nondecreasing on } (0,\infty) \text{ for some } \gamma > 1, \\ \nu(t^2+\mu^2)^{\frac{\gamma-2}{2}} \le g_1(t) & \text{for } t \ge T, \end{cases} \tag{1.42}$$

where $\nu > 0$ and $0 \le \mu \le 1$ are fixed constants. The number γ here plays the role of $\tau + 2$ in $(1.33)_1$, according to $(1.33)_2$, as we are going to take taking $\beta_0 = 1$ in comparison to (1.33). The main result in the vectorial case, and actually containing all the precise assumptions, is the following:

THEOREM 1.13 (General vectorial estimates). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ be a local minimizer of the functional in* (1.1)*, and assume that there exists a function $\widetilde{F}\colon [0,\infty) \to [0,\infty)$ of class $C^1_{\mathrm{loc}}[0,\infty) \cap C^2_{\mathrm{loc}}(0,\infty)$ such that*

$$F(z) = \widetilde{F}(|z|) \quad \forall z \in \mathbb{R}^{N\times n} \tag{1.43}$$

holds. Moreover, assume that (1.31)–(1.32) *and* (1.42) *hold together with*

$$\frac{g_2(t)}{g_1(t)} \le c_b \min\left\{ \left(\int_T^t g_1(s)s\,ds \right)^{\frac{2-\sigma}{(1+\vartheta)n}}, \left(\frac{1}{t^{1/\beta_1}} \int_T^t g_1(s)s\,ds \right)^{\frac{4\beta_1}{(1+\vartheta)(n-2)}} \right\} + c_b, \tag{1.44}$$

for every $t \ge T$, where $\sigma \in (0,2]$, $1/\gamma \le \beta_1 < 1$, $c_b \ge 1$. The number ϑ is such that $\vartheta = 0$ if $\gamma \ge 2$ in (1.42) (*and, consequently, $t \mapsto \widetilde{F}'(t)/t$ is nondecreasing*) *and $\vartheta = 1$ otherwise.*

- *If $f \in L(n,1)(\Omega;\mathbb{R}^N)$ and $n > 2$, then $Du \in L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^{N\times n})$. Moreover, estimate* (1.35) *holds with $\beta_0 = 1$ and $\tau = \gamma - 2$; the constant c depends only on n, N, ν, c_a, c_b, γ, σ, β_1, and $\widetilde{a}(1)$.*
- *If $f \equiv 0$, then the same result holds replacing* (1.44) *by*

$$\frac{g_2(t)}{g_1(t)} \le c_b \left(\int_T^t g_1(s)s\,ds \right)^{\frac{2-\sigma}{(1+\vartheta)n}} + c_b, \quad \forall t \ge T. \tag{1.45}$$

- *If $n = 2$, assume $f \in \mathrm{L}^2(\mathrm{Log\,L})^\alpha(\Omega;\mathbb{R}^N)$ for some $\alpha > 2$. Then $Du \in L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^{N\times n})$ holds replacing* (1.44) *by* (1.45) *and estimate* (1.38) *holds for every $\theta \in (0,\sigma)$, where $\sigma \le 1$ and $\tau = \gamma - 2$. The constant c here also depends on α.*

Notice that the convexity of F coming from $(1.32)_1$ implies that $\widetilde{a}(\cdot)$ is nonnegative (see (5.4) below). Assumption (1.42) is natural and is satisfied by the p-Laplacian system (1.4) by taking $\widetilde{a}(t) \equiv t^{p-2}$, $\gamma \equiv p > 1$ and $\mu \equiv 0$; it holds for $\gamma = i_a + 2 > 1$ for the case of (1.2) and (1.5) when (1.3) is assumed. In the setting of (p,q)-growth conditions from Section 1.1, the fact that $t \mapsto \widetilde{F}'(t)/t$ is nondecreasing typically corresponds to the case $p \ge 2$. When no information is available on $t \mapsto \widetilde{F}'(t)/t$ and $f \equiv 0$ and/or $\gamma < 2$, the required bound becomes $q/p < 1 + 1/n$, and it coincides with the known bounds when $p < 2$; see [26]. In comparison with Theorem 1.9, here we are taking $1/\gamma \leq \beta_1 < 1$ rather than $\beta_1 \in (0,1)$; this is by no means restrictive as in this setting we try to take β_1 as close to 1 as possible since this is going to weaken the main assumption (1.44). When dealing with functionals with fast growth, in most of the cases it happens that $t \mapsto \widetilde{a}'(t)$ and $\gamma \geq 2$. In that case we often have that $g_1(t) \approx \widetilde{a}(t)$ and $(1.42)_2$ follows from $(1.42)_1$ with $\nu \approx \widetilde{a}(T)/(T^2 + \mu^2)^{(\gamma-2)/2}$; see (5.6) below. See also Section 6 below for several examples in this respect.

Remark 1.14. When $f \equiv 0$, comparing Theorems 1.9 and 1.13 and those of Marcellini [45–48] reveals that the main model examples considered in our and Marcellini's papers can be covered by both theories. However, there are some differences and a somewhat detailed comparison is attempted in Section 6.6 below.

1.5 Uniform Ellipticity Revisited

Theorem 1.13 also yields new results in the standard uniformly elliptic case (1.2), when looking at vectorial problems.

THEOREM 1.15 (Natural growth estimates with $\triangle_2$-condition). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ be a local minimizer of the functional in* (1.5). *Assume that the uniformly ellipticity condition* (1.3) *holds and that $f \in L(n,1)(\Omega;\mathbb{R}^N)$ for $n > 2$. Then $Du \in L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^{N\times n})$ and the estimate*

$$\|Du\|_{L^\infty(B/2)} \leq cA^{-1}\left(\fint_B A(|Du|)dx\right) + cA^{-1}\left(\|f\|^{\frac{i_a+2}{i_a+1}}_{L(n,1)(B)}\right) + c_2 \tag{1.46}$$

holds for every ball $B \Subset \Omega$, for constants c, c_2 depending only on n, N, i_a, s_a, and $\widetilde{a}(1)$. When $n = 2$ a similar result holds assuming that $f \in \mathrm{L}^2(\mathrm{Log\,L})^\alpha(\Omega;\mathbb{R}^N)$ for some $\alpha > 2$. In the case it is

$$i_2 := \liminf_{t\to 0} \frac{\widetilde{a}(t)}{t^{i_a}} > 0, \tag{1.47}$$

then we can take $c_2 = 0$ in (1.46) *and the constant c depends also on i_2.*

Theorem 1.15 and estimate (1.46) extend the results of Baroni [1] and Lieberman [39] to the nonhomogeneous vector-valued case and also provides a local analogue to the global bounds of Cianchi and Maz′ya [15, 17]. Estimate (1.46) is the best possible estimate in this setting. Testing it in the p-Laplacian case (1.4), where $\widetilde{a}(t) \equiv t^{p-2}$, $i_a = s_a = p-2$, and $i_2 = 1$, we get back the optimal local estimate in (1.17). The situation in which $A(\cdot)$ does not satisfy the $\triangle_2$-condition (1.6) is more delicate, and it is treated in the next section.

1.6 Natural Estimates When the Δ_2-Condition Fails

Optimal growth estimates of the form in (1.27), (1.28), and (1.46) are natural when considering general functionals defined in Orlicz spaces as

$$w \mapsto \int_\Omega A(|Dw|)dx, \tag{1.48}$$

when, typically, $A(\cdot)$ is an N-function (see Section 2 and (2.4) below for the definition). Such estimates are known to hold in several cases, but only provided the so-called Δ_2-condition (1.6) is satisfied (this is not the case for functionals as in (1.24)–(1.26)). Their validity has remained a delicate open question otherwise. The next theorem fills this gap in the general case and, in fact, it implies estimates (1.27) and (1.28) when applied to the functionals in (1.26).

THEOREM 1.16 (Natural growth estimates when the $\triangle_2$-condition fails). *Let $u \in W^{1,1}_{\rm loc}(\Omega;\mathbb{R}^N)$ be a local minimizer of the functional in* (1.48) *such that the function $A\colon [0,\infty) \to [0,\infty)$ is $C^1_{\rm loc}[0,\infty) \cap C^2_{\rm loc}(0,\infty)$-regular. Assume that for some $T > 0$ the integrand $F(z) := A(|z|)$ satisfies* (1.31)–(1.32) *and* (1.42) (*with $\widetilde{a}(t) = A'(t)/t$*), *and that* (1.45) *holds, where $\vartheta = 0$ if $\gamma \geq 2$ and $\vartheta = 1$ otherwise. Moreover, assume that*

$$A(t) \leq \Lambda\left(\int_T^t g_1(s)s\,ds\right) + \Lambda \qquad \forall t \geq T \tag{1.49}$$

holds for some $\Lambda \geq 1$ and that there exist three constants $\upsilon, c_\upsilon, d_\upsilon > 1$ such that

$$A^{-1}(t^\upsilon) \leq c_\upsilon A^{-1}(t) + d_\upsilon \quad \textit{holds for every } t \geq A(T). \tag{1.50}$$

Then, for every ball $B \Subset \Omega$, the local a priori estimate

$$\|Du\|_{L^\infty(B/2)} \leq cA^{-1}\left(\fint_B A(|Du|)dx\right) + cA^{-1}(c) \tag{1.51}$$

holds for a constant c depending only on n, N, ν, Λ, υ, c_υ, d_υ, c_a, c_b, γ, σ, T, and $A'(1)$, with the convention that $A^{-1}(t) = T$ for every $t \in [0, A(T)]$.

Comments are in order. Condition (1.49) says that the problem is well-posed in the Orlicz space generated by functions that are asymptotically equivalent to $t \mapsto \int^t g_1(s)s\,ds$; see Section 2 below for the precise definitions. Condition (1.50) exactly relates to the typical situation where the Δ_2-condition on $A(\cdot)$ fails. As a matter of fact, assumption (1.50) is satisfied by integrands with fast growth conditions but fails when tested on integrands with polynomial growth. From the assumptions it follows that $t \mapsto A(t)$ is increasing, and therefore invertible, on $[T,\infty)$; the inverse $A^{-1}(\cdot)$ is defined only on $[A(T),\infty)$. For this reason, estimate (1.51) makes sense only under the specification that $A^{-1}(t) = T$ on $[0, A(T)]$. Notice that in Theorem 1.16 we are not necessarily considering a function $A(\cdot)$ that is an N-function in the sense of Section 2 below.

1.7 Existence and Regularity Results

Our methods can be used to prove existence and regularity results for Dirichlet problems, not necessarily arising from integral functionals, of the type

$$\begin{cases} -\operatorname{div} a(Du) = f & \text{in } \Omega, \\ u \equiv u_0 & \text{on } \partial\Omega, \end{cases} \qquad u_0 \in W^{1,\frac{p(q-1)}{p-1}}(\Omega). \tag{1.52}$$

Here we additionally assume that Ω is a bounded and Lipschitz regular domain of $\mathbb{R}^n$. The vector field $a\colon \mathbb{R}^n \mapsto \mathbb{R}^n$ is assumed to be C^1-regular outside the origin and such that

$$\begin{cases} |a(z)| + (|z|^2+\mu^2)^{1/2}|\partial a(z)| \\ \quad \le \Lambda(|z|^2+\mu^2)^{(q-1)/2} + \Lambda(|z|^2+\mu^2)^{(p-1)/2}, \\ \nu(|z|^2+\mu^2)^{(p-2)/2}|\xi|^2 \le \langle \partial a(z)\xi, \xi\rangle, \end{cases} \tag{1.53}$$

hold, as in (1.12), for every choice of $z, \xi \in \mathbb{R}^n$ with $z \neq 0$, exponents $1 \le p \le q$, ellipticity constants $0 < \nu \le 1 \le L$, and $0 \le \mu \le 1$. Marcellini's by-now classical result [45] states the existence of locally Lipschitz-continuous solutions to (1.53) assuming that $f \in L^\infty(\Omega)$. This can be upgraded to the following optimal version:

THEOREM 1.17 (Existence of locally Lipschitz solutions). *With* $1 < p \le q$, $n > 2$, *and under assumptions* (1.19) *and* (1.53)*, there exists a solution* $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega) \cap W^{1,p}(\Omega)$ *to the Dirichlet problem in* (1.52)*. Moreover, the a priori estimate*

$$\begin{aligned} \|Du\|_{L^\infty(B/2)} \le c\frac{\mathscr{D}}{|B|} + c\left(\frac{\mathscr{D}}{|B|}\right)^{\frac{p}{p-(q-p)n}} + c\|f\|_{L(n,1)(B)}^{\frac{1}{p-1}} \\ + c\|f\|_{L(n,1)(B)}^{\frac{2}{2(p-1)-(q-p)(n-2)}} \end{aligned} \tag{1.54}$$

holds for every ball $B \Subset \Omega$*, where* $c \equiv c(n,p,q,\nu,\Lambda)$*, and*

$$\mathscr{D}^p := \int_\Omega (|Du_0|^2+1)^{\frac{p(q-1)}{2(p-1)}}\,dx + c|\Omega|\|f\|_{L^n(\Omega)}^{p/(p-1)}. \tag{1.55}$$

When $n = 2$ *and* $f \in \mathrm{L}^2(\mathrm{Log\,L})^\alpha(\Omega)$ *holds for some* $\alpha > 2$*, again* Du *is locally bounded in* Ω *and an estimate similar to* (1.54) *holds upon replacing* $\|f\|_{L(n,1)(B)}$ *by* $\|f\|_{\mathrm{L}^2(\mathrm{Log\,L})^\alpha(B)}$.

We notice that in the case when $p = q$ and Ω coincides with a ball B, estimate (1.54) gives

$$\|Du\|_{L^\infty(B/2)}^p \lesssim \fint_B (|Du_0|^p+1)dx + \|f\|_{L(n,1)(B)}^{p/(p-1)},$$

which is the usual a priori local estimate for the Dirichlet problems; see [36] and compare with (1.17).

1.8 Technical Novelties

The approach of this paper departs from previous ones on nonuniformly elliptic equations, as it is based on potential theoretic-type arguments, and it is new already in the homogeneous case $f \equiv 0$. To briefly explain it, we consider Theorem 1.9. The main new idea is that, by taking the so-called second variation equation, the Euler-Lagrange equation of the functional in (1.1) can be treated as if it were uniformly elliptic. This is achieved by controlling the ellipticity ratio $\mathscr{R}(Du)$ in (1.13) by $g_2(M)/g_1(M)$, where $M \equiv \|Du\|_{L^\infty}$; see Section 4.2 below. To such an equation we apply a delicate iteration argument coming from nonlinear potential theory [31, 52], and that in turn finds its origins in the insights of De Giorgi [22]. This is in Lemma 3.1 below.

Notice that such potential theoretic-type arguments work well when applied to uniformly elliptic equations; that's why we perform the uniformization procedure mentioned above. The drawback is that all the constants involved depend on $g_2(M)/g_1(M)$. For this reason, Lemma 3.1 is devised to carefully track the quantitative dependence of the final constants on the various parameters involved in the iteration. This subtle dependence surprisingly plays a crucial role as it allows us to control the way energy estimates are affected by the quantity $g_2(M)/g_1(M)$. This leads to the final estimate (1.16) via an iteration argument; see Lemma 4.8. The delicate interplay between Lemmas 3.1 and 4.8 eventually gives back the bounds (1.14) and (1.30) known for the case $f \equiv 0$. As for the right-hand side f, we notice that Lemma 3.1 leads us to consider a nonlinear potential of f, that is, the one defined in (2.2). As explained in Section 2 below, this potential plays a role similar to that of the standard Riesz potential [36]. Accordingly, it easily allows us to exploit the assumed regularity properties of f and derive the corresponding implied estimates on Du.

It is worth emphasizing that the above line of proof only works at the level of a priori estimates, that is, by assuming that Du is locally bounded and differentiable. To eventually overcome this point and commute a priori estimates in real regularity results, the whole procedure must be embedded in a delicate approximation scheme; see Section 4.1 below. This is aimed at approximating the original functional with more regular ones satisfying growth and ellipticity conditions compatible with those in (1.32) and (1.34). A different type of approximation is needed when dealing with vector-valued minimizers in Theorem 1.13. In this case the approximating functionals exhibit a polynomial behavior at infinity, and some less-known regularity properties of minimizers of functionals with polynomial growth must be employed. These are listed and partly proved in the final Section 8.

2 Lorentz Spaces, Orlicz Spaces, and Nonlinear Potentials; Notation

Let us start with some notation. In this paper we denote by $\Omega \subset \mathbb{R}^n$ an open domain; additional restrictions can be considered. Since our estimates will be local, we shall always assume, w.l.o.g., that Ω is also bounded. We denote by c a general constant larger than 1. Different occurrences from line to line will be still denoted by c. Special occurrences will be denoted by c_1, c_2, or $\widetilde{c}$ or something similar. Important dependencies on parameters will be as usual emphasized by putting them in parentheses. We shall denote by $\mathbb{N}$ the set of positive integers and set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. As usual, we denote by $B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$ the open ball with center x_0 and radius $r > 0$; when it is clear from the context, we omit denoting the center, i.e., $B_r \equiv B_r(x_0)$. When not otherwise stated, different balls in the same context will share the same center. We shall also denote $B_1 = B_1(0)$ if not differently specified. Finally, with B being a given ball with radius r and θ being a positive number, we denote by θB the concentric ball with radius θr and by $B/\theta \equiv (1/\theta)B$. In denoting several function spaces like $L^p(\Omega)$ and $W^{1,p}(\Omega)$, we shall denote the vector-valued version by $L^p(\Omega;\mathbb{R}^k)$ and $W^{1,p}(\Omega;\mathbb{R}^k)$ in the case the maps considered take values in $\mathbb{R}^k$, $k \in \mathbb{N}$. We shall often abbreviate $L^p(\Omega;\mathbb{R}^k) \equiv L^p(\Omega)$ and $W^{1,p}(\Omega;\mathbb{R}^k) \equiv W^{1,p}(\Omega)$. With $\mathscr{B} \subset \mathbb{R}^n$ being a measurable subset with bounded positive measure $0 < |\mathscr{B}| < \infty$, and with $g\colon \mathscr{B} \to \mathbb{R}^k$, $k \geq 1$, being a measurable map, we shall denote the integral average of g over $\mathscr{B}$ by

$$(g)_{\mathscr{B}} \equiv \fint_{\mathscr{B}} g(x)dx := \frac{1}{|\mathscr{B}|}\int_{\mathscr{B}} g(x)dx.$$

For the rest of the paper we shall keep the following notation:

$$H(t) := (t^2 + \mu^2)^{1/2}, \quad H_\varepsilon(t) := [t^2 + (\mu + \varepsilon)^2]^{1/2}, \tag{2.1}$$

for $t > 0$, $\mu \in [0,1]$, $\varepsilon \in (0,1]$. The role of the constant μ will be clear from the context; it will usually be the number introduced in (1.12), (1.33), etc.

We next fix some notation that will be especially useful in the vectorial case. We denote by $\{e^\alpha\}_{\alpha=1}^N$ and $\{e_i\}_{j=1}^n$ standard bases for $\mathbb{R}^N$ and $\mathbb{R}^n$, respectively; we shall always assume $n \geq 2$ and $N \geq 1$. The general second-order tensor of size (N,n) as $\zeta = \zeta_i^\alpha e^\alpha \otimes e_i$ is identified with an element of $\mathbb{R}^{N\times n}$ (here we use the standard convention on the sum of repeated indices). The Frobenius product of second-order tensors z and ξ is defined as $\langle z, \xi\rangle = z_i^\alpha \xi_i^\alpha$; it follows that $\langle \xi,\xi\rangle = |\xi|^2$, and in the rest of the paper we shall use the classical Frobenius norm for matrices and tensors. We shall sometimes use the symbol $\langle\,\cdot\,,\,\cdot\,\rangle$ also to denote the scalar product in $\mathbb{R}^n$. The gradient of a map $u = u^\alpha e^\alpha$ is thus defined as $Du = \partial_{x_i} u^\alpha e^\alpha \otimes e_i$, and the divergence of a tensor $\zeta = \zeta_i^\alpha e^\alpha \otimes e_i$ as $\operatorname{div}\zeta = \partial_{x_i}\zeta_i^\alpha e^\alpha$. When dealing with the integrands of the type $F\colon \mathbb{R}^{N\times n} \to [0,\infty)$ of the type considered in Section 1, we interpret the second differential of $\partial^2 F(z)$ as a fourth-order tensor defined as $\partial^2 F(z) = \partial_{z_j^\beta}\partial_{z_i^\alpha} F(z)(e^\alpha \otimes e_i) \otimes (e^\beta \otimes e_j)$ whenever $z \in \mathbb{R}^{N\times n}$.

We now describe a relevant nonlinear theoretic potential quantity that will play a crucial role in our estimates, with related function spaces. The modified nonlinear

Riesz potential of a map $g \in L^2_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^k)$ is defined as

$$\mathbf{P}_1^g(x_0,R) := \int_0^R \left(\varrho^2 \fint_{B_\varrho(x_0)} |g|^2\,dx\right)^{1/2} \frac{d\varrho}{\varrho} \tag{2.2}$$

for $x_0 \in \mathbb{R}^n$ and $R > 0$. The potential $\mathbf{P}_1^g(\cdot,R)$ plays in the present context the role of the (truncated) Riesz potential, which is instead defined by

$$\mathbf{I}_1^g(x_0,R) := \int_0^R \frac{|g|(B_\varrho(x_0))}{\varrho^{n-1}}\frac{d\varrho}{\varrho} = \int_0^R \frac{1}{\varrho^n}\int_{B_\varrho(x_0)} |g|dx\,d\varrho,$$

provided the argument function g is at least locally L^1-regular. This can be easily seen by using the Hölder inequality to estimate

$$\mathbf{I}_1^g(x_0,R) = |B_1|\int_0^R \fint_{B_\varrho(x_0)} |g|dx\,d\varrho \le |B_1|\mathbf{P}_1^g(x_0,R).$$

The potential in (2.2) is actually a nonlinear one. This is the right one to use when dealing with a large class of problems, from degenerate elliptic and parabolic ones to fully nonlinear equations; see, for instance, [20, 33] and references therein. The potential $\mathbf{P}_1^g$ defines an operator whose behavior is well-known in various function spaces. Here we concentrate on the Lorentz space $L(n,1)$ defined via (1.7) when $n > 2$, and, in the two-dimensional case $n = 2$, on the Orlicz space $\mathrm{L}^2(\mathrm{Log\,L})^\alpha$, which is defined via condition (1.21). Specifically, with $B_R(x_0) \subset \mathbb{R}^n$ we have that $\mathbf{P}_1^g(x_0,R) \le c(n)\|g\|_{L(n,1)(B_R(x_0))}$ holds (see, for instance, [33, lemma 2.3 and lemma 2.4] for a proof of this fact), so that

$$\left\|\mathbf{P}_1^g(\cdot,R)\right\|_{L^\infty(B_R)} \le c\|g\|_{L(n,1)(B_{2R})} \quad \text{holds for every ball } B_R \subset \mathbb{R}^n. \tag{2.3}$$

For the corresponding estimates in $\mathrm{L}^2(\mathrm{Log\,L})^\alpha$ we need a few more preliminaries. We recall that $A\colon[0,\infty) \to [0,\infty)$ is called an N-function provided it is convex, and it is such that

$$A(0) = 0, \quad \lim_{t\to 0}\frac{A(t)}{t} = 0, \quad \text{and} \quad \lim_{t\to\infty}\frac{A(t)}{t} = \infty. \tag{2.4}$$

In this case its convex conjugate defined by $\widetilde{A}(t) := \sup\{st - A(s) : s \ge 0\}$, for every $t \ge 0$, is again an N-function. With $\Omega \subset \mathbb{R}^n$, the Orlicz space $L^A(\Omega)$ is defined as the vector space of measurable maps $w\colon \Omega \to \mathbb{R}^k$ such that the following Luxemburg norm is finite:

$$\|w\|_{L^A(\Omega)} := \inf\left\{\lambda > 0 : \int_\Omega A\left(\frac{|w|}{\lambda}\right)dx \le 1\right\}. \tag{2.5}$$

This is a Banach space. We refer to [55] for more on such spaces. The main information we need is the following:

$$\left|\int_\Omega wv\,dx\right| \le 2\|w\|_{L^A(\Omega)}\|v\|_{L^{\widetilde{A}}(\Omega)} \quad \text{and} \quad \|\mathbb{1}_B\|_{L^{\widetilde{A}}(\Omega)} = \frac{1}{(\widetilde{A})^{-1}(|B|^{-1})}, \tag{2.6}$$

for every ball $B \subset \Omega$. The first relation appearing in the above display is the so-called Hölder inequality in Orlicz spaces. The second relation in (2.6) is a straightforward consequence of the definition in (2.5). Obviously, to determine the space L^A, only the asymptotic behavior of $A(\cdot)$ matters. We are here interested in the special case $A(t) \equiv A_{p,\alpha}(t) := t^p \log^\alpha(1+t)$, for $\alpha > 2$ and $p \ge 1$, with notation $L^A \equiv \mathrm{L}^p(\mathrm{Log\,L})^\alpha$. It is now straightforward to show that

$$\||g|^2\|_{\mathrm{L}(\mathrm{Log\,L})^\alpha(\Omega)} \le 2^\alpha \|g\|^2_{\mathrm{L}^2(\mathrm{Log\,L})^\alpha(\Omega)}. \tag{2.7}$$

Indeed, let us take $\lambda_s > 0$ such that $\int_\Omega (|g|^2/\lambda_s^2)\log^\alpha(1+|g|/\lambda_s)dx \le 1$. Then, with $K \ge 1$, we have

$$\begin{aligned}\int_\Omega \left[\frac{|g|}{K\lambda_s}\right]^2 \log^\alpha\left(1+\left[\frac{|g|}{K\lambda_s}\right]^2\right)dx &\le \int_\Omega \left[\frac{|g|}{K\lambda_s}\right]^2 \log^\alpha\left(1+\frac{|g|}{K\lambda_s}\right)^2 dx \\ &\le \frac{2^\alpha}{K^2}\int_\Omega \frac{|g|^2}{\lambda_s^2}\log^\alpha\left(1+\frac{|g|}{\lambda_s}\right)dx \le 1,\end{aligned}$$

provided we take $K^2 = 2^\alpha$. This shows (2.7).

Concerning $\widetilde{A}_{1,\alpha}(t)$, it is easy to see from the definition of the convex conjugate that it is for all $t \ge 0$ nonnegative and that the supremum is in fact attained for some $s \le \exp(t^{1/\alpha})$ (otherwise the term $st - A_{1,\alpha}(s)$ is negative); hence we obtain $\widetilde{A}_{1,\alpha}(t) \le t\exp(t^{1/\alpha}) \le \exp((1+\alpha)t^{1/\alpha}) =: \mathscr{A}_\alpha(t)$ for every $t \ge 0$, and thus also $\mathscr{A}_\alpha^{-1}(t) \le \widetilde{A}_{1,\alpha}^{-1}(t)$ holds for all $t \ge 1$. Therefore, whenever $B_\varrho \subset \Omega \subset \mathbb{R}^2$ is a ball with radius $\varrho < 1/\sqrt{\pi}$, we conclude via the second relation in (2.6) that

$$\|\mathbb{1}_{B_\varrho}\|_{L^{\widetilde{A}_{1,\alpha}}(\Omega)} = \frac{1}{(\widetilde{A}_{1,\alpha})^{-1}(|B_\varrho|^{-1})} \le \frac{1}{(\mathscr{A}_\alpha)^{-1}(|B_\varrho|^{-1})} \le \frac{c}{\log^\alpha(1/\varrho)} \tag{2.8}$$

for a constant $c = c(\alpha)$. To proceed, we estimate

$$\begin{aligned}\left(\int_{B_\varrho}|g|^2\,dx\right)^{1/2} &\overset{(2.6)}{\le} \sqrt{2}\left(\||g|^2\|_{\mathrm{L}(\mathrm{Log\,L})^\alpha(B_\varrho)}\|\mathbb{1}_{B_\varrho}\|_{L^{\widetilde{A}_{1,\alpha}}(\Omega)}\right)^{1/2} \\ &\overset{(2.7)}{\le} c\left(\|g\|^2_{\mathrm{L}^2(\mathrm{Log\,L})^\alpha(B_\varrho)}\|\mathbb{1}_{B_\varrho}\|_{L^{\widetilde{A}_{1,\alpha}}(\Omega)}\right)^{1/2} \\ &\overset{(2.8)}{\le} \frac{c\|g\|_{\mathrm{L}^2(\mathrm{Log\,L})^\alpha(B_\varrho)}}{\log^{\alpha/2}(1/\varrho)}.\end{aligned}$$

Let us now consider a map $g \in L^2_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^k)$, where we assume that $g \in \mathrm{L}^2(\mathrm{Log\,L})^\alpha(B)$ for every ball $B \subset \mathbb{R}^2$. For an arbitrary ball $B_R(x_0) \subset \mathbb{R}^2$ with $R < 1/\sqrt{\pi}$, we then find

$$\begin{aligned}\mathbf{P}_1^g(x_0,R) &= \int_0^R \frac{1}{\sqrt{\pi}}\left(\int_{B_\varrho(x_0)} |g|^2\,dx\right)^{1/2}\frac{d\varrho}{\varrho}\\ &\le c\|g\|_{\mathrm{L}^2(\mathrm{Log\,L})^\alpha(B_R(x_0))}\int_0^R \frac{d\varrho}{\varrho\log^{\alpha/2}(1/\varrho)},\end{aligned}$$

where the integral on the right-hand side blows up when $\alpha \searrow 2$. Therefore, we finally conclude with the following inequality that holds for all $\alpha > 2$ whenever $B_R \subset \mathbb{R}^2$ is a ball with radius $R \le 1/2$:

$$\mathbf{P}_1^g(x_0,R) \le c(\alpha)\|g\|_{\mathrm{L}^2(\mathrm{Log\,L})^\alpha(B_{2R})} \text{ for every } x_0 \in B_R \text{ and } R \le \tfrac{1}{2}, \tag{2.9}$$

where $c(\alpha) \to \infty$ when $\alpha \searrow 2$.

3 A Nonlinear Iteration

Here we shall exploit in detail some nonlinear potential theoretic arguments developed by Kilpeläinen and Malý [31], who originally obtained pointwise potential estimates building on the fundamental iteration scheme of De Giorgi [22]. The version proposed here is instead closer to the one given in [33]. For related estimates and approaches, we also refer to [34, 35, 52]. The main emphasis here is on the precise dependence of the various constants involved. This will eventually lead to recovering the precise bounds appearing in Theorems 1.2 and 1.9 (see Section 4.2 below).

LEMMA 3.1. *Let $B_{R_0}(x_0) \subset \mathbb{R}^n$, $n \ge 2$, $\kappa \in (0,1/2)$ and let $v \in W^{1,2}(B_{R_0}(x_0))$ be nonnegative, and $f \in L^2(\mathbb{R}^n)$; assume that there exist positive constants M_1, M_2, c_m with $M_1 \ge 1$ and a number $k_0 \ge 0$ such that for all $k \ge k_0$ and every ball $B_r(x_0) \subset B_{R_0}(x_0)$ the inequality*

$$\begin{aligned}\int_{B_{r/2}(x_0)} |D(v-k)_+|^2\,dx &\le \frac{c_m M_1^2}{r^2}\int_{B_r(x_0)} (v-k)_+^2\,dx\\ &\quad + c_m M_2^2\int_{B_r(x_0)} |f|^2\,dx\end{aligned} \tag{3.1}$$

holds. Here we have denoted $(v-k)_+ := \max\{v-k,0\}$. If x_0 is a Lebesgue point of v, then

$$\begin{aligned}v(x_0) \le k_0 &+ cM_1^{1+\max\{\kappa,\frac{n-2}{2}\}}\left(\fint_{B_{R_0}(x_0)} (v-k_0)_+^2\,dx\right)^{1/2}\\ &+ cM_1^{\max\{\kappa,\frac{n-2}{2}\}}M_2\mathbf{P}_1^f(x_0,2R_0)\end{aligned} \tag{3.2}$$

holds for a constant c, which depends only on n and c_m when $n > 2$ and on κ and c_m when $n = 2$.

PROOF. We start by defining a strictly decreasing sequence of radii $\{r_j\}_{j\in\mathbb{N}_0}$, via setting $r_j := 2^{-j}R_0$ for every nonnegative integer j (so that $r_0 = R_0$), and a sequence of numbers $\{W_j\}_{j\in\mathbb{N}_0}$, via

$$W_j := r_j \left(\fint_{B_{r_j}(x_0)} |f|^2\, dx \right)^{1/2}$$

for $j \in \mathbb{N}_0$. Next we recursively define a nondecreasing sequence of numbers $\{k_j\}_{j\in\mathbb{N}_0}$, with k_0 as in the statement, and a sequence of numbers $\{V_j\}_{j\in\mathbb{N}_0}$ in the following way: for each $j \in \mathbb{N}$ we set

$$V_j := \left(\fint_{B_{r_j}(x_0)} (v-k_j)_+^2\, dx \right)^{1/2} \tag{3.3}$$

and choose the successor k_{j+1} such that

$$k_{j+1} = k_j + \frac{V_j}{\delta} \tag{3.4}$$

for some $\delta > 0$ to be determined later. By the above definition we have $V_{j+1} \le 2^{n/2}V_j$, and therefore

$$k_{j+2} - k_{j+1} \le 2^{\frac{n}{2}}(k_{j+1} - k_j) \tag{3.5}$$

for each $j \in \mathbb{N}_0$. We now recall the classical Sobolev inequality and for this, with $\kappa \in (0, 1/2)$ as in the statement, we set

$$1 < \chi := \begin{cases} \frac{n}{n-2} & \text{if } n > 2, \\ 1 + \frac{1}{\kappa} & \text{if } n = 2. \end{cases} \tag{3.6}$$

Also using (3.1), we write

$$\begin{aligned} &\left(\fint_{B_{r_{j+1}}(x_0)} (v-k_j)_+^{2\chi}\, dx \right)^{\frac{1}{\chi}} \\ &\quad\le c r_{j+1}^2 \fint_{B_{r_{j+1}}(x_0)} |D(v-k_j)_+|^2\, dx + c \fint_{B_{r_{j+1}}(x_0)} (v-k_j)_+^2\, dx \\ &\quad\le c M_1^2 \fint_{B_{r_j}(x_0)} (v-k_j)_+^2\, dx + c M_2^2 r_j^2 \fint_{B_{r_j}(x_0)} |f|^2\, dx \end{aligned}$$

for a constant c depending only on χ and c_m. On the other hand, we have in view of (3.3)

$$\begin{aligned} (k_{j+1}-k_j)^{\frac{\chi-1}{\chi}} (V_{j+1})^{\frac{1}{\chi}} &= (k_{j+1}-k_j)^{\frac{\chi-1}{\chi}} \left(\fint_{B_{r_{j+1}}(x_0)} (v-k_{j+1})_+^2\, dx \right)^{\frac{1}{2\chi}} \\ &\le \left(\fint_{B_{r_{j+1}}(x_0)} (v-k_j)^{2(\chi-1)} (v-k_{j+1})_+^2\, dx \right)^{\frac{1}{2\chi}} \le \end{aligned}$$

$$\leq \left(\fint_{B_{r_{j+1}}(x_0)} (v - k_j)_+^{2\chi} \, dx \right)^{\frac{1}{2\chi}}.$$

Combining the estimates in the last two displays yields

$$(k_{j+1} - k_j)^{\frac{\chi-1}{\chi}} (V_{j+1})^{\frac{1}{\chi}} \leq c_0(\chi, c_m)(M_1 V_j + M_2 W_j). \tag{3.7}$$

This is precisely the starting point of the iteration scheme, for which the ultimate goal is to show that in the limit $j \to \infty$ the sequence $\{V_j\}_{j \in \mathbb{N}_0}$ vanishes, while the sequence $\{k_j\}_{j \in \mathbb{N}_0}$ remains bounded. This in turn will imply boundedness of v with a corresponding estimate (provided that x_0 is a Lebesgue point). Concerning a suitable choice of $\delta > 0$ from (3.4) (which shall be independent of the index j), we first assume that

$$k_{j+2} - k_{j+1} \geq \frac{1}{2}(k_{j+1} - k_j) \tag{3.8}$$

holds. By definition of the sequence $\{k_j\}_{j \in \mathbb{N}}$, we then deduce from (3.4) and (3.7) the estimate

$$\begin{aligned} 2^{-\frac{1}{\chi}} \delta^{\frac{1}{\chi}} (k_{j+1} - k_j) &\leq \delta^{\frac{1}{\chi}} (k_{j+1} - k_j)^{\frac{\chi-1}{\chi}} (k_{j+2} - k_{j+1})^{\frac{1}{\chi}} \\ &\leq (k_{j+1} - k_j)^{\frac{\chi-1}{\chi}} (V_{j+1})^{\frac{1}{\chi}} \\ &\leq c_0 M_1 \delta (k_{j+1} - k_j) + c_0 M_2 W_j. \end{aligned} \tag{3.9}$$

Therefore, if

$$2^{-\frac{1}{\chi}} \delta^{\frac{1}{\chi}} = 2 c_0 M_1 \delta, \quad \text{i.e.,} \quad \delta := 2^{-\frac{\chi+1}{\chi-1}} c_0^{-\frac{\chi}{\chi-1}} M_1^{-\frac{\chi}{\chi-1}},$$

then (3.9) reduces with (3.5) to

$$k_{j+2} - k_{j+1} \leq 2^{\frac{n}{2}} (k_{j+1} - k_j) \leq c_1 M_1^{\frac{1}{\chi-1}} M_2 W_j$$

for some constant c_1 still depending only on χ and c_m. This last inequality holds provided (3.8) is satisfied. Consequently, we may work in any case with the inequality

$$k_{j+2} - k_{j+1} \leq \frac{1}{2}(k_{j+1} - k_j) + c_1 M_1^{\frac{1}{\chi-1}} M_2 W_j$$

for all $j \in \mathbb{N}_0$. Summing up the above inequalities, we infer first

$$\sum_{j=0}^{\infty} (k_{j+2} - k_{j+1}) \leq k_1 - k_0 + 2 c_1 M_1^{\frac{1}{\chi-1}} M_2 \sum_{j=0}^{\infty} W_j$$

and then, telescoping, we get

$$\begin{aligned}\lim_{j\to\infty} k_j &= \sum_{j=0}^{\infty}(k_{j+2}-k_{j+1}) + k_1\\ &\le k_0 + 2(k_1-k_0) + 2c_1 M_1^{\frac{1}{\chi-1}} M_2 \sum_{j=0}^{\infty} W_j\\ &\le k_0 + \frac{2}{\delta}V_0 + 2c_1 M_1^{\frac{1}{\chi-1}} M_2 \sum_{j=0}^{\infty} W_j\\ &\le k_0 + c_2 M_1^{\frac{\chi}{\chi-1}} V_0 + c_2 M_1^{\frac{1}{\chi-1}} M_2 \sum_{j=0}^{\infty} W_j\end{aligned} \tag{3.10}$$

with c_2 depending only on χ and c_m.

Next, we estimate the latter sum (setting $r_{-1} := 2R_0$) in terms of the potential of f via

$$\begin{aligned}\sum_{j=0}^{\infty} W_j &= \sum_{j=0}^{\infty} r_j \left(\fint_{B_{r_j}(x_0)} |f|^2\,dx\right)^{1/2}\\ &= \sum_{j=0}^{\infty}\int_{r_j}^{r_{j-1}} d\varrho \left(\fint_{B_{r_j}(x_0)} |f|^2\,dx\right)^{1/2}\\ &\le 2^{\frac{n}{2}} \sum_{j=0}^{\infty}\int_{r_j}^{r_{j-1}} \left(\varrho^2 \fint_{B_\varrho(x_0)} |f|^2\,dx\right)^{1/2}\frac{d\varrho}{\varrho}\\ &\le 2^{\frac{n}{2}} \int_0^{2R_0} \left(\varrho^2 \fint_{B_\varrho(x_0)} |f|^2\,dx\right)^{1/2}\frac{d\varrho}{\varrho} = 2^{\frac{n}{2}}\mathbf{P}_1^f(x_0,2R_0).\end{aligned} \tag{3.11}$$

If $\mathbf{P}_1^f(x_0,2R_0)$ is finite (otherwise, there is nothing to prove), then $\{k_j\}_{j\in\mathbb{N}_0}$ is a nondecreasing, bounded sequence and consequently converges. In turn, by (3.4) this allows us to conclude the convergence $V_j \searrow 0$ as $j\to\infty$. Then, since x_0 is a Lebesgue point for v, we have

$$\begin{aligned}v(x_0) &= \lim_{j\to\infty}\fint_{B_{r_j}(x_0)} v\,dx \le \limsup_{j\to\infty}\left(\fint_{B_{r_j}(x_0)} v^2\,dx\right)^{1/2}\\ &\le \lim_{j\to\infty} V_j + \lim_{j\to\infty} k_j\\ &= \lim_{j\to\infty} k_j \overset{(3.10)}{\le} k_0 + c_2 M_1^{\frac{\chi}{\chi-1}} V_0 + c_2 M_1^{\frac{1}{\chi-1}} M_2 \sum_{j=0}^{\infty} W_j.\end{aligned}$$

In order to estimate the right-hand side in the last display we use the definition of V_0 in (3.3) and (3.11), thereby concluding the proof of the lemma by recalling the definition of χ in (3.6). □

4 Scalar Estimates and Theorems 1.9 and 1.11

Here we give the proofs of Theorems 1.9 and 1.11; therefore, in Section 4 we shall consider the corresponding assumptions to be in force. The proof goes in several steps, which are described in Sections 4.1–4.4. We notice that some of the forthcoming lemmas also hold in the two-dimensional case $n = 2$, and this will be emphasized in the corresponding statements. In the same way, we shall keep a level of generality that is larger than the one needed in the proof of Theorem 1.9; this will be useful when proving Theorem 1.13 later on. Upon setting $f \equiv 0$ outside Ω, in the following we can assume that $f \in L(n,1)(\mathbb{R}^n)$; in particular, we have $f \in L^2(\mathbb{R}^n)$.

4.1 Approximation Tools Without the Δ_2-Condition

In this section we build up a family of integrands $\{F_\varepsilon\}$ aimed at approximating the original integrand $z \mapsto F(z)$ considered in (1.1) with globally C^2-regular and strictly convex integrands. This means we are aiming at correcting the potential lack of strict convexity and smoothness of $z \mapsto F(z)$ on the set $\{|z| \le T\}$ expressed in condition $(1.32)_3$. The first idea would be to take a smoothed version F_ε of F via convolution with ε-mollifiers. However, this strategy might not work immediately since it would not be completely clear how to get uniform estimates of the type in (1.32) on the approximation functionals. Therefore, we take a different path and modify $z \mapsto F(z)$ essentially only on the set $\{|z| \le T\}$. The details are as follows.

We start by choosing a number $\overline{T}$ such that

$$T < \overline{T} \le T + \min\{1, T\}, \tag{4.1}$$

where T is the number fixed in (1.31)–(1.33). We then consider a family of standard, nonnegative, smooth, and radially symmetric mollifiers $\{\phi_\varepsilon\}_\varepsilon$ so that

$$\phi \in C_0^\infty(B_1), \quad \|\phi\|_{L^1} = 1, \quad \phi_\varepsilon(x) := \varepsilon^{-n}\phi(x/\varepsilon), \tag{4.2}$$

while in the following we shall always consider numbers ε that satisfy

$$0 < \varepsilon < \frac{\min\{g_2(T)T, \overline{T} - T\}}{8\sqrt{n}(1 + c_a)} =: \bar{\varepsilon}_0 < 1. \tag{4.3}$$

Here the constant c_a is the one appearing in (1.31). We remark that the right-hand side in the above display is indeed positive, which comes from the fact that $g_1(T), g_2(T) > 0$, which is one of the initial assumptions on $g_1(\cdot), g_2(\cdot)$. Furthermore, we take a cutoff function $\eta \in C_0^\infty([0,\overline{T}); [0,1])$ such that $\eta \equiv 1$ on $[0, (T+\overline{T})/2]$ and $|\eta'|^2 + |\eta''| \le 400/(\overline{T} - T)^2$.

Now, we set up the construction of the C^2-regular and strictly convex integrands. To this end, we define the new integrand

$$L(z) \equiv \widetilde{L}(|z|) := \sqrt{|z|^2+1} - 1,$$

which has linear growth at infinity and is strictly convex. Moreover, it satisfies

$$\frac{|\xi|^2}{(|z|^2+1)^{3/2}} \le \langle \partial^2 L(z)\xi, \xi\rangle \quad \text{and} \quad |\partial^2 L(z)| \le \frac{2\sqrt{n}}{\sqrt{|z|^2+1}} \tag{4.4}$$

for every choice of $z, \xi \in \mathbb{R}^n$. Finally, for every ε as in (4.3), we define

$$F_\varepsilon(z) := \eta(|z|)\bar{F}_\varepsilon(z) + [1-\eta(|z|)]F(z) + \varepsilon L(z) \tag{4.5}$$

where

$$\bar{F}_\varepsilon(z) := (F * \phi_\varepsilon)(z) = \int_{B_1} F(z+\varepsilon y)\phi(y)\,dy \tag{4.6}$$

is the standard mollification of $z \mapsto F(z)$ via ϕ_ε. Hence, each approximated integrand F_ε is a regularized version of the original integrand F in B_T (where it is only known to be convex and thus locally Lipschitz-continuous), and it coincides with $F + \varepsilon L(\cdot)$ outside of $B_{\overline{T}}$, which ensures a quantified strict convexity also in B_T. Since by (1.32) we also have that $z \mapsto F(z)$ is locally C^2-regular outside B_T, we therefore have the following:

LEMMA 4.1. *Let $F\colon \mathbb{R}^n \to \mathbb{R}$ be the integrand of Theorem* 1.9 *with $n \ge 2$, and let $\{F_\varepsilon\}_{\varepsilon\in(0,\bar{\varepsilon}_0)}$ be the family of integrands introduced in* (4.5). *The following properties hold*:

$$\begin{cases} F_\varepsilon & \text{is strictly convex and globally } C^2\text{-regular},\\ F_\varepsilon \equiv F + \varepsilon L & \text{on } \{|z| \ge \overline{T}\},\\ F_\varepsilon \to F & \text{uniformly on compact sets of } \mathbb{R}^n,\\ \partial F_\varepsilon \to \partial F & \text{uniformly in the annulus } (T+\overline{T})/2 \le |z| \le \overline{T}. \end{cases} \tag{4.7}$$

Notice that all the properties in (4.7) are a straightforward consequence of the definitions except for the convexity of F_ε. This will be proved in Lemma 4.2 below, together with a quantitative estimate. For every $\varepsilon \in [0,1]$ we further define

$$g_{2,\varepsilon}(t) := g_2(t) + \frac{2\sqrt{n}\varepsilon}{\sqrt{t^2+1}} \quad \text{for every } t > 0. \tag{4.8}$$

We now want to derive the growth and ellipticity conditions satisfied by the regularized integrands F_ε in terms of the functions $g_1(\cdot), g_{2,\varepsilon}(\cdot)$, especially in comparison to those satisfied by the original integrand F, that is, to (1.32). We have the following:

LEMMA 4.2. *Let $F\colon \mathbb{R}^n \to \mathbb{R}$ be the integrand of Theorem* 1.9 *with $n \ge 2$, and let $\{F_\varepsilon\}_{\varepsilon \le \bar{\varepsilon}_0}$ be the family of integrands introduced in* (4.5), *where $\overline{T}$ is defined in* (4.1) *and $\bar{\varepsilon}_0$ is defined in* (4.3). *There exists a positive number*

$$\bar{\varepsilon} \equiv \bar{\varepsilon}\big(g_1(\cdot), g_2(\cdot), F(\cdot), \partial F(\cdot), T, \overline{T}, c_a, \nu, \tau, \beta_0\big) < \bar{\varepsilon}_0 < 1 \tag{4.9}$$

such that, if $0 < \varepsilon < \bar{\varepsilon}$*, then the following growth and ellipticity properties hold*:

$$
\begin{cases} |\partial^2 F_\varepsilon(z)| \le g_{2,\varepsilon}(|z|) & \text{on } \{|z| \ge \overline{T}\}, \\ g_1(|z|)|\xi|^2 \le \langle \partial^2 F_\varepsilon(z)\xi, \xi\rangle & \text{on } \{|z| \ge \overline{T}\} \text{ for every } \xi \in \mathbb{R}^n, \\ \frac{\varepsilon}{(1+|\overline{T}|^2)^{3/2}}|\xi|^2 \le \langle \partial^2 F_\varepsilon(z)\xi, \xi\rangle & \text{on } \{|z| < \overline{T}\} \text{ for every } \xi \in \mathbb{R}^n. \end{cases} \tag{4.10}
$$

Moreover, the following coercivity properties hold:

$$
\begin{aligned} F_\varepsilon(z) &\ge F(z) \\ &\ge \frac{\nu^{\beta_0}}{(\tau+2)^{\beta_0}}\big[(|z|^2+\mu^2)^{\gamma/2} - (T^2+\mu^2 f)^{\gamma/2}\big] \quad \text{on } \{|z| \ge \overline{T}\}. \end{aligned} \tag{4.11}
$$

Finally, we also have

$$
F_\varepsilon(z) \le 2F(z) + (\overline{T}^\gamma + 1) + \sup_{|\xi| \le 2\overline{T}} F(\xi) \quad \text{for all } z \in \mathbb{R}^n. \tag{4.12}
$$

PROOF. We initially consider positive numbers ε such that $\varepsilon < \bar{\varepsilon}_0$, with $\bar{\varepsilon}_0$ as defined in (4.3); further restrictions will be made later. We start with the proof of the inequalities $(4.10)_{1,2}$. In these cases we work on the set $\{|z| \ge \overline{T}\}$, where $F_\varepsilon \equiv F + \varepsilon L$. It follows by $(1.32)_3$ and (4.4) that for every $\varepsilon < \bar{\varepsilon}_0$ it holds that

$$
|\partial^2 F_\varepsilon(z)| \le |\partial^2 F(z)| + \varepsilon|\partial^2 L(z)| \le g_{2,\varepsilon}(|z|) \quad \text{provided } |z| \ge \overline{T},
$$

that is, $(4.10)_1$. Next, via $(1.32)_4$ and (4.4), we have

$$
g_1(|z|)|\xi|^2 \le \langle \partial^2 F(z)\xi, \xi\rangle + \varepsilon\langle \partial^2 L_\varepsilon(z)\xi, \xi\rangle = \langle \partial^2 F_\varepsilon(z)\xi, \xi\rangle \quad \text{provided } |z| \ge \overline{T}
$$

for every $\xi \in \mathbb{R}^n$, that is, $(4.10)_2$. In order to prove estimate $(4.10)_3$, we first observe the identity

$$
\begin{aligned} \partial^2 F_\varepsilon(z) &= \varepsilon\partial^2 L(z) + \eta(|z|)\partial^2 \bar{F}_\varepsilon(z) + [1-\eta(|z|)]\partial^2 F(z) \\ &\quad + 2\eta'(|z|)\frac{z}{|z|} \otimes [\partial \bar{F}_\varepsilon(z) - \partial F(z)] \\ &\quad + \eta''(|z|)[\bar{F}_\varepsilon(z) - F(z)]\frac{z}{|z|} \otimes \frac{z}{|z|} \\ &\quad + \frac{\eta'(|z|)}{|z|}[\bar{F}_\varepsilon(z) - F(z)]\bigg[\mathbb{I}_n - \frac{z}{|z|} \otimes \frac{z}{|z|}\bigg], \end{aligned} \tag{4.13}
$$

where $\mathbb{I}_n$ denotes the identity $n \times n$ matrix. Recalling that by our assumptions we have that $g_1(\cdot)$ is continuous on $[T, \infty)$ with $g_1(T) > 0$, we then introduce the positive quantities

$$
\mathscr{K}_{\mathrm{i},\varepsilon} := \inf_{(T+\overline{T})/2 \le |z| \le \overline{T}} \frac{\int_{B_1} g_1(|z+\varepsilon y|)\phi(y)\,dy}{g_1(|z|)} > 0
$$

and

$$
\mathscr{K}_{\mathrm{i}} := \inf_{(T+\overline{T})/2 \le |z| \le \overline{T}} g_1(|z|) > 0.
$$

By the continuity of $g_1(\cdot)$, we have $\mathscr{K}_{\mathrm{i},\varepsilon} \to 1$ as $\varepsilon \to 0$. Therefore, we can choose a positive number $\bar{\varepsilon}_1 \equiv \bar{\varepsilon}_1(g_1(\cdot), g_2(\cdot), T, \overline{T}, c_a) < \bar{\varepsilon}_0$ such that

$$0 < \varepsilon < \bar{\varepsilon}_1 \quad \Longrightarrow \quad \mathscr{K}_{\mathrm{i},\varepsilon} \geq 1/2. \tag{4.14}$$

From the definitions in (4.6) and $(1.32)_4$, it follows that

$$\int_{B_1} g_1(|z+\varepsilon y|)\phi(y)dy|\xi|^2 \leq \langle \partial^2 \bar{F}_\varepsilon(z)\xi, \xi\rangle \tag{4.15}$$

holds for all $z, \xi \in \mathbb{R}^n$ with $|z| \geq (T+\overline{T})/2$. Notice that here we have again employed the fact that $\varepsilon \leq \bar{\varepsilon}_0 \leq (\overline{T}-T)/8$, which ensures $|z+\varepsilon y| > T$ whenever $|z| \geq (T+\overline{T})/2$ and $|y| \leq 1$ so that we can use $(1.32)_4$ with z replaced by $z+\varepsilon y$ for the estimate from below in (4.15). For the same reason we deduce from $(1.33)_2$ that

$$\int_{B_1} \left(\int_T^{|z+\varepsilon y|} g_1(s)s\, ds \right)^{\beta_0} \phi(y)dy \leq \bar{F}_\varepsilon(z)$$

holds for all $z \in \mathbb{R}^n$ with $|z| \geq (T+\overline{T})/2$. With these prerequisites, we now come to the proof of $(4.10)_3$ for all $z, \xi \in \mathbb{R}^n$ such that $(T+\overline{T})/2 \leq |z| \leq \overline{T}$. By using again (4.4) and (4.15) in the identity (4.13), and with $\overline{T}-T \leq T$ coming from (4.1), we find

$$\begin{aligned}
\langle \partial^2 F_\varepsilon(z)\xi, \xi\rangle &\geq \varepsilon\langle \partial^2 L(z)\xi, \xi\rangle + \eta(|z|)\langle \partial^2 \bar{F}_\varepsilon(z)\xi, \xi\rangle + [1-\eta(|z|)]\langle \partial^2 F(z)\xi, \xi\rangle \\
&\quad - \frac{10^5}{(\overline{T}-T)^2}\big[\big|\partial \bar{F}_\varepsilon(z) - \partial F(z)\big| + \big|\bar{F}_\varepsilon(z) - F(z)\big|\big]|\xi|^2 \\
&\geq \frac{\varepsilon|\xi|^2}{(|z|^2+1)^{3/2}} + \eta(|z|)\int_{B_1} g_1(|z+\varepsilon y|)\phi(y)dy\, |\xi|^2 \\
&\quad + [1-\eta(|z|)]g_1(|z|)|\xi|^2 \\
&\quad - \frac{10^5}{(\overline{T}-T)^2}\big[\big|\partial \bar{F}_\varepsilon(z) - \partial F(z)\big| + \big|\bar{F}_\varepsilon(z) - F(z)\big|\big]|\xi|^2 \\
&\geq \frac{\varepsilon|\xi|^2}{(|z|^2+1)^{3/2}} + \min\{\mathscr{K}_{\mathrm{i},\varepsilon}, 1\}\mathscr{K}_{\mathrm{i}}|\xi|^2 \\
&\quad - \frac{10^5}{(\overline{T}-T)^2}\big[\big|\partial \bar{F}_\varepsilon(z) - \partial F(z)\big| + \big|\bar{F}_\varepsilon(z) - F(z)\big|\big]|\xi|^2.
\end{aligned}$$

With $\bar{\varepsilon}_1$ determined according to (4.14), we can now define another positive number $\bar{\varepsilon}_2 \leq \bar{\varepsilon}_1$ by choosing it—by means of (4.7) and depending on $g_1(\cdot)$, $g_2(\cdot), F(\cdot), \partial F(\cdot), T, \overline{T}, c_a$—in such a way that

$$\begin{aligned}
&\frac{10^5}{(\overline{T}-T)^2} \sup_{(T+\overline{T})/2 \leq |z| \leq \overline{T}} \big[\big|\partial \bar{F}_\varepsilon(z) - \partial F(z)\big| + \big|\bar{F}_\varepsilon(z) - F(z)\big|\big] \\
&\quad \leq \frac{1}{2}\mathscr{K}_{\mathrm{i}} \overset{(4.14)}{\leq} \min\{\mathscr{K}_{\mathrm{i},\varepsilon}, 1\}\mathscr{K}_{\mathrm{i}}
\end{aligned}$$

holds for all $\varepsilon \in (0, \bar{\varepsilon}_2]$. Therefore, we conclude with

$$\frac{\varepsilon|\xi|^2}{(|z|^2+1)^{3/2}} \le \langle \partial^2 F_\varepsilon(z)\xi, \xi\rangle \quad \text{for } (T+\overline{T})/2 \le |z| \le \overline{T} \tag{4.16}$$

and every $\xi \in \mathbb{R}^n$. In the remaining case, when $|z| \le (T+\overline{T})/2$, we have $\eta(|z|) = 1$, and since $\bar{F}_\varepsilon$ is C^2-regular and convex, we have

$$\begin{aligned}\frac{\varepsilon|\xi|^2}{(|z|^2+1)^{3/2}} &\overset{4.4}{\le} \varepsilon\langle \partial^2 L(z)\xi, \xi\rangle \\ &\le \langle \partial^2 F_\varepsilon(z)\xi, \xi\rangle \quad \text{for } |z| \le (T+\overline{T})/2,\end{aligned} \tag{4.17}$$

every $\xi \in \mathbb{R}^n$ and all $\varepsilon \in (0, \bar{\varepsilon}_0)$. Now $(4.10)_3$ follows by taking into account both inequalities (4.16) and (4.17). Next, the assertion (4.11) follows immediately by (1.33) (recall that $\beta_0 \le 1$) in combination with $F_\varepsilon \equiv F + \varepsilon L$ on $\{|z| \ge \overline{T}\}$ stated in $(4.7)_2$. Finally, with the help of the definition of $L(\cdot)$, the last claim (4.12) is obvious for $|z| \le \overline{T}$, while for $|z| > \overline{T}$ it follows from $(4.7)_2$ and (4.11) provided we take $\varepsilon < \bar{\varepsilon} \equiv \bar{\varepsilon}(\nu, \tau, \beta_0) \le \bar{\varepsilon}_2$. This finally determines the number $\bar{\varepsilon}$ appearing in (4.9), and with the specified dependence on the constants in that display (that also takes into account the dependence on the constants of the previously defined numbers $\bar{\varepsilon}_0$, $\bar{\varepsilon}_1$, and $\bar{\varepsilon}_2$). The proof of the lemma is complete. □

Remark 4.3. By the properties verified in Lemmas 4.1 and 4.2, and in particular by $(4.7)_2$ and (4.12), it trivially follows that, for any given $w \in W^{1,1}_{\mathrm{loc}}(\Omega)$, we have $F(Dw) \in L^1_{\mathrm{loc}}(\Omega)$ if and only if $F_\varepsilon(Dw) \in L^1_{\mathrm{loc}}(\Omega)$.

We conclude this section with yet another technical lemma concerning the properties of the newly defined function $g_{2,\varepsilon}(\cdot)$. Specifically, together with the original function $g_1(\cdot)$, the function $g_{2,\varepsilon}(\cdot)$ is found to satisfy suitable versions of the assumptions (1.31) and (1.34) from Theorem 1.9.

LEMMA 4.4. *For every $\varepsilon \in (0, \bar{\varepsilon}_0)$ as in* (4.3) *the following statements are true*:

- *The functions $[T, \infty) \ni t \mapsto g_{2,\varepsilon}(t)/g_1(t)$ and $t \mapsto g_1(t)t$ are almost nondecreasing and nondecreasing, respectively, in the sense that*

$$T \le s \le t \quad \Longrightarrow \quad \frac{g_{2,\varepsilon}(s)}{g_1(s)} \le 2c_a \frac{g_{2,\varepsilon}(t)}{g_1(t)} \quad \text{and} \quad g_1(s)s \le g_1(t)t. \tag{4.18}$$

- *If* (1.34) *is in force for $n > 2$, then also the following inequality holds for every choice of t and $\overline{T}$ such that $t \ge \overline{T} \ge T$*:

$$\begin{aligned}\frac{g_{2,\varepsilon}(t)}{g_1(t)} \le 2^5 c_b \min\Bigg\{&\left(\int_{\overline{T}}^t g_1(s)s\,ds\right)^{\frac{2\beta_0-\sigma}{n}}, \\ &\left(\frac{1}{t^{1/\beta_1}}\int_{\overline{T}}^t g_1(s)s\,ds\right)^{\frac{4\beta_1}{n-2}}\Bigg\} + 2^5 c_{b,\overline{T}}\end{aligned} \tag{4.19}$$

where

$$c_{b,\overline{T}} := c_b + c_b\big[g_1(\overline{T})\overline{T}(\overline{T}-T)\big]^{\frac{2\beta_0-\sigma}{n}} + c_b\big[g_1(\overline{T})\overline{T}^{1-1/\beta_1}(\overline{T}-T)\big]^{\frac{4\beta_1}{n-2}}, \tag{4.20}$$

and therefore $c_{b,\overline{T}} \to c_b$ *as* $\overline{T} \to T$ *in the case* $\sigma < 2\beta_0$*, and* $c_{b,\overline{T}} \to 2c_b$ *when* $\sigma = 2\beta_0$*.*

- *If* (1.36) *holds, then the following inequality holds for every choice of* t *and* $\overline{T}$ *such that* $t \geq \overline{T} \geq T$:

$$\frac{g_{2,\varepsilon}(t)}{g_1(t)} \leq 4c_b\left(\int_{\overline{T}}^t g_1(s)s\,ds\right)^{\frac{2\beta_0-\sigma}{n}} + 4c_{b,\overline{T}} \tag{4.21}$$

where $c_{b,\overline{T}} = c_b + c_b[g_1(\overline{T})\overline{T}(\overline{T}-T)]^{(2\beta_0-\sigma)/n}$ *and therefore* $c_{b,\overline{T}} \to c_b$ *as* $\overline{T} \to T$ *in the case* $\sigma < 2\beta_0$*, and* $c_{b,\overline{T}} \to 2c_b$ *when* $\sigma = 2\beta_0$*.*

- *For every* $t \geq T$ *it holds that*

$$\frac{g_1(t)}{g_{2,\varepsilon}(t)} \leq 1. \tag{4.22}$$

PROOF. Since $t \mapsto g_1(t)t$ is nondecreasing on $[T,\infty)$ by assumption (1.31), we first observe

$$T \leq s \quad \Longrightarrow \quad g_1(s) \geq \frac{g_1(T)T}{s} \geq \frac{g_1(T)T}{\sqrt{s^2+1}}.$$

Similarly, since $t \mapsto g_2(t)/g_1(t)$ is almost nondecreasing by again (1.31) in combination with the above inequality, we have

$$T \leq s \quad \Longrightarrow \quad g_2(s) \geq \frac{1}{c_a}\frac{g_2(T)}{g_1(T)}g_1(s) \geq \frac{1}{c_a}\frac{g_2(T)T}{\sqrt{s^2+1}}. \tag{4.23}$$

Therefore, for $T \leq s \leq t$, we can estimate

$$\begin{aligned}\frac{g_{2,\varepsilon}(s)}{g_1(s)} &= \frac{g_2(s) + \frac{2\sqrt{n}\varepsilon}{\sqrt{s^2+1}}}{g_1(s)} \overset{(4.3)}{\leq} \frac{g_2(s) + \frac{1}{c_a}\frac{g_2(T)T}{\sqrt{s^2+1}}}{g_1(s)} \\ &\overset{(4.23)}{\leq} 2\frac{g_2(s)}{g_1(s)} \overset{(1.31)}{\leq} 2c_a\frac{g_2(t)}{g_1(t)} \leq \frac{2c_a\Big[g_2(t) + \frac{2\sqrt{n}\varepsilon}{\sqrt{t^2+1}}\Big]}{g_1(t)} = 2c_a\frac{g_{2,\varepsilon}(t)}{g_1(t)},\end{aligned}$$

which proves (4.18). Notice that in fact we have proved that

$$T \leq s \quad \Longrightarrow \quad \frac{g_{2,\varepsilon}(s)}{g_1(s)} \leq 2\frac{g_2(s)}{g_1(s)}. \tag{4.24}$$

As a consequence, we immediately arrive at the claims (4.19) and (4.21), which follow straightaway from (1.34) and (1.36), respectively, by recalling that $t \mapsto g_1(t)t$ is nondecreasing. The final claim (4.22) follows directly from the observation that by $(1.32)_{3,4}$ with $|z| = t \geq T$ we have $g_1(t)/g_2(t) \leq 1$. □

4.2 A Priori Estimates Using Additional Regularity

Again in the setting and under the assumptions of Theorem 1.9, in this section we consider the special situation in which we have a *regular* (weak) solution $u \in W^{1,\infty}_{\mathrm{loc}}(B)$ of the equation

$$-\operatorname{div} a(Du) = f \quad \text{in } B \subset \mathbb{R}^n,\ f \in L^\infty(\mathbb{R}^n), \tag{4.25}$$

where B is a fixed ball and the vector field $a\colon \mathbb{R}^n \to \mathbb{R}^n$ is C^1-regular and satisfies the following assumptions:

$$\begin{cases} |\partial a(z)| \le g_{2,\varepsilon}(|z|) & \text{on } \{|z| \ge \overline{T}\}, \\ g_1(|z|)|\xi|^2 \le \langle \partial a(z)\xi, \xi\rangle & \text{on } \{|z| \ge \overline{T}\} \text{ for every } \xi \in \mathbb{R}^n, \\ \partial a(z) & \text{is strictly positive definite on } \{|z| \le \overline{T}\}. \end{cases} \tag{4.26}$$

Here the function $g_1(\cdot)$ is from our original setup, $g_{2,\varepsilon}(\cdot)$ is as defined in (4.8), and we take ε as in (4.3), so that, in particular, this gives us that the results of Lemma 4.4 are available. Indeed, we shall use (4.18) repeatedly. The number $\overline{T} > T$ is the one initially chosen in (4.1). Before going on, it will be useful to work, for every $T_0 \ge T$, with the integral function $G_{T_0}\colon [0,\infty) \to [0,\infty)$ defined as

$$G_{T_0}(t) := \int_{T_0}^{\max\{t,T_0\}} g_1(s)s\,ds \quad \text{so that} \quad \lim_{t\to\infty} G_{T_0}(t) = \infty. \tag{4.27}$$

We then have the following:

LEMMA 4.5 (Caccioppoli-type inequality). *Let $u \in W^{1,\infty}_{\mathrm{loc}}(B)$ be a weak solution to* (4.25) *under the assumptions* (4.26) *and $n \ge 2$. Moreover, assume that $\partial a(z)$ is symmetric, i.e.,*

$$\partial_{z_i} a_j(z) = \partial_{z_j} a_i(z) \quad \text{for all } z \in \mathbb{R}^n \text{ and } i,j \in \{1,\dots,n\}. \tag{4.28}$$

Let $B_r(x_0) \Subset B$ be another ball and M be such that $\|Du\|_{L^\infty(B_r(x_0))} \le M$ and $\overline{T} \le M$. Then, for each $k \ge 0$, the inequality

$$\begin{aligned} &\int_{B_{r/2}(x_0)} \left|D(G_{\overline{T}}(|Du|) - k)_+\right|^2 dx \\ &\quad\le \frac{c}{r^2}\left[\frac{g_{2,\varepsilon}(M)}{g_1(M)}\right]\int_{B_r(x_0)} (G_{\overline{T}}(|Du|) - k)_+^2\,dx \\ &\qquad + cM^2\int_{B_r(x_0)\cap\{G_{\overline{T}}(|Du|)>k\}} |f|^2\,dx \end{aligned} \tag{4.29}$$

holds for a constant c depending only on c_a, but otherwise independent of M, k, T, $\overline{T}$, and ε.

PROOF. We first observe that under the assumptions considered, that is (4.26), we find numbers $\bar{\nu}$, $\bar{L}$ depending only on M, $\overline{T}$, and $a(\cdot)$ such that

$$|\partial a(z)| \le \bar{L} \quad \text{and} \quad \bar{\nu}|\xi|^2 \le \langle \partial a(z)\xi, \xi\rangle \tag{4.30}$$

provided $|z| \le \overline{T} + M$, where

$$\bar{L} = \max_{|z|\le \overline{T}+M} |\partial a(z)| \quad \text{and} \quad \bar{\nu} := \min_{|z|\le \overline{T}+M, \xi\in\mathbb{R}^n} \frac{\langle \partial a(z)\xi, \xi\rangle}{|\xi|^2}.$$

Notice that $\bar{L}$ is finite since $a(\cdot)$ is C^1-regular and $(4.26)_1$ is in force. Moreover, $\bar{\nu} > 0$ by $(4.26)_{2,3}$ and the positivity of g_1 on $\{s \in [\overline{T}, \overline{T} + M]\}$. Since we assume $\|Du\|_{L^\infty(B_r(x_0))} \le M$, standard regularity theory applies and it follows that

$$\begin{cases} u \in W^{2,2}(B_r(x_0)), \\ u \in C^{1,\alpha}(B_r(x_0)) \text{ for some } \alpha \in (0,1), \\ a(Du) \in W^{1,2}(B_r(x_0); \mathbb{R}^n). \end{cases} \tag{4.31}$$

For this see, for instance, [26, 34]. We can therefore differentiate the equation (4.25). This means that, whenever $s \in \{1, \dots, n\}$, we have that

$$\int_B \langle \partial a(Du) DD_s u, D\varphi\rangle dx = -\int_B f D_s\varphi \, dx \tag{4.32}$$

holds for every choice of $\varphi \in C_0^\infty(B_r(x_0))$ and for every $\varphi \in W_0^{1,2}(B_r(x_0))$ with compact support in $B_r(x_0)$. In the identity (4.32) we now choose $\varphi \equiv \varphi_s := \eta^2(G_{\overline{T}}(|Du|) - k)_+ D_s u$, $k \ge 0$, where $\eta \in C_0^\infty(B_r(x_0), [0,1])$ is a localization function satisfying $\mathbb{1}_{B_{r/2}(x_0)} \le \eta \le \mathbb{1}_{B_r(x_0)}$ and $|D\eta| \le 4/r$. By the regularity properties of u in (4.31), φ is an admissible test function. Notice that by the very definition in (4.27) it follows, in particular, that $\|Du\|_{L^\infty(B_r(x_0))} \le \overline{T}$ implies that $\varphi_s = 0$. Furthermore, we have

$$\begin{aligned} D\varphi_s = {} & \eta^2(G_{\overline{T}}(|Du|) - k)_+ DD_s u + \eta^2 D_s u D(G_{\overline{T}}(|Du|) - k)_+ \\ & + 2\eta(G_{\overline{T}}(|Du|) - k)_+ D_s u D\eta, \end{aligned} \tag{4.33}$$

and whenever $G_{\overline{T}}(|Du|) > k$ (which is only possible for $|Du| > \overline{T}$), then there holds

$$D(G_{\overline{T}}(|Du|) - k)_+ = D[G_{\overline{T}}(|Du|)] = g_1(|Du|) \sum_{s=1}^n D_s u DD_s u \quad \text{and} \quad g_1(|Du|) > 0.$$

Using the identities in the last two displays, from (4.32) with $\varphi \equiv \varphi_s$ and summing over $s \in \{1, \dots, n\}$, we get

$$
\begin{aligned}
&\sum_{s=1}^{n} \int_B \langle \partial a(Du) DD_s u, DD_s u\rangle (G_{\overline{T}}(|Du|) - k)_+ \eta^2 \, dx \\
&\quad + \int_B [g_1(|Du|)]^{-1} \\
&\qquad \cdot \langle \partial a(Du) D(G_{\overline{T}}(|Du|) - k)_+, D(G_{\overline{T}}(|Du|) - k)_+\rangle \eta^2 \, dx \\
&= \sum_{s=1}^{n} \int_B \langle \partial a(Du) DD_s u, (G_{\overline{T}}(|Du|) - k)_+ DD_s u\rangle \eta^2 \, dx \\
&\quad + \sum_{s=1}^{n} \int_B \langle \partial a(Du) DD_s u, D_s u D(G_{\overline{T}}(|Du|) - k)_+\rangle \eta^2 \, dx \\
&= -2 \sum_{s=1}^{n} \int_B \langle \partial a(Du) DD_s u, D\eta\rangle D_s u (G_{\overline{T}}(|Du|) - k)_+ \eta \, dx \\
&\quad - \sum_{s=1}^{n} \int_B f D_s \varphi_s \, dx \\
&= -2 \int_B [g_1(|Du|)]^{-1} \langle \partial a(Du) D(G_{\overline{T}}(|Du|) - k)_+, D\eta\rangle \\
&\qquad \cdot (G_{\overline{T}}(|Du|) - k)_+ \eta \, dx \\
&\quad - \sum_{s=1}^{n} \int_B f D_s \varphi_s \, dx.
\end{aligned}
\tag{4.34}
$$

By the assumed symmetry on ∂a in (4.28), we can apply the Cauchy-Schwarz inequality to estimate the first integral on the right-hand side from above via

$$
\begin{aligned}
&-2 \int_B [g_1(|Du|)]^{-1} \big\langle \partial a(Du) D(G_{\overline{T}}(|Du|) - k)_+, D\eta \big\rangle (G_{\overline{T}}(|Du|) - k)_+ \eta \, dx \\
&\quad \le \frac{1}{2} \int_B [g_1(|Du|)]^{-1} \\
&\qquad \cdot \big\langle \partial a(Du) D(G_{\overline{T}}(|Du|) - k)_+, D(G_{\overline{T}}(|Du|) - k)_+ \big\rangle \eta^2 \, dx \\
&\quad + 8 \int_B [g_1(|Du|)]^{-1} \langle \partial a(Du) D\eta, D\eta\rangle (G_{\overline{T}}(|Du|) - k)_+^2 \, dx.
\end{aligned}
\tag{4.35}
$$

Combining (4.34) and (4.35), reabsorbing terms, and using (4.26) (recall that $(G_{\overline{T}}(|Du|)-k)_+ > 0$ implies in particular $|Du| > \overline{T}$), we then deduce

$$\begin{aligned}
&\int_B \big(g_1(|Du|)(G_{\overline{T}}(|Du|)-k)_+|D^2u|^2 + \big|D(G_{\overline{T}}(|Du|)-k)_+\big|^2\big)\eta^2\,dx \\
&\qquad \le c\int_B \left[\frac{g_{2,\varepsilon}(|Du|)}{g_1(|Du|)}\right](G_{\overline{T}}(|Du|)-k)_+^2|D\eta|^2\,dx \\
&\qquad\quad + 2\sum_{s=1}^n \int_B |f||D_s\varphi_s|\,dx.
\end{aligned} \tag{4.36}$$

Recalling (4.33), for the latter integral the Young inequality yields

$$\begin{aligned}
&2\sum_{s=1}^n \int_B |f||D_s\varphi_s|dx \\
&\quad \le \frac{1}{2}\int_B \big[g_1(|Du|)(G_{\overline{T}}(|Du|)-k)_+|D^2u|^2 + \big|D(G_{\overline{T}}(|Du|)-k)_+\big|^2\big]\eta^2\,dx \\
&\qquad + c\int_{B\cap\{G_{\overline{T}}(|Du|)>k\}} (G_{\overline{T}}(|Du|)-k)_+^2|D\eta|^2\,dx \\
&\qquad + c\int_{B\cap\{G_{\overline{T}}(|Du|)>k\}} |f|^2\big\{[g_1(|Du|)]^{-1}(G_{\overline{T}}(|Du|)-k)_+ + |Du|^2\big\}\eta^2\,dx.
\end{aligned}$$

We notice that all the terms in the above display make sense as $g_1(t) > 0$ for $t \ge T$, and all the terms are evaluated only where $|Du| > \overline{T} > T$. Moreover, we observe from (4.18)

$$G_{\overline{T}}(t) = \int_{\overline{T}}^t g_1(s)s\,ds \le g_1(t)t\int_{\overline{T}}^t ds = g_1(t)t(t-\overline{T}) \le g_1(t)t^2$$

for all $t \ge \overline{T}$ so that

$$\begin{aligned}
&\int_{B\cap\{G_{\overline{T}}(|Du|)>k\}} |f|^2\big\{[g_1(|Du|)]^{-1}(G_{\overline{T}}(|Du|)-k)_+ + |Du|^2\big\}\eta^2\,dx \\
&\qquad \le 2\int_{B\cap\{G_{\overline{T}}(|Du|)>k\}} |f|^2|Du|^2\eta^2\,dx.
\end{aligned}$$

Gathering the content of the last four displays and reabsorbing terms once again, we find

$$\begin{aligned}
&\int_B \big(g_1(|Du|)(G_{\overline{T}}(|Du|)-k)_+|D^2u|^2 + \big|D(G_{\overline{T}}(|Du|)-k)_+\big|^2\big)\eta^2\,dx \\
&\qquad \le c\int_B \left[\frac{g_{2,\varepsilon}(|Du|)}{g_1(|Du|)}+1\right](G_{\overline{T}}(|Du|)-k)_+^2|D\eta|^2\,dx \\
&\qquad\quad + c\int_{B\cap\{G_{\overline{T}}(|Du|)>k\}} |f|^2|Du|^2\eta^2\,dx,
\end{aligned}$$

where c is an absolute constant. By estimating (when $|Du| \geq \overline{T}$)

$$\left[\frac{g_{2,\varepsilon}(|Du|)}{g_1(|Du|)} + 1\right] \overset{(4.18)}{\leq} 2c_a\left[\frac{g_{2,\varepsilon}(M)}{g_1(M)}\right] + 1 \overset{(4.22)}{\leq} 3c_a\left[\frac{g_{2,\varepsilon}(M)}{g_1(M)}\right]$$

and taking into account the properties of the localization function η, we conclude with (4.29), and the proof of the lemma is complete. □

A corresponding version of the Caccioppoli-type inequality, with different constants, holds if the symmetry assumption (4.28) on $\partial a(\cdot)$ is dropped.

LEMMA 4.6. *Let $u \in W^{1,\infty}_{\mathrm{loc}}(B)$ be a solution to* (4.25) *under the assumptions* (4.26)*, with $n \geq 2$. Let $B_r(x_0) \Subset B$ be another ball and M be such that $\|Du\|_{L^\infty(B_r(x_0))} \leq M$ and $\overline{T} \leq M$. Then, for each $k \geq 0$, the inequality*

$$\begin{aligned}
&\int_{B_{r/2}(x_0)} \left|D(G_{\overline{T}}(|Du|) - k)_+\right|^2 dx \\
&\quad\leq \frac{c}{r^2}\left[\frac{g_{2,\varepsilon}(M)}{g_1(M)}\right]^2 \int_{B_r(x_0)} (G_{\overline{T}}(|Du|) - k)_+^2 \, dx \\
&\qquad + cM^2 \int_{B_r(x_0)\cap\{G_{\overline{T}}(|Du|)>k\}} |f|^2 \, dx
\end{aligned} \tag{4.37}$$

holds for a constant c depending only on c_a, but otherwise independent of M, k, T, $\overline{T}$, and ε.

PROOF. The proof follows that given for Lemma 4.5, with some minor modifications due to the fact that (4.28) is not in force here. We proceed as for Lemma 4.5 and arrive at (4.34). At this point we directly use assumption $(4.26)_1$ in order to estimate the right-hand side. The outcome is

$$\begin{aligned}
&\int_B \left(g_1(|Du|)(G_{\overline{T}}(|Du|) - k)_+|D^2u|^2 + \left|D(G_{\overline{T}}(|Du|) - k)_+\right|^2\right)\eta^2 \, dx \\
&\quad\leq 2\int_B \frac{g_{2,\varepsilon}(|Du|)}{g_1(|Du|)}|D(G_{\overline{T}}(|Du|) - k)_+|(G_{\overline{T}}(|Du|) - k)_+|D\eta|\eta \, dx \\
&\qquad + \sum_{s=1}^n \int_B |f||D_s\varphi_s| \, dx.
\end{aligned}$$

Using the Young inequality and reabsorbing terms yields

$$\begin{aligned}
&\int_B \left(g_1(|Du|)(G_{\overline{T}}(|Du|) - k)_+|D^2u|^2 + \left|D(G_{\overline{T}}(|Du|) - k)_+\right|^2\right)\eta^2 \, dx \\
&\quad\leq c\int_B \left[\frac{g_{2,\varepsilon}(|Du|)}{g_1(|Du|)}\right]^2 (G_{\overline{T}}(|Du|) - k)_+^2|D\eta|^2 \, dx \\
&\qquad + c\sum_{s=1}^n \int_B |f||D_s\varphi_s| dx.
\end{aligned}$$

The last integral can be then treated as for Lemma 4.5—see (4.36) and subsequent estimates—which leads to

$$\begin{aligned}\int_B \big(g_1(|Du|)(G_{\overline{T}}(|Du|)-k)_+|D^2u|^2 + \big|D(G_{\overline{T}}(|Du|)-k)_+\big|^2\big)\eta^2\,dx \\ \le c\int_B \bigg(\bigg[\frac{g_{2,\varepsilon}(|Du|)}{g_1(|Du|)}\bigg]^2+1\bigg)(G_{\overline{T}}(|Du|)-k)_+^2|D\eta|^2\,dx \\ + c\int_{B\cap\{G_{\overline{T}}(|Du|)>k\}} |f|^2|Du|^2\eta^2\,dx.\end{aligned}$$

From this inequality, the assertion (4.37) follows as for Lemma 4.5. □

As a consequence of the Caccioppoli-type inequalities (4.29) and (4.37), we can now give a first quantitative L^∞-estimate for the gradient, still formulated in terms of the intrinsic function $G_{\overline{T}}$ defined in (4.27).

LEMMA 4.7. *Let $u \in W^{1,\infty}_{\mathrm{loc}}(B)$ be a solution to* (4.25) *under the assumptions* (4.26)*, with $n \ge 2$. Let $\kappa \in (0,1/2)$ be a number, let $B_{R_0}(x_0) \Subset B$ be another ball, and M be such that $\|Du\|_{L^\infty(B_{R_0}(x_0))} \le M$ and $\overline{T} \le M$. Then the inequality*

$$\begin{aligned}G_{\overline{T}}(|Du(x_0)|) \\ \le k + c\bigg[\frac{g_{2,\varepsilon}(M)}{g_1(M)}\bigg]^{\left(\frac{1+\vartheta}{2}\right)\left(1+\max\left\{\kappa,\frac{n-2}{2}\right\}\right)} \\ \cdot\bigg(\fint_{B_{R_0}(x_0)}(G_{\overline{T}}(|Du|)-k)_+^2\,dx\bigg)^{1/2} \\ + c\bigg[\frac{g_{2,\varepsilon}(M)}{g_1(M)}\bigg]^{\left(\frac{1+\vartheta}{2}\right)\max\left\{\kappa,\frac{n-2}{2}\right\}} M\mathbf{P}_1^f(x_0,2R_0)\end{aligned} \tag{4.38}$$

holds for every $k \ge 0$, where the constant c depends only on n and c_a when $n > 2$ and on κ and c_a when $n = 2$ (but is independent of M, k, T, $\overline{T}$, and ε). Then the number ϑ is such that $\vartheta = 0$ in the case where the symmetry assumptions (4.28) *are in force and $\vartheta = 1$ otherwise.*

PROOF. Using (4.29) and (4.37), we are able to verify (3.2) with the obvious choices $v \equiv G_{\overline{T}}(|Du|)$, $M_1 := [g_{2,\varepsilon}(M)/g_1(M)]^{(1+\vartheta)/2} \ge 1$, and $M_2 := M$, and for a proper constant c_m that depends only on c_a; at this point the claim (4.38) follows directly by (3.2) from Lemma 3.1. Notice that Lemma 3.1 applies, as by (4.31) we have that every point $x_0 \in B$ is actually a Lebesgue point for Du and therefore also for $G_{\overline{T}}(|Du|)$. □

We can finally conclude with the first basic a priori estimate. For this, recall the notation introduced in (2.1).

LEMMA 4.8. *Let $u \in W^{1,\infty}_{\mathrm{loc}}(B)$ be a solution to* (4.25) *under the assumptions* (4.26) *and* (4.28) *and with $n \geq 3$. Let $\{F_\varepsilon\}$ be the family of approximating integrands introduced in Lemma* 4.2 (*and recall that* (1.31), (1.33), *and* (1.34) *hold as in the statement of Theorem* 1.9 *for some $\sigma \in (0, 2\beta_0]$ and $\beta_1 \in (0,1)$*). *Let $B_R \Subset B$ be a ball and denote $f_{B_R} := \mathbb{1}_{B_R} f$ so that $f_{B_R} \in L^2(\mathbb{R}^n)$. Then the inequality*

$$
\begin{aligned}
&G_{\overline{T}}(\|Du\|_{L^\infty(B_{R/2})})\\
&\quad \leq c\left(\fint_{B_R} F_\varepsilon(Du)dx\right)^{\frac{2}{\sigma}} + c\left(\fint_{B_R} F_\varepsilon(Du)dx\right)^{\frac{1}{\beta_0}}\\
&\qquad + c\left\|\mathbf{P}_1^{f_{B_R}}(\cdot, R)\right\|_{L^\infty(B_R)}^{\frac{1}{1-\beta_1}} + c\left\|\mathbf{P}_1^{f_{B_R}}(\cdot, R)\right\|_{L^\infty(B_R)}^{\frac{\tau+2}{\tau+1}}\\
&\qquad + cH(\overline{T})\left\|\mathbf{P}_1^{f_{B_R}}(\cdot, R)\right\|_{L^\infty(B_R)}
\end{aligned}
\tag{4.39}
$$

holds for a constant $c \equiv c(n, \nu, \tau, c_a, c_{b,\overline{T}}, \sigma, \beta_0, \beta_1)$. The constant $c_{b,\overline{T}}$ was defined in (4.20) *and c_b was introduced in* (1.34).

PROOF. The main ingredient in the proof is the local a priori estimate (4.38) that is going to be applied under the symmetry assumption (4.28) with $\vartheta = 0$. Indeed, later on we shall use Lemma 4.8 with the choice $a(\cdot) \equiv \partial F_\varepsilon$, where $\{F_\varepsilon\}$ is the family of approximating integrands introduced in Section 4.1 (see Lemmas 4.1 and 4.2); this justifies the assumption made here that ∂a is symmetric. However, for the first part of the proof we shall keep formally $(1+\vartheta)/2 = 1/2$ and general dimensions $n \geq 2$, as this will give us the opportunity to use some computations made here for later proofs. It is sufficient to consider the case $\|Du\|_{L^\infty(B_{R/2})} \geq \overline{T}$; otherwise, (4.39) is trivial by the very definition of $G_{\overline{T}}(\cdot)$.

To proceed and use Lemma 4.7 in a proper way, with B_R denoting the ball from the statement of Lemma 4.8, let us observe that we can immediately localize everything to B_R, so that we can consider $u \in W^{1,\infty}(B_R)$ as a solution to the equation $-\operatorname{div} a(Du) = f = f_{B_R}$ in B_R, so that Lemma 4.7 applies with f replaced by f_{B_R} and $B \equiv B_R$. To begin the proof, we now consider concentric balls $B_{R/2} \Subset B_s \Subset B_t \Subset B_R$, a point $x_0 \in B_s$, and $R_0 = t - s$ such that $B_{R_0}(x_0) \subset B_t$. We then apply estimate (4.38) with $k = 0$ and $\kappa \in (0, 1/2)$, thereby obtaining via $\|Du\|_{L^\infty(B_{R_0}(x_0))} \leq \|Du\|_{L^\infty(B_t)} = M$

$$
\begin{aligned}
&G_{\overline{T}}(|Du(x_0)|)\\
&\quad \leq c\left[\frac{g_{2,\varepsilon}(\|Du\|_{L^\infty(B_t)})}{g_1(\|Du\|_{L^\infty(B_t)})}\right]^{\left(\frac{1+\vartheta}{2}\right)\left(1+\max\left\{\kappa, \frac{n-2}{2}\right\}\right)}\\
&\qquad \cdot \left(\fint_{B_{R_0}(x_0)} [G_{\overline{T}}(|Du|)]^2\, dx\right)^{1/2}\\
&\qquad + c\left[\frac{g_{2,\varepsilon}\left(\|Du\|_{L^\infty(B_t)}\right)}{g_1\left(\|Du\|_{L^\infty(B_t)}\right)}\right]^{\left(\frac{1+\vartheta}{2}\right)\max\left\{\kappa, \frac{n-2}{2}\right\}} \|Du\|_{L^\infty(B_t)} \mathbf{P}_1^{f_{B_R}}(x_0, 2R_0) \leq
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{c}{(t-s)^{n/2}}\left[\frac{g_{2,\varepsilon}(\|Du\|_{L^\infty(B_t)})}{g_1\big(\|Du\|_{L^\infty(B_t)}\big)}\right]^{\left(\frac{1+\vartheta}{2}\right)\left(1+\max\left\{\kappa,\frac{n-2}{2}\right\}\right)} \\
&\quad\cdot\left(\int_{B_t}[G_{\overline{T}}(|Du|)]^2\,dx\right)^{1/2} \\
&\quad+c\left[\frac{g_{2,\varepsilon}(\|Du\|_{L^\infty(B_t)})}{g_1(\|Du\|_{L^\infty(B_t)})}\right]^{\left(\frac{1+\vartheta}{2}\right)\max\left\{\kappa,\frac{n-2}{2}\right\}} \\
&\quad\cdot\|Du\|_{L^\infty(B_t)}\left\|\mathbf{P}_1^{f_{B_R}}(\,\cdot\,,2(t-s))\right\|_{L^\infty(B_t)}
\end{aligned}
$$

with c depending only on n and c_a (and also on κ when $n=2$). By definition of $G_{\overline{T}}(\cdot)$ and since $y\mapsto G_{\overline{T}}(y)$ is nondecreasing, we further estimate by means of $(1.33)_2$

$$
\begin{aligned}
&\int_{B_t}[G_{\overline{T}}(|Du|)]^2\,dx \\
&\quad\leq\int_{B_t\cap\{|Du|>\overline{T}\}}\left(\int_{\overline{T}}^{|Du|}g_1(y)y\,dy\right)^2dx \\
&\quad\leq[G_{\overline{T}}(\|Du\|_{L^\infty(B_t)})]^{2-\beta_0}\int_{B_t\cap\{|Du|>\overline{T}\}}\left(\int_{\overline{T}}^{|Du|}g_1(y)y\,dy\right)^{\beta_0}dx \\
&\quad\leq[G_{\overline{T}}(\|Du\|_{L^\infty(B_t)})]^{2-\beta_0}\int_{B_t}F_\varepsilon(Du)dx,
\end{aligned}
\tag{4.40}
$$

where we have also used $(4.7)_2$. Combining the last two inequalities and recalling that the point $x_0\in B_s$ is arbitrary, we then have

$$
\begin{aligned}
&G_{\overline{T}}(\|Du\|_{L^\infty(B_s)}) \\
&\quad\leq\frac{c}{(t-s)^{n/2}}\left[\frac{g_{2,\varepsilon}(\|Du\|_{L^\infty(B_t)})}{g_1\big(\|Du\|_{L^\infty(B_t)}\big)}\right]^{\left(\frac{1+\vartheta}{2}\right)\left(1+\max\left\{\kappa,\frac{n-2}{2}\right\}\right)} \\
&\qquad\cdot[G_{\overline{T}}(\|Du\|_{L^\infty(B_t)})]^{1-\frac{\beta_0}{2}}\left(\int_{B_R}F_\varepsilon(Du)dx\right)^{\frac{1}{2}} \\
&\qquad+c\left[\frac{g_{2,\varepsilon}(\|Du\|_{L^\infty(B_t)})}{g_1(\|Du\|_{L^\infty(B_t)})}\right]^{\left(\frac{1+\vartheta}{2}\right)\max\left\{\kappa,\frac{n-2}{2}\right\}} \\
&\qquad\quad\cdot\|Du\|_{L^\infty(B_t)}\left\|\mathbf{P}_1^{f_{B_R}}(\,\cdot\,,2(t-s))\right\|_{L^\infty(B_t)},
\end{aligned}
\tag{4.41}
$$

with c depending now in addition on ν.

Recalling that $\tau + 2 \geq 1$ and the notation in (2.1), we have

$$\begin{aligned}[H(\|Du\|_{L^\infty(B_s)}) - H(\overline{T})]^{\tau+2} &\\ \leq [H(\|Du\|_{L^\infty(B_s)})]^{\tau+2} - [H(\overline{T})]^{\tau+2} &\\ \leq \frac{\tau+2}{\nu}\int_{\overline{T}}^{\|Du\|_{L^\infty(B_s)}} g_1(y)y\,dy = \frac{\tau+2}{\nu} G_{\overline{T}}\big(\|Du\|_{L^\infty(B_s)}\big),&\end{aligned} \tag{4.42}$$

which is a consequence of $(1.33)_1$. We now specialize to the case $n > 2$, which is the one of interest for Theorem 1.9, and notice that in this case we have that $1 + \max\{\kappa, (n-2)/2\} = n/2$. Recalling that here $\vartheta = 0$, we may now combine the estimates (4.41) and (4.42) and bound the right-hand side of (4.41) via (4.19) and $\|Du\|_{L^\infty(B_t)} \leq H(\|Du\|_{L^\infty(B_t)})$. In this way we find

$$\begin{aligned}G_{\overline{T}}(\|Du\|_{L^\infty(B_s)}) + [H(\|Du\|_{L^\infty(B_s)}) - H(\overline{T})]^{\tau+2} &\\ \leq \frac{c}{(t-s)^{n/2}}\big[G_{\overline{T}}(\|Du\|_{L^\infty(B_t)})\big]^{1-\sigma/4}\bigg(\int_{B_R} F_\varepsilon(Du)dx\bigg)^{1/2} &\\ + \frac{c}{(t-s)^{n/2}}\big[G_{\overline{T}}\big(\|Du\|_{L^\infty(B_t)}\big)\big]^{1-\beta_0/2}\bigg(\int_{B_R} F_\varepsilon(Du)dx\bigg)^{1/2} &\\ + c\big[G_{\overline{T}}\big(\|Du\|_{L^\infty(B_t)}\big)\big]^{\beta_1}\Big\|\mathbf{P}_1^{f_{B_R}}(\,\cdot\,, 2(t-s))\Big\|_{L^\infty(B_t)} &\\ + c\big[H\big(\|Du\|_{L^\infty(B_t)}\big) - H\big(\overline{T}\big)\big]\Big\|\mathbf{P}_1^{f_{B_R}}(\,\cdot\,, 2(t-s))\Big\|_{L^\infty(B_t)} &\\ + cH\big(\overline{T}\big)\Big\|\mathbf{P}_1^{f_{B_R}}(\,\cdot\,, 2(t-s))\Big\|_{L^\infty(B_t)},&\end{aligned} \tag{4.43}$$

with $c \equiv c(n, \nu, \tau, c_a, c_{b,\overline{T}})$. We next apply the Young inequality four times to estimate the first four terms on the right-hand side of (4.43), with conjugate exponents $(4/(4-\sigma), 4/\sigma)$, $(2/(2-\beta_0), 2/\beta_0)$, $(1/\beta_1, 1/(1-\beta_1))$, and finally $(\tau+2, (\tau+2)/(\tau+1))$; the outcome is

$$\begin{aligned}G_{\overline{T}}(\|Du\|_{L^\infty(B_s)}) + [H(\|Du\|_{L^\infty(B_s)}) - H(\overline{T})]^{\tau+2} &\\ \leq \frac{1}{2}G_{\overline{T}}(\|Du\|_{L^\infty(B_t)}) + \frac{1}{2}[H(\|Du\|_{L^\infty(B_t)}) - H(\overline{T})]^{\tau+2} &\\ + \frac{c}{(t-s)^{2n/\sigma}}\bigg(\int_{B_R} F_\varepsilon(Du)dx\bigg)^{\frac{2}{\sigma}} &\\ + \frac{c}{(t-s)^{n/\beta_0}}\bigg(\int_{B_R} F_\varepsilon(Du)dx\bigg)^{\frac{1}{\beta_0}} &\\ + c\Big\|\mathbf{P}_1^{f_{B_R}}(\,\cdot\,, R)\Big\|_{L^\infty(B_R)}^{\frac{1}{1-\beta_1}} + c\Big\|\mathbf{P}_1^{f_{B_R}}(\,\cdot\,, R)\Big\|_{L^\infty(B_R)}^{\frac{\tau+2}{\tau+1}} &\\ + cH(\overline{T})\Big\|\mathbf{P}_1^{f_{B_R}}(\,\cdot\,, R)\Big\|_{L^\infty(B_R)},&\end{aligned} \tag{4.44}$$

where $c \equiv c(n, \nu, \tau, c_a, c_{b,\overline{T}}, \sigma, \beta_0, \beta_1)$. We are now able to apply Lemma 4.9 below with the obvious choice

$$\phi(y) := G_{\overline{T}}(\|Du\|_{L^\infty(B_y)}) + [H(\|Du\|_{L^\infty(B_y)}) - H(\overline{T})]^{\tau+2} \geq 0 \tag{4.45}$$

for $y \in [R/2, R]$. This completes the proof of Lemma 4.8 by establishing the claim (4.39) with the asserted dependencies of the constants. □

The following lemma, which has been used in the above proof, is a standard variant of a classical iteration result [29, chap. 6, lemma 6.1].

LEMMA 4.9. *Let $\phi\colon [R/2, R] \to \mathbb{R}$ be a nonnegative, bounded function. Assume that for all s, t such that $R/2 \leq s < t \leq R$ it holds that*

$$\phi(s) \leq \frac{1}{2}\phi(t) + \frac{a_1}{(t-s)^{\gamma_1}} + \frac{a_2}{(t-s)^{\gamma_2}} + b$$

for some nonnegative constants a_1, a_2, b and nonnegative exponents $\gamma_1 \geq \gamma_2$. Then the following inequality holds with $c \equiv c(\gamma_1)$:

$$\phi(R/2) \leq c\left[\frac{a_1}{R^{\gamma_1}} + \frac{a_2}{R^{\gamma_2}} + b\right].$$

4.3 A Theorem of Bousquet and Brasco Revisited

In this section we restate a nice result of Bousquet and Brasco [6] concerning the existence of Lipschitz solutions to variational problems satisfying the so-called bounded slope condition. The restatement below follows from the original formulation of the main result in [6] and is adapted to the special case we are considering. We include some details for the sake of completeness.

THEOREM 4.10 ([6, Main Theorem]). *Consider the Dirichlet minimization problem*

$$\min_{w \in u_0 + W^{1,1}_0(B)} \int_B [F(Dw) - fw]dx$$

where $B \subset \mathbb{R}^n$ is a ball, $n \geq 2$, $u_0 \in C^2(B)$, and $f \in L^\infty(B)$. Furthermore, we assume that the integrand $F\colon \mathbb{R}^n \to \mathbb{R}$ satisfies $(1.32)_{1,2}$ and $(1.32)_4$ for a function $g_1\colon [0,\infty) \to \mathbb{R}$ such that $(1.33)_1$ holds. Then the problem admits at least one solution and every solution u satisfies $Du \in L^\infty(B;\mathbb{R}^n)$.

PROOF. Notice that the assumption $u_0 \in C^2(B)$ ensures that u_0 satisfies the bounded slope condition on B; see [29, chap. 1, theorem 1.1]. It is therefore sufficient to check that the assumptions of [6, Main Theorem] are satisfied; we therefore adopt the terminology of [6]. For this purpose, we first observe that $u_0|_{\partial B}$ satisfies, for some $K \geq 0$, the bounded slope condition of rank K. Second, we need to verify that the following type of strictly uniform convexity condition

$$\begin{aligned} &\theta F(z_1) + (1-\theta)F(z_2) - F(\theta z_1 + (1-\theta)z_2) \\ &\quad \geq \theta(1-\theta)\Psi_1(|z_1| + |z_2|)|z_1 - z_2|^2 \end{aligned} \tag{4.46}$$

holds for all $\theta \in [0,1]$ whenever the segment $[z_1, z_2]$ lies in $\{|z| > T\}$ for some $T > 0$ and a function $\Psi_1\colon [0,\infty) \to [0,\infty)$ satisfying $t\Psi_1(t) \to \infty$ when $t \to \infty$. Indeed, we can take T from (1.32) and $\Psi_1(t) \approx (t^2+\mu^2)^{\tau/2}$, where τ, μ are from (1.33). To see (4.46), we introduce the vectors $\overline{\theta} := \theta z_1 + (1-\theta) z_2$, $\overline{\theta}_1(t) := t z_1 + (1-t)\overline{\theta}$, and $\overline{\theta}_2(t) := t\overline{\theta} + (1-t) z_2$ and observe that the segment $[\overline{\theta}_1(t), \overline{\theta}_2(t)]$ lies in $[z_1, z_2] \subset \{|z| > T\}$ for all $t \in [0,1]$. Thus, we have

$$
\begin{aligned}
&\theta F(z_1) + (1-\theta) F(z_2) - F(\theta z_1 + (1-\theta) z_2)\\
&\quad = \theta[F(z_1) - F(\overline{\theta})] - (1-\theta)[F(\overline{\theta}) - F(z_2)]\\
&\quad = \theta(1-\theta)\int_0^1 \langle \partial F(t z_1 + (1-t)\overline{\theta}) dt, z_1 - z_2\rangle\\
&\qquad - \theta(1-\theta)\int_0^1 \langle \partial F(t\overline{\theta} + (1-t) z_2) dt, z_1 - z_2\rangle\\
&\quad = \theta(1-\theta)\int_0^1\int_0^1 [t(1-\theta) + (1-t)\theta]\\
&\qquad \cdot \langle \partial^2 F(s\overline{\theta}_1(t) + (1-s)\overline{\theta}_2(t))(z_1 - z_2), z_1 - z_2\rangle ds\, dt\\
&\quad \ge \theta(1-\theta)\int_0^1\int_0^1 [t(1-\theta) + (1-t)\theta]\\
&\qquad \cdot g_1(s\overline{\theta}_1(t) + (1-s)\overline{\theta}_2(t)) ds\, dt\, |z_1 - z_2|^2\\
&\quad \ge \nu\theta(1-\theta)\int_0^1\int_0^1 [t(1-\theta) + (1-t)\theta]\\
&\qquad \cdot \big(|s\overline{\theta}_1(t) + (1-s)\overline{\theta}_2(t)|^2 + \mu^2\big)^{\tau/2} ds\, dt\, |z_1 - z_2|^2.
\end{aligned}
\tag{4.47}
$$

Observe that the estimation in the last line is a consequence of assumptions $(1.32)_4$ and $(1.33)_1$ and of the fact that $T < |s\overline{\theta}_1(t) + (1-s)\overline{\theta}_2(t)| \le |z_1| + |z_2|$ for all $s \in [0,1]$. Finally, using a standard algebraic inequality (see, for instance, [30]) and that $\tau > -1$, we conclude with

$$
\begin{aligned}
&\theta F(z_1) + (1-\theta) F(z_2) - F(\theta z_1 + (1-\theta) z_2)\\
&\quad \ge \frac{\nu\theta(1-\theta)}{c(\tau)}[(|z_1| + |z_2|)^2 + \mu^2]^{\tau/2} |z_1 - z_2|^2,
\end{aligned}
$$

which is precisely the strict convexity condition (4.46) with the choice $\Psi_1(t) = \nu(t^2+\mu^2)^{\tau/2}/c(\tau)$. Again, since $\tau > -1$, it then follows that $t\Psi_1(t) \to \infty$ when $t \to \infty$. We are therefore able to apply [6, Main Theorem], which yields the boundedness of u in $W^{1,\infty}(B)$ in terms of a constant depending only on n, T, ν, τ, $\|u_0\|_{C^2(B)}$, $\|f\|_{L^\infty}$, and B. We finally note that, from the computation in (4.47), it follows, whenever $z_1, z_2 \in \mathbb{R}^n$ are such that $[z_1, z_2] \cap \mathbb{R}^n \setminus B_T \neq \emptyset$, there exists a set $\mathscr{C} \subset [0,1]$ with positive measure such that

$$
\theta \in \mathscr{C} \quad\Longrightarrow\quad \theta F(z_1) + (1-\theta) F(z_2) - F(\theta z_1 + (1-\theta) z_2) > 0.
$$

The proof is complete. □

4.4 Passage to the Limit and Proof of Theorem 1.9

We are now ready to complete the proof of Theorem 1.9. With $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ being the local minimizer from the statement of Theorem 1.9, as observed immediately after Definition 1.1, we have that $F(Du) \in L^1_{\mathrm{loc}}(\Omega)$. Then, as a consequence of (4.11), this implies that $u \in W^{1,\gamma}_{\mathrm{loc}}(\Omega)$. Now, let $B \Subset \Omega$ be a fixed ball. We take a sequence $\{\varepsilon_m\}$ of positive numbers such that $\varepsilon_m \to 0$ as $m \to \infty$ and $\varepsilon_m < \min\{\mathrm{dist}(B, \partial\Omega), |B|^{1/n}, \overline{\varepsilon}\} < 1$ for all $m \in \mathbb{N}$, where $\overline{\varepsilon}$ is the number introduced in (4.9) in Lemma 4.2. We consider the usual family of mollifiers $\{\phi_\varepsilon\}$ in (4.2) and take the regularized functions

$$\overline{u}_m(x) := (u * \phi_{\varepsilon_m})(x) := \int_{B_1} u(x + \varepsilon_m y)\phi(y)dy \tag{4.48}$$

for $m \in \mathbb{N}$. The sequence $\{u_m\}$ is bounded in $W^{1,\gamma}(B)$. We then define, for every $m \in \mathbb{N}$, $f_m(x) := \min\{\max\{f(x), -m\}, m\}$, so that $f_m \in L^\infty(B)$ and $|f_m| \le \min\{|f|, m\}$, and we consider the functional

$$\mathscr{F}_m(w; B) := \int_B [F_m(Dw) - f_m w]dx, \tag{4.49}$$

where $F_m := F_{\varepsilon_m}$ is the regularized integrand that was introduced in (4.5) and described in Lemmas 4.1 and 4.2. Note that $\mathscr{F}_m$ is defined for every $w \in W^{1,1}(B)$. With $\gamma := \beta_0(\tau + 2) > 1$ as defined in (1.33), we look at the variational Dirichlet problem

$$\min_{w \in \overline{u}_m + W^{1,1}_0(B)} \int_B [F_m(Dw) - f_m w]dx. \tag{4.50}$$

This problem is of the type considered in Theorem 4.10, which in fact ensures the existence of a globally Lipschitz-continuous solution u_m. As a consequence, it is a distributional solution to the Euler-Lagrange equation

$$-\operatorname{div} \partial F_m(Du_m) = f_m, \quad u_m \in \big(\overline{u}_m + W^{1,1}_0(B)\big) \cap W^{1,\infty}(B). \tag{4.51}$$

In what follows, we prove that the sequence $\{u_m\}$ is uniformly bounded in $W^{1,\infty}_{\mathrm{loc}}(B)$ with suitable uniform local estimates. Finally, we pass to the limit $m \to \infty$, recovering in the end a minimizer $\overline{u}$ of the original functional $\mathscr{F}$ in (1.1), which coincides with the original minimizer u for large values of the gradients Du and $D\overline{u}$, and for which we additionally have established the desired $W^{1,\infty}$-estimate. Let us denote by γ^* the Sobolev conjugate of γ in the sense that $\gamma^* = n\gamma/(n - \gamma)$ if $\gamma < n$ and $\gamma^* = 2n/(n-1)$ otherwise. Hölder and Sobolev inequalities then give

$$
\begin{aligned}
&\fint_B |f_m(u_m - \overline{u}_m)|dx \\
&\quad \le \left(\fint_B |f_m|^n\,dx\right)^{1/n} \left(\fint_B |u_m - \overline{u}_m|^{\frac{n}{n-1}}\,dx\right)^{1-1/n} \\
&\quad \le \left(\fint_B |f_m|^n\,dx\right)^{1/n} \left(\fint_B |u_m - \overline{u}_m|^{\gamma^*}\,dx\right)^{1/\gamma^*} \\
&\quad \le c(n,\gamma)\|f_m\|_{L^n(B)} \left(\fint_B |Du_m - D\overline{u}_m|^{\gamma}\,dx\right)^{1/\gamma} \\
&\quad \le c\|f_m\|_{L^n(B)} \\
&\qquad \cdot \left(\fint_B F_m(Du_m)dx + \fint_B F_m(D\overline{u}_m)dx + \overline{T}^{\gamma} + \mu^{\gamma}\right)^{1/\gamma},
\end{aligned}
\tag{4.52}
$$

with a constant c depending only on n, ν, β_0, and τ, and where in the last line we have used (4.11) (with $\varepsilon \equiv \varepsilon_m$). Since $F_m(\cdot)$ is convex, the Jensen inequality gives

$$
\int_B F_m(D\overline{u}_m)dx \le \int_{B+\varepsilon_m B_1} F_m(Du)dx. \tag{4.53}
$$

Notice that the right-hand side is finite by Remark 4.3) so that by the Young inequality we find

$$
\begin{aligned}
&\fint_B |f_m(u_m - \overline{u}_m)|dx \\
&\quad \le c\|f_m\|_{L^n(B)} \\
&\qquad \cdot \left(\fint_B F_m(Du_m)dx \right.\\
&\qquad\quad \left. + \frac{|B+\varepsilon_m B_1|}{|B|} \fint_{B+\varepsilon_m B_1} F_m(Du)dx + \overline{T}^{\gamma} + \mu^{\gamma}\right)^{1/\gamma} \\
&\quad \le \frac{1}{2}\fint_B F_m(Du_m)dx + \frac{|B+\varepsilon_m B_1|}{|B|} \fint_{B+\varepsilon_m B_1} F_m(Du)dx \\
&\qquad + c\|f\|_{L^n(B)}^{\frac{\gamma}{\gamma-1}} + c\overline{T}^{\gamma} + c\mu^{\gamma},
\end{aligned}
\tag{4.54}
$$

with a constant c depending only on n, ν, τ, β_0, and γ. To proceed, we notice that the minimality of u_m and inequalities (4.53) and (4.54) yield

$$
\begin{aligned}
&\fint_B F_m(Du_m)dx \\
&\quad \le \fint_B F_m(D\overline{u}_m)dx + \fint_B f_m(u_m - \overline{u}_m)dx \le
\end{aligned}
$$

$$\begin{aligned}&\le \frac{|B+\varepsilon_m B_1|}{|B|}\fint_{B+\varepsilon_m B_1} F_m(Du)dx + \fint_B |f_m(u_m-\bar{u}_m)|dx\\ &\le \frac{1}{2}\fint_B F_m(Du_m)dx + c\fint_{B+\varepsilon_m B_1} F_m(Du)dx + c\|f\|^{\frac{\gamma}{\gamma-1}}_{L^n(B)} + c\overline{T}^{\gamma} + c\mu^{\gamma},\end{aligned}$$

where in the last inequality we have also used that $\varepsilon_m \le |B|^{1/n}$. Reabsorbing terms we arrive at

$$\fint_B F_m(Du_m)dx \le c\fint_{B+\varepsilon_m B_1} F_m(Du)dx + c\|f\|^{\frac{\gamma}{\gamma-1}}_{L^n(B)} + c\overline{T}^{\gamma} + c\mu^{\gamma}, \tag{4.55}$$

for a constant c depending only on n, ν, τ, and γ. Next, let us fix another ball B_R of radius R and with $B_R \Subset B$. Thanks to (4.51), which is of the type in (4.25)–(4.26) by Lemmas 4.1 and 4.2, and keeping in mind the Lipschitz regularity of u_m, we are in the setting of Section 4.2. We can therefore apply to u_m the a priori estimates stated there. By Lemma 4.8 in combination with the estimate in (2.3) and the previous inequality (4.55), we deduce

$$\begin{aligned}&G_{\overline{T}}(\|Du_m\|_{L^\infty(B_{R/2})})\\ &\le c\left[\frac{|B|}{|B_R|}\left(\fint_{B+\varepsilon_m B_1} F_m(Du)dx + \|f\|^{\frac{\gamma}{\gamma-1}}_{L^n(B)} + \overline{T}^{\gamma} + \mu^{\gamma}\right)\right]^{\frac{2}{\sigma}}\\ &\quad + c\left[\frac{|B|}{|B_R|}\left(\fint_{B+\varepsilon_m B_1} F_m(Du)dx + \|f\|^{\frac{\gamma}{\gamma-1}}_{L^n(B)} + \overline{T}^{\gamma} + \mu^{\gamma}\right)\right]^{\frac{1}{\beta_0}}\\ &\quad + c\|f\|^{\frac{1}{1-\beta_1}}_{L(n,1)(B_R)} + c\|f\|^{\frac{\tau+2}{\tau+1}}_{L(n,1)(B_R)} + cH(\overline{T})\|f\|_{L(n,1)(B_R)}.\end{aligned} \tag{4.56}$$

The constant c depends only on n, ν, τ, c_a, $c_{b,\overline{T}}$, σ, β_0, and β_1. Notice that we have applied (2.3) in order to estimate

$$\left\|\mathbf{P}_1^{f_{B_R}}(\cdot,R)\right\|_{L^\infty(B_R)} \le c\|f_{B_R}\|_{L(n,1)(B_{2R})} = c\|f\|_{L(n,1)(B_R)}.$$

Let us collect what we have established so far. We first notice that, by (4.12), the sequence $\{\|F_m(Du)\|_{L^1(B+\varepsilon_m B_1)}\}$ is bounded. Therefore, by (4.55), the sequence $\{\|F_m(Du_m)\|_{L^1(B)}\}$ is also bounded. In turn, this implies that the sequence $\{u_m\}$ is bounded in $W^{1,\gamma}(B)$ via (4.11). Moreover, by (4.56) and the fact that $G_{\overline{T}}(t) \to \infty$ as $t \to \infty$, we also have boundedness of $\{u_m\}$ in $W^{1,\infty}(B_{R/2})$. As a consequence, we conclude that, up to a not-relabeled subsequence, there exists $\bar{u} \in u + W^{1,\gamma}_0(B)$ such that

$$\begin{cases} u_m \rightharpoonup \bar{u} & \text{weakly in } W^{1,\gamma}(B),\\ u_m \to \bar{u} & \text{strongly in } L^{n/(n-1)}(B),\\ u_m \rightharpoonup \bar{u} & \text{weakly-}* \text{ in } W^{1,\infty}(B_{R/2}).\end{cases} \tag{4.57}$$

Since the integrand $z \mapsto F(z)$ is convex, we then infer that

$$\int_B F(D\bar{u})dx \le \liminf_{m\to\infty} \int_B F(Du_m)\, dx. \tag{4.58}$$

On one hand, the boundedness of $\{Du_m\}$ in $L^\gamma(B)$ and the linear growth of the integrand L give $\varepsilon_m \int_B L(Du_m)dx \to 0$ as $m \to \infty$. Therefore, by $(4.7)_2$, we can compute

$$\begin{aligned}
&\lim_{m\to\infty} \int_B [F_m(Du_m) - F(Du_m)]dx \\
&\quad = \lim_{m\to\infty} \int_{B\cap\{|Du_m|\le \bar{T}\}} [F_m(Du_m) - F(Du_m)]dx \\
&\qquad + \lim_{m\to\infty} \varepsilon_m \int_B L(Du_m)dx = 0,
\end{aligned}$$

where we have employed the fact that by construction the convergence $F_m \to F$ is uniform on compact sets; see $(4.7)_3$. Using the last identity together with (4.58) we conclude with

$$\int_B F(D\bar{u})dx \le \liminf_{m\to\infty} \int_B F_m(Du_m)dx.$$

On the other hand, since we have $u_m \to \bar{u}$ as well as $\bar{u}_m \to u$ strongly in $L^{n/(n-1)}(B)$ and $f_m \to f$ strongly in $L^n(B)$, we also get, relying once again on the minimality of u_m, (4.53), and (4.7), that

$$\begin{aligned}
\mathscr{F}(\bar{u}; B) &\le \liminf_{m\to\infty} \mathscr{F}_m(u_m; B) \le \liminf_{m\to\infty} \mathscr{F}_m(\bar{u}_m; B) \\
&\le \lim_{m\to\infty} \int_{B+\varepsilon_m B_1} F_m(Du)dx - \lim_{m\to\infty} \int_B f_m \bar{u}_m\, dx = \mathscr{F}(u; B).
\end{aligned}$$

The minimality of u then implies

$$\mathscr{F}(u; B) = \mathscr{F}(\bar{u}; B), \tag{4.59}$$

while we also record the obvious identity

$$\lim_{m\to\infty} \int_{B+\varepsilon_m B_1} F_m(Du)dx = \int_B F(Du)dx. \tag{4.60}$$

Let us continue by recalling a standard convexity argument (see, for instance, [6, 47]), reported here for the sake of completeness. Setting $w_\theta := (1-\theta)u + \theta\bar{u}$ for $\theta \in [0,1]$, we get $\mathscr{F}(w_\theta; B) \le (1-\theta)\mathscr{F}(u; B) + \theta\mathscr{F}(\bar{u}; B)$ by convexity, which in turn yields $\mathscr{F}(w_\theta; B) = \mathscr{F}(u; B)$ via (4.59) and thus

$$\int_B F(Dw_\theta)dx = \int_B \big[(1-\theta)F(Du) + \theta F(D\bar{u})\big]dx.$$

Since $z \mapsto F(z)$ is convex, we even get the pointwise equality of the integrands

$$F(Dw_\theta) = (1-\theta)F(Du) + \theta F(D\bar{u}) \quad \text{a.e. in } B. \tag{4.61}$$

We now take a point x with equality for all rational $\theta \in [0,1]$. If one of the two vectors $Du(x)$ and $D\overline{u}(x)$ does not belong to $\overline{B}_T$, then we conclude $Du(x) = D\overline{u}(x)$, since otherwise a nonempty subsegment of $[Du(x), D\overline{u}(x)]$ would lie in $\mathbb{R}^n \setminus \overline{B}_T$, where we have strict convexity of F (see, for instance, the computation (4.47)), and this would be in contradiction to the equality in (4.61) for a suitable rational $\theta \in (0,1)$. Hence, for a.e. $x \in B$ we have

$$\text{either } Du(x) = D\overline{u}(x) \quad \text{or} \quad \max\{|Du(x)|, |D\overline{u}(x)|\} \le T \,. \tag{4.62}$$

We finally want to let $m \to \infty$ in (4.56). By the fact that $s \mapsto g_1(s)s$ in nondecreasing on $[T,\infty)$ by assumption and by $\overline{T} > T$, the map $s \mapsto G_{\overline{T}}(s)$ is convex on the same set. For this reason, for any $q > 1$, also the function $z \mapsto [G_{\overline{T}}(|z|)]^q$ is still convex, so that, by lower semicontinuity of convex integral functionals with respect to the weak convergence in Sobolev spaces, we get

$$\left(\fint_{B_{R/2}} [G_{\overline{T}}(|D\overline{u}|)]^q \, dx\right)^{1/q} \le \liminf_{m\to\infty} \left(\fint_{B_{R/2}} [G_{\overline{T}}(|Du_m|)]^q \, dx\right)^{1/q}.$$

Therefore, since $G_{\overline{T}}(\cdot)$ is nondecreasing and continuous on $(0,\infty)$, we first find

$$\left(\fint_{B_{R/2}} [G_{\overline{T}}(|D\overline{u}|)]^q \, dx\right)^{1/q} \le \liminf_{m\to\infty} G_{\overline{T}}(\|Du_m\|_{L^\infty(B_{R/2})})$$

and then, letting $q \to \infty$, we conclude with

$$G_{\overline{T}}(\|D\overline{u}\|_{L^\infty(B_{R/2})}) \le \liminf_{m\to\infty} G_{\overline{T}}(\|Du_m\|_{L^\infty(B_{R/2})}).$$

Using this information in (4.56), as well as the convergence in (4.60), we first let $m \to \infty$, and then with $\overline{T} \to T$ we finally obtain

$$\begin{aligned}
&G_T(\|D\overline{u}\|_{L^\infty(B_{R/2})}) \\
&\quad \le c\left[\frac{|B|}{|B_R|}\left(\fint_B F(Du)dx + \|f\|_{L^n(B)}^{\frac{\gamma}{\gamma-1}} + T^\gamma + \mu^\gamma\right)\right]^{\frac{2}{\sigma}} \\
&\qquad + c\left[\frac{|B|}{|B_R|}\left(\fint_B F(Du)dx + \|f\|_{L^n(B)}^{\frac{\gamma}{\gamma-1}} + T^\gamma + \mu^\gamma\right)\right]^{\frac{1}{\beta_0}} \\
&\qquad + c\|f\|_{L(n,1)(B_R)}^{\frac{1}{1-\beta_1}} + c\|f\|_{L(n,1)(B_R)}^{\frac{\tau+2}{\tau+1}} + cH(T)\|f\|_{L(n,1)(B_R)},
\end{aligned}$$

where we have also used the definition of $c_{b,\overline{T}}$ in (4.20) and $c \equiv c(n,\nu,\tau,c_a, c_b,\sigma,\beta_0,\beta_1)$. Using this last inequality in combination with (4.62) and the definition of $G_T(\cdot)$ in (4.27), we then obtain (1.35) by taking B_R concentric with B and eventually letting $B_R \to B$. In order to finish the proof of Theorem 1.9, it only remains to justify the last assertion concerning (1.36). To this end, it is sufficient to note that the need for the full assumption (1.34) only enters when $f \neq 0$. In the case when $f = 0$, the last term in the last line of (4.41) does not appear and we do not need the inequality (coming from (4.19))

$$\frac{g_{2,\varepsilon}(t)}{g_1(t)} \lesssim c_b \left(\frac{1}{t^{1/\beta_1}} \int_{\overline{T}}^t g_1(s) s \, ds \right)^{\frac{4\beta_1}{n-2}} + c_{b,\overline{T}}$$

to estimate it. □

4.5 Regularity in Two Dimensions and Theorem 1.11

The proof of Theorem 1.11 is a modification of the one given for Theorem 1.9 above and proceeds essentially unchanged up to estimate (4.42) in the proof of Lemma 4.8. Then differences arise. Taking into account (4.42) and recalling the notation in (2.1), that we deal with the case $\vartheta = 0$, and that now we have $n = 2$, estimate (4.41) now reads, for a constant $c \equiv c(\nu, c_a, \kappa)$ and $\kappa \in (0, 1/2)$ to be chosen later, as

$$\begin{aligned} &G_{\overline{T}}(\|Du\|_{L^\infty(B_s)}) + [H(\|Du\|_{L^\infty(B_s)}) - H(\overline{T})]^{\tau+2} \\ &\quad \le \frac{c}{t-s} \left[\frac{g_{2,\varepsilon}\big(\|Du\|_{L^\infty(B_t)}\big)}{g_1\big(\|Du\|_{L^\infty(B_t)}\big)} \right]^{\frac{1+\vartheta}{2}(1+\kappa)} \\ &\qquad \cdot \big[G_{\overline{T}}\big(\|Du\|_{L^\infty(B_t)}\big)\big]^{1-\frac{\beta_0}{2}} \left(\int_{B_R} F_\varepsilon(Du) dx \right)^{\frac{1}{2}} \\ &\qquad + c \left[\frac{g_{2,\varepsilon}(\|Du\|_{L^\infty(B_t)})}{g_1(\|Du\|_{L^\infty(B_t)})} \right]^{\frac{1+\vartheta}{2}\kappa} \\ &\qquad \cdot [H(\|Du\|_{L^\infty(B_t)}) - H(\overline{T})] \big\| \mathbf{P}_1^{f_{B_R}}(\cdot, 2(t-s)) \big\|_{L^\infty(B_t)} \\ &\qquad + c \left[\frac{g_{2,\varepsilon}(\|Du\|_{L^\infty(B_t)})}{g_1(\|Du\|_{L^\infty(B_t)})} \right]^{\frac{1+\vartheta}{2}\kappa} H(\overline{T}) \big\| \mathbf{P}_1^{f_{B_R}}(\cdot, 2(t-s)) \big\|_{L^\infty(B_t)}. \end{aligned} \tag{4.63}$$

In view of (4.21) (which replaces (4.19) and was used before in the proof of Theorem 1.9, and that is in force here since in the two-dimensional case we are assuming (1.37)), recalling that here it is $\vartheta = 0$, we then get

$$\begin{aligned} &G_{\overline{T}}(\|Du\|_{L^\infty(B_s)}) + [H(\|Du\|_{L^\infty(B_s)}) - H(\overline{T})]^{\tau+2} \\ &\quad \le \frac{c}{t-s} [G_{\overline{T}}(\|Du\|_{L^\infty(B_t)})]^{\left(\frac{1+\kappa}{2}\right)(\beta_0 - \frac{\sigma}{2}) + 1 - \frac{\beta_0}{2}} \left(\int_{B_R} F_\varepsilon(Du) dx \right)^{\frac{1}{2}} \\ &\qquad + \frac{c}{t-s} [G_{\overline{T}}(\|Du\|_{L^\infty(B_t)})]^{1-\frac{\beta_0}{2}} \left(\int_{B_R} F_\varepsilon(Du) dx \right)^{\frac{1}{2}} + \end{aligned} \tag{4.64}$$

$$
\begin{aligned}
&+ c[G_{\overline{T}}(\|Du\|_{L^\infty(B_t)})]^{\frac{\kappa}{2}(\beta_0-\frac{\sigma}{2})} \\
&\quad \cdot [H(\|Du\|_{L^\infty(B_t)}) - H(\overline{T})]\big\|\mathbf{P}_1^{f_{B_R}}(\cdot, R)\big\|_{L^\infty(B_t)} \\
&+ c[H(\|Du\|_{L^\infty(B_t)}) - H(\overline{T})]\big\|\mathbf{P}_1^{f_{B_R}}(\cdot, R)\big\|_{L^\infty(B_t)} \\
&+ c[G_{\overline{T}}(\|Du\|_{L^\infty(B_t)})]^{\frac{\kappa}{2}(\beta_0-\frac{\sigma}{2})} H(\overline{T})\big\|\mathbf{P}_1^{f_{B_R}}(\cdot, R)\big\|_{L^\infty(B_t)} \\
&+ cH(\overline{T})\big\|\mathbf{P}_1^{f_{B_R}}(\cdot, R)\big\|_{L^\infty(B_t)}
\end{aligned}
$$

with $c \equiv c(\nu, \tau, c_a, c_{b,\overline{T}}, \kappa)$. As prescribed in the statement of Theorem 1.11, we fix $\theta \in (0, \sigma)$ and then take $\kappa \in (0, 1/2)$ sufficiently small in order to have

$$
\sigma - \theta \le (1+\kappa)\sigma - 2\kappa\beta_0 \Longleftrightarrow \left(\frac{1+\kappa}{2}\right)\left(\beta_0 - \frac{\sigma}{2}\right) + 1 - \frac{\beta_0}{2} \le 1 - \frac{\sigma - \theta}{4}. \tag{4.65}
$$

By possibly choosing κ smaller, we may also assume that

$$
0 < \frac{\kappa}{2}\left(\beta_0 - \frac{\sigma}{2}\right)\left(\frac{1+\theta}{\theta}\right) < \frac{\kappa}{2}\left(\beta_0 - \frac{\sigma}{2}\right)\left(\frac{\tau+2}{\tau+1}\right)\left(\frac{\tau+2+\theta}{\theta}\right) \le \frac{1}{2} \tag{4.66}
$$

(recall it is $\tau > -1$). The choice in the last two displays determines the value of $\kappa \equiv \kappa(\theta, \tau, \sigma, \beta_0)$ and therefore the value of the constant c in inequality (4.64).

Taking (4.65)–(4.66) into account, recalling the definition of the function ϕ introduced in (4.45), and using Young inequality repeatedly (compare with (4.44)), we find

$$
\begin{aligned}
\phi(s) &\le \frac{1}{2}\phi(t) \\
&\quad + \frac{c}{(t-s)^{\frac{4}{\sigma-\theta}}}\left(\int_{B_R} F_\varepsilon(Du)dx\right)^{\frac{2}{\sigma-\theta}} + \frac{c}{(t-s)^{\frac{2}{\beta_0}}}\left(\int_{B_R} F_\varepsilon(Du)dx\right)^{\frac{1}{\beta_0}} \\
&\quad + c\left\|\mathbf{P}_1^{f_{B_R}}(\cdot, R)\right\|_{L^\infty(B_R)}^{\frac{\tau+2+\theta}{\tau+1}} + c\Big[H(\overline{T})\Big\|\mathbf{P}_1^{f_{B_R}}(\cdot, R)\Big\|_{L^\infty(B_R)}\Big]^{1+\theta} + c,
\end{aligned}
$$

where c depends only on ν, τ, c_a, $c_{b,\overline{T}}$, σ, β_0, and θ, but can be taken otherwise independently of $\overline{T}$ and ε. We are now in position to use Lemma 4.9 as in the proof of Lemma 4.8; this yields

$$
\begin{aligned}
&G_{\overline{T}}(\|Du\|_{L^\infty(B_{R/2})}) \\
&\quad \le c\left(\fint_{B_R} F_\varepsilon(Du)dx\right)^{\frac{2}{\sigma-\theta}} + c\left\|\mathbf{P}_1^{f_{B_R}}(\cdot, R)\right\|_{L^\infty(B_R)}^{\frac{\tau+2+\theta}{\tau+1}} \\
&\qquad + c\Big[H(\overline{T})\Big\|\mathbf{P}_1^{f_{B_R}}(\cdot, R)\Big\|_{L^\infty(B_R)}\Big]^{1+\theta} + c,
\end{aligned} \tag{4.67}
$$

where we have also used that $2/(\sigma - \theta) \geq 1/\beta_0$ and where the constant c still has the same dependencies. This last estimate is the two-dimensional analogue of (4.39); from this point on, the rest of the proof follows the one for Theorem 1.9 (from Section 4.3 on) with minor variants, most importantly, with (2.9) used instead of (2.3) in order to control the Riesz potential term of f. This finally leads to the assertion (1.38) by again using Young's inequality. Note that we are also using that, as $\sigma \leq \beta_0 \leq 1$, we have $2/(\sigma - \theta) \geq 1 + \theta$.

5 Vectorial Estimates and Theorem 1.13

The proof breaks down to three different steps, distributed through Sections 5.1–5.3. We first find a suitable approximation of the original problem, this time via functionals with polynomial growth. Then, in the second step, we prove uniform a priori estimates for the gradient. In both steps the structure assumption (1.43) is essential. Finally, we combine the local a priori estimates and the approximation method to recover the local Lipschitz regularity result for the original minimizer.

Notice that, by replacing $\widetilde{F}$ by $\widetilde{F} - \widetilde{F}(0)$, in the rest of the proof we can assume without loss of generality that $\widetilde{F}(0) = 0$; notice also that $\widetilde{F}'(0) = 0$ follows by $(1.42)_1$. From now on, we assume that all the assumptions of Theorem 1.13 are in force; moreover, as in the case of the proof of Theorem 1.9, we shall assume with no loss of generality that $f \in L(n,1)(\mathbb{R}^n;\mathbb{R}^n)$ for $n > 2$ and $f \in \mathrm{L}^2(\mathrm{Log\,L})^{\alpha}(\mathbb{R}^n;\mathbb{R}^N)$ when $n = 2$. We refer the reader to the notation about tensors fixed in Section 2.

5.1 Approximation in the Vectorial Case

Here we implement an approximation scheme that plays, in the vectorial case, the role of the one developed in Section 4.1 in the scalar case. Indeed, as in Section 4.1, given the original integrand $z \mapsto F(z)$ from Theorem 1.13, we construct an approximating family $\{F_\varepsilon\}$ of integrands with standard growth and ellipticity conditions that is uniformly converging to F on compact subsets. The main point is that the integrands F_ε must share the structure property in (1.43), so we shall construct a family $\{\widetilde{F}_\varepsilon\}$ such that $F_\varepsilon(z) := \widetilde{F}_\varepsilon(|z|)$ for some $\widetilde{F}_\varepsilon\colon [0,\infty) \to [0,\infty)$. The functions $\widetilde{F}_\varepsilon$ will be locally C^1, but only piecewise C^2-regular; this will involve additional technical complications. For approximation methods in this setting, see also [46, 48].

To start with the construction of the approximating integrands, let us note that from the notation introduced in (1.41) we have

$$\partial a(z) = \partial(\widetilde{a}(|z|)z) = \widetilde{a}(|z|)\mathbb{I}_{N\times n} + \widetilde{a}'(|z|)|z|\frac{z \otimes z}{|z|^2}, \quad z \in \mathbb{R}^{N\times n},\ |z| \neq 0, \tag{5.1}$$

and (recalling the free assumption $\widetilde{F}(0) = 0$)

$$F(z) = \widetilde{F}(|z|) \quad \text{where} \quad \widetilde{F}(t) := \int_0^t \widetilde{a}(s)s\,ds. \tag{5.2}$$

The symbol $\mathbb{I}_{N\times n}$ here denotes the fourth-order tensor

$$\mathbb{I}_{N\times n} = \delta_{ij}\delta_{\alpha\beta}(e^\alpha \otimes e_i) \otimes (e^\beta \otimes e_j) \tag{5.3}$$

for $i, j \in \{1, \dots, n\}$, $\alpha, \beta \in \{1, \dots, N\}$, where δ denotes the usual Kronecker's symbol. Using (5.1)–(5.2) and applying $(1.32)_4$ and $(1.32)_1$ for $\xi \perp z$ and for $\xi \parallel z$, respectively, we find

$$\begin{cases} \widetilde{a}(|z|) \geq g_1(|z|) & \text{for every } z \in \mathbb{R}^{N\times n} \text{ with } |z| > T, \\ \widetilde{a}(|z|) + \widetilde{a}'(|z|)|z| \geq g_1(|z|) & \text{for every } z \in \mathbb{R}^{N\times n} \text{ with } |z| > T, \\ \widetilde{a}(|z|) \geq 0 & \text{for every } z \in \mathbb{R}^{N\times n} \text{ with } |z| > 0, \end{cases} \tag{5.4}$$

and, analogously, by means of $(1.32)_3$, we get

$$\begin{cases} \widetilde{a}(|z|) \leq g_2(|z|) \\ \widetilde{a}(|z|) + \widetilde{a}'(|z|)|z| \leq g_2(|z|) \end{cases} \quad \text{for every } z \in \mathbb{R}^{N\times n} \text{ with } |z| > T. \tag{5.5}$$

Notice that an immediate consequence of assumption (1.42) is

$$\begin{cases} t \geq T \implies \widetilde{\nu}(t^2 + \mu^2)^{\frac{\gamma-2}{2}} \leq \widetilde{a}(t), \\ \widetilde{\nu} := \dfrac{\widetilde{a}(T)}{(T^2 + \mu^2)^{\frac{\gamma-2}{2}}} \overset{(5.4)_1}{\geq} \dfrac{g_1(T)}{(T^2 + \mu^2)^{\frac{\gamma-2}{2}}} \overset{(1.42)_2}{\geq} \nu. \end{cases} \tag{5.6}$$

In the following we use a parameter ε such that $0 < \varepsilon < \min\{1, T\}/4$. We now start to construct the approximating family of integrands by introducing

$$\mu_\varepsilon = \mu + \varepsilon, \quad T_\varepsilon := T + \frac{1}{\varepsilon}, \tag{5.7}$$

and $\widetilde{a}_\varepsilon \colon [0, \infty) \to [0, \infty)$ as

$$\widetilde{a}_\varepsilon(t) := \begin{cases} \dfrac{\widetilde{a}(\varepsilon)}{(\varepsilon^2 + \mu_\varepsilon^2)^{\frac{\gamma-2}{2}}}(t^2 + \mu_\varepsilon^2)^{\frac{\gamma-2}{2}} & \text{if } 0 \leq t < \varepsilon, \\ \widetilde{a}(t) & \text{if } \varepsilon \leq t < T_\varepsilon, \\ \dfrac{\widetilde{a}(T_\varepsilon)}{(T_\varepsilon^2 + \mu_\varepsilon^2)^{\frac{\gamma-2}{2}}}(t^2 + \mu_\varepsilon^2)^{\frac{\gamma-2}{2}} & \text{if } T_\varepsilon \leq t. \end{cases} \tag{5.8}$$

Notice that, thanks to $(1.42)_1$, this function is nondecreasing when $\gamma \geq 2$. Then, for $t > 0$, we define

$$F_\varepsilon(z) := \widetilde{F}_\varepsilon(|z|) \quad \text{where} \quad \widetilde{F}_\varepsilon(t) := \int_0^t \widetilde{a}_\varepsilon(s)s\,ds + \varepsilon L_{\gamma,\varepsilon}(t) \tag{5.9}$$

and

$$L_{\gamma,\varepsilon}(t) := \frac{1}{\gamma}\big[\big(t^2+\mu_\varepsilon^2\big)^{\gamma/2}-\mu_\varepsilon^\gamma\big] = \int_0^t \big(s^2+\mu_\varepsilon^2\big)^{\frac{\gamma-2}{2}} s\,ds, \tag{5.10}$$

so that, in view of (1.42) and (5.2)–(5.5), it follows that

$$\begin{cases} \widetilde{F}_\varepsilon \in C^1_{\mathrm{loc}}[0,\infty)\cap W^{2,\infty}_{\mathrm{loc}}[0,\infty)\cap C^2_{\mathrm{loc}}([0,\infty)\setminus\{\varepsilon,T_\varepsilon\}),\\ t\mapsto \dfrac{\widetilde{F}'_\varepsilon(t)}{t},\ \widetilde{a}_\varepsilon \in W^{1,\infty}_{\mathrm{loc}}[0,\infty)\cap C^1_{\mathrm{loc}}([0,\infty)\setminus\{\varepsilon,T_\varepsilon\}),\\ F_\varepsilon \text{ is strictly convex},\\ t\mapsto \widetilde{F}(t),\ t\mapsto \widetilde{F}_\varepsilon(t) \text{ are nondecreasing},\\ F_\varepsilon \to F \text{ uniformly on compact subsets of } \mathbb{R}^{N\times n}. \end{cases} \tag{5.11}$$

The above definitions then lead to introducing the related control functions $g_{1,\varepsilon}(\cdot)$ and $g_{2,\varepsilon}(\cdot)$ as

$$g_{1,\varepsilon}(t) := \mathfrak{g}_1 \begin{cases} \dfrac{g_1(\varepsilon)}{(\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}}(t^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} & \text{if } 0<t<\varepsilon,\\ g_1(t) & \text{if } \varepsilon\le t<T_\varepsilon,\\ \dfrac{g_1(T_\varepsilon)}{(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}}(t^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} & \text{if } T_\varepsilon\le t, \end{cases} \tag{5.12}$$

and

$$g_{2,\varepsilon}(t) := \mathfrak{g}_2 \begin{cases} \left[\dfrac{g_2(\varepsilon)}{(\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}}+\varepsilon\right](t^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} & \text{if } 0<t<\varepsilon,\\ g_2(t)+\varepsilon(t^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} & \text{if } \varepsilon\le t<T_\varepsilon,\\ \left[\dfrac{g_2(T_\varepsilon)}{(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}}+\varepsilon\right](t^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} & \text{if } T_\varepsilon\le t, \end{cases} \tag{5.13}$$

respectively, where the constants $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are defined by

$$\min\{1,\gamma-1\} =: \mathfrak{g}_1 \le 1 \le \mathfrak{g}_2 := 4\sqrt{Nn}+4\gamma. \tag{5.14}$$

We then have the following analogue of Lemma 4.2:

LEMMA 5.1. *Let* $F\colon\mathbb{R}^n\to\mathbb{R}$ *be the integrand of Theorem* 1.13 *and* $\{F_\varepsilon\}$ *the family of integrands introduced in* (5.9)*, with* $0<\varepsilon<\min\{1,T\}/4$. *Then there exists a constant* $\Gamma_\varepsilon\ge 1$ *such that the following growth and ellipticity properties* *hold*:

$$
\begin{cases}
|\partial^2 F_\varepsilon(z)| \le g_{2,\varepsilon}(|z|) & \textit{on } \{|z| > T\} \textit{ with } |z| \ne T_\varepsilon,\\
g_{1,\varepsilon}(|z|)|\xi|^2 \le \langle \partial^2 F_\varepsilon(z)\xi,\xi\rangle & \textit{on } \{|z| > T\} \textit{ with } |z| \ne T_\varepsilon,\\
|\partial^2 F_\varepsilon(z)| \le \Gamma_\varepsilon(|z|^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} & \textit{on } \{|z| \notin \{\varepsilon, T_\varepsilon\}\},\\
\varepsilon \mathfrak{g}_1(|z|^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}|\xi|^2 \le \langle \partial^2 F_\varepsilon(z)\xi,\xi\rangle & \textit{on } \{|z| \notin \{\varepsilon, T_\varepsilon\}\},
\end{cases} \tag{5.15}
$$

for every $\xi \in \mathbb{R}^{N\times n}$ *and* $\mathfrak{g}_1$ *as defined in* (5.14). *Moreover, the following inequalities hold on* $\{|z| > T\}$:

$$
F(z) \ge \int_T^{|z|} g_1(s)s\,ds, \qquad F_\varepsilon(z) \ge \int_T^{|z|} g_{1,\varepsilon}(s)s\,ds, \tag{5.16}
$$

$$
\begin{cases}
F(z) \ge \dfrac{\nu}{\gamma}\big[(|z|^2+\mu^2)^{\gamma/2} - (T^2+\mu^2)^{\gamma/2}\big]\\[2mm]
F_\varepsilon(z) \ge \dfrac{\nu}{\gamma 2^{1+\gamma}}\big[(|z|^2+\mu^2)^{\gamma/2} - (T^2+\mu^2)^{\gamma/2}\big],
\end{cases} \tag{5.17}
$$

$$
[H_\varepsilon(t)]^\gamma - [H_\varepsilon(T)]^\gamma \le \frac{\gamma 2^{1+\gamma}}{\mathfrak{g}_1 \nu}\int_T^t g_{1,\varepsilon}(s)s\,ds \quad \textit{for } t \ge T, \tag{5.18}
$$

where ν *has been introduced in* (1.42) *and* H_ε *in* (2.1). *Finally, if* $\gamma \ge 2$ *in* (1.42), *then* $t \mapsto \widetilde{F}_\varepsilon'(t)/t$ *is nondecreasing on* $[0,\infty)$.

PROOF. By the definition in (5.9) and recalling the notation in (5.3), we first notice that the $\partial^2 F_\varepsilon(z)$ exists provided $|z| \notin \{\varepsilon, T_\varepsilon\}$, with

$$
\partial^2 F_\varepsilon(z) =
\begin{cases}
(|z|^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}\left[\dfrac{\widetilde{a}(\varepsilon)}{(\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}} + \varepsilon\right]\mathscr{D}(z) & \text{if } |z| < \varepsilon,\\[3mm]
\partial^2 F(z) + \varepsilon(|z|^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}\mathscr{D}(z) & \text{if } \varepsilon < |z| < T_\varepsilon,\\[3mm]
(|z|^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}\left[\dfrac{\widetilde{a}(T_\varepsilon)}{(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}} + \varepsilon\right]\mathscr{D}(z) & \text{if } T_\varepsilon < |z|,
\end{cases}
$$

where

$$
\mathscr{D}(z) := \mathbb{I}_{N\times n} + (\gamma-2)\frac{z\otimes z}{|z|^2+\mu_\varepsilon^2} \qquad \forall z \in \mathbb{R}^{N\times n}.
$$

Then, $(5.15)_{1,2}$ follows directly from (1.32), the definitions in (5.12)–(5.13) and (5.4)–(5.5), and (5.14). As for $(5.15)_{3,4}$, these again follow from the explicit expression of $\partial^2 F_\varepsilon(z)$ in the above display; in particular, using $(1.42)_1$, we see that $(5.15)_3$ holds for any Γ_ε such that

$$
\Gamma_\varepsilon \ge \mathfrak{g}_2 \max\left\{\frac{\widetilde{a}(T_\varepsilon)}{(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}},\ \sup_{\varepsilon\le|z|\le T_\varepsilon}\frac{|\partial^2 F(z)|}{(|z|^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}}\right\} + \mathfrak{g}_2\varepsilon,
$$

and the constant $\mathfrak{g}_2$ is as defined in (5.14). The two inequalities in (5.16) are a straightforward consequence of (5.4) and of the definitions in (5.7)–(5.8), (5.9), and (5.12).

For the proof of $(5.17)_1$ it is sufficient to estimate

$$F(z) \geq \int_T^{|z|} \widetilde{a}(s) s\, ds \overset{(5.6)}{\geq} \widetilde{\nu} \int_T^{|z|} (s^2+\mu^2)^{\frac{\gamma-2}{2}} s\, ds \tag{5.19}$$

and recall that $\widetilde{\nu} \geq \nu$ by (5.6). Concerning $(5.17)_2$, the proof is as in (5.19) for the case $|z| < T_\varepsilon$. In the case $|z| \geq T_\varepsilon$, we estimate via the positivity of $L_{\gamma,\varepsilon}(|z|)$, (5.7)–(5.8), and (5.6)

$$\begin{aligned} F_\varepsilon(z) &\geq \int_T^{T_\varepsilon} \widetilde{a}(s) s\, ds + \frac{\widetilde{a}(T_\varepsilon)}{(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}} \int_{T_\varepsilon}^{|z|} \left(s^2+\mu_\varepsilon^2\right)^{\frac{\gamma-2}{2}} s\, ds \\ &\geq \widetilde{\nu} \int_T^{T_\varepsilon} (s^2+\mu^2)^{\frac{\gamma-2}{2}} s\, ds + \widetilde{\nu} \frac{(T_\varepsilon^2+\mu^2)^{\frac{\gamma-2}{2}}}{(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}} \int_{T_\varepsilon}^{|z|} \left(s^2+\mu_\varepsilon^2\right)^{\frac{\gamma-2}{2}} s\, ds \\ &\geq \nu \min\left\{1, \frac{(T_\varepsilon^2+\mu^2)^{\frac{\gamma-2}{2}}}{(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}}\right\} \int_T^{|z|} (s^2+\mu^2)^{\frac{\gamma-2}{2}} s\, ds, \end{aligned}$$

so that $(5.17)_2$ follows via elementary estimations (recall that $\varepsilon < T/4$). Arguing in a similar way, and using the definition in (5.12) and $(1.42)_2$, we get that $2^{1+\gamma} g_{1,\varepsilon}(s) \geq \mathfrak{g}_1 \nu (s^2+\mu_\varepsilon^2)^{(\gamma-2)/2}$ for $s \geq T$, so that (5.18) follows after integration. Finally, observe that the last assertion concerning the fact that $t \mapsto \widetilde{F}_\varepsilon'(t)/t$ is nondecreasing is a straightforward consequence of the fact that $\gamma \geq 2$ and the definitions in (5.8)–(5.9) (recall that $\gamma \geq 2$ in (1.42) implies that $t \mapsto \widetilde{F}'(t)/t = \widetilde{a}(t)$ is itself nondecreasing). The proof of the lemma is now complete. □

Furthermore, we need another technical lemma; the peculiar notation concerning the number c_σ below is motivated by later applications (see Section 5.3 below).

LEMMA 5.2. *For every ε such that $0 < \varepsilon < \min\{1, T\}/4$, the following statements hold*:

- *There exists a constant $c \equiv c(\widetilde{a}(1), \nu, \gamma)$ such that*

$$F_\varepsilon(z) \leq c\left[F(z) + T^\gamma + \mu_\varepsilon^\gamma\right] \quad \text{for all } z \in \mathbb{R}^{N\times n}. \tag{5.20}$$

- *For every positive number c_σ there exists $\bar{\varepsilon} \equiv \bar{\varepsilon}(c_\sigma) < \min\{1, T\}/4$ such that if $0 < \varepsilon_1 \leq \varepsilon_2 < \bar{\varepsilon}$, then*

$$\begin{aligned} |F_{\varepsilon_1}(z) - F_{\varepsilon_2}(z)| \leq c[L_{\gamma,\varepsilon_1}(\varepsilon_1) + L_{\gamma,\varepsilon_2}(\varepsilon_2)] + \varepsilon_1 L_{\gamma,\varepsilon_1}(|z|) \\ + \varepsilon_2 L_{\gamma,\varepsilon_2}(|z|) + c\varepsilon_2, \end{aligned} \tag{5.21}$$

holds whenever $|z| \leq c_\sigma$, where $L_{\gamma,\varepsilon}$ has been defined in (5.10) *and $c \equiv c(\widetilde{a}(1), \gamma)$.*

PROOF. We start with the proof of (5.20); in the following we shall repeatedly use the fact that the functions $t \mapsto \widetilde{F}(t)$ and $t \mapsto \widetilde{F}_\varepsilon(t)$ are nondecreasing. For $|z| \leq \varepsilon$, by the definitions in (5.7)–(5.8) we have

$$
\begin{aligned}
F_\varepsilon(z) &\leq \widetilde{F}_\varepsilon(\varepsilon) \\
&\leq 2\int_0^\varepsilon \frac{\widetilde{a}(\varepsilon)(s^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} s}{(\varepsilon^2+\mu^2)^{\frac{\gamma-2}{2}}}\, ds + \varepsilon \int_0^\varepsilon (s^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} s\, ds \\
&\overset{(1.42)}{\leq} 2\left[\frac{\widetilde{a}(1)}{(1+\mu^2)^{\frac{\gamma-2}{2}}} + 1\right] \int_0^\varepsilon (s^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} s\, ds \\
&\leq c(\gamma)[\widetilde{a}(1)+1]\mu_\varepsilon^\gamma,
\end{aligned} \tag{5.22}
$$

so that the assertion follows in this case. Next, for $\varepsilon < |z| \leq T_\varepsilon$, by recalling that $\widetilde{a}(\cdot) \geq 0$, by the above inequality, (5.2), and (5.7)–(5.8), we find

$$
F_\varepsilon(z) \leq \int_\varepsilon^{|z|} \widetilde{a}(s) s\, ds + \varepsilon L_{\gamma,\varepsilon}(|z|) + \widetilde{F}_\varepsilon(\varepsilon) \leq F(z) + c(\widetilde{a}(1),\gamma)\big(|z|^\gamma + \mu_\varepsilon^\gamma\big).
$$

It remains to treat the case when $|z| \geq T_\varepsilon$. Since the previous estimate in particular implies $\widetilde{F}_\varepsilon(T_\varepsilon) \leq c(\widetilde{a}(1),\gamma)[F(z) + |z|^\gamma + \mu_\varepsilon^\gamma]$, we find similarly to the above

$$
\begin{aligned}
F_\varepsilon(z) &= \int_{T_\varepsilon}^{|z|} \frac{\widetilde{a}(T_\varepsilon)(s^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} s}{(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}}\, ds + \varepsilon L_{\gamma,\varepsilon}(|z|) + \widetilde{F}_\varepsilon(T_\varepsilon) \\
&\leq c(\gamma)\int_{T_\varepsilon}^{|z|} \widetilde{a}(s) s\, ds + c\big[F(z) + |z|^\gamma + \mu_\varepsilon^\gamma\big] \\
&\leq c(\widetilde{a}(1),\gamma)\big[F(z) + |z|^\gamma + \mu_\varepsilon^\gamma\big].
\end{aligned}
$$

Notice that we have also used $(1.42)_1$ to estimate

$$
\frac{\widetilde{a}(T_\varepsilon)(s^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}}{(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}} \leq c(\gamma)\widetilde{a}(s)
$$

when $T_\varepsilon \leq s$. Employing once again $(5.17)_1$, the proof of (5.20) is then complete. We next show (5.21). For this, we consider $\bar{\varepsilon} > 0$ with $\bar{\varepsilon} < \min\{1, T\}/4$ such that $T_{\bar{\varepsilon}} > c_\sigma$. By the very definition in (5.8) we notice that $\widetilde{a}_{\varepsilon_1}(t) = \widetilde{a}_{\varepsilon_2}(t)$ for

$\varepsilon_2 \le t \le c_\sigma$. Therefore, for $c_\sigma \ge |z| \ge \varepsilon_2$, as for (5.22) we get

$$\begin{aligned}
&|F_{\varepsilon_1}(z) - F_{\varepsilon_2}(z)| \\
&\quad\le 2\int_0^{\varepsilon_1} \frac{\widetilde{a}(\varepsilon_1)(s^2+\mu_{\varepsilon_1}^2)^{\frac{\gamma-2}{2}} s}{(\varepsilon_1^2+\mu^2)^{\frac{\gamma-2}{2}}}\, ds + \int_{\varepsilon_1}^{\varepsilon_2} \widetilde{a}(s) s\, ds \\
&\qquad + 2\int_0^{\varepsilon_2} \frac{\widetilde{a}(\varepsilon_2)(s^2+\mu_{\varepsilon_2}^2)^{\frac{\gamma-2}{2}} s}{(\varepsilon_2^2+\mu^2)^{\frac{\gamma-2}{2}}}\, ds + \varepsilon_1 L_{\gamma,\varepsilon_1}(|z|) + \varepsilon_2 L_{\gamma,\varepsilon_2}(|z|) \qquad (5.23)\\
&\quad\le c\widetilde{a}(1)\int_0^{\varepsilon_1} (s^2+\mu_{\varepsilon_1}^2)^{\frac{\gamma-2}{2}} s\, ds + c\widetilde{a}(1)\int_{\varepsilon_1}^{\varepsilon_2} (s^2+\mu^2)^{\frac{\gamma-2}{2}} s\, ds \\
&\qquad + c\widetilde{a}(1)\int_0^{\varepsilon_2} (s^2+\mu_{\varepsilon_2}^2)^{\frac{\gamma-2}{2}} s\, ds + \varepsilon_1 L_{\gamma,\varepsilon_1}(|z|) + \varepsilon_2 L_{\gamma,\varepsilon_2}(|z|).
\end{aligned}$$

Thus, (5.21) easily follows in this case, also recalling (5.10). If $|z| \le \varepsilon_1$, by simply estimating (look also at the previous display),

$$\begin{aligned}
&|F_{\varepsilon_1}(z) - F_{\varepsilon_2}(z)| \\
&\quad\le |F_{\varepsilon_1}(z)| + |F_{\varepsilon_2}(z)| \\
&\quad\le c\widetilde{a}(1)\int_0^{\varepsilon_1} (s^2+\mu_{\varepsilon_1}^2)^{\frac{\gamma-2}{2}} s\, ds + c\widetilde{a}(1)\int_0^{\varepsilon_2} (s^2+\mu_{\varepsilon_2}^2)^{\frac{\gamma-2}{2}} s\, ds \\
&\qquad + \varepsilon_1 L_{\gamma,\varepsilon_1}(|z|) + \varepsilon_2 L_{\gamma,\varepsilon_2}(|z|).
\end{aligned}$$

Finally, when $\varepsilon_1 < |z| < \varepsilon_2$ we again estimate $|F_{\varepsilon_1}(z) - F_{\varepsilon_2}(z)| \le |F_{\varepsilon_1}(z)| + |F_{\varepsilon_2}(z)|$ and come up with (5.23). In conclusion, we have proved that (5.21) holds in any case and the proof of the lemma is complete. □

As for the newly defined functions $g_{1,\varepsilon}(\cdot)$ and $g_{2,\varepsilon}(\cdot)$ displayed in (5.12) and (5.13), respectively, their basic properties can be summarized in the following analogue of Lemma 4.4:

LEMMA 5.3. *For every ε such that $0 < \varepsilon < \min\{1, T\}/4$, the following statements are true*:

- *The functions $[T, \infty) \ni t \mapsto g_{2,\varepsilon}(t)/g_{1,\varepsilon}(t)$ and $t \mapsto g_{1,\varepsilon}(t)t$ are almost nondecreasing and nondecreasing, respectively, in the sense that*

$$\begin{cases} T \le s \le t \quad \Longrightarrow \quad \dfrac{g_{2,\varepsilon}(s)}{g_{1,\varepsilon}(s)} \le c\,\dfrac{g_{2,\varepsilon}(t)}{g_{1,\varepsilon}(t)} \quad \textit{for } c \equiv c(n, N, \nu, c_a, \gamma), \\ g_{1,\varepsilon}(s)s \le g_{1,\varepsilon}(t)t. \end{cases} \tag{5.24}$$

- *If* (1.44) *holds, then also the following inequality holds for all* $t \geq T$:

$$\frac{g_{2,\varepsilon}(t)}{g_{1,\varepsilon}(t)} \leq cc_b \min\left\{\left(\int_T^t g_{1,\varepsilon}(s)s\,ds\right)^{\frac{2-\sigma}{(1+\vartheta)n}}, \left(\frac{1}{t^{1/\beta_1}}\int_T^t g_{1,\varepsilon}(s)s\,ds\right)^{\frac{4\beta_1}{(1+\vartheta)(n-2)}}\right\} + cc_b, \tag{5.25}$$

where $c \equiv c(n, N, \nu, \gamma)$ *and* ϑ *is defined as in the statement of Theorem* 1.13.

- *If* (1.45) *holds, then also the following inequality holds for all* $t \geq T$:

$$\frac{g_{2,\varepsilon}(t)}{g_{1,\varepsilon}(t)} \leq cc_b\left(\int_T^t g_{1,\varepsilon}(s)s\,ds\right)^{\frac{2-\sigma}{(1+\vartheta)n}} + cc_b, \tag{5.26}$$

with $c \equiv c(n, N, \nu, \gamma)$.

- *For every* $t \geq T$ *it holds that* $g_{1,\varepsilon}(t)/g_{2,\varepsilon}(t) \leq 1$.

PROOF. In order to prove that $y \mapsto g_{1,\varepsilon}(y)y$ is nondecreasing on $[T, \infty)$, we take $T \leq s < t$ and show that $g_{1,\varepsilon}(s)s \leq g_{1,\varepsilon}(t)t$. If $T \leq s < t \leq T_\varepsilon$, by definition (5.12) of $g_{1,\varepsilon}(\cdot)$ this simply means $g_1(s)s \leq g_1(t)t$, which is true by assumption (1.31). Next, if $T_\varepsilon \leq s < t$, we trivially obtain $g_{1,\varepsilon}(s)s \leq g_{1,\varepsilon}(t)t$ since the function $y \mapsto (y^2 + \mu_\varepsilon^2)^{(\gamma-2)/2}y$ is nondecreasing (recall that $\gamma > 1$). Finally, if $T \leq s \leq T_\varepsilon \leq t$, by the previous two cases we have $g_{1,\varepsilon}(s)s \leq g_{1,\varepsilon}(T_\varepsilon)T_\varepsilon \leq g_{1,\varepsilon}(t)t$, and the second inequality in (5.24) is shown. For the proof of the first inequality in (5.24), we similarly first consider the case $T \leq s \leq t \leq T_\varepsilon$ and distinguish two different situations. If $\gamma \geq 2$, we notice that $4\varepsilon \leq T \leq s$ implies that $(s^2 + \mu_\varepsilon^2) \leq 2(s^2 + \mu^2)$ and therefore, we can estimate

$$\begin{aligned} \frac{g_{2,\varepsilon}(s)}{g_{1,\varepsilon}(s)} &\leq \frac{\mathfrak{g}_2}{\mathfrak{g}_1}\frac{g_2(s) + \varepsilon 2^{\frac{\gamma-2}{2}}(s^2+\mu^2)^{\frac{\gamma-2}{2}}}{g_1(s)} \\ &\overset{(5.6)}{\leq} \frac{\mathfrak{g}_2}{\mathfrak{g}_1}\frac{g_2(s) + \varepsilon 2^{\frac{\gamma-2}{2}}\widetilde{\nu}^{-1}\widetilde{a}(s)}{g_1(s)} \\ &\overset{(5.5)}{\leq} c\,\frac{g_2(s)}{g_1(s)} \overset{(1.31)}{\leq} c\,\frac{g_2(t)}{g_1(t)} \leq c\,\frac{g_{2,\varepsilon}(t)}{g_1(t)} = c\,\frac{g_{2,\varepsilon}(t)}{g_{1,\varepsilon}(t)} \end{aligned} \tag{5.27}$$

where $c \equiv c(n, N, \nu, c_a, \gamma)$ (recall (5.14) and $\nu \leq \widetilde{\nu}$ by (5.6)). Otherwise, if $\gamma < 2$, we can estimate directly $(s^2 + \mu_\varepsilon^2)^{(\gamma-2)/2} \leq (s^2 + \mu^2)^{(\gamma-2)/2}$ and proceed as in the last display. Notice that we have indeed also proved that

$$T \leq s \leq T_\varepsilon \quad \Longrightarrow \quad \frac{g_{2,\varepsilon}(s)}{g_{1,\varepsilon}(s)} \leq c\,\frac{g_2(s)}{g_1(s)}, \quad c \equiv c(n, N, \nu, \gamma). \tag{5.28}$$

Next, in the case $T_\varepsilon \leq s < t$, the assertion follows after observing that

$$\frac{g_{2,\varepsilon}(y)}{g_{1,\varepsilon}(y)} = \frac{g_{2,\varepsilon}(T_\varepsilon)}{g_{1,\varepsilon}(T_\varepsilon)} \quad \text{for all } y \geq T_\varepsilon. \tag{5.29}$$

In the remaining case $T \le s \le T_\varepsilon \le t$, we again use the last two cases and (5.29) to get $g_{2,\varepsilon}(s)/g_{1,\varepsilon}(s) \le c g_{2,\varepsilon}(T_\varepsilon)/g_{1,\varepsilon}(T_\varepsilon) = c g_{2,\varepsilon}(t)/g_{1,\varepsilon}(t)$ and the proof of (5.24) is complete. Next, we prove the second claim (5.25). This follows straightaway from the assumption (1.44) and (5.28) when $T \le t \le T_\varepsilon$. A different reasoning is needed for the case $T_\varepsilon < t$. Here we may use the result from the first case and start estimating via (5.29)

$$\begin{aligned} \frac{g_{2,\varepsilon}(t)}{g_{1,\varepsilon}(t)} &= \frac{g_{2,\varepsilon}(T_\varepsilon)}{g_{1,\varepsilon}(T_\varepsilon)} \\ &\le c c_b \min\left\{\left(\int_T^{T_\varepsilon} g_{1,\varepsilon}(s) s\, ds\right)^{\frac{2-\sigma}{(1+\vartheta)n}}, \right. \\ &\qquad \left. \left(\frac{1}{T_\varepsilon^{1/\beta_1}} \int_T^{T_\varepsilon} g_{1,\varepsilon}(s) s\, ds\right)^{\frac{4\beta_1}{(1+\vartheta)(n-2)}}\right\} + c c_b. \end{aligned} \tag{5.30}$$

Then it obviously follows that

$$\frac{g_{2,\varepsilon}(t)}{g_{1,\varepsilon}(t)} \le c c_b \left(\int_T^t g_{1,\varepsilon}(s) s\, ds\right)^{\frac{2-\sigma}{(1+\vartheta)n}} + c c_b \tag{5.31}$$

as $t \ge T_\varepsilon$ for $c \equiv c(n, N, \nu, \gamma)$. On the other hand, let us now introduce the nonnegative quantity

$$\begin{aligned} \mathcal{Q}_\varepsilon &:= \left(\frac{t}{T_\varepsilon}\right)^{1/\beta_1} \frac{\displaystyle\int_T^{T_\varepsilon} g_{1,\varepsilon}(s) s\, ds}{\displaystyle\int_T^t g_{1,\varepsilon}(s) s\, ds} \\ &= \left(\frac{t}{T_\varepsilon}\right)^{1/\beta_1} \frac{I_\varepsilon}{I_\varepsilon + \dfrac{\mathfrak{g}_1 g_1(T_\varepsilon)}{(T_\varepsilon^2 + \mu_\varepsilon^2)^{\frac{\gamma-2}{2}}} \displaystyle\int_{T_\varepsilon}^t (s^2 + \mu_\varepsilon^2)^{\frac{\gamma-2}{2}} s\, ds} \\ &= \frac{t_\varepsilon^{1/\beta_1} I_\varepsilon}{I_\varepsilon + \dfrac{\mathfrak{g}_1 g_1(T_\varepsilon) T_\varepsilon^\gamma}{\gamma (T_\varepsilon^2 + \mu_\varepsilon^2)^{\frac{\gamma-2}{2}}} \left\{\left[t_\varepsilon^2 + \left(\dfrac{\mu_\varepsilon}{T_\varepsilon}\right)^2\right]^{\gamma/2} - \left[1 + \left(\dfrac{\mu_\varepsilon}{T_\varepsilon}\right)^2\right]^{\gamma/2}\right\}} \end{aligned} \tag{5.32}$$

where $t_\varepsilon := t/T_\varepsilon$ and where, thanks to the fact that $t \mapsto g_1(t)t$ is nondecreasing on $[T, \infty)$, we have

$$I_\varepsilon := \mathfrak{g}_1 \int_T^{T_\varepsilon} g_1(s) s\, ds \le \mathfrak{g}_1 g_1(T_\varepsilon) T_\varepsilon^2. \tag{5.33}$$

We now distinguish two cases. If $t_\varepsilon \le 1000$, then it easily follows that $\mathcal{Q}_\varepsilon \le 1000^{1/\beta_1}$. Otherwise, if $t_\varepsilon > 1000$, then by recalling that $1/\beta_1 \le \gamma$ and $\mu_\varepsilon/T_\varepsilon \le 2$

we instead have

$$\mathscr{Q}_\varepsilon \overset{(5.33)}{\leq} c(\gamma)\frac{\mathfrak{g}_1 t_\varepsilon^\gamma g_1(T_\varepsilon)T_\varepsilon^2}{I_\varepsilon + \mathfrak{g}_1 t_\varepsilon^\gamma g_1(T_\varepsilon)T_\varepsilon^2}.$$

In any case we conclude that $\mathscr{Q}_\varepsilon \leq c(\gamma)$, uniformly with respect to ε, and therefore we can bound

$$\frac{g_{2,\varepsilon}(t)}{g_{1,\varepsilon}(t)} \leq cc_b\Big[\sup_\varepsilon \mathscr{Q}_\varepsilon\Big]^{\frac{4\beta_1}{(1+\vartheta)(n-2)}}\left(\frac{1}{t^{1/\beta_1}}\int_T^t g_{1,\varepsilon}(s)s\,ds\right)^{\frac{4\beta_1}{(1+\vartheta)(n-2)}} + cc_b.$$

Then (5.25) follows from (5.31) and the above display. Similarly, the third claim (5.26) follows. Finally, we observe that (5.4)–(5.5) imply $g_1(t) \leq g_2(t)$ for all $t \geq T$, which in turn shows the final claim $g_{1,\varepsilon}(t) \leq g_{2,\varepsilon}(t)$ for every $t \geq T$, by the very definitions of $g_{1,\varepsilon}$ and $g_{2,\varepsilon}$ given in (5.12) and (5.13), respectively. This completes the proof of the lemma. □

5.2 A Priori Estimates

In this section we consider again a functional of the type in (1.1) under additional assumptions on the integrand $z \mapsto F(z) \equiv \tilde{F}(|z|)$ and then apply the corresponding estimates to the case of the approximating functionals defined in the previous section. Specifically, with $B \subset \mathbb{R}^n$ being a fixed ball with $n \geq 2$, we start considering a vector-valued weak solution $u \in W^{1,\gamma}(B;\mathbb{R}^N)$ to the system

$$-\operatorname{div} a(Du) = f \quad \text{in } B \subset \mathbb{R}^n, \qquad f \in L^\infty(\mathbb{R}^n;\mathbb{R}^N), \tag{5.34}$$

with $a\colon \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ being such that

$$a(z) = \tilde{a}(|z|)z \quad \text{for all } z \in \mathbb{R}^{N\times n}, \tag{5.35}$$

where $\tilde{a}\colon [0,\infty) \to [0,\infty)$ is of class $W^{1,\infty}_{\mathrm{loc}}[0,\infty) \cap C^1_{\mathrm{loc}}([0,\infty)\setminus \mathscr{N}\mathscr{D})$; i.e., it is locally C^1-regular outside a finite set $\mathscr{N}\mathscr{D} \subset (0,\infty)$, and it is such that $\tilde{a}'(0) = 0$. This implies that $a(\cdot) \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^{N\times n})$. The prototype we have in mind is of course given by the function $\tilde{a}_\varepsilon(\cdot)$ in (5.8). Note that this structure assumption implies that $\partial a(z)$ is a symmetric nonnegative bilinear form on $\mathbb{R}^{N\times n}$ whenever it makes sense. For a fixed $\varepsilon \in (0,1)$ such that $0 < \varepsilon < \min\{1,T\}/4$ as in Section 5.1, we then assume that the following growth and ellipticity conditions are satisfied whenever $\partial a(z)$ makes sense (this in fact happens whenever $|z| \notin \mathscr{N}\mathscr{D}$):

$$\begin{cases} |\partial a(z)| \leq g_{2,\varepsilon}(|z|) & \text{on } \{|z| > T, |z| \notin \mathscr{N}\mathscr{D}\}, \\ g_{1,\varepsilon}(|z|)|\xi|^2 \leq \langle \partial a(z)\xi,\xi\rangle & \text{on } \{|z| > T, |z| \notin \mathscr{N}\mathscr{D}\}, \\ |\partial a(z)| \leq \Gamma(|z|^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} & \text{on } \{|z| \notin \mathscr{N}\mathscr{D}\}, \\ \nu_0(|z|^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}|\xi|^2 \leq \langle \partial a(z)\xi,\xi\rangle & \text{on } \{|z| \notin \mathscr{N}\mathscr{D}\}, \end{cases} \tag{5.36}$$

for every $\xi \in \mathbb{R}^{N\times n}$, where the functions $g_{1,\varepsilon}, g_{2,\varepsilon}\colon (0,\infty) \to (0,\infty)$ have been defined in (5.12) and (5.13), respectively, while $\mu_\varepsilon = \mu + \varepsilon > 0$, as defined in (5.7). Here $0 < \nu_0 \leq 1 \leq \Gamma$ denote fixed constants that are not going to play any quantitative role in the forthcoming a priori estimates. Exactly as for (5.4)–(5.5), we find from (5.35)–(5.36) and from the definitions in (5.8) and (5.12)–(5.13) that for any $z \in \mathbb{R}^{N\times n}$ such that $|z| > T$ it holds that

$$\widetilde{a}(|z|) \geq g_{1,\varepsilon}(|z|) \quad \text{and} \quad \widetilde{a}(|z|) \leq g_{2,\varepsilon}(|z|) \tag{5.37}$$

and

$$\begin{cases} \widetilde{a}(|z|) + \widetilde{a}'(|z|)|z| \geq g_{1,\varepsilon}(|z|) \\ \widetilde{a}(|z|) + \widetilde{a}'(|z|)|z| \leq g_{2,\varepsilon}(|z|) \end{cases} \qquad \text{if, in addition, } |z| \notin \mathscr{N}\mathscr{D}. \tag{5.38}$$

A trivial consequence of (5.37)–(5.38) is

$$|\widetilde{a}'(|z|)||z| \leq g_{2,\varepsilon}(|z|) \quad \text{if } |z| > T \text{ and } |z| \notin \mathscr{N}\mathscr{D}, \tag{5.39}$$

while, similarly to (5.37), we have that

$$\widetilde{a}(|z|) \geq \nu_0(|z|^2 + \mu_\varepsilon^2)^{\frac{\gamma-2}{2}} \quad \text{and} \quad \widetilde{a}(|z|) \leq \Gamma(|z|^2 + \mu_\varepsilon^2)^{\frac{\gamma-2}{2}} \tag{5.40}$$

hold this time whenever $|z| \geq 0$ (recall that $\widetilde{a}(\cdot)$ is continuous). We notice that, upon defining

$$F(z) := \widetilde{F}(|z|) := \int_0^{|z|} \widetilde{a}(s)s\,ds, \tag{5.41}$$

which by (5.36) is a strictly convex integrand, by (5.34) we have that u is a local minimizer of the functional

$$w \mapsto \int_B [\widetilde{F}(|Dw|) - fw]dx = \int_B [F(Dw) - fw]dx \tag{5.42}$$

in the sense of Definition 1.1. By (5.2) we get that

$$\frac{\nu_0}{\gamma}(t^2 + \mu_\varepsilon^2)^{\gamma/2} - \frac{\nu_0\mu_\varepsilon^\gamma}{\gamma} \leq \widetilde{F}(t) \leq \Gamma(t^2 + \mu_\varepsilon^2)^{\gamma/2} \quad \forall t \geq 0. \tag{5.43}$$

Assumptions $(5.36)_{3,4}$ made on $a(\cdot)$ and (5.43) allow us to verify that assumptions (8.2) below are satisfied by the integrand in (5.41) (for a suitable choice of parameters ν and Λ adopted there). Therefore the regularity results in (8.3) apply to u and read

$$\begin{gathered} Du \in L^\infty_{\mathrm{loc}}(B;\mathbb{R}^{N\times n}), \\ u \in W^{2,2}_{\mathrm{loc}}(B;\mathbb{R}^N), \quad a(Du) \in W^{1,2}_{\mathrm{loc}}(B;\mathbb{R}^{N\times n}). \end{gathered} \tag{5.44}$$

We start with a technical lemma, exploiting the structure assumption (5.35) (see also Remark 5.5 below).

LEMMA 5.4. *Let* $a\colon \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ *be the vector field considered in* (5.34) *and therefore satisfy conditions* (5.35) *and* (5.36). *Then, for every* $w \in W^{2,2}_{\mathrm{loc}}(B;\mathbb{R}^N) \cap W^{1,\infty}_{\mathrm{loc}}(B;\mathbb{R}^N)$*, the inequality*

$$\sum_{s=1}^n \left\langle \frac{\partial a(Dw)}{|Dw|} DD_s w, D_s w \otimes D|Dw| \right\rangle \geq g_{1,\varepsilon}(|Dw|)\big|D|Dw|\big|^2 \tag{5.45}$$

holds at almost every point $x \in B$ such that $|Dw(x)| > T$ and $|Dw(x)| \notin \mathcal{N}\mathcal{D}$. Furthermore, if $H \in L^1_{\rm loc}(B)$ and $\eta \in W^{1,1}_{\rm loc}(B)$ are two functions, again at almost every point $x \in B$ such that $|Dw(x)| > T$ and $|Dw(x)| \notin \mathcal{N}\mathcal{D}$, the inequality

$$\sum_{s=1}^{n}\left\langle \frac{\partial a(Dw)}{|Dw|} DD_s w, D_s w \otimes \left(\eta^2 D|Dw| + H\eta D\eta\right)\right\rangle \geq \frac{1}{2} g_{1,\varepsilon}(|Dw|)\big|D|Dw|\big|^2\eta^2 - \frac{1}{2} g_{2,\varepsilon}(|Dw|)H^2|D\eta|^2 \tag{5.46}$$

holds provided $\tilde{a}(\cdot)$ is nondecreasing on $[T,\infty)$.

PROOF. Recalling (5.1) and (5.3) we have

$$\left(\frac{\partial a(z)}{|z|}\right)^{\alpha,\beta}_{i,j} = \frac{\partial_{z_j^\beta} a_i^\alpha}{|z|} = \frac{\tilde{a}(|z|)}{|z|}\delta_{ij}\delta_{\alpha\beta} + \tilde{a}'(|z|)\frac{z_i^\alpha z_j^\beta}{|z|^2} \tag{5.47}$$

for $|z| \neq 0$, $|z| \notin \mathcal{N}\mathcal{D}$. We calculate (recall the notation in (5.3))

$$\begin{aligned} \sum_{s=1}^{n} \frac{\langle \mathbb{I}_{N\times n} DD_s w, D_s w \otimes D|Dw|\rangle}{|Dw|} &= \sum_{i,s=1}^{n}\sum_{\alpha=1}^{N} \frac{D_i D_s w^\alpha D_s w^\alpha D_i |Dw|}{|Dw|} \\ &= \sum_{i=1}^{n} \big|D_i|Dw|\big|^2 = \big|D|Dw|\big|^2 \end{aligned} \tag{5.48}$$

and

$$\begin{aligned} \sum_{s=1}^{n} \frac{\langle (Dw\otimes Dw) DD_s w, D_s w \otimes D|Dw|\rangle}{|Dw|^2} &= \sum_{i,j,s=1}^{n}\sum_{\alpha,\beta=1}^{N} \frac{D_i w^\alpha D_j w^\beta D_j D_s w^\beta D_s w^\alpha D_i|Dw|}{|Dw|^2} \\ &= \sum_{i,s=1}^{n}\sum_{\alpha=1}^{N} \frac{D_i w^\alpha D_s w^\alpha D_s|Dw| D_i|Dw|}{|Dw|} \\ &= \sum_{\alpha=1}^{N} \frac{|\langle Dw^\alpha, D|Dw|\rangle|^2}{|Dw|}. \end{aligned} \tag{5.49}$$

Merging (5.47)–(5.49) yields

$$\begin{aligned}(5.50)\quad &\sum_{s=1}^{n}\Big\langle \frac{\partial a(Dw)}{|Dw|} DD_s w, D_s w \otimes \eta^2 D|Dw|\Big\rangle \\ &= \eta^2 \widetilde{a}(|Dw|)\big|D|Dw|\big|^2 + \eta^2 \widetilde{a}'(|Dw|)|Dw| \sum_{\alpha=1}^{N} \frac{\big|\langle Dw^\alpha, D|Dw|\rangle\big|^2}{|Dw|^2}\,.\end{aligned}$$

Proof of (5.45). The assertion in (5.45) follows immediately from (5.50) and the first inequality in (5.37), whenever $\widetilde{a}'(|Dw(x)|)$ is non-negative. Otherwise, when $\widetilde{a}'(|Dw|)$ is negative, we estimate

$$(5.51)\qquad \sum_{\alpha=1}^{N} \frac{\big|\langle Dw^\alpha, D|Dw|\rangle\big|^2}{|Dw|^2} \le \big|D|Dw|\big|^2$$

so that, by (5.50), as we are evaluating Dw at those points x where $|Dw(x)| > T$ and $|Dw(x)| \notin \mathscr{N}\mathscr{D}$, we obtain

$$\begin{aligned}&\sum_{s=1}^{n}\Big\langle \frac{\partial a(Dw)}{|Dw|} DD_s w, D_s w \otimes D|Dw|\Big\rangle \\ &\overset{(5.51)}{\ge} \big[\widetilde{a}(|Dw|) + \widetilde{a}'(|Dw|)|Dw|\big]\big|D|Dw|\big|^2 \\ &\overset{(5.38)_1}{\ge} g_{1,\varepsilon}(|Dw|)\big|D|Dw|\big|^2\end{aligned}$$

and (5.45) is completely proved.

Proof of (5.46). Proceeding as for (5.48), we here find

$$\begin{aligned}H\eta \sum_{s=1}^{n} \frac{\langle \mathbb{I}_{N\times n} DD_s w, D_s w \otimes D\eta\rangle}{|Dw|} &= H\eta \sum_{i,s=1}^{n}\sum_{\alpha=1}^{N} \frac{D_i D_s w^\alpha D_s w^\alpha D_i \eta}{|Dw|} \\ &= H\eta \langle D|Dw|, D\eta\rangle\end{aligned}$$

and, as for (5.49), we get

$$\begin{aligned}&H\eta \sum_{s=1}^{n} \frac{\langle (Dw\otimes Dw) DD_s w, D_s w \otimes D\eta\rangle}{|Dw|^3} \\ &= H\eta \sum_{i,j,s=1}^{n}\sum_{\alpha,\beta=1}^{N} \frac{D_i w^\alpha D_j w^\beta D_j D_s w^\beta D_s w^\alpha D_i \eta}{|Dw|^3} \\ &= H\eta \sum_{i,s=1}^{n}\sum_{\alpha=1}^{N} \frac{D_i w^\alpha D_s w^\alpha D_s |Dw| D_i \eta}{|Dw|^2} \\ &= H\eta \sum_{\alpha=1}^{N} \frac{\langle Dw^\alpha, D|Dw|\rangle \langle Dw^\alpha, D\eta\rangle}{|Dw|^2}\,.\end{aligned}$$

Recalling that $\widetilde{a}'(\cdot) \geq 0$ on $[T, \infty)$, it follows that

$$
\begin{aligned}
\sum_{s=1}^{n} & \left\langle \frac{\partial a(Dw)}{|Dw|} DD_s w, D_s w \otimes H\eta D\eta \right\rangle \\
&= \widetilde{a}(|Dw|)H\eta\langle D|Dw|, D\eta\rangle \\
&\quad + \widetilde{a}'(|Dw|)|Dw|H\eta \sum_{\alpha=1}^{N} \frac{\langle Dw^{\alpha}, D|Dw|\rangle\langle Dw^{\alpha}, D\eta\rangle}{|Dw|^2} \\
&\geq \widetilde{a}(|Dw|)\left[-\frac{1}{2}|D|Dw||^2\eta^2 - \frac{1}{2}H^2|D\eta|^2\right] \\
&\quad + \widetilde{a}'(|Dw|)|Dw| \\
&\qquad \cdot \left[-\frac{\eta^2}{2}\sum_{\alpha=1}^{N} \frac{\left|\langle Dw^{\alpha}, D|Dw|\rangle\right|^2}{|Dw|^2} - \frac{1}{2}\frac{H^2|D\eta|^2}{|Dw|^2}\sum_{\alpha=1}^{N}|Dw^{\alpha}|^2\right].
\end{aligned}
\tag{5.52}
$$

Adding up (5.52) and (5.50), and using $(5.38)_1$, finally yields (5.46). □

Remark 5.5. Lemma 5.4 continues to hold a.e. in $\{|Dw| \in \mathcal{N}\mathcal{D}\} \cap \{|Dw| > T\}$ provided some conventions are made. Indeed, as $|Dw| \in W^{1,2}_{\mathrm{loc}}(B)$, by (5.35) the standard chain rule applies and gives that $a(Dw) \in W^{1,2}_{\mathrm{loc}}(B;\mathbb{R}^{N\times n})$. Moreover, for any $s \in \{1,\dots,n\}$ it holds that

$$
\begin{aligned}
D_s[a(Dw)] &= \partial a(Dw)DD_s w \\
&= \widetilde{a}(|Dw|)DD_s w + \widetilde{a}'(|Dw|)D_s|Dw|Dw
\end{aligned}
\tag{5.53}
$$

while the co-area formula gives that $|D|Dw|| \equiv 0$ a.e. on $\{|Dw| \in \mathcal{N}\mathcal{D}\} \cap \{|Dw| > T\}$. This fact allows us to give meaning to the right-hand side of (5.53) a.e. on $\{|Dw| \in \mathcal{N}\mathcal{D}\}$. Specifically, we interpret (5.53) as $\partial a(Dw)DD_s w = \widetilde{a}(|Dw|)DD_s w$; that is, we set (recall the notation in (5.3))

$$
\partial a(Dw) = \widetilde{a}(|Dw|)\mathbb{I}_{N\times n} \quad \text{a.e. in } \{|Dw| \in \mathcal{N}\mathcal{D}\}. \tag{5.54}
$$

With such notation, it is now straightforward to note that both (5.45) and (5.46) also hold a.e. on $\{|Dw| \in \mathcal{N}\mathcal{D}\} \cap \{|Dw| > T\}$. In particular, (5.46) turns out to be valid independently of whether $\widetilde{a}(\cdot)$ is nondecreasing or not, as the proof works in this case as if it were $\widetilde{a}'(|Dw(x)|) \equiv 0$.

In the following, with $0 < \varepsilon < \min\{1, T\}/4$, accordingly to what was done in (4.27), we define

$$
G_{T,\varepsilon}(t) := \int_T^{\max\{t,T\}} g_{1,\varepsilon}(s)s\,ds. \tag{5.55}
$$

We then have the following lemma:

LEMMA 5.6. *Let* $u \in W^{1,\gamma}_{\mathrm{loc}}(B;\mathbb{R}^N)$ *be a weak solution to* (5.34) *under the assumptions* (5.35) *and* (5.36) *and with* $n \geq 2$. *Let* $B_r(x_0) \Subset B$ *be another ball and*

M be such that $\|Du\|_{L^\infty(B_r(x_0))} \le M$ and $T \le M$. Then, for each $k \ge 0$, the inequality

$$\begin{aligned}\int_{B_{r/2}(x_0)} &\left|D(G_{T,\varepsilon}(|Du|)-k)_+\right|^2 dx \\ &\le \frac{c}{r^2}\left[\frac{g_{2,\varepsilon}(M)}{g_{1,\varepsilon}(M)}\right]^{1+\vartheta}\int_{B_r(x_0)}(G_{T,\varepsilon}(|Du|)-k)_+^2\,dx \\ &\quad + cM^2\int_{B_r(x_0)\cap\{G_{T,\varepsilon}(|Du|)>k\}}|f|^2\,dx\end{aligned} \tag{5.56}$$

holds for $c \equiv c(n,N,\nu,c_a,\gamma)$, with $\vartheta = 0$ if $\tilde{a}(\cdot)$ in (5.35) is a nondecreasing function on $[T,\infty)$ and $\vartheta = 1$ otherwise.

PROOF. We recall that u enjoys the regularity properties in (5.44). We follow the same strategy as for the proof of Lemma 4.5, and hence we adopt the same notation used there (in particular, for the localization function η). Throughout the proof we use the simplified notation $G(\cdot) \equiv G_{T,\varepsilon}(\cdot)$, and we keep in mind the convention fixed in Remark 5.5 to treat those points x where $|Du(x)| \in \mathscr{N}\mathscr{D}$ and $|Du(x)| > T$. Specifically, we use (5.54) with $Dw \equiv Du$. As in the scalar case we have

$$\begin{aligned}D_i(G(|Du|)-k)_+ &= g_{1,\varepsilon}(|Du|)\sum_{s=1}^{n}\sum_{\alpha=1}^{N} D_iD_su^\alpha D_su^\alpha \\ &= g_{1,\varepsilon}(|Du|)|Du|D_i|Du| \quad \text{and} \quad g_{1,\varepsilon}(|Du|) > 0\end{aligned} \tag{5.57}$$

for every $i \in \{1,\dots,n\}$ and whenever $G(|Du|) > k$. The differentiated form of the system (5.34), that is,

$$\int_B \langle \partial a(Du)DD_su, D\varphi\rangle dx = -\int_B fD_s\varphi\,dx \tag{5.58}$$

(as in (4.32)) for $s \in \{1,\dots,n\}$ can be tested with

$$\varphi \equiv \varphi_s := \eta^2(G(|Du|)-k)_+D_su \in W_0^{1,2}(B;\mathbb{R}^n)$$

so that

$$\begin{aligned}D\varphi_s &= \eta^2(G(|Du|)-k)_+DD_su + \eta^2D_su \otimes D(G(|Du|)-k)_+ \\ &\quad + 2\eta(G(|Du|)-k)_+D_su \otimes D\eta.\end{aligned} \tag{5.59}$$

Summing over $s \in \{1,\dots,n\}$ accordingly, three terms appear on the left-hand side of (5.58). For the first one we use $(5.36)_2$ for those points where $|Dw| \notin \mathscr{N}\mathscr{D}$, while for a.e. point where $|Dw| \in \mathscr{N}\mathscr{D}$, we use (5.54) with $Dw \equiv Du$ and the first inequality in (5.37) (the remaining set of points is negligible); summarizing, we get the nonnegative contribution

$$\sum_{s=1}^{n}\int_B \langle \partial a(Du)DD_s u, DD_s u\rangle (G(|Du|)-k)_+\eta^2\,dx \\ \geq \frac{1}{2}\int_B g_{1,\varepsilon}(|Du|)(G(|Du|)-k)_+|D^2u|^2\eta^2\,dx.$$

To estimate the remaining two terms, we use Lemma 5.4, taking into account also the content of Remark 5.5. We first consider the situation when $\widetilde{a}(\cdot)$ is nondecreasing (and therefore $\widetilde{a}'(\cdot)\geq 0$). In this case we apply (5.46) with the choice

$$H \equiv \mathbb{1}_{\{|Du|>T\}}\frac{2(G(|Du|)-k)_+}{g_{1,\varepsilon}(|Du|)|Du|}\in L^\infty(B_r)$$

in order to estimate

$$\begin{aligned}
&\sum_{s=1}^{n}\int_B \Big\langle \partial a(Du)DD_s u, \\
&\qquad D_s u\otimes\big[\eta^2 D(G(|Du|)-k)_+ + 2(G(|Du|)-k)_+\eta D\eta\big]\Big\rangle\,dx \\
&= \sum_{s=1}^{n}\int_{B\cap\{G(|Du|)>k\}} g_{1,\varepsilon}(|Du|)|Du|\Big\langle \partial a(Du)DD_s u, \\
&\qquad\qquad D_s u\otimes\left[\eta^2 D|Du| + \frac{2(G(|Du|)-k)_+}{g_{1,\varepsilon}(|Du|)|Du|}\eta D\eta\right]\Big\rangle dx \\
&\geq \frac{1}{2}\int_{B\cap\{G(|Du|)>k\}}[g_{1,\varepsilon}(|Du|)]^2|Du|^2|D|Du||^2\eta^2\,dx \\
&\quad - 2\int_B \frac{g_{2,\varepsilon}(|Du|)}{g_{1,\varepsilon}(|Du|)}(G(|Du|)-k)_+^2|D\eta|^2\,dx \\
&\geq \frac{1}{2}\int_{B\cap\{G(|Du|)>k\}}\big|D(G(|Du|)-k)_+\big|^2\eta^2\,dx \\
&\quad - \frac{c}{r^2}\left[\frac{g_{2,\varepsilon}(M)}{g_{1,\varepsilon}(M)}\right]\int_{B_r(x_0)}(G(|Du|)-k)_+^2\,dx,
\end{aligned}$$

with $c\equiv c(n,N,\nu,c_a,\gamma)$; recall (5.57). Notice that in the last estimation we have also used (5.24). This concludes the estimates for the right-hand side terms in (5.58) when $\widetilde{a}(\cdot)$ is a nondecreasing function on $[T,\infty)$. In the general case, when no additional information on $\widetilde{a}(\cdot)$ is available, we instead use (5.45) (again keeping Remark 5.5 in mind) to have only an estimate for the second term arising from

(5.59); that is, recalling (5.57), we here find

$$\begin{aligned}
&\sum_{s=1}^{n}\int \langle \partial a(Du)DD_s u, D_s u \otimes D(G(|Du|)-k)_+\rangle \eta^2\,dx\\
&\quad= \sum_{s=1}^{n}\int_{B\cap\{G(|Du|)>k\}} g_{1,\varepsilon}(|Du|)|Du|\langle \partial a(Du)DD_s u, D_s u\otimes D|Du|\rangle\eta^2\,dx\\
&\quad\geq \int_{B\cap\{G(|Du|)>k\}} [g_{1,\varepsilon}(|Du|)]^2|Du|^2\big|D|Du|\big|^2\eta^2\,dx\\
&\quad= \int_B \big|D(G(|Du|)-k)_+\big|^2\eta^2\,dx.
\end{aligned}$$

Finally, we treat the third term coming from (5.59). Recalling (5.47) and (5.57), we use the first identity in (5.52) with the choice $H \equiv 2(G(|Du|)-k)_+$, getting

$$\begin{aligned}
&\sum_{s=1}^{n}\langle \partial a(Du)DD_s u, D_s u\otimes H\eta D\eta\rangle\\
&\quad= \widetilde{a}(|Du|)H\eta\sum_{s,i=1}^{n}\sum_{\alpha=1}^{N} D_i D_s u^\alpha D_s u^\alpha D_i\eta\\
&\qquad+ \widetilde{a}'(|Du|)|Du|H\eta\sum_{i,j,s=1}^{n}\sum_{\alpha,\beta=1}^{N}\frac{D_i u^\alpha D_j u^\beta D_j D_s u^\beta D_s u^\alpha D_i\eta}{|Du|^2}\\
&\quad= \frac{\widetilde{a}(|Du|)}{g_{2,\varepsilon}(|Du|)}\frac{g_{2,\varepsilon}(|Du|)}{g_{1,\varepsilon}(|Du|)}(G(|Du|)-k)_+ 2\eta\sum_{i=1}^{n} D_i(G(|Du|)-k)_+ D_i\eta\\
&\qquad+ \frac{\widetilde{a}'(|Du|)|Du|}{g_{2,\varepsilon}(|Du|)}\frac{g_{2,\varepsilon}(|Du|)}{g_{1,\varepsilon}(|Du|)}(G(|Du|)-k)_+\\
&\qquad\quad\cdot 2\eta\sum_{i,s=1}^{n}\sum_{\alpha=1}^{N}\frac{D_s(G(|Du|)-k)_+ D_i u^\alpha D_s u^\alpha D_i\eta}{|Du|^2}.
\end{aligned}$$

By the above identity and using $(5.37)_2$ and (5.39), and eventually also the Young inequality, we get

$$\begin{aligned}
&2\left|\sum_{s=1}^{n}\int_B \langle \partial a(Du)DD_s u, (G(|Du|)-k)_+ D_s u\otimes D\eta\rangle\eta\,dx\right|\\
&\quad\leq \frac{1}{2}\int_B \big|D(G(|Du|)-k)_+\big|^2\eta^2\,dx + \frac{c}{r^2}\left[\frac{g_{2,\varepsilon}(M)}{g_{1,\varepsilon}(M)}\right]^2\int_B (G(|Du|)-k)_+^2\,dx,
\end{aligned}$$

again with $c \equiv c(n, N, \nu, c_a, \gamma)$. Connecting all the estimates in the displays coming after (5.59), in any case we have

$$\int_B \big(g_{1,\varepsilon}(|Du|)(G(|Du|)-k)_+|D^2u|^2 + \big|D(G(|Du|)-k)_+\big|^2\big)\eta^2\,dx$$
$$\le \frac{c}{r^2}\bigg[\frac{g_{2,\varepsilon}(M)}{g_{1,\varepsilon}(M)}\bigg]^{1+\vartheta}\int_B (G(|Du|)-k)_+^2\,dx + c\sum_{s=1}^n \int_B |f||D_s\varphi_s|dx,$$

where $c \equiv c(n, N, \nu, c_a, \gamma)$. The last term in the above display can be estimated exactly as it was done in the scalar Lemma 4.5 (compare with (4.36) and subsequent estimates, and use the last point in Lemma 5.3), which then leads to the claim (5.56). □

We now apply the above results to the setting of Section 5.1, that is, to the case that $\widetilde{a}(\cdot)$ in (5.35) is such that $\widetilde{a}(s) \equiv \widetilde{a}_\varepsilon(s) + \varepsilon(s^2+\mu_\varepsilon^2)^{(\gamma-2)/2}$ and therefore that the functionals in (5.42) have integrands $F \equiv F_\varepsilon$ as in (5.9) while the functions $\widetilde{a}_\varepsilon$ are defined in (5.8). We recall that Lemma 5.1 gives that $t \mapsto \widetilde{F}_\varepsilon'(t)/t$ is nondecreasing if $\gamma \ge 2$ in (1.42). The inequalities deriving from (5.56) with the choices $\vartheta = 0$ ($\gamma \ge 2$) and $\vartheta = 1$ ($\gamma < 2$) are formally equal to those displayed in Lemmas 4.5 and 4.6, respectively. Here we are taking $\beta_0 = 1$ in (1.34) while $g_1(\cdot), g_{2,\varepsilon}(\cdot)$ in this setting are replaced by the functions $g_{1,\varepsilon}(\cdot), g_{2,\varepsilon}(\cdot)$ in (5.12)–(5.13). The only difference is that the value of ϑ now depends on the behavior of $t \mapsto \widetilde{F}'(t)/t = \widetilde{a}(t)$ (and therefore on the value of γ in (1.42)), and it is no longer fixed as $\vartheta = 0$ as for Theorem 1.9. Using inequality (5.56) we can therefore proceed as in Lemma 4.7, thereby getting the quantitative L^∞-estimate (4.38) for the gradient. In turn, with (4.38) at our disposal, we can repeat the proof of Lemma 4.8 and arrive at (4.41), with the only further modification that we have to use the second inequality in (5.16) in order to get the last inequality in the estimate (4.40) (with $\beta_0 = 1$). After having established (4.41) (again with $\beta_0 = 1$ and using (5.18) to estimate as in (4.42) with $\gamma \equiv \tau + 2$), we need to distinguish between the situation with $n \ge 3$ and $n = 2$. In the case $n \ge 3$, we now have to use (5.25) instead of (4.19), according to whether $t \mapsto \widetilde{F}'(t)/t$ is nondecreasing or not, and therefore if $\gamma \ge 2$ or not in (1.42). These precisely yield inequality (4.43) with $\tau = \gamma - 2$ in any case ($\vartheta = 0, 1$), from which we then arrive at (4.39) with $\beta_0 = 1$, that is

$$\begin{aligned} &G_{T,\varepsilon}\big(\|Du\|_{L^\infty(B_{R/2})}\big)\\ &\le c\bigg(\fint_{B_R} F_\varepsilon(Du)dx\bigg)^{\frac{2}{\sigma}} + c\fint_{B_R} F_\varepsilon(Du)dx\\ &\quad + c\big\|\mathbf{P}_1^{f_{B_R}}(\cdot,R)\big\|_{L^\infty(B_R)}^{\frac{1}{1-\beta_1}} + c\big\|\mathbf{P}_1^{f_{B_R}}(\cdot,R)\big\|_{L^\infty(B_R)}^{\frac{\gamma}{\gamma-1}}\\ &\quad + cH_\varepsilon(T)\big\|\mathbf{P}_1^{f_{B_R}}(\cdot,R)\big\|_{L^\infty(B_R)}, \end{aligned} \tag{5.60}$$

for a constant c depending only on n, N, ν, c_a, c_b, γ, σ, and β_1. If $f \equiv 0$, then it is easy to see that the bound in (5.25) can be replaced by the one in (5.26) (which holds upon assuming (1.45)) and (5.60) follows without the potential terms. Finally, in the case $n = 2$, estimate (4.41) implies (4.63) (still with $\beta_0 = 1$, $\tau + 2 = \gamma$, and $\theta = 0, 1$, as above depending on the behavior of $t \mapsto \widetilde{F}'(t)/t$). By then taking advantage of (5.26) instead of (4.21), fixing an arbitrary $\theta \in (0, \sigma)$ and choosing $\kappa \in (0, 1/2)$ such that (4.65) and (4.66) hold, we then arrive at (4.67), which now reads as

$$\begin{aligned} &G_{T,\varepsilon}\big(\|Du\|_{L^\infty(B_{R/2})}\big) \\ &\quad \le c\left(\fint_{B_R} F_\varepsilon(Du)\,dx\right)^{\frac{2}{\sigma-\theta}} + c\big\|\mathbf{P}_1^{f_{B_R}}(\cdot, R)\big\|_{L^\infty(B_R)}^{\frac{\gamma+\theta}{\gamma-1}} \\ &\qquad + c\Big[H_\varepsilon(T)\Big\|\mathbf{P}_1^{f_{B_R}}(\cdot, R)\Big\|_{L^\infty(B_R)}\Big]^{1+\theta} + c \end{aligned} \tag{5.61}$$

for a constant c depending only on N, ν, c_a, c_b, γ, σ, β_1, and θ. The a priori estimates in (5.60) and (5.61) are now needed to conclude with the proof of Theorem 1.13 via the approximation argument from the next section.

5.3 Passage to the Limit and Conclusion

We shall follow the strategy of Section 4.4, to which we shall often refer, but with several important modifications. We start by fixing a ball $B \Subset \Omega$, and we define a decreasing sequence $\{\varepsilon_m\}$ of positive numbers such that $\varepsilon_m \to 0$ as $m \to \infty$ and $0 < \varepsilon_m < \min\{\operatorname{dist}(B, \partial\Omega), |B|^{1/n}, 1, T\}/4$ for all $m \in \mathbb{N}$; in addition, we define a sequence of positive numbers $\{\mu_m\}$ as $\mu_m := \mu_{\varepsilon_m}$ (recall the notation of μ_ε in (5.7)) and a sequence of integrands $\{L_m\}$ as $L_m := L_{\gamma,\varepsilon_m}$ as in (5.10). In what follows, we denote by u the local minimizer from the statement of Theorem 1.13, for which (5.17) implies $u \in W^{1,\gamma}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$. Then, we define the sequence of regularized functions $\{\bar{u}_m\}$ in $W^{1,\gamma}(B; \mathbb{R}^N)$ as in (4.48), where the family of mollifiers $\{\phi_\varepsilon\}$ is as in (4.2). The functionals $\mathscr{F}_m$ are then defined as

$$\mathscr{F}_m(w; B) := \int_B [F_m(Dw) - f_m w]\,dx,$$

as in (4.49), where this time the family of integrands $\{F_m\}$ is defined through $F_m := F_{\varepsilon_m}$ via (5.9), while the functions $\{f_m\}$ are defined as

$$f_m := \begin{cases} f & \text{if } |f| \le m, \\ \dfrac{mf}{|f|}, & \text{if } |f| > m. \end{cases} \tag{5.62}$$

In this way we have that $|f_m| \le \min\{|f|, m\}$, and thus $f_m \in L^\infty(\Omega; \mathbb{R}^N)$ for every $m \in \mathbb{N}$, and, as a consequence of the fact that $f \in L^n(\Omega; \mathbb{R}^N)$ by assumption, we also have $f_m \to f$ in $L^n(\Omega; \mathbb{R}^N)$. We remark that we are in the situation of Section 5.2 with $\widetilde{F} \equiv \widetilde{F}_m$ (and in particular of (5.41)), $\mathscr{N}\mathscr{D} \equiv \{\varepsilon_m, T_{\varepsilon_m}\}$, $\widetilde{a}(\cdot) \equiv \widetilde{a}_m(\cdot)$, and $a(\cdot) \equiv \partial F_m(\cdot)$, where

$$F_m(z) = \widetilde{F}_m(|z|) = \int_0^{|z|} \widetilde{a}_m(s)s\,ds, \quad \widetilde{a}_m(s) := \widetilde{a}_{\varepsilon_m}(s) + \varepsilon_m(s^2 + \mu_\varepsilon^2)^{\frac{\gamma-2}{2}}.$$

Therefore, conditions (5.36) are satisfied by (5.15), and the results of Section 5.2 can be applied to local minimizers of the functional $\mathscr{F}_m$. In particular, the local estimates (5.60) and (5.61) apply. Notice also that using Lemma 5.2 and $(5.17)_2$, we get

$$\sup_m \|F_m(Du)\|_{L^1(B+\varepsilon_m B_1)} + \|D\overline{u}_m\|_{L^\gamma(B)} < \infty. \tag{5.63}$$

Finally, we define the sequence $\{u_m\}$, $u_m \in \overline{u}_m + W_0^{1,\gamma}(B;\mathbb{R}^N)$, as that of the solutions to the Dirichlet problems in (4.50). Also in this case direct methods apply as all the functionals $\mathscr{F}_m$ are lower-semicontinuous and coercive in the Dirichlet class $\overline{u}_m + W_0^{1,\gamma}(B;\mathbb{R}^N)$. This last fact can be checked by noticing that in (4.52)–(4.54) one can replace u_m by any other function $w \in \overline{u}_m + W_0^{1,\gamma}(B;\mathbb{R}^N)$. Indeed, by also using $(5.17)_2$ in (4.54) (with u_m replaced by w) we get

$$\begin{aligned}\frac{\mathscr{F}_m(w;B)}{|B|} &\geq \frac{1}{c}\fint_B |Dw|^\gamma\,dx - \fint_B |f_m\overline{u}_m|dx \\ &\quad - \frac{|B+\varepsilon_m B_1|}{|B|}\fint_{B+\varepsilon_m B_1} F_m(Du)dx - c\|f\|_{L^n(B)}^{\frac{\gamma}{\gamma-1}} - cT^\gamma - c\mu^\gamma\end{aligned}$$

for a constant c still depending only on n, N, ν, and γ. This easily gives coercivity of $\mathscr{F}_m$ in the Dirichlet class $\overline{u}_m + W_0^{1,\gamma}(B;\mathbb{R}^N)$ and therefore yields the existence of the needed minimizer u_m for (4.50). Next, proceeding as in (4.52)–(4.54), we can then arrive at (4.55), which, combined with Lemma 5.2 (use (5.20) with $F_\varepsilon \equiv F_m$) and $(5.17)_2$, gives

$$\begin{aligned}&\fint_B (|Du_m|^2 + \mu_m^2)^{\gamma/2}\,dx + \fint_B F_m(Du_m)dx \\ &\quad \leq c\fint_{B+\varepsilon_m B_1} F(Du)dx + c\|f\|_{L^n(B)}^{\frac{\gamma}{\gamma-1}} + cT^\gamma + c\mu^\gamma,\end{aligned}$$

with $c \equiv c(n, N, \nu, \widetilde{a}(1), \gamma)$. Using this last inequality in (5.60) (again with $F_\varepsilon \equiv F_m$, $u \equiv u_m$), and proceeding as in the proof of Theorem 1.9, we again arrive at (4.56), where on the right-hand side $F_m(Du)$ is replaced by $F(Du)$ and $\overline{T}$ by T, while on the left-hand side we find $G_{T,\varepsilon}(\cdot)$ as defined in (5.55). By a standard covering argument, we then find for every $\sigma < 1$ a constant c_σ, also depending on σ but independent of m, such that $\|Du_m\|_{L^\infty(\sigma B)} \leq c_\sigma$ for every index $m \in \mathbb{N}$. Therefore, we can proceed by extracting a converging subsequence such that the convergences in (4.57) hold, with limit function $\overline{u} \in u + W_0^{1,\gamma}(B;\mathbb{R}^N)$, which, by lower semicontinuity, also satisfies $\|D\overline{u}\|_{L^\infty(\sigma B)} \leq c_\sigma$. Accordingly, using Lemma 5.2, and in particular (5.21), we have that for every $\sigma \in (0,1)$, there exists

$\overline{m} \in \mathbb{N}$, depending on c_σ and therefore ultimately on σ, such that the following holds:

$$
\begin{aligned}
(5.64)\qquad & |z| \le c_\sigma \quad \text{and} \quad \overline{m} \le m_0 \le m \\
& \quad \Longrightarrow \quad |F_m(z) - F_{m_0}(z)| \le c\varepsilon_{m_0}(c_\sigma + \mu + 1)^\gamma + \mathbf{o}(m_0) + \mathbf{o}(m),
\end{aligned}
$$

where $\mathbf{o}(m_0), \mathbf{o}(m) \to 0$ for $m_0, m \to \infty$, respectively, and the constant c is independent of m, m_0. Using the definitions of $\overline{u}_m$ and f_m in (4.48) and (5.62), respectively, and recalling $(4.57)_2$, we infer that

$$
(5.65)\qquad f_m u_m \to f\overline{u} \quad \text{and} \quad f_m \overline{u}_m \to f u \quad \text{strongly in } L^1(B).
$$

As every integrand F_{m_0} is convex, with m_0 being fixed, by lower semicontinuity we have

$$
(5.66)\qquad \int_{\sigma B} [F_{m_0}(D\overline{u}) - f\overline{u}]dx \le \liminf_{m\to\infty} \int_{\sigma B} [F_{m_0}(Du_m) - f_m u_m]dx.
$$

On the other hand, with $m \ge m_0 \ge \overline{m}$, employing, in order, (5.64), the minimality of u_m, and (4.53), we have that

$$
\begin{aligned}
&\int_{\sigma B} [F_{m_0}(Du_m) - f_m u_m]dx \\
&\quad \le \int_{\sigma B} [F_m(Du_m) - f_m u_m]dx + c\varepsilon_{m_0}(c_\sigma + \mu + 1)^\gamma + \mathbf{o}(m_0) + \mathbf{o}(m) \\
&\quad \le \int_B [F_m(D\overline{u}_m) - f_m\overline{u}_m]dx + \int_{B\setminus\sigma B} f_m u_m \, dx \\
&\qquad\quad + c\varepsilon_{m_0}(c_\sigma + \mu + 1)^\gamma + \mathbf{o}(m_0) + \mathbf{o}(m) \\
&\quad \le \int_{B+\varepsilon_m B_1} F_m(Du)dx - \int_B f_m \overline{u}_m \, dx + \int_{B\setminus\sigma B} f_m u_m \, dx \\
&\qquad\qquad + c\varepsilon_{m_0}(c_\sigma + \mu + 1)^\gamma + \mathbf{o}(m_0) + \mathbf{o}(m).
\end{aligned}
$$

Using (5.63) and (5.65), the above inequality implies

$$
\begin{aligned}
&\limsup_{m_0\to\infty} \liminf_{m\to\infty} \int_{\sigma B} [F_{m_0}(Du_m) - f_m u_m]dx \\
&\quad \le \limsup_{m_0\to\infty} \limsup_{m\to\infty} \int_{\sigma B} [F_{m_0}(Du_m) - f_m u_m]dx \le \mathscr{F}(u; B) + \int_{B\setminus\sigma B} f\overline{u}\, dx.
\end{aligned}
$$

Notice that here we have used (5.20) and Lebesgue's dominated convergence theorem. Connecting the last display with (5.66), recalling that $D\overline{u}$ is bounded on σB, and taking advantage of $(5.11)_3$, we arrive at $\mathscr{F}(\overline{u}; \sigma B) \le \mathscr{F}(u; B)$. Letting $\sigma \to 1$, we finally conclude that $\mathscr{F}(\overline{u}; B) \le \mathscr{F}(u; B)$. We have therefore obtained (4.59) again. From this point on, recalling that now (4.56) holds in this case too (with $F_m(Du)$ replaced by $F(Du)$ and $\overline{T}$ by T, $\tau \equiv \gamma - 2$, and $\beta_0 = 1$), the rest of the proof, and in particular the a priori estimates for the vectorial case, that is, (1.35) for $n \geq 3$ and (1.38) for $n = 2$, follows as in Section 4.4 in the setting of Theorem 1.9 and as in Section 4.5 in the setting of Theorem 1.11, respectively.

Let us finally comment on the case $f \equiv 0$. Here, exactly as in the scalar case, the second term in estimate (4.41) does not appear. Hence, only assumption (1.45) (which implies the corresponding version (5.26) for $g_{2,\varepsilon}/g_{1,\varepsilon}$) instead of (1.44) is needed to prove a local quantitative L^∞-estimate for the gradients Du_m and to then pass to the limit.

Remark 5.7. A careful analysis of the proof of Theorem 1.13 reveals that the constant c appearing in estimate (1.35) depends on ν in a such a way that $c \to \infty$ when $\nu \to 0$; see in particular (5.17) in Lemma 5.1.

6 Applications and Theorems 1.2–1.8 and 1.16

Here we show how to obtain Theorems 1.2–1.8 and 1.16 from Theorems 1.9, 1.11, and 1.13.

6.1 Proof of Theorems 1.2–1.5

We here prove all results on minimizers of functionals with (p, q)-growth by showing the applicability of Theorems 1.9 and 1.11 in the scalar setting and of Theorem 1.13 in the vectorial setting. Since (1.12) is assumed throughout Theorems 1.2–1.5, in what follows, we shall permanently work with the choice

$$\begin{cases} g_1(t) = \nu(t^2 + \mu^2)^{(p-2)/2}, \\ g_2(t) = \Lambda(t^2 + \mu^2)^{(q-2)/2} + \Lambda(t^2 + \mu^2)^{(p-2)/2}, \end{cases} \tag{6.1}$$

for $t \in (0, \infty)$. In this way, since the functions $t \mapsto g_2(t)/g_1(t)$ and $t \mapsto g_1(t)t$ are continuous and nondecreasing on $(0, \infty)$, assumption (1.31) is fulfilled with constant $c_a = 1$ for any $T > 0$, while the assumptions in (1.32) are automatically satisfied. Moreover, we shall also fix the parameter $\tau = p - 2$. We start with the proof of Theorem 1.2 as a consequence of Theorem 1.9; here it is $n > 2$. We fix the parameters β_0, β_1, σ as

$$\beta_0 = 1, \quad \beta_1 = \frac{4 + (n-2)(q-p)}{4p}, \quad \text{and} \quad \sigma = 2 - n\left(\frac{q}{p} - 1\right). \tag{6.2}$$

We next determine the constants ν, c_a, c_b in order to meet the requirements of Theorems 1.9; all of the following computations will hold for any choice of $T > 0$, and we shall always consider $T \leq 1$. We note that $(1.33)_1$ is true for our choice $\tau = p - 2$ for any $T > 0$. Moreover, with

$$p \int_T^t g_1(s)s\, ds = p\nu \int_T^t (s^2 + \mu^2)^{(p-2)/2} s\, ds = \nu(t^2 + \mu^2)^{p/2} - \nu(T^2 + \mu^2)^{p/2}$$

for all $t \geq T > 0$, also $(1.33)_2$ is verified, due to $(1.12)_1$.

In order to deal with the final assumption (1.34), let us first notice that $q \geq p$ implies $\sigma \leq 2 = 2\beta_0$, while we also have the implications

$$\frac{q}{p} < 1 + \frac{2}{n} \Longrightarrow \sigma > 0 \quad \text{and} \quad \frac{q}{p} < 1 + \frac{4(p-1)}{p(n-2)} \Longrightarrow \beta_1 < 1.$$

Thus, the choices of σ and β_1 in (6.2) are admissible, according to assumption (1.15). Next we observe that, for all $t \geq T > 0$, the calculation above yields, with $(q-p)/p = (2\beta_0 - \sigma)/n \leq 1$ by the choice of the exponent σ,

$$\begin{aligned} \frac{g_2(t)}{g_1(t)} &= \frac{\Lambda}{\nu}(t^2+\mu^2)^{(q-p)/2} + \frac{\Lambda}{\nu} \\ &\leq \frac{\Lambda}{\nu}[(t^2+\mu^2)^{p/2} - (T^2+\mu^2)^{p/2}]^{\frac{q-p}{p}} \\ &\quad + \frac{\Lambda}{\nu}(T^2+\mu^2)^{(q-p)/2} + \frac{\Lambda}{\nu} \\ &\leq p^{\frac{q-p}{p}} \frac{\Lambda}{\nu^{q/p}} \left(\int_T^t g_1(s) s \, ds \right)^{\frac{2\beta_0 - \sigma}{n}} + \frac{\Lambda}{\nu}(T^2+\mu^2)^{(q-p)/2} + \frac{\Lambda}{\nu}. \end{aligned} \tag{6.3}$$

Similarly, we find for all $t \geq T > 0$, with $\frac{4+(n-2)(q-p)}{(n-2)p} = \frac{4\beta_1}{n-2} \leq 5$ by the choice of the exponent β_1,

$$\begin{aligned} \frac{g_2(t)}{g_1(t)} &\leq 2^5 \frac{\Lambda}{\nu}(t^2+\mu^2)^{-\frac{2}{n-2}}[(t^2+\mu^2)^{p/2} - (T^2+\mu^2)^{p/2}]^{\frac{4+(n-2)(q-p)}{(n-2)p}} \\ &\quad + 2^5 \frac{\Lambda}{\nu}(T^2+\mu^2)^{(q-p)/2} + \frac{\Lambda}{\nu} \\ &\leq (2p)^5 \frac{\Lambda}{\nu^6} \left[\frac{1}{t^{1/\beta_1}} \int_T^t g_1(s) s \, ds \right]^{\frac{4\beta_1}{n-2}} + (1 + 2^{5+(q-p)/2}) \frac{\Lambda}{\nu}. \end{aligned}$$

Therefore, the assumption (1.34) is verified with the choice of the constants

$$c_b = \max \left\{ (2p)^5 \frac{\Lambda}{\nu^6}, \left(1 + 2^{5+(q-p)/2}\right) \frac{\Lambda}{\nu} \right\}. \tag{6.4}$$

In conclusion, we have proved that all assumptions (1.31)–(1.34) are satisfied in the setting of Theorem 1.2. We can thus apply Theorem 1.9, getting estimate (1.35), that with $\tau = p - 2$ and the values displayed in (6.2) becomes

$$\begin{aligned} &\max\left\{ \left(\|Du\|_{L^\infty(B/2)}^2 + \mu^2 \right)^{p/2} - (T^2+\mu^2)^{p/2}, 0 \right\} \\ &\quad \leq c \left(\fint_B F(Du) dx + \|f\|_{L^n(B)}^{\frac{p}{p-1}} + T^p + \mu^p \right) \\ &\qquad + c \left(\fint_B F(Du) dx + \|f\|_{L^n(B)}^{\frac{p}{p-1}} + T^p + \mu^p \right)^{\frac{2p}{(n+2)p - nq}} \\ &\qquad + c\|f\|_{L(n,1)(B)}^{\frac{p}{p-1}} + c\|f\|_{L(n,1)(B)}^{\frac{4p}{4(p-1)-(n-2)(q-p)}} + c(T+\mu)\|f\|_{L(n,1)(B)}. \end{aligned}$$

By (6.4) this holds for $c \equiv c(n, p, \nu, \Lambda)$, independently of $T \in (0,1]$. Letting $T \to 0$, noting $F(z) \geq \nu\mu^p$, and finally using the Young inequality with conjugate exponents $(p, p/(p-1))$ to estimate the last term, we get the assertion (1.16), also using the obvious inequality $\|f\|_{L^n} \lesssim \|f\|_{L(n,1)}$.

Finally, we comment on the assertion that condition (1.15) can be replaced by the weaker one (1.14) when $p \geq 2 - 4/(n+2)$ or when $f \equiv 0$. In the first case $p \geq 2-4/(n+2)$, this is trivial since here we have $2/n \leq 4(p-1)/(p(n-2))$. In the second case $f \equiv 0$, one easily checks from the above computations that (1.36) is ensured by (1.14) for the choice of the constant $c_b = 4^{q-p+1}(\Lambda/\nu)^{q/p}$, and the conclusion then follows as above, after applying Theorem 1.9 in the version with $f \equiv 0$.

The proofs of Theorems 1.3 and 1.5 follow from Theorems 1.13 and 1.11 in a similar fashion, and we therefore only comment on the suitable choices and necessary modifications. The remaining details are left to the interested reader.

We continue with the proof of Theorem 1.3, which is found as a consequence of Theorem 1.13 (for $n \geq 2$). Here the conditions in (1.42) are satisfied by assumptions (1.20) and (1.12) with $\gamma = p$. As a main difference, one needs to replace the definition of the parameters β_1 and σ in (6.2) by

$$\beta_1 = \frac{4 + (1+\vartheta)(n-2)(q-p)}{4p} \quad \text{and} \quad \sigma = 2 - (1+\vartheta)\frac{n(q-p)}{p}, \tag{6.5}$$

where once again we have set $\vartheta = 0$ if $p = \gamma \geq 2$ and $\vartheta = 1$ otherwise. With these choices one can follow exactly the computations from above to show that assumption (1.44) is satisfied. Furthermore, we observe that the parameters in (6.5) are indeed admissible: the assumptions $\beta_1 \geq 1/p = 1/\gamma$ and $\sigma \leq 2$ are trivially satisfied in view of $q \geq p$, while $\beta_1 < 1$ and $\sigma > 0$ follow precisely from the assumptions (1.15) and (1.19), for the cases $\vartheta = 0$ and $\vartheta = 1$, respectively. Thus, all assumptions of Theorem 1.13 are satisfied and the local boundedness of Du follows (with an estimate coming from (1.35) with $\beta_0 = 1$ and $T = 0$).

Next, we comment on how to derive Theorem 1.4 from Theorem 1.11; here we have $n = 2$. We may fix the parameters β_0 and σ as in (6.2), which again ensures that the assumptions in (1.33) are satisfied. Moreover, the computation in (6.3) shows that also (1.37) holds true. Concerning the admissibility of σ, we note that $\sigma > 0$ is implied by the assumption $q < 2p$ in (1.22), and $\sigma \leq \beta_0 = 1$ can be guaranteed by possibly taking q larger such that $q \geq 3p/2$ is satisfied. (Note that this does not change the validity of the other conditions, for the reason that if the assumptions (1.12) are satisfied for some q, then the same assumptions are satisfied for larger values of q.) Thus, all assumptions of Theorem 1.11 are satisfied and the local boundedness of Du follows (with an estimate coming from (1.38) with $T = 0$, which is at this point similar to the one in (1.16)).

Finally, we deduce Theorem 1.5 from Theorem 1.13 (for $n = 2$). Again, condition (1.42) is satisfied by assumption, while (1.45) is true when choosing σ as in (6.5) (distinguishing by the value of ϑ once again the settings when $p = \gamma \geq 2$ or not). The choice of σ is once again admissible, since $\sigma > 0$ is true by the assumptions (1.22) and (1.23) for the cases $\vartheta = 0$ and $\vartheta = 1$, respectively, and $\sigma \leq 1$ can be guaranteed by possibly choosing q larger such that $q \geq (1 + 1/(2(1+\vartheta)))p$. Thus, all assumptions of Theorem 1.13 for $n = 2$ are satisfied and the local boundedness of Du follows (with an estimate coming from (1.38) and $T = 0$).

6.2 Proof of Theorem 1.7

We deduce Theorem 1.7 on minimizers of functionals with exponential growth from Theorem 1.13 by making a suitable choice of the structure functions $g_1(\cdot)$ and $g_2(\cdot)$ and of the parameters σ, β_1, c_a, c_b, ν, μ, ϑ, and γ and for $T = 1$. We consider $n > 2$; the two-dimensional case $n = 2$ can be obtained in a similar way from Theorem 1.11. The functions $g_1(\cdot)$ and $g_2(\cdot)$, defined below, provide lower and upper bounds on the second-order derivative of the integrands $\mathbf{e}_k(\cdot)$, which are given by

$$\partial^2 \mathbf{e}_k(|z|) = \mathbf{e}_k''(|z|)\frac{z \otimes z}{|z|^2} + \mathbf{e}_k'(|z|)\left[\frac{\mathbb{I}_{N\times n}}{|z|} - \frac{z \otimes z}{|z|^3}\right] \tag{6.6}$$

for every $z \in \mathbb{R}^{N\times n}$ such that $|z| \neq 0$ (again recall the notation in (5.3)). By a straightforward computation, (6.6) gives

$$\left|\partial^2 \mathbf{e}_k(|z|)\right|^2 = \left[\mathbf{e}_k''(|z|)\right]^2 + \left[\frac{\mathbf{e}_k'(|z|)}{|z|}\right]^2 (Nn - 1), \quad |z| \neq 0. \tag{6.7}$$

In order to evaluate and bound the expressions in the last two displays, we first infer by induction that

$$\begin{aligned} \mathbf{e}_0'(t) &= p_0 t^{p_0 - 1}\mathbf{e}_0(t), \\ \mathbf{e}_k'(t) &= p_k t^{p_0 - 1}\mathbf{e}_k(t) \prod_{j=0}^{k-1} p_j [\mathbf{e}_j(t)]^{p_{j+1}} \quad \text{for } k \geq 1, \end{aligned} \tag{6.8}$$

holds for every $t \in (0, \infty)$, and we note that $\mathbf{e}_k'(\cdot)$ is always positive. By differentiating the identities in (6.8), we obtain, for $k \geq 2$, that

$$\begin{aligned} \mathbf{e}_k''(t) = t^{p_0 - 1}\mathbf{e}_k'(t) \Bigg\{ & p_k \prod_{j=0}^{k-1} p_j [\mathbf{e}_j(t)]^{p_{j+1}} \\ & + \sum_{s=0}^{k-2} p_{s+2} p_{s+1} \prod_{j=0}^{s} p_j [\mathbf{e}_j(t)]^{p_{j+1}} \Bigg\} \\ & + t^{p_0 - 1}\mathbf{e}_k'(t)\left\{ p_1 p_0 + \frac{p_0 - 1}{t^{p_0}} \right\}, \end{aligned} \tag{6.9}$$

while, in the remaining cases $k = 1, 0$, we instead have

$$\begin{cases} \mathbf{e}_1''(t) = t^{p_0-1}\mathbf{e}_1'(t)\left\{p_1 p_0[\mathbf{e}_0(t)]^{p_1} + p_1 p_0 + \dfrac{p_0-1}{t^{p_0}}\right\}, \\ \mathbf{e}_0''(t) = t^{p_0-1}\mathbf{e}_0'(t)\left\{p_0 + \dfrac{p_0-1}{t^{p_0}}\right\}, \end{cases} \tag{6.10}$$

respectively. This in turn implies that both matrices on the right-hand side of (6.6) are positive definite (recall that we are assuming that $p_0 > 1$), and we have that

$$\langle \partial^2 \mathbf{e}_k(|z|)\xi, \xi\rangle \geq \min\left\{\mathbf{e}_k''(|z|), \frac{\mathbf{e}_k'(|z|)}{|z|}\right\}|\xi|^2 \tag{6.11}$$

holds for every choice of $z, \xi \in \mathbb{R}^{N\times n}$ with $z \neq 0$. By (6.9)–(6.10), and essentially using only the second-to-last terms appearing in the curly brackets, for $k \geq 1$ we find

$$t \geq 1 \implies \mathbf{e}_k''(t) \geq p_1 p_0 \frac{\mathbf{e}_k'(t)}{t} \quad \text{and} \quad \mathbf{e}_0''(t) \geq p_0 \frac{\mathbf{e}_0'(t)}{t}. \tag{6.12}$$

We now consider the situation where $k \geq 1$ and find upper and lower bounds for the eigenvalues of the matrix $\partial^2 \mathbf{e}_k$ using (6.11). We next define $g_1(\cdot) \equiv g_{1,k}(\cdot)$ and $g_2(\cdot) \equiv g_{2,k}(\cdot)$ as

$$\begin{cases} g_1(t) := \mathbb{1}_{\{t\geq 1\}} \min\{p_1, 1\}\dfrac{\mathbf{e}_k'(t)}{t}, \\ g_2(t) := \mathbb{1}_{\{t\geq 1\}} 10\hat{p}_k^{k+1}\left[\sqrt{k+2} + \sqrt{Nn}\right]t^{p_0-1}\mathbf{e}_k'(t) \\ \qquad\qquad \cdot \prod_{j=0}^{k-1} \mathbf{e}_j(t)]^{p_{j+1}}, \end{cases} \tag{6.13}$$

for every $t > 0$ (recall that $p_0 > 1$), respectively, where we have set $\hat{p}_k := 1 + \max_{0\leq j\leq k}\{p_0, \dots, p_k\}$. The definition of the lower-bound function $g_1(\cdot)$ is justified by the first inequality in (6.12), while the definition of the upper-bound function $g_2(\cdot)$ follows by taking into account (6.9)–$(6.10)_1$. In the case $k = 0$, by the second inequality in (6.12) and by $(6.10)_2$, we set analogously

$$\begin{cases} g_1(t) := \mathbb{1}_{\{t\geq 1\}} \dfrac{\mathbf{e}_0'(t)}{t}, \\ g_2(t) := \mathbb{1}_{\{t\geq 1\}} 10 p_0\left[2 + \sqrt{Nn}\right]t^{p_0-1}\mathbf{e}_0'(t). \end{cases} \tag{6.14}$$

Now, let us verify that with the above settings all assumptions of Theorem 1.13 are satisfied for suitable choices of the parameters, in particular, for any $\sigma \in (0, 2)$, $\gamma = p_0$, $\mu = 0$, $\beta_1 \in [1/p_0, 1) = [1/\gamma, 1)$, $\vartheta = 1$, $\nu = \min\{p_1, 1\}\min\{p_0 p_1 \cdots p_k, 1\}$, and finally $T = 1$. We shall treat the case $k \geq 1$; the case $k = 0$ can be treated in a similar way, also by recalling (6.14). We first observe that the functions $t \mapsto g_2(t)/g_1(t)$ and $t \mapsto g_1(t)t$ are continuous and increasing on $[1, \infty)$, which shows that assumption (1.31) is fulfilled with constant $c_a = 1$. Next, note that, by (6.8) and (6.11)–(6.13), the growth and ellipticity assumptions (1.32) are satisfied.

In order to verify (1.42) with $\mu = 0$ and $\gamma = p_0$, it is sufficient to take into account that $\tilde{a}(t) \equiv \mathbf{e}'_k(t)/t$ and the identity in (6.8). We finally check (1.44). For this, noting that the first identity in (6.13) implies

$$\int_1^t g_1(s)s\,ds = \min\{p_1, 1\}[\mathbf{e}_k(t) - \mathbf{e}_k(1)] \tag{6.15}$$

for every $k \geq 1$ and $t \geq 1$. Then, recalling the definitions in (6.13) and (1.44) follows with any $\sigma \in (0,2)$ and $\beta_1 \in [1/p_0, 1)$ provided that the constant c_b is chosen large enough as a function of $k, p_0, p_1, \ldots, p_k, \sigma, \beta_1$. With these choices all the assumptions of Theorem 1.13 are satisfied. We therefore conclude with the local Lipschitz continuity of minimizers of the functional in (1.26) whenever we have $f \in L(n,1)(\Omega;\mathbb{R}^N)$ for $n > 2$ and, in an analogous way, whenever we have $f \in \mathrm{L}^2(\mathrm{Log\,L})^\alpha(\Omega;\mathbb{R}^N)$ for $\alpha > 2$ for $n = 2$.

Finally, in order to deal with the case $f \equiv 0$ and prove estimate (1.27), we apply Theorem 1.16. Assumption (1.49) is obviously satisfied thanks to (6.15). By the definition in (1.25) it follows that, for every integer $k \geq 0$,

$$\begin{cases} \mathbf{e}_{k+1}^{-1}(t) := \mathbf{e}_k^{-1}[(\log t)^{1/p_{k+1}}], \\ \mathbf{e}_0^{-1}(t) := (\log t)^{1/p_0}, \end{cases}$$

with every $\mathbf{e}_k^{-1}(\cdot)$, which is defined on $[\mathbf{e}_k(0), \infty)$. It is therefore easy to see that each function $\mathbf{e}_k^{-1}(\cdot)$ satisfies (1.50) for every $\upsilon > 1$ and suitable constants $c_\upsilon, d_\upsilon > 1$ depending on $\upsilon, k, p_0, p_1, \ldots, p_k$. We can apply estimate (1.51), which in turn yields (1.27), and the proof is complete.

6.3 Proof of Theorem 1.8

Theorem 1.8 concerning the anisotropic functionals $\mathscr{E}_a$ can be obtained as a corollary of Theorem 1.9, and again we restrict to the case $n > 2$ for brevity. The proof is similar to that of Theorem 1.7. This time, again with $T = 1$, it is sufficient to choose

$$\begin{cases} g_1(t) := \mathbb{1}_{\{t \geq 1\}} \min\{(A_0 p)^2, 1\} t^{p-2} \exp(A_0 t^p), \\ g_2(t) := \mathbb{1}_{\{t \geq 1\}} 10[p(1 + A_n)(n+1)]^2\, t^{2p-2} \exp(A_n t^p). \end{cases}$$

The functions $t \mapsto g_2(t)/g_1(t)$ and $t \mapsto g_1(t)t$ are continuous and nondecreasing on $[1,\infty)$, and thus (1.31) holds with $c_a = 1$, and (1.32) is satisfied. By $\int_1^t g_1(s)s\,ds \approx \exp(A_0 t^p) - \exp(A_0)$ for $t \geq 1$, and, as for Theorem 1.7, we check that (1.34) and (1.36) are satisfied provided (1.30) holds, $\beta_0 = 1$, σ is sufficiently close to 0, and β_1 close to 1 for a suitable choice of c_b depending on n, p, A_0, A_n, β_1, and σ. Similarly, (1.33) holds for $\tau = p - 2$, $\mu = 0$, and $\nu = \min\{(A_0 p)^2, 1\}$. Theorem 1.9 applies, thereby yielding the local Lipschitz continuity of minimizers of $\mathscr{E}_a$.

6.4 Proof of Theorem 1.16

The proof builds on an application of Theorem 1.13 and the arguments developed for its proof, where we take $\widetilde{F}(|z|) \equiv A(|z|)$; needless to say, we can assume that $A(0) = 0$. Notice that by (5.2) and (5.4) it follows that $t \mapsto A(t)$ is a nondecreasing function, which is increasing and therefore invertible on $[T, \infty)$. By the extension specified in the statement, i.e., $A^{-1}(t) \equiv T$ for every $t \in [0, A(T)]$, the function $t \mapsto A^{-1}(t)$ is also nondecreasing (and actually increasing on $[A(T), \infty)$).

We start by recalling a few immediate consequences of (1.50). First, by a simple iteration argument we see that the validity of (1.50), for some couple (υ, c_υ) such that $\upsilon, c_\upsilon > 1$ actually implies that, *for every* $\bar{\upsilon} > 1$, there exists constants $c_{\bar{\upsilon}}, d_{\bar{\upsilon}} > 1$ such that

$$A^{-1}(t^{\bar{\upsilon}}) \le c_{\bar{\upsilon}} A^{-1}(t) + d_{\bar{\upsilon}} \quad \forall t \ge A(T). \tag{6.16}$$

(Indeed, fix $\bar{\upsilon}$ and take $t \ge A(T)$. Let us first consider the case $t > 1$; if $\bar{\upsilon} \le \upsilon$, then, obviously $A^{-1}(t^{\bar{\upsilon}}) \le A^{-1}(t^{\upsilon}) \le c_\upsilon A^{-1}(t) + d_\upsilon$. Otherwise, if $\bar{\upsilon} > \upsilon$, let $m \equiv m(\bar{\upsilon}, \upsilon) \ge 1$ be the smallest integer such that $\upsilon^m \ge \bar{\upsilon}$; then, iterating (1.50) we find

$$A^{-1}(t^{\bar{\upsilon}}) \le A^{-1}(t^{\upsilon^m})h \le c_\upsilon^m A^{-1}(t) + d_\upsilon \sum_{k=0}^{m-1} c_\upsilon^k =: c_{\bar{\upsilon}} A^{-1}(t) + d_{\bar{\upsilon}},$$

that is, the claim (6.16). Next, if $t \le 1$ there is nothing to prove as $A^{-1}(t^{\bar{\upsilon}}) \le A^{-1}(t) \le c_{\bar{\upsilon}} A^{-1}(t) + d_{\bar{\upsilon}}$). Moreover, we have the estimates

$$\begin{cases} A^{-1}(a+b) \le A^{-1}(2a) + A^{-1}(2b) & \forall a, b \ge 0, \\ A^{-1}(ab) \le c_2\big[A^{-1}(a) + A^{-1}(b)\big] & \forall a, b \ge 0 \\ & \text{such that} \max\{a, b\} \ge A(T). \end{cases} \tag{6.17}$$

(Indeed, $(6.17)_1$ just follows from the fact that $t \mapsto A^{-1}(t)$ is nondecreasing. For $(6.17)_2$ note that we can assume $a, b \ge 1$, since otherwise we can simply use the fact that $t \mapsto A^{-1}(t)$ is nondecreasing. Also, recalling that $\max\{a, b\} \ge A(T)$, we can confine ourselves to treat the case $a \ge A(T) \ge b \ge 1$; thus $A^{-1}(a) \ge T = A^{-1}(b)$, so that the others will follow similarly. In that case, using (6.16) we have $A^{-1}(ab) \le A^{-1}(a^2) \le c_2 A^{-1}(a) + d_2$). To proceed, we observe that the current assumptions allow Theorem 1.13 to be applied with $f \equiv 0$ and $\widetilde{F} \equiv A$. We can assume that $\|Du\|_{L^\infty(B/2)} > T$; otherwise (1.51) follows immediately, provided c is suitably chosen. Then, combining (1.35) (used with $\beta_0 = 1$) and (1.49) we get, after a few simple manipulations

$$A(\|Du\|_{L^\infty(B/2)}) \le c\left(1 + A(T) + T^\gamma + \fint_B A(|Du|)dx\right)^{2/\sigma} + \Lambda,$$

where the constant c depends only on n, N, ν, Λ, c_a, c_b, γ, σ, and $A'(1)$; we may of course assume that $c \ge A(T)$. Applying A^{-1} to both sides of the previous inequality, and using (6.16) and (6.17) repeatedly, we arrive at (1.51) with the asserted dependence of the constant c.

6.5 Proof of Theorem 1.15

The natural growth estimates under a $\triangle_2$-condition stated in Theorem 1.15 are deduced as a consequence of Theorem 1.13 applied with $F(z) \equiv A(|z|)$. To this end, we will work with lower- and upper-bound functions

$$g_1(t) = (i_a + 1)\widetilde{a}(t) \quad \text{and} \quad g_2(t) = (s_a + 1)\widetilde{a}(t) \tag{6.18}$$

and now verify the assumptions of Theorem 1.13 by relying on the uniform ellipticity assumption (1.3). First, notice that (1.3) implies that $\widetilde{a}(1)t^{i_a+1} \le \widetilde{a}(t)t \le \widetilde{a}(1)t^{s_a+1}$ for $t > 0$, and therefore $A \in C^1_{\text{loc}}[0,\infty) \cap C^2_{\text{loc}}(0,\infty)$. Needless to say, we have that $g_1(t) > 0$ for $t > 0$. Obviously, $t \mapsto g_2(t)/g_1(t)$ is constant, so in particular it is nondecreasing, while $t \mapsto g_1(t)t$ is increasing on $(0,\infty)$ because of (1.3), which yields (1.31) for every $T > 0$. Next, (1.3) is used once again to see, in view of (5.1), that (1.32) holds for every $T > 0$, and that condition (1.42) is satisfied with $\mu = 0$, $\gamma = i_a + 2 > 1$, and $\nu = (i_a + 1)\widetilde{a}(T)/T^{i_a}$. We finally verify (1.44) for $c_b = (s_a + 1)/(i_a + 1)$ and $\sigma = 2$, $\beta_1 = 1/(i_a + 2)$ (these parameters provide the best estimates; the value of ϑ is irrelevant here and we can formally take $\vartheta = 1$). Thus, Theorem 1.13 applies and yields the estimate (1.35) with $\beta_0 = 1$ and $\tau = i_a$ for any choice of $T \in (0,1]$ and a constant depending on the quantity $\widetilde{a}(T)/T^{i_a}$; eventually applying Young's inequality, we get

$$A(\|Du\|_{L^\infty(B/2)}) \le c \fint_B A(|Du|)\,dx + c\|f\|_{L(n,1)(B)}^{\frac{i_a+2}{i_a+1}} + cT^{i_a+2}. \tag{6.19}$$

The constant here depends on n, N, i_a, s_a, and $\widetilde{a}(T)/T^{i_a}$.

In order to conclude with the desired estimate (1.46), we choose $T = 1$ and employ a few elementary arguments. We need to prove that for every $c_1 \ge 1$, it holds that

$$A^{-1}(c_1 s) \le c_1 A^{-1}(s) \quad \forall s > 0. \tag{6.20}$$

For this, notice that the lower bound in (1.3) implies that the function $s \mapsto \widetilde{a}(s)s$ is nondecreasing; therefore, changing variables, we have that

$$\int_0^t \widetilde{a}(s)s\,ds = \int_0^{c_1 t} \widetilde{a}\left(\frac{s}{c_1}\right)\frac{s}{c_1}\frac{ds}{c_1} \le \frac{1}{c_1}\int_0^{c_1 t} \widetilde{a}(s)s\,ds.$$

By the definition in (1.5) we have therefore proved that $c_1 A(t) \le A(c_1 t)$, for every $t > 0$, which in turn implies (6.20). Finally, applying $A^{-1}(\cdot)$ to both sides of (6.19) and combining (6.20) with $(6.17)_1$ (which is still valid in this case as $A(\cdot)$ is nondecreasing), we easily get the first claim (1.46) of the theorem. We then consider the second claim, asserting that if (1.47) is in force, then the estimate (1.46) holds with $c_2 = 0$. To this end, let us choose a sequence $\{T_k\}$ in $(0,1]$ such that $T_k \to 0$ and $\widetilde{a}(T_k)/T_k^{i_a} \to i_2$. We then apply (6.19) with $T \equiv T_k$ for every $k \in \mathbb{N}$, getting a corresponding estimate. By assumption (1.47), the constants involved now depend also on i_2 but are independent of k. Therefore, letting $k \to \infty$, we arrive at (6.19) without the final T term, and the rest of the proof is as in the case where (1.47) is not in force. This completes the proof of Theorem 1.15.

6.6 Comparisons with Marcellini's Theory

Following Remark 1.14, we briefly compare our assumptions with those in Marcellini's seminal papers [46, 47] when $f \equiv 0$. The overall outcome is that our assumptions are not weaker than those in [46] for the vectorial case and for functionals with superquadratic growth, while they are essentially equivalent to those in [47] for the scalar one. Let us remark that the assumptions in [48] remarkably capture simultaneously fast-growth conditions as well as linear-type growth conditions. These are not considered here, mainly due to the presence of the right-hand side f, but they will be the object of future investigation.

Let us now compare the results with the vectorial case, thereby considering Theorem 1.13. For simplicity we shall consider the case that $t \mapsto \widetilde{F}'(t)/t$ is nondecreasing (implied by $\gamma \geq 2$ and in turn implying that $\widetilde{F}'(t)/t \leq \widetilde{F}''(t)$). In this case Theorem 1.13 involves the following bound:

$$\frac{g_2(t)}{g_1(t)} \lesssim \left(\int_0^t g_1(s) s \, ds \right)^{\frac{2-\sigma}{n}} + 1 \tag{6.21}$$

for large values of t and some $\sigma < 2$ (we are formally taking $T \equiv 0$ as here only the behavior for large values of the gradient variable matters). By looking at (5.1)–(5.4), and recalling the definitions in (1.41), with our notation we have $g_1(t) \approx \widetilde{F}'(t)/t$ and $g_2(t) \approx \widetilde{F}'(t)/t + \widetilde{F}''(t)$, so that (6.21) amounts to requiring that

$$\frac{\widetilde{F}''(t)t}{\widetilde{F}'(t)} \lesssim [\widetilde{F}(t)]^{\frac{2-\sigma}{n}} + 1 =: \mathscr{R}_2(t). \tag{6.22}$$

On the other hand, the main assumption in [46] (see [46, lemma 2.3]) prescribes that

$$\lim_{t \to \infty} \frac{\widetilde{F}''(t) t^{1+\kappa}}{[\widetilde{F}'(t)]^{1+\kappa}} < \infty \quad \forall \kappa > 0, \tag{6.23}$$

and in particular that this limit exists. In turn, the latter assumption implies that

$$\frac{\widetilde{F}''(t)t}{\widetilde{F}'(t)} \lesssim \left[\frac{\widetilde{F}'(t)}{t} \right]^{\kappa} =: \mathscr{R}_1(t) \quad \forall \kappa > 0, \tag{6.24}$$

holds for large values of t; see [46, 48]. The right-hand side of (6.22) cannot be controlled by the one in (6.24). This can be easily seen by observing that positive solutions to the differential inequality $t[\widetilde{F}(t)]^{\beta} \lesssim \widetilde{F}'(t)$ for $1 < \beta \approx (2-\sigma)/(n\kappa)$

exhibit a finite-time blowup. We can conclude that in the case of vectorial functionals such that $t \mapsto \widetilde{F}'(t)/t$ is nondecreasing, our assumptions are not covered by those in [46]. Moreover, notice that comparing (6.22) and (6.24) in the case of the exponential-type functionals in (1.25)–(1.26) leads us to the expressions for $\widetilde{F}'(t)$ and $\widetilde{F}''(t)$ computed in (6.8) and (6.9), respectively. We then get $[\mathbf{e}_k(t)]^{(2/n)-\varepsilon} \lesssim \mathscr{R}_2(t)/\mathscr{R}_1(t)$ for every $\varepsilon > 0$ and for t suitably large, depending on σ, n, N, $p_0, \dots, p_k$, and ε. Therefore, the right-hand side in (6.22) is asymptotically much larger than the right-hand side in (6.24) whenever $\sigma \in (0,2)$ for all $\kappa < (2-\sigma)/n$. So, in the case of fast-growth conditions, assumption (6.21) seems to be weaker than (6.23) from [46], although a direct comparison is difficult.

We then switch to the case of abstract (p,q)-growth conditions, that is, when the only available information is in (1.12). As noted in Section 1, the assumptions considered here recover in the scalar case the bounds in [45]. On the other hand, notice that conditions of the type (6.24) and in [46, 48] do not apply to this case unless additional structure assumptions are satisfied (consider, for instance, the oscillating integrand in [48, (2.10)], where in fact no bound is required on q/p). Indeed, (6.24) would prescribe in the general (p,q)-growth case that $q/p = 1$, thereby yielding no result. We can conclude that the assumptions considered here provide a unified approach to both exponential and (p,q)-growth conditions in the vectorial case.

We finally turn to the scalar case, where our main reference is [47]. A direct comparison in this case is not straightforward, as Marcellini's assumptions involve a wide set of parameters. Anyway, an optimal choice of such parameters, as detailed in [47, remark 2.1], determines as assumptions $g_1'(t)t \lesssim g_1(t)$ for every $t > 0$, and $g_2(|z|)|z|^2 \lesssim \min\{g_1(|z|)|z|^2, F(z)\}^{1+(2-\sigma)/n} + 1$ for every $z \in \mathbb{R}^n$ and some $\sigma \in (0,2]$. Notice that in [47], we have that $g_1(\cdot)$ and $g_2(\cdot)$ are defined on $[0,\infty)$. The above assumption on $g_1(\cdot)$ gives $g_1(t)t^2 \lesssim \int_0^t g_1(s)s\,ds$, and therefore the assumptions in [46] amount to requiring

$$\frac{g_2(|z|)}{g_1(|z|)} \lesssim \min\left\{\int_0^{|z|} g_1(s)s\,ds, F(z)\right\}^{\frac{2-\sigma}{n}} + 1 \quad \text{for some } \sigma \in (0,2].$$

These are essentially equivalent to the assumptions (1.34) of Theorem 1.9, when $\beta_0 = 1$ and taking $(1.33)_2$ into account. Notice that here the comparison is complete and is not only restricted to the case of functionals with superquadratic growth as done above in the vectorial case. Notice also that we are assuming here the additional superlinear growth in $(1.33)_1$. This is avoidable at several stages and is linked to the presence of the right-hand side f in Theorem 1.9.

7 General Equations and Theorem 1.17

For the proof of Theorem 1.17 we shall confine ourselves to the case $n > 2$. The proof in the case $n = 2$ can be obtained easily as the proof in the higher-dimensional case and looking at the proof of Theorem 1.11. As usual, we assume that $f \in L(n,1)(\mathbb{R}^n;\mathbb{R}^N)$ by letting $f \equiv 0$ outside Ω. We start the proof by modifying an approximation scheme that appears in various forms in the literature. For instance, we refer to [26, 45]. For any $\varepsilon \in (0,1]$, we denote $\mu_\varepsilon := \mu + \varepsilon$ as in (2.1), and recalling (4.2), we define the mollified vector field $\overline{a}_\varepsilon\colon \mathbb{R}^n \to \mathbb{R}^n$ and the truncated functions f_ε as follows:

$$\begin{cases} \overline{a}_\varepsilon(z) := (a * \phi_\varepsilon)(z) = \displaystyle\int_{B_1} a(z+\varepsilon y)\phi(y)\,dy, \\ f_\varepsilon(x) := \min\{\max\{f(x), -1/\varepsilon\}, 1/\varepsilon\}. \end{cases} \tag{7.1}$$

Thanks to (1.53), and using calculations similar to those in [26, sec. 3], we get that $\overline{a}_\varepsilon\colon \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following conditions:

$$\begin{cases} |\overline{a}_\varepsilon(z)| + H_\varepsilon(|z|)|\partial\overline{a}_\varepsilon(z)| \le c[H_\varepsilon(|z|)]^{q-1} + c[H_\varepsilon(|z|)]^{p-1}, \\ [H_\varepsilon(|z|)]^{p-2}|\xi|^2 \le c\langle \partial\overline{a}_\varepsilon(z)\xi, \xi\rangle, \end{cases} \tag{7.2}$$

for a constant $c \equiv c(n,p,q,\nu,\Lambda)$ that is independent of $\varepsilon \in (0,1]$; recall the notation of H_ε in (2.1). Finally, we define the q-growth vector field $a_\varepsilon\colon \mathbb{R}^n \to \mathbb{R}^n$ as $a_\varepsilon(z) := \overline{a}_\varepsilon(z) + \varepsilon[H_\varepsilon(|z|)]^{q-2}z$ for every $z \in \mathbb{R}^n$. The newly defined family $\{a_\varepsilon\}$ of vector fields is such that

$$a_\varepsilon(\cdot) \to a(\cdot) \quad \text{uniformly on compact subsets of } \mathbb{R}^n \text{ as } \varepsilon \to 0. \tag{7.3}$$

Moreover, by (7.2) it follows that for every $\varepsilon \in (0,1]$ there exists a constant $c_\varepsilon \ge 1$ such that the following growth and ellipticity conditions are satisfied:

$$\begin{cases} |a_\varepsilon(z)| + H_\varepsilon(|z|)|\partial a_\varepsilon(z)| \le c[H_\varepsilon(|z|)]^{q-1} + c[H_\varepsilon(|z|)]^{p-1}, \\ [H_\varepsilon(|z|)]^{p-2}|\xi|^2 + \varepsilon[H_\varepsilon(|z|)]^{q-2}|\xi|^2 \le c\langle \partial a_\varepsilon(z)\xi, \xi\rangle, \\ |a_\varepsilon(z)| + H_\varepsilon(|z|)|\partial a_\varepsilon(z)| \le c_\varepsilon[H_\varepsilon(|z|)]^{q-1}, \\ [H_\varepsilon(|z|)]^{q-2}|\xi|^2 \le c_\varepsilon\langle \partial a_\varepsilon(z)\xi, \xi\rangle, \end{cases} \tag{7.4}$$

whenever $z, \xi \in \mathbb{R}^n$, for another constant $c \equiv c(n,p,q,\nu,\Lambda) \ge 1$, which is this time independent of ε. The lower bound in $(7.4)_2$ implies that for every choice of $z_1, z_2 \in \mathbb{R}^n$ the following monotonicity inequality holds:

$$\begin{aligned} &(|z_1|^2 + |z_2|^2 + \mu_\varepsilon^2)^{(p-2)/2}|z_1 - z_2|^2 \\ &\quad + \varepsilon(|z_1|^2 + |z_2|^2 + \mu_\varepsilon^2)^{(q-2)/2}|z_1 - z_2|^2 \\ &\quad\quad \le c\langle a_\varepsilon(z_1) - a_\varepsilon(z_2), z_1 - z_2\rangle, \end{aligned} \tag{7.5}$$

for a constant $c \equiv c(n,p,q,\nu,\Lambda)$ that is independent of $\varepsilon \in (0,1]$. In turn, using this last inequality and recalling again the notation in (2.1), the next estimate can now be obtained via minor variants of the arguments developed for [45, lemma 4.4]:

$$[H_\varepsilon(|z_1|)]^p + \varepsilon[H_\varepsilon(|z_1|)]^q \le c\big(|z_2|^2 + 1\big)^{\frac{p(q-1)}{2(p-1)}} + c\langle a_\varepsilon(z_1), z_1 - z_2\rangle. \tag{7.6}$$

This holds uniformly in $\varepsilon \in (0,1]$ and for every choice of $z_1, z_2 \in \mathbb{R}^n$, where $c \equiv c(n, p, q, \nu, \Lambda)$. We now define $u_\varepsilon \in u_0 + W_0^{1,q}(\Omega)$ as the (unique) solution to the regularized Dirichlet problem

$$\begin{cases} -\operatorname{div} a_\varepsilon(Du_\varepsilon) = f_\varepsilon & \text{in } \Omega, \\ u \equiv u_0 & \text{on } \partial\Omega. \end{cases} \tag{7.7}$$

Existence follows by standard monotonicity methods in the Dirichlet class $u_0 + W_0^{1,q}(\Omega)$. Notice also that $p(q-1)/(p-1) \geq q$ as $p \leq q$, so that, in particular, it is $u_0 \in W^{1,q}(\Omega)$. We then have the following:

LEMMA 7.1. *Let $u_\varepsilon \in W^{1,q}(\Omega)$ be the solution to the Dirichlet problem in* (7.7); *then the inequality*

$$\int_\Omega \{[H_\varepsilon(Du_\varepsilon)]^p + \varepsilon[H_\varepsilon(Du_\varepsilon)]^q\}\,dx \leq c\mathscr{D}^p \tag{7.8}$$

holds for a constant c depending only on n, p, q, ν, and Λ, uniformly with respect to $\varepsilon \in (0,1]$. We recall that $\mathscr{D}$ has been defined in (1.55).

PROOF. Using (7.6) and the fact that u_ε solves (7.7), we get

$$\begin{aligned} &\int_\Omega \{[H_\varepsilon(Du_\varepsilon)]^p + \varepsilon[H_\varepsilon(Du_\varepsilon)]^q\}\,dx \\ &\quad\leq c\int_\Omega \big(|Du_0|^2+1\big)^{\frac{p(q-1)}{2(p-1)}}\,dx + \tilde{c}\int_\Omega |f_\varepsilon(u_\varepsilon - u_0)|\,dx \end{aligned} \tag{7.9}$$

for $c, \tilde{c} \equiv c, \tilde{c}(n, p, q, \nu, \Lambda)$, and it remains to estimate the last integral in the right-hand side of the above inequality. For this we distinguish two cases. If $p < n$, we set $p^* := np/(n-p)$, so that $(p^*)' \leq n$. Then, using the first Sobolev inequality and then the Hölder inequality, and recalling the definition of f_ε in (7.1), we have

$$\begin{aligned} \int_\Omega |f_\varepsilon(u_\varepsilon - u_0)|\,dx &\leq \|u_\varepsilon - u_0\|_{L^{p^*}(\Omega)}\|f_\varepsilon\|_{L^{(p^*)'}(\Omega)} \\ &\leq c\|Du_\varepsilon - Du_0\|_{L^p(\Omega)}\|f\|_{L^{(p^*)'}(\Omega)} \\ &\leq c|\Omega|^{\frac{p-1}{p}}\|Du_\varepsilon - Du_0\|_{L^p(\Omega)}\|f\|_{L^n(\Omega)}. \end{aligned}$$

Otherwise, if $p \geq n$, we similarly find

$$\begin{aligned} \int_\Omega |f_\varepsilon(u_\varepsilon - u_0)|\,dx &\leq \|u_\varepsilon - u_0\|_{L^{\frac{n}{n-1}}(\Omega)}\|f_\varepsilon\|_{L^n(\Omega)} \\ &\leq c|\Omega|^{\frac{1}{n}}\|Du_\varepsilon - Du_0\|_{L^{\frac{n}{n-1}}(\Omega)}\|f\|_{L^n(\Omega)} \\ &\leq c|\Omega|^{\frac{p-1}{p}}\|Du_\varepsilon - Du_0\|_{L^p(\Omega)}\|f\|_{L^n(\Omega)}. \end{aligned}$$

Taking into account the content of the last two displays and using the Young inequality, we get

$$\begin{aligned}\widetilde{c}\int_{\Omega}|f_{\varepsilon}(u_{\varepsilon}-u_0)|dx \\ \le \frac{1}{2}\int_{\Omega}[H_{\varepsilon}(Du_{\varepsilon})]^p\,dx + c|\Omega|\|f\|_{L^n(\Omega)}^{p/(p-1)} + c\int_{\Omega}\big(|Du_0|^2+\mu^2\big)^{p/2}\,dx\end{aligned}$$

for a new constant $c = c(n,p,q,\nu,\Lambda)$, where $\widetilde{c}$ is the constant appearing in (7.9). Combining this last estimate with the one in (7.9) yields (7.8) and the proof is complete. □

We are ready for the crucial uniform a priori estimate, which is a counterpart of the one contained in Lemma 4.8.

LEMMA 7.2. *Let $u_{\varepsilon} \in W^{1,q}(\Omega)$ be the solution to the Dirichlet problem in* (7.7)*; there exists a constant $c \equiv c(n,p,q,\nu,\Lambda)$, but otherwise independent of $\varepsilon \in (0,1]$, such that the inequality*

$$\begin{aligned}\|H_{\varepsilon}(|Du_{\varepsilon}|)\|_{L^{\infty}(B_{R/2})}^p \le c\left(\frac{\mathscr{D}^p}{|B_R|}\right)^{\frac{p}{p-(q-p)n}} + c\frac{\mathscr{D}^p}{|B_R|} \\ + c\|f\|_{L(n,1)(B_R)}^{\frac{p}{p-1}} + c\|f\|_{L(n,1)(B_R)}^{\frac{2p}{2(p-1)-(q-p)(n-2)}}\end{aligned} \tag{7.10}$$

holds for any ball $B_R \Subset \Omega$, where the quantity $\mathscr{D}$ has been defined in (1.55)*.*

PROOF. By $(7.4)_{3,4}$ the vector field $a_{\varepsilon}\colon \mathbb{R}^n \to \mathbb{R}^n$ has standard and nondegenerate q-growth and ellipticity, so by standard regularity theory it follows that u_{ε} satisfies the regularity properties displayed in (4.31). Thanks to $(7.4)_{1,2}$, we then intend to adapt to the setting of Section 4.2 with the choices

$$\begin{aligned}\widetilde{a}(\cdot) \equiv a_{\varepsilon}(\cdot), \quad g_1(t) \equiv g_{1,\varepsilon}(t) := [H_{\varepsilon}(t)]^{p-2}/c, \\ g_{2,\varepsilon}(t) := c[H_{\varepsilon}(t)]^{p-2} + c[H_{\varepsilon}(t)]^{q-2},\end{aligned} \tag{7.11}$$

for a suitable constant $c \equiv c(n,p,q,\nu,\Lambda)$ and, finally, with $\overline{T} = T = 0$. In particular, notice that the conditions in (4.26) are satisfied with the choice in (7.11), provided c is chosen suitably large. Notice also that we shall use several a priori estimates derived in Section 4.2 for the choice $\overline{T} = T = 0$ although these have been derived for the case $0 < T < \overline{T}$. Indeed, all the estimates are uniform with respect to $\overline{T}$, and therefore we may choose a small $\overline{T}$ and eventually let $\overline{T} \to 0$. We are therefore able to proceed as for the Caccioppoli inequalities in Lemmas 4.5 and 4.6, and eventually we arrive at the quantitative L^{∞}-estimate (4.38). We use this estimate as done in Lemma 4.8 but with $\vartheta = 1$ since the vector field $a_{\varepsilon}\colon \mathbb{R}^n \to \mathbb{R}^n$ is not assumed to be symmetric.

Let's see the details. Notice that with the choice (7.11) (and $\overline{T} = 0$), and recalling (2.1), we also have that $G_0(t) = [H_{\varepsilon}(t)]^p/(cp)$. Estimate (4.38) with $\vartheta = 1$ applied in the present context (again we now consider concentric balls

$B_{R/2} \Subset B_s \Subset B_t \Subset B_R$, $x_0 \in B_t$, and $R_0 = t - s$ such that $B_{R_0}(x_0) \subset B_t$ and $k = 0$ as done for Lemma 4.8), using computations similar to those in (6.3), and finally taking also (7.8) into account, now gives

$$\begin{aligned}
&[H_\varepsilon(|Du_\varepsilon(x_0)|)]^p \\
&\quad \le \frac{c}{(t-s)^{n/2}}\big[\|H_\varepsilon(|Du_\varepsilon|)\|_{L^\infty(B_t)} + 1\big]^{\frac{(q-p)n}{2}} \|H_\varepsilon(|Du_\varepsilon|)\|^{p/2}_{L^\infty(B_t)} \mathscr{D}^{p/2} \\
&\qquad + c\big[\|H_\varepsilon(|Du_\varepsilon|)\|_{L^\infty(B_t)} + 1\big]^{\frac{(q-p)(n-2)}{2}} \|H_\varepsilon(|Du_\varepsilon|)\|_{L^\infty(B_t)} \mathbf{P}_1^{f_{B_R}}(x_0, 2R_0),
\end{aligned}$$

with $c \equiv c(n,p,q,\nu,\Lambda)$ being independent of ε. Notice that, in proceeding as in Lemma 4.8, we have used the obvious pointwise inequality $|(f_\varepsilon)_{B_R}| \le |f_{B_R}|$, which is an immediate consequence of the definition in (7.1). Recalling that $x_0 \in B_t$ is arbitrary, the previous estimate implies

$$\begin{aligned}
&\|H_\varepsilon(|Du_\varepsilon|)\|^p_{L^\infty(B_s)} \\
&\quad \le \frac{c}{(t-s)^{n/2}}\Big[\|H_\varepsilon(|Du_\varepsilon|)\|^{\frac{(q-p)n+p}{2}}_{L^\infty(B_t)} + \|H_\varepsilon(|Du_\varepsilon|)\|^{p/2}_{L^\infty(B_t)}\Big]\mathscr{D}^{p/2} \\
&\qquad + c\big[\|H_\varepsilon(|Du_\varepsilon|)\|_{L^\infty(B_t)}\big]^{\frac{(q-p)(n-2)+2}{2}} \Big\|\mathbf{P}_1^{f_{B_R}}(\cdot, 2(t-s))\Big\|_{L^\infty(B_t)} \\
&\qquad + c\|H_\varepsilon(|Du_\varepsilon|)\|_{L^\infty(B_t)} \Big\|\mathbf{P}_1^{f_{B_R}}(\cdot, 2(t-s))\Big\|_{L^\infty(B_t)}.
\end{aligned}$$

The bound on q/p assumed in (1.19) (recall we are considering the case $n > 2$) implies $[(q-p)n+p]/2 < p$ and $[(q-p)(n-2)+2]/2 < p$. We may therefore apply the Young inequality to get

$$\begin{aligned}
&\|H_\varepsilon(|Du_\varepsilon|)\|^p_{L^\infty(B_s)} \\
&\quad \le \frac{1}{2}\|H_\varepsilon(|Du_\varepsilon|)\|^p_{L^\infty(B_t)} + c\frac{\mathscr{D}^p}{(t-s)^n} + c\left[\frac{\mathscr{D}^p}{(t-s)^n}\right]^{\frac{p}{p-(q-p)n}} \\
&\qquad + c\Big\|\mathbf{P}_1^{f_{B_R}}(\cdot, 2(t-s))\Big\|^{\frac{p}{p-1}}_{L^\infty(B_t)} + c\Big\|\mathbf{P}_1^{f_{B_R}}(\cdot, 2(t-s))\Big\|^{\frac{2p}{2(p-1)-(q-p)(n-2)}}_{L^\infty(B_t)},
\end{aligned}$$

which again holds for a constant $c \equiv c(n,p,q,\nu,\Lambda)$ that is independent of ε. We are now again able to apply the iteration Lemma 4.9 as was done at the end of the proof of Lemma 4.8, this time with $\phi(y) := \|H_\varepsilon(Du_\varepsilon)\|^p_{L^\infty(B_y)}$ for $y \in [R/2, R]$. This and (2.3) finally yields (7.10) with the dependence of the constant c described in the statement. □

We complete the proof of Theorem 1.17 with a convergence argument. We take a decreasing sequence $\{\varepsilon_m\}$ in $(0,1]$ such that $\varepsilon_m \to 0$ and denote $\{Du_m\} := \{Du_{\varepsilon_m}\}$. By using (7.8) with $u_\varepsilon \equiv u_m$, we infer that the sequence $\{u_m\}$ is bounded in $W^{1,p}(\Omega)$ and therefore, up to a not-relabeled subsequence, we may assume that there exists $u \in u_0 + W^{1,p}_0(\Omega)$ such that $u_m \rightharpoonup u$ in $W^{1,p}(\Omega)$. We are now going to prove that u is a distributional solution to $-\operatorname{div} a(Du) = f$ and satisfies the conditions stated in Theorem 1.17, thereby concluding the proof. Applying Lemma 7.2 and a standard covering argument, we infer for every bounded $\widetilde{\Omega} \Subset \Omega$, there exists a constant $M \equiv M(\operatorname{dist}(\widetilde{\Omega}, \partial\Omega)) \geq 1$ such that

$$\|Du_m\|_{L^\infty(\widetilde{\Omega})} \leq M \qquad \forall m \in \mathbb{N}. \tag{7.12}$$

By using (7.12) we can essentially reduce to the standard case $p = q$ in (7.2) (with the involved constant c additionally depending on M^{q-p}), as in this case the problem becomes uniformly elliptic.

In this case several estimates are already available. When $p \geq 2$ one can use the results in [57] and [25, theorem 5.2] (formally with $p = q$), or directly combine the estimates in [50, sec. 4] with those of [25, theorem 5.2]. The outcome, after a standard diagonalization and localization argument, is that the sequence $\{Du_m\}$ is bounded in $W^{\sigma,p}_{\mathrm{loc}}(\widetilde{\Omega}; \mathbb{R}^n)$ for any $\sigma < 1/(p-1)$. In the case $p < 2$ we can use more directly the information in (7.12) to deduce that $\{Du_m\}$ is bounded in $W^{1,2}_{\mathrm{loc}}(\widetilde{\Omega}; \mathbb{R}^n)$. This is more classical; see, for instance, step 3 in Section 8 below. In any case, as $\widetilde{\Omega} \Subset \Omega$ is arbitrary, again up to a subsequence we deduce that $Du_m \to Du$ in $L^1_{\mathrm{loc}}(\Omega)$, which, together with (7.12), implies that $Du_m \to Du$ strongly in $L^t_{\mathrm{loc}}(\Omega)$ for every $t < \infty$. This allows us to let $m \to \infty$ in the distributional form of $(7.7)_1$, thereby getting that u solves the Dirichlet problem (1.52) (recall that $a_m(\cdot) \to a(\cdot)$ as $m \to \infty$, uniformly on compact subset of $\mathbb{R}^n$ and $f_m \to f$ in L^n). Finally, estimate (1.54) follows letting $m \to \infty$ in (7.10) ($\varepsilon \equiv \varepsilon_m$) and using lower semicontinuity. The proof of Theorem 1.17 is therefore complete.

8 Regularity for Irregular Functionals with Polynomial Growth

In this final section we justify (5.44); therefore we use the notation adopted in Section 5 and proceed via another approximation argument. We could have incorporated it into the one used in Section 5.1, but this would have led to several additional complications while several of the arguments needed are already available elsewhere. Indeed, the result follows as a combination of various hidden facts scattered in the literature, and we could not find any explicit reference to what we needed. Since we also believe that the facts reported here could be useful somewhere else in the future, we briefly report the proofs here.

We consider a functional of the type in (1.1), where now it is $f \in L^\infty(\Omega; \mathbb{R}^N)$ and the integrand $F(z)$ satisfies the structure condition in (1.43) for some $\widetilde{F} \in C^1[0,\infty) \cap W^{2,\infty}_{\mathrm{loc}}[0,+\infty)$ such that

$$\tilde{a}(\cdot) \in W^{1,\infty}_{\mathrm{loc}}[0,+\infty), \qquad \widetilde{F}(t) := \int_0^t \tilde{a}(s)s\,ds, \tag{8.1}$$

and furthermore satisfies the bounds

$$\begin{cases} \nu(t^2+\mu^2)^{\gamma/2} - \Lambda\mu^{\gamma/2} \le \widetilde{F}(t) \le \Lambda(t^2+\mu^2)^{\gamma/2}, \\ |\partial^2 F(z)| \le \Lambda(|z|^2+\mu^2)^{\frac{\gamma-2}{2}}, \\ \nu(|z|^2+\mu^2)^{\frac{\gamma-2}{2}}|\xi|^2 \le \langle \partial^2 F(z)\xi,\xi\rangle, \end{cases} \tag{8.2}$$

for every choice of $t \ge 0$ and $z,\xi \in \mathbb{R}^{N\times n}$ (such that $\partial^2 F(z)$ exists), where $0 < \nu \le 1 \le \Lambda$ and $\mu > 0$ (this is crucial here) are fixed, positive constants. (Notice that, in fact, $(8.2)_1$ can be derived from $(8.2)_{2,3}$ via the argument used for (5.43), modulo adjusting the constant ν. It also follows that $\widetilde{F}'(0) = 0$). Due to $(8.2)_1$, the functional in (1.1) is then naturally defined in $W^{1,\gamma}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and, in what follows, we consider a local minimizer $u \in W^{1,\gamma}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ of $\mathscr{F}$ in the sense of Definition 1.1. Our aim is to prove, in several steps and via an approximation argument, that

$$Du \in L^{\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^{N\times n}),\ \ u \in W^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N),\ \ a(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^{N\times n}), \tag{8.3}$$

where $a(z) := \widetilde{a}(|z|)z = \partial F(z)$ for every $z \in \mathbb{R}^{N\times n}$. This provides a justification to (5.44). To this end, because of standard covering arguments, it is actually sufficient to prove that

$$\begin{gathered} Du \in L^{\infty}(B/2;\mathbb{R}^{N\times n}), \quad u \in W^{2,2}(B/8;\mathbb{R}^N), \\ a(Du) \in W^{1,2}(B/8;\mathbb{R}^{N\times n}), \end{gathered} \tag{8.4}$$

holds for any fixed ball $B \Subset \Omega$ such that $|B| \le 1$.

Step 1. *Introduction of approximate problems*. We are able to use the arguments [26, lemmas 3.1 and 3.2] to get a sequence of approximating and $C^2_{\mathrm{loc}}[0,\infty)$-regular functions $\widetilde{F}_k$ such that

$$\begin{cases} \frac{1}{c_*}(t^2+\mu_k^2)^{\gamma/2} - c_*\mu_k^{\gamma} \le \widetilde{F}_k(t) \le c_*(t^2+\mu_k^2)^{\gamma/2}, \\ |\partial^2 F_k(z)| \le c_*(|z|^2+\mu_k^2)^{\frac{\gamma-2}{2}}, \\ \frac{1}{c_*}(|z|^2+\mu_k^2)^{\frac{\gamma-2}{2}}|\xi|^2 \le \langle \partial^2 F_k(z)\xi,\xi\rangle, \end{cases} \tag{8.5}$$

for every $z,\xi \in \mathbb{R}^{N\times n}$, $t \ge 0$, where $c_* \equiv c_*(n,N,\nu,\Lambda,\gamma) \ge 1$ is independent of k, $\mu_k := \mu + 1/k$, and $F_k(z) := \widetilde{F}_k(|z|)$. The uniform convergence $F_k \to F$ of these integrands takes place on compact subsets of $\mathbb{R}^{N\times n}$. Here we need to remark that the arguments in [26, lemmas 3.1 and 3.2] work in a more general situation, where no upper bound on $\partial^2 F$ as in (8.2) is in force. Accordingly, these lemmas yield (8.5) without the upper bound on $\partial^2 F_k$ displayed in $(8.5)_3$. Essentially, what we can do here is use [26, lemmas 3.1 and 3.2] with the choice $p = q$ (with the notation used in [26]; these correspond to γ in the present setting) and $(8.2)_{2,3}$ being in force. This makes several of the constructions in [26, lemmas 3.1 and

3.2] superfluous; essentially, the initial mollification procedure in those lemmas suffices. We then consider, for every $k \in \mathbb{N}$, the variational problem

$$\min_{w \in u + W_0^{1,\gamma}(B)} \int_B [F_k(Dw) - fw] dx \tag{8.6}$$

and denote by $u_k \in u + W_0^{1,\gamma}(B;\mathbb{R}^N)$ its unique solution.

Step 2. $Du \in L^\infty(B/2;\mathbb{R}^{N\times n})$. By the growth conditions of F_k in (8.5), the minimality of u_k, and using an argument that is very similar to the one in Section 4.4 (in particular, see estimate (4.55)), we get

$$\fint_B \left(|Du_k|^2 + \mu_k^2\right)^{\gamma/2} dx \le c \fint_B \left(|Du|^2 + \mu_k^2\right)^{\gamma/2} dx + c\|f\|_{L^n(B)}^{\frac{\gamma}{\gamma-1}} \tag{8.7}$$

with $c \equiv c(n, N, \nu, \Lambda, \gamma)$. By the arguments of [26, sec. 5] or those of Section 4.4, it follows that (up to the choice of a subsequence) $u_k \rightharpoonup u$ weakly in $W^{1,\gamma}(B;\mathbb{R}^N)$. By known regularity theory for standard growth functionals (see [24, 33, 36]), we get the following local estimate for u_k:

$$\|Du_k\|_{L^\infty(B/2)} \le c\left(\fint_B (|Du_k|^2 + \mu_k^2)^{\gamma/2} dx\right)^{\frac{1}{\gamma}} + c[|B|^{1/n}\|f\|_{L^\infty(B)}]^{\frac{1}{\gamma-1}},$$

still with $c \equiv c(n, N, \nu, \Lambda, \gamma)$. Letting $k \to \infty$ and taking advantage of (8.7), we get

$$\|Du\|_{L^\infty(B/2)} \le c\left(\fint_B (|Du|^2 + \mu^2)^{\gamma/2} dx\right)^{\frac{1}{\gamma}} + c[|B|^{1/n}\|f\|_{L^\infty(B)}]^{\frac{1}{\gamma-1}}.$$

Step 3. $Du \in W^{1,2}(B/8;\mathbb{R}^{N\times n})$. A crucial point here is that all the numbers $\{\mu_k\}, \mu$ are uniformly bounded away from 0 as we are assuming that $\mu > 0$ holds from the beginning. We use standard difference quotient arguments. We refer to [29, chap. 8] for the basic properties of difference quotient operators $\Delta_{s,h}\varphi(x) := [\varphi(x + he_s) - \varphi(x)]/h$ (here $|h| > 0$ and $\{e_s\}$ is the standard basis of $\mathbb{R}^n$) and their use. In the Euler-Lagrange equation of the functional in (8.6),

$$\int_B \langle \partial F_k(Du_k), D\varphi\rangle dx = \int_B f\varphi\, dx,$$

which holds for every $\varphi \in C_0^\infty(B;\mathbb{R}^N)$, we take $\Delta_{s,-h}\varphi$ instead of φ, for $0 < |h| < \min\{\text{dist}(\text{spt}(\varphi), \partial B), r/8\}$, where r denotes the radius of B. Integration by parts for finite differences yields

$$\int_B \langle \Delta_{s,h}[\partial F_k(Du_k)], D\varphi\rangle dx = -\int_B f\Delta_{s,-h}\varphi\, dx. \tag{8.8}$$

We then take a standard cutoff function $\eta \in C_0^\infty(B/4, [0, 1])$ with $\eta \equiv 1$ in $B/8$ and $|D\eta| \lesssim 1/r$ and define $\varphi \equiv \varphi_s := \eta^2\Delta_{s,h}u_k$ so that $\varphi \in W_0^{1,2}(B/4;\mathbb{R}^N)$.

Using this test function in (8.8) and employing the bounds in (8.5), and also the by-now classical methods explained in [29, theorem 8.1], we first find

$$\begin{aligned}\int_{B/4}(|Du_k|^2+\mu_k^2)^{\frac{\gamma-2}{2}}|\Delta_{s,h}Du_k|^2\eta^2\,dx\\ \le c\int_{B/4}(|Du_k|^2+\mu_k^2)^{\frac{\gamma-2}{2}}|\Delta_{s,h}Du_k||\Delta_{s,h}u_k||D\eta|\eta\,dx\\ + c\int_{B/4}|f||\Delta_{s,-h}\varphi|dx.\end{aligned}$$

By the uniform estimates from Step 2, $M := \sup_k \|Du_k\|_{L^\infty(B/2)}$ is finite, so that we have

$$\begin{aligned}\int_{B/4}|f||\Delta_{s,-h}\varphi|dx &\le \|f\|_{L^\infty(\Omega)}\int_{B/4}|D\varphi|dx\\ &\le c\|f\|_{L^\infty(\Omega)}\left[Mr^{n-1}+\int_{B/4}|\Delta_{s,h}Du_k|\eta^2\,dx\right].\end{aligned} \tag{8.9}$$

Then, after using standard norm estimates for difference quotients and Young's inequality and proceeding in a standard way by reabsorbing terms, we arrive at

$$\begin{aligned}\int_{B/4}(|Du_k|^2+\mu_k^2)^{\frac{\gamma-2}{2}}|\Delta_{s,h}Du_k|^2\eta^2\,dx\\ \le c\int_{B/4}(|Du_k|^2+\mu_k^2)^{\frac{\gamma-2}{2}}|\Delta_{s,h}u_k|^2|D\eta|^2\,dx\\ + c\|f\|_{L^\infty(\Omega)}\left[M+\int_{B/4}|\Delta_{s,h}Du_k|\eta^2\,dx\right].\end{aligned} \tag{8.10}$$

In the case $\gamma \ge 2$, the above estimate and a further use of the Young inequality implies

$$\begin{aligned}\mu^{\gamma-2}\int_{B/4}|\Delta_{s,h}Du_k|^2\eta^2\,dx\\ \le c\big(M^2+\mu_k^2\big)^{\gamma/2}+c\big[\|f\|^2_{L^\infty(\Omega)}(1+\mu^{2-\gamma})+M^2\big].\end{aligned} \tag{8.11}$$

Otherwise, (8.10) implies

$$\begin{aligned}\big(M^2+\mu_k^2\big)^{\frac{\gamma-2}{2}}\int_{B/4}|\Delta_{s,h}Du_k|^2\eta^2\,dx\\ \le c\mu^{\gamma-2}M^2+c\Big\{\|f\|^2_{L^\infty(\Omega)}\Big[1+(M^2+\mu_k^2)^{\frac{2-\gamma}{2}}\Big]+M^2\Big\}.\end{aligned} \tag{8.12}$$

In both cases the involved constant c only depends on n, N, ν, Λ, and γ and is otherwise independent of $k \in \mathbb{N}$. Since h (small) and $s \in \{1,\dots,n\}$ are arbitrary, the fact that $\eta \equiv 1$ in $B/8$ shows $Du_k \in W^{1,2}(B/8;\mathbb{R}^{N\times n})$. Furthermore, in view of $\Delta_{s,h} Du_k \to D_s Du_k$ strongly in $L^2(B/8;\mathbb{R}^{N\times n})$, we may let $h \to 0$ in (8.11)–(8.12), and this yields

$$\int_{B/8} |D^2 u_k|^2\,dx \le c(n, N, \nu, \Lambda, \gamma, \mu, M, \|f\|_{L^\infty(\Omega)}) \tag{8.13}$$

for each $k \in \mathbb{N}$, and the constant is independent of $k \in \mathbb{N}$. As a consequence of this uniform boundedness of the sequence $\{D^2 u_k\}$ in $L^2(B/8;\mathbb{R}^{N\times n})$ in combination with the weak convergence $u_k \rightharpoonup u$ in $W^{1,\gamma}(B;\mathbb{R}^N)$ established in Step 2, we then find (again up to a subsequence) the weak convergence $D^2 u_k \rightharpoonup D^2 u$ in $L^2(B/8;\mathbb{R}^{N\times n})$, and $D^2 u$ satisfies, by lower semicontinuity, the corresponding estimate. We have therefore proved that $Du \in W^{1,2}(B/8;\mathbb{R}^{N\times n})$. A similar argument actually gives $u_k \in W^{2,2}_{\mathrm{loc}}(B/2;\mathbb{R}^N)$.

In this respect, when $\gamma \le 2$, we can further improve (8.13). In fact, we have that $|f||\Delta_{s,-h}\varphi| \le \varepsilon|\Delta_{s,-h}\varphi|^2 + |f|^2/\varepsilon$ holds for $\varepsilon \in (0,1)$. We use this estimate to replace the last term in (8.10); then, letting $h \to 0$, summing on s, and choosing $\varepsilon \equiv \varepsilon(M)$ small enough to reabsorb terms, we arrive at (8.13) but with the right-hand side independent of μ, and where $\|f\|_{L^\infty}$ is now replaced by $\|f\|_{L^2}$. As a consequence, a similar estimate holds for Du after letting $k \to \infty$. In turn, if we a priori know that $\{Du_k\}$ is locally uniformly bounded, we can infer that $u \in W^{2,2}_{\mathrm{loc}}$ under the only assumption that $f \in L^2$ and including the case $\mu = 0$. This can be used for equations with standard γ-growth and ellipticity, as done in the final convergence argument at the end of Section 7.

Step 4. $a(Du) \in W^{1,2}(B/8;\mathbb{R}^{N\times n})$. Note that the previous step also implies $|Du| \in W^{1,2}(B/8)$, and so, in view of (8.1) and $\partial F(Du) = a(Du) = \tilde{a}(|Du|)Du$, the chain rule for Sobolev functions gives

$$a(Du) \in W^{1,2}(B/8;\mathbb{R}^{N\times n})$$

(see also Remark 5.5). Thus, all the assertions in (8.4) are established and the proof is complete.

Acknowledgments. We would like to thank the referee for providing a number of observations on the originally submitted manuscript that eventually led to a better version of the paper. We also would like to thank Cristiana De Filippis, Francesco Leonetti, and Jihoon Ok for a careful reading of former versions of the manuscript and related important remarks and discussions.

Bibliography

[1] Baroni, P. Riesz potential estimates for a general class of quasilinear equations. *Calc. Var. Partial Differential Equations* **53** (2015), no. 3-4, 803–846. doi:10.1007/s00526-014-0768-z

[2] Baroni, P.; Colombo, M.; Mingione, G. Regularity for general functionals with double phase. *Calc. Var. Partial Differential Equations* **57** (2018), no. 2, Art. 62, 48 pp. doi:10.1007/s00526-018-1332-z

[3] Beck, L.; Schmidt, T. On the Dirichlet problem for variational integrals in BV. *J. Reine Angew. Math.* **674** (2013), 113–194.

[4] Beck, L.; Schmidt, T. Interior gradient regularity for BV minimizers of singular variational problems. *Nonlinear Anal.* **120** (2015), 86–106. doi:10.1016/j.na.2015.02.011

[5] Bildhauer, M.; Fuchs, M. Partial regularity for variational integrals with (s, μ, q)-growth. *Calc. Var. Partial Differential Equations* **13** (2001), no. 4, 537–560. doi:10.1007/s005260100090

[6] Bousquet, P.; Brasco, L. Global Lipschitz continuity for minima of degenerate problems. *Math. Ann.* **366** (2016), no. 3-4, 1403–1450. doi:10.1007/s00208-016-1362-9

[7] Byun, S.-S.; Oh, J. Global gradient estimates for non-uniformly elliptic equations. *Calc. Var. Partial Differential Equations* **56** (2017), no. 2, Art. 46, 36 pp. doi:10.1007/s00526-017-1148-2

[8] Byun, S.-S.; Youn, Y. Riesz potential estimates for a class of double phase problems. *J. Differential Equations* **264** (2018), no. 2, 1263–1316. doi:10.1016/j.jde.2017.09.038

[9] Carozza, M.; Kristensen, J.; Passarelli di Napoli, A. Higher differentiability of minimizers of convex variational integrals. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **28** (2011), no. 3, 395–411. doi:10.1016/j.anihpc.2011.02.005

[10] Carozza, M.; Kristensen, J.; Passarelli di Napoli, A. Regularity of minimizers of autonomous convex variational integrals. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **13** (2014), no. 4, 1065–1089.

[11] Cellina, A. Strict convexity and the regularity of solutions to variational problems. *ESAIM Control Optim. Calc. Var.* **22** (2016), no. 3, 862–871. doi:10.1051/cocv/2015034

[12] Cellina, A.; Staicu, V. On the higher differentiability of solutions to a class of variational problems of fast growth. *Calc. Var. Partial Differential Equations* **57** (2018), no. 2, Art. 66, 13 pp. doi:10.1007/s00526-018-1323-0

[13] Cianchi, A. Maximizing the L^∞ norm of the gradient of solutions to the Poisson equation. *J. Geom. Anal.* **2** (1992), no. 6, 499–515. doi:10.1007/BF02921575

[14] Cianchi, A.; Maz′ya, V. G. Global Lipschitz regularity for a class of quasilinear elliptic equations. *Comm. Partial Differential Equations* **36** (2011), no. 1, 100–133. doi:10.1080/03605301003657843

[15] Cianchi, A.; Maz′ya, V. G. Global boundedness of the gradient for a class of nonlinear elliptic systems. *Arch. Ration. Mech. Anal.* **212** (2014), no. 1, 129–177. doi:10.1007/s00205-013-0705-x

[16] Cianchi, A.; Maz′ya, V. G. Gradient regularity via rearrangements for p-Laplacian type elliptic boundary value problems. *J. Eur. Math. Soc. (JEMS)* **16** (2014), no. 3, 571–595. doi:10.4171/JEMS/440

[17] Cianchi, A.; Maz′ya, V. G. Global gradient estimates in elliptic problems under minimal data and domain regularity. *Commun. Pure Appl. Anal.* **14** (2015), no. 1, 285–311. doi:10.3934/cpaa.2015.14.285

[18] Colombo, M.; Mingione, G. Bounded minimisers of double phase variational integrals. *Arch. Ration. Mech. Anal.* **218** (2015), no. 1, 219–273. doi:10.1007/s00205-015-0859-9

[19] Colombo, M.; Mingione, G. Calderón-Zygmund estimates and non-uniformly elliptic operators. *J. Funct. Anal.* **270** (2016), no. 4, 1416–1478. doi:10.1016/j.jfa.2015.06.022

[20] Daskalopoulos, P.; Kuusi, T.; Mingione, G. Borderline estimates for fully nonlinear elliptic equations. *Comm. Partial Differential Equations* **39** (2014), no. 3, 574–590. doi:10.1080/03605302.2013.866959

[21] De Filippis, C. Higher integrability for constrained minimizers of integral functionals with (p, q)-growth in low dimension. *Nonlinear Anal.* **170** (2018), 1–20. doi:10.1016/j.na.2017.12.007

[22] De Giorgi, E. Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3)* **3** (1957), 25–43.

[23] Duc, D. M.; Eells, J. Regularity of exponentially harmonic functions. *Internat. J. Math.* **2** (1991), no. 4, 395–408. doi:10.1142/S0129167X91000223

[24] Duzaar, F.; Mingione, G. Local Lipschitz regularity for degenerate elliptic systems. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **27** (2010), no. 6, 1361–1396. doi:10.1016/j.anihpc.2010.07.002

[25] Esposito, L.; Leonetti, F.; Mingione, G. Higher integrability for minimizers of integral functionals with (p,q) growth. *J. Differential Equations* **157** (1999), no. 2, 414–438. doi:10.1006/jdeq.1998.3614

[26] Esposito, L.; Leonetti, F.; Mingione, G. Regularity results for minimizers of irregular integrals with (p,q) growth. *Forum Math.* **14** (2002), no. 2, 245–272. doi:10.1515/form.2002.011

[27] Fuchs, M.; Mingione, G. Full $C^{1,\alpha}$-regularity for free and constrained local minimizers of elliptic variational integrals with nearly linear growth. *Manuscripta Math.* **102** (2000), no. 2, 227–250. doi:10.1007/s002291020227

[28] Giaquinta, M. Growth conditions and regularity, a counterexample. *Manuscripta Math.* **59** (1987), no. 2, 245–248. doi:10.1007/BF01158049

[29] Giusti, E. *Direct methods in the calculus of variations*. World Scientific, River Edge, N.J., 2003. doi:10.1142/9789812795557

[30] Hamburger, C. Regularity of differential forms minimizing degenerate elliptic functionals. *J. Reine Angew. Math.* **431** (1992), 7–64. doi:10.1515/crll.1992.431.7

[31] Kilpeläinen, T.; Malý, J. The Wiener test and potential estimates for quasilinear elliptic equations. *Acta Math.* **172** (1994), no. 1, 137–161. doi:10.1007/BF02392793

[32] Kristensen, J.; Mingione, G. The singular set of Lipschitzian minima of multiple integrals. *Arch. Ration. Mech. Anal.* **184** (2007), no. 2, 341–369. doi:10.1007/s00205-006-0036-2

[33] Kuusi, T.; Mingione, G. Potential estimates and gradient boundedness for nonlinear parabolic systems. *Rev. Mat. Iberoam.* **28** (2012), no. 2, 535–576.

[34] Kuusi, T.; Mingione, G. Guide to nonlinear potential estimates. *Bull. Math. Sci.* **4** (2014), no. 1, 1–82. doi:10.1007/s13373-013-0048-9

[35] Kuusi, T.; Mingione, G. Linear potentials in nonlinear potential theory. *Arch. Ration. Mech. Anal.* **207** (2013), no. 1, 215–246. doi:10.1007/s00205-012-0562-z

[36] Kuusi, T.; Mingione, G. Vectorial nonlinear potential theory. *J. Eur. Math. Soc. (JEMS)* **20** (2018), no. 4, 929–1004. doi:10.4171/JEMS/780

[37] Ladyzhenskaya, O. A.; Ural′tseva, N. N. Local estimates for gradients of solutions of non-uniformly elliptic and parabolic equations. *Comm. Pure Appl. Math.* **23** (1970), 677–703. doi:10.1002/cpa.3160230409

[38] Leonetti, F.; Mascolo, E.; Siepe, F. Everywhere regularity for a class of vectorial functionals under subquadratic general growth conditions. *J. Math. Anal. Appl.* **287** (2003), no. 2, 593–608. doi:10.1016/S0022-247X(03)00584-5

[39] Lieberman, G. M. The natural generalization of the natural conditions of Ladyzhenskaya and Ural′tseva for elliptic equations. *Comm. Partial Differential Equations* **16** (1991), no. 2-3, 311–361. doi:10.1080/03605309108820761

[40] Lieberman, G. M. On the regularity of the minimizer of a functional with exponential growth. *Comment. Math. Univ. Carolin.* **33** (1992), no. 1, 45–49.

[41] Lieberman, G. M. Gradient estimates for a class of elliptic systems. *Ann. Mat. Pura Appl. (4)* **164** (1993), 103–120. doi:10.1007/BF01759317

[42] Manfredi, J. J. Regularity of the gradient for a class of nonlinear possibly degenerate elliptic equations. Ph.D. thesis, Washington University in St. Louis, 1986.

[43] Manfredi, J. J. Regularity for minima of functionals with p-growth. *J. Differential Equations* **76** (1988), no. 2, 203–212. doi:10.1016/0022-0396(88)90070-8

[44] Marcellini, P. Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions. *Arch. Rational Mech. Anal.* **105** (1989), no. 3, 267–284. doi:10.1007/BF00251503

[45] Marcellini, P. Regularity and existence of solutions of elliptic equations with p,q-growth conditions. *J. Differential Equations* **90** (1991), no. 1, 1–30. doi:10.1016/0022-0396(91)90158-6

[46] Marcellini, P. Everywhere regularity for a class of elliptic systems without growth conditions. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **23** (1996), no. 1, 1–25.

[47] Marcellini, P. Regularity for some scalar variational problems under general growth conditions. *J. Optim. Theory Appl.* **90** (1996), no. 1, 161–181. doi:10.1007/BF02192251
[48] Marcellini, P.; Papi, G. Nonlinear elliptic systems with general growth. *J. Differential Equations* **221** (2006), no. 2, 412–443. doi:10.1016/j.jde.2004.11.011
[49] Maz′ja, V. G.; Havin, V. P. A nonlinear potential theory. *Russ. Math. Surveys* **27** (1972), 67–138.
[50] Mingione, G. Bounds for the singular set of solutions to non linear elliptic systems. *Calc. Var. Partial Differential Equations* **18** (2003), no. 4, 373–400. doi:10.1007/s00526-003-0209-x
[51] Mingione, G. Regularity of minima: an invitation to the dark side of the calculus of variations. *Appl. Math.* **51** (2006), no. 4, 355–426. doi:10.1007/s10778-006-0110-3
[52] Mingione, G. Gradient potential estimates. *J. Eur. Math. Soc. (JEMS)* **13** (2011), no. 2, 459–486. doi:10.4171/JEMS/258
[53] Ok, J. Regularity of ω-minimizers for a class of functionals with non-standard growth. *Calc. Var. Partial Differential Equations* **56** (2017), no. 2, Art. 48, 31 pp. doi:10.1007/s00526-017-1137-5
[54] Rădulescu, V. D.; Repovš, D. D. *Partial differential equations with variable exponents.* Variational methods and qualitative analysis. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, Fla., 2015. doi:10.1201/b18601
[55] Rao, M. M.; Ren, Z. D. *Theory of Orlicz spaces.* Monographs and Textbooks in Pure and Applied Mathematics, 146. Marcel Dekker, New York, 1991.
[56] Simon, L. Interior gradient bounds for non-uniformly elliptic equations. *Indiana Univ. Math. J.* **25** (1976), no. 9, 821–855. doi:10.1512/iumj.1976.25.25066
[57] Simon, J. Régularité de solutions de problèmes nonlinéaires. *C. R. Acad. Sci. Paris Sér. A-B* **282** (1976), no. 23, Aii, A1351–A1354.
[58] Schmidt, T. Regularity of relaxed minimizers of quasiconvex variational integrals with (p,q)-growth. *Arch. Ration. Mech. Anal.* **193** (2009), no. 2, 311–337. doi:10.1007/s00205-008-0162-0
[59] Stein, E. M. Editor's note: the differentiability of functions in $\mathbf{R}^n$. *Ann. of Math. (2)* **113** (1981), no. 2, 383–385.
[60] Šverák, V.; Yan, X. Non-Lipschitz minimizers of smooth uniformly convex variational integrals. *Proc. Natl. Acad. Sci. USA* **99** (2002), no. 24, 15269–15276. doi:10.1073/pnas.222494699
[61] Uhlenbeck, K. Regularity for a class of non-linear elliptic systems. *Acta Math.* **138** (1977), no. 3-4, 219–240. doi:10.1007/BF02392316
[62] Ural′ceva, N. N. Degenerate quasilinear elliptic systems. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **7** (1968), 184–222.
[63] Ural′tseva, N. N.; Urdaletova, A. B. The boundedness of the gradients of generalized solutions of degenerate quasilinear nonuniformly elliptic equations. *Vestnik Leningrad. Univ. Mat. Mekh. Astronom.* **1983**, vyp. 4, 50–56; translated in *Vestnik Leningrad Univ. Math.* **16** (1984), 263–270.
[64] Zhikov, V. V. Averaging of functionals of the calculus of variations and elasticity theory. *Izv. Akad. Nauk SSSR Ser. Mat.* **50** (1986), no. 4, 675–710, 877.

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Received May 2018.