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The ϵ-function and the coefficients of its asymptotic expansion for constant scalar curvature Kähler metrics on noncompact manifolds

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Introduction

An important open problem in Kähler geometry consists in characterizing projectively induced metrics in view of the properties of their curvatures. A Kähler metric g on a complex manifold M is said to be *projectively induced* if there exists a *local* Kähler immersion (i.e. holomorphic and isometric) into a finite or infinite dimensional complex projective space $\mathbb{C}\mathrm{P}^N$ endowed with the Fubini-Study metric.

The study of Kähler immersions of a (real analytic) Kähler manifold into a finite or infinite dimensional complex space form originates from the work of E. Calabi in a paper entitled "Isometric imbeddings of complex manifolds" ([16]) from 1953. Here Calabi defined a special local potential, called *diastasis function* which allows to obtain necessary and sufficient conditions for a neighbourhood of a point to be locally Kähler immersed into a finite or infinite complex space form. However, due to the general difficulty of applicability of this criterion, a complete classification of Kähler manifolds admitting a Kähler immersion into complex space forms is not known, not even when the Kähler manifolds involved are of great interest, such as when they are Kähler–Einstein or homogeneous spaces.

Of course many examples of projectively induced metrics can be constructed by taking the pull-back of the Fubini–Study metric on holomorphic submanifolds of $\mathbb{C}\mathrm{P}^N$, although, it is harder to find projectively induced metrics with prescribed curvature. In this context, the topology and the geometry of the complex manifolds give obstructions to the existence of such immersions. For instance, D. Hulin in [35] proved that the scalar curvature of a compact Kähler–Einstein manifold Kähler immersed into $\mathbb{C}\mathrm{P}^N$, is forced to be positive. In particular there are not Calabi-Yau (i.e. compact and Ricci-flat) submanifolds of $\mathbb{C}\mathrm{P}^N$, with $N < +\infty$. Moreover, compact Kähler submanifolds of $\mathbb{C}\mathrm{P}^\infty$ actually live in a finite dimensional complex projective space. Although, this holds true only for *global* Kähler immersions, as shown by the flat torus which is an example of compact manifold that is *locally* projectively induced in $\mathbb{C}\mathrm{P}^\infty$ but does not admit any Kähler immersion in $\mathbb{C}\mathrm{P}^N$ for finite N, as follows by the above mentioned result of Hulin or by Calabi's rigidity Theorem in [16, Th. 9] (see also [49] for an overview of Calabi's work). Recently in [1], C. Arezzo, C. Li and A. Loi, extending Hulin's result, proved that there are not Ricci–flat submanifolds of $\mathbb{C}\mathrm{P}^N$ with $N < \infty$. It is important to emphasize that when the ambient space is taken to be infinite dimensional the situation could be much different, for example in [47] Kähler-Einstein submanifolds of $\mathbb{C}\mathrm{P}^\infty$ with negative scalar curvature are given. It is still an open question if there exists a Ricci–flat (nonflat) Kähler submanifold of $\mathbb{C}\mathrm{P}^\infty$. In [44] A. Loi, F. Salis and F. Zuddas conjectured that the only possibility is the flat one, i.e. that *a Ricci-flat* *projectively induced metric is flat.* They validate this conjecture in the case in which the metric is *radial* and the immersion is *stable* (see also [50, 51, 68] for other results in the same context).

Somehow the requirement that a Kähler metric is projectively induced is a very strong assumption. Thus one could try to approximate an (integral) Kähler metric g on a complex manifold M with projectively induced ones, through the following construction, based on the theory of geometric quantization.

A *geometric quantization* (L, h) of a n-dimensional Kähler manifold (M, ω) consists of an hermitian holomorphic line bundle L over M such that the first Chern class of L is represented by ω and its curvature $\mathrm{Ric}(h) := -i\partial\bar{\partial}\log h$ satisfies $\mathrm{Ric}(h) = \omega$. A necessary and sufficient condition that guarantees the existence of such a geometric quantization is ω to be integral. For each positive integer m one can consider the holomorphic line bundle $L^m = L^{\otimes m}$ endowed with the hermitian metric h_m induced on L^m by h, such that $\mathrm{Ric}(h_m) = m\omega$. Let $\mathcal{H}_m$ be the space of global holomorphic sections of L^m which have limited norm with respect to the following scalar product:

$$\langle s, s\rangle_{h_m} := \int_M h_m(s(x), s(x))\frac{\omega^n}{n!}.$$

When $\mathcal{H}_m \neq \{0\}$ (condition that is always satisfied for $m >> 0$ by Kodaira's theorem when M is compact and in which case with finite dimension) we can take an orthonormal basis $\{s_j\}_{j=0,\dots,d_m}$ of $\mathcal{H}_m$, and define a function on M (that we call *ϵ-function*) by:

$$\epsilon_{mg}(x) := \sum_{j=0}^{d_m} h_m(s_j(x), s_j(x)). \tag{1}$$

This function is a tool to evaluate how a Kähler metric differs from being projectivey induced. More precisely, if there exists m sufficiently large such that the line bundle is basepoint-free, namely for each point $x \in M$ there exists $s \in \mathcal{H}_m$ non vanishing at x (condition always ensured in the compact case by Kodaira's theory), one can construct a holomorphic map $F_m : M \to \mathbb{C}\mathrm{P}^{d_m}$, $(d_m \leq +\infty)$, called the *coherent states map*, by:

$$F_m : M \to \mathbb{C}\mathrm{P}^{d_m} \quad ; \quad x \mapsto [s_0(x) : \cdots : s_{d_m}(x)].$$

which satisfies (see e.g. [2]):

$$F_m^*(\omega_{FS}) = m\omega + \frac{i}{2}\partial\bar{\partial}\log\epsilon_{mg}. \tag{2}$$

In particular one has that when ϵ_{mg} is constant, F_m is a holomorphic and isometric immersion.

Although not all Kähler metrics are projectively induced, G. Tian ([62]) and W-D. Ruan ([59]) solved a conjecture by S-T. Yau proving that any polarized metric on a compact complex manifold is the C^∞-limit of projectively induced ones, that is

$$\lim_{m\to\infty}\frac{F_m^* g_{\mathrm{FS}}}{m} = g.$$

Further, generalizing the Tian-Ruan theorem, D. Catlin ([21]) and S. Zelditch ([69]) proved that, when M is compact the function ϵ_{mg} admits an asymptotic expansion, the so called *Tian-Yau-Catlin-Zelditch expansion* (TYCZ-expansion for short):

$$\epsilon_{mg}(x) \sim \sum_{j=0}^{\infty} a_j(x) m^{n-j}, \tag{3}$$

where $a_0(x) \equiv 1$ and the $a_j(x)$, $j = 1, 2, \ldots$ are smooth functions on M depending on the curvature and on its covariant derivatives at x of g. For this asymptotic expansion it is meant that, for every integers l, r

$$\left\| \epsilon_{mg}(x) - \sum_{j=0}^{l} a_j(x) m^{n-j} \right\|_{C^r} \leq \frac{C(l,r)}{m^{l+1}}, \tag{4}$$

for some constant $C(l, r) > 0$. In particular, Z. Lu [52] computed the first three coefficients, that read:

$$\begin{cases} a_1 = -\frac{1}{2} scal_g \\ a_2 = -\frac{1}{3}\Delta\ scal_g + \frac{1}{24}\left(|R|^2 - 4|\mathrm{Ric}|^2 + 3scal_g^2\right) \\ a_3 = -\dfrac{1}{8}\Delta\Delta\ scal_g + \dfrac{1}{24}\mathrm{divdiv}(R, \mathrm{Ric}) - \dfrac{1}{6}\mathrm{div}(\mathrm{div}(scal_g \mathrm{Ric})) + \\ \qquad + \dfrac{1}{48}\Delta(|R|^2 - 4|\mathrm{Ric}|^2 + 8scal_g^2) - \dfrac{1}{48} scal_g (scal_g^2 + |R|^2 - 4|\mathrm{Ric}|^2) + \\ \qquad - \dfrac{1}{24}(\sigma_3(\mathrm{Ric}) - \mathrm{Ric}(R, R) + R(\mathrm{Ric}, \mathrm{Ric})) \end{cases} \tag{5}$$

where $scal_g$, Ric_g and R_g denote respectively the scalar curvature, the Ricci tensor and the curvature tensor of g, and the norms are taken with respect to g.

When M is noncompact, the tools used in the compact setting are not available and many complications may occur, for example because the volume of M could be infinite. In this case, $\mathcal{H}_m$ could reduce to be $\{0\}$ and thus we can not construct the ϵ-function. If we assume $\mathcal{H}_m \neq \{0\}$, we say that a TYCZ–expansion (3) exists if (2.2) holds for any compact subset of $K \subset M$, as introduced in [2] by C. Arezzo and A. Loi. In constrast with the compact setting, the existence of such an expansion is not guaranteed and only partial results are given. In [30] M. Engliš proved the existence of a TYCZ–expansion in the case of strongly pseudoconvex domains of $\mathbb{C}^n$ with real analytic boundary, and computed the a_j's coefficients obtaining the same results as Lu. A more general result can be obtained by the work of X. Ma and G. Marinescu [54, Thm. 6.1.1], described in Section 3.1 for Kähler-Einstein metrics and in Section 4.5 for Hwang–Singer metrics.

Studying metrics with the TYCZ coefficients being prescribed is a very natural generalization of the problem of finding Kähler metrics with constant scalar curvature on a Kähler manifold. The vanishing of the coefficients a_k for $k \geq n+1$ turns out to be related to some important problems in the theory of pseudoconvex manifolds (cf. [53, 5, 46]). In the noncompact setting, one can find in [48] a characterization of the flat metric among locally hermitian symmetric spaces as the only one with vanishing a_1 and a_2, while in [33] Z. Feng and Z. Tu solve a conjecture formulated in [67] by showing that the complex hyperbolic space is the only Cartan-Hartogs domain where the coefficient a_2 is constant. Another characterization of the flat metric can be found in [50], where it is given as a Taub-NUT metric with $a_3 = 0$.

During my Ph.D., I have been interested in studying the third coefficient arising from the TYCZ-expansion of the ϵ-function associated to a Kähler-Einstein metric and the consequences of its vanishing. The first result of this thesis is the following theorem.

Theorem 1. *Let (M, g) be a n-dimensional Kähler–Einstein manifold with integral Kähler form ω. If $\mathcal{H}_m \neq \{0\}$, there exists a TYCZ–expansion for ϵ_{mg} whose coefficients satisfy the following:*

1. *if $n = 2$ then $a_3 = 0$ if and only if $\Delta|R|^2 = 0$;*
2. *if $n \geq 3$ then $a_3 = 0$ implies g is Ricci-flat.*

In [44, Thm. 1.1], A. Loi, F. Salis and F. Zuddas prove that *the coefficient a_3 of a radial constant scalar curvature Kähler metric is constant if and only if a_2 is constant.* Combining this result with Thm. 1 we give a characterization of radial Kähler metrics with vanishing a_3:

Corollary 1. *Let (M, g) be a Kähler–Einstein manifold endowed with a radial Kähler metric g. Then $a_3 = 0$ if and only if (M, g) is biholomorphically isometric to $\mathbb{C}^n$, $\mathbb{C}\mathrm{H}^2$ or $\mathbb{C}\mathrm{P}^2$ with (a multiple of) their standard metric.*

In [44], Loi, Salis, and Zuddas, conjecture that a Ricci–flat metric on a n-dimensional complex manifold such that $a_{n+1} = 0$ is forced to be flat. By Theorem 1 such conjecture is equivalent, in the $n = 2$ case, to proving that a Ricci–flat surface with harmonic $|R|^2$ is flat. Furthermore, it follows from the proof of Theorem 1 that Ricci–flat metrics or Kähler–Einstein metrics on surfaces which are either homogeneous or regular have vanishing a_3. Here homogeneous means that the group of isometric automorphisms of (M, g) acts transitively on M, while a regular metric is a metric whose ϵ-function ϵ_{mg} is constant for all large enough m. It is an open question to understand if regular Kähler–Einstein metrics are homogeneous. If one drops the Kähler–Einstein assumption one gets a negative answer, at least for noncompact manifolds, as in [18] F. Cannas Aghedu and A. Loi proved that the scalar flat Burn–Simanca metric is a regular nonhomogeneous metric on the blow-up of $\mathbb{C}^2$ at one point. The question if a regular Kähler manifold is homogeneous is still an interesting and open question in the compact setting, where the manifolds involved are projective algebraic.

This leads us to the following question:

Question 1. *Is a Kähler-Einstein manifold with vanishing a_3 homogeneous?*

In view of Theorem 1, a positive answer to this question would imply that the only nonflat Kähler-Einstein manifolds with vanishing a_3 are homogeneous surfaces. Observe that we can construct many examples of metrics with vanishing a_3 by taking the Kähler product of a bounded symmetric domain with its Bergman metric times its compact dual (see [48]), however the product metric is not Kähler-Einstein. Notice also that the family of Taub-NUT metrics on $\mathbb{C}^2$ are an example of Ricci-flat nonhomogeneous metrics with $a_3 \neq 0$ (see [50]).

As second result of this thesis, we compute the coefficient a_3 of a locally nonhomogeneous complete Kähler–Einstein manifold constructed by Calabi in [15], proving the following:

Theorem 2. *The ϵ-function associated to Calabi's metric admits a TYCZ–expansion with $a_3 \neq 0$.*

A natural way to weaken the Ricci–flatness condition is to request the Kähler metric to have constant scalar curvature. However, very little is known for constant scalar curvature Kähler metrics. With this generalization, the previous conjecture, which states that the only Ricci-flat projectively induced metric is flat, has a negative answer, as shown by F. Cannas–Aghedu and A. Loi in [20], where they prove that the Burns-Simanca metric on the blow-up of $\mathbb{C}^2$ at one point, an example of scalar flat (nonflat) complete metric, is projectively induced. In the finite dimensional context, it is conjectured by A. Loi, F. Salis, F. Zuddas in [45] that the only projectively induced constant scalar curvature Kähler metrics lie on flag manifolds (actually their conjecture includes also extremal Kähler metrics).

In particular they prove that under suitable assumptions on the Kähler potential (namely *radial* and *well-behaved*), a constant scalar curvature Kähler manifold Kähler immersed in $\mathbb{C}\mathrm{P}^\infty$ is a complex space form with non positive holomorphic sectional curvature or it is an open subset of $\mathbb{C}\mathrm{P}^N$ with a multiple of the Fubini-Study metric.

In view of a better understanding of the geometry of scalar flat Kähler metrics, during my Ph.D., I have been interested in the study of two families of scalar flat Kähler metrics constructed in [36] by A. D. Hwang and M. A. Singer on $\mathbb{C}^{n+1}$ and on $\mathcal{O}(-k)$. For the metrics in both the families, we prove the existence of an asymptotic expansion for their ϵ-functions and we show that they can be approximated by a sequence of projectively induced Kähler metrics. Further, we show that the metrics on $\mathbb{C}^{n+1}$ are not projectively induced, and that the Burns–Simanca metric is characterized among the scalar flat metrics on $\mathcal{O}(-k)$ to be the only projectively induced one as well as the only one whose second coefficient in the asymptotic expansion of the ϵ-function vanishes. More precisely, for the metrics in $\mathbb{C}^{n+1}$ we have

Theorem 3. *Let g_β be the Kähler metric on $\mathbb{C}^{n+1}$ arising from Hwang–Singer construction. Then cg_β is not projectively induced for any value of $c > 0$ and $\beta < 0$, but it can be approximated by a sequence of projectively induced metrics.*

While for the ones in $\mathcal{O}(-k)$, we have

Theorem 4. *Let g_k be the Kähler metric arising from Hwang–Singer construction on $\mathcal{O}(-k)$. Then g_k is projectively induced if and only if its second coefficient vanishes identically, that is if and only if it is the Burns–Simanca metric on the blow-up of $\mathbb{C}^2$ at one point. Moreover g_k can be approximated by a sequence of projectively induced metrics.*

The thesis is organized as follows.

In Chapter 1 we deal with preliminary notions about Kähler manifolds (Section 1.1) and present the work of Calabi on the existence of Kähler immersions of a complex manifold into a complex space form (Section 1.2). In the last section we introduce canonical metrics, that is Kähler metrics with special curvature properties, focusing on Kähler–Einstein and constant scalar curvature metrics, giving examples that will be found later in the thesis.

In Chapter 2 we deal with the theory of geometric quantization. Section 2.1 provides introductory material on holomorphic hermitian line bundles, which allows to give the definition of geometric quantization in Section 2.2. In the last section we touch the important notion of the ϵ-function giving an overview of the problematic of its asymptotic expansion.

In Chapter 3 we deal with the asymptotic expansion of the ϵ-function for Kähler–Einstein manifolds. The chapter consists of two sections. In the first one we compute the coefficients a_2 and a_3 for Kähler–Einstein manifolds, from Lu's formulas, proving Theorem 1 and Corollary 1. Last section is devoted to Calabi's inhomogeneous metric and the proof of Theorem 2.

In Chapter 4 we deal with scalar-flat Kähler metrics. The chapter is organized as follows. In Section 4.1 we recall what we need about Hwang-Singer construction restricted to polarized Kähler–Einstein manifolds. In Section 4.2 we give an overview of Calabi's criterion, deriving a necessary condition for the Hwang-Singer metrics to be projectively induced. Sections 4.3 and 4.4 are devoted respectively to the description of Hwang-Singer metrics on $\mathbb{C}^{n+1}$ and $\mathcal{O}(-k)$. Section 4.5 contains the existence results for the ϵ-function associated to the Hwang-Singer metrics on $\mathbb{C}^{n+1}$ and $\mathcal{O}(-k)$, and for its asymptotic expansion, and the proofs of theorems 3 and 4. Finally, Section 4.6 includes some computations regarding the a_2 coefficient.

Contents

Chapter 1

Preliminaries on Kähler geometry

Kähler geometry lies in the overlap of Riemannian, symplectic and complex geometry and owes its name to the German mathematician Erich Kähler, who introduced the concept of Kähler structures in his article "Über eine bemerkenswerte Hermitesche Metrik" ([38]) in 1933. Section 1.1 is devoted to recall definitions and standard properties of Kähler manifolds (we mainly refer to [56]). Section 1.2 briefly summarizes the theory of Kähler immersions into complex space forms developed by E. Calabi in [16] (see also [49] for a more recent overview of Calabi's work). Section 1.3 deals with the problem of finding canonical metrics in a given cohomology class of Kähler metrics, focusing on Kähler–Einstein and constant scalar curvature Kähler metrics (see [61] and [63] for further details).

1.1 Kähler manifolds

Let (M, J) be a complex manifold, where J denotes the complex structure.

Definition 1.1.1. An *hermitian metric* on M is a Riemannian metric g such that J is an orthogonal transformation of the tangent bundle, that is $g(X, Y) = g(JX, JY)$ for all $X, Y \in TM$. The *fundamental form* of an hermitian metric is the 2-form of type $(1, 1)$ defined by $\omega(X, Y) := g(JX, Y)$. An *hermitian manifold* is a couple (M, g) where M is a complex manifold and g an hermitian metric on M.

Let z_α be holomorphic coordinates on an open set $U \subset M$ and denote by $g_{\alpha\overline{\beta}} := g(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \overline{z}_\beta})$ the coefficients of the metric tensor in these coordinates. The fundamental form is given by

$$\omega = \frac{i}{2} \sum_{\alpha,\beta=1}^{n} g_{\alpha\overline{\beta}}\, dz_\alpha \wedge d\overline{z}_\beta$$

Definition 1.1.2. An hermitian metric g on M is a *Kähler metric* if the associated 2-form ω is closed, i.e. $d\omega = 0$. A *Kähler manifold* is a couple (M, g) where M is a complex manifold and g is a Kähler metric.

One of the most surprising properties of Kähler metrics is that they can locally be described simply by real-valued functions:

Lemma 1.1.3 ($\partial\overline{\partial}$-lemma)**.** Let ω be a Kähler form on M. Then for every point $x \in M$ there exists an open neighbourhood $x \in U \subset M$ and a smooth real-valued function $\varphi : U \to \mathbb{R}$ such that

$$\omega|_U = \frac{i}{2}\partial\overline{\partial}\varphi$$

that is, in local coordinates,

$$g_{\alpha\overline{\beta}} = \frac{\partial^2\varphi}{\partial z_\alpha \partial \overline{z}_\beta}.$$

The function φ is called *local Kähler potential.*

Remark 1.1.4. The Kähler potential φ is uniquely defined up to the sum with the real part of a holomorphic function. In fact, if $\varphi' : U \to \mathbb{R}$ is another Kähler potential, in local coordinates we have

$$\frac{\partial^2(\varphi - \varphi')}{\partial z_\alpha \partial \overline{z}_\beta} = 0,$$

that implies

$$\varphi = \varphi' + h + \overline{h}$$

for some holomorphic function h on M.

The main examples of Kähler manifolds are the *complex space forms*, namely Kähler manifolds with finite or infinite dimension of constant holomorphic sectional curvature b, that if we assume to be complete and simply connected, they are, up to holomorphic isometries, one of the following spaces (according to the sign of b):

Example 1.1.5 (Complex space forms). They are:

- The complex Euclidean space $\mathbb{C}^N$ of complex dimension $N \leq +\infty$ endowed with the flat metric g_0, where

  $$\mathbb{C}^\infty = l^2(\mathbb{C}) = \left\{ (z_1, z_2, \dots, z_j, \dots) \mid \sum_{j=1}^{+\infty} |z_j|^2 < +\infty,\ z_j \in \mathbb{C} \right\}.$$

  The associated Kähler form ω_0 is given by

  $$\omega_0 = \frac{i}{2}\sum_{j=1}^{N} dz_j \wedge d\overline{z}_j = \frac{i}{2}\partial\overline{\partial}|z|^2$$

  thus

  $$\varphi : \mathbb{C}^N \to \mathbb{R}\ ; \quad z \mapsto |z|^2$$

  is a global Kähler potential for g_0. The holomorphic sectional curvature of the Euclidean metric g_0 is 0.

- The complex projective space $\mathbb{C}\mathrm{P}^N$ of complex dimension $N \leq +\infty$ endowed with the Fubini-Study metric g_{FS}. Let $[Z_0 : \cdots : Z_N]$ be homogeneous coordinates and $(z_1, \ldots, z_N)$ the respective affine coordinates on $U_0 = \{Z_0 \neq 0\}$ defined by $z_k = \frac{Z_k}{Z_0}$, where

  $$\mathbb{C}\mathrm{P}^\infty = \frac{l^2(\mathbb{C})\backslash\{0\}}{\sim}$$

  and $\sim$ is the usual equivalent relation defining projective spaces. In homogeneous coordinates the fundamental form is given by

  $$\omega_{\mathrm{FS}} = \frac{i}{2}\partial\bar{\partial}\log(|Z_0|^2 + \cdots + |Z_N|^2).$$

  Thus

  $$\varphi : U_0 \to \mathbb{R}\ ; \quad z \mapsto \log\left(1 + \sum_{j=1}^N |z_j|^2\right)$$

  is a local Kähler potential for the Fubini-Study metric on $U_0 \subset \mathbb{C}\mathrm{P}^N$ with respect to the affine coordinate z_j. The holomorphic sectional curvature of the Fubini-Study metric g_{FS} is 4.

- The complex hyperbolic space $\mathbb{C}\mathrm{H}^N$ of complex dimension $N \leq +\infty$ defined by

  $$\mathbb{C}\mathrm{H}^N = \left\{(z_1, \ldots, z_N) \in \mathbb{C}^N \mid \sum_{j=1}^N |z_j|^2 < 1\right\}$$

  endowed with the hyperbolic metric g_{hyp}. For this metric there exists a global Kähler potential given by

  $$\varphi : \mathbb{C}\mathrm{H}^N \to \mathbb{R}\ ; \quad z \mapsto -\log\left(1 - \sum_{j=1}^N |z_j|^2\right)$$

  thus the Kähler form ω_{hyp} is given by

  $$\omega_{\mathrm{hyp}} = \frac{i}{2}\partial\bar{\partial}\log\left(\frac{1}{1 - \sum_{j=1}^N |z_j|^2}\right).$$

  The holomorphic sectional curvature of the hyperbolic metric g_{hyp} is -4.

1.2 Kähler immersions

Unlike riemannian manifolds, which by a famous theorem of J. Nash ([57]) can be realized as riemannian submanifolds of $(\mathbb{R}^n, g_0)$, i.e. every riemannian manifold admits an isometric imbedding into the Euclidean space $\mathbb{R}^n$ for n sufficiently large, Kähler manifolds not always can be holomorphically and isometrically embedded into the complex Euclidean space $\mathbb{C}^N$ (not even if we allow N to be infinity). Many obstructions occur in the complex setting: for instance, as a consequence of the maximum principle for holomorphic functions, a compact complex manifold cannot be embedded into $\mathbb{C}^N$. The problem of the existence and uniqueness of holomorphic and isometric immersions of a Kähler manifold into a complex space form was theoretically solved by E. Calabi in his seminal paper [16]. Let $F(N,b)$ be an N-dimensional complex space form of holomorphic sectional curvature $4b$, with $b=0,1,-1$ as in Example 1.1.5.

Definition 1.2.1. A Kähler metric g on a Kähler manifold M admits a *local Kähler immersion* into $F(N,b)$ if there exist a point $p\in M$, a neighbourhood U of p and a map $f:(U,g)\to(F(N,b),g_b)$ such that

- f is holomorphic;
- f is isometric, i.e. $f^*g_b=g$.

If $F(N,b)=\mathbb{C}\mathrm{P}^N$, we also say that the metric g is *projectively induced* and we say that the map f is *full* if the image $f(M)\subset F(N,b)$ is not contained in any totally geodesic subspace of $F(N,b)$.

Remark 1.2.2. If g admits a local Kähler immersion in $F(N,b)$, then g must be real analytic, being the pull-back through a holomorphic map of real analytic Kähler potentials, that is, fixed a local coordinate system $z=(z_1,\dots,z_n)$ on a neighbourhood U of a point $p\in M$, it can be described on U by a real analytic Kähler potential $\varphi:U\to\mathbb{R}$ (equivalently φ admits an expansion in power series of the form $\varphi=\sum_{j,k=1}^{\infty}\varphi_{jk}z^{m_j}\overline{z}^{m_k}$, with $\varphi_{jk}\in\mathbb{C}$). In this case, the potential $\varphi(z)$ can be analytically extended to a function $\tilde\varphi$ defined on an open neighbourhood $W\subset U\times\overline{U}$ of the diagonal containing $(p,\overline{p})\in M\times\overline{M}$ (here $\overline{M}$ denotes the manifold conjugated to M), that is

$$\tilde\varphi:U\times\overline{U}\to\mathbb{R}\ ;\quad (z,\overline{w})\mapsto\tilde\varphi(z,\overline{w})$$

such that $\tilde\varphi(z,\overline{z})=\varphi(z)$. We are adopting the following convention: we arrange every n-tuple of nonnegative integers as the sequence $m_j=(m_1^j,m_2^j,\dots,m_n^j)$ such that $m_0=(0,\dots,0)$, $|m_j|\le|m_{j+1}|$ for all positive integers j and all the m_j's with the same lenght $|m_j|$ using lexicographic order and $z^{m_j}=\prod_{i=1}^n z_i^{m_{ji}}$.

The fundamental tool introduced by Calabi to study Kähler immersions of a Kähler manifold into a complex space form is the *diastasis function*:

Definition 1.2.3. The *diastasis function* is defined by

$$D(z,w)=\tilde\varphi(z,\overline{z})+\tilde\varphi(w,\overline{w})-\tilde\varphi(z,\overline{w})-\tilde\varphi(w,\overline{z}),\quad \forall\, z,w\in W. \tag{1.1}$$

Since the Kähler potential is independent of the coordinate system chosen, so is the diastasis D. Moreover, while a Kähler potential is defined up to the addition with the real part of a holomorphic function, the diastasis is independent from the potential chosen and so D depends only on the Kähler metric g. The diastasis is real valued and symmetric in z and w.

When in a coordinate neighbourhood we fix a point $p\in M$ with coordinates $w=(w_1,\dots,w_n)$ we write $D_p(z)$ for the diastasis centred at this point and if p is the origin of the coordinate system, we write $D_0(z)$. Notice that, once fixed one of its two entries, D is a Kähler potential for g.

Example 1.2.4 (complex space forms). The Kähler potentials of the complex space forms (Example 1.1.5) are actually their diastasis functions centred in the origin.

The following gives a characterization of the diastasis in terms of its power expansion.

Theorem 1.2.5 (Characterization of the diastasis, [16])**.** Among all the Kähler potentials, the diastasis $D_p(z)$ is characterized by the fact that in every coordinate system $(z_1, \dots, z_n)$ centred in p, the $\infty \times \infty$ matrix of coefficients (a_{jk}) of its power expansion around the origin

$$D_p(z) = \sum_{j,k=0}^{\infty} a_{jk} z^{m_j} \overline{z}^{m_k}$$

doesn't contain pure holomorphic or antiholomorphic terms, i.e. it satisfies $a_{j0} = a_{0j} = 0$ for every nonnegative integer j.

The importance of the diastasis function is expressed by the following proposition, that is the key ingredient in the proof of the Calabi's criterion (Theorem 1.2.7).

Proposition 1.2.6 ([16], Proposition 6)**.** Let (M, g) and (S, G) be real analytic Kähler manifolds. If there exists a Kähler immersion $f : (M, g) \to (S, G)$, then $D^S_{f(p)} \circ f = D^M_p$ on $f^{-1}(V) \cap U$, where $D^M_p : U \to \mathbb{R}$ and $D^S_{f(p)} : V \to \mathbb{R}$ are the diastasis functions of M and S around p and $f(p)$ respectively.

While the pullback through a Kähler immersion $f : (M, \varphi) \to (S, \Phi)$ of a Kähler potential is still a Kähler potential with a factor of indeterminacy given by the sum of the real part of a holomorphic function, that is

$$f^*\Phi = \varphi + h + \overline{h},$$

the above proposition states that the diastasis is invariant by pullback through a Kähler immersion.

The diastasis function allows to obtain necessary and sufficient conditions for a neighbourhood of a point to be locally Kähler immersed into a complex space form. This is the content of Calabi's criterion expressed in the following.

Theorem 1.2.7 (local Calabi's criterion ([16], Thms 3,8))**.** Let (M, g) be a real analytic Kähler manifold. Then there exists a neighbourhood U of a point $p \in M$ that admits a Kähler immersion into a complex space form $F(N, b)$ (with $b = 0, 1, -1$) if and only if the $\infty \times \infty$ matrix of coefficients (a_{jk}) defined by

- if $b = 0$:

$$D_0(z) = \sum_{j,k=1}^{\infty} a_{jk} (z - p)^{m_j} (\overline{z} - \overline{p})^{m_k}; \tag{1.2}$$

- if $b = 1$:

$$e^{D_0(z)} - 1 = \sum_{j,k=1}^{\infty} a_{jk} (z - p)^{m_j} (\overline{z} - \overline{p})^{m_k}; \tag{1.3}$$

- if $b = -1$:

$$1 - e^{-D_0(z)} = \sum_{j,k=1}^{\infty} a_{jk} (z - p)^{m_j} (\overline{z} - \overline{p})^{m_k} \tag{1.4}$$

is semipositive definite of rank at most N at p. Furthermore, if the rank is exactly N, the immersion is full.

The following two results permit to state a global version of Calabi's criterion.

Theorem 1.2.8 (rigidity ([16], Thm 9))**.** If a neighbourhood U of a point p admits a full Kähler immersion into $(F(N,b), g_b)$, then N is uniquely determined by the metric and the immersion is unique up to rigid motions of $F(N,b)$.

Theorem 1.2.9 (global character of Kähler immersions ([16], Thm 10))**.** If a neighbourhood of a point p of a connected Kähler manifold (M,g) admits a local Kähler immersion into $F(N,b)$, then any other point can be locally Kähler immersed in $F(N,b)$.

Due to the previous theorem, we can say that a Kähler manifold admits a local Kähler immersion into a complex space form without specifying the point for which there exists a neighbourhood that can be Kähler immersed. If M is simply connected, then it is possible to extend the immersion to the whole manifold:

Theorem 1.2.10 (global Calabi's theorem ([16], Thm 11))**.** If a Kähler metric is defined on a simply connected manifold M, then a local Kähler immersion $f : U \subset M \to F(N,b)$ can be extended to a global one. This immersion is also injective if and only if $D(z,w) = 0$ only for $z = w$.

Remark 1.2.11. As concern projectively induced metrics, we notice that

- Let $f : (M,\omega) \to (\mathbb{C}\mathrm{P}^\infty, \omega_{\mathrm{FS}})$ be a global Kähler immersion of a compact Kähler manifold M into $\mathbb{C}\mathrm{P}^\infty$. Then f cannot be full, that is, if a compact Kähler manifold admits a full isometric immersion into $\mathbb{C}\mathrm{P}^N$, then N must be finite.

- Being the pullback of the integral Fubini-Study form of $\mathbb{C}\mathrm{P}^N$ through a holomorphic map, the Kähler metric $\omega = f^*\omega_{\mathrm{FS}}$ is forced to be integral, i.e. $\omega \in H^2(M,\mathbb{Z})$. Thus a necessary condition for a Kähler metric ω to be projectively induced is that ω is integral.

1.3 Canonical Kähler metrics

Let (M,g) be an hermitian manifold and Ric be the Ricci curvature of M. The *Ricci form* ρ associated to Ric is the (closed) 2-form on M of type $(1,1)$ defined by

$$\rho(X,Y) = \mathrm{Ric}(JX,Y) \quad \forall X,Y \in \Gamma(TM).$$

For the special case of a Kähler manifold, the Ricci form has a simple expression in terms of the metric tensor. If $\omega = \frac{i}{2}\sum_{\alpha,\beta=1}^n g_{\alpha,\bar\beta}\ dz_\alpha \wedge d\bar z_\beta$ is the local expression of the fundamental form associated with g on an open set U with local coordinates $z = (z_1,\dots,z_n)$, then the Ricci form can be written as

$$\rho = -i\partial\bar\partial \log\det(g_{\alpha\bar\beta}). \tag{1.5}$$

Definition 1.3.1. A Kähler metric g on a complex manifold M is *Einstein* if there exists $\lambda \in \mathbb{R}$, called the *Einstein constant* such that

$$\rho = \lambda\omega. \tag{1.6}$$

The pair (M, g) is called *Kähler–Einstein manifold.*

Example 1.3.2 (complex space forms)*.* The complex space forms $\mathbb{C}^N, \mathbb{C}\mathrm{P}^N, \mathbb{C}\mathrm{H}^N$ given in Example 1.1.5 are Kähler–Einstein manifolds with Einstein constant $0, 2(N+1)$ and $-2(N+1)$ respectively.

Proposition 1.3.3 (cf [7])**.** Every Kähler–Einstein metric is real analytic.

Let (M, g) be a Kähler–Eintein manifold. Expressing equation 1.6 in local coordinates in a neighbourhood U of a point $z \in M$, we have

$$\partial\overline{\partial}\left(\log\det(g_{\alpha\overline{\beta}}) + \frac{\lambda}{2}\varphi\right) = 0$$

and hence

$$\log\det(g_{\alpha\overline{\beta}}) = -\frac{\lambda}{2}\varphi + h + \overline{h}$$

for some holomorphic function h on U. Thus a Kähler metric g is Einstein if and only if locally it satisfies the Monge-Ampère equation

$$\det(g_{\alpha\overline{\beta}}) = e^{-\frac{\lambda}{2}\varphi + h + \overline{h}}.$$

We recall that given a real analytic Kähler manifold (M, g) and a point $p \in M$, there exists ([10]) a coordinate system $(z_1, \ldots, z_n)$ around p such that

$$D_p(z) = \sum_{j=1}^{n} |z_j|^2 + \psi_{2,2}$$

where $\psi_{2,2}$ is a power series with degree ≥ 2 in both the variables z and $\overline{z}$. These coordinates, uniquely defined up to unitary transformations ([16]), are called *Bochner's coordinates* around the point p. By Proposition 1.3.3 follows that we can choose Bochner's coordinates on Kähler–Einstein manifolds. In particular we have the following characterization of Kähler–Einstein metrics:

Proposition 1.3.4 (cf [3])**.** A Kähler manifold (M, g) is Einstein if and only if by choosing Bochner's coordinates on a neighborhood U of any point $z \in M$, the diastasis function $D_0(z)$ satisfies the Monge-Ampère equation

$$\det\left(\frac{\partial^2 D_0}{\partial z_j \partial \overline{z}_j}\right) = e^{-\frac{\lambda}{2} D_0(z)}.$$

Let M be a complex manifold and ω_1, ω_2 two Kähler metrics on M with Ricci forms ρ_1, ρ_2 respectively. From the local expression of the Ricci form (1.5) of a Kähler metric and the $\partial\overline{\partial}$-lemma 1.1.3, it can be shown that the difference $\rho_1 - \rho_2$ is an exact form globally defined on M. Thus the cohomology class $[\rho]$ is independent of the choice of the Kähler metric and we have that $\rho/2\pi$ represents the first Chern class $c_1(M)$ of M, i.e.

$$c_1(M) = \frac{1}{2\pi}[\rho] \in H^2(M, \mathbb{R}).$$

Actually, it is possible to show that $c_1(M)$ is an integral cohomology class, i.e. $c_1(M) \in H^2(M, \mathbb{Z})$.

In [14] and [17], Calabi posed the following problem. Let (M, ω) be a compact Kähler manifold. Suppose $\tilde{\rho}$ is a closed, real $(1,1)$ form on M with $[\tilde{\rho}] = 2\pi c_1(M)$. Is it possible to find a Kähler metric $\tilde{\omega}$ on M such that $[\tilde{\omega}] = [\omega] \in H^2(M, \mathbb{R})$ and such that the Ricci form of $\tilde{\omega}$ is $\tilde{\rho}$?

This longstanding problem was finally solved and can be expressed as follows.

Theorem 1.3.5 (Calabi's conjecture)**.** *Let M be a compact Kähler manifold with Kähler form Ω and Ricci form ρ. Then for every closed real $(1,1)$-form $\rho_1 \in 2\pi c_1(M)$, there exists a unique Kähler metric with Kähler form $\Omega_1 \in [\Omega]$, whose Ricci form is exactly ρ_1. In particular, if the first Chern class of a compact Kähler manifold vanishes, then M admits a Ricci-flat Kähler metric.*

The uniqueness was proved by Calabi in [17], while the existence of ω in each Kähler class was proved by S.T.Yau in 1976 ([66] and [65]). This problem is closely related to the problem of the existence of Kähler–Einstein metrics on compact complex manifolds. Let (M, ω) be a compact Kähler–Einstein manifold. Since $\rho = \lambda\omega$, according to the sign of the Einstein constant, the first Chern class $c_1(M)$ must either vanish or have a representative which is negative ($c_1(M) < 0$) or positive ($c_1(M) > 0$) definite. We have the following situation:

- If $c_1(M) = 0$, then M admits a unique Ricci-flat metric on each Kähler class: in fact, in this case $c_1(M) = [0]$ and the Calabi's conjecture guarantees the existence of a Kähler form whose Ricci form is vanishing on each Kähler class $[\omega]$.

- If $c_1(M) < 0$, T. Aubin ([6]) and S. T. Yau ([66]) independently proved that a compact Kähler manifold admits a unique Kähler-Einstein metric with Einstein constant $\lambda = -1$.

- If $c_1(M) > 0$, the existence of a Kähler–Einstein metric with Einstein constant $\lambda = 1$ is not guaranteed and there are obstructions to those metrics to exist (for instance in terms of reductivity of the Lie algebra of holomorphic vector fields). In particular, the blow-up of $\mathbb{C}\mathrm{P}^2$ at one point has positive first Chern class but by a theorem of Y. Matsushima (see [55] or [56], Thm 19.4) it follows that this manifold cannot admit a Kähler–Einstein metric. Nowadays, the existence of such metrics is related to some algebro-geometric notions of stability (see [22],[23], [24]).

We now give some examples of Kähler–Einstein manifolds.

Example 1.3.6 (Taub-NUT metric). In [41], C. Lebrun constructs the following family of Kähler forms on $\mathbb{C}^2$ defined by $\omega_m = \frac{i}{2}\partial\bar{\partial}\Phi_m$ where

$$\Phi_m(u, v) = u^2 + v^2 + m(u^4 + v^4), \quad m \geq 0$$

and u and v are implicitly defined by

$$|z_1| = e^{m(u^2 - v^2)}u, \ |z_2| = e^{m(v^2 - u^2)}v.$$

For $m = 0$ one gets the flat metric, while for $m > 0$ each of the metrics ω_m of this family are complete Ricci-flat (non-flat) metrics on $\mathbb{C}^2$ having the same volume form of the flat metric ω_0. In [50], the authors prove that the metric αg_m is not projectively induced for $m > \frac{\alpha}{2}$ and conjecture that αg_m is not projectively induced also for $0 < m \leq \frac{\alpha}{2}$.

Example 1.3.7 (Calabi's Ricci-flat metric). In [18], Calabi constructs the following example of Ricci-flat Kähler metric. Let (M, g) be a compact Kähler–Einstein manifold of complex dimension $n-1$ and with associated Kähler form ω_g and Einstein constant $k_0 > 0$. Let $\pi : \Lambda^{n-1}M \to M$ be the canonical line bundle over M. Calabi shows that there exists a smooth function $u : [0, \infty) \to \mathbb{R}$ such that if $\omega_g = \frac{i}{2}\partial\overline{\partial}\Phi$ on U, then the function $\Psi : \pi^{-1}(U) \to \mathbb{R}$ defined by

$$\Psi = \Phi \circ \pi + u(\det(g)^{-1}|\xi|^2)$$

is a Kähler potential on $\pi^{-1}(U)$ for a Ricci-flat and complete metric g_C on $\Lambda^{n-1}M$. In [51], the authors prove that the metric g_C is not projectively induced.

Example 1.3.8 (Eguchi-Hanson metric). It is the complete and Ricci-flat Kähler metric g_{EH} defined on the blow up $\hat{\mathbb{C}}^2$ of $\mathbb{C}^2$ at the origin given in $\mathbb{C}^2\backslash\{0\}$ by the following Kähler potential

$$\Phi_{\mathrm{EH}} = \sqrt{|z|^4 + 1} + \log|z|^2 - \log(1 + \sqrt{|z|^4 + 1}).$$

It can be shown that the Kähler form $\omega = \frac{i}{2}\partial\overline{\partial}\Phi_{\mathrm{EH}}$ a priori defined on $\mathbb{C}^2\backslash\{0\}$ can be extended to a Kähler metric g_{EH} on the whole $\hat{\mathbb{C}}^2$. In [44], the authors prove that g_{EH} is not projectively induced. Moreover in [51], it is proved that every integer multiple mg_{EH} of the Eguchi-Hanson metric is not projectively induced.

Let M be a compact Kähler manifold with a Kähler class $\Omega \in H^2(M, \mathbb{R})$. A natural question is to find canonical metrics representing the Kähler class Ω, namely metrics with nice curvature properties. This question generalizes the problem of the existence of Kähler–Einstein metrics. The best candidate to represent the Kähler class was proposed by Calabi in [19], where he defined extremal metrics as natural generalizations of these metrics to arbitrary Kähler classes on compact Kähler manifolds. In order to define extremal metrics, we recall that the *scalar curvature* of a Kähler manifold (M, g) is the trace of the Ricci tensor

$$scal_g := Tr(\mathrm{Ric})$$

whose expression in local coordinates is given by

$$scal_g = \sum_{\alpha,\beta=1}^{n} g^{\alpha\overline{\beta}}\mathrm{Ric}_{\alpha\overline{\beta}}$$

where $(g^{\alpha\overline{\beta}})$ denotes the inverse matrix of $(g_{\alpha\overline{\beta}})$.

Remark 1.3.9. For a Kähler–Einstein metric we have

$$\mathrm{Ric} = \lambda g$$

and taking the trace with respect to g we get

$$scal_g = n\lambda.$$

Definition 1.3.10. An *extremal metric* on M in the class Ω is a critical point of the functional

$$Cal(\omega) = \int_M scal_\omega^2 \, \omega^n$$

where $\omega \in \Omega$. This functional is called the *Calabi functional.*

The following gives a characterization of extremal metrics. Let $f : M \to \mathbb{R}$ be a function on a Kähler manifold and write $grad^{1,0} f = g^{j\bar{k}} \partial_{\bar{k}} f \frac{\partial}{\partial z_j}$. We have that $grad^{1,0} f \in \Gamma(T^{1,0}M)$.

Proposition 1.3.11. A metric ω on M is extremal if and only if $grad^{1,0} scal_\omega$ is a holomorphic vector field.

The most important examples of extremal metrics are constant scalar curvature Kähler metrics. In fact, most compact Kähler manifolds doesn't admit non-zero holomorphic vector fields, and so on these manifolds an extremal metric must have constant scalar curvature. In particular, Kähler–Einsten metrics have constant scalar curvature and thus are examples of extremal metrics. However, there are examples of extremal metrics with non-constant scalar curvature. We conclude this section giving an example of scalar–flat Kähler metric and an example of extremal metric with non-constant scalar curvature.

Example 1.3.12 (Burns-Simanca metric)*.* In [60], Simanca constructs a scalar flat Kähler complete (not Ricci-flat) metric g_S on $\hat{\mathbb{C}}^2$ whose Kähler potential on $\hat{\mathbb{C}}^2 \setminus \{0\}$ can be written as

$$\Phi_S(|z|^2) = |z|^2 + \log |z|^2, \quad |z|^2 = |z_1|^2 + |z_2|^2.$$

In [20], the authors prove that g_S can be Kähler immersed in $\mathbb{C}\mathrm{P}^\infty$.

Example 1.3.13 (ruled surface, (see ([61] Sec. 4.4 for details))*.* Let (Σ, ω_Σ) be a curve of genus $g = 2$ endowed with a Kähler metric with constant scalar curvature $scal_\omega = -2$. Let $\pi : (L, h) \to (\Sigma, \omega_\Sigma)$ be an hermitian holomorphic line bundle on Σ with degree -1 (i.e. $c_1(L) = -1$) such that the curvature form of h is $\gamma = -\omega_\Sigma$. Consider the $\mathbb{C}\mathrm{P}^1$-bundle $X = \mathbb{P}(L \oplus \mathcal{O})$ over Σ. Consider the following Kähler form on X, whose local expression is

$$\omega = (1 + f'(t))\pi^*\omega_\Sigma + i\, f''(t)\frac{d\xi \wedge d\bar{\xi}}{|\xi|^2}$$

where $t = \frac{1}{2}\log |z|_h^2$ and f is a strictly convex function which makes ω positive definite and ξ is a fibre coordinate on L. Then the Ricci form of ω is

$$\rho = -i\partial\bar{\partial}\log[(1 + f'(t))f''(t)] - 2\pi^*\omega_\Sigma.$$

By momentum construction (see Section 4.1) one obtains

$$\omega = (1 + \tau)\pi^*\omega_\Sigma + i\, \varphi(\tau)\frac{d\xi \wedge d\bar{\xi}}{|\xi|^2}$$

and

$$\rho = -i\partial\bar{\partial}\log[(1 + \tau)\varphi(\tau)] - 2\pi^*\omega_\Sigma$$

where $\tau = f'(t)$ and $\varphi(\tau) = f''(t)$. It follows that

$$scal_\omega = -\frac{2}{1+\tau} - \frac{1}{1+\tau}[(1+\tau)\varphi]''.$$

Thus, by Proposition 1.3.11, ω is extremal if and only if $grad^{1,0}scal_\omega$ is holomorphic. Since

$$grad^{1,0}scal_\omega = scal'_\omega(\tau)\xi\frac{\partial}{\partial\xi},$$

which is holomorphic if and only if $scal'_\omega(\tau)$ is constant, we have that ω is extremal if and only if $scal''_\omega(\tau) = 0$.

Chapter 2

Geometric quantization of Kähler manifolds

The modern theory of geometric quantization was developed by B. Kostant and J-M. Souriau in the 1970's, giving valuable insights into the relationship between classical and quantum systems. Section 2.1 is devoted to recall definitions and standard facts about hermitian holomorphic line bundles needed in the following (we mainly refer to [56]). Section 2.2 gives the central definition of a geometric quantization of a Kähler manifold. Section 2.3 introduces the ϵ-function, a fundamental tool in the interaction between a geometric quantization of a Kähler manifold and its realization as a Kähler submanifold (through the coherent state map) of a complex projective space endowed with the Fubini–Study metric. Then we give the definitions of balanced metrics and that of regular quantizations, concluding with the problem of the existence of an asymptotic expansion for the ϵ-function.

2.1 Hermitian holomorphic line bundles

Let M be a complex manifold and let $\pi : L \to M$ be a complex line bundle (i.e. each fibre $\pi^{-1}(x) = L_x$ is a 1-dimensional vector space over $\mathbb{C}$).

Definition 2.1.1. L is a *holomorphic line bundle* if it admits a trivialization with holomorphic transition functions.

This means that there exists an open cover $\mathcal{U}$ of M and for each $U \in \mathcal{U}$ a diffeomorphism $\psi_U : \pi^{-1}(U) \to U \times \mathbb{C}$ such that

- the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\psi_U"] \arrow[d, "\pi"'] & U \times \mathbb{C} \arrow[dl, "pr_U"] \\
U &
\end{tikzcd}$$

- for every intersecting U_α and U_β one has

$$\psi_\alpha \circ \psi_\beta^{-1}(x, v) = (x, g_{\alpha\beta}(x)v),$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}\backslash\{0\}$; $x \mapsto (\psi_\alpha \circ \psi_\beta^{-1})(x)$ are nonvanishing holomorphic functions, called *transition functions*, satisfying the cocycle relations

$$g_{\alpha\beta} \cdot g_{\beta\alpha} = 1 \text{ on } U_\alpha \cap U_\beta, \quad g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma} \text{ on } U_\alpha \cap U_\beta \cap U_\gamma.$$

Remark 2.1.2. A holomorphic line bundle can be described by its transition functions, that is, given functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}\backslash\{0\}$ satisfying the cocycle relations, then there exists a unique (up to isomorphisms) line bundle L over M with transition functions $\{g_{\alpha\beta}\}$.

The set of holomorphic line bundles has a natural commutative group structure: multiplication is given by tensor product, the neutral element is given by the trivial line bundle and the inverse of L is given by the dual bundle L^{-1}, where if L is given by data $\{g_{\alpha\beta}\}$, L' by $\{g'_{\alpha\beta}\}$, we have

$$L \otimes L' \sim \{g_{\alpha\beta} \cdot g'_{\alpha\beta}\}, \quad L^{-1} \sim \{g_{\alpha\beta}^{-1}\}$$

This group is called the *Picard group* of M and denoted by $Pic(M)$.

Definition 2.1.3. Two holomorphic line bundles $\pi_i : L_i \to M$, $i = 1, 2$, over M are said to be *isomorphic* if there exists a holomorphic map $\Psi : L_1 \to L_2$ such that $\pi_2 \circ \Psi = \pi_1$, which is a linear homomorphism on the fibers. The isomorphism class of L is denoted by $[L]$.

Example 2.1.4 (Trivial line bundle). Let M be a complex manifold. The *trivial line bundle* is defined taking the projection on the first component, i.e. $\pi : M \times \mathbb{C} \to M$; $(p, z) \mapsto p$ whose fiber L_p over some point $p \in M$ is $\pi^{-1}(p) = \{p\} \times \mathbb{C} \cong \mathbb{C}$.

Example 2.1.5 (bundles $\mathcal{O}(k)$ on $\mathbb{C}\mathrm{P}^N$). Let M be the complex projective space $\mathbb{C}\mathrm{P}^N$. The *tautological line bundle* $\mathcal{O}(-1)$ over $\mathbb{C}\mathrm{P}^N$ is defined as the complex line bundle $\pi : \mathcal{O}(-1) \to \mathbb{C}\mathrm{P}^N$ whose fiber $L_{[z]}$ over some point $l = [z] = [z_0 : \cdots : z_N] \in \mathbb{C}\mathrm{P}^N$ is the complex line $< z >$ in $\mathbb{C}^{N+1}$. More precisely, $\mathcal{O}(-1) = \{(l, z) \in \mathbb{C}\mathrm{P}^N \times \mathbb{C}^{N+1} \mid z \in l\} \subset \mathbb{C}\mathrm{P}^N \times \mathbb{C}^{N+1}$ and the projection $\pi : \mathcal{O}(-1) \to \mathbb{C}\mathrm{P}^N$ is given by projecting on the first factor $(l, z) \mapsto l$. We consider the canonical holomorphic charts (U_i, φ_i) on $\mathbb{C}\mathrm{P}^N$ and the local trivializations $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ of $\mathcal{O}(-1)$ defined by $(l, z) \mapsto (l, z_i)$. It follows that the transitions functions are

$$\psi_i \circ \psi_j^{-1}(l, \lambda) = (l, g_{ij}(l)\lambda), \quad \text{where } g_{ij}(l) = \frac{z_i}{z_j}.$$

This shows that the tautological bundle $\mathcal{O}(-1)$ of $\mathbb{C}\mathrm{P}^N$ is holomorphic. The dual of the tautological bundle of $\mathbb{C}\mathrm{P}^N$ is called the *hyperplane line bundle* of $\mathbb{C}\mathrm{P}^N$ and denoted by $\mathcal{O}(1)$. The fiber of $\pi : \mathcal{O}(1) \to \mathbb{C}\mathrm{P}^N$ over some point $l = [z] \in \mathbb{C}\mathrm{P}^N$ is l^*, that is the set of $\mathbb{C}$-linear maps on the line that determines $[z] \in \mathbb{C}\mathrm{P}^N$. Taking the tensor products we define the following line bundles over the complex projective space $\mathbb{C}\mathrm{P}^N$:

- for $k > 0$, $\mathcal{O}(k) := \mathcal{O}(1) \otimes \cdots \otimes \mathcal{O}(1)$ k-times,
- for $k < 0$, $\mathcal{O}(k) := \mathcal{O}(-1) \otimes \cdots \otimes \mathcal{O}(-1)$ k-times,

- for $k = 0$, $\mathcal{O}(0) := \mathbb{C}\mathrm{P}^N \times \mathbb{C}$ is the trivial line bundle.

In particular $Pic(\mathbb{C}\mathrm{P}^N) = \mathbb{Z}$ and one can show that every holomorphic line bundle over $\mathbb{C}\mathrm{P}^N$ is of the form $\mathcal{O}(k)$ for some $k \in \mathbb{Z}$.

Let L be a complex line bundle over a complex manifold M. We denote by $\Lambda^{p,q}(L) := \Lambda^{p,q}M \otimes L$ and by $\Omega^{p,q}(L)$ the space of its sections, called *L-valued differential forms of type* (p, q). Chosing a local frame s of L, a section $\sigma \in \Omega^{p,q}(L)$ can be written as $\sigma = \omega \otimes s$ in this trivialization, where ω is a local (p, q)-form on M.

Definition 2.1.6. A *holomorphic structure* $\overline{\partial}$ on a complex line bundle is an operator

$$\overline{\partial} : \Omega^{p,q}(L) \to \Omega^{p,q+1}(L)$$

satisfying the Leibniz rule and such that $\overline{\partial}^2 = 0$. A section $s \in \Gamma(L)$ is said to be *holomorphic* if $\overline{\partial} s = 0$. The space of global holomorphic sections on a holomorphic line bundle L is denoted by $H^0(L)$.

The existence of such an operator $\overline{\partial}$ characterizes holomorphic line bundles among complex line bundles, i.e. we have the following

Theorem 2.1.7 ([56], Thm 9.2)**.** *A complex line bundle $L \to M$ is holomorphic if and only if it has a holomorphic structure $\overline{\partial}$.*

2.1.1 Connection and curvature

Let $\pi : L \to M$ be a line bundle over a smooth manifold M (i.e. each fibre is a 1-dimensional vector space over $\mathbb{R}$).

Definition 2.1.8. A *connection* on L is a $\mathbb{C}$-linear differential operator

$$\nabla : \Gamma(L) \to \Omega^1(L) = \Gamma(\Lambda^1_{\mathbb{C}} M \otimes L)$$

satisfying the Leibniz rule

$$\nabla(fs) = f\nabla s + df \otimes s$$

for every $s \in \Gamma(L)$ and $f : M \to \mathbb{C}$.

Let $\sigma : U \to L$ be a trivializing section over an open set $U \subset M$, then we can write

$$\nabla\sigma = \theta \otimes \sigma$$

where θ is a 1-form, called the *connection form* (with respect to the frame σ).

We can extend any connection to the bundles $\Omega^p(L)$, $p \geq 1$, by imposing the Leibniz rule

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^p \omega \wedge \nabla s$$

for all $\omega \in \Omega^p(M)$ and $s \in \Gamma(L)$. We can now give the following definition.

Definition 2.1.9. The *curvature* of the connection ∇ is the $C^\infty(M)$-linear operator defined by

$$\nabla^2 = \nabla \circ \nabla : \Gamma(L) \to \Omega^2(L)$$

for every section $s \in \Gamma(L)$.

Let $\sigma : U \to L$ be a trivializing section over an open set $U \subset M$, then we can write

$$\nabla^2 \sigma = \gamma \otimes \sigma$$

where γ is a 2-form, called the *curvature form* (with respect to the frame σ).

We can express the curvature form in terms of the connection form: if $\nabla\sigma = \theta \otimes \sigma$, then we have

$$\nabla^2\sigma = \nabla(\theta\otimes\sigma) = d\theta\otimes\sigma - \theta\wedge\nabla\sigma = d\theta\otimes\sigma - \theta\wedge(\theta\otimes\sigma) = (d\theta - \theta\wedge\theta)\otimes\sigma = d\theta\otimes\sigma.$$

Moreover, one can show that the de Rham cohomology class of $\frac{i}{2\pi}\gamma$ is the first Chern class of L, i.e.

$$c_1(L) = \left[\frac{i}{2\pi}\gamma\right].$$

Proposition 2.1.10 ([34])**.** Two holomorphic line bundles L_1 and L_2 over a simply connected complex manifold M are isomorphic if and only if $c_1(L_1) = c_1(L_2)$.

2.1.2 Hermitian metrics

Let $L \to M$ be a complex line bundle over a complex manifold.

Definition 2.1.11. A *hermitian metric* h on L is a smooth field of hermitian products on the fibres of L, that is, for every $x \in M$, the map $h : L_x \times L_x \to \mathbb{C}$ satisfies

- $h(u, v)$ is $\mathbb{C}$-linear in u for every $v \in L_x$;
- $h(u, v) = \overline{h(v, u)}$ for every $u, v \in L_x$;
- $h(u, u) > 0$, $\forall u \neq 0$;
- $h(u, v)$ is a smooth function on M for every $u, v \in \Gamma(L)$, that is if $\sigma : U \to \mathbb{C}$ is a local frame over an open set $U \subset M$, then the function $h(x) := h(\sigma(x), \sigma(x)) = |\sigma(x)|^2$ is smooth.

A complex line bundle endowed with an hermitian metric is called *hermitian line bundle.* A holomorphic line bundle endowed with an hermitian metric is called *hermitian holomorphic line bundle.*

Using partition of unity one can prove that

Proposition 2.1.12 ([56], p.78)**.** Any complex line bundle admits an hermitian metric.

Definition 2.1.13. Let (L, h) be an hermitian line bundle. A connection $\nabla : \Gamma(L) \to \Omega^1(L)$ is *compatible with the metric* if

$$Xh(s,t)(x) = h(\nabla_X s, t)(x) + h(s, \nabla_X t)(x)$$

for every $s, t \in \Gamma(L)$ and every vector field $X \in \Gamma(TM)$, where we write $h(s,t)(x)$ for the function $h(s(x), t(x))$, and $\nabla_X s$ for $(\nabla s)(X) \in \Gamma(L)$.

Proposition 2.1.14. Any hermitian line bundle admits a connection compatible with the metric.

The decomposition of 1-forms into types $(1,0)$ and $(0,1)$ induces a decomposition of the connection $\nabla : \Gamma(L) \to \Omega^1(L)$ as $\nabla = \nabla^{1,0} + \nabla^{0,1}$, where

$$\nabla^{1,0} : \Gamma(L) \to \Omega^{1,0}(L) \quad ; \quad \nabla^{0,1} : \Gamma(L) \to \Omega^{0,1}(L).$$

Definition 2.1.15. A connection ∇ on a hermitian holomorphic line bundle with holomorphic structure $\overline{\partial}$ is said to be *compatible with the complex structure* if $\nabla^{0,1} = \overline{\partial}$.

Theorem 2.1.16 ([56], Thm 10.3)**.** If (L,h) is a hermitian holomorphic line bundle over M, there exists a unique connection ∇ compatible with both the metric and the complex structure. This connection is called the *Chern connection.*

Remark 2.1.17. If $(L,h) \to M$ is an hermitian holomorphic line bundle over M with connection ∇, then

- if ∇ is compatible with the complex structure, then the curvature form γ has no $(0,2)$-part, i.e. $\gamma^{0,2} = 0$;
- if ∇ is compatible with the hermitian metric, then the connection form θ is purely imaginary, i.e. $\overline{\theta} = -\theta$ and so is the curvature form γ;
- if ∇ is the Chern connection, then the curvature form γ is of type $(1,1)$ (and purely imaginary).

Notice that the curvature of the Chern connection is a purely imaginary 2-form of type $(1,1)$, that is $\gamma \in \Omega^2(M) \otimes \mathbb{C}$ and $\overline{\gamma} = -\gamma$. From now on we adopt the convention that the curvature is a real form, namely we consider $i\gamma \in \Omega^2(M)$ instead of γ. We denote this 2-form on M with $\mathrm{Ric}(h)$.

Proposition 2.1.18 ([8] Prop. 3.2.1)**.** The curvature of the Chern connection equals

$$-i\partial\overline{\partial} \log h(\sigma(x), \sigma(x))$$

where $\sigma : U \to L\backslash\{0\}$ is a trivializing holomorphic section.

2.2 Geometric quantization of Kähler manifolds

Let M be a Kähler manifold and (L,h) be an hermitian holomorphic line bundle over M. By theorem 2.1.16, the space of the connections compatible with the metric and the space of the connections compatible with the holomorphic structure intersect in one point: the Chern connection. In the following we consider on L the Chern connection ∇.

Definition 2.2.1. A *geometric quantization* (L,h) of a n-dimensional Kähler manifold (M,ω) consists of an hermitian holomorphic line bundle (L,h) over M such that its curvature $\mathrm{Ric}(h) = -i\partial\overline{\partial}\log h$ satisfies

$$\mathrm{Ric}(h) = \omega.$$

The line bundle L is called the *quantum line bundle* of M. A Kähler manifold M is said to be *quantizable* if it admits a geometric quantization.

By Proposition 2.1.18, we can check if a holomorphic hermitian line bundle (L, h) over (M, ω) is a geometric quantization simply by choosing a trivialising holomorphic section $\sigma : U \to L$ and verifying if

$$\omega = \frac{i}{2}\partial\overline{\partial}\log(h(\sigma(x), \sigma(x))).$$

Not all Kähler manifolds admit a geometric quantization. In terms of cohomology classes a necessary and sufficient condition is expressed by the following theorem.

Theorem 2.2.2 ([40]). A Kähler manifold (M, ω) admits a geometric quantization (L, h) if and only if $c_1(L) = [\omega]$ (i.e. if and only if ω is integral, i.e. the cohomology class $[\omega]_{\mathrm{dR}}$ of ω in the de Rham group, is in the image of the natural map $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{C})$).

Example 2.2.3 (Complex space forms). The complex space forms admit a geometric quantization.

- (Flat space) Let $\mathbb{C}^N$ be the complex Euclidean space endowed with the Kähler form $\omega_0 = \frac{i}{2}\partial\overline{\partial}|z|^2$. Consider the trivial line bundle $L = \mathbb{C}^N \times \mathbb{C} \to \mathbb{C}^N$ and for each $z \in \mathbb{C}^N$ define the map

  $$h : L_z \times L_z \to \mathbb{C} \quad , \quad ((z, \xi_1), (z, \xi_2)) \mapsto e^{-\frac{1}{2}|z|^2}\xi_1\overline{\xi}_2$$

  The above map induces an hermitian structure on L that defines a geometric quantization of $(\mathbb{C}^N, \omega_0)$. Indeed, if $\sigma(z) = (z, f(z))$ is a global holomorphic section on L, where $f : \mathbb{C}^N \to \mathbb{C}$ is a holomorphic function, then one obtains

  $$\mathrm{Ric}(h) = -i\partial\overline{\partial}\log h = -i\partial\overline{\partial}\log(e^{-\frac{1}{2}|z|^2}|f(z)|^2) = \omega_0$$

- (The projective space) Let $\mathbb{C}\mathrm{P}^N$ be the complex projective space endowed with the Fubini-Study form $\omega_{\mathrm{FS}} = \frac{i}{2}\partial\overline{\partial}\log(|Z_0|^2 + \cdots + |Z_N|^2)$. One can show that ω_{FS} is integral and so by Theorem 2.2.2 there exists a hermitian line bundle (L, h) such that $\mathrm{Ric}(h) = \omega_{\mathrm{FS}}$. This line bundle is the tautological line bundle $\mathcal{O}(-1)$ with hermitian metric $h(z) = 1 + ||z||^2$, where z is an affine coordinate on a coordinate chart U.

- (The hyperbolic space) Let $\mathbb{C}H^N$ be the complex hyperbolic space endowed with the hyperbolic form $\omega_{\mathrm{hyp}} = \frac{i}{2}\partial\overline{\partial}\log(1 - |z|^2)^{-1}$. Consider the trivial bundle $L = \mathbb{C}H^N \times \mathbb{C} \to \mathbb{C}H^N$ and for each $z \in \mathbb{C}H^N$ define the map

  $$h : L_z \times L_z \to \mathbb{C} \quad , \quad ((z, \xi_1), (z, \xi_2)) \mapsto (1 - |z|^2)^{\frac{1}{2}}\xi_1\overline{\xi}_2$$

  The above map induces an hermitian structure on L that defines a geometric quantization of $(\mathbb{C}H^N, \omega_{\mathrm{hyp}})$. Indeed, if $\sigma(z) = (z, f(z))$ is a global holomorphic section on L, where $f : \mathbb{C}^N \to \mathbb{C}$ is a holomorphic function, then one obtains

  $$\mathrm{Ric}(h) = -i\partial\overline{\partial}\log h = -i\partial\overline{\partial}\log((1 - |z|^2)^{\frac{1}{2}}|f(z)|^2) = \omega_{\mathrm{hyp}}.$$

Definition 2.2.4. Two holomorphic hermitian line bundles (L_i, h_i), $i = 1, 2$, over the same Kähler manifold (M, ω) are said to be *equivalent* if there exists an isomorphism of holomorphic line bundles $\Psi : L_1 \to L_2$ such that $\Psi^* h_2 = h_1$. The equivalence class of (L, h) is denoted by $[(L, h)]$.

Let $\mathcal{L}(M,\omega)$ be the set of all geometric quantizations of a Kähler manifold (M,ω). For a class $[(L,h)]$ we can define

$$\gamma\left([(L,h)]\right) := \gamma(L,h).$$

This is well defined, i.e. it doesn't depend on the representative in the equivalence class $[(L,h)]$, namely the curvature form in an equivalence class is unique up to isomorphisms:

$$-i\partial\overline{\partial}\log h_1 = -i\partial\overline{\partial}\log \Psi^*(h_2) = \Psi^*\left(-i\partial\overline{\partial}\log h_2\right).$$

Then the set $\mathcal{L}(M,\omega)$ can be partitioned in equivalence classes $[(L,h)]$. If M is a simply connected Kähler manifold, then all geometric quantizations are equivalent ([42]) and thus $\mathcal{L}(M,\omega)$ consists of a single equivalence class.

Let $(L,h)\in[(L,h)]$ be a geometric quantization of a Kähler manifold (M,ω). We use the following notations:

- $Aut(M) := \{f : M \to M \mid f \text{ is biholomorphic}\}$;
- $Isom(M,\omega) := \{f : (M,\omega) \to (M,\omega) \mid f \in C^\infty(M),\ f^*\omega = \omega\}$;
- $Aut(L,h) := \{\hat{f} : L \to L \mid \hat{f} \text{ is biholomorphic, } \mathbb{C}\text{-linear on fibers, } \hat{f}^*h = h\}$

Definition 2.2.5. A *lifting* of a map $f \in Aut(M)\cap Isom(M,\omega)$ is a map $\hat{f}\in Aut(L,h)$ such that the following diagram is commutative

$$\begin{tikzcd}
(L,h) \arrow[r, "\hat{f}"] \arrow[d, "\pi"'] & (L,h) \arrow[d, "\pi"] \\
(M,\omega) \arrow[r, "f"] & (M,\omega)
\end{tikzcd}$$

The group of all maps f which admit a lifting $\hat{f}$ is denoted by $D_{[(L,h)]}(M) \subset Aut(M)\cap Isom(M,\omega)$.

For simply connected Kähler manifolds we have

Proposition 2.2.6 ([42] Prop. 1.5.1)**.** Let $(L,h)\in[(L,h)]$ be a geometric quantization of a simply connected Kähler manifold (M,ω). Then

$$D_{[(L,h)]}(M) = Aut(M)\cap Isom(M,\omega).$$

2.3 The epsilon function

Let (L,h) be a geometric quantization of a Kähler manifold (M,ω). Consider the space $\mathcal{H}$

$$\mathcal{H} = \left\{ s\in H^0(L) \mid \int_M h(s(x),s(x))\frac{\omega^n}{n!} < +\infty \right\}$$

of global holomorphic sections of L which are bounded with respect to the scalar product $\langle\cdot,\cdot\rangle_h$:

$$\langle s,t\rangle_h := \int_M h(s(x),t(x))\frac{\omega^n}{n!}$$

for $s, t \in H^0(L)$. It is possible to prove that $(\mathcal{H}, <>_h)$ is a complex separable Hilbert space ([11]), i.e. $(\mathcal{H}, <>_h)$ admits an orthonormal basis $\{s_j\}_{j=0,\dots,N}$ ($\dim \mathcal{H} = N+1 \leq +\infty$).

When M is compact, the space $\mathcal{H}$ coincides with the space of global holomorphic sections $H^0(L)$, thus $\dim \mathcal{H}$ is finite. Instead, if M in noncompact $\mathcal{H}$ may even contain just the zero section.

Definition 2.3.1. Let (L, h) be a geometric quantization of a Kähler manifold (M, ω) and $\{s_j\}_{j=0,\dots,N}$ (with $\dim \mathcal{H} = N+1 \leq \infty$) be an orthonormal basis for the Hilbert space $(\mathcal{H}, <>_h)$. The *ϵ-function* of (L, h) is a smooth real valued function on M defined, for any $x \in M$, by

$$\epsilon_{(L,h)}(x) = \sum_{j=0}^{N} h(s_j(x), s_j(x)). \tag{2.1}$$

In literature this ϵ-function was first introduced under the name of *η-function* by J. Rawnsley in [58], later renamed as *θ-function* in [11] followed by the *distortion function* of G. R. Kempf [39] and S. Ji [37], for the special case of Abelian varieties and of S. Zhang [70] for complex projective varieties.

Proposition 2.3.2. The ϵ-function does not depend on the choice of the orthonormal basis of $\mathcal{H}$.

Proof. Let $x \in M$ and $q \in L \setminus \{0\}$ be a fixed point of the fiber over x, i.e. $\pi(q) = x$. If one evaluates $s \in \mathcal{H}$ at x, one gets a multiple $\delta_q(s)$ of q, i.e. $s(x) = \delta_q(s)q$. It can be shown ([11]) that $\delta_q : \mathcal{H} \to \mathbb{C}$ is a linear continuous functional of s, thus from the Riesz's theorem there exists a unique $e_q \in \mathcal{H}$ such that

$$\delta_q(s) = < s, e_q >_h$$

for every $s \in \mathcal{H}$. Thus every element s_j of an orthonormal basis of $\mathcal{H}$ satisfies

$$s_j(x) = < s_j, e_q > q$$

and substituting we get

$$\epsilon_{(L,h)}(x) = h(q, q) ||e_q||_h^2$$

proving the independence of the choice of the basis. □

When M is compact, the dimension of $\mathcal{H} = H^0(L)$ is finite and one has

$$\int_M \epsilon_{(L,h)}(x) \frac{\omega^n(x)}{n!} = \dim \mathcal{H}.$$

Thus the ϵ-function in well defined, i.e.

- it doesn't depend on the chosen representative (L, h) in the class $[(L, h)] \in \mathcal{L}(M, \omega)$;
- it doesn't depend on the hermitian metric h;
- In the case of simply connected manifolds, since $\mathcal{L}(M, \omega)$ consists of a single equivalence class, $\epsilon_{(L,h)}$ depends only on the Kähler form ω and we can write simply ϵ_g.

Lemma 2.3.3 ([11])**.** The function $\epsilon_{(L,h)}$ is invariant under the group $D_{[(L,h)]}(M)$, i.e. $F^*(\epsilon_{(L,h)}) = \epsilon_{(L,h)}$, for every $F \in D_{[(L,h)]}(M)$.

Definition 2.3.4. A Kähler manifold is said to be *homogeneous* if the group $Aut(M) \cap Isom(M, \omega)$ acts transitively on M.

Lemma 2.3.5. Let M be a homogeneous Kähler manifold and $g : M \to \mathbb{R}$ be a real valued function on M invariant under the group $Aut(M) \cap Isom(M, \omega)$. Then g is constant.

Proof. By hypothesis, g is invariant under the group $Aut(M) \cap Isom(M, \omega)$, that is

$$g(f(x)) = g(x)$$

for all $f \in Aut(M) \cap Isom(M, \omega)$ and for all $x \in M$. Since M is homogeneous, for any $x, y \in M$ there exist a map $f \in Aut(M) \cap Isom(M, \omega)$ such that $f(x) = y$. Thus $g(x) = g(y)$ for any $x, y \in M$, so g is constant. □

Corollary 2.3.6. Let (L, h) be a geometric quantization of a simply connected homogeneous Kähler manifold (M, ω). Then the function $\epsilon_{(L,h)}$ is constant.

Proof. Since M is simply connected, it follows by Proposition 2.2.6 that $D_{[(L,h)]} = Aut(M) \cap Isom(M, \omega)$ and thus by Lemma 2.3.3 the ϵ-function $\epsilon_{(L,h)}$ is invariant by the group $Aut(M) \cap Isom(M, \omega)$. Since M is homogeneous, by Lemma 2.3.5, $\epsilon_{(L,h)}$ is constant. □

2.3.1 Coherent state map

Let (L, h) be a geometric quantization of a Kähler (M, ω) manifold and $\{s_j\}_{j=0,\dots,N}$ be an orthonormal basis for $(H, < \, , \, >_h)$ and let $\sigma : U \to L\backslash\{0\}$ be a trivializing holomorphic section. Suppose that for all $x \in M$ there exists $s_a \in \{s_j\}_{j=0,\dots,N}$ such that $s_a(x) \neq 0$, namely is (L, h) is *base point free.* Under these assumptions, one can define the holomorphic map

$$\phi_\sigma : U \to \mathbb{C}^{N+1}\backslash\{0\} \quad , \quad x \mapsto \left(\frac{s_0(x)}{\sigma(x)}, \dots, \frac{s_N(x)}{\sigma(x)}\right)$$

This map glues coherently in the intersections $U \cap V$ and so induces a holomorphic map on the whole M, called the *coherent state map*

$$\phi : M \to \mathbb{C}\mathrm{P}^N \quad , \quad x \mapsto [s_0(x), \dots, s_N(x)]$$

whose local expression in the open set U is ϕ_σ.

Remark 2.3.7. More in general, we can assume that there exists α sufficiently large such that for each point $x \in M$ there exists $s \in \mathcal{H}_\alpha$, the space of global holomorphic sections of L^α that are L^2-limited in norm with respect the hermitian metric h_α induced by h on L^α, not vanishing at x. In the compact case such an α exists by standard algebraic geometry methods and corresponds to the free-based point condition in the Kodaira's theory.

The following theorem relates the coherent state map and the ϵ-function.

Theorem 2.3.8 ([11],[58],[42])**.** Let ω_{FS} be the Fubini-Study form on $\mathbb{C}\mathrm{P}^N$ and let ϕ be the coherent state map associated to (M, ω). Then

$$\phi^*(\omega_{\mathrm{FS}}) = \omega + \frac{i}{2}\partial\bar{\partial} \log \epsilon_{(L,h)}.$$

Thus the ϵ-function measures how the Kähler metric ω is not projectively induced via the coherent state map.

Corollary 2.3.9. Let (L, h) be a geometric quantization of a Kähler manifold (M, ω). If $\epsilon_{(L,h)}$ is a positive constant, then the coherent state map is a Kähler immersion.

Proof. If $\epsilon_{(L,h)}$ is a positive constant, then for all $x \in M$ there exists $s_{j_0} \in \{s_j\}_{j=0,\dots,N}$ such that $s_{j_0} \neq 0$, i.e. (L, h) is base point free and thus the coherent state map can be defined. By construction the coherent state map is holomorphic and by the above theorem is isometric, i.e. $\phi^*(\omega_{\mathrm{FS}}) = \omega$, concluding the proof. □

Corollary 2.3.10. Let (L, h) be a quantization of a compact Kähler manifold (M, g). Then the function $\epsilon_{(L,h)}$ is a positive constant if and only if the coherent state map is a full Kähler immersion in $(\mathbb{C}\mathrm{P}^N, \omega_{\mathrm{FS}})$.

Proof. Since M is Kähler, we have $\partial\bar{\partial} \log \epsilon_{(L,h)} = 0$ if and only if $\log \epsilon_{(L,h)}$ is an harmonic function on M and thus constant being M compact. Finally the immersion is full by Remark 1.2.11. □

Although not all Kähler metrics are projectively induced, Tian and Ruan, by means of peak section method, solved a conjectured by Yau by proving that any polarized metric on a compact complex manifold is the C^∞-limit of normalized projectively induced metrics $g_m = \frac{\phi_m^* g_{\mathrm{FS}}}{m}$. More precisely, Tian proved the following

Theorem 2.3.11 ([62])**.** Let M be an algebraic manifold polarized by a line bundle L and let g be a polarized Kähler metric. Then

$$||g_m - g||_{C^2} = O\left(\frac{1}{\sqrt{m}}\right),$$

i.e. the metrics g_m are C^2 convergent to the metric g.

In the same paper Tian conjectures that the metrics g_m converge to g in the C^∞-topology. This was later proved by Ruan:

Theorem 2.3.12 ([59])**.** For any $l \in \mathbb{Z}_+$

$$||g_m - g||_{C^l} = O\left(\frac{1}{m}\right)$$

i.e. g_m are C^∞ convergent to the original metric g.

2.3.2 Balanced metrics

Let (L, h) be a geometric quantization of a Kähler manifold (M, g).

Definition 2.3.13. The metric g on M is called *balanced* if $\epsilon_{(L,h)}$ is a positive constant.

The definition of balanced metric was originally given by Donaldson [28] in the case of a compact polarized Kähler manifold. If M is compact, then $\mathcal{H} = H^0(L)$ and so in this case $\dim \mathcal{H} < \infty$ and so the ϵ-function is a finite sum. The definition was generalized in the noncaompact case by C. Arezzo and A. Loi in [2] ispired by the geometric quantization theory.

The two fundamental results about the existence and uniqueness of balanced metrics in the compact case are summarized in the following two theorems.

Theorem 2.3.14 ([28])**.** Let (L, h) be a geometric quantization of a compact Kähler manifold (M, g) (i.e. $c_1(L) = [\omega]$) such that $scal_g$ is constant. Assume that $\frac{Aut(M,L)}{\mathbb{C}^*}$ is discrete. Then, for all sufficiently large integers α, there exists a unique balanced metric $\tilde{g}_\alpha$ on M, with respect the geometric quantization (L^α, h_α) (i.e. $c_1(L) = [\tilde{\omega}_\alpha]$), such that $\frac{\tilde{g}_\alpha}{\alpha}$ converges C^∞ to g. Moreover, if $\tilde{g}_\alpha$ is a sequence of balanced metrics on M with associated Kähler forms $\tilde{\omega}_\alpha \in c_1(L^\alpha)$ such that $\frac{\tilde{g}_\alpha}{\alpha}$ converges C^∞ to g, then g has constant scalar curvature.

Here, the quotient space $\frac{Aut(M,L)}{\mathbb{C}^*}$ denotes the biholomorphisms group of M which lift to holomorphic bundles maps $L \to L$ modulo the trivial automorphism group $\mathbb{C}^*$. Notice also that the assumption on the automorphism group cannot be dropped. Indeed a result of Della Vedova and Zuddas ([27]) shows that there exists a large class of geometric quantizations (L, h) of (M, g) with $scal_g$ constant and for α sufficiently large it cannot exists a balanced metric g_α as in the thoerem. Regarding the uniqueness of balanced metrics we have the following

Proposition 2.3.15 ([4])**.** Let g and $\tilde{g}$ be two balanced metrics whose associated Kähler forms are cohomologous, Then g and $\tilde{g}$ are isometric, i.e. there exists $F \in Aut(M)$ such that $F^*\tilde{g} = g$.

One can calculate the function $\epsilon_{(L^m,h_m)}$ for every natural number m. More precisely, consider the Kähler form $m\omega$ on M, where $m \in \mathbb{Z}_+$. If ω is integral, then $m\omega$ is integral for any positive integer. Therefore one can consider the quantum line bundle (L^m, h_m) for $(M, m\omega)$, where $L^m = L \otimes \cdots \otimes L$ m-times, is the m-tensor power of L and h_m is the m-tensor power of h defined by extending the definition of h:

$$h_m(s_1 \otimes \ldots s_m, t_1 \otimes \ldots t_m)(x) := h(s_1, t_1)(x) \cdots h(s_m, t_m)(x)$$

for any $s_1 \otimes \ldots s_m, t_1 \otimes \ldots t_m \in \Gamma(L^m)$ and for all $x \in M$. In this context is interesting to study the balanced condition for the metric mg when m varies, namely study the constancy of the epsilon function ϵ_{mg} for every positive integer m. The fact that g is balanced doesn't imply that mg is.

Definition 2.3.16. A geometric quantization is called *regular* if mg is balanced for any (sufficiently large) natural number m, i.e. if $\epsilon_{(L^m,h_m)}$ is a positive constant for any (sufficiently large) natural number m.

Example 2.3.17 (Regular metrics). Some examples of regular metrics are the following:

- Let $(\mathbb{C}^N, \omega_0)$ be the complex Euclidean space endowed with the flat metric. It can be proven ([42]) that $\epsilon_{mg_0} = \frac{N!}{m^n}$. Thus $(\mathbb{C}^N, \omega_0)$ admits a regular quantization.

- Let $(\tilde{\mathbb{C}}^2, g_{BS})$ be the blow-up of $\mathbb{C}^2$ at the origin endowed with the Burns-Simanca metric (see Example 1.3.12). In [20], the authors proved that $\epsilon_{mg_{BS}} = m^2$ and thus $(\tilde{\mathbb{C}}^2, g_{BS})$ admits a regular quantization.
- A geometric quantization of a homogeneous and simply connected Kähler manifold is regular as follows by Corollary 2.3.6,

Regular metrics have nice geometric properties, in fact:

Theorem 2.3.18 ([43])**.** A Kähler metric which admits a regular quantization is a constant scalar curvature metric.

Remark 2.3.19. Not all homogeneous manifolds admit a regular quantization. An example is given by the complex torus V/Λ equipped with the flat form ω_0. Suppose that (L, h) is a regular quantization of $(V/\Lambda, \omega_0)$. Then, from Corollary 2.3.9, the flat metric ω_0 would be projectively induced by the coherent state map, namely $\phi : (V/\Lambda, \omega_0) \to (\mathbb{C}\mathrm{P}^N, \omega_{\mathrm{FS}})$ would be a full Kähler immersion being V/Λ compact. However it can be proven that V/Λ is not projectively induced (take the universal covering and apply Calabi's Rigidity theorem 1.2.8).

2.3.3 Asymptotic expansion of the epsilon function

In the following we distinguish between the compact and the noncompact case.

The compact case. If M is a compact polarized manifold, there exists a complete asymptotic expansion of the ϵ-function (that we call *TYCZ-expansion*) introduced by D. Catlin [21] and S. Zelditch [69] independently

$$\epsilon_{(L^m, h_m)}(x) \sim \sum_{j=0}^{\infty} a_j(x) m^{n-j}$$

where $a_0(x) = 1$ and $a_j(x)$ are smooth functions on M. This means that, for any non negative integers r, k, the following estimate holds

$$||\epsilon_{(L^m, h_m)}(x) - \sum_{j=0}^{k} a_j(x) m^{n-j}||_{C^r} \leq C_{k,r} m^{n-k-1} \tag{2.2}$$

where $C_{k,r}$ are constants depending on k, r and on the Kähler form ω and $||\ ||_{C^r}$ denotes the C^r norm. This expansion is called *Tian-Yau-Catlin-Zelditch expansion.* Later, Z. Lu [52] proved that each of the coefficients $a_j(x)$ is a polynomial of the curvature and its covariant derivatives at x of the metric g. In particular the first three are

$$\begin{cases} a_1 = -\frac{1}{2} scal_g \\ a_2 = -\frac{1}{3}\Delta scal_g + \frac{1}{24}\left(|R|^2 - 4|\mathrm{Ric}|^2 + 3 scal_g^2\right) \\ a_3 = -\dfrac{1}{8}\Delta\Delta scal_g + \dfrac{1}{24}\mathrm{div}\mathrm{div}(R, \mathrm{Ric}) + \\ \qquad - \dfrac{1}{6}\mathrm{div}(\mathrm{div}(scal_g\ \mathrm{Ric})) + \dfrac{1}{48}\Delta(|R|^2 - 4|\mathrm{Ric}|^2 + 8 scal_g^2) + \\ \qquad - \dfrac{1}{48} scal_g(scal_g^2 + |R|^2 - 4|\mathrm{Ric}|^2) - \dfrac{1}{24}(\sigma_3(\mathrm{Ric}) - \mathrm{Ric}(R, R) + R(\mathrm{Ric}, \mathrm{Ric})) \end{cases} \tag{2.3}$$

where R denotes the Riemannian tensor of g, given in local coordinates by:

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} - \sum_{p,q=1}^{n} g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l},$$

Ric denotes the Ricci tensor, that (in contrast with Lu's notation that has the opposite sign) reads:

$$\mathrm{Ric}_{i\bar{j}} = g^{l\bar{m}} R_{i\bar{j}l\bar{m}},$$

and:

$$scal_g = g^{i\bar{j}} \mathrm{Ric}_{i\bar{j}},$$

is the scalar curvature. All the norms are taken with respect to g, and we further are using the following notations (here again the signs has be changed accordingly to our notation):

$$\begin{aligned}
&|D' scal_g|^2 = g^{j\bar{i}} \frac{\partial\ scal_g}{\partial z_i} \frac{\partial\ scal_g}{\partial \overline{z_j}},\\
&|D'\mathrm{Ric}|^2 = g^{\alpha\bar{i}} g^{j\bar{\beta}} g^{\gamma\bar{k}} Ric_{i\bar{j},k} \overline{\mathrm{Ric}_{\alpha\bar{\beta},\gamma}},\\
&|D'R|^2 = g^{\alpha\bar{i}} g^{j\bar{\beta}} g^{\gamma\bar{k}} g^{l\bar{\delta}} g^{\epsilon\bar{p}} R_{i\bar{j}k\bar{l},p} \overline{R_{\alpha\bar{\beta}\gamma\bar{\delta},\epsilon}},\\
&\mathrm{divdiv}(scal_g\ \mathrm{Ric}) = 2|D' scal_g|^2 + g^{\beta\bar{i}} g^{j\bar{\alpha}} \mathrm{Ric}_{i\bar{j}} \frac{\partial^2 scal_g}{\partial z_\alpha \partial \overline{z_\beta}} + scal_g \Delta scal_g,\\
&\sigma_3(\mathrm{Ric}) = g^{\delta\bar{i}} g^{j\bar{\alpha}} g^{\beta\bar{\gamma}} \mathrm{Ric}_{i\bar{j}} \mathrm{Ric}_{\alpha\bar{\beta}} \mathrm{Ric}_{\gamma\bar{\delta}},\\
&R(\mathrm{Ric}, \mathrm{Ric}) = g^{\alpha\bar{i}} g^{j\bar{\beta}} g^{\gamma\bar{k}} g^{l\bar{\delta}} R_{i\bar{j}k\bar{l}} \mathrm{Ric}_{\beta\bar{\alpha}} \mathrm{Ric}_{\delta\bar{\gamma}},\\
&\mathrm{Ric}(R, R) = g^{\alpha\bar{i}} g^{j\bar{\beta}} g^{\gamma\bar{k}} g^{\delta\bar{p}} g^{q\bar{\epsilon}} \mathrm{Ric}_{i\bar{j}} R_{\beta\bar{\gamma}p\bar{q}} R_{k\bar{\alpha}\epsilon\bar{\delta}},\\
&\mathrm{divdiv}(R, \mathrm{Ric}) = -g^{\beta\bar{i}} g^{j\bar{\alpha}} \mathrm{Ric}_{i\bar{j}} \frac{\partial^2 scal_g}{\partial z_\alpha \partial \overline{z_\beta}} - 2|D'\mathrm{Ric}|^2 - g^{\alpha\bar{i}} g^{j\bar{\beta}} g^{\gamma\bar{k}} g^{l\bar{\delta}} \mathrm{Ric}_{i\bar{j},k\bar{l}} R_{\beta\bar{\alpha}\delta\bar{\gamma}} +\\
&\qquad\qquad - R(\mathrm{Ric}, \mathrm{Ric}) + \sigma_3(\mathrm{Ric}),
\end{aligned} \tag{2.4}$$

where ”$,p$” represents the covariant derivative with respect to $\frac{\partial}{\partial z_p}$ and Δ represents the Laplace operator

$$\Delta = \sum_{i,j=1}^{n} g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}. \tag{2.5}$$

Prescribing geometric structures of a complex manifold often introduces interesting and important partial differential equations. A typical example of this kind is the problem of finding the Kähler metrics with constant scalar curvature on a Kähler manifold. Such a problem defines a fourth order elliptic partial differential equation. The study of these partial differential equations, including the Kähler–Einstein equations, forms one of the richest topics in complex geometry. In view of Donaldson's work [28],[29] (see also [2]) in the compact setting and the theory of quantization [9],[12],[13] in the noncompact one, it is natural to believe that prescribed coefficients encode geometric properties of the Kähler metric. Thus it is natural to study metrics with the coefficients of the TYCZ-expansion being prescribed. In this regards Z. Lu and G. Tian [53] prove that the PDEs $a_j = f$ ($j \geq 2$ and f smooth function on M) are elliptic. The study of these PDEs makes sense despite to the existence of an TYCZ-expansion and so given any Kähler manifold (M, g) it makes sense to call the a_j 's the *coefficients associated to metric g.* Thus prescribing a_j gives an interesting set of new elliptic equations.

The noncompact case. In this case, for all real number $m > 0$ (not necessarily integer), the metric mg is polarized with respect to the trivial holomorphic line bundle $L^m = M \times \mathbb{C}$. Fix $m > 0$. Consider the hermitian metric h_m on L^m defined by

$$h_m(x, v) = e^{-m\varphi(x)}|v|^2$$

where φ is a Kähler potential for the Kähler metric g. We see that $\mathrm{Ric}(h_m) = m\omega$. Therefore, in this case the Hilbert space $\mathcal{H}_m$ equals the weighted Hilbert space $\mathcal{H}_{mg}$ consisting of square integrable holomorphic functions on M, with weight $e^{-m\varphi}$, namely

$$\mathcal{H}_{mg} = \left\{ f \in Hol(M) \mid \int_M e^{-m\varphi}|f|^2 \frac{\omega^n}{n!} < +\infty \right\}.$$

If $\mathcal{H}_{mg} \neq \{0\}$, we can pick an orthonormal basis $\{f_j^m\}$ and define its reproducing kernel

$$K_{m\varphi}(x, y) = \sum_{j=0}^{N} f_j^m(x)\overline{f_j^m(y)}, \quad x, y \in M$$

where $N + 1 = \dim \mathcal{H}_{mg}$. In particular, in this case the ϵ-function reads

$$\epsilon_{mg}(x) = e^{-m\varphi(x)} K_{m\varphi}(x, x)$$

and can be verified that this function depends only on the metric mg and not on the choice of the Kähler potential or the orthonormal basis chosen. In this particular case, the coherent state map is simply the Kähler immersion

$$F_m : M \to \mathbb{C}\mathrm{P}^\infty \quad x \mapsto [f_0^m(x), \dots, f_j^m(x), \dots]$$

(F_m is well defined since ϵ_{mg} is a positive constant and hence for all $x \in M$ there exists $\phi \in \mathcal{H}_{mg}$ such that $\phi(x) \neq 0$) and still holds Theorem 2.3.8 in the noncompact case, that is

$$\begin{aligned} F_m^* \omega_{\mathrm{FS}} &= \frac{i}{2}\partial\overline{\partial} \log \sum_{j=0}^{N} |f_j(z)|^2 \\ &= \frac{i}{2}\partial\overline{\partial} \log K_{mg}(x, y) \\ &= \frac{i}{2}\partial\overline{\partial} \log \epsilon_{mg} + \frac{i}{2}\partial\overline{\partial} \log e^{m\phi} \\ &= \frac{i}{2}\partial\overline{\partial} \log \epsilon_{mg} + m\omega. \end{aligned}$$

The tools developed in the compact case for the study of existence and uniqueness of balanced metrics are not avaible in the noncompact one (for instance the volume of M can be infinite).

When M is not compact, there is not a general theorem which assures the existence of an asymptotic expansion of the ϵ-function ϵ_{mg} for $m \to +\infty$ and only partial results are given. However, if the expansion exists, in [30] M. Engliš computed the a_j's coefficients obtaining the same results as Lu. This expansion exists for example when M is a strongly pseudoconvex bounded domain of $\mathbb{C}^n$ with real analytic boundary or when M is a bounded symmetric domain ([31]). Furthermore, X. Ma and G. Marinescu proved this existence under some boundness assumptions of the curvature of the bundles involved. More precisely, let (X, g) be a complete hermitian manifold and let ω be the $(1, 1)$-form associated to g. Let $(L, h_L), (E, h_E)$ be holomorphic hermitian vector bundles on X with $rk(L) = 1$. Consider the space $H^0_{(2)}(X, L^p \otimes E)$ of holomorphic sections of $L^p \otimes E$ which have L^2-limited norm with respect the product

$$(s_1, s_2)_{L^2} = \int_X < s_1(x), s_2(x) > d\nu_X(x)$$

where $< \, , >$ denotes the hermitian product on $L^p \otimes E$ induced by h_L and h_E. Let $P_p(x, y)$, $(x, y \in X)$ be the Schwartz kernel of the orthogonal projection $P_p : L^p \otimes E \to H^0_{(2)}(X, L^p \otimes E)$. Denote by $R^E, R^L, R^{\det}$ the curvature of E (of the connection induced by h_E), the curvature of L (of the connection induced by h_L) and the curvature of $\det(T^{(1,0)}X)$ (of the connection associated to the metric induced by g) respectively. Then we have the following

Theorem 2.3.20 ([54], Thm 6.1.1.)**.** Suppose that there exist $t > 0$, $c > 0$ such that:

$$iR^L < t\omega, \quad i(R^{\det} + R^E) > -c\,\omega\, Id_E, \quad |\partial\omega|_g < c, \tag{2.6}$$

then the kernel $P_p(x, y)$ has an asymptotic expansion as $p \to \infty$. Especially, it holds

$$|\frac{P_p(x, x)}{p^n} - \sum_{r=0}^{k} b_r(x) p^{-r}|_{C^l(K)} \leq C_{k,l,K}\, p^{-k-1}$$

uniformly for any $x, y \in K$, a compact set of X.

We will see a version of this theorem adapted to specific settings in Section 3.1 (for Kähler-Einstein metrics) and in Section 4.5 (for Hwang–Singer metrics), (cf. also [50, Theorem 7]).

Chapter 3

The third coefficient in the TYCZ–expansion of the epsilon function of Kähler–Einstein manifolds

In this chapter we present the material of [26]. Here we study the third coefficient arising from the TYCZ-expansion of the ϵ-function associated to a Kähler–Einstein metric and discuss the consequences of its vanishing. The main results of this chapter are the following.

Theorem 3.0.1. Let (M, g) be a n-dimensional Kähler–Einstein manifold with integral Kähler form ω. If $\mathcal{H}_\alpha \neq \{0\}$, there exists a TYCZ–expansion for $\epsilon_{\alpha g}$ whose coefficients satisfy the following:

1. if $n = 2$ then $a_3 = 0$ if and only if $\Delta|R|^2 = 0$;
2. if $n \geq 3$ then $a_3 = 0$ implies g is Ricci-flat.

Moreover we show that the Calabi nonhomogeneous metrics on tubular domains of $\mathbb{C}^n$ are examples of Kähler–Einstein manifolds which answer positively to Question 1. More precisely we prove

Theorem 3.0.2. The ϵ-function associated to Calabi's metric admits a TYCZ–expansion with $a_3 \neq 0$.

3.1 The vanishing of a_3 for Kähler–Einstein manifolds

Let (M, ω) be a Kähler–Einstein manifold, that is $\rho = \lambda\omega$. We note that in this setting the existence of a TYCZ-expansion for $\varepsilon_{\alpha g}$ is always guaranteed, by the result of X. Ma and G. Marinescu [54, Theorem 6.1.1]. More precisely we have

Theorem 3.1.1. Let (M, g) be a Kähler–Einstein manifold polarized by a holomorphic line bundle. Then, the ϵ–function associated to g, if it exists, admits a TYCZ–expansion.

Proof. Let us check that conditions (2.6) hold when (M, g) is a Kähler–Einstein manifold polarized by $L \to M$. The first condition is satisfied for $l \in (0, 1)$ since

$$iR^L = -i\partial\overline{\partial} \log h = -i\partial\overline{\partial} \log e^{-\frac{1}{2}\varphi} = \omega.$$

The second condition is satisfied since:

$$iR^{\det} + c\omega_n = \rho + c\omega = (\lambda + c)\omega > 0,$$

for any $c > \lambda$. Finally, the third condition is satisfied for every positive $c > 0$, since $\partial\omega = 0$, being the metric Kähler. ☐

However, observe that to construct the ε–function we need $\mathcal{H}_\alpha \neq \{0\}$. Such condition is satisfied for example when M has finite volume, since in this case the constant functions belongs to $\mathcal{H}_\alpha$, but of course there are manifolds with infinite volume such that $\mathcal{H}_\alpha \neq \{0\}$ (see e.g. Calabi's manifold in the next Section).

In order to prove Theorem 3.0.1 we need the formulas 2.3, computed by Z. Lu [52] for compact manifolds and by M. Engliš [30] for noncompact ones. For a Kähler–Einstein manifold (M, g) with Einstein constant λ, we have

$$\mathrm{Ric} = \lambda g, \qquad scal_g = n\lambda.$$

Further

$$\mathrm{divdiv}(R, \mathrm{Ric}) = -R(\mathrm{Ric}, \mathrm{Ric}) + \sigma_3(\mathrm{Ric}), \tag{3.1}$$

since by (2.4), using that $scal_g$ is constant, we get

$$\mathrm{divdiv}(R, \mathrm{Ric}) = -2|D'\mathrm{Ric}|^2 - g^{\alpha\bar{i}} g^{j\bar{\beta}} g^{\gamma\bar{k}} g^{l\bar{\delta}} \mathrm{Ric}_{i\bar{j},k\bar{l}} R_{\beta\overline{\alpha}\delta\overline{\gamma}} - R(\mathrm{Ric}, \mathrm{Ric}) + \sigma_3(\mathrm{Ric}),$$

and both the terms $|D'\mathrm{Ric}|^2$ and $g^{\alpha\bar{i}} g^{j\bar{\beta}} g^{\gamma\bar{k}} g^{l\bar{\delta}} \mathrm{Ric}_{i\bar{j},k\bar{l}} R_{\beta\overline{\alpha}\delta\overline{\gamma}}$ involve the covariant derivatives of the Ricci tensor, that vanish since the metric is Kähler–Einstein ($\mathrm{Ric} = \lambda g$ and $\nabla g = 0$).

The following lemma is a key step in the proof of Theorem 3.0.1.

Lemma 3.1.2. Let (M, g) be a n-dimensional Kähler–Einstein manifold with Einstein constant λ. Then:

1. $a_2 = \frac{1}{24}\left(|R|^2 + n\lambda^2(3n - 4)\right)$;
2. $a_3 = \frac{1}{48}\left(\Delta|R|^2 - \lambda(n - 2)(\lambda^2 n(n - 2) + |R|^2)\right)$.

Proof. Setting $scal_g = \lambda n$ and since

$$|\mathrm{Ric}|^2 = g^{i\bar{k}} g^{l\bar{j}} \mathrm{Ric}_{i\bar{j}} \overline{Ric_{k\bar{l}}} = g^{i\bar{k}} g^{l\bar{j}} \mathrm{Ric}_{i\bar{j}} \mathrm{Ric}_{l\bar{k}} = \lambda^2 \, g^{i\bar{k}} g^{l\bar{j}} g_{i\bar{j}} g_{l\bar{k}} = \lambda^2 n,$$

(2.3) reads:

$$a_2 = \frac{1}{24}\left(|R|^2 + n\lambda^2(3n - 4)\right);$$

$$\begin{aligned} a_3 = &\frac{1}{24}\mathrm{divdiv}(R, \mathrm{Ric}) + \frac{1}{48}\Delta|R|^2 - \frac{1}{48}\lambda n(\lambda^2 n^2 + |R|^2 - 4\lambda^2 n) + \\ &- \frac{1}{24}(\sigma_3(\mathrm{Ric}) - \mathrm{Ric}(R, R) + R(\mathrm{Ric}, \mathrm{Ric})). \end{aligned} \tag{3.2}$$

By (3.1) and since:

$$\begin{aligned}
\mathrm{Ric}(R,R) =& g^{\alpha\bar{i}} g^{j\bar{\beta}} g^{\gamma\bar{k}} g^{\delta\bar{p}} g^{q\bar{\epsilon}} \mathrm{Ric}_{i\bar{j}} R_{\beta\bar{\gamma}p\bar{q}} R_{k\bar{\alpha}\epsilon\bar{\delta}} \\
=& \lambda g^{j\bar{\beta}} g^{\gamma\bar{k}} g^{\delta\bar{p}} g^{q\bar{\epsilon}} R_{\beta\bar{\gamma}p\bar{q}} R_{k\bar{j}\epsilon\bar{\delta}} \\
=& \lambda |R|^2, \\
R(\mathrm{Ric},\mathrm{Ric}) =& g^{\alpha\bar{i}} g^{j\bar{\beta}} g^{\gamma\bar{k}} g^{l\bar{\delta}} R_{i\bar{j}k\bar{l}} \mathrm{Ric}_{\beta\bar{\alpha}} \mathrm{Ric}_{\delta\bar{\gamma}} \\
=& \lambda^2 g^{j\bar{\beta}} g^{\gamma\bar{k}} R_{\beta\bar{j}k\bar{\gamma}} = \lambda^2 g^{\gamma\bar{k}} \mathrm{Ric}_{k\bar{\gamma}} \\
=& \lambda^2 scal_g = n\lambda^3,
\end{aligned} \tag{3.3}$$

substituting in 3.2, we obtain

$$\begin{aligned}
a_3 =& \frac{1}{24}(-R(\mathrm{Ric},\mathrm{Ric}) + \sigma_3(Ric)) + \frac{1}{48}\Delta(|R|^2) - \frac{1}{48}\lambda n(\lambda^2 n^2 + |R|^2 - 4\lambda^2 n) + \\
& - \frac{1}{24}(\sigma_3(\mathrm{Ric}) - \mathrm{Ric}(R,R) + R(\mathrm{Ric},\mathrm{Ric})) \\
=& \frac{1}{24}(-2R(\mathrm{Ric},\mathrm{Ric}) + \mathrm{Ric}(R,R)) + \frac{1}{48}\Delta(|R|^2) - \frac{1}{48}\lambda n(\lambda^2 n^2 + |R|^2 - 4\lambda^2 n) \\
=& \frac{1}{24}(-2n\lambda^3 + \lambda|R|^2) + \frac{1}{48}\Delta(|R|^2) - \frac{1}{48}\lambda n(\lambda^2 n^2 + |R|^2 - 4\lambda^2 n) \\
=& \frac{1}{48}\left(\Delta|R|^2 - \lambda(n-2)(\lambda^2 n(n-2) + |R|^2)\right),
\end{aligned}$$

concluding the proof. □

The proof of Theorem 3.0.1 follows now by Theorem 3.1.1 and by Lemma 3.1.2. More precisely:

Proof of Theorem 3.0.1. The existence of a TYCZ–expansion follows directly from Theorem 3.1.1, while (1) and (2) follow readily from Lemma 3.1.2. To prove (3) assume $n > 2$. Then:

$$0 = \int_M a_3 \frac{\omega^n}{n!} = -\frac{1}{48}\lambda(n-2)\int_M \left(\lambda^2 n(n-2) + |R|^2\right)\frac{\omega^n}{n!},$$

and since the integrand function is nonnegative, to the right hand side to be zero we need $\lambda = 0$. □

Observe that the vanishing of a_2 (for $n \geq 2$) readily implies that the metric is flat. In [44, Thm. 1.1], A. Loi, F. Salis and F. Zuddas prove the following

Theorem 3.1.3 (A. Loi, F. Salis, F. Zuddas)**.** The third coefficient a_3 of a radial constant scalar curvature Kähler metric is constant if and only if the second coefficient a_2 is constant.

Here for a *radial metric* we mean a Kähler metric ω on a Kähler manifold M such that for any point $p \in M$ there exists a coordinate neighborhood U of p such that $\omega|_U$ can be described by a Kähler potential which depends only on the sum $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$ of the moduli of the local coordinates. Combining this result with Thm. 3.0.1 we obtain the following:

Corollary 3.1.4. Let (M, g) be a Kähler–Einstein manifold endowed with a radial Kähler metric g. Then $a_3 = 0$ if and only if (M, g) is biholomorphically isometric to $\mathbb{C}^n$, $\mathbb{C}\mathrm{H}^2$ or $\mathbb{C}\mathrm{P}^2$ with (a multiple of) their standard metric.

Proof of Corollary 3.1.4. Assume (M, ω) to be Kähler–Einstein and radial. If $a_3 = 0$, then by Theorem 3.1.3 a_2 is constant. From Z. Feng classification [32] (see also [44, Thm. 2.1]) of radial constant scalar curvature potentials with constant a_2, the only Kähler–Einstein are (a multiple of) the standard metrics on $\mathbb{C}^n$, $\mathbb{C}\mathrm{H}^n$ or $\mathbb{C}\mathrm{P}^n$. The case $\mathbb{C}\mathrm{H}^n$ and $\mathbb{C}\mathrm{P}^n$, with $n \geq 3$, are excluded by Theorem 3.0.1, since they are not Ricci-flat. ☐

We conclude this section with some consequences of Lemma 3.1.2. First of all, it follows readily from Theorem 3.1.2 that, for Kähler–Einstein manifolds, if a_2 is constant also a_3 is. Notice also that in the Kähler–Einstein case, $a_2 = 0$ if and only if the metric is flat, since $|R|^2 = n\lambda^2(4 - 3n)$ implies $\lambda = |R|^2 \equiv 0$. For the Ricci-flat case we have:

Corollary 3.1.5. Let (M, g) be a Ricci–flat Kähler manifold. Then the following hold:

1. if a_2 is constant then $a_3 = 0$. If in addition M is compact, the converse is also true.
2. if (M, g) is either regular or homogeneous then $a_3 = 0$.

Notice that the constancy of a_2 cannot be dropped, as the Taub–NUT metric is an example of Ricci–flat metric on $\mathbb{C}^2$ with nonvanishing a_3 (see [50]).

The same holds for Kähler–Einstein surfaces:

Corollary 3.1.6. Let (M, g) be a Kähler–Einstein surface. Then the following hold:

1. if a_2 is constant then $a_3 = 0$. If in addition M is compact, the converse is also true.
2. if (M, g) is either regular or homogeneous then $a_3 = 0$.

Finally, for Calabi-Yau manifolds we have:

Corollary 3.1.7. Let M be a Calabi-Yau manifold. Then $\int_M a_3\omega^n = 0$ for any Kähler metric on M.

Proof. $\int_M a_3\omega^n$ is a cohomologial invariant. By Yau Theorem, there exists a (unique) Kähler–Einstein metric in any Kähler class on M and $\int_M \Delta|R|^2\omega^n = 0$ on a Kähler manifold. ☐

3.2 The coefficient a_3 for Calabi's inhomogeneous metric

In [15], E. Calabi constructs the following complete not locally homogeneous Kähler–Einstein metric. Consider the complex tubular domains $M_n := \frac{1}{2}D \oplus$

$i\mathbb{R}^n \subset \mathbb{C}^n$, where D is an open ball in $\mathbb{R}^n$ centred in the origin and of radius a. Let g_n be the Kähler metric on M_n whose Kähler form is given by

$$\omega_n = \frac{i}{2}\partial\overline{\partial}F(z,\overline{z}),$$

with

$$F(z,\overline{z}) = f(z_1+\overline{z_1},\ldots,z_n+\overline{z_n})$$

where $f : D \to \mathbb{R}$ is a strongly convex, differentiable and radial function, i.e. $f(x_1,\ldots,x_n) = y(r)$, with $r = \sqrt{\sum_{j=1}^n x_j^2}$, that diverges uniformly at $+\infty$ at the boundary of D and $x_\alpha = z_\alpha + \overline{z_\alpha}$, for $\alpha = 1,\ldots,n$. The function f satisfies the following Cauchy problem

$$\begin{cases} (\frac{y'}{r})^{n-1}y'' = e^y \\ y'(0) = 0 \\ y''(0) = e^{\frac{y(0)}{n}}. \end{cases} \tag{3.4}$$

The metric so constructed is the first example of Kähler–Einstein metric (with Einstein constant $\lambda = -1$) which is not locally homogeneous (see also [64] for an alternative proof of the fact that it is not locally homogeneous using Lie groups).

Observe that we can see the tubular domain $M_n = \frac{1}{2}D \oplus i\mathbb{R}^n$ as the open submanifold of $\mathbb{C}^n$ given by

$$M_n = \left\{ z = (z_1,\ldots,z_n) \in \mathbb{C}^n \mid \sum_{j=1}^n (z_j+\overline{z_j})^2 < a^2 \right\},$$

where a is the upper bound of the domain of regularity of $y(r)$.

Remark 3.2.1. By recursion from (3.4), one obtains that for all $j \in \mathbb{N}$

$$y^{(2j+1)}(0) = 0,$$

thus the power expansion around the origin of the function $y(r)$ is of the form

$$y(r) = y(0) + \sum_{j=1}^{\infty} b_{2j}r^{2j} \tag{3.5}$$

where $b_{2j} = \frac{y^{(2j)}(0)}{(2j)!}$. In particular, the first three coefficients of the expansion are

$$b_2 = \frac{1}{2}e^{y(0)/2}, \quad b_4 = \frac{1}{32}e^{y(0)}, \quad b_6 = \frac{7}{2304}e^{3y(0)/2}. \tag{3.6}$$

In order to prove Theorem 4.0.2, we first have to show that $\mathcal{H}_\alpha \neq \{0\}$ for (M_n, g_n), ensuring the existence of the ε-function. This is done in Lemma 3.2.2 below.

Lemma 3.2.2. In the notation above, $\mathcal{H}_\alpha \neq \{0\}$ for (M_n, g_n).

Proof. First, notice that the volume form of g_n is given by

$$\frac{\omega_n^n}{n!} = \det(H)\frac{\omega_0^n}{n!} = e^f\frac{\omega_0^n}{n!}$$

where $H = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ is the hessian of the Kähler potential f, ω_0 is the standard euclidean form of $\mathbb{C}^n$, and the second equality follows since by construction we have $\det(f_{jk}) = e^f$ (see [15] p.19). Consider now the holomorphic function

$$h(z) = \prod_{j=1}^{n} \frac{1}{z_j - 2a}$$

over $M_n = \frac{1}{2}D \oplus i\mathbb{R}^n$, where a is the radius of the ball $D \subset \mathbb{R}^n$. For $j = 1, \dots, n$, denote $x_j := 2\mathrm{Re}(z_j)$ (as before) and $u_j := 2\mathrm{Im}(z_j)$. Observe that in this notation,

$$\frac{\omega_0^n}{n!} = \left(\frac{i}{2}\right)^n \prod_{j=1}^{n} dz_j \wedge d\overline{z_j} = \frac{1}{2^{2n}} \prod_{j=1}^{n} dx_j \wedge du_j.$$

We have

$$|h(z)|^2 = \prod_{j=1}^{n} \frac{1}{z_j - 2a} \frac{1}{\overline{z_j} - 2a} = \prod_{j=1}^{n} \frac{1}{\frac{x_j^2}{4} + \frac{u_j^2}{4} - 2ax_j + 4a^2}.$$

Thus,

$$\begin{aligned}
\int_{M_n} e^{-\alpha f} |h|^2 \frac{\omega_n^n}{n!} &= \int_{M_n} e^{(1-\alpha)f} |h|^2 \frac{\omega_0^n}{n!} = \int_{\frac{1}{2}D} \int_{\mathbb{R}^n} e^{(1-\alpha)f} |h|^2 \frac{\omega_0^n}{n!} \\
&= \frac{1}{2^{2n}} \int_{\frac{1}{2}D} \int_{\mathbb{R}^n} e^{(1-\alpha)f} \prod_{j=1}^{n} \frac{1}{\frac{x_j^2}{4} + \frac{y_j^2}{4} - 2ax_j + 4a^2} \prod_{j=1}^{n} dx_j \wedge dy_j \\
&= \frac{1}{2^{2n}} \int_{\frac{1}{2}D} e^{(1-\alpha)f} \prod_{j=1}^{n} dx_j \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \frac{1}{\frac{x_j^2}{4} + \frac{y_j^2}{4} - 2ax_j + 4a^2} \prod_{j=1}^{n} dy_j \\
&= \pi^n \int_{\frac{1}{2}D} \frac{e^{(1-\alpha)f}}{\prod_{j=1}^{n} |x_j - 4a|} \prod_{j=1}^{n} dx_j.
\end{aligned}$$

Now, the function $\frac{1}{\prod_{j=1}^{n} |x_j - 4a|}$ is bounded on $\frac{1}{2}D$ since D has radius a and similarly $e^{(1-\alpha)f}$ is bounded since f is a smooth function diverging at $+\infty$ on the boundary of D, i.e. $e^{(1-\alpha)f} \to 0$ as $p \to \partial D$, for $\alpha > 1$. ☐

By Theorem 3.1.1, this proves the first part of Theorem 3.0.2. To show the second part, namely that the a_3 coefficient of the TYCZ-expansion of the ε-function of (M_n, g_n) is not zero, we start by describing more in details Calabi's metric for $n = 2$. By definition, g_2 is given by:

$$(g_2)_{i\bar{j}} = \frac{\partial^2 y(r)}{\partial x_i \partial x_j} = \frac{y'}{r} \delta_{ij} + \left(y'' - \frac{y'}{r}\right) \frac{x_i x_j}{r^2},$$

while its inverse is described by

$$g_{i\bar{j}}^{-1} = \frac{r}{y'} \delta_{ij} + \left(\frac{1}{y''} - \frac{r}{y'}\right) \frac{x_i x_j}{r^2}$$

that is, the matrix representing g_2 is

$$G_2 = \begin{bmatrix} \frac{y'}{r} + (y'' - \frac{y'}{r}) \frac{x_1^2}{r^2} & (y'' - \frac{y'}{r}) \frac{x_1 x_2}{r^2} \\ (y'' - \frac{y'}{r}) \frac{x_1 x_2}{r^2} & \frac{y'}{r} + (y'' - \frac{y'}{r}) \frac{x_2^2}{r^2} \end{bmatrix} \tag{3.7}$$

while its inverse is

$$G_2^{-1} = \begin{bmatrix} \frac{r}{y'} + (\frac{1}{y''} - \frac{r}{y'})\frac{x_1^2}{r^2} & (\frac{1}{y''} - \frac{r}{y'})\frac{x_1 x_2}{r^2} \\ (\frac{1}{y''} - \frac{r}{y'})\frac{x_1 x_2}{r^2} & \frac{r}{y'} + (\frac{1}{y''} - \frac{r}{y'})\frac{x_2^2}{r^2} \end{bmatrix}. \tag{3.8}$$

Let us prove Theorem 3.0.2.

Proof of Theorem 3.0.2. Calabi's metric is Kähler–Einstein with Einstein constant -1, thus by Theorem 3.0.1 a_3 is different from zero for all $n \geq 3$. Set $n = 2$.

We divide the proof in 5 steps. In Step 1, Step 2 and Step 3 we prove that the vanishing of a_3 is equivalent to $y'\partial_r|R|^2 = 0$, that is, since $y'(r) > 0$ for any $r > 0$, it is equivalent to $\partial_r|R|^2 = 0$ for $r \in (0, a)$. To conclude the proof we show in Step 4 and Step 5 that $|R|^2$ cannot be constant in $(0, a)$ since $\lim_{r\to 0}|R|^2 \neq \lim_{r\to a}|R|^2$.

Step 1: $\Delta|R|^2 = \frac{1}{y''}\partial_r^2|R|^2 + \frac{1}{y'}\partial_r|R|^2$.

A straighforward computation using the formula for the Riemannian tensor for g_n computed by Calabi (see [16], p. 23), we get:

$$\frac{1}{2}|R|^2 = 2 - \frac{8re^y}{y'^3} + \frac{y'^4}{r^4e^{2y}} + \frac{4}{ry'} + \frac{12r^2e^{2y}}{y'^6} - \frac{2y'}{r^3e^y} - \frac{12e^y}{y'^4} + \frac{6y'^3}{r^5e^{2y}} - \frac{12}{r^4e^y} + \frac{12y'^2}{r^6e^{2y}}. \tag{3.9}$$

In particular, $|R|^2$ is a function of r, and by (2.5) we have:

$$\begin{aligned}
\Delta|R|^2 =& g^{11}\frac{\partial^2}{\partial z_1\partial\bar{z}_1}|R|^2 + g^{12}\frac{\partial^2}{\partial z_1\partial\bar{z}_2}|R|^2 + g^{21}\frac{\partial^2}{\partial z_2\partial\bar{z}_1}|R|^2 + g^{22}\frac{\partial^2}{\partial z_2\partial\bar{z}_2}|R|^2 = \\
=& g^{11}\frac{\partial^2}{\partial x_1^2}|R|^2 + 2g^{12}\frac{\partial^2}{\partial x_1\partial x_2}|R|^2 + g^{22}\frac{\partial^2}{\partial x_2^2}|R|^2 \\
=& g^{11}\frac{\partial}{\partial x_1}\left(\partial_r|R|^2\frac{x_1}{r}\right) + 2g^{12}\frac{\partial}{\partial x_1}\left(\partial_r|R|^2\frac{x_2}{r}\right) + g^{22}\frac{\partial}{\partial x_2}\left(\partial_r|R|^2\frac{x_2}{r}\right) \\
=& g^{11}\left(\partial_r^2|R|^2\frac{x_1^2}{r^2} + \frac{1}{r}\partial_r|R|^2\left(1 - \frac{x_1^2}{r^2}\right)\right) + 2g^{12}\left(\partial_r^2|R|^2\frac{x_1x_2}{r^2} - \frac{1}{r}\partial_r|R|^2\frac{x_1x_2}{r^2}\right) + \\
&+ g^{22}\left(\partial_r^2|R|^2\frac{x_2^2}{r^2} + \frac{1}{r}\partial_r|R|^2\left(1 - \frac{x_2^2}{r^2}\right)\right) \\
=& \partial_r^2|R|^2\left(g^{11}\frac{x_1^2}{r^2} + 2g^{12}\frac{x_1x_2}{r^2} + g^{22}\frac{x_2^2}{r^2}\right) + \\
&\qquad + \frac{1}{r}\partial_r|R|^2\left(g^{11} + g^{22} - \left(g^{11}\frac{x_1^2}{r^2} + 2g^{12}\frac{x_1x_2}{r^2} + g^{22}\frac{x_2^2}{r^2}\right)\right) \\
=& \frac{1}{y''}\partial_r^2|R|^2 + \frac{1}{y'}\partial_r|R|^2.
\end{aligned}$$

where g_{jk} and g^{jk}, j, $k = 1, 2$, represent respectively the entries of G_2 and G_2^{-1}, given in (3.7) and (3.8).

Step 2: $\Delta|R|^2 = 0$ if and only if $y'\partial_r|R|^2$ is constant.

By Step 1, $\Delta|R|^2 = 0$ if and only if:

$$\frac{y''}{y'} = -\frac{\partial_r^2|R|^2}{\partial_r|R|^2}$$

that is:

$$\partial_r\left(\log(y')\right) = -\partial_r\left(\log(\partial_r|R|^2)\right),$$

and thus for a constant c:

$$\log(\partial_r|R|^2) = -\log(y') + c,$$

that is:

$$\partial_r|R|^2 = \frac{c}{y'}.$$

Step 3: $\lim_{r\to a} y'\partial_r|R|^2 = 0$.

From (3.9) it follows:

$$\begin{aligned}\frac{1}{4}\partial_r|R|^2 = &-\frac{4re^y}{y'^2} + \frac{24r^2e^{2y}}{y'^5} - \frac{36r^3e^{3y}}{y'^8} - \frac{8y'^4}{r^5e^{2y}} - \frac{y'^5}{r^4e^{2y}} + \frac{3y'^2}{r^3e^y} - \frac{3}{r^2y'} + \\ &+ \frac{18y'}{r^4e^y} - \frac{27y'^3}{r^6e^{2y}} + \frac{36}{r^5e^y} - \frac{36y'^2}{r^7e^{2y}} + \frac{36re^{2y}}{y'^6} - \frac{12e^y}{y'^3}.\end{aligned} \tag{3.10}$$

Let us write:

$$\frac{1}{4}y'\partial_r|R|^2 = A + B + C,$$

where:

$$\begin{aligned}A &:= -\frac{8y'^5}{r^5e^{2y}} + \frac{18y'^2}{r^4e^y} - \frac{27y'^4}{r^6e^{2y}} + \frac{36y'}{r^5e^y} - \frac{36y'^3}{r^7e^{2y}},\\ B &:= -\frac{y'^6}{r^4e^{2y}} + \frac{3y'^3}{r^3e^y} - \frac{3}{r^2},\\ C &:= -\frac{4re^y}{y'} + \frac{24r^2e^{2y}}{y'^4} - \frac{36r^3e^{3y}}{y'^7} + \frac{36re^{2y}}{y'^5} - \frac{12e^y}{y'^2}.\end{aligned}$$

In order to compute the limits for $r \to a$ of A, B and C, we first observe that by construction (see [15, p. 21]):

$$\lim_{r\to a} y = +\infty, \quad \lim_{r\to a} y' = +\infty,$$

and by (3.4):

$$\lim_{r\to a} y'' = \lim_{r\to a}\frac{e^y r}{y'} = \lim_{r\to a}\frac{(1+ry')y'}{r} = +\infty, \tag{3.11}$$

$$\lim_{r\to a}\frac{e^y}{y'^3} = \lim_{r\to a}\frac{e^y y'}{3y'^2y''} = \lim_{r\to a}\frac{1}{3r} = \frac{1}{3a}, \tag{3.12}$$

Further, $\lim_{r\to a}(y'^3 - 3re^y)$ is not finite, in fact

$$\lim_{r\to a}(y'^3 - 3re^y) = \lim_{r\to a} y'^3\left(1 - \frac{3re^y}{y'^3}\right) = \lim_{r\to a}\frac{1-\frac{3re^y}{y'^3}}{\frac{1}{y'^3}} = \frac{0}{0}$$

and applying de l'Hopital we get:

$$\lim_{r\to a} \frac{1-\frac{3re^y}{y'^3}}{\frac{1}{y'^3}} = \lim_{r\to a} \frac{\frac{-y'^3(3e^y+3re^y y')+9y'^2y''re^y}{y'^6}}{-\frac{3y'^2y''}{y'^6}} = \frac{y'^2}{r} + y'^3 - 3re^y.$$

Thus, if $\lim_{r\to a}\left(y'^3 - 3re^y\right) = c$, with $c \in \mathbb{R}$, then we get the contradiction:

$$\lim_{r\to a}\left(y'^3 - 3re^y\right) = \lim_{r\to a}\left(\frac{y'^2}{r} + y'^3 - 3re^y\right) = +\infty.$$

Therefore we can apply de l'Hopital to $\lim_{r\to a} \frac{y'^3-3re^y}{y'^2}$ and we get:

$$\begin{aligned}\lim_{r\to a} \frac{y'^3 - 3re^y}{y'^2} &= \lim_{r\to a} \frac{3y'^2y'' - 3e^y - 3re^y y'}{2y'y''} = \\ &= \lim_{r\to a} \frac{3y're^y - 3e^y - 3re^y y'}{2re^y} = -\frac{3}{2a}.\end{aligned} \tag{3.13}$$

By (3.11) and (3.12) we have:

$$\lim_{r\to a} A = \lim_{r\to a}\left(-\frac{8y'^5}{r^5e^{2y}} + \frac{18y'^2}{r^4e^y} - \frac{27y'^4}{r^6e^{2y}} + \frac{36y'}{r^5e^y} - \frac{36y'^3}{r^7e^{2y}}\right) = 0,$$

$$\lim_{r\to a} B = \lim_{r\to a}\left(-\frac{y'^6}{r^4e^{2y}} + \frac{3y'^3}{r^3e^y} - \frac{3}{r^2}\right) = -\frac{3}{a^2},$$

Finally by (3.12) and (3.13) we have:

$$\begin{aligned}\lim_{r\to a} C &= \lim_{r\to a}\left(-\frac{4re^y}{y'} + \frac{24r^2e^{2y}}{y'^4} - \frac{36r^3e^{3y}}{y'^7} + \frac{36re^{2y}}{y'^5} - \frac{12e^y}{y'^2}\right) \\ &= \lim_{r\to a}\left[-y'^2\left(4 - \frac{24re^y}{y'^3} + \frac{36r^2e^{2y}}{y'^6}\right) + 12\left(\frac{3e^y}{y'^3} - \frac{1}{r}\right)y'\right]\frac{re^y}{y'^3} \\ &= \lim_{r\to a}\left[-\left(\frac{y'^3 - 3re^y}{y'^2}\right)^2 - \frac{3}{r}\frac{y'^3 - 3re^y}{y'^2}\right]\frac{4re^y}{y'^3} \\ &= \left[-\frac{9}{4a^2} + \frac{9}{2a^2}\right]\frac{4}{3} = \frac{3}{a^2},\end{aligned}$$

and we are done.

Step 4: $\lim_{r\to a} |R|^2 = \frac{4}{3}$**.**
This follows directly from (3.9).

Step 5: $\lim_{r\to 0} |R|^2 = \frac{3}{2}$**.**

Using (3.9), let us write:

$$\begin{aligned}\frac{1}{2}|R|^2 =& 2 - \frac{8re^y}{y'^3} + \frac{y'^4}{r^4e^{2y}} + \frac{4}{ry'} + \frac{12r^2e^{2y}}{y'^6} - \frac{2y'}{r^3e^y} - \frac{12e^y}{y'^4} + \frac{6y'^3}{r^5e^{2y}} - \frac{12}{r^4e^y} + \frac{12y'^2}{r^6e^{2y}} \\ =& 2 + \frac{y'^4}{r^4e^{2y}} + \frac{r^3}{y'^3}\left(-8e^{3y} + 4\frac{y'^2e^{2y}}{r^2} - \frac{2y'^4e^y}{r^4} + \frac{6y'^6}{r^6}\right)\frac{1}{r^2}\frac{1}{e^{2y}} + \\ & 12\frac{r^6}{y'^6}\left(e^{4y} - \frac{e^{3y}y'^2}{r^2} - \frac{e^y y'^6}{r^6} + \frac{y'^8}{r^8}\right)\frac{1}{r^4}\frac{1}{e^{2y}}.\end{aligned}$$

Since

$$\lim_{r\to 0} \frac{y'^4}{r^4 e^{2y}} = 1,$$

we are done by showing that:

$$\lim_{r\to 0} \frac{r^3}{y'^3} \left(-8e^{3y} + 4\frac{y'^2 e^{2y}}{r^2} - \frac{2y'^4 e^y}{r^4} + \frac{6y'^6}{r^6} \right) \frac{1}{r^2}\frac{1}{e^{2y}} = -\frac{9}{2}, \tag{3.14}$$

$$\lim_{r\to 0} \frac{r^6}{y'^6} \left(e^{4y} - \frac{e^{3y} y'^2}{r^2} - \frac{e^y y'^6}{r^6} + \frac{y'^8}{r^8} \right) \frac{1}{r^4}\frac{1}{e^{2y}} = \frac{3}{16}. \tag{3.15}$$

By (3.5) and (3.6) we have:

$$\frac{y'(r)}{r} = e^{y(0)/2} + \sum_{j=1}^{\infty} c_{2j} r^{2j}, \tag{3.16}$$

with $c_2 = \frac{1}{8}e^{y(0)}$ and $c_4 = \frac{7}{384}e^{3y(0)/2}$. For shorten the notation let us write:

$$\begin{aligned} P :=& \frac{y'}{r} = e^{y(0)/2} + \sum_{j=1}^{\infty} c_{2j} r^{2j}, \\ Q :=& \frac{P'}{r} = 2c_2 + \sum_{j=2}^{\infty} 2j c_{2j} r^{2j-2}, \\ S :=& \frac{Q'}{r} = 8c_4 + \sum_{j=3}^{\infty} 2j(2j-2) c_{2j} r^{2j-4} \end{aligned} \tag{3.17}$$

To compute (3.14) let us start with the following:

$$\begin{aligned} &\lim_{r\to 0} \left(-8e^{3y} + 4\frac{y'^2 e^{2y}}{r^2} - \frac{2y'^4 e^y}{r^4} + \frac{6y'^6}{r^6} \right) \frac{1}{r^2} \\ &= \lim_{r\to 0} \frac{-8e^{3y} + 4e^{2y}P^2 - 2e^y P^4 + 6P^6}{r^2} \\ &= \lim_{r\to 0} \left(\frac{-24e^{3y}y' + 8e^{2y}y'P^2 + 8e^{2y}PP' - 8e^y P^3 P' - 2e^y y' P^4 + 36P^5 P'}{2r} \right) \\ &= \frac{1}{2} \lim_{r\to 0} \left(-24e^{3y}P + 8e^{2y}P^3 + 8e^{2y}PQ - 8e^y P^3 Q - 2e^y P^5 + 36P^5 Q \right) \\ &= -\frac{9}{2} e^{7y(0)/2}, \end{aligned}$$

where we applied de l'Hopital and used (3.17). Plugging the result into (3.14) we get:

$$\lim_{r\to 0} \frac{r^3}{y'^3} \left(-8e^{3y} + 4\frac{y'^2 e^{2y}}{r^2} - \frac{2y'^4 e^y}{r^4} + \frac{6y'^6}{r^6} \right) \frac{1}{r^2}\frac{1}{e^{2y}} = -\frac{9}{2}.$$

To compute (3.15), we need to apply de l'Hopital twice to the following limit:

$$
\begin{aligned}
&\lim_{r\to 0}\left(e^{4y}-\frac{e^{3y}y'^2}{r^2}-\frac{e^y y'^6}{r^6}+\frac{y'^8}{r^8}\right)\frac{1}{r^4}=\lim_{r\to 0}\frac{e^{4y}-e^{3y}P^2-e^yP^6+P^8}{r^4}\\
&=\lim_{r\to 0}\left(\frac{4e^{4y}y'-3e^{3y}y'P^2-2e^{3y}PP'-e^yy'P^6-6e^yP^5P'+8P^7P'}{4r^3}\right)\\
&=\frac{1}{4}\lim_{r\to 0}\frac{1}{r^2}\big(4e^{4y}P-3e^{3y}P^3-2e^{3y}PQ-e^yP^7-6e^yP^5Q+8P^7Q\big)\\
&=\frac{1}{8}\lim_{r\to 0}\frac{1}{r}\big(16y'e^{4y}P+4e^{4y}P'-9y'e^{3y}P^3-9e^{3y}P^2P'-6y'e^{3y}PQ+\\
&\qquad -2e^{3y}P'Q-2e^{3y}PQ'-y'e^yP^7-7e^yP^6P'-6y'e^yP^5Q+\\
&\qquad -30e^yP^4P'Q-6e^yP^5Q'+56P^6P'Q+8P^7Q'\big)\\
&=\frac{1}{8}\lim_{r\to 0}\big(16e^{4y}P^2+4e^{4y}Q-9e^{3y}P^4-9e^{3y}P^2Q-6e^{3y}P^2Q-2e^{3y}Q^2+\\
&\qquad -2e^{3y}PS-e^yP^8-7e^yP^6Q-6e^yP^6Q-30e^yP^4Q^2+\\
&\qquad -6e^yP^5S+56P^6Q^2+8P^7S\big)\\
&=\frac{3}{16}e^{5y(0)},
\end{aligned}
$$

thus

$$
\lim_{r\to 0}\frac{r^6}{y'^6}\left(e^{4y}-\frac{e^{3y}y'^2}{r^2}-\frac{e^y y'^6}{r^6}+\frac{y'^8}{r^8}\right)\frac{1}{r^4}\frac{1}{e^{2y}}=\frac{3}{16},
$$

concluding the proof.

$\square$

Chapter 4

Hwang–Singer metrics

This chapter presents the material of [25]. Here we study families of scalar flat metrics constructed via Calabi ansatz on the total space of a hermitian line bundle over Kähler–Einstein manifolds by Andrew D. Hwang and Michael A. Singer in [36]. The necessary hypotheses for the existence of scalar flat metrics include the so called sigma constancy, condition that is automatically satisfied by polarized Kähler manifolds, which is the case we are interested in. In particular we consider the following families:

(A) the 1-parameter family of nontrivial scalar flat Kähler metrics g_β on $\mathbb{C}^{n+1}$, $\beta < 0$ (described in Section 4.3);

(B) the scalar flat metrics g_k on $\mathcal{O}(-k)$ for integers $k > 0$ (described in Section 4.4).

Observe that the metrics g_k in (B) reduce to the Burns–Simanca metric for $k = 1$, and to the Ricci–flat Eguchi–Hanson metric for $k = 2$.

With reference to (A) and (B) we have the following results. Firstly,

Theorem 4.0.1. Let g_β be the Kähler metric on $\mathbb{C}^{n+1}$ arising from Hwang–Singer construction. Then cg_β is not projectively induced for any value of $c > 0$ and $\beta < 0$, but it can be approximated by a sequence of projectively induced metrics.

Secondly, we give a characterization of the Burns–Simanca metric among the Hwang–Singer family g_k on $\mathcal{O}(-k)$. More precisely we prove the following:

Theorem 4.0.2. Let g_k be the Kähler metric arising from Hwang–Singer construction on $\mathcal{O}(-k)$. Then g_k is projectively induced if and only if its second coefficient vanishes identically, that is if and only if it is the Burns–Simanca metric on the blow-up of $\mathbb{C}^2$ at one point. Moreover g_k can be approximated by a sequence of projectively induced metrics.

4.1 Momentum construction

A technique to produce complete Kähler metrics with good curvature properties is known as *Calabi ansatz*, firstly introduced by E. Calabi in [18] and later adopted by several authors. Andrew D. Hwang and Michael A. Singer generalized this construction on the total space of an hermitian holomorphic line bundle $\pi : L \to M$ with "*σ-constant curvature*" over a Kähler manifold (M, g_M). In this section we summarize Hwang–Singer construction, restricting our attention to the case of polarized manifolds, where these hypothesis are automatically satisfied.

Let $\pi : (L, h) \to (M, \omega_M)$ be a polarized hermitian holomorphic line bundle with curvature form $\mathrm{Ric}(h) = -i\partial\overline{\partial} \log h \in \Omega^2(M)$ such that $\mathrm{Ric}(h) = \beta\omega_M$ over a Kähler–Einstein manifold of complex dimension n, that is $\rho_M = \lambda\omega_M$, where ρ_M is the Ricci form associated to g_M. This method, also known as *momentum construction*, gives rise to *bundle-adapted metrics* on L, that is Kähler metrics $g_{\varphi,\beta}$ whose Kähler form arises from the Calabi ansatz

$$\omega_{\varphi,\beta} = \pi^*\omega_M + 2i\partial\overline{\partial} f(t),$$

where t is the logarithm of the norm function defined by h and $f : (-\infty, +\infty) \to [0, +\infty)$ is an increasing and strictly convex function of one real variable which makes $\omega_{\varphi,\beta}$ positive definite.

In a coordinate chart $U \subset M$ over which L is trivial, i.e. $\pi^{-1}(U) \cong U \times \mathbb{C}$, there exists a local coordinate system $\tilde{z} = (z, \xi) = (z^1, \dots, z^n, \xi)$ for L where $\xi = \rho e^{i\theta}$ is a fibre coordinate and $z = (z^1, \dots, z^n)$ are pullbacks of coordinates on M, i.e., if $q \in M$ is a point with coordinates z, then every point in the fiber $\pi^{-1}(q)$ can be described by coordinates $\tilde{z}$. In such a chart there is a smooth positive function $h : U \subset M \to \mathbb{R}$ such that

$$t := \log ||\tilde{z}|| = \frac{1}{2} \log \left(|\xi|^2 h(z)\right).$$

As explained in [36], to simplify the construction of scalar flat Kähler metrics on L, it is advantageous to change coordinates. Setting

$$\tau = f'(t) \quad , \quad \varphi(\tau) = f''(t) \ ,$$

so that f satisfies the differential equation

$$\begin{cases} f''(t) = \varphi(\tau) \\ f'(0) = \mu_0 > 0 \end{cases} ,$$

the Kähler metric $\omega_{\varphi,\beta}$ reads as

$$\omega_{\varphi,\beta} = \pi^*\omega_M - \tau\pi^*\mathrm{Ric}(h) + \frac{1}{\varphi}\, d\tau \wedge d^c\tau, \tag{4.1}$$

and along the fibre L_x over $x \in M$, restricts to

$$\omega_{\varphi,\beta}|_{\mathrm{fibre}} = \frac{\varphi(\tau)}{|\xi|^2}\, d\xi \wedge d\overline{\xi} \ .$$

The explicit expression for the profile function φ for scalar-flat polarized metrics is:

$$\varphi(\tau) = \frac{2}{(1-\beta\tau)^n}\left(\tau + \frac{\lambda\big((1-\beta\tau)^{n+1} - (1-\beta\tau) + \beta n\tau\big)}{\beta^2(n+1)}\right). \tag{4.2}$$

Observe that the factor $(1-\beta\tau)^n$ arises as the determinant of the endomorphism $\mathrm{Id} - \tau\mathrm{B}$, since $\mathrm{B} := \omega_M^{-1}\gamma = \beta\,\mathrm{Id}$ for $\mathrm{Ric}(h) = \beta\omega_M$.

Remark 4.1.1. The function $f' : (-\infty, +\infty) \to I := (0, +\infty)$ is an increasing and surjective function (see Prop. 1.4. in [36]). In particular

$$\lim_{t \to -\infty} f'(t) = 0.$$

Remark 4.1.2. The derivatives of the function $f(t)$ are expressed recursively in the variable τ as

$$f^{(n)}(t) = \varphi(\tau)(f^{(n-1)}(t))$$

for $n \geq 3$. In particular we have

$$\begin{aligned} f'''(t) &= \varphi(\tau)\varphi'(\tau), \\ f^{(iv)}(t) &= \varphi(\tau)(\varphi(\tau)\varphi''(\tau) + (\varphi'(\tau))^2). \end{aligned} \tag{4.3}$$

Proposition 4.1.3. Let $c > 0$ be a positive real number. If f is a solution for the ODE $y'' = \varphi(y')$ with φ given by (4.2), then $\hat{f} := cf$ is a solution to $y'' = \hat{\varphi}(y')$, where we denote with $\hat{\varphi}$ the profile function with parameters $\hat{\beta} = \frac{\beta}{c}$ and $\hat{\lambda} = \frac{\lambda}{c}$.

Proof. It follows by noticing that

$$\begin{aligned} \varphi(y') &= \frac{2}{(1 - \frac{\beta}{c}(cy)')^n} \left(\frac{1}{c}(cy)' + \frac{1}{c^2} \frac{\lambda\big((1 - \frac{\beta}{c}(cy)')^{n+1} - (1 - \frac{\beta}{c}(cy)') + n\frac{\beta}{c}(cy)'\big)}{\frac{\beta^2}{c^2}(n+1)} \right) \\ &= \frac{1}{c} \frac{2}{(1 - \hat{\beta}(cy)')^n} \left(y' + \frac{\hat{\lambda}\big((1 - \hat{\beta}(cy)')^{n+1} - (1 - \hat{\beta}(cy)') + \hat{\beta}n(cy)'\big)}{\hat{\beta}^2(n+1)} \right). \end{aligned}$$

Thus

$$c\varphi(y') = \frac{2}{(1 - \hat{\beta}(cy)')^n} \left(y' + \frac{\hat{\lambda}\big((1 - \hat{\beta}(cy)')^{n+1} - (1 - \hat{\beta}(cy)') + \hat{\beta}n(cy)'\big)}{\hat{\beta}^2(n+1)} \right).$$

□

In this setting [36, Theorem B] by A. D. Hwang and M. A. Singer, reads:

Theorem 4.1.4. Let $\pi : (L, h) \to (M, \omega_M)$ be a polarized hermitian holomorphic line bundle over a complete Kähler–Einstein manifold (M, ω_M) such that $\mathrm{Ric}(h) = \beta\omega_M$, with $\beta < 0$. Then the metric $g_{\varphi,\beta}$ on the total space of L is a complete scalar flat Kähler metric. Moreover, the metric $g_{\varphi,\beta}$ is Ricci–flat if and only if $\rho_M = -\mathrm{Ric}(h)$.

Remark 4.1.5. For $\mathrm{Ric}(h) = 0$ we have local product metrics since they are bundle-adapted metrics on flat-bundles, see ([36], Remark 1.6).

4.2 Calabi's Criterion applied to $g_{\varphi,\beta}$

Let (M, ω_M) be a Kähler-Einstein manifold with Einstein constant λ, that is $\rho_M = \lambda\omega_M$. As described in Section 4.1, the momentum construction gives a 1-parameter family of scalar flat Kähler metrics $\omega_{\varphi,\beta}$ on the polarized line bundle (L, h) described by the Kähler potential:

$$\Psi(z, \xi) = \Phi(z) + 4f\left(\frac{1}{2}\log[|\xi|^2 h(z)]\right), \tag{4.4}$$

where we can take as Φ the diastasis function for ω_M, centred at $z = 0$. We now describe the diastasis function for the metrics $g_{\varphi,\beta}$ and give a necessary condition for these metrics to be projectively induced, which follows directly by applying the Calabi's criterion 1.2.7 to them.

By (1.2.3), the diastasis function associated to $\omega_{\varphi,\beta}$, centred at $p = (0, s)$ with $s \in \mathbb{R}^+$ is:

$$\begin{aligned} D(z,\xi)|_p = \Phi(z) + 4f\left(\frac{1}{2}\log\left(|\xi|^2 h(z)\right)\right) + 4f\left(\frac{1}{2}\log s^2\right) + \\ - 4f\left(\frac{1}{2}\log(\xi s)\right) - 4f\left(\frac{1}{2}\log(\bar{\xi} s)\right), \end{aligned} \tag{4.5}$$

where we set $h(0) = 1$.

In particular, for the fibre metric we have:

$$D_p(\xi)|_{\text{fibre}} = 4f\left(\frac{1}{2}\log\left(|\xi|^2\right)\right) + 4f\left(\frac{1}{2}\log s^2\right) - 4f\left(\frac{1}{2}\log(\xi s)\right) - 4f\left(\frac{1}{2}\log(\bar{\xi} s)\right). \tag{4.6}$$

Proposition 4.2.1. *Let $c > 0$ be a positive real number. Then the metric $c\,\omega_{\varphi,\beta} = \omega_{\hat{\varphi},\hat{\beta}}$, where $\hat{\varphi}$ is the profile function defined by (4.2) with parameters $\hat{\beta} := \beta/c$ and $\hat{\lambda} := \lambda/c$.*

Proof. Observe that

$$c\omega_{\varphi,\beta} = c\omega_M + 2i\partial\bar{\partial} cf(t),$$

and $c\omega_M$ is a Kähler–Einstein metric with Einstein constant $\frac{\lambda}{c}$. Conclusion follows since by Proposition 4.1.3 $\hat{f} = cf$ is a solution to $y'' = \hat{\varphi}(y')$. ☐

Remark 4.2.2. We note that if g is a scalar flat projectively induced Kähler metric, then its (scalar flat) multiples cg may not be so. If the base manifold is Kähler–Einstein with Einstein constant λ, we find a close connection between the parameters c and λ (as in the previous proposition). Namely, it turns out that varying the parameter c over the positive real line, corresponds to construct the Hwang–Singer metrics on the same line bundle over a rescaled Kähler-Einstein manifold with Einstein constant $\frac{\lambda}{c}$. Thus it is equivalent to study the metric $c\omega_\varphi$ as c varies and the metric ω_φ as λ varies in the base manifold.

Lemma 4.2.3. *In the notation above, a necessary condition for the metric $\omega_{\varphi,\beta}$ to be projectively induced is that:*

$$n\,(\lambda + 2\beta) \geq -4. \tag{4.7}$$

Proof. By Calabi's Criterion Theorem 1.2.7, since $\frac{\partial^4(e^{D_p}-1)}{\partial\xi^2\bar{\partial}\bar{\xi}^2}|_p$ is an element on the diagonal of the matrix (b_{jk}) in (1.3), a necessary condition for the metric $\omega_{\varphi,\beta}$ to be projectively induced is that:

$$\begin{aligned} \frac{\partial^4(e^{D_p}-1)}{\partial\xi^2\bar{\partial}\bar{\xi}^2}|_p = \frac{1}{s^4}\left(\frac{1}{4}f^{(4)}\left(\frac{\log s^2}{2}\right) - f^{(3)}\left(\frac{\log s^2}{2}\right) + \right. \\ \left. + 2f''\left(\frac{\log s^2}{2}\right)^2 + f''\left(\frac{\log s^2}{2}\right)\right) \geq 0. \end{aligned} \tag{4.8}$$

By (4.3) and since $\varphi(\tau) > 0$ for every $\tau \in \mathbb{R}^+$, (4.8) is equivalent to:

$$4 + 8\varphi(\mu_0) - 4\varphi'(\mu_0) + \varphi'(\mu_0)^2 + \varphi(\mu_0)\varphi''(\mu_0) \geq 0,$$

where $\mu_0 := f'\left(\frac{\log s^2}{2}\right)$, i.e.:

$$(2 - \varphi'(\mu_0))^2 + \varphi(\mu_0)\left(8 + \varphi''(\mu_0)\right) \geq 0,$$

that is:

$$\varphi''(\mu_0) \geq -8 - \frac{(2 - \varphi'(\mu_0))^2}{\varphi(\mu_0)}. \tag{4.9}$$

By the definition of φ (4.2),

$$\varphi'(\mu_0) = \frac{2\left((n+1)\beta + \lambda\left(1 - (1 - \beta\mu_0)^n\right) + \left((\beta^2+1)(n^2-1) + \lambda((1-\beta\mu_0)^n)\right)\mu_0\right)}{(n+1)\beta(1-\beta\mu_0)^{n+1}},$$

$$\varphi''(\mu_0) = \frac{2n}{(1-\beta\mu_0)^{n+2}}\left(\lambda + 2\beta + (n-1)\beta(\beta+\lambda)\mu_0\right).$$

Since we can choose $s > 0$ arbitrarily small, then (4.9) must hold for $\mu_0 \to 0$ (see Remark 4.1.1). It is not hard to see that as $\mu_0 \to 0$,

$$\frac{(2 - \varphi'(\mu_0))^2}{\varphi(\mu_0)} \to 0,$$

and

$$\varphi''(\mu_0) \to 2n\left(\lambda + 2\beta\right).$$

Thus (4.9) implies

$$n\left(\lambda + 2\beta\right) \geq -4,$$

as wished. □

Remark 4.2.4. Considering the j-th derivatives $\frac{\partial^{2j}(e^{D_p}-1)}{\partial\xi^j\partial\bar{\xi}^j}|_p$, we get sharper necessary conditions for the metric $g_{\varphi,\beta}$ to be projectively induced. Although, such conditions will always depend on the choice of β and λ.

Remark 4.2.5. When $\beta = -\lambda$, the metric $g_{\varphi,\beta}$ is Ricci-flat. In this case condition (4.7) gives that $g_{\varphi,\beta}$ is not projectively induced for any $\lambda > \frac{4}{n}$. This estimate can be improved to $\lambda \geq 1$ also for $n = 2$, 3, and 4, by computing the 4-th derivative $\frac{\partial^8(e^{D_p}-1)}{\partial\xi^4\partial\bar{\xi}^4}|_p$, evaluated at $\mu_0 = \frac{1}{100\lambda}$. Further, observe that when $n = 1$, $g_{\varphi,\beta}$ is the Eguchi–Hanson metric on $\mathbb{C}\mathrm{P}^1$, which has been proven to be not projectively induced in [51]. As before, observe that such condition can be improved considering higher derivatives but will always depend on the choice of λ, as in the above remark.

4.3 Hwang–Singer metrics on $\mathbb{C}^{n+1}$

Let $(M, \omega_M) = (\mathbb{C}^n, \omega_0)$, where ω_0 is the canonical flat metric, i.e. $\omega_0 = \frac{i}{2}\partial\bar{\partial}||z||^2$. The momentum construction in this case gives a 1-parameter family of scalar flat Kähler metrics $\omega_{\varphi,\beta}$ on $\mathbb{C}^{n+1}$ described by the Kähler potential (see 4.4):

$$\Phi(z,\xi) := ||z||^2 + 4f\left(\frac{1}{2}\log\left[|\xi|^2 e^{-\frac{\beta}{2}||z||^2}\right]\right), \tag{4.10}$$

obtained setting $h(z) = e^{-\frac{\beta}{2}||z||^2}$ in (4.4), for $\beta < 0$, so that $\mathrm{Ric}(h) = -i\partial\bar\partial\log h(z) = \beta\frac{i}{2}\partial\bar\partial||z||^2 = \beta\omega_M$ and by (4.5) the diastasis function for $\omega_{\varphi,\beta}$ centred at $(z,\xi) = (0,s)$ reads:

$$\begin{aligned} D_{(s,0)}(z,\xi) = ||z||^2 + 4f\left(\frac{1}{2}\log\left[|\xi|^2 e^{-\frac{\beta}{2}||z||^2}\right]\right) + 4f\left(\frac{1}{2}\log s^2\right) + \\ - 4f\left(\frac{1}{2}\log\left[\xi s\right]\right) - 4f\left(\frac{1}{2}\log\left[\bar\xi s\right]\right). \end{aligned} \tag{4.11}$$

The profile function, obtained setting $\lambda = 0$ in (4.2) is given by:

$$\varphi(\tau) = \frac{2\tau}{(1-\beta\tau)^n}, \tag{4.12}$$

for $\tau \in [0, +\infty)$.

In order to prove the first part of Theorem 4.0.1, i.e. that $(\mathbb{C}^{n+1}, c\omega_{\varphi,\beta})$ is not projectively induced for any c and β, let us first show how to drop the dependence on the parameters β and c.

Lemma 4.3.1. Up to an affine change of coordinates on $\mathbb{C}^n$, the metric $c\omega_{\varphi,\beta}$ on $\mathbb{C}^{n+1}$ is equivalent to $\omega_{\varphi,-1}$.

Proof. Let us first deal with β. The metric $\omega_{\varphi,-1}$ is obtained by a momentum construction on $(\mathbb{C}^n, \omega_0)$ with profile $\varphi(\tau) = \frac{2\tau}{(1+\tau)^n}$. Perform a change of coordinates on $\mathbb{C}^n$ by setting $z' = \frac{1}{\sqrt{-\beta}}z$. Then:

$$\omega_0 = \frac{i}{2}\partial\bar\partial||z||^2 = -\beta\frac{i}{2}\partial\bar\partial||z'||^2,$$

Observe that while in the z coordinates $\mathrm{Ric}(h) = -\omega_0$, in the coordinates z', $\mathrm{Ric}(h) = \beta\omega_0$. The determinant of the endomorphism $\mathrm{Id} - \tau\mathrm{B}$ (see Section 4.1 after formula (4.2)), that in the z coordinates was $(1+\tau)^n$, now in z' reads $(1-\beta\tau)^n$. Thus the change of coordinates, transforms the metric $\omega_{\varphi,-1}$ on $\mathbb{C}^{n+1}$ in the metric $\omega_{\varphi,\beta}$.

Let us now prove that the multiplication of $\omega_{\varphi,-1}$ by $c > 0$ is equivalent to consider $\omega_{\varphi,-\frac{1}{c}}$. By (4.10) a Kähler potential for $c\omega_{\varphi,-1}$ is given by

$$c\Phi(z,\xi) = c||z||^2 + 4cf\left(\frac{1}{2}\log\left[|\xi|^2 e^{\frac{1}{2}||z||^2}\right]\right).$$

Performing a change of coordinates $z' = \sqrt{c}z$ we get

$$c\Phi(z',\xi) = ||z'||^2 + 4cf\left(\frac{1}{2}\log\left[|\xi|^2 e^{\frac{1}{2c}||z'||^2}\right]\right).$$

Conclusion follows observing that by Proposition 4.1.3, if f satisfies the ODE given by $\varphi(\tau) = \frac{2\tau}{(1+\tau)^n}$, then cf satisfies the ODE given by $\varphi(\tau) = \frac{2\tau}{(1+\frac{1}{c}\tau)^n}$. □

4.4 Hwang–Singer metrics on line bundles over $\mathbb{C}P^1$

Let L be a holomorphic line bundle over $\mathbb{C}P^1$ endowed with the Fubini-Study metric normalized so that $\lambda = 1$, that is, in affine coordinates $z = \frac{Z_1}{\frac{1}{2}Z_0}$ on $U_0 = \{Z_0 \neq 0\}$

$$\omega_{\mathrm{FS}} = \frac{i}{2}\partial\overline{\partial}\, 4\log(1 + \frac{1}{4}|z|^2).$$

Since L is a holomorphic line bundle over $\mathbb{C}P^1$, then L is of the form $\mathcal{O}(-k)$, $k \in \mathbb{Z}$.

The natural hermitian metric on the line bundle $\mathcal{O}(-1)$ on $\mathbb{C}P^1$ (see Example 2.1.5) is given by restricting the hermitian metric of $\mathbb{C}^2$ to each fiber $l = L_x \subset \mathbb{C}^2$. So if $\{U_0, U_1\}$ is a cover of $\mathbb{C}P^1$ with

$$U_\alpha = \{Z_\alpha \neq 0\} \quad , \quad \alpha = 0, 1,$$

then

$$h_\alpha = \frac{|Z_0|^2 + |Z_1|^2}{|Z_\alpha|^2} \quad , \quad \alpha = 0, 1.$$

So, on each open set U_α, if we take z as local coordinate, we have

$$h(z) = 1 + \frac{1}{4}|z|^2.$$

For $k > 0$, the line bundles $\mathcal{O}(-k) := \mathcal{O}(k)^* = \mathcal{O}(1)^* \otimes \cdots \otimes \mathcal{O}(1)^*$ inherit natural hermitian structures given by

$$h_k(z) = \left(1 + \frac{1}{4}|z|^2\right)^k.$$

The curvature form is then

$$\mathrm{Ric}(h_k) = -i\partial\overline{\partial}\log\left(1 + \frac{1}{4}|z|^2\right)^k = -\frac{k}{2}\omega_{\mathrm{FS}}$$

and the line bundle is polarized, with $\lambda = 1$ and $\beta = -\frac{k}{2}$, with k a positive integer.

So the profile 4.2 reads

$$\varphi_k(\tau) = \frac{2\tau + \tau^2}{1 + \frac{k}{2}\tau},$$

and the momentum construction gives a 1-parameter family $\omega_k := \omega_{\varphi, -\frac{k}{2}}$ of scalar flat Kähler metrics on the polarized line bundle $\mathcal{O}(-k)$ described by the potentials

$$\Psi(z, \xi) = 4\log\left(1 + \frac{1}{4}|z|^2\right) + 4f\left(\frac{1}{2}\log\left[|\xi|^2\left(1 + \frac{1}{4}|z|^2\right)^k\right]\right). \tag{4.13}$$

Remark 4.4.1. For $k = 0$, the metric ω_k reduces to the local product metric on $\mathbb{C}P^1 \times \mathbb{C}$, see Remark 4.1.5.

In the proof of Theorem 4.0.2, we need the following lemma.

Lemma 4.4.2. The metric ω_k on $\mathbb{C}\mathrm{P}^1$ is not projectively induced for any $k \geq 3$.

Proof. Let $p \in \mathcal{O}(-k)$ be the point of coordinates $(s, 0)$ and let D_p be the diastasis function for the metric ω_k as in (4.6). The fourth derivative of $e^{D_p} - 1$ evaluated at p is given by:

$$\begin{aligned}\frac{\partial^8(e^{D_p}-1)}{\partial\xi^4\bar{\partial}\xi^4}|_p =& \frac{1}{s^8}\Bigg(+ 24f''^4 + 216f''^3 + f'''(3f^{(5)} - 45f^{(4)} - 66) + \\ & + (f^{(6)} - 18f^{(5)} + 125f^{(4)} + 36f'''^2 - 396f''' + 36)f'' + 114f'''^2 \\ & + \frac{1}{64}(f^{(8)} - 24f^{(7)} + 232f^{(6)} - 1152f^{(5)} + 136(f^{(4)})^2 + 3088f^{(4)}) + \\ & + (18f^{(4)} - 216f''' + 242)f''^2\Bigg)\left(\frac{\log s^2}{2}\right),\end{aligned}$$

that written in terms of $\varphi(\mu_0)$ with $\mu_0 = f'\left(\frac{\log s^2}{2}\right)$, up to the multiplication by the positive constant $\frac{1}{s^8}$, reads $\frac{1}{64}\varphi(\mu_0)A(\varphi(\mu_0))$, with (to simplify the notation we drop the dependence from μ_0 in $\varphi(\mu_0)$ and its derivatives):

$$\begin{aligned}&A(\varphi(\mu_0)) = \varphi^{(6)}\varphi^5 + ((\varphi')^3 - 12(\varphi')^2 + 44\varphi' - 48)^2 + \varphi(\varphi') - 2)(-8(193\varphi'' + 968) + \\ &(\varphi')^3(57\varphi'' + 392) - 2(\varphi'))^2(255\varphi'' + 1624) + 4\varphi'(383\varphi'' + 2200)) + 2\varphi^2(16(-36\varphi^{(3)} \\ &+ 29(\varphi'')^2 + 250\varphi'' + 432) + 61\varphi^{(3)}(\varphi')^3 + 2(\varphi')^2(96(9 - 2\varphi^{(3)} + 45(\varphi'')^2 + 436\varphi'') + \\ &- 4\varphi'(-203\varphi^{(3)} + 102(\varphi'')^2 + 936\varphi'' + 1728)) + 2\varphi^3(17(\varphi'')^3 + 196(\varphi'')^2 \\ &+ 2\varphi^{(4)}(19(\varphi')^2 - 66\varphi' + 58) + 64\varphi^{(3)}(5\varphi' - 9) + 12(\varphi^{(3)}(8\varphi' - 15) + 48)\varphi'' + 768) + \\ &\varphi^4(15(\varphi^{(3)})^2 + 8\varphi^{(5)}(2\varphi' - 3) + \varphi^{(4)}(26\varphi'' + 64)),\end{aligned}$$

since $\varphi(\mu_0)$ is positive, the sign of $\frac{\partial^8(e^{D_p}-1)}{\partial\xi^4\bar{\partial}\xi^4}|_p$ is the same as that of $A(\varphi(\mu_0))$. From (4.2), we get:

$$\varphi_k^{(j)}(\tau) = (-1)^{j+1}\frac{8j!(k-1)k^{j-2}}{(2+k\tau)^{j+1}},$$

that substituted into the expression of $A(\varphi(\mu_0))$ gives:

$$\frac{\mu_0^3}{2^7 3(2+k\mu_0)^{12}}P_k(\mu_0),$$

where $P_k(\mu_0)$ is the polynomial in μ_0:

$$P_k(\mu_0) = 105 - 113k + 48k^2 - 8k^3 + \sum_{s=1}^{12} q_s(k)\mu_0^s,$$

for given $q_s(k)$ that are not relevant for our analysis. Since μ_0 can be chosen small enough in $[0, +\infty)$ taking $s \to 0$, the sign of $A(\varphi(\mu_0))$ is the same as the sign of $P_k(\mu_0)$ for positive values of μ_0. Conclusion follows by noticing that

$$\lim_{\mu_0 \to 0} P_k(\mu_0) = 105 - 113k + 48k^2 - 8k^3,$$

and the right hand side is negative for any $k \geq 3$. ☐

4.5 Asymptotic expansion of $\omega_{\varphi,\beta}$ and proofs of Theorem 4.0.1 and Theorem 4.0.2

Throughout this section let us write $(X, \omega_{\varphi,\beta})$ for either $X = \mathbb{C}^{n+1}$ or $X = \mathcal{O}(-k)$. Let $\hat{L}$ be a holomorphic line bundle over X and let $h_{\hat{L}}$ be an hermitian metric on $\hat{L}$ such that $\mathrm{Ric}(\hat{h}) = \omega_{\varphi,\beta}$. Notice that such an $(\hat{L}, \hat{h})$ exists if and only if $\omega_{\varphi,\beta}$ is integral. In our case, this occurs since the base metric ω_M is an integral form and:

$$[\omega_{\varphi,\beta}] = [\pi^*\omega_M + 2i\partial\bar{\partial} f(t)] = [\pi^*\omega_M],$$

where $\pi : L \to M$ is the projection given in Section 4.1 (here $M = \mathbb{C}^n$ or $\mathbb{C}\mathrm{P}^1$). Consider the tensor power $(\hat{L}^\alpha, \hat{h}_\alpha)$ and let $\mathcal{H}_\alpha$ be the space of global holomorphic sections of $\hat{L}^\alpha$. In order to define the ϵ-function for $(X, \omega_{\varphi,\beta})$ we first need to show that $\mathcal{H}_\alpha \neq \{0\}$.

Lemma 4.5.1. In the notation above, $1 \in \mathcal{H}_\alpha$ for either $X = (\mathbb{C}^{n+1}, \omega_{\varphi,\beta})$ or $X = (\mathcal{O}(-k), \omega_k))$.

Proof. Observe that by formula (2.21) in [36] we have:

$$\frac{\omega_{\varphi,\beta}^{n+1}}{(n+1)!} = \varphi Q \det(g_M) \frac{1}{|\xi|^2} \left(\frac{i}{2}\right)^{n+1} d\xi \wedge d\bar{\xi} \prod_{j=1}^{n} dz_j \wedge d\bar{z}_j.$$

where Q is the determinant of the endomorphism $\mathrm{Id} - \tau \mathrm{B}$, as after equation (4.2) of the profile .

Let us deal first with the case $X = \mathbb{C}^{n+1}$. In this case, $\mathcal{H}_\alpha$ is the weighted Hilbert space of global holomorphic functions over $\mathbb{C}^{n+1}$ that are L^2 limited in norm, namely:

$$\mathcal{H}_\alpha = \left\{ u \in \mathrm{Hol}(\mathbb{C}^{n+1}) | \int_{\mathbb{C}^{n+1}} |u|^2 e^{-\alpha\Phi} \frac{\omega_{\varphi,\beta}^{n+1}}{(n+1)!} < +\infty \right\},$$

where Φ is given by (4.10).

Due to Lemma 4.3.1, we can set $\beta = -1$. In order to prove that $1 \in \mathcal{H}_\alpha$, it is enough to check the convergence of the integral:

$$\begin{aligned} &\int_{\mathbb{C}^{n+1}} e^{-\alpha(||z||^2 + 4f(t)))} \frac{2f'(t)}{|\xi|^2} \left(\frac{i}{2}\right)^{n+1} d\xi \wedge d\bar{\xi} \prod_{j=1}^{n} dz_j \wedge d\bar{z}_j \\ =&\pi^{n+1} \int_0^\infty \cdots \int_0^\infty \int_0^\infty e^{-\alpha(\sum_j r_j + 4f(\hat{t})))} \frac{2f'(\hat{t})}{r_0} dr_0 \prod_{j=1}^{n} dr_j, \end{aligned} \tag{4.14}$$

where we set polar coordinates $\xi := \rho_0 e^{i\theta_0}$, $z_j := \rho_j e^{i\theta_j}$ and $r_j := \rho_j^2$, $j = 0, \dots, n$, and we denote $\hat{t} = \frac{1}{2}(\log r_0 + \frac{1}{2}\sum_j r_j)$. The function under the integral is positive and smooth, since $\frac{\varphi(t)}{|\xi|^2} \to g_{0\bar{0}}$ as $|\xi|^2 \to 0$, so its integral converges inside any closed ball of ray $R > 0$ centered at the origin. Thus, it is enough to check that the integral outside the ball is finite. Using that the function under the integral is positive and that:

$$e^{-\alpha(\sum_j r_j + 4f(\hat{t})))} \frac{2f'(\hat{t})}{r_0} \leq e^{-\alpha(4f(\hat{t})))} \frac{2f'(\hat{t})}{r_0} = -\frac{1}{\alpha} \frac{d}{dr_0} e^{-4\alpha f(\hat{t})},$$

we have:

$$\begin{aligned} &\int_R^\infty \cdots \int_R^\infty \int_R^\infty e^{-\alpha(\sum_j r_j+4f(\hat{t})))} \frac{2f'(\hat{t})}{r_0} dr_0 \prod_{j=1}^n dr_j \\ &\leq -\frac{1}{\alpha} \int_R^\infty \cdots \int_R^\infty \int_R^\infty \frac{d}{dr_0} e^{-4\alpha f(\hat{t})} dr_0 \prod_{j=1}^n dr_j \\ &= \frac{1}{\alpha} \int_R^\infty \cdots \int_R^\infty e^{-4\alpha f(\frac{1}{2}\log R+\frac{1}{4}\sum_j r_j)} \prod_{j=1}^n dr_j . \end{aligned} \tag{4.15}$$

The last integral in (4.15) converges at least for $\alpha > n/4$ and for a large enough R, since:

$$e^{-4\alpha f(\frac{1}{2}\log R+\frac{1}{4}\sum_j r_j))} \leq \frac{1}{\left(\frac{1}{4}\right)^n r_1^{4\alpha/n} \cdots r_n^{4\alpha/n}}.$$

More precisely, since f is an increasing function and $\frac{\partial^2}{\partial r_j^2} f = \frac{1}{4} f'' > 0$, there exists $R \in \mathbb{R}$ such that for $r_j > R$, $j = 1, \ldots, n$,

$$f\left(\frac{1}{2}\log R + \frac{1}{4}\sum_j r_j\right) \geq f\left(\frac{1}{4} r_j\right) \geq \log\left(\frac{1}{4} r_j\right),$$

thus

$$f\left(\frac{1}{2}\log R + \frac{1}{4}\sum_j r_j\right) \geq \frac{1}{n}\sum_{j=1}^n \log\left(\frac{1}{4} r_j\right).$$

Let us now deal with $X = \mathcal{O}(-k)$ over $\mathbb{CP}^1$. In this case it is enough to check the convergence of the following integral over the chart $\mathcal{U}_0 \times \mathbb{C} \simeq \mathbb{C}^2$:

$$\begin{aligned} &\int_{\mathbb{C}^2} \frac{e^{-4\alpha f(t)}}{(1+\frac{1}{4}|z|^2)^{4\alpha+2}} \frac{2f'(t)+f'(t)^2}{|\xi|^2} \left(\frac{i}{2}\right)^2 d\xi \wedge d\bar{\xi} \wedge dz \wedge d\bar{z} \\ &=\pi^2 \int_0^\infty \int_0^\infty \frac{e^{-4\alpha f(\hat{t})}}{(1+\frac{1}{4}r_1)^{4\alpha+2}} \frac{2f'(\hat{t})+f'(\hat{t})^2}{r_0} dr_0 dr_1, \end{aligned}$$

where we set polar coordinates $\xi = \rho_0 e^{i\theta_0}$, $z_1 = \rho_1 e^{i\theta_1}$, $r_j := \rho_j^2$, $j = 0, 1$, and set $\hat{t} := \frac{1}{2}\log r_0 + \frac{k}{2}\log(1+\frac{1}{4}r_1)$. As before, since the function we are integrating is smooth on any closed ball of ray $R > 0$ (since $\frac{\varphi(t)}{|\xi|^2} \to g_{0\bar{0}}$ as $|\xi|^2 \to 0$), we reduce to check that the integral converges outside such ball. First observe that

$$I_1 := \int_R^\infty -e^{-4\alpha f(\hat{t}(r_0))} \frac{f'(\hat{t}(r_0))}{r_0} dr_0 = \frac{1}{2\alpha} \int_R^\infty \frac{d}{dr_0} e^{-4\alpha f} = \frac{1}{2\alpha} \left[e^{-4\alpha f}\right]_R^\infty < \infty$$

and

$$\begin{aligned} I_2 &:= \int_R^\infty \int_R^\infty \frac{e^{-4\alpha f(\hat{t})}}{(1+\frac{1}{4}r_1)^2} \frac{2f'(\hat{t})}{r_0} dr_0 dr_1 = -\frac{1}{\alpha} \int_R^\infty \int_R^\infty \frac{1}{(1+\frac{1}{4}r_1)^2} \frac{d}{dr_0} e^{-4\alpha f} dr_0 dr_1 \\ &= -\frac{1}{\alpha} \int_R^\infty \frac{1}{(1+\frac{1}{4}r_1)^2} \left[e^{-4\alpha f}\right]_R^\infty dr_1 = \frac{1}{\alpha} \int_R^\infty \frac{e^{-4\alpha f(\hat{t}(r_1))}}{(1+\frac{1}{4}r_1)^2} dr_1 < \infty. \end{aligned}$$

So, since $\alpha > 0$, we have

$$\begin{aligned}\int_R^\infty \int_R^\infty \frac{e^{-4\alpha f(\hat{t})}}{(1+\frac{1}{4}r_1)^{4\alpha+2}} \frac{2f'(\hat{t})+f'(\hat{t})^2}{r_0} dr_0 dr_1 &= \int_R^\infty \int_R^\infty \frac{e^{-4\alpha f(\hat{t})}}{(1+\frac{1}{4}r_1)^{4\alpha+2}} \frac{2f'(\hat{t})}{r_0} dr_0 dr_1 + \\ &\quad + \int_R^\infty \int_R^\infty \frac{e^{-4\alpha f(\hat{t})}}{(1+\frac{1}{4}r_1)^{4\alpha+2}} \frac{f'(\hat{t})^2}{r_0} dr_0 dr_1 \\ &\leq I_2 + \int_R^\infty \int_R^\infty \frac{e^{-4\alpha f(\hat{t})}}{(1+\frac{1}{4}r_1)^{2}} \frac{f'(\hat{t})^2}{r_0} dr_0 dr_1 .\end{aligned}$$

It remains to check that:

$$I := \int_R^\infty \int_R^\infty \frac{e^{-4\alpha f(\hat{t})}}{(1+\frac{1}{4}r_1)^{2}} \frac{f'(\hat{t})^2}{r_0} dr_0 dr_1$$

converges. Integrating by parts, since:

$$\frac{e^{-4\alpha f(\hat{t})}}{(1+\frac{1}{4}r_1)^{2}} \frac{f'(\hat{t})^2}{r_0} = -\frac{2}{k\alpha} \frac{d}{dr_1} e^{-4\alpha f(\hat{t})} \frac{f'(\hat{t})}{r_0(1+\frac{1}{4}r_1)} .$$

we get:

$$\begin{aligned} I &= -\frac{2}{k\alpha} \int_R^\infty \int_R^\infty \frac{d}{dr_1} e^{-4\alpha f(\hat{t})} \frac{f'(\hat{t})}{r_0(1+\frac{1}{4}r_1)} dr_0 dr_1 \\ &= -\frac{2}{k\alpha} \left\{ \int_R^\infty \left[e^{-4\alpha f} \frac{f'}{r_0(1+\frac{1}{4}r_1)} \Big|_R^\infty - \frac{1}{r_0} \int_R^\infty e^{-4\alpha f} \frac{f'' \frac{k}{8} - \frac{f'}{4}}{(1+\frac{1}{4}r_1)^2} dr_1 \right] dr_0 \right\} \\ &= -\frac{2}{k\alpha} \left\{ \frac{I_1}{(1+\frac{1}{4}R)} - \frac{k}{8} \int_R^\infty \int_R^\infty \frac{e^{-4\alpha f} f''}{r_0(1+\frac{1}{4}r_1)^2} dr_1 dr_0 + \frac{1}{8} \int_R^\infty \int_R^\infty \frac{2f' e^{-4\alpha f}}{r_0(1+\frac{1}{4}r_1)^2} dr_1 dr_0 \right\} \\ &= -\frac{2}{k\alpha} \left\{ \frac{I_1}{(1+\frac{1}{4}R)} - \frac{k}{8} \int_R^\infty \int_R^\infty \frac{e^{-4\alpha f}(2f' + (f')^2)}{r_0(1+\frac{1}{4}r_1)^2(1+\frac{k}{2}f')} dr_1 dr_0 + \frac{I_2}{8} \right\} \\ &\leq -\frac{2}{k\alpha} \left\{ \frac{I_1}{(1+\frac{1}{4}R)} - \frac{k}{8} \int_R^\infty \int_R^\infty \frac{e^{-4\alpha f}(2f' + (f')^2)}{r_0(1+\frac{1}{4}r_1)^2} dr_1 dr_0 + \frac{I_2}{8} \right\} \\ &= -\frac{2}{k\alpha} \left\{ \frac{I_1}{(1+\frac{1}{4}R)} - \frac{k}{8} I_2 - \frac{k}{8} I + \frac{I_2}{8} \right\} \end{aligned} \tag{4.16}$$

where in the second equality we used that

$$\lim_{r_1 \to +\infty} \frac{f' e^{-4\alpha f}}{(1+\frac{1}{4}r_1)} = 0$$

as follows by applying de l'Hopital and using that $f'' = \frac{2f'+f'^2}{1+\frac{k}{2}f'}$. Further the inequality follows by $(1+\frac{k}{2}f') > 1$, since f' is a positive function. From (4.16) we obtain:

$$\left(1 - \frac{1}{4\alpha}\right) I \leq C$$

for a suitable constant $C \in \mathbb{R}$. In particular I converges at least for $\alpha > \frac{1}{4}$. □

In the following theorem we prove that conditions (2.6) hold for Hwang–Singer metrics based on a Kähler–Einstein polarized manifold. Recall that from [36] Section 2, the Ricci form ρ_φ of $\omega_{\varphi,\beta}$ is given by

$$\rho_\varphi = \pi^*\rho_M + \frac{1}{2Q}(\varphi Q)'(\tau)\pi^*(\mathrm{Ric}(h)) - \frac{1}{2\varphi}[\frac{1}{Q}(\varphi Q)']'d\tau \wedge d^c\tau,$$

thus, when ω_M is polarized we have:

$$\begin{aligned}\rho_\varphi &= \pi^*(\lambda\omega_M) + \frac{1}{2Q}(\varphi Q)'(\tau)\pi^*(\beta\omega_M) - \frac{1}{2\varphi}[\frac{1}{Q}(\varphi Q)']'d\tau \wedge d^c\tau \\ &= \big(\lambda + \frac{\beta}{2Q}(\varphi Q)'\big)\pi^*\omega_M - \frac{1}{2\varphi}\big[\frac{1}{Q}(\varphi Q)'\big]'d\tau \wedge d^c\tau.\end{aligned} \tag{4.17}$$

Theorem 4.5.2. Let $\omega_{\varphi,\beta}$ be the the Hwang–Singer metric on a polarized line bundle over a Kähler–Einstein manifold with integral Kähler form. Then, if $\mathcal{H}_\alpha \neq \{0\}$, the Engliš expansion of the function $\epsilon_{\alpha g_{\varphi,\beta}}$ exists and the coefficients a_j are given by 2.3.

Proof. Let us check that conditions (2.6) hold for $\omega_{\varphi,\beta}$. The first condition is satisfied for $l \in (0,1)$ since

$$iR^{\hat{L}} = -i\partial\overline{\partial}\log\hat{h} = -i\partial\overline{\partial}\log e^{-\frac{1}{2}\Psi} = \omega_{\varphi,\beta}\ ,$$

while the third condition is satisfied for every positive $c > 0$, since $\partial\omega_{\varphi,\beta} = 0$, being the metric Kähler. Let us now deal with the second condition. We want to show that there exists a positive $c > 0$ such that the form given by

$$iR^{\det} + c\omega_{\varphi,\beta} = \rho_\varphi + c\omega_{\varphi,\beta}$$

is positive. Thus, using 4.17 and 4.1, it is sufficient to show that there exists $c > 0$ such that

$$\left(\lambda + \frac{\beta}{2Q}(\varphi Q)' + c(1-\tau\beta)\right)\omega_M + \left(-\frac{1}{2\varphi}\big[\frac{1}{Q}(\varphi Q)'\big]' + \frac{c}{\varphi}\right)d\tau \wedge d^c\tau > 0.$$

Being $\lambda \geq 0$ and $\varphi > 0$, we show that there exists $c > 0$ such that

$$\begin{cases}\frac{\beta}{2Q}(\varphi Q)' + c(1-\tau\beta) > 0 \\ -\frac{1}{2}\big[\frac{1}{Q}(\varphi Q)'\big]' + c \geq 0\ ,\end{cases}$$

namely, we want a positive c that satisfies

$$\begin{cases}c > -\frac{\beta}{2Q}(\varphi Q)'\frac{1}{1-\tau\beta} \\ c \geq \frac{1}{2}\big(\frac{1}{Q}(\varphi Q)'\big)'\ .\end{cases}$$

Since $\frac{1}{1-\tau\beta} \leq 1$, we reduce to prove that

$$\frac{(\varphi Q)'}{Q}\quad ,\quad \left(\frac{(\varphi Q)'}{Q}\right)'$$

are limited functions, proving the existence of such a c. Using the expression of the profile function 4.2 and since $Q(\tau) = (1-\beta\tau)^n$, we get

$$\begin{aligned}\frac{(\varphi Q)'}{Q} &= \frac{2\lambda}{\beta(1-\beta\tau)^n} - \frac{2\lambda}{\beta} + \frac{2}{(1-\beta\tau)^n}\\ &< -\frac{2\lambda}{\beta} + \frac{2}{(1-\beta\tau)^n}\\ &< -\frac{2\lambda}{\beta} + 2,\end{aligned}$$

and

$$\left(\frac{(\varphi Q)'}{Q}\right)' = \frac{2n(\beta+\lambda)}{(1-\beta\tau)^{n+1}} \leq 2n(\beta+\lambda),$$

concluding the proof. ☐

From the existence of an asymptotic expansion of the ϵ-function it follows that the metric can be approximated by a sequence of projectively induced ones in the following way (cf. [50, Corollary 9]).

Lemma 4.5.3. Let (M, g) be a polarized Kähler manifold such that the $1 \in \mathcal{H}$, where $\mathcal{H}$ is the weighted Hilbert space of holomorphic functions on M limited in norm. Then the ϵ-function associated to g exists and, if it admits an asymptotic expansion whose coefficients are given by (2.3), then g can be approximated by a sequence of projectively induced Kähler metrics.

Proof. Denote by ω the Kähler form associated to g. Let $F_\alpha : M \to \mathbb{C}\mathrm{P}^{d_\alpha}$ be the coherent states map, i.e. $F_\alpha(x) = [\sigma_0(x) : \cdots : \sigma_j(x) : \ldots]$, where $\{\sigma_j\}_{j=0,1,\ldots}$ is an orthonormal basis of $\mathcal{H}$ such that $\sigma_0 \equiv 1$. Since $\mathcal{H} \neq \{0\}$, we can define the ϵ-function for g by (2.1), and we have:

$$F_\alpha^*\omega_{FS} = \alpha\omega + \frac{i}{2}\partial\bar{\partial}\log\epsilon_{\alpha g}.$$

By (2.2), since $a_0 = 1$, we have that $\lim_{\alpha\to\infty} \frac{1}{\alpha} F_\alpha^*\, g_{FS} = g$. ☐

Remark 4.5.4. Observe that the assumption $1 \in \mathcal{H}$ is needed to define the coherent states map. When M is a compact polarized Kähler manifold, the existence of F_α is guaranteed by Kodaira's Theorem. In the noncompact case one can always define the map F_α for example when g is regular, i.e. when $\epsilon_{\alpha g}$ is constant. In this case (2.1) implies that for each $x \in M$ there exists a nonvanishing $\sigma_j(x)$.

We are now in the position of proving Theorem 4.0.1.

Proof of Theorem 4.0.1. By Lemma 4.3.1 we can reduce ourselves to prove that $\omega_{\varphi,\beta}$ is not projectively induced for a given value of β. By Lemma 4.2.3, a necessary condition for the metric $\omega_{\varphi,\beta}$ on $\mathbb{C}^{n+1}$ to be projectively induced is that $\beta n \geq -2$. Thus, it is enough to set $\beta < -\frac{2}{n}$.

The second part follows by Lemma 4.5.1, Theorem 4.5.2 and Lemma 4.5.3. ☐

Let us now complete the proof of Theorem 4.0.2.

Proof of Theorem 4.0.2. By Lemma 4.4.2, ω_k is not projectively induced for any $k \geq 3$. For $k = 2$, ω_2 is the Eguchi-Hanson metric on the canonical line bundle $\mathcal{O}(-2)$, that is not projectively induced as shown by A. Loi, M. Zedda, F. Zuddas in [51]. For $k = 1$, ω_1 is the Burns-Simanca metric that is projectiveley induced as shown by F. Cannas Aghedu and A. Loi in [20]. The second part follows by Lemma 4.5.1, Theorem 4.5.2 and Lemma 4.5.3. Finally, a direct computation (see Section 4.6 below), gives:

$$a_2 = -\frac{48(k-1)\big(k^2\tau - 2k\tau - 2\big)}{(k\tau+2)^6},$$

that is identically zero if and only if $k = 1$, concluding the proof. ☐

Remark 4.5.5. In [20], F. Cannas Aghedu and A. Loi showed that the Simanca metric g_1 is projectively induced, and this implies that any of its integer multiples kg_1 also are. We note here that these are the only possible multiples that can be Kähler immersed in $\mathbb{C}\mathrm{P}^\infty$. In fact, by momentum construction, the Simanca metric on $\mathcal{O}(-1)$ arises as a metric on a line bundle over $\mathbb{C}\mathrm{P}^1$. In particular, $\mathbb{C}\mathrm{P}^1$ is a Kähler submanifold of $\mathcal{O}(-1)$ (obtained setting the fibre coordinate $\xi = 0$) and the Fubini–Study form is not integral when multiplied by a noninteger factor.

4.6 Computations of a_2

We compute here the a_2 coefficients for the metrics $\omega_{\varphi,\beta}$ in the case where the base manifold M is the complex projective line $\mathbb{C}\mathrm{P}^1$, completing the proofs of Theorem 4.0.2.

From (4.13), the metric g_k reads:

$$g_k = \begin{pmatrix} \frac{k^2|z|^2 f''(t)+8kf'(t)+16}{(|z|^2+4)^2} & \frac{kzf''(t)}{\xi(|z|^2+4)} \\ \frac{k\bar{z}f''(t)}{\bar{\xi}(|z|^2+4)} & \frac{f''(t)}{|\xi|^2} \end{pmatrix}.$$

It follows that

$$\det(g_k) = \frac{(1+\frac{k}{2}\tau)\varphi(\tau)}{|\xi|^2(1+\frac{1}{4}|z|^2)},$$

and

$$g_k^{-1} = \begin{pmatrix} \frac{\left(|z|^2+4\right)^2}{8(kf'(t)+2)} & -\frac{k\bar{\xi}z\left(|z|^2+4\right)}{8(kf'(t)+2)} \\ -\frac{k\bar{z}\xi\left(|z|^2+4\right)}{8(kf'(t)+2)} & \frac{|\xi|^2\left(k^2|z|^2f''(t)+8kf'(t)+16\right)}{8f''(t)(kf'(t)+2)} \end{pmatrix}.$$

The norms of the Riemann and Ricci tensors are

$$\begin{aligned} |R|^2 = \frac{1}{16}\bigg(&\frac{f^{(4)}(t)^2}{f''(t)^4} + \frac{f^{(3)}(t)^4}{f''(t)^6} - \frac{8(k^3 f^{(3)}(t) - 2kf'(t) - 4)}{(kf'(t)+2)^3} + \frac{8k^4 f''(t)^2}{(kf'(t)+2)^4} \\ &- \frac{16k^2 f''(t)}{(kf'(t)+2)^3} - \frac{2f^{(3)}(t)^2 f^{(4)}(t)}{f''(t)^5} + \frac{4k^2 f^{(3)}(t)^2}{f''(t)^2(kf'(t)+2)^2}\bigg), \end{aligned}$$

and

$$|\mathrm{Ric}|^2 = \frac{1}{16}\Bigg(\frac{16}{(kf'(t)+2)^2} + \frac{f^{(3)}(t)^4}{f''(t)^6} + \frac{2k^4 f''(t)^2}{(kf'(t)+2)^4} - \frac{8k^2 f''(t)}{(kf'(t)+2)^3} - \frac{2f^{(3)}(t)^2 f^{(4)}(t)}{f''(t)^5}$$
$$+ \frac{4k^2 f^{(3)}(t)^2}{f''(t)^2(kf'(t)+2)^2} + \frac{2kf^{(3)}(t)f^{(4)}(t)}{f''(t)^3(kf'(t)+2)} - \frac{2k(kf^{(4)}(t)+4f^{(3)}(t))}{f''(t)(kf'(t)+2)^2} + \frac{f^{(4)}(t)^2}{f''(t)^4} +$$
$$- \frac{2kf^{(3)}(t)^3}{(kf'(t)+2)f''(t)^4}\Bigg).$$

By (4.3) with $\varphi(\tau) = \frac{2\tau+\tau^2}{1+\frac{k}{2}\tau}$, the a_2 coefficient for the metrics ω_k on $\mathcal{O}(-k)$ is given by:

$$a_2 = -\frac{2(k-1)\left(k^2\tau - 2k\tau - 2\right)}{(k\tau+2)^6}. \tag{4.18}$$

Remark 4.6.1. A similar computation for the Hwang–Singer metric on $\mathbb{C}^{n+1}$ gives:

$$a_2(0,1) = \frac{\beta^2}{4(1-\beta\tau)^{2(n+2)}}\big(\beta^2 n^4\tau^2 + n(\beta^2 2^n\tau^2 + 2\beta(2^n+4)\tau + 2^n - 4) + 2^n(1-\beta^2\tau^2) +$$
$$+ \beta n^3\tau(\beta(2^n-2)\tau + 4) + \beta n^2\tau(\beta(2^n-3)\tau + 2\,(2^n+2))\big).$$

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