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KKT-based primal-dual exactness conditions for the Shor relaxation

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Abstract

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- ³ more generally, diagonalizable) QCQPs, which extend the conditions introduced in different
- ⁴ recent papers about the same topic. It is shown that the Shor relaxation is equivalent to two
- 5 convex quadratic relaxations. Then, sufficient conditions for the exactness of the relaxations
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Keywords Quadratically Constrained Quadratic Programming · Shor relaxation · Convex relaxations · Exactness conditions

12 1 Introduction

In the recent literature different results about the exactness of the Shor relaxation (see [17]) 13 for Quadratically Constrained Quadratic Programming (QCQP in what follows) problems 14 have been proposed. The Shor relaxation can be proved to be exact for the Generalized Trust 15 Region Subproblem (GTRS), where a single (not necessarily convex) quadratic inequality 16 constraint is present. The exactness proof can be derived from a result discussed in [11]. 17 For other QCQPs the Shor relaxation is not always exact and different papers introduce 18 conditions under which exactness holds for sub-classes of QCQPs. Some exactness results 19 for the case of QCQPs with two quadratic constraints have been presented in [21], while in 20 [1] a necessary and sufficient condition for the exactness of the related Lagrangian dual has 21 been given. Note that the case with two quadratic constraints, which includes the well known 22 Celis-Dennis-Tapia (CDT) problem, has been recently proved to be polynomially solvable in 23 different works [5, 10, 16]. However, both the polynomial approaches proposed in [10, 16], 24 based on the enumeration of all KKT points via the solution of bivariate polynomial systems, 25

and the polynomial approach proposed in [5], based on Barvinok's construction, have a

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limited practical applicability due to the large exponent of the polynomials appearing in the 27 complexity result. For QCQPs with a single unit ball constraint and further linear constraints, 28 in [13] a dimension condition establishing exactness of the Shor relaxation is introduced. 29 In [3] a Second Order Cone Programming (SOCP) relaxation for the same problem has 30 been discussed, while in [15] it has been shown that such relaxation is equivalent to the 31 Shor relaxation. By the analysis of the KKT conditions for the SOCP relaxation, in [15] a 32 condition more general than the dimension condition presented in [13] has been given. Note 33 that in [6, 19] an exact convex relaxation, obtained by adding to the Shor relaxation a so 34 called SOC-RLT constraint, has been introduced for the case of a single linear constraint, 35 while in [7] the result has been extended to a generic number of linear constraints provided 36 that these constraints have an empty intersection inside the unit ball. It is also worthwhile 37 to mention that a polynomial-time algorithm for the solution of this problem (possibly also 38 with the addition of further ball and reverse ball constraints) has been proposed under the 39 assumption that the overall number of constraints is fixed (see [4]). The approach is based 40 on an enumeration of all possible KKT points. 41

In this paper we are interested in deriving exactness conditions of the Shor relaxation in 42 case of diagonal QCQPs, i.e., quadratic problems where the Hessian of all quadratic functions 43 is diagonal or can be made diagonal after a change of variables (the Hessian matrices are 44 simultaneously diagonalizable). In what follows we assume that the QCQP problem is already 45 given in diagonal form. Throughout the paper $N = \{1, ..., n\}$ will be the index set of the 46 variables, and $M = \{1, \dots, m\}$ will be the index set of the constraints. For a given symmetric 47 matrix Y, the notation $Y \succeq O$ means that the matrix is positive semidefinite. By diag(Y) we 48 will denote the vector whose entries are the diagonal entries of matrix Y. 49

⁵⁰ A diagonal QCQP problem is defined as follows: $c^{\star} = \min_{\mathbf{x}} \mathbf{x}^{\top} \mathbf{D} \mathbf{x} + 2\mathbf{c}^{\top} \mathbf{x}$

$$^{\star} = \min_{\mathbf{x}} \mathbf{x}^{\top} \mathbf{D} \mathbf{x} + 2\mathbf{c}^{\top} \mathbf{x} \mathbf{x}^{\top} \mathbf{A}^{i} \mathbf{x} + 2\mathbf{a}_{i}^{\top} \mathbf{x} \le b_{i} \quad i \in M,$$
 (1)

where matrix **D** and all matrices A_i , $i \in M$, are diagonal. The classical Shor relaxation for this problem is:

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- $v^{\star} = \min_{\mathbf{x}, \mathbf{X}} \mathbf{D} \bullet \mathbf{X} + 2\mathbf{c}^{\top} \mathbf{x}$ $\mathbf{A}^{i} \bullet \mathbf{X} + 2\mathbf{a}_{i}^{\top} \mathbf{x} \leq b_{i} \ i \in M$ $\mathbf{X} - \mathbf{x} \mathbf{x}^{\top} \succeq \mathbf{O}.$ (2)
- ⁵⁵ The existence of minimizers and, thus, the use of min rather than inf in problems (1) and (2)
- is guaranteed under the following suitable assumptions, introduced in [8]:
- 57 Assumption 1 The following hold:
- The feasible region of (1) is nonempty;
- ⁵⁹ $\exists \bar{\mathbf{y}} \ge \mathbf{0}$ such that $\sum_{i \in M} \bar{y}_i \mathbf{A}_i \succ \mathbf{O}$;
- The interior of the feasible region of (2) is nonempty.
- In particular, note that these assumptions imply that the feasible region of problem (1) is bounded.
- ⁶³ This assumption will be maintained throughout the paper.
- ⁶⁴ In [8] some sufficient conditions are introduced under which there exists an optimal rank-one
- solution for the Shor relaxation, which is equivalent to proving that the Shor relaxation is exact, i.e., $v^* = c^*$. More precisely, for $k \in N$, let:

$$\mathcal{L}_{k} = \left\{ \boldsymbol{\mu} \ge \mathbf{0} : D_{kk} + \sum_{i \in M} \mu_{i} A_{kk}^{i} = 0, \ c_{k} + \sum_{i \in M} \mu_{i} a_{ik} = 0 \right\},$$
(3)

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and for $j \in N$: 68

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$$\mathcal{H}_j = \left\{ \boldsymbol{\mu} : D_{jj} + \sum_{i \in M} \mu_i A^i_{jj} \ge 0 \right\}.$$
(4)

It is proved that the Shor relaxation is exact if for each $k \in N$ the following polyhedral set 70 is empty: 71

$$S_k = \mathcal{L}_k \cap \left[\cap_{j \in N \setminus \{k\}} \mathcal{H}_j \right].$$
(5)

This result allows to re-derive a sign-definiteness condition presented in [18], stating that 73 exactness holds if for all $j \in N$, c_i and a_{ii} , $i \in M$, are all nonpositive or all nonnegative. 74

Moreover, for the relevant special case when $A_i \in \{I, -I, O\}$ for each $i \in M$, i.e., when 75 all constraints are ball, reverse ball, and linear constraints, in [8] it is shown that exactness 76 holds when the sign-definite condition is only satisfied by the variable corresponding to 77 the lowest diagonal entry of matrix **D**. Note that this special case is addressed also in [2], 78 where a branch-and-bound approach for its solution is proposed and an application to source 79 localization problems is presented. 80

A further very recent result has been proved in [23], where a class of problems larger than 81 the class of diagonal QCQPs is considered. We briefly discuss the condition introduced in 82 that paper, only in the case of inequality constraints, although also equality constraints may 83 be included. Note that in this case matrices **D** and A_i , $i \in M$, are not necessarily diagonal. 84

Let 85

$$\mathbf{A}(\boldsymbol{\gamma}) = \mathbf{D} + \sum_{i \in M} \gamma_i \mathbf{A}_i, \quad \mathbf{b}(\boldsymbol{\gamma}) = \mathbf{c} + \sum_{i \in M} \gamma_i \mathbf{a}_i$$

Let 87

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$$\Gamma = \{ \boldsymbol{\gamma} : \mathbf{A}(\boldsymbol{\gamma}) \succeq \mathbf{O}, \ \boldsymbol{\gamma} \ge \boldsymbol{0} \}.$$

A face \mathcal{F} of Γ which does not contain any γ such that $\mathbf{A}(\gamma) \succ \mathbf{O}$ is called a semidefinite 89 face, and the zero eigenspace of \mathcal{F} is 90

$$\mathcal{V}(\mathcal{F}) = \{\mathbf{x} : \mathbf{A}(\boldsymbol{\gamma})\mathbf{x} = \mathbf{0}, \ \forall \boldsymbol{\gamma} \in \mathcal{F}\}.$$

In [23] it is assumed that Γ is a polyhedral set. While this assumption is always fulfilled for 92 diagonal OCOPs, it is also shown that it may hold also for non-diagonal OCOPs, but it is 93 pointed out that it is coNP-hard to decide whether the assumption holds. Exactness of the 94 Shor relaxation is proved under the condition that there exists some infinite sequence $\{\mathbf{h}^k\}$ 95 such that $\mathbf{h}^k \to \mathbf{0}$ (see the perturbation argument below) and for any k and any semidefinite 96 face \mathcal{F} it holds that: 97

$$\mathbf{0} \notin \{ Proj_{\mathcal{V}(\mathcal{F})}(b(\boldsymbol{\gamma}) + \mathbf{h}^k) : \boldsymbol{\gamma} \ge \mathbf{0} \}.$$
(6)

Note that in the same paper also some conditions are discussed under which the convex hull 99 of the epigraph of the QCQP is given by the projection of the epigraph of its Shor relaxation. 100 Another recent result about this topic can be found in [14]. In that work minimax QCQPs 101 are considered, namely, the following problems are addressed 102

$$\min_{\mathbf{x}} \max_{r \in R} \mathbf{x}^{\top} \mathbf{D}^{r} \mathbf{x} + 2\mathbf{c}_{r}^{\top} \mathbf{x} + c_{0r}$$

$$\mathbf{x}^{\top} \mathbf{A}^{i} \mathbf{x} + 2\mathbf{a}_{i}^{\top} \mathbf{x} \le b_{i} \qquad i \in M,$$
(7)

where all matrices \mathbf{D}^r , $r \in R$, \mathbf{A}^i , $i \in M$, are diagonal, possibly obtained after the simultane-104 ous diagonalization of all the Hessian matrices. Note that this class of problems is equivalent 105

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to the class of problems (1). Indeed, each problem (1) can be viewed as a special case of (7) by 106 taking |R| = 1, while, on the other hand, each problem (7) can be converted into an instance 107 of problem (1) after the addition of a variable y, which becomes the objective function to 108 be minimized, and of the related constraints $y \ge \mathbf{x}^\top \mathbf{D}^r \mathbf{x} + 2\mathbf{c}_r^\top \mathbf{x} + c_{0r}$ for each $r \in R$. In 109 [14] a SOCP relaxation of problem (7) is introduced which is equivalent to the Lagrangian 110 dual of this problem and, thus, also to the Shor relaxation (recall that the Lagrangian dual 111 and the Shor relaxation are dual to each other and, thus, have the same optimal value if a 112 constraint qualification holds). In [14] an exactness condition is introduced based on the so 113 called epigraphical set, defined as follows: 114

$$E = \left\{ (\mathbf{w}, \mathbf{v}) \in \mathbb{R}^{|R| + |M|} : \exists \mathbf{x} \in \mathbb{R}^{|N|} : \mathbf{x}^\top \mathbf{D}^r \mathbf{x} + 2\mathbf{c}_r^\top \mathbf{x} + c_{0r} \le w_r, r \in R, \\ \mathbf{x}^\top \mathbf{A}^i \mathbf{x} + 2\mathbf{a}_i^\top \mathbf{x} \le v_i, i \in M \right\}.$$
(8)

¹¹⁶ It is shown that the SOCP relaxation is exact if the epigraphical set is closed and convex.

Some applications of diagonal QCQPs In the literature there are different applications of
 diagonal QCQPs. Here we briefly review a few of them.

The extended trust region subproblem (extended TRS) is the trust region problem with 119 additional linear constraints. After diagonalizing the objective function, this becomes a diag-120 onal QCQP where $A_i \in \{I, O\}$ for each $i \in M$ (more precisely, all matrices A_i are null, except 121 one which is equal to the identity matrix). As outlined in [13], such problem arises from the 122 application of the trust region method in the context of linearly constrained problems, from 123 nonlinear optimization problems with discrete variables, and from robust optimization prob-124 lems. Moreover, QP problems whose feasible region is a polytope can be reformulated as an 125 extended TRS after the addition of a ball constraint (a ball enclosing the feasible polytope). 126 In [14] the max dispersion problem is presented as an application of diagonal QCQPs. In 127 this problem, given a finite set of location positions \mathbf{u}_i , $i = 1, \dots, p$, and a further point \mathbf{x}_0 , 128

we aim at identifying the position of a new location which maximizes the minimal distance from all the other locations. The new position is subject to a ball constraint, i.e., it must belong to a sphere centered at \mathbf{x}_0 , and is possibly subject to further linear constraints.

The problem of minimizing a quadratic function over a 'Swiss cheese' domain, i.e., a feasible region defined by ball, reverse ball, and linear constraints, has been discussed, e.g., in [4] (see also [23] for an exactness result when the objective function to be minimized is the Euclidean norm). After diagonalization of the objective function, this problem belongs to the special case of diagonal QCQPs with $A_i \in \{I, -I, O\}$ for each $i \in M$. The latter special case is also the focus of paper [2], where a branch and bound approach is proposed and an application to sparse source localization problems is presented.

Finally, in [8] it is shown that general QCQPs can be reformulated as diagonal QCQPs with additional variables.

Statement of contribution The main contribution of this work lies in the derivation of 141 exactness conditions of the Shor relaxation for diagonal QCQPs through an approach different 142 with respect to the existing, recent, literature, in particular, with respect to [8, 14, 23]. The 143 conditions are derived from the KKT conditions of an equivalent SOCP reformulation of the 144 Shor relaxation. They are primal-dual conditions, while the other conditions in the literature 145 appear to be dual conditions. As we will see through a simple example, besides being derived 146 in a different way, the new conditions also allow to establish exactness results which cannot 147 be established by the existing conditions. The new conditions are particularly significant 148

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when $A_i \in \{I, -I, O\}$ for each $i \in M$, which, according to the previous discussion, is a relevant subcase of diagonal QCQPs.

Outline of the paper In this paper we first state in Sect. 2, by a straightforward extension 151 of a result proved in [15], that for diagonal QCQPs the Shor relaxation is equivalent to a 152 quadratic convex relaxation of problem (1). Next, in Sect. 3 the exactness condition related 153 to the emptiness of the sets (5) is re-derived through an analysis of the KKT conditions of the 154 convex relaxation. Moreover, in Sect. 4, it is shown how to strengthen the exactness condition 155 in some cases and, in particular, in the already mentioned case when $A^i \in \{I, -I, O\}$ for each 156 $i \in M$. It is shown through an example that the new condition can be stronger than those 157 discussed in [8, 14, 23]. Finally, in Sect. 5 a further equivalent convex relaxation is introduced 158 and it is shown that KKT conditions for this relaxation allow to define an exactness condition 159 which can be more efficiently checked. 160

2 A convex relaxation equivalent to the Shor relaxation

Before proceeding, we subdivide the class of diagonal QCQPs in some subclasses on the basis of a partition N_h , $h \in H$, of the set N, such that each set N_h contains indexes of variables whose coefficients of the quadratic terms are all equal throughout the constraints (but not necessarily in the objective function). Formally:

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$$\forall j,k \in N_h, \ \forall h \in H, \ \forall i \in M : \ A^i_{jj} = A^i_{kk} = \xi^{ih}.$$
(9)

Note that the general case is a special case where |H| = |N| and each set N_h is a singleton. In the special case, discussed in [8], when $A_i \in \{I, -I, O\}$ for all $i \in M$, we have that |H| = 1. In fact, when |H| = 1 the problem can always be rewritten in such a way that $A_i \in \{I, -I, O\}$ for all $i \in M$. We introduce the following assumption.

Assumption 2 For each $h \in H$, the set $\arg \min_{j \in N_h} D_{jj}$ is a singleton. We denote by j_h its single member and by d_h^* the minimum diagonal entry D_{jj} for $j \in N_h$, i.e.:

 $j_h = \arg\min_{j \in N_h} D_{jj}, \quad d_h^* = \min_{j \in N_h} D_{jj}.$ (10)

Later on we will show that removing this assumption allows to derive an even more general exactness condition. But, in order to simplify the presentation, we will impose that the assumption holds.

In what follows we employ set $N_H = \{j_h : h \in H\} \subseteq N$.

Exploiting the fact that all matrices are diagonal, following [3], a convex relaxation of problem
(1) is:

$$p^{\star} = \min_{(\mathbf{x}, \mathbf{z}) \in \mathcal{X}} \sum_{j \in N} D_{jj} z_j + 2 \sum_{j \in N} c_j x_j, \tag{11}$$

181 where:

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$$\mathcal{X} = \left\{ (\mathbf{x}, \mathbf{z}) : \sum_{h \in H} \sum_{j \in N_h} \xi^{ih} z_j + 2 \sum_{j \in N} a_{ij} x_j \le b_i, \ i \in M, \ x_j^2 \le z_j, \ j \in N \right\}.$$

Note that this is a relaxation since the same problem with constraints $x_j^2 \le z_j$, $j \in N$, replaced by equations $x_j^2 = z_j$ is an equivalent reformulation of problem (1). In [15] the

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equivalence was proven between this relaxation and the Shor relaxation when $A_1 = I$, $a_1 = 0$, $A_i = O$ for all $i \in M \setminus \{1\}$. The result can be extended in a quite straightforward way to the general problem (1) (see also the proof in [23] and note that the result can also be obtained as a special case of some results on sparse semidefinite programming problems presented in [12]).

Theorem 1 It holds that $p^* = v^*$, i.e., the optimal values of the Shor relaxation (2) and of the convex relaxation (11) are equal.

Now, this equivalence result can be employed in order to establish exactness conditions for
 the Shor relaxation by the analysis of the KKT conditions of the convex relaxation. This will
 be the topic of the next section.

Before proceeding we briefly introduce the perturbation argument already adopted in [8, 15,
23] (see, e.g., the discussion following Proposition 1 in [8], Theorem 3.1 in [15], and (6) in
[23]). We will make extensive use of this argument in the following sections.

Proposition 1 Let Assumption 1 hold. Exactness of the Shor relaxation is verified for a problem with data ($\mathbf{D}, \mathbf{A}_i, \mathbf{a}_i, \mathbf{c}, \mathbf{b}$) if it is verified for an infinite sequence of problems with perturbed data ($\mathbf{D} + \Delta \mathbf{D}^k, \mathbf{A}_i, \mathbf{a}_i, \mathbf{c} + \Delta \mathbf{c}^k, \mathbf{b}$) such that

$$||\Delta \mathbf{D}^k||, ||\Delta \mathbf{c}^k|| \to 0.$$

Proof The result holds true for perturbations $\Delta \mathbf{D}^k$ and $\Delta \mathbf{c}^k$ in the objective function, since, by continuity and by boundedness of the feasible region implied by Assumption 1, the optimal values of problem (1) with the perturbed data converge to the optimal value of the unperturbed problem, and the same holds for the optimal values of the corresponding Shor relaxations. \Box

3 Sufficient conditions for exactness of the Shor relaxation

Theorem 1 implies that proving exactness of the Shor relaxation is equivalent to prove 207 exactness of the convex relaxation (11). Under Assumption 1, which we recall is maintained 208 throughout the paper, optimal solutions of the convex problem (11) fulfill the corresponding 209 KKT conditions. In particular, we notice that existence of an interior feasible solution $(\bar{\mathbf{X}}, \bar{\mathbf{x}})$ 210 for problem (2) implies that also the convex relaxation (11) admits an interior feasible point. 211 Indeed, it is enough to consider the point $(diag(\mathbf{X}), \bar{\mathbf{x}})$. Then, Slater's condition holds and 212 we can search the minimizer of problem (11) among the KKT points of the same problem. 213 The KKT conditions are the following: 214

215
$$D_{jj} + \sum_{i \in M} \mu_i \xi^{ih} - \nu_j = 0 \quad j \in N_h, \ h \in H$$
(12a)

1

$$c_j + \sum_{i \in M} \mu_i a_{ij} + \nu_j x_j = 0 \quad j \in N$$
(12b)

$$\mu_i \left(b_i - \sum_{h \in H} \sum_{j \in N_h} \xi^{ih} z_j - 2 \sum_{j \in N} a_{ij} x_j \right) = 0 \quad i \in M$$
(12c)

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$$p_j(z_j - x_j^2) = 0 \quad j \in N$$
 (12d)

$$(\mathbf{x}, \mathbf{z}) \in \mathcal{X}, \ \boldsymbol{\mu}, \boldsymbol{\nu} \ge \mathbf{0}. \tag{12e}$$

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Note that in view of equations (12a), for each $h \in H$: 220

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$$\nu_j - \nu_{j_h} = D_{jj} - d_h^* \quad \forall j \in N_h.$$
⁽¹³⁾

In view of the definition of j_h , we have, under Assumption 2, $v_i > 0$ for all $j \in N_h \setminus \{j_h\}$. 222 223

Now, it obviously holds that the relaxation is exact if all constraints $x_i^2 \leq z_i, j \in N$, 224 are active at the optimal solution of (11). In view of the complementarity conditions (12d), 225 this certainly holds if $v_i > 0$ for all $j \in N$. 226

Let us denote by \mathcal{W} the set of vectors $(\mathbf{x}, \mathbf{z}, \boldsymbol{\mu}, \boldsymbol{\nu})$ which fulfill the KKT conditions (12). 227 Since, as previously observed, for each $h \in H$, $v_i > 0$ for all $j \in N_h \setminus \{j_h\}$, then 228

$$\mathcal{W}_h = \mathcal{W} \cap \{ \mathbf{v} : v_{j_h} = 0 \} = \emptyset \quad \forall h \in H,$$
(14)

is an exactness condition for the Shor relaxation. Indeed, if (14) holds, it follows that no KKT 230 point with some $v_i = 0, i \in N$, exists. However, in general emptiness of these sets cannot 231 be easily checked. 232

Each set \mathcal{W}_h for $h \in H$ can be rewritten as follows. Since $v_{j_h} = 0$, from (12a)-(12b) and 233 from (13) with $v_{j_h} = 0$, we can derive the following expressions for $x_j, j \in N \setminus N_H$, in 234 terms of μ : 235

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$$x_{j}^{h}(\boldsymbol{\mu}) = \begin{cases} -\frac{c_{j} + \sum_{i \in M} \mu_{i} a_{ij}}{D_{jj} - d_{h}^{*}} & \forall j \in N_{h} \setminus \{j_{h}\} \\ -\frac{c_{j} + \sum_{i \in M} \mu_{i} a_{ij}}{D_{jj} + \sum_{i \in M} \xi^{ir} \mu_{i}} & \forall j \in N_{r} \setminus \{j_{r}\}, \ r \neq h. \end{cases}$$
(15)

It also follows from (12a)-(12b) and from (12d) that for $r \in H \setminus \{h\}$: 237

238
$$(d_r^* + \sum_{i \in M} \mu_i \xi^{ir}) x_{j_r} = -(c_{j_r} + \sum_{i \in M} \mu_i a_{ij_r})$$
(16a)

$$(d_r^* + \sum_{i \in M} \mu_i \xi^{ir}) z_{j_r} = -(c_{j_r} + \sum_{i \in M} \mu_i a_{ij_r}) x_{j_r}.$$
 (16b)

 $i \in M$

We denote by \mathcal{M}_{i_r} the set of vectors (x_{i_r}, z_{i_r}, μ) which fulfill these two equations. Then, 240 the set W_h , i.e., the set of KKT points with $v_{j_h} = 0$, is defined by the following constraints, 241 where \mathcal{L}_{ih} is defined in (3): 242

$$\mu \in \mathcal{L}_{j_h} \cap \left[\cap_{r \in H \setminus \{h\}} \mathcal{H}_{j_r} \right]$$

$$(x_{j_r}, z_{j_r}, \mu) \in \mathcal{M}_{j_r}$$

$$r \in H \setminus \{h\}$$

$$\sum_{245} \left[\xi^{ir} z_{j_r} + 2a_{ij_r} x_{j_r} \right] + \sum_{j_r} \left[x_j^h(\mu)^2 + 2a_{ij} x_j^h(\mu) \right] \le b_i \qquad i \in M \quad (17a)$$

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$$r \in H \qquad \qquad j \in N \setminus N_H$$

$$x_{j_r}^2 \le z_{j_r} \qquad \qquad r \in H. \quad (17b)$$

Note that for $j \notin N_H$, $v_j > 0$, so that we could replace z_j with $x_j^h(\boldsymbol{\mu})^2$. Taking into account 247 that the values for $x_i^h(\mu)$ are given in (15), the above sets can be seen as solution sets of a 248 system of polynomial equations and inequalities, where the degree of the polynomials is at 249 most 2n. Unfortunately, establishing whether these systems admit no solution or, equivalently, 250 that the Shor relaxation is exact is, in general, a hard task. 251

However, in the next section we will discuss cases for which the condition can be efficiently 252 checked. Moreover, if a set $\mathcal{W}' \supseteq \mathcal{W}$ is available, a valid exactness condition is 253

$$\mathcal{W}_{h}' = \mathcal{W}' \cap \{ \mathbf{v} : v_{j_{h}} = 0 \} = \emptyset \quad \forall h \in H,$$
(18)

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and for proper choices of W' emptiness can be checked efficiently. For instance, the exactness condition stated in Theorem 1 of [8], derived in that work by showing existence of a rank-one solution for the Shor relaxation when the condition holds, here it is derived in a different way, by choosing W' as the set defined by the constraints (12a) and (12b) and by $\mu \ge 0$. In this case we have that

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$$\mathcal{W}'_h = \mathcal{L}_{j_h} \cap \left[\cap_{r \in H \setminus \{h\}} \mathcal{H}_{j_r} \right],$$

so that exactness is guaranteed if the above polyhedral sets are empty for all $h \in H$.

The condition presented in [8], as well as the one discussed in [23], can be viewed, in 262 terms of the KKT conditions (12) for the convex problem (11), as *dual* exactness conditions, 263 since they only involve the Lagrange multipliers associated to the constraints or, stated in 264 another way, we consider a set \mathcal{W}' only depending on the dual variables. But the KKT system 265 also involves the original, primal, variables. So the question is whether we can include, at 266 least in some special cases, both the original variables and the Lagrange multipliers in order 267 to define *primal-dual* exactness conditions, but in such a way that the conditions can be 268 efficiently checked. This will be the topic of the next sections. 269

We finally note that in case Assumption 2 is not fulfilled, then we have a further degree of freedom. Indeed, if $\arg \min_{j \in N_h} D_{jj}$ is not a singleton, by using the perturbation argument stated in Proposition 1, we can choose any member $j_h \in \arg \min_{j \in N_h} D_{jj}$ and add a small positive perturbation to values D_{jj} for all other members $j \in \arg \min_{j \in N_h} D_{jj}$. Then, given a set $W' \supseteq W$, exactness is guaranteed if for each $h \in H$

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$$\exists j_h \in \arg\min_{i \in N_h} D_{jj} : \mathcal{W}'_h = \emptyset.$$

4 Some applications of a primal-dual exactness condition

²⁷⁷ In this section we present some cases where the exactness condition (14), based on the ²⁷⁸ emptiness of the sets defined by constraints (17), can be checked in an efficient way.

279 4.1 The cases |M| = 1 and |M| = 2

We briefly discuss the case |M| = 1. This is the already mentioned GTRS problem for which 280 it is well known that the Shor relaxation is always exact. Exactness can be viewed as an 281 immediate consequence of the fact that, for each $h \in H$, the two equations in the definition 282 of the set \mathcal{L}_{i_h} , possibly after the application of the perturbation argument stated in Proposition 283 1 (either perturb d_h^* or c_{j_h}), cannot be fulfilled at the same time, so that the set \mathcal{W}_h is empty. 284 When |M| = 2 exactness does not always hold but the condition (14) can be easily checked. 285 For each $h \in H$, in order to check emptiness of the set \mathcal{W}_h , we need to proceed as follows. 286 First note that: 287

- either the two equations in the definition of the set \mathcal{L}_{j_h} are linearly dependent, in which case we can apply the perturbation argument, perturbing, e.g., c_{j_h} so that the two equations become incompatible and emptiness of \mathcal{W}_h is guaranteed for arbitrarily small perturbations of the objective coefficients;

- or they are linearly independent, in which case the corresponding system admits a unique solution $\bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2)$.

²⁹⁴ In the latter case we may have:

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- min{ $\bar{\mu}_1, \bar{\mu}_2$ } < 0: then nonnegativity of the μ values is violated and, again, W_h is guaranteed to be empty.

- min{ $\bar{\mu}_1, \bar{\mu}_2$ } = 0: by the usual perturbation argument, we can introduce a perturbation either of d_h^* or of c_{j_h} in order to have a negative μ value and, consequently, emptiness of \mathcal{W}_h for arbitrarily small perturbations of the objective coefficients holds;

- min{ $\bar{\mu}_1, \bar{\mu}_2$ } > 0: in this case we can convert all inequalities (17a) into equations by exploiting the complementarity conditions (12c). Moreover:

- otherwise, if one of the inequality defining the half-spaces \mathcal{H}_{j_r} for some $r \in H \setminus \{h\}$ is active, then we must have, by (16a), that:

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$$c_{j_r} + \sum_{i \in M} \bar{\mu}_i a_{ij_r} = d_r^* + \sum_{i \in M} \bar{\mu}_i \xi^{ir} = 0.$$

³⁰⁷ By the perturbation argument, e.g., by slightly increasing d_r^* , we have that $(\bar{\mu}_1, \bar{\mu}_2)$ ³⁰⁸ violates one of the two equations above, so that emptiness of W_h for arbitrarily small ³⁰⁹ perturbations of the objective coefficients holds.

- otherwise, when all the inequalities defining the half-spaces $\mathcal{H}_{j_r}, r \in H \setminus \{h\}$, are sastified and not active at $(\bar{\mu}_1, \bar{\mu}_2)$, then by (16a) and (16b) we can set for each $r \in H \setminus \{h\}$:

$$x_{j_r} = -\frac{c_{j_r} + \sum_{i \in M} \bar{\mu}_i a_{ij_r}}{d_r^* + \sum_{i \in M} \bar{\mu}_i \xi^{ir}}, \quad z_{j_r} = x_{j_r}^2.$$

This way, in the two equations (17a) we just have the two unknowns z_{j_h} and x_{j_h} . Once we have solved the linear system and computed the values of these unknowns, we can conclude that the set W_h is empty if $x_{j_h}^2 > z_{j_h}$ holds for all possible solutions of the system.

For the sake of illustration we derive the exactness condition in the case of trust region problems with one additional linear constraint.

4.1.1 The case of trust region problems with a single additional linear constraint

As already mentioned, for this problem in [6, 19] an exact SOC-RLT relaxation is proposed. The Shor relaxation is not always exact but its exactness can be checked by a very simple

³²³ condition. The problem can always be converted into an instance of diagonal QCQP:

$$\min \sum_{j \in N} D_{jj} x_j^2 + 2 \sum_{j \in N} c_j x_j$$
$$\sum_{j \in N} x_j^2 \le 1$$
$$2 \sum_{j \in N} a_j x_j \le b.$$

Note that we can take |H| = 1 in this case.

Exactness certainly holds if $c_{j_1}a_{j_1} \ge 0$ (sign-definiteness condition). If $c_{j_1}a_{j_1} < 0$, we have $\bar{\mu}_1 = -d_1^*$ and $\bar{\mu}_2 = -\frac{c_{j_1}}{a_{j_1}}$. Then,

$$x_{j}^{1}(\bar{\mu}_{1},\bar{\mu}_{2}) = -\frac{c_{j} - a_{j}\frac{c_{j_{1}}}{a_{j_{1}}}}{D_{jj} - d_{1}^{*}} \quad \forall j \in N \setminus \{j_{1}\}.$$

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⁻ if $(\bar{\mu}_1, \bar{\mu}_2)$ violates one of the inequalities defining the half-spaces \mathcal{H}_{j_r} for some $r \in H \setminus \{h\}$, then \mathcal{W}_h is empty;

For convenience, let $\bar{x}_j = x_i^1(\bar{\mu}_1, \bar{\mu}_2)$. Then,

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$$x_{j_1}^1(\bar{\mu}_1,\bar{\mu}_2) = \frac{b - 2\sum_{j \in N \setminus \{j_1\}} a_j \bar{x}_j}{a_{j_1}}.$$

Again, for convenience, set $\bar{x}_{j_1} = x_{j_1}^1(\bar{\mu}_1, \bar{\mu}_2)$. Finally, exactness of the convex relaxation holds if

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$$\sum_{j \in N} \bar{x}_j^2 \ge 1. \tag{19}$$

Actually, the exactness condition holds if the above inequality is strict. However, we can also include the equality case, e.g., by the perturbation argument. Indeed, we can perturb c_j for some $j \in N \setminus \{j_1\}$ so that the equality becomes a strict inequality.

Remark 1 In [9], where a correction of Theorem 3 in [8] is given, it is proved that for a 337 class of random diagonal QCQPs the probability of having an exact semidefinite relaxation 338 converges to 1 as $|N| \rightarrow \infty$. For QCQPs with a single quadratic constraint and a single linear 339 constraint this fact emerges quite clearly from the above exactness condition. Indeed, under 340 very mild assumptions on the random generation of the data, for some $j \in N \setminus \{j_1\}$ there is a 341 strictly positive probability $\ell > 0$ that $\bar{x}_i \notin (-1, 1)$, and this is enough to guarantee that the 342 exactness condition (19) holds. Therefore, under the assumption of independent generation 343 of the data, the probability of fulfilling the exactness condition is at least $1 - (1 - \ell)^{|N|-1}$, 344 which converges to 1 as $|N| \to \infty$. 345

346 4.2 The case |M| = 3

With a little more effort, exactness conditions can also be given for |M| = 3.

For each $h \in H$ we need to proceed as follows. We first notice that we can consider only 348 points for which none of the inequalities defining the half-spaces $\mathcal{H}_{i_r}, r \in H \setminus \{h\}$, is active. 349 Indeed, if one of them were active, then by (16a) we should also have $c_{i_r} + \sum_{i \in M} \mu_i a_{i_{j_r}} = 0$, 350 i.e., the three μ variables should fulfill four equations which, possibly after applying the 351 perturbation argument, is not possible. Indeed, if the four equations do not admit any solution, 352 we are done (emptiness of \mathcal{W}_h holds). If they admit a solution, then one of the equations 353 can be obtained as a linear combination of the other three equations. Then, we can add a 354 small perturbation to one of the coefficients $c_{j_h}, c_{j_r}, d_h^*, d_r^*$ in order to make the linearly 355 dependent equation incompatible with the three other equations, thus causing emptiness of 356 \mathcal{W}_h for arbitrarily small perturbations of the objective coefficients. 357

If none of the inequalities defining the half-spaces \mathcal{H}_{j_r} , $r \in H \setminus \{h\}$, is active, by the two equations in the definition of the set \mathcal{L}_{j_h} , we have that at least two μ variables must be positive. Indeed, in case at least two μ variables were equal to 0, we would be left with two equations (those in \mathcal{L}_{j_h}) with a single unknown, which could be made incompatible by the usual perturbation argument applied, e.g., to the coefficient c_{j_h} .

Thus, we can consider four distinct cases: (i) $\mu_1, \mu_2 > 0, \mu_3 = 0$; (ii) $\mu_1, \mu_3 > 0, \mu_2 = 0$; (iii) $\mu_2, \mu_3 > 0, \mu_1 = 0$; (iv) $\mu_1, \mu_2, \mu_3 > 0$.

³⁶⁵ If case i) holds, then we can:

- derive μ_1, μ_2 from the two equations in the definition of the set \mathcal{L}_{i_h} ;

- check whether the computed values (together with $\mu_3 = 0$) fulfill the inequalities defining the half-spaces \mathcal{H}_{i_r} , $r \in H \setminus \{h\}$, and the positivity constraints $\mu_1, \mu_2 > 0$;

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- if not, emptiness of \mathcal{W}_h holds (possibly after applying the perturbation argument, e.g., in 360 case either μ_1 or μ_2 is equal to 0); 370

- if yes, then: 371
- derive $x_i, j \in N_h \setminus \{j_h\}$ from (16a) and z_i from (16b); 372
- impose, in view of (12c), that equality holds for constraints (17a) for i = 1, 2; 373
- derive the solution(s) x_{j_h} and z_{j_h} of the system obtained from these two equations; 374
- finally, if $x_{j_h}^2 > z_{j_h}$ for all such solutions, then $\mathcal{W}_h = \emptyset$. 375
- In a completely similar way we can deal with cases ii) and iii). 376
- In case iv), we proceed as follows: 377
- in view of (12c) we notice that all three constraints (17a) must be active; 378
- then we have a system of three equations with two unknowns x_{ih} and z_{ih} , which can be 379 fulfilled only if one of the three equations can be obtained as a linear combination of 380 the other two equations. In particular, this imply that the right-hand side of one of the 381
- equations is a given linear combination of the right-hand sides of the other two equations; 382
- in the equation obtained by imposing the equality between the right-hand side of one 383 of the equations and a given linear combination of the right-hand sides of the other two 384 equations, replace two of the three μ variables, say μ_1 and μ_2 , by affine functions of the 384
- remaining one μ_3 obtained through the two equations in the definition of the set \mathcal{L}_{i_b} ; 386
- the resulting equation turns out to be an univariate polynomial equation with variable μ_3 387 and its roots can be efficiently computed; 388
- 389
- for each root $\bar{\mu}_3 > 0$, compute the corresponding values of $\bar{\mu}_1$, $\bar{\mu}_2$ and of \bar{x}_{jh} , \bar{z}_{jh} ; finally, if for each root $\bar{\mu}_3 > 0$ either $\bar{\mu}_1 \le 0$, or $\bar{\mu}_2 \le 0$, or $\bar{x}_{jh}^2 > \bar{z}_{jh}$, then $\mathcal{W}_h = \emptyset$. 390

In principle, we could proceed in the same way for larger |M| values, but the resulting 391 procedure tends to become quite inefficient with the need of solving multivariate polynomial 392 systems. 393

4.3 The case |H| = 1, |M| arbitrary 394

We discuss the special case when |M| is arbitrary but |H| = 1, so that for each $i \in M$, 395 $A_{ij}^i = \xi_i$ for all $j \in N$. The case when $A_i \in \{I, -I, O\}$ for each $i \in M$, discussed in [8], 396 corresponds to $\xi^i \in \{0, -1, 1\}$, for each $i \in M$. Based on the previous discussion, we have 397 from (17) that the single set whose emptiness guarantees exactness of the Shor relaxation is: 398

$$\left\{ (x_{j_1}, z_{j_1}, \boldsymbol{\mu}) : \boldsymbol{\mu} \in \mathcal{L}_{j_1}, \ x_{j_1}^2 \le z_{j_1}, \ \xi^i z_{j_1} + 2a_{ij_1} x_{j_1} + \sum_{j \ne j_1} [\xi^i x_j(\boldsymbol{\mu})^2 + 2a_{ij} x_j(\boldsymbol{\mu})] \le b_i \ \forall i \in M \right\},$$
(20)

where 400

$$x_{j}(\boldsymbol{\mu}) = -\frac{c_{j} + \sum_{i \in M} \mu_{i} a_{ij}}{D_{jj} - d_{1}^{*}}.$$
(21)

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- A drawback of the above condition is that the set (20), defined by linear and quadratic 402 inequalities, is not convex if $\xi_i < 0$ for at least one $i \in M$. 403
- In the next section, we will introduce a further condition, at least as strong as this one, but 404 only involving convex sets, so that the condition can be checked in polynomial time. Before 405 that, in what follows we present a simple example where exactness can be established by the 406 new condition but not through the conditions introduced in [8], [14] and [23]. 407

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Example 1 Let us consider the following problem parameterized with respect to the right-hand side of the second constraint:

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$$\min -x_1^2 - \frac{1}{2}x_2^2 + x_2 x_1^2 + x_2^2 + x_1 - x_2 \le 2 -x_1 + x_2 \le \xi.$$
(22)

The feasible set has a nonempty interior for $\xi \in (1 - \sqrt{5}, +\infty)$. Now, the set defined by constraints (20) in this case is:

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$$\left\{ (x_1, z_1, \mu_1, \mu_2) : -1 + \mu_1 = 0, \ \mu_1 - \mu_2 = 0, \ z_1 + 1 + x_1 + 1 \le 2, x_1 - 1 \le \xi, \ x_1^2 \le z_1 \right\},$$

which can be seen to be empty for $\xi < -1$, so that exactness of the convex relaxation (11) is established in these cases, while it is not empty (consider, e.g., $x_1 = z_1 = 0$, $\mu_1 = \mu_2 = 1$) for $\xi \ge -1$. But exactness cannot be established by the conditions proposed in [8], [14] and [23]. Indeed, regarding the condition proposed in [8], we notice that for k = 1 the set (5) is:

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$$\{(\mu_1, \mu_2) : -1 + \mu_1 = 0, \ \mu_1 - \mu_2 = 0, \ \mu_1, \mu_2 \ge 0\},\$$

which is not empty. Regarding the condition introduced in [14], in this case the epigraphical set (8) is

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$$E = \{(w_1, v_1, v_2) : \exists (x_1, x_2) : -x_1^2 - \frac{1}{2}x_2^2 + x_2 \le w_1, x_1^2 + x_2^2 + x_1 - x_2 \le v_1, -x_1 + x_2 \le v_2\}.$$

It can be seen that the points $\left(-\frac{5}{2}, 4, -2\right)$ and $\left(-\frac{5}{2}, 2, 0\right)$ belong to *E* (consider $x_1 = 1, x_2 = -1$ and $x_1 = x_2 = -1$, respectively). But their midpoint $\left(-\frac{5}{2}, 3, -1\right)$ does not belong to *E*, so that *E* is not convex. Regarding the condition introduced in [23], we notice that in this case we have

A26
$$\mathbf{A}(\gamma_1, \gamma_2) = \begin{pmatrix} -1 + \gamma_1 & 0\\ 0 & -\frac{1}{2} + \gamma_1 \end{pmatrix}, \quad \mathbf{b}(\gamma_1, \gamma_2) = \begin{pmatrix} \gamma_1 - \gamma_2\\ 1 - \gamma_1 + \gamma_2 \end{pmatrix}.$$

⁴²⁷ We also have the following semidefinite face:

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$$\mathcal{F} = \{(\gamma_1, \gamma_2) : \gamma_1 = 1, \gamma_2 \ge 0\},\$$

429 so that

$$\mathcal{V}(\mathcal{F}) = \{(t,0) : t \in \mathbb{R}\}.$$

Then, the condition introduced in [23] requires that for some sequence $\{h^k\}$, with $h^k \to 0$, we have that

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$$0 \notin \{1 - \gamma_2 + h^k, \ \gamma_2 \ge 0\},$$

which, however, does not hold. Note that the exactness conditions in [8], [14] and [23] do not depend on the right-hand sides of the constraints. Thus, in this example all three conditions are not fulfilled for all possible ξ values.

437 5 A further convex relaxation

The convex relaxation (11) can be further simplified when the set of variables can be partitioned as indicated in (9), where each set N_h collects variables whose quadratic terms are

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equal throughout all the constraints. Recalling the definitions of j_h and d_h^* given in (10), the new convex relaxation is the following:

$$\min_{\mathbf{x}, \mathbf{w} \in \mathcal{X}'} \sum_{h \in H} d_h^* w_h + \sum_{h \in H} \sum_{j \in N_h} (D_{jj} - d_h^*) x_j^2 + 2 \sum_{h \in H} \sum_{j \in N_h} c_j x_j,$$
(23)

443 where

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444
$$\mathcal{X}' = \left\{ \sum_{h \in H} \xi_{ih} w_h + 2 \sum_{h \in H} \sum_{j \in N_h} a_{ij} x_j \le b_i, \ i \in M, \ \sum_{j \in N_h} x_j^2 \le w_h, \ h \in H \right\}.$$

⁴⁴⁵ Note that for |H| = |N| this is the same as the convex relaxation (11). But for |H| < |N|⁴⁴⁶ this relaxation requires the addition of a lower number of variables and of related convex ⁴⁴⁷ quadratic constraints. The KKT conditions for such relaxation are:

$$d_h^* + \sum_{i \in M} \mu_i \xi^{ih} - \gamma_h = 0 \quad h \in H$$
(24a)

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$$(D_{jj} - d_h^*)x_j + c_j + \sum_{i \in M} \mu_i a_{ij} + \gamma_h x_j = 0 \quad j \in N_h, \ h \in H$$
 (24b)

$$\mu_i\left(b_i - \sum_{h \in H} \xi^{ih} w_h - 2\sum_{h \in H} \sum_{j \in N_h} a_{ij} x_j\right) = 0 \quad i \in M$$
(24c)

$$\gamma_h(w_h - \sum_{j \in N_h} x_j^2) = 0 \quad h \in H$$
(24d)

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We prove the following proposition stating that the optimal value of the new convex relaxation

 $(\mathbf{w},\mathbf{x})\in\mathcal{X}',\ \boldsymbol{\mu},\boldsymbol{\gamma}>\mathbf{0}.$

(23) is equal to the optimal value of the original convex relaxation (11) (and, as a consequence,
 also of the Shor relaxation).

⁴⁵⁶ **Proposition 2** *The optimal values of the convex relaxations (11) and (23) are equal.*

⁴⁵⁷ *Proof* Let (\mathbf{x}^* , \mathbf{w}^*) be an optimal solution of (23). For each *h* ∈ *H*, let

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$$\bar{z}_j = \begin{cases} x_j^{\star 2} & j \neq j_h \\ w^{\star} - \sum_{j \in N_h \setminus \{j_h\}} x_j^{\star 2} & j = j_h. \end{cases}$$

⁴⁵⁹ it turns out that $(\mathbf{x}^*, \bar{\mathbf{z}})$ is feasible for (11) and its objective function value is equal to that of ⁴⁶⁰ $(\mathbf{x}^*, \mathbf{w}^*)$. Then, the optimal value of (11) is not larger than the optimal value of (23). To prove ⁴⁶¹ equivalence, we only need to show that also the opposite is true. Let $(\mathbf{x}^*, \mathbf{z}^*)$ be an optimal ⁴⁶² solution of (11). For each $h \in H$, let

$$\bar{w}_h = \sum_{j \in N_h} z_j^\star.$$

Then, $(\mathbf{x}^{\star}, \bar{\mathbf{w}})$ is feasible for (23) and its objective function value is not larger than that of ($\mathbf{x}^{\star}, \mathbf{z}^{\star}$). Then, the optimal value of (23) is not larger than the optimal value of (11) and equivalence is proved.

⁴⁶⁷ If we consider the special case |H| = 1, which includes (in fact, is equivalent to) the case ⁴⁶⁸ when $A_i \in \{I, -I, O\}$, then only a single additional variable w_1 needs to be introduced.

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(24e)

Without loss of generality, we assume that $j_1 = 1$. As in Sect. 3, we denote by \mathcal{W} the set of KKT points, while we denote by $\mathcal{W}' \supseteq \mathcal{W}$ the set of points fulfilling (24) except the complementarity conditions (24c). Then, exactness holds if $\mathcal{W}' \cap \{\gamma_1 = 0\} = \emptyset$. Let us consider the following half-spaces for $i \in M$:

$$H_i^{\leq} = \{ (x_1, w_1, \boldsymbol{\mu}) : \xi_i w_1 + 2a_{i1}x_1 + 2\sum_{j \in N \setminus \{1\}} a_{ij}x_j(\boldsymbol{\mu}) \le b_i \},$$

where $x_j(\mu)$ is defined in (21), while $H_i^=$ is the hyper-plane defined in the same way but with the equality replacing the inequality. Then, we have the following result.

Proposition 3 For |H| = 1 the convex relaxation (23) is exact if the following convex set is empty:

$$Q_1 = \left\{ (x_1, w_1, \boldsymbol{\mu}) \in \bigcap_{i \in M} H_i^{\leq} : \boldsymbol{\mu} \in \mathcal{L}_1, \ x_1^2 + \sum_{j \in N \setminus \{1\}} x_j(\boldsymbol{\mu})^2 \leq w_1 \right\}.$$
 (25)

479 **Proof** It is enough to observe that $\mathcal{W}' \cap \{\gamma_1 = 0\} = \mathcal{Q}_1$.

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Note that this condition can be checked more efficiently than the one stated in Sect. 4.3 (with $j_1 = 1$), since (25) is a convex set, and is at least as strong as that condition. Indeed, if $(\bar{\mu}, \bar{\mathbf{x}}, \bar{w}_1)$ belongs to the set (25), then $(\bar{\mu}, \bar{\mathbf{x}}, \bar{\mathbf{z}})$, where

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$$\bar{z}_j = \bar{x}_j^2, \quad j \neq 1, \quad \bar{z}_1 = \bar{w}_1 - \sum_{j \neq 1} \bar{x}_j^2,$$

⁴⁸⁴ belongs to the set (20).

In fact, in (25) we could replace $\sum_{j \in N} x_j^2 \leq w$ with $\sum_{j \in N} x_j^2 < w$. Indeed, if the set defined in (25) is not empty but only contains points for which equality holds, then the relaxation is still exact. Thus, we could reformulate Proposition 3 in this slightly stronger way.

Proposition 4 For |H| = 1 the convex relaxation (23) is exact if the following convex problem has a nonnegative optimal value.

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 $\min_{(x_1,w_1,\mu)\in\cap_{i\in M}H_i^{\leq}} : \mu\in\mathcal{L}_1 \quad x_1^2 + \sum_{j\in N\setminus\{1\}} x_j(\mu)^2 - w_1.$ (26)

⁴⁹² Up to now we have basically ignored the complementarity conditions (24c). We can strengthen
 ⁴⁹³ the exactness result stated in Proposition 3 by taking them into account.

We first notice that, possibly after the application of the perturbation argument, we must have that at least two μ values are strictly positive. Indeed, both the equation $d_1^* + \sum_{i \in M} \xi_i \mu_i = 0$ and the equation $c_1 + \sum_{i \in M} a_{i1}\mu_i = 0$ must be fulfilled and, possibly after an arbitrarily small perturbation of d_1^* or c_1 , such equations can not be fulfilled if all but one of the μ values are equal to 0.

Then, by complete enumeration of all subsets $I \subseteq M$ with $|I| \ge 2$, we have that $\mathcal{W} = \bigcup_{I \subseteq M, |I| \ge 2} \mathcal{W}_I$, where

$$\mathcal{W}_I = [\cap_{i \in M \setminus I} H_i^{\leq}] \cap [\cap_{i \in I} H_i^{=}] \cap \{(x_1, w_1, \mu) : \mu \in \mathcal{L}_1, \ \mu_i = 0 \ \forall i \in M \setminus I\}.$$

Therefore, the relaxation is exact if for each $I \subseteq M$, $|I| \ge 2$, we have $W_I \cap {\gamma_1 = 0} = \emptyset$ or, equivalently, if the following convex problem has empty feasible region or has nonnegative optimal value:

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This condition is strong and can be applied when |M| is low (in fact, we have already applied it in Sects. 4.1 and 4.2 not only for the case |H| = 1 but also for the general case). But its obvious drawback is that it becomes unpractical when |M| is large, since the number of convex problems grows exponentially with |M|.

An alternative condition, which can be checked in polynomial time, is based on the following cover $\bigcup_{I \subseteq M, |I|=2} \mathcal{W}'_I \supseteq \mathcal{W}$, where $\mathcal{W}'_I = [\bigcap_{i \in M \setminus I} H_i^{\leq}] \cap [\bigcap_{i \in I} H_i^{=}] \cap \{(x_1, w_1, \mu) : \mu \in \mathcal{L}_1\}$. Thus, we have the following exactness condition.

Proposition 5 For |H| = 1 the Shor relaxation is exact if for each $I \subset M$ with |I| = 2, it holds that $W'_I \cap \{\gamma_1 = 0\} = \emptyset$ or, equivalently, that the following convex problem either has empty feasible region or has nonnegative optimal value:

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$$\min_{x_1, w_1, \mu \in \mathcal{L}_1} \sum_{\substack{j \in N \setminus \{1\} \\ (x_1, w_1, \mu) \in [\cap_{i \in M \setminus I} H_i^{\leq}] \cap [\cap_{i \in I} H_i^{=}]}} \sum_{(27)}$$

Notice that this condition is stronger than the one stated in Proposition 3 since the feasible region of each problem (27) is a subset of the feasible region of problem (26), and over it the objective functions of the two problems are equal.

In what follows we provide an example where exactness cannot be established by the result stated in Sect. 4.3 but can be established by Proposition 5.

Example 2 Let us consider again problem (22) from Example 1. The convex relaxation (23)
 of that problem is:

1

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$$\min -w_1 + \frac{1}{2}x_2^2 + x_2$$

$$w_1 + x_1 - x_2 \le 2$$

$$-x_1 + x_2 \le \xi$$

$$x_1^2 + x_2^2 \le w_1.$$

As already discussed, the exactness condition stated in Section 4.3 does not hold for all 525 $\xi \geq -1$. Also recall that exactness cannot be established by the conditions proposed in [8], 526 [14] and [23] for all possible ξ values, since these conditions do not depend on the right-527 hand sides of the constraints. Regarding Proposition 5, we first notice that we can only take 528 $I = \{1, 2\}$, so that in problem (27), after deriving x_1 and w_1 as a function of μ_1, μ_2 , we 529 have that $M \setminus I = \emptyset$, while $\mu_1 = \mu_2 = 1$, $x_1(\mu_1, \mu_2) = -1 - \xi$, $x_2(\mu_1, \mu_2) = -1$, and 530 $w_1(\mu_1, \mu_2) = 2 + \xi$. Then, the optimal value of problem (27) is equal to $\xi^2 + \xi$ and, thus, 531 exactness holds for all $\xi \leq -1$ and all $\xi \geq 0$. Note that, since |M| = 2, here we could also have 532 employed the exactness condition stated in Sect. 4.1. For $\xi \in (-1, 0)$ the exactness condition 533 does not hold but, actually, this happens since the bound provided by the convex relaxation in 534 these cases is not tight. Indeed, the optimal value of the convex relaxation is equal to $-\frac{5}{2}-\xi$, 535 attained at the given point $x_1 = -1 - \xi$, $x_2 = -1$, $w_1 = 2 + \xi$, while the optimal value of problem (22) can be seen to be equal to $\left[-6 - 2\xi - (2 + \xi)\sqrt{4 + 2\xi - \xi^2}\right]/4$, attained 536 537 at the following point where both constraints are active: $x_1^* = \left[-\xi - \sqrt{4 + 2\xi - \xi^2}\right]/2$, 538 $x_2^* = \left[\xi - \sqrt{4 + 2\xi - \xi^2}\right]/2.$ 539

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540 6 Conclusion

In this work we have shown that exactness results for the Shor relaxation of diagonal QCQPs 541 can be derived by first proving the equivalence of this relaxation with two convex quadratic 542 relaxations, and then by analyzing the KKT systems of these convex relaxations. All this 543 allows to re-derive previous exactness results in the literature and, in some cases, to strengthen 544 them into primal-dual exactness conditions, i.e., conditions based both on the original (primal) 545 variables of the convex relaxations and on the dual variables (Lagrange multipliers). As a 546 possible topic for future research we mention the possibility of extending the exactness results 547 to non-diagonal QCQPs. In fact, as already mentioned, the result in [23] already covers some 548 non-diagonal cases. It could be interesting to see whether the derivation discussed in this 549 paper could be extended, e.g., to block diagonal QCQPs, by first proving the equivalence 550 between the Shor relaxation and a convex program where a distinct semidefinite condition 551 is imposed for each distinct block, and then deriving optimality conditions for the convex 552 problem. 553

554 **Data Availability** There are no data that support the findings of this study.

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