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Introduction: From Social Ontology to Mathematical Practice, and Back Again

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In this introductory essay we compare different strategies to study the possibility of applying philosophical theories of social ontology to mathematical practice and vice versa. Analyzing the contributions to the special issue *Mathematical practice and social ontology*, we distinguish four main strands: (1) to verify whether the very act of producing mathematical knowledge is an intersubjective activity; (2) to explain how the intersubjective nature of mathematics relates to mathematical objectivity; (3) to show how this intersubjectivity-based objectivity is the result of social practice; (4) to understand whether, given the social nature of intersubjectivity-based mathematical objectivity, mathematical objects can be described by analogy with social facts as institutions.

1 Mathematical Practice and Social Ontology

The relationship between mathematics and social ontology is often guided by the question of the possibility of applying mathematics to social sciences, especially economics. As interesting as these questions may be, they neglect the inverse possibility of applying a conceptual analysis derived from social ontology to mathematics. This issue will be devoted to the question of whether the distinction between social object and social fact, on the one hand, and between different theoretical approaches to the notion of social fact, on the other, can be successfully applied to mathematical practice.

There is a well-established tendency in the contemporary philosophy of mathematics to emphasize the importance of scientific practice in answering certain epistemological questions such as visualization, the use of diagrams, reasoning, explanation, purity of evidence, concept formation, the analysis of definitions, and so on. While some of the approaches to mathematical practice are based on Lakatos's interpretation of mathematics as a quasi-empirical science, this special issue takes this statement a step further, as it relies on the idea that the objectivity of mathematical concepts might be the result of a social constitution.[1]

What theory of social facts and social objects could explain the characteristics of mathematical objectivity? Are there new ontological or epistemological perspectives that can be developed in this social philosophy of mathematics? The introduction will analyze these four problematic nodes, enucleating the originality of the solutions proposed by the authors and highlighting new directions for research opened up by these attempts to bring the philosophy of mathematical practice into dialogue with social ontology.

The project of this special issue is not a renewal of David Bloor's (1976) research, aimed at a sociological study of mathematics. It is rather a study of the possibility of applying philosophical theories of social objectivity to mathematical objects. This is a new topic that requires the search for adequate mathematical examples to satisfy the objectivity constraints proposed by the philosophy of social ontology. Tendencies in this direction can be traced, but no general survey has been offered. For example, Feferman

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1 We prefer to speak of 'social constitution' rather than 'social construction', since the term construction is strongly connoted both in the philosophy of mathematics, where it refers to constructive mathematics only, and in social ontology, where it refers primarily to theories that explain how agents construct some kind of objects (representations and facts about nature and about humans, including the emotions, gender, race, sex, homo- and heterosexuality, mental illness, technology, quarks, facts, reality, and truth). See e.g., Mallon 2016.

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(2011) characterized mathematical objectivity as a special case of intersubjective social objectivity. Two other authors included in this issue had already contributed to an analysis of mathematics that approaches or explicitly connects mathematical practice with social ontology. José Ferreirós (2016) defined mathematical practice as an activity supported by individual and social agents and characterized by stability, reliability, and intersubjectivity. Julian C. Cole (2013, 2015) described mathematical objects as institutional rather than mental objects, referring to Searle's theory of collective intentionality.

As editors we have constructed the call for papers around some key questions, typical of more recent results in philosophy of mathematical practice, but rarely related to developments in social ontology. We thank the contributors for taking these solicitations seriously and will try in this introduction to highlight the original developments that this discussion has made possible. The purpose of the issue was not to determine which theory among the many alternatives offered by contemporary social ontology is best applied to the construction of mathematical objectivity, but rather to verify whether new epistemological and ontological issues might emerge from the effort to explain the kind of objectivity that is proper to mathematics.

The articles collected in this volume explain the intersubjective origin of mathematical practices from different points of view: Peirce's semiotics and pragmatism, Husserl's phenomenology, Wittgenstein's philosophy of language, Searle's social ontology, Brandom's inferentialism, embodied cognitive science and enculturation theory. They explain how objectivity comes in different degrees, and how practices based on some intersubjective social constitution might account for mathematical objectivity. Some papers address the relationship between sociology, history of mathematics, the social ontology of processes and conversations, and formal ontology, showing advantages and difficulties that emerge from the construction of an ontology of mathematics that takes into account social, historical and formal language elements.

Despite these different goals, all the articles in this special issue address the question of whether and to what extent mathematics can be explained as a social intersubjective activity that produces objective knowledge. To analyze the different strategies and highlight the problematic cores around which the problem of comparing social ontology and mathematical practice is structured, we will distinguish four main strands in this general question and check which of these points each article actually answers. (1) At the first level it is to show that mathematics is an intersubjective practice: by this we mean not only that mathematical activity is, like any other human activity, linked to social exchanges for its transmission, but also that the very act of producing mathematical knowledge is an intersubjective activity. (2) At the second level, it is to explain how the intersubjective nature of mathematics relates to mathematical objectivity. Comparing different notions of objectivity in use in the literature of contemporary mathematics, an attempt will be made to explain whether and with which of them an intersubjectivity-based idea of mathematical objectivity can be linked. (3) At the third level, it is a matter of showing how this intersubjectivity-based objectivity is the result of social practice. Even assuming that some form of mathematical objectivity can be grounded in intersubjectivity, the question remains as to what the social nature of such intersubjectivity consists of: is it linguistic, symbolic, or preverbal in nature? Does it include activities of cultural coordination, planning, and transmission? (4) Finally, the question emerges as to whether, given the social nature of intersubjectivity-based mathematical objectivity, mathematical objects can be described by analogy with social facts as institutions. Are mathematical objects institutional facts? What are the goals that guide their constitution? Do they differ from those of other institutional facts such as marriage, money, and private property?

We will discuss the different articles sometimes within one and sometimes within several sections in order to highlight the contributions made to each problem and to compare the prerequisites of each approach, without, however, expecting the authors to identify with our presentation of the problem.

2 Mathematics as an Intersubjective Practice

The philosophy of mathematical practice today is a diverse set of research unified by the common goal of investigating mathematics as a scientific practice, focusing attention on its history, specific case studies, and non-elementary theories. This strand of research arose in part out of opposition to the predominantly foundational and ontological interest of analytic philosophy, and it focused on epistemological issues related to explanation and visualization in mathematics. Recent surveys have detailed the origins and the history of the study of mathematical practice, recalling that it is not the prerogative of philosophy alone, but is also studied by sociology, educational theory, ethnology, evolutionary biology and cognitive psychology (van Bendegem 2016). Various approaches including agent-based, epistemological and historical have been distinguished (Carter 2019), and several key issues of investigation into mathematical knowledge have been highlighted, including the components, the role of agents, mathematical values, mathematical practice, the history and the relations to other disciplines (Hamami and Morris 2020), as well as dynamic, genetic and heuristic aspects and different definitions of 'mathematical practice' (Giardino 2017).

The philosophy of mathematical practice now encompasses numerous approaches emphasizing that mathematics is primarily an activity and must therefore be described by considering its historical development, its cultural components, the context of production, the agents involved in this practice, and the goals that guide it. However, it is one thing to hold that mathematics is an activity to be described by considering cultural and social elements, and quite another to hold that the products of such a practice are grounded in an inherently intersubjective activity. The articles collected in this volume measure up to this second aspect of mathematics as a practice.

In "Degrees of Objectivity? Mathemata and Social Objects" José Ferreirós analyzes mathematical practice as an intersubjective semiotic practice, based on language and logical symbolism, and more precisely as a practice having strong cognitive roots and linked to the analysis of relational patterns in the world of our experience and action. Ferreirós, following a tradition inaugurated by Kreisel, Putnam and taken up by Feferman (see also Ferreirós 2022), insists on the need to explain first the (for him intersubjective) origin of mathematical objectivity, and only later "the practice of taking objects as surrogates to aid us with representational activities".

A semiotic approach is also found in the paper "Mathematical Practice, Fictionalism and Social Ontology" by Jessica Carter, who combines an agent-based approach to mathematics as an intersubjective activity of human agents engaged in different mathematical tasks with an approach to mathematics as a practice consisting in a body of mathematical theories and in the historical and social process of their development. Furthermore, following C.S. Peirce, Carter argues that such an intersubjective activity consists in a practice of reasoning whose character, in agreement with Ferreirós's view, is hypothetical but not simply arbitrary and conventional (see also Carter 2014).

Peirce's semiotics and pragmatism are further developed in the paper "C.S. Peirce on Mathematical Practice: Objectivity and the Community of Inquirers" by Maria Regina Brioschi, who builds on an analysis of two complementary definitions of mathematics proposed by Peirce: one derived from his father Benjamin - mathematics as "the science which draws necessary conclusions"- and one developed by Peirce himself to show the priority of mathematics among the heuristic sciences - mathematics as "the science of hypothesis". Brioschi shows how intersubjectivity intervenes at various levels in the formulation of hypotheses (although partly subjective, it is the result of the mathematician's membership of a community), in the correction of errors and in the dialogic nature of deduction. On the other hand, mathematics, while not a positive science aimed at the study of existing objects, is nevertheless an observational science, because its objects are signs, and the mathematical method, like the scientific method in general in Peirce's pragmatist approach, is based on the collaboration of a community of inquirers.

Robert Brandom's philosophy offers a philosophical reading of the origin of intersubjectivity that focuses on the inferential origin of semantics rather than on the iconic value of mathematical symbols. In "Of Marriage and Mathematics: Inferentialism and Social Ontology", J.H. Collin considers mathematics in the light of an inferentialist semantics based on Brandom's normative pragmatics: it is a linguistic practice whose meaning is instituted by norms that presuppose intersubjectivity. Grasping and communicating mathematical meanings requires intersubjectivity, because it requires that one acknowledges entitlements and commitments to claims, keeping track of further entitlements and commitments that arise as a result of the intersubjective linguistic practice.

Valeria Giardino in the paper "The Practice of Mathematics: Cognitive Resources and Conceptual Content" adopts a pluralist and interdisciplinary approach that characterizes mathematical practice as based on both mathematical symbols and mathematical inferences. She combines insights coming from the philosophy of mind of Edwin Hutchins, and its descriptive and naturalist approach to mathematics as a symbolic activity emerging from the natural habitat of culturally constituted beings, distributed across a community of practitioners. But she also includes insights from Brandom's pragmatist philosophy of language and its normative approach to mathematics as an inferential practice based on deontic states such as commitments and entitlements. Following Ferreirós and more broadly the pragmatist focus on conceptual competence as a kind of doing (know-how), mathematics can thus be characterized as an interplay of practices rather than as an a priori body of necessary truths. Inferentialism can further contribute to specifying more precisely the conceptual and normative core of such an interplay in terms of reasoning practices. Focusing on the primacy of material inference over formal inference, it can help to go beyond the formalist paradigm in the foundations of mathematics.

If Valeria Giardino had addressed the problem of the relationship between nature and culture, this theme becomes central in Markus Pantsar's paper "From Maximal Intersubjectivity to Objectivity. An Argument from the Development of Arithmetical Cognition", which draws on empirical evidence to argue that arithmetic is the result of cultural development partially based on proto-arithmetic skills that evolved biologically. In what sense then can arithmetic be considered intersubjective? If intersubjectivity simply means the possibility for two people to share a cognitive state or subjective experience, then proto arithmetic is intersubjective in the sense that different cultures share subitizing and estimating abilities, to be considered therefore as universal abilities for neurotypical humans. The question is whether arithmetic knowledge is also intersubjective, given that it can vary from culture to culture. Pantsar uses accumulated empirical evidence in cognitive science to argue that arithmetic is the result of a process of enculturation, that is a process in which cultural factors contribute to changes in individual cognitive abilities. For example, in the case of arithmetic, the change occurs through the introduction of new numeral words for new tallying and finger counting practices. The intersubjective nature of arithmetic would then be explained by the fact that cultural evolution develops universal proto-arithmetic capacities, and thus leads to convergent arithmetic systems even in different cultures. In this sense, the intersubjectivity of arithmetic while being intracultural would in fact end up being also transcultural, and thus being, in Pantsar's words, a maximal intersubjectivity.

The role of history and culture is further analyzed in the paper "No Magic: From Phenomenology of Practice to Social Ontology of Mathematics" by Mirja Hartimo and Jenny Rytilä, who develop an approach to mathematical practice based on an interpretation of Husserl's phenomenological approach (see Hartimo 2021). The authors rely especially on later writings, such as *Formal and Transcendental Logic* (1929), where Husserl considers science, and mathematics, as an intersubjective, historically developing cultural formation. Accordingly, the phenomenological method, combining sense-investigation and transcendental reflection, is aimed at elucidating and clarifying the practitioners' point of view, offering an analysis of the mathematicians' aims, their implicit presuppositions, given kinds of evidence, basic concepts and principles.

The specific role of agents, emphasized by Ferreirós, who makes it one of the defining elements of the notion of practice, but also by Carter and Giardino, who insist on knowledge related to know-how, is central to Michel Le Du's approach, which refers to Wittgenstein's rule following. In the paper "No place for Private Practice" Le Du interprets mathematics as a rule-based activity and claims that what has intersubjective origin is not only transmission or learning but the content and necessity of mathematics. To criticize the mythology that hypostatizes rules, the author analyzes an arithmetic example: adding one or two to a number series does not presuppose knowledge of the entire number series, and thus the existence of a mathematical structure independent of the acts of the agents who compute the result of the operation. The presence of social institutions characterized as a set of rules does not presuppose the existence of a social structure independent of individuals, but only structural properties that exist in space and time as they are instantiated through the actions of cognitive agents. Similarly, mathematical rules acquire concrete existence only in the course of procedures enacted by calculating agents.

While all authors agree on the intersubjective nature of mathematical practice, the underlying notion of practice is constructed by emphasizing distinct elements: the iconic function of mathematical symbols, the manner in which agents construct relational patterns, the contribution of an inferentialist semantics, the role of know-how and rule-following, and the presence of transcultural proto-arithmetic skills evolving on the basis of different historical and cultural traditions. The question now is whether the insistence on the role of agents, having individual goals and values, leads to some form of relativism or is able to explain the convergence towards some kind of objective truth. Are mathematical practices governed solely by their historicity, or are there some rational constraints imposed by their intersubjective nature?

3 Intersubjectivity and Objectivity of Mathematical Knowledge

Showing that mathematics is an intersubjective activity still does not imply that mathematical knowledge is objective, nor that it is objective by virtue of its intersubjective nature. On the other hand, the very notion of mathematical objectivity is multifarious and is often used to mean different things in the literature. This special issue does not intend to explicitly thematize mathematical objectivity, which was extensively discussed in a recent special issue of the journal *Noesis* (Cantù et al. 2022). There, Platonic realism in its various strong and weak forms, formalism and constructivism, semantic objectivism and that generated by declarative acts, structural invariance and objectivity of fictional entities, and also historical and stylistic objectivity of the presentation of a theory are analyzed. And yet, to be able to claim that mathematical objectivity is based on intersubjectivity, the authors find themselves needing to draw certain distinctions.

An extremely interesting result of this special issue is the way in which several authors converged on the idea that mathematical objectivity is not an all-or-nothing matter, but rather a notion that appears by degrees, and which might have different answers depending on whether one is questioning the objectivity of mathematical knowledge or that of mathematical objects. Moreover, different mathematical propositions or objects appear to be endowed with different degrees of objectivity, which are in some sense a measure of their degree of intersubjective complexity (from numbers originating from subitizing and computational activities to abstract structures based on relational patterns that are simpler because they are based on less complex or less culturally determined intersubjective practices).

Another question that can be answered by comparing different articles is what differences it would make to ground intersubjective mathematical objectivity (1) on the relation between agents and some external reality or some goal to which the practice is applied to, or on some cognitive abilities, (2) on semantic inferences, (3) on intentions (phenomenological or shared intentions), (4) or on rules. We will discuss the first three points here and defer discussion of the last point to the next section.

The relationships between the different philosophical paradigms on which the authors base their analysis and the results they arrive at emerge more clearly: we merely point out a certain convergence between pragmatist and inferentialist approaches, in which objectivity is ultimately grounded in constraints imposed by reasoning, while other approaches based on representational semantics end up maintaining a relationship to an activity that applies to something external or more primary.

It is very interesting here to compare the role of some mathematical objects, such as natural and real numbers in the papers by Cole, Ferreirós, Pantsar, Carter, and Brioschi. Even if mathematical propositions are taken to be hypothetical, their conventional nature does not prevent them from having objectivity in a stronger sense, one that binds the intersubjectivity of practices to the function they serve in relation to pre- or transcultural human activities or to some 'primary substances'. The role of mathematical language and symbolism changes accordingly: it is not directly representational, but rather performs some sort of representational or surrogacy function, to take up Cole's notion (2017; 2015), according to which mathematical objects would play a surrogate role in aiding representational activities like inquiring and reasoning: activities that would become much easier when their subject matter is being treated as an object with properties and relations.

For Ferreirós, the objectivity of mathematics is based on two distinct but related ideas: on the one hand, mathematical objects are objective because they have a thin existence; on the other hand, mathematical knowledge is objective because it is intersubjective and applicable. To say that mathematical objects are thin objects means for Ferreirós that they are the semantic correlates of numeral and other relational terms of language, and therefore independent of the mental processes of intelligent agents (but not of the agents themselves). In this sense, the status of mathematical objects is that of objects characterized only by non-contradiction, appearing in the conceptual analysis typical of structuralism, aimed at investigating structures as hypothetical conceptions. In what, however, does the objectivity of mathematical knowledge consist? "Thinning out" Linnebo's definition of mathematical Platonism, Ferreirós defines it through the following two conditions: cognitive intersubjectivity, and the link with the analysis of relational patterns embedded in the natural world and situated in the world of our actions. In other words, it is cognitive intersubjectivity and applicability that guarantee the objectivity of mathematical knowledge.

In the paper "Some Preliminary Notes on the Objectivity of Mathematics" Julian C. Cole assumes that intersubjective collective agreements are often responsible for there being socially constituted contents, and account for these contents to have different degrees of objectivity, depending on the kind of agreement and its level of explicit codification in social practices. On this basis he argues that it is not necessary to assume that, in order to account for the semantic objectivity that most ascribe to mathematical contents, one needs to ground them in the ontological objectivity of facets of reality they would represent. Cole counters the idea that socially constructed contents lack objectivity. He argues that something can be socially constituted - ontologically subjective - and be semantically objective. Yet, he revises here in light of Rytilä's criticism (2021) his previous account of this strongly constrained mathematical objectivity: an account that was suggesting that the source of this highly constrained objectivity are not facts but logically possible relations. Cole now argues that this source of what puts strong constraints on the truth values of various mathematical contents is rather to be found in the intended application of a practice, arising primarily from past and future applicability in reasoning "about the relations among the cardinalities of actual finite collections of stable spatio-temporal objects". For instance, natural numbers are then understood as surrogacy functions that are constituted by acts of collective agreement, and that serve to represent primary facets of reality we interact with: tools that in everyday activities like counting help us in our dealings with small collections of concrete things.

According to Carter, intersubjective practices hold some kind of objectivity: along with Peirce's pragmatic maxim, mathematical hypotheses rather than corresponding to existing abstract entities - as ontological realism holds - would be "pragmatically real", that is, would be hypotheses whose reality - as well as Cole's surrogacy functions - depends on the validity of statements about assumed more primary substances at a lower level. Mathematics and reality are seen as a dynamic and multi-layered view of an evolving process. While Carter shares with fictionalist approaches such as Thomasson's (1999) the idea that mathematical entities are introduced by human agents through postulation, she distances herself from fictionalism insofar as she - by using as a case study the development of K-theory - underlines the ways in which such entities are introduced in mathematics. Unlike in fiction, the introduction of entities in mathematics is more constrained by previous methods, motivated by local and global concerns, and more strongly related to the activity of reasoning and its strive for completeness.

Brioschi, who shares with Carter a pragmatist starting point, seeks to explain precisely how objectivity à la Peirce can be distinguished from strong and weak Platonic objectivity, and from conventional and fictional objectivity. And this, not only because pragmatism seeks to avoid false dichotomies, but mainly because for Peirce there is no world of mathematical ideas independent of us. Objectivity does not depend on the existence of certain objects, but on the constraints imposed by reasoning about certain hypothetical relations. The objectivity of mathematics is thus compatible with the idea that it is a fallible, corrigible, tentative, and evolving knowledge. If one wants to interpret Peirce's theory as social constructivism, then collective intentionality should be considered as an expression of a broader rationality that is diffused through nature and the distinction between natural and social reality would fall away, given that human beings need to be construed as symbols as well.

Pantsar, too, builds on Platonic objectivity, seen here as the underlying problem to be accounted for from an intersubjective perspective that is clearly anti-Platonic. Pantsar contrasts the standard Platonist notion of objectivity, based on the idea that there are abstract objects which are timeless and mind-independent, with the criteria proposed by Wright and Shapiro to characterize what he calls robust objectivity. Mathematical discourse is objective if (1) it can contain true propositions whose truth we ignore (epistemic constraint), if (2) it does not contain blameless disagreements, that is, disagreements that can be resolved by considering a divergence of information between speakers (cognitive command), and if (3) it has explanatory value even outside the mathematical domain (wider cosmological role). Even those who do not attribute Platonic objectivity to mathematics must, according to Pantsar, account for the fact that mathematics appears as objective knowledge according to such criteria. His main aim is to justify the apparent objectivity of mathematics from a non-Platonist perspective by showing that arithmetic enjoys maximal intersubjectivity, because it is based on a social (cultural) transformation of inherited proto-arithmetical abilities, which therefore enjoy a kind of cross-cultural objectivity.

Collin introduces an even more complex classification among five distinct types of objectivity. (1) A claim is objective if it is not equivalent with one's or everybody's acknowledgement of it. (2) A claim is objective if it is about objects. (3) A claim is objective if there can be disagreement on it. (4) A claim about objects whose meaning has been introduced conventionally (by introduction and elimination rules of a term) is objective if what follows from such rules is not up to us. (5) When an object is introduced axiomatically, and a certain claim does not follow inferentially from the set of axioms, it is still possible to say whether the claim holds, because there is some objective fact of the matter about it. Collin's goal is then to explain how inferentialism can grant objectivity to mathematics according to some but not all senses of objectivity. (1) A mathematical claim is objective because it is not equivalent with one's or everybody's acknowledgement of it: one can gain entitlement to one's self-acknowledgement of a claim by introspectively surveying one's acknowledged commitments, but this is not enough to gain entitlement to the claim itself. (2) A mathematical claim is objective because it is about objects, even if not physical objects, but rather objects introduced by conventions, by means of introduction and elimination rules for terms. (3) A mathematical claim is objective because there can be disagreement on it, e.g., when a commitment is judged to be incorrect depending on the use of some anaphoric chain. (4) A claim about objects whose meaning has been introduced conventionally (by introduction and elimination rules of a term) is objective because what follows from such rules is not up to us, as certain consequences of a set of axioms and rules. (5) But there is a sense in which some mathematical claims are not objective. When an object is introduced axiomatically, and a certain claim does not follow inferentially from the set of axioms, it is not possible to say whether the claim holds, because there is no objective fact of the matter about it. So, accepting the Zermelo-Fraenkel axioms for set theory, then neither the continuum hypothesis nor its negation holds true.

Hartimo and Rytilä argue that a socio-ontological approach to mathematics as an intersubjective practice can also account for the objectivity of mathematical entities, that is the fact that, even if socially constructed, they have objective features. In line with Cole, Carter and Ferreirós, they argue that the objectivity of mathematical entities, in comparison with other social constructions, does not amount only to its intersubjective character, but is moreover highly constrained. Furthermore, they argue that these constraints are plural (normative, inter-theoretical, biological, physical), some of which are independent of human activities, and know many degrees (from strongly physically constrained elementary mathematics up to the more free-floating axioms of choice). This can account socio-ontologically for the seeming necessity of mathematical facts, that is for the fact that, after being introduced by mathematical practices, mathematical entities are experienced from the point of view of mathematicians as necessary and timeless abstract things that exist externally to individual minds and that, like social patterns, can have features which are difficult to discover.

Curiously, more than in Hartimo and Rytilä 's paper, which takes its point of departure from phenomenology (but referring mainly to the later Husserl), it is Matteo Bianchetti and Giorgio Venturi, through their study of formal ontology, who explicitly set out to take into account the intentions of agents. In the paper "Formal Ontology and Mathematics. A Case Study on the Identity of Proof" the authors investigate objectivity through a formal ontological approach, which they propose as a preliminary investigation to the study of the intentional aspects of mathematical practices. The article does not investigate the mechanisms by which the objectivity of mathematical proofs would be grounded in intersubjective practices, and yet the chosen ontological perspective (formal ontology as the study of semantically structured databases), sheds light on the intersubjective work by which the criteria for identifying the ontological commitments of mathematical propositions are constructed and redefined. The formal ontology-focused approach is interested neither in the metaphysical problem of the nature and existence of abstract mathematical objects (as in much recent philosophy of mathematics) nor in the problem of the cognitive access we may have to such entities (as in the study of the validity of proofs implemented by automated prover or proof-assistant projects), but in the semantic classification of the objects to which mathematical propositions are committed. The article analyzes a specific case, namely that of mathematical proofs, which are considered as sets of propositions. The authors distinguish two problems: to understand when two proofs express the same ontological content (noetic challenge), and to understand how a demonstration carves out a background ontology (ontological challenge). They focus only on the second, explaining how the determination of the meaning of the sentences composing a proof requires us to consider not only the explicit definitions possibly provided by its author, but also other conceptions by the author or mathematical traditions. In this sense, the formal ontological approach seeks to offer an objective and detailed account of intentional practices in the linguistic presentation of a proof that might escape proof-assistant approaches based on a preliminary formalization. As an example, the authors claim that Euclid's and Proclus's proofs that "two angles of a triangle are less than two right angles" differ in ampliative steps, that is, in changes in the proof ontology attesting the introduction of new objects.

4 The Social Origin of Intersubjectivity-based Objectivity

Different strategies are deployed to argue that social constitution grounds the intersubjectivity-based objectivity of mathematics, or at least part of it. One strategy consists in relating the social constitution of intersubjectivity-based objectivity to those naturally evolved cognitive skills that allow interaction in the biophysical environment and in looking at how cognitive skills and intersubjective practice are enculturated (see Ferreirós, Pantsar) and objectified in the social world, being implemented in material anchors, tools, and technical practices (see Giardino). A second strategy consists in rooting such an objectivity in the communicative structure of interaction (see Bianchetti and Venturi, Livet). A third strategy consists in grounding such an objectivity in an externalist semantics (see Gandon). The authors contributing to this issue present different combinations of these strategies.

While Pantsar grounds the (apparent) objectivity of arithmetic on maximal intersubjectivity, which has a biological origin, but results from a social (cultural) transformation of inherited proto-arithmetical abilities, Ferreirós discusses the social origin of mathematical objectivity on two levels. On the one hand, he considers objectivity as based on what he calls the strong intersubjectivity of mathematics and originating in the interaction between basic cognitive abilities linked with action in the biophysical environment, social interactions, technical practices such as counting, drawing designs, or measuring, and symbolic or semiotic representations (including notations and diagrams). On the other hand, Ferreirós asserts that the tendency to regard mathematical objects as having an abstract existence that is not merely thin also has a social origin, because it is precisely the intersubjective representational activities, which can be simplified through the reification or hypostatization of a logical-linguistic phenomenon (a procedure Cole calls surrogation), that cause us to regard correlates of linguistic terms as abstract logical objects.

Combining in a pragmatist vein Hutchins's ecological, distributed cognition framework with inferentialist semantics allows Giardino to broaden inferentialism beyond Brandom's linguistic, propositionalist framework and anti-naturalistic stance, so as to apply it to symbols in general and to human activities as "cultural cognitive systems" naturally situated, socially distributed, and implemented in material tools (Hutchins 2013). Moreover, one can now understand the practices of mathematics as both determined by social practices based on inferential skills and going beyond the realm of the discursive. This includes the conceptual role played in mathematics by the use of figures - such as the figure of a triangle - that serve as 'material anchors' that can be manipulated so as to apply some operations on particular conceptual models and to extend the network of inferences that are activated by them.

Bianchetti and Venturi assume rather than explicitly claim that the objectivity of formal ontologies is based on intersubjectivity, given the intentional and social character of semantic artifacts, as well as the communicative goal of the Web Ontology Language (OWL 2) they use to represent the structure of knowledge conveyed by mathematical proofs. However, the article highlights an aspect of mathematical practice that has been little explored in the other contributions: the communication of mathematical knowledge. The article exposes numerous problems and difficulties to be faced when one wants to encode the knowledge offered by mathematical proofs into language that can express communicable but also modifiable data on the Web. Besides, the open-ended questions proposed by the authors at the end of the article suggest a promising line of research to investigate subjective or agent-based aspects of mathematics. The comparison of different ways of carving out the ontology of objects and relations underlying mathematical linguistic proof practices clarifies the intentional content of proofs, some of its related properties, such as purity, creativity, or the ability to modify the standard representation of a problem, and informal or pre-formal activities of conceptual analysis and clarification.

In the paper "Process Ontology: conversations and argumentations, controversies in mathematics and mathematics as socialization" Pierre Livet presents mathematical actions, practices and communications as intersubjective processes, as elements of a dynamic ontology based on different modes of generation of virtualities, i.e., possible worlds viewed from the perspective of different participants in communicative social interactions. In mathematics, as in any social process, the starting point is virtualities, and the social interaction might occur only if there is some compatibility between different virtualities. The difference between mathematical and non-mathematical social interactions is just a matter of degree: social interactions require compatibility between some perspectives only (those of the participants in the interaction), whereas mathematics tends to the closure of operations, i.e., tends to require not only compatibility between all perspectives, but also the possibility of passing from one to the other. This is the case in the construction of mathematical space: any perceptual perspective differs from another (three-quarters face or three-quarters back), but in the mathematical space each perspective can be transformed into another. An ordinary conversation can be seen as a socialization if it is possible to combine some impossibility (the differences between the points of view of the different participants in dialogue) with the compatibility of at least some virtualities of connections between partners. In a mathematical conversation this is possible even when the virtualities appear to be contradictory, through the creation of new virtualities, as in the creation of complex numbers to solve the impossibility of calculating the square root of negative numbers. In doing so, the author emphasizes both the possibility of reconstructing mathematical ontology in terms of an ontology of processes and connections between virtualities, but also in terms of explaining the peculiarity introduced by mathematical formalism, which enables the construction of new frameworks of virtualities. Using the language of processes and virtualities, which the author discussed from the perspective of social ontology in a volume written in collaboration with Bernard Conein on social processes and interaction types (Livet and Conein 2020), Pierre Livet reconstructs as intersubjective practices several mathematical actions (success, definition of method, conditions for the application of that method, preliminary definition of the limitations of that method, possibility of constructing a new method from the difficulties encountered by other methods) and several argumentation strategies that are usually characterized as value-based pragmatic reasoning (generality, extensibility, simplicity).

In the paper "Sheldon Smith on Newton's derivative. Retrospective assignation, externalism and the history of mathematics," Sébastien Gandon addresses the issue of the referent of mathematical terms, building on a detailed analysis of the opposing theses of Sheldon Smith and Tyler Burge. Both develop a semantic theory centered on the use of mathematical terms to determine referent and share the belief that ontological issues cannot be separated from an analysis of the history of mathematics, but they have different views on historical continuity and on the notion of use that grounds the semantics of mathematical terms. For Smith, mathematical terms that denote concepts, such as that of Newton's Derivative, change over time, and later definitions are not extensively equivalent to each other (e.g., the symmetric derivative and the Weierstrass' derivative) or to earlier definitions. This is the thesis of reference indeterminacy of mathematical terms that Smith advances to show how ontological questions can and should be related to the historical investigation of mathematical practices. Identifying the reference of a concept as it is defined in contemporary mathematics with the reference of a concept developed in earlier stages is possible only because the historian identifies a canonical use of the historical concept and then fixes the reference of the term on the basis of that canonical use. For Burge, who takes an externalist approach to meaning, the identification of the same reference for two distinct mathematical concepts associated in different eras with the same term is possible because the use that fixes the reference is not only individual use, but also environmental and social use. Indeed, in the case of mathematics, social use matters most of all. It is precisely by taking into account historically and socially situated mathematical practices that the problem of reference indeterminacy finds a solution. The author, by critiquing Sheldon's thesis building on an analysis of some historiographical practices related to mathematics, raises several fundamental problems for any philosophical approach to mathematical practice: how to take historical transformations into account in the construction of a mathematical ontology, what kind of semantics best lends itself to the purpose, and what conceptions of mathematics as a socio-historical practice are really able to dialogue with the methodological and historiographical results recently developed in the history of mathematics?

5 Mathematical Objects as Social Institutions

Some authors do not merely account for the objectivity of mathematical knowledge or the objectivity of mathematical ontology through intersubjective social practices but hold that such practices produce institutional facts. Different strategies are deployed to show that mathematical facts are instituted or institutional facts. On the one hand, one needs to account for how the constrained objectivity of mathematical facts can be explained on the basis of those practices of attribution and codification of social statuses that also constitute institutional facts such as money, marriages, laws (see Bianchetti and Venturi, Hartimo and Rytilä, Collin). On the other hand, once we acknowledge that mathematical facts pertain to the genus of institutional facts, we should also be able to account for their specificity, and in particular for how, due to their more strongly constrained objectivity, they differ under some important aspects from other species of institutional facts (see Cole, Carter, Pantsar and Ferreirós). The crucial question, which is of more general epistemological interest is whether it is possible for mathematical objects to have the same intersubjective objectivity of social facts, or is there a fundamental difference between social facts, that are present in all cultures but usually differ in form, as e.g., marriage, private property or money, and at least some arithmetical objects, e.g., the natural numbers, which have more or less the same structure in any culture.

Bianchetti and Venturi discuss a central point for understanding the relationship between institutional communicative practices and the mathematical content they convey: at what level does conceptual analysis occur? Whereas in the formalization and symbolization of mathematical language, conceptual analysis precedes the transformation or interpretation of mathematical language and does not depend significantly on extra mathematical factors, in the definition of the formal ontology of a proof the level of analysis depends on social factors such as context and communicative goals, which in turn may depend on institutionalized social practices.

Given the metaphysical neutrality of the phenomenology of mathematical practice, Hartimo and Rytilä claim that this analysis needs to be complemented by a philosophical perspective which fully accounts for the ontological implications of the practices of mathematicians. For this purpose, they complement the phenomenology of mathematical practice with an ontological approach that locates mathematical objects in the region of being shared with social reality. They assume that a socio-ontological approach is compatible with Husserl's understanding of mathematics as an historically developing practice and with what mathematicians do within their intersubjective practice and its varied and context-dependent development. Accordingly, mathematical objects and structures are understood as intended or unintended products of shared mathematical practices, that is, as intersubjectively shared and socially constituted, and put in the region of being of those institutional entities whose existence depends on shared practices, such as money, marriages, universities, and so forth. Mathematical objects and structures, like institutional entities, are practice dependent and nonetheless taken to genuinely exist, but still not as concrete physical objects but rather as abstract entities.

According to Collin, a semantic inferentialist account of meaning can make sense of social ontology, explaining both social facts and mathematical facts as socially instituted. The first step is to show how semantic inferentialism can explain how we can talk about objects having properties that transcend our and others' commitments. Thanks to the three layers of inference, substitution and anaphora, inferentialism can account for communication as an activity of deontic scorekeeping. The latter is based on the human ability to keep track of one's own commitments and of other people's commitments and of their inferential relationships of incompatibility, commitment-preservation and entitlement-preservation; involves the capacity to understand when one subsentential part of a sentence can be substituted with another, and moreover when a tokening is a recurrence of a previous tokening. On a similar basis, one can understand how the position in a space of reasons assigns a physical thing a social status (money, marriage, university), producing quantificational commitments over and above physical quantificational commitments ("A is the same prime minister as B" is a quantificational commitment different from "A is the same individuum as B"). But mathematical objects can then be considered as socially constituted too, because they occupy a position in a space of reasons. Yet, how can mathematical objects be introduced, given that there is no physical object to which we can attribute a normative status? Abstraction principles are considered as introduction and elimination rules and shown to play this role, thereby determining the norms that sentences about the newly introduced objects (e.g., cardinal numbers) should obey. So, the difference between mathematical objects and other social facts does not concern their social constitution, but the object to which one attributes a normative status: abstraction principles for mathematical facts and physical objects for social facts.

According to Cole, mathematical contents can be intersubjectively constituted and semantically objective; still, mathematical contents, unlike other social constructs such as gender - that are highly context sensitive, vary widely and involve many expectations not explicitly codified - are more similar to institutional facts such as codified legal contents, and typically show a highly constrained objectivity, involving features that are often not arbitrary. Following Searle's account (1995, 2010), Cole argues that mathematical entities, like institutional facts such as marriages, depend on there being constitutive rules for their existence as status functions, but are endowed with a stronger objectivity than other species of institutional phenomena. This is due to the fact that they play some kind of representational function in relation to facets of reality that are not the entities in question (see also Cole 2013) and pose constraint on their social construction - think of the representational function that natural numbers play with respect to their applicability to spatio-temporal facets of reality that we find around us.

It is exactly an account of the constitution of human beings at the evolutionary stage which according to Pantsar can help to address the criticism of Cole's account of the objectivity of mathematics in terms of its applicability function (see Rytilä 2021). In this way Pantsar offers an account of the connection between numbers as institutional entities and our cognitive constitution by looking at how cognitive representational skills evolve biologically, are intersubjectively structured, and socially enculturated both in individual ontogeny as a result of the enculturated development based on proto-arithmetical abilities, and in cultural history and phylogeny by cumulative cultural evolution. Acculturation theories, while explaining how cultural learning transforms neural resources related to universally shared proto-arithmetical capacities by using them for culturally specific arithmetical learning (see for instance Menary 2015), could in this sense account for the fact that arithmetic can be granted a sort of strong intersubjectivity which Pantsar labels as "maximal" - spanning across cultures, languages and culturally developed practices - and that could be accepted by many as being in this sense objective. This strategy would account for the representational function and applicability that Cole grants to mathematical entities while allowing that they, unlike constitutive rules such as rules of chess and other games, are not just conventions, but rather institutional facts rooted in our both biologically and culturally evolved cognitive architecture.

Carter shares with socio-ontological approaches such as Cole's (2013) the idea that collective agreements and imposition of status functions play the main role in the way mathematical entities are introduced as a special kind of social institutions, and that the purpose of this introduction is to allow us to reason about things that ultimately concern the physical world. Yet, she distances herself from Cole's assumption (2013) that objects so introduced are genuine entities, atemporal and exist by necessity, and offers a Peircean account of the status of mathematical entities as only pragmatically real but nevertheless objective.

In contrast to Cole, who regards mathematical objects as institutional facts, Ferreirós believes that there are important disanalogies between mathematical objects and institutions: essentially a greater independence from socio-cultural context, manifested in a more ubiquitous presence, in a closer relationship with certain innate capacities such as subitizing, in less diversification across different societies, in the occurrence even in organizationally very loosely complex societies, and perhaps also in the direction of fit, less directed towards collective intentionality and more aimed at representing relational patterns in experience and action. However, the difference could ultimately be conceived more as a difference in degree than in kind: mathematical objects are not only intersubjective, but strongly intersubjective, so their surrogate objectivity is, at least in the case of natural numbers, of a greater degree than that of other institutional social facts such as money or marriage.

6 New Directions for Research

If the philosophy of mathematical practice arose at least in part as a reaction to certain analytic philosophy, which aimed to develop philosophical reasons in favor of a certain mathematical ontology and normatively determine how mathematics should be done, it might at first seem contradictory or at least bizarre to invoke social ontology, which is nowadays mostly developed as a metaphysical analytical theory, to analyze the concept of mathematical practice. And yet one of the risks of the philosophy of mathematical practice, invoked for example by Gandon (2013), is precisely that of becoming a mere externalist (historical and social) analysis of mathematical activity, failing to offer a philosophical and normative analysis of its constitution. The challenge of this volume is, on the one hand, to bring into dialogue two traditions that are far apart and yet have many concepts in common, such as the notions of practice, agent, rule, and on the other hand, to suggest new horizons for both social ontology and the philosophy of mathematical practice.

For example, mathematical objectivity, which has traditionally been associated with the axiomatic formulation of theories, now presents itself as a matter of degrees of construction from simpler surrogacy or representational functions up to more complex functions, or mediated by particular principles that allow the intersubjective construction of social artifacts of greater abstraction. Instead of the classical opposition between real and ideal, based on the piecemeal construction of theories through the modification of their axioms, the notion of varying degrees of objectivity, obtained through gradually more complex processes of construction, becomes central, even going so far as to question the social and institutional nature of axiomatics itself (Cantù 2022).

Another interesting result is that the mathematical objectivity at issue is still often modeled on a certain strong or robust or maximal objectivity analogous to Platonic objectivity but achieved through social constructions of a different kind. This has the effect of transposing the ontological debate to a different level. While a certain constructivist approach forms a common thread in the discourses of social ontology, the central issue concerns whether there are differences in the objectivity of mathematics and social sciences, and in the role played by shared or collective intentions. If social ontology theories consider paradigmatic examples as test cases (e.g., marriage, private property and money, as in Guala 2016, p. xxi), should the same hold for mathematical ontology? What would the paradigmatic examples be?

Open questions that are not discussed here but which could now be addressed with new tools concern the details of the constitution of mathematical objects as social objects. Does the distinction between grounding and anchoring apply to mathematical objects? Are the instantiations and identity conditions of a mathematical property and of a social kind significantly different (Epstein 2014, pp. 2–3)? Does the distinction between constitutive and regulative rules apply to mathematical practices too? Should the constituents of mathematical practices be sought in language, in agents' attitudes, in causal patterns that practices seek to account for? What properly sanctions the institutionalization of a mathematical practice? An intersubjective agreement, a convention to which surrogacy functions are attributed, an attempt to coordinate human practices with cognitive problems generated by the encounter with an external environment?

In contrast, from the perspective of social ontology, what observations or new problems may emerge from the comparison with the analysis of mathematical practices? On the one hand, one may argue, relying on the results of accounts such as Ferreirós, Pantsar and Giardino, that socio-ontological models based on an embodied and situated model of cognition seem more apt than representationalist ones to understand mathematical practice and the socio-ontological constitution of mathematical facts through evolutive processes of enculturation. Could a socio-ontological model based on embodied, ecological processes of habit formation (see Testa 2021) rather than on intentionality better account for both the multilayered and scaffolding character of the natural, social, and institutional dimensions of the objectivity of mathematical entities and the place that their strong objectivity plays within the genus of social and institutional objects?

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