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# Sublinear Longest Path Transversals

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## Abstract

We show that connected graphs admit sublinear longest path transversals. This improves an earlier result of Rautenbach and Sereni and is related to the fifty-year-old question of whether connected graphs admit longest path transversals of constant size. The same technique allows us to show that 2-connected graphs admit sublinear longest cycle transversals.

## 1 Introduction

A classical exercise in graph theory is to show that if  $P$  and  $Q$  are longest paths in a connected graph, then the vertex sets of  $P$  and  $Q$  have non-empty intersection (see [8], exercise 1.2.40). In 1966, Gallai [2] asked whether this result could be strengthened to assert that the family of all longest paths in a connected graph  $G$  has non-empty intersection. It turns out the answer is no, as shown by Walther [6] with a 25-vertex counterexample. A 12-vertex counterexample, due to Walther and Voss [7] and independently Zamfirescu [10], is obtained from the Petersen graph by replacing one vertex  $v$  with an independent set  $\{v_1, v_2, v_3\}$  such that each  $v_i$  becomes an endpoint of an edge incident to  $v$  (see Figure 1).

Since Gallai's question has a negative answer, a single vertex is generally insufficient to meet every longest path in a connected graph  $G$ . A *longest path transversal* in  $G$  is a set of vertices that intersects every longest path. Such a set is a transversal in the hypergraph on  $V(G)$  whose edges are the vertex sets of longest paths in  $G$ . Let  $\text{lpt}(G)$  be the minimum size of a longest path transversal in  $G$ . The graph  $G_0$  in Figure 1 is a connected 12-vertex graph with  $\text{lpt}(G_0) = 2$ . Grünbaum [3] constructed a connected 324-vertex graph  $G$  with  $\text{lpt}(G) = 3$ . Soon afterward, Zamfirescu [10] found such a graph with 270 vertices. Walther [6] and Zamfirescu [9] asked if  $\text{lpt}(G)$  is bounded for connected graphs  $G$ , and this remains

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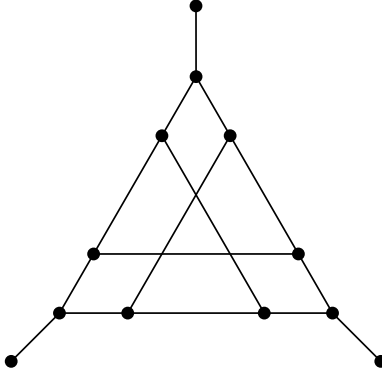


Figure 1: The graph  $G_0$ : a 12-vertex graph with  $\text{lpt}(G_0) = 2$ .

open. In fact, it is not known whether there is a connected graph  $G$  with  $\text{lpt}(G) \geq 4$ . Let  $G$  be a connected graph. Since a connected graph does not contain vertex-disjoint longest paths, every partition of  $V(G)$  into two sets has a part that contains no longest path in  $G$ , forcing the other part to be a longest path transversal. Applying this to a partition of  $V(G)$  into two parts of nearly equal size gives  $\text{lpt}(G) \leq \lceil n/2 \rceil$  when  $G$  is an  $n$ -vertex connected graph. It is not too difficult to improve this argument to obtain  $\text{lpt}(G) \leq \lceil n/4 \rceil$ . Rautenbach and Sereni [4] showed that  $\text{lpt}(G) \leq \lceil \frac{n}{4} - \frac{n^{2/3}}{90} \rceil$  for every connected  $n$ -vertex graph  $G$ . We show that  $\text{lpt}(G) \leq 8n^{3/4}$  when  $G$  is an  $n$ -vertex connected graph, implying that connected graphs have sublinear longest path transversals.

Let  $\text{lct}(G)$  be the minimum size of a set of vertices  $S$  such that  $S$  intersects every longest cycle in  $G$ . Analogously to the case of longest paths in 1-connected graphs, every pair of longest cycles in a 2-connected graph intersect. The Petersen graph  $G$  is 2-connected and  $\text{lct}(G) = 2$ . With no connectivity assumptions, Thomassen [5] showed that  $\text{lct}(G) \leq \lceil n/3 \rceil$  for each  $n$ -vertex graph  $G$ . The bound is sharp when  $G$  is a disjoint union of triangles and nearly sharp in the 1-connected case when  $G$  is obtained from a star with  $(n-1)/3$  leaves by replacing each leaf with a triangle. On the other hand, Rautenbach and Sereni [4] proved that if  $G$  is 2-connected, then  $\text{lct}(G) \leq \lceil \frac{n}{3} - \frac{n^{2/3}}{36} \rceil$ . We show that  $\text{lct}(G) \leq 20n^{3/4}$  when  $G$  is 2-connected (Corollary 2).

The problems of finding small longest path transversals and small longest cycle transversals are special cases of a general problem that we aim to address. Given a multigraph  $F$  and an edge  $e \in E(F)$  with endpoints  $u$  and  $v$ , the *subdivision operation* produces a new multigraph  $F'$  in which  $e$  is replaced by a path  $uvw$  through a new vertex  $w$  in  $F'$ . A *subdivision* of  $F$  is a graph obtained from  $F$  via a sequence of zero or more subdivision operations. For a multigraph  $R$  and a graph  $G$ , an  $R$ -subdivision in  $G$  is a subgraph of  $G$  isomorphic to a subdivision of  $R$ . We ask for a small set of vertices in  $G$  that intersects every  $R$ -subdivision in  $G$  of maximum size. The cases of longest path transversals and longest cycle transversals arise as  $R = P_2$  and  $R = C_2$  (the multigraph 2-vertex cycle), respectively. We prove that for each connected multigraph  $R$ , if the family  $\mathcal{F}$  of maximum  $R$ -subdivisions in  $G$  is pairwise intersecting, then  $\mathcal{F}$  admits a transversal of size at most  $Cn^{3/4}$ , where  $C$  is a constant depending on  $R$ .

## 2 Maximum subdivision transversals

Let  $R$  be a multigraph. Recall that an  $R$ -subdivision in  $G$  is a subgraph of  $G$  isomorphic to a subdivision of  $R$ , and a *maximum  $R$ -subdivision* is an  $R$ -subdivision  $F$  in  $G$  that maximizes  $|V(F)|$ . An  $R$ -*transversal* of  $G$  is a set of vertices intersecting each maximum  $R$ -subdivision. Let  $\tau_R(G)$  be the minimum size of an  $R$ -transversal in  $G$ .

Given sets of vertices  $X$  and  $Y$  of  $G$ , an  $(X, Y)$ -*separator* is a set of vertices  $S$  such that no path in  $G - S$  has one endpoint in  $X$  and the other endpoint in  $Y$ . We allow an  $(X, Y)$ -separator to contain vertices in  $X$  and  $Y$ . An  $(X, Y)$ -*connector* is a collection of vertex-disjoint paths  $\{P_1, \dots, P_k\}$  such that each  $P_i$  has one endpoint in  $X$ , the other endpoint in  $Y$ , and the interior vertices of  $P_i$  are outside  $X \cup Y$ . A variant of Menger's Theorem asserts that the minimum size of an  $(X, Y)$ -separator equals the maximum size of an  $(X, Y)$ -connector (see, e.g., Theorem 3.3.1 in [1]).

Our next result shows that when the maximum  $R$ -subdivisions in a graph  $G$  pairwise intersect,  $G$  has sublinear  $R$ -transversals. We make no attempt to optimize the multiplicative constant 8 or the dependence on  $m$ .

**Theorem 1.** *Let  $R$  be a connected  $m$ -edge multigraph with  $m \geq 1$  and let  $G$  be an  $n$ -vertex graph. If the maximum  $R$ -subdivisions in  $G$  pairwise intersect, then  $\tau_R(G) \leq 8m^{5/4}n^{3/4}$ .*

*Proof.* Let  $m = |E(R)|$  and let  $\varepsilon = 2(m/n)^{1/4}$ . We may assume that  $m \leq n$ , since otherwise we may take  $V(G)$  as our  $R$ -transversal. Let  $\mathcal{F}$  be the family of maximum  $R$ -subdivisions in  $G$ . An  $\varepsilon$ -*partial transversal* is a triple  $(H, X, Y)$  such that  $H$  is a subgraph of  $G$ ,  $X = V(G) - V(H)$ ,  $Y \subseteq X$  with  $|Y| \leq \varepsilon|X|$ , and each  $F \in \mathcal{F}$  is a subgraph of  $H$  or contains a vertex in  $Y$ . Given an  $\varepsilon$ -partial transversal  $(H, X, Y)$ , we either obtain an  $\varepsilon$ -partial transversal  $(H', X', Y')$  with  $|V(H')| < |V(H)|$  or we produce an  $R$ -transversal with at most  $8m^{5/4}n^{3/4}$  vertices. Starting with  $(H, X, Y) = (G, \emptyset, \emptyset)$  and iterating gives the result.

Let  $(H, X, Y)$  be an  $\varepsilon$ -partial transversal, and let  $\mathcal{F}_0$  be the set of  $F \in \mathcal{F}$  such that  $F$  is a subgraph of  $H$ . We may assume that  $H$  contains vertex-disjoint paths  $P_1$  and  $P_2$  each of size  $\lceil \varepsilon n \rceil$ . Otherwise, every path in  $H$  has size less than  $2 \lceil \varepsilon n \rceil$ , and so each  $F \in \mathcal{F}_0$  has at most  $2m \lceil \varepsilon n \rceil$  vertices. Since  $\mathcal{F}_0$  is pairwise intersecting, we have that  $V(F) \cup Y$  is an  $R$ -transversal for each  $F \in \mathcal{F}_0$ . It follows that  $\tau_R(G) \leq |Y| + 2m \lceil \varepsilon n \rceil \leq \varepsilon n + 2m \lceil \varepsilon n \rceil \leq (2m + 1)\varepsilon n + 2m \leq (2m + 2)\varepsilon n \leq 4m\varepsilon n = 8m^{5/4}n^{3/4}$ .

Suppose that  $H$  has a  $(V(P_1), V(P_2))$ -separator  $S$  of size at most  $\varepsilon^2 n$ . Since graphs in  $\mathcal{F}_0$  are connected, each  $F \in \mathcal{F}_0$  has a vertex in  $S$  or is contained in some component of  $H - S$ . Also, since  $\mathcal{F}_0$  is pairwise intersecting, at most one component  $H'$  of  $H - S$  contains graphs in  $\mathcal{F}_0$ . Since  $S$  is a separator,  $H'$  is disjoint from at least one of  $\{P_1, P_2\}$ . With  $X' = V(G) - V(H')$  and  $Y' = Y \cup S$ , we have  $|X'| - |X| \geq \varepsilon n$  and  $|Y'| = |Y| + |S| \leq \varepsilon|X| + \varepsilon^2 n \leq \varepsilon|X'| + \varepsilon(|X'| - |X|) \leq \varepsilon|X'|$ . It follows that  $(H', X', Y')$  is an  $\varepsilon$ -partial transversal. Also  $|V(H')| < |V(H)|$  since  $|X'| > |X|$ .

Otherwise, by Menger's Theorem,  $H$  has a  $(V(P_1), V(P_2))$ -connector  $\mathcal{P}$  with  $|\mathcal{P}| \geq \varepsilon^2 n$ . Let  $\mathcal{P}'$  be the set of paths in  $\mathcal{P}$  of size at most  $2/\varepsilon^2$ . Note that  $|\mathcal{P}'| \geq |\mathcal{P}|/2$ , or else  $\mathcal{P}$  has at least  $(\varepsilon^2 n)/2$  paths of size more than  $2/\varepsilon^2$ , contradicting that the paths in  $\mathcal{P}$  are disjoint. So we have  $|\mathcal{P}'| \geq |\mathcal{P}|/2 \geq (\varepsilon^2/2)n = 2m^{1/2}n^{1/2} \geq 2$ . Combining  $P_1$  with two paths in  $\mathcal{P}'$  whose endpoints in  $V(P_1)$  are as far apart as possible and a segment of  $P_2$  gives a cycle  $C_0$  such that  $(\varepsilon^2/2)n \leq |V(C_0)| \leq 2 \lceil \varepsilon n \rceil + 4/\varepsilon^2 - 4 \leq 2\varepsilon n + 4/\varepsilon^2$ , where the lower bound

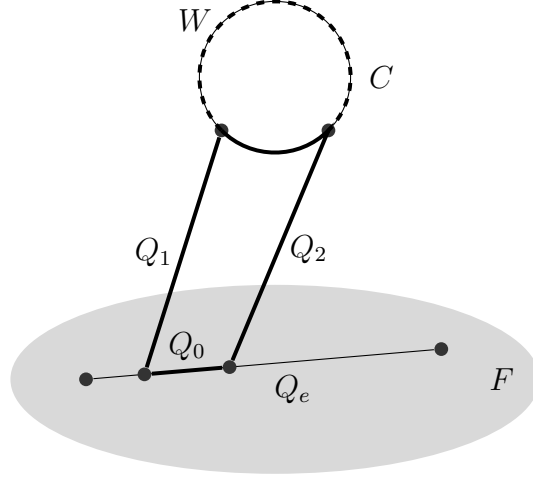


Figure 2:  $(V(C), V(F))$ -connector case. The subpath  $W$  of the cycle  $C$  is dashed, and the cycle  $D$  is displayed in bold.

counts vertices in  $V(P_1) \cap V(C_0)$  and the upper bound counts at most  $2 \lceil \varepsilon n \rceil$  vertices in  $(V(P_1) \cup V(P_2)) \cap V(C_0)$ , at most  $4/\varepsilon^2$  vertices on the paths in  $\mathcal{P}'$  linking  $P_1$  and  $P_2$ , and observing that the 4 endpoints of the linking paths are counted twice.

Let  $C$  be a longest cycle in  $H$  subject to  $|V(C)| \leq 2\varepsilon n + 4/\varepsilon^2$ , let  $\ell = |V(C)|$ , and note that  $\ell \geq |V(C_0)| \geq (\varepsilon^2/2)n$ . If  $V(C)$  intersects each subgraph in  $\mathcal{F}_0$ , then  $Y \cup V(C)$  witnesses  $\tau_R(G) \leq |V(C)| + |Y| \leq (2\varepsilon n + 4/\varepsilon^2) + \varepsilon n = 3\varepsilon n + (n/m)^{1/2} < 8m^{5/4}n^{3/4}$ . Otherwise, choose  $F \in \mathcal{F}_0$  that is disjoint from  $C$ . We may assume  $|V(F)| \geq \ell$ , or else  $Y \cup V(F)$  witnesses that  $\tau_R(G) \leq |V(F)| + |Y| < (2\varepsilon n + 4/\varepsilon^2) + \varepsilon n < 8m^{5/4}n^{3/4}$ .

If  $H$  has a  $(V(C), V(F))$ -separator  $T$  of size at most  $\varepsilon\ell$ , then we obtain an  $\varepsilon$ -partial transversal as follows. At most one component  $H'$  of  $H - T$  contains graphs in  $\mathcal{F}_0$ . Let  $X' = V(G) - V(H')$  and let  $Y' = Y \cup T$ . Since  $H'$  is disjoint from one of  $\{C, F\}$ , it follows that  $|X'| - |X| \geq \ell$ . We compute  $|Y'| = |Y| + |T| \leq \varepsilon|X| + \varepsilon\ell \leq \varepsilon|X| + \varepsilon(|X'| - |X|) \leq \varepsilon|X'|$ . Hence  $(H', X', Y')$  is an  $\varepsilon$ -partial transversal with  $|V(H')| < |V(H)|$ .

Otherwise,  $H$  has a  $(V(C), V(F))$ -connector  $\mathcal{Q}$  with  $|\mathcal{Q}| \geq \varepsilon\ell$ . We use  $\mathcal{Q}$  to obtain a contradiction. For  $e \in E(R)$ , let  $Q_e$  be the path in  $F$  corresponding to  $e$ , and let  $\mathcal{Q}_e$  be the set of paths in  $\mathcal{Q}$  which have an endpoint in  $Q_e$ . Since  $|E(R)| = m$ , it follows that  $|\mathcal{Q}_e| \geq |\mathcal{Q}|/m \geq \varepsilon\ell/m$  for some edge  $e \in E(R)$ . Let  $\mathcal{Q}'$  be the set of paths in  $\mathcal{Q}_e$  of size at most  $\frac{2mn}{\varepsilon\ell}$ , and note that  $|\mathcal{Q}'| \geq |\mathcal{Q}_e|/2 \geq \frac{\varepsilon\ell}{2m}$ , or else  $\mathcal{Q}_e$  has at least  $\frac{\varepsilon\ell}{2m}$  paths of size more than  $\frac{2mn}{\varepsilon\ell}$ , a contradiction. The endpoints of paths in  $\mathcal{Q}'$  divide  $Q_e$  into  $|\mathcal{Q}'| - 1$  edge-disjoint subpaths. Choose  $Q_1, Q_2 \in \mathcal{Q}'$  to minimize the length of such a subpath  $Q_0$  of  $Q_e$ , and note that  $Q_0$  has length at most  $\frac{n-1}{|\mathcal{Q}'|-1}$ ; see Figure 2. Since  $m \leq n$ , we have  $2m \leq 2m^{3/4}n^{1/4} = \frac{\varepsilon^3}{4}n \leq \frac{\varepsilon\ell}{2}$ , and hence  $\frac{n-1}{|\mathcal{Q}'|-1} < \frac{n}{\frac{\varepsilon\ell}{2m}-1} = \frac{2mn}{\varepsilon\ell-2m} \leq \frac{4mn}{\varepsilon\ell}$ .

The endpoints of  $Q_1$  and  $Q_2$  on  $C$  partition  $C$  into two subpaths; let  $W$  be the longer subpath. If  $|E(W)| \geq |E(Q_0)|$ , then we would obtain a larger  $R$ -subdivision by using  $Q_1$ ,  $W$ , and  $Q_2$  to bypass  $Q_0$ . Since  $F$  is a maximum  $R$ -subdivision, we have  $|E(W)| < |E(Q_0)|$ . Therefore using  $Q_1$ ,  $Q_0$ , and  $Q_2$  to bypass  $W$  gives a cycle  $D$  with  $|E(D)| > |E(C)|$ . By the extremal choice of  $C$ , it follows that  $|V(D)| > 2\varepsilon n + 4/\varepsilon^2$ . On the other hand,  $|V(D)| =$

$$|E(D)| \leq \frac{\ell}{2} + |E(Q_1)| + |E(Q_0)| + |E(Q_2)| \leq \frac{\ell}{2} + \frac{2mn}{\varepsilon\ell} + \frac{4mn}{\varepsilon\ell} + \frac{2mn}{\varepsilon\ell} = \frac{\ell}{2} + \frac{8mn}{\varepsilon\ell}.$$

Therefore  $2\varepsilon n + \frac{4}{\varepsilon^2} < |V(D)| \leq \frac{\ell}{2} + \frac{8mn}{\varepsilon\ell} \leq \varepsilon n + \frac{2}{\varepsilon^2} + \frac{8mn}{\varepsilon\ell} \leq \varepsilon n + \frac{2}{\varepsilon^2} + \frac{16m}{\varepsilon^3}$ , where the last inequality uses  $\ell \geq (\varepsilon^2/2)n$ . Simplifying gives  $\varepsilon n < \frac{16m}{\varepsilon^3} - \frac{2}{\varepsilon^2} < \frac{16m}{\varepsilon^3}$ , and this inequality is violated when  $\varepsilon \geq (16m/n)^{1/4}$ .  $\square$

Applying Theorem 1, we obtain the following corollary.

**Corollary 2.** *Let  $G$  be an  $n$ -vertex graph. If  $G$  is connected, then  $\text{lpt}(G) \leq 8n^{3/4}$ . If  $G$  is 2-connected, then  $\text{lct}(G) \leq 20n^{3/4}$ .*

*Proof.* When  $R = P_2$ , an  $R$ -transversal is a longest path transversal. It is well known that if  $G$  is connected, then the longest paths pairwise intersect. By Theorem 1, we have  $\text{lpt}(G) = \tau_R(G) \leq 8n^{3/4}$ .

Similarly, when  $R = C_2$ , an  $R$ -transversal is a longest cycle transversal. If  $G$  is 2-connected, then the longest cycles pairwise intersect. By Theorem 1, we have  $\text{lct}(G) = \tau_R(G) \leq 8 \cdot 2^{5/4} \cdot n^{3/4} \leq 20n^{3/4}$ .  $\square$

We do not know whether the assumption in Theorem 1 that  $R$  is connected is necessary to obtain sublinear  $R$ -transversals. To obtain analogues of Corollary 2 for general  $R$ , we show that the maximum  $R$ -subdivisions pairwise intersect when the connectivity of  $G$  is sufficiently large. Recall that a graph  $G$  is  $k$ -connected if  $|V(G)| > k$  and  $G - S$  is connected for each  $S \subseteq V(G)$  with  $|S| < k$ . Moreover, the *connectivity* of  $G$ , denoted  $\kappa(G)$ , is the maximum  $k$  such that  $G$  is  $k$ -connected.

**Lemma 3.** *Let  $R$  be a connected  $m$ -edge multigraph with  $m \geq 1$ . If  $\kappa(G) > m^2$ , then the maximum  $R$ -subdivisions in  $G$  are pairwise intersecting.*

*Proof.* Suppose for a contradiction that  $G$  has disjoint maximum  $R$ -subdivisions  $F_1$  and  $F_2$ , and let  $k = |V(F_1)| = |V(F_2)|$ . By Menger's Theorem, there is an  $(V(F_1), V(F_2))$ -connector  $\mathcal{P}$  with  $|\mathcal{P}| = \min\{k, m^2 + 1\}$ . If  $|\mathcal{P}| = k$ , then every vertex in  $F_1$  is an endpoint of a path in  $\mathcal{P}$ , and we obtain an  $R$ -subdivision of size more than  $k$  by replacing an edge  $uv \in E(F_1)$  with a path in  $\mathcal{P}$  having  $u$  as an endpoint, a path in  $\mathcal{P}$  having  $v$  as an endpoint, and an appropriate path in the connected subgraph  $F_2$ .

So we may assume  $|\mathcal{P}| = m^2 + 1$ . For each  $e \in E(R)$ , let  $F_i(e)$  be the path in  $F_i$  corresponding to  $e$ . Since  $R$  has no isolated vertices, we may associate each  $P \in \mathcal{P}$  with an ordered pair of edges  $(e_1, e_2) \in (E(R))^2$  such that  $P$  has its endpoint in  $F_1$  in  $F_1(e_1)$  and its endpoint in  $F_2$  in  $F_2(e_2)$ . Since  $|\mathcal{P}| > m^2$ , some pair  $(e_1, e_2)$  is associated with distinct paths  $P, Q \in \mathcal{P}$ . Let  $W_i$  be the subpath of  $F_i(e_i)$  whose endpoints are in  $V(P) \cup V(Q)$ . If  $|E(W_1)| \geq |E(W_2)|$ , then we modify  $F_2$  to obtain a larger  $R$ -subdivision by using  $P$ ,  $W_1$ , and  $Q$  to bypass  $W_2$ . Similarly, if  $|E(W_2)| \geq |E(W_1)|$ , then we modify  $F_1$  to obtain a larger  $R$ -subdivision by using  $P$ ,  $W_2$ , and  $Q$  to bypass  $W_1$ .  $\square$

**Corollary 4.** *Let  $R$  be a connected  $m$ -edge multigraph. If  $G$  is an  $n$ -vertex graph with  $\kappa(G) > m^2$ , then  $\tau_R(G) \leq 8m^{5/4}n^{3/4}$ .*

As it is not known whether there exists a connected graph  $G$  with  $\text{lpt}(G) > 3$ , reducing the gap between our sublinear upper bound on  $\text{lpt}(G)$  and the constant lower bound remains a major open problem in the area of longest path transversals.

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