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Pluriclosed and Strominger Kähler–like metrics compatible with abelian complex structures

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Abstract

We show that the existence of a left-invariant pluriclosed Hermitian metric on a unimodular Lie group with a left-invariant abelian complex structure forces the group to be 2-step nilpotent. Moreover, we prove that the pluriclosed flow starting from a left-invariant Hermitian metric on a 2-step nilpotent Lie group preserves the Strominger Kähler–like condition.

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1 | INTRODUCTION

A Hermitian metric g on a complex manifold (M, J) is called *pluriclosed* (or *SKT*) if its fundamental form $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$ satisfies

$$dJd\omega = 0\,. \tag{1}$$

The pluriclosed condition (1) can be characterized in terms of the torsion of the Bismut (or Strominger) connection ∇^B. Indeed, in [10] Bismut proved that on a Hermitian manifold (M, J, g) there is a unique Hermitian connection ∇^B whose torsion T^B, once regarded as a (3,0)-tensor via g, is skew-symmetric. The pluriclosed condition is equivalent to $dT^B = 0$. If $T^B = 0$, the Bismut connection ∇^B coincides withe the Levi–Civita condition and the metric g is Kähler.

By [30] a Hermitian metric g is pluriclosed and satisfies the condition $\nabla^B T^B = 0$ if and only if its Bismut curvature R^B satisfies the first Bianchi identity

$$\sigma_{x,y,z} R^B(x, y, z) = 0 \tag{2}$$

and the type condition

$$R^B(x, y, z) = R^B(Jx, Jy, z), \tag{3}$$

for any tangent vectors x, y, z in M. Hermitian metrics satisfying (2) and (3) are called in literature *Strominger Kähler–like* and have been studied recently in [5, 17, 29, 30].

An important tool in the geometry of pluriclosed metrics is the so-called *pluriclosed flow*, defined by the equation

$$\frac{\partial}{\partial t}\omega(t) = -\left(\rho^B\right)^{1,1}, \qquad \omega(0) = \omega_0,$$

where $(\rho^B)^{1,1}$ denotes the (1,1)-part of the Ricci form of the Bismut connection and ω_0 is a fixed Hermitian metric. This is a parabolic flow of Hermitian metrics which preserves the pluriclosed condition [26, 27]. A natural question is to see if the Strominger Kähler–like condition is preserved by the flow.

Every conformal class of any Hermitian metric on a compact complex surface admits a pluriclosed metric, but in higher dimensions, the existence of a pluriclosed metric is not automatically guaranteed anymore. Looking at the existence of left-invariant pluriclosed metrics on 6-dimensional nilpotent Lie groups endowed with a left-invariant complex structure, only 4 out of the 34 isomorphism classes admit pluriclosed metrics and they are all 2-step nilpotent, leading to the question whether this is a general feature in arbitrary dimensions [18]. It turns out that 2 of the 4 classes in dimension six admit Strominger Kähler–like metrics [5] and that the complex structure is abelian. More in general, a characterization of 2-step nilpotent Lie algebras admitting Strominger Kähler–like metrics have been obtained in [31], showing in particular that the left-invariant complex structure has to be abelian.

We recall that a left-invariant complex structure on a real Lie group G of real dimension $2n$ is completely determined by a complex structure J on the Lie algebra $\mathfrak{g}$ of G, that is, by an endomorphism satisfying $J^2 = -\mathrm{Id}$ and the integrability condition

$$J[x, y] - [Jx, y] - [x, Jy] - J[Jx, Jy] = 0, \quad \forall x, y \in \mathfrak{g}\,.$$

The complex structure J is called *abelian* if

$$[Jx, Jy] = [x, y], \quad \forall x, y \in \mathfrak{g}\,, \tag{4}$$

or equivalently if the i-eigenspace of J, denoted with $\mathfrak{g}^{1,0}$, is an abelian subalgebra of $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ (that motivates the terminology introduced in [8]). By [23] a Lie algebra admitting an abelian complex structure has abelian commutator, thus, it is 2-step solvable.

Recent results about the existence of pluriclosed metrics on solvable Lie groups have been obtained in [7, 14, 15, 19, 22].

The purpose of this paper is twofold. On one hand we study the existence of a pluriclosed metric on a unimodular Lie group with an abelian complex structure and on the other hand we investigate the interplay between the Strominger Kähler–like condition and the pluriclosed flow. We recall that a Lie group G is unimodular if and only if $|det(Ad_g)| = 1$, for every $g \in G$, where Ad is the adjoint representation. For a connected Lie group G this is equivalent to requiring that $tr(ad_X) = 0$, for every $X \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G.

The existence of other types of Hermitian inner products compatible with abelian complex structures, like for instance *Kähler* [3], *balanced* [4] and *locally conformally Kähler* inner products [4], has been already studied in literature. In [13] the second author and the third author, in collaboration with H. Kasuya, proved that on non-abelian Lie algebras with an abelian complex structure there are no Hermitian-symplectic structures. The latter can be regarded as special pluriclosed inner products and the natural follow-up is focusing on the existence of pluriclosed metrics compatible with abelian complex structures.

Our first result is the following

Theorem 1.1. *Let $\mathfrak{g}$ be a unimodular Lie algebra with an abelian complex structure J. If $(\mathfrak{g}, J)$ admits a pluriclosed inner product, then $\mathfrak{g}$ is 2-step nilpotent.*

In the particular case when the commutator of $\mathfrak{g}$ is totally real the result follows from [19, Corollary 5.7], but our proof does not make use of the argument in [19]. Moreover, Theorem 1.1 generalizes [13, Proposition 6.1].

Next we focus on the existence of Strominger Kähler–like metrics in relation to the pluriclosed flow. By using the characterization in [31] of left-invariant Strominger Kähler–like metrics on 2-step nilpotent Lie groups, we prove the following

Theorem 1.2. *Let (G, J, g_0) be a 2-step nilpotent Lie group with a left-invariant Strominger Kähler–like Hermitian structure and let g_t be the solution to the pluriclosed flow starting from g_0. Then g_t is Strominger Kähler–like for every t.*

2 | PROOF OF THEOREM 1.1

We first need the following

Lemma 2.1. *Let $\mathfrak{g}$ be a Lie algebra with an abelian complex structure J and an Hermitian inner product g. Then, the torsion 3-form T^B of the Bismut connection of $(\mathfrak{g}, J, g)$ satisfies*

$$T^B(x, y, z) = -g([x, y], z) - g([y, z], x) - g([z, x], y),$$

for every $x, y, z \in \mathfrak{g}$.

Proof. Let ω be the fundamental form of g. Let $x, y, z, w \in \mathfrak{g}$, then $T^B(x, y, z) = -d\omega(Jx, Jy, Jz)$ and we directly compute

$$\begin{aligned} d\omega(Jx, Jy, Jz) &= -\omega([Jx, Jy], Jz) - \omega([Jy, Jz], Jx) - \omega([Jz, Jx], Jy) \\ &= -\omega([x, y], Jz) - \omega([y, z], Jx) - \omega([z, x], Jy)\,. \end{aligned}$$

Hence the claim follows. □

As a consequence we have the following:

Proposition 2.2. *Let $(\mathfrak{g}, J)$ be a Lie algebra with an abelian complex structure. A Hermitian inner product g on $(\mathfrak{g}, J)$ is pluriclosed if and only if*

$$g([y, z], [w, x]) - g([x, z], [w, y]) + g([x, y], [w, z]) = 0 \tag{5}$$

for every $x, y, z, w \in \mathfrak{g}$.

Proof. We recall that g is pluriclosed if and only if $dT^B = 0$. Let $x, y, z, w \in \mathfrak{g}$, then, by the previous Lemma,

$$\begin{aligned} dT^B(w, x, y, z) &= -T^B([w, x], y, z) + T^B([w, y], x, z) - T^B([w, z], x, y) \\ &\quad - T^B([x, y], w, z) + T^B([x, z], w, y) - T^B([y, z], w, x) \\ &= g([[w, x], y], z) + g([y, z], [w, x]) + g([z, [w, x]], y) \\ &\quad - g([[w, y], x], z) - g([x, z], [w, y]) - g([z, [w, y]], x) \\ &\quad + g([[w, z], x], y) + g([x, y], [w, z]) + g([y, [w, z]], x) \\ &\quad + g([[x, y], w], z) + g([w, z], [x, y]) + g([z, [x, y]], w) \\ &\quad - g([[x, z], w], y) - g([w, y], [x, z]) - g([y, [x, z]], w) \\ &\quad + g([[y, z], w], x) + g([w, x], [y, z]) + g([x, [y, z]], w) \\ &= 2(g([y, z], [w, x]) - g([x, z], [w, y]) + g([x, y], [w, z])), \end{aligned}$$

where, in the last equality, we used the Jacobi identity.

Therefore, g is pluriclosed if and only if

$$g([y, z], [w, x]) - g([x, z], [w, y]) + g([x, y], [w, z]) = 0\,,$$

as required. □

Remark 2.3. Note that from the complex point of view, condition (5) is equivalent to

$$g([z_1, \bar{z}_2], [z_3, \bar{z}_4]) = g([z_1, \bar{z}_4], [z_3, \bar{z}_2]),$$

for every $z_1, z_2, z_3, z_4 \in \mathfrak{g}^{1,0}$.

From now on, for a Lie algebra $\mathfrak{g}$ with an abelian complex structure J we will denote by ζ the center of $\mathfrak{g}$ and by $\mathfrak{g}^1_J$ the ideal

$$\mathfrak{g}^1_J = \mathfrak{g}^1 + J\mathfrak{g}^1\,,$$

where $\mathfrak{g}^1 := [\mathfrak{g}, \mathfrak{g}]$. Note that $\mathfrak{g}^1_J$ is a J-invariant Lie subalgebra of $\mathfrak{g}$.

Under the hypothesis of Proposition 2.2 we obtain the following characterization in terms of the center ζ of $\mathfrak{g}$.

Corollary 2.4. *Let $(\mathfrak{g}, J, g)$ be a Lie algebra with an abelian complex structure and a pluriclosed inner product. Then*

$$\|[x,y]\|^2 + \|[x,Jy]\|^2 = g([x,Jx],[y,Jy])$$

for every $x, y \in \mathfrak{g}$. In particular, $x \in \mathfrak{g}$ lies in the center of $\mathfrak{g}$ if and only if

$$[x, Jx] = 0\,,$$

that is,

$$\zeta = \{x \in \mathfrak{g} \;:\; [x,Jx] = 0\}\,.$$

Proof. By using (5) for $x, y \in \mathfrak{g}$ we have

$$\begin{aligned}\|[x,y]\|^2 + \|[x,Jy]\|^2 &= g([x,y],[x,y]) + g([x,Jy],[x,Jy]) \\ &= g([Jx,Jy],[x,y]) - g([Jx,y],[x,Jy]) = g([y,Jy],[x,Jx])\,,\end{aligned}$$

and the claim follows. ☐

We will need the following:

Lemma 2.5. *Let $\mathfrak{g}$ be a unimodular Lie algebra with an abelian complex structure J. Then,*

$$\mathfrak{g}^1_J \neq \mathfrak{g}\,.$$

Proof. By contradiction, assume that $\mathfrak{g}^1_J = \mathfrak{g}$. Then, since by hypothesis $\mathfrak{g}^1$ is an abelian ideal in $\mathfrak{g}$, by [9, Proposition 4.1] $(\mathfrak{g}/\zeta, J)$ is holomorphically isomorphic to $\mathfrak{aff}(\mathcal{A})$ for some commutative algebra $\mathcal{A}$. Since, $\mathfrak{g}$ is unimodular, also $\mathfrak{g}/\zeta$ is unimodular, and so $\mathfrak{aff}(\mathcal{A})$ is unimodular. So, by [4, Lemma 2.6], $\mathcal{A}$ is nilpotent and $\mathfrak{aff}(\mathcal{A})$ is a nilpotent Lie algebra. As a consequence, we have that $\mathfrak{g}/\zeta$ is also nilpotent implying that $\mathfrak{g}$ is nilpotent too. But, this is absurd since by [25] for a nilpotent Lie algebra $\mathfrak{g}$ we have $\mathfrak{g}^1_J \neq \mathfrak{g}$. ☐

Proposition 2.6. *Let $\mathfrak{g}$ be a Lie algebra with an abelian complex structure J. Assume that $(\mathfrak{g}, J)$ has a pluriclosed inner product g and $\mathfrak{g}^1_J$ is 2-step nilpotent. Then $\mathfrak{g}$ is 2-step nilpotent.*

Proof. Write

$$\mathfrak{g} = (\mathfrak{g}_J^1)^\perp \oplus \mathfrak{g}_J^1$$

with respect to the inner product g. Since $\mathfrak{g}_J^1$ is nilpotent and has a pluriclosed inner product, its center $\mathfrak{u}$ is J-invariant. We write

$$\mathfrak{g}_J^1 = \mathfrak{u}^\perp \oplus \mathfrak{u}.$$

The key observation is that $\mathfrak{u}$ is contained in the center of $\mathfrak{g}$. Indeed, if $x \in \mathfrak{u}$, then in particular we have $[x, Jx] = 0$ and Corollary 2.4 implies that x belongs to the center of $\mathfrak{g}$.

Now let $f \in (\mathfrak{g}_J^1)^\perp$. We show that $[f, x]$ lies in the center of $\mathfrak{g}$, for every $x \in \mathfrak{g}$.

Set $D := \mathrm{ad}_f : \mathfrak{g}_J^1 \to \mathfrak{g}_J^1$. Since J is abelian, $ad_f J = -ad_{Jf}$ and therefore, D and DJ are both derivations. Moreover, we observe that

$$D[x, y] = 0 \quad \text{for every} \quad x, y \in \mathfrak{g}_J^1\,;$$

indeed from the 2-step nilpotency of $\mathfrak{g}_J^1$, we have that $[x, y] \in \mathfrak{u}$, for every $x, y \in \mathfrak{g}_J^1$, and that $\mathfrak{u} \subset \zeta$.

Let $x \in \mathfrak{g}_J^1$ and $y \in \mathfrak{g}$. We first show that

$$[Dx, JDx] = [DJx, Dx]\,.$$

Since D is a derivation,

$$[Dx, JDx] = D[x, JDx] - [x, DJDx]\,,$$

now, $x, JDx \in \mathfrak{g}_J^1$ because $\mathfrak{g}_J^1$ is J-invariant, and so, by the previous observation, $D[x, JDx] = 0$.

Hence, now using that also DJ is a derivation we get

$$[Dx, JDx] = -[x, DJDx] = -DJ[x, Dx] + [DJx, Dx]\,.$$

Similarly, $x, Dx \in \mathfrak{g}_J^1$, and by the 2-step nilpotency of $\mathfrak{g}_J^1$, $[x, Dx] \in \mathfrak{u}$. Since $\mathfrak{u}$ is J-invariant, $J[x, Dx] \in \mathfrak{u} \subset \zeta$, therefore $DJ[x, Dx] = 0$, showing the claim.

Then, taking into account that $\mathfrak{g}$ is 2-step solvable, Corollary 2.4 yields that

$$\|[Dx, y]\|^2 + \|[Dx, Jy]\|^2 = g([Dx, JDx], [y, Jy]) = g([DJx, Dx], [y, Jy]) = 0\,,$$

from which we deduce that $[f, x]$ is in the center of $\mathfrak{g}$ for all $x \in \mathfrak{g}_J^1$.

Now let $f_1, f_2 \in (\mathfrak{g}_J^1)^\perp$. By Jacobi identity

$$[[f_1, f_2], x] = 0$$

for every $x \in \mathfrak{g}_J^1$. Hence $[f_1, f_2] \in \mathfrak{u}$ and so in the center of $\mathfrak{g}$, as required. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We work by induction on the complex dimension n of $\mathfrak{g}$. The base case $n = 1$ is trivial and we assume that the statement holds up to complex dimension $n-1$. Let $(\mathfrak{g}, J, g)$ be a Lie algebra of complex dimension n with an abelian complex structure and a pluriclosed inner product. In view of Lemma 2.5, $\mathfrak{g}^1_J$ is a proper Lie subalgebra and inherits an abelian complex structure and a pluriclosed inner product. By induction, assumption $\mathfrak{g}^1_J$ is 2-step nilpotent. Hence Proposition 2.6 implies that $\mathfrak{g}$ is 2-step nilpotent and the claim follows. □

Remark 2.7. By Theorem 1.1, if $\mathfrak{g}$ is a unimodular Lie algebra with an abelian complex structure J and a pluriclosed inner product, then $\mathfrak{g}$ is 2-step nilpotent. In particular, notice that $\mathfrak{g}^1_J$ is abelian. Indeed, since J is abelian, $\mathfrak{g}$ is 2-step solvable and

$$[\mathfrak{g}^1, \mathfrak{g}^1] = [J\mathfrak{g}^1, J\mathfrak{g}^1] = 0\,.$$

Moreover from the 2-step nilpotency of $\mathfrak{g}$ we infer that also

$$[\mathfrak{g}^1, J\mathfrak{g}^1] = 0\,.$$

As a consequence, if $X = \Gamma\backslash G$ is a nilmanifold endowed with an invariant abelian complex structure J and a pluriclosed metric g, then by [18, Theorem A], X is a total space of a principal holomorphic torus bundle over a torus.

Notice that from Theorem 1.1 in particular follows that a nilpotent Lie algebra with an abelian complex structure and admitting a pluriclosed inner product is necessarily 2-step. This partially confirms the conjecture that the existence of a pluriclosed inner product on a nilpotent Lie algebra $\mathfrak{g}$ with a complex structure forces $\mathfrak{g}$ to be 2-step.

Moreover, it is quite natural to wonder how rigid is the existence of another kind of special inner products on a Lie algebra with a complex structure. In particular, the so-called *astheno-Kähler* metrics introduced by Jost and Yau in [20], which are characterized by the condition

$$\partial\bar{\partial}\omega^{n-2} = 0\,.$$

Clearly, on a complex surface any Hermitian metric is astheno-Kähler and in complex dimension 3 the notion of astheno-Kähler metric coincides with that of pluriclosed. Here we observe that in the nilpotent case the existence of a astheno-Kähler inner product on a Lie algebra compatible with an abelian complex structure does not force the 2-step condition in contrast to Theorem 1.1 for the pluriclosed case.

Example 2.8. In view of [21, Corollary 5.1.9] we consider the 8-dimensional 3-step nilpotent Lie algebra $\mathfrak{g}$ with complex structure equations

$$d\varphi^1 = d\varphi^2 = 0\,, \qquad d\varphi^3 = \varphi^{1\bar{1}}\,, \qquad d\varphi^4 = B_{1\bar{1}}\varphi^{1\bar{1}} + B_{1\bar{3}}(\varphi^{1\bar{2}} + \varphi^{1\bar{3}}) + D_{3\bar{1}}(\varphi^{2\bar{1}} + \varphi^{3\bar{1}})\,,$$

with $D_{3\bar{1}} \neq 0$. In particular, the complex structure J is abelian.

Let

$$\omega = \sum_{k=1}^{3} i x_{k\bar{k}} \varphi^{k\bar{k}} + \sum_{1\leqslant k<l\leqslant 3} (x_{k\bar{l}}\varphi^{k\bar{l}} - \bar{x}_{k\bar{l}}\varphi^{l\bar{k}}) + \frac{i}{2}\varphi^{4\bar{4}}.$$

If $ix_{2\bar{2}} + ix_{3\bar{3}} + 2\Im m\,(x_{2\bar{3}}) = 0$, then ω defines an astheno Kähler metric on $(\mathfrak{g}, J)$.

3 | PROOF OF THEOREM 1.2

Let G be a 2-step nilpotent Lie group with a left-invariant Hermitian structure (g, J) and denote by $\mathfrak{g}$ its Lie algebra. Assume further that g is pluriclosed. In view of [31], the metric g is Strominger Kähler–like if and only if there exists an orthonormal basis $\{x_i\}_{i=1}^{s}$ of $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ and an orthonormal basis $\{\epsilon_i\}_{i=1}^{2n}$ of $\mathfrak{g}$ such that

1. $J\epsilon_i = \epsilon_{i+n}\,, \quad i = 1, \dots, n;$
2. $\mathfrak{g}^1 + J\mathfrak{g}^1 = \mathrm{span}\{\epsilon_{r+1}, \dots, \epsilon_n, \epsilon_{n+r+1}, \dots, \epsilon_{2n}\};$
3. the only non-trivial brackets under $\{\epsilon_i\}$ are

$$[\epsilon_i, \epsilon_{n+i}] = \lambda_i x_i\,, \quad i = 1, \dots, s,$$

   for some positive numbers $\{\lambda_i\}_{i=1}^{s}$ and $n - r \leqslant s \leqslant \min\{r, 2(n-r)\}$.

Note, that in particular J has to be abelian.

If $\{\epsilon^i\}$ is the dual basis to $\{\epsilon_i\}$, then the metric g writes as

$$g = \sum_{k=1}^{2n} (\epsilon^k)^2.$$

From [31] it follows that every other left-invariant pluriclosed metric h taking the diagonal form

$$h = \sum_{k=1}^{2n} a_k \epsilon^k \epsilon^k\,, \quad a_k > 0, a_k = a_{n+k}, \text{ for every } k = 1, \dots, n\,,$$

is Strominger Kähler–like since we can modify the basis $\{\epsilon_k\}$ to

$$\tilde{\epsilon}_k = \frac{1}{\sqrt{a_k}}\epsilon_k$$

which still satisfies items 1–3.

Moreover, in view of [12], the Ricci form of the Bismut connection of h takes the following expression:

$$\rho_h^B(x, y) = \frac{1}{2} \sum_{k=1}^{s} \frac{1}{a_k} h([\epsilon_k, \epsilon_{k+n}], [x, y])\,,$$

which implies that ρ_h^B takes the diagonal form

$$\rho_h^B = \sum_{k=1}^{s} b_k \epsilon^k \wedge \epsilon^{n+k}\,.$$

It follows that, by uniqueness, the solution to the pluriclosed flow starting from g_0 is diagonal for every t and the claim of Theorem 1.2 follows.

Remark 3.1. Notice that we can give a more explicit expression for the Ricci form $\rho_{g_t}^B$ of the Bismut connection ∇^B of the metric g_t. Let $\{\epsilon_i\}$ be a basis satisfying items 1–3 and

$$g_0 = \sum (\epsilon^k)^2\,.$$

Consider the solution to the pluriclosed flow

$$g_t = \sum a_k^t (\epsilon^k)^2\,.$$

If $\{\epsilon_k^t\}$ is a g_t-orthonormal basis satisfying items 1–3, namely

$$\epsilon_k^t = \frac{1}{\sqrt{a_k^t}} \epsilon_k\,,$$

then

$$\rho_{g_t}^B(x,y) = \frac{1}{2} \sum g_t([\epsilon_k^t, \epsilon_{n+k}^t], [x,y])\,.$$

We have

$$[\epsilon_k^t, \epsilon_{n+k}^t] = \frac{1}{\sqrt{a_k^t}} \frac{1}{\sqrt{a_{n+k}^t}} [\epsilon_k, \epsilon_{n+k}] = \frac{1}{a_k^t} \lambda_k x_k$$

and

$$[\epsilon_k^t, \epsilon_{n+k}^t] = \lambda_k^t x_k^t$$

with $\{x_k^t\}$ g_t-orthonormal.

Hence

$$\rho_{g_t}^B(x,y) = \frac{1}{2} \sum \lambda_k^t g_t(x_k^t, [x,y])\,.$$

Now

$$\rho_{g_t}^B(\epsilon_i, \epsilon_{n+i}) = \frac{1}{2} \sum_{k=1}^{s} \lambda_k^t g_t(x_k^t, [\epsilon_i, \epsilon_{n+i}])\,.$$

Since

$$[\epsilon_i, \epsilon_{n+i}] = a_i^t[\epsilon_i^t, \epsilon_{n+i}^t] = a_i \lambda_i^t x_i^t$$

we get

$$\rho_{g_t}^B(\epsilon_i, \epsilon_{n+i}) = \frac{1}{2}\sum_{k=1}^{s} \lambda_k^t g_t(x_k^t, [\epsilon_i, \epsilon_{n+i}]) = \frac{1}{2}\sum_{k=1}^{s} \lambda_k^t a_i^t g_t(x_k^t, \lambda_i^t x_i^t) = \frac{1}{2}(\lambda_i^t)^2 a_i^t.$$

Therefore,

$$\rho_{g_t}^B = \frac{1}{2}\sum_{k=1}^{s}(\lambda_k^t)^2 a_k^t \epsilon^k \wedge \epsilon^{n+k}.$$

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