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TESI DI DOTTORATO

On non autonomous Ornstein-Uhlenbeck evolution operators in infinite dimension

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Riassunto

La presente tesi riguarda equazioni di evoluzione non autonome associate a operatori di Ornstein-Uhlenbeck dipendenti dal tempo in uno spazio di Hilbert separabile. Si presentano come equazioni di Kolmogorov di equazioni di evoluzione stocastiche lineari a derivate parziali (SPDEs) con rumore Gaussiano. Nel caso autonomo, le SPDEs e le loro equazioni di Kolmogorov sono state ampiamente studiate; uno degli approcci più fruttuosi è quello di Da Prato-Zabczyk, che sfrutta la teoria dei semigruppi. Nel caso non autonomo i semigruppi sono sostituiti da operatori di evoluzione la cui teoria non è completa e soddisfacente come nel caso autonomo. Effettivamente, lo studio delle equazioni di Kolmogorov non autonome in dimensione infinita è solo agli inizi.

Nel primo capitolo di questa tesi riassumiamo alcune nozioni di analisi funzionale, di analisi in dimensione infinita e riguardo problemi parabolici astratti non autonomi. Nel secondo capitolo consideriamo equazioni di Kolmogorov lineari retrograde non autonome e le loro soluzioni mild, espresse attraverso l'operatore di evoluzione di Ornstein-Uhlenbeck $P_{s,t}$. Inoltre studiamo le proprietà di regolarizzazione di $P_{s,t}$ lungo opportune direzioni e nel capitolo tre le utilizziamo per dimostrare risultati di massima regolarità Hölderiana per le soluzioni mild. Nel quarto capitolo studiamo il comportamento di $P_{s,t}$ su un particolare spazio lineare di funzioni cilindriche e regolari che sono essenziali per ulteriori risultati. Nel quinto capitolo estendiamo $P_{s,t}$ ad un operatore continuo e limitato tra spazi L^p rispetto ad opportune misure di probabilità. Per essere più precisi, sotto ipotesi naturali mostriamo l'esistenza di un sistema di evoluzione di misure Gaussiane, cioè una famiglia $\{\gamma_r\}_{r\in\mathbb{R}}$ di misure Gaussiane tale che per ogni $s \leqslant t$ e per ogni $f : X \mapsto \mathbb{R}$ continua e limitata abbiamo

$$\int_X P_{s,t} f \, d\gamma_s = \int_X f \, d\gamma_t.$$

In questo caso, $P_{s,t}$ può essere esteso ad un operatore lineare limitato da $L^p(X, \gamma_t)$ a $L^p(X, \gamma_s)$, con $s \leqslant t$. Dimostriamo una famiglia di disuguaglianze di tipo Log-Sobolev e le usiamo per dimostrare un risultato di ipercontrattività per l'operatore di evoluzione di Ornstein-Uhlenbeck. Nel sesto capitolo dimostriamo una formula di rappresentazione per $P_{s,t}$ tramite l'operatore di seconda quantizzazione e la usiamo per dimostrare ulteriori proprietà di $P_{s,t}$.

Abstract

The present thesis concerns non autonomous evolution equations driven by time depending Ornstein-Uhlenbeck operators in an infinite dimensional separable Hilbert space. They arise as Kolmogorov equations of linear stochastic evolution PDEs with Gaussian noise, rewritten as stochastic evolution equation in suitable Hilbert spaces. In the autonomous case, stochastic PDEs and their Kolmogorov equations are widely studied; one of the most fruitful approaches is the Da Prato-Zabczyk one, that exploits semigroup theory. In the non autonomous case, semigroups of operators are replaced by evolution operators whose theory is not complete and satisfying as in the autonomous case. In fact, the study of non autonomous Kolmogorov equations in infinite dimensions is just at the beginning.

In the first chapter of this thesis we recall some preliminaries about functional analysis, infinite dimensional analysis and about abstract non autonomous parabolic problems. In chapter two we consider backward linear non autonomous Kolmogorov equations and their mild solutions, expressed through the Ornstein-Uhlenbeck evolution operator $P_{s,t}$. Moreover we study smoothing properties of $P_{s,t}$ along suitable directions and in chapter three we use these results to prove maximal Hölder regularity for the mild solutions. In chapter four we study the behavior of $P_{s,t}$ on a linear space of cylindrical and smooth functions that are essential for further results. In chapter five we extend $P_{s,t}$ to a continuous and bounded operator between L^p spaces with respect to suitable probability measures. To be more precise, under natural assumptions we show the existence of an evolution system of Gaussian measures, namely a family $\{\gamma_r\}_{r\in\mathbb{R}}$ of Gaussian measures such that for every $s \leqslant t$ and for every continuous and bounded $f : X \mapsto \mathbb{R}$ we have

$$\int_X P_{s,t} f \, d\gamma_s = \int_X f \, d\gamma_t.$$

In this case, $P_{s,t}$ can be extended to a linear bounded operator from $L^p(X,\gamma_t)$ to $L^p(X,\gamma_s)$, with $s \leqslant t$. We prove a family of Logarithmic-Sobolev inequalities, and we use them to prove a hypercontractivity result for the evolution operator. In chapter six we prove a representation formula for $P_{s,t}$ as a second quantization operator and we use it to prove further properties of $P_{s,t}$.

Introduction

This thesis deals with Orstein-Uhlenbeck evolution operators $P_{s,t}$ in a separable Hilbert space $(X, \langle\cdot,\cdot\rangle, \|\cdot\|_X)$. They arise as transition evolution operators of a general class of stochastic differential equations with Gaussian noise, that are abstract models of certain stochastic partial differential equations. However, our approach is purely deterministic, which is made possible by the Mehler type representation formula (3) for $P_{s,t}$.

The autonomous case is well understood. We recall that Ornstein-Uhlenbeck semigroups P_t are defined in the space $B_b(X)$ of the bounded Borel measurable functions $\varphi : X \longrightarrow \mathbb{R}$ by $P_0\varphi = \varphi$ and

$$P_t\varphi(x) = \int_X \varphi(y)\mathcal{N}_{e^{tA}x, Q(t)}(dy), \quad t > 0, \tag{1}$$

where $A : D(A) \subset X \mapsto X$ is the infinitesimal generator of a strongly continuous semigroup e^{tA}, $Q(t) = \int_0^t e^{sA}Qe^{sA^\star}ds$, $Q \in \mathcal{L}(X)$ being self-adjoint and non negative, and $Q(t) \in \mathcal{L}_1(X)$ (namely, it has finite trace) for every $t > 0$. Here and in the following, $\mathcal{N}_{m,R}$ denotes the Gaussian measure with mean $m \in X$ and covariance $R \in \mathcal{L}_1(X)$.

Detailed results about smoothing properties, asymptotic behavior, existence (and uniqueness) of invariant measures γ, properties of the realizations of P_t in the spaces $L^p(X, \gamma)$ such as analyticity, hypercontractivity, characterizations of the domains of the infinitesimal generators, etc., are well known. See e.g [DZ02, Chapters 6, 10] and the bibliographies of the survey papers [GN03; LP20b]. If X is finite dimensional, even more general transition evolution operators have been widely studied, see e.g. [Add13; AAL17; AL14a; AL14b; ALL13; DL07; GL08; GL09; KLL10].

In contrast to this rich theory, only a few papers, such as [Knä11; De 22; BD23; CL21; OR16a; AD] have been devoted to the non autonomous case in infinite dimension.

This thesis is a step forward filling the gap. Its main results are smoothing properties of $P_{s,t}$ along suitable directions and optimal Schauder type theorems, still along suitable directions, contained in Chapters 2 and 3, differentiability of $P_{s,t}\varphi(x)$ with respect to t and s for a class of good functions φ, contained in Chapter 4, logarithmic Sobolev inequalities and hypercontractivity of $P_{s,t}$ in L^p spaces with respect to suitable families of measures, contained in Chapter 5, which rely on the results of Chapter 4. Most of these results are in the papers [De 22; BD23; AD].

The Ornstein-Uhlenbeck evolution operator $\{P_{s,t}\}_{s\leqslant t}$ is defined on $B_b(X)$ by

$$P_{t,t} = I \quad \text{for any} \quad t \in \mathbb{R}, \tag{2}$$

$$\begin{aligned} P_{s,t}\varphi(x) &= \int_X \varphi(y)\, \mathcal{N}_{m^x(t,s),Q(t,s)}(dy) \\ &= \int_X \varphi(y + m^x(t,s))\, \mathcal{N}_{0,Q(t,s)}(dy), \quad s < t, \end{aligned} \tag{3}$$

where (3) is a non autonomous Mehler formula. Here, for any $s \geqslant t$ we have

$$Q(t,s) = \int_s^t U(t,r)Q(r)U(t,r)^\star\, dr, \tag{4}$$

$$m^x(t,s) = U(t,s)x + g(t,s), \tag{5}$$

$$g(t,s) = \int_s^t U(t,r)f(r)\, dr. \tag{6}$$

where $\{U(t,r)\}_{t\geqslant r}$ is a strongly continuous evolution operator in X. Again, we need that the operators $Q(t,s)$ defined by (4.8) have finite trace. Clearly $P_{s,t}$ coincides with P_{t-s} defined in (1) if $f \equiv 0$, $Q(t) = Q$ constant and $U(t,s) = e^{(t-s)A}$.

As mentioned above $P_{s,t}$ is the transition evolution operator associated to stochastic differential equation. More precisely, let Y be a separable Hilbert space and $\{B(t)\}_{t\in[0,T]} \subseteq \mathcal{L}(Y;X)$. We consider the following forward stochastic differential equation

$$\begin{cases} dX_t(s,x) = \big(A(t)X_t(s,x) + f(t)\big)dt + B(t)dW_t, & t > s, \\ X_s(s,x) = x \in X, \end{cases} \tag{7}$$

where W_t is a Y-valued cylindrical Wiener process. If $Q(t) = B(t)B(t)^\star$, $P_{s,t}$ is the transition evolution operator associated to (7), namely

$$P_{s,t}\varphi(x) = \mathbb{E}\big(\varphi(X_t(s,x))\big), \quad s \leqslant t.$$

For a proof of this fact in the autonomous case see [DZ14]. For the non asutonomous case we refer to [DIT82; Tub82; VZ08; Sei93; SS03].

Let us give a detailed description of any chapter.

Chapter 1. Here we define the function spaces and we recall the results of functional analysis that are used throughout the thesis. In particular, sections 1.5 and 1.7 are concerned with Gaussian measures in infinite dimensional Hilbert spaces and to abstract non autonomous Cauchy problems, respectively.

Chapter 2. This chapter is devoted to the study of $P_{s,t}$ in spaces of continuous functions and we study smoothing properties of $P_{s,t}$ along suitable directions. A part of our results are extensions to the non autonomous case of some theorems from [GN03, Sect. 3]. It is well known that in infinite dimension the classical Ornstein-Uhlenbeck semigroup is smoothing only along suitable directions, see [Bog98], as well as the heat semigroup generated by the Gross Laplacian, see [DZ02]. Non autonomous Ornstein Uhlenbeck equations in infinite dimension were studied in [CL21] under the assumption that $U(t,s)(X) \subseteq Q(t,s)^{\frac{1}{2}}(X)$. In this case, $P_{s,t}$ is strong Feller, namely it maps $B_b(X)$ (the set of bounded Borel measurable functions) into $C_b(X)$. A more general class of non autonomous evolution families $P_{s,t}$ associated to stochastic differential equations with Levy noise are studied in [OR16b] again in the Strong Feller case. Others authors looked for smoothing results along suitable directions for perturbations of autonomous Ornstein-Uhlenbeck semigroups. For further readings we refer to [BF22a; BF23a; BF23b] and [Mas07].

Chapter 3. Let $T > 0$. Given $t \in [0, T]$, in this chapter we consider the function

$$u(s,x) := P_{s,t}\varphi(x) - \int_s^t \big(P_{s,\sigma}\psi(\sigma,\cdot)\big)(x)\,d\sigma, \quad \varphi \in C_b(X),\ \psi \in C_b\big([0,t]\times X\big). \tag{8}$$

(8) turns out to be the mild solution to a class of backward non autonomous initial value problems,

$$\begin{cases} \partial_s u(s,x) + L(s)u(s,x) = \psi(s,x), & s < t,\ x \in X, \\ u(t,x) = \varphi(x), \quad x \in X, \end{cases} \tag{9}$$

where the operators $L(t)$ are of Ornstein-Uhlenbeck type

$$L(t)\varphi(x) = \frac{1}{2}\mathrm{Tr}\Big(Q(t)D^2\varphi(x)\Big) + \langle x, A(t)^\star \nabla\varphi(x)\rangle_X + \langle f(t), \nabla\varphi(x)\rangle_X, \tag{10}$$

the family $\{A(t)\}_{t\in[0,T]}$ is associated to $U(t,s)$ and $Q(t)$ is a self-adjoint nonnegative operator for every $t \in [0,T]$. In this chapter we use smoothing properties of $P_{s,t}$ to prove Schauder type theorems for u along suitable directions. In [CL21], $P_{s,t}$ maps $B_b(X)$ into $C_b^k(X)$ for every $k \in \mathbb{N}$ and it is proven that maximal Hölder regularity for (8) holds along any direction. In the autonomous case in [CL19] Schauder type theorems along suitable directions are proven in the case of the classical Ornstein-Uhlenbeck semigroup. Still in the autonomous case, the first Schauder estimates for Ornstein-Uhlenbeck type equations were proven in [DL95; Lun97] in finite dimension and in [CD96] in some infinite dimensional equations. Maximal Hölder regularity along suitable directions was proved in [LR21] for a general class of autonomous equations. In finite dimension Schauder theorems were proved in [LSU68; KKL75] for equations with bounded coefficients.

In the next chapters we assume $X = Y$ as Hilbert spaces and $f = 0$ in (3).

Chapter 4. The main purpose of this chapter is to prove that for every $s < t$ and $x \in X$ we have

$$\frac{\partial}{\partial s}P_{s,t}\varphi(x) = -L(s)P_{s,t}\varphi(x), \tag{11}$$

$$\frac{\partial}{\partial t}P_{s,t}\varphi(x) = P_{s,t}L(t)\varphi(x), \tag{12}$$

where φ belongs to the space of smooth cylindrical functions $\mathcal{E}_t(X)$ defined in (4.27). Although (11) and (12) are expected formulas, their proof is a non trivial issue. Even if X is finite dimensional, the realizations of the the operators $\{P_{s,t}\}_{(s,t)\in\Delta}$ in $B_b(X)$, $C_b(X)$ and others function spaces, do not satisfy the classical conditions of [Acq88; AT87; AT92; Sch04; Tan79; Paz83], and the techniques used in [KLL10] to overcome this problem are not extendable to the infinite dimensional case since they heavily rely on local compactness, see Remark 4.2.18 for details. So we employ a completely new procedure. It is worth to mention that it could be possible to derive (11) and (12) via the Itô formula, such stochastic approach involves significant technical complexity compared to the autonomous case, see Remark 4.2.17. So we decided to present a more direct and simple deterministic proof.

Chapter 5. The main result of this chapter is the hypercontractivity of $P_{s,t}$ in Lebesgue spaces with respect to suitable measures. As noted e.g. in [DR08] a single invariant measure in general does not exist in this setting, being replaced by an evolution system of measures, namely a family of Borel probability measures $\{\gamma_r\}_{r\in\mathbb{R}}$ in X such that

$$\int_X P_{s,t}\varphi(x)\gamma_s(dx) = \int_X \varphi(x)\gamma_t(dx),$$

for every $s < t$ and for every bounded and continuous $\varphi : X \to \mathbb{R}$. So, an obviuous difficulty arises, namely the spaces $L^p(X, \gamma_r)$ depend explicitly on r and we cannot set our problem in a fixed L^p space. In our case, under natural assumption, an evolution system of measure $\{\gamma_t\}_{t\in\mathbb{R}}$ for $P_{s,t}$ exists. This result was proven in [OR16a] in a more general setting; see also [GL08] [DR08] for the case $X = \mathbb{R}^n$. It is easily seen that for $t > s$ $P_{s,t}$ has an extension (still denoted by $P_{s,t}$) belonging to $\mathcal{L}(L^p(X, \gamma_t); L^p(X, \gamma_s))$ whith unit norm. We show that in fact $P_{s,t}$ is a contraction from $L^q(X, \gamma_t)$ to $L^p(X, \gamma_s)$ with

$$p \leqslant (q-1)e^{\frac{t-s}{2\kappa}} + 1,$$

where k is the constant appearing in (13). The proof relies on a family of logaritmic Sobolev inequalities,

$$\int_X |\varphi|^p \log\left(|\varphi|^p\right) \, d\gamma_t \leqslant m_t(|\varphi|^p) \log\left(m_t\left(|\varphi|^p\right)\right) + \kappa p^2 \int_X |\varphi|^{p-2} \left\| Q\left(t\right)^{\frac{1}{2}} \nabla\varphi \right\|^2 \mathbb{1}_{\{\varphi\neq 0\}} d\gamma_t, \tag{13}$$

where $\varphi \in C_b^1(X)$, $t \in \mathbb{R}$, $p \in (1, +\infty)$, $m_t(\varphi) := \displaystyle\int_X \varphi \, d\nu_t$ and κ is an explicit positive constant (see Theorem 5.2.2). If $X = \mathbb{R}^n$ (13) was proven in [ALL13]. In infinite dimension a hyperboundedness result for $\{P_{s,t}\}_{(s,t)\in\overline{\Delta}}$ was proven in [OR16a, Thm 5.9] as a consequence of a Harnack type inequality; however the constant is not necessarily 1

Chapter 6. In this Chapter we give an alternative representation formula of $P_{s,t}$ as a second quantized operator, in the case that there exists the evolution system of Gaussian measures above mentioned. More precisely we prove that for every $s \leqslant t$ and $f \in B_b(X)$ we have

$$(P_{s,t}f) \circ (I - P_s) = \Gamma_{\gamma_s,\gamma_t}(L^\star)f,$$

where $L := Q(t, -\infty)^{-\frac{1}{2}} U(t, s) Q(s, -\infty)^{\frac{1}{2}}$ belongs to $\mathcal{L}(X)$ and P_s is the orthogonal projection on $\ker Q(s, -\infty)$. Here $\Gamma_{\gamma_s,\gamma_t}$ is a non autonomous version of the second quantization operator. Other consequences of such a representation formula are still under study. We refer to [CG96; Jan97; Sim74]. Besides its own interest such a formula allows to prove that

$$\|P_{s,t}f - m_t(f)\|_{L^p(X,\gamma_s)} \leqslant c_p e^{-\omega(t-s)} \|f\|_{L^p(X,\gamma_t)} \qquad \forall f \in L^p(X, \gamma_t),$$

where c_p and ω are two positive constants (see Proposition 6.2.5). Moreover we give an alternative proof of the hypercontractivity of $P_{s,t}$ in L^p spaces with respect to (5.18).

The results of Chapter 2 and Chapter 3 are contained in [De 22] and the results of Chapter 4 and Chapter 5 are in [BD23]. Chapter 6 is a collection of partial results of [AD].

Contents

Chapter 1

Preliminaries

1.1 Function spaces

If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are real Banach spaces we denote by $\mathcal{L}(X;Y)$ the space of bounded linear operators from X to Y. If $Y = \mathbb{R}$ we simply write $X^\star$ instead of $\mathcal{L}(X;\mathbb{R})$. For $k \geqslant 2$, $\mathcal{L}^k(X;Y)$ is the space of the k-linear bounded operators $T : X^k \longrightarrow Y$ endowed with the norm

$$\|T\|_{\mathcal{L}^k(X;Y)} = \sup\left\{\frac{\|T(x_1, ..., x_k)\|_Y}{\|x_1\|_X \cdots \|x_k\|_X} : \ x_1, ..., x_k \in X \setminus \{0\}\right\}.$$

We set $\mathcal{L}^0(X) := X$ and $\mathcal{L}(X) := \mathcal{L}(X;X)$. By $B(X;Y)$, $B_b(X;Y)$ and $C_b(X;Y)$ we denote the space of bounded functions from X to Y, the space of bounded Borel functions from X to Y and the space of bounded and continuous functions from X to Y, respectively. We endow them with the sup norm

$$\|F\|_\infty = \sup_{x \in X} \|F(x)\|_Y .$$

If $Y = \mathbb{R}$, we simply write $B_b(X)$ and $C_b(X)$ instead of $B_b(X;\mathbb{R})$ and $C_b(X;\mathbb{R})$, respectively.

1.1.1 Hölder, Zygmund and C^k spaces

For $\alpha \in (0,1)$ we define the Hölder spaces as

$$C_b^\alpha(X;Y) = \left\{F \in C_b(X;Y) : \ [F]_{C_b^\alpha(X;Y)} = \sup_{\substack{x \in X, \\ h \in X \setminus \{0\}}} \frac{\|F(x+h) - F(x)\|_Y}{\|h\|_X^\alpha} < +\infty\right\},$$

$$\|F\|_{C_b^\alpha(X;Y)} = \sup_{x \in X} \|F(x)\|_Y + [F]_{C_b^\alpha(X;Y)},$$

and the Lipschitz space as

$$\mathrm{Lip_b}(X;Y) = \left\{F \in C_b(X;Y) : \ [F]_{\mathrm{Lip_b}(X;Y)} = \sup_{\substack{x \in X, \\ h \in X \setminus \{0\}}} \frac{\|F(x+h) - F(x)\|_Y}{\|h\|_X} < +\infty\right\},$$

$$\|F\|_{\mathrm{Lip_b}(X;Y)} = \sup_{x\in X} \|F(x)\|_Y + [F]_{\mathrm{Lip_b}(X;Y)}.$$

Again, if $Y = \mathbb{R}$ we write $C_b^\alpha(X)$ and $\mathrm{Lip_b}(X)$ instead of $C_b^\alpha(X;\mathbb{R})$ and $\mathrm{Lip_b}(X;\mathbb{R})$, respectively. $Z_b^1(X;Y)$ is the Zygmund space. It is defined by

$$Z_b^1(X;Y) = \Bigg\{ F \in C_b(X;Y) \ : \\ [F]_{Z_b^1(X;Y)} = \sup_{\substack{x\in X,\\ h\in X\smallsetminus\{0\}}} \frac{\|F(x+2h) - 2F(x+h) + F(x)\|_Y}{\|h\|_X} < +\infty \Bigg\},$$

and it is endowed with the norm

$$\|F\|_{Z_b^1(X;Y)} = \|F\|_\infty + [F]_{Z_b^1(X;Y)}.$$

We say that a map $f : X \longrightarrow \mathbb{R}$ is Gâteaux differentiable at $x \in X$ if there exists a bounded linear operator $l_x : X \longrightarrow \mathbb{R}$ such that for every $h \in X$ we have

$$\lim_{t\to 0} \frac{f(x+th) - f(x) - l_x(h)}{t} = 0. \tag{1.1}$$

l_x is the Gâteaux differential of f at x and we set $l_x = D^G f(x)$.

We say that a map $f : X \longrightarrow \mathbb{R}$ is Fréchet differentiable at $x \in X$ if there exists a bounded linear operator $t_x : X \longrightarrow \mathbb{R}$ such that

$$\lim_{\|h\|_X\to 0} \frac{f(x+h) - f(x) - t_x(h)}{\|h\|_X} = 0. \tag{1.2}$$

t_x is the Fréchet differential of f at x and we set $t_x = Df(x)$.

More generally, for every $F : X \longrightarrow Y$, we say that F is Gâteaux differentiable at x if there exists $L_x \in \mathcal{L}(X,Y)$ such that for every $h \in X$, we have

$$\lim_{t\to 0} \frac{\|F(x+th) - F(x) - L_x(h)\|_Y}{t} = 0. \tag{1.3}$$

L_x is the Gâteaux differential of F at x and we denote it by $D^G F(x)$. We say that F is Fréchet differentiable at x if there exists $T_x \in \mathcal{L}(X,Y)$ such that

$$\lim_{\|h\|_X\to 0} \frac{\|F(x+h) - F(x) - T_x(h)\|_Y}{\|h\|_X} = 0. \tag{1.4}$$

T_x is the Fréchet differential of F at x and we denote it by $DF(x)$.

If $f : X \longrightarrow \mathbb{R}$ is Fréchet differentiable at x, we say that f is twice Fréchet differentiable at x if $Df : X \longrightarrow X^\star$ is Fréchet differentiable at x. We denote by $D^2\varphi$ the unique element of $\mathcal{L}^2(X;\mathbb{R})$ such that

$$D^2\varphi(x)(k,h) := (T_x k)(h), \quad h,k \in X,$$

where T_x is the operator in (1.4) with F replaced by $D\varphi$ and $Y = X^\star$.

If f is $(k-1)$-times Fréchet differentiable at x with $k \geqslant 2$, we say that f is k-times Fréchet differentiable at x if $D^{k-1}f : X \longrightarrow \mathcal{L}^{k-1}(X;\mathbb{R})$ is Fréchet differentiable at x. In this case we denote by $D^k\varphi$ the unique element of $\mathcal{L}^k(X;\mathbb{R})$ such that

$$D^k f(x)(h_1, ..., h_k) := (T_x h_1)(h_2, ..., h_k)$$

where T_x is the operator in (1.4) with $F(x)$ replaced by $D^{k-1}f(x)$ and $Y = \mathcal{L}^{k-1}(X;\mathbb{R})$.

For every $k \in \mathbb{N}$, $C_b^k(X)$ is the subspace of $C_b(X)$ consisting of all functions $f : X \longrightarrow \mathbb{R}$ k-times Fréchet differentiable at any point with $D^j f$ continuous and bounded from X to $\mathcal{L}^j(X;\mathbb{R})$ for $j \leqslant k$. $C_b^k(X)$ is endowed with the norm

$$\|f\|_{C_b^k(X)} := \|F\|_\infty + \sum_{j=1}^k \sup_{x\in X} \left\|D^j f(x)\right\|_{\mathcal{L}^j(X;\mathbb{R})}. \tag{1.5}$$

Assume now that X is a separable Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_X$.

Let $\varphi \in C_b^k(X)$. By the Riesz representation theorem, for every $j = 1, ..., k$, for every $x \in X$ there are unique $S_x^j \in \mathcal{L}^{j-1}(X)$ such that

$$D^j\varphi(x)(h_1, ..., h_j) = \langle S_x^j(h_1, ..., h_{j-1}), h_j\rangle_X, \qquad h_1, ..., h_j \in X.$$

We set $\nabla^j\varphi(x) := S_x^j$ and we call $\nabla\varphi(x)$ and $\nabla^2\varphi(x)$ the gradient and Hessian operator of φ at $x \in X$, respectively. Moreover

$$\|\varphi\|_{C_b^k(X)} = \|\varphi\|_\infty + \sum_{j=1}^k \sup_{x\in X} \left\|\nabla^j\varphi(x)\right\|_{\mathcal{L}^{j-1}(X)}.$$

Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of X and let $\varphi : X \to$ be a k-times Fréchet differentiable function. As in the finite dimensional case, for all $j = 1, ..., k$ we define the partial derivatives of φ of order j at $x \in X$ along the directions of such basis. Moreover it can be shown that for all $j = 1, ..., k$

$$\frac{\partial^j\varphi}{\partial e_{i_1}, ..., \partial e_{i_j}}(x) := \langle \nabla^j\varphi(x)(e_{i_1}, ..., e_{i_{j-1}}), e_{i_j}\rangle_X, \qquad i_1, ..., i_j \in \mathbb{N}.$$

Higher order Hölder and Zygmund spaces are defined as follows. For $\alpha \in (0,1)$ and $n \in \mathbb{N}$, we set

$$C_b^{n+\alpha}(X) := \left\{ f \in C_b^n(X) \ : \ D^n f \in C_b^\alpha(X;\mathcal{L}^n(X;\mathbb{R})) \right\},$$

$$\|f\|_{C_b^{n+\alpha}(X)} := \|f\|_{C_b^n(X)} + [D^n f]_{C_b^\alpha(X;\mathcal{L}^n(X;\mathbb{R}))}$$

and for $n \geqslant 2$,

$$Z_b^n(X) = \left\{ f \in C_b^{n-1}(X) \ : \ D^{n-1} f \in Z_b^1(X;\mathcal{L}^n(X;\mathbb{R})) \right\},$$

$$\|f\|_{Z_b^n(X)} := \|f\|_{C_b^{n-1}(X)} + [D^{n-1}F]_{Z_b^1(X;\mathcal{L}^{n-1}(X;\mathbb{R}))}.$$

Finally we introduce spaces of functions depending both on time and space variables. For every $a, b \in \mathbb{R}$, $a < b$ and $\alpha > 0$, we denote by $C_b^{0,\alpha}\big([a,b] \times X\big)$ the space of all bounded continuous functions $\psi : [a,b] \times X \longrightarrow \mathbb{R}$ such that $\psi(s,\cdot) \in C_b^\alpha(X)$, for every $s \in [a,b]$, with

$$\|\psi\|_{C_b^{0,\alpha}([a,b]\times X)} = \sup_{s\in[a,b]} \|\psi(s,\cdot)\|_{C_b^\alpha(X)} < +\infty.$$

If $\alpha \geqslant 1$ we also require that the mappings

$$(s,x) \longrightarrow \frac{\partial \psi}{\partial h_1 ... \partial h_k}(s,x)$$

are continuous in $[a,b] \times X$, for every $h_1, ..., h_k \in X$ with $k \leqslant [\alpha]$. For every $k \in \mathbb{N}$ we denote by $Z_b^{0,k}\big([a,b] \times X\big)$ the space of all bounded continuous functions $\psi : [a,b] \times X \longrightarrow \mathbb{R}$ such that $\psi(s,\cdot) \in Z_b^k(X)$, for every $s \in [a,b]$, with

$$\|\psi\|_{Z_b^{0,k}([a,b]\times X)} = \sup_{s\in[a,b]} \|\psi(s,\cdot)\|_{Z_b^k(X)} < +\infty.$$

If $k \geqslant 2$, we also require that the mapping

$$(s,x) \longrightarrow \frac{\partial \psi}{\partial h_1 ... \partial h_i}(s,x)$$

are continuous in $[a,b] \times X$, for every $h_1, ..., h_i \in X$ with $i \leqslant k-1$.

1.1.2 Hölder, Zygmund and C^k spaces along a subspace E

Let $(E, \|\cdot\|_E)$ be a normed space such that $E \subseteq X$ with continuous embedding. For $\alpha \in (0,1)$ we define the Hölder spaces along E as

$$C_E^\alpha(X;Y) = \left\{ F \in C_b(X;Y) : \ [F]_{C_E^\alpha(X;Y)} = \sup_{\substack{x\in X,\\ h\in E\smallsetminus\{0\}}} \frac{\|F(x+h)-F(x)\|_Y}{\|h\|_E^\alpha} < +\infty \right\},$$

$$\|F\|_{C_E^\alpha(X;Y)} = \sup_{x\in X} \|F(x)\|_Y + [F]_{C_E^\alpha(X:Y)},$$

and the Lipschitz space along E as

$$\mathrm{Lip_E}(X;Y) = \left\{ F \in C_b(X;Y) : \ [F]_{\mathrm{Lip_E}(X;Y)} = \sup_{\substack{x\in X,\\ h\in E\smallsetminus\{0\}}} \frac{\|F(x+h)-F(x)\|_Y}{\|h\|_E} < +\infty \right\},$$

$$\|F\|_{\mathrm{Lip_E}(X;Y)} = \sup_{x\in X} \|F(x)\|_Y + [F]_{\mathrm{Lip_E}(X;Y)}.$$

Again, if $Y = \mathbb{R}$ we write $C_E^\alpha(X)$ and $\mathrm{Lip_E}(X)$ instead of $C_E^\alpha(X;\mathbb{R})$ and $\mathrm{Lip_E}(X;\mathbb{R})$, respectively. $Z_E^1(X;Y)$ is the Zygmund space along E. It is defined by

$$Z_E^1(X;Y) = \Bigg\{ F \in C_b(X;Y) \ : \ [F]_{Z_E^1(X;Y)} = \sup_{\substack{x\in X,\\ h\in E\smallsetminus\{0\}}} \frac{\|F(x+2h)-2F(x+h)+F(x)\|_Y}{\|h\|_E} < +\infty \Bigg\},$$

and it is endowed with the norm

$$\|F\|_{Z^1_E(X;Y)} = \|F\|_\infty + [F]_{Z^1_E(X;Y)}.$$

We say that a map $f : X \longrightarrow \mathbb{R}$ is E-Gâteaux differentiable at $x \in X$ if there exists a bounded linear operator $l_x : E \longrightarrow \mathbb{R}$ such that for every $h \in E$ we have

$$\lim_{t\to 0} \frac{f(x+th) - f(x) - l_x(h)}{t} = 0. \tag{1.6}$$

l_x is the E-Gâteaux differential of f at x and we set $l_x = D^G_E f(x)$.

We say that a map $f : X \longrightarrow \mathbb{R}$ is E-Fréchet differentiable at $x \in X$ if there exists a bounded linear operator $t_x : E \longrightarrow \mathbb{R}$ such that

$$\lim_{\|h\|_E\to 0} \frac{f(x+h) - f(x) - t_x(h)}{\|h\|_E} = 0. \tag{1.7}$$

t_x is the E-Fréchet differential of f at x and we set $t_x = D_E f(x)$. Clearly, if f is Fréchet differentiable at x then f is E-Fréchet differentiable at x and this is due to the continuous embedding of E in X.

More generally, for every $F : X \longrightarrow Y$, we say that F is E-Gâteaux differentiable at x if there exists $L_x \in \mathcal{L}(E, Y)$ such that for every $h \in E$, we have

$$\lim_{t\to 0} \frac{\|F(x+th) - F(x) - L_x(h)\|_Y}{t} = 0. \tag{1.8}$$

L_x is the E-Gâteaux differential of F at x and we denote it by $D^G_E F(x)$. We say that F is E-Fréchet differentiable at x if there exists $T_x \in \mathcal{L}(E, Y)$ such that

$$\lim_{\|h\|_E\to 0} \frac{\|F(x+h) - F(x) - T_x(h)\|_Y}{\|h\|_E} = 0. \tag{1.9}$$

T_x is the E- Fréchet differential of F at x and we denote it by $D_E F(x)$. Clearly, if $E = X$ the notions of E-Fréchet differentiability and E-Gâteaux differentiability coincide with the usual ones. In this case we omit the subindex E in the above notations. For instance, we write DF instead of $D_E F$ for the Fréchet derivative.

If $f : X \longrightarrow \mathbb{R}$ is E-Fréchet differentiable at x, we say that f is twice E-Fréchet differentiable at x if $D_E f : X \longrightarrow E^\star$ is E-Fréchet differentiable at x. We denote by $D^2_E\varphi$ the unique element of $\mathcal{L}^2(E;\mathbb{R})$ such that

$$D^2_E f(x)(k,h) := (T_x k)(h),$$

where T_x is the operator in (1.9) with $F(x)$ replaced by $D_E f$ and $Y = E^\star$.

If f is $(k-1)$-times E-Fréchet differentiable at x with $k \geqslant 2$, we say that f is k-times E-Fréchet differentiable at x if $D^{k-1}_E f : X \longrightarrow \mathcal{L}^{k-1}(E;\mathbb{R})$ is E-Fréchet differentiable at x. We denote by $D^k_E\varphi$ the unique element of $\mathcal{L}^k(E;\mathbb{R})$ such that

$$D^k_E f(x)(h_1, ..., h_k) := (T_x h_1)(h_2, ..., h_k)$$

where T_x is the operator in (1.9) with $F(x)$ replaced by $D^{k-1}_E f(x)$ and $Y = \mathcal{L}^{k-1}(E;\mathbb{R})$.

For every $k \in \mathbb{N}$, $C^k_E(X)$ is the subspace of $C_b(X)$ consisting of all functions $f : X \longrightarrow \mathbb{R}$ k-times E-Fréchet differentiable at any point with $D^j_E f$ continuous and bounded from X to $\mathcal{L}^j(E;\mathbb{R})$ for $j \leqslant k$. $C^k_E(X)$ is endowed with the norm

$$\|f\|_{C^k_E(X)} := \|F\|_\infty + \sum_{j=1}^{k} \sup_{x\in X} \left\| D^j_E f(x) \right\|_{\mathcal{L}^j(E;\mathbb{R})}. \tag{1.10}$$

Assume now that E is a separable Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_E$.

Let $\varphi \in C^k_E(X)$. By the Riesz representation theorem, for every $j = 1, ..., k$, for every $x \in X$ there are unique $S^j_x \in \mathcal{L}^{j-1}(E)$ such that

$$D^j_E \varphi(x)(h_1, ..., h_j) = \langle S^j_x(h_1, ..., h_{j-1}), h_j \rangle_E, \qquad h_1, ..., h_j \in E.$$

We set $\nabla^j_E \varphi(x) := S^j_x$ and we call $\nabla_E \varphi(x)$ and $\nabla^2_E \varphi(x)$ the E-gradient and E-Hessian operator of φ at $x \in X$, respectively. Moreover

$$\|\varphi\|_{C^k_E(X)} = \|\varphi\|_\infty + \sum_{j=1}^{k} \sup_{x\in X} \left\| \nabla^j_E \varphi(x) \right\|_{\mathcal{L}^{j-1}(E)}.$$

Let $\{e_k\}_{k\in\mathbb{N}}$ be a Hilbert basis of E and let $\varphi : X \to$ be a k-times E-Fréchet differentiable function. As in the finite dimensional case, for all $j = 1, ..., k$ we define the partial derivatives of φ of order j at $x \in X$ along the directions of such basis. Moreover it can be shown that for all $j = 1, ..., k$

$$\frac{\partial^j \varphi}{\partial e_{i_1}, ..., \partial e_{i_j}}(x) := \langle \nabla^j_E \varphi(x)(e_{i_1}, ..., e_{i_{j-1}}), e_{i_j} \rangle_X, \qquad i_1, ..., i_j \in \mathbb{N}.$$

Higher order Hölder and Zygmund spaces along E are defined as in the case $E = X$. Namely, for $\alpha \in (0, 1)$ and $n \in \mathbb{N}$, we set

$$C^{n+\alpha}_E(X) := \left\{ f \in C^n_E(X) \ : \ D^n_E f \in C^\alpha_E(X; \mathcal{L}^n(E;\mathbb{R})) \right\},$$

$$\|f\|_{C^{n+\alpha}_E(X)} := \|f\|_{C^n_E(X)} + [D^n_E f]_{C^\alpha_E(X;\mathcal{L}^n(E;\mathbb{R}))}$$

and for $n \geqslant 2$,

$$Z^n_E(X) = \left\{ f \in C^{n-1}_E(X) \ : \ D^{n-1}_E f \in Z^1_E(X; \mathcal{L}^n(E;\mathbb{R})) \right\},$$

$$\|f\|_{Z^n_E(X)} := \|f\|_{C^{n-1}_E(X)} + [D^{n-1}_E F]_{Z^1_E(X;\mathcal{L}^{n-1}(E;\mathbb{R}))}.$$

Clearly if $E = X$ the functional spaces above coincide with the usual ones.

Finally we introduce spaces of functions depending both on time and space variables. For every $a, b \in \mathbb{R}$, $a < b$ and $\alpha > 0$, we denote by $C^{0,\alpha}_E\big([a, b] \times X\big)$ the space of all bounded continuous functions $\psi : [a, b] \times X \longrightarrow \mathbb{R}$ such that $\psi(s, \cdot) \in C^\alpha_E(X)$, for every $s \in [a, b]$, with

$$\|\psi\|_{C^{0,\alpha}_E([a,b]\times X)} = \sup_{s\in[a,b]} \|\psi(s, \cdot)\|_{C^\alpha_E(X)} < +\infty.$$

If $\alpha \geqslant 1$ we also require that the mappings

$$(s,x) \longrightarrow \frac{\partial \psi}{\partial h_1 ... \partial h_k}(s,x)$$

are continuous in $[a,b] \times X$, for every $h_1, ..., h_k \in E$ with $k \leqslant [\alpha]$. For every $k \in \mathbb{N}$ we denote by $Z_E^{0,k}\big([a,b] \times X\big)$ the space of all bounded continuous functions $\psi : [a,b] \times X \longrightarrow \mathbb{R}$ such that $\psi(s,\cdot) \in Z_E^k(X)$, for every $s \in [a,b]$, with

$$\|\psi\|_{Z_E^{0,k}([a,b]\times X)} = \sup_{s\in[a,b]} \|\psi(s,\cdot)\|_{Z_E^k(X)} < +\infty.$$

If $k \geqslant 2$, we also require that the mapping

$$(s,x) \longrightarrow \frac{\partial \psi}{\partial h_1 ... \partial h_i}(s,x)$$

are continuous in $[a,b] \times X$, for every $h_1, ..., h_i \in E$ with $i \leqslant k-1$.

1.2 Linear operators in a Banach space

In this section we recall some basic definitions and results about the theory of linear operators.

Let $(X, \|\cdot\|_X)$ be a real Banach space.

Let $A : \mathrm{Dom}(A) \subseteq X \longrightarrow X$ be a linear operator. We say that A is closed if for every sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathrm{Dom}(A)$ such that $x_n \longrightarrow x$ and $Ax_n \longrightarrow y$ we have $x \in \mathrm{Dom}(A)$ and $y = Ax$. If A is closed operator then $D(A)$ is a Banach spaces endowed with the graph norm

$$\|x\|_{D(A)} = \|x\|_X + \|Ax\|_X\,, \quad x \in D(A).$$

We say that A is closable if for every $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathrm{Dom}(A)$ such that $x_n \longrightarrow 0$ and $Ax_n \longrightarrow y$ we have $y = 0$. In this case we define the closure of A in the following way

$$\mathrm{Dom}(\overline{A}) := \{x \in X \mid \exists \{x_n\}_{n\in\mathbb{N}} \subseteq \mathrm{Dom}(A) \mid x_n \longrightarrow x,\ \{Ax_n\}_{n\in\mathbb{N}} \text{ converges in } X\},$$
$$\overline{A}x = \lim_{n\longrightarrow+\infty} Ax_n, \quad x \in \mathrm{Dom}(\overline{A}).$$

Given $A : D(A) \subseteq X \longrightarrow X$ and a closed subspace of $V \subseteq X$, we call part of A in V the operator $\widetilde{A}$ with domain $D(\widetilde{A}) = \{x \in D(A) \cap V \mid Ax \in V\}$ defined by $\widetilde{A}x = Ax$ for all $x \in D(\widetilde{A})$.

Let $A : \mathrm{Dom}(A) \subseteq X \longrightarrow X$ be a linear operator such that $\mathrm{Dom}(A)$ is dense in X. The adjoint of A is defined as the operator in $X^\star$ with domain

$$\mathrm{Dom}(A^\star) := \{x^\star \in X^\star \mid x \in D(A) \longrightarrow \langle Ax, x^\star\rangle \text{ has a linear continuous extension to the whole X}\}.$$

Since $D(A)$ is dense in X the continuous extension of $\langle A\cdot, x^\star\rangle$ is unique and by definition it is an element of $X^\star$ denoted by $A^\star x^\star$. Then we have

$$\langle Ax, x^\star\rangle = \langle x, A^\star x^\star\rangle, \quad \forall x \in \mathrm{Dom}(A),\ x^\star \in \mathrm{Dom}(A^\star).$$

To define the resolvent set and the spectrum of an operator, we introduce the complexification $\widetilde{X}$ of X defined by $\widetilde{X} = \{x + iy \,|\, x, y \in X\}$ and we endow $\widetilde{X}$ with the norm

$$\|x + iy\|_{\widetilde{X}} = \sup_{-\pi \leqslant \theta \leqslant \pi} \|x \cos\theta + y \sin\theta\|_X .$$

In the following by abuse of notation we still denote the complexification of X by X.

Let $A : \mathrm{Dom}(A) \subseteq X \longrightarrow X$ be a linear operator. We define the resolvent set of A as

$$\rho(A) := \{\lambda \in \mathbb{C} \;: \exists (A - \lambda \mathrm{Id}_X)^{-1} \in \mathcal{L}(X)\}.$$

$\sigma(A) := \mathbb{C} \smallsetminus \rho(A)$ is the spectrum of A and the elements $\lambda \in \sigma(A)$ such that $A - \lambda\mathrm{Id}_X$ is not injective are called eigenvalues. The elements $x \in \mathrm{Dom}(A)$ such that there exists an eigenvalue λ such that $Ax = \lambda x$ are called eigenvectors associated λ. We denote by $\sigma_p(A)$ the set of eigenvalues of A and it is called point spectrum of A. Moreover for every $\lambda \in \rho(A)$ we define the resolvent operator of A as

$$R(\lambda, A) = (A - \lambda\mathrm{Id}_X)^{-1}.$$

1.3 Linear operators in a Hilbert space

In this section we assume that $(X, \|\cdot\|, \langle\cdot,\cdot\rangle_X)$ is a real separable Hilbert space. We say that $T \in \mathcal{L}(X)$ is non-negative (respectively negative, non-positive, positive) if for every $x \in X \smallsetminus \{0\}$

$$\langle Tx, x\rangle_X \geqslant 0 \ (< 0,\ \geqslant,\ > 0).$$

Moreover we say that T is self-adjoint if $T = T^\star$.

Let $A : \mathrm{Dom}(A) \subseteq X \longrightarrow X$ be a linear operator. If there exists a unique operator $B : \mathrm{Dom}(B) \subseteq X \longrightarrow X$ such that $\mathrm{Dom}(B^2) = \mathrm{Dom}(A)$ and $A = B^2$ then we call B square root of A and we denote it by $\sqrt{A}$ or $A^{\frac{1}{2}}$. Clearly a general linear operator may not have a square root. A sufficient condition is given by the following proposition.

We refer to [RS80] for the following results.

Proposition 1.3.1. *Let $T \in \mathcal{L}(X)$ be a non-negative operator and let $G \in \mathcal{L}(X)$. The following statements hold true.*

1. *If G is non-negative and G and T commute then GT is a non-negative operator.*
2. *T has a unique square root $\sqrt{T}$. Moreover*
   - *(a) if G commutes with T then G commutes with $\sqrt{T}$,*
   - *(b) if T is bijective then $\sqrt{T}$ is bijective.*

Let $T \in \mathcal{L}(X)$. It easy to see that $T^\star T$ is a non-negative (self-adjoint) operator. Then we define the absolute value of T as

$$|T| := \sqrt{T^\star T} \tag{1.11}$$

Let $R \in \mathcal{L}(X)$. We denote by $\ker R$ the kernel of R and by $(\ker R)^\perp$ its orthogonal subspace in X.

We denote by $H_R := R(X)$ the range of the operator R and we recall that $(\ker R)^\perp = \overline{R(X)}$. In order to provide H_R with a Hilbert structure, we recall that the restriction $R_{|_{(\ker R)^\perp}}$ is a injective operator, and so

$$R_{|_{(\ker R)^\perp}} : (\ker R)^\perp \subseteq X \longrightarrow H_R$$

is bijective. We call pseudo-inverse of R the linear bounded operator $R^{-1} : H_R \longrightarrow X$ where for all $y \in H_R$, $R^{-1}y$ is the unique $x \in (\ker R)^\perp$ such that $Rx = y$, see [LR15, Appendix C]. We introduce the scalar product

$$\langle x, y\rangle_{H_R} := \langle R^{-1}x, R^{-1}y\rangle_X, \quad x, y \in H_R \tag{1.12}$$

and its associated norm $\|x\|_{H_R} := \|R^{-1}x\|_X$. Then $(H_R, \|\cdot\|_{H_R}, \langle\cdot,\cdot\rangle_{H_R})$ is a separable Hilbert space and a Borel subset of X (see [Kec12, Theorem 15.1]). A possible orthonormal basis of H_R is given by $\{Re_k\}_{k\in\mathbb{N}}$, where $\{e_k\}_{k\in\mathbb{N}}$ is any orthonormal basis of $(\ker R)^\perp$. Denoting by P the orthogonal projection on $\ker R$, we recall that

$$RR^{-1} = \mathrm{Id}_{H_R}, \qquad R^{-1}R = \mathrm{Id}_X - P. \tag{1.13}$$

Notice that for every $x \in H_R$

$$\|x\|_X = \|RR^{-1}x\|_X \leqslant \|R\|_{\mathcal{L}(X)} \|R^{-1}x\|_X \leqslant \|R\|_{\mathcal{L}(X)} \|x\|_{H_R}.$$

Proposition 1.3.2. *If $\varphi \in C_b^1(X)$, then $\varphi \in C^1_{H_R}(X)$. Moreover for every $x \in X$ $\nabla_{H_R}\varphi(x) = R^2\nabla\varphi(x)$ and $\|\nabla_{H_R}\varphi(x)\|_{H_R} = \|R\nabla\varphi(x)\|_X$.*

Proof. Let $\varphi \in C_b^1(X)$, $x \in X$ and $h \in H_R \smallsetminus \{0\}$. We have

$$\left|\frac{\varphi(x+h) - \varphi(x) - \langle\nabla\varphi(x), h\rangle_X}{\|h\|_{H_R}}\right| = \left|\frac{\varphi(x+h) - \varphi(x) - \langle\nabla\varphi(x), h\rangle_X}{\|h\|_X}\right| \frac{\|h\|_X}{\|h\|_{H_R}}. \tag{1.14}$$

Since H_R is continuously embedded in X, $\dfrac{\|h\|_X}{\|h\|_{H_R}}$ is bounded and φ is H_R-differentiable at $x \in X$. Moreover $\langle\nabla\varphi(x), h\rangle_X = \langle\nabla_{H_R}\varphi(x), h\rangle_{H_R}$ for $h \in H_R$. Denoting by P the orthonormal projection on $\ker R$, we get

$$\begin{aligned}
\langle\nabla_{H_R}\varphi(x), h\rangle_{H_R} &= \langle\nabla\varphi(x), h\rangle_X = \langle\nabla\varphi(x), (I-P)h\rangle_X = \langle\nabla\varphi(x), R^{-2}R^2h\rangle_X\\
&= \langle R^2\nabla\varphi(x), h\rangle_{H_R},\\
\|\nabla_{H_R}\varphi(x)\|^2_{H_R} &= \langle R^2\nabla\varphi(x), R^2\nabla\varphi(x)\rangle_{H_R} = \langle R^{-1}R^2\nabla\varphi(x), R^{-1}R^2\nabla\varphi(x)\rangle_X\\
&= \langle(I-P)R\,\nabla\varphi(x), (I-P)R\,\nabla\varphi(x)\rangle_X = \|R\,\nabla\varphi(x)\|^2_X.
\end{aligned}$$

Since $\nabla\varphi$ is continuous at every $x \in X$, we obtain that φ belongs to $C^1_{H_R}(X)$. $\square$

We recall the following useful result.

Proposition 1.3.3. *Let $(X, \|\cdot\|_X, \langle\cdot,\cdot\rangle_X)$, $(X_1, \|\cdot\|_{X_1}, \langle\cdot,\cdot\rangle_{X_1})$ and $(X_2, \|\cdot\|_{X_2}, \langle\cdot,\cdot\rangle_{X_2})$ be Hilbert spaces and let $L_1 : X_1 \longrightarrow X$, $L_2 : X_2 \longrightarrow X$ be linear bounded operators. The following statements hold.*

1. Range $L_1 \subseteq$ Range L_2 *if and only if there exists a constant* $C > 0$ *such that*

$$\|L_1^\star x\|_{X_1} \leqslant C \|L_2^\star x\|_{X_2}, \quad x \in X. \tag{1.15}$$

In this case $\left\|L_2^{-1} L_1\right\|_{\mathcal{L}(X_1, X_2)} \leqslant C$*; more precisely*

$$\left\|L_2^{-1} L_1\right\|_{\mathcal{L}(X_1, X_2)} = \inf\{C > 0 \text{ s.t. } (1.15) \text{ holds}\} \tag{1.16}$$

2. *If* $\|L_1^\star x\|_{X_1} = \|L_2^\star x\|_{X_2}$ *for every* $x \in X$ *then* Range $L_1 =$ Range L_2 *and* $\left\|L_1^{-1} x\right\|_{X_1} = \left\|L_2^{-1} x\right\|_{X_2}$ *for every* $x \in X$.

Proof. See Proposition B.1 in Appendix B in [DZ14, pag. 429]. □

1.4 Compact operators

Here we summarize basic facts on compact operators. We refer to [Sim15; DS88; RS80] for more details.

Let $(X, \|\cdot\|_X, \langle\cdot,\cdot\rangle_X)$ and $(Y, \|\cdot\|_Y, \langle\cdot,\cdot\rangle_Y)$ be separable Hilbert spaces.

Definition 1.4.1. Let $T \in \mathcal{L}(X; Y)$. We say that T is a compact operator if for every bounded $M \subseteq X$, the subset $T(M)$ is relatively compact in Y.

Therefore T is compact if and only if for every sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$, the sequence $\{Tx_n\}_{n\in\mathbb{N}} \subseteq Y$ has some converging subsequence. We denote by $\mathcal{K}(X; Y)$ the subspace of $\mathcal{L}(X)$ of the compact operators from X to Y. If $X = Y$ we simply write $\mathcal{K}(X)$ instead of $\mathcal{K}(X; X)$.

It can be shown easily that $\mathcal{K}(X; Y)$ is a closed subspace of $\mathcal{L}(X; Y)$ and that $\mathcal{K}(X)$ is a two-sided ideal, namely for all $T \in \mathcal{L}(X)$ and $S \in \mathcal{K}(X)$ we have $TS, ST \in \mathcal{K}(X)$.

We have the following basic results about compact operators.

Proposition 1.4.2. *Let* $T \in \mathcal{L}(X)$*. The following statements holds true.*

1. T *is compact if and only if* $T^\star$ *is compact.*
2. T *is compact if and only if it maps weakly converging sequences into norm converging sequences.*

We finally recall results on the spectral theory of self-adjoint operators.

Theorem 1.4.3. *Let* $T \in \mathcal{K}(X)$ *be a self-adjoint operator. The following statements hold.*

1. *for every* $\lambda \in \sigma_p(T)$ *such that* $\lambda \neq 0$*, the eigenspace* X_λ *with eigenvalue* λ *is finite dimensional.*
2. $\sigma_p(T) \subseteq \mathbb{R}$ *is not empty and it is at most countable.*
3. *If* $\sigma_p(T)$ *has an accumulation point* λ_0 *then* $\lambda_0 = 0$.
4. $\|T\|_{\mathcal{L}(X)} = \sup\{|\lambda| \ : \ \lambda \in \sigma_p(T)\}$.

5. There exists an orthonormal basis of X consisting of eigenvectors of T.

Proposition 1.4.4. *Let $T \in \mathcal{L}(X)$ be a self-adjoint operator. T is compact if and only if there exist a Hilbert basis $\{e_k\}_{k\in\mathbb{N}}$ of X and a sequence $\{\lambda_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}$ converging to 0 such that*

$$Tx = \sum_{k\in\mathbb{N}} \lambda_k \langle x, e_k\rangle e_k, \quad x \in X,$$

where the series converges in $\mathcal{L}(X)$. In this case $\{e_k\}_{k\in\mathbb{N}}$ and $\{\lambda_k\}_{k\in\mathbb{N}}$ are the eigenvectors and the eigenvalues of T respectively.

By Theorem 1.4.3, $\sigma_p(T)$ is finite if and only if the range of T is a finite dimensional space. In general the range of T is not dense in X. However we have the following corollary.

Corollary 1.4.5. *Let $T \in \mathcal{K}(X)$ be a self-adjoint and injective operator. Then the range of T is dense in X.*

1.4.1 Hilbert-Schmidt operators

Now we introduce the Hilbert-Schmidt operators.

Definition 1.4.6. Let $T \in \mathcal{L}(X;Y)$. We say that T is a Hilbert-Schmidt operator if there exists a Hilbert basis $\{e_k\}_{k\in\mathbb{N}}$ of X such that

$$\sum_{k=1}^{+\infty} \|Te_k\|_Y^2 < +\infty. \tag{1.17}$$

It can be proven easily that the series in (1.17) does not depend on the choice of the Hilbert basis of X. We denote by $\mathcal{H}(X;Y)$ the space of the Hilbert-Schmidt operators from X to Y, if $Y = X$ we set $\mathcal{H}(X;Y) = \mathcal{H}(X)$.

$\mathcal{H}(X;Y)$ is endowed with the norm

$$\|T\|_{\mathcal{H}(X;Y)} := \sqrt{\sum_{k=1}^{+\infty} \|Te_k\|_Y^2}, \tag{1.18}$$

for every $T \in \mathcal{H}(X;Y)$ and for every Hilbert basis $\{e_k\}_{k\in\mathbb{N}}$ of X. (1.18) comes from the inner product

$$\langle T, S\rangle_{\mathcal{H}(X;Y)} := \sum_{k=1}^{+\infty} \langle Te_k, Se_k\rangle_Y, \tag{1.19}$$

where for every $S, T \in \mathcal{H}(X;Y)$, the series on the right hand side of (1.19) does not depend on the choice of the Hilbert basis $\{e_k\}_{k\in\mathbb{N}}$ of X. Hence we can state the following theorem

Theorem 1.4.7. $(\mathcal{H}(X;Y), \|\cdot\|_{\mathcal{H}(X;Y)}, \langle\cdot,\cdot\rangle_{\mathcal{H}(X;Y)})$ *is a Hilbert space. Moreover $\mathcal{H}(X;Y)$ is continuously embedded in $\mathcal{L}(X;Y)$, every Hilbert-Schmidt operator is compact and the finite rank operators are dense in $\mathcal{H}(X;Y)$.*

We have the following characterization of the Hilbert-Schmidt self-adjoint operators.

Proposition 1.4.8. *Let $T \in \mathcal{K}(X)$ be a self-adjoint operator. $T \in \mathcal{H}(X)$ if and only if*

$$\sum_{k \in \mathbb{N}} \lambda_k^2 < +\infty,$$

where $\{\lambda_k\}_{k \in \mathbb{N}}$ are the eigenvalues of T. Moreover

$$\|T\|_{\mathcal{H}(X,Y)} = \sqrt{\sum_{k=1}^{+\infty} \lambda_k^2}.$$

1.4.2 Trace class operators

Finally we define the trace class (or nuclear) operators.

Definition 1.4.9. Let $T \in \mathcal{L}(X)$. We say that T is a trace class (or nuclear) operator if there exists a Hilbert basis $\{e_k\}_{k \in \mathbb{N}}$ of X such that

$$\sum_{k \in \mathbb{N}} \langle |T| \, e_k, e_k \rangle_X < +\infty, \tag{1.20}$$

where $|T|$ is defined in (1.11). In this case the trace of T

$$\mathrm{Tr}[T] := \sum_{k \in \mathbb{N}} \langle T e_k, e_k \rangle_X.$$

is well defined and it does not depend on the choice of the basis $\{e_k\}_{k \in \mathbb{N}}$. Clearly T is a trace class operator if and only if the operator $\sqrt{|T|}$ is a Hilbert-Schmidt operator and

$$\mathrm{Tr}\,|T| = \left\| \sqrt{|T|} \right\|^2_{\mathcal{H}(X)}.$$

We denote by $\mathcal{L}_1(X)$ the space of the trace class operators. It is complete with respect to the norm

$$\|T\|_{\mathcal{L}_1(X)} := \mathrm{Tr}\,|T|, \quad T \in \mathcal{L}_1(X).$$

Moreover $\mathcal{L}_1(X)$ is continuously embedded in $\mathcal{L}(X)$; every trace class operator is compact and the set of finite rank operators is dense in $(\mathcal{L}_1(X), \|\cdot\|_{\mathcal{L}_1(X)})$. Moreover $\mathcal{L}_1(X)$ is a two-sided ideal and $\|T\|_{\mathcal{L}_1(X)} = \|T^\star\|_{\mathcal{L}_1(X)}$ for all $T \in \mathcal{L}_1(X)$.

Proposition 1.4.10. *Let $T \in \mathcal{K}(X)$ be a self-adjoint operator. T is a trace class operator if and only if*

$$\sum_{k \in \mathbb{N}} |\lambda_k| < +\infty,$$

where $\{\lambda_k\}_{k \in \mathbb{N}}$ are the eigenvalues of T. Moreover

$$\mathrm{Tr}\,[T] = \sum_{k \in \mathbb{N}} \lambda_k \in \mathbb{R}.$$

We say that a sequence $\{L_k\}_{k\in\mathbb{N}} \subseteq \mathcal{L}(X)$ is increasing if $L_{k+1} - L_k$ is nonnegative for every $k \in \mathbb{N}$, i.e., $\langle (L_{k+1} - L_k)x, x\rangle_X \geqslant 0$ for every $x \in X$ and for every $k \in \mathbb{N}$.

Here we recall a Proposition on increasing sequences of trace class operators.

Proposition 1.4.11. *Let $\{L_k\}_{k\in\mathbb{N}} \subseteq \mathcal{L}_1(X)$ be an increasing sequence of self-adjoint non-negative nuclear operators such that $\sup\limits_{k\in\mathbb{N}} \|L_k\|_{\mathcal{L}_1(X)} < \infty$. Then there exists $L \in \mathcal{L}_1(X)$ such that $L_k \longrightarrow L$ in $\mathcal{L}_1(X)$.*

Proof. We first show that there is $L \in \mathcal{L}(X)$ such that $L_k \xrightarrow{k\to+\infty} L$ in $\mathcal{L}(X)$. By hypothesis, we have

$$\phi(x) := \lim_{k\to\infty} \langle x, L_k x\rangle_X \leqslant \sup_{k\to\infty} \|L_k\|_{\mathcal{L}(X)} =: c < \infty$$

whenever $x \in X$ and $\|x\| \leqslant 1$. Set now $L_{nm} = L_n - L_m$ for $m < n$, so that the L_{nm} are non-negative, and observe that for every $x, y \in X$ we have

$$\begin{aligned} |\langle y, L_{nm}x\rangle_X| &= |\langle L_{nm}^{\frac{1}{2}}x, L_{nm}^{\frac{1}{2}}y\rangle_X| \leqslant \|L_{nm}^{\frac{1}{2}}x\|_X \|L_{nm}^{\frac{1}{2}}y\|_X \\ &= \langle x, L_{nm}x\rangle_X \langle y, L_{nm}y\rangle_X \leqslant \phi(y)\langle x, L_{nm}x\rangle_X. \end{aligned}$$

Therefore,

$$\|L_{nm}x\|_X = \sup_{\|y\|_X \leqslant 1} |\langle y, L_{nm}x\rangle_X| \leqslant c\langle x, L_{nm}x\rangle_X \quad \text{for every} \quad n, m \in \mathbb{N}. \tag{1.21}$$

We claim that the right hand side of (1.21) vanishes as $n, m \to \infty$. Indeed, if this were not the case there would exist $x \in X$ and $\alpha > 0$ such that for every $m \in \mathbb{N}$ there is $p(m) \in \mathbb{N}$ such that $\langle (L_{m+p(m)} - L_m)x, x\rangle_X \geqslant \alpha > 0$. Then, defining $m_1 = 1, p_1 = p(m_1)$ and by iteration $m_{j+1} = m_j + p_j$ we would have

$$\langle (L_{m_N+p_N} - L_1)x, x\rangle_X = \sum_{j=1}^{N} \langle (L_{m_j+p_j} - L_{m_j})x, x\rangle_X \geqslant N\alpha \longrightarrow +\infty,$$

which contradicts $\phi(x) \leqslant c$ for every $x \in X$. Summarizing, $\{L_k x\}_{k\in\mathbb{N}}$ is a Cauchy sequence for every $x \in X$ and defines the self-adjoint nonnegative limit operator $Lx = \lim\limits_{k\to+\infty} L_k x$.

Let us show that $L \in \mathcal{L}_1(X)$ and that $L_k \xrightarrow{k\to+\infty} L$ in $\mathcal{L}_1(X)$. For an arbitrary orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ by the Fatou lemma applied to the series we have

$$\mathrm{Tr}(L) = \sum_{k=1}^{\infty} \langle \lim_{n\to\infty} L_n e_k, e_k\rangle_X \leqslant \liminf_{n\to\infty} \sum_{k=1}^{\infty} \langle L_n e_k, e_k\rangle_X = \liminf_{n\to\infty} \mathrm{Tr}(L_n)$$

and $L \in \mathcal{L}_1(X)$. To prove the convergence, notice that by monotonicity $\langle L_k e_n, e_n\rangle_X \leqslant \langle L e_n, e_n\rangle_X$ for every $k, n \in \mathbb{N}$, whence the thesis follows by the Dominated Convergence Theorem applied to series

$$\mathrm{Tr}\,(L - L_k) = \sum_{n=1}^{\infty} \langle (L - L_k)e_n, e_n\rangle \longrightarrow 0 \quad \text{as } k \to \infty.$$

□

1.5 Gaussian measures

Let $(X, \|\cdot\|_X)$ be a real separable Banach space. In this section we summarize basic facts about Gaussian measures on X. We refer to [Bog98] for a detailed overview of this topic. We start by setting some standard notations from probability theory.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. We say that $\xi : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (X, \mathcal{B}(X))$ is a random variable if ξ is a Borel measurable function. We call law of ξ in X the probability measure on $(X, \mathcal{B}(X))$ given by

$$\mathbb{P} \circ \xi^{-1}(A) := \mathbb{P}(\xi^{-1}(A)) = \mathbb{P}(\{\omega \in \Omega \ : \ \xi(\omega) \in A\}), \quad A \in \mathcal{B}(X).$$

The following change of variable formula is very useful and will be used frequently. If $u \in L^1(X, \mathbb{P} \circ \xi^{-1})$, then $u \circ \xi \in L^1(\Omega, \mathbb{P})$ and

$$\int_X u(x)\, \mathbb{P} \circ \xi^{-1}(dx) = \int_\Omega (u \circ \xi)(\omega)\, (d\omega).$$

Moreover we denote by

$$\mathbb{E}[\xi] := \int_\Omega \xi(w)\ \mathbb{P}(d\omega) = \int_X x\ \mathbb{P} \circ \xi^{-1}(dx)$$

the expectation of ξ with respect to $\mathbb{P}$.

Given a probability measure γ on $(X, \mathcal{B}(X))$, the characteristic function of γ is a mapping $\widehat{\gamma} : X^\star \longrightarrow \mathbb{R}$ given by

$$\widehat{\gamma}(f) := \int_X e^{if(x)} \gamma(dx), \quad f \in X^\star.$$

As in the finite dimensional case the following result holds.

Proposition 1.5.1. *Let $(X, \|\cdot\|_X)$ be a real separable Banach space and let γ_1, γ_2 two probability measures on $(X, \mathcal{B}(X))$. If $\widehat{\gamma}_1 \equiv \widehat{\gamma}_2$ then $\gamma_1 = \gamma_2$.*

Let γ be a probability measure on $(X, \mathcal{B}(X))$. The support of γ is the intersection of all closed sets C such that $\gamma(C) = \gamma(X)$ and we say that γ is concentrate on $E \subseteq X$ if $\gamma(X \setminus E) = 0$. Moreover we say that γ is non degenerate if $\gamma(A) > 0$ for all open set $A \subseteq X$. If not, we say that γ is degenerate.

Let $\{\mu_j\}_j \in \mathbb{N}$ be a sequence of measures on $(X, \mathcal{B}(X))$. We say that μ_j weakly converges to a Borel measure μ and we write $\mu_j \longrightarrow \mu$ if and only if

$$\int_X f \mu_j(dx) \longrightarrow \int_X f \mu_j(dx), \quad f \in C_b(X).$$

Let γ_1 and γ_2 be two probability measures on $(X, \mathcal{B}(X))$. We say that γ_1 is absolutely continuous with respect to γ_2 if for every $B \in \mathcal{B}(X)$, $\gamma_2(B) = 0$ implies that $\gamma_1(B) = 0$. In this case we write $\gamma_1 \ll \gamma_2$. If $\gamma_1 \ll \gamma_2$ and $\gamma_2 \ll \gamma_1$ we say that γ_1 and γ_2 are equivalent and we write $\gamma_1 \approx \gamma_2$. We say that γ_1 and γ_2 are mutually singular and we write $\gamma_1 \perp \gamma_2$ if there exists $B \in \mathcal{B}(X)$ such that $\gamma_1(B) = 0$ and $\gamma_2(B) = 1$.

Let γ and ν be two probability measures on $(X, \mathcal{B}(X))$. By the Radon-Nikodym Theorem there exist $\gamma_a \ll \nu$ and $\gamma_s \perp \nu$ such that $\gamma = \gamma_a + \gamma_s$. Moreover there exists $f \in L^1(X, \nu)$ such that $\gamma_a = f\nu$. The function f is called density or Radon-Nikodym derivative of γ with respect to ν.

A Borel probability measure γ is called non degenerate Gaussian measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ if there exist $m \in \mathbb{R}$ and $\sigma > 0$ such that

$$\gamma(A) = \frac{1}{\sqrt{2\pi}} \int_A e^{\frac{|x-m|^2}{2\sigma^2}} dx, \quad A \in \mathcal{B}(\mathbb{R}),$$

where m is the mean of γ and σ^2 is the variance of γ. In this case we denote γ by $\mathcal{N}_{a,\sigma^2}$ or by $\mathcal{N}(a, \sigma^2)$. Instead we say that γ is a degenerate Gaussian measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ if there exist $m \in \mathbb{R}$ such that

$$\gamma(A) = \delta_m(A), \quad A \in \mathcal{B}(\mathbb{R}),$$

where $\delta_m(\cdot)$ is the Dirac measure defined by

$$\delta_m(A) = \begin{cases} 1 & m \in A \\ 0 & m \notin A, \end{cases} \qquad A \in \mathcal{B}(\mathbb{R}).$$

We say that $\xi : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a real Gaussian random variable if ξ is a random variable and the law of ξ in $\mathbb{R}$ is a Gaussian measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We say that ξ is non degenerate (degenerate) if the law of ξ is non degenerate (degenerate).

Definition 1.5.2. Let γ be a probability measure on $(X, \mathcal{B}(X))$. We say that γ is a Gaussian measure on X if for every $f \in X^\star$ the probability measure $\gamma \circ f^{-1}$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a Gaussian measure.

Remark 1.5.3. Any Gaussian measure γ on X is non degenerate if and only if for every $f \in X^\star$ such that $f \neq 0$ the probability measure $\gamma \circ f^{-1}$ is a non degenerate Gaussian measure.

Remark 1.5.4. If $f \in X^\star$ then $f \in L^p(X, \gamma)$ for all $p \geqslant 1$. Indeed

$$\int_X |f(x)|^p \, \gamma(dx) = \int_{\mathbb{R}} |t|^p \, \gamma \circ f^{-1}(dt)$$

and since the law of f in $\mathbb{R}$ is a Gaussian measure in $\mathbb{R}$ we conclude.

Definition 1.5.5. Let $\xi : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (X, \mathcal{B}(X))$ be a random variable. We say that ξ is a Gaussian random variable if for every $f \in X^\star$ the random variable $f \circ \xi$ is a real Gaussian random variable. We say that ξ is a non degenerate Gaussian random variable if for every $f \in X^\star \setminus \{0\}$ the random variable $f \circ \xi$ is a real non degenerate Gaussian random variable.

We can define the mean and the covariance of γ.

Definition 1.5.6. Let γ be a Gaussian measure on X. The functions $a_\gamma : X^\star \longrightarrow \mathbb{R}$ and $B_\gamma : X^\star \times X^\star \longrightarrow \mathbb{R}$ defined by

$$a_\gamma(f) := \int_{\mathbb{R}} f(x)\gamma(dx), \qquad f \in X^\star, \tag{1.22}$$

$$B_\gamma(f, g) := \int_R (f(x) - a_\gamma(f))(g(x) - a_\gamma(g))\gamma(dx), \quad f, g \in X^\star. \tag{1.23}$$

are called mean and covariance of γ, respectively. We say that γ is centered if $a_\gamma \equiv 0$.

Clearly the mapping $f \in X^\star \longmapsto a_\gamma(f)$ is linear and $(f, g) \in X^\star \times X^\star \longmapsto B_\gamma(f, g)$ is bilinear. Moreover $B_\gamma(f, f) = \|f - a_\gamma(f)\|^2_{L^2(X,\gamma)}$ for every $f \in X^\star$.

Remark 1.5.7. Let $\xi : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (X, \mathcal{B}(X))$ be a Gaussian random variable with law γ. For every $f \in X^\star$, $f \circ \xi$ is a real Gaussian random variable. In particular ξ is a non degenerate Gaussian random variable if and only if $B_\gamma(f, f) \neq 0$ for every $f \in X^\star$.

As in the finite dimensional case we have a characterization for the Gaussian measures via their Fourier transforms.

Proposition 1.5.8. *Let γ be a probability measure on $(X, \mathcal{B}(X))$. γ is a Gaussian measure if and only if there exist $a_\gamma : X^\star \longrightarrow \mathbb{R}$ linear and $B_\gamma : X^\star \times X^\star \longrightarrow \mathbb{R}$ bilinear, symmetric and non-negative such that*

$$\widehat{\gamma}(f) = e^{ia_\gamma(f) - \frac{1}{2}B_\gamma(f,f)}, \quad f \in X^\star.$$

Now we state the Fernique Theorem whose consequences are relevant.

Theorem 1.5.9 (Fernique)**.** *Let γ be a Gaussian measure on X.*
Then there exists $\alpha > 0$ such that

$$\int_X e^{\alpha \|x\|_X^2} \gamma(dx) < +\infty. \tag{1.24}$$

A first consequence of this theorem is that

$$\int_X \|x\|_X^p \, \gamma(dx) < +\infty$$

for all $p \geqslant 1$. Moreover the following proposition holds true.

Proposition 1.5.10. *If γ is a Gaussian measure on X then a_γ and b_γ given by (1.22) and (1.23) respectively, are continuous. Moreover there exists $a \in X$ such that $a_\gamma(f) = f(a)$ for all $f \in X^\star$.*

1.5.1 The reproducing Kernel and the Cameron-Martin Space

Now we introduce the Cameron-Martin space that has a fundamental role in Gaussian analysis.

Let γ be a Gaussian measure on X. By Remark 1.5.4 $X^\star$ is contained in $L^2(X, \gamma)$. Moreover the map $j : X^\star \longrightarrow L^2(X, \gamma)$ defined by

$$j(f) = f - a_\gamma(f), \quad f \in X^\star,$$

is continuous.

Definition 1.5.11. Let γ be a Gaussian measure on X. The reproducing kernel of γ is the space defined by

$$X_\gamma^\star := \overline{j(X^\star)}^{L^2(X,\gamma)}.$$

It is not difficult to prove that a_γ and B_γ can be continuously extended to $X_\gamma^\star$. Even the Fourier transform can be extended to $X_\gamma^\star$. We have the following result

Proposition 1.5.12. *Let γ be a Gaussian measure on X. Then*

$$\widehat{\gamma}(f) = e^{-\frac{1}{2}\|f\|_{L^2(X,\gamma)}^2}, \quad f \in X_\gamma^\star. \tag{1.25}$$

Therefore every $f \in X_\gamma^\star$ is a Gaussian random variable and the law of f in $\mathbb{R}$ is given by

$$\gamma \circ f^{-1} = \mathcal{N}_{0,\|f\|^2_{L^2(X,\gamma)}}.$$

In particular $f \in L^p(X,\gamma)$ for every $p \in [1,+\infty)$.

Now we define the operator $R_\gamma : X_\gamma^\star \longrightarrow (X^\star)'$ by

$$R_\gamma(f)(g) := \int_X f(x)[g(x) - a_\gamma(g)]\,\gamma(dx) = \langle f, g - a_\gamma(g)\rangle_{L^2(X,\gamma)}, \quad f \in X_\gamma^\star,\ g \in X^\star.$$

Since X is separable, it can be shown that the range of R_γ is contained in X. Namely, for every $f \in X_\gamma^\star$ there exists a unique $x_f \in X$ such that

$$R_\gamma(f)(g) = g(x_f), \quad g \in X^\star.$$

Hence, from here on, we identify $R_\gamma(f)$ with the element $x_f \in X$ and we write

$$R_\gamma(f)(g) = g(R_\gamma(f)), \quad g \in X^\star.$$

Definition 1.5.13. Let γ be a Gaussian measure on X. Setting

$$\|h\|_{H_\gamma} := \sup\Big\{ f(h) : f \in X^\star,\ \|j(f)\|_{L^2(X,\gamma)} \leqslant 1 \Big\},$$

the Cameron–Martin space of the measure γ is the space defined by

$$H_\gamma := \Big\{ h \in X \ :\ \|h\|_{H_\gamma} < +\infty \Big\}.$$

Proposition 1.5.14. *The Cameron-Martin space is the range of R_γ. Then $h \in H_\gamma$ if and only if there exists $\widehat{h} \in X_\gamma^\star$ such that $h = R_\gamma(\widehat{h})$. Moreover*

$$\|h\|_{H_\gamma} = \|\widehat{h}\|_{L^2(X,\gamma)}.$$

Therefore R_γ is an isometry from $X_\gamma^\star$ to H_γ and H_γ is a Hilbert space equipped with the inner product

$$\langle h, k\rangle_{H_\gamma} := \langle \widehat{h}, \widehat{k}\rangle_{L^2(X,\gamma)}.$$

Moreover every $\widehat{h} \in X_\gamma^\star$ is a Gaussian random variable and the law of $\widehat{h}$ in $\mathbb{R}$ is given by

$$\gamma \circ \widehat{h}^{-1} = \mathcal{N}_{0,\|\widehat{h}\|^2_{L^2(X,\gamma)}} = \mathcal{N}_{0,\|\widehat{h}\|^2_{H_\gamma}}.$$

In particular, $\widehat{h} \in L^p(X,\gamma)$ for every $p \in [1,\infty)$, with

$$\left\| \widehat{h} \right\|_{L^p(X,\gamma)} \leqslant c_p \|h\|_{H_\gamma} \tag{1.26}$$

Let $h \in X$. We define the probability measure

$$\gamma_h(B) = \gamma(B - h), \quad B \in \mathcal{B}(X).$$

If $X = \mathbb{R}^n$, it is easy to see that if γ is non degenerate, then γ_h and γ are equivalent. However in the infinite dimensional case this is not true in general, even if γ is non degenerate. The Cameron-Martin theorem shows exactly for which $h \in X$ this fact occurs.

Theorem 1.5.15. *If $h \in H_\gamma$, then γ and γ_h are equivalent, and we have*

$$\gamma_h = e^{\widehat{h} - \frac{1}{2}\|h\|^2_{H_\gamma}}\gamma.$$

If $h \notin H_\gamma$, then γ and γ_h are singular.

1.5.2 The Hilbert case

In this subsection we consider the special case in which X is a separable Hilbert space. We refer to the books [DZ14; LR15; Kuk20] for more details.

Let $(X, \|\cdot\|_X, \langle\cdot,\cdot\rangle_X)$ be a separable Hilbert space. Let γ be a Gaussian measure on X with mean a_γ and covariance operator B_γ. After the canonical identification of X with $X^\star$, there exist $a \in X$ and a non-negative operator $Q \in \mathcal{L}(X)$ such that

$$a_\gamma(f) = \langle f, a\rangle_X, \quad f \in X,$$
$$B_\gamma(f,g) = \langle Qf, g\rangle_X, \quad f,g \in X.$$

We have the following characterization for the Gaussian measures and for their Cameron-Martin spaces.

Proposition 1.5.16. *Let γ be a probability measure on $(X, \mathcal{B}(X))$. γ is Gaussian if and only if there exist $a \in \mathbb{R}$ and a linear, non-negative self-adjoint trace class operator $Q : X \longrightarrow X$ such that*

$$\widehat{\gamma}(x) = e^{i\langle x,a\rangle_X - \frac{1}{2}\|Q^{\frac{1}{2}}x\|_X^2}, \quad x \in X.$$

In this case we denote γ by $\mathcal{N}_{a,Q}$ or by $\mathcal{N}(a,Q)$.

We fix $a \in X$ and a non-negative and trace class operator Q. By Theorem 1.4.3, there exists a Hilbert basis $\{e_k\}_{k\in\mathbb{N}}$ of X such that

$$Qe_k = \lambda_k e_k, \quad k \in \mathbb{N},$$

where $\{\lambda_k\}_{k\in\mathbb{N}}$ are the eigenvalues of Q. In particular, since Q is non-negative then $\lambda_k \geqslant 0$ for every $k \in \mathbb{N}$.

We get a useful characterization for the random Gaussian variables.

Proposition 1.5.17. *Let $\xi : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (X, \mathcal{B}(X))$ be a random variable. Let $a \in \mathbb{R}$ and let $Q \in \mathcal{L}(X)$ be non-negative and trace class operator. The law of ξ in X is $\mathcal{N}(a,Q)$ if and only if*

$$\xi = \sum_{k\in\mathbb{N}} \beta_k \sqrt{\lambda_k} e_k + a, \tag{1.27}$$

where $\{\beta_k\}_{k\in\mathbb{N}}$ are independent real Gaussian random variables with mean 0 and variance 1. Moreover the series in (1.27) converges in $L^2((\Omega, \mathcal{F}, \mathbb{P}), (X, \mathcal{B}(X)))$ and we have

$$\mathbb{E}[\langle \xi, u\rangle_X] = \langle m, u\rangle_X, \quad u \in X,$$
$$\mathbb{E}[\langle \xi - a, u\rangle_X \langle \xi - a, v\rangle_X] = \langle Qu, v\rangle_X, \quad u, v \in X,$$
$$\mathbb{E}[\|\xi - a\|_X^2] = \mathrm{Tr}\,[Q]\,.$$

Proposition 1.5.18. *Let X be a separable Hilbert space endowed with a Gaussian measure γ, with mean a and covariance Q.*

(i) If Y is a separable Hilbert space endowed with a Gaussian measure μ with mean b and covariance R, then $\gamma \otimes \mu$ is a Gaussian measure on $X \times Y$ with mean (a,b) and covariance $Q_{\gamma\otimes\mu}$ defined by

$$Q_{\gamma\otimes\mu}(f,g) = Q(f) + R(g), \quad f \in X,\ g \in Y.$$

(ii) If μ is another Gaussian measure on X with mean b and covariance R, then the convolution measure $\gamma * \mu$, defined as the image measure in X of $\gamma \otimes \mu$ on $X \times X$ under the map $(x, y) \mapsto x + y$, is a Gaussian measure with mean $a + b$ and covariance $Q + R$.

(iii) If Y is a separable Hilbert space and $A \in \mathcal{L}(X, Y)$, $b \in Y$, setting $T(x) = Ax + b$ for $x \in X$, the measure $\mu = \gamma \circ T^{-1}$ is a Gaussian measure in Y with mean $Aa + b$ and covariance $AQA^\star$.

We list in the next proposition the main properties of the Cameron Martin space in the Hilbert setting.

Proposition 1.5.19. *Let $\gamma = \mathcal{N}_{a,Q}$ be the Gaussian measure with mean a and covariance Q. Let $\{e_k\}_{k\in\mathbb{N}} \subseteq X$ be a Hilbert basis of X consisting of eigenvectors of Q, $Qe_k = \lambda_k e_k$ for all $k \in \mathbb{N}$. Moreover for all $x \in X$ we set $x_k := \langle x, e_k\rangle_X$.*

For every $z \in X$ the series

$$f_n(x) = \sum_{1\leqslant k\leqslant n:\, \lambda_k\neq 0} (x_k - a_k) z_k \lambda_k^{-\frac{1}{2}} \tag{1.28}$$

converges in $L^p(X, \gamma)$ for any $p \in [1, \infty)$. The reproducing kernel of γ is given by

$$X^\star_\gamma := \left\{ f : X \longrightarrow \mathbb{R} \ :\ f(x) = \sum_{k\in\mathbb{N}:\, \lambda_k\neq 0} (x_k - a_k) z_k \lambda_k^{-\frac{1}{2}} \right\}. \tag{1.29}$$

The Cameron-Martin space H_γ of γ is the subspace $Q^{\frac{1}{2}}(X)$, namely

$$H_\gamma := \left\{ x \in X \ :\ \sum_{k\in\mathbb{N}:\, \lambda_k\neq 0} x_k^2 \lambda_k^{-1} < +\infty \right\}. \tag{1.30}$$

and they have the same inner product, namely

$$\langle h, l\rangle_{H_\gamma} = \langle Q^{-\frac{1}{2}} h, Q^{-\frac{1}{2}} k\rangle_X = \sum_{k\in\mathbb{N}:\, \lambda_k\neq 0} h_k l_k \lambda_k^{-1}, \quad h, l \in Q^{\frac{1}{2}}(X),$$

where $Q^{-\frac{1}{2}}$ is the pseudo-inverse of $Q^{\frac{1}{2}}$.

$\widehat{h}$ is given by

$$\widehat{h}(y) = \sum_{\substack{k\in\mathbb{N},\\ \lambda_k\neq 0}} (y_k - a_k) \Big(Q^{-\frac{1}{2}} h\Big)_k \lambda_k^{-\frac{1}{2}} = \sum_{\substack{k\in\mathbb{N},\\ \lambda_k\neq 0}} (y_k - a_k)\, h_k\, \lambda_k^{-1}. \tag{1.31}$$

Moreover the following statements hold.

1. *If γ is non degenerate then for every $\alpha < \inf\left\{ \dfrac{1}{2\lambda_k} \ :\ k \in \mathbb{N} \right\}$, (1.24) is verified.*

2. *$\left\{ \lambda_k^{\frac{1}{2}} e_k :\ k \in \mathbb{N} \text{ and } \lambda_k \neq 0 \right\}$ is an orthonormal basis of the Cameron-Martin space $H_\gamma = Q^{\frac{1}{2}}(X)$.*

3. *The Cameron-Martin space H_γ has finite dimension $\Leftrightarrow$ the set $\sigma_p(Q)$ of the eigenvalues of Q is finite.*

4. *The Cameron-Martin space H_γ is dense in X $\Leftrightarrow$ $\lambda_k > 0$ for every $k \in \mathbb{N}$ $\Leftrightarrow$ Q is positive $\Leftrightarrow$ γ is a non degenerate Gaussian measure in X.*

In the Hilbert case the Cameron-Martin theorem can be generalized. Indeed we have the following result that states necessary and sufficient conditions such that two Gaussian measure are equivalent.

Theorem 1.5.20 (Feldman-Hajek)**.** *Let $\gamma_1 = \mathcal{N}_{m_1,Q_1}$ and $\gamma_2 = \mathcal{N}_{m_2,Q_2}$ be two Gaussian measures on X. γ_1 and γ_2 are equivalent if and only if the following conditions are verified.*

1. $H := Q_1^{\frac{1}{2}}(X) = Q_2^{\frac{1}{2}}(X)$.

2. $m_1 - m_2 \in H$

3. $(Q_1^{-\frac{1}{2}} Q_2^{\frac{1}{2}})(Q_1^{-\frac{1}{2}} Q_2^{\frac{1}{2}})^\star - \mathrm{Id}$ *is a Hilbert-Schmidt operator on* $\overline{H}^X$.

In the Hilbert case it is possible to characterize the weak convergence of Gaussian measures in a precise way. We recall the following Proposition proved in [Bog98, Example 3.8.15 page 135].

Proposition 1.5.21. *For all $n \in \mathbb{N}$, let $\gamma_n = \mathcal{N}_{a_n,Q_n}$ be the Gaussian measure on X with mean a_n and covariance Q_n. Moreover let $a \in X$ and let Q be a non-negative trace class operator. The following statements are equivalent.*

1. *The sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ converges weakly to $\gamma = \mathcal{N}_{a,Q}$.*

2. $a_n \longrightarrow a$ *in* X *and* $Q_n^{\frac{1}{2}} \longrightarrow Q^{\frac{1}{2}}$ *in* $\mathcal{H}(X)$.

3. $a_n \longrightarrow a$ *in* X, $Q_n \longrightarrow Q$ *in* $\mathcal{L}^1(X)$.

Remark 1.5.22. We recall that $Q_n \longrightarrow Q$ in $\mathcal{L}^1(X)$ if and only if

$$\mathrm{Tr}\,[Q_n] = \int_X \|x\|_X^2\, \gamma_n(dx) \longrightarrow \mathrm{Tr}\,[Q] = \int_X \|x\|_X^2\, \gamma(dx).$$

1.6 Evolution families

In this section the state basic notions on evolution operators.

Let $(X, \|\cdot\|_X)$ be a real Banach space. Let $J \subseteq \mathbb{R}$ be a closed interval and let

$$\Delta_J = \big\{(s,t) \in J^2 : \ s < t\big\}.$$

In the next chapters we set $\Delta := \Delta_{\mathbb{R}}$ and $\Delta_T = \Delta_{[0,T]}$ for every $T > 0$.

1.6.1 Forward evolution operators

Definition 1.6.1. $\{U(t,s)\}_{(s,t)\in\overline{\Delta}_J} \subseteq \mathcal{L}(X)$ is a forward evolution operator if

1. $U(t,t) = I$ for any $t \in J$,
2. $U(t,s) = U(t,r)U(r,s)$ for $s,r,t \in J$ such that $s \leqslant r \leqslant t$.

Moreover we say that $\{U(t,s)\}_{(s,t)\in\overline{\Delta}_J}$ is strongly continuous if for every $x \in X$ the map

$$(s,t) \in \overline{\Delta}_J \longmapsto U(t,s)x \in X, \tag{1.32}$$

is continuous.

If no confusion may arise we omit the adjective "forward" and we simply write $U(t,s)$ instead of $\{U(t,s)\}_{(s,t)\in\overline{\Delta}_J}$.

We consider now an evolution operator $\{U(t,s)\}_{(s,t)\in\overline{\Delta}_J}$ acting on $B_b(X)$.

1. We say that $U(t,s)$ is Feller, if for every $s < t$ we have
$$U(t,s)\left(C_b(X)\right) \subseteq C_b(X).$$
2. We say that $U(t,s)$ is Strong Feller, if for every $s < t$ we have
$$U(t,s)\left(B_b(X)\right) \subseteq C_b(X).$$
3. We say that $U(t,s)$ is contractive, if for every $s < t$ and $\varphi \in B_b(X)$ we have
$$\|U(t,s)\varphi\|_\infty \leqslant \|\varphi\|_\infty .$$

Definition 1.6.2. Let $\{\mu_t\}_{t\in\mathbb{R}}$ be a family of probability measures on $(X, \mathcal{B}(X))$. We say that $\{\mu_t\}_{t\in\mathbb{R}}$ is an evolution system of measures for $U(t,s)$ in $\mathbb{R}$ if, for every $\varphi \in C_b(X)$ and $s < t$, we have
$$\int_X U(t,s)\varphi(x)\nu_t(dx) = \int_X \varphi(x)\nu_s(dx).$$

1.6.2 Backward evolution operators

Definition 1.6.3. $\{V_{s,t}\}_{(s,t)\in\overline{\Delta}_J} \subseteq \mathcal{L}(X)$ is a backward evolution operator if

1. $V_{t,t} = I$ for any $t \in J$,
2. $V_{s,t} = V_{s,r}V_{r,t}$ for $s \leqslant r \leqslant t$.

Moreover we say that $\{V_{s,t}\}_{(s,t)\in\overline{\Delta}_J} \subseteq \mathcal{L}(X)$ is strongly continuous if for every $x \in X$ the map

$$(s,t) \in \overline{\Delta}_J \longmapsto V_{s,t}x \in X, \tag{1.33}$$

is continuous.

Again we often omit the adjective "backward" and we simply write $V_{s,t}$ instead of $\{V_{s,t}\}_{(s,t)\in\overline{\Delta}_J}$.

For a backward evolution operator $\{V_{s,t}\}_{(s,t)\in\overline{\Delta}_J}$ acting on $B_b(X)$, the definitions of Feller, strong Feller and contractive evolution operator are the same as the forward case.

Definition 1.6.4. Let $\{\mu_t\}_{t\in\mathbb{R}}$ be a family of probability measures on $(X, \mathcal{B}(X))$. We say that $\{\mu_t\}_{t\in\mathbb{R}}$ is an evolution system of measures for $V_{s,t}$ in $\mathbb{R}$ if, for every $\varphi \in C_b(X)$ and $s < t$, we have

$$\int_X V_{s,t}\varphi(x)\nu_s(dx) = \int_X \varphi(x)\nu_t(dx).$$

1.7 Abstract evolution problems

In this section we summarize results on abstract parabolic problems.

Let $(X, \|\cdot\|_X)$ be a real Banach space and $T > 0$. We consider the problem

$$\begin{cases} u'(t) = A(t)u(t) & t > s, \\ u(s) = x, \end{cases} \tag{1.34}$$

where $s \in [0, T)$, $x \in X$, $u : [0, T] \longrightarrow X$ and $\{A(t)\}_{t\in[0,T]}$ is a family of linear closed operators, possibly unbounded and with domains depending on t.

Definition 1.7.1.

1. A strict solution of (1.34) is a function u belonging to $C^1([s, T); X]$ such that $u(t) \in D(A(t))$ for every $t \in [s, T]$ and $u(s) = x$ and $u'(t) = A(t)u(t)$ for $t \in [s, T]$.
2. A classical solution of (1.34) is a function u belonging to $C([s, T]; X) \cap C^1((s, T]; X)$ such that $u(t) \in D(A(t))$ for every $t \in (s, T]$, $u(s) = x$ and $u'(t) = A(t)u(t)$ for $t \in (s, T]$.

Many authors gave suitable conditions on the operators $A(t)$ to prove the existence of an evolution operator $U(t, s)$ such that for suitable $x \in X$ (tipically $x \in D(A(s))$ or $x \in \overline{D(A(s))}$ and $A(s)x \in \overline{D(A(s))}$) $u(t) := U(t, s)x$ is a classical or strict solution of (1.34). In such cases we say that $U(t, s)$ is associated to $\{A(t)\}_{t\in[0,T]}$. We refer to [Acq88; AT87; Tan79; Yag76; Sob61; Paz83] for many results in this direction. However the theory of evolution operators is not simple and complete as in the autonomous case; for instance given an evolution operator $U(t, s)$, there are no theories that generalize the Hille-Yoshida theorem and there are no connections between the spectra of the operators $A(t)$ and the asymptotic behavior of $U(t, s)$ as $t - s \to +\infty$ (if $T = +\infty$).

In this section we state the hypotheses of Acquistapace-Terreni who built an evolution operator for (1.34) for abstract parabolic equations in a very general setting. In such case each $A(t)$ is the generator of an analytic semigroup. For what follows we refer to [Acq88; AT87; Sch04].

Hypothesis 1.7.2.

AT1 For every $t \in \mathbb{R}$, $A(t) : D(A(t)) \subseteq X \longrightarrow X$ is a linear operator and there exist $M \geqslant 0$, $\omega \in \mathbb{R}$ and $\phi \in \left(\frac{\pi}{2}, \pi\right)$ such that

$$\rho(A(t)) \supseteq S_{\omega,\phi} := \{\omega\} \cup \{\lambda \in \mathbb{C} \setminus \{\omega\} : \ |\arg(\lambda - \omega)| < \phi\},$$

and
$$\|R(\lambda, A(t))\|_{\mathcal{L}(X)} \leqslant \frac{M}{1+|\lambda-\omega|}.$$

AT2 There exist $L \geqslant 0$, $\mu, \nu \in (0,1]$ such that $\mu+\nu>1$ and
$$|\lambda|^{\nu} \left\|A_{\omega}(t)R(\lambda, A_{\omega}(t))(A_{\omega}(t)^{-1}-A_{\omega}(s)^{-1})\right\|_{\mathcal{L}(X)} \leqslant L\,|t-s|^{\mu}, \quad |\arg\lambda| \leqslant \phi,$$

where $A_{\omega}(t) := A(t)-\omega$.

Theorem 1.7.3 ([Acq88, Thm. 2.3],[AT92, Thm.6.1])**.** *Assume that $\{A(t)\}_{t\in[0,T]}$ satisfies Hypothesis 1.7.2. Then there exists a unique (forward) evolution operator $U(t,s)_{(s,t)\in\overline{\Delta}_T}$ such that*

1. *the mapping $(s,t) \longmapsto U(t,s)$ belongs to $B(\overline{\Delta}_T; \mathcal{L}(X)) \cap C(\Delta_T; \mathcal{L}(X))$;*
2. *$\lim_{s\nearrow t} \|U(t,s)x-x\|_X = 0$ if and only if $x \in \overline{D(A(t))}$;*
3. *$\lim_{s\searrow t} \|U(t,s)x-x\|_X = 0$ if and only if $x \in \overline{D(A(s))}$;*
4. *$U(t,s)$ maps X into $D(A(t))$ for every $(s,t) \in \Delta_T$ and there exists $C_1 > 0$ such that*
$$\|U(t,s)\|_{\mathcal{L}(X;D(A(t)))} = \|A(t)U(t,s)\|_{\mathcal{L}(X)} \leqslant \frac{C_1}{t-s}; \tag{1.35}$$
5. *the mapping $t \longrightarrow U(t,s)$ belongs to $C^1((s,T];\mathcal{L}(X)) \cap C((s,T];\mathcal{L}(X;D(A(t))))$ and*
$$\frac{\partial}{\partial t}U(t,s) = A(t)U(t,s); \tag{1.36}$$
6. *U(t,s) maps $D(A(s))$ in $D(A(t))$ and there exists $C_2 > 0$ such that*
$$\left\|A(t)U(t,s)(\omega\mathrm{Id}-A(s))^{-1}\right\|_{\mathcal{L}(X)} \leqslant C_2; \tag{1.37}$$
7. *the mapping $(s,t) \longrightarrow A(t)U(t,s)(\omega\mathrm{Id}-A(s))^{-1}$ belongs to $C(\overline{\Delta}_T;\mathcal{L}(X)) \cap C(\Delta;\mathcal{L}(X))$;*
8. *the mapping $s \in [0,t) \longrightarrow U(t,s)x$ with $x \in D(A(s))$ is right differentiable and*
$$\frac{\partial^+}{\partial s}U(t,s)x = -U(t,s)A(s)x, \quad x \in D(A(s)). \tag{1.38}$$

In this thesis we assume that $D(A(t))$ is dense in X for every $t \in [0,T]$ so that by Theorem 1.7.3 2 and 3, $U(t,s)$ turns to be strongly continuous. Moreover we consider the following (backward) adjoint problem

$$\begin{cases} v'(s) = -A(s)^{\star}v(s) & s<t, \\ v(t) = x^{\star}, \end{cases} \tag{1.39}$$

where $x^{\star} \in X^{\star}$ and $v : [0,T] \longrightarrow X^{\star}$.

If $\{A(t)^{\star}\}$ satisfies Hypothesis 1.7.2 there exists a unique (backward) evolution operator $\{V_{s,t}\}_{(s,t)\in\overline{\Delta}_T}$ with similar properties to the ones stated in Theorem 1.7.3. By [AT92, Rmk. 6.2] and [AFT91, Prop. 2.9], $V_{s,t} = U(t,s)^{\star}$ for all $(s,t) \in \overline{\Delta}_T$. Moreover we have the following result.

Proposition 1.7.4 ([AT92, Thm.6.4]). *Assume that* $\{A(t)\}_{t\in[0,T]}$ *and* $\{A(t)^\star\}_{t\in[0,T]}$ *satisfy Hypothesis 1.7.2. Let* $J : X \longrightarrow X^{\star\star}$ *be the canonical injection. Then the operator*

$$R(t,s) = -J^{-1}(A(s)^\star U(t,s)^\star)^\star J$$

is such that

1. $\dfrac{\partial}{\partial s}U(t,s) = R(t,s)$ *and* $\dfrac{\partial}{\partial s}U(t,s)x = -U(t,s)A(s)x, \quad x \in D(A(s))$*;*
2. *there exists* $C_3 > 0$ *such that*

$$\|R(t,s)\|_{\mathcal{L}(X)} \leqslant \frac{C_3}{t-s}. \tag{1.40}$$

Chapter 2

The Ornstein-Uhlenbeck evolution operator in $C_b(X)$

The (backward) evolution operator $\{P_{s,t}\}_{(s,t)\in\overline{\Delta}_T}$ is defined by

$$\begin{aligned}
P_{t,t} &= I \quad \text{for any} \quad t \in [0,T],\\
P_{s,t}\varphi(x) &= \int_X \varphi(y)\,\mathcal{N}_{m^x(t,s),Q(t,s)}(dy)\\
&= \int_X \varphi(y+m^x(t,s))\,\mathcal{N}_{0,Q(t,s)}(dy), \quad (s,t)\in\overline{\Delta}_T,\ \varphi\in C_b(X)
\end{aligned} \tag{2.1}$$

where we denote by $\mathcal{N}_{m,Q}$ the Gaussian measure in X with mean m and covariance Q. Here, for any $(s,t)\in\overline{\Delta}_T$ we have

$$Q(t,s) = \int_s^t U(t,r)Q(r)U(t,r)^\star\,dr, \tag{2.2}$$

$$m^x(t,s) = U(t,s)x + g(t,s), \tag{2.3}$$

$$g(t,s) = \int_s^t U(t,r)f(r)\,dr. \tag{2.4}$$

where $\{U(t,s)\}_{(t,s)\in\overline{\Delta}_T}$ is a strongly continuous evolution operator in X, $f:[0,T]\longrightarrow X$, $Q(t)=B(t)B(t)^\star$ with $B(t)\in\mathcal{L}(X)$ for all $t\in[0,T]$, satisfy Hypothesis 2.0.1.

Fixed $(s,t)\in\Delta_T$, we are interested in smoothing properties of $P_{s,t}$ along some normed spaces E, continuously embedded in X. Such properties will be proved under the assumption

$$U(t,s)(E)\subseteq\mathcal{H}_{t,s}:=Q(t,s)^{\frac{1}{2}}(X) \text{ and } U(t,s)_{|E}\in\mathcal{L}(E,\mathcal{H}_{t,s}). \tag{2.5}$$

In this case we define the operator

$$\Lambda(t,s) = Q(t,s)^{-\frac{1}{2}}U(t,s), \quad (s,t)\in\Delta_T,$$

where $Q(t,s)^{-\frac{1}{2}}$ is the pseudo-inverse of $Q(t,s)^{\frac{1}{2}}$.

If (2.5) holds, we prove that for all $k \in \mathbb{N}$, $P_{s,t}$ maps $C_b(X)$ (the space of continuous and bounded functions from X to $\mathbb{R}$, see Sect. 2) into $C^k_E(X)$ (the space of the functions from X to $\mathbb{R}$ k-times Frèchet differentiable along E having bounded Frèchet differentials along E see Sect. 2). Moreover, we give an explicit formula for the Frèchet derivatives of $P_{s,t}$ of any order along E, that involves the operators $\Lambda(t, s)$ and allows to estimate such derivatives.

In Section 2.1 we study the smoothing properties of $P_{s,t}$. In particular we prove that, if (2.5) holds, $P_{s,t}$ maps $C_b(X)$ in $C^k_E(X)$ for every $k \in \mathbb{N}$ and moreover there exists a constant $C_k > 0$ such that

$$\sup_{x\in X} \left\| D^k_E(P_{s,t}\varphi)(x)\right\|_{\mathcal{L}^k(E)} \leqslant C_k \left\|\Lambda(t,s)\right\|^k_{\mathcal{L}(E;X)} \|\varphi\|_\infty . \tag{2.6}$$

In Section 2.2 we extend to our non autonomous case a result of [GN03, Sect. 3], giving sufficient conditions in order that $E = Q(s)^{\frac{1}{2}}(X)$ satisfies (2.5) for $s \in (0, t)$.

Smoothing estimates such as (2.6) are basic tools for several developments of the theory besides Hölder maximal regularity. For instance we use them in the study of hypercontractivity properties of $P_{s,t}$ in L^p spaces with respect to suitable measures (see Chapter 5).

Our basic assumptions are the following.

Hypothesis 2.0.1. Let $(X, \|\cdot\|_X, \langle\cdot,\cdot\rangle_X)$, $(Y, \|\cdot\|_Y, \langle\cdot,\cdot\rangle_Y)$ be two real separable Hilbert spaces.

(1) $\{U(t,s)\}_{(s,t)\in\overline{\Delta}_T} \subseteq \mathcal{L}(X)$ is a strongly continuous evolution operator, namely for every $x \in X$ the map

$$(s,t) \in \overline{\Delta}_T \longmapsto U(t,s)x \in X, \tag{2.7}$$

is continuous and

  (a) $U(t,t) = I$ for any $t \in [0,T]$,
  (b) $U(t,r)U(r,s) = U(t,s)$ for $s \leqslant r \leqslant t$.

(2) The family of operators $\{B(t)\}_{t\in[0,T]} \subseteq \mathcal{L}(Y;X)$ is bounded and strongly measurable, namely

  (a) there exists $K > 0$ such that

$$\sup_{t\in[0,T]} \|B(t)\|_{\mathcal{L}(Y;X)} \leqslant K, \tag{2.8}$$

  (b) the map

$$t \in [0,T] \mapsto B(t)x \in X \tag{2.9}$$

  is measurable for any $x \in X$.

(3) The map $f : [0,T] \longrightarrow X$ is bounded and measurable.

(4) Setting $Q(t) = B(t)B(t)^\star$ for all $t \in [0,T]$, the trace of the operator $Q(t,s)$, defined in (2.2), is finite for every $s < t$.

By the Uniform Boundedness Theorem, there exists $N > 0$ such that

$$\|U(t,s)\|_{\mathcal{L}(X)} \leqslant N, \quad (s,t) \in \overline{\Delta}_T. \tag{2.10}$$

2.1 Smoothing properties of $P_{s,t}$

If Hypothesis 2.0.1 holds, the evolution family

$$P_{s,t}\varphi(x) = \int_X \varphi(y + m^x(t,s))\, \mathcal{N}_{0,Q(t,s)}(dy), \quad x \in X,\ (s,t) \in \Delta_T,\ \varphi \in C_b(X),$$

is well defined.

Definition 2.1.1. For every $(s,t) \in \Delta_T$ we denote by $\mathcal{H}_{t,s}$ the Cameron-Martin space of the measure $\mathcal{N}_{0,Q(t,s)}$. Moreover, for every $t \in \mathbb{R}$ we denote by H_t the space $Q(t)^{\frac{1}{2}}(X)$ endowed with the scalar product

$$\langle h, k\rangle_{\mathrm{H}_t} = \langle Q(t)^{-\frac{1}{2}} h, Q(t)^{-\frac{1}{2}} k\rangle_X, \quad h, k \in \mathrm{H}_t,$$

where $Q(t)^{-\frac{1}{2}}$ is the pseudo-inverse of $Q(t)^{\frac{1}{2}}$.

We recall here some basic results proved in [CL21].

Proposition 2.1.2 ([CL21, Cor. 2.2])**.** *We assume that Hypothesis 2.0.1 holds true. Then for every fixed $t \in [0,T]$ the mappings*

$$s \in [0,t] \longmapsto Q(t,s) \in \mathcal{L}(X), \quad s \in [0,t] \longmapsto \mathrm{Tr}\,[Q(t,s)] \in \mathbb{R}$$

are both continuous.

Proposition 2.1.3 ([CL21][Lemma 2.3])**.** *We assume that Hypothesis 2.0.1 holds true. Then for every $\varphi \in C_b(X)$ the mapping*

$$(s,t,x) \in \overline{\Delta}_T \times X \longmapsto P_{s,t}\varphi(x) \in \mathbb{R}$$

is measurable. Moreover if for every $t \in (0,T]$ the mapping

$$(s,x) \in [0,t] \times X \longmapsto P_{s,t}\varphi(x) \in \mathbb{R}$$

is continuous. If in addition the mapping

$$(s,t) \in \Delta_T \longmapsto U(t,s) \in \mathcal{L}(X)$$

is continuous, then the mapping

$$(s,t,x) \in \overline{\Delta}_T \times X \longmapsto P_{s,t}\varphi(x) \in \mathbb{R}$$

is continuous.

In [CL21] it was proved that $P_{s,t}$ maps $C^1_b(X)$ into itself, and

$$\nabla(P_{s,t}\varphi)(x) = U(t,s)^\star P_{s,t}\nabla\varphi(x), \quad (s,t) \in \Delta_T,\ x \in X,\ \varphi \in C^1_b(X),$$

so that

$$\sup_{x\in X} \|D(P_{s,t}\varphi)(x)\|_{X^\star} \leqslant \|U(t,s)\|_{\mathcal{L}(X)} \|D\varphi\|_\infty \leqslant M \|\varphi\|_{C^1_b(X)}, \quad (s,t) \in \Delta_T,\ \varphi \in C^1_b(X).$$

More generally it was proved that $P_{s,t}$ maps $C^k_b(X)$ into itself for every $k \in \mathbb{N}$. Here we extend this results working along the directions of a suitable subspace E.

Lemma 2.1.4. *We assume that Hypothesis 2.0.1 holds and that there exists a normed space $(E, \|\cdot\|_E)$ continuously embedded in X such that for some $(s,t) \in \Delta_T$*

$$U(t,s)(E) \subseteq E, \quad U(t,s)_{|E} \in \mathcal{L}(E) \text{ and } \exists\ M > 0 \text{ such that } \left\|U(t,s)_{|E}\right\|_{\mathcal{L}(E)} \leqslant M. \tag{2.11}$$

Then $P_{s,t}$ maps $C^k_E(X)$ into itself for every $k \in \mathbb{N}$ and

$$D^k_E(P_{s,t}\varphi)(x)(h_1, ..., h_k) = P_{s,t}\big(D^k_E\varphi(\cdot)(U(t,s)h_1, ..., U(t,s)h_k)\big)(x), \tag{2.12}$$

for any $x \in X$ and $h_1, ..., h_k \in E$. In particular for every $\varphi \in C^k_b(X)$

$$\left|D^k_E P_{s,t}\varphi(x)(h_1, ..., h_k)\right| \leqslant P_{s,t}\Big(\left|D^k_E\varphi(\cdot)(U(t,s)h_1, ..., U(t,s)h_k)\right|\Big), \tag{2.13}$$

$$\sup_{x\in X}\left\|D^k_E(P_{s,t}\varphi)\right\|_{\mathcal{L}^k(E)} \leqslant \|U(t,s)\|^k_{\mathcal{L}(E)} \sup_{x\in X}\left\|D^k_E\varphi\right\|_{\mathcal{L}^k(E)} \leqslant M^k \sup_{x\in X}\left\|D^k_E\varphi\right\|_{\mathcal{L}^k(E)}. \tag{2.14}$$

If $\varphi \in C^\alpha_E(X)$, where $\alpha = k + \sigma$, $k \in \mathbb{N} \cup \{0\}$ and $\sigma \in (0,1)$, then $P_{s,t}\varphi \in C^\alpha_E(X)$ and

$$[D^k_E(P_{s,t}\varphi)]_{C^\sigma_E(X)} \leqslant \|U(t,s)\|^k_{\mathcal{L}(E)}\, [D^k_E\varphi]_{C^\sigma_E(X)} \leqslant M^k[D^k\varphi]_{C^\sigma_E(X)}. \tag{2.15}$$

If $\varphi \in Z^k_E(X)$, where $k \in \mathbb{N}$, then $P_{s,t}\varphi \in Z^k_E(X)$ and

$$[D^{k-1}_E(P_{s,t}\varphi)]_{Z^k_E(X)} \leqslant \|U(t,s)\|^{k-1}_{\mathcal{L}(E)}\, [D^k_E\varphi]_{Z^k_E(X)} \leqslant M^{k-1}[D^k_E\varphi]_{Z^k_E(X)}. \tag{2.16}$$

Remark 2.1.5. By abuse of language, we will often write $U(t,s)$ instead of $U(t,s)_{|E}$.

Proof. We start proving (2.12) and (2.14) by recurrence over k. If $k = 1$, let $x \in X$, $h \in E$ and $\varepsilon > 0$, we have

$$\frac{P_{s,t}\varphi(x+\varepsilon h) - P_{s,t}\varphi(x)}{\varepsilon} \leqslant \int_X \frac{\left|\varphi\big(y + m^{x+\varepsilon h}(t,s)\big) - \varphi\big(y + m^x(t,s)\big)\right|}{\varepsilon}\, \mathcal{N}_{0,Q(t,s)}(dy). \tag{2.17}$$

Since

$$\frac{\left|\varphi\big(y + m^{x+\varepsilon h}(t,s)\big) - \varphi\big(y + m^x(t,s)\big)\right|}{\varepsilon} \leqslant \|D_E\varphi\|_\infty \|U(t,s)h\|_E \leqslant \|D_E\varphi\|_\infty \|U(t,s)\|_{\mathcal{L}(E)} \|h\|_E\,,$$

by the Dominated Convergence Theorem, as $\varepsilon \to 0^+$, we get

$$D_E P_{s,t}\varphi(x)(h) = P_{s,t}D_E\varphi(\cdot)(U(t,s)h). \tag{2.18}$$

Moreover

$$\begin{aligned} |D_E P_{s,t}\varphi(x)(h)| &\leqslant \int_X |D_E\varphi(y + m^x(t,s))(U(t,s)h)|\, \mathcal{N}_{0,Q(t,s)}(dy) \\ &\leqslant \sup_{x\in X}\|D_E\varphi\|_{E^\star} \|U(t,s)\|_{\mathcal{L}(E)} \|h\|_E\,, \end{aligned}$$

and

$$\|D_E P_{s,t}\varphi(x)\|_{E^\star} \leqslant \|D_E\varphi\|_\infty \|U(t,s)\|_{\mathcal{L}(E)}\,. \tag{2.19}$$

Hence

$$\sup_{x\in X}\|D_E P_{s,t}\varphi\|_{E^\star} \leqslant \sup_{x\in X}\|D_E\varphi\|_{E^\star}\|U(t,s)\|_{\mathcal{L}(E)} \leqslant M\|\varphi\|_{C^1_E(X)}. \tag{2.20}$$

We assume now $\varphi \in C^k_E(X)$ and that (2.12) and (2.14) hold. Let $x \in X$, $h_1, ..., h_{k+1} \in E$ and $\varepsilon > 0$. Then we have

$$\begin{aligned}
&\frac{1}{\varepsilon}\Big(P_{s,t}\big(D^k_E\varphi(\cdot)(U(t,s)h_1, ..., U(t,s)h_k)\big)(x+\varepsilon h_{k+1}) - P_{s,t}\big(D^k_E\varphi(\cdot)(U(t,s)h_1, ..., U(t,s)h_k)\big)(x)\Big)\\
&\leqslant \frac{1}{\varepsilon}\int_X\Big|D^k_E\varphi\big(y+m^{x+\varepsilon h_{k+1}}(t,s)\big)(U(t,s)h_1, ..., U(t,s)h_k)\\
&- D^k_E\varphi\big(y+m^x(t,s)\big)(U(t,s)h_1, ..., U(t,s)h_k)\Big|\mathcal{N}_{0,Q(t,s)}(dy).
\end{aligned}$$

Since

$$\begin{aligned}
&\frac{1}{\varepsilon}\left|D^k_E\varphi\big(y+m^{x+\varepsilon h_{k+1}}(t,s)\big)(U(t,s)h_1, ..., U(t,s)h_k) - D^k_E\varphi\big(y+m^x(t,s)\big)(U(t,s)h_1, ..., U(t,s)h_k)\right|\\
&\leqslant \frac{1}{\varepsilon}\left\|D^k_E\varphi\big(y+m^{x+\varepsilon h_{k+1}}(t,s)\big) - D^k_E\varphi\big(y+m^x(t,s)\big)\right\|_{\mathcal{L}^{k+1}(E)}\prod_{i=1}^k\|U(t,s)h_i\|_E\\
&\leqslant \sup_{x\in X}\left\|D^{k+1}_E\varphi\right\|_{\mathcal{L}^{k+1}(E)}\|U(t,s)\|^{k+1}_{\mathcal{L}(E)}\prod_{i=1}^{k+1}\|h_i\|_E
\end{aligned}$$

by the Dominated Convergence Theorem, as $\varepsilon \to 0^+$, we get

$$D^{k+1}_E(P_{s,t}\varphi)(x)(h_1, ..., h_{k+1}) = P_{s,t}\big(D^{k+1}_E\varphi(\cdot)(U(t,s)h_1, ..., U(t,s)h_{k+1})\big)(x). \tag{2.21}$$

Moreover

$$\begin{aligned}
\left|D^{k+1}_E(P_{s,t}\varphi)(x)(h_1, ..., h_{k+1})\right| &\leqslant \int_X\left|D^{k+1}_E\varphi(y+m^x(t,s))(U(t,s)h_1, ..., U(t,s)h_{k+1})\right|\mathcal{N}_{0,Q(t,s)}(dy)\\
&\leqslant \sup_{x\in X}\left\|D^{k+1}_E\varphi\right\|_{\mathcal{L}^{k+1}(E)}\|U(t,s)\|^{k+1}_{\mathcal{L}(E)}\prod_{i=1}^{k+1}\|h_i\|_E,
\end{aligned}$$

so that

$$\left\|D^{k+1}_E(P_{s,t}\varphi)(x)\right\|_{\mathcal{L}^k(E)} \leqslant \sup_{x\in X}\left\|D^{k+1}_E\varphi\right\|_{\mathcal{L}^{k+1}(E)}\|U(t,s)\|^{k+1}_{\mathcal{L}(E)}. \tag{2.22}$$

Hence

$$\sup_{x\in X}\left\|D^{k+1}_E(P_{s,t}\varphi)\right\|_{\mathcal{L}^{k+1}(E)} \leqslant \|U(t,s)\|^{k+1}_{\mathcal{L}(E)}\sup_{x\in X}\left\|D^{k+1}_E\varphi\right\|_{\mathcal{L}^{k+1}(E)} \leqslant M^{k+1}\|\varphi\|_{C^{k+1}_E(X)}. \tag{2.23}$$

Finally (2.15), (2.16) are consequences of (2.12) and of the definition of $P_{s,t}$. □

Remark 2.1.6. If E and F are Banach spaces continuously embedded in X such that $U(t,s)(E) \subseteq F$, then $U(t,s)_{|E} \in \mathcal{L}(E;F)$. Indeed, if $x_n \xrightarrow[n\to+\infty]{E} x$ and $U(t,s)_{|E}x_n \xrightarrow[n\to+\infty]{F} y$, then $x_n \xrightarrow[n\to+\infty]{X} x$ and $U(t,s)x_n \xrightarrow[n\to+\infty]{X} y$, since the embeddings of E in X and of F in X are continuous. Since $U(t,s) \in \mathcal{L}(X)$ then $y = U(t,s)x$ and by the Closed Graph Theorem $U(t,s)_{|E} \in \mathcal{L}(E,F)$.

Lemma 2.1.7. *We assume that Hypothesis 2.0.1 holds and that (2.11) holds for every $(s,t) \in \Delta_T$. Then, for every fixed $t \in \mathbb{R}$, $\varphi \in C_E^k(X)$ and $h_1, \dots, h_k \in X$, the mapping*

$$(s,x) \in (0,t] \times X \longmapsto D_E^k(P_{s,t}\varphi)(x)(h_1, \dots, h_k) \in \mathbb{R}, \tag{2.24}$$

is continuous.

If in addition the mapping $(s,t) \in \Delta_T \longmapsto U(t,s) \in \mathcal{L}(X)$ is continuous, then the function

$$(s,t,x) \in \overline{\Delta}_T \times X \longmapsto D_E^k(P_{s,t}\varphi)(x)(h_1, \dots, h_k) \in \mathbb{R}, \tag{2.25}$$

is continuous.

Proof. The proofs of these continuity properties follow from (2.12) and Proposition 2.1.3. ☐

Corollary 2.1.8. *We assume that Hypothesis 2.0.1 holds and that there exists a normed space $(E, \|\cdot\|_E)$ continuously embedded in X such that (2.11) holds for every $(s,t) \in \Delta_T$. Then, for every $\alpha > 0$ and for every $\varphi \in C_E^\alpha(X)$, $t \in (0,T]$, the function*

$$(s,x) \in [0,t] \times X \longmapsto u_0(s,x) := P_{s,t}\varphi(x) \in \mathbb{R},$$

belongs to $C_E^{0,\alpha}\big([0,t] \times X\big)$, and there exists $C = C(\alpha, T) > 0$ such that

$$\|u_0\|_{C_E^{0,\alpha}([0,t]\times X)} \leqslant C \|\varphi\|_{C_E^\alpha(X)}. \tag{2.26}$$

Similarly, if $\varphi \in Z_E^k(X)$ for some $k \in \mathbb{N}$, then the function u_0 belongs to $Z_E^{0,k}\big([0,t] \times X\big)$, and there exists $C = C(k,T) > 0$ such that

$$\|u_0\|_{Z_E^{0,k}([0,t]\times X)} \leqslant C \|\varphi\|_{Z_E^k(X)}. \tag{2.27}$$

Proof. It follows from Lemma 2.1.4 and by Lemma 2.1.7. ☐

Theorem 2.1.9. *Assuming that Hypothesis 2.0.1 holds and there exists a normed space $(E, \|\cdot\|_E)$ continuously embedded in X such that (2.5) holds, $P_{s,t}\varphi \in \bigcap\limits_{k\in\mathbb{N}} C_E^k(X)$ for every $\varphi \in C_b(X)$ and*

$$D_E(P_{s,t}\varphi)(x)(h) = \int_X \varphi\big(y + m^x(t,s)\big)\, \widehat{U(t,s)h}(y)\, \mathcal{N}_{0,Q(t,s)}(dy), \quad h \in E, \tag{2.28}$$

where $\widehat{\cdot}$ is with respect to the Gaussian measure $\mathcal{N}_{0,Q(t,s)}$ (see Proposition 1.5.19).

Moreover, there exists $C > 0$ such that

$$\sup_{x\in X} \|D_E(P_{s,t}\varphi)(x)\|_{E^\star} \leqslant C \|\Lambda(t,s)\|_{\mathcal{L}(E;X)} \|\varphi\|_\infty. \tag{2.29}$$

For $n \geqslant 2$

$$D_E^n(P_{s,t}\varphi)(x)(h_1, \dots, h_n) = \int_X \varphi\big(y + m^x(t,s)\big)\, I_n(t,s)(y)(h_1, \dots, h_n)\, \mathcal{N}_{0,Q(t,s)}(dy), \quad h_1, \dots, h_n \in E, \tag{2.30}$$

where

$$I_n(t,s)(y)(h_1,...,h_n) := \prod_{i=1}^{n} \widehat{U(t,s)h_i}(y)$$

$$+\sum_{s=1}^{r_n}(-1)^s \sum_{\substack{i_1,\dots i_{2s},\\ i_{2k-1}<i_{2k},\\ i_{2k-1}<i_{2k+1}}}^{n} \prod_{i=1}^{s}\langle\Lambda(t,s)h_{i_{2k-1}},\Lambda(t,s)h_{i_{2k}}\rangle_X \prod_{\substack{i_m=1,\\ i_m\neq i_1,\dots,i_{2s}}}^{n} \widehat{U(t,s)h_{i_m}}(y) \tag{2.31}$$

and

$$r_n = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,}\\ \frac{n-1}{2} & \text{if } n \text{ is odd.}\end{cases}$$

It follows that for every $n \geqslant 2$ there exists $C_n > 0$ such that

$$\sup_{x\in X}\|D_E^n(P_{s,t}\varphi)(x)\|_{\mathcal{L}^n(E)} \leqslant C_n\|\Lambda(t,s)\|^n_{\mathcal{L}(E;X)}\|\varphi\|_\infty . \tag{2.32}$$

Proof.

Step 1. We prove that $P_{s,t}\varphi(x) \in C_b(X)$ for every $\varphi \in C_b(X)$. For $x, x_0 \in X$ we have

$$|P_{s,t}\varphi(x) - P_{s,t}\varphi(x_0)| \leqslant \int_X \big|\varphi\big(y+m^x(t,s)\big) - \varphi\big(y+m^{x_0}(t,s)\big)\big|\,\mathcal{N}_{0,Q(t,s)}(dy). \tag{2.33}$$

Since

$$\big|\varphi\big(y+m^x(t,s)\big) - \varphi\big(y+m^{x_0}(t,s)\big)\big| \leqslant 2\|\varphi\|_\infty$$

the statement follows letting $x \to x_0$, by the Dominated Convergence theorem .

Step 2. We prove that $P_{s,t}\varphi(x) \in C^1_E(X)$ for every $\varphi \in C_b(X)$ and that (2.28) holds.
For every $\varepsilon \in (0,1)$, $x \in X$ and $h \in E$ we have

$$\begin{aligned}\frac{P_{s,t}\varphi(x+\varepsilon h) - P_{s,t}\varphi(x)}{\varepsilon} =& \frac{1}{\varepsilon}\bigg(\int_X \varphi(y+m^x(t,s))\,\mathcal{N}_{\varepsilon U(t,s)h,Q(t,s)}(dy)\\ &-\int_X \varphi(y+m^x(t,s))\,\mathcal{N}_{0,Q(t,s)}(dy)\bigg).\end{aligned}$$

Since $\varepsilon\, U(t,s)h \in \mathcal{H}_{t,s}$, thanks to the Cameron-Martin formula we get

$$\begin{aligned}\mathcal{N}_{\varepsilon U(t,s)h,Q(t,s)}(dy) &= \exp\bigg(-\frac{1}{2}\|\varepsilon U(t,s)h\|^2_{\mathcal{H}_{t,s}} + \widehat{\varepsilon U(t,s)h}(y)\bigg)\mathcal{N}_{0,Q(t,s)}(dy)\\ &= \exp\bigg(-\frac{1}{2}\varepsilon^2\|\Lambda(t,s)h\|^2_X + \varepsilon\widehat{U(t,s)h}(y)\bigg)\mathcal{N}_{0,Q(t,s)}(dy).\end{aligned}$$

Therefore, setting

$$f_\varepsilon(y) = -\frac{1}{2}\varepsilon\|\Lambda(t,s)h\|^2_X + \widehat{U(t,s)h}(y), \tag{2.34}$$

we get

$$\frac{P_{s,t}\varphi(x+\varepsilon h) - P_{s,t}\varphi(x)}{\varepsilon} = \int_X \frac{\exp(\varepsilon f_\varepsilon(y)) - 1}{\varepsilon}\,\varphi\big(y+m^x(t,s)\big)\,\mathcal{N}_{0,Q(t,s)}(dy). \tag{2.35}$$

Now

$$\lim_{\varepsilon\to 0} f_\varepsilon(y) = \widehat{U(t,s)h}(y) \quad \text{for a.e. } y$$

$$\left|\frac{\exp\Big(\varepsilon f_\varepsilon(y)\Big)-1}{\varepsilon}\,\varphi\big(y+m^x(t,s)\big)\right| \leqslant C\,|f_\varepsilon(y)|\,\Big(\exp|f_\varepsilon(y)|+1\Big)\,\|\varphi\|_\infty$$

$$y \mapsto \widehat{U(t,s)h}(y)\exp\Big(\widehat{U(t,s)h}(y)\Big) \in L^1(X,\mathcal{N}_{0,Q(t,s)}).$$

Hence by the Dominated Convergence theorem we obtain

$$\lim_{\varepsilon\to 0}\frac{P_{s,t}\varphi(x+\varepsilon h)-P_{s,t}\varphi(x)}{\varepsilon} = \int_X \varphi(y+m^x(t,s))\widehat{U(t,s)h}(y)\,\mathcal{N}_{0,Q(t,s)}(dy). \tag{2.36}$$

Moreover, by (1.26) with $p=1$ we get

$$\begin{aligned}\left|\int_X \varphi(y+m^x(t,s))\widehat{U(t,s)h}(y)\,\mathcal{N}_{0,Q(t,s)}(dy)\right| &\leqslant \|\varphi\|_\infty \left\|\widehat{U(t,s)h}(\cdot)\right\|_{L^1(X,\mathcal{N}_{0,Q(t,s)})}\\ &\leqslant \|\varphi\|_\infty\, c_1\, \|\Lambda(t,s)\|_{\mathcal{L}(E;X)}\,\|h\|_E\,,\end{aligned}$$

so that $P_{s,t}\varphi$ is E-Gâteaux differentiable at x and

$$D_E^G(P_{s,t}\varphi)(x) = \int_X \varphi(y+m^x(t,s))\widehat{U(t,s)h}(y)\,\mathcal{N}_{0,Q(t,s)}(dy).$$

To conclude we prove that $D_E^G(P_{s,t}\varphi) : X \longrightarrow E^\star$ is continuous. Indeed for $x, x_0 \in X$ we apply (1.26) with $p=2$ and we have

$$\begin{aligned}&\left|D_E^G(P_{s,t}\varphi)(x)(h) - D_E^G(P_{s,t}\varphi)(x_0)(h)\right|\\ &\leqslant \int_X |\varphi(y+m^x(t,s))-\varphi(y+m^{x_0}(t,s))|\left|\widehat{U(t,s)h}(y)\right|\,\mathcal{N}(0,Q(t,s))(dy)\\ &\leqslant c_2\,\|\varphi(y+m^x(t,s))-\varphi(y+m^{x_0}(t,s))\|_{L^2(X,\mathcal{N}_{0,Q(t,s)})}\,\|\Lambda(t,s)\|_{\mathcal{L}(E;X)}\,\|h\|_E\,.\end{aligned}$$

Hence

$$\left\|D_E^G(P_{s,t}\varphi)(x) - D_E^G(P_{s,t}\varphi)(x_0)\right\|_{E^\star} \leqslant \bar{c}\,\|\varphi(y+m^x(t,s))-\varphi(y+m^{x_0}(t,s))\|_{L^2(X,\mathcal{N}_{0,Q(t,s)})}$$

and since $\left|\varphi\big(y+m^x(t,s)\big)-\varphi\big(y+m^{x_0}(t,s)\big)\right| \leqslant 2\,\|\varphi\|_\infty$, the statement follows from the Dominated Convergence Theorem letting $x\to x_0$.

Step 3. We prove that $P_{s,t}\varphi \in C_E^2(X)$. We show first that for every $h\in E$, the mapping

$$D_E(P_{s,t}\varphi)(\cdot)(h) : X \longrightarrow \mathbb{R}$$

is E-Gâteaux differentiable. Let $\widetilde{h}\in E$, we set

$$f_\varepsilon(y) = -\frac{1}{2}\varepsilon\left\|\Lambda(t,s)\widetilde{h}\right\|_X^2 + \widehat{U(t,s)\widetilde{h}}(y),$$

and due to (2.28) and using again the Cameron-Martin formula, for every $\varepsilon > 0$ we have

$$
\begin{aligned}
&\frac{D_E(P_{s,t}\varphi)(x+\varepsilon\widetilde{h})(h) - D_E(P_{s,t}\varphi)(x)(h)}{\varepsilon} \\
&= \frac{1}{\varepsilon}\Big[\int_X \varphi\big(y+m^x(t,s)\big)\,\widehat{U(t,s)h}\big(y-\varepsilon U(t,s)\widetilde{h}\big)\,\mathcal{N}_{\varepsilon U(t,s)\widetilde{h},Q(t,s)}(dy) \\
&- \int_X \varphi\big(y+m^x(t,s)\big)\,\widehat{U(t,s)h}(y)\,\mathcal{N}_{0,Q(t,s)}(dy)\Big] \\
&= \frac{1}{\varepsilon}\int_X \varphi\big(y+m^x(t,s)\big)\widehat{U(t,s)h}(y)\big(\exp(\varepsilon f_\varepsilon(x))-1\big)\mathcal{N}_{0,Q(t,s)}(dy) \\
&- \int_X \varphi(y+m^x(t,s))\langle\Lambda(t,s)h,\Lambda(t,s)\widetilde{h}\rangle_X \exp(\varepsilon f_\varepsilon(y))\mathcal{N}_{0,Q(t,s)}(dy).
\end{aligned}
$$

Proceeding as in Step 2, we obtain

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{D_E(P_{s,t}\varphi)(x+\varepsilon\widetilde{h})h - D_E(P_{s,t}\varphi)(x)h}{\varepsilon} \\
&= \int_X \varphi\big(y+m^x(t,s)\big)\,\widehat{U(t,s)h}(y)\,\widehat{U(t,s)\widetilde{h}}(y)\,\mathcal{N}_{0,Q(t,s)}(dy) \\
&- \langle\Lambda(t,s)h,\Lambda(t,s)\widetilde{h}\rangle_X P_{s,t}\varphi(x).
\end{aligned}
\tag{2.37}
$$

The right-hand side of 2.37 is the Gâteaux derivative of $D_E P_{s,t}\varphi(\cdot)h$ at x. In the same way we did in Step 2, we can show that the mapping $D_E^G(D_E(P_{s,t}\varphi)(\cdot)h) : X \longrightarrow E^\star$ is continuous, so that we conclude that $P_{s,t}\varphi \in C_E^2(X)$.

Step 4. We prove that $P_{s,t}\varphi \in C_E^n(X)$ for every $n \in \mathbb{N}$ and that (2.30) holds. We proceed by recurrence, so we assume that $P_{s,t}\varphi \in C_E^{n-1}(X)$ and that formula (2.30) holds for $D_E^{n-1}(P_{s,t}\varphi)$, for some $n \geqslant 3$.

We first show that $D_E^{n-1}(P_{s,t})\varphi$ is E- Gâteaux differentiable. For $x \in X$, $h_1, ..., h_{n-1}, h_n \in E$ and $\varepsilon > 0$ we set

$$
f_\varepsilon(y) = -\frac{1}{2}\varepsilon\,\|\Lambda(t,s)h_n\|_X^2 + \widehat{U(t,s)h_n}(y).
$$

and due to the Cameron-Martin Formula we have

$$
\begin{aligned}
&D_E^{n-1}(P_{s,t}\varphi)(x+\varepsilon h_n)(h_1,...,h_{n-1}) \\
&= \int_X \varphi(m^x(t,s)+y)I_{n-1}(t,s)(y-\varepsilon U(t,s)h_n)\exp(\varepsilon f_\varepsilon(y))\mathcal{N}_{0,Q(t,s)}(dy).
\end{aligned}
$$

Moreover we have

$$
\begin{aligned}
I_{n-1}(t,s)(y-\varepsilon U(t,s)h_n)(h_1,\dots,h_{n-1}) &= I_{n-1}(t,s)(y)(h_1,\dots,h_{n-1}) \\
&- \varepsilon \sum_{i=1}^{n-1} \langle \Lambda(t,s)h_i, \Lambda(t,s)h_n\rangle_X \prod_{\substack{j=1\\ j\neq i}}^{n-1} \widehat{U(t,s)h_j}(y) \\
&- \varepsilon \sum_{s=1}^{r_{n-1}} (-1)^s \sum_{\substack{i_1,\dots i_{2s},\\ i_{2k-1}<i_{2k},\\ i_{2k-1}<i_{2k+1}}}^{n} \prod_{i=1}^{s} \langle \Lambda(t,s)h_{i_{2k-1}}, \Lambda(t,s)h_{i_{2k}}\rangle_X \\
&\times \sum_{\substack{i_m=1,\\ i_m\neq i_1,\dots,i_{2s}}}^{n-1} \langle \Lambda(t,s)h_{i_m}, \Lambda(t,s)h_n\rangle_X \prod_{\substack{i_j=1\\ i_j\neq i_m,i_1,\dots,i_{2s}}}^{n-1} \widehat{U(t,s)h_{i_j}}(y) + O(\varepsilon^2).
\end{aligned}
$$

In the same way we did in Step 2 and in Step 3, by the Dominated Convergence theorem we have

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{D_E^{n-1}(P_{s,t}\varphi)(x+\varepsilon h_n)(h_1,\dots,h_{n-1}) - D_E^{n-1}(P_{s,t}\varphi)(x)(h_1,\dots,h_{n-1})}{\varepsilon} = \\
&= \int_X \varphi(m^x(t,s)+y) I_{n-1}(t,s)(y)(h_1,\dots,h_{n-1})\, \widehat{U(t,s)h_n}(y)\, \mathcal{N}_{0,Q(t,s)}(dy) + \\
&- \int_X \varphi(m^x(t,s)+y) \Bigg[\sum_{i=1}^{n-1} \langle \Lambda(t,s)h_i, \Lambda(t,s)h_n\rangle_X \prod_{\substack{j=1\\ j\neq i}}^{n-1} \widehat{U(t,s)h_j}(y) + \\
&+ \sum_{s=1}^{r_{n-1}} (-1)^s \sum_{\substack{i_1,\dots i_{2s},\\ i_{2k-1}<i_{2k},\\ i_{2k-1}<i_{2k+1}}}^{n} \prod_{i=1}^{s} \langle \Lambda(t,s)h_{i_{2k-1}}, \Lambda(t,s)h_{i_{2k}}\rangle_X \times \\
&\times \sum_{\substack{i_m=1,\\ i_m\neq i_1,\dots,i_{2s}}}^{n-1} \langle \Lambda(t,s)h_{i_m}, \Lambda(t,s)h_n\rangle_X \prod_{\substack{i_j=1\\ i_j\neq i_m,i_1,\dots,i_{2s}}}^{n-1} \widehat{U(t,s)h_{i_j}}(y) \Bigg] \mathcal{N}_{0,Q(t,s)}(dy).
\end{aligned}
$$

It is easy to show that the right hand side of the equation above coincides with the expression of $D_E^n(P_{s,t}\varphi)(h_1,\dots,h_n)$ given in (2.30). The same arguments of Step 2 and Step 3 imply that $P_{s,t}\varphi \in C_E^n(X)$.

□

Remark 2.1.10. We assume that the hypotheses of Theorem 2.1.9 hold and that E is a Banach space. Then $U(t,s)_{|E} \in \mathcal{L}(E,\mathcal{H}_{t,s})$ if and only if $\Lambda(t,s) \in \mathcal{L}(E;X)$. Indeed, $\|U(t,s)\|_{\mathcal{L}(E,\mathcal{H}_{t,s})} = \|\Lambda(t,s)\|_{\mathcal{L}(E,X)}$ and by Remark 2.1.6 $U(t,s)$ belongs to $\mathcal{L}(E,\mathcal{H}_{t,s})$.

Remark 2.1.11. If $E = X$ in Theorem 2.1.9, then $P_{s,t}$ is strong Feller as proved in [CL21; OR16b].

Remark 2.1.12. Similarly to [CL21], fixing $(s,t) \in \Delta_T$ we define the operator $L : L^2\big((s,t);Y\big) \longrightarrow X$ given by

$$
Ly := \int_s^t U(t,\sigma)B(\sigma)y(\sigma)\,d\sigma, \quad y \in L^2\big((s,t);Y\big). \tag{2.38}
$$

The adjoint operator $L^\star : X \longrightarrow L^2\big((s,t);Y\big)$ satisfies $\langle Ly, x\rangle_X = \langle y, L^\star x\rangle_{L^2((s,t);Y)}$ for every $x \in X$ and $y \in L^2\big((s,t);Y\big)$, which means

$$\int_s^t \langle U(t,\sigma)B(\sigma)y(\sigma), x\rangle_X \, d\sigma = \int_s^t \langle y(\sigma), (L^\star x)(\sigma)\rangle_Y \, d\sigma, \quad x \in X, y \in L^2\big((s,t);Y\big),$$

so that

$$(L^\star x)(\sigma) = B^\star(\sigma)U^\star(t,\sigma)x, \quad x \in X, \text{ a.e } \sigma \in (s,t)$$

and we get

$$LL^\star x = \int_s^t U(t,\sigma)B(\sigma)B^\star(\sigma)U^\star(t,\sigma)x \, d\sigma = Q(t,s)x, \quad x \in X.$$

Therefore, by the general theory of linear operators in Hilbert spaces (e.g., [DZ14, Cor. B3]), we get

$$\text{(2.39)} \qquad \begin{cases} \operatorname{Range} L = \operatorname{Range} Q\,(t,s)^{\frac{1}{2}} \\ \left\| Q\,(t,s)^{-\frac{1}{2}}\, x \right\|_X = \left\| L^{-1}x \right\|_{L^2((s,t);Y)}, \quad x \in \mathcal{H}_{t,s}. \end{cases}$$

Since in general L is not invertible, L^{-1} is meant as the pseudo-inverse of L. Hence

$$\text{(2.40)} \qquad \left\| L^{-1}x \right\|_{L^2((s,t);Y)} = \min\Big\{ \|y\|_{L^2((s,t);Y)} \ : \ Ly = x \Big\}, \quad x \in \mathcal{H}_{t,s}.$$

The range of L is the set of the traces at time t of the mild solutions of the evolution problems

$$\text{(2.41)} \qquad \begin{cases} u'(r) = A(r)u(r) + B(r)y(r), \quad s < r < t, \\ u(s) = 0 \end{cases}$$

where y varies in $L^2((s,t);Y)$. So (2.5) may be reformulated requiring that $U(t,s)$ maps E in the trace space.

Now we combine Lemma 2.1.4 and Theorem 2.1.9 to obtain the following result.

Corollary 2.1.13. *We assume that Hypotheses 2.0.1 holds and there exists a normed space* $(E, \|\cdot\|_E)$ *continuously embedded in* X *satisfying* (2.5) *and* (2.11). *Then for every* $k, n \in \mathbb{N} \cup \{0\}$ *such that* $k + n \geqslant 1$, $\varphi \in C^k_E(X)$, $P_{s,t}\varphi \in C^{k+n}_E(X)$ *and*

$$\text{(2.42)} \qquad \begin{aligned} &D^{k+n}_E(P_{s,t}\varphi)(x)(h_1, ..., h_{k+n}) \\ &= \int_X D^k_E \varphi\big(U(t,s)x + y\big)\big(U(t,s)h_1, ..., U(t,s)h_k\big) I_n(t,s)(y)(h_{k+1}, ..., h_{k+n}) \mathcal{N}_{0,Q(t,s)}(dy). \end{aligned}$$

For every $\alpha_2 \geqslant \alpha_1 \geqslant 0$ *there exists* $C = C(\alpha_1, \alpha_2)$ *such that*

$$\text{(2.43)} \qquad \|P_{s,t}\varphi\|_{C^{\alpha_2}_E(X)} \leqslant C\big(\|\Lambda(t,s)\|^{\alpha_2-\alpha_1}_{\mathcal{L}(E;X)} + 1\big) \|\varphi\|_{C^{\alpha_1}_E(X)}, \quad \varphi \in C^{\alpha_1}_E(X).$$

Proof. Formula (2.42) follows applying (2.12) and (2.30). Moreover, by (2.14) and (2.32) there exists $C = C(k,n) > 0$ such that

$$\text{(2.44)} \qquad \sup_{x \in X} \left\| D^{k+n}_E(P_{s,t}\varphi)(x) \right\|_{\mathcal{L}^{k+n}(E)} \leqslant C \, \|\Lambda(t,s)\|^n_{\mathcal{L}(E;X)} \, \|\varphi\|_{C^k_E(X)}, \quad \varphi \in C^k_E(X),$$

and for every $\alpha \in (0,1)$

$$\left\| D_E^{k+n}(P_{s,t}\varphi) \right\|_{C_E^{\alpha}(X,\mathcal{L}^{n+k}(E))} \leqslant C \left\| \Lambda(t,s) \right\|_{\mathcal{L}(E;X)}^{n} \left\| \varphi \right\|_{C_E^k(X)}, \quad \varphi \in C_E^{k+\alpha}(X). \tag{2.45}$$

We point out that if $\alpha_1 = \alpha_2$, then (2.43) follows from Lemma 2.1.4; so we may assume $\alpha_2 > \alpha_1$. Now if $\alpha_2 - \alpha_1 = n \in \mathbb{N}$, we set $\alpha_1 = m + \sigma$, where $m \in \mathbb{N}$ and $\sigma \in (0,1)$. We have

$$\begin{aligned}
&\left\| P_{s,t}\varphi \right\|_{C_E^{\alpha_2}} = \left\| P_{s,t}\varphi \right\|_{\infty} + \sum_{j=1}^{m+n} \sup_{x\in X} \left\| D_E^j(P_{s,t}\varphi)(x) \right\|_{\mathcal{L}^j(E)} + [D_E^{m+n} P_{s,t}\varphi]_{C_E^{\sigma}(X)} \\
&\leqslant \left\| \varphi \right\|_{\infty} + \sum_{j=1}^{m} \sup_{x\in X} \left\| D_E^j \varphi(x) \right\|_{\mathcal{L}^j(E)} + \sum_{j=1}^{n} C_j \left\| \Lambda(t,s) \right\|_{\mathcal{L}(E;X)}^{j} \sup_{x\in X} \left\| D_E^m \varphi(x) \right\|_{\mathcal{L}^m(E)} + \\
&+ C \left\| \Lambda(t,s) \right\|_{\mathcal{L}(E;X)}^{n} [D_E^m \varphi]_{C_E^{\sigma}(X)} \leqslant C(\alpha_1,\alpha_2) \big(\left\| \Lambda(t,s) \right\|_{\mathcal{L}(E;X)}^{\alpha_2-\alpha_1} + 1 \big) \left\| \varphi \right\|_{C_E^{\alpha_1}(X)}.
\end{aligned}$$

If $\alpha_2 - \alpha_1 \notin \mathbb{N}$, we set $\alpha_2 = \alpha_1 + n + \sigma$ with $n \in \mathbb{N} \cup \{0\}$ and $\sigma \in (0,1)$. We apply the following interpolation inequality

$$\left\| \psi \right\|_{C_E^{\alpha_2}(X)} \leqslant \left\| \psi \right\|_{C_E^{\alpha_1+n}(X)}^{1-\sigma} \left\| \psi \right\|_{C_E^{\alpha_1+n+1}(X)}^{\sigma}, \quad \psi \in C_E^{\alpha_1+n+1}(X) \tag{2.46}$$

to $\psi = P_{s,t}\varphi$. Then, due to (2.43) with α_1 replaced by $\alpha_1 + n$ and α_2 by $\alpha_1 + n + 1$, we have (2.43) in the general case. □

Remark 2.1.14. The interpolation inequality (2.46) is proved in [CL21, Prop. 2.8] in the case $E = X$ with equivalent norms. Anyway the proof of the general case is the same.

Corollary 2.1.15. *We assume that Hypotheses 2.0.1 holds and there exists a normed space $(E, \|\cdot\|_E)$ continuously embedded in X satisfying (2.5) and (2.11) for all $(s,t) \in \Delta_T$. Then for every $j \in \mathbb{N}$, $h_1, ..., h_j \in E$, $t \in [0,T]$ and $\varphi \in C_E(X)$ the mapping*

$$(s,x) \in [0,t) \times X \longmapsto D_E^j P_{s,t}\varphi(x)(h_1,...,h_j) \in \mathbb{R}$$

is continuous.

If in addition the mapping $(s,t) \in \Delta_T \longmapsto U(t,s) \in \mathcal{L}(X)$ is continuous, then the function

$$(s,t,x) \in \Delta_T \times X \longmapsto D_E^j(P_{s,t}\varphi)(x)(h_1,...,h_j) \in \mathbb{R}, \tag{2.47}$$

is continuous.

Proof. The statement about the continuity of the derivatives is a consequence of Lemma 2.1.4 and Theorem 2.1.9. Indeed for a fixed $t > 0$ and $\varepsilon \in (0,t)$, we have

$$P_{s,t}\varphi = P_{s,t-\varepsilon} P_{t-\varepsilon,t}, \quad 0 \leqslant s < t - \varepsilon \leqslant T.$$

Since $\psi = P_{t-\varepsilon,t}\varphi \in C_E^k(X)$ by Theorem 2.1.9, the function

$$(s,x) \in [0,t-\varepsilon] \times X \longmapsto D_E^k P_{s,t-\varepsilon}\psi(x)(h_1,...,h_j) = D_E^k P_{s,t}\varphi(h)(h_1,...,h_j)$$

is continuous by Lemma 2.1.4. The proof of the last claim is similar. □

2.2 Examples of smoothing directions

In the following propositions we give some examples of Hilbert spaces E satisfying (2.5) and (2.11). We start with two preliminary results.

Lemma 2.2.1. *Under Hypothesis 2.0.1, the following properties hold.*

1. *the mapping $s \longmapsto Q(t,s)$ is Lipschitz continuous with values in $\mathcal{L}(X)$ and it is decreasing, namely*
$$\langle Q(t,s_1)x,x\rangle_X \geqslant \langle Q(t,s_2)x,x\rangle_X \quad \text{for any} \quad 0 \leqslant s_1 \leqslant s_2 < t \leqslant T.$$
2. *$\mathcal{H}_{t,s_2} \subseteq \mathcal{H}_{t,s_1}$ for every $0 \leqslant s_1 < s_2 < t \leqslant T$ and the norm of the embedding is $\leqslant 1$.*
3. *$\operatorname{Ker}(Q(t,s_1)) \subseteq \operatorname{Ker}(Q(t,s_2)) \subseteq \operatorname{Ker}(Q(t))$ for any $0 \leqslant s_1 \leqslant s_2 < t \leqslant T$.*

Proof.

1. Let $0 \leqslant s_1 \leqslant s_2 < t \leqslant T$ and $x \in X$. We have
$$\|Q(t,s_1)x - Q(t,s_2)x\|_X = \left\| \int_{s_1}^{s_2} U(t,r)Q(r)U(t,r)^\star x dr \right\|_X \leqslant M^2K^2 \|x\|_X |s_2 - s_1|$$
so that
$$\|Q(t,s_1) - Q(t,s_2)\|_{\mathcal{L}(X)} \leqslant N^2K^2 |s_2 - s_1|,$$
where N and K are the constants defined in Hypothesis 2.0.1 and in (2.8). Moreover
$$\begin{aligned}\langle Q(t,s_2)x,x\rangle_X &= \int_{s_2}^t \langle U(t,r)Q(r)^{\frac{1}{2}} Q(r)^{\frac{1}{2}} U(t,r)^\star x, x\rangle_X \, dr = \\ &= \int_{s_2}^t \left\| Q(r)^{\frac{1}{2}} U(t,r)^\star x \right\|_X^2 dr \leqslant \\ &\leqslant \int_{s_1}^t \left\| Q(r)^{\frac{1}{2}} U(t,r)^\star x \right\|_X^2 dr = \langle Q(t,s_1)x,x\rangle_X.\end{aligned}$$
2. The continuous embedding $\mathcal{H}_{t,s_2} \subseteq \mathcal{H}_{t,s_1}$ is an immediate consequence of proposition 1.3.3 and statement 1, observing that $\left\| Q(t,s)^{\frac{1}{2}} x \right\|_X^2 = \langle Q(t,s)x,x\rangle_X$.
3. Let $x \in \operatorname{Ker}(Q(t,s_1))$. Since
$$\langle Q(t,s_1)x,x\rangle_X = \int_{s_1}^t \left\| Q(r)^{\frac{1}{2}} U(t,r)^\star x \right\|_X^2 dr,$$
we obtain $Q(r)^{\frac{1}{2}} U(t,r)^\star x = 0$ for almost every $r \in (s_1,t)$, so that $x \in \operatorname{Ker}(Q(t,s_2))$.
Let $x \in \operatorname{Ker}(Q(t,s_2))$. Then $Q(r)^{\frac{1}{2}} U(t,r)^\star x = 0$ for almost every $r \in (s_2,t)$ and for $y \in X$ we have
$$0 = \langle Q(r)^{\frac{1}{2}} U(t,r)^\star x, y\rangle_X = \langle x, U(t,r)Q(r)^{\frac{1}{2}} y\rangle_X, \quad \text{a.e. } s_2 < r < t.$$

Letting $(r_n)_{n\in\mathbb{N}} \subset \{r \in [0,T] \mid s_2 < r < t$ and (2.48) holds$\}$ be such that $r_n \longrightarrow t$, we obtain

$$0 = \langle x, Q\,(t)^{\frac{1}{2}}\, y\rangle_X = \langle Q\,(t)^{\frac{1}{2}}\, x, y\rangle_X, \quad \text{for all} \;\; y \in X. \tag{2.48}$$

Hence $Q\,(t)^{\frac{1}{2}}\, x = 0$ and since $\operatorname{Ker}(Q(t)) = \operatorname{Ker}\left(Q\,(t)^{\frac{1}{2}}\right)$, the statement holds.

□

Here we extend some results of [GN03, Sect. 3] to the non autonomous case.

Proposition 2.2.2. *Let Hypothesis 2.0.1 hold. We assume that for a fixed $s \in [0,T)$ there exists a normed space $E \subseteq X$ with continuous embedding and that $U(t,s)(E) \subseteq \mathrm{H}_t$ for every $t \in (s,T]$ and that there exist $M > 0$ and $\alpha \in \left[0, \frac{1}{2}\right)$ such that*

$$\|U(t,s)\|_{\mathcal{L}(E,\mathrm{H}_t)} \leqslant \frac{M}{(t-s)^\alpha}, \quad t \in (s,T]. \tag{2.49}$$

Then for every $t \in (s,T]$, $U(t,s)(E) \subseteq \mathcal{H}_{t,s}$ and

$$\|U(t,s)\|_{\mathcal{L}(E,\mathcal{H}_{t,s})} = \left\|Q\,(t,s)^{-\frac{1}{2}}\, U(t,s)\right\|_{\mathcal{L}(E,X)} \leqslant \frac{M}{(t-s)^{\frac{1}{2}+\alpha}}. \tag{2.50}$$

Proof. We consider the operator L from remark 2.1.12 with $Q\,(\sigma)^{-\frac{1}{2}}$ instead of $B(\sigma)$. We know that $\operatorname{Range} L = \operatorname{Range} Q\,(t,s)^{\frac{1}{2}}$. On the other hand, for every $x \in E$ we have

$$\begin{aligned}(t-s)U(t,s)x &= \int_s^t U(t,s)x\, d\sigma = \int_s^t U(t,\sigma)U(\sigma,s)x\, d\sigma = \\ &= \int_s^t U(t,\sigma)Q\,(\sigma)^{\frac{1}{2}}\, Q\,(\sigma)^{-\frac{1}{2}}\, U(\sigma,s)x\, d\sigma.\end{aligned} \tag{2.51}$$

where $y(\sigma) := Q\,(\sigma)^{-\frac{1}{2}}\, U(\sigma,s)x$ belongs to $L^2\big((s,t);X\big)$. Hence $U(t,s)x$ belongs to the range of L, and since $\alpha \in \left[0, \frac{1}{2}\right)$ by (2.40) we get

$$\begin{aligned}\left\|Q\,(t,s)^{-\frac{1}{2}}\, ((t-s)U(t,s)x)\right\|_X &\leqslant \|y\|_{L^2((s,t);X)} \leqslant \left(\int_s^t \left\|Q\,(\sigma)^{-\frac{1}{2}}\, U(\sigma,s)x\right\|_X^2 d\sigma\right)^{\frac{1}{2}} \leqslant \\ &\leqslant \left(\int_s^t \frac{M^2}{(\sigma-s)^{2\alpha}}\, \|x\|_E^2\, d\sigma\right)^{\frac{1}{2}} \leqslant (t-s)^{\frac{1}{2}-\alpha} \frac{M}{(1-2\alpha)^{\frac{1}{2}}}\, \|x\|_E\,,\end{aligned}$$

which yields (2.50). □

Theorem 2.2.3. *Let Hypothesis 2.0.1 hold. We assume that $U(t,s)(\mathrm{H}_s) \subseteq \mathrm{H}_t$ for any $(s,t) \in \Delta_T$ and that there exist $M > 0$ and $\alpha \in \left[0, \frac{1}{2}\right)$ such that*

$$\|U(t,s)\|_{\mathcal{L}(\mathrm{H}_s,\mathrm{H}_t)} \leqslant \frac{M}{(t-s)^\alpha}, \quad (s,t) \in \Delta_T. \tag{2.52}$$

Then the following statements hold.

1. *For every* $0 \leqslant s < t \leqslant T$, $U(t,s)(\mathrm{H}_s) \subseteq \mathcal{H}_{t,s}$ *and*

$$\|U(t,s)\|_{\mathcal{L}(\mathrm{H}_s,\mathcal{H}_{t,s})} = \left\|Q\,(t,s)^{-\frac{1}{2}}\,U(t,s)\right\|_{\mathcal{L}(\mathrm{H}_s,X)} \leqslant \frac{M}{(t-s)^{\frac{1}{2}-\alpha}}. \tag{2.53}$$

2. *For every* $0 \leqslant s < t \leqslant T$, $U(t,s)^\star\big(\mathrm{Ker}\,(Q(t))\big) \subseteq \mathrm{Ker}\,(Q(s))$ *and* $\mathrm{Ker}\,(Q(t,s)) = \mathrm{Ker}\,(Q(t))$.
3. *For every* $0 \leqslant s_1 < s_2 < t \leqslant T$, $\mathcal{H}_{t,s_1} = \mathcal{H}_{t,s_2}$ *and their norms are equivalent.*
4. *For every* $0 \leqslant s < t \leqslant T$, $\mathcal{H}_{t,s}$ *is continuously embedded in* H_t.

Proof.

1. Apply Proposition 2.2.2 with $E = \mathrm{H}_s$.
2. We first remark some facts.
   (a) Since $U(t,s)(\mathrm{H}_s) \subseteq \mathrm{H}_t$, by continuity we get $U(t,s)(\overline{\mathrm{H}_s}) \subseteq \overline{\mathrm{H}_t}$.
   (b) $\mathrm{Ker}\left(Q\,(t)^{\frac{1}{2}}\right) = \mathrm{Ker}\,(Q(t))$, $\big(\mathrm{Ker}\,(Q(t))\big)^\perp = \overline{\mathrm{H}_t}$ and consequently $\overline{\mathrm{H}_t} = (I - P_t)(X)$, where we denote by P_t the orthogonal projection on $\mathrm{Ker}\,(Q(t))$.
   (c) By statements (a) and (b) we have $U(t,s)^\star(\mathrm{Ker}\,(Q(t))) \subseteq \mathrm{Ker}\,(Q(s))$. In fact

$$\begin{aligned} U(t,s)(\overline{\mathrm{H}_s}) \subseteq \overline{\mathrm{H}_t} &\iff U(t,s)(I-P_s)(X) \subseteq (I-P_t)(X) \\ &\iff P_t U(t,s)(I-P_s) = 0. \end{aligned}$$

   Hence

$$(I-P_s)^\star U(t,s)^\star P_t^\star = (I-P_s)U(t,s)^\star P_t = 0, \tag{2.54}$$

   namely $U(t,s)^\star(\mathrm{Ker}\,(Q(t))) \subseteq \mathrm{Ker}\,(Q(s))$.

   For every $x \in \mathrm{Ker}\,(Q(t))$, we have $U(t,r)^\star x \in \mathrm{Ker}\,(Q(r)) = \mathrm{Ker}\left(Q\,(r)^{\frac{1}{2}}\right)$ for every $r \in (s,t)$ and therefore

$$\left\|Q\,(t,s)^{\frac{1}{2}}\,x\right\|_X^2 = \langle Q(t,s)x, x\rangle_X = \int_s^t \left\|Q\,(r)^{\frac{1}{2}}\,U(t,r)^\star x\right\|_X^2\,dr,$$

   so that $x \in \mathrm{Ker}\,(Q(t,s))$. Conversely, the inclusion of $\mathrm{Ker}\,(Q(t,s))$ in $\mathrm{Ker}\,(Q(t))$ follows from statement 3 of lemma 2.2.1.
3. The continuous embedding $\mathcal{H}_{t,s_2} \subseteq \mathcal{H}_{t,s_1}$ is statement 2 of lemma 2.2.1. Concerning the reverse embedding, we first point out that the adjoint of the operator $U(t,s)_{|\mathrm{H}_s} : \mathrm{H}_s \longrightarrow \mathrm{H}_t$ is the operator $(U(t,s)_{|\mathrm{H}_s})^\star : \mathrm{H}_t \longrightarrow \mathrm{H}_s$ such that

$$\langle x, (U(t,s)_{|\mathrm{H}_s})^\star y\rangle_{\mathrm{H}_s} = \langle U(t,s)_{|\mathrm{H}_s} x, y\rangle_{\mathrm{H}_t}, \quad \text{for all } x \in \mathrm{H}_s,\ y \in \mathrm{H}_t.$$

   Now we claim that

$$(U(t,s)_{|\mathrm{H}_s})^\star Q(t)x = Q(s)U(t,s)^\star x, \quad x \in X,\ (s,t) \in \Delta_T, \tag{2.55}$$

where $U(t,s)^\star$ denotes the adjoint operator of $U(t,s) \in \mathcal{L}(X)$. Indeed for all $h \in \mathrm{H}_s$ we have $(I-P_s)h = h$, $(I-P_t)U(t,s)h = U(t,s)h$ and

$$\begin{aligned}
\langle (U(t,s)_{|\mathrm{H}_s})^\star Q(t)x, h\rangle_{\mathrm{H}_s} &= \langle Q(t)x, U(t,s)_{|\mathrm{H}_s} h\rangle_{\mathrm{H}_t} = \langle (I-P_t)x, U(t,s)h\rangle_X \\
&= \langle x, (I-P_t)U(t,s)h\rangle_X = \langle U(t,s)^\star x, h\rangle_X \\
&= \langle U(t,s)^\star x, (I-P_s)h\rangle_X = \langle Q(s)U(t,s)^\star x, h\rangle_{\mathrm{H}_s} \\
&= \langle Q(s)U(t,s)^\star x, h\rangle_{\mathrm{H}_s},
\end{aligned}$$

where the last equality follows from statement 2.
Hence we get for $0 \leqslant s_1 < s_2 < t \leqslant T$

$$\begin{aligned}
\left\| Q(t,s_1)^{\frac{1}{2}} x \right\|_X^2 &= \int_{s_1}^t \left\| Q(r)^{\frac{1}{2}} U(t,r)^\star x \right\|_X^2 dr \\
&= \int_{s_1}^{s_2} \left\| Q(r)^{\frac{1}{2}} U(t,r)^\star x \right\|_X^2 dr + \int_{s_2}^t \left\| Q(r)^{\frac{1}{2}} U(t,r)^\star x \right\|_X^2 dr \\
&= \int_{s_2}^{2s_2-s_1} \left\| Q(\sigma_0+s_1-s_2)^{\frac{1}{2}} U(t,\sigma_0+s_1-s_2)^\star x \right\|_X^2 d\sigma_0 + \left\| Q(t,s_2)^{\frac{1}{2}} x \right\|_X^2,
\end{aligned}$$

where $\sigma_0 = r - s_1 + s_2$ in the first integral.
If $t > 2s_2 - s_1$, by (2.55) we have

$$\begin{aligned}
&\int_{s_2}^{2s_2-s_1} \left\| Q(\sigma_0+s_1-s_2)^{\frac{1}{2}} U(t,\sigma_0+s_1-s_2)^\star x \right\|_X^2 d\sigma_0 \\
&\leqslant \int_{s_2}^{t} \left\| Q(\sigma_0+s_1-s_2)^{\frac{1}{2}} U(t,\sigma_0+s_1-s_2)^\star x \right\|_X^2 d\sigma_0 \\
&= \int_{s_2}^{t} \left\| Q(\sigma_0+s_1-s_2) U(\sigma_0,\sigma_0+s_1-s_2)^\star U(t,\sigma_0)^\star x \right\|_{H_{\sigma_0+s_1-s_2}}^2 d\sigma_0 \\
&= \int_{s_2}^{t} \left\| (U(\sigma_0,\sigma_0+s_1-s_2)_{|H_{\sigma_0+s_1-s_2}})^\star Q(\sigma_0) U(t,\sigma_0)^\star x \right\|_{H_{\sigma_0+s_1-s_2}}^2 d\sigma_0 \\
&\leqslant M^2 \int_{s_2}^{t} \left\| Q(\sigma_0) U(t,\sigma_0)^\star x \right\|_{H_{\sigma_0+s_1-s_2}}^2 d\sigma_0 \\
&= M^2 \int_{s_2}^{t} \left\| Q(\sigma_0)^{\frac{1}{2}} U(t,\sigma_0)^\star x \right\|_X^2 d\sigma_0 = M^2 \left\| Q(t,s_2)^{\frac{1}{2}} x \right\|_X^2 .
\end{aligned}$$

Hence

$$\left\| Q(t,s_1)^{\frac{1}{2}} x \right\|_X^2 \leqslant (1+M^2) \left\| Q(t,s_2)^{\frac{1}{2}} x \right\|_X^2 . \tag{2.56}$$

If $s_2 < t \leqslant 2s_2 - s_1$, by (2.56) we have

$$\begin{aligned}
&\int_{s_2}^{2s_2-s_1} \left\| Q(\sigma_0+s_1-s_2)^{\frac{1}{2}} U(t,\sigma_0+s_1-s_2)^\star x \right\|_X^2 d\sigma_0 \\
&= \int_{s_2}^{t} + \int_{t}^{2s_2-s_1} \left\| Q(\sigma_0+s_1-s_2)^{\frac{1}{2}} U(t,\sigma_0+s_1-s_2)^\star x \right\|_X^2 d\sigma_0 \\
&\leqslant M^2 \left\| Q(t,s_2)^{\frac{1}{2}} x \right\|_X^2 + \int_{s_2}^{3s_2-s_1-t} \left\| Q(\sigma_1+t+s_1-2s_2)^{\frac{1}{2}} U(t,\sigma_1+t+s_1-2s_2)^\star x \right\|_X^2 d\sigma_1,
\end{aligned}$$

where $\sigma_1 = \sigma_0 - t + s_2$.

If $t > 3s_2 - s_1 - t \iff t > \frac{3s_2 - s_1}{2}$, hence for any $t \in \left(\frac{3s_2-s_1}{2}, 2s_2 - s_1\right]$ we have

$$\begin{aligned}
&\int_{s_2}^{3s_2-s_1-t} \left\| Q\left(\sigma_1 + t + s_1 - 2s_2\right)^{\frac{1}{2}} U(t, \sigma_1 + t + s_1 - 2s_2)^\star x \right\|_X^2 \, d\sigma_0 \\
&\leqslant \int_{s_2}^{t} \left\| Q\left(\sigma_1 + t + s_1 - 2s_2\right)^{\frac{1}{2}} U(t, \sigma_1 + t + s_1 - 2s_2)^\star x \right\|_X^2 \, d\sigma_1
\end{aligned}$$

and by (2.55)

$$\begin{aligned}
&\int_{s_2}^{t} \left\| Q\left(\sigma_1 + t + s_1 - 2s_2\right)^{\frac{1}{2}} U(t, \sigma_1 + t + s_1 - 2s_2)^\star x \right\|_X^2 \, d\sigma_1 \\
&= \int_{s_2}^{t} \| Q(\sigma_1 + t + s_1 - 2s_2) U(\sigma_1, \sigma_1 + t + s_1 - 2s_2)^\star U(t, \sigma_1)^\star x \|_{H_{\sigma_1+t+s_1-2s_2}}^2 \, d\sigma_1 \\
&= \int_{s_2}^{t} \| (U(\sigma_1, \sigma_1 + t + s_1 - 2s_2)_{H_{\sigma_1+t+s_1-2s_2}})^\star Q(\sigma_1) U^\star(t, \sigma_1) x \|_{H_{\sigma_1+t+s_1-2s_2}}^2 \, d\sigma_1 \\
&\leqslant M^2 \left\| Q\left(t, s_2\right)^{\frac{1}{2}} \right\|_X^2 .
\end{aligned}$$

Adding up,

$$\left\| Q\left(t, s_1\right)^{\frac{1}{2}} x \right\|_X^2 \leqslant (1 + 2M^2) \left\| Q\left(t, s_2\right)^{\frac{1}{2}} x \right\|_X^2 . \tag{2.57}$$

Instead, if $s_2 < t \leqslant \dfrac{3s_2 - s_1}{2}$

$$\begin{aligned}
&\int_{s_2}^{3s_2-s_1-t} \left\| Q\left(\sigma_1 + t + s_1 - 2s_2\right)^{\frac{1}{2}} U^\star(t, \sigma_1 + t + s_1 - 2s_2) x \right\|_X^2 \, d\sigma_1 \\
&= \int_{s_2}^{t} + \int_{t}^{3s_2-s_1-t} \left\| Q\left(\sigma_1 + t + s_1 - 2s_2\right)^{\frac{1}{2}} U^\star(t, \sigma_1 + t + s_1 - 2s_2) x \right\|_X^2 \, d\sigma_1 \\
&\leqslant \int_{s_2}^{4s_2-s_1-2t} \left\| Q\left(\sigma_2 + 2t + s_1 - 3s_2\right)^{\frac{1}{2}} U^\star(t, \sigma_2 + 2t + s_1 - 3s_2) x \right\|_X^2 \, d\sigma_2 \\
&+ M^2 \left\| Q\left(t, s_2\right)^{\frac{1}{2}} x \right\|_X^2 ,
\end{aligned} \tag{2.58}$$

where $\sigma_2 = \sigma_1 - t + s_2$.

Let $k \in \mathbb{N}$. Now we prove the following.

- If $t \in \left(\frac{(k+2)s_2 - s_1}{k+1}, \frac{(k+1)s_2 - s_1}{k}\right]$, then

$$\left\| Q\left(t, s_1\right)^{\frac{1}{2}} x \right\|_X^2 \leqslant \left[1 + (k+1)M^2\right] \left\| Q\left(t, s_2\right)^{\frac{1}{2}} x \right\|_X^2 . \tag{2.59}$$

- If $t \in \left(s_2, \frac{(k+2)s_2 - s_1}{k+1}\right]$, then

$$\begin{aligned}
&\int_{s_2}^{s_k} \left\| Q\left(t_k\right)^{\frac{1}{2}} U^\star\big(t, t_k\big) \right\|_X^2 \, d\sigma_k \leqslant \int_{s_2}^{s_{k+1}} \left\| Q\left(t_{k+1}\right)^{\frac{1}{2}} U^\star\big(t, t_{k+1}\big) \right\|_X^2 \, d\sigma_{k+1} \\
&+ M^2 \left\| Q\left(t, s_2\right)^{\frac{1}{2}} \right\|_X^2 ,
\end{aligned} \tag{2.60}$$

where

$$s_k = (k+2)s_2 - s_1 - kt,$$
$$t_k = \sigma_k + kt + s_1 - (k+1)s_2,$$
$$\sigma_{k+1} = \sigma_k - t + s_2.$$

We proceed by recurrence over k.

For $k = 1$ (2.59) and (2.60) have been already proved in (2.57) and (2.58).

We assume now that (2.59) and (2.60) hold for a given $k \in \mathbb{N}$ and we prove that they hold for $k+1$.

Let $t \in \left(s_2, \frac{(k+2)s_2 - s_1}{k+1}\right]$.

If $t > s_{k+1} \iff t > \frac{(k+3)s_2 - s_1}{k+2}$, for any $t \in \left(\frac{(k+3)s_2-s_1}{k+2}, \frac{(k+2)s_2-s_1}{k+1}\right]$ we have

$$\int_{s_2}^{s_{k+1}} \left\| Q\left(t_{k+1}\right)^{\frac{1}{2}} U^\star\left(t, t_{k+1}\right)\right\|_X^2 \, d\sigma_{k+1} \leqslant \int_{s_2}^{t} \left\| Q\left(t_{k+1}\right)^{\frac{1}{2}} U^\star\left(t, t_{k+1}\right)\right\|_X^2 \, d\sigma_{k+1}$$

and by (2.55)

$$\int_{s_2}^{t} \left\| Q\left(t_{k+1}\right)^{\frac{1}{2}}) U(t, t_{k+1})^\star\right\|_X^2 \, d\sigma_{k+1} = \int_{s_2}^{t} \left\| Q(t_{k+1}) U(t, t_{k+1})^\star x\right\|_{H_{t_{k+1}}}^2 \, d\sigma_{k+1} =$$
$$= \int_{s_2}^{t} \left\| (U(\sigma_{k+1}, t_{k+1})_{|H_{t_{k+1}}})^\star Q(\sigma_{k+1}) U(t, \sigma_{k+1})^\star x\right\|_{H_{t_{k+1}}}^2 \, d\sigma_{k+1} =$$
$$\leqslant M^2 \left\| Q\left(t, s_2\right)^{\frac{1}{2}}\right\|_X^2.$$

Moreover

$$\left\| Q\left(t, s_1\right)^{\frac{1}{2}} x\right\|_X^2 \leqslant \left[1 + (k+2)M^2\right] \left\| Q\left(t, s_2\right)^{\frac{1}{2}} x\right\|_X^2,$$

and (2.59) holds for $k+1$.

Now if $t \in \left(s_2, \frac{(k+3)s_2-s_1}{k+2}\right]$, then

$$\int_{s_2}^{s_{k+1}} \left\| Q\left(t_{k+1}\right)^{\frac{1}{2}} U(t, t_{k+1})^\star\right\|_X^2 \, d\sigma_{k+1} = \int_{s_2}^{t} + \int_{t}^{s_{k+1}} \left\| Q\left(t_{k+1}\right)^{\frac{1}{2}} U(t, t_{k+1})^\star\right\|_X^2 \, d\sigma_{k+1}$$
$$\leqslant \int_{s_2}^{s_{k+2}} \left\| Q\left(t_{k+2}\right)^{\frac{1}{2}}) U(t, t_{k+2})^\star\right\|_X^2 \, d\sigma_{k+2} + M^2 \left\| Q\left(t, s_2\right)^{\frac{1}{2}}\right\|_X^2.$$

and (2.60) holds for $k+1$.

Therefore if $k \in \mathbb{N}$ and $t \in \left(\frac{(k+2)s_2-s_1}{k+1}, \frac{(k+1)s_2-s_1}{k}\right]$,

$$\left\| Q\left(t, s_1\right)^{\frac{1}{2}} x\right\|_X^2 \leqslant \left[1 + (k+1)M^2\right] \left\| Q\left(t, s_2\right)^{\frac{1}{2}} x\right\|_X^2.$$

Since

$$\bigcup_{k \in \mathbb{N}} \left(\frac{(k+2)s_2 - s_1}{k+1}, \frac{(k+1)s_2 - s_1}{k}\right] = (s_2, 2s_2 - s_1]$$

and (2.56) holds for $t \in (2s_2 - s_1, +\infty)$, statement 3 follows.

4. For $(s,t) \in \Delta_T$ and $x \in X$ we use again (2.55), to get

$$\begin{aligned}\left\| Q\left(x\right)^{\frac{1}{2}} \right\|_X &= \int_s^t \left\| Q\left(r\right)^{\frac{1}{2}} U(t,r)^\star x \right\|_X^2 \, dr = \int_s^t \left\| Q(r) U(t,r)^\star x \right\|_{H_r}^2 \, dr \\ &= \int_s^t \left\| (U(t,r)_{|_{H_r}})^\star Q(t) x \right\|_{H_r}^2 \, dr \leqslant M^2 (t-s) \left\| Q\left(t\right)^{\frac{1}{2}} x \right\|_X^2 .\end{aligned}$$

By Proposition 1.3.3 the statement follows.

□

Chapter 3

Maximal Hölder regularity

In this Chapter we assume that Hypothesis 2.0.1 holds true.

Let $t \in [0,T]$, let $\varphi \in C_b(X)$ and let $\psi \in C_b\big([0,t] \times X\big)$. We consider the function

$$u(s,x) := P_{s,t}\varphi(x) - \int_s^t \big(P_{s,\sigma}\psi(\sigma,\cdot)\big)(x)\, d\sigma, \tag{3.1}$$

and we prove Schauder type theorems for such u. If $U(t,s)$ is associated to a family of linear closed operators $\{A(t)\}_{t\in[0,T]}$, then (3.1) is the mild solution to a class of backward non autonomous initial value problems,

$$\begin{cases} \partial_s u(s,x) + L(s)u(s,x) = \psi(s,x), & (s,t) \in \Delta_T,\ x \in X, \\ u(t,x) = \varphi(x), \quad x \in X, \end{cases} \tag{3.2}$$

where the operators $L(t)$ are of Ornstein-Uhlenbeck type

$$L(t)\varphi(x) = \frac{1}{2}\mathrm{Tr}\Big(Q(t)D^2\varphi(x)\Big) + \langle x, A(t)^\star \nabla\varphi(x)\rangle_X + \langle f(t), \nabla\varphi(x)\rangle_X. \tag{3.3}$$

Here $Q(t)$ is a self-adjoint nonnegative operator for every $t \in [0,T]$.

In Chapter 2 we have studied smoothing properties of $P_{s,t}$ along a subspace $E \subseteq X$ such that (2.5) holds for every $(s,t) \in \Delta_T$. Estimates (2.14) and (2.43) depend on $\|\Lambda(t,s)\|_{\mathcal{L}(E,X)}$ that, in general, blows up as $t-s \to 0^+$. Here we assume (2.5) holds and in addition that $\|\Lambda(t,s)\|_{\mathcal{L}(E,X)}$ has a powerlike bound, namely there exist $\theta, C > 0$ such that

$$\|\Lambda(t,s)\|_{\mathcal{L}(E,X)} \leqslant \frac{C}{(t-s)^\theta}, \quad (s,t) \in \Delta_T. \tag{3.4}$$

In this case we prove Hölder maximal regularity of (3.1) along E. More precisely, if $\alpha \in [0,1)$, $\alpha + \dfrac{1}{\theta} \notin \mathbb{N}$, $\varphi \in C_E^{\alpha+\frac{1}{\theta}}(X)$ and $\psi \in C_E^{0,\alpha}\big([0,t] \times X\big)$, then $u(s,\cdot) \in C_E^{0,\alpha+\frac{1}{\theta}}\big([0,t] \times X\big)$. Moreover there exists $C = C(T,\alpha) > 0$, independent of φ and ψ, such that

$$\|u\|_{C_E^{0,\alpha+\frac{1}{\theta}}([0,t]\times X)} \leqslant C\big(\|\varphi\|_{C_E^{\alpha+\frac{1}{\theta}}(X)} + \|\psi\|_{C_E^{0,\alpha}([0,t]\times X)}\big). \tag{3.5}$$

If $\alpha + \dfrac{1}{\theta}$ is an integer, $u(s,\cdot)$ only belongs to the Zygmund space $Z_E^{\alpha+\frac{1}{\theta}}(X)$ for every s. For the definitions of the spaces $C_E^{\gamma}(X)$, $Z_E^k(X)$ and $C_E^{0,\gamma}([0,t]\times X)$ see Sect. 2. Zygmund regularity in place of C^k regularity is not due to the infinite dimensional setting nor to the time dependence of the data. Indeed, we have the same result even in finite dimension for the heat equation $u_s + \Delta u = \psi$.

Section 3.2 concerns four genuinely non autonomous examples. In the first example $A(t)$ and $B(t)$ are diagonal operators with respect to the same Hilbert basis of X. This allows to give easily necessary and sufficient conditions for (2.5) and (3.4) to hold.

In the second example we consider $A(t) = a(t)I$, where a is a continuous function. We get a non autonomous version of the Ornstein-Uhlenbeck semigroup used in the Malliavin calculus and we extend to such non autonomous case the results of [CL19].

In the third example the operator $A(t)$ is the realization of a second order elliptic differential operator in $X = L^2(\mathcal{O})$ with Dirichlet, Neumann or Robin boundary conditions and smooth enough coefficients, $\mathcal{O}$ is a bounded open smooth subset of $\mathbb{R}^d$.

3.1 Schauder type theorems

Besides Hypotheses 2.0.1 we assume also the following hypothesis.

Hypothesis 3.1.1.

1. There exists a normed space $(E, \|\cdot\|_E)$ continuously embedded in X satisfying (2.5) and (2.11) for all $(s,t) \in \Delta_T$.

2. There exist $C, \theta > 0$ such that

$$\|\Lambda(t,s)\|_{\mathcal{L}(E;X)} \leqslant \frac{C}{(t-s)^{\theta}}, \quad \text{for all } (s,t) \in \Delta_T. \tag{3.6}$$

We prove maximal Hölder regularity for the mild solution of problem (3.2), that is given by (3.1). We argue as in the papers [LR21] and [CL21].

We have immediately the following

Corollary 3.1.2. *Let Hypotheses 2.0.1 and 3.1.1 hold. For every $n \in \mathbb{N}$ there exists $K_n > 0$ such that*

$$\|D_E^n P_{s,t}\varphi(x)\|_{\mathcal{L}^n(E)} \leqslant \frac{K_n}{(t-s)^{n\theta}} \|\varphi\|_\infty, \quad (s,t) \in \Delta_T,\ \varphi \in C_b(X). \tag{3.7}$$

For every $\alpha \in (0,1)$ and $n \in \mathbb{N}$ there exists $K_{n,\alpha} > 0$ such that

$$\|D_E^n P_{s,t}\varphi(x)\|_{\mathcal{L}^n(E)} \leqslant \frac{K_{n,\alpha}}{(t-s)^{(n-\alpha)\theta}} \|\varphi\|_{C_b^\alpha(X)}, \quad (s,t) \in \Delta_T,\ \varphi \in C_b(X). \tag{3.8}$$

Proof. Estimates (3.7) and (3.8) follow from Theorem 2.1.9 and Corollary 2.1.13 taking into account Hypothesis 3.1.1. □

We set

$$u_0(s,x) = P_{s,t}\varphi(x), \quad (s,t) \in \Delta_T,\ x \in X \tag{3.9}$$

$$u_1(s,x) = -\int_s^t \big(P_{s,\sigma}\psi(\sigma,\cdot)\big)(x)\, d\sigma, \quad (s,t) \in \Delta_T,\ x \in X, \tag{3.10}$$

and we split $u = u_0 + u_1$.

Proposition 3.1.3. *For every $t \in [0,T]$ and $\psi \in C_b\big([0,t] \times X\big)$, u_1 is continuous and*

$$\|u_1\|_\infty \leqslant t\,\|\psi\|_\infty . \tag{3.11}$$

Moreover the following statements hold.

(1) Let $\theta < 1$. For every $n \in \mathbb{N}$ such that $n < \dfrac{1}{\theta}$, the function u_1 belongs to $C_E^{0,n}\big([0,t] \times X\big)$ and there exists $C > 0$, independent of ψ and t, such that

$$\|u_1\|_{C_E^{0,n}([0,t]\times X)} \leqslant C\,\|\psi\|_\infty . \tag{3.12}$$

(2) Let $\alpha \in (0,1)$ be such that $\alpha + \dfrac{1}{\theta} > 1$. For every $\psi \in C_E^{\alpha}(X)$ and for every $n \in \mathbb{N}$ such that $n < \alpha + \dfrac{1}{\theta}$, the function u_1 belongs to $C_E^{0,n}\big([0,t] \times X\big)$ and there exists $C > 0$, independent of ψ and t, such that

$$\|u_1\|_{C_E^{0,n}([0,t]\times X)} \leqslant C\,\|\psi\|_{C_E^{0,\alpha}([0,t]\times X)} . \tag{3.13}$$

Proof. Estimate (3.11) is obvious and the proof that u is continuous is in [CL21, Prop. 3.2].

Concerning statements (1) and (2), due to Corollary 3.1.2 we can differentiate in the integral that defines u_1 to obtain

$$\frac{\partial u_1}{\partial h_1, ..., \partial h_k}(s,x) = \int_s^t D_E^k P_{s,\sigma}\psi(\sigma,\cdot)(x)(h_1, ..., h_k)\, d\sigma, \quad s \in [0,t),\ x \in X, \tag{3.14}$$

for every $h_1, ..., h_k \in E$ and $k \in \{1, ..., n\}$. We prove now that the mapping

$$(s,x) \in [0,t) \times X \longmapsto D_E^k u_1(s,\cdot)(x)(h_1, ..., h_k) \in \mathbb{R}, \tag{3.15}$$

is continuous for every $h_1, ..., h_k \in E$ and $k \in \{1, ..., n\}$.

For $0 \leqslant s_0 \leqslant s < t \leqslant T$ and $x, x_0 \in X$, we write

$$\begin{aligned}\left|\frac{\partial u_1}{\partial h_1, ..., \partial h_k}(s,x) - \frac{\partial u_1}{\partial h_1, ..., \partial h_k}(s_0,x_0)\right| &= \int_{s_0}^s \left|D_E^k P_{s_0,\sigma}\psi(\sigma,\cdot)(x_0)(h_1, ..., h_k)\right|\, d\sigma \\ &+ \int_{s_0}^t \mathbb{1}_{[s,t]}(\sigma)\left|D_E^k P_{s,\sigma}\psi(\sigma,\cdot)(x)(h_1, ..., h_k) - D_E^k P_{s_0,\sigma}\psi(\sigma,\cdot)(x)(h_1, ..., h_k)\right|\, d\sigma.\end{aligned} \tag{3.16}$$

Since for every $\sigma \geqslant s_0$ the mapping

$$(s,x) \in [0,T] \times X \longmapsto \mathbb{1}_{[s,t]}(\sigma) D_E^k P_{s,\sigma}\psi(\sigma,\cdot)(x)(h_1, ..., h_k) \in \mathbb{R},$$

is continuous by Lemma 2.1.7 and by Lemma 2.1.4 and

$$\left|D^k_E P_{s,\sigma}\psi(\sigma,\cdot)(x)(h_1,...,h_k) - D^k_E P_{s_0,\sigma}\psi(\sigma,\cdot)(x_0)(h_1,...,h_k)\right| \leqslant 2M^k \sup_{x\in X}\left\|D^k_E\psi\right\|_{\mathcal{L}^k E}\prod_{i=1}^k \|h_i\|_E\,,$$

by the Dominated Convergence Theorem the first integral in 3.16 vanishes as $s \to s_0^+$ and $x \to x_0$. Moreover, since

$$\int_{s_0}^s \left|D^k_E P_{s_0,\sigma}\psi(\sigma,\cdot)(x_0)(h_1,...,h_k)\right| d\sigma \leqslant M^k \left\|D^k_E\psi\right\|_\infty \prod_{i=1}^k \|h_i\|_E\,(s-s_0),$$

even the second integral in 3.16 vanishes as $s \to s_0^+$ and $x \to x_0$.

If $s < s_0$ we split

$$\begin{aligned}\left|\frac{\partial u_1}{\partial h_1,...,\partial h_k}(s,x) - \frac{\partial u_1}{\partial h_1,...,\partial h_k}(s_0,x_0)\right| &\leqslant \int_s^{s_0}\left|D^k_E P_{s_0,\sigma}\psi(\sigma,\cdot)(x_0)(h_1,...,h_k)\right| d\sigma\\ + \int_{s_0}^t \left|D^k_E P_{s,\sigma}\psi(\sigma,\cdot)(x)(h_1,...,h_k) - D^k_E P_{s_0,\sigma}\psi(\sigma,\cdot)(x_0)(h_1,...,h_k)\right| d\sigma\end{aligned}$$

and we proceed as before to show that $\dfrac{\partial u_1}{\partial h_1,...,\partial h_k}$ is continuous.

□

Theorem 3.1.4. *Assume that Hypotheses 2.0.1 and 3.1.1 hold. Let $\varphi \in C_b(X)$, $\psi \in C_b\big([0,t]\times X\big)$ and let u be defined by (3.1).*

(1) If $\frac{1}{\theta} \notin \mathbb{N}$, $\varphi \in C_E^{\frac{1}{\theta}}(X)$ and $\psi \in C_b\big([0,t]\times X\big)$, then $u \in C_E^{0,\frac{1}{\theta}}\big([0,t]\times X\big)$. Moreover there exists $C = C(T) > 0$, independent of φ and ψ, such that

$$\|u\|_{C_E^{0,\frac{1}{\theta}}([0,t]\times X)} \leqslant C\big(\|\varphi\|_{C_E^{\frac{1}{\theta}}(X)} + \|\psi\|_\infty\big). \tag{3.17}$$

(2) If $\alpha \in (0,1)$, $\alpha + \frac{1}{\theta} \notin \mathbb{N}$, $\varphi \in C_E^{\alpha+\frac{1}{\theta}}(X)$ and $\psi \in C_E^{0,\alpha}\big([0,t]\times X\big)$, then $u \in C_E^{0,\alpha+\frac{1}{\theta}}\big([0,t]\times X\big)$. Moreover there exists $C = C(T,\alpha) > 0$, independent of φ and ψ, such that

$$\|u\|_{C_E^{0,\alpha+\frac{1}{\theta}}([0,t]\times X)} \leqslant C\big(\|\varphi\|_{C_E^{\alpha+\frac{1}{\theta}}(X)} + \|\psi\|_{C_E^{0,\alpha}([0,t]\times X)}\big). \tag{3.18}$$

Proof. We can assume that $\varphi \equiv 0$ since, by Corollary 2.1.8, we know that for every $\gamma > 0$ u_0 belongs to $C_E^{\gamma}([0,t]\times X)$, if $\varphi \in C_E^\gamma(X)$; moreover there exists $C = C(\gamma,T) > 0$ such that

$$\|u_0\|_{C_E^{0,\gamma}([0,t]\times X)} \leqslant C\,\|\varphi\|_{C_E^\gamma(X)}\,. \tag{3.19}$$

We already know that $\psi \in C_b([s,t]\times X)$.

Let us prove statement (1). Setting $n := \left[\dfrac{1}{\theta}\right]$, we have

$$n\theta \in (0,1), \quad (n+1)\theta > 1.$$

We have to show that $u_1(s,\cdot) \in C_E^{\frac{1}{\theta}}(X)$ for every $s \in [0,t]$.

If $n > 0$, we know from Proposition 3.1.3 that $u_1 \in C_b^{0,n}\big([0,t] \times X\big)$. Hence, we have to prove that $D_E^n u_1(s,\cdot)$ is $\left(\dfrac{1}{\theta} - n\right)$-Hölder continuous with values in $\mathcal{L}^n(E)$, with Hölder constant independent of s.

Let $h, h_1, ...h_n \in E$. We split every partial derivative $D_E^n u_1(s,y)(h_1,...h_n)$ into $a_h(s,y) + b_h(s,y)$ where

$$a_h(s,y) := -\int_s^{(s+\|h\|_E^{\frac{1}{\theta}})\wedge t} D_E^n P_{s,\sigma}\psi(\sigma,\cdot)(y)(h_1,...,h_n)\, d\sigma, \quad s \in [0,t],\ y \in X, \tag{3.20}$$

$$b_h(s,y) := -\int_{(s+\|h\|_E^{\frac{1}{\theta}})\wedge t}^t D_E^n P_{s,\sigma}\psi(\sigma,\cdot)(y)(h_1,...,h_n)\, d\sigma, \quad s \in [0,t],\ y \in X. \tag{3.21}$$

Due to (3.7), we obtain

$$\begin{aligned}
&|a_h(s,x+h) - a_h(s,x)| \leqslant |a_h(s,x+h)| + |a_h(s,x)| \\
&\leqslant 2K_n \|\psi\|_\infty \prod_{j=1}^n \|h_j\|_E \int_s^{(s+\|h\|_E^{\frac{1}{\theta}})\wedge t} (\sigma - s)^{-n\sigma}\, d\sigma. \\
&\leqslant \frac{2K_n}{1-n\theta} \|h\|_E^{\frac{1-n\theta}{\theta}} \|\psi\|_\infty \prod_{j=1}^n \|h_j\|_E .
\end{aligned}$$

Since if $\|h\|_E^{\frac{1}{\theta}} \geqslant t-s$ $b_h(s,\cdot)$ vanishes, we estimate $|b_h(s,x+h) - b_h(s,x)|$ when $\|h\|_E^{\frac{1}{\theta}} < t-s$. Again, by (3.7) we get

$$\begin{aligned}
\|D_E^n P_{s,\sigma}\psi(\sigma,\cdot)(x+h) - D_E^n P_{s,\sigma}\psi(\sigma,\cdot)(x)\|_{\mathcal{L}^n(E)} &\leqslant \sup_{y\in X} \left\|D_E^{n+1} P_{s,\sigma}\psi(\sigma,\cdot)(y)\right\|_{\mathcal{L}^n(E)} \|h\|_E \\
&\leqslant \frac{K_{n+1}}{(\sigma-s)^{(n+1)\theta}} \|\psi\|_\infty \|h\|_E \ \ \sigma \in (s,t)
\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}
&|b_h(s,x+h) - b_h(s,x)| \leqslant K_{n+1} \|\psi\|_\infty \|h\|_E \prod_{j=1}^n \|h_j\|_E \int_{(s+\|h\|_E^{\frac{1}{\theta}})\wedge t}^t (\sigma - s)^{-(n+1)\theta}\, d\sigma. \\
&\leqslant \frac{K_{n+1}}{(n+1)\theta - 1} \|h\|_E^{\frac{1}{\theta}-n} \|\psi\|_\infty \prod_{j=1}^n \|h_j\|_E .
\end{aligned}$$

Summing up we get

$$\left|\big(D_E^n u_1(s,x+h) - D_E^n u_1(s,x)\big)(h_1,...,h_n)\right| \leqslant C \|h\|_E^{\frac{1}{\theta}-n} \|\psi\|_\infty \prod_{j=1}^n \|h_j\|_E$$

with

$$C = \frac{2K_n}{1-n\theta} + \frac{K_{n+1}}{(n+1)\theta - 1}.$$

Therefore,

$$[D_E^n u_1(s,\cdot)]_{C_E^{\frac{1}{\theta}-n}(X;\mathcal{L}^n(E))} \leqslant C\,\|\psi\|_\infty .$$

The case $n=0$ is analogous to the previous one where instead of (3.7) we use (3.11).

Let us prove statement (2). Setting $n := \left[\alpha + \dfrac{1}{\theta}\right]$, we have

$$(n-\alpha)\theta \in (0,1), \quad (n+1-\alpha)\theta > 1.$$

We already know that $u_1 \in C_E^{0,n}(X)$ by Proposition 3.1.3 (2) and we have to show that $u_1(s,\cdot) \in C_E^{\alpha+\frac{1}{\theta}}(X)$ for every $s \in [0,t]$.

If $n>0$, we have to prove that $D_E^n u_1(s,\cdot)$ is $\left(\alpha + \dfrac{1}{\theta} - n\right)$-Hölder continuous with values in $\mathcal{L}^n(E)$, with Hölder constant independent of s.

Let $h, h_1, ...h_n \in E$. We split every partial derivative $D_E^n u_1(s,y)(h_1,...h_n)$ into $a_h(s,y) + b_h(s,y)$ where

$$a_h(s,y) := -\int_s^{(s+\|h\|_E^{\frac{1}{\theta}})\wedge t} D_E^n P_{s,\sigma}\psi(\sigma,\cdot)(y)(h_1,...,h_n)\,d\sigma, \quad s\in[0,t],\; y\in X, \tag{3.23}$$

$$b_h(s,y) := -\int_{(s+\|h\|_E^{\frac{1}{\theta}})\wedge t}^{t} D_E^n P_{s,\sigma}\psi(\sigma,\cdot)(y)(h_1,...,h_n)\,d\sigma, \quad s\in[0,t],\; y\in X. \tag{3.24}$$

Due to (3.8), we obtain

$$\begin{aligned}
&|a_h(s,x+h) - a_h(s,x)| \leqslant |a_h(s,x+h)| + |a_h(s,x)| \\
&\leqslant 2K_{n,\alpha}\,\|\psi\|_{C_E^\alpha([0,t]\times X)} \prod_{j=1}^n \|h_j\|_E \int_s^{(s+\|h\|_E^{\frac{1}{\theta}})\wedge t} (\sigma-s)^{-(n-\alpha)\sigma}\,d\sigma. \\
&\leqslant \frac{2K_{n,\alpha}}{1-(n-\alpha)\theta}\,\|h\|_E^{\frac{1-(n-\alpha)\theta}{\theta}}\,\|\psi\|_{C_E^\alpha([0,t]\times X)} \prod_{j=1}^n \|h_j\|_E .
\end{aligned}$$

We observe that if $\|h\|_E^{\frac{1}{\theta}} \geqslant t-s$, $b_h(s,\cdot)$ vanishes. Hence, we estimate now $|b_h(s,x+h) - b_h(s,x)|$ when $\|h\|_E^{\frac{1}{\theta}} < t-s$. Again, by (3.8) we get

$$\begin{aligned}
&\|D_E^n P_{s,\sigma}\psi(\sigma,\cdot)(x+h) - D_E^n P_{s,\sigma}\psi(\sigma,\cdot)(x)\|_{\mathcal{L}^n(E)} \leqslant \sup_{y\in X} \left\|D_E^{n+1} P_{s,\sigma}\psi(\sigma,\cdot)(y)\right\|_{\mathcal{L}^n(E)} \|h\|_E \\
&\leqslant \frac{K_{n+1,\alpha}}{(\sigma-s)^{(n+1-\alpha)\theta}}\,\|\psi\|_{C_E^\alpha([0,t]\times X)}\,\|h\|_E\,,\;\; \sigma\in(s,t)
\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}
&|b_h(s,x+h) - b_h(s,x)| \leqslant K_{n+1,\alpha}\,\|\psi\|_{C_E^\alpha([0,t]\times X)}\,\|h\|_E \prod_{j=1}^n \|h_j\|_E \int_{(s+\|h\|_E^{\frac{1}{\theta}})\wedge t}^{t} (\sigma-s)^{-(n+1-\alpha)\theta}\,d\sigma. \\
&\leqslant \frac{K_{n+1,\alpha}}{(n+1-\alpha)\theta - 1}\,\|h\|_E^{\frac{1}{\theta}+\alpha-n}\,\|\psi\|_{C_E^\alpha([0,t]\times X)} \prod_{j=1}^n \|h_j\|_E .
\end{aligned}$$

Summing up we get

$$\left|\left(D_E^n u_1(s,x+h)-D_E^n u_1(s,x)\right)(h_1,...,h_n)\right| \leqslant C\,\|h\|_E^{\frac{1}{\theta}+\alpha-n}\,\|\psi\|_{C_E^\alpha([0,t]\times X)}\prod_{j=1}^n \|h_j\|_E$$

with

$$C=\frac{2K_{n,\alpha}}{1-(n-\alpha)\theta}+\frac{K_{n+1,\alpha}}{(n+1-\alpha)\theta-1}.$$

Therefore,

$$[D_E^n u_1(s,\cdot)]_{C_E^{\frac{1}{\theta}+\alpha-n}(X;\mathcal{L}^n(E))} \leqslant C\,\|\psi\|_{C_E^\alpha([0,t]\times X)}\,.$$

The case $n=0$ is analogous to the previous one where instead of (3.8) we use (3.11).

□

Theorem 3.1.5. *Assume that Hypotheses 2.0.1 and 3.1.1 hold. Let $\varphi\in C_b(X)$, $\psi\in C_b\big([0,t]\times X\big)$ and let u be defined by (3.1).*

(1) If $\dfrac{1}{\theta}=k\in\mathbb{N}$ and if $\varphi\in Z_E^k(X)$, then $u\in Z_E^{0,k}\big([0,t]\times X\big)$. Moreover there exists $C=C(T)>0$, independent of φ and ψ, such that

$$\|u\|_{Z_E^{0,k}([0,t]\times X)} \leqslant C\big(\|\varphi\|_{Z_E^k(X)}+\|\psi\|_\infty\big). \tag{3.26}$$

(2) If $\alpha\in(0,1)$ and $\alpha+\dfrac{1}{\theta}=k\in\mathbb{N}$ and if $\varphi\in Z_E^k(X)$ and $\psi\in C_E^{0,\alpha}\big([0,t]\times X\big)$, then $u\in Z_E^{0,k}\big([0,t]\times X\big)$. Moreover there exists $C=C(T,\alpha)>0$, independent of φ and ψ, such that

$$\|u\|_{Z_E^{0,k}([0,t]\times X)} \leqslant C\big(\|\varphi\|_{Z_E^k(X)}+\|\psi\|_{C_E^{0,\alpha}([0,t]\times X)}\big). \tag{3.27}$$

Proof. We can assume that $\varphi\equiv 0$ since, by Corollary 2.1.8, we know that for every $k\in\mathbb{N}$ u_0 belongs to $Z_E^k([0,t]\times X)$, if $\varphi\in Z_E^k(X)$; moreover there exists $C=C(k,T)>0$ such that

$$\|u_0\|_{Z_E^{0,k}([0,t]\times X)} \leqslant C\,\|\varphi\|_{Z_E^k(X)}\,. \tag{3.28}$$

Let us prove statement (1). In this case we have $\psi\in C_b\big([s,t]\times X\big)$ and $k=\dfrac{1}{\theta}$. If $k\geqslant 2$, we know from Proposition 3.1.3 that $u_1\in C_E^{0,k-1}\big([0,t]\times X\big)$. So we have to show that $[D_E^{k-1}u_1(s,\cdot)]_{Z_E^1(X;\mathcal{L}^{k-1}(E))}$ is bounded by a constant independent of s.

Let $h,h_1,...h_n\in E$. We split every partial derivative $D_E^{k-1}u_1(s,y)(h_1,...h_n)$ into the sum $a_h(s,y)+b_h(s,y)$ defined in (3.23) and (3.24), as we did in the Theorem 3.1.4. Due to (3.7), we obtain

$$
\begin{aligned}
&|a_h(s,x+2h)-2a_h(s,x+h)+a_h(s,x)| \leqslant |a_h(s,x+2h)|+2\,|a_h(s,x+h)|+|a_h(s,x)|\\
&\leqslant 4K_{k-1}\,\|\psi\|_\infty \prod_{j=1}^{k-1}\|h_j\|_E \int_s^{(s+\|h\|_E^{\frac{1}{\theta}})\wedge t} (\sigma-s)^{-(k-1)\sigma}\,d\sigma.\\
&\leqslant \frac{4K_{k-1}}{1-(k-1)\theta}\,\|h\|_E^{\frac{1-(k-1)\theta}{\theta}}\,\|\psi\|_\infty \prod_{j=1}^{n}\|h_j\|_E\,.\\
&= 4kK_{k-1}\,\|h\|_E\,\|\psi\|_\infty \prod_{j=1}^{n}\|h_j\|_E\,.
\end{aligned}
$$

We observe that if $\|h\|_E^{\frac{1}{\theta}} \geqslant t-s$, $b_h(s,\cdot)$ vanishes. Hence we estimate

$$|b_h(s,x+2h)-2b_h(s,x+h)+b_h(s,x)|$$

when $\|h\|_E^{\frac{1}{\theta}} < t-s$. Again, by (3.7) we get

$$
\begin{aligned}
&|b_h(s,x+2h)-2b_h(s,x+h)+b_h(s,x)|\\
&\leqslant \int_{s+\|h\|_E^{\frac{1}{\theta}}}^{t} \Big| \big(D_E^{k-1}P_{s,\sigma}\psi(\sigma,\cdot)(x+2h)-2D_E^{k-1}P_{s,\sigma}\psi(\sigma,\cdot)(x+h)\\
&+D_E^{k-1}P_{s,\sigma}\psi(\sigma,\cdot)(x)\big)(h_1,...,h_{k-1})\Big|\,d\sigma\\
&\leqslant \|h\|_E^2 \prod_{j=1}^{k-1}\|h_j\|_E \int_{s+\|h\|_E^{\frac{1}{\theta}}}^{t} \sup_{y\in X}\left\|D_E^{k+1}P_{s,\sigma}\psi(\sigma,\cdot)(y)\right\|_{\mathcal{L}^{k+1}}\,d\sigma\\
&\leqslant K_{k+1}\,\|\psi\|_\infty\,\|h\|_E^2 \prod_{j=1}^{k-1}\|h_j\|_E \int_{s+\|h\|_E^{\frac{1}{\theta}}}^{t}(\sigma-s)^{-(k+1)\theta}\,d\sigma \leqslant kK_{k+1}\,\|\psi\|_\infty\,\|h\|_E \prod_{j=1}^{k-1}\|h_j\|_E\,.
\end{aligned}
$$

Summing up we get

$$[D_E^{k-1}u_1(s,\cdot)]_{Z_E^1(X;\mathcal{L}^{k-1}(E))} \leqslant k(4K_{k-1}+K_{k+1})\,\|\psi\|_\infty\,.$$

The case $k=\theta=1$ is analogous to the previous one where instead of (3.7) we use (3.11).

In the case of statement (2) we have $\psi \in C_E^{0,\alpha}\big([s,t]\times X\big)$ and $k=\alpha+\dfrac{1}{\theta}$. If $k\geqslant 2$, we know from Proposition 3.1.3 that $u_1 \in C_E^{0,k-1}\big([0,t]\times X\big)$ and so we have to show that $[D_E^{k-1}u_1(s,\cdot)]_{Z_E^1(X;\mathcal{L}^{k-1}(E))}$ is bounded by a constant independent of s.

Let $h,h_1,...h_n \in E$. We split every partial derivative $D_E^{k-1}u_1(s,y)(h_1,...h_n)$ into $a_h(s,y)+$

$b_h(s,y)$ as we did in the proof of Theorem 3.1.4. Due to (3.8), we obtain

$$\begin{aligned}
&|a_h(s,x+2h) - 2a_h(s,x+h) + a_h(s,x)| \leqslant |a_h(s,x+2h)| + 2\,|a_h(s,x+h)| + |a_h(s,x)| \\
&\leqslant 4K_{k-1,\alpha}\,\|\psi\|_{C_E^{0,\alpha}([0,t]\times X)} \prod_{j=1}^{k-1} \|h_j\|_E \int_s^{(s+\|h\|_E^{\frac{1}{\theta}})\wedge t} (\sigma - s)^{-(k-1-\alpha)\sigma}\, d\sigma. \\
&\leqslant \frac{4K_{k-1,\alpha}}{1-(k-1-\alpha)\theta}\,\|h\|_E^{\frac{1-(k-1-\alpha)\theta}{\theta}}\,\|\psi\|_{C_E^{0,\alpha}([0,t]\times X)} \prod_{j=1}^{n} \|h_j\|_E\,. \\
&= 4(k-\alpha)K_{k-1,\alpha}\,\|h\|_E\,\|\psi\|_{C_E^{0,\alpha}([0,t]\times X)} \prod_{j=1}^{n} \|h_j\|_E\,.
\end{aligned}$$

We observe that if $\|h\|_E^{\frac{1}{\theta}} \geqslant t-s$, $b_h(s,\cdot)$ vanishes. Hence we estimate

$$|b_h(s,x+2h) - 2b_h(s,x+h) + b_h(s,x)|$$

when $\|h\|_E^{\frac{1}{\theta}} < t-s$. Again, by (3.8) we get

$$\begin{aligned}
&|b_h(s,x+2h) - 2b_h(s,x+h) + b_h(s,x)| \\
&\int_{s+\|h\|_E^{\frac{1}{\theta}}}^t \left|\left(D_E^{k-1}P_{s,\sigma}\psi(\sigma,\cdot)(x+2h) - 2D_E^{k-1}P_{s,\sigma}\psi(\sigma,\cdot)(x+h) + D_E^{k-1}P_{s,\sigma}\psi(\sigma,\cdot)(x)\right)(h_1,\dots,h_{k-1})\right| d\sigma \\
&\leqslant \|h\|_E^2 \prod_{j=1}^{k-1} \|h_j\|_E \int_{s+\|h\|_E^{\frac{1}{\theta}}}^t \sup_{y\in X} \left\|D^{k+1}P_{s,\sigma}\psi(\sigma,\cdot)(y)\right\|_{\mathcal{L}^{k+1}}\, d\sigma \\
&\leqslant K_{k+1,\alpha}\,\|\psi\|_{C_E^{0,\alpha}([0,t]\times X)}\,\|h\|_E^2 \prod_{j=1}^{k-1} \|h_j\|_E \int_{s+\|h\|_E^{\frac{1}{\theta}}}^t (\sigma - s)^{-(k+1-\alpha)\theta}\, d\sigma \\
&\leqslant (k-\alpha)K_{k+1,\alpha}\,\|\psi\|_{C_E^{0,\alpha}([0,t]\times X)}\,\|h\|_E \prod_{j=1}^{k-1} \|h_j\|_E\,.
\end{aligned}$$

Summing up we get

$$[D_E^{k-1}u_1(s,\cdot)]_{Z_E^1(X;\mathcal{L}^{k-1}(E))} \leqslant (k-\alpha)(4K_{k-1,\alpha} + K_{k+1,\alpha})\,\|\psi\|_\infty\,.$$

The case $k = \theta = 1$ is analogous to the previous one where instead of (3.8) we use (3.11). □

3.2 Examples

In this section we give three genuinely non autonomous examples.

3.2.1 Diagonal operators

Let $T > 0$ and let $A(t)$, $B(t)$ be self-adjoint operators in diagonal form with respect to the same Hilbert basis $\{e_k:\ k \in \mathbb{N}\}$, namely

$$A(t)e_k = a_k(t)e_k, \quad B(t)e_k = b_k(t)e_k \quad 0 \leqslant t \leqslant T, \quad k \in \mathbb{N},$$

with continuous coefficients a_k, b_k. We set

$$\mu_k = \min_{t\in[0,T]} a_k(t), \quad \lambda_k = \max_{t\in[0,T]} a_k(t)$$

and we assume that there exists $\lambda_0 > 0$ such that

$$\lambda_k \leqslant \lambda_0, \quad \forall\ k \in \mathbb{N}. \tag{3.29}$$

In this setting the operator $U(t,s)$ is defined by

$$U(t,s)e_k = \exp\bigg(\int_s^t a_k(\tau)\,d\tau\bigg)e_k, \quad (s,t) \in \Delta_T, \quad k \in \mathbb{N}$$

is the strongly continuous evolution operator associated to the family $\{A(t)\}_{t\in[0,T]}$. Moreover we assume that there exists $K > 0$ such that

$$|b_k(t)| \leqslant K, \quad t \in [0,T],\ k \in \mathbb{N}. \tag{3.30}$$

Hence $B(t) \in \mathcal{L}(X)$ for all $t \in [0,T]$, the function $B : [0,T] \longmapsto \mathcal{L}(X)$ is continuous and

$$\sup_{t\in[0,T]} \|B(t)\|_{\mathcal{L}(X)} \leqslant K.$$

The operators $Q(t,s)$ are given by

$$Q(t,s)e_k = \int_s^t \exp\bigg(2\int_\sigma^t a_k(\tau)\,d\tau\bigg)(b_k(\sigma))^2\,d\sigma\, e_k =: q_k(t,s)e_k, \quad (s,t) \in \Delta_T,\ k \in \mathbb{N}.$$

Hypothesis 2.0.1 is fullfilled if

$$\sum_{k=0}^{\infty} q_k(t,s) < +\infty, \quad (s,t) \in \Delta_T. \tag{3.31}$$

We give now a sufficient condition for (3.31) to hold. We assume that λ_k is eventually nonzero (say for $k \geqslant k_0$). Given $(s,t) \in \Delta_T$, we have

$$\begin{aligned} &\bigg|\int_s^t \exp\bigg(2\int_\sigma^t a_k(\tau)\,d\tau\bigg)(b_k(\sigma))^2\,d\sigma\bigg| \leqslant \|b_k\|_\infty^2 \bigg|\int_s^t \exp\big(2\lambda_k(t-\sigma)\big)\,d\sigma\bigg| \\ &= \frac{\|b_k\|_\infty^2}{2\,|\lambda_k|}\,|1 - \exp(2\lambda_k(t-s))| \leqslant \frac{\|b_k\|_\infty^2}{2\,|\lambda_k|}\big(1 + \exp(2\lambda_0 T)\big). \end{aligned} \tag{3.32}$$

Hence (3.31) holds if we require

$$\sum_{k=k_0}^{\infty} \frac{\|b_k\|_\infty^2}{|\lambda_k|} < +\infty. \tag{3.33}$$

In [CL21] sufficient conditions for $U(t,s)(X) \subseteq \mathcal{H}_{t,s}$ were given. We look now for a condition such that there exists $(s,t) \in \Delta_T$ such that

$$U(t,s)(X) \nsubseteq \mathcal{H}_{t,s}. \tag{3.34}$$

Since $Q(t,s)^{\frac{1}{2}} e_k = (q_k(t,s))^{\frac{1}{2}} e_k$ for every $k \in \mathbb{N}$, (3.34) holds if either some of the $q_k(t,s)$ vanishes (so that the measure $\mathcal{N}_{0,Q(t,s)}$ is degenerate) or if none of the $q_k(t,s)$ vanishes, but

$$\sup_{k\in\mathbb{N}} \frac{\exp\left(\int_s^t 2a_k(\tau)\,d\tau\right)}{q_k(t,s)} = +\infty. \tag{3.35}$$

We obtain a sufficient condition for (3.35) if we assume that λ_k is eventually negative (say for $k \geqslant k_1 \geqslant k_0$). We have

$$\begin{aligned}
&\frac{\exp\left(\int_s^t 2a_k(\tau)\,d\tau\right)}{\int_s^t \exp\left(2\int_\sigma^t a_k(\tau)\,d\tau\right)(b_k(\sigma))^2\,d\sigma} \geqslant \frac{\exp\big(2\mu_k(t-s)\big)}{\|b_k\|_\infty^2 \int_s^t \exp\big(2\lambda_k(t-\sigma)\big)\,d\sigma} \\
&= \frac{\exp\big(2\mu_k(t-s)\big)}{\|b_k\|_\infty^2 \dfrac{1}{-2\lambda_k}\big(1-\exp(2\lambda_k(t-s))\big)} = \\
&= \frac{2\,|\lambda_k|}{\|b_k\|_\infty^2 \left[\exp\big(-2\mu_k(t-s)\big) - \exp\big(2(\lambda_k-\mu_k)(t-s)\big)\right]}.
\end{aligned}$$

So (3.35) is fullfilled if

$$\sup_{k\geqslant k_1} \frac{2\,|\lambda_k|}{\|b_k\|_\infty^2 \left(\exp\big(-2\mu_k(t-s)\big) - \exp\big(-2(\mu_k-\lambda_k)(t-s)\big)\right)} = +\infty, \quad \text{for some } (s,t)\in\Delta_T. \tag{3.36}$$

Now we want to find some conditions such that the hypotheses of Theorem 2.2.3 are satisfied. Let $(s,t)\in\Delta_T$, we assume that $b_k(t)$ is nonzero for all $k\in\mathbb{N}$. Hence we have $U(t,s)H_s \subseteq H_t$ if

$$\sup_{k\in\mathbb{N}} \frac{b_k^2(s)}{b_k^2(t)} \exp\left(\int_s^t 2a_k(\tau)\,d\tau\right) < +\infty. \tag{3.37}$$

Since

$$\frac{b_k^2(s)}{b_k^2(t)} \exp\left(\int_s^t 2a_k(\tau)\right) \leqslant \frac{b_k^2(s)}{b_k^2(t)} \exp(2\lambda_k T), \quad k\in\mathbb{N}$$

a sufficient condition for (3.37) to hold is

$$\sup_{k\in\mathbb{N}} \frac{b_k^2(s)}{b_k^2(t)} \exp(2\lambda_k T) < +\infty. \tag{3.38}$$

Now we investigate when (2.52) holds. We observe first that for $y = Q(s)^{\frac{1}{2}} x$ with $x\in H_s$, we have

$$\begin{aligned}
\|U(t,s)y\|_{H_t}^2 &= \left\|Q(t)^{-\frac{1}{2}} U(t,s) Q(s)^{\frac{1}{2}} x\right\|_X^2 = \sum_{k\in\mathbb{N}} \left(\frac{|b_k(s)|}{|b_k(t)|} \exp\left(\int_s^t a_k(\tau)\right)\langle x, e_k\rangle\right)^2 \\
&= \sum_{k\in\mathbb{N}} \left(\frac{|b_k(s)|}{|b_k(t)|} \exp\left(\int_s^t a_k(\tau)\right)\frac{\langle y, e_k\rangle}{|b_k(s)|}\right)^2 = \sum_{k\in\mathbb{N}} \left(\frac{1}{|b_k(t)|} \exp\left(\int_s^t a_k(\tau)\langle y, e_k\rangle_{H_s}\right)\right)^2.
\end{aligned}$$

We assume that there exist $L \geqslant 0$ such that

$$|b_k(t)| \geqslant L, \quad \text{for all } k \in \mathbb{N}. \tag{3.39}$$

Hence for any $k \in \mathbb{N}$, we have

$$\frac{1}{b_k^2(t)} \exp\left(\int_s^t 2a_k(\tau)\, d\tau\right) \leqslant \frac{1}{L^2} \exp\bigl(2\lambda_0 T\bigr), \tag{3.40}$$

and

$$\|U(t,s)\|_{\mathcal{L}(H_s,H_t)} \leqslant \frac{1}{L} \exp(\lambda_0 T), \quad 0 \leqslant s < t \leqslant T. \tag{3.41}$$

Therefore, by Theorem 2.2.3 there exists $M > 0$ such that

$$\|U(t,s)\|_{\mathcal{L}(H_s;\mathcal{H}_{t,s})} \leqslant \frac{M}{(t-s)^{\frac{1}{2}}} \tag{3.42}$$

Moreover, $H_{t_1} = H_{t_2}$ as Hilbert spaces for every $t_1, t_2 \in [0,T]$. Indeed, let $y \in H_{t_1}$, we know that

$$Q\,(t_1)^{-\frac{1}{2}}\, e_k = \frac{b_k(t_2)}{b_k(t_1)} Q\,(t_2)^{-\frac{1}{2}}\, e_k, \quad \text{for any } k \in \mathbb{N} \tag{3.43}$$

and

$$\|y\|_{H_{t_1}} \leqslant \frac{K}{L}\, \|y\|_{H_{t_2}}. \tag{3.44}$$

So, Hypothesis 3.1.1 is satisfied by H_t for every $t \in [0,T]$ with $\theta = \frac{1}{2}$. Fixing $E = H_{t_0}$ with $t_0 \in [0,T]$, Theorems 3.1.4(2) and 3.1.5(1) may be applied to obtain maximal Hölder regularity of the mild solution to (3.2).

We summarize what we obtained in the following theorem.

Theorem 3.2.1. *Let X be a real separable Hilbert space. Let $T > 0$ and let $A(t)$, $B(t)$ be self-adjoint operators in diagonal form with respect to the same Hilbert basis $\{e_k\}_{k\in\mathbb{N}}$, namely*

$$A(t)e_k = a_k(t)e_k, \quad B(t)e_k = b_k(t)e_k \quad 0 \leqslant t \leqslant T, \quad k \in \mathbb{N},$$

with continuous coefficients a_k, b_k. Under conditions (3.29) and (3.30), Hypothesis 2.0.1 is verified. Moreover if (3.33), holds, there exists $(s,t) \in \Delta_T$ such that $U(t,s)(X) \nsubseteq \mathcal{H}_{t,s}$ and if in addition (3.36), (3.38) and (3.39) hold, the spaces H_t coincides with equivalent norms. Hence $E = H_0$ satisfies 3.1.1 with $\theta = \frac{1}{2}$. We can apply Theorems 3.1.4 and 3.1.5 with $E = H_0$.

Remark 3.2.2. As an explicit example we can choose $a_k(t) = -k^2(t^k + 1)$ and $b_k(t) = \sin(kt) + c$, $c > 1$ for all $t \in [0,T]$.

3.2.2 A non autonomous version of the classical Ornstein-Uhlenbeck operator

Let $A(t) = a(t)I$, where a is a continuous real valued map on $[0,T]$ and set $\|a\|_\infty = a_0$. Hence

$$U(t,s) = \exp\left(\int_s^t a(\tau)\, d\tau\right) I, \quad (s,t) \in \Delta_T$$

is the strongly continuous evolution operator associated to the family $\{A(t)\}_{t\in[0,T]}$.

Let $\{B(t)\}_{t\in[0,T]} \subseteq \mathcal{L}(X)$ be a family of operators satisfying Hypothesis 2.0.1 (2). Since

$$Q(t,s) = \int_s^t \exp\left(2\int_\sigma^t a(\tau)\,d\tau\right)Q(\sigma)\,d\sigma \quad (s,t)\in\Delta_T,$$

where $Q(\sigma) = B(\sigma)B(\sigma)^\star$, an obvious sufficient condition for $\mathrm{Tr}\,[Q(t,s)] < +\infty$ is that the mapping $\sigma \longmapsto \mathrm{Tr}\,[Q(\sigma)]$ belongs to $L^1(0,T)$. In this case, $P_{s,t}$ is a non autonomous generalization of the classical Ornstein-Uhlenbeck semigroup widely used in the Malliavin Calculus. We refer to [CL19] for Schauder theorems in such autonomous case.

Since $U(t,s)$ is a multiple of the identity, if X is infinite dimensional $U(t,s)$ cannot map X into $\mathcal{H}_{t,s}$ for any $(s,t)\in\Delta_T$ and $U(t,s)(H_s) = H_s$ for all $(s,t)\in\Delta_T$. We look for sufficient conditions such that the hypotheses of Theorem 2.2.3 are satisfied.

In addition to the above assumptions on the trace of the operators $Q(\sigma)$, we require that for every $(s,t)\in\Delta_T$ there exists $C>0$ such that

$$\|B(s)\|_{\mathcal{L}(X)} \leqslant C\,\|B(t)\|_{\mathcal{L}(X)}\,. \tag{3.45}$$

It follows

$$\left\|Q\,(s)^{\frac{1}{2}}\right\|^2_{\mathcal{L}(X)} = \|B(s)^\star\|^2_{\mathcal{L}(X)} \leqslant C\,\|B(t)^\star\|^2_{\mathcal{L}(X)} = C\left\|Q\,(t)^{\frac{1}{2}}\right\|^2_{\mathcal{L}(X)}. \tag{3.46}$$

Then, by Proposition 1.3.3, $H_s \subseteq H_t$ with continuous embedding and for $x\in H_s$ we have

$$\|U(t,s)x\|_{H_t} = \left\|\exp\left(\int_s^t a(\tau)\,d\tau\right)x\right\|_{H_t} \leqslant e^{a_0T}\sqrt{C}\,\|x\|_{H_s}\,. \tag{3.47}$$

Hence, we can take $E = H_s$ in Theorem 2.1.9.

In order to have (3.45), we assume now that there exists $m>0$ such that

$$0 < m \leqslant \|B(t)\|_{\mathcal{L}(X)} \leqslant K, \quad \text{for any } t\in[0,T]. \tag{3.48}$$

In this case, for every $t_1,t_2\in[0,T]$

$$\|B(t_1)\|_{\mathcal{L}(X)} \leqslant \frac{K}{m}\,\|B(t_2)\|_{\mathcal{L}(X)} \tag{3.49}$$

and (3.45) holds with $C = \dfrac{K}{m}$. Moreover $\mathrm{H}_{t_1} = \mathrm{H}_{t_2}$ as Hilbert spaces for every $t_1,t_2\in[0,T]$. Hence Hypothesis 3.1.1 is satisfied by choosing as $E = \mathrm{H}_0$ with $\theta = \dfrac{1}{2}$ and we can apply Theorems 3.1.4 and 3.1.5.

We can state the following theorem.

Theorem 3.2.3. *Let X be a real separable Hilbert space and let $T>0$. Let $A(t) = a(t)I$, where a is a continuous real valued map on $[0,T]$. Let $\{B(t)\}_{t\in[0,T]} \subseteq \mathcal{L}(X)$ be a family of operators such that*

1. *there exists $K>0$ such that*

$$\sup_{t\in\mathbb{R}} \|B(t)\|_{\mathcal{L}(X)} \leqslant K, \tag{3.50}$$

2. *the map*

$$t \in \mathbb{R} \mapsto B(t)x \in X \tag{3.51}$$

is measurable for any $x \in X$.

3. $t \longmapsto \operatorname{Tr}[B(t)B(t)^\star] \in L^1(0,T)$.

Then Hypothesis 2.0.1 is satisfied and $U(t,s)(X) \nsubseteq \mathcal{H}_{t,s}$ *for all* $(s,t) \in \Delta_T$*. If in addition* (3.48) *holds, then* $\mathrm{H}_{t_1} = \mathrm{H}_{t_2}$ *as Hilbert spaces for every* $t_1, t_2 \in [0,T]$*, Hypothesis 3.1.1 is satisfied by choosing as* $E = \mathrm{H}_0$ *with* $\theta = \frac{1}{2}$ *and we can apply Theorems 3.1.4 and 3.1.5.*

3.2.3 A non autonomous parabolic problem

We generalize now Example 2 of [CL21]. Let $T > 0$ and let $\mathcal{O} \subseteq \mathbb{R}^d$ be a bounded open set with smooth boundary. We consider the evolution operator $U(t,s)$ in $X := L^2(\mathcal{O})$ associated to an evolution equation of parabolic type,

$$\begin{cases} u_t(t,x) = \mathcal{A}(t)u(t,\cdot)(x), & (t,x) \in (s,T) \times \mathcal{O}, \\ \mathcal{B}(t)u(t,\cdot)(x) = 0, & (t,x) \in (s,T) \times \partial\mathcal{O}. \end{cases} \tag{3.52}$$

The differential operators $\mathcal{A}(t)$ are defined by

$$\mathcal{A}(t)\varphi(x) = \sum_{i,j=1}^d a_{ij}(t,x)D_{ij}\varphi(x) + \sum_{i=1}^d a_i(t,x)D_i\varphi(x) + a_0(t,x)\varphi(x), t \in [0,T],\ x \in \mathcal{O} \tag{3.53}$$

and the family of the boundary operators $\{\mathcal{B}(t)\}_{t\in[0,T]}$ is either of Dirichlet, Neumann or Robin type, namely

$$\mathcal{B}(t)u = \begin{cases} u & \text{(Dirichlet)}, \\ \displaystyle\sum_{i,j=1}^d a_{ij}(x,t)D_i u\,\nu_j & \text{(Neumann)}, \\ \displaystyle\sum_{i,j=1}^d a_{ij}(x,t)D_i u\,\nu_j + b_0(x,t)u & \text{(Robin)}, \end{cases} \tag{3.54}$$

where $\nu = (\nu_1, ..., \nu_d)$ is the unit outer normal vector at the boundary of Ω.

We make the following assumptions.

Hypothesis 3.2.4. We assume $a_{ij} = a_{ji}$ and for some $\rho \in \left(\frac{1}{2}, 1\right)$, $a_{ij}, b_0 \in C^{\rho,1}\left([0,T] \times \overline{\mathcal{O}}\right)$, $a_i, a_0 \in C^{\rho,0}\left([0,T] \times \overline{\mathcal{O}}\right)$. Moreover, we assume that there exists $\nu > 0$ such that for all $\xi \in \mathbb{R}^d$

$$\sum_{i,j=1}^d a_{ij}(t,x)\xi_i\xi_j \geqslant \nu\,|\xi|^2, \quad t \in [0,T],\ x \in \mathcal{O}. \tag{3.55}$$

Under Hypothesis 3.2.4, for all $t \in [0,T]$, we define

$$D(A(t)) = \left\{u \in H^2(\mathcal{O}) : \ \mathcal{B}(t)u = 0\right\}, \qquad A(t)u = \mathcal{A}(t)u, \quad u \in D(A(t)). \tag{3.56}$$

In [Dan00, Thm 2.4] it is proved that for all $u_0 \in X$ there exists a unique weak solution u of (3.52). Setting $U(t,s)u_0 =: u(t)$, $U(t,s)$ turns out to be an evolution operator in $L^2(\mathcal{O})$. Moreover in [Dan00] it is shown that $U(t,s)$ has an extension (still denoted by $U(t,s)$) to the whole space $L^1(\mathcal{O})$ that belongs to $\mathcal{L}(L^1(\mathcal{O}); L^\infty(\mathcal{O}))$ and it is represented by

$$U(t,s)\varphi(x) = \int_{\mathcal{O}} k(x,y,t,s)\varphi(y)\,dy, \quad \varphi \in L^1(\mathcal{O}) \tag{3.57}$$

where for every $0 \leqslant s < t \leqslant T$, $k(\cdot,\cdot,t,s) \in L^\infty(\mathcal{O}\times\mathcal{O})$ and there exist $M, m > 0$ such that

$$|k(x,y,t,s)| \leqslant \frac{M}{(t-s)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{m(t-s)}}, \quad x,y \in \mathcal{O},\ 0 \leqslant s < t \leqslant T. \tag{3.58}$$

Moreover in [Sch04, Ex. 2.8 and Ex. 2.9] it is proven that the family $\{A(t)\}_{t\in[0,T]}$ satisfies the assumptions of [Acq88; AT87]. Then by Theorem 1.7.3, $U(t,s) \in \mathcal{L}\left(D(A(s); D(A(t)))\right) \cap \mathcal{L}(X; D(A(t)))$ with norms bounded by a constant independent of s and t. Hence, there exists $C > 0$ such that for every $(s,t) \in \Delta_T$ we have

$$\|U(t,s)\|_{\mathcal{L}(X)} \leqslant C, \tag{3.59}$$

$$\|U(t,s)\|_{\mathcal{L}(X;D(A(t)))} \leqslant \frac{C}{t-s}, \tag{3.60}$$

$$\|U(t,s)\|_{\mathcal{L}(D(A(s));D(A(t)))} \leqslant C. \tag{3.61}$$

By (3.59) and (3.61) for every $(s,t) \in \Delta_T$ and $\beta \in (0,1)$,

$$U(t,s) \in \mathcal{L}\left((X, D(A(s))_{\beta,2}; (X, D(A(t))_{\beta,2})\right),$$

where for every $r \in [0,T]$, $(X, D(A(r))_{\beta,2}$ is the standard real interpolation space. Moreover there exists a constant $C_1 = C_1(\beta) > 0$ such that

$$\|U(t,s)\|_{\mathcal{L}((X,D(A(s))_{\beta,2};(X,D(A(t))_{\beta,2}))} \leqslant C_1, \quad (s,t) \in \Delta_T. \tag{3.62}$$

We consider first Neumann and Robin boundary conditions for $A(t)$. Due to [Gui91, Thm. 3.5, Thm. 4.15] for all $0 \leqslant t \leqslant T$ and $0 < \gamma < 2$, we have

$$(X, D(A(t))_{\frac{\gamma}{2},2} = \begin{cases} H^\gamma(\mathcal{O}) & \text{if } 0 < \gamma < \frac{3}{2} \\ \left\{u \in H^{\frac{3}{2}}(\mathcal{O}) \mid \widetilde{\mathcal{B}}(t)u \in \mathring{H}^{\frac{1}{2}}(\mathcal{O})\right\} & \text{if } \gamma = \frac{3}{2} \\ \{u \in H^\gamma(\mathcal{O}) \mid \mathcal{B}(t)u = 0\} & \text{if } \frac{3}{2} < \gamma < 1 \end{cases}, \tag{3.63}$$

where

$$\widetilde{\mathcal{B}}(t)u = \begin{cases} \displaystyle\sum_{i,j=1}^d a_{ij}(x,t) D_i u\, \widetilde{\nu}_j & \text{(Neumann)} \\ \displaystyle\sum_{i,j=1}^d a_{ij}(x,t) D_i u\, \widetilde{\nu}_j + b_0(x,t)u & \text{(Robin)} \end{cases}, \tag{3.64}$$

$\widetilde{\nu}$ is a smooth enough extension of ν to $\overline{\mathcal{O}}$ and $\mathring{H}^{\frac{1}{2}}(\mathcal{O})$ consists on all the elements $\varphi \in H^{\frac{1}{2}}(\mathcal{O})$ whose null extension outside $\overline{\mathcal{O}}$ belongs to $H^{\frac{1}{2}}(\mathbb{R})$.

One simple example of the family $\{B(t)\}_{t\in[0,T]}$ is given by $B(t) = (\mathrm{Id} - \Delta)^{-\frac{\gamma}{2}}$ for all $t \in [0,T]$, where Δ is the realization of the Laplace operator in $L^2(\mathcal{O})$ with Neumann boundary conditions. We have

$$D\left((\mathrm{Id}-\Delta)^{\frac{\gamma}{2}}\right) = (L^2(\mathcal{O}), D(\Delta))_{\frac{\gamma}{2},2}$$

$$= \begin{cases} H^\gamma(\mathcal{O}) & \text{if } 0 < \gamma < \frac{3}{2} \\ \left\{u \in H^{\frac{3}{2}}(\mathcal{O}) \mid \langle \nabla u, \widetilde{\nu}\rangle \in \mathring{H}^{\frac{1}{2}}(\mathcal{O})\right\} & \text{if } \gamma = \frac{3}{2} \\ \left\{u \in H^\gamma(\mathcal{O}) \mid \langle \nabla u, \nu\rangle_{|\partial\mathcal{O}} = 0\right\} & \text{if } \frac{3}{2} < \gamma < 2 \end{cases}, \tag{3.65}$$

where $\langle \nabla u, \nu\rangle_{|\partial\mathcal{O}}$ is the trace of $\langle \nabla u, \nu\rangle$ at $\partial\mathcal{O}$.

In [CL21, Lemma 4.3] it is proven that if $B(t) \in \mathcal{L}(L^2(\mathcal{O}); L^q(\mathcal{O}))$ with $q \geqslant \max\{2, d\}$ and $\sup_{t\in[0,T]} B(t) < +\infty$, then $\mathrm{Tr}\,[Q(t,s)] < +\infty$ for all $(s,t) \in \Delta_T$. In our case for every $q \geqslant 2$, the embedding of $D\left((\mathrm{Id}-\Delta)^{\frac{\gamma}{2}}\right)$ in $L^q(\mathcal{O})$ is continuous for $\gamma \geqslant d\left(\frac{1}{2} - \frac{1}{q}\right)$. Hence for such choices of γ, $(\mathrm{Id}-\Delta)^{-\frac{\gamma}{2}} \in \mathcal{L}(L^2(\mathcal{O}); L^q(\mathcal{O}))$. In the same way, we can prove that if $q > d$, then $Q(t,s)$ has finite trace for all $0 \leqslant s < t \leqslant T$ and, in this case, Hypothesis 2.0.1 holds.

We check now that the space $E = H^\gamma(\mathcal{O})$ with $0 < \gamma < \frac{3}{2}$ satisfies Hypothesis 3.1.1. In fact by (3.63) and (3.65) we have $E = (X, D(A(t)))_{\frac{\gamma}{2},2} = \mathrm{H}_t$ for all $t \in [0,T]$. Then $U(t,s) \in \mathcal{L}(E, \mathrm{H}_t)$ for all $0 \leqslant s < t \leqslant T$ with norm bounded by a constant independent of s and t and (2.49) holds with $\alpha = 0$. By Proposition 2.2.2, Hypothesis 3.1.1 is satisfied by $E = (X, D(A(t)))_{\frac{\gamma}{2},2}$ with $\theta = \frac{1}{2}$.

Summarizing, we need $q \geqslant 2$, $q > d$ and $d\left(\dfrac{1}{2} - \dfrac{1}{q}\right) \leqslant \gamma < \dfrac{3}{2}$. Namely,

$$\begin{cases} q \geqslant 2, & \frac{q-2}{2q} \leqslant \gamma < \frac{3}{2} \ \text{ if } d = 1 \\ q > 2, & \frac{q-2}{q} \leqslant \gamma < \frac{3}{2} \ \text{ if } d = 2 \\ q > 3, & 3\frac{q-2}{2q} \leqslant \gamma < \frac{3}{2} \ \text{ if } d = 3 \\ 4 < q < 8, & 2\frac{q-2}{q} \leqslant \gamma < \frac{3}{2} \ \text{ if } d = 4 \end{cases}. \tag{3.66}$$

So we can take

$$\begin{cases} 0 \leqslant \gamma < \frac{3}{2} & \text{if } d = 1 \\ 0 < \gamma < \frac{3}{2} & \text{if } d = 2 \\ \frac{1}{2} < \gamma < \frac{3}{2} & \text{if } d = 3 \\ 1 < \gamma < \frac{3}{2} & \text{if } d = 4 \end{cases}. \tag{3.67}$$

With the same choices of γ and d, we can also take $E = L^2(\mathcal{O})$. Indeed, since $U(t,s) \in \mathcal{L}\big(L^2(\mathcal{O})\big) \cap \mathcal{L}\big(L^2(\mathcal{O}); D(A(t))\big)$, then by interpolation $U(t,s) \in \mathcal{L}\left(L^2(\mathcal{O}), H^\gamma(\mathcal{O})\right)$. Moreover there exists $C_2 > 0$ such that

$$\|U(t,s)\|_{\mathcal{L}(L^2(\mathcal{O}), H^\gamma(\mathcal{O}))} \leqslant \frac{C_2}{(t-s)^{\frac{\gamma}{2}}}, \quad (s,t) \in \Delta_T. \tag{3.68}$$

Hence, choosing $E = L^2(\mathcal{O})$ and noting that $\mathrm{H}_t = H^\gamma(\mathcal{O})$ for all $t \in [0,T]$, (2.49) holds for $\alpha = \dfrac{\gamma}{2}$ and therefore by Proposition 2.2.2, Hypothesis 3.1.1 holds with $\theta = \dfrac{1}{2} + \dfrac{\gamma}{2}$. Moreover, since for all $p > 2$ $L^p(\mathcal{O}) \subseteq L^2(\mathcal{O})$ with continuous embedding, we can choose $E = L^p(\mathcal{O})$. Hence for $0 < \gamma < 2$, (2.49) holds for $\alpha = \dfrac{\gamma}{2}$ and therefore by Proposition 2.2.2, Hypothesis 3.1.1 holds with $\theta = \dfrac{1}{2} + \dfrac{\gamma}{2}$.

We consider now Dirichlet boundary conditions for $A(t)$. In this case $D(A(t)) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$ for all $t \in [0,T]$.

One simple example of the family $\{B(t)\}_{t\in[0,T]}$ is given by $B(t) = (-\Delta)^{-\frac{\gamma}{2}}$ for all $t \in [0,T]$ and $\gamma \in (0,2)$, where Δ is the realization of the Laplace operator in $L^2(\mathcal{O})$ with Dirichlet boundary conditions. Due to [Gui91, Thm. 3.5, Thm 4.15] we have

$$D\left((-\Delta)^{\frac{\gamma}{2}}\right) = \left(L^2(\mathcal{O}), H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})\right)_{\frac{\gamma}{2},2} = \begin{cases} H^\gamma(\mathcal{O}) & \text{if } 0 < \gamma < \frac{1}{2} \\ \mathring{H}^{\frac{1}{2}}(\mathcal{O}) & \text{if } \gamma = \frac{1}{2} \\ \left\{u \in H^\gamma(\mathcal{O}) \mid u_{|\partial\mathcal{O}} = 0\right\} & \text{if } \frac{1}{2} < \gamma < 2 \end{cases}. \tag{3.69}$$

For every $q \geqslant 2$, the embedding of $D\left((-\Delta)^{\frac{\gamma}{2}}\right)$ in $L^q(\mathcal{O})$ is continuous for $\gamma \geqslant d\left(\frac{1}{2} - \frac{1}{q}\right)$. Hence for such choices of γ, $(-\Delta)^{-\frac{\gamma}{2}} \in \mathcal{L}(L^2(\mathcal{O}); L^q(\mathcal{O}))$. Similarly to [CL21, Lemma 4.3], we can prove that if $q > d$, then $Q(t,s)$ has finite trace for all $0 \leqslant s < t \leqslant T$ and, in this case, Hypothesis 2.0.1 holds.

We check now that the space $E = D\left((-\Delta)^{\frac{\gamma}{2}}\right)$ still for every $0 < \gamma < 2$ satisfies Hypothesis 3.1.1. In fact by (3.63) and (3.69) we have $E = D\left((-\Delta)^{\frac{\gamma}{2}}\right) = \mathrm{H}_t$ for all $t \in [0,T]$. Then $U(t,s) \in \mathcal{L}(E, \mathrm{H}_t)$ for all $0 \leqslant s < t \leqslant T$ with norm bounded by a constant independent of s and t and (2.49) holds with $\alpha = 0$. By Proposition 2.2.2, Hypothesis 3.1.1 is satisfied by $E = D\left((-\Delta)^{\frac{\gamma}{2}}\right)$ with $\theta = \frac{1}{2}$.

Summarizing, we need $q \geqslant 2$, $q > d$ and $d\left(\dfrac{1}{2} - \dfrac{1}{q}\right) \leqslant \gamma < 2$. Namely,

$$\begin{cases} q \geqslant 2, & \frac{q-2}{2q} \leqslant \gamma < 2 & \text{if } d = 1 \\ q > 2, & \frac{q-2}{q} \leqslant \gamma < 2 & \text{if } d = 2 \\ q > 3, & 3\frac{q-2}{2q} \leqslant \gamma < 2 & \text{if } d = 3 \\ q > 4, & 2\frac{q-2}{q} \leqslant \gamma < 2 & \text{if } d = 4 \\ 5 < q < 10, & 5\frac{q-2}{2q} \leqslant \gamma < 2 & \text{if } d = 5 \end{cases}. \tag{3.70}$$

So we can take

$$
(3.71) \qquad \begin{cases} 0 \leqslant \gamma < 2 & \text{if } d = 1 \\ 0 < \gamma < 2 & \text{if } d = 2 \\ \frac{1}{2} < \gamma < 2 & \text{if } d = 3 \\ 1 < \gamma < 2 & \text{if } d = 4 \\ \frac{3}{2} < \gamma < 2 & \text{if } d = 5 \end{cases} .
$$

We prove now that we can take $E = L^2(\mathcal{O})$ for $0 < \gamma < 2$. Indeed, since $U(t,s) \in \mathcal{L}\big(L^2(\mathcal{O})\big) \cap \mathcal{L}\big(L^2(\mathcal{O}); H^2(\mathcal{O}) \cap H^1_0(\mathcal{O})\big)$, then by interpolation $U(t,s) \in \mathcal{L}\left(L^2(\mathcal{O}), H^\gamma(\mathcal{O})\right)$ for all $0 < \gamma < 2$. Moreover there exists $C_3 > 0$ such that

$$
(3.72) \qquad \|U(t,s)\|_{\mathcal{L}(L^2(\mathcal{O}),H^\gamma(\mathcal{O}))} \leqslant \frac{C_3}{(t-s)^{\frac{\gamma}{2}}}, \quad (s,t) \in \Delta_T.
$$

Hence, choosing $E = L^2(\mathcal{O})$, (2.49) holds for $\alpha = \dfrac{\gamma}{2}$ and therefore by Proposition 2.2.2, Hypothesis 3.1.1 holds with $\theta = \dfrac{1}{2} + \dfrac{\gamma}{2}$. Moreover, since for all $p > 2$ $L^p(\mathcal{O}) \in L^2(\mathcal{O})$ with continuous embedding, we can choose $E = L^p(\mathcal{O})$. Hence for $0 < \gamma < 2$, (2.49) holds for $\alpha = \dfrac{\gamma}{2}$ and therefore by Proposition 2.2.2, Hypothesis 3.1.1 holds with $\theta = \dfrac{1}{2} + \dfrac{\gamma}{2}$.

Now, we fix $d = 1$. In this case $\mathcal{O}$ is a bounded open interval I and we can choose $\gamma = 0$, namely $B(t) = Id$ for all $t \in [0,T]$. We take now $E = E_{\alpha,p} := \big(L^2(I), H^2(I) \cap H^1_0(I)\big)_{\alpha,p}$ for every $0 < \alpha < \frac{1}{2}$ and $p \in [1,+\infty)$.

In this particular case we can characterize the Cameron Martin space $\mathcal{H}_{ts}$ for every $0 \leqslant s < t \leqslant T$ as the space $(L^2(I), H^2(I) \cap H^1_0(I))_{\frac{1}{2},2}) = H^1_0(I)$ by (3.69). The proof is in [CL21].

Since $U(t,s) \in \mathcal{L}\left(H^2(I) \cap H^1_0(I)\right) \cap \mathcal{L}\left(L^2(I)\right)$ with norms bounded by a constant independent of s, t and (3.60) holds, we obtain that $U(t,s) \in \mathcal{L}\left(H^1_0(I)\right)$ with norms bounded by a constant independent of s, t, $U(t,s) \in \mathcal{L}\left(L^2(I); H^1_0(I)\right)$ and there exists $C_4 > 0$ such that

$$
(3.73) \qquad \|U(t,s)\|_{\mathcal{L}\left(L^2(I);H^1_0(I)\right)} \leqslant \frac{C_4}{(t-s)^{\frac{1}{2}}}, \quad (s,t) \in \Delta_T.
$$

By reiteration, for every $0 < \alpha < \frac{1}{2}$ and $p \in [1,+\infty)$, we get

$$
\left(L^2(I), H^1_0(I)\right)_{2\alpha,p} = \left(L^2(I), H^2(I) \cap H^1_0(I)\right)_{\alpha,p},
$$

for every $0 < \alpha < \frac{1}{2}$, $p \in [1,+\infty)$.

Hence, by interpolation

$$
U(t,s) \in \mathcal{L}\big(E_{\alpha,p}; H^1_0(I)\big)
$$

and there exists a constant $C_5 > 0$ independent of s and t such that

$$
(3.74) \qquad \|U(t,s)\|_{\mathcal{L}(E_{\alpha,p};H^1_0(I))} \leqslant \frac{C_4}{(t-s)^{\frac{1}{2}-\alpha}}, \quad (s,t) \in \Delta_T.
$$

Hypothesis 3.1.1 is satisfied by $E_{\alpha,p}$ with $\theta = \frac{1}{2} - \alpha$.

We recall the following characterization of the spaces $E_{\alpha,p}$ as Besov spaces with (possibly) boundary conditions. For all $q \in [1,+\infty)$ we have

$$E_{\alpha,p} = (L^q(I), W^{2,q}(I) \cap W_0^{1,q}(I))_{\alpha,p} = B^{2\alpha}_{q,p,0}(I), \tag{3.75}$$

where

$$B^{2\alpha}_{q,p,0}(I) = \begin{cases} B^{2\alpha}_{q,p}(I) & \text{if } \ 0 < \alpha < \frac{1}{2q} \\ \mathring{B}^{\frac{1}{q}}_{p,q}(I) & \text{if } \alpha = \frac{1}{2q} \\ \left\{u \in B^{2\alpha}_{q,p}(I) \mid u_{|\partial I} = 0\right\} & \text{if } \frac{1}{2q} < \alpha < \frac{1}{2} \end{cases},$$

where $\mathring{B}^{\frac{1}{q}}_{p,q}(I)$ consists on all the elements $u \in B^{\frac{1}{q}}_{q,p}(I)$ whose null extension outside $\overline{I}$ belongs to $B^{\frac{1}{q}}_{p,q}(\mathbb{R})$, and for $\alpha > \frac{1}{2q}$ $u_{|\partial I}$ means the trace of u at ∂I. See [Gui91, Thm. 3.5, Thm 4.15]. We refer to [Tri95, Ch. 2, 4] for the standard theory of the Besov spaces $B^k_{p,q}(\mathbb{R})$, $B^k_{p,q}(I)$.

We summarize what we obtained in the following theorems.

Theorem 3.2.5 (Neumann or Robin boundary conditions)**.** *Assume the following conditions hold true.*

1. *$X := L^2(\mathcal{O})$ where $\mathcal{O} \subseteq \mathbb{R}^d$ is a bounded open set with smooth boundary and $d = 1, 2, 3, 4$.*
2. *The operators $\mathcal{A}(r)$ given by (3.53) verify Hypothesis 3.2.4 with Neumann or Robin boundary conditions.*
3. *For every $r \in [0,T]$ we have*

$$B(r) = (\mathrm{Id} - \Delta)^{-\frac{\gamma}{2}}, \quad r \in [0,T],$$

where Δ is the realization of the Laplacian operator in $L^2(\mathcal{O})$ with Neumann boundary conditions and $\gamma \in [0,2]$.

Then Hypothesis 2.0.1 holds and Hypothesis 3.1.1 is satisfied by $E = H^\gamma(\mathcal{O})$ with $\theta = \frac{1}{2}$ for all $0 < \gamma < \frac{3}{2}$.

Moreover, if we choose γ as in (3.67), then Hypothesis 3.1.1 is satisfied by $E = L^p(\mathcal{O})$, $p \geqslant 2$ with $\theta = \frac{1}{2} + \frac{\gamma}{2}$.

So we can apply Theorem 3.1.4 and 3.1.5 with such choices of E.

Theorem 3.2.6 (Dirichlet boundary condition)**.** *Assume the following conditions hold true.*

1. *$X := L^2(\mathcal{O})$ where $\mathcal{O} \subseteq \mathbb{R}^d$ is a bounded open set with smooth boundary and $d = 1, 2, 3, 4, 5$.*
2. *The operators $\mathcal{A}(r)$ given by (3.53) verify Hypothesis 3.2.4 with Dirichlet boundary conditions.*
3. *For every $r \in [0,T]$ we have*

$$B(r) = (-\Delta)^{-\frac{\gamma}{2}}, \quad r \in [0,T],$$

where Δ is the realization of the Laplacian operator in $L^2(\mathcal{O})$ with Dirichlet boundary conditions and $\gamma \in [0,2]$.

Then Hypothesis 2.0.1 holds and Hypothesis 3.1.1 is satisfied by $E = D((-\Delta)^{\frac{\gamma}{2}})$ with $\theta = \frac{1}{2}$ for all $0 < \gamma < 2$.

Moreover, if we choose γ as in (3.71), then Hypothesis 3.1.1 is satisfied by $E = L^p(\mathcal{O})$, $p \geqslant 2$ with $\theta = \frac{1}{2} + \frac{\gamma}{2}$.

So we can apply Theorem 3.1.4 and 3.1.5 with such choices of E.

Theorem 3.2.7 (Dirichlet boundary condition)**.** *Assume the following conditions hold true.*

1. *$X := L^2(I)$ where $I \subseteq \mathbb{R}^d$ is a bounded open interval.*
2. *The operators $\mathcal{A}(r)$ given by (3.53) verify Hypothesis 3.2.4 with Dirichlet boundary conditions.*
3. *For every $r \in [0, T]$ we have*
$$B(r) = \mathrm{Id}, \quad r \in [0, T].$$

Then Hypothesis 2.0.1 holds and Hypothesis 3.1.1 is satisfied by $E = B^{2\alpha}_{q,p,0}(I)$, $p, q \in [1, +\infty)$ and $0 < \alpha < \frac{1}{2}$, with $\theta = \frac{1}{2} - \alpha$.

So we can apply Theorem 3.1.4 and 3.1.5 with such choices of E.

Chapter 4

Time-differentiation formulas for $P_{s,t}$

In this Chapter we study the case $f = 0$.

Let $\{U(t,s)\}_{(t,s)\in\overline{\Delta}}$ be an evolution operator in X and let $\{B(r)\}_{r\in\mathbb{R}}$ be a strongly continuous family of linear bounded operators on X. In this chapter we consider the Ornstein-Uhlenbeck evolution operator $\{P_{s,t}\}_{(s,t)\in\overline{\Delta}}$ on $B_b(X)$ defined by

$$\begin{aligned} &P_{r,r} = I, \qquad \forall\, r \in \mathbb{R}, \\ &P_{s,t}\varphi(x) = \int_X \varphi(y)\, \mathcal{N}_{U(t,s)x,Q(t,s)}(dy), \quad (s,t) \in \Delta, \varphi \in B_b(X), x \in X \end{aligned} \tag{4.1}$$

where $\mathcal{N}_{U(t,s)x,Q(t,s)}$ is the Gaussian measure in X with mean $U(t,s)x$ and covariance operator

$$Q(t,s) = \int_s^t U(t,r)Q(r)U(t,r)^\star\, dr, \qquad Q(r) := B(r)B(r)^\star. \tag{4.2}$$

The starting point of our analysis are basic (but not trivial in this setting) results on the relation between $P_{s,t}$ and the family of operators $L(t)$ given by (3.3) with $f = 0$.

In Section 4.2 we prove that for every $(s,t) \in \Delta$ and $x \in X$ we have

$$\frac{\partial}{\partial s} P_{s,t}\varphi(x) = -L(s)P_{s,t}\varphi(x), \tag{4.3}$$

$$\frac{\partial}{\partial t} P_{s,t}\varphi(x) = P_{s,t}L(t)\varphi(x), \tag{4.4}$$

where φ belongs to the space of smooth cylindrical functions $\mathcal{E}_t(X)$ defined in (4.27). Besides their use in the proof of Log-Sobolev inequalities and hypercontractivity, (4.3) and (4.4) have independent interest and may be used as a starting point for other results, for instance in the study of classical and strict solutions to evolution equations driven by the operators $L(t)$.

Our basic assumptions are stated in the following hypothesis.

Hypothesis 4.0.1.

1. $\{U(t,s)\}_{(s,t)\in\overline{\Delta}} \subseteq \mathcal{L}(X)$ is a (forward) strongly continuous evolution operator.Moreover we assume that there exist $M > 0$ such that

$$\|U(t,s)\|_{\mathcal{L}(X)} \leqslant M, \quad (s,t) \in \overline{\Delta}. \tag{4.5}$$

2. $\{B(t)\}_{t\in\mathbb{R}} \subseteq \mathcal{L}(X)$ is a bounded family of strongly continuous linear and bounded operators, namely
   (a) there exists $K > 0$ such that

   $$\sup_{t\in\mathbb{R}} \|B(t)\|_{\mathcal{L}(X)} \leqslant K, \tag{4.6}$$

   (b) the map

   $$t \in \mathbb{R} \mapsto B(t)x \in X \tag{4.7}$$

   is continuous for every $x \in X$.

3. For every $(s,t) \in \Delta$ the operator $Q(t,s) : X \to X$ given by

$$Q(t,s) = \int_s^t U(t,r)B(r)B(r)^\star U(t,r)^\star \, dr, \tag{4.8}$$

has finite trace.

Remark 4.0.2. In the next chapters we assume $f \equiv 0$ only for the presentation. All the results of this chapters are extendable to the general case assuming that the function $g(t,s)$ given by (6) is differentiable. More precisely we need that $g(t,s)$ is a strict solution of

$$\begin{cases} u'(t) = A(t)u(t), & t > s, \\ u(s) = 0. \end{cases} \tag{4.9}$$

4.1 Preliminary results

We notice that, by the Fernique Theorem, it is possible to define $P_{s,t}$ on Borel measurable functions with power growth, namely for Borel measurable functions $\varphi : X \longrightarrow \mathbb{R}$ such that there exists $C, m > 0$ such that

$$|\varphi(x)| \leqslant C(1 + \|x\|_X^m), \quad x \in X. \tag{4.10}$$

Moreover $P_{s,t}$ leaves invariant the space of Borel measurable functions having fixed power growth $m > 0$. Indeed, by Theorem 2.1.3 $P_{s,t}\varphi$ is a Borel measurable. Moreover, if φ satisfies (4.10), then for every $x \in X$ we have

$$\begin{aligned} |P_{s,t}\varphi(x)| &\leqslant \int_X |\varphi(y + U(t,s)x)| \; N_{0,Q(t,s)}(dy) \leqslant C \int_X [1 + (\|y\|_X + \|U(t,s)x\|_X)^m] \; N_{0,Q(t,s)}(dy) \\ &\leqslant C_m \left[1 + \|U(t,s)\|_{\mathcal{L}(X)}^m \|x\|_X^m + \int_X \|y\|_X^m \; \mathcal{N}_{0,Q(t,s)}(dy)\right], \end{aligned}$$

where C_m is a positive constant.

Lemma 4.1.1. *Let $\varphi : X \longrightarrow \mathbb{R}$ be a continuous function verifying (4.10). The mapping $(s,t,x) \in \overline{\Delta} \times X \to P_{s,t}\varphi(x)$ is continuous.*

Proof. Let $\{(s_k,t_k,x_k)\}_{k\in\mathbb{N}} \subseteq \overline{\Delta} \times X$ be a sequence convergent to (s,t,x). The claim holds true if we prove that

$$\lim_{k\to+\infty} P_{s_k,t_k}\varphi(x_k) = P_{s,t}\varphi(x). \tag{4.11}$$

By (4.10) we get

$$K := \sup_{k\in\mathbb{N}} \int_X |\varphi(y)|\, \mathcal{N}_{U(t_k,s_k)x_k,Q(t_k,s_k)}(dy) \leqslant C\left(1 + \sup_{k\in\mathbb{N}} \int_X \|y\|_X^m\, \mathcal{N}_{U(t_k,s_k)x_k,Q(t_k,s_k)}(dy)\right), \tag{4.12}$$

where C and m are the constants given by (4.10). By [CL21, Lemma 2.3], $\mathcal{N}_{U(t_k,s_k)x_k,Q(t_k,s_k)}$ weakly converges to $\mathcal{N}_{U(t,s)x,Q(t,s)}$ as $k \to +\infty$, so by [Bog98, Remark 3.8.16(iii)]

$$\sup_{k\in\mathbb{N}} \int_X \|y\|_X^m\, \mathcal{N}_{U(t_k,s_k)x_k,Q(t_k,s_k)}(dy) < +\infty, \tag{4.13}$$

and K is finite. For every $R > 0$, by the Chebyshev inequality (e.g. [Bog07, Theorem 2.5.3]) and (4.12) we get

$$\int_{|\varphi|\geqslant R} |\varphi(y)|\, \mu_k(dy) \leqslant \frac{1}{R}\int_X |\varphi(y)|\, \mu_k(dy) \leqslant \frac{K}{R}. \tag{4.14}$$

Hence by (4.14) we have

$$\lim_{R\to+\infty} \sup_{k\in\mathbb{N}} \int_{|\varphi|>R} |\varphi(y)|\, \mu_k(dy) = 0. \tag{4.15}$$

Since $\mathcal{N}_{U(t_k,s_k)x_k,Q(t_k,s_k)}$ weakly converges to $\mathcal{N}_{U(t,s)x,Q(t,s)}$ as $k \to +\infty$, (4.15) and [Bog07, Lemma 8.4.3] yield that (4.11) holds true. □

We conclude this section studying some regularization properties of $P_{s,t}$. In the autonomous case the smoothing properties of Ornstein-Uhlenbeck semigroups are well known, see for instance [BF23b; BF22b; Mas07; CL19; LR21; MP23; GN03; DZ02; LP20c]. Time dependency of diffusion operator $(Q(t) := B(t)B(t)^\star)$ yelds significant differences in the regularity properties of $P_{s,t}$.

Let E be a subspace of X. In the following we often denote $U(t,s)_{|E}$ by $U(t,s)$ by abuse of language. Moreover we recall that $\mathrm{H}_t = Q(t)^{\frac{1}{2}}(X)$ for all $t \in \mathbb{R}$. We state here a similar result to Proposition 2.1.4.

Proposition 4.1.2. *Assume that Hypothesis 4.0.1 holds true and that for every $(s,t) \in \overline{\Delta}$*

$$U(t,s) \in \mathcal{L}(\mathrm{H}_s; \mathrm{H}_t).$$

Then for every $(s,t) \in \overline{\Delta}$ we have

$$P_{s,t}(C^1_{\mathrm{H}_t}(X)) \subseteq C^1_{\mathrm{H}_s}(X).$$

Moreover for every $\varphi \in C^1_{H_t}(X)$, $x \in X$ and $h \in H$

$$D_{\mathrm{H}_s}(P_{s,t}\varphi)(x)h = P_{s,t}\big(D_{\mathrm{H}_t}\varphi(\cdot)U(t,s)h\big)(x), \tag{4.16}$$

$$\|\nabla_{\mathrm{H}_s} P_{s,t}\varphi(x)\|_{\mathrm{H}_s} \leqslant \|U(t,s)\|_{\mathcal{L}(\mathrm{H}_s;\mathrm{H}_t)}\, P_{s,t}\left(\|\nabla_{\mathrm{H}_t}\varphi(\cdot)\|_{\mathrm{H}_t}\right)(x). \tag{4.17}$$

Proof. Let $x \in X$, $h \in \mathrm{H}_s$ and $\varepsilon > 0$. For all $(s,t) \in \Delta$, we have

$$\frac{P_{s,t}\varphi(x+\varepsilon h) - P_{s,t}\varphi(x)}{\varepsilon} \leqslant \int_X \frac{\left|\varphi\big(y+U(t,s)(x+\varepsilon h)\big) - \varphi\big(y+U(t,s)x\big)\right|}{\varepsilon} \mathcal{N}_{0,Q(t,s)}(dy). \tag{4.18}$$

Since

$$\begin{aligned}
\frac{\left|\varphi\big(y+U(t,s)(x+\varepsilon h)\big) - \varphi\big(y+U(t,s)x\big)\right|}{\varepsilon} &\leqslant \sup_{x\in X} \|D_{\mathrm{H}_t}\varphi\|_{\mathrm{H}_t} \|U(t,s)h\|_{\mathrm{H}_t} \\
&\leqslant \sup_{x\in X} \|D_{\mathrm{H}_t}\varphi\|_{\mathrm{H}_t} \|U(t,s)\|_{\mathcal{L}(\mathrm{H}_s;\mathrm{H}_t)} \|h\|_{\mathrm{H}_s},
\end{aligned}$$

by the Dominated Convergence Theorem, as $\varepsilon \to 0^+$, we get

$$D_{\mathrm{H}_s} P_{s,t}\varphi(x)h = P_{s,t}\left(D_{\mathrm{H}_t}\varphi(\cdot)U(t,s)h\right)(x). \tag{4.19}$$

Moreover we get

$$\begin{aligned}
|\langle \nabla_{\mathrm{H}_s} P_{s,t}\varphi(x), h\rangle_{\mathrm{H}_s}| = |D_{H_s}P_{s,t}\varphi(x)h| &= \left|\int_X D_{\mathrm{H}_t}\varphi(y+m^x(t,s))U(t,s)h\, \mathcal{N}_{0,Q(t,s)}(dy)\right| \\
&= \left|\int_X \langle \nabla_{\mathrm{H}_t}\varphi(y+m^x(t,s)), U(t,s)h\rangle_{\mathrm{H}_t} \mathcal{N}_{0,Q(t,s)}(dy)\right| \\
&\leqslant \|U(t,s)\|_{\mathcal{L}(\mathrm{H}_s;\mathrm{H}_t)} \|h\|_{\mathrm{H}_s} P_{s,t}\left(\|\nabla_{\mathrm{H}_t}\varphi(\cdot)\|_{\mathrm{H}_t}\right)(x),
\end{aligned}$$

so (4.17) holds true. ☐

Remark 4.1.3. In view of Proposition 1.3.2 for every $\varphi \in C_b^1(X)$, $x \in X$ and $(s,t) \in \Delta$, inequality (4.17) reads as

$$\left\|Q(s)^{\frac{1}{2}}\nabla P_{s,t}\varphi(x)\right\|_X \leqslant \|U(t,s)\|_{\mathcal{L}(\mathrm{H}_s;\mathrm{H}_t)} P_{s,t}\left(\left\|Q(t)^{\frac{1}{2}}\nabla\varphi(\cdot)\right\|_X\right)(x). \tag{4.20}$$

4.2 Connections between $\{P_{s,t}\}_{(s,t)\in\overline{\Delta}}$ and $\{L(r)\}_{r\in\mathbb{R}}$

One of the main issues working on non autonomous problems is the lack of similar theories to the ones of strongly continuous or analytic semigroups for evolution operators. We cannot even define the weak genertor of $P_{s,t}$ via Laplace transform as in the case of Ornstein-Uhlenbeck semigroups in [Cer94; Lun22; Pri99]. In this section we prove that for suitable functions $\varphi : X \to \mathbb{R}$ we have

$$\frac{\partial}{\partial s}P_{s,t}\varphi(x) = -L(s)P_{s,t}\varphi(x), \quad (s,t) \in \Delta,\ x \in X, \tag{4.21}$$

$$\frac{\partial}{\partial t}P_{s,t}\varphi(x) = P_{s,t}L(t)\varphi(x), \quad (s,t) \in \Delta,\ x \in X, \tag{4.22}$$

where $\{L(r)\}_{r\in\mathbb{R}}$ is the family of operators given by

$$L(r)\varphi(x) = \frac{1}{2}\mathrm{Tr}\Big(Q(r)D^2\varphi(x)\Big) + \langle x, A(r)^\star\nabla\varphi(x)\rangle_X. \tag{4.23}$$

Hypothesis 4.2.1. Assume that Hypothesis 4.0.1 holds true and that $\{V_{s,t}\}_{(s,t)\in\overline{\Delta}} := \{U(t,s)^\star\}_{(s,t)\in\overline{\Delta}} \subseteq \mathcal{L}(X)$ is a (backward) strongly continuous evolution operator and there exists a family of closed linear operators $A(r) : D(A(r)) \subseteq X \to X$, $r \in \mathbb{R}$, satisfying

i) $D(A(r))$ and $D(A(r)^\star)$ are dense in X for every $r \in \mathbb{R}$;

ii) for every $(s,t) \in \Delta$ we have

$$U(t,s)^\star D(A(t)^\star) \subseteq D(A(s)^\star)$$

iii) for every $t \in \mathbb{R}$ and $h \in D(A(t)^\star)$ the function $s \longrightarrow A(s)^\star U(t,s)^\star h$ belongs to $C((-\infty,t);X)$;

iv) for every $(s,t) \in \Delta$ and $x \in D(A(t)^\star)$ we have

$$\frac{\partial}{\partial s}U(t,s)^\star x = -A(s)^\star U(t,s)^\star x, \tag{4.24}$$

$$\frac{\partial}{\partial t}U(t,s)^\star x = U(t,s)^\star A(t)^\star x. \tag{4.25}$$

Remark 4.2.2. In Section 5.3 we consider a family $\{A(r)\}_{r\in\mathbb{R}}$ associated to a non autonomous abstract parabolic problem in the sense of Acquistapace-Terreni ([Acq88; AT87]) and such that $D(A(r))$ is dense for all $r \in \mathbb{R}$. We show that such a family satisfies Hypothesis 4.2.1.

4.2.1 Cylindrical functions

In this subsection we define a space of smooth cylindrical functions such that (4.21) and (4.22) hold true. We define suitable trigonometric polynomials on X and we introduce the space of Bohr almost periodic functions that will be crucial in Section 5.2.

Throughout this subsection we fix $r \in \mathbb{R}$.

Definition 4.2.3 (Trigonometric polynomials)**.** Let V be a subspace of X. We denote by $\mathcal{E}(X;V)$ the linear span of all real and imaginary parts of the functions

$$x \to \varphi^{(h)}(x) := e^{i\langle x,h\rangle_X}, \tag{4.26}$$

where $h \in V$ (we shall omit h from the notation $\varphi^{(h)}$ when it is not necessary).

Remark 4.2.4. We note that $\mathrm{Trig}(\mathbb{R}^n) := \mathcal{E}(\mathbb{R}^n;\mathbb{R}^n)$ is the usual space of trigonometric polynomials on $\mathbb{R}^n$.

We set

$$\mathcal{E}_r(X) := \mathcal{E}(X; D(A(r)^\star)). \tag{4.27}$$

Remark 4.2.5. The set of functions $\mathcal{E}_r(X)$ is often used in the autonomous case in which it represents a core for Ornstein-Uhlenbeck type operators in L^p spaces with respect to the invariant measure, see for instance [DZ02]. However in general there is not a dense subspace $\mathfrak{D} \subseteq \bigcap_{r\in\mathbb{R}} D(A(r)^\star)$, which prevents from using a unique space independent of r.

Let $L(r)$ be the operator defined in (4.23). If $h \in D(A(r)^\star)$ and $\varphi^{(h)}$ is defined by (4.26), then we have

$$L(r)\varphi^{(h)}(x) = \left[i\langle x, A(r)^\star h\rangle_X - \frac{1}{2}\left\| Q(r)^{\frac{1}{2}} h\right\|_X^2 \right] \varphi^{(h)}(x), \quad x \in X. \tag{4.28}$$

Here we introduce a space of functions that contains $\mathcal{E}_r(X)$ and that will be used in the proofs of Section 5.2.

Definition 4.2.6. Let $k \in \mathbb{N} \cup \{0\}$. We denote by $\mathcal{F}_r C_b^k(X)$ the space of functions φ such that there exists $n \in \mathbb{N}$, $\psi \in C_b^k(\mathbb{R}^n)$ and $h_1, \dots h_n \in D(A(r)^\star)$ orthonormal such that

$$\varphi(x) = \psi\left(\langle x, h_1\rangle, \dots, \langle x, h_n\rangle\right), \qquad x \in X. \tag{4.29}$$

If $k = 0$ we write $\mathcal{F}_r C_b(X)$ instead of $\mathcal{F}_r C_b^0(X)$.

Remark 4.2.7. Let $\varphi \in \mathcal{F}_r C_b^2(X)$ be given by (4.29). Since $h_1, \dots h_n \in D(A(r)^\star)$ are orthonormal then

$$L(r)\varphi(x) = \frac{1}{2}\sum_{i=1}^n \langle Q(r)\nabla^2\varphi(x)h_i, h_i\rangle_X + \sum_{i=1}^n \langle x, A(r)^\star h_i\rangle_X \langle \nabla\varphi(x), h_i\rangle_X, \quad x \in X, \tag{4.30}$$

where $L(r)$ is the operator defined in (4.23). Moreover if $I \subseteq X$ is an interval, $g \in C_b^2(I;\mathbb{R})$ and φ takes values in I then

$$(L(r)(g \circ \varphi))(x) = g'(\varphi(x))L(r)\varphi(x) + g''(\varphi(x)) \left\| Q(r)^{\frac{1}{2}} \nabla\varphi(x)\right\|, \qquad x \in X. \tag{4.31}$$

By (4.27) and Definitions 4.2.3 and 4.2.6 it follows immediately that

$$\mathcal{E}_r(X) \subseteq \mathcal{F}_r C_b^k(X), \quad \forall\, k \in \mathbb{N}. \tag{4.32}$$

Definition 4.2.8. Let $n \in \mathbb{N}$ and $\varphi \in C_b(\mathbb{R}^n)$. We say that φ is Bohr almost periodic if for every $\varepsilon > 0$ there exists $\rho > 0$ such that for all $x_0 \in \mathbb{R}^n$ there exists $\tau \in B(x_0, \rho)$ such that

$$|\varphi(x+\tau) - \varphi(x)| < \varepsilon, \qquad \forall\, x \in \mathbb{R}^n. \tag{4.33}$$

We denote by $AP_b(\mathbb{R}^n)$ the subspace of $C_b(\mathbb{R}^n)$ of Bohr almost periodic functions from $\mathbb{R}^n$ to $\mathbb{R}$. Moreover we denote by $AP_b^2(\mathbb{R}^n)$ the subspace of $C_b^2(\mathbb{R}^n) \cap AP_b(\mathbb{R}^n)$ of the functions $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that the partial derivatives $\dfrac{\partial\varphi}{\partial x_i}$ and $\dfrac{\partial^2\varphi}{\partial x_i \partial x_j}$ belongs to $AP_b(\mathbb{R}^n)$, for every $i, j \in \mathbb{N}$.

For more details about Bohr almost periodic functions in several variables we refer to [Pan90; Chá+23; Kos22].

Definition 4.2.9. We denote by $\mathfrak{B}_r^2(X)$ the subspace of $\mathcal{F}_r C_b^2(X)$ of the functions $\varphi : X \to \mathbb{R}$ given by (4.29) with $\psi \in AP_b^2(\mathbb{R}^n)$, for some $n \in \mathbb{N}$.

Proposition 4.2.10. *Let $h \in \mathbb{N} \cup \{0\}$. For every $\varphi \in \mathcal{F}_r C_b^h(X)$ there exist a sequence $\{\varphi_k\}_{k\in\mathbb{N}} \subseteq \mathcal{E}_r(X)$ and $C > 0$ such that*

$$\|\varphi_k\|_{C_b^h(X)} \leqslant C\, \|\varphi\|_{C_b^h(X)}, \qquad k \in \mathbb{N}, \tag{4.34}$$

and for every $x \in X$ we have

$$\lim_{k\to+\infty}\Big(|\varphi_k(x)-\varphi(x)|+\sum_{j=1}^{h}\|\nabla^j\varphi_k(x)-\nabla^j\varphi(x)\|_{\mathcal{L}^{j-1}(X)}\Big)=0. \tag{4.35}$$

Moreover if $\varphi \in \mathfrak{B}^2_r(X)$ then

$$\lim_{k\to+\infty}\left(\|\varphi_k-\varphi\|_{C^2_b(X)}+\sup_{x\in X}\frac{|L(r)\varphi_k(x)-L(r)\varphi(x)|}{1+\|x\|_X}\right)=0. \tag{4.36}$$

Proof. Let $h \in \mathbb{N}\cup\{0\}$ and let $\varphi \in \mathcal{F}_rC^h_b(X)$. There exist $n \in \mathbb{N}$, $\psi \in C^h_b(\mathbb{R}^n)$ and $h_1,\ldots h_n \in D(A(r)^\star)$ orthonormal such that

$$\varphi(x)=\psi\left(\langle x,h_1\rangle,\ldots,\langle x,h_n\rangle\right),\qquad x\in X. \tag{4.37}$$

We define the orthogonal projection P^r_n on $\mathrm{span}\{h_1,\ldots,h_n\}$, namely

$$P^r_n x:=\sum_{k=1}^{n}\langle x,h_k\rangle h_k. \tag{4.38}$$

We denote by $\mathfrak{I}^r_n : P^r_n(X)\to\mathbb{R}^n$ the canonical isometry given by

$$\mathfrak{I}^r_n h_j=e_j,\qquad j\in\{1,\ldots,n\}, \tag{4.39}$$

where $e_1,\ldots e_n$ is the canonical basis of $\mathbb{R}^n$. By the definitions of P^r_n and $\mathfrak{I}^r_n$ formula (4.37) reads as

$$\varphi(x)=\psi\left(\mathfrak{I}^r_nP^r_nx\right),\qquad x\in X.$$

By [DLT18, Lemma 8.1], there exists $C>0$ (depending only on n) and a sequence of trigonometric polynomials $\{\psi_k\}_{k\in\mathbb{N}}\subseteq \mathrm{Trig}(\mathbb{R}^n)$ such that

$$\begin{aligned}
&\|\psi_k\|_{C^h_b(\mathbb{R}^n)}\leqslant C\,\|\varphi\|_{C^h_b(\mathbb{R}^n)}, && k\in\mathbb{N},\\
&\lim_{k\to+\infty}\Big(|\psi_k(x)-\psi(x)|+\sum_{j=1}^{h}\|\nabla^j\psi_k(x)-\nabla^j\psi(x)\|_{\mathcal{L}^{j-1}(\mathbb{R}^n)}\Big)=0, && x\in\mathbb{R}^n.
\end{aligned}$$

Setting $\varphi_k(\cdot):=\psi_k(\mathfrak{I}^r_nP^r_n(\cdot))$ for every $k\in\mathbb{N}$ and recalling that $\|P^r_n\|_{\mathcal{L}(X)}=\|\mathfrak{I}^r_n\|_{\mathcal{L}(X)}=1$ we obtain (4.34) and (4.35). Now we prove that $\psi_k\in\mathrm{Trig}(\mathbb{R}^n)$ implies $\varphi_k\in\mathcal{E}_r(X)$, for every $k\in\mathbb{N}$. Indeed, for every $h\in\mathbb{R}^n$ and setting

$$\Phi^{(h)}(\xi)=e^{\langle h,\xi\rangle_{\mathbb{R}^n}},\quad \xi\in\mathbb{R}^n,$$

we get

$$\Phi^{(h)}(\mathfrak{I}^r_nP^r_nx)=e^{i\langle\mathfrak{I}^r_nP^r_n(x),h\rangle_X}=e^{i\langle x,\mathfrak{I}^{r\star}_n h\rangle_X},\qquad x\in X. \tag{4.40}$$

Since $\mathfrak{I}^{r\star}_n h\in P^r_n(X)$ and $h_1,\ldots h_n\in D(A(r)^\star)$ then $\mathfrak{I}^{r\star}_n h\in D(A(r)^\star)$ and so $\varphi_k\in\mathcal{E}_r(X)$, for every $k\in\mathbb{N}$ (see Definition 4.2.3 and (4.27)).

Now we prove 4.36. Let $\varphi\in\mathfrak{B}^2_r(X)$ given by

$$\varphi(x)=\psi\left(\langle x,h_1\rangle_X,\ldots,\langle x,h_n\rangle_X\right),\quad x\in X,$$

where $\psi \in AP^2(\mathbb{R}^n)$ and $h_1, ..., h_n \in D(A(r)^\star)$. By [Pan90, Prop. 6.1] $AP_b^2(\mathbb{R}^n)$ is the closure in $C_b^2(\mathbb{R}^n)$ of $\mathrm{Trig}(\mathbb{R}^n)$, then there exists $\{\psi_k\}_{k\in\mathbb{N}} \subseteq \mathrm{Trig}(\mathbb{R}^n)$ such that

$$\lim_{k\to+\infty} \|\psi_k - \psi\|_{C_b^2(\mathbb{R}^n)} = 0.$$

Hence we get

$$\lim_{k\to+\infty} \|\varphi_k - \varphi\|_{C_b^2(X)} = 0, \tag{4.41}$$

where the functions φ_k are defined as in the previous approximation procedure. Finally combining (4.30) and (4.41) we obtain (4.36). □

It is well know that trigonometric polynomials are not dense in $C_b(X)$ even if $X = \mathbb{R}$, however it is possible to prove the following weaker approximation result.

Proposition 4.2.11. *Let $h \in \mathbb{N}\cup\{0\}$. For every $\varphi \in C_b^h(X)$ there exist a 2-sequence $\{\varphi_{n,m}\}_{n,m\in\mathbb{N}} \subseteq \mathcal{E}_r(X)$, $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}_r C_b^h(X)$ and $\{c_n\}_{n\in\mathbb{N}} \subseteq [0, +\infty)$ such that*

$$\|\varphi_{n,m}\|_{C_b^h(X)} \leqslant c_n \|\varphi_n\|_{C_b^h(X)}, \qquad n, m \in \mathbb{N}, \tag{4.42}$$

$$\|\varphi_n\|_{C_b^h(X)} \leqslant \|\varphi\|_{C_b^h(X)}, \qquad n \in \mathbb{N}, \tag{4.43}$$

$$\lim_{m\to+\infty} \Big(|\varphi_{n,m}(x) - \varphi_n(x)| + \sum_{j=1}^{h} \|\nabla^j \varphi_{n,m}(x) - \nabla^j \varphi_n(x)\|_{\mathcal{L}^{j-1}(X)}\Big) = 0, \quad x \in X,\ n \in \mathbb{N}, \tag{4.44}$$

$$\lim_{n\to+\infty} \Big(|\varphi_n(x)-\varphi(x)| + \sum_{j=1}^{h} \|\nabla^j \varphi_n(x) - \nabla^j \varphi(x)\|_{\mathcal{L}^{j-1}(X)}\Big) = 0, \qquad x \in X. \tag{4.45}$$

Proof. Since $D(A(r)^\star)$ is dense in X there exists an orthonormal basis $\{e_k^r\}_{k\in\mathbb{N}}$ of X such that $e_k^r \in D(A(r)^\star)$, for every $k \in \mathbb{N}$. Let $n \in \mathbb{N}$ and let P_n^r be the orthogonal projection on $\mathrm{span}\{e_1^r, \dots e_n^r\}$. Let $h \in \mathbb{N} \cup \{0\}$ and let $\varphi \in C_b^h(X)$. We define

$$\varphi_n(x) := \varphi(P_n^r x), \quad x \in X.$$

Since $\varphi \in C_b^h(X)$ and $\|P_n^r\|_{\mathcal{L}(X)} = 1$, we obtain (4.43) and (4.45). Fixed $n \in \mathbb{N}$ the function φ_n belongs to $\mathcal{F}_r C_b(X)$ so by Proposition 4.2.10 there exists a sequence $\{\varphi_{n,m}\}_{m\in\mathbb{N}} \subseteq \mathcal{E}_r(X)$ and $c_n > 0$ such that (4.42) and (4.44) are verified. □

Remark 4.2.12. Let γ be a Borel probability measure on X. By Proposition 4.2.11 $\mathcal{E}_r(X)$ is dense in $L^p(X, \gamma)$ for all $p \geqslant 1$.

4.2.2 Differentiation formulas for $P_{s,t}$

In this subsection we prove formulas (4.21) and (4.22).

Lemma 4.2.13. *Assume that Hypothesis 4.2.1 holds true. Fix $(s,t) \in \Delta$, $h \in D(A(t)^\star)$ and let $\varphi^{(h)}$ be defined by (4.26). Then*

$$P_{s,t}\varphi^{(h)}(x) = e^{-\frac{1}{2}\langle Q(t,s)h,h\rangle_X} \varphi^{(U(t,s)^\star h)}(x). \tag{4.46}$$

It follows that $P_{s,t}(\mathcal{E}_t(X)) \subseteq \mathcal{E}_s(X)$. Furthermore for all $x \in X$ we have

$$L(s)P_{s,t}\varphi^{(h)}(x) = \left[i\langle x, A(s)^\star U(t,s)^\star h\rangle_X - \frac{1}{2}\left\|Q(s)^{\frac{1}{2}}U(t,s)^\star h\right\|_X^2\right]P_{s,t}\varphi^{(h)}(x), \tag{4.47}$$

$$P_{s,t}L(t)\varphi^{(h)}(x) = \left[i\langle x, U(t,s)^\star A(t)^\star h\rangle_X - \langle Q(t,s)A(t)^\star h, h\rangle_X - \frac{1}{2}\left\|Q(t)^{\frac{1}{2}}h\right\|_X^2\right]P_{s,t}\varphi^{(h)}(x). \tag{4.48}$$

Proof. Let $h \in X^t$ and let $\varphi^{(h)}$ be defined by (4.26). For all $x \in X$ we get

$$\begin{aligned} P_{s,t}\varphi^{(h)}(x) &= \int_X e^{i\langle h,y\rangle_X}\,\mathcal{N}_{U(t,s)x,Q(t,s)}(dy) = \widehat{\mathcal{N}}_{U(t,s)x,Q(t,s)}(h) \\ &= e^{i\langle h,U(t,s)x\rangle_X}e^{-\frac{1}{2}\langle Q(t,s)h,h\rangle_X} = e^{-\frac{1}{2}\langle Q(t,s)h,h\rangle_X}\varphi^{(U(t,s)^\star h)}(x), \end{aligned}$$

so that (4.46) holds and $P_{s,t}$ maps $\mathcal{E}_t(X)$ into $\mathcal{E}_s(X)$ for every $(s,t) \in \Delta$.

Recalling that by Hypothesis 4.2.1 $U(t,s)^\star h \in X^s$, we have $P_{s,t}(\mathcal{E}_t(X)) \subseteq \mathcal{E}_s(X)$. Let us prove (4.47). By (4.28) for all $x \in X$ we have

$$\begin{aligned} L(s)P_{s,t}\varphi^{(h)}(x) &= L(s)\left(e^{-\frac{1}{2}\langle Q(t,s)h,h\rangle_X}\varphi^{(U(t,s)^\star h)}\right)(x) \\ &= e^{-\frac{1}{2}\langle Q(t,s)h,h\rangle_X}\left[i\langle x, A(s)^\star U(t,s)^\star h\rangle_X - \frac{1}{2}\left\|Q(s)^{\frac{1}{2}}U(t,s)^\star h\right\|_X^2\right]\varphi^{(U(t,s)^\star h)}(x) \\ &= \left[i\langle x, A(s)^\star U(t,s)^\star h\rangle_X - \frac{1}{2}\left\|Q(s)^{\frac{1}{2}}U(t,s)^\star h\right\|_X^2\right]P_{s,t}\varphi^{(h)}(x) \end{aligned}$$

and (4.47) holds. To prove (4.48), we combine (4.28) and (4.46) and for all $x \in X$ we get

$$\begin{aligned} P_{s,t}L(t)\varphi^{(h)}(x) &= P_{s,t}\left(\left[i\langle \cdot, A(t)^\star h\rangle_X - \frac{1}{2}\left\|Q(t)^{\frac{1}{2}}h\right\|_X^2\right]\varphi^{(h)}(\cdot)\right)(x) \\ &= \int_X \left(i\langle U(t,s)x+y, A(t)^\star h\rangle_X\varphi^{(h)}\left(U(t,s)x+y\right)\right)\mathcal{N}_{0,Q(t,s)}(dy) - \frac{1}{2}\left\|Q(t)^{\frac{1}{2}}h\right\|_X^2 P_{s,t}\varphi^{(h)}(x) \\ &=: I_1 - \frac{1}{2}\left\|Q(t)^{\frac{1}{2}}h\right\|_X^2 P_{s,t}\varphi^{(h)}(x). \end{aligned}$$

We note that

$$\begin{aligned} I_1 &= \int_X \left(i\langle U(t,s)x+y, A(t)^\star h\rangle_X\varphi_h\left(U(t,s)x+y\right)\right)\mathcal{N}_{0,Q(t,s)}(dy) \\ &= \int_X \frac{\partial}{\partial\left(A(t)^\star h\right)}e^{i\langle U(t,s)x+y,h\rangle_X}\,\mathcal{N}_{0,Q(t,s)}(dy) = \frac{\partial}{\partial\left(A(t)^\star h\right)}\int_X e^{i\langle y,h\rangle_X}\,\mathcal{N}_{U(t,s)x,Q(t,s)}(dy) \\ &= \frac{\partial}{\partial\left(A(t)^\star h\right)}\widehat{\mathcal{N}}_{U(t,s)x,Q(t,s)}(h) = \frac{\partial}{\partial(A(t)^\star h)}\left(e^{-\frac{1}{2}\langle Q(t,s)h,h\rangle_X}e^{i\langle x,U(t,s)^\star h\rangle_X}\right) \\ &= \left[i\langle x, U(t,s)^\star A(t)^\star h\rangle_X - \langle Q(t,s)A(t)^\star h, h\rangle_X\right]e^{-\frac{1}{2}\langle Q(t,s)h,h\rangle_X}e^{i\langle x,U(t,s)^\star h\rangle_X} \\ &= \left[i\langle x, U(t,s)^\star A(t)^\star h\rangle_X - \langle Q(t,s)A(t)^\star h, h\rangle_X\right]P_{s,t}\varphi^{(h)}(x). \end{aligned}$$

Summing up, (4.48) follows. ☐

By Hypotheses 4.2.1 and Lemma (4.2.13) we deduce the following remark.

Remark 4.2.14. Fix $(s,t) \in \Delta$, $h \in D(A(t)^\star)$ and let $\varphi^{(h)}$ be defined by (4.26). There exists $C' > 0$ such that

$$\left| L(r)P_{r,t}\varphi^{(h)}(x) \right| \leqslant C'(1 + \|x\|_X), \qquad r \in [s,t],\ x \in X. \tag{4.49}$$

Let $\varphi \in \mathcal{E}_t(X)$. By definition, φ is a linear combination of real and imaginary parts of functions $\varphi^{(h_i)}$ defined by (4.26) with $h_i \in D(A(t)^*)$ and $i = 1, \dots n$ for some $n \in \mathbb{N}$. Hence by (4.49), there exists $C > 0$ such that

$$|L(r)P_{r,t}\varphi(x)| \leqslant C(1 + \|x\|_X), \qquad r \in [s,t],\ x \in X. \tag{4.50}$$

Lemma 4.2.15. *Assume that Hypothesis 4.2.1 holds true. For each $s_0, t_0 \in \mathbb{R}$, we have*

$$\left(\frac{\partial}{\partial t}\langle Q(t,s)h,h\rangle_X \right)\bigg|_{t=t_0} = \left\| Q\,(t_0)^{\frac{1}{2}}\, h \right\|_X^{\frac{1}{2}} + 2\langle Q(t_0,s)A(t_0)^\star h, h\rangle_X, \quad h \in D(A(t_0)^\star),\ s \leqslant t, t_0, \tag{4.51}$$

$$\left(\frac{\partial}{\partial s}\langle Q(t,s)x,x\rangle_X \right)\bigg|_{s=s_0} = -\langle U(t,s_0)Q(s_0)U(t,s_0)^\star x, x\rangle_X, \quad x \in X,\ t \geqslant s, s_0. \tag{4.52}$$

Proof. We start by proving (4.51). Let $h \in D(A(t_0)^\star)$ and $\varepsilon > 0$. Then for $s \leqslant t_0 - \varepsilon$ we have

$$\begin{aligned}
&- \frac{\langle Q(t_0-\varepsilon,s)h,h\rangle_X - \langle Q(t_0,s)h,h\rangle_X}{\varepsilon} \\
&= -\frac{1}{\varepsilon}\left(\int_s^{t_0-\varepsilon} \langle U(t_0-\varepsilon,r)Q(r)U(t_0-\varepsilon,r)^\star h,h\rangle_X\,dr - \int_s^{t_0} \langle U(t_0,r)Q(r)U(t_0,r)^\star h,h\rangle_X\,dr \right) \\
&= -\frac{1}{\varepsilon}\bigg[\int_s^{t_0-\varepsilon} \left(\langle U(t_0-\varepsilon,r)Q(r)U(t_0-\varepsilon,r)^\star h,h\rangle_X - \langle U(t_0,r)Q(r)U(t_0,r)^\star h,h\rangle_X \right) dr \\
&- \int_{t_0-\varepsilon}^{t_0} \langle U(t_0,r)Q(r)U(t_0,r)^\star h,h\rangle_X\,dr \bigg] =: -\frac{I_\varepsilon - J_\varepsilon}{\varepsilon}.
\end{aligned}$$

Since for every $h \in D(A(t_0)^\star)$ the mapping

$$r \in [t_0-\varepsilon, t_0] \longmapsto \langle U(t_0,r)Q(r)U(t_0,r)^\star h,h\rangle_X \in \mathbb{R}$$

is continuous, by the Mean Value Theorem for integrals there exists $r_\varepsilon \in (t_0-\varepsilon, t_0)$ such that

$$\frac{J_\varepsilon}{\varepsilon} = \langle U(t_0,r_\varepsilon)Q(r_\varepsilon)U(t_0,r_\varepsilon)^\star h,h\rangle_X$$

and

$$\lim_{\varepsilon\to 0^+} \frac{J_\varepsilon}{\varepsilon} = \lim_{\varepsilon\to 0^+} \left(\langle U(t_0,r_\varepsilon)Q(r)U(t_0,r_\varepsilon)^\star h,h\rangle_X \right) = \left\| Q\,(t)^{\frac{1}{2}}\, h \right\|_X^2.$$

Let us consider I_ε. We have

$$\begin{aligned}
-\frac{I_\varepsilon}{\varepsilon} &= -\frac{1}{\varepsilon}\bigg(\int_s^{t_0-\varepsilon} \langle \left(U(t_0-\varepsilon,r)Q(r)U(t_0-\varepsilon,r)^\star - U(t_0-\varepsilon,r)Q(r)U(t_0,r)^\star \right) h,h\rangle_X\,dr \\
&+ \int_s^{t_0-\varepsilon} \langle \left(U(t_0-\varepsilon,r)Q(r)U(t_0,r)^\star - U(t_0,r)Q(r)U(t_0,r)^\star \right) h,h\rangle_X\,dr \bigg) \\
&= \int_s^{t_0-\varepsilon} \left\langle \frac{U(t_0-\varepsilon,r)^\star - U(t_0,r)^\star}{-\varepsilon} h, Q(r)U(t_0-\varepsilon,r)^\star h \right\rangle_X dr \\
&+ \int_s^{t_0-\varepsilon} \langle \frac{U(t_0-\varepsilon,r)^\star - U(t_0,r)^\star}{-\varepsilon} h, Q(r)U(t_0,r)^\star h\rangle_X\,dr.
\end{aligned}$$

By the Dominated convergence Theorem we get

$$\lim_{\varepsilon\to 0^+}\frac{I_\varepsilon}{\varepsilon}=2\langle Q(t_0,s)A(t_0)^\star h,h\rangle_X. \tag{4.53}$$

Since we can compute with similar arguments the right derivative, (4.51) follows. (4.52)is an immediate consequence of the Fundamental Theorem of Calculus. $\square$

Theorem 4.2.16. *Assume that Hypothesis 4.2.1 holds true. For every $(s,t)\in\Delta$ and $\varphi\in\mathfrak{B}^2_t(X)$ we have*

$$P_{s,t}\varphi\in\mathfrak{B}^2_s(X), \tag{4.54}$$

$$\frac{\partial}{\partial s}P_{s,t}\varphi(x)=-L(s)P_{s,t}\varphi(x),\quad x\in X, \tag{4.55}$$

$$\frac{\partial}{\partial t}P_{s,t}\varphi(x)=P_{s,t}L(t)\varphi(x),\quad x\in X. \tag{4.56}$$

In addition, if $\varphi\in\mathcal{E}_t(X)$ then, for every $x\in X$ the function $s\to\frac{\partial}{\partial s}P_{s,t}\varphi(x)=L(s)P_{s,t}\varphi(x)$ is continuous from $(-\infty,t]$ in $\mathbb{R}$.

Proof. We start by proving the statements of the theorem for $\varphi^{(h)}$ defined by (4.26) with $h\in D(A(t)^\star)$. Lemmas 4.2.13 and 4.2.15 yield (4.55) and (4.56). Moreover, by Hypothesis 4.2.1 and (4.47) we deduce that the function $s\to L(s)P_{s,t}\varphi^{(h)}(x)$ is continuous from $(-\infty,t]$ in $\mathbb{R}$.
Now we consider $\varphi\in\mathcal{E}_t(X)$. The statements of the theorem hold true for such φ, since by definition, φ is a linear combination of real and imaginary parts of functions $\varphi^{(h_i)}$ defined by (4.26) with $h_i\in D(A(t)^*)$ and $i=1,\dots n$ for some $n\in\mathbb{N}$.

Let us now prove the statements for $\varphi\in\mathfrak{B}^2_t(X)$, in which case

$$\varphi(x):=\psi(\mathfrak{I}^t_nP^t_nx),\quad x\in X,$$

where $n\in\mathbb{N}$, $\psi\in AP^2(\mathbb{R}^n)$ and the operators $\mathfrak{I}^r_n$ and P^r_n are defined in (4.38) and (4.39), respectively. Let $\{\varphi_k\}_{k\in\mathbb{N}}\subseteq\mathcal{E}_t(X)$ be the approximation built in the proof of Proposition 4.2.10, namely

$$\varphi_k(x):=\psi_k(\mathfrak{I}^t_nP^t_nx),\quad x\in X,$$

where $\psi_k\in\mathrm{Trig}(\mathbb{R}^n)$. We recall that since $\psi_k\in\mathrm{Trig}(\mathbb{R}^n)$ then $\varphi_k\in\mathcal{E}_t(X)$, for every $k\in\mathbb{N}$ (see (4.40)). Let $h\in\mathbb{R}^n$ we set

$$\Phi^{(h)}(\xi)=e^{\langle h,\xi\rangle_{\mathbb{R}^n}},\quad \xi\in\mathbb{R}^n.$$

So by (4.40) and (4.46) we have

$$P_{s,t}\Phi^{(h)}(\mathfrak{I}^t_nP^r_nx)=e^{-\frac{1}{2}\langle Q(t,s)h,h\rangle_X}e^{i\langle x,U(t,s)^\star\mathfrak{I}^{t\,\star}_n h\rangle_X},\qquad x\in X.$$

Recalling that $\mathfrak{I}^{t\,\star}_n h\in D(A(t)^\star)$ and that $U(t,s)^\star(D(A(t)^\star))\subseteq D(A(s)^\star)$, for every $k\in\mathbb{N}$ we deduce

$$P_{s,t}\varphi_k\in\mathcal{E}_s(X),$$

so by the first part of this proof for every $k\in\mathbb{N}$ we have

$$\frac{\partial}{\partial s}P_{s,t}\varphi_k(x)=-L(s)P_{s,t}\varphi_k(x),\quad x\in X, \tag{4.57}$$

$$\frac{\partial}{\partial t}P_{s,t}\varphi_k(x)=P_{s,t}L(t)\varphi_k(x),\quad x\in X. \tag{4.58}$$

So by (4.36) letting $k\to+\infty$ in (4.57) and (4.58) we obtain (4.55) and (4.56), respectively. $\square$

Remark 4.2.17. Let $x \in X$ and $s \geqslant 0$. We consider the stochastic partial differential equation

$$\begin{cases} dZ(t) = A(t)Z(t)dt + B(t)dW(t), & t > s, \\ Z(s) = x, \end{cases} \tag{4.59}$$

where the operators $A(t)$ satisfies Hypothesis 4.2.1 and $\{W(t)\}_{t\geqslant 0}$ is a X-cylindrical Wiener process. We assume that, for every $s \geqslant 0$ and $x \in X$, (4.59) has a unique mild solution to (4.59) having continuous trajectories, namely a X-valued stochastic process $\{Z_s(t,x)\}_{t\geqslant s}$ such that

$$Z_s(t,x) = U(t,s)x + \int_s^t U(t,r)B(r)dW(r), \quad t > s, \quad \mathbb{P}\text{–a.s}, \tag{4.60}$$

where $U(t,s)$ is the evolution operator assocuiated to $A(t)$. We note that by Hypothesis 4.2.1 the first term in the right-hand side of (4.60) is well defined, instead additional assumptions are needed to ensure that the stochastic convolution

$$\left\{\int_s^t U(t,r)B(r)dW(r)\right\}_{t\geqslant s}$$

is a X-valued stochastic process having continuous trajectories, see for instance [Sei93; DIT82; VZ08]. Under such assumptions $\{P_{s,t}\}_{0\leqslant s\leqslant t}$ is the Markov evolution operator associated to (4.59). In this framework one could expect to derive formulas (4.55) and (4.56) via the Itô formula. We discuss some different cases.

1. If $X = \mathbb{R}^n$, then the calculations are standard since (4.60) is a strong solution to (4.59), namely for every $s \geqslant 0$, $x \in X$ and $t \geqslant s$

$$Z_s(t,x) = x + \int_s^t A(r)Z(r,x)dr + \int_s^t B(r)dW(r), \tag{4.61}$$

   see e.g. [Bal17].

2. If for every $r \geqslant 0$ $A(r)$ and $B(r)$ are diagonal operators with respect to the same Hilbert basis $\{e_k\}_{k\in\mathbb{N}}$ of X, then we can consider the time independent cylindrical functions φ defined by

$$\varphi(x) = \psi(P_n x), \quad x \in X, \tag{4.62}$$

   where $\psi \in C_b^2(\mathbb{R}^n;\mathbb{R})$, P_n is the orthogonal projection on $\operatorname{span}\{e_1,\ldots,e_n\}$ and $n \in \mathbb{N}$. In this case, for every $s \geqslant 0$ and $x \in X$ the process $\{P_n Z_s(t,x)\}_{t>s}$ is the unique strong solution of the following finite dimensional approximation of (4.59)

$$\begin{cases} dZ_n(t) = P_n A(t)P_n Z_n(t)dt + P_n B(t)P_n dW(t), & t > s \\ Z_n(s) = x. \end{cases} \tag{4.63}$$

   In view of 1. we can deduce formulas (4.55) and (4.56) for cylindrical functions of the type (4.62) tracing back to the finite-dimensional case.

3. In the general case a natural idea is to replace the finite dimensional approximations by suitable Yosida approximations at least in the case $D(A(t)) = D(A(0))$ for $t \geqslant 0$. However

this has not been performed up to now to prove (4.55) and (4.56). Indeed it has to be proved that the sequence $\{U_n(t,s)\}_{n\in\mathbb{N}}$ given by the Yosida approxiamations converges to $U(t,s)$ in a suitable sense. If the domains vary with time and $\bigcap_{r\geqslant 0} D(A(r))$ is not dense in X, there are no results in this direction.

Taking into account the above considerations, we preferred to avoid the use of the Itô formula by proposing a more direct procedure, in the same spirit of [DZ14, Prop. 3.19] in the autonomous case.

Remark 4.2.18. In the autonomous case, where $P_{s,t} = P_{0,t-s}$ is a semigroup, formulas similar to (4.55) and (4.56) are known for suitable functions φ accordingly to the theory of weakly continuous semigroups (see [Cer94; Lun22; Pri99]). Currently in the non-autonomous case there is no similar theory that can be exploited. We remark that we cannot use the abstract results on evolution operators of [AT87; Tan79], since the family of realizations of the operators $\{L(r)\}_{r\in\mathbb{R}}$ in spaces of bounded and continuous functions from X to $\mathbb{R}$ does not satisfy their assumptions. If $X = \mathbb{R}^n$, then (4.56) and (4.55) were proven in [KLL10] for smooth functions with compact support, but in infinite dimension compactly supported functions are not relevant and must be replaced by other classes of functions such as $\mathcal{E}_t(X)$ defined in (4.27), which is dense in $L^p(X,\gamma)$ for every $p \in (1,+\infty)$ and for every Borel probability measure γ in X. Moreover, such spaces depend explicitly on t, so it is not possible to use a technique similar to the one presented in [KLL10] to prove (4.56) and (4.55).

The following proposition is a technical result needed to prove Theorem 5.2.2.

Proposition 4.2.19. *Assume that Hypothesis 4.2.1 holds true. Let $I \subseteq \mathbb{R}$ be an open interval and $g \in C^3_b(I;\mathbb{R})$. Let $(s,t) \in \Delta$ and let $\varphi \in \mathfrak{B}^2_t(X)$ be a function taking values in I. For every $x \in X$ the function $r \to P_{s,r}(g(P_{r,t}\varphi))(x)$ is differentiable from (s,t) to $\mathbb{R}$ and*

$$\frac{\partial}{\partial r} P_{s,r}(g(P_{r,t}\varphi))(x) = P_{s,r}\left(-g'(P_{r,t}\varphi)L(r)P_{r,t}\varphi\right)(x) + P_{s,r}\left(L(r)g(P_{r,t}\varphi)\right)(x). \tag{4.64}$$

Proof. Let $(s,t) \in \Delta$ and $\varphi \in \mathcal{E}_t(X)$. We start by proving that (4.64) holds true for such a φ. We set

$$\psi(r,x) := g(P_{r,t}\varphi(x)), \qquad r \in [s,t],\ x \in X.$$

Since g is continuously differentiable, by Theorem 4.2.16, for every $x \in X$ the function $r \to \psi(r,x)$ is continuously differentiable from $[s,t]$ to $\mathbb{R}$ and

$$\frac{\partial}{\partial r}\psi(r,x) = -g'(P_{r,t}\varphi(x))L(r)P_{r,t})\varphi(x). \tag{4.65}$$

Fix $r \in (s,t)$ and $x \in X$. For every $v \in (s-r, t-r)$ we have

$$\begin{aligned}\frac{P_{s,r+v}[\psi(r+v,\cdot)](x) - P_{s,r}[\psi(r,\cdot)](x)}{v} &= \frac{P_{s,r+v}[\psi(r,\cdot)](x) - P_{s,r}[\psi(r,\cdot)](x)}{v} \\ &\quad + \frac{P_{s,r+v}[\psi(r+v,\cdot) - \psi(r,\cdot)](x)}{v}.\end{aligned}$$

Since $g \in C^2(I;\mathbb{R})$ then $\psi(\sigma,\cdot) \in \mathfrak{B}^2_\sigma(X)$, for every $\sigma \in (s,t)$. By (4.56) we know that

$$\lim_{v\to 0} \frac{P_{s,r+v}[\psi(r,\cdot)](x) - P_{s,r}[\psi(r,\cdot)](x)}{v} = P_{s,r}[L(r)\psi(r,\cdot)](x).$$

We only have to prove that

$$\lim_{v\to 0} \frac{P_{s,r+v}[\psi(r+v,\cdot)-\psi(r,\cdot)](x)}{v} = P_{s,r}\left[\frac{\partial}{\partial r}\psi(r,\cdot)\right](x). \tag{4.66}$$

We note that

$$\left|\frac{P_{s,r+v}[\psi(r+v,\cdot)-\psi(r,\cdot)](x)}{v} - P_{s,r}\left[\frac{\partial}{\partial r}\psi(r,\cdot)\right](x)\right| \leqslant I_1(v)+I_2(v)$$

where

$$\begin{aligned} I_1(v) &= P_{s,r+v}\left[\left|\frac{\psi(r+v,\cdot)-\psi(r,\cdot)}{v} - \frac{\partial}{\partial r}\psi(r,\cdot)\right|\right](x), \\ I_2(v) &= \left|P_{s,r+v}\left[\frac{\partial}{\partial r}\psi(r,\cdot)\right](x) - P_{s,r}\left[\frac{\partial}{\partial r}\psi(r,\cdot)\right](x)\right|. \end{aligned} \tag{4.67}$$

By (4.50), (4.65) and Remark 4.1.1 we get

$$\lim_{v\to 0} I_2(v) = 0.$$

Hence if

$$\lim_{v\to 0} I_1(v) = 0, \tag{4.68}$$

then (4.64) follows. By Theorem 4.2.16, for every $y \in X$ the function $\psi(\cdot, y) \in \mathbb{R}$ is continuously differentiable, so that

$$\left|\frac{\psi(r+v,y)-\psi(r,y)}{v} - \frac{\partial}{\partial r}\psi(r,y)\right| \leqslant \frac{1}{v}\int_r^{r+v} \left|\frac{\partial}{\partial r}\psi(\sigma,y) - \frac{\partial}{\partial r}\psi(r,y)\right| d\sigma. \tag{4.69}$$

By (4.65), for every $\sigma \in [r+0\wedge v, r+0\vee v]$ we get

$$\begin{aligned} \left|\frac{\partial}{\partial r}\psi(\sigma,y) - \frac{\partial}{\partial r}\psi(r,y)\right| &= \left|g'(P_{\sigma,t}\varphi(y))L(\sigma)P_{\sigma,t}\varphi(y) - g'(P_{r,t}\varphi(y))L(r)P_{r,t}\varphi(y)\right| \\ &\leqslant \|g\|_{C_b^2(I;\mathbb{R})} |L(r)P_{r,t}\varphi(y)| \, |P_{\sigma,t}\varphi(y) - P_{r,t}\varphi(y)| \\ &\quad + \|g\|_{C_b^2(I;\mathbb{R})} |L(\sigma)P_{\sigma,t}\varphi(y) - L(r)P_{r,t}\varphi(y)|. \end{aligned}$$

By (4.50), there exists $K > 0$ independent of r and y such that

$$\left|\frac{\partial}{\partial r}\psi(\sigma,y) - \frac{\partial}{\partial r}\psi(r,y)\right| \leqslant K(1+\|x\|_X)R_1(\sigma) + KR_2(\sigma), \tag{4.70}$$

where

$$\begin{aligned} R_1(\sigma) &:= |P_{\sigma,t}\varphi(y) - P_{r,t}\varphi(y)| \\ R_2(\sigma) &:= |L(\sigma)(P_{\sigma,t}\varphi(\cdot))(y) - L(r)(P_{r,t}\varphi(\cdot))(y)|. \end{aligned}$$

We begin to estimate R_1 and R_2 with φ replaced by $\varphi^{(h)}$ defined by (4.26) with $h \in D(A(t)^\star)$. By (4.46)

$$\begin{aligned}\left|P_{\sigma,t}\varphi^{(h)}(y) - P_{r,t}\varphi^{(h)}(y)\right| &= \left|e^{-\frac{1}{2}\langle Q(t,\sigma)h,h\rangle_X}e^{i\langle y,U(t,\sigma)\rangle^\star_X h} - e^{-\frac{1}{2}\langle Q(t,r)h,h\rangle_X}e^{i\langle x,U(t,r)^\star h\rangle_X}\right|\\ &\leqslant \left|e^{i\langle y,U(t,\sigma)^\star h\rangle_X}\right|\left|e^{-\frac{1}{2}\langle Q(t,\sigma)h,h\rangle_X} - e^{-\frac{1}{2}\langle Q(t,r)h,h\rangle_X}\right|\\ &+\left|e^{-\frac{1}{2}\langle Q(t,r)h,h\rangle_X}\right|\left|e^{i\langle y,U(t,\sigma)^\star h\rangle_X}\right|\left|e^{i\langle y,(U(t,r)^\star - U(t,\sigma)^\star)h\rangle_X}-1\right|,\end{aligned}$$

recalling that $Q(t,r)$ is a non-negative operator and $\left|e^{i\lambda}-1\right| \leqslant |\lambda|$, we have

$$\begin{aligned}\left|P_{\sigma,t}\varphi^{(h)}(y) - P_{r,t}\varphi^{(h)}(y)\right| &\leqslant \left|e^{-\frac{1}{2}\langle Q(t,\sigma)h,h\rangle_X} - e^{-\frac{1}{2}\langle Q(t,r)h,h\rangle_X}\right|\\ &+\|y\|_X\left\|(U(t,r)^\star - U(t,\sigma)^\star)h\right\|_X,\end{aligned}\tag{4.71}$$

By (4.47)

$$\begin{aligned}&|L(\sigma)(P_{\sigma,t}\varphi^{(h)}(\cdot))(y) - L(r)(P_{r,t}\varphi^{(h)}(\cdot))(y)|\\ &\quad\leqslant \left|P_{\sigma,t}\varphi^{(h)}(y)\right| |i\langle y,(A(\sigma)^\star U(t,\sigma)^\star h - A(r)^\star U(t,r)^\star)h\rangle_X|\\ &\quad+\frac{1}{2}\left|P_{\sigma,t}\varphi^{(h)}(y)\right|\left|\left\|Q(\sigma)^{\frac{1}{2}}U(t,\sigma)^\star h\right\|^2_X - \left\|Q(r)^{\frac{1}{2}}U(t,r)^\star h\right\|^2_X\right|\\ &\quad+\left|i\langle y,A(r)^\star U(t,r)^\star h\rangle_X - \frac{1}{2}\left\|Q(r)^{\frac{1}{2}}U(t,r)^\star h\right\|^2_X\right|\left|P_{\sigma,t}\varphi^{(h)}(y) - P_{r,t}\varphi^{(h)}(y)\right|,\end{aligned}$$

and by Hypothesis 4.2.1 and (4.71) we have

$$\begin{aligned}|L(\sigma)(P_{\sigma,t}\varphi^{(h)}(\cdot))(y) - L(r)(P_{r,t}\varphi^{(h)}(\cdot))(y)| &\leqslant \|y\|_X\left\|(A(\sigma)^\star U(t,\sigma)^\star h - A(r)^\star U(t,r)^\star)h\right\|_X\\ &+\left|\left\|Q(\sigma)^{\frac{1}{2}}U(t,\sigma)^\star h\right\|^2_X - \left\|Q(\sigma)^{\frac{1}{2}}U(t,r)^\star h\right\|^2_X\right|\\ &+C(1+\|y\|_X)\left|e^{-\frac{1}{2}\langle Q(t,\sigma)h,h\rangle_X} - e^{-\frac{1}{2}\langle Q(t,r)h,h\rangle_X}\right|\\ &+C(1+\|y\|_X)\|y\|_X\left\|(U(t,\sigma)^\star - U(t,r)^\star)h\right\|_X,\end{aligned}\tag{4.72}$$

for some constant $C := C_{s,t} > 0$. Now we can estimate R_1 and R_2. By definition, φ is a linear combination of real and imaginary parts of functions $\varphi^{(h_i)}$ defined by (4.26) with $h_i \in D(A(t)^*)$ and $i = 1,\dots n$ for some $n \in \mathbb{N}$. Hence by (4.71) and (4.72)

$$R_1(\sigma) \leqslant C_1(1+\|y\|_X)\sum_{i=1}^n\left|e^{-\frac{1}{2}\langle Q(t,\sigma)h_i,h_i\rangle_X} - e^{-\frac{1}{2}\langle Q(t,r)h_i,h_i\rangle_X}\right| + \left\|(U(t,r)^\star - U(t,\sigma)^\star)h_i\right\|_X\tag{4.73}$$

$$R_2(\sigma) \leqslant C_2(1+\|y\|^2_X)\sum_{i=1}^n\rho_i(\sigma)\tag{4.74}$$

where C_1 and C_2 are two positive constants and

$$\begin{aligned}\rho_i(\sigma) := \left\|(A(\sigma_0)^\star U(t,\sigma)^\star - A(r)^\star U(t,r)^\star)h_i\right\|_X + \left|e^{-\frac{1}{2}\langle Q(t,\sigma)h_i,h_i\rangle_X} - e^{-\frac{1}{2}\langle Q(t,r)h_i,h_i\rangle_X}\right|\\ +\left\|(U(t,\sigma)^\star - U(t,r)^\star)h_i\right\|_X + \left|\left\|Q(\sigma)^{\frac{1}{2}}U(t,\sigma)^\star h_i\right\|^2_X - \left\|Q(\sigma)^{\frac{1}{2}}U(t,r)^\star h_i\right\|^2_X\right|.\end{aligned}$$

By (4.70), (4.73), (4.74) for every $\sigma \in [r + 0 \wedge v, r + 0 \vee v]$ and $y \in X$ we have

$$\left|\frac{\partial}{\partial r}\psi(\sigma, y) - \frac{\partial}{\partial r}\psi(r, y)\right| \leqslant M(1 + \|y\|_X^2)\rho(\sigma) \tag{4.75}$$

where $M := M_{s,t}$ is a positive constant and

$$\rho(\sigma) := \sum_{i=1}^{n} \rho_i(\sigma).$$

Using (4.75) in (4.69) we get

$$\left|\frac{\psi(r+v, y) - \psi(r, y)}{v} - \frac{\partial}{\partial r}\psi(r, y)\right| \leqslant M(1 + \|y\|_X^2)\frac{1}{v}\int_r^{r+v} \rho(\sigma)d\sigma. \tag{4.76}$$

Now exploiting (4.76) in (4.67), by (4.13) we get

$$I_1(v) \leqslant K\frac{1}{v}\int_r^{r+v} \rho(\sigma)d\sigma.$$

where $K = M[1 + \sup_{l\in[s,t]} P_{s,l}(\|\cdot\|_X^2)(x)]$. By Hypothesis 4.2.1 the function $\sigma \to \rho(\sigma)$ is continuous from $[r + 0 \wedge v, r + 0 \vee v]$ to $\mathbb{R}$ and $\rho(r) = 0$, so by the integral mean value theorem (4.68) is verified and (4.64) holds true.

Now let $\varphi \in \mathfrak{B}_t^2(X)$. By Proposition (4.2.10) there exists $\{\varphi_k\}_{k\in\mathbb{N}} \subseteq \mathcal{E}_t(X)$ such that

$$\lim_{k\to+\infty} \|\varphi_k - \varphi\|_{C_b^2(X)} = 0,$$

hence we get

$$\lim_{k\to+\infty} \sup_{r\in[s,t]} \|P_{r,t}\varphi_k - P_{r,t}\varphi\|_{C_b^2(X)} = 0. \tag{4.77}$$

Since $g \in C_b^3(I; \mathbb{R})$ and I is open, by (4.77) we get

$$\lim_{k\to+\infty} \sup_{r\in[s,t]} \|g(P_{r,t}\varphi_k) - g(P_{r,t}\varphi)\|_{C_b^2(X)} = 0. \tag{4.78}$$

By the first part of this proof, for every $k \in \mathbb{N}$ and $x \in X$ we have

$$\frac{\partial}{\partial r}P_{s,r}(g(P_{r,t}\varphi_k(\cdot)))(x) = P_{s,r}\left(-g'(P_{r,t}\varphi_k(\cdot))L(r)P_{r,t}\varphi_k(\cdot)\right)(x) + P_{s,r}\left(L(r)g(P_{r,t}\varphi_k(\cdot))\right)(x). \tag{4.79}$$

By (4.54), for every $r \in (s, t)$ the function $x \to g(P_{r,t}\varphi(x))$ belongs to $\mathfrak{B}_r^2(X)$ and so, by (4.31), (4.36), (4.78) and (4.79) we obtain the statement letting $k \to +\infty$ in (4.79). $\square$

Chapter 5

The Ornstein-Uhlenbeck evolution operator in L^p spaces with respect to evolution systems of measures

We consider the Ornstein-Uhlenbeck evolution operator given by 4.1 and we extend it as a bounded operator between suitable L^p spaces. We recall that in the autonomous case, $P_{s,t} = P_{0,t-s}$ is a semigroup and it is usually settled in Lebesgue spaces with respect to its invariant measure, that exists and is unique under suitable assumptions.

In the non autonomous case a single invariant measure does not exist in general, being replaced by evolution systems of measures, namely families of Borel probability measures $\{\gamma_r\}_{r\in\mathbb{R}}$ in X such that

$$\int_X P_{s,t}\varphi(x)\gamma_s(dx) = \int_X \varphi(x)\gamma_t(dx),$$

for every $s < t$ and for every bounded and continuous $\varphi : X \to \mathbb{R}$. So, an obviuous difficulty arises, namely the spaces $L^p(X,\gamma_r)$ depend explicitly on r and we cannot set our problem in a fixed L^p space.

In Section 5.1 we provide conditions that guarantee existence of an evolution system of measures $\{\gamma_r\}_{r\in\mathbb{R}}$ where γ_r is a Gaussian measure with mean zero and it satisfies

$$\lim_{s\to-\infty} P_{s,t}\varphi(x) = \int_X \varphi(y)\ \gamma_t(dy), \quad t \in \mathbb{R},\ x \in X,$$

for every bounded and continuous function $\varphi : X \to \mathbb{R}$. Such results are already contained in [OR16a] (see also [GL08] [DR08] for the case $X = \mathbb{R}^n$), however we give the proofs for the convenience of the readers.

As in the finite dimensional setting, if an evolution system of measures $\{\gamma_r\}_{r\in\mathbb{R}}$ exists, then it is possible to extend each operator $P_{s,t}$ to a linear bounded operator from $L^p(X,\gamma_t)$ to $L^p(X,\gamma_s)$, denoted by $P^{(p)}_{s,t}$, for every $p \geqslant 1$. Such operators are consistent, i.e. for every $s < t$ and $f \in L^p(X,\gamma_t) \cap L^q(X,\gamma_t)$ it holds $P^{(p)}_{s,t} f = P^{(q)}_{s,t} f$. For this reason, we omit the index p if no confusion can arise, and we denote them by $P_{s,t}$.

In Section 5.2, for the Gaussian evolution system of measures constructed in Section 5.1, we prove a family of logarithmic Sobolev inequalities,

$$\int_X |\varphi|^p \log\left(|\varphi|^p\right) d\gamma_t \leqslant m_t(|\varphi|^p)\log\left(m_t\left(|\varphi|^p\right)\right) + \kappa p^2 \int_X |\varphi|^{p-2} \left\| Q\left(t\right)^{\frac{1}{2}} \nabla\varphi \right\|^2 \mathbb{1}_{\{\varphi\neq 0\}} d\gamma_t. \tag{5.1}$$

where $\varphi \in C_b^1(X)$, $t \in \mathbb{R}$, $p \in (1, +\infty)$, $m_t(\varphi) := \displaystyle\int_X \varphi\, d\nu_t$ and κ is an explicit positive constant (see Theorem 5.2.2).

Exploiting (5.1) we prove a hypercontractivity result for $\{P_{s,t}\}_{(s,t)\in\overline{\Delta}}$, namely for every $(s,t) \in \overline{\Delta}$, $q > 1$ and $p \leqslant c(s,t,q) := (q-1)e^{\frac{t-s}{2\kappa}} + 1$ we have

$$\|P_{s,t}\varphi\|_{L^p(X,\,\gamma_s)} \leqslant \|\varphi\|_{L^q(X,\,\gamma_t)}\,, \quad \varphi \in L^q(X,\,\gamma_t), \tag{5.2}$$

where κ is the constant appearing in (5.1) and we still denote by $P_{s,t}$ the extension of $P_{s,t}$ from $L^g(X,\gamma_t)$ to $L^g(X,\gamma_s)$ for every $g > 1$. We stress that $c(s,t,q)$ is the non autonomous version of the optimal constant in the autonomous case, see Remark 5.3.10.

Finally in Section 5.3 we present different examples of $\{P_{s,t}\}_{(s,t)\in\overline{\Delta}}$ that verify our assumptions. In our most significant example, $A(r)$ is the realization of a negative second order elliptic differential operator in $X = L^2(\mathcal{O})$ with Dirichlet or Robin boundary conditions and smooth enough coefficients; $\mathcal{O}$ is a bounded open smooth subset of $\mathbb{R}^d$. $\{U(t,s)\}_{(s,t)\in\overline{\Delta}}$ is the evolution operator associated to $\{A(r)\}_{r\in\mathbb{R}}$ according to Acquistapace-Terreni [Acq88; AT87] and $B(r) := (-A(r))^{-\gamma}$ with $\gamma \geqslant 0$.

5.1 Evolution systems of measures

In this section we will investigate existence of an evolution system of measures for $\{P_{s,t}\}_{(s,t)\in\overline{\Delta}}$.

Definition 5.1.1. An evolution system of measures for $\{P_{s,t}\}_{(s,t)\in\overline{\Delta}}$ is a family of Borel probability measures $\{\nu_r\}_{r\in\mathbb{R}}$ such that

$$\int_X P_{s,t}\varphi(x)\nu_s(dx) = \int_X \varphi(x)\nu_t(dx), \quad s \leqslant t,\ \varphi \in C_b(X). \tag{5.3}$$

First of all we prove the following useful characterization of evolution system of measures.

Proposition 5.1.2. *Assume that Hypothesis 4.0.1 holds true. Then $\{\nu_r\}_{r\in\mathbb{R}}$ is an evolution system of measures for $\{P_{s,t}\}_{(s,t)\in\overline{\Delta}}$ if and only if*

$$\widehat{\nu}_t(h) = e^{-\frac{1}{2}\langle Q(t,s)h,h\rangle_X}\,\widehat{\nu}_s(U(t,s)^\star h), \quad s \leqslant t. \tag{5.4}$$

Proof. Let $h \in X$ and let $s \leqslant t$. If $\{\nu_r\}_{r\in\mathbb{R}}$ is an evolution system of measures for $\{P_{s,t}\}_{(s,t)\in\overline{\Delta}}$, then we have

$$\begin{aligned}
\widehat{\nu}_t(h) &= \int_X e^{i\langle h,x\rangle_X}\,\nu_t(dx) = \int_X P_{s,t}(e^{i\langle h,\cdot\rangle_X})(x)\,\nu_s(dx)\\
&= \int_X\int_X e^{i\langle h,y\rangle_X}\,\mathcal{N}_{U(t,s)x,Q(t,s)}(dy)\nu_s(dx) = \int_X \widehat{\mathcal{N}}_{U(t,s)x,Q(t,s)}(h)\nu_s(dx)\\
&= \int_X e^{i\langle h,U(t,s)x\rangle_X}e^{-\frac{1}{2}\langle Q(t,s)h,h\rangle_X}\nu_s(dx) = e^{-\frac{1}{2}\langle Q(t,s)h,h\rangle_X}\int_X e^{i\langle h,U(t,s)x\rangle_X}\,\nu_s(dx)\\
&= e^{-\frac{1}{2}\langle Q(t,s)h,h\rangle_X}\widehat{\nu}_s(U(t,s)^\star h),
\end{aligned}$$

and (5.4) holds.

Conversely, we assume that (5.4) holds true. Given $h\in X$, we first show that (5.3) holds for $\varphi(x)=e^{i\langle h,x\rangle_X}$, $x\in X$. If $s\leqslant t$, we have by definition

$$\widehat{\nu}_t(h) = \int_X e^{i\langle h,x\rangle_X}\,\nu_t(dx), \tag{5.5}$$

and by (5.4) we have

$$\begin{aligned}
\widehat{\nu}_t(h) &= e^{-\frac{1}{2}\langle Q(t,s)h,h\rangle_X}\widehat{\nu}_s(U(t,s)^\star h) = e^{-\frac{1}{2}\langle Q(t,s)h,h\rangle_X}\int_X e^{i\langle h,U(t,s)x\rangle_X}\,\nu_s(dx)\\
&= \int_X e^{i\langle h,U(t,s)x\rangle_X}e^{-\frac{1}{2}\langle Q(t,s)h,h\rangle_X}\nu_s(dx)\\
&= \int_X\int_X e^{i\langle h,y\rangle_X}\,\mathcal{N}_{U(t,s)x,Q(t,s)}(dy)\nu_s(dx) = \int_X \widehat{\mathcal{N}}_{U(t,s)x,Q(t,s)}(h)\nu_s(dx)\\
&= \int_X P_{s,t}(e^{i\langle h,\cdot\rangle_X})(x)\,\nu_s(dx). \qquad (5.6)
\end{aligned}$$

By (5.5) and (5.6), (5.3) holds for $\varphi(x)=e^{i\langle h,x\rangle_X}$.

If $\varphi\in C_b(X)$ by Proposition 4.2.11 there exist a 2-sequence $\{\varphi_{n,m}\}_{n,m\in\mathbb{N}}\subseteq\mathcal{E}_r(X)$, $\{\varphi_n\}_{n\in\mathbb{N}}\subseteq\mathcal{F}_rC_b(X)$ and $\{c_n\}_{n\in\mathbb{N}}\subseteq[0,+\infty)$ such that

$$\|\varphi_{n,m}\|_{C_b(X)}\leqslant c_n\,\|\varphi_n\|_{C_b(X)}\,,\qquad n,m\in\mathbb{N}, \tag{5.7}$$

$$\|\varphi_n\|_{C_b(X)}\leqslant \|\varphi\|_{C_b(X)}\,,\qquad n\in\mathbb{N}, \tag{5.8}$$

$$\lim_{m\to+\infty}|\varphi_{n,m}(x)-\varphi_n(x)|=0,\qquad n\in\mathbb{N},\ x\in X, \tag{5.9}$$

$$\lim_{n\to+\infty}|\varphi_n(x)-\varphi(x)|=0,\qquad x\in X. \tag{5.10}$$

Of course $\varphi_{n,m}$ satisfies (5.3) for every $n,m\in\mathbb{N}$. To prove that φ satisfies (5.3) we apply two times the dominated convergence theorem: the first time as $m\to+\infty$ thanks to (5.7) and (5.9) and the second as $n\to+\infty$ thanks to (5.8) and (5.10).

$\square$

Remark 5.1.3. We consider the problem

$$\begin{cases}\dfrac{\partial u}{\partial t}(t,x)=A(t)u(t,\cdot)(x), & s<t\\ u(s,x)=y\in X\end{cases} \tag{5.11}$$

whose corresponding transition evolution operator $\{V_{s,t}\}_{(s,t)\in\overline{\Delta}}$ is given by

$$V_{s,t}\varphi(x) = \varphi(U(t,s)x), \quad \varphi \in C_b(X),\ x \in X,\ s \leqslant t. \tag{5.12}$$

In this case (5.3) reads as

$$\int_X \varphi(U(t,s)x)\,\mu_s(dx) = \int_X \varphi(x)\,\mu_t(dx), \quad s \leqslant t \tag{5.13}$$

and by Proposition 5.1.2 $\{\mu_r\}_{r\in\mathbb{R}}$ is an evolution system of measures for $\{V_{s,t}\}_{(s,t)\in\overline{\Delta}}$ if and only if

$$\widehat{\mu}_t(h) = \widehat{\mu}_s(U(t,s)^\star h), \quad s \leqslant t,\ h \in X. \tag{5.14}$$

We recall the infinite dimensional version of the Bochner theorem for the characteristic functions of Borel probability measures in separable Hilbert spaces (see [DZ14, Thm 2.27 pag. 61]).

Theorem 5.1.4 (Bochner)**.** *Let X be a real separable Hilbert space and let $\varphi : X \longrightarrow \mathbb{C}$. φ is the characteristic function of a probability measure μ on $(X, \mathcal{B}(X))$ if and only if*

(1) φ is continuous and $\varphi(0) = 1$,

(2) φ is a positive definite function, namely for every $k \in \mathbb{N}$ and every choice of $x_1, ..., x_k \in X$ and $c_1, .., c_k \in \mathbb{C}$ we have

$$\sum_{i,j=1}^{k} c_i\overline{c_j}\varphi(x_i - x_j) \geqslant 0, \tag{5.15}$$

(3) for every $\varepsilon > 0$ there exists a non-negative nuclear operator S_ε such that

$$1 - \mathfrak{Re}\varphi(x) \leqslant \varepsilon, \tag{5.16}$$

for all $x \in X$ satisfying $\langle S_\varepsilon x, x\rangle_X \leqslant 1$.

Remark 5.1.5. Condition (3) is related to the continuity with respect to the Sazonov topology that is relevant in our setting (see [Bog07, Thm 7.13.7]). If $X = \mathbb{R}^n$ then condition (3) is verified and Theorem 5.1.4 is the Bochner Theorem (see [Bog07, Thm. 7.13.1]).

Theorem 5.1.6. *Assume that Hypothesis 4.0.1 holds true. If for every $t \in \mathbb{R}$*

$$\sup_{s<t} [\mathrm{Tr}\,[Q(t,s)]] < +\infty \tag{5.17}$$

then the operator

$$Q(t,-\infty) := \int_{-\infty}^{t} U(t,r)Q(r)U(t,r)^\star dr$$

is well defined and it has finite trace for every $t \in \mathbb{R}$. The family of measures $\{\gamma_t\}_{t\in\mathbb{R}}$ given by

$$\gamma_t := \mathcal{N}_{0,Q(t,-\infty)}, \quad t \in \mathbb{R}, \tag{5.18}$$

is an evolution system of measures for $\{P_{s,t}\}_{(s,t)\in\overline{\Delta}}$.

Moreover $\{\nu_t\}_{t\in\mathbb{R}}$ is an evolution system of measures for $\{P_{s,t}\}_{(s,t)\in\overline{\Delta}}$ if and only if

$$\nu_t = \gamma_t \star \mu_t, \quad t \in \mathbb{R},$$

where $\{\mu_t\}_{t\in\mathbb{R}}$ is an evolution system of measures for $\{V_{s,t}\}_{(s,t)\in\overline{\Delta}}$ and $\{V_{s,t}\}_{(s,t)\in\overline{\Delta}}$ is given by (5.12)

Proof. By Proposition 1.4.11 $Q(t,-\infty)$ is well defined and it belongs to $\mathcal{L}_1^+(X)$. Now we show that the measures γ_t satisfy (5.4). Let $s \leqslant t$ and $h \in X$, then

$$U(t,s)Q(s,-\infty)U(t,s)^\star h = \int_{-\infty}^{s} U(t,s)U(s,r)Q(r)U^\star(s,r)U(t,s)^\star h\, dr$$
$$= \int_{-\infty}^{s} U(t,r)Q(r)U(t,r)^\star h\, dr = Q(t,-\infty)h - Q(t,s)h.$$

Moreover

$$\widehat{\gamma}_t(h) = e^{-\frac{1}{2}\langle Q(t,-\infty)h,h\rangle_X}$$
$$\widehat{\gamma}_t(U(t,s)^\star h) = e^{-\frac{1}{2}\langle Q(s,-\infty)U(t,s)^\star h, U(t,s)^\star h\rangle_X}.$$

Therefore, we have

$$\begin{aligned}\langle Q(s,-\infty)U(t,s)^\star h, U(t,s)^\star h\rangle_X &= \langle U(t,s)Q(s,-\infty)U(t,s)^\star h, h\rangle_X \\ &= \langle Q(t,-\infty)h - Q(t,s)h, h\rangle_X.\end{aligned} \tag{5.19}$$

Hence, we get

$$\widehat{\gamma}_s(U(t,s)^\star h) = e^{i\langle g(t,-\infty)-g(t,s),h\rangle_X} e^{-\frac{1}{2}\langle Q(t,-\infty)h,h\rangle_X} e^{\frac{1}{2}\langle Q(t,s)h,h\rangle_X} \tag{5.20}$$

and

$$\widehat{\gamma}_t(h) = e^{-\frac{1}{2}\langle Q(t,s)h,h\rangle_X}\widehat{\gamma_s}(U(t,s)^\star h). \tag{5.21}$$

Now we prove the last statement. Let $\{\mu_t\}_{t\in\mathbb{R}}$ be an evolution system of measures for $V_{s,t}$, we set $\nu_t = \gamma_t \star \mu_t$ and we show that $\{\nu_t\}_{t\in\mathbb{R}}$ is an evolution system of measures for $P_{s,t}$, namely we show that for all ν_t satisfies (5.4) for all $t \in \mathbb{R}$. Let $t \in \mathbb{R}$ and $h \in X$, then for every $s \leqslant t$ by (5.19) we have

$$\begin{aligned}\widehat{\nu}_t(h) &= \widehat{\gamma}_t(h)\widehat{\mu}_t(h) = e^{-\frac{1}{2}\langle Q(t,-\infty)h,h\rangle_X}\widehat{\mu}_s(U(t,s)^\star h) \\ &= e^{-\frac{1}{2}\langle Q(t,s)h,h\rangle_X} e^{-\frac{1}{2}\langle Q(s,-\infty)U(t,s)^\star h, U(t,s)^\star h\rangle_X}\widehat{\mu}_s(U(t,s)^\star h) \\ &= e^{-\frac{1}{2}\langle Q(t,s)h,h\rangle_X}\widehat{\nu}_s(U(t,s)^\star h).\end{aligned}$$

Conversely, if $\{\nu_t\}_{t\in\mathbb{R}}$ is an evolution system of measures for $P_{s,t}$, then

$$\widehat{\nu}_t(h) = e^{-\frac{1}{2}\langle Q(t,s)h,h\rangle_X}\widehat{\nu}_s(U(t,s)^\star h), \quad h \in X, \; s \leqslant t. \tag{5.22}$$

We set

$$\psi_t(h) = \lim_{s\to-\infty}\widehat{\nu}_s(U(t,s)^\star h) = e^{\frac{1}{2}\langle Q(t,-\infty)h,h\rangle_X}\widehat{\nu}_t(h), \tag{5.23}$$

and so

$$\widehat{\nu}_t(h) = e^{-\frac{1}{2}\langle Q(t,-\infty)h,h\rangle_X}\lim_{s\to-\infty}\widehat{\nu}_s(U(t,s)^\star h) = \widehat{\gamma}_t(h)\psi_t(h). \tag{5.24}$$

To conclude the proof is sufficient to show that

(a) for all $t \in \mathbb{R}$, ψ_t is the characteristic function of a Borel probability measure μ_t;

(b) the family $\{\mu_t\}_{t\in\mathbb{R}}$ is an evolution system of measure for $V_{s,t}$.

To prove (a), we note first that $e^{-i\langle g(t,-\infty),h\rangle_X}\widehat{\nu}_t(h)$ is the characteristic function of the measure $\zeta_t := \delta_{g(t,-\infty)} \star \nu_t$. Then, we apply Theorem 5.1.4 to ζ_t and we obtain that

1. $\widehat{\zeta}_t(0) = 1$,
2. $\widehat{\zeta}_t$ is positive definite,
3. for every $\varepsilon > 0$ there exists a non-negative nuclear operator S_ε such that

   $$1 - \mathfrak{Re}\,\widehat{\zeta}_t(h) \leqslant \varepsilon, \tag{5.25}$$

   for all $h \in X$ satisfying $\langle S_\varepsilon h, h\rangle_X \leqslant 1$.

Since

$$\psi_t(h) = \frac{\widehat{\nu}_t(h)}{\widehat{\gamma}_t(h)} = e^{\frac{1}{2}\langle Q(t,-\infty)h,h\rangle_X}\widehat{\zeta}_t(h), \quad h \in X,$$

then $\psi_t(0) = 1$, ψ_t is positive definite and

$$1 - \mathfrak{Re}\,\psi_t(h) \leqslant 1 - \mathfrak{Re}\,\widehat{\zeta}_t(h) \leqslant \varepsilon, \tag{5.26}$$

for all $h \in X$ satisfying $\langle S_\varepsilon h, h\rangle_X \leqslant 1$. Hence, by theorem 5.1.4 for all $t \in \mathbb{R}$ there exists a Borel probability measure μ_t such that $\widehat{\mu}_t \equiv \psi_t$ and for all $h \in X$ we get

$$\begin{aligned}\widehat{\mu}_s(U(t,s)^\star h) = \psi_s(U(t,s)^\star h) &= \lim_{\sigma\to-\infty}\widehat{\nu}_r(U(s,\sigma)^\star U(t,s)^\star h)\\ &= \lim_{\sigma\to-\infty}\widehat{\nu}_r(U(t,\sigma)^\star h) = \widehat{\mu}_t(h).\end{aligned}$$

Hence $\{\mu_t\}_{t\in\mathbb{R}}$ is an evolution system of measure for $V_{s,t}$ and the statement follows. ☐

Theorem 5.1.7. *Assume that Hypothesis 4.0.1 holds true. If for every $t \in \mathbb{R}$*

$$\sup_{s<t}\,[\mathrm{Tr}\,[Q(t,s)]] < +\infty$$

and for every $t \in \mathbb{R}$ and $x \in X$

$$\lim_{s\to-\infty} U(t,s)x = 0, \tag{5.27}$$

then for every $f \in C_b(X)$ we have

$$\lim_{s\to-\infty} P_{s,t}f(x) = m_t(f) := \int_X f(y)\ \gamma_t(dy), \quad t \in \mathbb{R},\ x \in X. \tag{5.28}$$

where $\{\gamma_t\}_{t\in\mathbb{R}}$ is the evolution system of measures for $\{P_{s,t}\}_{(s,t)\in\overline{\Delta}}$ given by (5.18).

Proof. The statement follows from [Bog98, Example 3.8.15 page 135] since $U(t,s)x \xrightarrow{s\to-\infty} 0$ in X and $Q(t,s) \xrightarrow{s\to-\infty} Q(t,-\infty)$ in $\mathcal{L}_1(X)$. ☐

Remark 5.1.8. We note that if $\{U(t,s)\}_{(s,t)\in\overline{\Delta}}$ verifies (4.5) with $\zeta > 0$ then it verifies (5.27) and by [OR16a, Cor. 4.12] $\{\gamma_t\}_{t\in\mathbb{R}}$ is the unique evolution system of measures uniformly tight for $\{P_{s,t}\}_{(s,t)\in\overline{\Delta}}$. Moreover, if (5.27) does not hold then there may exist many evolution systems of measures for $P_{s,t}$, see Remark 5.3.8.

Corollary 5.1.9. *Assume that Hypothesis 4.0.1 holds true. If for every $t \in \mathbb{R}$*

$$\sup_{s<t}\left[\mathrm{Tr}\left[Q(t,s)\right]\right] < +\infty$$

and for every $t \in \mathbb{R}$ and $x \in X$

$$\lim_{s\to-\infty} U(t,s)x = 0,$$

then for every $\varphi \in C_b(X)$ with strictly positive infimum and $t \in \mathbb{R}$ we have

$$\lim_{s\to-\infty}\int_X P_{s,t}\varphi(x)\log P_{s,t}\varphi(x)\gamma_t(dy) = m_t(\varphi)\log m_t(\varphi). \tag{5.29}$$

Proof. Let $\varphi \in C_b(X)$ have strictly positive infimum. Then $P_{s,t}\varphi$ belongs to $C_b(X)$ and has positive infimum. Since the mapping $y \longmapsto y\log y$ is $\frac{1}{2}$- Hölder continuous on bounded sets of $(0,+\infty)$, we get

$$\begin{aligned}
&\left|\int_X P_{s,t}\varphi(x)\log P_{s,t}\varphi(x)\,\gamma_t(dy) - m_t(\varphi)\log m_t(\varphi)\right| \\
&= \left|\int_X \left(P_{s,t}\varphi(x)\log P_{s,t}\varphi(x) - m_t(\varphi)\log m_t(\varphi)\right)\,\gamma_t(dy)\right| \\
&\leqslant C\int_X \left|P_{s,t}\varphi(x) - m_t(\varphi)\right|^{\frac{1}{2}}\,\gamma_t(dy),
\end{aligned}$$

for some constant $C > 0$. By the Dominated Convergence Theorem and (5.28) the statement follows. ☐

5.2 Logarithmic Sobolev inequalities and the hypercontractivity property of $P_{s,t}$

In this Section we need all the results proved in the previous sections, so we assume the following hypothesis.

Hypothesis 5.2.1. Assume that Hypothesis 4.2.1 holds true and that (5.17) and (5.27) are satisfied.

Let $\{\gamma_t\}_{t\in\mathbb{R}}$ be the evolution system of measures for $\{P_{s,t}\}_{(s,t)\in\overline{\Delta}}$ given by (5.18).

Now we can prove one of the main result of this chapter.

Theorem 5.2.2. *Assume that Hypothesis 5.2.1 holds true. Moreover we assume that $U(t,s)\mathrm{H}_s \subseteq \mathrm{H}_t$ for every $(s,t) \in \Delta$ and that there exist $C, \eta > 0$ and $\alpha \in \left[0,\dfrac{1}{2}\right)$ such that*

$$\left\|U(t,s)_{|\mathrm{H}_s}\right\|_{\mathcal{L}(\mathrm{H}_s;\mathrm{H}_t)} \leqslant C\,\frac{e^{-\eta(t-s)}}{(t-s)^{\alpha}}, \qquad \forall\ (s,t) \in \Delta. \tag{5.30}$$

Then, for every $\varphi \in C_b^1(X)$, $t \in \mathbb{R}$ *and* $p \in (1, +\infty)$ *we have*

$$\int_X |\varphi|^p \log\left(|\varphi|^p\right) d\gamma_t - m_t(|\varphi|^p)\log\left(m_t\left(|\varphi|^p\right)\right) \leqslant \kappa p^2 \int_X |\varphi|^{p-2} \left\| Q\left(t\right)^{\frac{1}{2}} \nabla\varphi \right\|_X^2 \mathbb{1}_{\{\varphi\neq 0\}} d\gamma_t. \tag{5.31}$$

where

$$\kappa = C(2\eta)^{2\alpha-1}\Gamma(1-2\alpha), \tag{5.32}$$

and Γ *is the Euler Gamma function.*

Proof. Let $p \in (1, +\infty)$ and $r \in \mathbb{R}$. We prove first that (5.31) holds for $\varphi \in \mathfrak{B}_t^2(X)$ such that $\inf_{x\in X} \varphi(x) > \varepsilon$ for some $\varepsilon > 0$. Let $(s,t) \in \Delta$ and $s < r < t$; we define a mapping $\psi : [s,t] \times X \to \mathbb{R}$ by

$$\psi(r,x) := P_{r,t}\varphi^p(x). \tag{5.33}$$

By (4.54), $\psi(r,\cdot) \in \mathfrak{B}_r^2(X)$ for every $r \in [s,t]$ and

$$\inf_{r\in[s,t]} \inf_{x\in X} \psi(r,x) > \varepsilon^p, \tag{5.34}$$

and thanks to (4.55) we get

$$\frac{\partial\psi}{\partial r}(r,x) = -L(r)\psi(r,\cdot)(x), \quad r \in [s,t],\ x \in X. \tag{5.35}$$

Now we consider the function $G : [s,t] \to \mathbb{R}$ defined by

$$G(r) := \int_X \psi(r,x)\log(\psi(r,x))\,\gamma_r(dx) = \int_X P_{s,r}\left(\psi(r,\cdot)\log(\psi(r,\cdot))\right)(x)\,\gamma_s(dx).$$

We claim that

$$\frac{\partial}{\partial r}G(r) = \int_X \frac{\partial}{\partial r}P_{s,r}\left(\psi(r,\cdot)\log(\psi(r,\cdot))\right)(x)\,\gamma_s(dx). \tag{5.36}$$

To prove (5.36)we argue as follows.

Since $\psi(r,\cdot) \in \mathfrak{B}_r^2(X)$ has positive infimum independent of $r \in [s,t]$, then $\log(\psi(r,\cdot)) \in \mathfrak{B}_r^2(X)$ for every $r \in [s,t]$. Hence by (4.64), with $g(\xi) = \xi\log\xi$ where $\xi \in \left(\frac{\varepsilon^p}{2}, 2\left\|\varphi\right\|_\infty^p\right)$, we get

$$\begin{aligned}\int_X \frac{\partial}{\partial r}P_{s,r}\left(\psi(r,\cdot)\log(\psi(r,\cdot))\right)(x)\,\gamma_s(dx) &= \int_X L(r)\left(\psi(r,\cdot)\log(\psi(r,\cdot))\right)(x)\,\gamma_r(dx)\\ &\quad - \int_X L(r)\psi(r,\cdot)(x)\log(\psi(r,x))\,\gamma_r(dx)\\ &\quad - \int_X L(r)\psi(r,\cdot)(x)\,\gamma_r(dx).\end{aligned} \tag{5.37}$$

Let ∇ be the gradient with respect to the variable x. By (4.30) and (4.31), for every $r \in [s,t]$ and $F, \Psi, \Phi \in \mathcal{F}_r C_b^2(X)$ with F having positive infimum we have

$$L(r)(\Phi\Psi) = \Phi L(r)\Psi + \Psi L(r)\Phi + \langle Q\left(r\right)^{\frac{1}{2}}\nabla\Phi, Q\left(r\right)^{\frac{1}{2}}\nabla\Psi\rangle_X, \tag{5.38}$$

$$L(r)(\log F) = \frac{1}{F}L(r)F - \frac{1}{2}\frac{\left\|Q\left(r\right)^{\frac{1}{2}}\nabla F\right\|_X^2}{F^2}. \tag{5.39}$$

Using (5.38) in (5.37) we obtain

$$
\begin{aligned}
&\int_X \frac{\partial}{\partial r} P_{s,r}\left(\psi(r,\cdot)\log(\psi(r,\cdot))\right)(x)\,\gamma_s(dx) = -\int_X L(r)\psi(r,\cdot)(x)\log(\psi(r,x))\,\gamma_r(dx)\\
&-\int_X L(r)\psi(r,\cdot)(x)\,\gamma_r(dx) + \int_X \log\psi(r,x)L(r)\psi(r,\cdot)(x)\,\gamma_r(dx)\\
&+\int_X \psi(r,x)L(r)\log(\psi(r,\cdot))(x)\,\gamma_r(dx) + \int_X \langle Q(r)^{\frac12}\nabla\psi(r,x), Q(r)^{\frac12}\nabla\log\psi(r,x)\rangle_X\,\gamma_r(dx).
\end{aligned}
$$

Hence by (5.39) we have

(5.40)

$$
\int_X \frac{\partial}{\partial r} P_{s,r}\left(\psi(r,\cdot)\log(\psi(r,\cdot))\right)(x)\,\gamma_s(dx) = \int_X \left(\frac{\left\|Q(r)^{\frac12}\nabla\psi(r,x)\right\|^2}{\psi(r,x)} - \frac12\frac{\left\|Q(r)^{\frac12}\nabla\psi(r,x)\right\|^2}{\psi^2(r,x)}\right)\gamma_r(dx).
$$

We prove now that (5.36) follows from the dominated convergence theorem. Indeed we rewrite (5.40) as

(5.41)

$$
\int_X \frac{\partial}{\partial r} P_{s,r}\left(\psi(r,\cdot)\log(\psi(r,\cdot))\right)(x)\,\gamma_s(dx) = \int_X P_{s,r}\left(\frac{\left\|Q(r)^{\frac12}\nabla\psi(r,\cdot)\right\|^2}{\psi(r,\cdot)} - \frac12\frac{\left\|Q(r)^{\frac12}\nabla\psi(r,\cdot)\right\|^2}{\psi^2(r,\cdot)}\right)(x)\,\gamma_s(dx).
$$

We recall that as an easy consequence of [CL21, Lemma 2.4], for all $h \in C_b^1(X)$ we have

$$
\nabla P_{s,t}h(x) = \int_X U(t,s)^\star \nabla h(y + U(t,s)x)\,\mathcal{N}_{0,Q(t,s)}(dy), \quad (s,t)\in\Delta,\ x\in X. \tag{5.42}
$$

By (4.6), (5.34) and (5.42) we get

$$
\begin{aligned}
&\left|P_{s,r}\left(\frac{\left\|Q(r)^{\frac12}\nabla\psi(r,\cdot)\right\|_X^2}{\psi(r,\cdot)} - \frac12\frac{\left\|Q(r)^{\frac12}\nabla\psi(r,\cdot)\right\|_X^2}{\psi^2(r,\cdot)}\right)(x)\right| \leqslant \frac{(2\|\varphi^p\|_\infty+1)K^2}{2\varepsilon^{2p}} P_{s,r}\left(\|\nabla\psi(r,\cdot)\|_X^2\right)(x)\\
&= \frac{(2\|\varphi^p\|_\infty+1)K^2}{2\varepsilon^{2p}} P_{s,r}\left(\|\nabla P_{r,t}\varphi^p\|_X^2\right)(x) \leqslant \frac{(2\|\varphi^p\|_\infty+1)K^2}{2\varepsilon^{2p}} \|U(t,r)^\star\|_{\mathcal{L}(X)} P_{s,r}\left(P_{r,t}\|\nabla\varphi^p\|_X^2\right)(x)\\
&\leqslant \frac{(2\|\varphi^p\|_\infty+1)K^2M}{2\varepsilon^{2p}} P_{s,t}\left(\|\nabla\varphi^p\|_X^2\right)(x),
\end{aligned}
$$

and the function in the right side belongs to $L^1(X,\gamma_s)$.

Since the second summand in the right hand side of (5.40) is negative, we get

$$
G'(r) \leqslant \int_X \frac{\left\|Q(r)^{\frac12}\nabla\psi(r,x)\right\|_X^2}{\psi(r,x)}\,\gamma_r(dx) = \int_X \frac{\left\|Q(r)^{\frac12}\nabla(P_{r,t}\varphi^p)(x)\right\|_X^2}{(P_{r,t}\varphi^p)(x)}\,\gamma_r(dx).
$$

Applying (4.20), we obtain

$$G'(r) \leqslant C\frac{e^{-2\eta(t-r)}}{(t-r)^{2\alpha}}\int_X \frac{\left(P_{r,t}\left\|Q(t)^{\frac{1}{2}}\nabla\varphi^p\right\|_X\right)^2(x)}{(P_{r,t}\varphi^p)(x)}\gamma_r(dx) \tag{5.43}$$

and by the Hölder inequality we get

$$P_{r,t}\left(\left\|Q(t)^{\frac{1}{2}}\nabla\varphi^p\right\|_X\right) \leqslant \left(P_{r,t}\left(\frac{\left\|Q(t)^{\frac{1}{2}}\nabla\varphi^p\right\|_X^2}{\varphi^p}\right)\right)^{1/2}(P_{r,t}(\varphi^p))^{1/2}. \tag{5.44}$$

Applying (5.44) to (5.43), by (5.3) we get

$$\begin{aligned}
G'(r) &\leqslant C\frac{e^{-2\eta(t-r)}}{(t-r)^{2\alpha}}\int_X P_{r,t}\frac{\left\|Q(t)^{\frac{1}{2}}\nabla\varphi^p\right\|_X^2}{\varphi^p}(x)\,\gamma_r(dx)\\
&= C\frac{e^{-2\eta(t-r)}}{(t-r)^{2\alpha}}\int_X \frac{\left\|Q(t)^{\frac{1}{2}}\nabla\varphi^p(x)\right\|_X^2}{\varphi^p(x)}\gamma_t(dx)\\
&= p^2C\frac{e^{-2\eta(t-r)}}{(t-r)^{2\alpha}}\int_X \varphi(x)^{p-2}\left\|Q(t)^{\frac{1}{2}}\nabla\varphi(x)\right\|_X^2\gamma_t(dx).
\end{aligned}$$

Integrating with respect to r over $[s,t]$ we obtain

$$\int_X \varphi^p\log\varphi^p\,d\gamma_t - \int_X P_{s,t}\varphi^p\log P_{s,t}\varphi^p\,d\gamma_s \leqslant p^2\left(C\int_s^t\frac{e^{-2\eta(t-r)}}{(t-r)^{2\alpha}}dr\right)\int_X\varphi^{p-2}\left\|Q(t)^{\frac{1}{2}}\nabla\varphi\right\|_X^2 d\gamma_t,$$

Letting $s\to-\infty$ and using Proposition 5.1.9 we conclude

$$\int_X\varphi^p\log\varphi^p\,\gamma_t(dx) \leqslant m_t(\varphi^p)m_t(\log\varphi^p) + p^2\kappa\int_X\varphi^{p-2}\left\|Q(t)^{\frac{1}{2}}\nabla\varphi\right\|_X^2\gamma_t(dx), \tag{5.45}$$

where

$$\kappa := C\int_0^{+\infty}\frac{e^{-2\eta r}}{r^{2\alpha}}dr = C(2\eta)^{2\alpha-1}\Gamma(1-2\alpha). \tag{5.46}$$

We obtain (5.31) for every $\varphi\in\mathfrak{B}_t^2(X)$ applying (5.45) to the standard approximation $\varphi_n = \sqrt{\varphi^2+\frac{1}{n^2}}$ and letting $n\to+\infty$ in (5.45). We remark that, fixed $n\in\mathbb{N}$, the function φ_n belongs to $\mathfrak{B}_t^2(X)$ since the function $h(x)=\sqrt{x^2+\frac{1}{n}}$ has continuous and bounded derivatives of every order.

Finally, if $\varphi\in C_b^1(X)$ by Proposition 4.2.11 there exist a 2-sequence $\{\varphi_{n,m}\}_{n,m\in\mathbb{N}}\subseteq\mathcal{E}_r(X)$, $\{\varphi_n\}_{n\in\mathbb{N}}\subseteq\mathcal{F}_rC_b^1(X)$ and $\{c_n\}_{n\in\mathbb{N}}\subseteq[0,+\infty)$ such that

$$\|\varphi_{n,m}\|_{C_b^1(X)} \leqslant c_n\|\varphi_n\|_{C_b^1(X)}, \qquad n,m\in\mathbb{N}, \tag{5.47}$$

$$\|\varphi_n\|_{C_b^1(X)} \leqslant \|\varphi\|_{C_b^1(X)}, \qquad n\in\mathbb{N}, \tag{5.48}$$

$$\lim_{m\to+\infty}(|\varphi_{n,m}(x)-\varphi_n(x)|+|\nabla\varphi_{n,m}(x)-\nabla\varphi_n(x)|)=0, \qquad n\in\mathbb{N},\ x\in X, \tag{5.49}$$

$$\lim_{n\to+\infty}(|\varphi_n(x)-\varphi(x)|+|\nabla\varphi_n(x)-\nabla\varphi(x)|)=0, \qquad x\in X. \tag{5.50}$$

Noting that $\mathcal{E}_t(X) \subseteq \mathfrak{B}_t^2(X)$, $\varphi_{n,m}$ satisfies (5.31) for every $n, m \in \mathbb{N}$. To prove that also φ satisfies (5.31), we apply two times the Dominated Convergence Theorem: the first time as $m \to +\infty$ thanks to (5.47) and (5.49) and the second time as $n \to +\infty$ thanks to (5.48) and (5.50). $\square$

Remark 5.2.3. Let us compare the Logarithmic Sobolev Inequality provided by L. Gross in [Gro75; Gro93; Gro67] with the one in (5.31). We fix $r \in \mathbb{R}$ and we consider the operator $L(r)$ defined by (4.23). We assume that $A(r)$ is the infinitesimal generator of a strongly continuous semigroup $\{T^{(r)}(t)\}_{t\geqslant 0}$ such that

$$\left\|T^{(r)}(t)\right\|_{\mathcal{L}(X)} \leqslant e^{-c^{(r)}t}, \quad t > 0, \tag{5.51}$$

for some positive constant $c^{(r)}$ and

$$\int_0^{+\infty} \mathrm{Tr}\left[T^{(r)}(t)B(r)B(r)^\star\left(T^{(r)}(t)\right)^\star\right]dt < +\infty.$$

Under these assumptions we consider the Ornstein-Uhlenbeck semigroups $\{R^{(r)}(t)\}_{t\geqslant 0}$, given by

$$R^{(r)}(t)\varphi(x) = \int_X \varphi(y)\mathcal{N}_{T^{(r)}(t)x, Q_t^{(r)}}(dy), \qquad t > 0,\, x \in X,\, \varphi \in C_b(X),$$

where

$$Q_t^{(r)} := \int_0^t T^{(r)}(s)B(r)B(r)^\star T^{(r)}(s)^\star ds.$$

Setting $Q_\infty^{(r)} := \int_0^{+\infty} T^{(r)}(s)B(r)B(r)^\star\left(T^{(r)}(s)\right)^\star ds$, the Gaussian measure $\mu^{(r)} = \mathcal{N}_{0,Q_\infty^{(r)}}$ is the unique invariant measure of $\{R^{(r)}(t)\}_{t\geqslant 0}$. Moreover it is well known that $\{R^{(r)}(t)\}_{t\geqslant 0}$ is uniquely extendable to a strongly continuous semigroup in $L^2(X, \mu^{(r)})$, still denoted by $\{R^{(r)}(t)\}_{t\geqslant 0}$. Its infinitesimal generator is the closure in $L^2(X, \mu^{(r)})$ of the second order Kolmogorov operator $L(r)$ given by (4.23) defined on $\mathcal{E}_r(X)$ (see [DZ02; Big21]). We still denote it by $L(r)$.

We note that $\mu^{(r)}$ is not necessarily the Gaussian measure γ_r of our evolution system of measures $\{\gamma_t\}_{t\in\mathbb{R}}$ of $\{P_{s,t}\}_{(s,t)\in\overline{\Delta}}$. Moreover by [Gro75; Gro93; Gro67] the measure $\mu^{(r)}$ verifies the following logarithmic Sobolev inequality

$$\int_X |\varphi|^p \log\left(|\varphi|^2\right)\, d\mu^{(r)} - m^{(r)}(|\varphi|^2)\log\left(m^{(r)}\left(|\varphi|^2\right)\right) \leqslant -\frac{4}{c^{(r)}}\int_X \varphi\, L(r)\varphi\, d\mu^{(r)}, \tag{5.52}$$

where $\varphi \in D(L(r))$, $c^{(r)}$ is the constant in (5.51),

$$m^{(r)}(\psi) := \int_X \psi(x)\mu^{(r)}(dx), \quad \psi \in L^1(X, \mu^{(r)}).$$

We recall that under suitable assumptions on $A(r)$ and $B(r)$, we have

$$\int_X \varphi\, L(r)\varphi\, d\mu^{(r)} = -\frac{1}{2}\int_X \left\|Q\,(r)^{\frac{1}{2}}\,\nabla\varphi\right\|_X^2 d\mu.$$

So (5.31) does not coincide in general with (5.52). In the next theorem we will see that (5.31) implies a hypercontractivity result for $\{P_{s,t}\}_{(s,t)\in\overline{\Delta}}$.

Lemma 5.2.4. *Assume that there exists a unique evolution system of measures $\{\gamma_t\}_{t\in\mathbb{R}}$ for $P_{s,t}$. Then for any $p \geqslant 1$ and $(s,t) \in \overline{\Delta}$ the operator $P_{s,t}$ is extendable to a linear bounded operator from $L^p(X, \gamma_t)$ to $L^p(X, \gamma_s)$, with unit norm.*

Proof. For every $f \in C_b(X)$, $(s,t) \in \overline{\Delta}$ and $x \in X$ we have

$$|P_{s,t}f(x)|^p \leqslant \int_X |f(y + U(t,s)x)|^p d\gamma_t = P_{s,t}(|f|^p)(x).$$

Integrating over X and recalling (5.3) we obtain

$$\int_X |P_{s,t}f|^p d\gamma_t \leqslant \int_X P_{s,t}(|f|^p)\, d\gamma_t = \int_X |f|^p d\gamma_s.$$

Since $C_b(X)$ is dense in $L^p(X,\gamma)$, $P_{s,t}$ has a unique bounded extension from the whole $L^p(X,\gamma_t)$ into $L^p(X,\gamma_s)$, with norm $\leqslant 1$. Taking $f \equiv 1$, $P_{s,t}f \equiv 1$ so that the norm of the extension is 1. □

Definition 5.2.5. We still denote by $P_{s,t}$ the extension of the Ornstein-Uhlenbeck evolution operator to $L^p(X, \gamma_t)$.

Remark 5.2.6. In general the spaces $L^p(X, \gamma_t)$ and $L^p(X, \gamma_s)$ are different if $t \neq s$, and the classical theory of evolution operators in fixed Banach spaces cannot be used.

Theorem 5.2.7. *Assume that Hypothesis 5.2.1 holds true. Moreover we assume that $U(t,s)\mathrm{H}_s \subseteq \mathrm{H}_t$ for every $(s,t) \in \Delta$ and that there exist $C, \eta > 0$ and $\alpha \in \left[0, \frac{1}{2}\right)$ such that*

$$\left\| U(t,s)_{|\mathrm{H}_s} \right\|_{\mathcal{L}(\mathrm{H}_s;\mathrm{H}_t)} \leqslant C\, \frac{e^{-\eta(t-s)}}{(t-s)^{\alpha}}, \qquad \forall\ (s,t) \in \Delta.$$

Then, for every $(s,t) \in \Delta$, $q \in (1, +\infty)$ and $p \leqslant (q-1)e^{\frac{t-s}{2\kappa}} + 1$ we have

$$\|P_{s,t}\varphi\|_{L^p(X,\,\gamma_s)} \leqslant \|\varphi\|_{L^q(X,\,\gamma_t)}\,, \quad \varphi \in L^q(X,\, \gamma_t). \tag{5.53}$$

where κ is given by (5.32).

Proof. Let $(s,t) \in \Delta$ and let $r \in [s,t]$. We prove first that (5.53) holds for $\varphi \in \mathfrak{B}^2_t(X)$ such that $\inf_{x\in X} \varphi(x) > \varepsilon$ for some $\varepsilon > 0$. We consider the mapping $\psi : [s,t] \times X \to \mathbb{R}$ given by

$$\psi(r,x) := P_{r,t}\varphi(x). \tag{5.54}$$

By Lemma 4.54, $\psi(r,\cdot) \in \mathfrak{B}^2_r(X)$ for every $r \in [s,t]$ and

$$\inf_{r\in[s,t]} \inf_{x\in X} \psi(r,x) > \varepsilon \tag{5.55}$$

and thanks to (4.55) we get

$$\frac{\partial \psi}{\partial r}(r,x) = -L(r)\psi(r,\cdot)(x), \quad r \in [s,t],\ x \in X. \tag{5.56}$$

Now we define the functions $G : [s,t] \to \mathbb{R}$ and $H : [s,t] \to \mathbb{R}$ by

$$G(r) := \int_X \psi(r,x)^{p(r)}\,\gamma_r(dx), \qquad H(r) = G(r)^{\frac{1}{p(r)}}, \tag{5.57}$$

where

$$p(r) := (q-1)e^{\frac{t-r}{2\kappa}} + 1. \tag{5.58}$$

$$\frac{\partial}{\partial r}G(r) = \int_X \frac{\partial}{\partial r}P_{s,r}\left(\psi(r,\cdot)^{p(r)}\right)(x)\,\gamma_s(dx). \tag{5.59}$$

Since $\psi(r,\cdot) \in \mathfrak{B}^2_r(X)$ has positive infimum independent of $r \in [s,t]$, since $\log(\cdot) : [\varepsilon, +\infty) \to \mathbb{R}$ has continuous and bounded derivatives of every order, then $\psi(r,\cdot)^{p(r)} \in \mathfrak{B}^2_r(X)$ for every $r \in [s,t]$. Hence by a slight modification of (4.64), where $g(\xi)$ is replaced by $g(r,\xi) = \xi^{p(r)}$ with $\xi \in \left(\frac{\varepsilon}{2}, 2\|\varphi\|_\infty\right))$, we get

$$\begin{aligned}\int_X \frac{\partial}{\partial r}P_{s,r}\left(\psi(r,\cdot)^{p(r)}\right)(x)\,\gamma_s(dx) &= \int_X \psi(r,x)^{p(r)}\log\psi(r,x)\,p'(r)\,\gamma_r(dx)\\ &\quad - \int_X \psi(r,x)^{p(r)-1}p(r)L(r)\psi(r,\cdot)(x)\,\gamma_r(dx)\\ &\quad + \int_X L(r)\psi(r,\cdot)^{p(r)}(x)\,\gamma_r(dx).\end{aligned} \tag{5.60}$$

For every $r \in [s,t]$, $p \geqslant 1$ and $\Phi \in \mathcal{F}_r C^2_b(X)$, by (4.31) we have

$$L(r)\Phi^p = \frac{1}{2}p(p-1)\Phi^{p-2}\left\|Q(r)^{\frac{1}{2}}\nabla\Phi\right\|^2_X + p\Phi^{p-1}L(r)\Phi, \tag{5.61}$$

and applying (5.61) to (5.60), we get

$$\begin{aligned}&\int_X \frac{\partial}{\partial r}P_{s,r}\left(\psi(r,\cdot)^{p(r)}\right)(x)\,\gamma_s(dx) = p'(r)\int_X \psi(r,x)^{p(r)}\log\psi(r,x)\,\gamma_r(dx)\\ &+ \frac{1}{2}p(r)(p(r)-1)\int_X \psi(r,x)^{p(r)-2}\left\|Q(r)^{\frac{1}{2}}\nabla\psi(r,x)\right\|^2_X\,\gamma_r(dx).\end{aligned} \tag{5.62}$$

We prove now that (5.59) follows from the dominated convergence theorem. Indeed we rewrite (5.62) as

$$\begin{aligned}&\int_X \frac{\partial}{\partial r}P_{s,r}\left(\psi(r,\cdot)^{p(r)}\right)(x)\,\gamma_s(dx) = p'(r)\int_X P_{s,r}\left(\psi(r,\cdot)^{p(r)}\log\psi(r,\cdot)\right)(x)\,\gamma_s(dx)\\ &+ \frac{1}{2}p(r)(p(r)-1)\int_X P_{s,r}\left(\psi(r,\cdot)^{p(r)-2}\left\|Q(r)^{\frac{1}{2}}\nabla\psi(r,\cdot)\right\|^2_X\right)(x)\,\gamma_s(dx).\end{aligned} \tag{5.63}$$

By (4.6), (5.58) and (5.42)we obtain

$$\begin{aligned}
&\left|p'(r)P_{s,r}\left(\psi(r,\cdot)^{p(r)}\log\psi(r,\cdot)\right)(x)+\frac{1}{2}p(r)(p(r)-1)P_{s,r}\left(\psi(r,\cdot)^{p(r)-2}\left\|Q\,(r)^{\frac{1}{2}}\,\nabla\psi(r,\cdot)\right\|_X^2\right)(x)\right|\\
&\leqslant\frac{q-1}{2\kappa}e^{\frac{t-s}{2\kappa}}\,\|\varphi\|_\infty^{p(t)}\log\|\varphi\|_\infty+\frac{1}{2}p(t)(p(t)+1)K^2P_{s,r}\left(\|\nabla\psi(r,\cdot)\|_X^2\right)(x)\\
&=\frac{q-1}{2\kappa}e^{\frac{t-s}{2\kappa}}\,\|\varphi\|_\infty^{p(t)}\log\|\varphi\|_\infty+\frac{1}{2}p(t)(p(t)+1)K^2P_{s,r}\left(\|\nabla P_{r,t}\varphi(\cdot)\|_X^2\right)(x)\\
&\leqslant\frac{q-1}{2\kappa}e^{\frac{t-s}{2\kappa}}\,\|\varphi\|_\infty^{p(t)}\log\|\varphi\|_\infty+\frac{1}{2}p(t)(p(t)+1)K^2\,\|U(t,r)^\star\|_{\mathcal{L}(X)}\,P_{s,r}\left(P_{r,t}\,\|\nabla\varphi(\cdot)\|_X^2\right)(x)\\
&\leqslant\frac{q-1}{2\kappa}e^{\frac{t-s}{2\kappa}}\,\|\varphi\|_\infty^{p(t)}\log\|\varphi\|_\infty+\frac{1}{2}p(t)(p(t)+1)K^2M\,P_{s,t}\,\|\nabla\varphi(\cdot)\|_X^2\,(x)\in L^1(X,\gamma_s).
\end{aligned}$$

By (5.59) and (5.62) we get

$$\begin{aligned}
\frac{\partial}{\partial r}(\log H(r))&=\frac{1}{p(r)G(r)}G'(r)-\frac{p'(r)}{p(r)^2}\log G(r)\\
&=\frac{p'(r)}{p(r)\int_X\psi(r,x)^{p(r)}\,\gamma_r(dx)}\int_X\psi(r,x)^{p(r)}\log\psi(r,x)\,\gamma_r(dx)\\
&+\frac{p(r)-1}{2\int_X\psi(r,x)^{p(r)}\,\gamma_r(dx)}\int_X\psi(r,x)^{p-2}\left\|Q\,(r)^{\frac{1}{2}}\,\nabla\psi(r,x)\right\|_X^2\,\gamma_r(dx)\\
&-\frac{p'(r)}{p(r)^2}\log\left(\int_X\psi(r,x)^{p(r)}\,\gamma_r(dx)\right)\\
&=\frac{p'(r)}{p(r)^2G(r)}\left[\int_X\psi(r,x)^{p(r)}\log\psi(r,x)^{p(r)}\,\gamma_r(dx)-m_r(\psi(r,\cdot)^p)\log m_r(\psi(r,\cdot)^p)\right]\\
&+\frac{p(r)-1}{2G(r)}\int_X\psi(r,x)^{p-2}\left\|Q\,(r)^{\frac{1}{2}}\,\nabla\psi(r,x)\right\|_X^2\,\gamma_r(dx).
\end{aligned}$$

Since $p(r)$ is given by (5.58) and $p'(r)<0$, by (5.31) we get

(5.64)
$$\frac{\partial}{\partial r}(\log H(r))\geqslant\frac{1}{2G(r)}\left(2p'(r)\kappa+p(r)-1\right)\int_X\psi(r,x)^{p-2}\left\|Q\,(r)^{\frac{1}{2}}\,\nabla\psi(r,x)\right\|_X^2\,\gamma_r(dx)=0,$$

and $\log H(r)$ is a non decreasing function. Hence also $H(r)$ is a non decreasing function and (5.53) holds true for every $\varphi\in\mathfrak{B}_t^2(X)$ with positive infimum.

Arguing as in Theorem 5.2.2, we obtain (5.53) for every $\varphi\in\mathfrak{B}_t^2(X)$ applying (5.53) to the standard approximation $\varphi_n=\sqrt{\varphi^2+\dfrac{1}{n^2}}$ and letting $n\to+\infty$ in (5.53). Since $\mathcal{E}_t(X)\subseteq\mathfrak{B}_t^2(X)$ (5.53) holds for all $\varphi\in L^q(X,\gamma_t)$ by Remark 4.2.12. $\square$

5.3 Examples

In this section we consider again examples of Section 3.2 with Δ_T replaced by Δ and with suitable assumptions that allow to prove the existence of an evolution system of Gaussian measures and to apply Theorem 5.2.2 and Theorem 5.2.7.

5.3.1 A non autonomous parabolic problem

Let $d \in \mathbb{N}$ and let $\mathcal{O} \subseteq \mathbb{R}^d$ be a bounded open set with smooth boundary. We consider the evolution operator $\{U(t,s)\}_{(s,t)\in\overline{\Delta}}$ in $X := L^2(\mathcal{O})$ associated to an evolution equation of parabolic type,

$$
\begin{cases} u_t(t,x) = \mathcal{A}(t)u(t,\cdot)(x), & (t,x) \in (s,+\infty) \times \mathcal{O}, \\ \mathcal{B}(t)u(t,\cdot)(x) = 0, & (t,x) \in (s,+\infty) \times \partial\mathcal{O}. \end{cases} \tag{5.65}
$$

The differential operators $\mathcal{A}(r)$ are now written in divergence form as

$$
\mathcal{A}(r)\varphi(x) = \sum_{i,j=1}^{d} D_i\left(a_{ij}(r,x)D_i\varphi(x)\right) + \sum_{i=1}^{d} a_i(r,x)D_i\varphi(x) + a_0(r,x)\varphi(x), \quad r \in \mathbb{R},\ x \in \mathcal{O} \tag{5.66}
$$

and the family of the boundary operators $\{\mathcal{B}(r)\}_{r\in\mathbb{R}}$ is either of Dirichlet, Neumann or Robin type, namely

$$
\mathcal{B}(r)u = \begin{cases} u & \text{(Dirichlet)}, \\ \displaystyle\sum_{i,j=1}^{d} a_{ij}(x,t)D_iu\,\nu_j & \text{(Neumann)}, \\ \displaystyle\sum_{i,j=1}^{d} a_{ij}(x,t)D_iu\,\nu_j + b_0(x,t)u & \text{(Robin)}, \end{cases} \tag{5.67}
$$

where $\nu = (\nu_1, ..., \nu_d)$ is the unit outer normal vector at the boundary of Ω.

We make the following assumptions.

Hypothesis 5.3.1. We assume $a_{ij} = a_{ji}$ and for some $\rho \in \left(\frac{1}{2}, 1\right)$, $a_{ij} \in C_b^{\rho,2}\left(\mathbb{R} \times \overline{\mathcal{O}}\right)$, $b_0 \in C_b^{\rho,1}\left(\mathbb{R} \times \overline{\mathcal{O}}\right)$, $a_i, a_0 \in C_b^{\rho,0}\left(\mathbb{R} \times \overline{\mathcal{O}}\right)$. Moreover, we assume that there exist $\nu, \omega > 0$ such that

$$
\sum_{i,j=1}^{d} a_{ij}(r,x)\xi_i\xi_j \geqslant \nu\,|\xi|^2, \quad r \in \mathbb{R},\ x \in \mathcal{O},\ \xi \in \mathbb{R}^d, \tag{5.68}
$$

$$
\sup_{(r,x)\in\mathbb{R}\times\mathcal{O}} a_0(r,x) \leqslant -\omega, \tag{5.69}
$$

$$
\delta_0 - \omega < 0, \tag{5.70}
$$

where $\delta_0 = \dfrac{1}{\nu}\left(\displaystyle\sum_{i=1}^{d} \|a_i\|_\infty^2\right)^{\frac{1}{2}}$.

For every $r \in \mathbb{R}$ we denote by $A(r)$ the realization in $L^2(\mathcal{O})$ of $\mathcal{A}(r)$ with one of the boundary conditions 5.67. In [Sch04, Ex. 2.8, Ex. 2.9, Rmk 3.19 and Ex. 4.9] it is proven that the families $\{A(r)\}_{r\in\mathbb{R}}$ and $\{A(r)^\star\}_{r\in\mathbb{R}}$ satisfy the assumptions of [Acq88; AT87] and of [AT92, Sect. 6], so by [Acq88, Thm. 2.3] and [AT92, Prop. 6.1, Thm 6.4] there exists an evolution operator $\{U(t,s)\}_{s\leqslant t}$ on X such that Hypothesis 4.2.1 holds true.

Proposition 5.3.2. *Under Hypothesis 5.3.1, we have*

$$\|U(t,s)\|_{\mathcal{L}(X)} \leqslant e^{-(\omega-\delta_0)(t-s)}, \quad (s,t) \in \Delta.$$

Proof. We consider the family of operators $\left\{-\widetilde{A}(r)\right\}_{r\in\mathbb{R}}$ defined by

$$-\widetilde{A}(r) = -A(r) + \omega \mathrm{Id}_X, \quad r \in \mathbb{R}. \tag{5.71}$$

Since $\{A(r)\}_{r\in\mathbb{R}}$ verifies Hypothesis 4.2.1 then $\left\{\widetilde{A}(r)\right\}_{r\in\mathbb{R}}$ verifies Hypothesis 4.2.1 and it is associated to the evolution operator $\{\widetilde{U}(t,s)\}_{s\leqslant t}$, given by

$$\widetilde{U}(t,s) = e^{\omega(t-s)} U(t,s), \quad s \leqslant t. \tag{5.72}$$

Moreover $\left\{-\widetilde{A}(r)\right\}_{r\in\mathbb{R}}$ satisfies all hypotheses of [Dan00] and by [Dan00, Thm 5.1] we have

$$\left\|\widetilde{U}(t,s)\right\|_{\mathcal{L}(X)} \leqslant e^{\delta_0(t-s)}, \quad s \leqslant t, \tag{5.73}$$

and

$$\|U(t,s)\|_{\mathcal{L}(X)} = e^{-\omega(t-s)} \left\|\widetilde{U}(t,s)\right\|_{\mathcal{L}(X)} \leqslant e^{-(\omega-\delta_0)(t-s)}, \quad s \leqslant t. \tag{5.74}$$

□

Proposition 5.3.3. *Assume that Hypothesis 5.3.1 holds true. Let $\{B(r)\}_{r\in\mathbb{R}}$ be a family of operators such that*

1. *for every $r \in \mathbb{R}$, $B(r) \in \mathcal{L}(L^2(\mathcal{O}), L^q(\mathcal{O}))$ for some $q \in [2,+\infty) \cap (d,+\infty)$ and*
$$\sup_{r\in\mathbb{R}} \|B(r)\|_{\mathcal{L}(L^2(\mathcal{O});L^q(\mathcal{O}))} < +\infty;$$
2. *for every $\varphi \in L^2(\mathcal{O})$ the mapping $r \in \mathbb{R} \longmapsto B(r)\varphi \in L^q(\mathcal{O})$ is continuous.*

Then

$$\mathrm{Tr}\left[Q(t,s)\right] \leqslant C^2 \left|\mathcal{O}\right| M_{q'}^{\frac{2}{q'}} \int_s^t \frac{e^{-\omega(t-r)}}{(t-r)^{\frac{d}{q}}}\, dr. \tag{5.75}$$

and the operator $Q(t,s)$ has finite trace for all $s < t$. Moreover there exists a unique evolution system of measures $\{\gamma_t\}_{t\in\mathbb{R}}$ for $P_{s,t}$ given by (5.18).

Proof. We adapt to our setting the arguments of [CL21, Lemma 4.3].

By [Dan00] $\widetilde{U}(t,s)$ defined in (5.72) may be exended to the whole $L^1(\mathcal{O})$, and the extension (still denoted by $\widetilde{U}(t,s)$) belongs to $\mathcal{L}\left(L^1(\mathcal{O}); L^\infty(\mathcal{O})\right)$. Moreover $\widetilde{U}(t,s)$ is represented by

$$\widetilde{U}(t,s)\varphi(x) = \int_{\mathcal{O}} k(x,y,t,r)\varphi(y)\, dy, \quad \varphi \in L^1(\mathcal{O}),\ s < t, \tag{5.76}$$

where $k(\cdot,\cdot,t,s)$ belongs to $L^\infty(\mathcal{O}\times\mathcal{O})$ and by [Dan00, thm. 6.1] there exist $M, m > 0$ such that

$$|k(x,y,t,r)| \leqslant \frac{M}{(t-s)^{\frac{d}{2}}}\, e^{-\frac{|x-y|^2}{m(t-s)}}, \quad x,y\in\mathcal{O},\ s<t, \tag{5.77}$$

and recalling (5.72), we get

$$U(t,s)\varphi(x) = e^{-\omega(t-s)}\int_{\mathcal{O}} k(x,y,t,r)\varphi(y)\,dy, \quad \varphi\in L^1(\mathcal{O}),\ \ s<t,\ x\in\mathcal{O}. \tag{5.78}$$

Let $\{e_k\}_{k\in\mathbb{N}}$ be a Hilbert basis of X. Then

$$\mathrm{Tr}\,[Q(t,s)] = \int_s^t \sum_{k=1}^\infty \|B(r)^\star U(t,r)^\star e_k\|^2_{L^2(\mathcal{O})}\,dr, \quad s<t. \tag{5.79}$$

By the representation formula (5.78), we get

$$(U(t,r)^\star e_k)(y) = e^{-\omega(t-r)}\int_{\mathcal{O}} k(x,y,t,r)e_k(x)\,dx, \quad \text{a.e. } y\in\mathcal{O}, \tag{5.80}$$

and then

$$(B(r)^\star U(t,r)^\star e_k)(y) = e^{-\omega(t-r)}\int_{\mathcal{O}} \left(B(r)^\star k(x,\cdot,t,r)\right)(y)\,e_k(x)\,dx, \quad \text{a.e. } y\in\mathcal{O}. \tag{5.81}$$

We obtain

$$\begin{aligned}
\mathrm{Tr}\,[Q(t,s)] &= \int_s^t e^{-\omega(t-r)}\int_{\mathcal{O}}\sum_{k=1}^\infty\left(\int_{\mathcal{O}}\left(B(r)^\star k(x,\cdot,t,r)\right)(y)\,e_k(x)\,dx\right)^2 dydr\\
&= \int_s^t e^{-\omega(t-r)}\int_{\mathcal{O}}\int_{\mathcal{O}}\left(\left(B(r)^\star k(x,\cdot,t,r)\right)(y)\right)^2\,dxdydr\\
&= \int_s^t e^{-\omega(t-r)}\int_{\mathcal{O}}\int_{\mathcal{O}}\left(\left(B(r)^\star k(x,\cdot,t,r)\right)(y)\right)^2\,dydxdr.
\end{aligned} \tag{5.82}$$

Since $B(r)\in\mathcal{L}(L^2(\mathcal{O});L^q(\mathcal{O}))$ then $B(r)^\star\in\mathcal{L}(L^{q'}(\mathcal{O});L^2(\mathcal{O}))$ and there exists $C>0$ such that for every $x\in\mathcal{O}$ we have

$$\begin{aligned}
\|B(r)^\star k(x,\cdot,t,r)\|_{L^2(\mathcal{O})} &\leqslant \|B(r)^\star\|_{\mathcal{L}(L^{q'}(\mathcal{O});L^2(\mathcal{O}))}\,\|k(x,\cdot,t,r)\|_{L^{q'}(\mathcal{O})}\\
&\leqslant C\,\|k(x,\cdot,t,r)\|_{L^{q'}(\mathcal{O})}\,.
\end{aligned} \tag{5.83}$$

By (5.77), for every $p>1$ there exists $M_p>0$ independent of x such that

$$\|k(x,\cdot,t,r)\|^p_{L^p(\mathcal{O})} \leqslant \frac{M}{(t-r)^{\frac{dp}{2}}}\int_{\mathbb{R}^d} e^{-\frac{p}{m}\frac{|x-y|}{t-r}}\,dy = \frac{M_p}{(t-r)^{\frac{d(p-1)}{2}}}. \tag{5.84}$$

Choosing $p=q'$ we obtain

$$\|B(r)^\star k(x,\cdot,t,r)\|^2_{L^2(\mathcal{O})} \leqslant \frac{C^2 M_{q'}^{\frac{2}{q'}}}{(t-r)^{\frac{d}{q}}}. \tag{5.85}$$

Combining (5.82) and (5.85) we obtain

$$\operatorname{Tr}[Q(t,s)] \leqslant C^2\, |\mathcal{O}|\, M_{q'}^{\frac{2}{q'}} \int_s^t \frac{e^{-\omega(t-r)}}{(t-r)^{\frac{d}{q}}}\, dr. \tag{5.86}$$

where $|\mathcal{O}|$ is the d-dimensional Lebesgue measure of $\mathcal{O}$.

Since $q > d$, (5.75) implies that the trace of $Q(t,s)$ is finite for every $s < t$ and

$$\sup_{s<t} [\operatorname{Tr}[Q(t,s)]] < +\infty.$$

By Corollary 5.1.7, Remark 5.1.8 and (5.74), there exists a unique evolution system of measures for $P_{s,t}$ and it is given by (5.18). ☐

We just have to give sufficient conditions guaranteeing that (5.30) holds true. To this aim we need to recall some preliminary results. By [Acq88, Thm. 2.3], $U(t,s) \in \mathcal{L}(X; D(A(t)))$ and there exists $C_1 > 0$ such that for every $0 < t-s \leqslant 1$ we have

$$\|U(t,s)\|_{\mathcal{L}(X)} + \|A(t)U(t,s)\|_{\mathcal{L}(X)} = \|U(t,s)\|_{\mathcal{L}(X;D(A(t)))} \leqslant \frac{C_1}{t-s}, \tag{5.87}$$

$$\|U(t,s)\|_{\mathcal{L}(D(A(s));D(A(t)))} \leqslant C_1. \tag{5.88}$$

By Proposition 5.3.2 and (5.87), for every $t-s > 1$ we have

$$\begin{aligned}
\|U(t,s)\|_{\mathcal{L}(D(A(s));D(A(t)))} &\leqslant \|U(t,s)\|_{\mathcal{L}(X;D(A(t)))} \\
&= \|U(t,t-1)U(t-1,s)\|_{\mathcal{L}(X)} + \|A(t)U(t,t-1)U(t-1,s)\|_{\mathcal{L}(X)} \\
&\leqslant \Big(\|U(t,t-1)\|_{\mathcal{L}(X)} + \|A(t)U(t,t-1)\|_{\mathcal{L}(X)}\Big) \|U(t-1,s)\|_{\mathcal{L}(X)} \\
&\leqslant C_1 e^{\omega-\delta_0} e^{-(\omega-\delta_0)(t-s)}
\end{aligned} \tag{5.89}$$

so by Proposition 5.3.2 and combining (5.87), (5.88) and (5.89) there exists $C > 0$ such that for every $s < t$ we have

$$\|U(t,s)\|_{\mathcal{L}(X)} \leqslant C\, e^{-(\omega-\delta_0)(t-s)}, \tag{5.90}$$

$$\|U(t,s)\|_{\mathcal{L}(X;D(A(t)))} \leqslant C \max\left\{1, (t-s)^{-1}\right\} e^{-(\omega-\delta_0)(t-s)}, \tag{5.91}$$

$$\|U(t,s)\|_{\mathcal{L}(D(A(s));D(A(t)))} \leqslant C\, e^{-(\omega-\delta_0)(t-s)}. \tag{5.92}$$

By (5.90) and (5.92) for every $0 < \theta < 1$ and for every $s < t$ we have

$$\|U(t,s)\|_{\mathcal{L}((X,D(A(s))_{\theta,2};(X,D(A(t))_{\theta,2}))} \leqslant C\, e^{-(\omega-\delta_0)(t-s)}, \tag{5.93}$$

where, for every $r \in \mathbb{R}$, $(X, D(A(r))_{\theta,2}$ is the standard real interpolation space between X and $D(A(r))$. Moreover by (5.90) and (5.91) for every $0 < \theta < 1$ and for every $s < t$ we have

$$\|U(t,s)\|_{\mathcal{L}(X;(X,D(A(t))_{\theta,2}))} \leqslant C \max\{1, (t-s)^{-\theta}\}\, e^{-(\omega-\delta_0)(t-s)}. \tag{5.94}$$

Combining (5.93) and (5.94) for every $0 < \sigma < 1$ and $s < t$ we get

$$\|U(t,s)\|_{\mathcal{L}((X,(X,D(A(s))_{\theta,2})_{\sigma,2};(X,D(A(t))_{\theta,2}))} \leqslant C \max\left\{1, (t-s)^{-\theta(1-\sigma)}\right\} e^{-(\omega-\delta_0)(t-s)}. \tag{5.95}$$

Recalling that by reiteration $(X,(X,D(A(s))_{\theta,2})_{\sigma,2} = (X,D(A(s))_{\theta\sigma,2}$, for every $\theta \in (0,1)$, $\rho \in (0,\theta]$ and $s < t$ we get

$$\|U(t,s)\|_{\mathcal{L}((X,D(A(s))_{\rho,2};(X,D(A(t))_{\theta,2}))} \leqslant C\, \max\{1,(t-s)^{-(\theta-\rho)}\}\, e^{-(\omega-\delta_0)(t-s)}, \tag{5.96}$$

where we have chosen $\sigma = \dfrac{\rho}{\theta}$ in (5.95). Now we recall the characterization of $D(A(r))$ for every $r \in \mathbb{R}$.

- **Neumann or Robin boundary condition** In this case for every $r \in \mathbb{R}$ we have

  $$D(A(r)) = \left\{u \in H^2(\mathcal{O}) :\ \mathcal{B}(r)u = 0\right\}. \tag{5.97}$$

  Moreover by e.g. [Gui91, Thm. 3.5, Thm. 4.15] for every $r \in \mathbb{R}$ and $0 < \gamma < 1$, we have

  $$(X,D(A(r)))_{\gamma,2} = \begin{cases} H^{2\gamma}(\mathcal{O}) & \text{if } 0<\gamma<\frac{3}{4} \\ \left\{u \in H^{\frac{3}{2}}(\mathcal{O}) \mid \widetilde{\mathcal{B}}(r)u \in \mathring{H}^{\frac{1}{2}}(\mathcal{O})\right\} & \text{if } \gamma = \frac{3}{4} \\ \left\{u \in H^{2\gamma}(\mathcal{O}) \mid \mathcal{B}(r)u = 0\right\} & \text{if } \frac{3}{4}<\gamma<1 \end{cases}, \tag{5.98}$$

  where

  $$\widetilde{\mathcal{B}}(r)u = \begin{cases} \displaystyle\sum_{i,j=1}^{d} a_{ij}(x,r)D_iu\,\widetilde{\nu_j} & \text{(Neumann)} \\ \displaystyle\sum_{i,j=1}^{d} a_{ij}(x,r)D_iu\,\widetilde{\nu_j} + b_0(x,r)u, & \text{(Robin)} \end{cases}, \tag{5.99}$$

  $\widetilde{\nu}$ is a smooth enough extension of ν to $\overline{\mathcal{O}}$ and $\mathring{H}^{\frac{1}{2}}(\mathcal{O})$ consists on all the elements $\varphi \in H^{\frac{1}{2}}(\mathcal{O})$ whose null extension outside $\overline{\mathcal{O}}$ belongs to $H^{\frac{1}{2}}(\mathbb{R}^d)$.

- **Dirichlet boundary condition** In this case for every $r \in \mathbb{R}$ we have

  $$D(A(r)) = H^2(\mathcal{O}) \cap H^1_0(\mathcal{O}). \tag{5.100}$$

  Moreover by [Gui91, Thm. 3.5, Thm. 4.15] for every $r \in \mathbb{R}$ and $0 < \gamma < 1$, we have

  $$\left(L^2(\mathcal{O}), H^2(\mathcal{O}) \cap H^1_0(\mathcal{O})\right)_{\gamma,2} = \begin{cases} H^{2\gamma}(\mathcal{O}) & \text{if } 0<\gamma<\frac{1}{4} \\ \mathring{H}^{\frac{1}{2}}(\mathcal{O}) & \text{if } \gamma=\frac{1}{4} \\ \left\{u \in H^{2\gamma}(\mathcal{O}) \mid u_{|\partial\mathcal{O}} = 0\right\} & \text{if } \frac{1}{4}<\gamma<1 \end{cases}. \tag{5.101}$$

Now we present two explicit examples of $\{B(r)\}_{r\in\mathbb{R}}$ where all the hypotheses of Theorems 5.2.2 and 5.2.7 are verified.

Theorem 5.3.4. *Assume the following conditions hold true.*

1. *$X := L^2(\mathcal{O})$ where $\mathcal{O} \subseteq \mathbb{R}^d$ is a bounded open set with smooth boundary and $d = 1,2,3,4,5$.*
2. *The operators $\mathcal{A}(r)$ given by (5.66) verify Hypothesis 5.3.1.*
3. *The realization $A(r)$ in $L^2(\mathcal{O})$ of $\mathcal{A}(r)$ with one of the boundary conditions 5.67 is a negative operator for every $r \in \mathbb{R}$.*

4. For every $r \in \mathbb{R}$ we have

$$B(r) = (-A(r))^{-\gamma}, \quad r \in \mathbb{R},$$

for some $\gamma \geqslant 0$.

In the following cases

$$\begin{cases} 0 \leqslant \gamma < 1 & \text{if } d = 1 \\ 0 < \gamma < 1 & \text{if } d = 2 \\ \frac{1}{4} < \gamma < 1 & \text{if } d = 3 \\ \frac{1}{2} < \gamma < 1 & \text{if } d = 4 \\ \frac{3}{4} < \gamma < 1 & \text{if } d = 5 \end{cases}, \tag{5.102}$$

there exists and evolution operator $U(t,s)$ satisfing Hypothesis 5.2.1. Moreover for every $s < t$ we have

$$\|U(t,s)\|_{\mathcal{L}(\mathrm{H}_s,\mathrm{H}_t)} \leqslant Ce^{-(\omega-\delta_0)(t-s)}, \tag{5.103}$$

where C, ω and δ_0 are the constants in (5.93).

Proof. Let us show that Hypothesis 5.2.1 is satisfied. We have already mentioned that Hypothesis 4.0.1 holds. Moreover Proposition 5.3.2 yields (5.27), since $\delta_0 - \omega < 0$ by Hypothesis 5.3.1. Let us check that the operator $B(r) = (-A(r))^{-\gamma}$ verify the assumptions of Proposition 5.3.3 for every $r \in \mathbb{R}$. By [Lun18, Thm. 4.36] we get

$$\mathrm{H}_r := Q(r)^{\frac{1}{2}}(X) = D((-A(r))^{\gamma}) = (X, D(A(r)))_{\gamma,2}. \tag{5.104}$$

It follows that for every $q \geqslant 2$, the embedding of $D((-A(r))^{\gamma})$ in $L^q(\mathcal{O})$ is continuous for $\gamma \geqslant 2d\left(\frac{1}{2} - \frac{1}{q}\right)$. Hence for such choices of γ, $B(r) \in \mathcal{L}(L^2(\mathcal{O}); L^q(\mathcal{O}))$. The rest of the assumptions of Proposition 5.3.3 follows by the representation formula of $(-A(r))^{-\gamma}$ through a Dunford integral, see e.g. [Lun18, Def. 4.3]. Hence, by Proposition 5.3.3, (5.17) holds. Estimate (5.103) follows by (5.93) and (5.104). □

Remark 5.3.5. If $a_i \equiv 0$ for all $i = 1,..,d$ hypothesis *3* of Theorem 5.3.4 is satisfied. If not, by standard arguments, one can find sufficient conditions on the coefficients a_i such that hypothesis *3* holds.

So by Proposition 5.3.3 and recalling (5.98) (5.101), for every $s < t$ the operator $Q(t,s)$ given by (2.2) has finite trace in all the cases (5.102). Hypothesis 5.2.1 holds true in view of Proposition 5.3.2. Finally (5.103) holds true by (5.93) and (5.104) with $\theta = \gamma$.

Theorem 5.3.6. *Assume the following conditions hold true.*

1. *$X := L^2(\mathcal{O})$ where $\mathcal{O} \subseteq \mathbb{R}^d$ is a bounded open set with smooth boundary and $d = 1, 2, 3$.*
2. *The operators $\mathcal{A}(r)$ given by (5.66) verify Hypothesis 5.3.1.*

3. For every $r \in \mathbb{R}$ we have

$$B(r) = (-\Delta)^{-\gamma(r)}, \quad r \in \mathbb{R},$$

where Δ is the realization of the Laplacian operator in $L^2(\mathcal{O})$ with Dirichlet (Neumann or Robin) boundary conditions, $\gamma : \mathbb{R} \to [0, \alpha]$ is a non-decreasing continuous function and $0 < \alpha < \dfrac{1}{2}$.

In the following cases

$$\begin{cases} \inf_{r\in\mathbb{R}} \gamma(r) \geqslant 0 & \text{if } d = 1 \\ \inf_{r\in\mathbb{R}} \gamma(r) > 0 & \text{if } d = 2 \\ \inf_{r\in\mathbb{R}} \gamma(r) > \dfrac{1}{4} & \text{if } d = 3 \end{cases}, \tag{5.105}$$

there exists and evolution operator $U(t, s)$ satisfing Hypothesis 5.2.1. Moreover for every $s < t$ we have

$$\|U(t, s)\|_{\mathcal{L}(\mathrm{H}_s,\mathrm{H}_t)} \leqslant C \max\{1, (t - s)^{-\alpha}\} e^{-(\omega-\delta_0)(t-s)}, \tag{5.106}$$

where C, ω, δ_0 are the constants appearing in (5.96).

Proof. We prove the statement in the case where $\{A(r)\}_{r\in\mathbb{R}}$ is the realization in $L^2(\mathcal{O})$ of $\mathcal{A}(r)$ with Robin boundary condition, the case with Dirichlet or Neumann boundary conditions can be treated in the same way.

The same arguments used in the proof of Theorem 5.3.4 show that Hypothesis 5.2.1 holds true.

Moreover, since $\sup_{r\in\mathbb{R}} \gamma(r) < \dfrac{1}{2}$ by (5.98) and (5.101) for every $r \in \mathbb{R}$ we have

$$\mathrm{H}_r := Q(r)^{\frac{1}{2}}(X) = D((-\Delta)^{\gamma(r)}) = (X, D(A(r)))_{\gamma,2} = H^{2\gamma}(\mathcal{O}). \tag{5.107}$$

It follows that for every $q \geqslant 2$, the embedding of $D((-A(r))^{\gamma})$ in $L^q(\mathcal{O})$ is continuous for $\gamma \geqslant 2d\left(\frac{1}{2} - \frac{1}{q}\right)$. Hence for such choices of γ, $(-A(r))^{-\gamma} \in \mathcal{L}(L^2(\mathcal{O}); L^q(\mathcal{O}))$.

So by Proposition 5.3.3, for every $s < t$ the operator $Q(t, s)$ given by (2.2) has finite trace in all cases (5.105). Hypothesis 5.2.1 holds true in view of Proposition 5.3.2 and for every $s < t$, by (5.96)(with $\theta = \gamma(t)$ and $\rho = \gamma(s)$) and (5.107) we have

$$\|U(t, s)\|_{\mathcal{L}(\mathrm{H}_s,\mathrm{H}_t)} \leqslant C \max\{1, (t - s)^{-(\gamma(t)-\gamma(s))}\} e^{-(\omega-\delta_0)(t-s)},$$

so recalling that $\gamma : \mathbb{R} \to [0, \alpha]$ is a non-decreasing continuous function and $0 < \alpha < \dfrac{1}{2}$ we obtain (5.106). □

Under the assumptions of Theorem 5.3.4 or Theorem 5.3.6, by Theorem 5.1.6 there exists an evolution system of measures $\{\gamma_t\}_{t\in\mathbb{R}}$ for $P_{s,t}$ given by (5.18). Such evolution system of measures satisfies the Logaritmic-Sobolev inequalities (5.31) and $P_{s,t}$ is hypercontractive as stated in Theorem 5.2.7.

5.3.2 Diagonal operators

Let $(X, \|\cdot\|_X, \langle\cdot,\cdot\rangle_X)$ be a real separable Hilbert space. Let $t \in \mathbb{R}$ and let $A(t)$, $B(t)$ be self-adjoint operators in diagonal form with respect to the same Hilbert basis $\{e_k : k \in \mathbb{N}\}$, namely

$$A(t)e_k = a_k(t)e_k, \quad B(t)e_k = b_k(t)e_k \quad t \in \mathbb{R}, \quad k \in \mathbb{N},$$

with continuous coefficients a_k, b_k. We set $\lambda_k = \sup_{t\in\mathbb{R}} a_k(t)$ and we assume that there exists $\lambda_0 \in \mathbb{R}$ such that $\lambda_k \leqslant \lambda_0, \quad \forall\, k \in \mathbb{N}$.

In this setting the operator $U(t,s)$ defined by

$$U(t,s)e_k = \exp\left(\int_s^t a_k(\tau)\,d\tau\right)e_k, \quad (s,t) \in \Delta, \quad k \in \mathbb{N}, \tag{5.108}$$

is the strongly continuous evolution operator associated to the family $\{A(t)\}_{t\in\mathbb{R}}$. Moreover we assume that there exists $K > 0$ such that

$$|b_k(t)| \leqslant K, \quad t \in \mathbb{R},\ k \in \mathbb{N}.$$

Hence $B(t) \in \mathcal{L}(X)$ for all $t \in \mathbb{R}$, the function $B : \mathbb{R} \longmapsto \mathcal{L}(X)$ is continuous and

$$\sup_{t\in\mathbb{R}} \|B(t)\|_{\mathcal{L}(X)} \leqslant K. \tag{5.109}$$

The operators $Q(t,s)$ are given by

$$Q(t,s)e_k = \int_s^t \exp\left(2\int_\sigma^t a_k(\tau)\,d\tau\right)(b_k(\sigma))^2\,d\sigma\, e_k =: q_k(t,s)e_k, \quad (s,t) \in \Delta,\ k \in \mathbb{N}.$$

Hypothesis 4.0.1 is fulfilled if

$$\sum_{k=0}^{\infty} q_k(t,s) < +\infty, \quad (s,t) \in \Delta. \tag{5.110}$$

We give now a sufficient condition for (5.110) to hold. We assume that λ_k is eventually nonzero (say for $k \geqslant k_0$). Given $(s,t) \in \Delta$, we have

$$\begin{aligned} &\left|\int_s^t \exp\left(2\int_\sigma^t a_k(\tau)\,d\tau\right)(b_k(\sigma))^2\,d\sigma\right| \leqslant \|b_k\|_\infty^2 \left|\int_s^t \exp\big(2\lambda_k(t-\sigma)\big)\,d\sigma\right| \\ &= \frac{\|b_k\|_\infty^2}{2\,|\lambda_k|}\,|1 - \exp(2\lambda_k(t-s))| \leqslant \frac{\|b_k\|_\infty^2}{2\,|\lambda_k|}\big(1 + \exp(2\lambda_0(t-s))\big). \end{aligned} \tag{5.111}$$

Hence (5.110) holds if we require

$$\sum_{k=k_0}^{\infty} \frac{\|b_k\|_\infty^2}{|\lambda_k|} < +\infty. \tag{5.112}$$

We note that $D(A(t)) = D(A(t)^\star) = X$ for all $t \in \mathbb{R}$ and Hypothesis 4.2.1 is easily checked.

In order to have existence of an evolution system of measures for $P_{s,t}$, the hypotheses of Theorem 5.1.6 are satisfied if we require $\lambda_0 \leqslant 0$ in (5.111). Moreover if we require $\lambda_0 < 0$, (4.5) holds with $\zeta < 0$ and by Theorem 5.1.7 there exists a unique system of measures for $P_{s,t}$.

Now we investigate when (5.30) holds. We observe first that for $y = Q(s)^{\frac{1}{2}} x$ with $x \in H_s$, we have

$$\|U(t,s)y\|_{H_t}^2 = \left\|Q(t)^{-\frac{1}{2}} U(t,s) Q(s)^{\frac{1}{2}} x\right\|_X^2 = \sum_{k\in\mathbb{N}} \left(\frac{|b_k(s)|}{|b_k(t)|} \exp\left(\int_s^t a_k(\tau) \right) \langle x, e_k \rangle \right)^2$$
$$= \sum_{k\in\mathbb{N}} \left(\frac{|b_k(s)|}{|b_k(t)|} \exp\left(\int_s^t a_k(\tau) \right) \frac{\langle y, e_k\rangle}{|b_k(s)|} \right)^2 = \sum_{k\in\mathbb{N}} \left(\frac{1}{|b_k(t)|} \exp\left(\int_s^t a_k(\tau) \langle y, e_k\rangle_{H_s} \right) \right)^2.$$

We assume that there exist $L > 0$ such that

$$|b_k(t)| \geqslant L, \quad k \in \mathbb{N}, \quad t \in \mathbb{R}. \tag{5.113}$$

Hence for any $k \in \mathbb{N}$, we have

$$\frac{1}{b_k^2(t)} \exp\left(\int_s^t 2a_k(\tau)\, d\tau \right) \leqslant \frac{1}{L^2} e^{2\lambda_0 (t-s)}, \tag{5.114}$$

and

$$\|U(t,s)\|_{\mathcal{L}(H_s, H_t)} \leqslant \frac{1}{L} e^{\lambda_0 (t-s)}, \quad (s,t) \in \Delta. \tag{5.115}$$

Since $\lambda_0 < 0$, hypotheses of Theorems 5.2.2 and 5.2.7 are satisfied.

Theorem 5.3.7. *Let $(X, \|\cdot\|_X, \langle\cdot,\cdot\rangle_X)$ be a separable Hilbert space. Let $A(t)$, $B(t)$ be self-adjoint operators in diagonal form with respect to the same Hilbert basis $\{e_k\}_{k\in\mathbb{N}}$, namely*

$$A(t)e_k = a_k(t)e_k, \quad B(t)e_k = b_k(t)e_k \quad t \in \mathbb{R}, \quad k \in \mathbb{N},$$

with continuous coefficients a_k, b_k. We set $\lambda_k = \sup_{t\in\mathbb{R}} a_k(t)$. If that there exists $\lambda_0 \in \mathbb{R}$ such that $\lambda_k \leqslant \lambda_0$, $\forall\, k \in \mathbb{N}$ and (5.109), (5.112) hold, then Hypothesis 4.0.1 and Hypothesis 4.2.1 are satisfied with $U(t,s)$ given by (5.108). If in addition $\lambda_0 < 0$, then there exists an evolution system of measure for $P_{s,t}$ given by (5.18). Finally assuming also (5.113), hypotheses of Theorems 5.2.2 and 5.2.7 are satisfied.

As an explicit example we can choose $a_k(t) = -\dfrac{k^2 + c_1}{t^{2k} + 1}$, $c_1 > 0$ and $b_k(t) = \sin(kt) + c_2$, $c_2 > 1$ for all $t \in \mathbb{R}$.

Remark 5.3.8. We assume now that $\max_{t\in\mathbb{R}} a_1(t) = 0$, $a_1 \in L^1(\mathbb{R})$ and there exists $c > 0$ such that $\lambda_k < -c$ for $k \geqslant 2$. In this case $\lambda_0 = 0$ and we can show that there exist at least two different evolution system of measures uniformly tight for $P_{s,t}$. Setting

$$V_{s,t}\varphi(x) = \varphi(U(t,s)x), \quad \varphi \in C_b(X), \; x \in X, \; s \leqslant t, \tag{5.116}$$

we show first that there exist at least two evolution system of measures for $V_{s,t}$. We claim that the families $\{\mu_t^{(1)}\}_{t\in\mathbb{R}}$ and $\{\mu_t^{(2)}\}_{t\in\mathbb{R}}$ defined by $\mu_t^{(1)} \equiv \delta_0$ and $\mu_t^{(2)} = \delta_{\frac{e_1}{m_t}}$ with $m_t = e^{\int_{-\infty}^t a_1(\tau)\, d\tau}$ for all $t \in \mathbb{R}$, are evolution system of measures for $V_{s,t}$. Indeed given $\varphi \in C_b(X)$, we have

$$\int_X \varphi\left(U(t,s)x\right)\, \delta_0(dx) = \varphi\left(U(t,s)0\right) = \varphi(0) = \int_X \varphi\left(x\right)\, \delta_0(dx),$$
$$\int_X \varphi\left(U(t,s)x\right)\, \delta_{\frac{e_1}{m_t}}(dx) = \varphi\left(U(t,s)\frac{e_1}{m_t} \right) = \varphi\left(\frac{e_1}{m_s} \right) = \int_X \varphi\left(x\right)\, \delta_{\frac{e_1}{m_s}}(dx).$$

Hence, by Theorem 5.1.6, $\{\gamma_t\}_{t\in\mathbb{R}}$ with γ_t given by (5.18) and $\{\nu_t\}_{t\in\mathbb{R}}$ with $\nu_t = \gamma_t \star \mu_t^{(2)}$ are evolutions systems of measures for $P_{s,t}$.

5.3.3 A non autonomous version of the classical Ornstein-Uhlenbeck operator

Let $A(t) = a(t)I$, where a is a continuous and bounded real valued map on $\mathbb{R}$ and set $\sup\limits_{t\in\mathbb{R}} a_0(t) = a_0$. Hence

$$U(t,s) = \exp\left(\int_s^t a(\tau)\,d\tau\right)I, \quad (s,t) \in \mathbb{R}^2 \tag{5.117}$$

is continuous with values in $\mathcal{L}(X)$ and it is associated to the family $\{A(t)\}_{t\in\mathbb{R}}$.

Let $\{B(t)\}_{t\in\mathbb{R}} \subseteq \mathcal{L}(X)$ be a family of operators satisfying Hypothesis 4.0.1 (2). Since

$$Q(t,s) = \int_s^t \exp\left(2\int_\sigma^t a(\tau)\,d\tau\right)Q(\sigma)\,d\sigma \quad (s,t) \in \Delta,$$

where $Q(\sigma) = B(\sigma)B(\sigma)^\star$, a sufficient and obvious condition for $\mathrm{Tr}\,[Q(t,s)] < +\infty$ is that the mapping $\sigma \longmapsto \mathrm{Tr}\,[Q(\sigma)]$ belongs to $L^1(\mathbb{R})$. Indeed, if $\{e_k\}_{k\in\mathbb{N}}$ is a Hilbert basis of X, we have

$$\langle Q(t,s)e_k, e_k\rangle_X \leqslant e^{2a_0(t-s)}\int_s^t \langle Q(\sigma)e_k, e_k\rangle_X\,d\sigma \quad (s,t) \in \Delta. \tag{5.118}$$

We note that $D(A(t)) = D(A(t)^\star) = X$ for all $t \in \mathbb{R}$ and Hypothesis 4.2.1 is obviously satisfied.

In addition to the above assumptions on the trace of the operators $Q(\sigma)$, we require that for all $t \in \mathbb{R}$ there exists $C_t > 0$ such that

$$\|B(s)\|_{\mathcal{L}(X)} \leqslant C_t\,\|B(t)\|_{\mathcal{L}(X)}, \quad \forall\, s < t. \tag{5.119}$$

Moreover we assume also that $a_0 < 0$.

By (5.118) and (5.119), (4.5) holds and

$$\|U(t,s)\|_{\mathcal{L}(X)} \leqslant e^{a_0(t-s)}, \quad (s,t \in \mathbb{R}^2).$$

For all $(s,t) \in \Delta$ we have

$$\begin{aligned}\langle Q(t,s)e_k, e_k\rangle_X &\leqslant e^{2a_0(t-s)}\int_s^t \langle Q(\sigma)e_k, e_k\rangle_X\,d\sigma = e^{2a_0(t-s)}\int_s^t \|B(\sigma)^\star e_k\|_X^2\,d\sigma\\ &\leqslant e^{2a_0(t-s)}(t-s)C_t^2\,\|B(t)^\star e_k\|_X^2 = C_t^2 e^{2a_0(t-s)}(t-s)\langle Q(t)e_k, e_k\rangle_X.\end{aligned} \tag{5.120}$$

Therefore

$$\sup_{s<t} \mathrm{Tr}\,[Q(t,s)] < +\infty$$

and by Theorem 5.1.7 there exists a unique evolution system of measures for $P_{s,t}$.

Moreover (5.30) holds. Indeed

$$\left\|Q\,(s)^{\frac{1}{2}}\right\|_{\mathcal{L}(X)}^2 = \|B(s)^\star\|_{\mathcal{L}(X)}^2 \leqslant C_t^2\,\|B(t)^\star\|_{\mathcal{L}(X)}^2 = C_t^2\left\|Q\,(t)^{\frac{1}{2}}\right\|_{\mathcal{L}(X)}^2. \tag{5.121}$$

Then, by Proposition B.1 in Appendix B in [DZ14, pag. 429], $H_s \subseteq H_t$ with continuous embedding and for $x \in H_s$ we have

$$\|U(t,s)x\|_{H_t} = \left\|\exp\left(\int_s^t a(\tau)\,d\tau\right)x\right\|_{H_t} \leqslant C_t e^{a_0(t-s)} \|x\|_{H_s}. \tag{5.122}$$

Since $a_0 < 0$, the hypotheses of Theorems 5.2.2 and 5.2.7 are satisfied.

Theorem 5.3.9. *Let $(X, \|\cdot\|_X, \langle\cdot,\cdot\rangle_X)$ be a separable Hilbert space. Let $A(t) = a(t)I$, where a is a continuous and bounded real valued map on $\mathbb{R}$ and set*

$$\sup_{t\in\mathbb{R}} a_0(t) = a_0.$$

Let $\{B(t)\}_{t\in\mathbb{R}} \subseteq \mathcal{L}(X)$ be a family of operators satisfying Hypothesis 4.0.1 (2).

If (5.119) holds, then Hypothesis 4.0.1 and Hypothesis 4.2.1 are satisfied with $U(t,s)$ given by (5.117). If in addition $a_0 < 0$, then there exists an evolution system of measure for $P_{s,t}$ given by (5.18). Moreover hypotheses of Theorems 5.2.2 and 5.2.7 are satisfied.

Remark 5.3.10. We state here some final remarks.

1. If $a(t) = -1$ and Range $\left(Q(t)^{\frac{1}{2}}\right)$ = Range $\left(Q(0)^{\frac{1}{2}}\right)$ for every $t \in \mathbb{R}$, then $a_0 = -1$ and $c_t = 1$ in (5.122) so κ defined in (5.32) is equal to $\dfrac{1}{2}$. Moreover (5.53) holds for every $(s,t) \in \Delta$ and $p \leqslant C(t,s,q) := (q-1)e^{t-s} + 1$. We stress that $C(t,s,q)$ is non autonomous version of the optimal constant given in [Gro75, Rmk. 3.4] and [Fuh98, p. 242 under (1.3)].

2. We now give an explicit example where Range$(B(t))$ = Range$(B(0))$ for every $t \in \mathbb{R}$. Let $d \in \mathbb{N}$, let $X = L^2(0,1)$. We choose $B(t) = (A(t))^{-\frac{1}{2}}$ where $A(t)$ is the realization of $\mathcal{A}(t)$ given by (5.66) with Dirichlet boundary conditions. The proof that $\mathrm{Tr}\left[Q(t,s)\right]$ is finite is the same as in Theorem 5.3.4. In this case Range$(B(t))$ = Range$(B(0)) = H_0^1(0,1)$ for every $t \in \mathbb{R}$.

Chapter 6

The Ornstein-Uhlenbeck evolution operator as a second quantized operator

In this Chapter we give an alternative representation formula of $P_{s,t}$ as a second quantized operator, provided that there exists the evolution system of measure $\{\gamma_t\}_{t\in\mathbb{R}}$ for $P_{s,t}$, where γ_t is given by (5.18). We start by defining the operator $L \in \mathcal{L}(X) \longrightarrow \Gamma_{\mu,\nu}(L)$ where μ and ν are two centered (possibly degenerate) Gaussian measure on X.

The second quantization operator was introduced in [Sim74] and we refer to [Nel73; FP94; CG96; Jan97] for several applications to the Ornstein-Uhlenbeck semigroup represented as a second quantized operator. In [Nel73; CG96; Jan97; Sim74] the second quantization operator is defined as a series using the Wiener Chaos decomposition. In this Chapter we follow the approach of [FP94] where the authors represented the second quantization operator by the so called generalized Mehler formula. Our aim is to make it simpler.

6.1 The second quantization operator

Let $\mu = \mathcal{N}(0, Q_\mu)$ and $\nu = \mathcal{N}(0, Q_\nu)$ be two centered Gaussian measures on X with Cameron-Martin spaces H_μ and H_ν, respectively. We recall that $H_\mu = Q_\mu^{\frac{1}{2}}(X)$ and $H_\nu = Q_\nu^{\frac{1}{2}}(X)$ and we denote by P_μ e P_ν the ortogonal projections on $\ker Q_\mu$ and on $\ker Q_\nu$, respectively. We fix $L \in \mathcal{L}(X)$ and we consider the linear operator $Q_\nu^{\frac{1}{2}} L Q_\mu^{-\frac{1}{2}} : H_\mu \to X$. If $\{e_n^\mu\}_{n\in\mathbb{N}}$ and $\{e_n^\nu\}_{n\in\mathbb{N}}$ are orthonormal bases of X consisting of eigenvectors of Q_μ and Q_ν, with corresponding eigenvalues $\{\lambda_n^\mu\}_{n\in\mathbb{N}}$ and $\{\lambda_n^\nu\}_{n\in\mathbb{N}}$, respectively, then for every $h \in H_\mu$ we get

$$Q_\nu^{\frac{1}{2}} L Q_\mu^{-\frac{1}{2}} h = \sum_{n=1}^{\infty} (\lambda_n^\nu)^{\frac{1}{2}} \sum_{\substack{k=1 \\ \lambda_k^\mu \neq 0}}^{\infty} \frac{\langle e_k^\mu, h\rangle_X}{(\lambda_k^\mu)^{\frac{1}{2}}} \langle e_n^\nu, L e_k^\mu\rangle_X e_n^\nu.$$

The function $h \longmapsto Q_\nu^{\frac{1}{2}} L Q_\mu^{-\frac{1}{2}} h$ belongs to $\mathcal{L}(H_\mu, X)$. We shall study an extension of it belonging to $L^2(X, \mu; X)$. For this purpose, we introduce the linear approximations $L^{\nu,\mu}_{\infty,m} : X \to X$, $m \in \mathbb{N}$, defined as

$$L^{\nu,\mu}_{\infty,m} x = \sum_{n=1}^{\infty} (\lambda^\nu_n)^{\frac{1}{2}} \sum_{\substack{k=1 \\ \lambda^\mu_k \neq 0}}^{m} \frac{\langle e^\mu_k, x\rangle_X}{(\lambda^\mu_k)^{\frac{1}{2}}} \langle e^\nu_n, L e^\mu_k\rangle_X e^\nu_n, \qquad \forall x \in X.$$

We notice that the operators $L^{\nu,\mu}_{\infty,m}$ are well-defined for every $m \in \mathbb{N}$. Further, the sequence $\{L^{\nu,\mu}_{\infty,m}\}_{m\in\mathbb{N}}$ converges in $L^2(X, \mu; X)$, as stated in the following result.

Proposition 6.1.1. *The sequence $\{L^{\nu,\mu}_{\infty,m}\}_{m\in\mathbb{N}}$ converges in $L^2(X, \mu; X)$ as m goes to ∞. Its limit $L^{\nu,\mu}_{\infty}$ is a centered X-valued Gaussian random variable with law*

$$\mu \circ (L^{\nu,\mu}_\infty)^{-1} = \mathcal{N}(0, Q_\nu^{\frac{1}{2}} L(I - P_\mu) L^\star Q_\nu^{\frac{1}{2}}). \tag{6.1}$$

Moreover

$$\int_X |L^{\nu,\mu}_\infty x|^2_X \mu(dx) \leqslant \mathrm{Tr}_X[Q_\nu L L^\star],$$

and for every $h \in X$ we get

$$\langle L^{\nu,\mu}_\infty \cdot, h\rangle_X = (Q_\mu^{\frac{1}{2}} L^\star Q_\nu^{\frac{1}{2}} h)\hat{}. \tag{6.2}$$

Proof. We show that $\{L^{\nu,\mu}_{\infty,m}\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $L^2(X, \mu; X)$. For every $m, h \in \mathbb{N}$ with $h > m$ we get

$$\begin{aligned}
\|L^{\nu,\mu}_{\infty,m} - L^{\nu,\mu}_{\infty,h}\|^2_{L^2(X,\mu;X)} &= \int_X \left\| \sum_{n=1}^{\infty} (\lambda^\nu_n)^{\frac{1}{2}} \left(\sum_{\substack{k=m+1, \\ \lambda^\mu_k \neq 0}}^{h} \frac{\langle e^\mu_k, x\rangle_X}{(\lambda^\mu_k)^{\frac{1}{2}}} \right) \langle e^\nu_n, L e^\mu_k\rangle_X e^\nu_n \right\|^2_X \mu(dx) \\
&= \int_X \sum_{n=1}^{\infty} \lambda^\nu_n \left(\sum_{\substack{k=m+1, \\ \lambda^\mu_k \neq 0}}^{h} \langle e^\nu_n, L e^\mu_k\rangle_X \frac{\langle e^\mu_k, x\rangle_X}{(\lambda^\mu_k)^{\frac{1}{2}}} \right)^2 \mu(dx), \qquad (6.3)
\end{aligned}$$

where we have used the fact that $\{e^\nu_n\}_{n\in\mathbb{N}}$ is an orthonormal system in X. Now we recall that

$$\int_X \langle e^\mu_k, x\rangle_X \langle e^\mu_j, x\rangle^2_X \mu(dx) = 0, \quad k \neq j, \tag{6.4}$$

$$\int_X \langle e^\mu_k, x\rangle^2_X \mu(dx) = \frac{1}{\sqrt{2\pi\lambda^\mu_k}} \int_{\mathbb{R}} \xi^2 e^{-\frac{1}{2\lambda^\mu_k}\xi^2} d\xi = \lambda^\mu_k, \quad \forall\, k \in \mathbb{N}. \tag{6.5}$$

By replacing (6.5) and (6.4) in (6.3) we infer

$$\|L^{\nu,\mu}_{\infty,m} - L^{\nu,\mu}_{\infty,h}\|^2_{L^2(X,\mu;X)} = \sum_{n=1}^{\infty} \lambda^\nu_n \sum_{\substack{k=m+1, \\ \lambda^\mu_k \neq 0}}^{h} \langle e^\nu_n, L e^\mu_k\rangle^2_X = \sum_{n=1}^{\infty} \lambda^\nu_n \sum_{\substack{k=m+1, \\ \lambda^\mu_k \neq 0}}^{h} \langle L^\star e^\nu_n, e^\mu_k\rangle^2_X.$$

Moreover

$$\sum_{\substack{k=m+1,\\ \lambda_k^\mu\neq 0}}^{h} \langle L^\star e_n^\nu, e_k^\mu\rangle_X^2 \leqslant \sum_{k=m+1}^{h} \langle L^\star e_n^\nu, e_k^\mu\rangle_X^2,$$

that vanishes as $m, n \to +\infty$. Since

$$\sum_{k=m+1}^{h} \langle L^\star e_n^\nu, e_k^\mu\rangle_X^2 \leqslant \sum_{k=1}^{h}\langle L^\star e_n^\nu, e_k^\mu\rangle_X^2 = \|L^\star e_h\|_X^2\,,$$

by the dominated convergence theorem for series we have that $\{L^{\nu,\mu}_{\infty,m}\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $L^2(X,\mu;X)$ and then it converges in $L^2(X,\mu;X)$.

Let us denote by $L^{\nu,\mu}_\infty$ its limit. Then, arguing as above and recalling that $(\lambda_n^\nu)^{\frac12} e_n^\nu = Q_\nu^{\frac12} e_n^\nu$ for every $n\in\mathbb{N}$, we get

$$\begin{aligned}\|L^{\nu,\mu}_\infty\|^2_{L^2(X,\mu;X)} &= \int_X |L^{\nu,\mu}_\infty x|_X^2\mu(dx) \leqslant \sum_{n=1}^\infty \lambda_n^\nu \sum_{k=1}^\infty\langle L^\star e_n^\nu, e_k^\mu\rangle_X^2 = \sum_{n=1}^\infty \lambda_n^\nu\, \|L^\star e_n^\nu\|_X^2\\ &= \sum_{n=1}^\infty \langle \lambda_n^\nu L L^\star e_n^\nu, e_n^\nu\rangle_X = \mathrm{Tr}[Q_\nu L L^\star].\end{aligned}$$

Let us fix $y\in X$. For all $n\in\mathbb{N}$, we set $y_n^\nu = \langle y, e_n^\nu\rangle_X$ and we have

$$\langle L^{\nu,\mu}_{\infty,m}x, h\rangle_X = \sum_{n=1}^\infty (\lambda_n^\nu)^{\frac12} \sum_{\substack{k=1\\ \lambda_k^\mu\neq 0}}^{m} \frac{\langle e_k^\mu, x\rangle_X}{(\lambda_k^\mu)^{\frac12}}\langle L^\star e_n^\nu, e_k^\mu\rangle_X y_n^\nu = \sum_{\substack{k=1\\ \lambda_k^\mu\neq 0}}^{m} \frac{\langle e_k^\mu, x\rangle_X}{(\lambda_k^\mu)^{\frac12}}\langle L^\star Q_\nu^{\frac12} y, e_k^\mu\rangle_X,$$

which converges to $(Q_\mu^{\frac12} L^\star Q_\nu^{\frac12} h)\hat{}$ in $L^2(X,\mu)$ by Proposition 1.5.19. Hence (6.2) follows, $\langle L^{\nu,\mu}_\infty\cdot, h\rangle_X$ equals $(Q_\mu^{\frac12} L^\star Q_\nu^{\frac12} h)\hat{}$, and so it is a centered X-valued Gaussian random variable with variance $\left\|Q_\mu^{\frac12} L^\star Q_\nu^{\frac12} h\right\|^2_{H_\mu} = \left\|(I-P_\mu)L^\star Q_\nu^{\frac12} h\right\|^2_X$.

To conclude, we prove that the law of $L^{\nu,\mu}_\infty$ is given by (6.1). Indeed, for every $h\in X$ we get

$$\begin{aligned}(\mu\circ \widehat{(L^{\nu,\mu}_\infty})^{-1})(h) &= \int_X e^{i\langle x,h\rangle_X}(\mu\circ (L^{\nu,\mu}_\infty)^{-1})(dx) = \int_X e^{i\langle L^{\nu,\mu}_\infty x,h\rangle_X}\mu(dx)\\ &= \exp\left(-\frac12 \left\|(Q_\mu^{\frac12} L^\star Q_\nu^{\frac12} h)\hat{}\right\|^2_{L^2(X,\mu)}\right) = \exp\left(-\frac12 \left\|Q_\mu^{\frac12} L^\star Q_\nu^{\frac12} h\right\|^2_{H_\mu}\right)\\ &= \exp\left(-\frac12 \left\|(I-P_\mu)L^\star Q_\nu^{\frac12} h\right\|^2_X\right) = \exp\left(-\frac12 \langle Q_\nu^{\frac12} L(I-P_\mu)L^\star Q_\nu^{\frac12} h, h\rangle_X\right).\end{aligned}$$

$\square$

Remark 6.1.2. From now on, if $\mu=\nu$ we simply write L^μ_∞ instead of $L^{\mu,\mu}_\infty$.

Let $L\in\mathcal{L}(X)$ with $\|L\|_{\mathcal{L}(X)}\leqslant 1$. Hence, $(I-L^\star L)^{\frac12}\in\mathcal{L}(X)$ is self-adjoint, nonnegative and has operator norm less than or equal to 1.

Definition 6.1.3. Let $L \in \mathcal{L}(X)$ with $\|L\|_{\mathcal{L}(X)} \leqslant 1$. For every $f \in B_b(X)$ we set

$$(\Gamma_{\mu,\nu}(L)f)(x) = \int_X f[(L^\star)^{\nu;\mu}_\infty x + ((I - L^\star L)^{\frac{1}{2}})^\nu_\infty y]\,\nu(dy), \quad \text{for } \mu \text{ a.e. } x \in X \tag{6.6}$$

If $\mu = \nu$, then $L \longmapsto \Gamma_\mu(L) := \Gamma_{\mu,\mu}(L)$ coincides with the second quantization operator on $L^2(X, \mu; X)$. Formula (6.6) is called generalized Mehler formula.

Setting $z = ((I - L^\star L)^{\frac{1}{2}})^\nu_\infty y$, by Proposition 6.1.1 we obtain

$$(\Gamma_{\mu,\nu}(L)f)(x) = \int_X f((L^\star)^{\nu;\mu}_\infty x + z)\,\gamma_{\nu,L}(dz), \tag{6.7}$$

where

$$\gamma_{\nu,L} = \mathcal{N}(0, Q_\nu^{\frac{1}{2}}(I - L^\star L)^{\frac{1}{2}}(I - P_\nu)(I - L^\star L)^{\frac{1}{2}}Q_\nu^{\frac{1}{2}}). \tag{6.8}$$

If Q_ν is one to one then $P_\nu = 0$ and $\gamma_{\nu,L} = \mathcal{N}(0, Q_\nu^{\frac{1}{2}}(I - L^\star L)Q_\nu^{\frac{1}{2}})$. The same holds if L vanishes on $\ker Q_\nu$. Indeed $LP_\nu = 0$ if and only if $P_\nu L^\star = 0$ and $I - L^\star L$ maps $(I - P_\nu)(X)$ into itself. Hence, even $(I - L^\star L)^{\frac{1}{2}}$ maps $(I - P_\nu)(X)$ into itself. So if $LP_\nu = 0$ we get

$$(\Gamma_{\mu,\nu}(L)f)(x) = \int_X f((L^\star)^{\nu;\mu}_\infty x + z)\,\mathcal{N}(0, Q_\nu^{\frac{1}{2}}(I - L^\star L)Q_\nu^{\frac{1}{2}})(dz), \tag{6.9}$$

Remark 6.1.4. We compute here the second quantization operator for two special operators L.

(i). If $L = 0$, then $(L^\star)^{\nu;\mu}_\infty = 0$ and $((I - L^\star L)^{\frac{1}{2}})^\nu_\infty = I - P_\nu$. Hence

$$(\Gamma_{\mu,\nu}(0)f)(x) = \int_X f((I - P_\nu)y)\,\nu(dy) = \int_X f(y)\,\nu(dy) \qquad \mu \text{ a.e. } x \in X,\ \forall f \in B_b(X),$$

i.e., $\Gamma_{\mu,\nu}(0)f$ is constant for μ *a.e.* $x \in X$. Such a constant coincides with the mean of f on X with respect to ν.

(ii). If $L = c\,I$ with $c \in (0,1]$, then $L^\mu_\infty = c\,(I - P_\mu)$ and $((I - L^\star L)^{\frac{1}{2}})^\mu_\infty = \sqrt{1 - c^2}(I - P_\mu)$. Therefore

$$(\Gamma_\mu(cI)f)(x) = \int_X f[c\,(I - P_\mu)x + \sqrt{1 - c^2}\,y]\,\mu(dy) \qquad \mu \text{ a.e. } x \in X,\ f \in B_b(X),$$

so that, if $c = 1$ we get $(\Gamma_\mu(I)f)(x) = f((I - P_\mu)x)$ for μ *a.e.* $x \in X$, while if $c \in (0,1)$ then

$$(\Gamma_\mu(cI)f)(x) = \int_X f(e^{-t}(I - P_\mu)x + \sqrt{1 - e^{-2t}}y)\mu(dy) \qquad \mu \text{ a.e. } x \in X,\ f \in B_b(X),$$

where $c = e^{-t}$. We get that $\Gamma_\mu(e^{-t})f = (P^\mu_t f) \circ (I - P_\mu)$ where P^μ_t is the classical Ornstein-Uhlenbeck semigroup in X with respect to the Gaussian measure μ, i.e.

$$P^\mu_t f(x) = \int_X f(e^{-t}x + \sqrt{1 - e^{-2t}}y)\mu(dy) \qquad \mu \text{ a.e. } x \in X,\ f \in B_b(X).$$

Proposition 6.1.5. *Let $L \in \mathcal{L}(X) \smallsetminus \{0\}$ be such that $\|L\|_{\mathcal{L}(X)} \leqslant 1$, $LP_\nu = 0$ and $P_\mu L = 0$. Then for every $p \in [1, +\infty)$ the operator $\Gamma_{\mu,\nu}(L)$ is extendable to a linear continuous operator from $L^p(X, \nu)$ to $L^p(X, \mu)$ with unit norm.*

Proof. Let $L \in \mathcal{L}(X)$ with $\|L\|_{\mathcal{L}(X)} \leqslant 1$, let us fix $p \in [1, +\infty)$ and let $f \in C_b(X)$. By (6.9), for all $x \in X$ we get

$$\begin{aligned}\int_X |\Gamma_{\mu,\nu}(L)f)(x)|^p \mu(dx) &= \int_X \left| \int_X f((L^\star)^{\nu,\mu}_\infty x + z)\gamma_{\nu,L}(dz) \right|^p \mu(dx) \\ &\leqslant \int_X \int_X |f((L^\star)^{\nu,\mu}_\infty x + z)|^p \gamma_{\nu,L}(dz)\, \mu(dx)\end{aligned}$$

where $\gamma_{\nu,L}$ is given by (6.8). Since $LP_\nu = 0$ and $P_\mu L = 0$, by Proposition 1.5.18 (i) we have

$$\int_X \int_X |f((L^\star)^{\nu,\mu}_\infty x + z)|^p \gamma_{\nu,L}(dz)\mu(dx) = \int_X |f(x)|^p \nu(dx).$$

This implies that $\|\Gamma_{\mu,\nu}(L)f\|_{L^p(X,\mu)} \leqslant \|f\|_{L^p(X,\nu)}$ for every $f \in C_b(X)$. The statement follows from the density of $C_b(X)$ in $L^p(X,\nu)$. Taking $f \equiv 1$, $\Gamma_{\mu,\nu}(L)f \equiv 1$ so that the norm of the extension is 1. $\square$

Definition 6.1.6. We still denote by $\Gamma_{\mu,\nu}$ the extension of the second quantization operator to $L^p(X,\nu)$.

Proposition 6.1.7. *Let $L \in \mathcal{L}(X) \smallsetminus \{0\}$ be such that $\|L\|_{\mathcal{L}(X)} \leqslant 1$, $LP_\nu = 0$. For every $c \in (0,1]$ and $f \in B_b(X)$, we have*

$$\Gamma_{\mu,\nu}(cL)f = \Gamma_{\mu,\nu}(L)\Gamma_\nu(cI)f. \tag{6.10}$$

If in addition $P_\mu L = 0$, then

$$\Gamma_{\mu,\nu}(cL)f = \Gamma_\mu(cI)\Gamma_{\mu,\nu}(L)f. \tag{6.11}$$

Moreover (6.10) and (6.11) hold also for every $f \in L^p(X,\nu)$.

Proof. Let $f \in B_b(X)$ and $x \in X$. We set

$$\gamma_{\nu,c} := \mathcal{N}(0, (1-c^2)Q_\nu), \quad \gamma_{\nu,L} := \mathcal{N}(0, Q_\nu^{\frac{1}{2}}(I - L^\star L)Q_\nu^{\frac{1}{2}}),$$

and by (6.9), we get

$$(\Gamma_\nu(cI)f)(x) = \int_X f(c(I-P_\nu)x + z)\gamma_{\nu,c}(dz),$$

$$(\Gamma_{\mu,\nu}(L)\Gamma_\nu(cI)f)(x) = \int_X \int_X f[c(I-P_\nu)(L^\star)^{\nu,\mu}_\infty x + c(I-P_\nu)w + z]\, \gamma_{\nu,c}(dz)\gamma_{\nu,L}(dw).$$

We notice that

$$\gamma_{\nu,L} \circ (c(I-P_\nu))^{-1} = \mathcal{N}(0, c^2 Q_\nu^{\frac{1}{2}}(I - L^\star L)Q_\nu^{\frac{1}{2}}) \tag{6.12}$$

and by Proposition 1.5.18 (ii) we have

$$\mathcal{N}(0,(1-c^2)Q_\nu) * \mathcal{N}(0, c^2 Q_\nu^{\frac{1}{2}}(I - L^\star L)Q_\nu^{\frac{1}{2}}) = \mathcal{N}(0, Q_\nu^{\frac{1}{2}}(I - c^2 L^\star L)Q_\nu^{\frac{1}{2}}).$$

By (6.12) and by Proposition 1.5.18 (ii) and (iii), we get

$$
\begin{aligned}
(\Gamma_{\mu,\nu}(L)\Gamma_\nu(cI)f)(x) =& \int_X\int_X f[c(I-P_\nu)(L^\star)^{\nu,\mu}_\infty x + c(I-P_\nu)w + z]\\
&\mathcal{N}(0,(1-c^2)Q_\nu)(dz)\,\mathcal{N}(0,Q_\nu^{\frac12}(I-L^\star L)Q_\nu^{\frac12})(dw)\\
=&\int_X\int_X f[c(L^\star)^{\nu,\mu}_\infty x + y + z]\\
&\mathcal{N}(0,(1-c^2)Q_\nu)(dz)\,\mathcal{N}(0,c^2Q_\nu^{\frac12}(I-L^\star L)Q_\nu^{\frac12})(dy)\\
=&\int_X f[c(L^\star)^{\nu,\mu}_\infty x + \xi]\,\mathcal{N}(0,Q_\nu^{\frac12}(I-c^2L^\star L)Q_\nu^{\frac12})(d\xi)\\
=&\Gamma_{\mu,\nu}(cL)f(x).
\end{aligned}
$$

To prove (6.11), we argue in the similar way assuming in addition that $P_\mu L = 0$. Let $f\in B_b(X)$ and $x\in X$. We get

$$
\begin{aligned}
&(\Gamma_\mu(cI)(\Gamma_{\mu,\nu}(L)f))(x)\\
=&\int_X (\Gamma_{\mu,\nu}(L)f)[c(I-P_\mu)x + \sqrt{1-c^2}y]\,\mu(dy)\\
=&\int_X f[(L^\star)^{\nu,\mu}_\infty[c(I-P_\mu)x+\sqrt{1-c^2}\,y] + ((I-L^\star L)^{\frac12})^{\nu}_\infty w]\,\nu(dw)\,\mu(dy)\\
=&\int_X f[c(L^\star)^{\nu,\mu}_\infty x + \sqrt{1-c^2}\,(L^\star)^{\nu,\mu}_\infty y + ((I-L^\star L)^{\frac12})^{\nu}_\infty w]\,\nu(dw)\,\mu(dy).
\end{aligned}
$$

We notice that

$$
\gamma_{\nu,c,L} = \mu\circ(\sqrt{1-c^2}\,(L^\star)^{\nu,\mu}_\infty)^{-1} = \mathcal{N}(0,(1-c^2)Q_\nu^{\frac12}L^\star LQ_\nu^{\frac12})
$$

and

$$
(\Gamma_\mu(cI)(\Gamma_{\mu,\nu}(L)f))(x) = \int_X f[c(L^\star)^{\nu,\mu}_\infty x + \xi + \eta]\,\gamma_{\nu,L}(d\eta)\,\gamma_{\nu,c,L}(d\xi).
$$

By Proposition 1.5.18 (ii) we have

$$
\gamma_{\nu,L}*\gamma_{\nu,c,L} = \mathcal{N}(0,Q_\nu^{\frac12}(I-c^2L^\star L)Q_\nu^{\frac12}),
$$

and (6.11) follows for all $f\in B_b(X)$. Finally if $P_\mu L=0$, then (6.10)and (6.11) hold also for $f\in L^p(X,\nu)$ by the density of $C_b(X)$ in $L^p(X,\nu)$.

□

Proposition 6.1.8. *Let $L\in\mathcal{L}(X)\smallsetminus\{0\}$ be such that $\|L\|_{\mathcal{L}(X)}<1$, $P_\mu L=0$ and $LP_\nu=0$. Then $\Gamma_{\mu,\nu}(L)$ is a contraction from $L^p(X,\nu)$ into $L^q(X,\mu)$, where $q=1+(p-1)\|L\|^{-2}_{\mathcal{L}(X)}$.*

Proof. Let $f\in L^p(X,\nu)$. We consider the operator $M=\|L\|^{-1}_{\mathcal{L}(X)}L$ which belongs to $\mathcal{L}(X)$ and $\|M\|_{\mathcal{L}(X)}=1$. By (6.10) we infer that

$$
\Gamma_{\mu,\nu}(L)f = \Gamma_{\mu,\nu}(M)(\Gamma_\nu(\|L\|_{\mathcal{L}(X)}I)f.
$$

By Remark 6.1.4(ii) $\Gamma_\nu(\|L\|_{\mathcal{L}(X)}I)f = P_t^\nu f \circ (I - P_\nu)$ with $t = -\log(\|L\|_{\mathcal{L}(X)})$. It is well-known (see e.g [Bog98, Thm. 1.6.2], [Nel73]) that P_t^ν is a contraction operator from $L^p(X,\nu)$ into $L^q(X,\nu)$ with $q = 1 + (p-1)e^{2t}$. Therefore, if we choose $q = 1 + (p-1)\|L\|_{\mathcal{L}(X)}^{-2}$, we get

$$\begin{aligned}
\|\Gamma_{\mu,\nu}(L)\|_{\mathcal{L}(L^p(X,\nu);L^q(X,\mu))} =&\|\Gamma_{\mu,\nu}(M)(\Gamma_\nu(\|L\|_{\mathcal{L}(X)}I)\|_{\mathcal{L}(L^p(X,\nu);L^q(X,\mu))}\\
\leqslant&\|\Gamma_{\mu,\nu}(M)\|_{\mathcal{L}(L^q(X,\nu);L^q(X,\mu))}\|\Gamma_\nu(\|L\|_{\mathcal{L}(X)}I)\|_{\mathcal{L}(L^p(X,\nu);L^q(X,\nu))}\\
=&\|\Gamma_{\mu,\nu}(M)\|_{\mathcal{L}(L^q(X,\nu);L^q(X,\mu))}\|P_t^\nu \circ (I-P_\nu)\|_{\mathcal{L}(L^p(X,\nu);L^q(X,\nu))}\\
=&\|\Gamma_{\mu,\nu}(M)\|_{\mathcal{L}(L^q(X,\nu);L^q(X,\mu))}\|P_t^\nu\|_{\mathcal{L}(L^p(X,\nu);L^q(X,\nu))}\\
\leqslant&1
\end{aligned}$$

with $t = -\log(\|L\|_{\mathcal{L}(X)})$. Since $\Gamma_{\mu,\nu}(M)$ is a contraction in $\mathcal{L}(L^q(X,\nu);L^q(X,\mu))$ for every $q \in [1,+\infty)$ and P_t^ν is a contraction from $L^p(X,\nu)$ into $L^q(X,\nu)$ with $q = 1 + (p-1)\|L\|_{\mathcal{L}(X)}^{-2} = 1 + (p-1)e^{2t}$. $\square$

6.2 Applications to the Ornstein-Uhlenbeck evolution operator

In this section we give a representation formula of $P_{s,t}$ in terms of the operator $\Gamma_{\gamma_s,\gamma_t}$ where γ_r is given by (5.18) for all $r \in \mathbb{R}$. Before we need a preliminary result.

Proposition 6.2.1. *We assume that Hypothesis 5.2.1 is verified. The following statements hold for every $(s,t) \in \Delta$.*

(1) $\mathcal{H}_{t,s} \subseteq \mathcal{H}_{t,-\infty}$ with continuous embedding and the norm of the embedding is $\leqslant 1$. Consequently, $\operatorname{Ker}(Q^{\frac{1}{2}}(t,-\infty)) \subseteq \operatorname{Ker}(Q^{\frac{1}{2}}(t,s))$.

(2) $U(t,s)$ maps $\mathcal{H}_{s,-\infty}$ into $\mathcal{H}_{t,-\infty}$ and

$$\|U(t,s)\|_{\mathcal{L}(\mathcal{H}_{s,-\infty};\mathcal{H}_{t,-\infty})} = \left\|Q^{-\frac{1}{2}}(t,-\infty)U(t,s)Q^{\frac{1}{2}}(s,-\infty)\right\|_{\mathcal{L}(X)} \leqslant 1. \tag{6.13}$$

(3) $\mathcal{H}_{t,s} = \mathcal{H}_{t,-\infty}$ if and only if

$$\|U(t,s)\|_{\mathcal{L}(\mathcal{H}_{s,-\infty};\mathcal{H}_{t,-\infty})} = \left\|Q^{-\frac{1}{2}}(t,-\infty)U(t,s)Q^{\frac{1}{2}}(s,-\infty)\right\|_{\mathcal{L}(X)} < 1. \tag{6.14}$$

Proof. The main tool to prove this proposition in Proposition 1.3.3 with $X_1 = X_2 = X$.

(1) Let $x \in X$. We have

$$\left\|Q^{\frac{1}{2}}(t,s)x\right\|_X^2 = \int_s^t \left\|Q^{\frac{1}{2}}(r)U(t,r)^\star x\right\|_X^2 dr \leqslant \int_{-\infty}^t \left\|Q^{\frac{1}{2}}(r)U(t,r)^\star x\right\|_X^2 dr. \tag{6.15}$$

Hence by Proposition 1.3.3 with $L_1 = Q^{\frac{1}{2}}(t,s)$ and $L_2 = Q^{\frac{1}{2}}(t,-\infty)$, (1) follows.

(2) We apply again Proposition 1.3.3 with $L_1 = U(t,s)Q^{\frac{1}{2}}(s,-\infty)$ and $L_2 = Q^{\frac{1}{2}}(t,-\infty)$. For every $x \in X$ we have

$$\begin{aligned}\left\|Q^{\frac{1}{2}}(s,-\infty)U(t,s)^\star x\right\|_X^2 &= \int_{-\infty}^s \langle U(t,r)Q(r)U(t,r)^\star x, x\rangle_X\, dr \\ &= \langle Q(t,-\infty)x,x\rangle_X - \langle Q(t,s)x,x\rangle_X \\ &= \left\|Q^{\frac{1}{2}}(t,-\infty)x\right\|_X^2 - \left\|Q^{\frac{1}{2}}(t,s)x\right\|_X^2 . \end{aligned} \tag{6.16}$$

Since

$$\left\|Q^{\frac{1}{2}}(s,-\infty)U(t,s)^\star x\right\|_X \leqslant \left\|Q^{\frac{1}{2}}(t,-\infty)x\right\|_X , \tag{6.17}$$

statement (2) follows.

(3) By Proposition 1.3.3, (6.14) holds if only if there exists $\alpha \in (0,1)$ such that

$$\left\|Q^{\frac{1}{2}}(s,-\infty)U(t,s)^\star x\right\|_X \leqslant \alpha \left\|Q^{\frac{1}{2}}(t,-\infty)x\right\|_X , \quad x \in X. \tag{6.18}$$

By (6.16) for every $x \in X$

$$\left\|Q^{\frac{1}{2}}(s,-\infty)U(t,s)^\star x\right\|_X^2 = \left\|Q^{\frac{1}{2}}(t,-\infty)x\right\|_X^2 - \left\|Q^{\frac{1}{2}}(t,s)x\right\|_X^2 . \tag{6.19}$$

(6.18) and (6.19) yield

$$\left\|Q^{\frac{1}{2}}(t,-\infty)x\right\|_X \leqslant \frac{1}{\sqrt{1-\alpha^2}} \left\|Q^{\frac{1}{2}}(t,s)x\right\|_X , \quad x \in X. \tag{6.20}$$

Still by Proposition 1.3.3, (6.20) is equivalent to $\mathcal{H}_{t,-\infty} \subseteq \mathcal{H}_{t,s}$.

□

Proposition 6.2.2. *We assume that Hypothesis 5.2.1 holds true. For every $(s,t) \in \overline{\Delta}$ and $f \in B_b(X)$ we have*

$$(P_{s,t}f) \circ (I - P_s) = \Gamma_{\gamma_s,\gamma_t}(L^\star)f,$$

where $L := Q(t,-\infty)^{-\frac{1}{2}}U(t,s)Q(s,-\infty)^{\frac{1}{2}}$ belongs to $\mathcal{L}(X)$.

Proof. Let $(s,t) \in \overline{\Delta}$, $f \in B_b(X)$ and $x \in X$. Since $L \in \mathcal{L}(X)$ and $P_tL = 0$, then $L^\star \in \mathcal{L}(X)$ and $L^\star P_s = 0$. Hence by (6.9) we have

(6.21)
$$(\Gamma_{\gamma_s,\gamma_t}(L^\star)f)(x) = \int_X f(L_\infty^{\gamma_t,\gamma_s}x + z)\,\mathcal{N}(0, Q_{\gamma_t}^{\frac{1}{2}}(I-LL^\star)Q_{\gamma_t}^{\frac{1}{2}})(dz), \quad f \in B_b(X), \quad \gamma_t \text{ a.e. } x \in X.$$

Moreover $L_\infty^{\gamma_s,\gamma_t} = U(t,s)(I-P_s)$ and

$$Q(t,-\infty)^{\frac{1}{2}}(I-LL^\star)Q(t,-\infty)^{\frac{1}{2}} = Q(t,-\infty) - U(t,s)Q(s,-\infty)U(t,s)^\star = Q(t,s),$$

from which it follows that

$$
\begin{aligned}
(\Gamma_{\gamma_s,\gamma_t}(L^\star)f)(x) &= \int_X f(L_\infty^{\gamma_s,\gamma_t}x+y)\,\mathcal{N}(0,Q(t,-\infty)^{\frac12}(I-LL^\star)Q(t,-\infty)^{\frac12})(dy)\\
&= \int_X f(U(t,s)(I-P_s)x+y)\,\mathcal{N}(0,Q(t,s))(dy)\\
&=(P_{s,t}f)\circ(I-P_s)(x).
\end{aligned}
$$

□

By Propositions 6.1.8 and 6.2.2 we obtain again the hypercontractivity of $P_{s,t}$. We remark that the result below is a bit different from Theorem 5.2.7 since we have different assumptions on $U(t,s)$.

Corollary 6.2.3. *We assume that Hypothesis 5.2.1 holds true and that there exists $(s,t)\in\Delta$ such that $\|Q(t,-\infty)^{-\frac12}U(t,s)Q(s,-\infty)^{\frac12}\|_{\mathcal{L}(X)}<1$. Given $p\geqslant 1$, the operator $P_{s,t}$ maps $L^p(X,\gamma_t)$ into $L^q(X,\gamma_s)$, for every $q\geqslant 1$ if $U(t,s)Q(s,-\infty)^{\frac12}=0$, for every $1\leqslant q\leqslant q_0$ with*

$$q_0=1+(p-1)\|Q(t,-\infty)^{-\frac12}U(t,s)Q(s,-\infty)^{\frac12}\|^{-2}_{\mathcal{L}(X)},$$

if $U(t,s)Q(s,-\infty)^{\frac12}\neq 0$. In any case

$$\|P_{s,t}f\|_{L^q(X,\gamma_s)}\leqslant\|f\|_{L^p(X,\gamma_t)},\quad f\in L^q(X,\gamma_t).$$

Proof. Let $(s,t)\in\Delta$. For every $f\in C_b(X)$, by Proposition 6.2.2 we get

$$\int_X|P_{s,t}f|^q\,d\gamma_s=\int_X|(P_{s,t}f)\circ(I-P_s)|^q\,d\gamma_s=\int_X|\Gamma(L^\star)f|^q\,d\gamma_s, \tag{6.22}$$

where $L=Q(t,-\infty)^{-\frac12}U(t,s)Q(s,-\infty)^{\frac12}$. Since $\|L^\star\|_{\mathcal{L}(X)}=\|L\|_{\mathcal{L}(X)}<1$ by Proposition 6.2.2 we have

$$\|P_{s,t}f\|_{L^q(X,\gamma_s)}\leqslant\|f\|_{L^p(X,\gamma_t)},\quad f\in C_b(X).$$

The statement follows by the density of $C_b(X)$ in $L^p(X,\gamma_t)$. □

Hypothesis 6.2.4. There exists $\omega>0$ such that $\|Q(t,-\infty)^{-1/2}U(t,s)Q(s,-\infty)^{1/2}\|_{\mathcal{L}(X)}<e^{-\omega(t-s)}$ for every $(s,t)\in\Delta$.

Proposition 6.2.5. *Assume that Hypothesis 5.2.1 and Hypothesis 6.2.4 holds true. Then for every $p\in[1,+\infty)$ there exists a positive constant c_p such that for every $(s,t)\in\Delta$*

$$\|P_{s,t}f-m_t(f)\|_{L^p(X,\gamma_s)}\leqslant c_pe^{-\omega(t-s)}\|f\|_{L^p(X,\gamma_t)}\qquad\forall f\in L^p(X,\gamma_t).$$

Proof. Let $(s,t)\in\Delta$ and let $f\in L^p(X,\gamma_t)$. We set $L=Q(t,-\infty)^{-\frac12}U(t,s)Q(s,-\infty)^{\frac12}$,$c=\|L\|_{\mathcal{L}(X)}$ and $\tau=-\log c$. From Remark 6.1.4(ii) and Proposition 6.1.7, for every $(s,t)\in\Delta$ we infer that

$$
\begin{aligned}
m_t(f)&=\int_X fd\gamma_t=\int_X P_{s,t}fd\gamma_s=\int_X P_{s,t}f\circ(I-P_s)d\gamma_s=\int_X\Gamma_{\gamma_s,\gamma_t}(L^\star)fd\gamma_s\\
&=\int_X\Gamma_{\gamma_s,\gamma_t}\left(cc^{-1}L^\star\right)fd\gamma_s=\int_X\Gamma_{\gamma_s}(cI)\Gamma_{\gamma_s,\gamma_t}\left(c^{-1}L^\star\right)fd\gamma_s\\
&=\int_X P^{\gamma_s}_\tau\Gamma_{\gamma_s,\gamma_t}\left(c^{-1}L^\star\right)f\circ(I-P_s)d\gamma_s=\int_X P^{\gamma_s}_\tau\Gamma_{\gamma_s,\gamma_t}\left(c^{-1}L^\star\right)fd\gamma_s.
\end{aligned}
$$

Recalling that for every $h \in L^p(X, \gamma_s)$ we have

$$\int_X P_\tau^{\gamma_s} h \, d\gamma_s = \int_X h \, d\gamma_s, \tag{6.23}$$

we obtain

$$m_t(f) = \int_X \Gamma_{\gamma_s,\gamma_t}\left(c^{-1}L^\star\right) f d\gamma_s. \tag{6.24}$$

Setting $g = \Gamma_{\gamma_s,\gamma_t}(c^{-1}L^\star)f$, for every $(s,t) \in \Delta$ we get

$$\begin{aligned}\|P_{s,t}f - m_t(f)\|_{L^p(X,\gamma_s)} =& \|(P_{s,t}f) \circ (I - P_s) - m_t(f)\|_{L^p(X,\gamma_s)} \\ =& \|\Gamma_{\gamma_s,\gamma_t}(L^\star)f - m_t(f)\|_{L^p(X,\gamma_s)} \\ =& \|\Gamma_{\gamma_s}(cI)g - m_s(g)\|_{L^p(X,\gamma_s)}.\end{aligned}$$

From Remark 6.1.4(ii) we deduce that

$$\begin{aligned}\|P_{s,t}f - m_t(f)\|_{L^p(X,\gamma_s)} &= \|(P_\sigma^{\gamma_s} g) \circ (I - P_s) - m_s(g)\|_{L^p(X,\gamma_s)} \\ &= \|P_\sigma^{\gamma_s} g - m_s(g)\|_{L^p(X,\gamma_s)},\end{aligned} \tag{6.25}$$

where $\sigma = -\log c$ and $\{P_\sigma^{\gamma_s}\}_{\sigma \geqslant 0}$ is the classical Ornstein-Uhlenbeck semigroup with respect to the measure γ_s. It is well-known (see for instance [Fuh98; LP20a]) that for every $p > 1$ there exists a positive constant c_p such that for every Gaussian measure μ we get

$$\|P_\sigma^\mu(\varphi) - m_s(\varphi)\|_{L^p(X,\mu)} \leqslant c_p e^{-\sigma} \|\varphi\|_{L^p(X,\mu)} \tag{6.26}$$

for every $\sigma > 0$. From (6.25) and (6.26) we infer that

$$\|P_{s,t}f - m_t(f)\|_{L^p(X,\gamma_s)} \leqslant c_p \, c \, \|g\|_{L^p(X,\gamma_s)},$$

which gives

$$\|P_{s,t}f - m_t(f)\|_{L^p(X,\gamma_s)} \leqslant c_p \, e^{-\omega(t-s)} \|f\|_{L^p(X,\gamma_t)}. \tag{6.27}$$

$\square$

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