Automatica 169 (2024) 111864

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Is time-optimal speed planning under jerk constraints a convex problem?☆

Luca Consolini *, Marco Locatelli

Dipartimento di Ingegneria e Architettura, Università di Parma, Italy

ARTICLE INFO

Article history:
Received 11 October 2023
Received in revised form 25 March 2024
Accepted 19 June 2024
Available online 16 August 2024

Keywords:
Industrial applications of optimal control
Optimization
Planning
Convex relaxations
Hidden convexity

ABSTRACT

We consider the speed planning problem for a vehicle moving along an assigned trajectory, under maximum speed, tangential and lateral acceleration, and jerk constraints. The problem is a nonconvex one, where nonconvexity is due to jerk constraints. We propose a convex relaxation, and we present various theoretical properties. In particular, we show that the relaxation is exact under some assumptions. Also, we rewrite the relaxation as a Second Order Cone Programming (SOCP) problem. This has a relevant practical impact, since solvers for SOCP problems are quite efficient and allow solving large instances within tenths of a second. We performed many numerical tests, and in all of them the relaxation turned out to be exact. For this reason, we conjecture that the convex relaxation is *always* exact, although we could not give a formal proof of this fact.

1. Introduction

Consider the problem of computing a minimum-time motion of a car-like vehicle from a start configuration to a target one, while avoiding collisions (obstacle avoidance), and satisfying kinematic, dynamic, and mechanical constraints (for instance, on velocities, accelerations and maximal steering angle). It is common to solve this problem in two steps. First, we use a geometric path planner to find a suitable path. Then, we perform minimum-time speed planning on the planned path (see, for instance, Bianco, 2006; Frego, Bertolazzi, Biral, Fontanelli, & Palopoli, 2016; Hauser & Saccon, 2006; Kant & Zucker, 1986; Velenis & Tsiotras, 2008). Clearly, this approach is sub-optimal with respect to a single-step procedure, in which we plan the geometric path and the speed law at the same time. However, this choice considerably simplifies the problem. In this paper, we assume that the path that joins the initial and final configurations is assigned, and we aim at finding the time-optimal speed law that satisfies some kinematic and dynamic constraints. The problem can be reformulated as an optimization problem, and it is quite relevant from a practical point of view. In particular, in automated warehouses, the speed laws of Automated Guided Vehicles (AGVs) are typically planned under acceleration and jerk constraints.

In our previous work (Consolini, Locatelli, Minari, & Piazzi, 2017), we proposed an optimal time-complexity algorithm for finding the time-optimal speed law that satisfies constraints on maximum velocity and tangential and normal acceleration. In the subsequent work (Cabassi, Consolini, & Locatelli, 2018), we included a bound on the derivative of the acceleration with respect to the arc-length, which results in a convex optimization problem. Then, in Consolini, Locatelli, and Minari (2022), we considered the presence of jerk constraints (constraints on the time derivative of the acceleration). The resulting optimization problem is nonconvex and, for this reason, is significantly more complex than the ones we discussed in Cabassi et al. (2018) and Consolini et al. (2017). This work addresses the same problem as Consolini et al. (2022). Namely, we compute a time-optimal control law, taking into account constraints on maximum speed, tangential and lateral acceleration, and jerk. However, we use a completely different approach. Basically, we propose a convex relaxation of the original nonconvex problem. Then, we show that, under some assumptions, this relaxation is exact. We reformulate the relaxed problem as a second order cone programming (SOCP) problem. This allows solving the problem very efficiently with modern solvers.

2. Problem description

This section is adapted from Consolini et al. (2022). For a more detailed discussion, we refer the reader to this reference. Let $\boldsymbol{\gamma} : [0, s_f] \to \mathbb{R}^2$ be a smooth function. The image set $\boldsymbol{\gamma}([0, s_f])$ is the path to be followed, $\boldsymbol{\gamma}(0)$ the initial configuration, and $\boldsymbol{\gamma}(s_f)$ the final one. Function $\boldsymbol{\gamma}$ has arc-length parameterization, that

☆ The material in this paper was not presented at any conference This paper was recommended for publication in revised form by Associate Editor Andrey V. Savkin under the direction of Editor Florian Dorfler.

* Corresponding author.
E-mail addresses: luca.consolini@unipr.it (L. Consolini), marco.locatelli@unipr.it (M. Locatelli).

https://doi.org/10.1016/j.automatica.2024.111864

is, it is such that $(\forall s \in [0, s_f])$, $\|\boldsymbol{\gamma}'(s)\| = 1$. In this way, s_f is the path length. We want to compute the speed-law that minimizes the overall transfer time (i.e., the time needed to go from $\boldsymbol{\gamma}(0)$ to $\boldsymbol{\gamma}(s_f)$). To this end, let $s : [0, t_f] \to [0, s_f]$ be a differentiable monotone strictly increasing function, that represents the vehicle's arc-length position along the curve as a function of time, and let $v : [0, s_f] \to [0, +\infty[$ be such that $(\forall t \in [0, t_f])$ $\dot{s}(t) = v(s(t))$. In this way, $v(s)$ is the derivative of the vehicle arc-length position, which corresponds to the norm of its velocity vector at position s. The position of the vehicle as a function of time is given by $\mathbf{x} : [0, t_f] \to \mathbb{R}^2$, $\mathbf{x}(t) = \boldsymbol{\gamma}(s(t))$. The velocity and acceleration are given, respectively, by

$$\dot{\mathbf{x}}(t) = \boldsymbol{\gamma}'(s(t))v(s(t)),$$
$$\ddot{\mathbf{x}}(t) = a_T(t)\boldsymbol{\gamma}'(s(t)) + a_N(t)\boldsymbol{\gamma}'^{\perp}(s(t)),$$

where $a_T(t) = v'(s(t))v(s(t))$, $a_N(t) = k(s(t))v(s(t))^2$ are, respectively, the tangential and normal components of the acceleration (i.e., the projections of the acceleration vector $\ddot{\mathbf{x}}$ on the tangent and the normal to the curve). Moreover, $\boldsymbol{\gamma}'^{\perp}(s) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \boldsymbol{\gamma}'(s)$ is the normal to vector $\boldsymbol{\gamma}'(s)$, the tangent of $\boldsymbol{\gamma}'$ at s. Here $k : [0, s_f] \to \mathbb{R}$ is the scalar curvature, defined as $k(s) = \langle \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}'(s)^{\perp} \rangle$. Note that $|k(s)| = \|\boldsymbol{\gamma}''(s)\|$. In the following, we assume that $k(s) \in \mathcal{C}^1([0, s_f], \mathbb{R})$. The total maneuver time, for a given velocity profile $v \in C^1([0, s_f], \mathbb{R})$, is returned by the functional

$$\mathcal{F} : C^1([0, s_f], \mathbb{R}) \to \mathbb{R}, \quad \mathcal{F}(v) = \int_0^{s_f} v^{-1}(s) ds. \tag{1}$$

We consider the following problem.

$$\min_{v \in \mathcal{V}} \mathcal{F}(v), \tag{2}$$

where the feasible region $\mathcal{V} \subset C^1([0, s_f], \mathbb{R})$ is defined by the following set of constraints

$$v(0) = 0, \ v(s_f) = 0, \tag{3a}$$
$$0 \le v(s) \le v_{\max}, \quad s \in \ ]0, s_f[, \tag{3b}$$
$$|v'(s)v(s)| \le \frac{A_T}{2}, \quad s \in [0, s_f], \tag{3c}$$
$$|k(s)|v(s)^2 \le A_N, \quad s \in [0, s_f], \tag{3d}$$
$$|v''(s)v(s)^2 + v'(s)^2 v(s)| \le \frac{J}{2}, \tag{3e}$$

where:

- Constraints (3a) are the initial and final interpolation conditions;
- k is the path curvature;
- $v_{\max}$, $\frac{A_T}{2}$, A_N, are upper bounds for the velocity, the tangential acceleration, and the normal acceleration, imposed through constraints (3b), (3c), (3d), respectively;
- Constraints (3e) impose an upper bound $\frac{J}{2}$ on the time derivative of the acceleration (also called "jerk"). Indeed, note that

$$\frac{d^2}{dt^2} v(s(t)) = \frac{d}{dt}(v'(s(t))\dot{s}(t)) = \frac{d}{dt}(v'(s(t))v(s(t)))$$
$$= v''(s(t))v(s(t))^2 + v'(s(t))^2 v(s(t))$$
$$= v''(s)v(s)^2 + v'(s)^2 v(s).$$

Observation 2.1. *Usually, term "jerk" denotes the time-derivative of the acceleration. In the context of speed planning, a bound on maximum jerk can be interpreted in two ways:*

- *As a bound on the third time-derivative of the path parameter s. This is the notion that we use in constraints (3e). Some works, such as Artuñedo, Villagra, and Godoy (2022) and Ma, Gao, teng Yan, Lv, and qing Hu (2021), make this choice.*
- *As a bound on the third time-derivative of the position vector $\gamma(s)$. For instance, this choice appears in Gasparetto, Lanzutti, Vidoni, and Zanotto (2012).*

In any case, note that enforcing bounds on maximum speed, acceleration, and jerk of path parameter s also allows bounding the third time-derivative of $\gamma(s)$. Some authors use the "pseudojerk", that is a linear approximation of the jerk constraint. For instance, one can substitute (3e) with the tightened constraint $|v''(s)v_{\max}(s)^2 + v'(s)^2 v_{\max}(s)| \le \frac{J}{2}$. This new constraint has the advantage of being convex (actually linear) with respect to v'' and v'. However, being obtained from a tightening of the original constraint, this approach only allows finding sub-optimal solutions. For instance, this choice is used in Shimizu, Horibe, Watanabe, and Kato (2022). We also considered this approach in our previous work (Cabassi et al., 2018).

After setting $w = v^2$, and noting that $w' = 2v'v$, $w'' = 2(v''v + v'^2)$, we end up with the following minimum-time problem:

Problem 1 (*Continuous Time Problem*)**.**

$$\min_{w \in C^2} \int_0^{s_f} w(s)^{-1/2} ds$$
$$w(0) = 0, \quad w(s_f) = 0,$$
$$0 \le w(s) \le \mu^+(s), \qquad s \in [0, s_f],$$
$$\frac{1}{2}\left|w'(s)\right| \le \frac{A_T}{2}, \qquad s \in [0, s_f],$$
$$\frac{1}{2}\left|w''(s)\sqrt{w(s)}\right| \le \frac{J}{2}, \qquad s \in [0, s_f], \tag{4}$$

where μ^+ is the square velocity upper bound, depending on the path curvature, i.e.,

$$\mu^+(s) = \min\left\{v_{\max}^2, \frac{A_N}{|k(s)|}\right\}.$$

The continuous problem is discretized as follows. We subdivide the path into $n-1$ intervals of equal length $h = \frac{s_f}{n}$ (i.e., we evaluate function w at points $s_i = \frac{(i-1)s_f}{n-1}$, $i = 1, \dots, n$), so that we have the following n-dimensional vector of variables

$$w = (w_1, w_2, \dots, w_n) = (w(s_1), w(s_2), \dots, w(s_n)) .$$

To simplify the notation, from now on we will denote A_T simply as A. Then, the finite dimensional version of the problem is:

Problem 2 (*Discretized Problem*)**.**

$$\min_{w \in \mathbb{R}^n} \sum_{i=2}^{n-1} \frac{h}{\sqrt{w_i}} \tag{5}$$
$$w_1 = w_n = 0$$
$$0 \le w_i \le w_i^{\max} \qquad i = 2, \dots, n-1,$$
$$w_{i+1} - w_i \le hA, \qquad i = 1, \dots, n-1, \tag{6}$$
$$w_i - w_{i+1} \le hA, \qquad i = 1, \dots, n-1, \tag{7}$$
$$(w_{i-1} - 2w_i + w_{i+1})\sqrt{w_i} \le h^2 J, \qquad i = 2, \dots, n-1, \tag{8}$$
$$-(w_{i-1} - 2w_i + w_{i+1})\sqrt{w_i} \le h^2 J, \qquad i = 2, \dots, n-1, \tag{9}$$

where $w_i^{\max} = \mu^+(s_i)$, for $i = 2, \dots, n-1$. The objective function (5) is an approximation of the objective function of Problem 1, given by a Riemann sum. Constraints (6) and (7) are obtained by a finite difference approximation of w'. Constraints (8) and (9) are obtained by using a second-order central finite difference to approximate w''. Due to jerk constraints (8) and (9), Problem 2 is nonconvex. Indeed, each constraint (8) (similar for (9)) can be rewritten as

$$w_{i-1} - 2w_i + w_{i+1} - \frac{h^2 J}{\sqrt{w_i}} \le 0.$$

Since the left-hand side is a concave function (the second derivative of $-\frac{h^2 J}{\sqrt{w_i}}$ is negative), the constraint defines a nonconvex region. In what follows, we refer to constraints (8) as *positive jerk constraints*, and to constraints (9) as *negative jerk constraints*. After a simple rewriting of the jerk constraints (8) and (9), we end up with the following equivalent formulation of Problem 2:

$$\begin{array}{lll} \min_w & f(w) = \sum_{i=2}^{n-1} \frac{h}{\sqrt{w_i}} & \\ & \frac{h}{\sqrt{w_i}} \geq \frac{w_{i-1}-2w_i+w_{i+1}}{hJ} & i=2,\ldots,n-1 \\ & \frac{h}{\sqrt{w_i}} \geq \frac{-w_{i-1}+2w_i-w_{i+1}}{hJ} & i=2,\ldots,n-1 \\ & w_{i+1}-w_i \leq Ah & i=2,\ldots,n-1 \\ & w_i-w_{i+1} \leq Ah & i=2,\ldots,n-1 \\ & w_1=w_n=0 & \\ & 0 \leq w_i \leq w_i^{\max} & i=1,\ldots,n. \end{array} \tag{10}$$

2.1. Main results

After adding variables t_i, $i = 2, \ldots, n-1$, and setting, again for each $i = 2, \ldots, n-1$:

$$t_i = \frac{h}{\sqrt{w_i}}, \quad \Delta w_i = \frac{w_{i-1} - 2w_i + w_{i+1}}{hJ}, \tag{11}$$

we have the following equivalent reformulation of problem (10):

$$\begin{array}{lll} \min_{w,t} & g(t) = \sum_{i=2}^{n-1} t_i & \\ & t_i \geq \Delta w_i & i=2,\ldots,n-1 \\ & t_i \geq -\Delta w_i & i=2,\ldots,n-1 \\ & t_i = \frac{h}{\sqrt{w_i}} & i=2,\ldots,n-1 \\ & w_{i+1}-w_i \leq Ah & i=2,\ldots,n-1 \\ & w_i-w_{i+1} \leq Ah & i=2,\ldots,n-1 \\ & w_1=w_n=0 & \\ & 0 \leq w_i \leq w_i^{\max} & i=1,\ldots,n, \end{array} \tag{12}$$

where each pair of jerk constraints is split into a triple of equivalent constraints. Nonconvexity of this formulation is restricted to the equality constraints $t_i = \frac{h}{\sqrt{w_i}}$ for $i = 2, \ldots, n-1$. Then, we can relax the problem into a convex one by replacing the equality constraints with inequalities $t_i \geq \frac{h}{\sqrt{w_i}}$:

$$\begin{array}{lll} \min_{w,t} & g(t) = \sum_{i=2}^{n-1} t_i & \\ & t_i \geq \Delta w_i & i=2,\ldots,n-1 \\ & t_i \geq -\Delta w_i & i=2,\ldots,n-1 \\ & t_i \geq \frac{h}{\sqrt{w_i}} & i=2,\ldots,n-1 \\ & w_{i+1}-w_i \leq Ah & i=2,\ldots,n-1 \\ & w_i-w_{i+1} \leq Ah & i=2,\ldots,n-1 \\ & w_1=w_n=0 & \\ & 0 \leq w_i \leq w_i^{\max} & i=1,\ldots,n. \end{array} \tag{13}$$

Note that at optimal solutions of (13), for each $i = 2, \ldots, n-1$, at least one of the three constraints involving variable t_i is active. Then, if for some i it holds that $t_i > \frac{h}{\sqrt{w_i}}$, either the negative jerk constraint (if constraint $t_i \geq -\Delta w_i$ is active), or the positive jerk constraint (if constraint $t_i \geq \Delta w_i$ is active) is violated. Constraint $t_i \geq \frac{h}{\sqrt{w_i}}$ is not well-defined if $w_i = 0$. However, we could rewrite this constraint as $\sqrt{w_i} t_i \geq h$. In this reformulation, it is clear that $w_i > 0$ in any feasible solution of (13). For this reason, in our setting, we do not need to require the strict inequality $w_i > 0$, as is done in other settings, such as Digani, Hsieh, Sabattini, and Secchi (2019), Frego et al. (2020) and Heinemann, Riedel, and Lechler (2019). Problem (13), and its relation with (12), is the main focus of this paper. Despite its simplicity, to our knowledge, relaxation (13) is new. Our main results are the following ones.

- We state some properties of relaxation (13). Namely, we show that a solution of (13) never violates negative jerk constraints (Proposition 4). We show that it can violate positive jerk constraints only if the velocity is equal to its upper bound (Proposition 5). Finally, we present a sufficient condition under which the relaxation is exact (Corollary 7).
- We discuss some properties related to the dual Lagrangian problem of relaxation (13). In particular, we present a sufficient exactness condition (Proposition 9).
- We propose a reformulation of relaxation (13) as a SOCP (see (27)). This can be efficiently handled by modern solvers.
- We present various numerical experiments. In these experiments, relaxation (13) is *always* exact. This led us to formulate Conjecture 4.1, is which we surmise that this is always the case. However, *we do not have a proof of this fact*.
- We present other numerical experiments, on a generalization of problem (10) (Section 7.2), in which acceleration and jerk constraints depend on step i (i.e., the position along the curve). We found that, in many cases, relaxation (13) is still exact. In all cases, it can be used to find a precise bound on the optimal value of (10). Further, the relaxed solution can be used as a starting point for a local search procedure for nonconvex problem (12).
- We extend the proposed approach to problems with additional convex constraints, through which it is possible to address the speed planning problem for road vehicles (see Section 9).

To our knowledge, all these results are new, since relaxation (13) and its properties have not been discussed in literature. From a practical point of view, we stress that the proposed nonconvex relaxation allows solving very efficiently speed planning problems with jerk constraints. Indeed, our computational times for problems with 1000 samples are in the order of 0.2 s (see Section 8.2).

2.2. Comparison with existing literature

For a summary of existing literature, we also refer the reader to our previous paper (Consolini et al., 2022) that, as said, addressed the same problem as this work. Various works consider problem (10), or similar ones, related to minimum-time speed planning in presence of jerk constraints. Since constraints on maximum jerk are nonconvex, these works often use iterative methods to find a local minimum. For instance, in Debrouwere et al. (2013), the authors observe that jerk constraints are nonconvex, but can be written as the difference of two convex functions. Based on this observation, the authors solve the problem by a sequence of convex subproblems, obtained by linearizing at the current point the concave part of the jerk constraints. In Singh and Krishna (2015), the authors reformulate the problem in such a way that its objective function is convex quadratic, while nonconvexity lies in difference-of-convex functions. The resulting problem is tackled through the solution of a sequence of convex subproblems obtained by linearizing the concave part of the nonconvex constraints. Other approaches dealing with jerk constraints do not rely on the solution of convex subproblems. For instance, in Macfarlane and Croft (2003), a concatenation of fifth-order polynomials is employed to provide smooth trajectories, which results in quadratic jerk profiles, while in Haschke, Weitnauer, and Ritter (2008) cubic polynomials are employed, resulting in piecewise constant jerk profiles. A recent and interesting approach to the problem with jerk constraints is Pham and Pham (2017). In this work an approach based on numerical integration is discussed. Numerical integration has been first applied under acceleration constraints in Bobrow, Dubowsky, and Gibson (1985) and Kang Shin and McKay (1985). In Pham and Pham (2017) jerk constraints are taken into account. The algorithm detects a position s along the trajectory where the jerk constraint is singular, that is, the jerk term disappears from one of the constraints. Then, it computes the speed profile up to s by computing two maximum jerk profiles and then connecting them by a minimum jerk profile, found by a shooting method. In general, the overall solution is composed of a sequence of various maximum and minimum jerk profiles. This approach does not guarantee reaching a local minimum of the traversal time.

Some algorithms use heuristics to quickly find suboptimal solutions of acceptable quality. For instance, Villagra, Milanés, Pérez, and Godoy (2012) propose an algorithm that applies to curves composed of clothoids, circles and straight lines. The algorithm does not guarantee local optimality of the solution. Reference Raineri and Guarino Lo Bianco (2019) presents an efficient heuristic algorithm. Also this method guarantees neither global nor local optimality. Various works in literature consider jerk bounds in the speed optimization problem for robotic manipulators instead of mobile vehicles. This is a slightly different problem, but mathematically similar to Problem 1. In particular, paper Dong, Ferreira, and Stori (2007) presents a method based on the solution of many nonlinear and nonconvex subproblems. The resulting algorithm is slow, due to the large number of subproblems; moreover, the authors do not prove its convergence. Reference Zhang, Yuan, Gao, and Li (2012) proposes a similar method that gives a continuous-time solution. Again, the method is computationally slow, since it is based on the numerical solution of many differential equations; moreover, the paper does not contain a proof of convergence or of local optimality. In Palleschi, Garabini, Caporale, and Pallottino (2019), the problem of speed planning for robotic manipulators with jerk constraints is reformulated in such a way that nonconvexity lies in simple bilinear terms. Such bilinear terms are replaced by the corresponding convex and concave envelopes, obtaining the so-called McCormick relaxation, which is the tightest possible convex relaxation of the nonconvex problem. Recent works Artuñedo et al. (2022) and Wang, Xiao, Liu, and Liu (2022) present efficient heuristics for computing speed profiles with limited jerk. However, they do not guarantee local or global optimality.

In general, all the above algorithms are able to find good quality solutions, but do not guarantee global (or even local) optimality of the found solution. In our recent work (Consolini et al., 2022), we were able to present an algorithm that guarantees local optimality. As said, to our knowledge, the present paper is the only one that guarantees (under some assumptions) global optimality of the found solution.

Some other works replace the jerk constraint with *pseudo-jerk*, that is the derivative of the acceleration with respect to arc-length, ending up with a convex optimization problem. For instance, Zhang et al. (2018) add to the objective function a pseudo-jerk penalizing term. This approach is computationally convenient, but may be overly restrictive at low speeds. Similarly, works (Palleschi et al., 2021; Shimizu et al., 2022) consider a convex problem obtained by linearizing, or approximating, the jerk constraint. These are convenient approaches from a computational point of view, but, obviously, do not provide the solution of the original problem.

3. An alternative way to derive the proposed convex relaxation

In this section, we show that relaxation (13) can be obtained from the Lagrangian relaxation (see, e.g., Boyd and Vandenberghe (2004)) of the jerk constraints in (10). We present the main idea in a slightly more general setting. Consider the following problem.

$$\begin{aligned} \min_{x\in\mathcal{C}} \quad & \textstyle\sum_{i\in I} f_i(x) \\ \text{s. t.} \quad & p_i(x) \le f_i(x),\ i\in I\,, \end{aligned} \tag{14}$$

where for $i \in I$, $f_i, p_i : \mathbb{R}^n \to \mathbb{R}$ are convex, and $\mathcal{C} \subset \mathbb{R}^n$ is convex and compact. Note that problem (10) falls into the class of problems (14) with:

- $I = \{2,\dots,n-1\}$ and $x = w$;
- $\mathcal{C} = \{w \;:\; w_1 = w_n = 0,\ |w_{i+1} - w_i| \le Ah,\ i = 2,\dots,n-1,\ 0 \le w_i \le w_i^{\max},\ i = 1,\dots,n\}$;
- $p_i(w) = \left|\frac{w_{i-1}-2w_i+w_{i+1}}{hJ}\right|$, $i = 2,\dots,n-1$;
- $f_i(w) = \frac{h}{\sqrt{w_i}}$, $i = 2,\dots,n-1$.

We denote by F^* the optimal value of problem (14). Note that this problem is nonconvex if (at least one of the) functions $p_i(x) - f_i(x)$ is not convex, which is usually the case since functions $-f_i$ are concave. In particular, it is the case for problem (10), since functions p_i are convex but not strictly convex, while functions $-f_i$ are strictly concave. We apply the Lagrangian relaxation to these constraints:

$$\min_{x\in\mathcal{C}} \quad \textstyle\sum_{i\in I} f_i(x) + \eta_i\,(p_i(x) - f_i(x))\,, \tag{15}$$

where η_i, $i \in I$ are non-negative Lagrange multipliers. For $\eta = (\eta_1,\dots,\eta_{|I|}) \in (\mathbb{R}^+)^{|I|}$, we denote by $F(\eta)$ the optimal value of (15), and by $x(\eta)$ a corresponding solution vector. For any $\eta \in (\mathbb{R}^+)^{|I|}$, $F(\eta) \le F^*$, since at each feasible solution of (14), which is also feasible for (15), it holds that

$$\sum_{i\in I} f_i(x) + \eta_i\,(p_i(x) - f_i(x)) \le \sum_{i\in I} f_i(x).$$

In other words, the optimal value of (15) is a lower bound for F^*. Rewrite (15) as

$$\min_{x\in\mathcal{C}} \quad \textstyle\sum_{i\in I} (1-\eta_i) f_i(x) + \eta_i p_i(x).$$

Note that, if $\eta_i \in [0, 1]$, $i \in I$, this is a convex problem. We consider the following convex relaxation of (14):

$$\max_{\eta\in[0,1]^{|I|}} \min_{x\in\mathcal{C}} \quad \textstyle\sum_{i\in I} (1-\eta_i) f_i(x) + \eta_i p_i(x). \tag{16}$$

Since $\mathcal{C}$ and $[0, 1]^{|I|}$ are compact, we can apply the Von Neumann-Fan minimax theorem, and exchange the maximum and minimum operations, obtaining the following equivalent convex relaxation:

$$\min_{x\in\mathcal{C}} \quad \textstyle\sum_{i\in I} \max\{f_i(x), p_i(x)\}, \tag{17}$$

which can also be written as follows:

$$\begin{aligned} &\min_{t\in\mathbb{R}^n, x\in\mathcal{C}} \textstyle\sum t_i \\ &t_i \ge f_i(x) \\ &t_i \ge p_i(x). \end{aligned} \tag{18}$$

Note that relaxation (13) is exactly the convex relaxation (18) for problem (10).

4. An exactness condition for the convex relaxation

Before proceeding, we introduce a slight modification in problem (10). To simplify the following mathematical analysis, it is worthwhile to modify definition (11) as follows:

$$\Delta w_i = \frac{w_{i-1} - (2+\rho(h))w_i + w_{i+1}}{hJ},$$

where $\rho(h) = o(h^2) > 0$. Provided that h is small,

$$\frac{h}{\sqrt{w_i}} \ge |\Delta w_i|\,, \quad i = 2,\dots,n-1,$$

still represents a correct discretization of the continuous constraints (4).

Now, we introduce a condition under which we guarantee that the optimal values of the problems (10) and (13) are equal. We first introduce the following simple lemma.

Lemma 3. *Any feasible solution w of the original problem (10) induces a feasible solution (w, t) with $t_i = \frac{h}{\sqrt{w_i}}$ of the relaxation (13). The two feasible solutions have the same objective function value, i.e., $f(w) = g(t)$.*

As a consequence of Lemma 3, we have the following primal condition for the exactness of the relaxation.

Observation 4.1. *If the optimal solution $(w^\star, t^\star)$ of the convex relaxation (13) is such that $w^\star$ is feasible for (10), then $w^\star$ is also optimal for (10), and the convex relaxation has the same optimal value of the original problem.*

Proof. In view of Lemma 3, the objective function value $g(t^\star)$ of (13) at $(w^\star, t^\star)$ is equal to the objective function value $f(w^\star)$ of (10) at $w^\star$. Then, given an optimal solution $\bar{w}_\star$ of (10), we must have

$$f(w^\star) = g(t^\star) \le f(\bar{w}_\star) \le f(w^\star),$$

where the first inequality comes from the fact that the optimal value of the relaxation is a lower bound of the optimal value of (10), while the second inequality comes from the fact that $\bar{w}_\star$ and $w^\star$ are an optimal and a feasible solution for problem (10), respectively. As a consequence, we must have that all inequalities are equalities, so that $w^\star$ is optimal for (10) and problems (10) and (13) have the same optimal value. □

The reverse of Lemma 3 is not true, i.e., given a feasible solution $(\tilde{w}, \tilde{t})$ of the convex relaxation (13), $\tilde{w}$ may violate some jerk constraint. E.g., for some $i \in \{2, \dots, n-1\}$, it may hold that:

$$\frac{h}{\sqrt{\tilde{w}_i}} < \frac{\tilde{w}_{i-1} - (2+\rho(h))\tilde{w}_i + \tilde{w}_{i+1}}{hJ}. \tag{19}$$

Such opportunity of violating jerk constraints has two conflicting effects on the objective function of the relaxation. On the one hand, violating jerk constraints allows enlarging the feasible set and, thus, to reduce the objective function. More precisely, the projection of the feasible set of (13) over the set of variables w, i.e., the set

$$\{w \ : \ (w, t) \text{ is feasible for (13)}\},$$

strictly contains the feasible region of (10). On the other hand, a violation has a cost. Indeed, if (19) holds, then we have that

$$\tilde{t}_i \ge \frac{\tilde{w}_{i-1} - (2+\rho(h))\tilde{w}_i + \tilde{w}_{i+1}}{hJ} > \frac{h}{\sqrt{\tilde{w}_i}},$$

i.e., the ith term of the objective function of the relaxation (13) is larger than the corresponding term of the objective function of the original problem (10). Therefore, the question is whether the gain obtained from the enlargement of the feasible region is able to counterbalance the cost of the violation. If not, optimal solutions of the relaxation (13) do not violate any jerk constraint and, consequently, are also optimal for (10), as stated in Observation 4.1.

4.1. Solution algorithm

Observation 4.1 motivates the following algorithm for solving (10). First, we solve convex relaxation (13). To this end, in Section 8.1, we will present an efficient method based on the reformulation of (12) as a second-order cone program (SOCP). If the found optimal solution $(w^\star, t^\star)$ is such that $w^\star$ is feasible, then, by Observation 4.1, $w^\star$ is optimal for (10).

In Section 5, Corollary 7, we will present a sufficient condition for the feasibility (and, hence, optimality) of $w^\star$. However, as we will discuss in more detail in the following (Section 8.2), this is always the case in our numerical tests, so that we make the following conjecture.

Conjecture 4.1. *The convex relaxation (13) is exact, i.e., its optimal value is equal to the optimal value of (10), and given an optimal solution $(w^\star, t^\star)$ for (13), $w^\star$ is feasible and optimal for (10).*

However, in spite of many attempts to give a formal proof of this conjecture, up to now we have not been able to derive it, apart under the mentioned sufficient condition that we will present in Corollary 7. To complete the algorithm, in case $w^\star$ is not feasible, $g(t^\star)$ is a lower bound for the solution of (10). Then, we use $(w^\star, t^\star)$ as the starting condition for a generic nonconvex solver for (12). We can summarize our solution algorithm as follows.

(1) Set $(w^\star, t^\star)$ as the solution of the convex problem (13).
(2) If $w^\star$ is feasible for (10), then $w^\star$ is the optimal solution of (10).
(3) If $w^\star$ is not feasible, use a generic nonconvex solver for (12), using initial condition $(w^\star, t^\star)$ and let $(\hat{w}, \hat{t})$ be the obtained solution. Then $\hat{w}$ is a suboptimal solution, and we can conclude that the true optimal solution $\bar{w}_\star$ of (10) satisfies $f(\bar{w}_\star) \in [g(t^\star), f(\hat{w})]$.

As said, in all our numerical tests, $w^\star$ is feasible, so that Step 3 is not necessary. But we cannot remove Step 3 since we do not have a formal proof of Conjecture 4.1. Moreover, Step 3 is necessary if we consider a generalization of problem (10), in which constraint parameters A, J vary with step i (see Section 8.2). As we will see, in these more general cases there exist instances where $w^\star$ is not feasible for (10).

5. Results on negative and positive jerk constraints

As previously mentioned, we do not have a formal proof of Conjecture 4.1. However, we can prove some strong theoretical properties for $w^\star$.

Let us denote by $X^\star$ the set of optimal solutions of the relaxed problem (13). We first prove this result.

Proposition 4. *Given the optimal solution $(w^\star, t^\star) \in X^\star$ of the convex relaxed problem (13), $w^\star$ does not violate any negative jerk constraint.*

Proof. Assume, by contradiction, that $(w^\star, t^\star) \in X^\star$ and for some $r \in \{2, \dots, n-1\}$:

$$t_r^\star = \frac{-w_{r-1}^\star + (2+\rho(h))w_r^\star - w_{r+1}^\star}{hJ} > \frac{h}{\sqrt{w_r^\star}}. \tag{20}$$

In other words, we are assuming that $w^\star$ violates the rth negative jerk constraint. Note that we must have $w_r^\star > 0$. Indeed, if $w_r^\star = 0$, then

$$-w_{r-1}^\star + (2+\rho(h))w_r^\star - w_{r+1}^\star > 0 \quad \Rightarrow \quad w_{r-1}^\star + w_{r+1}^\star < 0,$$

which is not possible. For $\delta > 0$ small enough, let us consider the new feasible solution $\bar{w}$ defined as follows:

$$\bar{w}_r = w_r^\star - \delta, \quad \bar{w}_i = w_i^\star, \quad i \neq r.$$

Obviously, the new solution does not violate the constraints $w_i \le w_i^{\max}$, and, for δ small enough, it does not violate the nonnegativity constraints and the positive jerk constraints. It also does not violate acceleration constraints. Indeed, by (20) we have that for h small enough, so that $\frac{Jh^2}{\sqrt{w_r^\star}} - \rho(h)w_r^\star > 0$, it holds that:

$$\begin{aligned} Ah &\ge w_r^\star - w_{r+1}^\star > w_{r-1}^\star - w_r^\star + \frac{Jh^2}{\sqrt{w_r^\star}} - \rho(h)w_r^\star \\ &> w_{r-1}^\star - w_r^\star \\ Ah &\ge w_r^\star - w_{r-1}^\star > w_{r+1}^\star - w_r^\star + \frac{Jh^2}{\sqrt{w_r^\star}} - \rho(h)w_r^\star \\ &> w_{r+1}^\star - w_r^\star, \end{aligned}$$

so that, for a small enough δ, it also holds that:

$$Ah > \bar{w}_r - \bar{w}_{r+1} > \bar{w}_{r-1} - \bar{w}_r$$
$$Ah > \bar{w}_r - \bar{w}_{r-1} > \bar{w}_{r+1} - \bar{w}_r,$$

and the solution $\bar{w}$ fulfills all the acceleration constraints. We first assume that $r > 2$. We have that

$$\bar{t}_r = \max\left\{\frac{w^\star_{r-1} - (2+\rho(h))w^\star_r + w^\star_{r+1} - \delta}{hJ}, \frac{-w^\star_{r-1} + (2+\rho(h))w^\star_r - w^\star_{r+1} + \delta}{hJ}, \frac{h}{\sqrt{w^\star_r}}\right\} = t^\star_r - \frac{(2+\rho(h))\delta}{hJ},$$

while

$$\bar{t}_{r-1} = \max\left\{\frac{w^\star_{r-2} - (2+\rho(h))w^\star_{r-1} + w^\star_r - \delta}{hJ}, \frac{-w^\star_{r-2} + (2+\rho(h))w^\star_{r-1} - w^\star_r + \delta}{hJ}, \frac{h}{\sqrt{w^\star_{r-1}}}\right\}$$
$$\bar{t}_{r+1} = \max\left\{\frac{w^\star_r - \delta - (2+\rho(h))w^\star_{r+1} + w^\star_{r+2}}{hJ}, \frac{-w^\star_r + \delta + (2+\rho(h))w^\star_{r+1} - w^\star_{r+2}}{hJ}, \frac{h}{\sqrt{w^\star_{r+1}}}\right\}.$$

Now, if

$$\frac{-w^\star_{r-2} + (2+\rho(h))w^\star_{r-1} - w^\star_r}{hJ} < t^\star_{r-1},$$

we have that, for δ small enough,

$$\bar{t}_{r-1} = t^\star_{r-1}.$$

Instead, if

$$\frac{-w^\star_{r-2} + (2+\rho(h))w^\star_{r-1} - w^\star_r}{hJ} = t^\star_{r-1},$$

then

$$\bar{t}_{r-1} = t^\star_{r-1} + \frac{\delta}{hJ},$$

and

$$\frac{-\bar{w}_{r-2} + (2+\rho(h))\bar{w}_{r-1} - \bar{w}_r}{hJ} > \frac{h}{\sqrt{\bar{w}_{r-1}}}.$$

Similar for $\bar{t}_{r+1}$. In all cases we have

$$\bar{t}_r + \bar{t}_{r-1} + \bar{t}_{r+1} \le t^\star_r + t^\star_{r-1} + t^\star_{r+1} - \frac{\rho(h)\delta}{hJ} < t^\star_r + t^\star_{r-1} + t^\star_{r+1},$$

so that optimality is contradicted. The case $r = 2$ can be dealt with in a completely analogous way: since $r = 2$, we do not have the updated term $\bar{t}_1$ but only the terms $\bar{t}_2$ and $\bar{t}_3$. In this case $\bar{t}_2 + \bar{t}_3 < t^\star_2 + t^\star_3$, which contradicts optimality. □

This theoretical property is strong. The negative jerk constraint is a nonconvex constraint, but, in spite of that, the optimal solution of the relaxed problem never violates it. If we could prove the same for the positive jerk constraint, equivalence between (10) and its convex relaxation (13) would be established. Unfortunately, we do not have such result. Still we can prove a weaker result which restricts the cases where a violation of the positive jerk constraint might occur.

Proposition 5. *Given the optimal solution $(w^\star, t^\star) \in X^\star$ of the convex relaxed problem (13), $w^\star$ can violate the ith positive jerk constraint only if $w^\star_i = w^{\max}_i$.*

Proof. We assume, again by contradiction, that $w^\star \in X^\star$ and for some $r \in \{2, \ldots, n-1\}$:

$$t^\star_r = \frac{w^\star_{r-1} - (2+\rho(h))w^\star_r + w^\star_{r+1}}{hJ} > \frac{h}{\sqrt{w^\star_r}}, \tag{21}$$

and $w^\star_r < w^{\max}_r$, i.e., we are assuming that $w^\star$ violates the rth positive jerk constraint and does not reach the maximum speed at r. For $\delta > 0$ small enough, let us consider the new feasible solution $\bar{w}$ defined as follows:

$$\bar{w}_r = w^\star_r + \delta, \quad \bar{w}_i = w^\star_i, \quad i \neq r.$$

For δ small enough the new solution does not violate the constraints $w_i \le w^{\max}_i$ and the negative jerk constraints, while it obviously does not violate the nonnegativity constraints. It also does not violate acceleration constraints. Indeed, by (21) we have that, for h small enough:

$$Ah \ge w^\star_{r-1} - w^\star_r > w^\star_r - w^\star_{r+1} + \frac{Jh^2}{\sqrt{w^\star_r}} + \rho(h)w^\star_r > w^\star_r - w^\star_{r+1}$$
$$Ah \ge w^\star_{r+1} - w^\star_r > w^\star_r - w^\star_{r-1} + \frac{Jh^2}{\sqrt{w^\star_r}} + \rho(h)w^\star_r > w^\star_r - w^\star_{r-1},$$

so that, for a small enough δ, it also holds that:

$$Ah > \bar{w}_{r-1} - \bar{w}_r > \bar{w}_r - \bar{w}_{r+1}$$
$$Ah > \bar{w}_{r+1} - \bar{w}_r > \bar{w}_r - \bar{w}_{r-1},$$

and the solution $\bar{w}$ fulfills all the acceleration constraints. We only discuss the case $r > 2$ (the case $r = 2$ can be dealt with in an analogous way). We have that

$$\bar{t}_r = \max\left\{\frac{w^\star_{r-1} - (2+\rho(h))w^\star_r + w^\star_{r+1} - \delta}{hJ}, \frac{-w^\star_{r-1} + (2+\rho(h))w^\star_r - w^\star_{r+1} + \delta}{hJ}, \frac{h}{\sqrt{w^\star_r}}\right\} = t^\star_r - \frac{(2+\rho(h))\delta}{hJ},$$

while

$$\bar{t}_{r-1} = \max\left\{\frac{w^\star_{r-2} - (2+\rho(h))w^\star_{r-1} + w^\star_r + \delta}{hJ}, \frac{-w^\star_{r-2} + (2+\rho(h))w^\star_{r-1} - w^\star_r - \delta}{hJ}, \frac{h}{\sqrt{w^\star_{r-1}}}\right\}$$
$$\bar{t}_{r+1} = \max\left\{\frac{w^\star_r + \delta - (2+\rho(h))w^\star_{r+1} + w^\star_{r+2}}{hJ}, \frac{-w^\star_r - \delta + (2+\rho(h))w^\star_{r+1} - w^\star_{r+2}}{hJ}, \frac{h}{\sqrt{w^\star_{r+1}}}\right\}.$$

Now, if

$$\frac{w^\star_{r-2} - (2+\rho(h))w^\star_{r-1} + w^\star_r}{hJ} < t^\star_{r-1},$$

we have that, for δ small enough,

$$\bar{t}_{r-1} = t^\star_{r-1}.$$

Instead, if

$$\frac{w^\star_{r-2} - (2+\rho(h))w^\star_{r-1} + w^\star_r}{hJ} = t^\star_{r-1},$$

then

$$\bar{t}_{r-1} = t^\star_{r-1} + \frac{\delta}{hJ},$$

and

$$\frac{\bar{w}_{r-2} - (2+\rho(h))\bar{w}_{r-1} + \bar{w}_r}{hJ} > \frac{h}{\sqrt{\bar{w}_{r-1}}}.$$

Similar for $\bar{t}_{r+1}$. In all cases we have

$$\bar{t}_r + \bar{t}_{r-1} + \bar{t}_{r+1} \le t_r^\star + t_{r-1}^\star + t_{r+1}^\star - \frac{\rho(h)\delta}{hJ} < t_r^\star + t_{r-1}^\star + t_{r+1}^\star,$$

so that optimality is contradicted. □

We can also prove the following proposition.

Proposition 6. *Given the optimal solution $(w^\star, t^\star)$ of problem (13), we have that $w^\star$ might violate the positive jerk constraint only at some i such that also the maximum speed $w^{\max}$ violates it.*

Proof. If $w^\star$ violates the ith positive jerk constraint we know that $w_i^\star = w_i^{\max}$, so that

$$\frac{w_{i-1}^\star - (2+\rho(h))w_i^{\max} + w_{i+1}^\star}{hJ} > \frac{h}{\sqrt{w_i^{\max}}}.$$

But $w_{i-1}^\star \le w_{i-1}^{\max}$ and $w_{i+1}^\star \le w_{i+1}^{\max}$ imply that

$$\frac{w_{i-1}^{\max} - (2+\rho(h))w_i^{\max} + w_{i+1}^{\max}}{hJ} > \frac{h}{\sqrt{w_i^{\max}}},$$

so that also the maximum speed $w^{\max}$ violates the ith positive jerk constraint. □

Then, we also proved the following corollary.

Corollary 7. *If the maximum speed $w^{\max}$ fulfills all positive jerk constraints, then the relaxation (13) is exact.*

6. Dual Lagrangian problem

In this section we discuss our problems from a dual perspective. Let:

$$W = \{w \;:\; 0 \le w_i \le w_i^{\max},\ i = 2, \dots, n-1,\ w_1 = w_n = 0\}.$$

The Lagrangian function for problem (13) is:

$$\begin{aligned}\mathcal{L}(w,t,\lambda,\phi,\xi,\beta,\alpha) = \sum_{i=2}^{n-1} &(1-\lambda_i-\phi_i-\xi_i)t_i \\ &+\lambda_i\Delta w_i - \phi_i\Delta w_i + \xi_i\frac{h}{\sqrt{w_i}} \\ +\alpha_i(w_{i+1} - w_i - Ah) &+ \beta_i(w_i - w_{i+1} - Ah),\end{aligned}$$

and the dual Lagrangian problem is:

$$\max_{\lambda,\phi,\xi,\beta,\alpha\ge 0} \min_{w\in W,t} \mathcal{L}(w,t,\lambda,\phi,\xi,\beta,\alpha). \tag{22}$$

We denote by $W^\star(\lambda,\phi,\alpha,\beta,\xi)$ the set of optimal solutions of the inner minimization problem. Since Slater's condition holds for the convex relaxation (13), i.e., its feasible region has a nonempty interior, the Lagrangian dual of the convex relaxation has the same optimal value as the convex relaxation itself. Since the inner minimization is unbounded from below for $1-\lambda_i-\phi_i-\xi_i \ne 0$, we can impose the equality or, equivalently, we can replace each ξ_i with $1-\lambda_i-\phi_i$, and remove variables t_i from the inner problem, so that we can define

$$\begin{aligned}\mathcal{L}'(w,\lambda,\phi,\beta,\alpha) = \sum_{i=2}^{n-1} &\frac{h}{\sqrt{w_i}} + \lambda_i\left(\Delta w_i - \frac{h}{\sqrt{w_i}}\right) \\ &+\phi_i\left(-\Delta w_i - \frac{h}{\sqrt{w_i}}\right) \\ +\alpha_i(w_{i+1} - w_i - Ah) &+ \beta_i(w_i - w_{i+1} - Ah),\end{aligned}$$

and the minimization problem reduces to:

$$\min_{w\in W} \mathcal{L}'(w,\lambda,\phi,\beta,\alpha), \tag{23}$$

whose optimal set is denoted by $W^\star(\lambda,\phi,\alpha,\beta)$. We make the following observation.

Observation 6.1. *The dual Lagrangian of the relaxed problem (13) is equivalent to the dual Lagrangian of the original problem (10).*

Proof. It is enough to notice that the Lagrangian function of problem (10) is equivalent to $\mathcal{L}'$. □

We observe that for $1-\lambda_i-\phi_i < 0$, the minimization problem is unbounded from below, so that we further impose that $\lambda_i+\phi_i \le 1$. After reorganizing the different terms, the objective function of the minimization problem can be written as the convex separable function:

$$\begin{aligned}\sum_{i=2}^{n-1} \Bigg[&(1-\lambda_i-\phi_i)\frac{h}{\sqrt{w_i}} \\ +(\Delta\lambda_i - \Delta\phi_i &+ \beta_i - \alpha_i - \beta_{i-1} + \alpha_{i-1})w_i \\ &-(\beta_i+\alpha_i)hA\Bigg],\end{aligned}$$

where:

$$\Delta\lambda_i = \frac{\lambda_{i-1} - (2+\rho(h))\lambda_i + \lambda_{i+1}}{hJ},$$

$$\Delta\phi_i = \frac{\phi_{i-1} - (2+\rho(h))\phi_i + \phi_{i+1}}{hJ}.$$

Now, let us denote by $\theta = (\lambda,\phi,\alpha,\beta)$ the vector of dual variables and set

$$\Gamma_i(\theta) = \Delta\lambda_i - \Delta\phi_i + \beta_i - \alpha_i - \beta_{i-1} + \alpha_{i-1}.$$

We consider the following five subsets:

$$\Theta_1^i = \left\{ \theta \ge 0 \;:\; \Gamma_i(\theta) > 0,\ \lambda_i + \phi_i < 1, \right.$$
$$\left. \text{and } h^{\frac{2}{3}}\left[\frac{1-\lambda_i-\phi_i}{2\Gamma_i(\theta)}\right]^{\frac{2}{3}} \le w_i^{\max} \right\}$$
$$\Theta_2^i = \left\{ \theta \ge 0 \;:\; (\Gamma_i(\theta) \le 0 \text{ and } \lambda_i+\phi_i < 1) \right.$$
$$\left. \text{or } \left(\Gamma_i(\theta) > 0 \text{ and } h^{\frac{2}{3}}\left[\frac{1-\lambda_i-\phi_i}{2\Gamma_i(\theta)}\right]^{\frac{2}{3}} > w_i^{\max}\right) \right\}$$
$$\Theta_3^i = \{\theta \ge 0 \;:\; \Gamma_i(\theta) < 0 \text{ and } \lambda_i+\phi_i = 1\}$$
$$\Theta_4^i = \{\theta \ge 0 \;:\; \Gamma_i(\theta) > 0 \text{ and } \lambda_i+\phi_i = 1\}$$
$$\Theta_5^i = \{\theta \ge 0 \;:\; \Gamma_i(\theta) = 0 \text{ and } \lambda_i+\phi_i = 1\}.$$

Then, the solution(s) of the minimization problem, belonging to the optimal set $W^\star(\theta)$, can be given in closed form:

$$w_i^\star(\theta) = \begin{cases} h^{\frac{2}{3}}\left[\frac{1-\lambda_i-\phi_i}{2\Gamma_i(\theta)}\right]^{\frac{2}{3}} & \text{if } \theta \in \Theta_1^i \\ w_i^{\max} & \text{if } \theta \in \Theta_2^i \cup \Theta_3^i \\ 0 & \text{if } \theta \in \Theta_4^i \\ [0, w_i^{\max}] & \text{if } \theta \in \Theta_5^i, \end{cases} \tag{24}$$

and its optimal value is:

$$F(\theta) = \sum_{i=2,\dots,n-1} F_i(\theta) - (\beta_i+\alpha_i)hA,$$

where

$$F_i(\theta) = \begin{cases} (1-\lambda_i-\phi_i)\frac{h}{\sqrt{w_i^{\max}}} + \Gamma_i(\theta)w_i^{\max} & \theta \in \Theta_2^i \cup \Theta_3^i \\ \frac{3}{2}h^{\frac{2}{3}}(1-\lambda_i-\phi_i)^{\frac{2}{3}}[2\Gamma_i(\theta)]^{\frac{1}{3}} & \text{otherwise.} \end{cases}$$

Then, the dual Lagrangian problem is:

$$\begin{array}{ll}\max_{\theta\geq 0} & F(\theta) \\ & \lambda_i+\phi_i\leq 1 \quad i=2,\ldots,n-1.\end{array} \tag{25}$$

Note that F is a continuous and concave function (see, e.g., Bazaraa, Sherali, and Shetty (1993)).

6.1. Dual exactness condition

We observe that from an optimal solution $\theta^\star$ of (25) we can derive an optimal solution $(\theta^\star,\xi^\star)$ of (22) by simply setting $\xi^\star = 1-\lambda_i^\star-\phi_i^\star$ for each $i\in\{2,\ldots,n-1\}$. It is well known (see, e.g., Theorem 6.2.5 in Bazaraa et al. (1993)) that since (13) fulfills the Slater's condition, it holds that $(\theta^\star,\xi^\star)$ is optimal for (22) and $(w^\star,t^\star)$ is an optimal solution of (13) if and only if:

- $(\theta^\star,\xi^\star)\geq 0$;
- $(w^\star,t^\star)$ is feasible for (13);
- $(w^\star,t^\star)\in W^\star(\theta^\star,\xi^\star)$ or, equivalently, $w^\star\in W^\star(\theta^\star)$;
- all complementarity conditions hold.

Note that, by optimality, we must have

$$t_i^\star=\max\left\{\frac{h}{\sqrt{w_i^\star}},\frac{w_{i-1}^\star-(2+\rho(h))w_i^\star+w_{i+1}^\star}{hJ},\right.$$
$$\left.\frac{-w_{i-1}^\star+(2+\rho(h))w_i^\star-w_{i+1}^\star}{hJ}\right\}.$$

Moreover, if $\xi_i^\star>0$ or, equivalently, $\lambda_i^\star+\phi_i^\star<1$, the corresponding complementarity condition $\xi_i^\star\left(t_i^\star-\frac{h}{\sqrt{w_i^\star}}\right)=0$ leads to $t_i^\star=\frac{h}{\sqrt{w_i^\star}}$, so that $w^\star$ fulfills the jerk constraints at i. Therefore, a dual exactness condition is the following.

Proposition 8. *If an optimal solution $\theta^\star$ of the dual Lagrangian problem (25) is such that*

$$\lambda_i^\star+\phi_i^\star<1,\quad i=2,\ldots,n-1,$$

then the optimal value of (25) is equal to the optimal value of (10).

6.2. Results on negative and positive jerk constraints (dual version)

Now, let us consider what happens when $\xi_i^\star=0$ for some i. In this case $w^\star$ might violate the ith jerk constraint. We first make the following observation.

Observation 6.2. *If $\theta^\star\in\Theta_4^i$ for some i, then $\theta^\star$ cannot be an optimal solution of (25).*

Proof. Assume by contradiction that there exists some optimal solution $\theta^\star$ of (25) such that $\theta^\star\in\Theta_4^i$ for some i. Since $\theta^\star\in\Theta_4^i$, we have that

$$\lambda_i^\star+\phi_i^\star=1,\quad \Gamma_i(\theta^\star)>0.$$

Let us consider a further feasible solution $\bar\theta$ such that $\bar\theta=\theta^\star$ except for

$$\bar\lambda_i=\lambda_i^\star-\delta,$$

for some $\delta>0$. We have that

$$F_i(\theta^\star)=0,\quad F_i(\bar\theta)\geq\frac{3}{2}h^{\frac{2}{3}}\left[2\Gamma_i(\theta^\star)\right]^{\frac{1}{3}}\delta^{\frac{2}{3}}.$$

Moreover:

- if $\theta^\star\in\Theta_4^{i-1}$, then $F_{i-1}(\bar\theta)=F_{i-1}(\theta^\star)=0$;
- if $\theta^\star\in\Theta_2^{i-1}\cup\Theta_3^{i-1}\cup\Theta_5^{i-1}$, then $F_{i-1}(\bar\theta)=F_{i-1}(\theta^\star)-\delta\frac{w_{i-1}^{\max}}{hJ}$;
- if $\theta^\star\in\Theta_1^{i-1}$, then $F_{i-1}(\bar\theta)\approx F_{i-1}(\theta^\star)-\delta\frac{h^{\frac{2}{3}}}{hJ}\left[\frac{1-\lambda_{i-1}^\star-\phi_{i-1}^\star}{2\Gamma_{i-1}(\theta^\star)}\right]^{\frac{2}{3}}$ $\geq F_{i-1}(\theta^\star)-\delta\frac{w_{i-1}^{\max}}{hJ}$.

Therefore, in every case $F_{i-1}(\bar\theta)\geq F_{i-1}(\theta^\star)-\delta\frac{w_{i-1}^{\max}}{hJ}$. Similarly, $F_{i+1}(\bar\theta)\geq F_{i+1}(\theta^\star)-\delta\frac{w_{i+1}^{\max}}{hJ}$. Now, since for $\delta>0$ and small enough

$$\begin{aligned}&F_{i-1}(\bar\theta)+F_i(\bar\theta)+F_{i+1}(\bar\theta)\\ &\geq F_{i-1}(\theta^\star)+F_i(\theta^\star)+F_{i+1}(\theta^\star)\\ &+\frac{3}{2}h^{\frac{2}{3}}\left[2\Gamma_i(\theta^\star)\right]^{\frac{1}{3}}\delta^{\frac{2}{3}}-\delta\frac{w_{i-1}^{\max}}{hJ}-\delta\frac{w_{i+1}^{\max}}{hJ}\\ &>F_{i-1}(\theta^\star)+F_i(\theta^\star)+F_{i+1}(\theta^\star),\end{aligned}$$

optimality of $\theta^\star$ is contradicted. □

Another implication of the complementarity conditions is that at optimal solutions $(\theta^\star,\xi^\star)$ of (22) it must hold that $\lambda_i^\star\phi_i^\star=0$ for all $i\in\{2,\ldots,n-1\}$. Indeed, if $\lambda_i^\star\phi_i^\star>0$, we must have $\lambda_i^\star,\phi_i^\star>0$ and, due to the complementarity conditions:

$$t_i^\star=\frac{w_{i-1}^\star-(2+\rho(h))w_i^\star+w_{i+1}^\star}{hJ}$$
$$t_i^\star=\frac{-w_{i-1}^\star+(2+\rho(h))w_i^\star-w_{i+1}^\star}{hJ},$$

which is possible only if $w_{i-1}^\star-(2+\rho(h))w_i^\star+w_{i+1}^\star=0$ and, consequently, $t_i^\star=0$, which is not possible since $t_i^\star\geq\frac{h}{\sqrt{w_i^{\max}}}$.

Now, assume that $\lambda_i^\star=1,\ \phi_i^\star=0$. Then, $t_i^\star=\frac{w_{i-1}^\star-(2+\rho(h))w_i^\star+w_{i+1}^\star}{hJ}>0$, so that:

$$\begin{aligned}hA\geq w_{i-1}^\star-w_i^\star>w_i^\star-w_{i+1}^\star+\rho(h)w_i^\star\quad&\Rightarrow\quad\beta_i^\star=0\\ hA\geq w_{i+1}^\star-w_i^\star>w_i^\star-w_{i-1}^\star+\rho(h)w_i^\star\quad&\Rightarrow\quad\alpha_{i-1}^\star=0,\end{aligned}$$

where the implications follow from the complementarity conditions. Recalling the definition of Γ_i we have that

$$\begin{aligned}\Gamma_i(\theta^\star)\;&=\;\frac{\lambda_{i-1}^\star-(2+\rho(h))\lambda_i^\star+\lambda_{i+1}^\star}{hJ}\\ &\quad+\frac{-\phi_{i-1}^\star+(2+\rho(h))\phi_i^\star-\phi_{i+1}^\star}{hJ}\\ &\quad+\beta_i^\star-\alpha_i^\star-\beta_{i-1}^\star+\alpha_{i-1}^\star\\ &=\;\frac{\lambda_{i-1}^\star-(2+\rho(h))+\lambda_{i+1}^\star}{hJ}+\frac{-\phi_{i-1}^\star-\phi_{i+1}^\star}{hJ}-\alpha_i^\star-\beta_{i-1}^\star\\ &\leq\frac{\lambda_{i-1}^\star-(2+\rho(h))+\lambda_{i+1}^\star}{hJ}<0,\end{aligned}$$

where the last inequality follows from $\lambda_{i-1}^\star,\lambda_{i+1}^\star\leq 1$. Therefore, $\theta^\star\in\Theta_3^i$ if $\lambda_i^\star=1$ and, consequently, $w_i^\star(\theta^\star)=w_i^{\max}$.

Similarly, assume that $\lambda_i^\star=0,\ \phi_i^\star=1$. Then, also recalling that $w_i^\star\leq w_i^{\max}$:

$$t_i^\star=\frac{-w_{i-1}^\star+(2+\rho(h))w_i^\star-w_{i+1}^\star}{hJ}\geq\frac{h}{\sqrt{w_i^\star}}\geq\frac{h}{\sqrt{w_i^{\max}}},$$

so that:

$$\begin{aligned}hA\geq w_i^\star-w_{i+1}^\star\geq w_{i-1}^\star-w_i^\star-\rho(h)w_i^\star+\frac{h}{\sqrt{w_i^{\max}}}\\ hA\geq w_i^\star-w_{i-1}^\star\geq w_{i+1}^\star-w_i^\star-\rho(h)w_i^\star+\frac{h}{\sqrt{w_i^{\max}}}.\end{aligned}$$

For h small enough we have $-\rho(h)w_i^\star+\frac{h}{\sqrt{w_i^{\max}}}>0$, so that

$$\begin{aligned}hA\geq w_i^\star-w_{i+1}^\star>w_{i-1}^\star-w_i^\star\quad&\Rightarrow\quad\beta_{i-1}^\star=0\\ hA\geq w_i^\star-w_{i-1}^\star>w_{i+1}^\star-w_i^\star\quad&\Rightarrow\quad\alpha_i^\star=0.\end{aligned}$$

Again, recalling the definition of Γ_i:

$$\begin{aligned}\Gamma_i(\theta^\star)\;&=\;\frac{\lambda_{i-1}^\star-(2+\rho(h))\lambda_i^\star+\lambda_{i+1}^\star}{hJ}+\frac{-\phi_{i-1}^\star+(2+\rho(h))\phi_i^\star-\phi_{i+1}^\star}{hJ}\\ &\quad+\beta_i^\star-\alpha_i^\star-\beta_{i-1}^\star+\alpha_{i-1}^\star\\ &=\;\frac{\lambda_{i-1}^\star+\lambda_{i+1}^\star}{hJ}+\frac{-\phi_{i-1}^\star+(2+\rho(h))-\phi_{i+1}^\star}{hJ}+\beta_i^\star+\alpha_{i-1}^\star\\ &\geq\frac{-\phi_{i-1}^\star+(2+\rho(h))-\phi_{i+1}^\star}{hJ}>0,\end{aligned}$$

where the last inequality follows from $\phi^\star_{i-1}, \phi^\star_{i+1} \leq 1$. Therefore, $\theta^\star \in \Theta^i_4$, and, as stated in Observation 6.2, $\theta^\star$ cannot be optimal. Then, we proved the following result.

Proposition 9. *The optimal value of (13) can be strictly lower than the optimal value of (10) only if at optimal solutions $\theta^\star$ of the dual Lagrangian (25), it holds that for some $i \in \{2, \ldots, n-1\}$*

$$\lambda^\star_i = 1 \quad \text{and} \quad \Gamma_i(\theta^\star) < 0,$$

i.e., $\theta^\star \in \Theta^i_3$.

7. Counterexamples for more general cases

Up to now we have not been able to generate any instance for which the optimal value of (13) is strictly lower than the optimal value of (10), i.e., we have not been able to show that Conjecture 4.1 is false. However, this is possible for problem classes more general than (10).

7.1. Minimum speed limits

If, in addition to maximum speed limits, we also add strictly positive minimum speed limits, then Conjecture 4.1 is false. A simple example is the following.

Example 7.1. Let us set $h, J = 1$ and $A = +\infty$. Let us assume that the following lower and upper bounds are imposed for the speeds at $i-1, i, i+1$:

$$\begin{array}{ll} w^{\min}_{i-1} = M & w^{\max}_{i-1} = M \\ w^{\min}_{i} = 0 & w^{\max}_{i} = 1 \\ w^{\min}_{i+1} = 1 & w^{\max}_{i+1} = 1. \end{array}$$

The positive jerk constraint

$$w_{i-1} - (2+\rho(h))w_i + w_{i+1} \leq \frac{1}{\sqrt{w_i}}$$

in (10) can only be fulfilled, for $M > 2$ large enough, when $w_i \leq \bar{w}_i$, where $\bar{w}_i < 1$ is such that

$$\frac{1}{\sqrt{\bar{w}_i}} = M - (2+\rho(h))\bar{w}_i + 1 > M - 1 - \rho(h).$$

Then, at an optimal solution of (10) we have $w_i = \bar{w}_i$ and the contribution of the i-term in the objective function is $\frac{1}{\sqrt{\bar{w}_i}}$. Instead, in (13) we can consider a feasible solution with $w_i = 1$ and, consequently, the ith term of the objective function is

$$t_i = M - 1 - \rho(h) < \frac{1}{\sqrt{\bar{w}_i}}.$$

The example above can be simply modified to show that (10) might have empty feasible region while (13) admits feasible solutions. To this end, it is enough to set $A = M - 1$ in the example. Indeed, to fulfill the jerk constraint we need to take $w_i = \bar{w}_i < 1$, so that $w_{i-1} - w_i > A$ and the acceleration constraint is violated. Instead, the solution $w_{i-1} = M$, $w_i = w_{i+1} = 1$ is feasible for (13). We can also prove the following simple observation about the feasibility of (13).

Observation 7.1. *Problem (13) is feasible if and only if problem (10) without the jerk constraints is feasible.*

Proof. The result immediately follows from the observation that the feasible region of (10) without the jerk constraints and the orthogonal projection of the feasible region of (13) over the set of variables $w_1, \ldots, w_n$, are identical sets. □

Given this observation, we remark that feasibility of (13) can be established as shown, e.g., in Consolini et al. (2017). Finally, we remark that conditions for the initial and final velocities $w_1 = w_{in}$ and $w_n = w_{fin}$ different from 0 can be imposed by setting the lower and upper bounds $w^{\min}_1 = w^{\max}_1 = w_{in}$ and $w^{\min}_n = w^{\max}_n = w_{fin}$. Therefore, the examples above with $n = 3$ and $i = 2$ show that for initial and final velocities different from 0 we cannot guarantee exactness of the relaxation and, when exactness does not hold, feasibility of the relaxation does not guarantee feasibility of (10).

7.2. Variable acceleration and jerk bounds

Up to now we have not been able to prove Conjecture 4.1. However, in Section 5 we have been able to prove that optimal solutions of the relaxed problem (13) never violate negative jerk constraints (see Proposition 4), while violations of positive jerk constraints might occur only at points where the maximum speed constraint is active (see Proposition 5). Both propositions are proved by contradiction. More precisely, under the assumption that the optimal solution of (13) violates a jerk constraint, it is shown that the optimal solution can be slightly perturbed in such a way that the perturbed solution is (i) still feasible, and (ii) with lower objective function value, which contradicts optimality.

As a next step we might wonder what happens if, in the definition of problem (10), we replace the constant acceleration bound A with variable bounds A_i, $i = 2, \ldots, n-1$. In this case, the proofs by contradiction of Propositions 4 and 5 cannot be applied any more. Indeed, the perturbed solutions employed in those proofs are not guaranteed to be feasible (some acceleration constraint might be violated). However, both proofs are still valid if we assume that the acceleration bounds do not vary too quickly. In particular, they are still valid if:

$$\left| \frac{A_{i+1} - A_i}{h} \right| \leq \frac{J}{\sqrt{w^{\max}_i}}, \tag{26}$$

i.e., if the variation of the acceleration bound is guaranteed to fulfill the jerk constraints.

Similarly, we might wonder what happens if we replace the constant jerk bound J with variable bounds J_i, $i = 2, \ldots, n-1$. Again, we cannot apply the proofs of Propositions 4 and 5. Indeed, in this case feasibility of the perturbed solution is maintained (provided that the acceleration bound is constant), but the perturbed solution might not have a lower objective function value with respect to the optimal one, thus not leading to a contradiction.

As we will see through the experiments in Section 8.2, in these more general cases we could detect instances where the optimal value of (13) is a strict lower bound of the optimal value of (10), except for the case with fixed jerk bound and variable acceleration bounds fulfilling condition (26). Thus, interestingly, the cases for which we have been able to find instances for which the lower bound is strict are also those to which we cannot extend the proofs of Propositions 4 and 5.

8. Numerical tests

8.1. SOCP reformulation

In the following numerical tests, we will apply the algorithm presented in Section 4.1 to various cases. For efficiency of computation it is convenient to reformulate the relaxed problem (13) as a SOCP (Second-Order Cone Programming), for which solvers more efficient than generic nonlinear solvers are available. To this end, note that constraint $t \geq \frac{h}{\sqrt{w}}$ is equivalent to $t^2 w \geq h^2$.

Since $t, w \geq 0$, following Alizadeh and Goldfarb (2003), this last constraint is equivalent to

$$x_2^2 \leq th$$
$$x_1^2 \leq twh$$
$$h^2 \leq x_1 x_2$$
$$x_1, x_2 \geq 0.$$

The quadratic constraints can be refomulated as the following SOCP constraints:

$$\left\| \begin{pmatrix} \frac{2x_2}{\sqrt{h}} \\ t-1 \end{pmatrix} \right\| \leq t+1, \quad \left\| \begin{pmatrix} \frac{2x_1}{\sqrt{h}} \\ t-w \end{pmatrix} \right\| \leq t+w,$$
$$\left\| \begin{pmatrix} 2h \\ x_2 - x_1 \end{pmatrix} \right\| \leq x_2 + x_1.$$

This leads to the following SOCP reformulation of (13):

$$\begin{array}{lll} \min_{w,t} & g(t) = \sum_{i=2}^{n-1} t_i & \\ & t_i \geq -\frac{w_{i-1}-2w_i+w_{i+1}}{hJ_i} & i = 2, \dots, n-1 \\ & t_i \geq \frac{w_{i-1}-2w_i+w_{i+1}}{hJ_i} & i = 2, \dots, n-1 \\ & x_{2,i}^2 \leq t_i h & i = 2, \dots, n-1 \\ & x_{1,i}^2 \leq t_i w_i h & i = 2, \dots, n-1 \\ & h^2 \leq x_{1,i} x_{2,i} & i = 2, \dots, n-1 \\ & x_{1,i}, x_{2,i} \geq 0 & i = 2, \dots, n-1 \\ & w_{i+1} - w_i \leq A_i h & i = 2, \dots, n-1 \\ & w_i - w_{i+1} \leq A_i h & i = 2, \dots, n-1 \\ & w_1 = w_n = 0 & \\ & 0 \leq w_i \leq w_i^{\max} & i = 1, \dots, n. \end{array} \tag{27}$$

We can solve (27) with efficient commercial solvers, such as MOSEK or GUROBI. Note that we are considering the generic case in which acceleration and jerk bounds depend on i.

8.2. Performed tests

We performed various numerical tests from randomly generated data. First, we comment on the parameters choice in (27). Define vectors $J = (J_2, \dots, J_{n-1}) \in \mathbb{R}^{n-2}, A = (A_2, \dots, A_{n-1}) \in \mathbb{R}^{n-2}$, and, similarly, $w^{\max}, w, t, x_1, x_2 \in \mathbb{R}^n$. We represent all parameters in (27) with set $(h, A, J, w^{\max})$. Define

$\mathcal{S}(h, A, J, w^{\max}) = \{(w, t, x_1, x_2) : (w, t, x_1, x_2)$ is a solution of (27) with parameters $h, A, J, w^{\max}\}$.

Then, by substitution, we can verify the following scaling property

$$(w, t, x_1, x_2) \in \mathcal{S}(h, A, J, w^{\max})$$
$$\Leftrightarrow (\forall \varrho \neq 0)(w, \varrho t, \varrho x_1, \varrho x_2) \in \mathcal{S}(\varrho h, \varrho^{-1} A, \varrho^{-2} J, w^{\max}).$$

Because of this property, it is not restrictive to assume $h = 1$, since we can always reduce to this case by scaling the parameters with $\varrho = h^{-1}$.

In our tests, we used $h = 1$ and $n = 1000$. We randomly generated $w^{\max}$ with the following possible strategies:

- A random vector, in which each component is uniformly distributed in interval [0.01, 100] (named `rnd` in Table 1).
- A piecewise constant vector, in which values $w^{\max}$ are constant for each $n/10$ consecutive components. Again, the values of $w^{\max}$ are random numbers, uniformly distributed in [0.01, 100] (`pw cnst`).
- A piecewise linear function, obtained by linearly interpolating random values at each $n/10$ samples, uniformly distributed in interval [0.1, 100] (`pw lin`).

Then, we generated A according to one of the three strategies:

- A random constant value, obtained from a uniform distribution in interval [0.1, 100] (`cnst`).
- A random vector, in which each component is uniformly distributed in interval [0.1, 100] (`rnd`).
- A regularized random vector, fulfilling condition (26) (`reg`).

Similarly, we generated J in two possible ways:

- A random constant value, obtained from a uniform distribution in interval [0.01, 100] (`cnst`).
- A random vector, in which each component is uniformly distributed in interval [0.01, 100] (`rnd`).

Fig. 1 represents three random problems with $n = 100$. We generated $w^{\max}$ with the three proposed strategies (random, piecewise constant, and piecewise linear), while A and J are constant. The dashed lines represent $w^{\max}$, while the solid line is the optimal solution w^*.

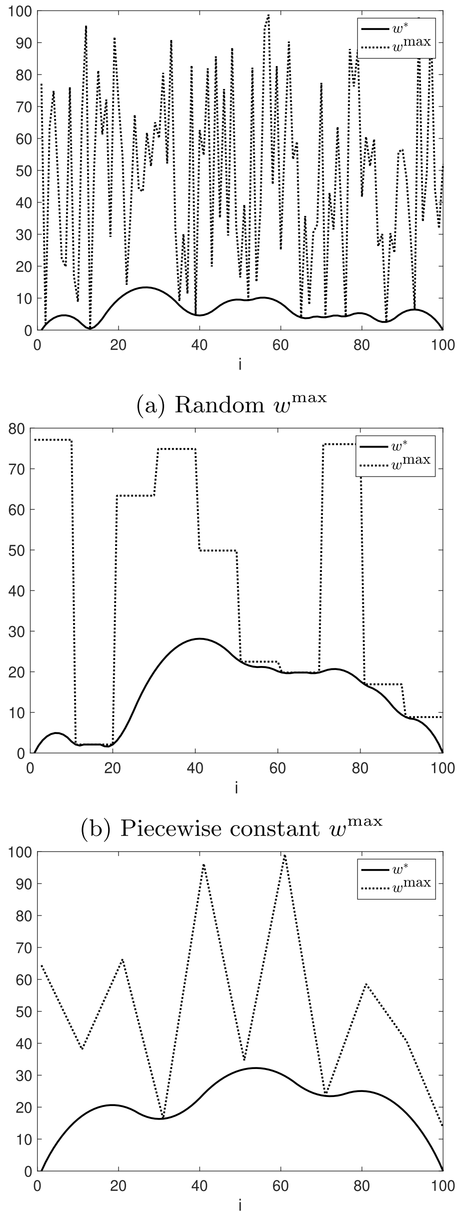
Given all possible choices, we have a total of $3 \times 3 \times 2 = 18$ test types. For each test type, we considered 1000 instances. We solved each instance with the algorithm presented in Section 4.1. For the convex relaxation, we used solver MOSEK on the SOCP reformulation (27). We considered a convex relaxation solution w^* a feasible one if

$$\max_{i=2,\dots,N} \left| w_{i-1}^* - 2w_i^* + w_{i+1}^* \right| - \frac{J_i}{\sqrt{w_i^*}} \leq 10^{-5}, \tag{28}$$

that is, if the violation of the jerk constraint does not exceed 10^{-5}. For those instances in which the convex relaxation solution w^* was not feasible, we used IPOPT to solve problem (10) with starting point w^*. Calling $\hat{w}$ the solution obtained by IPOPT, we then computed the relative gap $\frac{f(\hat{w})-g(t^*)}{g(t^*)}$. Recall that the (true) optimal solution $\bar{w}_\star$ of the non-relaxed problem satisfies $f(\bar{w}_\star) \in [g(t^*), f(\hat{w})]$. Hence, the relative gap is a measure of the remaining uncertainty on the true optimal solution.

Table 1 reports the obtained results. In particular, columns W, A, J represent the method used for generating the random $w^{\max}$, A, J parameter vectors. Then, for each case, we report the number of non-exact relaxations, that is the number of times (over the 1000 tests) in which the convex relaxation violated condition (28), and the maximum and mean values of the jerk error. Then, for those cases in which we have at least one non-exact relaxation, we report the maximum and mean gap. Finally, we report the MOSEK computation time for the convex relaxation. In these tests, we used a laptop with an 8 cores Apple M1 Pro processor and 16 GB of RAM. Note that the computational times are comparable to the ones reported in our paper (Consolini et al., 2022), where we used a completely different approach that only guarantees local optimality.

We stress the fact that, in all the 3000 instances with constant value for A and J, no failure has been observed. In fact, in an attempt to find a counterexample showing that the optimal values of (10) and (13) are not always equal, we performed many more tests with respect to the 3000 reported in Table 1, but no failure has ever been observed. No failure has ever been observed also in the case of constant J value and acceleration bounds fulfilling condition (26). Very few failures are observed when J is constant while A is a randomly generated vector. Instead, the number of failures becomes significantly large when J is a randomly generated vector, especially when also the maximum speed profile is generated in a fully random way. But even in cases where many failures occur, the percentage gap is never too large and never exceeds 1%.



(a) Random $w^{\max}$

(b) Piecewise constant $w^{\max}$

(c) Piecewise linear $w^{\max}$

Fig. 1. Three random problems with $w^{\max}$ generated by different strategies. Parameters J and A are constant, and $N = 100$. The dashed line represents $w^{\max}$, the solid one the optimal solution w^*.

9. Speed planning for road vehicles

We can apply the same method used for relaxation (13) to problems obtained by adding convex constraints to (10). For instance, we consider the longitudinal dynamics of a road vehicle moving along an assigned path:

$$Mv'(s)v(s) = T(s) - D_A v(s)^2 + r(s). \tag{29}$$

Here, $v'(s)v(s)$ is the acceleration at position s, M is the vehicle mass, $T(s)$ is the driving (if positive) or braking (if negative) force applied at position s, and D_A is the aerodynamic drag. Function $r(s)$ depends on the position s. It represents the longitudinal force due to gravity on a grade.

We bound T as $-F_B \le T(s) \le F_D$, where F_B is the maximum braking force, and F_D is the maximum driving force. After substituting $w(s) = v(s)^2$, (29) becomes

$$\frac{1}{2}Mw'(s) = T(s) - D_A w(s) + r(s). \tag{30}$$

Some references, such as Frego, Bertolazzi, Biral, Fontanelli, and Palopoli (2017), add to the right-hand side of (29) a term linear with respect to $v(s)$. Such term represents a drag force linear with speed. We do not consider such term since, after the substitution $w(s) = v(s)$, constraint (30) would become nonconvex.

To avoid slipping between the tyres and the ground, we add the friction ellipse constraint (see, for instance, Frego et al., 2017; Hauser & Saccon, 2006; Velenis & Tsiotras, 2008)

$$\left(\frac{T(s)}{Ma_x}\right)^2 + \left(\frac{\kappa(s)v(s)^2}{a_y}\right)^2 \le 1, \tag{31}$$

where a_x and a_y are the maximum longitudinal and lateral accelerations, and $\kappa(s)$ is the path curvature at position s.

After the substitution $w(s) = v(s)^2$, using (30) we rewrite (31) as

$$\left(\frac{\frac{M}{2}w'(s) + D_A w(s) - r(s)}{Ma_x}\right)^2 + \left(\frac{\kappa(s)w(s)}{a_y}\right)^2 \le 1.$$

This leads to the following adaptation of Problem 1.

Problem 10 (*Speed Planning for Road Vehicles*).

$$\begin{aligned}
&\min_{w\in C^2} \int_0^{s_f} w(s)^{-1/2}\, ds \\
&w(0) = 0, \quad w(s_f) = 0, \\
&0 \le w(s) \le \mu^+(s), && s \in [0, s_f], \\
&-F_B \le \frac{M}{2}w'(s) + D_A w(s) - r(s) \le F_D, && s \in [0, s_f], \\
&\left(\frac{\frac{M}{2}w'(s) + D_A w(s) - r(s)}{Ma_x}\right)^2 \\
&\qquad + \left(\frac{\kappa(s)w(s)}{a_y}\right)^2 \le 1, && s \in [0, s_f] \\
&\frac{1}{2}\left|w''(s)\sqrt{w(s)}\right| \le \frac{J}{2}, && s \in [0, s_f],
\end{aligned}$$

As we did for Problem 1, we discretize w.

Problem 11 (*Discretized Problem*).

$$\begin{aligned}
&\min_{w\in\mathbb{R}^n} \sum_{i=2}^{n-1} \frac{h}{\sqrt{w_i}} \\
&\quad w_1 = w_n = 0 \\
&\quad 0 \le w_i \le w_i^{\max} && i = 2, \dots, n-1, \\
&\quad -F_B \le \frac{M}{2}\frac{w_{i+1} - w_i}{h} \\
&\qquad - cw_i - r(s_i) \le F_D, && i = 1, \dots, n-1, \quad (32) \\
&\quad \left(\frac{\frac{M}{2}\frac{w_{i+1}-w_i}{h} + D_A w_i - r(s_i)}{Ma_x}\right)^2 \\
&\qquad + \left(\frac{\kappa(s_i)w_i}{a_y}\right)^2 \le 1, && i = 1, \dots, n-1 \quad (33) \\
&\quad (w_{i-1} - 2w_i + w_{i+1})\sqrt{w_i} \le h^2 J, && i = 2, \dots, n-1, \\
&\quad -(w_{i-1} - 2w_i + w_{i+1})\sqrt{w_i} \le h^2 J, && i = 2, \dots, n-1,
\end{aligned}$$

Table 1
Computational results for the different combinations of methods W, A, and J to generate parameters $w_{\max}$, A and J, respectively. In particular, the options for W are random (rnd), piecewise-constant (pw cnst), and piecewise-linear (pw lin). The options for A are constant (cnst), random regularized through condition (26) (reg), and random (rnd). Finally, the options for J are constant (cnst) and random (rnd).

W	A	J	Non-exact	Max jerk err	Mean jerk err	Max gap [%]	Mean gap [%]	Mean time [s]
rnd	cnst	cnst	0	8.5519e−08	8.5519e−11	N/A	N/A	0.17068
pw cnst	cnst	cnst	0	0	0	N/A	N/A	0.1533
pw lin	cnst	cnst	0	0	0	N/A	N/A	0.20595
rnd	rnd	cnst	0	0	0	N/A	N/A	0.17975
pw cnst	rnd	cnst	4	0.16594	0.0002596	0.004401	6.803e−06	0.1747
pw lin	rnd	cnst	0	7.9644e−08	9.7622e−11	N/A	N/A	0.1673
rnd	reg	cnst	0	0	0	N/A	N/A	0.19426
pw cnst	reg	cnst	0	0	0	N/A	N/A	0.18469
pw lin	reg	cnst	0	0	0	N/A	N/A	0.17357
rnd	cnst	rnd	756	1.0218	0.2061	0.17484	0.0098672	0.18201
pw cnst	cnst	rnd	45	0.59223	0.0080017	0.067108	0.00034977	0.15341
pw lin	cnst	rnd	0	0	0	N/A	N/A	0.15199
rnd	rnd	rnd	788	0.96156	0.20684	0.19155	0.010135	0.17649
pw cnst	rnd	rnd	251	2.0381	0.11864	0.64832	0.011838	0.21448
pw lin	rnd	rnd	191	2.3979	0.097524	0.80218	0.013251	0.15705
rnd	reg	rnd	808	1.1569	0.22148	0.27889	0.010681	0.19329
pw cnst	reg	rnd	42	0.70596	0.0080485	0.065445	0.00038971	0.19225
pw lin	reg	rnd	0	0	0	N/A	N/A	0.21015

With respect to Problem 2, we substituted the acceleration constraint with linear constraint (32), and we added convex constraint (33). Similarly to Problem 2, we can relax the nonconvex jerk constraints, leading to the following convex relaxation:

Problem 12 (*Convex Relaxation*)**.**

$$
\begin{aligned}
&\min_{w,t} g(t) = \sum_{i=2}^{n-1} t_i \\
&t_i \geq \Delta w_i && i = 2,\ldots,n-1 \\
&t_i \geq -\Delta w_i && i = 2,\ldots,n-1 \\
&t_i \geq \frac{h}{\sqrt{w_i}} && i = 2,\ldots,n-1 \\
&w_1 = w_n = 0 \\
&0 \leq w_i \leq w_i^{\max} && i = 2,\ldots,n-1, \\
&-F_B \leq \frac{M}{2}\frac{(w_{i+1}-w_i)}{h} \\
&\qquad + cw_i + r(s_i) \leq F_D, && i = 1,\ldots,n-1, \\
&\left(\frac{\frac{M}{2}\frac{w_{i+1}-w_i}{h} + cw_i + r(s_i)}{Ma_x}\right)^2 \\
&\qquad + \left(\frac{\kappa(s_i)w_i}{a_y}\right)^2 \leq 1, && i = 1,\ldots,n-1.
\end{aligned}
$$

Due to the analogous structure, we expect Problem 12 to have similar properties as (13). Indeed, form our numerical experiments, we found that, if term $r(s)$ is not present in (30), relaxation (13) is exact. It is not exact, in few cases, if term $r(s)$ is present. This is not surprising, since a non-constant term $r(s)$ is equivalent to have acceleration bounds that vary with s. Similarly to Section 8.2, we considered various instances of Problem 12, with $h = 1$, $n = 1000$, $M = 1000$. We randomly generated $w^{\max}$ in three different ways (random, piecewise constant, piecewise linear), as discussed in Section 8.2. For $\frac{F_B}{M}$, $\frac{F_D}{M}$ a_x, a_y, J, D_A we used random values obtained from a uniform distribution in interval [0.1, 100]. We generated each value of $k(s_i)$ from a random uniform distribution in interval [0.1, 100]. We generated $r(s_i)$ in two ways:

- We set $r(s_i) = 0$ (`zero`),
- We generated each value of $r(s_i)$ from a random uniform distribution in interval $[-\frac{F_D}{2}, \frac{F_B}{2}]$ (`rnd`). In this way, the braking and acceleration forces may be sufficiently large to overcome the force $r(s)$, due to gravity.

Table 2 reports the obtained results. In particular, columns W, R represent the method used for generating the $w^{\max}$ and r. The other columns have the same meaning as in Table 1.

Note that we did not observe any failure in all the 3000 instances with value r_i set equal to 0 for all i, while we observed a single failure, possibly due to numerical issues, when the r_i values are randomly generated.

10. Conclusions and future research

In this paper we addressed the speed planning problem along a given trajectory under maximum speed, tangential and lateral acceleration, and jerk constraints. The problem is reformulated as a nonconvex optimization problem, where nonconvexity is due to the jerk constraints. We derived a convex relaxation of such non-convex problem, exploiting the special structure of the jerk constraints, which can be rewritten in such a way that their left-hand side is equal to the terms of the objective function. Different strong theoretical properties of the convex relaxation are proved, in particular, the fact that its optimal solution never violates negative jerk constraints and can only violate positive jerk constraints at points where the maximum speed constraint is active, and the fact that the relaxation is exact if the maximum speed profile does not violate the positive jerk constraints. In fact, we conjectured that the convex relaxation is always exact, i.e., its optimal value is always equal to the optimal value of the speed planning problem, and an optimal solution of the latter can always be derived from an optimal solution of the former. We have been unable to prove this result, but after performing thousands of tests we have never found a counterexample to our conjecture. Instead, the conjecture turns out to be false as soon as we extend the class of problems, in particular, by allowing different bounds for the jerk along the trajectory. It has also been shown that the convex relaxation can be rewritten as a SOCP problem. This has a relevant practical impact, since solvers for SOCP problems are quite efficient and allow solving large instances within tenths of a second.

As a possible topic for future research, we are interested in evaluating the performance of the proposed approach to robotic manipulators. These problems can be reformulated in a way similar to the problems addressed in this paper. We do not expect to be able to extend Conjecture 4.1 to such problems, but we

Table 2
Computational results for road vehicles speed planning. Different combinations of the methods W and R to generate $w_{\max}$ and r, respectively, are tested. The options for W are random (rnd), piecewise-constant (pw cnst), and piecewise-linear (pw lin), while those for R are null (zero) or randomly generated in the interval $[-\frac{F_D}{2}, \frac{F_B}{2}]$ (rnd).

W	R	Non-exact	Max jerk err	Mean jerk err	Max gap [%]	Mean gap [%]	Mean time [s]
rnd	zero	0	0	0	N/A	N/A	0.17911
pw cnst	zero	0	0	0	N/A	N/A	0.162
pw lin	zero	0	0	0	N/A	N/A	0.16066
rnd	rnd	0	0	0	N/A	N/A	0.18549
pw cnst	rnd	0	0	0	N/A	N/A	0.17176
pw lin	rnd	1	3.687e−05	3.708e−08	0.0040357	4.0357e−06	0.16851

do expect that, even in cases where the convex relaxation is not exact, the final solution returned by the proposed solution algorithm has a small percentage gap with respect to the optimal one.

References

Alizadeh, F., & Goldfarb, D. (2003). Second-order cone programming. *Mathematical Programming*, *95*(1), 3–51.
Artuñedo, Antonio, Villagra, Jorge, & Godoy, Jorge (2022). Jerk-limited time-optimal speed planning for arbitrary paths. *IEEE Transactions on Intelligent Transportation Systems*, *23*(7), 8194–8208.
Bazaraa, M. S., Sherali, H. D., & Shetty, C. M. (1993). *Nonlinear programming theory and algorithms*. New York: John Wiley.
Bianco, Corrado Guarino Lo (2006). Optimal velocity planning for autonomous vehicles under kinematic constraints. *IFAC Proceedings Volumes*, *39*(15), 126–131.
Bobrow, J. E., Dubowsky, S., & Gibson, J. S. (1985). Time-optimal control of robotic manipulators along specified paths. *The International Journal of Robotics Research*, *4*(3), 3–17.
Boyd, Stephen, & Vandenberghe, Lieven (2004). *Convex optimization*. Cambridge University Press.
Cabassi, Federico, Consolini, Luca, & Locatelli, Marco (2018). Time-optimal velocity planning by a bound-tightening technique. *Computational Optimization and Applications*, *70*(1), 61–90.
Consolini, Luca, Locatelli, Marco, & Minari, Andrea (2022). A sequential algorithm for jerk limited speed planning. *IEEE Transactions on Automation Science and Engineering*, *19*(4), 3192–3209.
Consolini, Luca, Locatelli, Marco, Minari, Andrea, & Piazzi, Aurelio (2017). An optimal complexity algorithm for minimum-time velocity planning. *Systems & Control Letters*, *103*, 50–57.
Debrouwere, Frederik, Van Loock, Wannes, Pipeleers, Goele, Dinh, Quoc Tran, Diehl, Moritz, De Schutter, Joris, et al. (2013). Time-optimal path following for robots with convex–concave constraints using sequential convex programming. *IEEE Transactions on Robotics*, *29*(6), 1485–1495.
Digani, Valerio, Hsieh, M. Ani, Sabattini, Lorenzo, & Secchi, Cristian (2019). Coordination of multiple AGVs: a quadratic optimization method. *Autonomous Robots*, *43*, 539–555.
Dong, Jingyan, Ferreira, P. M., & Stori, J. A. (2007). Feed-rate optimization with jerk constraints for generating minimum-time trajectories. *International Journal of Machine Tools and Manufacture*, *47*(12), 1941–1955.
Frego, M., Bertolazzi, E., Biral, F., Fontanelli, D., & Palopoli, L. (2016). Semi-analytical minimum time solutions for a vehicle following clothoid-based trajectory subject to velocity constraints. In *2016 European control conference* (pp. 2221–2227).
Frego, Marco, Bertolazzi, Enrico, Biral, Francesco, Fontanelli, Daniele, & Palopoli, Luigi (2017). Semi-analytical minimum time solutions with velocity constraints for trajectory following of vehicles. *Automatica*, *86*, 18–28.
Frego, Marco, Bevilacqua, Paolo, Divan, Stefano, Zenatti, Fabiano, Palopoli, Luigi, Biral, Francesco, et al. (2020). Minimum time—Minimum jerk optimal traffic management for AGVs. *IEEE Robotics and Automation Letters*, *5*(4), 5307–5314.
Gasparetto, A., Lanzutti, A., Vidoni, R., & Zanotto, V. (2012). Experimental validation and comparative analysis of optimal time-jerk algorithms for trajectory planning. *Robotics and Computer-Integrated Manufacturing*, *28*(2), 164–181.
Haschke, R., Weitnauer, E., & Ritter, H. (2008). On-line planning of time-optimal, jerk-limited trajectories. In *2008 IEEE/RSJ international conference on intelligent robots and systems* (pp. 3248–3253).
Hauser, J., & Saccon, A. (2006). Motorcycle modeling for high-performance maneuvering. *IEEE Control Systems Magazine*, *26*(5), 89–105.
Heinemann, Tonja, Riedel, Oliver, & Lechler, Armin (2019). Generating smooth trajectories in local path planning for automated guided vehicles in production. *Procedia Manufacturing*, *39*, 98–105.
Kang Shin, & McKay, N. (1985). Minimum-time control of robotic manipulators with geometric path constraints. *IEEE Transactions on Automatic Control*, *30*(6), 531–541.
Kant, Kamal, & Zucker, Steven W. (1986). Toward efficient trajectory planning: The path-velocity decomposition. *The International Journal of Robotics Research*, *5*(3), 72–89.
Ma, Jian-wei, Gao, Song, teng Yan, Hui, Lv, Qi, & qing Hu, Guo (2021). A new approach to time-optimal trajectory planning with torque and jerk limits for robot. *Robotics and Autonomous Systems*, *140*, Article 103744.
Macfarlane, S., & Croft, E. A. (2003). Jerk-bounded manipulator trajectory planning: design for real-time applications. *IEEE Transactions on Robotics and Automation*, *19*(1), 42–52.
Palleschi, A., Garabini, M., Caporale, D., & Pallottino, L. (2019). Time-optimal path tracking for jerk controlled robots. *IEEE Robotics and Automation Letters*, *4*(4), 3932–3939.
Palleschi, Alessandro, Hamad, Mazin, Abdolshah, Saeed, Garabini, Manolo, Haddadin, Sami, & Pallottino, Lucia (2021). Fast and safe trajectory planning: Solving the cobot performance/safety trade-off in human-robot shared environments. *IEEE Robotics and Automation Letters*, *6*(3), 5445–5452.
Pham, Hung, & Pham, Quang-Cuong (2017). On the structure of the time-optimal path parameterization problem with third-order constraints. In *2017 IEEE international conference on robotics and automation* (pp. 679–686). IEEE.
Raineri, M., & Guarino Lo Bianco, C. (2019). Jerk limited planner for real-time applications requiring variable velocity bounds. In *2019 IEEE 15th international conference on automation science and engineering* (pp. 1611–1617).
Shimizu, Yutaka, Horibe, Takamasa, Watanabe, Fumiya, & Kato, Shinpei (2022). Jerk constrained velocity planning for an autonomous vehicle: Linear programming approach. In *2022 international conference on robotics and automation* (pp. 5814–5820).
Singh, A. K., & Krishna, K. M. (2015). A class of non-linear time scaling functions for smooth time optimal control along specified paths. In *2015 IEEE/RSJ international conference on intelligent robots and systems* (pp. 5809–5816).
Velenis, E., & Tsiotras, P. (2008). Minimum-time travel for a vehicle with acceleration limits: Theoretical analysis and receding-horizon implementation. *Journal of Optimization Theory and Applications*, *138*(2), 275–296.
Villagra, J., Milanés, V., Pérez, J., & Godoy, J. (2012). Smooth path and speed planning for an automated public transport vehicle. *Robotics and Autonomous Systems*, *60*, 252–265.
Wang, Mingli, Xiao, Juliang, Liu, Sijiang, & Liu, Haitao (2022). A third-order constrained approximate time-optimal feedrate planning algorithm. *IEEE Transactions on Robotics*, *38*(4), 2295–2307.
Zhang, Y., Chen, H., Waslander, S. L., Yang, T., Zhang, S., Xiong, G., et al. (2018). Speed planning for autonomous driving via convex optimization. In *2018 21st international conference on intelligent transportation systems* (pp. 1089–1094).
Zhang, Ke, Yuan, Chun-Ming, Gao, Xiao-Shan, & Li, Hongbo (2012). A greedy algorithm for feedrate planning of CNC machines along curved tool paths with confined jerk. *Robotics and Computer-Integrated Manufacturing*, *28*(4), 472–483.

Luca Consolini received the master's degree cum laude in Electronic Engineering from at the University of Parma in 2000, and the Ph.D. degree from the same university in 2005. From 2005 to 2009 he has been a postdoc at the University of Parma, where, from 2009 to 2014, he has been Assistant Professor. Since 2014, he is Associate Professor at the University of Parma. His main current research interests are nonlinear control, motion planning, and control of mechanical systems.

Marco Locatelli is Full Professor of Operations Research at the University of Parma. His main research interests are the theoretical, practical and applicative aspects of optimization. He has published more than one hundred papers in international journals and co-authored, with F. Schoen, the book Global Optimization: Theory, Algorithms, and Applications for the Society for Industrial and Applied Mathematics (SIAM). He has been nominated EUROPT Fellow in 2018. He is currently in the editorial board of the journals Computational Optimization and Applications, Journal of Global Optimization, Operations Research Forum, and Operations Research Letters.