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# Numerical modeling of wrinkled hyperelastic membranes with topologically complex internal boundary conditions

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#### Abstract

Several soft biological tissues and artificial materials are characterised by a mechanical behaviour described by two-dimensional structural systems sustaining in-plane forces. Within the framework of finite strain elasticity, in this paper the formulation and finite element implementation of a hyperelastic incompressible membrane is presented. Focus is placed on the behaviour of membranes presenting holes and internal cuts. A new efficient algorithm is presented to describe topologically complex internal boundaries along which dislocation-like distributions are prescribed, so as to allow a one-to-one progressive joining of boundary material points. The classical Ogden's model is modified into a relaxed version in order to accommodate the no-compression response of thin membranes due to wrinkling. Three applicative examples are presented to illustrate the potential of the method proposed.

Keywords: membrane; wrinkling; dislocation; internal boundary condition;

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#### 1 1. Introduction

Soft membranes can be encountered at different scales, both in natural 2 systems and in several engineering applications. In biology, membranes fulfil 3 crucial physiological needs. From cellular walls at the micro scale, to skin in living beings at the macro scale, biological membranes act as protection 5 against external hazards [1]. Thanks to osmosis, membranes also regulate 6 chemical exchanges between external and internal environment, maintaining 7 optimal conditions, e.g. ph, moisture and temperature, within the enclosed 8 domain [2]. Synthetic membranes are largely employed in medicine, for hy-9 giene devices, bio-mimicking tissues [3], tapes and tubes [4]. In recent years, 10 various kinds of membrane structures have been developed for applications 11 in new technologies to meet specific requirements, including, for instance, 12 batteries [5] and supercapacitors [6] for stretchable electronics. 13

By a mechanical point of view, soft membranes are lightweight structures 14 exposed to large deformations and relevant in-plane strains [7]. An area of 15 great interest concerns the behaviour of membranes presenting holes or cuts. 16 As these are usually the result of damage, they should be healed in order to 17 restore the continuity of the surface and the functionalities of the membrane 18 [8]. This can be done by applying patches of the same size of the loss using 19 external material, or by joining the boundaries of the discontinuity through 20 proper displacements in order to make them coincident in the final configu-21 ration. With the latter method, since no new material is supplied, the way 22 chosen for the closure procedure influences the rearrangement of the stress 23

and strain fields in the neighbourhood of the hole. Traditionally, in plastic 24 surgery the suturing of scars has explored the different ways in which the 25 joining process can be performed, in order to achieve the best aesthetic re-26 sult [9]. However, this concept can also be exploited in a reversed way, such 27 that desired distortions can be induced within the membrane by appropri-28 ately creating holes and related closures. In the field of metamaterials, for 29 instance, holes closure is used in kirigami tessellation. By creating a suitable 30 pattern of incisions on a flat sheet, the closure of holes produces a coordi-31 nated movement of interconnected tiles, which deploys the sheet in a different 32 and reversible shape [10-12]. 33

In mechanical terms, the process of inducing distortions through the ap-34 plication of internal boundary conditions in a soft membrane generates a 35 state of self-balanced stress within the domain, with higher stresses concen-36 trated in the neighborhood of the inner boundaries. The problem involves 37 both large strains and displacements, making numerical methods the most 38 appealing tool of analysis. A first attempt to study the mechanical behaviour 30 of biological membranes embedding closing holes was carried out by Larrabee 40 and Galt [13], who used the Finite Element (FE) method to simulate the su-41 turing of skin flaps. More recently, other authors have improved the analysis 42 by employing non-linear material models and adopting different approaches 43 to achieve closure. Lott-Crumpler and Chaudhry [14] and Flynn [15] achieved 44 closure of symmetric holes by approaching nodes lying on two separate edges 45 and making them coincide in the midway position. An alternative solution 46 was proposed by Rajabi et al. [16], who simulated the closure of unsymmetric 47 holes by coupling pairs of nodes on opposite edges with trusses, which were

<sup>49</sup> then shrunk through fictitious variations of temperature.

The main shortcoming of the available methods is the limited applicability 50 to complex shapes of the internal boundary, in which the final configuration 51 of the edges is a problem unknown, and no prescribed displacements can 52 be applied on them. In order to achieve a perfect coincidence between two 53 joining nodes, a suitable approach is to add kinematic equations in order to 54 constrain nodes to move toward each other until the gap is null, e.g. via 55 so-called multi-point constraints (MPCs) in FE models [17]. The approach 56 is similar to that of Rajabi et al. [16], but without the drawbacks of trusses, 57 which cannot reach vanishing lengths. Arbitrarily shaped cavities may in-58 clude several critical points, making meshing and definition of the constraints 59 difficult to be performed manually. Moreover, this task is further complicated 60 by the fact that discretisation along the hole boundaries must be consistent, 61 that is, nodes must be in equal number and uniformly distributed between 62 two pairing edges. In order to conciliate the requirements for a consistent 63 discretisation of domain boundaries with automatic meshing, the generation 64 of FE models using a custom pre-processing code is deemed to be convenient. 65 As membranes are thin solids, compressive internal forces may lead to 66 instabilities, resulting in out-of-plane displacements [18]. This phenomenon, 67 known as *wrinkling*, is well known in biological tissues [19, 20] as well as in 68 engineered materials [21–24]. In general, the study of wrinkling requires ge-69 ometrically non-linear analyses: in order to determine the exact out-of-plane 70 deflection of compressed membranes, the flexural buckling and post-buckling 71 behaviour should be analysed [25–28]. However, if no specific information 72 about wrinkle wavelength and amplitude is needed, a properly modified plane stress constitutive model can be adopted, in which the non-linearity induced by finite out-of-plane displacements is treated as a material non-linearity [29-33]. This represents a computationally efficient approach suitable to determine the in-plane tension field of the membrane, and precisely identify the distribution of wrinkled regions.

In this paper, a numerical model of soft membranes with topologically 79 complex internal boundaries is presented. The algorithm is developed within 80 the Matlab<sup>®</sup> environment and it is linked with the open source code DistMesh 81 developed by Persson and Strang [34], which is capable of efficiently meshing 82 any two-dimensional domain. The proposed algorithm allows general shapes 83 of boundaries to be generated and complex dislocations distributions to be 84 applied along them. Furthermore, a wrinkling constitutive model is proposed 85 and implemented in a FE formulation in order to take into account the no-86 compression behaviour of the soft membrane. In particular, the formation 87 of wrinkles is included through a relaxed strain-energy density, based on the 88 hyperelastic Ogden's function. Although the constitutive law is isotropic, 80 more general anisotropic formulations, which have been proposed for soft 90 biological membranes [35, 36], can also be included. 91

The outline of the paper is as follows. Sect. 2 presents the fundamentals of the finite strain theory of soft membranes in which the hyperelastic constitutive model is developed. Wrinkling is accounted for by a proper modification of the hyperelastic potential. The linearised form of the constitutive equations is also presented for the FE implementation. Sect. 3 describes the definition of general boundary conditions along topologically complex internal boundaries of the membrane, where general dislocation distributions

can be applied. Sect. 4 is devoted to the numerical implementation of the 99 model. In particular, algorithms for automatic generation of FE models 100 are presented, allowing the description of the topology of internal bound-101 aries, accurate refined meshing and the application of general dislocation 102 distributions along the boundaries. Sect. 5 presents three applicative cases 103 concerning different soft membranes geometries. Comparisons with existing 104 numerical and experimental data is included. Finally, Sect. 6 summarises 105 the potential and limits of the proposed approach and presents concluding 106 remarks. 107

#### <sup>108</sup> 2. Wrinkling hyperelastic membranes

#### 109 2.1. Kinematics

Given a material body  $\mathcal{B}$  occupying the region  $\Omega$  at time  $t_0 = 0$ , any mate-110 rial point  $P \in \mathcal{B}$  can be mapped from the reference position X to the current 111  $\mathbf{x}$  according to the unique biunivocal function  $\mathbf{x} = \chi(\mathbf{X}, t)$ . The resulting 112 deformation is described by the second-order tensor  $\mathbf{F} = \partial \chi (\mathbf{X}, t) / \partial \mathbf{X} =$ 113  $\partial \mathbf{X}/\partial \mathbf{x}$ , known as the deformation gradient. Membranes can be considered 114 two-dimensional structural elements enforcing a condition of plane stress. 115 In a co-rotational orthonormal system, defined by the in-plane basis vectors 116  $\mathbf{e}_1, \mathbf{e}_2$  and the normal vector  $\mathbf{e}_3$ , the deformation gradient is therefore written 117 as 118

$$\mathbf{F} = \begin{bmatrix} F_{11} & F_{12} & 0 \\ F_{21} & F_{22} & 0 \\ 0 & 0 & (F_{11}F_{22} - F_{12}F_{21})^{-1} \end{bmatrix}$$
(1)

where the assumption of incompressibility  $J = \det \mathbf{F} = 1$  was introduced.

For later use in the formulation of the constitutive model of the wrinkled membrane, the deformation gradient is decomposed according to  $\mathbf{F} = \mathbf{R}\mathbf{U} =$ **vR**, where **R** is an orthogonal rotation tensor, while **U** and **v** represent the symmetric stretch tensors, defined for the reference and the current configuration, respectively. Through spectral decomposition these tensors can be written as

$$\mathbf{U} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{N}}_a \otimes \hat{\mathbf{N}}_a, \quad \mathbf{v} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}_a \otimes \hat{\mathbf{n}}_a \tag{2}$$

in which the eigenvalues  $\lambda_a$ , a = 1, 3 represent the principal stretches, while the eigenvectors  $\hat{\mathbf{N}}_a$  and  $\hat{\mathbf{n}}_a$  represent the principal referential and spatial directions, respectively, and are related by  $\hat{\mathbf{n}}_a = \mathbf{R}\hat{\mathbf{N}}_a$ . Finally, we define the spatial velocity gradient as  $\mathbf{l} = \partial \mathbf{v}(\mathbf{x}, t)/\partial \mathbf{x} = \mathbf{d} + \mathbf{w}$ , where  $\mathbf{d}$  and  $\mathbf{w}$ represent the symmetric rate of strain tensor and the antisymmetric spin tensor, respectively [37].

#### 132 2.2. Constitutive model

In this work the membrane tissue is assumed to be homogeneous, hyperelastic and isotropic. A strain energy density function  $\Psi$ , provided by the well-known Ogden model [38], is introduced, which with the assumption of incompressibility  $(J = \lambda_1 \lambda_2 \lambda_3 = 1)$  is given by

$$\Psi = \sum_{i=1}^{N} \frac{\mu_i}{\alpha_i} \left( \lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3 \right) - p \left( J - 1 \right) = \Psi_O(\lambda_1, \lambda_2, \lambda_3) - p \left( J - 1 \right)$$
(3)

where  $\mu_i$  and  $\alpha_i$  are material properties, whereas p is an unknown hydrostatic pressure introduced as a Lagrange multiplier to enforce the incompressibility constraint. Usually, a single set of parameters (N = 1) gives a good approximation to the *J*-shaped stress-strain curves commonly encountered in soft tissues [39].

The second Piola-Kirchhoff (PKII) stress tensor is written in terms of the
principal stresses as

$$\mathbf{S} = \sum_{a=1}^{3} S_a \hat{\mathbf{N}}_a \otimes \hat{\mathbf{N}}_a \tag{4}$$

where the principal stress in the material configuration is  $S_a = \lambda_a^{-1} \partial \Psi / \partial \lambda_a$ . Through a standard push-forward operation to the current configuration, the Cauchy stress tensor reads

$$\boldsymbol{\sigma} = \sum_{a=1}^{3} \sigma_a \hat{\mathbf{n}}_a \otimes \hat{\mathbf{n}}_a, \quad \sigma_a = J^{-1} \lambda_a \frac{\partial \Psi}{\partial \lambda_a}$$
(5)

In order to compute the Lagrange multiplier p, the out-of-plane principal stress  $\sigma_3$  must be set to zero in Eq. (5), that is,

$$\sigma_3 = \lambda_3 \frac{\partial \Psi_O}{\partial \lambda_3} - p = 0 \tag{6}$$

149 from which we find

$$p = \lambda_3 \frac{\partial \Psi_O}{\partial \lambda_3} = \sum_{i=1}^N \mu_i \left(\lambda_1 \lambda_2\right)^{-\alpha_i} \tag{7}$$

<sup>150</sup> Combining Eqs. (5)-(7), the principal Cauchy stresses become

$$\sigma_a = \sum_i \mu_i \left( \lambda_a^{\alpha_i} - (\lambda_1 \lambda_2)^{-\alpha_i} \right), \quad a = 1, 2$$
(8)

#### 151 2.3. Wrinkling

The wrinkling behaviour when membranes are subjective to compressive states can be conveniently defined in terms of the principal stretches. Taking  $\lambda_1$  as the reference stretch, there are three possible strain configurations for the membrane, depending on  $\lambda_2$ . A modified strain-energy function  $\Psi_W$  can be defined, depending on the configuration. This procedure, which somehow treats the out-of-plane deflection as a constitutive material non-linearity, is known as quasi-convexification of  $\Psi$  [40].

<sup>159</sup> When  $\lambda_1 \geq 1$  and  $\lambda_2 \geq \lambda_1^{-1/2}$ , the lateral deformation  $\lambda_2$  is greater <sup>160</sup> than that due to Poisson contraction, obtained imposing  $\sigma_2 = 0$  in Eq. <sup>161</sup> (8). In this case, no wrinkling occurs as lateral stretches greater than  $\lambda_1^{-1/2}$ <sup>162</sup> provide positive stresses. This is the so-called *taut* condition (Fig. 1a), and <sup>163</sup> the relative strain-energy function  $\Psi_W = \Psi$  is given by Eq. (3), with the <sup>164</sup> incompressibility condition J = 1 (Fig. 1a).

If  $\lambda_1 \geq 1$  and  $\lambda_2 < \lambda_1^{-1/2}$ , the membrane is actually compressed in the 165 lateral direction, resulting in a wrinkling condition (Fig. 1b). Wrinkles 166 parallel to  $\lambda_1$  direction develop, carrying no loads along their orthogonal 167 direction. Keeping fixed  $\lambda_1$ , further reductions of  $\lambda_2$  do not increase the 168 strain-energy, which remains equal to the minimum reached in the uniaxial 169 stress state. Following the works of Evans [32] and Massabò and Gambarotta 170 [41],  $\Psi_W$  depends only on  $\lambda_1$ , and can be obtained substituting  $\lambda_2 = \lambda_1^{-1/2}$ 171 into Eq. (3). Thus, we find 172

$$\Psi_W = \sum_{i=1}^N \frac{\mu_i}{\alpha_i} \left( \lambda_1^{\alpha_i} + 2\lambda_1^{-\frac{\alpha_i}{2}} - 3 \right) \tag{9}$$

Finally, if  $\lambda_2 < \lambda_1 < 1$  the membrane is *slack* (Fig. 1c) and does not carry any load. No energy is stored within the material under loading, and hence the associated strain-energy function is set to zero.



**Figure 1.** Reference  $\Omega$  and deformed  $\omega$  configuration of a unit membrane element. (a) Taut, (b) wrinkled and (c) slack configurations.

The function  $\Psi_W$ , summarised in Tab. 1 for both cases of  $\lambda_1 > \lambda_2$  and  $\lambda_2 > \lambda_1$ , is called the relaxed strain-energy density function. Such a function is employed below to compute the stress and stiffness tensors in the linearised approximation of the governing equations.

## 180 2.4. Stress and elasticity tensors

<sup>181</sup> Numerical solution methods for nonlinear problems are based on an in-<sup>182</sup> cremental procedure, in which the principle of virtual work is consistently

Condition	Criteria	Relaxed strain-energy density
Taut (Fig. 1a)	$\lambda_1 \ge 1 \text{ and } \lambda_2 \ge \lambda_1^{-1/2}$ or $\lambda_2 \ge 1 \text{ and } \lambda_1 \ge \lambda_2^{-1/2}$	$\Psi_W = \sum_{i=1}^N \frac{\mu_i}{\alpha_i} \left( \lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + (\lambda_1 \lambda_2)^{-\alpha_i} - 3 \right)$
Wrinkled (Fig. 1b)	$\lambda_1 \ge 1$ and $\lambda_2 < \lambda_1^{-1/2}$ $\lambda_2 \ge 1$ and $\lambda_1 < \lambda_2^{-1/2}$	$\Psi_W = \sum_{i=1}^N \frac{\mu_i}{\alpha_i} \left( \lambda_1^{\alpha_i} + 2\lambda_1^{-\frac{\alpha_i}{2}} - 3 \right)$ $\Psi_W = \sum_{i=1}^N \frac{\mu_i}{\alpha_i} \left( \lambda_2^{\alpha_i} + 2\lambda_2^{-\frac{\alpha_i}{2}} - 3 \right)$
Slack (Fig. 1c)	$\lambda_1 < 1$ and $\lambda_2 < 1$	$\Psi_W = 0$

 Table 1. Relaxed strain-energy density function.

linearised with respect to displacements. This procedure also requires the linearisation of the constitutive relationship through the definition of a tangent stiffness matrix, in terms of the fourth-order elasticity tensor. In the spirit of an updated-Lagrangian method, where every increment is computed using the last equilibrium state as reference configuration, the elasticity tensor can be expressed in the spatial description through spectral decomposition as [37]

$$\mathbb{c}^{\boldsymbol{\tau}^{\circ}} = \sum_{a,b=1}^{3} \lambda_{a}^{2} \lambda_{b} \frac{\partial S_{a}}{\partial \lambda_{b}} \left( \hat{\mathbf{n}}_{a} \otimes \hat{\mathbf{n}}_{a} \otimes \hat{\mathbf{n}}_{b} \otimes \hat{\mathbf{n}}_{b} \right) + \sum_{a,b=1;a\neq b}^{3} \lambda_{a}^{2} \lambda_{b}^{2} \frac{S_{b} - S_{a}}{\lambda_{b}^{2} - \lambda_{a}^{2}} \left( \hat{\mathbf{n}}_{a} \otimes \hat{\mathbf{n}}_{b} \otimes \hat{\mathbf{n}}_{a} \otimes \hat{\mathbf{n}}_{b} + \hat{\mathbf{n}}_{a} \otimes \hat{\mathbf{n}}_{b} \otimes \hat{\mathbf{n}}_{b} \otimes \hat{\mathbf{n}}_{a} \right)$$
(10)

where  $S_{a,b}$  are the principal PKII stresses of Eq. (4). The superscript  $\tau^{\circ}$  in 189 the left term of Eq. (10) indicates that the elasticity tensor is in this way 190 expressed in terms of the Oldroyd rate of the Kirchhoff stress tensor  $\tau = J\sigma$ . 191 Indeed, the linearised constitutive relationship must satisfy the principle 192 of frame indifference. Different objective stress rates are available in the 193 literature, most of them being based on the Lie time derivative and its linear 194 combinations [42]. In particular, we here adopt the Jaumann rate of the 195 Kirchhoff stress  $\boldsymbol{\tau}^{\nabla} = \dot{\boldsymbol{\tau}} - \mathbf{w}\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{w} = \boldsymbol{\varepsilon}^{\boldsymbol{\tau}^{\nabla}} : \mathbf{d}$ , which is the co-rotational rate 196 required by the commercial FE software ABAQUS employed in the analyses. 197 Exploiting the relationship between Oldroyd and Jaumann rates, given by 198  $\boldsymbol{\tau}^{\triangledown} = \boldsymbol{\tau}^{\circ} + \mathbf{d}\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{d}$ , the correct elasticity tensor  $\mathbb{c}^{\boldsymbol{\tau}^{\triangledown}}$  is computed as follows 199

$$\mathbb{c}^{\boldsymbol{\tau}^{\nabla}} = \mathbb{c}^{\boldsymbol{\tau}^{\circ}} + \boldsymbol{\tau} \odot \mathbf{I} + \mathbf{I} \odot \boldsymbol{\tau}$$
(11)

in which the operator  $\odot$  denotes the symmetric dyadic product of secondorder tensors, defined as  $\{\bullet \odot \circ\}_{ijkl} = 1/2 \left(\{\bullet\}_{ik} \{\circ\}_{jl} + \{\bullet\}_{il} \{\circ\}_{jk}\right)$  [43].

From Eq. (10), it can be noticed that using the relaxed Ogden function 202  $\Psi_W$  (Tab. 1),  $\mathbb{C}^{\tau^{\circ}}$  would result in zero stiffness in compression for wrinkled 203 and slack regions, giving rise to ill-conditioning problems. This issue can 204 numerically be circumvented by adding a small stiffness in compression by 205 means of a fictitious tensor, computed from the standard strain-energy func-206 tion of Eq. (3). Accordingly, the stiffness  $c^{\tau^{\circ}}$  is additively decomposed as 207  $\mathbb{C}^{\boldsymbol{\tau}^{\circ}} = \mathbb{C}_{W}^{\boldsymbol{\tau}^{\circ}} + \tilde{\mathbb{C}}^{\boldsymbol{\tau}^{\circ}}$ , where  $\mathbb{C}_{W}^{\boldsymbol{\tau}^{\circ}}$  and  $\tilde{\mathbb{C}}^{\boldsymbol{\tau}^{\circ}}$  are the pure wrinkling and the fictitious 208 stiffness tensors derived from Eq. (10). Specifically, the latter is obtained 209 by considering the same constitutive Ogden model with a reduced stiffness, 210 which ought to be small enough to have negligible influence on the tension 211

field [32]. For the sake of simplicity, the stiffness of this sort of non-wrinkling fictitious layer is considered to depend on the first order coefficients of the original Ogden model, taking  $\tilde{\mu}$  as a small fraction of  $\mu_1$ , and  $\tilde{\alpha}$  equal to  $\alpha_1$ . This is equivalent to a uniform scaling of the original stress-stretch curve computed using the Ogden model. Test analyses are provided in Sect. 5 for a single plane stress element under uniaxial and biaxial strain conditions.

In the general case of an initial stress field  $\sigma_0$  acting across the domain  $\Omega$ , stresses can no longer be computed using Eq. (5), but must be updated at each increment from the tensor  $c^{\tau^{\vee}}$  in the spatial description. Following Hughes and Winget [44], at each increment stresses  $\sigma_{n+1}$  are obtained by rotating  $\sigma_n$  from the reference configuration at increment *n* to the current, and then adding the co-rotational stress increment, which is obtained from the Jaumann rate of the Cauchy stress, namely

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{Q}_n \boldsymbol{\sigma}_n \boldsymbol{Q}_n^T + \boldsymbol{\mathbb{C}}_n^{\boldsymbol{\sigma}^{\nabla}} : \delta \boldsymbol{\epsilon}_n$$
(12)

where  $Q_n$  is an incremental rotation tensor,  $\mathbb{C}_n^{\sigma^{\nabla}}$  is the elasticity tensor in terms of the Jaumann rate of the Cauchy stress, and  $\delta \epsilon_n$  is the incremental strain computed with the midpoint deformation rule from the rate of strain tensor **d** [45]. Knowing that  $\sigma^{\nabla} = J^{-1} \tau^{\nabla} - (\mathbf{d} : \mathbf{I}) \sigma$  [45], the elasticity tensor appearing in Eq. (12) is obtained as

$$\mathbb{c}_n^{\boldsymbol{\sigma}^{\nabla}} = J^{-1} \mathbb{c}_n^{\boldsymbol{\tau}^{\nabla}} - \boldsymbol{\sigma}_n \otimes \mathbf{I}$$
(13)

where  $\mathbb{C}_n^{\boldsymbol{\tau}^{\nabla}}$  is defined by Eq. (11) in combination with Eq. (10). Note that, in case of an initial pre-stress field  $\boldsymbol{\sigma}_0$ , at increment n = 0,  $\boldsymbol{\sigma}_n$  is equal to  $\boldsymbol{\sigma}_0$ 

and  $Q_n$  to the identity tensor in Eq. (12). 232

246

#### 3. Internal boundaries with prescribed dislocations 233

In order to represent discontinuities in the elastic membranes, we intro-234 duce internal boundary conditions in addition to the usual ones in terms of 235 prescribed tractions and displacements on the external boundaries. The type 236 of internal boundary conditions here considered can be accommodated within 237 the theory of *dislocations* (interested readers might refer to some fundamen-238 tal papers on this theory, as for instance [46, 47] and the memorial paper on 239 Eshelby's work [48]), which have to be intended as imposed relative displace-240 ments between points located on two different boundaries of a discontinuity. 241 In other words, we consider cuts and holes — the latter implying a subtrac-242 tion of material — embedded in the membrane, having general topologies, 243 and prescribe relative displacements such that the boundaries formed by the 244 cuts and holes are brought together and joined. The membrane will be in a 245 state of self-balanced stress as a result of these internal boundary conditions.

Let suppose we place a distribution of dislocations along a number of 247 internal boundaries  $\partial \Omega^{I,n}$  in the membrane, each partitioned in pairs of sub-248 sets  $\partial \Omega_i^{I,n+}$  and  $\partial \Omega_i^{I,n-}$  which are joined together in the final configuration. 249 Since, in general, the final topology of the subsets  $\partial \omega_i^{I,n+}$  and  $\partial \omega_i^{I,n-}$  is un-250 known, the prescribed displacements have to be expressed by introducing a 251 constraint function  $\Phi_i$ . Notice that subscripts and superscripts denote the 252 *n*-th internal boundary and the *i*-th couple of paired subsets. Given the map-253 ping function  $\Gamma$  :  $\partial \Omega_i^{I,n+} \to \partial \Omega_i^{I,n-}$ , which biunivocally relates every point 254  $\mathbf{X}_{i}^{+}(\xi) \in \partial \Omega_{i}^{I,n+}$  with its counterpart  $\mathbf{X}_{i}^{-}(\xi) \in \partial \Omega_{i}^{I,n-}$ , the constraint consists 255

in imposing the coincidence between the mapped points  $\mathbf{x}_{i}^{+}(\xi) \in \partial \omega_{i}^{I,n+}$  and  $\mathbf{x}_{i}^{-}(\xi) \in \partial \omega_{i}^{I,n-}$  after the motion  $\chi$  (Fig. 2), where  $\xi \in [0,1]$  biunivocally identifies the points  $\mathbf{X}_{i}^{+}, \mathbf{X}_{i}^{-}$  (and  $\mathbf{x}_{i}^{+}, \mathbf{x}_{i}^{-}$ ) in the subsets  $\partial \Omega_{i}^{I,n+}, \partial \Omega_{i}^{I,n-}$  (and  $\partial \omega_{i}^{I,n+}, \partial \omega_{i}^{I,n-}$ ). Knowing that  $\mathbf{x} = \chi(\mathbf{X}, t) = \mathbf{X} + \mathbf{u}(\mathbf{X}, t)$ , the constraint function is defined by



$$\Phi_i(\xi) = \mathbf{x}_i^+ - \mathbf{x}_i^- = \mathbf{X}_i^+ + \mathbf{u}(\mathbf{X}_i^+) - \Gamma(\mathbf{X}_i^+) - \mathbf{u}(\Gamma(\mathbf{X}_i^+)) = 0$$
(14)

Figure 2. Schematics of joining dislocations applied to an internal boundary.

In order to achieve a full closure of a hole or cut, an even number 2N subsets must be considered, on which N functions  $\Phi_i(\xi)$  are defined. Note that each *i*-th couple of subsets and the function  $\Gamma$  must be topologically consistent, i.e. they must not lead to knotted or intertwined surfaces. Then the discontinuity is closed by joining together the subsets encountered moving along the boundary from point  $A \in \partial \Omega^{I,n}$  to point  $B \in \partial \Omega^{I,n}$ , so that a continuous line AB results. Note that subsets are ordered counter-clockwise starting from A, so that  $\partial \Omega_i^{I,n+} = \partial \Omega_j^{I,n}$  and  $\partial \Omega_i^{I,n-} = \partial \Omega_{2N-j+1}^{I,n}$ , with j = 1, ..., N (Fig. 3).



**Figure 3.** Complete closure of a single hole following a unique line from A to B. The internal boundary in the reference configuration has been divided into 2N subsets, numbered with counterclockwise order from the point A. The closure is achieved matching together the subsets at the right and left of the point A, such that each couple can be defined by the pairing rule  $(\partial \Omega_{2N-j+1}^{I,n}, \partial \Omega_{j}^{I,n})$ .

The mapping bijective function  $\Gamma$  has to be defined in order to apply a joining dislocation distribution along the internal boundary which precisely describe the physical problem under consideration. Generally speaking, the function  $\Gamma$  can conveniently be chosen to be linear so that  $\|\mathbf{dX}_{i}^{I,n-}\| =$   $c \| \mathbf{dX}_{i}^{I,n+} \|$  for every  $\xi$ , where c is the ratio between the total lengths of  $\partial \Omega_{i}^{I,n-}$  and  $\partial \Omega_{i}^{I,n+}$ . Except for trivial cases, such as the stitching of two overlapping edges, this kind of function  $\Gamma$  always generates a self-balanced stress state along the joining boundaries, which depends on their shape in the reference configuration.

#### 279 4. Numerical implementation

The implementation of the model in a FE formulation, suitable to deal 280 with the highly non-linear nature of the problem, is here described in detail. 281 The membrane can be exposed to general displacements and tractions along 282 the external boundary, to an initial pre-stress field  $\sigma_0$ , and to complex dis-283 location distributions along the internal boundaries, as described in Sect. 3. 284 Specifically, closed internal boundaries describing both cuts and holes in the 285 membrane can be implemented. For the sake of simplicity, a single internal 286 boundary  $\partial \Omega^I$  will be considered, e.g. see Fig. 3. 287

In order to prescribe dislocation distributions along the internal boundary, the constraint represented by the function  $\Phi_i(\xi)$  of Eq. (14) can be addressed in the finite element framework by imposing kinematic Multi-Point Constraints (MPCs) between pairs of nodes in an incremental form. Accordingly, the function  $\Phi_i(\xi)$  is applied to each couple of nodes, belonging to  $\partial \Omega_i^{I-}$  and  $\partial \Omega_i^{I+}$ , respectively.

The FE geometry can be obtained by an automatic meshing of the plane domain of the membrane, by means of dedicated algorithms that specify the element density near the regions of interest. However, although it is possible to obtain well meshed domains, the internal boundary  $\partial \Omega^{I}$  may result in an uncontrolled distribution of nodes, raising issues in the MPC application when different number of nodes in the coupled boundaries  $\partial \Omega_i^{I-}$  and  $\partial \Omega_i^{I+}$  are encountered. Thus, the node spacing along the internal boundary becomes crucial when dislocation distributions are to be applied in a discretised manner. Therefore, a fully automatic procedure, which at the same time can generate the required dislocation distributions and ensure optimal meshing, is needed.

#### 305 4.1. Topology of internal boundaries

We here describe the algorithm developed to automatically compute generic 306 dislocation distributions along the internal boundaries. Let us assume that a 307 certain domain can be represented by a signed distance function  $d(\mathbf{X})$ , which 308 is the combination of closed geometrical entities obtained by multiple para-309 metric curves. The external boundary  $\partial \Omega^E$  usually consists of an elementary 310 shape, like a circle or a rectangle. The internal boundary  $\partial \Omega^I$  should be 311 able to describe any complex closed topology. We define a set  $\mathcal{V}_B^I$  of counter-312 clockwise vertices  $\mathbf{X}_{B,j} \in \partial \Omega^I$  and approximate the curve through a linear 313 piecewise function. Accordingly, the segment  $\mathbf{X}_{j}^{I}(\xi) \in \partial \Omega_{j}^{I}$  is expressed by 314 the linear Bézier curve 315

$$\mathbf{X}_{j}^{I}(\xi) = (1 - \xi)\mathbf{X}_{B,j} + \xi\mathbf{X}_{B,j+1}$$
(15)

with  $\xi \in [0, 1]$ . Note that the dislocation constraint  $\Phi_i$  in Eq. (14) is defined between two sets  $\partial \Omega_i^{I-}$  and  $\partial \Omega_i^{I+}$ , therefore  $\mathcal{V}_B^I$  must contain 2N + 1vertices, representing 2N subsets  $\partial \Omega_i^I$ .

In the case that  $\partial \Omega^I$  represents a cut in the membrane, the different

subsets are coincident and lie on the cut interface. This does not represent 320 an issue from a geometrical point of view, but it can create difficulties in the 321 auto-meshing, as elements might be generated across the cut. To circumvent 322 this problem, the original boundary  $\partial \Omega^I$  is offset externally by a quantity 323 s/2, such that two facing subsets of a cut become spaced by s. We call the 324 new boundary  $\partial \overline{\Omega}^{I}$ . The value of the spacing depends on the average element 325 size. So, the vertices  $\mathbf{X}_{S,j} \in \mathcal{V}_S^I$ , representing the new offset boundary (Fig. 326 4a), are given by 327

$$\mathbf{X}_{S,j} = \mathbf{X}_{B,j} + \mathbf{s}_j^- + \mathbf{s}_j^+ \tag{16}$$

328 with

$$\mathbf{s}_{j}^{-} = \bar{s} \frac{\mathbf{X}_{B,j} - \mathbf{X}_{B,j-1}}{\|\mathbf{X}_{B,j} - \mathbf{X}_{B,j-1}\|} \times \mathbf{e}_{3} \quad , \quad \mathbf{s}_{j}^{+} = \bar{s} \frac{\mathbf{X}_{B,j+1} - \mathbf{X}_{B,j}}{\|\mathbf{X}_{B,j+1} - \mathbf{X}_{B,j}\|} \times \mathbf{e}_{3}$$
(17)

where  $\bar{s} = s/(2(1 + \cos \Delta \theta))$ . Segments between two offset vertices are still defined by the linear curve in Eq.(15), using the new vertices  $\mathbf{X}_S$  in place of  $\mathbf{X}_B$ .

When  $\mathbf{X}_{B,j}$  represents a cut tip the preceding and following segments are parallel, characterized by  $\Delta \theta = \pi$ . This would make  $\mathbf{X}_{S,j}$  to be placed at an infinite distance, as  $\bar{s}$  degenerates to infinity for  $\Delta \theta \rightarrow \pi$ . In order to preserve the tip position, allowing us at the same time to create the offset of the internal boundary, the preceding and following segments can no longer be straight, and must be transformed into two curves which are parallel far from the tip, and converging in  $\mathbf{X}_{S,j} = \mathbf{X}_{B,j}$  near the tip. These curves can be parametrically represented using quadratic Bézier curves (Fig. 4b) for the subsets preceding and following the tip  $\mathbf{X}_{S,j}$ , respectively, namely

$$\overline{\mathbf{X}}_{j-1}^{I}(\xi) = (1-\xi)^{2} \mathbf{X}_{S,j-1} + 2(1-\xi)\xi \mathbf{X}_{S,j}^{-} + \xi^{2} \mathbf{X}_{S,j}$$

$$\overline{\mathbf{X}}_{j}^{I}(\xi) = (1-\xi)^{2} \mathbf{X}_{S,j} + 2(1-\xi)\xi \mathbf{X}_{S,j}^{+} + \xi^{2} \mathbf{X}_{S,j+1}$$
(18)

Note that the overbar symbol here indicates that the material point belongs to the offset internal boundary  $\partial \overline{\Omega}^{I}$ . The two auxiliary vertices  $\mathbf{X}_{S,j}^{-}$ and  $\mathbf{X}_{S,j}^{+}$  are computed as

$$\begin{cases} \mathbf{X}_{S,j}^{-} = \mathbf{X}_{B,j} + \mathbf{s}_{j}^{-} \\ \mathbf{X}_{S,j}^{+} = \mathbf{X}_{B,j} + \mathbf{s}_{j}^{+} + \mathbf{s}_{j}^{+} \end{cases}$$
(19)

344 where

$$\mathbf{s}_{j}^{\prime} = \frac{3s}{2} \frac{\left(\mathbf{X}_{B,j+1} - \mathbf{X}_{B,j}\right)}{\|\mathbf{X}_{B,j+1} - \mathbf{X}_{B,j}\|}$$
(20)

and  $\mathbf{s}_j^-$ ,  $\mathbf{s}_j^+$  are calculated by means of Eq. (17), with s/2 in place of  $\bar{s}$ . 345 The curves start and finish in the main vertices  $\mathbf{X}_{S,j-1}$ ,  $\mathbf{X}_{S,j}$ ,  $\mathbf{X}_{S,j+1}$ , 346 but do not pass through the auxiliary vertices. Moreover, the two ends of 347 the curves are tangent to the external polygons (fine dashed lines in Fig. 348 4b), making the opening angle at the tip equal to the angle formed by the 349 vertices  $\mathbf{X}_{S,j}^-, \mathbf{X}_{S,j}, \mathbf{X}_{S,j}^+$ . Thus, the position of the auxiliary vertices can be 350 adjusted in order to generate a desired tip opening angle. The modulus of 351  $\mathbf{s}_j',$  controlling the opening angle at tip, is taken as equal to 3s/2 to achieve 352 an acute angle. 353



**Figure 4.** Offset scheme of the internal boundary vertices  $\mathbf{X}_{B,j}$ . (a) Standard vertex  $\mathbf{X}_{B,j}$ , (b) vertex  $\mathbf{X}_{B,j}$  corresponding to a cut tip.

Among all the possible types of joining dislocations applied to the in-354 ternal boundary  $\partial \Omega^{I}$ , the simplest is the unique "glued" line starting from 355 point A and finishing at point B (see Fig. 3). Accordingly, each pairing 356  $(\partial \overline{\Omega}_i^{I-}, \partial \overline{\Omega}_i^{I+})$ , starting from the first element of  $\mathcal{V}_S^I$  (point A), is automati-357 cally defined by combining the first subset with the last and so forth, with the 358 rule  $(\partial \overline{\Omega}_{2N-j+1}^{I}, \partial \overline{\Omega}_{j}^{I}), j = 1, ..., N$ . This means that the closure is uniquely 359 defined by point A. In other words, any ordered permutation of the vertices 360 in  $\mathcal{V}_S^I$  describes the same inner boundary, but each of them generates a dif-361 ferent closure, having different vertices of  $\partial \overline{\Omega}^I$  as first element. Thus, since 362  $\mathcal{V}_S^I$  is derived from  $\mathcal{V}_B^I$ , the latter must be ordered by a proper permutation 363 of vertices, so that the first vertex coincides with the point A of the desired 364 closure. 365

### 366 4.2. Meshing

Element size across the domain is controlled by a function  $h(\mathbf{X})$  which provides the average element size for every point  $\mathbf{X} \in \Omega$ . Geometrical discontinuities represent critical points  $\mathbf{X}_P$  that need to be meshed with a suitable refinement. In the algorithm, each vertex of  $\partial \overline{\Omega}^I$  contained in the set  $\mathcal{V}_S^I$  is treated as a critical point. Accordingly, the global refinement function  $h(\mathbf{X})$ is given by  $h(\mathbf{X}) = \min_i [h_{P,i}(\mathbf{X})]$ , with  $h_{P,i}$  being defined as

$$h_{P,i}(\mathbf{X}) = h_{\max} - \frac{h_{\max} - h_{\min}}{\left(\frac{\|\mathbf{X} - \mathbf{X}_{P,i}\|^2}{c_k^2} + 1\right)^a}$$
(21)

where *a* is an exponent controlling mesh grading and  $c_k$  is a parameter which defines the mesh refinement extension. Note that the function  $h_{P,i}(\mathbf{X})$  is such that  $h_{P,i}(\mathbf{X}) = h_{\min}$  for  $\|\mathbf{X} - \mathbf{X}_{P,i}\| \to 0$ , and  $h_{P,i}(\mathbf{X}) = h_{\max}$  for  $\|\mathbf{X} - \mathbf{X}_{P,i}\| \to \infty$ , being  $h_{\min}$  and  $h_{\max}$  the minimum and the maximum element sizes, respectively.

#### 378 4.3. Nodal dislocations

The distribution of element nodes along the internal offset boundary  $\partial \overline{\Omega}^{I}$ has to fulfil the refinement function  $h(\mathbf{X})$  and the linear mapping function  $\Gamma: \partial \overline{\Omega}_{i}^{I+} \to \partial \overline{\Omega}_{i}^{I-}$ . In order to satisfy both conditions at the same time, nodes are created on the subset requiring the largest number of nodes, and then copied onto the opposite side using the function  $\Gamma$ .

The mean element size along a subset  $\partial \overline{\Omega}_i^I$  can be computed using the mean value theorem  $h_{m,i} = \int_0^1 h(\overline{\mathbf{X}}_i^I(\xi)) \|\partial \overline{\mathbf{X}}_i^I/\partial \xi\| d\xi$ . Accordingly, the number of elements along the boundary is given by the ratio between the length of the subset  $\partial \overline{\Omega}_i^I$  and the mean element size  $h_{m,i}$ , namely

$$N_{e,i} = \frac{1}{h_{m,i}} \int_0^1 \left\| \frac{\partial \overline{\mathbf{X}}_i^I}{\partial \xi} \right\| d\xi .$$
 (22)

Nodes are then placed starting from the most critical subset, i.e. that 388 requiring the highest number of elements. The first node,  $\mathbf{N}_{i,1} = \overline{\mathbf{X}}_i^I(\xi_1 =$ 389 0)  $\in \partial \overline{\Omega}_i^I$ , coincides with the vertex  $\mathbf{X}_{S,i}$ . From the second node onwards, 390 each node  $\mathbf{N}_{i,n}$  is placed along  $\partial \overline{\Omega}_i^I$  increasing progressively  $\xi_n$  by an arbitrary 391 increment  $\Delta \xi$ , until the length  $\|\mathbf{N}_{i,n} - \mathbf{N}_{i,n-1}\|$  attains the length given by 392  $h(\overline{\mathbf{X}}_{i}^{I}(\frac{\xi_{n}+\xi_{n-1}}{2}))$ , so that each element size perfectly matches the dimension 393 defined by the function  $h(\mathbf{X})$ . After this procedure, the nodes generated 394 are replicated on the pairing subset. So, if nodes have been created first 395 on  $\partial \overline{\Omega}_i^{I+}$  their counterparts are replicated on  $\partial \overline{\Omega}_i^{I-}$  using  $\mathbf{N}_{i,n}^- = \Gamma(\mathbf{N}_{i,n}^+)$ . 396 The obtained nodal distribution is now suitable for applying the dislocation 397 constraint, Eq. (14), in a discretised way to each couple of nodes  $(\mathbf{N}_{i,n}^{-}, \mathbf{N}_{i,n}^{+})$ , 398  $n = 1, 2, ..., N_{e,i}$ , for each couple of subsets  $(\partial \overline{\Omega}_i^{I-}, \partial \overline{\Omega}_i^{I+}), i = 1, ..., N$ . 399

#### 400 5. Applicative examples

The constitutive model of the hyperelastic wrinkling membrane (Sect. 2.2) has been implemented in the commercial FE code ABAQUS, through a user-defined material subroutine UMAT.

A preliminary verification was performed by comparing the wrinkling model to the standard Ogden model implemented in ABAQUS, under uniaxial tension and uniaxial and equibiaxial compression. As expected, the stress-strain response in uniaxial tension is not influenced by wrinkling (Fig. 5a), while the uniaxial and biaxial compression cases highlight a significant  $_{409}$   $\,$  reduction of the compressive stresses due to wrinkling, by one and two orders

<sup>410</sup> of magnitude, respectively (Fig. 5b-c).



Figure 5. Normalized stress-stretch responses on a single plane stress element analysed with the standard Ogden function  $\Psi$  (solid blue line) and that based on the relaxed function  $\Psi_W$ , summarized in Tab. 1 (solid red lines). Material parameters:  $\mu_1 = 200 \text{ Pa}$ ,  $\alpha_1 = 9$ . (a) Uniaxial tension, (b) uniaxial compression, and (c) equibiaxial compression. The slightly stiffer response in (a), as well as the non-zero response under compression in (b) and (c), is due to the introduction of the additional fictitious stiffnessbased on the Ogden function  $\Psi$ , with properties  $\tilde{\mu} = \mu_1/100$  and  $\tilde{\alpha} = \alpha_1$ , to prevent ill-conditioning during slackness and wrinkling.

In order to explore the capabilities of the implemented wrinkling model 411 in predicting the non-linear response of hyperelastic membranes, three rele-412 vant illustrative examples are presented below. The first one is related to a 413 rectangular polyethylene membrane under bending, which was introduced by 414 Barsotti et al. [49] and later employed by Massabò and Gambarotta [41] to 415 test their wrinkling model. The second example refers to a rectangular sili-416 cone membrane containing an elliptical hole under tension, which was tested 417 experimentally by Spagnoli et al. [50]. Finally, in order to verify the capa-418 bilities of the proposed algorithm in generating complex internal boundaries, 419 the case of a Z-shaped cut under joining dislocations is presented. This latter 420 case represents an archetypal topology in reconstructive surgery procedures 421 on the human skin, e.g. see Hove et al. [9]. 422

#### 423 5.1. Rectangular beam under bending

The geometry consists of a rectangular elastomeric beam with span l =424 75 mm, height h = 25 mm and thickness t = 1 mm, laterally constrained 425 along the shorter edge, with imposed displacements  $u_2 = -6$  mm applied 426 to the central fifth of the lower edge [49]. The model has been discretised 427 with a uniform mesh of 1700 3-nodes plane stress isoparametric elements 428 (CPS3), with an average size  $h_{avg} = 1.5$  mm. The mechanical behaviour 429 of the material was described by Massabò and Gambarotta [41] using an 430 exponential isotropic Fung model. The equivalent Ogden constants, fitted 431 under uniaxial tension, are  $\mu_1 = 749.18$  Pa and  $\alpha_1 = 17.14$ . For the fictitious 432 elasticity tensor  $\tilde{c}^{\boldsymbol{\tau}^{\circ}}$ ,  $\tilde{\mu} = \mu_1/100$  and  $\tilde{\alpha} = \alpha_1$  are used. 433

Fig. 6 shows the resulting stretch domains, obtained by comparing the wrinkling model with the standard Ogden one. Taut regions, characterised

by two positive principal stress components, are highlighted in red, while 436 wrinkled regions, with only one positive principal stress, have been coloured 437 in green. Slack regions where both the principal stresses are negative are blue 438 marked. Note that, using the standard Ogden function, regions are simply 439 identified through the value of the principal stretches, see Tab. 1. The main 440 differences between the two models can be found in the central portion of 441 the membrane. With the standard Ogden model (Fig. 6a), a wide central 442 taut region exists, which is separated from the top slack domain by a small 443 strip of wrinkled membrane. Along the lateral edges, the membrane is taut 444 at top and slack at bottom with a triangular shape. In the case with the 445 wrinkling model (Fig. 6b), the top slack domain extends downward to the 446 taut region, which is here narrower. Furthermore, taut zones near the lateral 447 edges disappear. The domains in Fig. 6b compare qualitatively good with 448 those observed in experiments, see Fig. 6c. 449



Figure 6. Coloured maps of the taut (red), wrinkled (green) and slack (blue) domains, plotted onto the deformed configuration, in a rectangular membrane analysed with Ogden function (a) and the wrinkling model (b). (c) Experimental test on a clamped rectangular polyethylene membrane conducted by Barsotti et al. [49].

450

In Fig. 7, the total vertical reaction force along the constrained lower

edge is shown against the imposed displacement, showing a relaxation in the
situation where the wrinkling model is used.



Figure 7. Normalized force-displacement curves of the rectangular beam under bending. The global force F has been divided by the loaded area  $l/5 \cdot t$  and normalized with respect to  $\mu_1$ .

#### 453 5.2. Notched sheet under tension

The geometry consists of a rectangular silicone polymer sheet with width 454 l = 117 mm, height h = 234 mm and thickness t = 2 mm, containing a 455 centred elliptical notch, with semi-axes  $r_1 = 20$  mm and  $r_2 = 5$  mm aligned 456 with  $\mathbf{e}_1$  and  $\mathbf{e}_2$ , respectively. The material constants are  $\mu_1 = 0.461$  MPa 457 and  $\alpha_1 = 2$  [50]. Constants for the fictitious elasticity tensor  $\tilde{c}^{\tau^{\circ}}$  are taken 458 as in the previous example with  $\tilde{\mu} = \mu_1/100$  and  $\tilde{\alpha} = \alpha_1$ . The membrane 459 is clamped on top and bottom edges and is stretched by applying a vertical 460 displacement  $u_2 = 60$  mm, corresponding to a remotely applied stretch of 461  $\lambda_0 \simeq 1.25$ . The model has been discretised with 3-nodes plane stress isopara-462

<sup>463</sup> metric elements (CPS3), with a minimum size  $h_{min} = 0.6$  mm near the notch, <sup>464</sup> and a maximum size  $h_{max} = 5$  mm at the constrained edges.

A comparison of the analyses, conducted using the standard Ogden model 465 and the wrinkling model, is shown in Fig. 8. Both models show a characteris-466 tic X-shaped wrinkling region, which seems to extend larger for the membrane 467 analysed with the wrinkling model (Fig. 8b). Slack regions are small and lo-468 calized near the top and bottom edges of the ellipse. Interestingly, a relevant 469 difference in the shape of the deformed elliptical notch can be appreciated. 470 Initially wider in the horizontal direction, the inner ellipse transforms into 471 a circle upon deformation in the first case (Fig. 8a), and in a vertically ex-472 tended ellipse in the second (Fig. 8b). This behaviour highlights the effective 473 reduction of the stiffness in compression with the wrinkling model, allowing 474 higher compressive stretches in the transversal direction to develop. From a 475 qualitative point of view, the wrinkling model shows an improved prediction 476 of the actual wrinkling area, which in Fig. 8c is represented by the blue 477 butterfly-like region. In fact, since the vertical  $E_{22}$  strain is mostly positive 478 across the domain (Fig. 8d), the sign of the  $E_{11}$  component is discriminating 479 in identifying wrinkling areas. 480



Figure 8. Coloured maps of the taut (red), wrinkled (green) and slack (blue) domains, plotted onto the deformed configuration, in a notched rectangular membrane analysed with Ogden function (a) and the wrinkling model (b). Dotted boxes in (a) and (b) highlight the same area shown in the experimental full-field map of the components  $E_{11}$  and  $E_{22}$  of the Green-Lagrange strain tensor [50] (c)-(d).

481 Similarly to the previous example, the total vertical reaction force on the
482 clamped top edge illustrated in Fig. 9 confirms a slight relaxation obtained
483 with the wrinkling model.



**Figure 9.** Normalized force-displacement curves of the notched sheet under tension. The global force F has been divided by the section area  $l \cdot t$  and normalized with respect to  $\mu_1$ .

#### <sup>484</sup> 5.3. Z-shaped cut under joining dislocations

A Z-shaped cut under joining dislocations is here analysed. The selected 485 problem is relevant in the field of reconstructive surgery of human skin, where 486 cutting, tissue rearrangement and suturing of skin are performed in order to 487 achieve a desired configuration within the skin membrane. The operation 488 consists in three incisions of equal length, forming a Z-shaped cut, in which 489 the lateral limbs are slanting  $60^{\circ}$  with respect to the central one [9]. Then, 490 skin is undermined from the subcutaneous tissues and the two resulting trian-491 gular flaps are transposed each other and sutured in place. Such a procedure 492 can be simulated by imposing joining dislocation distributions (describing 493 flaps transposition) along an internal boundary (corresponding to the sur-494 gical cut). This last example is the benchmark to test the combination of 495

the wrinkling model with the proposed algorithm for describing topologicallycomplex internal boundaries, as presented in Sect. 3.

A circular skin membrane of R = 100 mm, containing incisions of l =498 50 mm, is considered. The domain has been discretised using three-node 499 plane stress isoparametric elements (CPS3), with element size ranging be-500 tween  $h_{\text{max}} = l/5$ ,  $h_{\text{min}} = l/150$  and refinement parameters a = 8,  $c_k = l$ . 501 The cut offset, needed for numerical reasons, is assumed to be equal to 502  $s = h_{max}/20$  (Fig. 10a). The skin parameters are  $\mu_1 = 110$  Pa and  $\alpha_1 = 26$ , 503 taken from the *in-vivo* measurements of Mahmud et al. [51]. The circular ex-504 ternal boundary is kept fixed, while the flap transposition is achieved though 505 MPCs prescribed along the two incision sides, according to the dislocation 506 distribution of Eq. (14). 507



Figure 10. Schematics of the circular membrane with Z-shaped cut before and after the application of joining dislocations. (a) Geometrical description of the problem, with the offset on the cut already applied. Coupled subsets  $(\partial \overline{\Omega}_1^{I+} \equiv \partial \overline{\Omega}_1^{I})$  and  $\partial \overline{\Omega}_1^{I-} \equiv \partial \overline{\Omega}_6^{I}, \partial \overline{\Omega}_2^{I+} \equiv \partial \overline{\Omega}_2^{I}$  and  $\partial \overline{\Omega}_2^{I-} \equiv \partial \overline{\Omega}_5^{I}, \partial \overline{\Omega}_3^{I+} \equiv \partial \overline{\Omega}_3^{I}$  and  $\partial \overline{\Omega}_3^{I-} \equiv \partial \overline{\Omega}_4^{I})$  have been highlighted with same colors. (b) Discretised FE model in the reference configuration, and (c) FE model after the analysis with the wrinkling model.

The membrane has been analysed using both the standard Ogden function 508 and the wrinkling model. Contours of the stretch domains, reported in Fig. 509 11, show a noticeable difference between the two models, being the latter 510 closer to the actual no-compression mechanical behaviour of undermined skin 511 membranes during reconstructive surgery procedures. Slack domains close 512 to the ends of the internal boundaries are much smaller and localised in 513 the case analysed with the wrinkling model. Furthermore, the triangular 514 flaps are predominantly taut, in contrast to the wrinkling predicted by the 515 standard model. However, a narrow region of wrinkles remains throughout 516 the length of the central limb, which extends with two further drop-shaped 517 regions along the same direction. 518



**Figure 11.** Coloured maps of the taut (red), wrinkled (green) and slack (blue) domains, plotted onto the deformed configuration, of a Z-shaped cut analysed with Ogden function (a) and the wrinkling model (b).

The distribution of radial stresses along a circular path with r = 1.5l and a radial path for  $\theta = 90^{\circ}$  (refer to Fig. 10c), is reported in Fig. 12 for the two

analysed models. The distribution in Fig. 12a is symmetric and highlights 521 an effective reduction of compressive stresses around  $\theta = 90^{\circ}$  and  $\theta = 270^{\circ}$ . 522 Indeed, those regions are actually in the slack and wrinkled domain, and the 523 red curve remains at a constant stress value of  $\sigma_{rr} \simeq 0$ . Looking at the radial 524 distribution at  $\theta = 90^{\circ}$  (Fig. 12b), the membrane analysed with the standard 525 Ogden function shows negative stresses along the whole path, displaying a 526 peak at r = 35 mm representing the point B. The wrinkling model, instead, 527 presents an almost constant distribution around  $\sigma_{rr} = 0$ , with just a small 528 zone, near r = 10 mm, having positive stresses. 529



Figure 12. Normalized radial stress  $\sigma_{rr}$  distribution along (a) a circular path with radius R = 1.5l, and (b) a radial path with angle  $\theta = 90^{\circ}$  of the Z-shaped cut with the standard Ogden function and the wrinkling model.

#### 530 6. Discussion and conclusions

The analysis of soft tissue membranes using numerical methods is a fun-531 damental step in order to understand the mechanical behaviour of these 532 thin structures under particular loading conditions, such as closure of holes 533 and cuts. This is achieved by applying joining dislocations on the inner 534 boundaries, that is, enforcing two distinct edges to move towards each other, 535 with reciprocal forces, until the overlap is reached, while maintaining a self-536 balanced stress field during the whole process. In this work, a FE framework 537 is adopted, in which the closure is simulated with Multi-Point Constraints 538 (MPCs) applied to the nodes of the inner boundaries. To meet the require-539 ments of a refined discretisation near geometrical discontinuities, and the ne-540 cessities of MPCs to have evenly discretised coupling edges, a pre-processing 541 algorithm for automatic FE models generation has been formulated and im-542 plemented in Matlab<sup>®</sup> environment. The code developed offers five main 543 advantages with respect to existing solutions: (i) the automatic mesh gener-544 ating code offers wide control on discretisation management; (ii) the geome-545 tries are discretised with a high-quality mesh; (iii) the domain geometry can 546 be generated parametrically, minimising user-requiring inputs; (iv) nodes on 547 the holes and cuts boundary are consistently distributed for MPCs; finally, 548 (v) the time required to generate models is greatly reduced. 549

With respect to the mechanical behaviour of the soft tissues, this has been considered using the well-known isotropic Ogden's strain-energy function. In order to consider the instability of membranes subjected to compressive forces, also known as wrinkling, the function has been modified, treating the out-of-plane displacements of the actual wrinkles as material non-linearities. Although this approach cannot provide a detailed description of the actual waves and wrinkles, it represents a highly efficient way to obtain a good approximation of the overall tension field within the membrane, avoiding the problems of stability and convergence of buckling and post-buckling analyses.

The examples presented have been carefully selected in order to illustrate 559 the potential of the proposed algorithm, both in accurately simulating mem-560 branes undergoing wrinkling and in precisely describing the application of 561 general dislocation distributions along topologically complex internal bound-562 aries. All the examples showed a redistribution of the stress and strain fields 563 when analysed with the wrinkling function, observing a relevant improve-564 ment in the qualitative prediction of the taut-wrinkling-slack regions in the 565 example of the membrane tested by Barsotti et al. [49]. In the case of a 566 Z-shaped cut, the slack region is significantly reduced, highlighting the capa-567 bility of the wrinkling model to redistribute compression stresses into tensions 568 in other membrane regions to achieve equilibrium. The closure of the cut, 560 which induced the transposition of the triangular flaps, has been achieved 570 automatically during the FE computation, without convergence problems. 571

The proposed code is robust, and can generate cuts and holes of whichever 572 shape with high efficiency. However, further improvements are planned in 573 order to include the analysis of multiple holes, as well as layered and curved 574 membranes. This can be done without changing the basic theory of closing 575 holes herein presented, as it has been formulated for membranes placed in 576  $\mathbb{R}^3$  presenting n holes. As a result, the range of possible applications will be 577 further expanded, for instance to simulate kirigami tessellations [12], dorsal 578 closure of drosophila embryos [52], or the V-Y advancement flap in facial 579

<sup>580</sup> reconstructive surgery [53], to mention only a few relevant examples.

### 581 7. Declaration of Competing Interest

The authors declare no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

### <sup>584</sup> Appendix A. Structure of the algorithm

As described in Sect. 4, the proposed algorithm combines auto-meshing and generation of complex dislocation distributions in Matlab<sup>®</sup> environment, employing the open source auto-meshing tool DistMesh. The general structure of the code, composed of seven main functions, is shown in Tab. 1. The main functions are described in detail below.

#### Table 1: Structure of the code

590

```
: E_{shape}, I_{shape}
    geometry shapes
    geometry parameters: \{E_{p1}, E_{p2}, ...\}, \{I_{p1}, I_{p2}, ...\}, t, s
                                                  : h_{min}, h_{max}, a, c_k
    mesh parameters
    material properties
                                                 : \mu_i, \alpha_i
    pre-stress tensor
                                                  : \boldsymbol{\sigma}_0
1: \mathcal{V}_B^E, \mathcal{V}_B^I \leftarrow \textbf{call}
      GEOMBOUNDARIES (E_{shape}, I_{shape}, \{E_{p1}, E_{p2}, ...\}, \{I_{p1}, I_{p2}, ...\});
2: \mathcal{V}_{S}^{I} \leftarrow \text{call INTERNALOFFSET}(\mathcal{V}_{B}^{I},s);
3: d(\mathbf{X}) = SIGNEDDISTANCE(\mathbf{X}, \mathcal{V}_B^E, \mathcal{V}_S^I);
4: h(\mathbf{X}) = \text{ELEMENTSIZE}(\mathbf{X}, \mathcal{V}_S^I, h_{min}, h_{max}, a, c_k);
5: \mathcal{V}_N^{I-}, \mathcal{V}_N^{I+} \leftarrow \text{call InternalBoundaryNodes}(h(\mathbf{X}), \mathcal{V}_S^I, \mathcal{V}_B^I);
6: \mathcal{V}_N, \mathcal{C} \leftarrow \text{call DistMesh}(d(\mathbf{X}), h(\mathbf{X}), \{\mathcal{V}_N^{I-}, \mathcal{V}_B^{I+}, \mathcal{V}_B^E\});
7: call PRINTINPUTABQ(\mathcal{V}_N, \mathcal{C}, t, \mu_i, \alpha_i, \mathcal{V}_N^{I-}, \mathcal{V}_N^{I+}, \boldsymbol{\sigma}_0);
```

Initially, all the information about geometry, element size, material prop-591 erties and pre-stress field, is read. Note that this is the only code section 592 requiring user input. Depending on the shape chosen for the outer boundary 593 through the text variable  $E_{shape}$  (or  $I_{shape}$  for the inner boundary), the first 594 main function GEOMBOUNDARIES (Tab. 2) computes the set of external 595 boundary vertices  $\mathcal{V}_B^E$  (internal boundary vertices  $\mathcal{V}_B^I$ ), addressing the task 596 to the proper function ExtSHAPE-n (INTSHAPE-n) and taking the set of 597 geometrical parameters  $\{E_{p1}, E_{p2}, ...\}$  ( $\{I_{p1}, I_{p2}, ...\}$ ) as input. 598

1: Function

		GEOMBOUNDARIES ( $E_{shape}, I_{shape}, \{E_{p1}, E_{p2},\}, \{I_{p1}, I_{p2},\}$ )
	2:	switch $E_{shape}$ do compute the set $\mathcal{V}_B^E$
	3:	case (external shape 1) do
	4:	$\mathcal{V}_B^E \leftarrow \textbf{call ExtShape-1}(\{E_{p1}, E_{p2},\})$
	5:	case (external shape 2) do
	6:	$\mathcal{V}_B^E \leftarrow \textbf{call ExtShape-2}(\{E_{p1}, E_{p2},\})$
599	7:	
	8:	switch $I_{shape}$ do compute the set $\mathcal{V}_B^I$
	9:	case (internal shape 1) do
	10:	$\mathcal{V}_B^I \leftarrow \mathbf{call} \text{ IntShape-1}(\{I_{p1}, I_{p2},\})$
	11:	case (internal shape 2) do
	12:	$\mathcal{V}_B^I \leftarrow \mathbf{call} \text{ IntShape-2}(\{I_{p1}, I_{p2},\})$
	13:	
	14:	$\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ $

The second main function, INTERNALOFFSET (Tab. 3), performs the offset of the internal boundaries as described in Sect. 4.1, generating the set  $\mathcal{V}_S^I$  from  $\mathcal{V}_B^I$ .

	Ta	ble 3: Offset of internal boundary vertices		
	1: Function InternalOffset $(\mathcal{V}_B^I, s)$			
	2:	2: foreach $\left( \mathbf{X}_{B,j} \in \mathcal{V}_B^I  ight)$ do		
	3:	S: Compute the angle $\Delta \theta$ at the vertex $\mathbf{X}_{B,j}$ between the		
		segments $\mathbf{X}_{B,j} - \mathbf{X}_{B,j-1}$ and $\mathbf{X}_{B,j+1} - \mathbf{X}_{B,j}$ ;		
	4:	if $(\Delta \theta = \pi)$ then		
	5:	Compute $\mathbf{s}_j^-$ , $\mathbf{s}_j^+$ , $\mathbf{s}_j'$ using Eqs. (17) and (20) with $\bar{s} = s/2$ ;		
	6:	Set $\mathbf{X}_{S,j} \equiv \mathbf{X}_{B,j}$ ;		
603	7:	Compute $\mathbf{X}_{S,j}^-$ , $\mathbf{X}_{S,j}^+$ using Eqs. (19);		
	8:	Enqueue the set $\{\mathbf{X}_{S,j}^{-}, \mathbf{X}_{S,j}, \mathbf{X}_{S,j}^{+}\}$ into $\mathcal{V}_{S}^{I}$ ;		
	9:	else		
	10:	Compute $\mathbf{s}_j^-$ , $\mathbf{s}_j^+$ using Eq. (17) with		
		$\bar{s} = s/(2(1 + \cos \Delta\theta));$		
	11:	Compute $\mathbf{X}_{S,j}$ using Eq. (16);		
	12:	Enqueue $\mathbf{X}_{S,j}$ into $\mathcal{V}_S^I$ ;		
	13: $\begin{bmatrix}                                   $			

The third main function, SIGNEDDISTANCE, computes the signed dis-604 tance function  $d(\mathbf{X})$ , giving as output negative numbers for points  $\mathbf{X}$  inside 605 the domain, and positive elsewhere. It is a fundamental argument for the 606 DistMesh auto-meshing tool as it identifies the region to mesh, and it is 607 defined by subtraction of the two surfaces enclosed within the external and 608 internal boundaries. More details about this function can be found in [34]. 609 The fourth main function, **ELEMENTSIZE** (Tab. 4), provides  $h(\mathbf{X})$ , the ele-610 ment size parameter required in order to refine the mesh around geometrical 611

discontinuities. According to Sect. 4.2, the function refines the mesh around each vertex  $\mathbf{X}_{S,j} \in \mathcal{V}_S^I$ .

	Ta	ble 4: Mesh refinement	
	1: Function ELEMENTSIZE( $\mathbf{X}, \mathcal{V}_{S}^{I}, h_{min}, h_{max}, a, c_{k}$ )		
	2:	$\mathbf{foreach}\;ig(\mathbf{X}_{S,j}\in\mathcal{V}_S^Iig)\;\mathbf{do}$	
	3:	Compute element dimension $h_{S,j}$ in <b>X</b> according to the	
614		refinement function for the vertex $\mathbf{X}_{S,j}$ using Eq. (21);	
	4:	Compute element dimension in <b>X</b> as the minimum $h_{S,j}$ value:	
		$h = \min_{j} \left[ h_{S,j} \right];$	
	5:	return h	

Internal boundary nodes are generated following the procedure described in Sect. 4.3 through the main function INTERNALBOUNDARYNODES, and stored into the sets  $\mathcal{V}_N^{I-}$ ,  $\mathcal{V}_N^{I+}$ , pertaining to the left and right boundary sides, respectively.

	Ta	<b>ble 5:</b> Generation of internal boundary nodes
	1: <b>I</b>	Function InternalBoundaryNodes $(h(\mathbf{X}), \mathcal{V}_{S}^{I}, \mathcal{V}_{B}^{I})$
	2:	$N \leftarrow (\operatorname{size}(\mathcal{V}_B^I) - 1)/2$ number of coupled internal boundary
		subsets $(\partial \overline{\Omega}_i^{I-}, \partial \overline{\Omega}_i^{I+});$
	3:	Set as first elements of $\mathcal{V}_N^{I-}$ and $\mathcal{V}_N^{I+}$ the first vertex of $\mathcal{V}_S^{I}$ , i.e. A;
	4:	$\mathbf{N}_{k^-,n-1}, \mathbf{N}_{k^+,n-1} \leftarrow A;$
	5:	for $j \leftarrow 1$ to N do
	6:	Use the subset matching rule $(\partial \overline{\Omega}_{k^-}^I, \partial \overline{\Omega}_{k^+}^I)$ , where $k^+ = j$ and
		$k^- = 2N - j + 1;$
	7:	$\overline{\mathbf{X}}_{k^{-}}^{I}(\xi) = \mathbf{B} \acute{\mathbf{E}} \mathbf{Z} \mathbf{I} \mathbf{E} \mathbf{R} \mathbf{C} \mathbf{U} \mathbf{R} \mathbf{V} \mathbf{E} (\xi, k^{-}, \mathcal{V}_{S}^{I});$
	8:	$\overline{\mathbf{X}}_{k^+}^{I}(\xi) = \mathrm{B}\mathrm{\acute{e}zierCurve}(\xi, k^+, \mathcal{V}_S^{I});$
	9:	Compute the numbers of elements $N_{e,k^-}$ , $N_{e,k^+}$ required to
		discretize $\partial \overline{\Omega}_{k^-}^I$ and $\partial \overline{\Omega}_{k^+}^I$ , respectively, using Eq. (22);
	10:	$\xi_{n-1}, \xi_n \leftarrow 0;$
	11:	if $(N_{e,k^+} \ge N_{e,k^-})$ then
		generate nodes on $\partial \Omega_{k^+}^I$ and replicate them on $\partial \Omega_{k^-}^I$ :
619	12:	while $(\xi_n \leq 1)$ do
	13:	while $(\xi_n \leq 1)$ do
	14:	$ \begin{bmatrix} \xi_n \leftarrow \xi_n + \Delta \xi; \\ \xi_n \leftarrow \xi_n + \Delta \xi; \end{bmatrix} $
	15:	$\mathbf{N}_{k^+,n} \leftarrow \mathbf{X}_{k^+}^{\mathbf{I}}(\xi_n);$
	16:	$ \qquad \qquad$
		Exit loop;
	17:	$  \qquad   \qquad \mathbf{N}_{k^-,n} = \Gamma(\mathbf{N}_{k^+,n});$
	18:	Enqueue $\mathbf{N}_{k^+,n}$ and $\mathbf{N}_{k^-,n}$ into $\mathcal{V}_N^{I+}$ and $\mathcal{V}_N^{I-}$ ,
		respectively;
	19:	$\xi_{n-1} \leftarrow \xi_n;$
	20:	$ \begin{bmatrix} \mathbf{N}_{k^+,n-1} \leftarrow \mathbf{N}_{k^+,n} \text{ and } \mathbf{N}_{k^-,n-1} \leftarrow \mathbf{N}_{k^-,n}; \end{bmatrix}$
	21:	else if $(N_{e,k^+} < N_{e,k^-})$ then
		generate nodes on $\partial \overline{\Omega}_{k^-}^I$ and replicate them on $\partial \overline{\Omega}_{k^+}^I$ :
	22:	Do the same operations from line 12 to 20, but on the
		$k^{-}$ -th subset;
	23:	$\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ $

Parametric functions  $\overline{\mathbf{X}}_{k^-}^I(\xi)$ ,  $\overline{\mathbf{X}}_{k^+}^I(\xi)$ , in lines 8 and 7 of Tab. 5, are

621 defined by **BÉZIERCURVE**, shown in Tab. 6.

Ta	ble 6: Bézier curve definition	
1: <b>F</b>	1: Function BézierCurve( $\xi, k, \mathcal{V}_S^I$ )	
2:	if $\left(\mathbf{X}_{S,k+1}^{-} \text{ is between } \mathbf{X}_{S,k} \text{ and } \mathbf{X}_{S,k+1}\right)$ then	
	use quadratic Bézier curve from Eq. (18)a:	
3:	$\mathbf{X} = \overline{\mathbf{X}}_{k}^{I}(\xi) = (1 - \xi)^{2} \mathbf{X}_{S,k} + 2(1 - \xi)\xi \mathbf{X}_{S,k+1}^{-} + \xi^{2} \mathbf{X}_{S,k+1};$	
4:	else if $(\mathbf{X}_{S,k}^+$ is between $\mathbf{X}_{S,k}$ and $\mathbf{X}_{S,k+1}$ ) then	
	use quadratic Bézier curve from Eq. $(18)$ b:	
5:	$\mathbf{X} = \overline{\mathbf{X}}_k^I(\xi) = (1-\xi)^2 \mathbf{X}_{S,k} + 2(1-\xi)\xi \mathbf{X}_{S,k}^+ + \xi^2 \mathbf{X}_{S,k+1};$	
6:	else use linear Bézier curve from Eq. $(15)$ :	
7:	$\mathbf{X} = \overline{\mathbf{X}}_k^I(\xi) = (1 - \xi)\mathbf{X}_{S,k} + \xi\mathbf{X}_{S,k+1};$	
8:	return X	
	Ta           1:         I           2:         3:           4:         5:           6:         7:           8:         1	

After computing the required variables, automeshing begins. Nodes in  $\mathcal{V}_N^{I-}, \mathcal{V}_N^{I+}, \mathcal{V}_B^E$  are fixed, i.e. they do not change their position over DistMesh iterations to find the optimal discretization. Once convergence is reached, the set  $\mathcal{V}_N$  containing all nodes, and the connectivity matrix  $\mathcal{C}$ , are given as output. Then, all the information can be printed in an input file for any FE solver. In this work, the input file is written for ABAQUS commercial software using the PRINTINPUTABQ function.

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