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KKT-based primal-dual exactness conditions for the Shor relaxation

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Abstract

In this work we present some exactness conditions for the Shor relaxation of diagonal (or, more generally, diagonalizable) QCQPs, which extend the conditions introduced in different recent papers about the same topic. It is shown that the Shor relaxation is equivalent to two convex quadratic relaxations. Then, sufficient conditions for the exactness of the relaxations are derived from their KKT systems. It will be shown that, in some cases, by this derivation previous conditions in the literature, which can be viewed as *dual* conditions, since they only involve the Lagrange multipliers appearing in the KKT systems, can be extended to *primal-dual* conditions, which also involve the primal variables appearing in the KKT systems.

Keywords Quadratically Constrained Quadratic Programming · Shor relaxation · Convex relaxations · Exactness conditions

1 Introduction

In the recent literature different results about the exactness of the Shor relaxation (see [17]) for Quadratically Constrained Quadratic Programming (QCQP in what follows) problems have been proposed. The Shor relaxation can be proved to be exact for the Generalized Trust Region Subproblem (GTRS), where a single (not necessarily convex) quadratic inequality constraint is present. The exactness proof can be derived from a result discussed in [11]. For other QCQPs the Shor relaxation is not always exact and different papers introduce conditions under which exactness holds for sub-classes of QCQPs. Some exactness results for the case of QCQPs with two quadratic constraints have been presented in [21], while in [1] a necessary and sufficient condition for the exactness of the related Lagrangian dual has been given. Note that the case with two quadratic constraints, which includes the well known Celis-Dennis-Tapia (CDT) problem, has been recently proved to be polynomially solvable in different works [5, 10, 16]. However, both the polynomial approaches proposed in [10, 16], based on the enumeration of all KKT points via the solution of bivariate polynomial systems, and the polynomial approach proposed in [5], based on Barvinok's construction, have a

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SPI Journal: 10898 Article No.: 1258 TYPESET DISK LE CP Disp.:2022/11/28 Pages: 17 Layout: Small

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limited practical applicability due to the large exponent of the polynomials appearing in the complexity result. For QCQPs with a single unit ball constraint and further linear constraints, in [13] a dimension condition establishing exactness of the Shor relaxation is introduced. In [3] a Second Order Cone Programming (SOCP) relaxation for the same problem has been discussed, while in [15] it has been shown that such relaxation is equivalent to the Shor relaxation. By the analysis of the KKT conditions for the SOCP relaxation, in [15] a condition more general than the dimension condition presented in [13] has been given. Note that in [6, 19] an exact convex relaxation, obtained by adding to the Shor relaxation a so called SOC-RLT constraint, has been introduced for the case of a single linear constraint, while in [7] the result has been extended to a generic number of linear constraints provided that these constraints have an empty intersection inside the unit ball. It is also worthwhile to mention that a polynomial-time algorithm for the solution of this problem (possibly also with the addition of further ball and reverse ball constraints) has been proposed under the assumption that the overall number of constraints is fixed (see [4]). The approach is based on an enumeration of all possible KKT points.

In this paper we are interested in deriving exactness conditions of the Shor relaxation in case of diagonal QCQPs, i.e., quadratic problems where the Hessian of all quadratic functions is diagonal or can be made diagonal after a change of variables (the Hessian matrices are simultaneously diagonalizable). In what follows we assume that the QCQP problem is already given in diagonal form. Throughout the paper $N = \{1, \dots, n\}$ will be the index set of the variables, and $M = \{1, \dots, m\}$ will be the index set of the constraints. For a given symmetric matrix $\mathbf{Y}$, the notation $\mathbf{Y} \succeq \mathbf{O}$ means that the matrix is positive semidefinite. By $diag(\mathbf{Y})$ we will denote the vector whose entries are the diagonal entries of matrix $\mathbf{Y}$.

A diagonal QCQP problem is defined as follows:

$$\begin{array}{rl} c^\star = \min_{\mathbf{x}} & \mathbf{x}^\top \mathbf{D}\mathbf{x} + 2\mathbf{c}^\top \mathbf{x} \\ & \mathbf{x}^\top \mathbf{A}^i \mathbf{x} + 2\mathbf{a}_i^\top \mathbf{x} \le b_i \quad i \in M, \end{array} \tag{1}$$

where matrix $\mathbf{D}$ and all matrices $\mathbf{A}_i$, $i \in M$, are diagonal. The classical Shor relaxation for this problem is:

$$\begin{array}{rl} v^\star = \min_{\mathbf{x},\mathbf{X}} & \mathbf{D} \bullet \mathbf{X} + 2\mathbf{c}^\top \mathbf{x} \\ & \mathbf{A}^i \bullet \mathbf{X} + 2\mathbf{a}_i^\top \mathbf{x} \le b_i \ i \in M \\ & \mathbf{X} - \mathbf{x}\mathbf{x}^\top \succeq \mathbf{O}. \end{array} \tag{2}$$

The existence of minimizers and, thus, the use of min rather than inf in problems (1) and (2) is guaranteed under the following suitable assumptions, introduced in [8]:

Assumption 1 The following hold:

- The feasible region of (1) is nonempty;
- $\exists\, \bar{\mathbf{y}} \ge \mathbf{0}$ such that $\sum_{i \in M} \bar{y}_i \mathbf{A}_i \succ \mathbf{O}$;
- The interior of the feasible region of (2) is nonempty.

In particular, note that these assumptions imply that the feasible region of problem (1) is bounded.

This assumption will be maintained throughout the paper.

In [8] some sufficient conditions are introduced under which there exists an optimal rank-one solution for the Shor relaxation, which is equivalent to proving that the Shor relaxation is exact, i.e., $v^\star = c^\star$. More precisely, for $k \in N$, let:

$$\mathcal{L}_k = \left\{ \boldsymbol{\mu} \ge \mathbf{0} \ : \ D_{kk} + \sum_{i \in M} \mu_i A^i_{kk} = 0,\ c_k + \sum_{i \in M} \mu_i a_{ik} = 0 \right\}, \tag{3}$$

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SPI Journal: 10898 Article No.: 1258 TYPESET DISK LE CP Disp.:2022/11/28 Pages: 17 Layout: Small

and for $j \in N$:

$$\mathcal{H}_j = \left\{ \boldsymbol{\mu} \ : \ D_{jj} + \sum_{i \in M} \mu_i A^i_{jj} \geq 0 \right\}. \tag{4}$$

It is proved that the Shor relaxation is exact if for each $k \in N$ the following polyhedral set is empty:

$$\mathcal{S}_k = \mathcal{L}_k \cap \left[\cap_{j \in N \setminus \{k\}} \mathcal{H}_j \right]. \tag{5}$$

This result allows to re-derive a sign-definiteness condition presented in [18], stating that exactness holds if for all $j \in N$, c_j and a_{ij}, $i \in M$, are all nonpositive or all nonnegative.

Moreover, for the relevant special case when $\mathbf{A}_i \in \{\mathbf{I}, -\mathbf{I}, \mathbf{O}\}$ for each $i \in M$, i.e., when all constraints are ball, reverse ball, and linear constraints, in [8] it is shown that exactness holds when the sign-definite condition is only satisfied by the variable corresponding to the lowest diagonal entry of matrix $\mathbf{D}$. Note that this special case is addressed also in [2], where a branch-and-bound approach for its solution is proposed and an application to source localization problems is presented.

A further very recent result has been proved in [23], where a class of problems larger than the class of diagonal QCQPs is considered. We briefly discuss the condition introduced in that paper, only in the case of inequality constraints, although also equality constraints may be included. Note that in this case matrices $\mathbf{D}$ and $\mathbf{A}_i$, $i \in M$, are not necessarily diagonal. Let

$$\mathbf{A}(\boldsymbol{\gamma}) = \mathbf{D} + \sum_{i \in M} \gamma_i \mathbf{A}_i, \quad \mathbf{b}(\boldsymbol{\gamma}) = \mathbf{c} + \sum_{i \in M} \gamma_i \mathbf{a}_i.$$

Let

$$\Gamma = \{\boldsymbol{\gamma} \ : \ \mathbf{A}(\boldsymbol{\gamma}) \succeq \mathbf{O}, \ \boldsymbol{\gamma} \geq \mathbf{0}\}.$$

A face $\mathcal{F}$ of Γ which does not contain any $\boldsymbol{\gamma}$ such that $\mathbf{A}(\boldsymbol{\gamma}) \succ \mathbf{O}$ is called a semidefinite face, and the zero eigenspace of $\mathcal{F}$ is

$$\mathcal{V}(\mathcal{F}) = \{\mathbf{x} \ : \ \mathbf{A}(\boldsymbol{\gamma})\mathbf{x} = \mathbf{0}, \ \forall \boldsymbol{\gamma} \in \mathcal{F}\}.$$

In [23] it is assumed that Γ is a polyhedral set. While this assumption is always fulfilled for diagonal QCQPs, it is also shown that it may hold also for non-diagonal QCQPs, but it is pointed out that it is coNP-hard to decide whether the assumption holds. Exactness of the Shor relaxation is proved under the condition that there exists some infinite sequence $\{\mathbf{h}^k\}$ such that $\mathbf{h}^k \to \mathbf{0}$ (see the perturbation argument below) and for any k and any semidefinite face $\mathcal{F}$ it holds that:

$$\mathbf{0} \notin \{Proj_{\mathcal{V}(\mathcal{F})}(b(\boldsymbol{\gamma}) + \mathbf{h}^k) \ : \ \boldsymbol{\gamma} \geq \mathbf{0}\}. \tag{6}$$

Note that in the same paper also some conditions are discussed under which the convex hull of the epigraph of the QCQP is given by the projection of the epigraph of its Shor relaxation. Another recent result about this topic can be found in [14]. In that work minimax QCQPs are considered, namely, the following problems are addressed

$$\begin{array}{ll} \min_{\mathbf{x}} \max_{r \in R} \mathbf{x}^\top \mathbf{D}^r \mathbf{x} + 2\mathbf{c}_r^\top \mathbf{x} + c_{0r} & \\ \quad \mathbf{x}^\top \mathbf{A}^i \mathbf{x} + 2\mathbf{a}_i^\top \mathbf{x} \leq b_i & i \in M, \end{array} \tag{7}$$

where all matrices $\mathbf{D}^r$, $r \in R$, $\mathbf{A}^i$, $i \in M$, are diagonal, possibly obtained after the simultaneous diagonalization of all the Hessian matrices. Note that this class of problems is equivalent

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to the class of problems (1). Indeed, each problem (1) can be viewed as a special case of (7) by taking $|R| = 1$, while, on the other hand, each problem (7) can be converted into an instance of problem (1) after the addition of a variable y, which becomes the objective function to be minimized, and of the related constraints $y \geq \mathbf{x}^\top \mathbf{D}^r \mathbf{x} + 2\mathbf{c}_r^\top \mathbf{x} + c_{0r}$ for each $r \in R$. In [14] a SOCP relaxation of problem (7) is introduced which is equivalent to the Lagrangian dual of this problem and, thus, also to the Shor relaxation (recall that the Lagrangian dual and the Shor relaxation are dual to each other and, thus, have the same optimal value if a constraint qualification holds). In [14] an exactness condition is introduced based on the so called epigraphical set, defined as follows:

$$E = \Big\{(\mathbf{w}, \mathbf{v}) \in \mathbb{R}^{|R|+|M|} \ : \ \exists \mathbf{x} \in \mathbb{R}^{|N|} \ : \ \begin{array}{l} \mathbf{x}^\top \mathbf{D}^r \mathbf{x} + 2\mathbf{c}_r^\top \mathbf{x} + c_{0r} \leq w_r, \ r \in R, \\ \mathbf{x}^\top \mathbf{A}^i \mathbf{x} + 2\mathbf{a}_i^\top \mathbf{x} \leq v_i, \ i \in M \Big\}. \end{array} \tag{8}$$

It is shown that the SOCP relaxation is exact if the epigraphical set is closed and convex.

Some applications of diagonal QCQPs In the literature there are different applications of diagonal QCQPs. Here we briefly review a few of them.

The extended trust region subproblem (extended TRS) is the trust region problem with additional linear constraints. After diagonalizing the objective function, this becomes a diagonal QCQP where $\mathbf{A}_i \in \{\mathbf{I}, \mathbf{O}\}$ for each $i \in M$ (more precisely, all matrices $\mathbf{A}_i$ are null, except one which is equal to the identity matrix). As outlined in [13], such problem arises from the application of the trust region method in the context of linearly constrained problems, from nonlinear optimization problems with discrete variables, and from robust optimization problems. Moreover, QP problems whose feasible region is a polytope can be reformulated as an extended TRS after the addition of a ball constraint (a ball enclosing the feasible polytope).

In [14] the max dispersion problem is presented as an application of diagonal QCQPs. In this problem, given a finite set of location positions $\mathbf{u}_i$, $i = 1, \ldots, p$, and a further point $\mathbf{x}_0$, we aim at identifying the position of a new location which maximizes the minimal distance from all the other locations. The new position is subject to a ball constraint, i.e., it must belong to a sphere centered at $\mathbf{x}_0$, and is possibly subject to further linear constraints.

The problem of minimizing a quadratic function over a 'Swiss cheese' domain, i.e., a feasible region defined by ball, reverse ball, and linear constraints, has been discussed, e.g., in [4] (see also [23] for an exactness result when the objective function to be minimized is the Euclidean norm). After diagonalization of the objective function, this problem belongs to the special case of diagonal QCQPs with $\mathbf{A}_i \in \{\mathbf{I}, -\mathbf{I}, \mathbf{O}\}$ for each $i \in M$. The latter special case is also the focus of paper [2], where a branch and bound approach is proposed and an application to sparse source localization problems is presented.

Finally, in [8] it is shown that general QCQPs can be reformulated as diagonal QCQPs with additional variables.

Statement of contribution The main contribution of this work lies in the derivation of exactness conditions of the Shor relaxation for diagonal QCQPs through an approach different with respect to the existing, recent, literature, in particular, with respect to [8, 14, 23]. The conditions are derived from the KKT conditions of an equivalent SOCP reformulation of the Shor relaxation. They are primal-dual conditions, while the other conditions in the literature appear to be dual conditions. As we will see through a simple example, besides being derived in a different way, the new conditions also allow to establish exactness results which cannot be established by the existing conditions. The new conditions are particularly significant

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when $\mathbf{A}_i \in \{\mathbf{I}, -\mathbf{I}, \mathbf{O}\}$ for each $i \in M$, which, according to the previous discussion, is a relevant subcase of diagonal QCQPs.

Outline of the paper In this paper we first state in Sect. 2, by a straightforward extension of a result proved in [15], that for diagonal QCQPs the Shor relaxation is equivalent to a quadratic convex relaxation of problem (1). Next, in Sect. 3 the exactness condition related to the emptiness of the sets (5) is re-derived through an analysis of the KKT conditions of the convex relaxation. Moreover, in Sect. 4, it is shown how to strengthen the exactness condition in some cases and, in particular, in the already mentioned case when $\mathbf{A}^i \in \{\mathbf{I}, -\mathbf{I}, \mathbf{O}\}$ for each $i \in M$. It is shown through an example that the new condition can be stronger than those discussed in [8, 14, 23]. Finally, in Sect. 5 a further equivalent convex relaxation is introduced and it is shown that KKT conditions for this relaxation allow to define an exactness condition which can be more efficiently checked.

2 A convex relaxation equivalent to the Shor relaxation

Before proceeding, we subdivide the class of diagonal QCQPs in some subclasses on the basis of a partition N_h, $h \in H$, of the set N, such that each set N_h contains indexes of variables whose coefficients of the quadratic terms are all equal throughout the constraints (but not necessarily in the objective function). Formally:

$$\forall\, j, k \in N_h,\ \forall\, h \in H,\ \forall\, i \in M:\ A^i_{jj} = A^i_{kk} = \xi^{ih}. \tag{9}$$

Note that the general case is a special case where $|H| = |N|$ and each set N_h is a singleton. In the special case, discussed in [8], when $\mathbf{A}_i \in \{\mathbf{I}, -\mathbf{I}, \mathbf{O}\}$ for all $i \in M$, we have that $|H| = 1$. In fact, when $|H| = 1$ the problem can always be rewritten in such a way that $\mathbf{A}_i \in \{\mathbf{I}, -\mathbf{I}, \mathbf{O}\}$ for all $i \in M$. We introduce the following assumption.

Assumption 2 For each $h \in H$, the set $\arg\min_{j \in N_h} D_{jj}$ is a singleton. We denote by j_h its single member and by d_h^* the minimum diagonal entry D_{jj} for $j \in N_h$, i.e.:

$$j_h = \arg\min_{j \in N_h} D_{jj}, \quad d_h^* = \min_{j \in N_h} D_{jj}. \tag{10}$$

Later on we will show that removing this assumption allows to derive an even more general exactness condition. But, in order to simplify the presentation, we will impose that the assumption holds.

In what follows we employ set $N_H = \{j_h \ :\ h \in H\} \subseteq N$.

Exploiting the fact that all matrices are diagonal, following [3], a convex relaxation of problem (1) is:

$$p^\star = \min_{(\mathbf{x},\mathbf{z}) \in \mathcal{X}} \sum_{j \in N} D_{jj} z_j + 2 \sum_{j \in N} c_j x_j, \tag{11}$$

where:

$$\mathcal{X} = \left\{ (\mathbf{x}, \mathbf{z}) : \sum_{h \in H} \sum_{j \in N_h} \xi^{ih} z_j + 2 \sum_{j \in N} a_{ij} x_j \le b_i,\ i \in M,\ x_j^2 \le z_j,\ j \in N \right\}.$$

Note that this is a relaxation since the same problem with constraints $x_j^2 \le z_j$, $j \in N$, replaced by equations $x_j^2 = z_j$ is an equivalent reformulation of problem (1). In [15] the

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SPI Journal: 10898 Article No.: 1258 TYPESET DISK LE CP Disp.:2022/11/28 Pages: 17 Layout: Small

equivalence was proven between this relaxation and the Shor relaxation when $\mathbf{A}_1 = \mathbf{I}, \mathbf{a}_1 = \mathbf{0}$, $\mathbf{A}_i = \mathbf{O}$ for all $i \in M \setminus \{1\}$. The result can be extended in a quite straightforward way to the general problem (1) (see also the proof in [23] and note that the result can also be obtained as a special case of some results on sparse semidefinite programming problems presented in [12]).

Theorem 1 *It holds that $p^\star = v^\star$, i.e., the optimal values of the Shor relaxation (2) and of the convex relaxation (11) are equal.*

Now, this equivalence result can be employed in order to establish exactness conditions for the Shor relaxation by the analysis of the KKT conditions of the convex relaxation. This will be the topic of the next section.

Before proceeding we briefly introduce the perturbation argument already adopted in [8, 15, 23] (see, e.g., the discussion following Proposition 1 in [8], Theorem 3.1 in [15], and (6) in [23]). We will make extensive use of this argument in the following sections.

Proposition 1 *Let Assumption 1 hold. Exactness of the Shor relaxation is verified for a problem with data $(\mathbf{D}, \mathbf{A}_i, \mathbf{a}_i, \mathbf{c}, \mathbf{b})$ if it is verified for an infinite sequence of problems with perturbed data $(\mathbf{D} + \Delta\mathbf{D}^k, \mathbf{A}_i, \mathbf{a}_i, \mathbf{c} + \Delta\mathbf{c}^k, \mathbf{b})$ such that*

$$||\Delta\mathbf{D}^k||, ||\Delta\mathbf{c}^k|| \to 0.$$

Proof The result holds true for perturbations $\Delta\mathbf{D}^k$ and $\Delta\mathbf{c}^k$ in the objective function, since, by continuity and by boundedness of the feasible region implied by Assumption 1, the optimal values of problem (1) with the perturbed data converge to the optimal value of the unperturbed problem, and the same holds for the optimal values of the corresponding Shor relaxations. □

3 Sufficient conditions for exactness of the Shor relaxation

Theorem 1 implies that proving exactness of the Shor relaxation is equivalent to prove exactness of the convex relaxation (11). Under Assumption 1, which we recall is maintained throughout the paper, optimal solutions of the convex problem (11) fulfill the corresponding KKT conditions. In particular, we notice that existence of an interior feasible solution $(\bar{\mathbf{X}}, \bar{\mathbf{x}})$ for problem (2) implies that also the convex relaxation (11) admits an interior feasible point. Indeed, it is enough to consider the point $(diag(\bar{\mathbf{X}}), \bar{\mathbf{x}})$. Then, Slater's condition holds and we can search the minimizer of problem (11) among the KKT points of the same problem. The KKT conditions are the following:

$$D_{jj} + \sum_{i \in M} \mu_i \xi^{ih} - \nu_j = 0 \quad j \in N_h,\ h \in H \tag{12a}$$

$$c_j + \sum_{i \in M} \mu_i a_{ij} + \nu_j x_j = 0 \quad j \in N \tag{12b}$$

$$\mu_i \left(b_i - \sum_{h \in H} \sum_{j \in N_h} \xi^{ih} z_j - 2 \sum_{j \in N} a_{ij} x_j \right) = 0 \quad i \in M \tag{12c}$$

$$\nu_j (z_j - x_j^2) = 0 \quad j \in N \tag{12d}$$

$$(\mathbf{x}, \mathbf{z}) \in \mathcal{X},\ \boldsymbol{\mu}, \boldsymbol{\nu} \geq \mathbf{0}. \tag{12e}$$

SPI Journal: 10898 Article No.: 1258 TYPESET DISK LE CP Disp.:2022/11/28 Pages: 17 Layout: Small

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Note that in view of equations (12a), for each $h \in H$:

$$\nu_j - \nu_{j_h} = D_{jj} - d_h^* \quad \forall j \in N_h. \tag{13}$$

In view of the definition of j_h, we have, under Assumption 2, $\nu_j > 0$ for all $j \in N_h \setminus \{j_h\}$.

Now, it obviously holds that the relaxation is exact if all constraints $x_j^2 \leq z_j$, $j \in N$, are active at the optimal solution of (11). In view of the complementarity conditions (12d), this certainly holds if $\nu_j > 0$ for all $j \in N$.

Let us denote by $\mathcal{W}$ the set of vectors $(\mathbf{x}, \mathbf{z}, \boldsymbol{\mu}, \boldsymbol{\nu})$ which fulfill the KKT conditions (12). Since, as previously observed, for each $h \in H$, $\nu_j > 0$ for all $j \in N_h \setminus \{j_h\}$, then

$$\mathcal{W}_h = \mathcal{W} \cap \{\boldsymbol{\nu} \,:\, \nu_{j_h} = 0\} = \emptyset \quad \forall h \in H, \tag{14}$$

is an exactness condition for the Shor relaxation. Indeed, if (14) holds, it follows that no KKT point with some $\nu_j = 0$, $j \in N$, exists. However, in general emptiness of these sets cannot be easily checked.
Each set $\mathcal{W}_h$ for $h \in H$ can be rewritten as follows. Since $\nu_{j_h} = 0$, from (12a)-(12b) and from (13) with $\nu_{j_h} = 0$, we can derive the following expressions for x_j, $j \in N \setminus N_H$, in terms of $\boldsymbol{\mu}$:

$$x_j^h(\boldsymbol{\mu}) = \begin{cases} -\frac{c_j + \sum_{i \in M} \mu_i a_{ij}}{D_{jj} - d_h^*} & \forall j \in N_h \setminus \{j_h\} \\ -\frac{c_j + \sum_{i \in M} \mu_i a_{ij}}{D_{jj} + \sum_{i \in M} \xi^{ir} \mu_i} & \forall j \in N_r \setminus \{j_r\},\ r \neq h. \end{cases} \tag{15}$$

It also follows from (12a)-(12b) and from (12d) that for $r \in H \setminus \{h\}$:

$$(d_r^* + \sum_{i \in M} \mu_i \xi^{ir}) x_{j_r} = -(c_{j_r} + \sum_{i \in M} \mu_i a_{ij_r}) \tag{16a}$$

$$(d_r^* + \sum_{i \in M} \mu_i \xi^{ir}) z_{j_r} = -(c_{j_r} + \sum_{i \in M} \mu_i a_{ij_r}) x_{j_r}. \tag{16b}$$

We denote by $\mathcal{M}_{j_r}$ the set of vectors $(x_{j_r}, z_{j_r}, \boldsymbol{\mu})$ which fulfill these two equations. Then, the set $\mathcal{W}_h$, i.e., the set of KKT points with $\nu_{j_h} = 0$, is defined by the following constraints, where $\mathcal{L}_{j_h}$ is defined in (3):

$$\begin{aligned}
& \boldsymbol{\mu} \in \mathcal{L}_{j_h} \cap \left[\cap_{r \in H \setminus \{h\}} \mathcal{H}_{j_r}\right] & & \\
& (x_{j_r}, z_{j_r}, \boldsymbol{\mu}) \in \mathcal{M}_{j_r} & r \in H \setminus \{h\} & \\
& \sum_{r \in H} [\xi^{ir} z_{j_r} + 2a_{ij_r} x_{j_r}] + \sum_{j \in N \setminus N_H} [x_j^h(\boldsymbol{\mu})^2 + 2a_{ij} x_j^h(\boldsymbol{\mu})] \leq b_i & i \in M & \quad (17a) \\
& x_{j_r}^2 \leq z_{j_r} & r \in H. & \quad (17b)
\end{aligned}$$

Note that for $j \notin N_H$, $\nu_j > 0$, so that we could replace z_j with $x_j^h(\boldsymbol{\mu})^2$. Taking into account that the values for $x_j^h(\boldsymbol{\mu})$ are given in (15), the above sets can be seen as solution sets of a system of polynomial equations and inequalities, where the degree of the polynomials is at most $2n$. Unfortunately, establishing whether these systems admit no solution or, equivalently, that the Shor relaxation is exact is, in general, a hard task.

However, in the next section we will discuss cases for which the condition can be efficiently checked. Moreover, if a set $\mathcal{W}' \supseteq \mathcal{W}$ is available, a valid exactness condition is

$$\mathcal{W}'_h = \mathcal{W}' \cap \{\boldsymbol{\nu} \,:\, \nu_{j_h} = 0\} = \emptyset \quad \forall h \in H, \tag{18}$$

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SPI Journal: 10898 Article No.: 1258 TYPESET DISK LE CP Disp.:2022/11/28 Pages: 17 Layout: Small

and for proper choices of $\mathcal{W}'$ emptiness can be checked efficiently. For instance, the exactness condition stated in Theorem 1 of [8], derived in that work by showing existence of a rank-one solution for the Shor relaxation when the condition holds, here it is derived in a different way, by choosing $\mathcal{W}'$ as the set defined by the constraints (12a) and (12b) and by $\boldsymbol{\mu} \geq \mathbf{0}$. In this case we have that

$$\mathcal{W}_h' = \mathcal{L}_{j_h} \cap \left[\cap_{r \in H\setminus\{h\}} \mathcal{H}_{j_r}\right],$$

so that exactness is guaranteed if the above polyhedral sets are empty for all $h \in H$.

The condition presented in [8], as well as the one discussed in [23], can be viewed, in terms of the KKT conditions (12) for the convex problem (11), as *dual* exactness conditions, since they only involve the Lagrange multipliers associated to the constraints or, stated in another way, we consider a set $\mathcal{W}'$ only depending on the dual variables. But the KKT system also involves the original, primal, variables. So the question is whether we can include, at least in some special cases, both the original variables and the Lagrange multipliers in order to define *primal-dual* exactness conditions, but in such a way that the conditions can be efficiently checked. This will be the topic of the next sections.

We finally note that in case Assumption 2 is not fulfilled, then we have a further degree of freedom. Indeed, if $\arg\min_{j \in N_h} D_{jj}$ is not a singleton, by using the perturbation argument stated in Proposition 1, we can choose any member $j_h \in \arg\min_{j \in N_h} D_{jj}$ and add a small positive perturbation to values D_{jj} for all other members $j \in \arg\min_{j \in N_h} D_{jj}$. Then, given a set $\mathcal{W}' \supseteq \mathcal{W}$, exactness is guaranteed if for each $h \in H$

$$\exists j_h \in \arg\min_{j \in N_h} D_{jj} \ : \ \mathcal{W}_h' = \emptyset.$$

4 Some applications of a primal-dual exactness condition

In this section we present some cases where the exactness condition (14), based on the emptiness of the sets defined by constraints (17), can be checked in an efficient way.

4.1 The cases $|M| = 1$ and $|M| = 2$

We briefly discuss the case $|M| = 1$. This is the already mentioned GTRS problem for which it is well known that the Shor relaxation is always exact. Exactness can be viewed as an immediate consequence of the fact that, for each $h \in H$, the two equations in the definition of the set $\mathcal{L}_{j_h}$, possibly after the application of the perturbation argument stated in Proposition 1 (either perturb d_h^* or c_{j_h}), cannot be fulfilled at the same time, so that the set $\mathcal{W}_h$ is empty. When $|M| = 2$ exactness does not always hold but the condition (14) can be easily checked. For each $h \in H$, in order to check emptiness of the set $\mathcal{W}_h$, we need to proceed as follows. First note that:

- either the two equations in the definition of the set $\mathcal{L}_{j_h}$ are linearly dependent, in which case we can apply the perturbation argument, perturbing, e.g., c_{j_h} so that the two equations become incompatible and emptiness of $\mathcal{W}_h$ is guaranteed for arbitrarily small perturbations of the objective coefficients;
- or they are linearly independent, in which case the corresponding system admits a unique solution $\bar{\boldsymbol{\mu}} = (\bar{\mu}_1, \bar{\mu}_2)$.

In the latter case we may have:

SPI Journal: 10898 Article No.: 1258 TYPESET DISK LE CP Disp.:2022/11/28 Pages: 17 Layout: Small

- $\min\{\bar{\mu}_1, \bar{\mu}_2\} < 0$: then nonnegativity of the μ values is violated and, again, $\mathcal{W}_h$ is guaranteed to be empty.
- $\min\{\bar{\mu}_1, \bar{\mu}_2\} = 0$: by the usual perturbation argument, we can introduce a perturbation either of d_h^* or of c_{j_h} in order to have a negative μ value and, consequently, emptiness of $\mathcal{W}_h$ for arbitrarily small perturbations of the objective coefficients holds;
- $\min\{\bar{\mu}_1, \bar{\mu}_2\} > 0$: in this case we can convert all inequalities (17a) into equations by exploiting the complementarity conditions (12c). Moreover:
  - if $(\bar{\mu}_1, \bar{\mu}_2)$ violates one of the inequalities defining the half-spaces $\mathcal{H}_{j_r}$ for some $r \in H \setminus \{h\}$, then $\mathcal{W}_h$ is empty;
  - otherwise, if one of the inequality defining the half-spaces $\mathcal{H}_{j_r}$ for some $r \in H \setminus \{h\}$ is active, then we must have, by (16a), that:

$$c_{j_r} + \sum_{i \in M} \bar{\mu}_i a_{ij_r} = d_r^* + \sum_{i \in M} \bar{\mu}_i \xi^{ir} = 0.$$

    By the perturbation argument, e.g., by slightly increasing d_r^*, we have that $(\bar{\mu}_1, \bar{\mu}_2)$ violates one of the two equations above, so that emptiness of $\mathcal{W}_h$ for arbitrarily small perturbations of the objective coefficients holds.
  - otherwise, when all the inequalities defining the half-spaces $\mathcal{H}_{j_r}$, $r \in H \setminus \{h\}$, are sastified and not active at $(\bar{\mu}_1, \bar{\mu}_2)$, then by (16a) and (16b) we can set for each $r \in H \setminus \{h\}$:

$$x_{j_r} = -\frac{c_{j_r} + \sum_{i \in M} \bar{\mu}_i a_{ij_r}}{d_r^* + \sum_{i \in M} \bar{\mu}_i \xi^{ir}}, \quad z_{j_r} = x_{j_r}^2.$$

    This way, in the two equations (17a) we just have the two unknowns z_{j_h} and x_{j_h}. Once we have solved the linear system and computed the values of these unknowns, we can conclude that the set $\mathcal{W}_h$ is empty if $x_{j_h}^2 > z_{j_h}$ holds for all possible solutions of the system.

For the sake of illustration we derive the exactness condition in the case of trust region problems with one additional linear constraint.

4.1.1 The case of trust region problems with a single additional linear constraint

As already mentioned, for this problem in [6, 19] an exact SOC-RLT relaxation is proposed. The Shor relaxation is not always exact but its exactness can be checked by a very simple condition. The problem can always be converted into an instance of diagonal QCQP:

$$\begin{array}{l} \min \sum_{j \in N} D_{jj} x_j^2 + 2 \sum_{j \in N} c_j x_j \\ \quad\ \ \sum_{j \in N} x_j^2 \leq 1 \\ \quad\ \ 2 \sum_{j \in N} a_j x_j \leq b. \end{array}$$

Note that we can take $|H| = 1$ in this case.

Exactness certainly holds if $c_{j_1} a_{j_1} \geq 0$ (sign-definiteness condition). If $c_{j_1} a_{j_1} < 0$, we have $\bar{\mu}_1 = -d_1^*$ and $\bar{\mu}_2 = -\frac{c_{j_1}}{a_{j_1}}$. Then,

$$x_j^1(\bar{\mu}_1, \bar{\mu}_2) = -\frac{c_j - a_j \frac{c_{j_1}}{a_{j_1}}}{D_{jj} - d_1^*} \quad \forall j \in N \setminus \{j_1\}.$$

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SPI Journal: 10898 Article No.: 1258 TYPESET DISK LE CP Disp.:2022/11/28 Pages: 17 Layout: Small

For convenience, let $\bar{x}_j = x_j^1(\bar{\mu}_1, \bar{\mu}_2)$. Then,

$$x_{j_1}^1(\bar{\mu}_1, \bar{\mu}_2) = \frac{b - 2\sum_{j \in N\setminus\{j_1\}} a_j \bar{x}_j}{a_{j_1}}.$$

Again, for convenience, set $\bar{x}_{j_1} = x_{j_1}^1(\bar{\mu}_1, \bar{\mu}_2)$. Finally, exactness of the convex relaxation holds if

$$\sum_{j \in N} \bar{x}_j^2 \geq 1. \tag{19}$$

Actually, the exactness condition holds if the above inequality is strict. However, we can also include the equality case, e.g., by the perturbation argument. Indeed, we can perturb c_j for some $j \in N \setminus \{j_1\}$ so that the equality becomes a strict inequality.

Remark 1 In [9], where a correction of Theorem 3 in [8] is given, it is proved that for a class of random diagonal QCQPs the probability of having an exact semidefinite relaxation converges to 1 as $|N| \to \infty$. For QCQPs with a single quadratic constraint and a single linear constraint this fact emerges quite clearly from the above exactness condition. Indeed, under very mild assumptions on the random generation of the data, for some $j \in N \setminus \{j_1\}$ there is a strictly positive probability $\ell > 0$ that $\bar{x}_j \notin (-1, 1)$, and this is enough to guarantee that the exactness condition (19) holds. Therefore, under the assumption of independent generation of the data, the probability of fulfilling the exactness condition is at least $1 - (1 - \ell)^{|N|-1}$, which converges to 1 as $|N| \to \infty$.

4.2 The case $|M| = 3$

With a little more effort, exactness conditions can also be given for $|M| = 3$.

For each $h \in H$ we need to proceed as follows. We first notice that we can consider only points for which none of the inequalities defining the half-spaces $\mathcal{H}_{j_r}$, $r \in H \setminus \{h\}$, is active. Indeed, if one of them were active, then by (16a) we should also have $c_{j_r} + \sum_{i \in M} \mu_i a_{ij_r} = 0$, i.e., the three μ variables should fulfill four equations which, possibly after applying the perturbation argument, is not possible. Indeed, if the four equations do not admit any solution, we are done (emptiness of $\mathcal{W}_h$ holds). If they admit a solution, then one of the equations can be obtained as a linear combination of the other three equations. Then, we can add a small perturbation to one of the coefficients $c_{j_h}, c_{j_r}, d_h^*, d_r^*$ in order to make the linearly dependent equation incompatible with the three other equations, thus causing emptiness of $\mathcal{W}_h$ for arbitrarily small perturbations of the objective coefficients.

If none of the inequalities defining the half-spaces $\mathcal{H}_{j_r}$, $r \in H \setminus \{h\}$, is active, by the two equations in the definition of the set $\mathcal{L}_{j_h}$, we have that at least two μ variables must be positive. Indeed, in case at least two μ variables were equal to 0, we would be left with two equations (those in $\mathcal{L}_{j_h}$) with a single unknown, which could be made incompatible by the usual perturbation argument applied, e.g., to the coefficient c_{j_h}.

Thus, we can consider four distinct cases: (i) $\mu_1, \mu_2 > 0, \mu_3 = 0$; (ii) $\mu_1, \mu_3 > 0, \mu_2 = 0$; (iii) $\mu_2, \mu_3 > 0, \mu_1 = 0$; (iv) $\mu_1, \mu_2, \mu_3 > 0$.

If case i) holds, then we can:

- derive μ_1, μ_2 from the two equations in the definition of the set $\mathcal{L}_{j_h}$;
- check whether the computed values (together with $\mu_3 = 0$) fulfill the inequalities defining the half-spaces $\mathcal{H}_{j_r}$, $r \in H \setminus \{h\}$, and the positivity constraints $\mu_1, \mu_2 > 0$;

SPI Journal: 10898 Article No.: 1258 TYPESET DISK LE CP Disp.:2022/11/28 Pages: 17 Layout: Small

uncorrected proof

– if not, emptiness of $\mathcal{W}_h$ holds (possibly after applying the perturbation argument, e.g., in case either μ_1 or μ_2 is equal to 0);
– if yes, then:
  – derive x_j, $j \in N_h \setminus \{j_h\}$ from (16a) and z_j from (16b);
  – impose, in view of (12c), that equality holds for constraints (17a) for $i = 1, 2$;
  – derive the solution(s) x_{j_h} and z_{j_h} of the system obtained from these two equations;
  – finally, if $x_{j_h}^2 > z_{j_h}$ for all such solutions, then $\mathcal{W}_h = \emptyset$.

In a completely similar way we can deal with cases ii) and iii).

In case iv), we proceed as follows:

– in view of (12c) we notice that all three constraints (17a) must be active;
– then we have a system of three equations with two unknowns x_{j_h} and z_{j_h}, which can be fulfilled only if one of the three equations can be obtained as a linear combination of the other two equations. In particular, this imply that the right-hand side of one of the equations is a given linear combination of the right-hand sides of the other two equations;
– in the equation obtained by imposing the equality between the right-hand side of one of the equations and a given linear combination of the right-hand sides of the other two equations, replace two of the three μ variables, say μ_1 and μ_2, by affine functions of the remaining one μ_3 obtained through the two equations in the definition of the set $\mathcal{L}_{j_h}$;
– the resulting equation turns out to be an univariate polynomial equation with variable μ_3 and its roots can be efficiently computed;
– for each root $\bar{\mu}_3 > 0$, compute the corresponding values of $\bar{\mu}_1$, $\bar{\mu}_2$ and of $\bar{x}_{j_h}$, $\bar{z}_{j_h}$;
– finally, if for each root $\bar{\mu}_3 > 0$ either $\bar{\mu}_1 \le 0$, or $\bar{\mu}_2 \le 0$, or $\bar{x}_{j_h}^2 > \bar{z}_{j_h}$, then $\mathcal{W}_h = \emptyset$.

In principle, we could proceed in the same way for larger $|M|$ values, but the resulting procedure tends to become quite inefficient with the need of solving multivariate polynomial systems.

4.3 The case $|H| = 1$, $|M|$ arbitrary

We discuss the special case when $|M|$ is arbitrary but $|H| = 1$, so that for each $i \in M$, $A^i_{jj} = \xi_i$ for all $j \in N$. The case when $\mathbf{A}_i \in \{\mathbf{I}, -\mathbf{I}, \mathbf{O}\}$ for each $i \in M$, discussed in [8], corresponds to $\xi^i \in \{0, -1, 1\}$, for each $i \in M$. Based on the previous discussion, we have from (17) that the single set whose emptiness guarantees exactness of the Shor relaxation is:

$$\left\{(x_{j_1}, z_{j_1}, \boldsymbol{\mu}) : \boldsymbol{\mu} \in \mathcal{L}_{j_1},\ x_{j_1}^2 \le z_{j_1},\ \xi^i z_{j_1} + 2a_{ij_1}x_{j_1} + \sum_{j \ne j_1} [\xi^i x_j(\boldsymbol{\mu})^2 + 2a_{ij}x_j(\boldsymbol{\mu})] \le b_i\ \forall i \in M\right\}, \quad (20)$$

where

$$x_j(\boldsymbol{\mu}) = -\frac{c_j + \sum_{i \in M} \mu_i a_{ij}}{D_{jj} - d_1^*}. \quad (21)$$

A drawback of the above condition is that the set (20), defined by linear and quadratic inequalities, is not convex if $\xi_i < 0$ for at least one $i \in M$.

In the next section, we will introduce a further condition, at least as strong as this one, but only involving convex sets, so that the condition can be checked in polynomial time. Before that, in what follows we present a simple example where exactness can be established by the new condition but not through the conditions introduced in [8], [14] and [23].

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SPI Journal: 10898 Article No.: 1258 TYPESET DISK LE CP Disp.:2022/11/28 Pages: 17 Layout: Small

Example 1 Let us consider the following problem parameterized with respect to the right-hand side of the second constraint:

$$\begin{array}{l} \min\ -x_1^2 - \frac{1}{2}x_2^2 + x_2 \\ \quad\ \ x_1^2 + x_2^2 + x_1 - x_2 \leq 2 \\ \quad\ -x_1 + x_2 \leq \xi. \end{array} \tag{22}$$

The feasible set has a nonempty interior for $\xi \in (1-\sqrt{5}, +\infty)$. Now, the set defined by constraints (20) in this case is:

$$\left\{(x_1, z_1, \mu_1, \mu_2)\ :\ -1+\mu_1 = 0,\ \mu_1 - \mu_2 = 0,\ z_1 + 1 + x_1 + 1 \leq 2, x_1 - 1 \leq \xi,\ x_1^2 \leq z_1\right\},$$

which can be seen to be empty for $\xi < -1$, so that exactness of the convex relaxation (11) is established in these cases, while it is not empty (consider, e.g., $x_1 = z_1 = 0$, $\mu_1 = \mu_2 = 1$) for $\xi \geq -1$. But exactness cannot be established by the conditions proposed in [8], [14] and [23]. Indeed, regarding the condition proposed in [8], we notice that for $k = 1$ the set (5) is:

$$\{(\mu_1, \mu_2)\ :\ -1+\mu_1 = 0,\ \mu_1 - \mu_2 = 0,\ \mu_1, \mu_2 \geq 0\},$$

which is not empty. Regarding the condition introduced in [14], in this case the epigraphical set (8) is

$$E = \{(w_1, v_1, v_2) : \exists (x_1, x_2)\ :\ -x_1^2 - \frac{1}{2}x_2^2 + x_2 \leq w_1,\ x_1^2 + x_2^2 + x_1 - x_2 \leq v_1,\ -x_1 + x_2 \leq v_2\}.$$

It can be seen that the points $\left(-\frac{5}{2}, 4, -2\right)$ and $\left(-\frac{5}{2}, 2, 0\right)$ belong to E (consider $x_1 = 1, x_2 = -1$ and $x_1 = x_2 = -1$, respectively). But their midpoint $\left(-\frac{5}{2}, 3, -1\right)$ does not belong to E, so that E is not convex. Regarding the condition introduced in [23], we notice that in this case we have

$$\mathbf{A}(\gamma_1, \gamma_2) = \begin{pmatrix} -1+\gamma_1 & 0 \\ 0 & -\frac{1}{2}+\gamma_1 \end{pmatrix}, \quad \mathbf{b}(\gamma_1, \gamma_2) = \begin{pmatrix} \gamma_1 - \gamma_2 \\ 1 - \gamma_1 + \gamma_2 \end{pmatrix}.$$

We also have the following semidefinite face:

$$\mathcal{F} = \{(\gamma_1, \gamma_2)\ :\ \gamma_1 = 1, \quad \gamma_2 \geq 0\},$$

so that

$$\mathcal{V}(\mathcal{F}) = \{(t, 0)\ :\ t \in \mathbb{R}\}.$$

Then, the condition introduced in [23] requires that for some sequence $\{h^k\}$, with $h^k \to 0$, we have that

$$0 \notin \{1 - \gamma_2 + h^k, \ \ \gamma_2 \geq 0\},$$

which, however, does not hold. Note that the exactness conditions in [8], [14] and [23] do not depend on the right-hand sides of the constraints. Thus, in this example all three conditions are not fulfilled for all possible ξ values.

5 A further convex relaxation

The convex relaxation (11) can be further simplified when the set of variables can be partitioned as indicated in (9), where each set N_h collects variables whose quadratic terms are

uncorrected proof

SPI Journal: 10898 Article No.: 1258 TYPESET DISK LE CP Disp.:2022/11/28 Pages: 17 Layout: Small

equal throughout all the constraints. Recalling the definitions of j_h and d_h^* given in (10), the new convex relaxation is the following:

$$\min_{\mathbf{x},\mathbf{w}\in\mathcal{X}'} \sum_{h\in H} d_h^* w_h + \sum_{h\in H}\sum_{j\in N_h}(D_{jj}-d_h^*)x_j^2 + 2\sum_{h\in H}\sum_{j\in N_h} c_j x_j, \tag{23}$$

where

$$\mathcal{X}' = \left\{ \sum_{h\in H} \xi_{ih} w_h + 2\sum_{h\in H}\sum_{j\in N_h} a_{ij}x_j \le b_i,\ i\in M,\ \sum_{j\in N_h} x_j^2 \le w_h,\ h\in H \right\}.$$

Note that for $|H| = |N|$ this is the same as the convex relaxation (11). But for $|H| < |N|$ this relaxation requires the addition of a lower number of variables and of related convex quadratic constraints. The KKT conditions for such relaxation are:

$$d_h^* + \sum_{i\in M} \mu_i \xi^{ih} - \gamma_h = 0 \quad h\in H \tag{24a}$$

$$(D_{jj}-d_h^*)x_j + c_j + \sum_{i\in M}\mu_i a_{ij} + \gamma_h x_j = 0 \quad j\in N_h,\ h\in H \tag{24b}$$

$$\mu_i\left(b_i - \sum_{h\in H}\xi^{ih}w_h - 2\sum_{h\in H}\sum_{j\in N_h} a_{ij}x_j\right) = 0 \quad i\in M \tag{24c}$$

$$\gamma_h(w_h - \sum_{j\in N_h} x_j^2) = 0 \quad h\in H \tag{24d}$$

$$(\mathbf{w},\mathbf{x})\in\mathcal{X}',\ \boldsymbol{\mu},\boldsymbol{\gamma}\ge\mathbf{0}. \tag{24e}$$

We prove the following proposition stating that the optimal value of the new convex relaxation (23) is equal to the optimal value of the original convex relaxation (11) (and, as a consequence, also of the Shor relaxation).

Proposition 2 *The optimal values of the convex relaxations (11) and (23) are equal.*

Proof Let $(\mathbf{x}^\star,\mathbf{w}^\star)$ be an optimal solution of (23). For each $h\in H$, let

$$\bar{z}_j = \begin{cases} x_j^{\star 2} & j\ne j_h \\ w^\star - \sum_{j\in N_h\setminus\{j_h\}} x_j^{\star 2} & j = j_h. \end{cases}$$

it turns out that $(\mathbf{x}^\star,\bar{\mathbf{z}})$ is feasible for (11) and its objective function value is equal to that of $(\mathbf{x}^\star,\mathbf{w}^\star)$. Then, the optimal value of (11) is not larger than the optimal value of (23). To prove equivalence, we only need to show that also the opposite is true. Let $(\mathbf{x}^\star,\mathbf{z}^\star)$ be an optimal solution of (11). For each $h\in H$, let

$$\bar{w}_h = \sum_{j\in N_h} z_j^\star.$$

Then, $(\mathbf{x}^\star,\bar{\mathbf{w}})$ is feasible for (23) and its objective function value is not larger than that of $(\mathbf{x}^\star,\mathbf{z}^\star)$. Then, the optimal value of (23) is not larger than the optimal value of (11) and equivalence is proved. □

If we consider the special case $|H| = 1$, which includes (in fact, is equivalent to) the case when $\mathbf{A}_i \in \{\mathbf{I}, -\mathbf{I}, \mathbf{O}\}$, then only a single additional variable w_1 needs to be introduced.

uncorrected proof

SPI Journal: 10898 Article No.: 1258 TYPESET DISK LE CP Disp.:2022/11/28 Pages: 17 Layout: Small

Without loss of generality, we assume that $j_1 = 1$. As in Sect. 3, we denote by $\mathcal{W}$ the set of KKT points, while we denote by $\mathcal{W}' \supseteq \mathcal{W}$ the set of points fulfilling (24) except the complementarity conditions (24c). Then, exactness holds if $\mathcal{W}' \cap \{\gamma_1 = 0\} = \emptyset$. Let us consider the following half-spaces for $i \in M$:

$$H_i^{\leq} = \{(x_1, w_1, \boldsymbol{\mu}) : \ \xi_i w_1 + 2a_{i1}x_1 + 2 \sum_{j \in N \setminus \{1\}} a_{ij} x_j(\boldsymbol{\mu}) \leq b_i\},$$

where $x_j(\boldsymbol{\mu})$ is defined in (21), while $H_i^{=}$ is the hyper-plane defined in the same way but with the equality replacing the inequality. Then, we have the following result.

Proposition 3 *For $|H| = 1$ the convex relaxation (23) is exact if the following convex set is empty:*

$$\mathcal{Q}_1 = \left\{ (x_1, w_1, \boldsymbol{\mu}) \in \cap_{i \in M} H_i^{\leq} \ : \boldsymbol{\mu} \in \mathcal{L}_1, \ x_1^2 + \textstyle\sum_{j \in N \setminus \{1\}} x_j(\boldsymbol{\mu})^2 \leq w_1 \right\}. \qquad (25)$$

Proof It is enough to observe that $\mathcal{W}' \cap \{\gamma_1 = 0\} = \mathcal{Q}_1$. □

Note that this condition can be checked more efficiently than the one stated in Sect. 4.3 (with $j_1 = 1$), since (25) is a convex set, and is at least as strong as that condition. Indeed, if $(\bar{\boldsymbol{\mu}}, \bar{\mathbf{x}}, \bar{w}_1)$ belongs to the set (25), then $(\bar{\boldsymbol{\mu}}, \bar{\mathbf{x}}, \bar{\mathbf{z}})$, where

$$\bar{z}_j = \bar{x}_j^2, \quad j \neq 1, \quad \bar{z}_1 = \bar{w}_1 - \sum_{j \neq 1} \bar{x}_j^2,$$

belongs to the set (20).

In fact, in (25) we could replace $\sum_{j \in N} x_j^2 \leq w$ with $\sum_{j \in N} x_j^2 < w$. Indeed, if the set defined in (25) is not empty but only contains points for which equality holds, then the relaxation is still exact. Thus, we could reformulate Proposition 3 in this slightly stronger way.

Proposition 4 *For $|H| = 1$ the convex relaxation (23) is exact if the following convex problem has a nonnegative optimal value.*

$$\min_{(x_1, w_1, \boldsymbol{\mu}) \in \cap_{i \in M} H_i^{\leq} \ : \ \boldsymbol{\mu} \in \mathcal{L}_1} \quad x_1^2 + \textstyle\sum_{j \in N \setminus \{1\}} x_j(\boldsymbol{\mu})^2 - w_1. \qquad (26)$$

Up to now we have basically ignored the complementarity conditions (24c). We can strengthen the exactness result stated in Proposition 3 by taking them into account.

We first notice that, possibly after the application of the perturbation argument, we must have that at least two μ values are strictly positive. Indeed, both the equation $d_1^* + \sum_{i \in M} \xi_i \mu_i = 0$ and the equation $c_1 + \sum_{i \in M} a_{i1} \mu_i = 0$ must be fulfilled and, possibly after an arbitrarily small perturbation of d_1^* or c_1, such equations can not be fulfilled if all but one of the μ values are equal to 0.

Then, by complete enumeration of all subsets $I \subseteq M$ with $|I| \geq 2$, we have that $\mathcal{W} = \cup_{I \subseteq M, \ |I| \geq 2} \mathcal{W}_I$, where

$$\mathcal{W}_I = [\cap_{i \in M \setminus I} H_i^{\leq}] \cap [\cap_{i \in I} H_i^{=}] \cap \{(x_1, w_1, \boldsymbol{\mu}) : \ \boldsymbol{\mu} \in \mathcal{L}_1, \ \mu_i = 0 \ \forall i \in M \setminus I\}.$$

Therefore, the relaxation is exact if for each $I \subseteq M$, $|I| \geq 2$, we have $\mathcal{W}_I \cap \{\gamma_1 = 0\} = \emptyset$ or, equivalently, if the following convex problem has empty feasible region or has nonnegative optimal value:

uncorrected proof

SPI Journal: 10898 Article No.: 1258 TYPESET DISK LE CP Disp.:2022/11/28 Pages: 17 Layout: Small

$$\begin{array}{ll} \min_{x_1, w_1,\ \boldsymbol{\mu} \in \mathcal{L}_1,\ \mu_i = 0\ \forall i \in M \setminus I} & \sum_{j \in N \setminus \{1\}} x_j(\boldsymbol{\mu})^2 + x_1^2 - w_1 \\ & (x_1, w_1, \boldsymbol{\mu}) \in [\cap_{i \in M \setminus I} H_i^{\leq}] \cap [\cap_{i \in I} H_i^{=}]. \end{array}$$

This condition is strong and can be applied when $|M|$ is low (in fact, we have already applied it in Sects. 4.1 and 4.2 not only for the case $|H| = 1$ but also for the general case). But its obvious drawback is that it becomes unpractical when $|M|$ is large, since the number of convex problems grows exponentially with $|M|$.

An alternative condition, which can be checked in polynomial time, is based on the following cover $\cup_{I \subseteq M,\ |I|=2} \mathcal{W}_I' \supseteq \mathcal{W}$, where $\mathcal{W}_I' = [\cap_{i \in M \setminus I} H_i^{\leq}] \cap [\cap_{i \in I} H_i^{=}] \cap \{(x_1, w_1, \boldsymbol{\mu}) : \boldsymbol{\mu} \in \mathcal{L}_1\}$. Thus, we have the following exactness condition.

Proposition 5 *For $|H| = 1$ the Shor relaxation is exact if for each $I \subset M$ with $|I| = 2$, it holds that $\mathcal{W}_I' \cap \{\gamma_1 = 0\} = \emptyset$ or, equivalently, that the following convex problem either has empty feasible region or has nonnegative optimal value:*

$$\begin{array}{ll} \min_{x_1, w_1,\ \boldsymbol{\mu} \in \mathcal{L}_1} & \sum_{j \in N \setminus \{1\}} x_j(\boldsymbol{\mu})^2 + x_1^2 - w_1 \\ & (x_1, w_1, \boldsymbol{\mu}) \in [\cap_{i \in M \setminus I} H_i^{\leq}] \cap [\cap_{i \in I} H_i^{=}]. \end{array} \tag{27}$$

Notice that this condition is stronger than the one stated in Proposition 3 since the feasible region of each problem (27) is a subset of the feasible region of problem (26), and over it the objective functions of the two problems are equal.

In what follows we provide an example where exactness cannot be established by the result stated in Sect. 4.3 but can be established by Proposition 5.

Example 2 Let us consider again problem (22) from Example 1. The convex relaxation (23) of that problem is:

$$\begin{array}{ll} \min & -w_1 + \frac{1}{2}x_2^2 + x_2 \\ & w_1 + x_1 - x_2 \leq 2 \\ & -x_1 + x_2 \leq \xi \\ & x_1^2 + x_2^2 \leq w_1. \end{array}$$

As already discussed, the exactness condition stated in Section 4.3 does not hold for all $\xi \geq -1$. Also recall that exactness cannot be established by the conditions proposed in [8], [14] and [23] for all possible ξ values, since these conditions do not depend on the right-hand sides of the constraints. Regarding Proposition 5, we first notice that we can only take $I = \{1, 2\}$, so that in problem (27), after deriving x_1 and w_1 as a function of μ_1, μ_2, we have that $M \setminus I = \emptyset$, while $\mu_1 = \mu_2 = 1$, $x_1(\mu_1, \mu_2) = -1 - \xi$, $x_2(\mu_1, \mu_2) = -1$, and $w_1(\mu_1, \mu_2) = 2 + \xi$. Then, the optimal value of problem (27) is equal to $\xi^2 + \xi$ and, thus, exactness holds for all $\xi \leq -1$ and all $\xi \geq 0$. Note that, since $|M| = 2$, here we could also have employed the exactness condition stated in Sect. 4.1. For $\xi \in (-1, 0)$ the exactness condition does not hold but, actually, this happens since the bound provided by the convex relaxation in these cases is not tight. Indeed, the optimal value of the convex relaxation is equal to $-\frac{5}{2} - \xi$, attained at the given point $x_1 = -1 - \xi$, $x_2 = -1$, $w_1 = 2 + \xi$, while the optimal value of problem (22) can be seen to be equal to $\left[-6 - 2\xi - (2 + \xi)\sqrt{4 + 2\xi - \xi^2}\right]/4$, attained at the following point where both constraints are active: $x_1^* = \left[-\xi - \sqrt{4 + 2\xi - \xi^2}\right]/2$, $x_2^* = \left[\xi - \sqrt{4 + 2\xi - \xi^2}\right]/2$.

Uncorrected proof

SPI Journal: 10898 Article No.: 1258 TYPESET DISK LE CP Disp.:2022/11/28 Pages: 17 Layout: Small

6 Conclusion

In this work we have shown that exactness results for the Shor relaxation of diagonal QCQPs can be derived by first proving the equivalence of this relaxation with two convex quadratic relaxations, and then by analyzing the KKT systems of these convex relaxations. All this allows to re-derive previous exactness results in the literature and, in some cases, to strengthen them into primal-dual exactness conditions, i.e., conditions based both on the original (primal) variables of the convex relaxations and on the dual variables (Lagrange multipliers). As a possible topic for future research we mention the possibility of extending the exactness results to non-diagonal QCQPs. In fact, as already mentioned, the result in [23] already covers some non-diagonal cases. It could be interesting to see whether the derivation discussed in this paper could be extended, e.g., to block diagonal QCQPs, by first proving the equivalence between the Shor relaxation and a convex program where a distinct semidefinite condition is imposed for each distinct block, and then deriving optimality conditions for the convex problem.

Data Availability There are no data that support the findings of this study.

References

1. Ai, W., Zhang, S.: Strong duality for the CDT subproblem: a necessary and sufficient condition. SIAM J. Optim. **19**(4), 1735–1756 (2009)
2. Beck, A., Pan, D.: A branch and bound algorithm for nonconvex quadratic optimization with ball and linear constraint. J. Global Optim. **69**, 309–342 (2017)
3. Ben-Tal, A., den Hertog, D.: Hidden conic quadratic representation of some nonconvex quadratic optimization problems. Math. Program. **143**, 1–29 (2014)
4. Bienstock, D., Michalka, A.: Polynomial solvability of variants of the trust-region subproblem. In: SODA '14 Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms, 380–390 (2014)
5. Bienstock, D.: A note on polynomial solvability of the CDT problem. SIAM J. Optim. **26**, 488–498 (2016)
6. Burer, S., Anstreicher, K.M.: Second-order-cone constraints for extended trust-region subproblems. SIAM J. Optim. **23**(1), 432–451 (2013)
7. Burer, S., Yang, B.: The Trust Region Subproblem with non-intersecting linear constraints. Math. Program. **149**(1), 253–264 (2015)
8. Burer, S., Ye, Y.: Exact semidefinite formulations for a class of (random and non-random) nonconvex quadratic programs. Math. Program. **181**, 1–17 (2020)
9. Burer, S., Ye, Y.: Erratum to: exact semidefinite formulations for a class of (random and non-random) nonconvex quadratic programs. Math. Program. **190**(1), 845–848 (2021)
10. Consolini, L., Locatelli, M.: On the complexity of quadratic programming with two quadratic constraints. Math. Program. **164**, 91–128 (2017)
11. Fradkov, A.L., Yakubovich, V.A.: The S-procedure and duality relations in nonconvex problems of quadratic programming. Vestnik Leningrad Univ. Math. **6**, 101–109 (1979)
12. Fukuda, M., Kojima, M., Murota, K., Nakata, K.: Exploiting sparsity in semidefinite programming via matrix completion I: general framework. SIAM J. Optim. **11**(3), 647–674 (2000)
13. Jeyakumar, V., Li, G.: Trust-region problems with linear inequality constraints: exact SDP relaxation, global optimality and robust optimization. Math. Program. **147**(1), 171–206 (2014)
14. Jeyakumar, V., Li, G.: Exact second-order cone programming relaxations for some nonconvex minimax quadratic optimization problems. SIAM J. Optim. **28**(1), 760–787 (2018)
15. Locatelli, M.: Exactness conditions for an SDP relaxation of the extended trust region problem. Optim. Lett. **10**, 1141–1151 (2016)
16. Sakaue, S., Nakatsukasa, Y., Takeda, A., Iwata, S.: Solving generalized CDT problems via two-parameter eigenvalues. SIAM J. Optim. **26**, 1669–1694 (2016)
17. Shor, N.: Quadratic optimization problems. Soviet J. Comput. Syst. Sci. **25**, 1–11 (1987)
18. Sojoudi, S., Lavaei, J.: Exactness of semidefinite relaxations for nonlinear optimization problems with underlying graph structure. SIAM J. Optim. **24**(4), 1746–1778 (2014)

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19. Sturm, J.F., Zhang, S.: On cones of nonnegative quadratic functions. Math. Oper. Res. **28**(2), 246–267 (2003)
20. Yang, B., Burer, S.: A two-varable approach to the two-trust region subproblem. SIAM J. Optim. **26**(1), 661–680 (2016)
21. Ye, Y., Zhang, S.: New results on quadratic minimization. SIAM J. Optim. **14**, 245–267 (2003)
22. Yuan, J., Wang, M., Ai, W., Shuai, T.: New results on narrowing the duality gap on the extended Celis-Dennis-Tapia problem. SIAM J. Optim. **27**(2), 890–909 (2017)
23. Wang, A.L., Kilinç-Karzan, F.: On the tightness of SDP relaxations of QCQPs. Math. Program. **193**(1), 33–73 (2022)

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