Carleman estimate for the Navier-Stokes equations and applications

CARLEMAN ESTIMATE FOR THE NAVIER-STOKES EQUATIONS AND APPLICATIONS

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ABSTRACT. For linearized Navier-Stokes equations, we first derive a Carleman estimate with a regular weight function. Then we apply it to establish conditional stability for the lateral Cauchy problem and finally we prove conditional stability estimates for the the inverse source problem of determining a spatially varying divergence-free factor of a source term.

Key words. Navier-Stokes equations, Carleman estimate, inverse problem, lateral Cauchy problem.

AMS subject classifications. 35R30, 35R25, 35Q30

1. INTRODUCTION

Let $T > 0$ and $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. We deal with the following linearized Navier-Stokes equations:

$$(1.1) \qquad \begin{cases} \partial_t v(x,t) - \Delta v(x,t) + (A(x,t)\cdot\nabla)v(x,t) \\ +(v(x,t)\cdot\nabla)B(x,t) + \nabla p(x,t) = F(x,t), & (x,t)\in\Omega\times(0,T), \\ \operatorname{div} v(x,t) = 0, & (x,t)\in\Omega\times(0,T), \end{cases}$$

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where $A, B : \overline{\Omega} \times [0,T] \to \mathbb{R}^3$ are given functions. Throughout this article, let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\partial_i = \frac{\partial}{\partial x_i}$, $i = 1,2,3$, $\partial_t = \frac{\partial}{\partial t}$ and $\nabla = (\partial_1, \partial_2, \partial_3)$. By a^T we denote the transpose of a vector a under consideration, and we set

$$\operatorname{div} v = \sum_{j=1}^{3} \partial_j v_j \quad \text{for } v = (v_1, v_2, v_3)^T$$

and

$$(A \cdot \nabla)v = \left(\sum_{j=1}^{3} A_j \partial_j v_1, \sum_{j=1}^{3} A_j \partial_j v_2, \sum_{j=1}^{3} A_j \partial_j v_3 \right)^T$$

for $A = (A_1, A_2, A_3)^T$ and $v = (v_1, v_2, v_3)^T$.

Given a subboundary $\Gamma \subset \partial\Omega$ and a subdomain $\Omega_0 \subset \Omega$, we address the following two problems.

- **Continuation of solution by Cauchy data on lateral subboundary.** *Determine (v,p) in $\Omega_0 \times I$ by velocity field v and its derivatives on $\Gamma \times (0,T)$, where (v,p) satisfies* (1.1) *with $F = 0$ and I is a subinterval of $(0,T)$.*
- **Inverse source problem.** *Determine F by velocity field v and its derivatives on $\Gamma \times (0,T)$ and $v(\cdot, t_0)$ in Ω, where (v,p) satisfies* (1.1) *and t_0 is a fixed point in $(0,T)$.*

The key machinery for solving the above problems, is Carleman estimates. A pioneering paper Bukhgeim and Klibanov [5] proposed such a methodology and established the global uniqueness results for inverse problems. It turned out that their method is applicable also for proving stability estimates in determination of spatially varying functions in partial differential equations. Since then, the machinery developed in [5] has been used in many research publications on inverse problems, where physical processes are modelled by parabolic, hyperbolic or other types of evolution equations. We refer to Beilina and Klibanov [1], Bellassoued and Yamamoto [3], Imanuvilov and Yamamoto [14], [16], [17], Klibanov [19], Klibanov and Timonov [20], Yamamoto [22]. Here, we limit our references on related works. For additional literature, the reader may consult the references in the above articles.

On the other hand, there are not sufficient researches on inverse problems by a similar methodology for the Navier-Stokes equations in spite of the importance. We refer only to Imanuvilov and Yamamoto [15], Bellassoued, Imanuvilov and Yamamoto [2], Boulakia [4], Choulli, Imanuvilov, Puel and Yamamoto [6], Fabre [7], Fan, Di Cristo, Jiang and Nakamura

[8], Fan, Jiang and Nakamura [9]. See also Imanuvilov and Yamamoto [17], [18] as for inverse problems for the fluid equations.

Huang, Imanuvilov and Yamamoto [11] recently modified the method by [5] and, here, we apply the method developed in [11]. We remark that we use the same type of Carleman estimate for the Navier-Stokes equations as in [2], but the derivation of a Carleman estimate is simpler than in [2].

Thus, in this article, we aim at improving [2]. Compared to the existing results, the main achievements are:

- a more feasible Carleman estimate (Theorem 1) for the applications,
- an improvement of the conditional stability estimates for the continuation of solutions by Cauchy data,
- novel conditional stability estimates in determining force term $F(x,t)$ in view of the divergence-free component.

Now we state the key Carleman estimate, and to this end, we introduce some notations. Let $t_0 \in (0,T)$ be arbitrarily taken. We choose $\delta > 0$ such that $0 < t_0 - \delta < t_0 + \delta < T$ and we set

$$I = (t_0 - \delta,\, t_0 + \delta), \quad Q := \Omega \times I.$$

Let $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in (\mathbb{N} \setminus \{0\})^3$ and $|\gamma| = \gamma_1 + \gamma_2 + \gamma_3$. For each $k, \ell \in \mathbb{N} \cup \{0\}$, we set

$$H^{k,\ell}(Q) = \{v \in L^2(Q) : \partial_x^\gamma v \in L^2(Q),\, |\gamma| \le k,\, \partial_t^j v \in L^2(Q) \quad 0 \le j \le \ell\},$$

where $\partial_x^\gamma = \partial_1^{\gamma_1}\partial_2^{\gamma_2}\partial_3^{\gamma_3}$ for each $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in (\mathbb{N} \cup \{0\})^3$. Throughout this article, we use the following notations:

$$\operatorname{rot} v := (\partial_2 v_3 - \partial_3 v_2,\, \partial_3 v_1 - \partial_1 v_3,\, \partial_1 v_2 - \partial_2 v_1)^T, \quad \nabla_{x,t} = (\nabla,\, \partial_t),$$
$$\operatorname{rot}^0 v = v, \quad \nabla^0 v = v, \quad \nabla^0_{x,t} v = v,$$

and by $[u]_k$ with $k = 1,2,3$, we denote the k-th component of a vector $u \in \mathbb{R}^3$, and $\nabla(\operatorname{rot} v)$ means the 3×3 matrix whose (j,k)-element is $\partial_j([\operatorname{rot} v]_k)$ for a vector-valued function $v = (v_1, v_2, v_3)^T$.

Let a function $d \in C^2(\overline{\Omega})$ satisfy

$$|\nabla d| \ne 0 \quad \text{on } \overline{\Omega}. \tag{1.2}$$

Henceforth we fix a large constant $\lambda > 0$, and we omit the λ-dependency of the constants which will appear later. For arbitrarily chosen constant $\beta > 0$, we set

$$\varphi(x,t) = e^{\lambda(d(x)-\beta(t-t_0)^2)}, \quad (x,t) \in Q.$$

We here do not pursue any optimal regularity issues and accordingly we assume that for example, the $\mathbb{R}^3$-valued functions $A(x,t)$ and $B(x,t)$ in (1.1) are sufficiently smooth in order to concentrate on the inverse problems.

Then, the following is our first main result.

Theorem 1 (Carleman estimate for linearized Navier-Stokes equations). *Let a pair (v,p) satisfy (1.1) and $F, \mathrm{rot}\, F \in L^2(Q)$. There exist constants $s_0 > 0$ and $C > 0$, which are independent of s, such that*

$$\begin{aligned}
&\int_Q \Bigg\{ \frac{1}{s}(|\partial_t \mathrm{rot}\, v|^2 + |\Delta(\mathrm{rot}\, v)|^2) \\
&+ s(|\nabla(\mathrm{rot}\, v)|^2 + |\nabla v|^2 + |\Delta v|^2) + s^3(|\mathrm{rot}\, v|^2 + |v|^2)\Bigg\} e^{2s\varphi} dxdt \\
\le& C\int_Q |\mathrm{rot}\, F|^2 e^{2s\varphi} dxdt + Cs^3 \int_{\partial\Omega\times I} \sum_{j=0}^{1} (|\nabla^j_{x,t}(\mathrm{rot}\, v)|^2 + |\nabla^j v|^2) e^{2s\varphi} dSdt \\
&+ Cs^3 \sum_{\kappa=0}^{1} \int_\Omega \sum_{j=0}^{1} |\nabla^j(\mathrm{rot}\, v(x, t_0 + (-1)^\kappa \delta))|^2 e^{2s\varphi(x,t_0+\delta)} dx \qquad (1.3)
\end{aligned}$$

for all $s > s_0$ and (v,p) satisfying (1.1), $p \in L^2(I; H^2(\Omega))$ and $v, \mathrm{rot}\, v \in H^{2,1}(Q)$.

At the last term, we note that $\varphi(x, t_0 + \delta) = \varphi(x, t_0 - \delta)$, $x \in \Omega$.

Remark 1.

We can include also $\frac{1}{s}\int_Q |\nabla p|^2 e^{2s\varphi} dxdt$ on the left-hand side of (1.3), by solving the first equation in (1.1) with respect to ∇p and substituting (1.3). However, we do not need such an estimate for our purpose and we omit.

A similar Carleman estimate with the same type of weight function φ for linearized Navier-Stokes equations is derived in [2], but our derivation of the Carleman estimate is simpler. On the other hand, we can choose a different weight function with singularity at $t = 0, T$, which was created by Imanuvilov [12] for a single parabolic equation. See also Subsection 5.3 for further details. For Carleman estimates for linearized Navier-Stokes equations, we refer to Choulli, Imanuvilov, Puel and Yamamoto [6], Fan, Di Cristo, Jiang and Nakamura [8], Fan, Jiang and Nakamura [9], Fernández-Cara, Guerrero, Imanuvilov and Puel [10].

Next we formulate the main result for the continuation of solution. We consider the problem

$$\begin{cases} \partial_t v(x,t) - \Delta v + (A(x,t)\cdot\nabla)v + (v\cdot\nabla)B + \nabla p = 0, \\ \operatorname{div} v = 0, \quad (x,t) \in Q := \Omega \times I. \end{cases} \tag{1.4}$$

Then we are concerned with not only the uniqueness in the continuation of (v,p), but also the estimation of (v,p) in some subdomain by prescribed knowledge of v on $\Gamma \times (0,T)$, where $\Gamma \subset \partial\Omega$.

For an arbitrarily chosen subboundary $\Gamma \subset \partial\Omega$, we choose an arbitrary subdomain $\Omega_0 \subsetneqq \Omega$ such that

$$\overline{\Omega_0} \subset \Omega \cup \Gamma, \quad \overline{\Omega_0} \cap \partial\Omega \subset \text{ Int } (\Gamma).$$

Here $\operatorname{Int}(\Gamma)$ denotes the interior of Γ in the restriction of the Euclidean topology in $\mathbb{R}^3$ to $\partial\Omega$.

We are ready to state the main result for the continuation.

Theorem 2 (Stability of continuation of solution).
Let $(v,p) \in H^{2,1}(Q) \times L^2(0,T;H^1(\Omega))$ satisfy (1.4) and

$$\sum_{j,k=0}^{1} (\|\nabla_{x,t}^j(\operatorname{rot}\partial_t^k v)\|_{L^2(\partial\Omega\times(0,T))} + \|\nabla^j\partial_t^k v\|_{L^2(\partial\Omega\times(0,T))})$$

$$+ \sum_{k=0}^{1} \|\operatorname{rot}\partial_t^k v\|_{L^\infty(0,T;H^1(\Omega))} \le M \tag{1.5}$$

with arbitrarily fixed constant $M > 0$. Then for any $\varepsilon \in \left(0, \frac{T}{2}\right)$, there exist constants $C > 0$ and $\theta \in (0,1)$, which depend on $M, \varepsilon, \Gamma, \Omega_0$, such that

$$\sum_{\ell=0}^{1}(\|\Delta\operatorname{rot}^{\ell} v\|_{H^{0,1}(\Omega_0\times(\varepsilon,T-\varepsilon))} + \|\operatorname{rot}^{\ell} v\|_{H^1(\varepsilon,T-\varepsilon;H^1(\Omega_0))}) + \|\operatorname{rot} v\|_{H^{0,2}(\Omega_0\times(\varepsilon,T-\varepsilon))}$$

$$+\|\nabla p\|_{L^2(\Omega_0\times(\varepsilon,T-\varepsilon))} \le C(D(v) + D(v)^\theta),$$

where

$$D(v) = \sum_{j,k=0}^{1} (\|\nabla_{x,t}^j(\operatorname{rot}\partial_t^k v)\|_{L^2(\Gamma\times(0,T))} + \|\nabla^j\partial_t^k v\|_{L^2(\Gamma\times(0,T))}). \tag{1.6}$$

This theorem asserts stability in determining a solution (v,p) in some subdomain by boundary data of v on $\Gamma \times (0,T)$ under a priori bound (1.5).

A similar conditional stability for the continuation for linearized Navier-Stokes equations was proved in [2], but our proof is simplified by adapting the proof of Proposition 2 in Huang, Imanuvilov and Yamamoto [11] which is concerned with a single parabolic equation. In [11], differently from the conventional arguments by Carleman estimates, we do not use any cut-off function $\chi(x,t)$ which is compactly supported, and do not need to consider the function $\chi(x,t)v(x,t)$, but we argue directly on the function $v(x,t)$. In the case of the Navier-Stokes equations, the cut-off function destroys the original structure of the equations, that is, the equation $\mathrm{div}\,(\chi v) = 0$ does not hold. Consequently, if we introduce a cut-off function, then we have to prove a Carleman estimate for the solutions to linearized Navier-Stokes equations without $\mathrm{div}\, v = 0$. This causes an additional difficulty (see [2]). Since we do not here require any cut-off argument at all, the proofs of Theorems 1 - 3 are simplified.

Finally we state the main result for the inverse source problem. For it, we introduce the following three conditions on F:

$$\mathrm{div}\, F(x,t_0) = 0 \quad \text{for all } x \in \Omega. \tag{1.7}$$

There exists a constant $C > 0$ such that

$$|\partial_t^k \mathrm{rot}\, F(x,t)| \le C|\mathrm{rot}\, F(x,t_0)| \quad \text{for all } (x,t) \in Q \text{ and } k = 0,1. \tag{1.8}$$

There exists a constant $C > 0$ such that

$$|\partial_t^k \mathrm{rot}\, F(x,t)| \le C(|\nabla F(x,t_0)| + |F(x,t_0)|) \quad \text{for all } (x,t) \in Q \text{ and } k = 0,1,2. \tag{1.9}$$

Here $t_0 \in (0,T)$ is arbitrarily fixed.

We notice that, by right-hand side of (1.9), we can estimate the right-hand side of (1.8) from above if we change the constant $C > 0$ properly.

For example, in a special case where $F(x,t)$ is expressed by $R(x,t)f(x)$ with a sufficiently smooth 3×3 matrix-valued function R and a smooth $\mathbb{R}^3$-valued function f, we see that condition (1.8) implies (1.9), that is, condition (1.9) is more generous than (1.8).

We are ready to state two conditional stability estimates according to choices among (1.7) - (1.9).

Theorem 3 (Inverse source problem).

Let $\Gamma \subset \partial\Omega$ be an arbitrary subboundary and $\Omega_0 \subset \Omega$ be a subdomain satisfying the conditions $\overline{\Omega_0} \subset \Omega \cup \Gamma$ and $\overline{\Omega_0} \cap \partial\Omega \subset Int(\Gamma)$, and let $t_0 \in (0,T)$.

(i) We set

$$\mathcal{F}_1 := \{F \in L^2(Q);\ F, \mathrm{rot}\, F, \mathrm{div}\, F \in H^1(I; L^2(\Omega)),\ \textit{(1.8) is satisfied}\}$$

and

$$D_1(v) := \sum_{j,k=0}^{1} (\|\nabla_{x,t}^j \partial_t^k (\mathrm{rot}\, v)\|_{L^2(\Gamma\times I)} + \|\nabla^j \partial_t^k v\|_{L^2(\Gamma\times I)}) + \|v(\cdot, t_0)\|_{H^3(\Omega)}. \tag{1.10}$$

Let M_1 be an arbitrarily given positive constant and let $\mathcal{V}_{M_1}$ be the set of all the functions $(v,p,F) \in H^{2,1}(Q) \times H^{1,0}(Q) \times \mathcal{F}_1$ which satisfy (1.1), the regularity conditions $\partial_t \mathrm{rot}^{\,j} v \in H^{2,1}(Q)$ *with $j = 0, 1$, $\partial_t p \in H^{1,0}(Q)$ and*

$$E(v) := \sum_{j,k=0}^{1} (\|\nabla_{x,t}^j \partial_t^k (\mathrm{rot}\, v)\|_{L^2(\partial\Omega\times I)} + \|\nabla^j \partial_t^k v\|_{L^2(\partial\Omega\times I)})$$

$$+ \sum_{\kappa=0}^{1} \sum_{k=0}^{1} \|\partial_t^k \mathrm{rot}\, v(\cdot,\ t_0 + (-1)^\kappa \delta)\|_{H^1(\Omega)} \le M_1. \tag{1.11}$$

Then there exist constants $C > 0$ and $\theta \in (0,1)$, dependent on Ω_0, Γ, M_1, such that

$$\|\mathrm{rot}\, F(\cdot, t_0)\|_{L^2(\Omega_0)} \le C(D_1(v)^\theta + D_1(v)) \quad \text{for all } (v,p,F) \in \mathcal{V}_{M_1}.$$

(ii) Let $m_1 > 0$ be an arbitrarily chosen constant. We set

$$\mathcal{F}_2 := \{F \in H^2(I; H^2(\Omega));\ \|F(\cdot,t_0)\|_{H^2(\Omega)} \le m_1, \quad F(\cdot,t_0) = |\nabla F(\cdot,t_0)| = 0 \quad \text{on } \Gamma,$$
$$\textit{(1.7) and (1.9) are satisfied}\}$$

and

$$D_2(v) := \sum_{j=0}^{1} \sum_{k=0}^{2} (\|\nabla_{x,t}^j \partial_t^k (\mathrm{rot}\, v)\|_{L^2(\Gamma\times I)} + \|\nabla^j \partial_t^k v\|_{L^2(\Gamma\times I)}) + \|v(\cdot, t_0)\|_{H^4(\Omega)}. \tag{1.12}$$

Let M_2 be an arbitrarily given positive constant and let $\mathcal{W}_{M_2}$ be the set of all the functions $(v,p,F) \in H^{2,1}(Q) \times H^{1,0}(Q) \times \mathcal{F}_2$ which satisfy (1.1), the regularity conditions $\partial_t^k \mathrm{rot}^{\,j} v \in H^{2,1}(Q)$, *$\partial_t^k p \in H^{1,0}(Q)$ with $j = 0, 1$ and $k = 0, 1, 2$ and the a priori bound*

$$E(v) + E(\partial_t v) \le M_2. \tag{1.13}$$

Then there exist constants $C > 0$ and $\theta \in (0,1)$, dependent on $\Omega_0, \Gamma, m_1, M_2$, such that

$$\|F(\cdot, t_0)\|_{H^1(\Omega_0)} \le C(D_2(v)^\theta + D_2(v)) \quad \textit{for all } (v,p,F) \in \mathcal{W}_{M_2}.$$

We can rewrite (1.13) by

$$\sum_{j=0}^{1}\sum_{k=0}^{2}(\|\nabla^j_{x,t}\partial_t^k \mathrm{rot}\, v\|_{L^2(\partial\Omega\times I)} + \|\nabla^j\partial_t^k v\|_{L^2(\partial\Omega\times I)})$$
$$+\sum_{\kappa=0}^{1}\sum_{k=0}^{2}\|\partial_t^k \mathrm{rot}\, v(\cdot,\, t_0 + (-1)^\kappa\delta)\|_{H^1(\Omega)} \le M_2.$$

Theorem 3 asserts two kinds of conditional stability according to the admissible sets $\mathcal{F}_1$ and $\mathcal{F}_2$. Theorem 3 (i) determines only the rotation component of $F(x,t_0)$ and so it is not necessary to assume (1.7).

In Theorem 3, not assuming the boundary condition for v on the whole boundary of Ω, we establish stability in determining F in a subdomain of Ω. On the other hand, the stability estimate over Ω is proved with the boundary condition of v on the whole $\partial\Omega\times(0,T)$ in Choulli, Imanuvilov, Puel and Yamamoto [6] for the Navier-Stokes equations, and Fan, Jiang and Nakamura [9] for the Boussinesq equations.

Next we illustrate conditions (1.7) - (1.9). First, as the following example shows, we stress that the condition (1.7) is essential for the uniqueness in determining $F(x,t_0)$ in Theorem 3 (ii), if no data of p are observed.

Obstruction to the uniqueness in determining F:

We consider a simple case

$$\begin{cases} \partial_t v(x,t) - \Delta v + \nabla p = F(x), \\ \mathrm{div}\, v = 0, \quad (x,t) \in Q := \Omega \times I, \\ v(x,t_0) = 0 \quad \text{for } x \in \Omega, \quad \mathrm{supp}\ v \subset \Omega \times (0,T). \end{cases} \tag{1.14}$$

Here F is an $\mathbb{R}^3$-valued smooth function. It is trivial that $(v,p) = (0,0)$ satisfies (1.14) with $F = 0$. Let $\psi \in C_0^\infty(\Omega)$ satisfy $\nabla\psi \not\equiv 0$ in Ω. Then $(v,p) := (0,\, \psi)$ satisfies (1.14) with $F := \nabla\psi$. Since the velocity fields in both cases are identically equal to zero, for our inverse problem, the two distinct solutions $(v, p) = (0, 0)$ and $(v, p) = (0, \psi)$ generate the same data D_1 and D_2 as are given in Theorem 3. However, the corresponding sources are different, that is, 0 and $\nabla\psi \not\equiv 0$. Therefore, by the presence of the pressure field p in the Navier-Stokes equations, there is no possibility for the uniqueness in determining the component of F given by a scalar potential.

Next we provide several examples of the right-hand side F satisfying conditions (1.7) - (1.9).

Example 1. We see that (1.7) is replaced in terms of a vector potential:

$$F(x, t) := \mathrm{rot}\, q(x, t), \quad (x, t) \in Q,$$

where $q = q(x, t)$ is sufficiently smooth. Then (1.7) is automatically satisfied. Moreover, we see that (1.9) is equivalent to

$$|\mathrm{rot}\,\mathrm{rot}\,(\partial_t^k q(x, t))| \le C(|\nabla \mathrm{rot}\, q(x, t_0)| + |\mathrm{rot}\, q(x, t_0)|), \quad (x, t) \in Q,\ k = 0, 1, 2.$$

Example 2.

Let

$$F(x, t) = r(t)f(x), \quad x \in \Omega,\ 0 < t < T, \quad r(t_0) \ne 0, \quad \mathrm{div}\, f = 0 \quad \text{in } \Omega,$$

where $r \in C^2([0, T])$ is real-valued, $f = (f_1, f_2, f_3)^T$ and $f_1, f_2, f_3 \in C^1(\overline{\Omega})$. Then (1.7) and (1.8) are satisfied. Indeed we can directly verify $\mathrm{div}\, F(x, t_0) = r(t_0)\mathrm{div}\, f(x) = 0$ for $x \in \Omega$, which is (1.7). Moreover, we have

$$|\partial_t^k \mathrm{rot}\, F(x, t)| = \left| \frac{d^k r}{dt^k}(t) \mathrm{rot}\, f(x) \right| \le C|\mathrm{rot}\, f(x)|, \quad (x, t) \in Q, \quad k = 0, 1$$

and

$$|\mathrm{rot}\, F(x, t_0)| = |r(t_0)||\mathrm{rot}\, f(x)|, \quad (x, t) \in Q,$$

which is (1.8).

Example 3.

For more general source F unlike Example 2, it is not always easy to verify condition (1.8), but in the following case the condition (1.9) can be readily verified. Let

$$F(x, t) = R(x, t)f(x),$$

where $R(x,t) = (r_{jk}(x,t))_{1\le j,k\le 3}$ is a given matrix and each component r_{jk} is smooth in (x,t), and $f(x) = (f_1(x), f_2(x), f_3(x))^T$. Then, condition (1.9) is satisfied if there exists a constant $\delta_1 > 0$ such that

$$|\det R(x,t_0)| \ge \delta_1 > 0 \quad \text{for } x \in \overline{\Omega}.$$

Indeed, we have

$$|\partial_t^k \mathrm{rot}\, F(x,t)| \le C(|\nabla f(x)| + |f(x)|), \quad (x,t) \in \overline{Q}, \quad k = 0,1,2.$$

Therefore it is sufficient to verify

$$|\nabla f(x)| + |f(x)| \le C(|\nabla F(x,t_0)| + |F(x,t_0)|), \quad x \in \Omega. \tag{1.15}$$

Verification of (1.15). Since $|\det R(x,t_0)| \ne 0$ for $x \in \overline{\Omega}$, it follows that

$$|f(x)| \le C|F(x,t_0)|, \quad x \in \overline{\Omega}.$$

Next, setting $F = (F_1, F_2, F_3)^T$, we have

$$\partial_\ell F_k(x,t_0) = \sum_{j=1}^3 r_{kj}(x,t_0)\partial_\ell f_j(x) + \sum_{j=1}^3 (\partial_\ell r_{kj})(x,t_0) f_j(x), \quad \ell = 1,2,3,$$

and so

$$\sum_{j=1}^3 r_{kj}(x,t_0)\partial_\ell f_j(x) = \partial_\ell F_k(x,t_0) - \sum_{j=1}^3 (\partial_\ell r_{kj}(x,t_0)) f_j(x),$$

that is,

$$R(x,t_0)\partial_\ell f(x) = \partial_\ell F(x,t_0) - (\partial_\ell R(x,t_0)) f(x).$$

Therefore, by $|\det R(x,t_0)| \ge \delta_1 > 0$ for all $x \in \overline{\Omega}$, we deduce

$$|\partial_\ell f(x)| \le C(|\partial_\ell F(x,t_0)| + |F(x,t_0)|), \quad x \in \overline{\Omega}, \quad 1 \le \ell \le 3.$$

Thus (1.15) is verified.

If $F(x,t) = R(x,t)f(x)$ for $(x,t) \in Q$ with a 3×3 matrix $R(x,t)$, then we do not know a convenient sufficient condition for (1.8), but condition (1.9) can be more directly verified. Therefore, Theorem 3 (ii) is more feasible in view of applications.

Remark 2.

A special case $F(x,t) = f(x)r(x,t)$, where f is real-valued and r is $\mathbb{R}^3$-valued, is trivial by

(1.7), since (1.7) means that f satisfies a first-order transport equation $r(x,t_0)\cdot\nabla f(x) + f(x)\mathrm{div}\, r(x,t_0) = 0$ for $x\in\Omega$, and so f can be uniquely determined by additional boundary data of f under a certain condition on $r(x,t_0)$.

This article is composed of six sections. In Sections 2-4, we prove Theorems 1-3 respectively. Section 5 is devoted to concluding remarks and in Section 6, we derive Lemma 2 which is used for the proof of Theorem 1.

2. Proof of Theorem 1

Thanks to the large parameter $s>0$, it suffices to prove Theorem 1 for $B=0$. Indeed, suppose that the assertion (1.3) is true when $B=0$. Then, writing (1.1) with F being replaced by $F-(v\cdot\nabla)B$, and observing that

$$\mathrm{rot}\,(F-(v\cdot\nabla)B) = \mathrm{rot}\,F + G(\nabla v, v),$$

where $G(\nabla v, v)$ is a linear combination of $\nabla v, v$ with bounded coefficients in Q, we can easily show that

$$\int_Q |\mathrm{rot}\,(F-(v\cdot\nabla)B)|^2 e^{2s\varphi}dxdt \le C\int_Q |\mathrm{rot}\,F|^2 e^{2s\varphi}dxdt + C\int_Q (|v|^2+|\nabla v|^2)e^{2s\varphi}dxdt.$$

Hence, we obtain (1.3) with the extra term $C\int_Q(|v|^2+|\nabla v|^2)e^{2s\varphi}dxdt$ on its right-hand side. Taking s_0 larger, if needed, we can absorb this term on the left-hand side and obtain (1.3) in the case of $B\not\equiv 0$.

Now we proceed to the proof of Theorem 1. We set

$$z := \mathrm{rot}\, v.$$

Then, since $\mathrm{rot}\,\mathrm{rot}\, v = -\Delta v + \nabla(\mathrm{div}\, v) = -\Delta v$ by $\mathrm{div}\, v = 0$ in Q, system (1.1) is written as

$$\begin{cases} \partial_t z(x,t) - \Delta z + (A(x,t)\cdot\nabla)z = \mathrm{rot}\,F - \sum_{j=1}^3 \nabla A_j \times \partial_j v, \\ \Delta v = -\mathrm{rot}\, z, \quad x\in\Omega,\ 0<t<T. \end{cases} \tag{2.1}$$

The system (2.1) provides a decomposition of the Navier-Stokes equations into a parabolic equation in $z := \mathrm{rot}\, v$ and an elliptic equation in v, where p is eliminated.

For proving Theorem 1, we combine two Carleman estimates for a simple parabolic operator $\partial_t - \Delta$ and the Laplacian. We state such Carleman estimates. Let $\varphi = \varphi(x,t)$ be the same weight function as in Theorem 1. Henceforth $C > 0$ denotes generic constants which are independent of $s > 0$.

First we show a Carleman estimate for $\partial_t - \Delta$ in Q.

Lemma 1.

Let the function φ satisfies all the conditions of Theorem 1. There exist constants $C > 0$ and $s_0 > 0$ such that

$$\begin{aligned}
&\int_Q \left\{ \frac{1}{s}(|\partial_t u|^2 + |\Delta u|^2) + s|\nabla u|^2 + s^3|u|^2 \right\} e^{2s\varphi} dxdt \\
\leq &C \int_Q |\partial_t u - \Delta u|^2 e^{2s\varphi} dxdt + Cs^3 \int_{\partial\Omega\times I} (|\nabla_{x,t} u|^2 + |u|^2) e^{2s\varphi} dSdt \\
+&Cs^3 \sum_{\kappa=0}^{1} \int_\Omega (|\nabla u(x, t_0 + (-1)^\kappa \delta)|^2 + |u(x, t_0 + (-1)^\kappa \delta)|^2) e^{2s\varphi(x,t_0+\delta)} dx
\end{aligned}$$

for all $s > s_0$ and all $u \in H^{2,1}(Q)$.

Lemma 1 is a standard Carleman estimate for a single parabolic equation, and a direct proof can be found for example in Bellassoued and Yamamoto [3] (Lemma 7.1), Yamamoto [22] (Theorems 3.1 and 3.2).

Next we state two Carleman estimates for the Laplace operator.

Lemma 2.

Suppose that the function φ satisfies all the conditions of Theorem 1. Let $r \in L^2(I; H^2(\Omega))$ satisfy

$$-\Delta r(x,t) = g(x,t) \quad \textit{in } Q.$$

Then there exist constants $C > 0$ and $s_0 > 0$ such that

$$\begin{aligned}
&\int_Q (s|\nabla r(x,t)|^2 + s^3|r(x,t)|^2) e^{2s\varphi(x,t)} dxdt \\
\leq &C \int_Q |g(x,t)|^2 e^{2s\varphi(x,t)} dxdt + Cs^3 \int_{\partial\Omega\times I} (|\nabla r|^2 + |r|^2) e^{2s\varphi(x,t)} dSdt
\end{aligned}$$

for all $s > s_0$.

Lemma 3.

Suppose that the function φ satisfies all the conditions of Theorem 1. Let $t_0 \in (0,T)$ and let $w \in H^2(\Omega)$ satisfy

$$-\Delta w(x) = h(x), \quad x \in \Omega.$$

Then there exist constants $C > 0$ and $s_1 > 0$ such that

$$\begin{aligned}&\int_\Omega (s|\nabla w(x)|^2 + s^3|w(x)|^2)e^{2s\varphi(x,t_0)}dx\\ \le& C\int_\Omega |h(x)|^2 e^{2s\varphi(x,t_0)}dx + Cs^3\int_{\partial\Omega}(|\nabla w(x)|^2 + |w(x)|^2)e^{2s\varphi(x,t_0)}dS\end{aligned}$$

for all $s > s_1$.

The proof of Lemma 3 is standard and can be executed directly by integration by parts (e.g., Lemma 7.1 in [3]). Lemma 3 is used also for the proof of Theorem 3 in Section 4.

Lemma 2 is proved by integrating the standard elliptic Carleman estimate in Lemma 3 over the time interval I. For completeness, the derivation of Lemma 2 from Lemma 3 is provided in Appendix.

Now we complete the proof of Theorem 1. We apply Lemmata 1 and 2 to system (2.1), and obtain

$$\begin{aligned}&\int_Q \left\{\frac{1}{s}(|\partial_t z|^2 + |\Delta z|^2) + s|\nabla z|^2 + s^3|z|^2\right\} e^{2s\varphi}dxdt\\ \le& C\int_Q \left|\mathrm{rot}\, F - \sum_{j=1}^3 A_j \times (\partial_j v) - (A\cdot\nabla)z\right|^2 e^{2s\varphi}dxdt + CJ_1\end{aligned}$$

$$\le C\int_Q |\mathrm{rot}\, F|^2 e^{2s\varphi}dxdt + C\int_Q (|\nabla v|^2 + |v|^2)e^{2s\varphi}dxdt + C\int_Q |\nabla z|^2 e^{2s\varphi}dxdt + CJ_1 \quad (2.2)$$

and

$$\int_Q (s|\nabla v|^2 + s^3|v|^2)e^{2s\varphi}dxdt \le C\int_Q |\mathrm{rot}\, z|^2 e^{2s\varphi}dxdt + CJ_2. \quad (2.3)$$

Here and henceforth we set

$$\begin{aligned}J_1 :=& s^3\int_{\partial\Omega\times I}(|\nabla_{x,t}(\mathrm{rot}\, v)|^2 + |\mathrm{rot}\, v|^2)e^{2s\varphi}dSdt\\ &+s^3\sum_{\kappa=0}^1\int_\Omega (|\nabla(\mathrm{rot}\, v(x,t_0 + (-1)^\kappa\delta))|^2 + |\mathrm{rot}\, v(x,t_0+(-1)^\kappa\delta)|^2)e^{2s\varphi(x,t_0+\delta)}dx\end{aligned}$$

and

$$J_2 := s^3 \int_{\partial\Omega\times I} (|\nabla v|^2 + |v|^2) e^{2s\varphi} dSdt.$$

Hence, substituting (2.3) into the second term on the right-hand side of (2.2) we obtain

$$\begin{aligned}
&\int_Q \left\{ \frac{1}{s}(|\partial_t z|^2 + |\Delta z|^2) + s|\nabla z|^2 + s^3|z|^2 \right\} e^{2s\varphi} dxdt \\
\le& C\int_Q |\mathrm{rot}\, F|^2 e^{2s\varphi} dxdt + C\int_Q |\mathrm{rot}\, z|^2 e^{2s\varphi} dxdt + C\int_Q |\nabla z|^2 e^{2s\varphi} dxdt \\
&+ C(J_1 + J_2). \qquad (2.4)
\end{aligned}$$

Absorbing the second and the third terms on the right-hand side into the left-hand side of (2.4), we reach

$$\int_Q \left\{ \frac{1}{s}(|\partial_t z|^2 + |\Delta z|^2) + s|\nabla z|^2 + s^3|z|^2 \right\} e^{2s\varphi} dxdt \le C\int_Q |\mathrm{rot}\, F|^2 e^{2s\varphi} dxdt + C(J_1 + J_2). \qquad (2.5)$$

Combining (2.3) and (2.5), and using the equation $\Delta v = -\mathrm{rot}\, z$ in (2.1), we see that $|\Delta v| \le C|\nabla z|$ in Q, and we complete the proof of Theorem 1. ■

3. Proof of Theorem 2

Theorem 1 holds true with a weight function $\varphi(x,t) = e^{\lambda(d(x)-\beta(t-t_0)^2)}$ where $d \in C^2(\overline{\Omega})$ is an arbitrary function satisfying (1.2). However, in order to apply Theorem 1 for the proof of Theorems 2 and 3, we have to choose $d(x)$ closely related to the geometry of Γ as follows. First we construct a suitable domain Ω_1 which contains Ω. For a relatively open subset $\Gamma \subset \partial\Omega$, we choose a bounded domain Ω_1 with smooth boundary such that

$$\Omega \subset \Omega_1, \quad \overline{\Gamma} = \overline{\partial\Omega \cap \Omega_1}, \quad \partial\Omega_1 = (\partial\Omega_1 \setminus \partial\Omega) \cup (\partial\Omega \setminus \Gamma).$$

In particular, $\Omega_1 \setminus \overline{\Omega}$ contains some non-empty open subset. We note that Ω_1 can be constructed as the interior of a union of $\overline{\Omega}$ and the closure of a non-empty domain $\widehat{\Omega}$ satisfying the condition $\widehat{\Omega} \subset \mathbb{R}^n \setminus \overline{\Omega}$ and $\partial\widehat{\Omega} \cap \partial\Omega = \overline{\Gamma}$. We remark that we have to connect smooth subboundaries $\partial\Omega_1 \setminus \partial\Omega$ and $\partial\Omega \setminus \Gamma$ smoothly in $\partial\Gamma$, in order that $\partial\Omega_1$ is smooth.

We choose a domain ω such that $\overline{\omega} \subset \Omega_1 \setminus \overline{\Omega}$. Then, thanks to Imanuvilov [12], we can find $d \in C^2(\overline{\Omega_1})$ such that

$$d > 0 \quad \text{in } \Omega_1, \quad |\nabla d| > 0 \quad \text{on } \overline{\Omega_1 \setminus \omega}, \quad d = 0 \quad \text{on } \partial\Omega_1. \qquad (3.1)$$

We recall that a domain $\Omega_0 \subset \Omega$ satisfies $\overline{\partial\Omega_0 \cap \partial\Omega} \subset \mathrm{Int}(\Gamma)$ and $\overline{\Omega_0} \subset \Omega \cup \Gamma$.

Then, by $\overline{\Omega_0} \subset \Omega_1$ and $\partial\Omega \setminus \Gamma \subset \partial\Omega_1$, we see

$$d > 0 \quad \text{on } \overline{\Omega_0}, \quad d = 0 \quad \text{on } \partial\Omega \setminus \Gamma. \tag{3.2}$$

Henceforth we fix d satisfying (3.1) and a sufficiently large constants $\lambda > 0$ and define $\varphi(x,t) = e^{\lambda(d(x)-\beta(t-t_0)^2)}$ in Theorem 1. Later by (3.15), we choose a constant $\beta > 0$.

For given $t_0 \in (0,T)$ and $\delta_2 > 0$ satisfying $0 < t_0 - \delta_2 < t_0 + \delta_2 < T$, we apply the Carleman estimate (1.3) proved in Theorem 1 in the domain $\Omega \times (t_0 - \delta_2, t_0 + \delta_2)$ to system (1.4), so that we obtain

$$\begin{aligned}
&\int_{\Omega\times(t_0-\delta_2,t_0+\delta_2)} (|\mathrm{rot}\,\partial_t v|^2 + |\nabla(\mathrm{rot}\,v)|^2 + |\nabla v|^2 \\
&+|\Delta(\mathrm{rot}\,v)|^2 + |\Delta v|^2 + |\mathrm{rot}\,v|^2 + |v|^2)e^{2s\varphi}dxdt \\
&\le Cs^4 \int_{\partial\Omega\times(t_0-\delta_2,t_0+\delta_2)} \sum_{j=0}^{1} (|\nabla_{x,t}^j(\mathrm{rot}\,v)|^2 + |\nabla^j v|^2)e^{2s\varphi}dSdt \\
&+Cs^4 \sum_{\kappa=0}^{1} \int_\Omega \sum_{j=0}^{1} |\nabla^j(\mathrm{rot}\,v(x,t_0+(-1)^\kappa\delta_2))|^2 e^{2s\varphi(x,t_0+\delta_2)}dx
\end{aligned} \tag{3.3}$$

for all $s > s_0$.

Here we note that the constants $C > 0$ and $s_0 > 0$ are independent of t_0 because the Carleman estimate is invariant by the translation in time provided that the translated time interval is included in $(0,T)$.

Setting $v_1 := \partial_t v$, we have

$$\begin{cases}
& \partial_t v_1(x,t) - \Delta v_1 + (A(x,t)\cdot\nabla)v_1 + (v_1\cdot\nabla)B(x,t) + \nabla\partial_t p \\
= & -(\partial_t A\cdot\nabla)v - (v\cdot\nabla)\partial_t B, \\
& \mathrm{div}\,v_1(x,t) = 0, \quad x\in\Omega,\ 0<t<T.
\end{cases} \tag{3.4}$$

Here we have

$$\mathrm{rot}\,((\partial_t A\cdot\nabla)v) = (\partial_t A\cdot\nabla)\mathrm{rot}\,v \tag{3.5}$$
$$+\left(\sum_{j=1}^3(\partial_2 a_j)\partial_j v_3 - (\partial_3 a_j)\partial_j v_2,\ \sum_{j=1}^3(\partial_3 a_j)\partial_j v_1 - (\partial_1 a_j)\partial_j v_3,\ \sum_{j=1}^3(\partial_1 a_j)\partial_j v_2 - (\partial_2 a_j)\partial_j v_1\right)^T,$$

where we set $\partial_t A := (a_1, a_2, a_3)^T$. By (3.3) and (3.5), we have

$$\begin{aligned}
&\int_{\Omega\times(t_0-\delta_2,t_0+\delta_2)} (|\mathrm{rot}\,((\partial_t A\cdot\nabla)v)|^2 + |\mathrm{rot}\,((v\cdot\nabla)\partial_t B)|^2)e^{2s\varphi}dxdt\\
\le& C\int_{\Omega\times(t_0-\delta_2,t_0+\delta_2)} (|\nabla(\mathrm{rot}\,v)|^2 + |\nabla v|^2 + |v|^2)e^{2s\varphi}dxdt\\
\le& Cs^4\int_{\partial\Omega\times(t_0-\delta_2,t_0+\delta_2)} \sum_{j=0}^{1}(|\nabla^j_{x,t}(\mathrm{rot}\,v)|^2 + |\nabla^j v|^2)e^{2s\varphi}dSdt\\
+& Cs^4\sum_{\kappa=0}^{1}\int_\Omega\sum_{j=0}^{1}|\nabla^j(\mathrm{rot}\,v(x,t_0+(-1)^\kappa\delta_2))|^2e^{2s\varphi(x,t_0+\delta_2)}dx
\end{aligned}$$

for all $s > s_0$. Therefore, we apply Theorem 1 to system (3.4) to obtain

$$\begin{aligned}
&\int_{\Omega\times(t_0-\delta_2,t_0+\delta_2)} (|\partial_t^2\mathrm{rot}\,v|^2 + |\partial_t\mathrm{rot}\,v|^2 + |\nabla(\mathrm{rot}\,\partial_t v)|^2 + |\nabla\partial_t v|^2\\
+&|\Delta\partial_t v|^2 + |\Delta(\mathrm{rot}\,\partial_t v)|^2 + |\partial_t v|^2)e^{2s\varphi}dxdt\\
\le& Cs^4\int_{\partial\Omega\times(t_0-\delta_2,t_0+\delta_2)} \sum_{j=0}^{1}(|\nabla^j_{x,t}(\mathrm{rot}\,\partial_t v)|^2 + |\nabla^j\partial_t v|^2)e^{2s\varphi}dSdt\\
&+Cs^4\sum_{\kappa=0}^{1}\int_\Omega\sum_{j=0}^{1}|\nabla^j(\mathrm{rot}\,\partial_t v(x,t_0+(-1)^\kappa\delta_2)|^2e^{2s\varphi(x,t_0+\delta_2)}dx
\end{aligned} \tag{3.6}$$

for all $s > s_0$. Hence, applying (3.3) and (3.6), we obtain

$$\begin{aligned}
&\int_{\Omega\times(t_0-\delta_2,t_0+\delta_2)} \left(\sum_{k,\ell=0}^{1}(|\Delta(\mathrm{rot}\,^\ell\partial_t^k v)|^2 + |\nabla^\ell\partial_t^k v|^2 + |\nabla^\ell(\mathrm{rot}\,\partial_t^k v)|^2) + |\partial_t^2\mathrm{rot}\,v|^2\right)e^{2s\varphi}dxdt\\
\le& Cs^4\int_{\partial\Omega\times(t_0-\delta_2,t_0+\delta_2)} \sum_{j,k=0}^{1}(|\nabla^j_{x,t}\partial_t^k\mathrm{rot}\,v|^2 + |\nabla^j\partial_t^k v|^2)e^{2s\varphi}dSdt\\
&+Cs^4\sum_{\kappa=0}^{1}\int_\Omega\sum_{j,k=0}^{1}|\nabla^j(\mathrm{rot}\,\partial_t^k v(x,t_0+(-1)^\kappa\delta_2))|^2e^{2s\varphi(x,t_0+\delta_2)}dx
\end{aligned} \tag{3.7}$$

for all large $s > 0$.

We set

$$d_0 = \min_{x\in\overline{\Omega_0}} d(x), \quad d_1 = \max_{x\in\overline{\Omega}} d(x). \tag{3.8}$$

By $d > 0$ in Ω, we notice that $d_1 = \|d\|_{C(\overline{\Omega})}$ and $d_0 > 0$.

We fix $\widetilde{\varepsilon} > 0$ such that $0 < \widetilde{\varepsilon} < \delta_2$. Here we recall that $t_0 \in (0,T)$ and $\delta_2 > 0$ satisfies $0 < t_0 - \delta_2 < t_0 + \delta_2 < T$. Then, by (3.2), we have

$$\begin{cases} \max_{x\in\overline{\partial\Omega\setminus\Gamma}, t_0-\delta_2\le t\le t_0+\delta_2} \varphi(x,t) \le 1, \\ \max_{x\in\overline{\Omega}} \varphi(x, t_0-\delta_2) = \max_{x\in\overline{\Omega}} \varphi(x, t_0+\delta_2) \le e^{\lambda(d_1-\beta\delta_2^2)}, \\ \min_{x\in\overline{\Omega_0}, t_0-\widetilde{\varepsilon}\le t\le t_0+\widetilde{\varepsilon}} \varphi(x,t) \ge e^{\lambda(d_0-\beta\widetilde{\varepsilon}^2)}. \end{cases} \tag{3.9}$$

For concise descriptions, we set

$$\begin{aligned} &N(v; t_0-\widetilde{\varepsilon}, t_0+\widetilde{\varepsilon}) \\ :=& \sum_{\ell=0}^{1} (\|\Delta \mathrm{rot}^{\,\ell} v\|^2_{H^{0,1}(\Omega_0\times(t_0-\widetilde{\varepsilon},t_0+\widetilde{\varepsilon}))} + \|\mathrm{rot}^{\,\ell} v\|^2_{H^1(t_0-\widetilde{\varepsilon},t_0+\widetilde{\varepsilon};H^1(\Omega_0))}) + \|\mathrm{rot}\, v\|^2_{H^{0,2}(\Omega_0\times(t_0-\widetilde{\varepsilon},t_0+\widetilde{\varepsilon}))}. \end{aligned}$$

Therefore, shrinking the integral domain $\Omega \times (t_0-\delta_2, t_0+\delta_2)$ to $\Omega_0 \times (t_0-\widetilde{\varepsilon}, t_0+\widetilde{\varepsilon})$ on the left-hand side of (3.7), in terms of (1.5) and (3.9), we obtain

$$\exp(2se^{\lambda(d_0-\beta\widetilde{\varepsilon}^2)}) N(v; t_0-\widetilde{\varepsilon}, t_0+\widetilde{\varepsilon}) \tag{3.10}$$

$$\begin{aligned} \le& Cs^4 \left(\int_{\Gamma\times(t_0-\delta_2,t_0+\delta_2)} + \int_{(\partial\Omega\setminus\Gamma)\times(t_0-\delta_2,t_0+\delta_2)} \right) \sum_{j,k=0}^{1} (|\nabla^j_{x,t}(\partial_t^k \mathrm{rot}\, v)|^2 + |\nabla^j \partial_t^k v|^2) e^{2s\varphi} dSdt \\ &+ Cs^4 \sum_{\kappa=0}^{1} \int_\Omega \sum_{j,k=0}^{1} |\nabla^j \partial_t^k \mathrm{rot}\, v(x, t_0+(-1)^\kappa \delta_2)|^2 e^{2s\varphi(x,t_0+\delta_2)} dx \\ \le& Cs^4 e^{Cs} D(v)^2 + Cs^4 e^{2s} M^2 + Cs^4 M^2 \exp(2se^{\lambda(d_1-\beta\delta_2^2)}) \end{aligned}$$

for all $s \ge s_0$ and $\beta > 0$. Here M and $D(v)$ are defined by (1.5) and (1.6). Constant C in (3.11) is independent of s but might be dependent on β.

Now we will make a specific choice of $\beta > 0$, $\widetilde{\varepsilon} > 0$ and $\delta_2 > 0$ such that $0 < \widetilde{\varepsilon} < \delta_2$. First fix sufficiently large $N > 1$ such that

$$N - 1 > \frac{d_1 - d_0}{d_0}, \quad \text{that is,} \quad \frac{d_1}{d_0} < N. \tag{3.11}$$

For given $\varepsilon > 0$ in the statement of the theorem, we set

$$\widetilde{\varepsilon} := \frac{\varepsilon}{N-1}, \quad \delta_2 := N\widetilde{\varepsilon} = \frac{N\varepsilon}{N-1} > \widetilde{\varepsilon}. \tag{3.12}$$

Since it suffices to consider the case where $\varepsilon > 0$ is sufficiently small, we can assume that $\delta_2 > 0$ is small, in particular, $0 < \delta_2 < T - \delta_2 < T$. Then, we can prove

$$\frac{d_1 - d_0}{\delta_2^2 - \widetilde{\varepsilon}^2} < \frac{d_0}{\widetilde{\varepsilon}^2}. \tag{3.13}$$

Indeed, by the second equality in (3.12) and $N > 1$ we deduce

$$\frac{\delta_2^2}{\widetilde{\varepsilon}^2} = N^2 > N.$$

Using (3.11), we have

$$\frac{d_1}{d_0} < N < \frac{\delta_2^2}{\widetilde{\varepsilon}^2}, \quad \text{that is,} \quad \frac{d_1 - d_0}{d_0} < \frac{\delta_2^2 - \widetilde{\varepsilon}^2}{\widetilde{\varepsilon}^2},$$

which yields (3.13) by multiplying with $\frac{d_0}{\delta_2^2 - \widetilde{\varepsilon}^2} > 0$.

Therefore, we can choose $\beta > 0$ such that

$$\frac{d_1 - d_0}{\delta_2^2 - \widetilde{\varepsilon}^2} < \beta < \frac{d_0}{\widetilde{\varepsilon}^2},$$

which implies

$$d_0 - \beta\widetilde{\varepsilon}^2 > 0, \quad d_0 - \beta\widetilde{\varepsilon}^2 > d_1 - \beta\delta_2^2,$$

that is,

$$\mu_1 := e^{\lambda(d_0 - \beta\widetilde{\varepsilon}^2)} > \mu_2 := \max\{1,\, e^{\lambda(d_1 - \beta\delta_2^2)}\}.$$

We arbitrarily choose $t_0 \in (\delta_2, T - \delta_2)$. We notice that $(t_0 - \delta_2, t_0 + \delta_2) \subset (0, T)$. Hence (3.10) yields

$$N(v; t_0 - \widetilde{\varepsilon}, t_0 + \widetilde{\varepsilon}) \le Cs^4 M^2 e^{-2s\mu_0} + Cs^4 e^{Cs} D(v)^2$$

for all $s \ge s_0$. Here we note

$$\mu_0 := \mu_1 - \mu_2 > 0.$$

Replacing s and C by $s + s_0$ and e^{Cs_0} respectively, we see

$$N(v; t_0 - \widetilde{\varepsilon}, t_0 + \widetilde{\varepsilon}) \le C(s + s_0)^4 M^2 e^{-2s\mu_0} + C(s + s_0)^4 e^{Cs} D(v)^2$$

for all $s \ge 0$. Since $\mu_0 > 0$, we easily verify $\sup_{s>0}(s+s_0)^4 e^{-s\mu_0} < \infty$ and $(s+s_0)^4 e^{Cs} \le C_1 e^{C_1 s}$ with a large constant $C_1 > 0$. Hence we obtain

$$N(v; t_0 - \widetilde{\varepsilon}, t_0 + \widetilde{\varepsilon}) \le CM^2 e^{-s\mu_0} + Ce^{Cs} D(v)^2$$

for all $s \ge 0$. Here we note that $C > 0$ is a generic constant which is independent of $s > 0$. We make the right-hand side small by choosing $s \ge 0$ suitably. We now distinguish the two cases $D(v) < M$ and $D(v) \ge M$.

Case 1: $D(v) < M$**.**

We choose $s > 0$ such that

$$M^2 e^{-s\mu_0} = D(v)^2 e^{Cs},$$

that is,

$$s = \frac{2}{C+\mu_0} \log \frac{M}{D(v)} > 0.$$

Then

$$N(v; t_0 - \widetilde{\varepsilon}, t_0 + \widetilde{\varepsilon}) \le 2CM^{\frac{2C}{C+\mu_0}} D(v)^{\frac{2\mu_0}{C+\mu_0}}.$$

Case 2: $D(v) \ge M$**.**

Then we can directly estimate

$$N(v; t_0 - \widetilde{\varepsilon}, t_0 + \widetilde{\varepsilon}) \le CD(v)^2 (e^{-s\mu_0} + e^{Cs}).$$

Hence taking $s = 1$, we deduce that $N(v; t_0 - \widetilde{\varepsilon}, t_0 + \widetilde{\varepsilon}) \le CD(v)^2 e^C$.

Thus in both cases, we obtain

$$N(v; t_0 - \widetilde{\varepsilon}, t_0 + \widetilde{\varepsilon}) \le C(M)(D(v)^2 + D(v)^{2\theta}), \tag{3.14}$$

where $\theta = \frac{\mu_0}{C+\mu_0} \in (0,1)$. Here the constants C and θ are dependent on $\widetilde{\varepsilon}, \delta_2 > 0$, but independent of t_0. Varying t_0 over $(\delta_2,\, T - \delta_2)$, in view of (3.14) we reach

$$N(v; \delta_2 - \widetilde{\varepsilon}, T - \delta_2 + \widetilde{\varepsilon}) \le C(M)(D(v)^2 + D(v)^{2\theta}).$$

Since $\delta_2 - \widetilde{\varepsilon} = (N-1)\widetilde{\varepsilon} = \varepsilon$ and $T - \delta_2 + \widetilde{\varepsilon} = T - \varepsilon$, by (3.10) and observing that $\nabla p = \Delta v - \partial_t v - (A \cdot \nabla)v - (v \cdot \nabla)B$, we obtain

$$\|\nabla p\|^2_{L^2(\Omega_0 \times (\varepsilon, T-\varepsilon))} \le C(\|\partial_t v\|^2_{L^2(\Omega_0 \times (\varepsilon, T-\varepsilon))} + \|\Delta v\|^2_{L^2(\Omega_0 \times (\varepsilon, T-\varepsilon))} + \|v\|^2_{L^2(\varepsilon, T-\varepsilon; H^1(\Omega_0))}).$$

Thus we can complete the proof of Theorem 2. ■

In Theorem 2, in order to estimate ∇p, we have to apply the Carleman estimate not only for v but also for $\partial_t v$ to obtain (3.7). If we are not interested in estimating ∇p, then we can prove estimates of v with weaker norms of data.

4. Proof of Theorem 3

We divide the proof into three steps.

First Step: Inequalities obtained by the Carleman estimate.

We recall

$$I := (t_0 - \delta,\, t_0 + \delta)$$

with given constant $\delta > 0$, and that $C > 0$ denotes generic constants which are independent of $s > 0$ provided that $s > 0$ is sufficiently large.

We set

$$\begin{cases} \widetilde{D}_1(v) = \sum_{j,k=0}^{1}(\|\nabla_{x,t}^j \partial_t^k(\mathrm{rot}\, v)\|_{L^2(\Gamma\times I)} + \|\nabla^j \partial_t^k v\|_{L^2(\Gamma\times I)}), \\ \widetilde{D}_2(v) = \sum_{j=0}^{1}\sum_{k=0}^{2}(\|\nabla_{x,t}^j \partial_t^k(\mathrm{rot}\, v)\|_{L^2(\Gamma\times I)} + \|\nabla^j \partial_t^k v\|_{L^2(\Gamma\times I)}). \end{cases} \tag{4.1}$$

In this step, by Theorem 1 we will obtain two inequalities (4.7) and (4.10) stated below.

By (4.1), (1.10) and (1.12) we have

$$D_1(v) = \widetilde{D}_1(v) + \|v(\cdot,t_0)\|_{H^3(\Omega)}, \quad D_2(v) = \widetilde{D}_2(v) + \|v(\cdot,t_0)\|_{H^4(\Omega)}.$$

We choose the function $d \in C^2(\overline{\Omega})$ satisfying (3.1) and (3.2) and we set

$$\varphi(x,t) = e^{\lambda(d(x)-\beta(t-t_0)^2)}, \quad (x,t) \in Q$$

with sufficiently large $\lambda > 0$ and some parameter $\beta > 0$ which will be fixed later. Let the constants d_0 and d_1 be determined by (3.8). By (3.2) we have

$$\varphi(x,t) \le 1 \quad \text{for } (x,t) \in (\partial\Omega \setminus \Gamma) \times I, \quad d(x) \ge d_0 > 0 \quad \text{for } x \in \Omega_0. \tag{4.2}$$

We set

$$\mu_1 := e^{\lambda d_0}, \quad \mu_2 := \max\{1,\, e^{\lambda(d_1-\beta\delta^2)}\}. \tag{4.3}$$

For positive constants d_0, d_1 and δ, we can choose $\beta > 0$ large such that

$$d_0 > d_1 - \beta\delta^2. \tag{4.4}$$

Therefore, in view of (4.1) and (1.11), from (4.2) and (4.3) if follows that

$$\begin{cases} \int_{(\partial\Omega\setminus\Gamma)\times I} \sum_{j,k=0}^{1}(|\nabla_{x,t}^j \partial_t^k(\mathrm{rot}\, v)|^2 + |\nabla^j \partial_t^k v|^2)e^{2s\varphi}dSdt \le CE(v)^2 e^{2s\mu_2}, \\ \\ \int_{\Gamma\times I} \sum_{j,k=0}^{1}(|\nabla_{x,t}^j \partial_t^k(\mathrm{rot}\, v)|^2 + |\nabla^j \partial_t^k v|^2)e^{2s\varphi}dSdt \le Ce^{Cs}\widetilde{D}_1(v)^2. \end{cases} \tag{4.5}$$

Moreover, using (1.1) and (4.3) again, we obtain

$$\sum_{\kappa=0}^{1}\int_\Omega \sum_{j,k=0}^{1} |\nabla^j(\mathrm{rot}\,\partial_t^k v)(x,t_0+(-1)^\kappa\delta)|^2 e^{2s\varphi(x,t_0+\delta)}dx$$

$$\le CE(v)^2 e^{2se^{\lambda(d_1-\beta\delta^2)}} \le CE(v)^2e^{2s\mu_2}. \tag{4.6}$$

We apply Theorem 1 to system (1.1). By (4.5) and (4.6), we obtain

$$\int_Q (s|\nabla v|^2 + s|\nabla(\mathrm{rot}\,v)|^2 + s^3|v|^2)e^{2s\varphi}dxdt \le C\int_Q |\mathrm{rot}\,F|^2e^{2s\varphi}dxdt$$

$$+Cs^3E(v)^2e^{2s\mu_2} + Cs^3e^{Cs}\widetilde{D}_1(v)^2 \tag{4.7}$$

for all $s \ge s_0$.

Next we set $v_1 := \partial_t v$. Then

$$\begin{cases} & \partial_t v_1(x,t) - \Delta v_1 + (A(x,t)\cdot\nabla)v_1 + (v_1\cdot\nabla)B(x,t) + \nabla\partial_t p \\ = & \partial_t F(x,t) - (\partial_t A\cdot\nabla)v - (v\cdot\nabla)\partial_t B, \\ & \mathrm{div}\,v_1(x,t) = 0, \quad x\in\Omega,\ 0<t<T. \end{cases} \tag{4.8}$$

By (3.5) and (4.7), we have

$$\int_Q (|\mathrm{rot}\,((\partial_t A\cdot\nabla)v)|^2 + |\mathrm{rot}\,((v\cdot\nabla)\partial_t B)|^2)e^{2s\varphi}dxdt$$

$$\le C\int_Q (|\nabla(\mathrm{rot}\,v)|^2 + |\nabla v|^2 + |v|^2)e^{2s\varphi}dxdt$$

$$\le \frac{C}{s}\int_Q |\mathrm{rot}\,F|^2e^{2s\varphi}dxdt + Cs^2E(v)^2e^{2s\mu_2} + Cs^2e^{Cs}\widetilde{D}_1(v)^2. \tag{4.9}$$

Therefore, we apply Theorem 1 to system (4.8) in v_1 and use (4.5), (4.6) and (4.9) to obtain

$$\int_Q \left(\frac{1}{s}|\partial_t^2\mathrm{rot}\,v|^2 + s^3|\partial_t\mathrm{rot}\,v|^2 + s(|\nabla(\mathrm{rot}\,\partial_t v)|^2 + |\nabla\partial_t v|^2) + s^3|\partial_t v|^2\right)e^{2s\varphi}dxdt$$

$$\le C\int_Q \sum_{k=0}^{1}|\partial_t^k\mathrm{rot}\,F|^2e^{2s\varphi}dxdt + Cs^3E(v)^2e^{2s\mu_2} + Cs^3e^{Cs}\widetilde{D}_1(v)^2 \tag{4.10}$$

for all large $s \ge s_0$.

Second Step: Proof of Theorem 3 (i) .

Recalling that $E(v)$ is defined by (1.11), we estimate

$$\begin{aligned}
&\int_\Omega |\mathrm{rot}\,\partial_t v(x,t_0)|^2 e^{2s\varphi(x,t_0)}dx = \int_{t_0-\delta}^{t_0}\left(\frac{d}{dt}\int_\Omega |\mathrm{rot}\,\partial_t v(x,t)|^2 e^{2s\varphi(x,t)}dx\right)dt \\
&+\int_\Omega |\mathrm{rot}\,\partial_t v(x,t_0-\delta)|^2 e^{2s\varphi(x,t_0-\delta)}dx \\
&=\int_{t_0-\delta}^{t_0}\int_\Omega \left(2((\mathrm{rot}\,\partial_t v)\cdot(\mathrm{rot}\,\partial_t^2 v)) + 2s\partial_t\varphi|\mathrm{rot}\,\partial_t v|^2\right)e^{2s\varphi(x,t)}dxdt \\
&+\int_\Omega |\mathrm{rot}\,\partial_t v(x,t_0-\delta)|^2 e^{2s\varphi(x,t_0-\delta)}dx \\
&\le C\int_Q (s|\mathrm{rot}\,\partial_t v|^2 + |\mathrm{rot}\,\partial_t v||\mathrm{rot}\,\partial_t^2 v|)e^{2s\varphi(x,t)}dxdt + CE(v)^2 e^{2s\mu_2}.
\end{aligned}$$

Here we used

$$e^{2s\varphi(x,t_0-\delta)} = e^{2se^{\lambda(d(x)-\beta\delta^2)}} \le e^{2se^{\lambda(d_1-\beta\delta^2)}} \le e^{2s\mu_2}$$

by (4.3). We notice that $E(v) \le M_1$ by (1.11). Therefore,

$$\begin{aligned}
&\int_\Omega |\mathrm{rot}\,\partial_t v(x,t_0)|^2 e^{2s\varphi(x,t_0)}dx \\
&\le C\int_Q (s|\mathrm{rot}\,\partial_t v|^2 + |\mathrm{rot}\,\partial_t v||\mathrm{rot}\,\partial_t^2 v|)e^{2s\varphi(x,t)}dxdt + CM_1^2 e^{2s\mu_2}.
\end{aligned}$$

Moreover we have

$$\begin{aligned}
&|\mathrm{rot}\,\partial_t v||\mathrm{rot}\,\partial_t^2 v| = \left(\frac{1}{s}|\mathrm{rot}\,\partial_t^2 v|\right)(s|\mathrm{rot}\,\partial_t v|) \\
&\le \frac{1}{2}\left(\frac{1}{s^2}|\mathrm{rot}\,\partial_t^2 v|^2 + s^2|\mathrm{rot}\,\partial_t v|^2\right) = \frac{1}{2s}\left(\frac{1}{s}|\mathrm{rot}\,\partial_t^2 v|^2 + s^3|\mathrm{rot}\,\partial_t v|^2\right),
\end{aligned}$$

and so (4.10) implies

$$\int_Q |\mathrm{rot}\,\partial_t v||\mathrm{rot}\,\partial_t^2 v|e^{2s\varphi}dxdt \le \frac{C}{s}\int_Q \sum_{k=0}^1 |\partial_t^k \mathrm{rot}\,F|^2 e^{2s\varphi}dxdt + Cs^2M_1^2e^{2s\mu_2} + Cs^2e^{Cs}\widetilde{D}_1(v)^2$$

and

$$\int_Q s|\mathrm{rot}\,\partial_t v|^2 e^{2s\varphi}dxdt \le \frac{C}{s^2}\int_Q \sum_{k=0}^1 |\partial_t^k \mathrm{rot}\,F|^2 e^{2s\varphi}dxdt + CsM_1^2e^{2s\mu_2} + Cse^{Cs}\widetilde{D}_1(v)^2.$$

Hence,

$$\int_\Omega |\mathrm{rot}\,\partial_t v(x,t_0)|^2 e^{2s\varphi(x,t_0)}dx$$

$$\le \frac{C}{s}\int_Q \sum_{k=0}^{1} |\partial_t^k \mathrm{rot}\, F|^2 e^{2s\varphi} dxdt + Cs^2 M_1^2 e^{2s\mu_2} + Cs^2 e^{Cs} D_1(v)^2. \tag{4.11}$$

The a priori regularity assumption $\partial_t v \in H^{2,1}(Q)$ and the Sobolev embedding yield $\partial_t v \in C(\overline{I}; L^2(\Omega))$ and $v \in C(\overline{I}; H^2(\Omega))$. Therefore, by (1.1) we have

$$F(x,t_0) = \partial_t v(x,t_0) - \Delta v(x,t_0) + (A\cdot\nabla)v(x,t_0) + (v(x,t_0)\cdot\nabla)B(x,t_0) + \nabla p(x,t_0).$$

Moreover, since the a priori regularity assumption $\partial_t \mathrm{rot}\, v \in H^1(I; L^2(\Omega)) \subset C(\overline{I}; L^2(\Omega))$, we obtain

$$\mathrm{rot}\, F(x,t_0) = \mathrm{rot}\, \partial_t v(x,t_0) + a(x), \quad x \in \Omega, \tag{4.12}$$

where

$$a(x) := \mathrm{rot}\,(-\Delta v(x,t_0) + (A\cdot\nabla)v(x,t_0) + (v(x,t_0)\cdot\nabla)B(x,t_0)). \tag{4.13}$$

Observe that

$$\int_\Omega |a|^2 e^{2s\varphi(x,t_0)} dx \le C \sum_{|\alpha|\le 3} \int_\Omega |\partial_x^\alpha v(x,t_0)|^2 e^{2s\varphi(x,t_0)} dx \le Ce^{Cs} \|v(\cdot,t_0)\|^2_{H^3(\Omega)} \le Ce^{Cs} D_1(v)^2.$$

Consequently, from (4.11) and (4.12), there exists $s_1 > 0$ such that

$$\int_\Omega |\mathrm{rot}\, F(x,t_0)|^2 e^{2s\varphi(x,t_0)} dx$$

$$\le \frac{C}{s}\int_Q \sum_{k=0}^{1} |\partial_t^k \mathrm{rot}\, F|^2 e^{2s\varphi} dxdt + Cs^2 M_1^2 e^{2s\mu_2} + Cs^2 e^{Cs} D_1(v)^2 \tag{4.14}$$

for all $s \ge s_0$. By (1.8) and inequality $\varphi(x,t) \le \varphi(x,t_0)$ for $(x,t) \in Q$, we have

$$\frac{1}{s}\int_Q \sum_{k=0}^{1} |\mathrm{rot}\, \partial_t^k F(x,t)|^2 e^{2s\varphi(x,t)} dxdt \le \frac{C}{s}\int_\Omega |\mathrm{rot}\, F(x,t_0)|^2 e^{2s\varphi(x,t_0)} dx$$

for all $s \ge s_0$. Therefore,

$$\int_\Omega |\mathrm{rot}\, F(x,t_0)|^2 e^{2s\varphi(x,t_0)} dx \le \frac{C}{s}\int_\Omega |\mathrm{rot}\, F(x,t_0)|^2 e^{2s\varphi(x,t_0)} dx + Cs^2 M_1^2 e^{2s\mu_2} + Cs^2 e^{Cs} D_1(v)^2$$

for all large $s > s_0$.

Choosing $s_1 > 0$ large, we can absorb the first term on the right-hand side into the left-hand side, and obtain

$$\int_\Omega |\mathrm{rot}\, F(x,t_0)|^2 e^{2s\varphi(x,t_0)} dx \le Cs^2 M_1^2 e^{2s\mu_2} + Cs^2 e^{Cs} D_1(v)^2$$

for all $s > s_1$. Recalling that $d(x) \ge d_0 > 0$ for $x \in \Omega_0$ and $\Omega_0 \subset \Omega$, and noting $\varphi(x,t_0) = e^{\lambda d(x)}$, we see

$$e^{2se^{\lambda d_0}} \int_{\Omega_0} |\mathrm{rot}\, F(x,t_0)|^2 dx \le \int_\Omega |\mathrm{rot}\, F(x,t_0)|^2 e^{2s\varphi(x,t_0)} dx.$$

Therefore,

$$\int_{\Omega_0} |\mathrm{rot}\, F(x,t_0)|^2 dx \le Cs^2 M_1^2 e^{2s\mu_2} e^{-2se^{\lambda d_0}} + Cs^2 e^{Cs} D_1(v)^2.$$

By (4.3) and (4.4), we see $e^{\lambda d_0} > e^{\lambda(d_1 - \beta\delta^2)}$. Since $d_0 > 0$, we have $e^{\lambda d_0} > 1$. Therefore

$$\mu_1 = e^{\lambda d_0} > \max\{1,\, e^{\lambda(d_1-\beta\delta^2)}\} = \mu_2.$$

Consequently,

$$\mu_0 := \mu_1 - \mu_2 > 0.$$

Hence, we reach

$$\int_{\Omega_0} |\mathrm{rot}\, F(x,t_0)|^2 dx \le Cs^2 M_1^2 e^{-2s\mu_0} + Cs^2 e^{Cs} D_1(v)^2$$

for all $s > s_1$. By the same argument as in the final part of the proof of Theorem 2, we complete the proof of Theorem 3 (i).

Third Step: Proof of Theorem 3 (ii).

For short notation, we set

$$E_2(v) := E(v) + E(\partial_t v).$$

We choose the same weight function $\varphi(x,t) = e^{\lambda(d(x) - \beta(t-t_0)^2)}$ as in First Step.

Next, we set $v_2 := \partial_t^2 v$. Then

$$\begin{cases} & \partial_t v_2(x,t) - \Delta v_2 + (A(x,t)\cdot\nabla)v_2 + (v_2\cdot\nabla)B(x,t) + \nabla\partial_t^2 p \\ = & \partial_t^2 F(x,t) - 2(\partial_t A\cdot\nabla)v_1 - 2(v_1\cdot\nabla)\partial_t B - (\partial_t^2 A\cdot\nabla)v - (v\cdot\nabla)\partial_t^2 B, \\ & \mathrm{div}\, v_2(x,t) = 0, \quad x\in\Omega,\ 0<t<T. \end{cases} \tag{4.15}$$

We apply Theorem 1 to system (4.15) in v_2, and similarly to (4.5) and (4.6), in terms of $E_2(v)$ and $D_2(v)$, we estimate the boundary terms

$$\int_{\partial\Omega\times I} \left(\sum_{j=0}^{1}\sum_{k=0}^{2} |\nabla_{x,t}^j \partial_t^k (\mathrm{rot}\, v)|^2 + |\nabla^j \partial_t^k v|^2 \right) e^{2s\varphi} dSdt$$

and

$$\sum_{\kappa=0}^{1}\int_\Omega \sum_{j=0}^{1}\sum_{k=0}^{2} |\nabla^j(\mathrm{rot}\,\partial_t^k v)(x,t_0+(-1)^\kappa\delta)|^2 e^{2s\varphi(x,t_0+\delta)}dx.$$

These terms are of the same kind as (4.5) and (4.6) but the orders k of the t-derivatives change over $0,1,2$. Hence we obtain

$$\begin{aligned}\int_Q s|\nabla(\mathrm{rot}\,\partial_t^2 v)|^2 e^{2s\varphi}dxdt \le C\int_Q |\mathrm{rot}\,\partial_t^2 F|^2 e^{2s\varphi}dxdt\\ +C\int_Q (\left|\mathrm{rot}\ \{2(\partial_t A\cdot\nabla)v_1 + 2(v_1\cdot\nabla)\partial_t B\}\right|^2 + \left|\mathrm{rot}\ \left\{(\partial_t^2 A\cdot\nabla)v + (v\cdot\nabla)\partial_t^2 B\right\}\right|^2)e^{2s\varphi}dxdt\\ +Cs^3E_2(v)^2e^{2s\mu_2} + Cs^3e^{Cs}\widetilde{D}_2(v)^2\end{aligned} \tag{4.16}$$

for all large $s>0$. Now we need to estimate the second integral on the right-hand side of the inequality (4.16).

Since A and B are assumed to be sufficiently smooth (e.g., $A,B\in W^{2,\infty}(0,T;W^{1,\infty}(\Omega))$), in view of (3.5), we have

$$\begin{aligned}&\int_Q |\mathrm{rot}\,(2(\partial_t A\cdot\nabla)v_1 + 2(v_1\cdot\nabla)\partial_t B)|^2 e^{2s\varphi}dxdt\\ \le& C\int_Q (|\nabla(\mathrm{rot}\,v_1)|^2 + |\nabla v_1|^2 + |v_1|^2)e^{2s\varphi}dxdt\\ =& C\int_Q (|\nabla(\mathrm{rot}\,\partial_t v)|^2 + |\nabla\partial_t v|^2 + |\partial_t v|^2)e^{2s\varphi}dxdt.\end{aligned}$$

Therefore, by means of (4.10), we obtain

$$\begin{aligned}&\int_Q |\mathrm{rot}\,(2(\partial_t A\cdot\nabla)v_1 + 2(v_1\cdot\nabla)\partial_t B)|^2 e^{2s\varphi}dxdt\\ \le& \frac{C}{s}\int_Q \sum_{k=0}^{1}|\mathrm{rot}\,\partial_t^k F|^2 e^{2s\varphi}dxdt + Cs^2E(v)^2e^{2s\mu_2} + Cs^2e^{Cs}\widetilde{D}_1(v)^2.\end{aligned} \tag{4.17}$$

Similarly, using (3.5) and (4.7), we have

$$\begin{aligned}&\int_Q |\mathrm{rot}\,((\partial_t^2 A\cdot\nabla)v + (v\cdot\nabla)\partial_t^2 B)|^2 e^{2s\varphi}dxdt\\ \le& \frac{C}{s}\int_Q |\mathrm{rot}\,F|^2 e^{2s\varphi}dxdt + Cs^2E(v)^2e^{2s\mu_2} + Cs^2e^{Cs}\widetilde{D}_1(v)^2.\end{aligned} \tag{4.18}$$

Since $\widetilde{D}_1(v)\le\widetilde{D}_2(v)$ and $E(v)\le E_2(v)$, in terms of (1.13) we now replace $E(v)$ and $\widetilde{D}_1(v)$ by M_2 and $\widetilde{D}_2(v)$, respectively.

Substituting (4.17) and (4.18) into (4.16), we obtain

$$\int_Q s|\nabla(\mathrm{rot}\,\partial_t^2 v)|^2 e^{2s\varphi}dxdt$$
$$\le C\int_Q \sum_{k=0}^{2} |\mathrm{rot}\,\partial_t^k F|^2 e^{2s\varphi}dxdt + Cs^3M_2^2e^{2s\mu_2} + Cs^3e^{Cs}\widetilde{D}_2(v)^2.$$

Combining this estimate with (4.10), we deduce

$$\int_Q (s|\nabla(\mathrm{rot}\,\partial_t^2 v)|^2 + s|\nabla(\mathrm{rot}\,\partial_t v)|^2)e^{2s\varphi}dxdt$$
$$\le C\int_Q \sum_{k=0}^{2} |\mathrm{rot}\,\partial_t^k F|^2 e^{2s\varphi}dxdt + Cs^3M_2^2e^{2s\mu_2} + Cs^3e^{Cs}\widetilde{D}_2(v)^2 \tag{4.19}$$

for all large $s > 0$.

Henceforth we set

$$(\nabla(\mathrm{rot}\,\partial_t v)\cdot\nabla(\mathrm{rot}\,\partial_t^2 v)) := \sum_{j,k=1}^{3} (\partial_j[\mathrm{rot}\,\partial_t v]_k)\partial_j[\mathrm{rot}\,\partial_t^2 v]_k.$$

Now, arguing as in the Second Step, using (4.19) and (4.6), we see

$$\int_\Omega |\nabla(\mathrm{rot}\,\partial_t v(x,t_0))|^2 e^{2s\varphi(x,t_0)}dx = \int_{t_0-\delta}^{t_0}\left(\frac{d}{dt}\int_\Omega |\nabla(\mathrm{rot}\,\partial_t v(x,t))|^2 e^{2s\varphi(x,t)}dx\right)dt$$
$$+\int_\Omega |\nabla(\mathrm{rot}\,\partial_t v)(x,t_0-\delta)|^2 e^{2s\varphi(x,t_0-\delta)}dx$$
$$=\int_{t_0-\delta}^{t_0}\int_\Omega (2(\nabla(\mathrm{rot}\,\partial_t v)\cdot\nabla(\mathrm{rot}\,\partial_t^2 v)) + 2s\partial_t\varphi|\nabla(\mathrm{rot}\,\partial_t v)|^2)e^{2s\varphi(x,t)}dxdt$$
$$+\int_\Omega |\nabla(\mathrm{rot}\,\partial_t v)(x,t_0-\delta)|^2 e^{2s\varphi(x,t_0-\delta)}dx$$
$$\le C\int_Q (s|\nabla(\mathrm{rot}\,\partial_t v)|^2 + |\nabla(\mathrm{rot}\,\partial_t v)||\nabla(\mathrm{rot}\,\partial_t^2 v)|)e^{2s\varphi(x,t)}dxdt + CM_2^2e^{2s\mu_2}.$$

Since

$$|\nabla(\mathrm{rot}\,\partial_t v)||\nabla(\mathrm{rot}\,\partial_t^2 v)| \le \frac{1}{2}(|\nabla(\mathrm{rot}\,\partial_t v)|^2 + |\nabla(\mathrm{rot}\,\partial_t^2 v)|^2),$$

we apply (4.19) to obtain

$$\int_\Omega |\nabla(\mathrm{rot}\,\partial_t v(x,t_0))|^2 e^{2s\varphi(x,t_0)}dx$$
$$\le C\sum_{k=0}^{2}\int_Q |\mathrm{rot}\,\partial_t^k F|^2 e^{2s\varphi}dxdt + Cs^3M_2^2e^{2s\mu_2} + Cs^3e^{Cs}\widetilde{D}_2(v)^2. \tag{4.20}$$

Applying (1.9) to the first term on the right-hand side of (4.20), we obtain

$$\int_\Omega |\nabla(\mathrm{rot}\,\partial_t v(x,t_0))|^2 e^{2s\varphi(x,t_0)}dx \le C\int_Q (|F(x,t_0)|^2 + |\nabla F(x,t_0)|^2)e^{2s\varphi}dxdt$$
$$+Cs^3M_2^2e^{2s\mu_2} + Cs^3e^{Cs}\widetilde{D}_2(v)^2 \tag{4.21}$$

for all large $s > 0$.

By (1.7), we have

$$\mathrm{rot}\,\mathrm{rot}\,F(x,t_0) = -\Delta F(x,t_0) + \nabla(\mathrm{div}\,F(x,t_0)) = -\Delta F(x,t_0), \quad x \in \Omega$$

and (4.12) yields

$$\mathrm{rot}\,\mathrm{rot}\,F(x,t_0) = \mathrm{rot}\,\mathrm{rot}\,\partial_t v(x,t_0) + \mathrm{rot}\,a(x), \quad x \in \Omega,$$

where a is defined by (4.13). Hence,

$$-\Delta F(x,t_0) = \mathrm{rot}\,\mathrm{rot}\,(\partial_t v(x,t_0)) + \mathrm{rot}\,a(x), \quad x \in \Omega.$$

Therefore,

$$|\Delta F(x,t_0)| \le C(|\nabla(\mathrm{rot}\,\partial_t v(x,t_0))| + |\mathrm{rot}\,a(x)|), \quad x \in \Omega$$

and so (4.21) implies

$$\int_\Omega |\Delta F(x,t_0)|^2 e^{2s\varphi(x,t_0)}dx \le C\int_Q (|F(x,t_0)|^2 + |\nabla F(x,t_0)|^2)e^{2s\varphi(x,t)}dxdt$$
$$+Ce^{Cs}(s^3\widetilde{D}_2(v)^2 + \|v(\cdot,t_0)\|^2_{H^4(\Omega)}) + Cs^3M_2^2e^{2s\mu_2}$$

for all large $s > 0$. Since $\varphi(x,t) \le \varphi(x,t_0)$ for $(x,t) \in Q$, we obtain

$$\int_\Omega |\Delta F(x,t_0)|^2 e^{2s\varphi(x,t_0)}dx \le C\int_\Omega (|F(x,t_0)|^2 + |\nabla F(x,t_0)|^2)e^{2s\varphi(x,t_0)}dx$$
$$+Ce^{Cs}s^3D_2(v)^2 + Cs^3M_2^2e^{2s\mu_2} \tag{4.22}$$

for all large $s > 0$

Next we apply Lemma 3 proved already in Section 2, which is an elliptic Carleman estimate. Since $F(x,t_0) = |\nabla F(x,t_0)| = 0$ for $x \in \Gamma$, applying Lemma 3 to (4.22), we see

$$\int_\Omega (s|\nabla F(x,t_0)|^2 + s^3|F(x,t_0)|^2)e^{2s\varphi(x,t_0)}dx$$

$$\leq C\int_\Omega (|F(x,t_0)|^2+|\nabla F(x,t_0)|^2)e^{2s\varphi(x,t_0)}dx + Ce^{Cs}s^3D_2(v)^2+Cs^3M_2^2e^{2s\mu_2}$$

$$+Cs^3\int_{\partial\Omega\setminus\Gamma}(|\nabla F(x,t_0)|^2+|F(x,t_0)|^2)e^{2s\varphi(x,t_0)}dS \tag{4.23}$$

for all sufficiently large $s>0$. Recalling that $\varphi(x,t)\leq 1$ for $(x,t)\in(\partial\Omega\setminus\Gamma)\times I$ from (4.2), by $\|F(\cdot,t_0)\|_{H^2(\Omega)}\leq m_1$ and (4.3), we apply the trace theorem to obtain

$$\int_{\partial\Omega\setminus\Gamma}(|\nabla F(x,t_0)|^2+|F(x,t_0)|^2)e^{2s\varphi(x,t_0)}dS \leq m_1^2\int_{\partial\Omega\setminus\Gamma}e^{2s}dS\leq Cm_1^2e^{2s}\leq Cm_1^2e^{2s\mu_2}.$$

Henceforth we set $\widetilde{M_2}:=M_2+m_1$. Choosing $s>0$ large, we can absorb the first term on the right-hand side of (4.23) into the left-hand side, and so

$$\int_\Omega s(|\nabla F(x,t_0)|^2+|F(x,t_0)|^2)e^{2s\varphi(x,t_0)}dx\leq Ce^{Cs}s^3D_2(v)^2+Cs^3\widetilde{M_2}^2e^{2s\mu_2}$$

for all large $s>0$. Shrinking the integral domain Ω on the left-hand side to Ω_0 and using (4.2) and (4.3) and $e^{2s\varphi(x,t_0)}\geq e^{2se^{\lambda d_0}}=e^{2s\mu_1}$ for $x\in\Omega_0$, we obtain

$$e^{2s\mu_1}\|F(\cdot,t_0)\|^2_{H^1(\Omega_0)}\leq Ce^{Cs}s^3D_2(v)^2+Cs^3\widetilde{M_2}^2e^{2s\mu_2}$$

for all large $s>0$. By (4.4), we have $\mu_1>\mu_2$ and so

$$\|F(\cdot,t_0)\|^2_{H^1(\Omega_0)}\leq Ce^{Cs}s^3D_2(v)^2+Cs^3\widetilde{M_2}^2e^{-2s\mu_0}$$

for all large $s>0$. Here we set $\mu_0:=\mu_1-\mu_2>0$. Therefore, by the same argument as in the final part of the proof of Theorem 2, we can complete the proof of Theorem 3 (ii). ■

5. Concluding Remarks

5.1. The Navier-Stokes equations

In this article we mainly consider linearized Navier-Stokes equations. The original Navier-Stokes equations reads as

$$\partial_t v(x,t)-\Delta v(x,t)+(v\cdot\nabla)v(x,t)+\nabla p=F(x,t),\quad x\in\Omega,\ 0<t<T. \tag{5.1}$$

Let v_1 and v_2 satisfy (5.1) with F_1 and F_2 respectively. Then, setting $v:=v_1-v_2$, we have

$$\partial_t v(x,t)-\Delta v(x,t)+(v_1\cdot\nabla)v(x,t)+(v\cdot\nabla)v_2(x,t)+\nabla p=(F_1-F_2)(x,t),\quad x\in\Omega,\ 0<t<T,$$

which corresponds to (1.1) with $A=v_1$ and $B=v_2$. Although the existence of solutions v_1,v_2 to the initial boundary value problems in $\Omega\times(0,T)$ for the Navier-Stokes equations, is in general, not completely solved with our regularity assumptions, for our inverse problems we assume the existence of v_1, v_2 with such regularity. Moreover, we see by the proof that $T > 0$ can be arbitrarily small, and our inverse problem requires the existence of the solution (v, p) local in time. We further notice that for the inverse problems we consider the solutions only for positive time interval $t > t_0 - \delta > 0$, so that there is a possibility that we rely on the smoothing property in time of solutions to the Navier-Stokes equations, in order to gain the necessary regularity of solution for the inverse problem.

On the other hand, in our inverse source problems, we cannot choose $t_0 = 0$, in other words, our problem is not an inverse problem for any initial boundary value problem. In particular, we remark that we do not assume any initial conditions, but we need $v(\cdot, t_0)$ in Ω with some $t_0 > 0$. In general, the inverse problem with our formulation for parabolic equations as well as the Navier-Stokes equations has been a long-standing open problem for the case of $t_0 = 0$.

5.2. Data of pressure p.

In this article, we do not use any data of p. Therefore, in Theorem 3 (ii), we have to assume (1.7): $\operatorname{div} F(\cdot, t_0) = 0$ in Ω. With additional information of p, we do not need such assumption for unknown sources, but here we do not pursue this direction.

5.3. Available Carleman estimates.

As Carleman estimates for parabolic equations including the Navier-Stokes equations, we can obtain two types according to the following choices of weight functions:

- weight function without singularity:

$$\varphi(x,t) := e^{\lambda(d(x) - \beta(t-t_0)^2)}; \tag{5.2}$$

- weight function with singularities at $t = t_0 \pm \delta$:

$$\varphi(x,t) = \frac{e^{\lambda\eta(x)} - e^{2\lambda\|\eta\|_{C(\overline{\Omega})}}}{h(t)}, \tag{5.3}$$

  where $\lim_{t\to t_0 \pm \delta} h(t) = 0$.

Here $d, \eta \in C^2(\overline{\Omega})$ are chosen suitably.

The Carleman estimate with the weight (5.3) was proved firstly by Imanuvilov [12] for a parabolic equation.

As for related inverse problems for the Navier-Stokes equations, Choulli, Imanuvilov, Puel and Yamamoto [6], Fan, Di Cristo, Jiang and Nakamura [8], Fan, Jiang and Nakamura [9] used Carleman estimates with (5.3), while Bellassoued, Imanuvilov and Yamamoto [2] and the current article rely on (5.2).

Moreover, in order to derive Carleman estimates for the Navier-Stokes equations with (5.2) or (5.3), we have two options.

(1) First take the rotation operator rot to obtain a parabolic equation in $\mathrm{rot}\, v$ and then a Poisson equation in v. See [6], [8], [9] with the weight in form of (5.3).
(2) First take the divergence operator div to obtain a Poisson equation in p and then a parabolic equation in v. See also [2] with the weight in form of (5.2).

In the present article, the proof of the Carleman estimate is by the above option (1), so that our Carleman estimate established by Theorem 1 is different from [2] by option (2), although the weight function is the same.

5.4. Improvement of Theorem 3 (ii)

So far, we used an elliptic Carleman estimate with L^2-norm for non-homogeneous term. More precisely, for the Carleman estimate for the Laplace operator, we are restricted to Lemma 3 where the right-hand side of the Carleman estimate is taken in the weighted L^2-norm. On the other hand, we can apply an H^{-1}-Carleman estimate for Δ and improve Theorem 3 (ii).

Indeed we can relax condition (1.9) as

$$|\partial_t^k \mathrm{rot}\, F(x,t)| \le C(|\nabla F(x,t_0)| + |F(x,t_0)|), \quad (x,t) \in Q,\ k = 0,1. \tag{5.4}$$

Compared with (1.9), condition (5.4) does not require the second-order time derivative on the left-hand side. Then reducing the time regularity by order 1 in boundary data and a priori bound, we can improve the conclusion in Theorem 3 (ii) as follows. Here we recall that the functional $D_1(v)$ is defined by (1.10).

Proposition 1.
There exist constants $C > 0$ and $\theta \in (0,1)$, which are dependent on Ω_0, Γ, M_1 such that

$$\|F\|_{H^1(\Omega_0)} \le C(D_1(v) + D_1(v)^\theta) \tag{5.5}$$

for each

$$F \in \mathcal{F}_3 := \{F \in H^1(I; H^2(\Omega)) :\ \mathrm{supp}\, F(\cdot, t_0) \subset \Omega,\ \textit{(1.7) and (5.4) are satisfied}\},$$

provided that (1.11) holds with arbitrarily chosen constant $M_1 > 0$.

We can relax also the condition $\mathrm{supp}\, F(\cdot, t_0) \subset \Omega$, but for simplicity we keep this condition in $\mathcal{F}_3$ and here give

Sketch of the proof of Proposition 1.

Under the assumption, we can follow the argument in the proof of Theorem 3 (i) and reach (4.14). Application of (5.4) to the first term on the right-hand side of (4.14) yields

$$\int_\Omega |\mathrm{rot}\, F(x,t_0)|^2 e^{2s\varphi(x,t_0)} dx \le \frac{C}{s} \int_Q (|F(x,t_0)|^2 + |\nabla F(x,t_0)|^2) e^{2s\varphi(x,t)} dxdt$$
$$+ Cs^2 M_1^2 e^{2s\mu_2} + Ce^{Cs} s^2 D_1(v)^2 \tag{5.6}$$

for all large $s > 0$. We estimate the first term on the right-hand side of (5.6) as follows:

$$\int_Q (|F(x,t_0)|^2 + |\nabla F(x,t_0)|^2) e^{2s\varphi(x,t)} dxdt$$
$$= \int_\Omega (|F(x,t_0)|^2 + |\nabla F(x,t_0)|^2) e^{2s\varphi(x,t_0)} \left(\int_{t_0-\delta}^{t_0+\delta} e^{2s(\varphi(x,t)-\varphi(x,t_0))} dt \right) dx.$$

Since $d \ge 0$ on $\overline{\Omega}$, we obtain

$$\int_{t_0-\delta}^{t_0+\delta} e^{2s(\varphi(x,t)-\varphi(x,t_0))} dt = \int_{t_0-\delta}^{t_0+\delta} e^{-2se^{\lambda d(x)}(1-e^{-\lambda\beta(t-t_0)^2})} dt$$
$$\le \int_{t_0-\delta}^{t_0+\delta} e^{-2s(1-e^{-\lambda\beta(t-t_0)^2})} dt = o(1)$$

as $s \to \infty$ by the Lebesgue theorem. Therefore,

$$\int_Q (|F(x,t_0)|^2 + |\nabla F(x,t_0)|^2) e^{2s\varphi(x,t)} dxdt = o(1) \int_\Omega (|F(x,t_0)|^2 + |\nabla F(x,t_0)|^2) e^{2s\varphi(x,t_0)} dx$$

and so (5.6) yields

$$s \int_\Omega |\mathrm{rot}\, F(x,t_0)|^2 e^{2s\varphi(x,t_0)} dx$$
$$\le o(1) \int_\Omega (|F(x,t_0)|^2 + |\nabla F(x,t_0)|^2) e^{2s\varphi(x,t_0)} dx + Cs^3 M_1^2 e^{2s\mu_2} + Ce^{Cs} s^3 D_1(v)^2 \tag{5.7}$$

for all large $s > 0$.

Choosing a bounded domain $E \subset \mathbb{R}^n$ with $n \in \mathbb{N}$, we prove an H^{-1}-Carleman estimate for the operator Δ as follows.

Lemma 4.

Let $E, \widetilde{E} \subset \mathbb{R}^n$ be bounded domains with smooth boundaries ∂E and $\partial \widetilde{E}$ such that $\overline{E} \subset \widetilde{E}$, and $\omega \subset \widetilde{E} \setminus \overline{E}$ be an arbitrarily chosen non-empty open set. Let $\eta \in C^2(\overline{\widetilde{E}})$ satisfy

$$\eta = 0 \quad \text{on } \partial\widetilde{E}, \quad \eta > 0 \quad \text{in } \widetilde{E}, \quad |\nabla \eta| > 0 \quad \text{on } \overline{\widetilde{E} \setminus \omega}. \tag{5.8}$$

Let $g_1, ..., g_n \in H_0^1(\Omega)$ and $w \in H_0^2(E)$ satisfy

$$\Delta w(x) = \sum_{j=1}^n \partial_j g_j(x), \quad x \in E.$$

Then, for all sufficiently large $\lambda > 0$, setting $\varphi_0(x) := e^{\lambda\eta(x)}$, we can find constants $C > 0$, and $s_0 > 0$ such that

$$\int_E (|\nabla w(x)|^2 + s^2|w(x)|^2)e^{2s\varphi_0(x)}dx \le C\sum_{j=1}^n s\int_E |g_j(x)|^2 e^{2s\varphi_0(x)}dx$$

for all $s > s_0$.

This is a Carleman estimate where the right-hand side is estimated in the norm of H^{-1}-space, while Lemma 3 in Section 2 is a Carleman estimate whose right-hand side is estimated in the norm of L^2-space.

Proof of Lemma 4.

By the same letters w and g_j with $j = 1, ..., n$, we denote the zero extensions of the functions w and g_j from E to $\widetilde{E}$. Since $w \in H_0^2(E)$ and $g_1, ..., g_n \in H_0^1(E)$ we have $w \in H_0^2(\widetilde{E}), g_1, ..., g_n \in H_0^1(\widetilde{E})$ and $w = g_j = 0$ in $\widetilde{E}\setminus E$. Hence the equation $\Delta w = \sum_{j=1}^n \partial_j g_j$ holds true in the domain $\widetilde{E}$. Applying Theorem 2.2 in Imanuvilov and Puel [13], we obtain

$$\int_{\widetilde{E}} (|\nabla w(x)|^2 + s^2|w(x)|^2)e^{2s\varphi_0(x)}dx \le C\sum_{j=1}^n s\int_{\widetilde{E}} |g_j(x)|^2 e^{2s\varphi_0(x)}dx = C\sum_{j=1}^n s\int_E |g_j(x)|^2 e^{2s\varphi_0(x)}dx$$

for all large $s > 0$, which proves Lemma 4. ■

Now by (1.7), we have

$$-\Delta F(x, t_0) = \mathrm{rot}\,\mathrm{rot}\, F(x, t_0), \quad x \in \Omega, \tag{5.9}$$

and supp $F(\cdot, t_0) \subset \Omega$.

Here we recall that we fix sufficiently large $\lambda > 0$ and so in the Carleman estimate we can omit the dependence of the constants on λ.

Let Ω_2 be a subdomain of Ω such that $\overline{\Omega}_2 \subset \Omega$ and $\operatorname{supp} F(\cdot, t_0) \subset \Omega_2$. We apply Lemma 4 by setting $E = \Omega_2$ and $\widetilde{E} = \Omega_1$. Moreover, Ω_1 and $\eta(x) = d(x)$ satisfy (3.1). Moreover $\operatorname{rot}\operatorname{rot} F(x, t_0)$ is given by a linear combination of $\partial_j[\operatorname{rot} F(x, t_0)]_k$, $1 \le j, k \le 3$, where $[\cdot]_k$ denotes the k-th component of a vector under consideration. By noting that $\operatorname{supp} F(\cdot, x_0) \subset \Omega_2$, the application of Lemma 4 to (5.9) implies

$$\int_\Omega (|\nabla F(x, t_0)|^2 + s^2 |F(x, t_0)|^2) e^{2s\varphi(x,t_0)} dx \le Cs \int_\Omega |\operatorname{rot} F(x, t_0)|^2 e^{2s\varphi(x,t_0)} dx \tag{5.10}$$

for all large $s > 0$. Substituting (5.7) into (5.10), we obtain

$$\begin{aligned} &\int_\Omega (|\nabla F(x, t_0)|^2 + s^2 |F(x, t_0)|^2) e^{2s\varphi(x,t_0)} dx \\ \le\ & o(1) \int_\Omega (|\nabla F(x, t_0)|^2 + |F(x, t_0)|^2) e^{2s\varphi(x,t_0)} dx + Cs^3 M_1^2 e^{2s\mu_2} + Ce^{Cs} s^3 D_1(v)^2 \end{aligned} \tag{5.11}$$

for all large $s > 0$.

Choosing $s > 0$ sufficiently large, we can absorb the first term on the right-hand side of (5.11) into the left-hand side, and obtain

$$\int_\Omega (|\nabla F(x, t_0)|^2 + |F(x, t_0)|^2) e^{2s\varphi(x,t_0)} dx \le Cs^3 M_1^2 e^{2s\mu_2} + Ce^{Cs} s^3 D_1(v)^2$$

for all large $s > 0$. Then, by the same argument as in the final part of the proof of Theorem 3 (i), we can complete the proof of (5.5). ■

In place of Lemma 4, one may apply a Carleman estimate (e.g., Vogelsang [21]) for a system $\operatorname{rot} w = g$ and $\operatorname{div} w = 0$, but we do not exploit here.

6. Appendix. Derivation of Lemma 2 from Lemma 3.

Let $t \in I$ be arbitrarily fixed. Then, applying Lemma 3 to $-\Delta r(x, t) = g(x, t)$ in Q, we have that there exist constants C independents of s and constant s_1 such that

$$\begin{aligned} &\int_\Omega (s|\nabla r(x, t)|^2 + s^3 |r(x, t)|^2) e^{2\widetilde{s}\varphi(x,t_0)} dx \\ \le\ & C \int_\Omega |g(x, t)|^2 e^{2\widetilde{s}\varphi(x,t_0)} dx + C\widetilde{s}^3 \int_{\partial\Omega} (|\nabla r(x, t)|^2 + |r(x, t)|^2) e^{2\widetilde{s}\varphi(x,t_0)} dS \end{aligned} \tag{6.1}$$

for all $\widetilde{s} > s_1$.

We set $s_* := s_1 e^{\lambda\beta\delta^2}$. Let $s > s_*$. Then

$$se^{-\lambda\beta(t-t_0)^2} \ge se^{-\lambda\beta\delta^2} > s_1$$

for all $t \in \overline{I} = [t_0 - \delta,\, t_0 + \delta]$. Then in (6.1), taking $\widetilde{s} := se^{-\lambda\beta(t-t_0)^2}$, we obtain

$$\int_\Omega (se^{-\lambda\beta(t-t_0)^2}|\nabla r(x,t)|^2 + s^3e^{-3\lambda\beta(t-t_0)^2}|r(x,t)|^2)e^{2se^{-\lambda\beta(t-t_0)^2}\varphi(x,t_0)}dx$$

$$\le C\int_\Omega |g(x,t)|^2 e^{2se^{-\lambda\beta(t-t_0)^2}\varphi(x,t_0)}dx + Cs^3e^{-3\lambda\beta(t-t_0)^2}\int_{\partial\Omega}(|\nabla r|^2 + |r|^2)e^{2se^{-\lambda\beta(t-t_0)^2}\varphi(x,t_0)}dS.$$

Here, by $e^{-\lambda\beta(t-t_0)^2} \ge e^{-\lambda\beta\delta^2}$ for $t \in I$ and $e^{-\lambda\beta(t-t_0)^2}\varphi(x,t_0) = \varphi(x,t)$, we see

$$\int_\Omega (se^{-\lambda\beta\delta^2}|\nabla r(x,t)|^2 + s^3e^{-3\lambda\beta\delta^2}|r(x,t)|^2)e^{2s\varphi(x,t)}dx$$

$$\le C\int_\Omega |g(x,t)|^2 e^{2s\varphi(x,t)}dx + Cs^3\int_{\partial\Omega}(|\nabla r|^2 + |r|^2)e^{2s\varphi(x,t)}dS.$$

Multiplying with $e^{3\lambda\beta\delta^2}$ and integrating over $t \in I$, we complete the derivation of Lemma 2. ■

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