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Crystalline and Anisotropic, Nonlinear or Nonlocal Curvature Flows

Flots de la Courbure Cristalline et Anisotrope, Non Linéaire ou Non Locale

Soutenue par
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Ad Eugi.

Longtemps, je me suis couché de bonne heure.

Marcel Proust, *Du côté de chez Swann*

Non domandarci la formula che mondi possa aprirti
sì qualche storta sillaba e secca come un ramo.
Codesto solo oggi possiamo dirti . . .

Eugenio Montale, *Non chiederci la parola che squadri da ogni lato*

Crystalline and Anisotropic, Nonlinear or Nonlocal Curvature Flows

Abstract

This thesis is devoted to the study of geometric flows, with particular focus on the mean curvature flow. It is divided in two thematic parts. The first part, Part I, contains Chapters 2, 3 and 4, and concerns convergence results for the minimizing movements scheme, which is a variational procedure extending Euler's implicit scheme to evolutions having a gradient flow-like structure. We implement this scheme for anisotropic or crystalline, nonlocal or inhomogeneous curvature flows, in linear and nonlinear instances, and study its convergence towards weak solutions to the flows. In Chapter 4 we also pair this study with a discrete-to-continuum limit. The second part, Part II, is devoted to the study of asymptotic behaviour of volume-preserving curvature flows both in the discrete- and continuus-in-time instances. The main technical tool employed is a new Łojasiewicz-Simon inequality suited to the study of these kind of evolutions.

Keywords: Geometric Evolution Equations, Mean Curvature Flows, Crystalline Curvature Flows, Minimizing Movements

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Flot de la Courbure Cristalline et Anisotrope, Non Linéaire ou Non Locale

Résumé

Cette thèse est consacrée à l'étude de flots géométriques, avec un accent particulier sur le flot de la courbure moyenne. La thèse est divisée en deux parties thématiques. La première partie, Partie I, contient les Chapitres 2, 3 et 4, et concerne des résultats de convergence pour le schéma des mouvements minimisants, qui est une procédure variationnelle étendant le schéma implicite d'Euler aux évolutions ayant une structure de type flot gradient. Nous mettons en œuvre ce schéma pour des flots, linéaires ou non linéaires, de la courbure anisotrope ou cristalline, non locale ou inhomogène, et nous étudions sa convergence vers des solutions faibles. Au Chapitre 4, nous associons également cette étude à une limite discrète-continue. La deuxième partie, Partie II, est consacrée à l'étude du comportement asymptotique des flots de la courbure avec une contrainte de volume, à la fois en temps discret et en temps continu. Le principal outil technique utilisé est une nouvelle inégalité de Łojasiewicz-Simon adaptée à l'étude de ce type d'évolutions.

Mots clés : Equations d'évolution géométrique, Flot de la Courbure Moyenne, Mouvements Minimisants

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Résumé de la thése

Cette thèse est consacrée à l'étude de flots géométriques, et plus particulièrement au flots de la courbure moyenne et certaines de ses variantes. Il est divisé en deux parties thématiques. La première partie, Partie I, contient les Chapitres 2, 3 et 4, et concerne les résultats de convergence pour le schéma des mouvements minimisants, qui est une procédure variationnelle étendant le schéma d'Euler implicite aux évolutions avec une structure de type flot de gradient. Nous implémentons ce schéma pour des flots, linéaires ou non linéaires, de la courbure anisotrope ou cristalline, non locale ou non homogène, et étudions sa convergence vers des solutions faibles du flot de la courbure moyenne.

En particulier, dans les Chapitres 2 et 3, nous étudions la convergence du schéma des mouvements minimisants dans les cas où, respectivement, les énergies de surface sont d'un type non homogène (c'est-à-dire sans invariance par translations), ou le flot de la courbure moyenne est modifié par une non-linéarité. Nous nous intéressons à la convergence de ces schémas en temps discret vers des solutions faibles, de type viscosité dans les deux chapitres, et de type distributionnel dans le Chapitre 2. Dans le Chapitre 4, en revanche, nous considérons une seconde discrétisation du schéma de mouvement minimisant, qui s'avère être donc discrète en espace et en temps. Le principal résultat de ce chapitre est une limite discrète-continue lorsque les paramètres de discrétisation tendent vers zéro.

La seconde partie est consacrée à l'étude du comportement asymptotique de certains flots géométriques préservant le volume, à la fois dans le cas discrèt et continu en temps. Le principal outil technique utilisé est une inégalité de Łojasiewicz-Simon adaptée à l'étude de ce type d'évolution. Dans les Chapitres 5 et 5, nous nous intéressons au comportement asymptotique des discrétisations temporelles des flots de la courbure moyenne sous contrainte de volume, dans le cas périodique et le cas où l'énergie de surface considérée est fractionnaire. Le dernier chapitre, Chapitre 7, traite plutôt des flots continus en temps. Nous montrons ici des résultats d'existence en temps long et un comportement asymptotique pour le flot de la courbure moyenne sous contrainte de volume, ainsi que pour et le flot de diffusion de surface.

Riassunto del contenuto della tesi

Questo lavoro di tesi è dedicato allo studio di flussi geometrici, con particolare attenzione data al flusso per curvatura media. È diviso in due parti tematiche. La prima parte, Parte I, contiene i Capitoli 2, 3 e 4, e riguarda risultati di convergenza per lo schema dei movimenti minimizzanti, che è una procedura variazionale che estende lo schema implicito di Eulero a evoluzioni temporali con una struttura simile ad un flusso gradiente. Implementiamo questo schema per flussi di curvatura anisotropi o cristallini, non lineari, non locali o con curvatura non omogenea, e ne studiamo la convergenza verso soluzioni deboli del flusso della curvatura media.

In particolare, nei Capitoli 2 e 3 studiamo la convergenza dello schema dei movimenti minimizzanti nei casi in cui, rispettivamente, le energie superficiali siano di tipo non omogeneo (ovvero senza invarianza per traslazioni), oppure il moto per curvatura media sia modificato con una nonlinearità. Ci interessiamo alla convergenza di questi schemi discreti in tempo verso soluzioni deboli, di tipo viscoso in entrambi i capitoli, e di tipo distribuzionale nel Capitolo 2. Nel Capitolo 4, invece, consideriamo una seconda discretizzazione dello schema dei movimenti minimizzanti, che quindi risulta essere discreto in spazio e in tempo. Il risultato principale di questo capitolo è un limite discreto-continuo quando i parametri di discretizzazione tendono a zero.

La seconda parte, Parte II, è dedicata allo studio del comportamento asintotico di alcuni flussi geometrici con vincolo di volume, sia in istanze discrete che continue nel tempo. Lo strumento tecnico principale impiegato è una disuguaglianza di Łojasiewicz-Simon adatta allo studio di questi tipi di evoluzioni. Nei Capitoli 5 e 6 ci si interessa al comportamento asintotico di discretizzazioni in tempo del flusso per curvature media con vincolo di volume, nel caso, rispettivamente, periodico oppure in cui l'energia di superficie considerata è frazionaria. L'ultimo capitolo, il Capitolo 7, tratta invece flussi continui nel tempo. In questo caso mostriamo risultati di esistenza per tempi lunghi e comportamento asintotico per il flusso di curvatura medio con vincolo di volume e il flusso di diffusione superficiale, nel caso particolare in cui il dato iniziale sia una piccola deformazione normale di un insieme periodico strettamente stabile.

Chapter 1

Introduction

The primary focus of this thesis is the study of geometric evolutions, particularly those arising as gradient flows of surface energies. Our main emphasis will be on the mean curvature flow and some of its variants: anisotropic or crystalline, nonlinear or nonlocal, fractional, with forcing and mobility, and volume-preserving. Additionally, we will discuss the surface diffusion flow, albeit to a lesser extent.

1.1 General Introduction

1.1.1 Perimeter and Mean Curvature

Many physical and biological phenomena can be represented as problems of minimizing interfacial energies subject to various constraints. The geometric energies in these contexts typically depend on the surface area of interfaces or on higher-order geometric features like curvature. With surface area or perimeter of a smooth set $E \subseteq \mathbb{R}^N$ we mean the area of its boundary ∂E, that is, the $(N-1)-$Hausdorff measure of its boundary

$$P(E) = \mathcal{H}^{N-1}(\partial E).$$

This latter definition, valid for smooth enough sets, can then be extended to a wider category of sets thanks to the notion of distributional perimeter. Let us recall the distributional definition of the perimeter, referring to [145] for an introduction on the topic. Given a measurable set $E \subseteq \mathbb{R}^N$, its (distributional) perimeter is defined as

$$P(E) = \sup \left\{ -\int_{\partial^* E} \mathrm{div}\varphi \, \mathrm{d}\mathcal{H}^{N-1} \ : \ \varphi \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N), |\varphi| \leq 1 \right\}, \tag{1.1}$$

which is to say, it is the total variation of the distributional gradient of χ_E i.e. $P(E) = |D\chi_E|(\mathbb{R}^N)$.

The mean curvature can be defined for smooth enough sets, say C^2, starting from geometric properties. Indeed, given a smooth set E we denote $\mathrm{H}_1(E), \ldots, \mathrm{H}_{N-1}(E)$ its principal curvatures. We then denote $\mathrm{H}_E = \sum_{i=1,\ldots,N-1} \mathrm{H}_i(E)$, and, with a slight abuse of notation, refer to H_E as the mean curvature of the set E. This notion can be generalized starting from the notion of (distributional) perimeter, and considering its first variation. In our geometric setting, the first variation of the perimeter is defined as follows. Given a set of finite perimeter E and a vector field $X : \mathbb{R}^N \to \mathbb{R}^N$ of class C^2, consider the flow $\Phi : \mathbb{R}^N \times (0,1) \to \mathbb{R}^N$ associated to X, that is, the solution to

$$\frac{\partial}{\partial t}\Phi = X(\Phi), \qquad \Phi(\cdot, 0) = id.$$

Then, the first variation of the perimeter of E with respect to X is defined as

$$\delta P(E)[X] = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} P(\Phi(\cdot,t)(E)),$$

where $\Phi(\cdot,t)(E)$ denotes the image of E through $\Phi(\cdot,t)$. Analogously, one can define the notion of second variation of the perimeter.

Now, it is possible to show (see e.g. [145]) that the first variation of $P(E)$ can be represented

$$\delta P(E)[X] = \int_{\partial^* E} \mathrm{div}_\tau X \, \mathrm{d}\mathcal{H}^{N-1},$$

where $\partial^* E$ is the reduced boundary of E, and $\mathrm{div}_\tau X = \mathrm{div} X - (\nu_E \cdot X)\nu_E$, with ν_E being the (measure-theoretic) outer unit normal to E. Therefore, one can define the (distributional) mean curvature H_E of a set of finite perimeter E as the function $\mathrm{H}_E \in L^1_{loc}(\partial^* E; \mathrm{d}\mathcal{H}^{N-1})$ (if it exists) such that

$$\int_{\partial^* E} \mathrm{div}_\tau X \, \mathrm{d}\mathcal{H}^{N-1} = \int_{\partial^* E} X \cdot \nu_E \mathrm{H}_E \, \mathrm{d}\mathcal{H}^{N-1}.$$

One can then prove that this definition extends to sets of finite perimeter the notion of mean curvature valid for smooth sets.

1.1.2 The Mean Curvature Flow

This thesis is focused on evolutions of sets driven by their mean curvature, as defined in the previous section. In particular, we will treat the mean curvature flow and some of its variants, and in Chapter 7 the surface diffusion flow. The mean curvature flow is a geometric evolution of hypersurfaces $E_t \subseteq \mathbb{R}^N$ indexed by a time parameter $t \geq 0$, that are evolving according to the following normal velocity. For every point x on the boundary of E_t and every time $t > 0$, the motion law is

$$V(x,t) = -\mathrm{H}_{E_t}(x), \tag{1.2}$$

where V denotes the component of the velocity relative to the outer normal vector of ∂E_t, and H_E is the mean curvature of the set E.

This flow has applications in Mathematical Physics, where the mean curvature flow can describe the evolution of interfaces between different phases of a material. For instance, in the context of phase transitions, the mean curvature flow can model the dynamics of phase boundaries [40, 156]. The behaviour of biological membranes, such as cell walls, can be also modelled using mean curvature flow, as these membranes tend to minimize their surface tension. Analogously, in Material Sciences the (anisotropic or crystalline) mean curvature flow is used to model the process of grain growth in polycrystalline materials [114, 40, 156]. Grain boundaries, which are interfaces between different crystalline regions, tend to move in a way that reduces the overall surface energy, following the mean curvature flow [157, 115, 16]. In Image Processing and Computer Vision, the mean curvature flow is applied for image smoothing and denoising [11, 54]. By evolving the image contours according to the mean curvature flow, noise can be reduced while preserving important features of the image. More recently, interesting applications arose in the field of Data Analysis and Artificial Intelligence [33, 34].

This evolution is also particularly interesting from a mathematical perspective, as it can be formally seen as the gradient flow of the perimeter with respect to a suitable norm on the space of forms. This implies, for instance, that the perimeters of the sets E_t are non-increasing along the flow, which can be easily checked from (1.2) and using the first variation formula for the perimeter recalled above. Indeed

$$\frac{\mathrm{d}}{\mathrm{d}t} P(E(t)) = \int_{\partial E} V \, \mathrm{H}_{E(t)} \, \mathrm{d}\mathcal{H}^{N-1} = -\int_{\partial E} \mathrm{H}^2_{E(t)} \leq 0.$$

One the one hand, the mean curvature flow is one of the most studied geometric flows, mainly due to the availability of a comparison principle holding for (1.2). In the present geometric setting, with comparison principle we mean roughly the following statement. If two initial data satisfy $E_0 \subseteq F_0$, then the flows defined by (1.2) starting from E_0, F_0 will satisfy the same inclusion as long as they exist. For the evolution law (1.2), this property essentially follows from the remark that if $E \subseteq F$ and $x \in \partial E \cap \partial F$, then $\mathrm{H}_E(x) \geq \mathrm{H}_F(x)$. This property becomes crucial in many definitions of weak solutions, for instance allowing the use of the level-set method [162], which can be tackled by considering viscosity solutions [58, 87, 173, 104] or distributional solutions [55] to (1.2), as discussed later on. Note that, while the above discussion on the comparison principle largely holds for the anisotropic or crystalline mean curvature flow, it is no longer true if one considers a volume-preserving version of law (1.2). This modification can be achieved for instance by adding the forcing $\int_{\partial E_t} \mathrm{H}_{E_t} \, \mathrm{d}\mathcal{H}^{N-1}$ in (1.2). In this case the comparison principle does not hold anymore and the mathematical study of weak solutions to the flow becomes much harder.

On the other hand, the study of the evolution law (1.2) presents many mathematical difficulties, mainly caused by the possible appearance of singularities of different kinds, even in a finite time-span and even if the initial data is smooth. For example, we can see merging or collision of near sets, pinch-offs or shrinking of connected components to points [112, 148, 149]. After the onset of singularities, the classical or smooth formulation of the flow (1.2) ceases to hold and needs to be replaced by a weaker one.

Classical or smooth solutions to (1.2) can be defined solving a parabolic PDE associated to the evolution. The idea is that any initial datum E_0 can be parametrized over a fixed reference surface Σ by a height function $u_0 = u(\cdot, 0)$. Then, imposing that the evolving surfaces E_t are all parametrized over Σ by functions $u(\cdot, t)$, one finds that u satisfies a parabolic Cauchy problem. The principal part of the evolution operator is the Laplacian, so the flow enjoys nice smoothing properties and *a priori* estimates, but only for a short time. This procedure ensures that, whenever the initial datum is a smooth hypersurface, the flow (1.2) exists for a short time and it is composed of smooth sets. Smooth solutions are well-defined as long as singularities do not develop, after which one may invoke weak solutions instead. A nice reference for this classic subject is [147].

Concerning weak solutions to the mean curvature flow, we have by now many different notions of weak solutions. Without intending to be exhaustive, we cite Brakke's solutions [32], viscosity solutions [87, 58, 173, 104], flat flows [8, 144, 155] and BV solutions [144, 155], and distributional solutions [55, 52, 53]. Let us discuss some of these weak notions, as the first part of the thesis is closely related to an approximation procedure used to prove existence for these solutions.

1.1.3 BV Solutions and the Minimizing Movements Scheme

Following [144], denoting $\mathbb{R}^N_T := \mathbb{R}^N \times [0, T)$, we say that a map $\chi : \mathbb{R}^N_T \to \{0, 1\} \in L^\infty((0, T); BV(\mathbb{R}^N))$ is a BV solution to (1.2) if the following conditions hold: there exists $v : \mathbb{R}^N_T \to \mathbb{R} \in L^1((0, T); L^1(|D\chi(\cdot, t)|))$ such that for every $\varphi \in C^\infty_c(\mathbb{R}^N_T; \mathbb{R}^N)$ and $\eta \in C^\infty(\mathbb{R}^N_T)$ with $\eta(\cdot, T) = 0$, it holds

$$\begin{aligned} &\int_{\mathbb{R}^N_T} \mathrm{div}_\tau \varphi \, |D\chi| + v\varphi \cdot D\chi = 0 \\ &\int_{\mathbb{R}^N_T} \chi \partial_t \eta + \int_{E_0} \eta(0) = -\int_{\mathbb{R}^N_T} v\eta |D\chi|. \end{aligned} \tag{1.3}$$

The two equations above can be rewritten in a more transparent way denoting $E_t = \{\chi(\cdot, t) = 1\}$, so that (1.3) becomes

$$\begin{aligned} &\int_0^T \int_{\partial^* E_t} \mathrm{div}_\tau \varphi \, \mathrm{d}\mathcal{H}^{N-1} = -\int_0^T \int_{\partial^* E_t} v\varphi \cdot \nu_{E_t} \, \mathrm{d}\mathcal{H}^{N-1}, \\ &\int_0^T \int_{E_t} \partial_t \eta + \int_{E_0} \eta(0) = -\int_0^T \int_{\partial^* E_t} v\eta. \end{aligned}$$

It is then easy to see that the first equation in (1.3) is a weak formulation of $v = -\mathrm{H}_E$, while the second one is "$\partial_t \chi_{E_t} = v$".

A way of proving existence for equations (1.3) is provided by the notion of flat flows, which in turn are defined starting from an iterative minimization procedure known as the minimizing movements scheme, proposed in the present setting in [144]. We briefly recall this approximation procedure, which is essentially a reformulation of Euler's implicit scheme adapted to gradient flows

[70, 108]. Let us consider a classical ODE

$$\begin{cases} \dot{x}(t) = -\nabla \mathcal{F}(x(t)), \quad t \geq 0 \\ x(0) = x_0 \end{cases} \tag{1.4}$$

defined on the Euclidean space $\mathbb{R}^N$, which can be seen as a gradient flow for the energy $\mathcal{F}$. A classical approach to find solutions to (1.4) is Euler's implicit scheme. Given a time discretization parameter $h > 0$, we set $x_0^h = x_0$ and, for $n \in \mathbb{N}$, we define

$$x_{n+1}^h \in \operatorname{argmin} \left\{ \mathcal{F}(y) + \frac{1}{h}|x_n^h - y|^2 \ : \ y \in \mathbb{R}^N \right\}. \tag{1.5}$$

The link between this problem and (1.4) is the Euler-Lagrange equation of (1.5)

$$\frac{x_{n+1}^h - x_n^h}{h} = -\nabla \mathcal{F}(x_{n+1}^h),$$

holding for $\mathcal{F} \in C^1$, which is clearly a discretization ofthe evolution (1.4). If $\mathcal{F}$ is lower semicontinuous and non-negative, the variational problem above admits a minimizer, which is unique if $\mathcal{F}$ is convex. Under some hypothesis on $\mathcal{F}$, it is then possible to prove that the discrete-in-time scheme

$$x^h(t) := x_{[t/h]}^h$$

converges as $h \to 0$ to a solution of the gradient flow (1.4).

One can then remark that the iterative problem (1.5) may be formulated in a very general setting, for instance when $\mathcal{F}$ is not differentiable, or even in metric spaces [70, 108]. In this case, this approach usually is referred to as the minimizing movements scheme.

In the context of geometric flows, it is sometimes possible to properly define a gradient flow with respect to some carefully-chosen Riemannian structure on the space of hypersurfaces, but in the case of the mean curvature flow this fails [152, 158]. It is nonetheless possible to *formally* interpret the mean curvature flow as the gradient flow of the perimeter with respect to an L^2-Riemaniann structure. With this formal identification in mind, in the seminal papers [8, 144] the authors defined the following minimizing movements scheme[1]. Given an initial bounded set of finite perimeter E_0 and a parameter $h > 0$, we set $E_0^h = E_0$ and

$$E_{n+1}^h \in \operatorname{argmin} \left\{ P(F) + \frac{1}{h} \int_{F \triangle E_n^h} \operatorname{dist}_{\partial E_n^h} \ : \ F \text{ is of finite perimeter} \right\}, \tag{1.6}$$

where $\operatorname{dist}_{\partial E}$ denotes the distance to the boundary of E. Denoting sd_E the signed distance function to E ($\operatorname{sd}_E = \operatorname{dist}_E - \operatorname{dist}_{\mathbb{R}^N \setminus E}$), for bounded sets E_k^h the problem above is equivalent (adding the finite constant $\int_{E_n^h} \operatorname{sd}_{E_n^h}$) to

$$E_{n+1}^h \in \left\{ P(F) + \frac{1}{h} \int_F \operatorname{sd}_{E_n^h} \ : \ F \text{ is of finite perimeter} \right\}.$$

It is easy to see that the minimum problem above admits solutions (which may not be unique). The discrete-in-time flow (usually called the discrete flow) is then defined as $E^h(t) := E_{[t/h]}^h$, where $[\cdot]$ denotes the integer part of a real number. With some work, it is possible to prove that the family $\{E^h(t)\}_{t \geq 0}$ converges, up to subsequences, as $h \to 0$ and in L^1_{loc}, to a flow $E(t)$ which is usually called flat flow. In order to do so, one previously needs to prove some time-continuity of the discrete flows. Once the flat flows have been defined, the task becomes to characterize the limiting evolution as being a generalized solution to (1.2). This was indeed the main result of [144], which showed that flat flows satisfy (1.3), and thus they are BV solutions to (1.2), under some technical assumptions. One of the crucial point of [144] was indeed proving that the term

[1] To be precise, in [8] the authors were dealing with anisotropic and crystalline perimeters, whose definition will be recalled below.

$\mathrm{sd}_{E^h_n}$ approximates a sort of L^2-distance squared of ∂E^h_t to ∂E^h_{t+h}, so that the functional in (1.6) truly resembles a minimizing movements scheme for an L^2-metric structure.

This variational approach has the advantage of being extremely flexible, as it can be adapted *mutatitis mutandis* to the anisotropic or crystalline case, in the presence of forcing, and even in the nonlinear or nonlocal case. A more thorough discussion of these results will be given in the following chapters, especially in those of Part I, while in Part II we will deal with a modification of the scheme suited for volume-preserving evolutions.

1.1.4 Viscosity and Distributional Solution

Another commonly used notion of generalized solutions is provided by viscosity solutions (also known as level-set solutions). The main idea behind this approach goes back to [162], and is known as the level-set approach. The idea is quite simple yet very interesting, and crucially exploits the comparison principle holding for the evolution (1.2). We embed an initial datum E_0 as the (closed) $0-$superlevel set of a bounded, uniformly continuous function $u_0 \in BUC(\mathbb{R}^N)$. Then, we let each superlevel set of u_0 evolve by mean curvature and, taking advantage of the comparison principle, we note that they describe the superlevel sets of a function $u : \mathbb{R}^N \times [0,T) \to \mathbb{R}$ as long as all the flows exist. One can then determine the evolution law of the function u knowing that each superlevel sets evolves according to (1.2), and indeed finds that u solves the parabolic Cauchy problem

$$\begin{cases} \partial_t u = |\nabla u| \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) \\ u(\cdot,0) = u_0. \end{cases} \tag{1.7}$$

Even though (1.7) is parabolic, it is very degenerate. There is no diffusion effect in the normal direction to its level set since, by definition, each level set of u moves independently from the others. Moreover, the equation is singular at $|\nabla u| = 0$. This means that classical techniques and results in the theory of parabolic equations cannot be expected to apply. Additionally, one cannot expect in general to have global smooth solutions, even if the initial data are smooth. Briefly, the resolution of (1.7) with classical methods poses issues. Two main strategies are employed to solve (1.7): either elliptic regularization, in the spirit of [87, 126], or by a direct use of viscosity solutions. We will now sketch the second approach, as it will be used in the thesis, following [58].

Let us consider a parabolic equation of the form

$$\partial_t u + F(\nabla u, \nabla^2 u) = 0, \tag{1.8}$$

where $F = F(p, X)$ is degenerate elliptic

$$X \geq Y \implies F(p,X) \leq F(p,Y), \qquad \forall p \neq 0,$$

is geometric

$$F(\lambda p, \lambda X + \sigma p \otimes p) = F(p,X) \text{ for all } p \neq 0, \sigma, \lambda \in \mathbb{R},$$

and satisfies some continuity and non-degeneracy conditions

$$\begin{aligned} &F \text{ is continuous in } \mathbb{R}^N \setminus \{0\} \times Sym^{N\times N} \\ &-\infty < F_*(0,0) = F^*(0,0) < +\infty \\ &F^*(p,-I) \leq c_1(|p|), F_*(p,-I) \geq -c_2(|p|), \end{aligned}$$

where F_*, F^* are, respectively, the lower and upper semicontinuous relaxation of F, $c_1, c_2 \in C^1([0,+\infty))$ with $c_1, c_2 \geq c > 0$, and $Sym^{N\times N}$ denotes the set of symmetric matrices of size N. One can prove that (1.7) falls into this framework, see for instance [58, 101] for details. A viscosity subsolution to (1.8) is an upper semicontinuous function u such that, for every $\varphi \in C^2$ and local maximum point $\hat{z} = (\hat{x}, \hat{t})$ of $u - \varphi$, we have

$$\begin{aligned} &\partial_t \varphi(\hat{z}) + F(\nabla \varphi(\hat{z}), \nabla^2 \varphi(\hat{z})) \leq 0, && \text{if } |\nabla \varphi|(\hat{z}) \neq 0 \\ &\partial_t \varphi(\hat{z}) \leq 0, && \text{if } |\nabla \varphi|(\hat{z}) = 0 \end{aligned}$$

In turn, a viscosity supersolution to (1.8) is an lower semicontinuous function u such that, for every $\varphi \in C^2$ and local minimum point $\hat{z} = (\hat{x}, \hat{t})$ of $u - \varphi$, we have

$$\begin{aligned} &\partial_t \varphi(\hat{z}) + F(\nabla \varphi(\hat{z}), \nabla^2 \varphi(\hat{z})) \geq 0, && \text{if } |\nabla \varphi|(\hat{z}) \neq 0 \\ &\partial_t \varphi(\hat{z}) \geq 0, && \text{if } |\nabla \varphi|(\hat{z}) = 0. \end{aligned}$$

Finally, a continuous function u is a viscosity solution to (1.8) if it is both a super- and subsolution. The powerful theory of viscosity solutions [18, 58, 101, 102] ensures that, under the hypotheses above on F, there exists a unique viscosity solution to (1.8). Moreover, this notion extends to the more general case $F = F(t, x, r, p, X)$, as recalled in Chapter 2. The generalized solution to (1.2), known as viscosity solution, is the flow $E(t)$ defined by $E(t) = \{u(\cdot, t) \geq 0\}$, where u is the unique viscosity solution to (1.7). It is possible to show, see for instance [56] where more general results are shown, that flat flows are viscosity solutions. Some sections of Part I will be devoted to showing similar results for modifications of the minimizing movements scheme.

Lastly, we would like to recall the notion of distributional solutions to (1.2). The key remark here is that, if the function u appearing in (1.7) was a distance function, characterized by $|\nabla u| = 1$, then (1.7) is nothing but the heat equation. This can not be ensured in general, but if the evolving sets are smooth one can nonetheless make the following remark. Let E_t evolve according to (1.2), and let $d(\cdot, t)$ denote the signed distance function to E_t. Since we consider the distance function, the normal velocity of $x \in \{d(\cdot, t) = s\}$ for $s > 0$ is nothing but the normal velocity of its projection y on ∂E_t, and coincides with the derivative in time of the distance function. On the other hand, by comparison, the curvature of $\{d(\cdot, t) = s\}$ at x is less or equal to the curvature of $E_t = \{d(\cdot, t) = 0\}$ at y, therefore one finds

$$\partial_t d(x, t) = \partial_t d(y, t) = \Delta d(y, t) \geq \Delta d(x, t),$$

where we used the identity $\mathrm{H}_E = \Delta \mathrm{sd}_E$, holding for smooth enough sets. Reasoning in the same way for negative level sets, one is led to study the evolution of distance functions $d(x, t)$ that satisfy

$$\begin{cases} \partial_t d \geq \Delta d, & \text{in } \{d \geq 0\} \\ \partial_t d \leq \Delta d, & \text{in } \{d \leq 0\}. \end{cases} \tag{1.9}$$

Using the remark above as a starting point, in [173] the author shows that the viscosity solution d to (1.9) characterizes also viscosity solutions, meaning that for every time $t > 0$, it holds $\{d(\cdot, t) \geq 0\} = \{u(\cdot, t) \geq 0\}$ where u is the unique viscosity solution to (1.7). This approach may be extended to (smooth) anisotropic instances, where the evolution law considered is

$$V(x, t) = -\psi(\nu_{E(t)}(x))\mathrm{H}^\phi_{E(t)}(x), \qquad \text{for } x \in \partial E(t), t > 0,$$

where ψ, ϕ are 1-homogeneous, $\psi \in C^1$ is a mobility, and H^ϕ_E is the anisotropic ϕ-curvature $\mathrm{H}^\phi_E = \mathrm{div}(\nabla \phi(\nu_E))$. Here ϕ is assumed to be at least C^2, so that the curvature is well-defined. In this context, the system (1.9) becomes

$$\begin{cases} \partial_t d^\psi \geq \mathrm{div}(\nabla \phi(\nabla d^\psi)), & \text{in } \{d \geq 0\} \\ \partial_t d^\psi \leq \mathrm{div}(\nabla \phi(\nabla d^\psi)), & \text{in } \{d \leq 0\} \end{cases} \tag{1.10}$$

and d^ψ is ψ°-signed distance function satisfying (1.10) in the viscosity sense, where ψ° denotes the polar of ψ, defined as $\psi^\circ(\xi) = \sup\{\xi \cdot v : \psi(v) = 1\}$. . Anyhow, this approach is no longer viable is ϕ is less regular, as in the (purely) crystalline case, where the function ϕ is piecewise affine and the associated Wulff shape is a convex polytope. The main idea of [55] (extended to more general cases in [52, 53]) is that, while viscosity solutions to (1.10) can not be defined in the classical way if ϕ is non smooth, the system (1.10) can still be used to characterize the crystalline evolutions. Morally, one requires that (1.10) is satisfied just in the distributional sense. These generalized solutions are referred to as distributional solutions to (1.2). For a more precise definition of distributional solutions we refer to Chapter 4.

We now discuss in a more detailed way the main topics of the chapters comprising the present thesis, which arise from the publications [50, 67, 68, 69] and the preprints [49, 66]. Lastly, we will discuss some of the current projects and some possible perspectives.

1.2 Part I: Convergence Results for the Minimizing Movements Scheme

Part I of this thesis is devoted to showing the convergence of the minimizing movements scheme towards mean curvature-type flows, and contains contributions made in collaboration with the co-advisors Antonin Chambolle and Massimiliano Morini.

Chapters 2 and 3 are devoted to the study of the minimizing movements in two interesting variants of (1.2), namely the nonhomogeneous case, and the nonlinear case.

In Chapter 2 we consider (suitably smooth) anisotropic surface energies. Let us recall the definition of the anisotropic perimeter. An anisotropy is a function $\phi : \mathbb{R}^N \times S^1 \to \mathbb{R}$ which is 1−homogeneous in the second variable. The anisotropic perimeter P_ϕ associated to ϕ is defined for a set $E \subseteq \mathbb{R}^N$ of finite perimeter as follows

$$P_\phi(E) := \int_{\partial^* E} \phi(x, \nu_E(x)) \mathcal{H}^{N-1}(x). \tag{1.11}$$

In general, we impose strong regularity on ϕ in both variables, apart from Chapter 4. Note that if ϕ depends on the position, the functional P_ϕ is not translation invariant. We call these surface energies nonhomogeneous perimeters, and note that they arise naturally when one considers evolution equations on manifolds [23]. Anisotropies with nontrivial dependency on the position will be considered in Chapter 2, while in the rest of the thesis we will consider anisotropies ϕ, ψ depending on ν_E only in (1.11). We implement the minimizing movements scheme in this setting, where now (1.6) is substituted by

$$E^h_{n+1} \in \operatorname{argmin} \left\{ P_\phi(F) + \frac{1}{h} \int_F \mathrm{sd}^\psi_{E^h_n} + \int_F F_h(x,t)\,\mathrm{d}x \right\},$$

where ϕ, ψ are two suitably smooth anisotropies, sd^ψ_E denotes the geodesic signed distance function to E, and $F_h(x,t) = \fint_t^{t+h} f(x,s)\,\mathrm{d}s$, with f a (suitably smooth) forcing term. After proving some time-continuity for the flows, letting $h \to 0$ we recover flat flows as L^1_{loc}−limit points of the discrete flows $E^h(t) = E^h_{[t/h]}$. The main result of Chapter 2 is that flat flows are BV and viscosity solutions, as defined in the section above, for the following anisotropic version of (1.2)

$$V(x,t) = -\psi(\nu_{E_t}(x)) \left(\mathrm{H}^\phi_{E_t}(x) + f(x,t) \right), \tag{1.12}$$

where H^ϕ_E is the ϕ−curvature of E, i.e. the first variation of P_ϕ. While the general outline of the proof follows [56], nontrivial technical difficulties arise due to the lack of translation invariance in the functionals considered, which was one of the main assumptions in [56].

In Chapter 3 we instead consider a nonlinear variant of the mean curvature flow for smooth enough, translation invariant anisotropies ϕ, ψ, which now depend only on the normal vector to the hypersurface. The evolution speed is a modification of (1.2) by a nonlinear function G satisfying some structural assumptions, and takes the form

$$V(x,t) = \psi(\nu_{E_t}(x)) G\big(-\mathrm{H}^\phi_{E_t}(x) + f(t) \big), \tag{1.13}$$

where f is a bounded and continuous forcing term, and G is a continuous, monotone non-decreasing function, with $G(0) = 0$. It is interesting to note that no asymptotic behaviour of G is required, so choices like $G(s) = (s)^+$ are admissible. Also in this case we prove that the minimizing movements scheme converges toward viscosity solutions to the mean curvature flow. Note that in this case a notion of BV solutions is not available, as nonlinearities are not easily included in (1.3).

In Chapter 4 we deal with a full (both space- and time-)discretization of a class of crystalline flows, showing the convergence of the scheme. The starting point is a level-set reformulation of the scheme (1.6), originally due to Chambolle [45]. This scheme is defined as follows. Given an initial bounded set $E_0 \subseteq \Omega \subset\subset \mathbb{R}^N$, we set $u_0^h = \mathrm{sd}_{E_0}$ and let for $k \in \mathbb{N}$

$$u_{k+1}^h = \operatorname{argmin}\left\{ |Dv|(\Omega) + \frac{1}{2h}\|\mathrm{sd}_{\{u_k^h \le 0\}} - v\|^2_{L^2(\Omega)} \,:\, v \in BV(\Omega) \cap L^2(\Omega) \right\}, \tag{1.14}$$

where $|Dv|(\Omega) = TV(v;\Omega)$ is the total variation of the function of bounded variation v [13]. One can show that the closed (respectively, open) 0-sublevelset of u_{k+1}^h are the maximal (resp. minimal) solution to the problem (1.6) with $\{u_k^h \le 0\}$ substituting E_k^h. Furthermore, the problem (1.14) admits a unique minimizer (which is also 1-Lipschitz) by the strict convexity of the functional. In Chapter 4 we propose a discretization of (1.14) in order to study solutions to

$$V = -\phi(\nu_{E(t)})\mathrm{H}^{\phi}_{E(t)}, \tag{1.15}$$

for a class of (purely) crystalline anisotropies ϕ such that $\{\phi \le 1\}$ is a convex rational zonotope (as explained below). The setting is as follows. Given $\varepsilon > 0$, we work on the discrete grid $\varepsilon\mathbb{Z}^N$ and consider discrete functions $u : \varepsilon\mathbb{Z}^N \to \mathbb{R}$, $u_i := u(i)$. The total variation appearing in (1.14) is substituted by its discrete, crystalline version

$$TV^{\varepsilon}_{\beta}(u) = \varepsilon^{N-1} \sum_{i,j \in \varepsilon\mathbb{Z}^N} \beta\big(\tfrac{i}{\varepsilon} - \tfrac{j}{\varepsilon}\big)|u_i - u_j|,$$

where the function $\beta : \mathbb{Z}^N \to [0, +\infty)$ has finite support and characterizes the surface tension considered. Indeed, the anisotropic perimeter under study is the one associated to the (rational, crystalline) anisotropy $\phi(\nu) := \sum_{i\in\mathbb{Z}^N} \beta(i)|\nu \cdot i|$, whose Wulff shape is a convex polytope called a zonotope. We modify the minimizing movements scheme (1.6) to define $u_{k+1}^{h,\varepsilon}$ iteratively minimizing

$$TV^{\varepsilon}_{\beta}(u) + \frac{1}{2h} \sum_{i \in \varepsilon\mathbb{Z}^N} |\mathrm{sd}^{\varepsilon}(u_k^{h,\varepsilon})_i - u_i|^2$$

(or, to be precise, solving the associated Euler-Lagrange equation), where $\mathrm{sd}^{\varepsilon}(u^{h,\varepsilon})$ is a discrete version of the signed distance sd_{E_s} (properly defined in Chapter 4). This modification allows to pass to the limit $\varepsilon, h \to 0$ and recover distributional solutions to (1.15) as defined in [55]. The main improvement of this result with respect to the previous literature is that we are able to prove the convergence of our scheme towards weak solutions to the crystalline mean curvature flow in any regime $\varepsilon, h \to 0$, whereas previously the condition $h \gg \varepsilon$ had to be required, in order to avoid artificial pinning of the limiting interfaces. This is then reflected in some numerical computations we present, which show the consistency of our scheme. Furthermore, our results are the first holding in every dimension $N \ge 2$, whereas only the planar case could be addressed with the previous techniques.

One of the main new insight is the definition of $\mathrm{sd}^{\varepsilon}$, the redistancing operator, which builds discrete distance functions from the interfaces. Indeed, the mere restriction to the grid the classical signed distance function from the discrete sets would create a drift term in the scheme, which can then be seen in the limiting evolution law [28, 29]. Instead, our definition allows us to avoid these artifacts and to recover the correct limiting evolution law.

We conclude the chapter with an interesting observation that we made while working on the project. We noticed that our new distance function can be used to provide an easy proof of consistency of our discrete scheme in the isotropic case of (1.2). In this case we need to assume the Courant-Friedrichs-Lewy condition $h \approx \varepsilon^2$ in order to ensure both the convergence of the approximating operators and that the distance function creates an iterable error. Some more remarks are presented in the last section of this introduction.

1.3 Part 2: Long-Time Behaviour for Volume-Preserving Flows

Part II of the thesis is more focused on questions concerning the long-time behaviour of some volume-preserving flows. The chapters therein contain contributions made in collaboration with Antonia Diana, Andrea Kubin and Anna Kubin.

In this part we modify the minimizing movements scheme (1.6) imposing a volume constraint, in order to study the evolution of sets with fixed mass. This can be done in multiple ways [155, 130]. For instance, instead of minimizing iteratively (1.6) one can consider

$$E^h_{n+1} \in \operatorname{argmin}\left\{P(F) + \frac{1}{h}\int_F \mathrm{sd}_{E^h_n} \ :\ |F| = |E_0|\right\},$$

or, in order to enforce the volume constraint in a "soft" way, consider the penalized problem

$$E^h_{n+1} \in \operatorname{argmin}\left\{P(F) + \frac{1}{h}\int_F \mathrm{sd}_{E^h_n} + \frac{1}{\sqrt{h}}||F| - |E_0||\right\}.$$

In the end, the two schemes are equivalent [130]. These schemes define some volume-preserving discrete flows, whose long-time behaviour can be studied. One hopes that this study may provide good information for the limiting behaviour of the flat flows, as proven rigorously in [133, 132] in the two- and three-dimensional setting.

The general common idea of this part is the following. Formally viewing the mean curvature flow as a gradient flow, if one starts the flow close enough to a strictly stable set[2] for the perimeter, it is reasonable to expect that the evolution flow will exist for all time and converge to the subjacent stable set. In order to follow this idea, one needs some sort of stability estimate, that will plat the role of a Łojasiewicz–Simon inequality in our setting. In this geometric setting, we mention the quantitative Alexandrov inequality that has been proved for the first time in [135] with non-sharp exponents, and then improved in [154] with sharp exponents but for almost-spherical sets. In [69] (which forms Chapter 5) we devise a new proof of this result, which has proved to be flexible enough to be adapted to different cases (periodic setting, fractional energies...). This stability inequality essentially says the following (we refer to Chapter 5 for a more precise statement). Assume that a smooth set F is sufficiently close in C^1 to a strictly stable set E, with $|F| = |E|$. Then F can be parametrize as a normal deformation[3] of the set E, meaning that there exists $f : \partial E \to \mathbb{R}$ so that

$$\partial F = \{x + f(x)\nu_E(x) \ :\ x \in \partial E\}.$$

Then our stability estimate says that, up to replacing F with a small translate (and correspondingly, changing f), if f satisfies the additional bound $|\int_{\partial E} f\nu_E| \lesssim \|f\|^2_{L^2(\partial E)}$, then it holds:

$$\|f\|_{H^1(\partial E)} \le C\|\mathrm{H}_F(x + f(x)\nu_E(x)) - \fint_{\partial E} \mathrm{H}_F(y + f(y)\nu_E(y))\,\mathrm{d}\mathcal{H}^{N-1}(y)\|_{L^2(\partial E)}. \tag{1.16}$$

This result can be seen either as a Łojasiewicz–Simon inequality, if we regard the mean curvature flow as a gradient flow, or as a quantitative version of the Alexandrov theorem: under the hypotheses above, if the curvature of F is constant then F coincides with (a translate of) E.

Our first contributions in this line of research, contained in Chapters 5 and 6, analyse the long-time behaviour of the discrete volume-preserving flows in two cases. In the former, we consider the classical perimeter (1.1) in a periodic setting. This is somehow interesting as the possible limit point of the flows are not trivial as in the Euclidean setting (where they are only union of equal balls). The geometric characterization of critical points (and also strictly stable sets) of the perimeter in the periodic setting is still an open problem in every dimension. In this case, we prove the dynamical stability of strictly stable periodic sets, and a more precise convergence result in dimension $N = 2$. As mentioned before, one of the crucial parts of the proof is showing

[2]Morally, critical points for the perimeter having strictly positive second variation, but the precise definition is more involved and is recalled in Part II.

[3]Sometimes, this parametrization is known as Fermi's coordinates.

(1.16) in our setting, where an explicit characterization of the stable sets is missing. Our proof thus relies only on the strict minimality of the set E, and is thus promisingly general. Indeed, we have been able to prove similar results also in the fractional case (Chapter 6) and in a nonlocal case (discussed in the section 1.4).

The latter, Chapter 6, treats the case of the fractional perimeter but in the Euclidean setting. Let us recall the notion of fractional perimeter. Given a measurable set $E \subseteq \mathbb{R}^N$ and a parameter $s \in (0,1)$, the $s-$fractional perimeter of E is defined as

$$P^s(E) = \int_E \int_{\mathbb{R}^N \setminus E} \frac{1}{|x-y|^{N+s}} \, \mathrm{d}x \, \mathrm{d}y.$$

This perimeter, while nonlocal in nature, shares many important properties with the classical perimeter (and is an example of the generalized perimeters considered in [56]). In this case it is known that the only volume-constrained critical points for the fractional perimeter in the whole Euclidean space is a single ball [89], as the nonlocal interactions penalize disconnected components. In [68] we modify the minimizing movements scheme (1.6) as follows

$$E^h_{n+1} \in \operatorname{argmin} \left\{ P^s(F) + \frac{1}{h} \int_F \mathrm{sd}_{E^h_n} + \frac{1}{h^{\frac{s}{1+s}}} ||F| - |E_0|| \right\},$$

and thus build a discrete flow which is (almost) volume-preserving. We are then able to characterize the asymptotic behaviour of the discrete flow from any initial, bounded set of finite perimeter, and prove the exponential convergence of the discrete flows toward a single ball. Again, the proof is essentially based on a fractional version of the stability estimate (1.16), which we establish for the first time.

Our other contribution [67], contained in Chapter 7, treats similar questions but for smooth flows in the periodic setting. We consider both the volume-preserving mean curvature flow and the surface diffusion, and now work with classical solutions (as previously briefly introduced). Let us recall the definitions of the surface diffusion flow, which is a geometric evolution of hypersurfaces $E_t \subseteq \mathbb{R}^N$ evolving according to the following normal velocity

$$V(x,t) = \Delta_{E_t} \mathrm{H}_{E_t}(x), \tag{1.17}$$

where Δ_{E_t} denotes the Laplace-Beltrami operator of the hypersurface E_t. It is easy to see that this is another instance of volume-preserving flow, which also decreases the perimeter. Moreover, it can be seen (formally) as the $H^{-1}-$gradient flow of the perimeter. In Chapter 7, we work again in the periodic setting, and we address the question of global existence and long-time behaviour of the aforementioned flows. Since in general singularities may appear in finite time (also in dimension $N = 2$, contrary to the mean curvature flow), and both flows do not preserve convexity, an interesting question is if there are instances for which global existence of the flows can be ensured.

We thus consider initial data that are suitably close (in some norm) to a strictly stable set. Again, we invoke the stability inequality (1.16), which we now need to pair with some Schauder estimates for the smooth flows, in order to iterate the short-time existence result and prove global existence of the flows. Lastly, we manage to prove convergence of the flows toward (a translate of) the subjacent stable set.

1.4 Future perspectives

In this last section we would like to present some possible further developments of the subjects treated in the thesis, and sketch some of the present research work under progress.

In the spirit of Part I, we are now addressing further questions regarding discrete-to-continuum limits. In particular, inspired by some remarks sketched in Appendix 4.B, we are trying to further investigate some discrete schemes comprising an (explicit) diffusion step, and a redistancing operation. In the easier case of the Laplacian, as considered in Appendix 4.B, this scheme can be described as follows. Given an initial function $u_0 : \varepsilon\mathbb{Z}^N \to \mathbb{R}$, define $u_0^\varepsilon = u_0$ and iteratively set

$$u_{k+1}^\varepsilon = \mathrm{sd}^\varepsilon(K_h^\varepsilon * u_k^\varepsilon), \tag{1.18}$$

where K_h^ε is a kernel associated to the explicit Euler's scheme for the discrete heat equation, namely the kernel associated to $1 - h\Delta_\varepsilon$. Here, again, sd^ε denotes the discrete redistancing operator we introduced in [49], whose construction is recalled in Chapter 4. In this case, one essentially makes the choice $h = C\varepsilon^2$ (Courant-Friedrichs-Lewy condition), in order to ensure that the discrete Laplacian Δ_ε converges to the continuum Laplacian Δ as $\varepsilon \to 0$. Reasoning as in [49], we are able to pass to the limit $\varepsilon \to 0$ and prove that the functions $u_{[t/h]}^\varepsilon$ are converging to distance functions $d(t)$ satisfying (1.9) in the distributional and viscosity sense, which is equivalent to say that the sets $E_t = \{d(t) \leq 0\}$ are moving according to (1.2). A natural related question is how much we can expect to generalize these results. In particular, a natural idea is to consider now general convolution kernels K_h^ε, which are converging (in a suitable sense) towards the Laplacian as $\varepsilon, h \to 0$. This is the subject of a current investigation.

Another interesting extension of Part I has been suggested by Matteo Novaga, and concerns Chapter 3. Indeed, it is reasonable to expect that the methods developed in Chapter 3 extend to the abstract setting of variational curvatures introduced in [56]. This is a current work in progress.

Concerning Part II, we are now investigating two different extensions of the results presented. The first one is an adaptation of the work [133] to the periodic setting. In this work, the authors manage to characterize the asymptotic behaviour of volume-preserving flat flows in the plane, morally passing to the limit $h \to 0$ the study conducted in [154] for discrete flows. A work in progress with Vedansh Arya and Anna Kubin addresses exactly the same question in the $2-$dimensional flat torus, where the main new technical difficulty is given by the non-uniqueness of the limiting configuration.

Finally, with Anna Kubin we are working on some extensions of our work [67] to include different smooth flows. In particular, we have some preliminary dynamical stability results for the (modified) Mullins-Sekerka flow, and we are trying to extend this study to geometric flows (formally) arising as H^{-s}gradient flows of the perimeter, $s \in (0, 1)$. Thanks to Poincaré-type inequalities, the convergence proof should follow by the same strategy based on the stability estimate (1.16), provided one has sufficiently strong *a priori* Schauder estimates on the linearized evolution equations. This is the most technically demanding part, and it is a current work in progress.

Part I

Existence Via Minimizing Movements

Chapter 2

Minimizing Movements for Anisotropic and Inhomogeneous Mean Curvature Flows

Contents

1 Introduction

In this chapter we deal with the anisotropic, inhomogeneous mean curvature flow with forcing and mobility. By inhomogeneous we mean that the flow is driven by surface tensions depending on the position in addition to the orientation of the surface. The evolution of sets $t \mapsto E_t \subseteq \mathbb{R}^N$ considered is (formally) governed by the law

$$V(x,t) = \psi(x, \nu_{E_t}(x)) \left(-\mathrm{H}^{\phi}_{E_t}(x) + f(x,t) \right), \quad x \in \partial E_t,\ t \in (0,T), \tag{2.1}$$

where $V(x,t)$ is the (outer) normal velocity of the boundary ∂E_t at x, $\phi(x,p)$ is a given anisotropy representing the *surface tension*, H^ϕ is the *anisotropic mean curvature* of ∂E_t associated to ϕ, $\psi(x,p)$ is an anisotropy evaluated at the outer unit normal $\nu_{E_t}(x)$ to ∂E_t which represents a *velocity* modifier (also called the *mobility* term), and f is the *forcing* term. We will be mainly concerned with smooth anisotropies (and the regularity assumptions will be made precise later on): in this case, the curvature H^ϕ is the first variation of the anisotropic and inhomogeneous perimeter associated to the anisotropy ϕ (in short, $\phi-$perimeter) defined as

$$P_\phi(E) := \int_{\partial^* E} \phi(x, \nu_E(x)) \, \mathrm{d}\mathcal{H}^{N-1}(x)$$

for any set E of finite perimeter (where $\partial^* E$ denotes the reduced boundary of E) and, if E is sufficiently smooth, it takes the form

$$\mathrm{H}^\phi_E(x) = \mathrm{div}(\nabla_p \phi(x, \nu_E(x))),$$

where with ∇_p we denote the gradient made with respect to the second variable. Note that evolution (2.1) can be red as the motion of sets in $\mathbb{R}^N$, when the latter is endowed with the Finsler metric induced by the anisotropy (see Remark 2.45). Equation (2.1) is relevant in Material Sciences, Crystal Growth, Image Segmentation, Geometry Processing and other fields see e.g. [5, 76, 113, 172, 174].

The mathematical literature for inhomogeneous mean curvature flows is not as extensive as in the homogeneous case, mainly due to the difficulties arising from the lack of translational invariance. Indeed, assuming that the evolution is invariant under translations allows to simplify many arguments used in the classical proofs of, for example, comparison results and estimates on the speed of evolution. In the homogeneous case the well-posedness theory is nowadays well established and quite satisfactory, both in the local and nonlocal case, and even in the much more challenging crystalline case (that is, when the anisotropy ϕ is piecewise affine) see [6, 8, 23, 52, 54, 56, 58, 106, 139, 144, 155] to cite a few. Concerning the inhomogeneous mean curvature flow, we cite [121, 123] where the short time existence of smooth solutions on manifolds is shown, and [102, 126], where the viscosity level set approach (introduced for the homogeneous evolution in [58, 87]) is extended, respectively, to the equation (2.1) and to the Riemannian setting.

In this chapter we implement the minimizing movement approach à la Almgren-Taylor-Wang (in short, ATW scheme) [8] to prove existence via approximation of a level set solution to the generalized anisotropic and inhomogeneous motion (2.1). To carry on this scheme (which has only been sketched in [23], but lacks a formal proof) we gain insights from [56]. We also show that, under the additional hypothesis of convergence of the energies (2.2) and low dimension (2.11)(which are nowadays classical for this approach), the same approximate solutions provide in the limit a suitable notion of "BV-solutions", also termed distributional solutions, see [144, 155].

There are many more concepts of weak solution for the mean curvature flow. In particular, we cite the diffuse-interface approximation provided by the Allen-Cahn equation [86, 125, 118, 140] and the threshold dynamic scheme [151, 83] (see also the relative entropy methods of [139]). Other recent results concern the weak-strong uniqueness problem, which consists in proving that weak solutions coincide with the smooth ones as long as the latter exist. After classical works concerning viscosity solutions, a new definition of "BV-solution" (whose existence is proved via the Allen-Cahn approximation scheme) allows the authors in [118, 140] to prove weak-strong uniqueness for isotropic and anisotropic mean curvature flows. This result is based upon the so-called *optimal dissipation inequality* satisfied by their weak solution. In general, it is very difficult to say if the ATW scheme could satisfy such a property, mainly because of the “degeneracy” of the dissipation term in the incremental problem defined via the distance function. Even if all these results concern the translationally invariant case, a study of some of these properties in the inhomogeneous setting seems very interesting and challenging.

Other remarks on possible research directions are the following. To begin with, the new arguments which are used to compensate the lack of translation invariance are based on the locality of the anisotropic curvature H^ϕ associated with a smooth anisotropy ϕ. This implies that the proofs are not straightforwardly adaptable to the so-called “variational curvatures” considered in [56], which are non-local in nature. On the other hand, since the crystalline curvatures are highly nonlocal and degenerate operators (see e.g. [55, 52, 44]), they do not fall in the theory constructed in the present chapter. In principle, it would be possible to follow the same perturbative study conducted in [52] in order to prove at least existence for an inhomogeneous and crystalline mean curvature flow. However, a satisfactory characterization of the limiting motion equation bearing a comparison principle is lacking so far.

This work can be seen as a first step towards constructing a general theory of motions driven by non-translationally invariant and possibly nonlocal curvatures, in the spirit of [56].

1.1 Main results

Now briefly recall the minimizing movements procedure in order to state the main results of the chapter. Given an initial bounded set E_0 and a parameter $h > 0$, we define the *discrete flow* $E_t^{(h)} := T_{h,t-h}E_{t-h}^{(h)}$ for any $t \geq h$ and $E_t^{(h)} = E_0$ for $t \in [0, h)$, where the functional $T_{h,t}$ is defined for $t \geq 0$ as follows: for any bounded set E we set $T_{h,t}E$ (or, sometimes, $T_{h,t}^{-}E$) as the minimal solution to the problem

$$\min\left\{P_\phi(F) + \int_F \left(\frac{\mathrm{sd}_E^\psi(x)}{h} - \fint_{[\frac{t}{h}]h}^{[\frac{t}{h}]h+h} f(x,s)\,\mathrm{d}s\right)\mathrm{d}\mathcal{H}^{N-1}(x) \ : \ F \text{ is measurable}\right\},$$

where $\mathrm{sd}_E^\psi(x)$ is the signed geodesic distance between x and E induced by the anisotropy ψ (see (2.8) for the precise definition) and $[s] = \max\{n \leq s,\ n \in \mathbb{N}\cup\{0\}\}$ denotes the integer part of a non-negative real number $s \in [0, +\infty)$. We will then define $T_{h,t}^{+}E$ as the maximal solution to the problem above. Any L^1-limit point as $h \to 0$ of the family $\{E_t^{(h)}\}_{t\geq 0}$ will be called a *flat flow.* In the whole chapter we will assume that

$$\begin{aligned} &\phi \in \mathscr{E} \text{ (see Definition 2.6) and } \psi \text{ is an anisotropy as in Definition 2.5,}\\ &\forall t \in [0, +\infty) \text{ it holds } f(\cdot, t) \in C^0(\mathbb{R}^N),\ \|f\|_{L^\infty(\mathbb{R}^N\times[0,+\infty))} < \infty. \end{aligned} \tag{H0}$$

With more effort one could weaken a little the hypothesis on f (see [57]). For the sake of simplicity we will require the global-in-time boundedness. We prove existence and Hölder-in-time regularity for flat flows.

Theorem 2.1 (Existence of flat flows)**.** *Let E_0 be a bounded set of finite perimeter and ϕ, ψ, f satisfy* (H0)*. Fix $T > 0$. For any $h > 0$, let $\{E_t^{(h)}\}_{t\in[0,T)}$ be a discrete flow with initial datum E_0. Then, there exists a family of sets of finite perimeter $\{E_t\}_{t\in[0,T)}$ and a subsequence $h_k \searrow 0$ such that*

$$E_t^{(h)} \to E_t \quad \text{in } L^1,$$

for a.e. $t \in [0, T)$. Such flow satisfies the following regularity property: there exists a constant c, depending on T, such that for every $0 \leq s \leq t < T$,

$$\begin{aligned} |E_s \triangle E_t| &\leq c|t - s|^{1/2},\\ P_\phi(E_t) &\leq P_\phi(E_0) + c. \end{aligned}$$

Subsequently, we will show that flat flow s are distributional solutions, as defined in [144]. We will require additional hypothesis: firstly, low dimension (2.11) (linked to the complete regularity of the $\phi-$perimeter minimizer, see [144, 155]), moreover

$$\exists\, c_\psi > 0 \text{ s.t. } |\psi(x,v) - \psi(y,v)| \le c_\psi |x-y|, \quad \forall x, y \in \mathbb{R}^N, v \in S^{N-1}, \tag{H1}$$

$$f \in C^0(\mathbb{R}^N \times [0,\infty)]). \tag{H2}$$

Theorem 2.2 (Existence of distributional solutions)**.** *Assume* (H0)*,* (H1)*,* (H2) *and* (2.11)*. For any $T > 0$, if*

$$\lim_{k\to\infty} \int_0^T P_\phi(E_t^{(h_k)}) = \int_0^T P_\phi(E_t), \tag{2.2}$$

then $\{E_t\}_{t\in[0,T]}$ is a distributional solution (2.1) *with initial datum E_0 in the following sense:*

(1) for a.e. $t \in [0,T)$ he set E_t has weak $\phi-$curvature $\mathrm{H}^\phi_{E_t}$ (see (2.15) *for details) satisfying*

$$\int_0^T \int_{\partial^* E_t} |\mathrm{H}^\phi_{E_t}|^2 < \infty;$$

(2) there exist $v : \mathbb{R}^N \times (0,T) \to \mathbb{R}$ with $\int_0^T \int_{\partial^* E_t} v^2 \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}t < \infty$ and $v(\cdot,t)\big|_{\partial E_t} \in L^2(\partial E_t)$ for a.e. $t \in [0,T)$, such that

$$-\int_0^T \int_{\partial^* E_t} v\eta \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}t = \int_0^T \int_{\partial^* E_t} \left(\mathrm{H}^\phi_{E_t} - f\right) \eta \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}t \tag{2.3}$$

$$\int_0^T \int_{E_t} \partial_t \eta \, \mathrm{d}x \, \mathrm{d}t + \int_{E_0} \eta(\cdot,0) \, \mathrm{d}x = -\int_0^T \int_{\partial^* E_t} \psi(\cdot, \nu_{E_t}) v\eta \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}t, \tag{2.4}$$

for every $\eta \in C^1_c(\mathbb{R}^N \times [0,T))$.

The definitions $1), 2)$ extend to our case the definition of BV-solutions of [144] and the distributional solutions of [155]. We recall that hypothesis (2.2) ensures that the evolving sets avoid the so-called "fattening" phenomenon. It is known that this hypothesis is satisfied in the case of evolution of convex or mean-convex sets, see e.g. [44, 71, 91], but in general is not known under which general hypothesis it is valid. We also remark that the proof of the theorem above provides a detailed proof of [44, Theorem 3.2], which had only been sketched. Moreover, we bypass the use of a Bernstein-type result (which is usually employed) by a double blow-up technique.

In the second part of the chapter we will focus on the level set approach. Briefly, given an initial compact set E_0, we set u_0 such that $\{u_0 \ge 0\} = E_0$ and we look for a solution u in the viscosity sense (in a sense made precise in Definition 2.35) to

$$\begin{cases} \partial_t u + \psi(x, -\nabla u) \left(\mathrm{div} \nabla_p \phi(x, \nabla u(x)) - f(x,t)\right) = 0 \\ u(\cdot, t) = u_0. \end{cases} \tag{2.5}$$

Classical remarks ensure that any level set $\{u \ge s\}$ is evolving following the mean curvature flow (2.1). To prove existence for (2.5) we use an approximating procedure. For $h > 0$ and $t \in (0, +\infty)$ we set iteratively $u_h^\pm(\cdot,t) = u_0$ for $t \in [0,h)$ and for $t \ge h$

$$u_h^+(x,t) := \sup\left\{ s \in \mathbb{R} \ : \ x \in T^+_{h,t-h}\{u_h^+(\cdot, t-h) \ge s\}\right\}$$
$$u_h^-(x,t) := \sup\left\{ s \in \mathbb{R} \ : \ x \in T^-_{h,t-h}\{u_h^-(\cdot, t-h) > s\}\right\},$$

where the operator $T^\pm_{h,t}$ has been previously introduced. We remark that these are maps piecewise constant in time, since $T^\pm_{h,t} = T^\pm_{h,[t/h]h}$, which are only upper and lower semicontinuous in space respectively. Then, we will pass to the limit $h \to 0$ on the families $\{u_h^\pm\}_h$ to find functions u^+, u^- which are viscosity sub - and supersolution respectively of equation (2.5). Passing to the limit as $h \to 0$ in our case is not straightforward. The main issue is that we do not have an uniform estimate on the modulus of continuity of the functions u_h (compare [56]) and thus we can not pass to the (locally) uniform limit of the sequence. (More precisely, our best estimate contained in Lemma 2.40 decays too fast as $h \to 0$ to provide any useful information). Nonetheless, motivated by [19, 18, 20] we can define the half-relaxed limits

$$\begin{aligned} u^+(x,t) &:= \sup_{(x_h,t_h)\to(x,t)} \limsup_{h\to 0} u_h^+(x_h,t_h) \\ u^-(x,t) &:= \inf_{(x_h,t_h)\to(x,t)} \liminf_{h\to 0} u_h^-(x_h,t_h), \end{aligned} \tag{2.6}$$

and prove that the functions defined above are sub - and supersolutions, respectively, to (2.5). The main difficulty in this regard is that we need to work with just semicontinuous functions in space, as in the translationally invariant setting one can easily prove the uniform equicontinuity of the approximating sequence. We prove the following.

Theorem 2.3. *Assume* (H0), (H1) *and* $f \in C^0(\mathbb{R}^n \times [0,+\infty))$. *The function* u^+ *(respectively* u^-*) defined in* (2.6) *is a viscosity subsolution (respectively a viscosity supersolution) of* (2.5).

Thanks to the results of [58] we then prove that, under the additional hypothesis

$$\begin{aligned} &\nabla_x\nabla_p\phi(\cdot,p) \text{ and } \nabla_p^2\phi(\cdot,p) \text{ are Lipschitz, uniformly for } p\in S^{N-1} \\ &\nabla_p^2\phi^2(x,p) \text{ is uniformly elliptic in } p, \text{uniformly in } x \\ &\psi(\cdot,p) \text{ Lipschitz continuous, uniformly in } p \\ &f(\cdot,t) \text{ Lipschitz continuous, uniformly in } t, \end{aligned} \tag{H3}$$

the following uniqueness result holds.

Theorem 2.4. *Assume* (H0) *and* (H3). *If* u_0 *is a continuous function which is spatially constant outside a compact set, equation* (2.5) *with initial condition* u_0 *admits a unique continuous viscosity solution* u *given by* (2.6). *In particular,* $u^+ = u^- = u$ *is the unique continuous viscosity solution to* (2.5) *and* $u_h^\pm \to u$ *as* $h \to 0$*, locally uniformly.*

The previous result yields a proof of consistency between the level set approach and the minimizing movements one to study the evolution (2.1). We recall that it has been established for the classical mean curvature flow in [45], in the anisotropic but homogeneous case in [85] and in a very general nonlocal setting in [56].

2 Preliminaries

We start introducing some notations. We consider $0 \in \mathbb{N}$. We will use both $B_r(x)$ and $B(x,r)$ to denote the Euclidean ball in $\mathbb{R}^N$ centered in x and of radius r; with $B_r^{N-1}(x)$ we denote the Euclidean ball in $\mathbb{R}^{N-1}$ centered in x and of radius r; with S^{N-1} we denote the sphere $\partial B_1(0) \subseteq \mathbb{R}^N$; with Sym_N the symmetric real matrices of size $N \times N$. In the following, we will always speak about measurable sets and refer to a set as the union of all the points of density 1 of that set i.e. $E = E^{(1)}$. If not otherwise stated, we implicitly assume that the function spaces considered are defined on $\mathbb{R}^N$, e.g $L^\infty = L^\infty(\mathbb{R}^N)$; the space C^0 denotes the space of continuous functions. Moreover, we often drop the measure with respect to which we are integrating, if clear from the context. For $\delta \in \mathbb{R}$ we denote

$$E_\delta = \{x \in \mathbb{R}^N \ : \ \mathrm{sd}_E(x) \leq \delta\},$$

and use the notation $E_{-\infty} := \emptyset, E_{+\infty} := \mathbb{R}^N$.

Definition 2.5. We define *anisotropy* (sometimes defined as an *elliptic integrand*) a function ψ with the following properties: $\psi(x,p) : \mathbb{R}^N \times \mathbb{R}^N \setminus \{0\} \to [0,+\infty)$ is a continuous function, which is convex, positive, and positively 1-homogeneous in the second variable, such that

$$\frac{1}{c_\psi}|p| \leq \psi(x,p) \leq c_\psi|p|$$

for any point $x \in \mathbb{R}^N$ and vector $p \in \mathbb{R}^N$.

We remark that, as standard, we define a real function f positively 1-homogeneous if for any $\lambda \geq 0$, it holds $f(\lambda x) = \lambda f(x)$. In particular, the anisotropies that we will consider are not symmetric. In the following, we will always denote the gradient of an anisotropy with respect to the first (respectively second) variable as $\nabla_x \psi$ (respectively $\nabla_p \psi$). We then recall the definition of some well-known quantities (see [23]). Define the polar function of an anisotropy ψ, denoted with ψ°, as

$$\psi^\circ(\cdot, \xi) := \sup_{p \in \mathbb{R}^N} \{\xi \cdot p \ : \ \psi(\cdot, p) \leq 1\}. \tag{2.7}$$

Using the definition it is easy to see that for all $p, \xi \in \mathbb{R}^N$ it holds

$$\psi(\cdot, p)\psi^\circ(\cdot, \xi) \geq p \cdot \xi, \quad -\psi(\cdot, -p)\psi^\circ(\cdot, \xi) \leq p \cdot \xi.$$

Furthermore, one can prove that (see [23]) for $p \neq 0$

$$\psi^\circ(\nabla_p \psi) = 1, \ \psi(\nabla_p \psi^\circ) = 1, \ (\psi^\circ)^\circ = \psi.$$

We define for any $x, y \in \mathbb{R}^N$ the geodesic distance induced by ψ, or $\psi-$distance in short, as

$$\mathrm{dist}^\psi(x, y) := \inf \left\{ \int_0^1 \psi^\circ(\gamma(t), \dot{\gamma}(t)) \, \mathrm{d}t \ : \ \gamma \in W^{1,1}([0,1]; \mathbb{R}^N), \gamma(0) = x, \gamma(1) = y \right\}.$$

We remark that this function is not symmetric in general. We define the *signed distance function* from a closed set $E \subseteq \mathbb{R}^N$ as

$$\mathrm{sd}_E^\psi(x) := \inf_{y \in E} \mathrm{dist}^\psi(y, x) - \inf_{y \notin E} \mathrm{dist}^\psi(x, y), \tag{2.8}$$

so that $\mathrm{sd}_E^\psi \geq 0$ on E^c and $\mathrm{sd}_E^\psi \leq 0$ in E. We remark that the bounds stated in Definition 2.5 imply

$$\frac{1}{c_\psi} \mathrm{dist} \leq \mathrm{dist}^\psi \leq c_\psi \mathrm{dist}, \tag{2.9}$$

where here and in the following we will denote with $\mathrm{dist}, \mathrm{sd}$ the Euclidean distance and signed distance function respectively. We define the $\psi-$balls as the balls associated to the $\psi-$distance, that is

$$B_\rho^\psi(x) := \{y \in \mathbb{R}^N : \mathrm{dist}^\psi(y, x) < \rho\},$$

which in general are not convex nor symmetric.

Definition 2.6. We say that an anisotropy ϕ is a *regular elliptic integrand*, and write $\phi \in \mathscr{E}$, if $\phi \in C^2, 1(\mathbb{R}^N \times \mathbb{R}^N \setminus \{0\})$ there exists two constants $\lambda \geq 1$, $l \geq 0$ such that for every $x, y, e \in \mathbb{R}^N$, $\nu, \nu' \in S^{N-1}$ one has:

$$\begin{gathered}
\frac{1}{\lambda} \leq \phi(x, \nu) \leq \lambda, \\
|\phi(x, \nu) - \phi(y, \nu)| + |\nabla_p \phi(x, \nu) - \nabla_p \phi(y, \nu)| \leq l|x - y| \\
|\nabla_p \phi(x, \nu)| + \|\nabla_p^2 \phi(x, \nu)\| + \frac{\|\nabla_p^2 \phi(x, \nu) - \nabla_p^2 \phi(x, \nu')\|}{|\nu - \nu'|} \leq \lambda \\
e \cdot \nabla_p^2 \phi(x, \nu)[e] \geq \frac{|e - (e \cdot \nu)\nu|^2}{\lambda}.
\end{gathered}$$

Given any set of finite perimeter E, one can define the $\phi-$perimeter P_ϕ as follows

$$P_\phi(E) := \int_{\partial^* E} \phi(x, \nu_E(x)) \, \mathrm{d}\mathcal{H}^{N-1}(x),$$

where $\partial^* E$ is the reduced boundary of E and ν_E is the measure-theoretic outer normal, see [145] for further references on sets of finite perimeter. The $\phi-$perimeter of a set of finite perimeter E

in an open set A is defined as

$$P_\phi(E;A) := \int_{\partial^* E\cap A} \phi(x,\nu_E(x))\,\mathrm{d}\mathcal{H}^{N-1}(x).$$

We remark that, by definition of regular elliptic integrand, for any set E of finite perimeter it holds

$$\frac{1}{\lambda}P(E) \le P_\phi(E) \le \lambda P(E).$$

Some additional remarks on this definition can be found in [72]. We just recall the submodularity property of the $\phi-$perimeter, which can be proved for instance by using the formulae for the reduced boundary and measure-theoretic normal of union and intersection of sets of finite perimeter (see [145]).

Proposition 2.7 (Submodularity property)**.** *For any two sets $E, F \subseteq \mathbb{R}^N$ of finite perimeter, one has*

$$P_\phi(E\cup F) + P_\phi(E\cap F) \le P_\phi(E) + P_\phi(F). \tag{2.10}$$

Moreover, by homogeneity, (2.7) and recalling that for any set E of finite perimeter it holds $D\chi_E = -\nu_E\,\mathrm{d}\mathcal{H}^{N-1}|_{\partial^* E}$ we have the following equivalent definitions

$$\begin{aligned} P_\phi(E) &= \sup\left\{\int_{\mathbb{R}^N} -D\chi_E\cdot\xi \ : \ \xi\in C^1_c(\mathbb{R}^N;\mathbb{R}^N), \phi^\circ(\cdot,\xi)\le 1\right\} \\ &= \sup\left\{\int_E \operatorname{div}\xi\,\mathrm{d}\mathcal{H}^{N-1} \ : \ \xi\in C^1_c(\mathbb{R}^N;\mathbb{R}^N), \phi^\circ(\cdot,\xi)\le 1\right\}. \end{aligned}$$

Concerning the regularity property of the $\phi-$perimeter minimizers, we refer to [7]. We just recall the following results. Given two anisotropies $\phi,\psi\in\mathscr{E}$, we define the "distance" between them as

$$\begin{aligned} &\mathrm{dist}_{\mathscr{E}}(\phi,\psi) := \sup\{|\phi(x,p)-\psi(x,p)| \\ &+ |\nabla_p\phi(x,p)-\psi(x,p)| + |\nabla_p^2\phi(x,p)-\nabla_p^2\psi(x,p)| : x\in\mathbb{R}^N, p\in S^{N-1}\}, \end{aligned}$$

where $|\cdot|$ denotes the Euclidian norm. Given $\phi\in\mathscr{E}$, we recall that E is a $0-$minimizer for the $\phi-$perimeter if for any $x\in\mathbb{R}^N, r>0$

$$P_\phi(E;B_r(x)) \le P_\phi(F;B_r(x))$$

for every $F\subset\mathbb{R}^N$ such that $F\triangle E\subset\subset B_r$. Then, some regularity properties of minimizers of $\phi-$perimeter can be found in the theorems of part $II.7$ and $II.8$ in [7], which are recalled below.

Theorem 2.8. *Assume $\phi\in\mathscr{E}$. Then, for any 0-minimizer E of the $\phi-$perimeter, the reduced boundary $\partial^* E$ of the set E is of class $C^{1,1/2}$ and the singular set $\Sigma := \partial E\setminus\partial^* E$ satisfies*

$$\mathcal{H}^{N-3}(\Sigma) = 0.$$

Theorem 2.9. *Let $m>0, \alpha\in(0,1)$. Then, there exists $\varepsilon=\varepsilon(m,\alpha)>0$ with the following property: let $\phi=\phi(p)\in\mathscr{E}$, $\phi\in C^{3,\alpha}(\mathbb{R}^N\setminus\{0\})$ with*

$$\|\phi|_{S^{N-1}}\|_{C^{3,\alpha}} \le m \text{ and } \mathrm{dist}_{\mathscr{E}}(\phi,|\cdot|)\le\varepsilon.$$

Then, for any 0-minimizer E of the $\phi-$perimeter, the reduced boundary $\partial^* E$ of the set E is of class $C^{1,1/2}$ and the singular set $\Sigma := \partial E\setminus\partial^* E$ satisfies

$$\mathcal{H}^{N-7}(\Sigma) = 0.$$

We sum up these hypotheses that yield the complete regularity of minimizers of parametric elliptic integrands:

$$\begin{aligned} &\text{either } \phi\in\mathscr{E} \text{ and } N\le 3, \\ &\text{or } N\le 7 \text{ and the hypotheses of Theorem 2.9 are satisfied.} \end{aligned} \tag{2.11}$$

2.1 The first variation of the $\phi-$perimeter

In this section we compute the first variation of the ϕ-perimeter and define some additional operators associated to it.

Assume E is of class C^2. Let X be a smooth and compactly supported vector field and assume $\Psi(x,t) =: \Psi_t(x)$ is the associated flow. To simplify the notation, we write

$$\nu(x,t) = \nabla_x \mathrm{sd}_{\Psi(E,t)}(x).$$

By classical formulae (see e.g. [38]) we can compute the following. For the sake of brevity, we avoid writing the evaluation $\phi = \phi(x, \nu_E(x))$, if not otherwise specified, and assume that all the integrals are made with respect to the Hausdorff $(N-1)$-dimensional measure $\mathcal{H}^{N-1}$.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P_\phi(E_t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int_{\partial E} \phi(\Psi_t(x), \nu(\Psi_t(x),t)) J\Psi_t \\
&= \int_{\partial E} \nabla_x\phi \cdot X + \nabla_p\phi \cdot (-\nabla_\tau(X\cdot\nu) + D\nu[X]) + \phi\, \mathrm{div}_\tau X \qquad (2.12)\\
&= \int_{\partial E} \nabla_x\phi \cdot X + \nabla_p\phi \cdot (-\nabla_\tau(X\cdot\nu) + D\nu[X]) + \mathrm{div}_\tau(\phi X) - \nabla\phi\cdot X + (\nabla\phi\cdot\nu)(X\cdot\nu) \\
&= \int_{\partial E} \nabla_x\phi \cdot X + \nabla_p\phi \cdot (-\nabla_\tau(X\cdot\nu) + D\nu[X]) - \nabla_x\phi\cdot X - D\nu[\nabla_p\phi]\cdot X \\
&\quad + \mathrm{div}_\tau(\phi X) + (\nabla\phi\cdot\nu)(X\cdot\nu) \\
&= \int_{\partial E} -\nabla_p\phi\cdot\nabla_\tau(X\cdot\nu) + (\nabla_x\phi\cdot\nu)(X\cdot\nu) + (D\nu[\nabla_p\phi]\cdot\nu)\,(X\cdot\nu) + \mathrm{div}_\tau(\phi X) \\
&= \int_{\partial E} \mathrm{div}_\tau\left(\nabla_p\phi(X\cdot\nu)\right) - \nabla_p\phi\cdot\nabla_\tau(X\cdot\nu) + (X\cdot\nu)(\nabla_x\phi\cdot\nu) \\
&= \int_{\partial E} (\mathrm{div}_\tau\nabla_p\phi)(X\cdot\nu) + \nabla_p\phi\cdot\nabla_\tau(X\cdot\nu) - \nabla_p\phi\cdot\nabla_\tau(X\cdot\nu) + (\nabla_x\phi\cdot\nu)(X\cdot\nu) \\
&= \int_{\partial E} (X\cdot\nu)\,(\mathrm{div}_\tau\nabla_p\phi + \nabla_x\phi\cdot\nu) = \int_{\partial E} (X\cdot\nu)\,\mathrm{div}\nabla_p\phi
\end{aligned}$$

where the last equality follows from the definition of div_τ and the fact that ϕ is $1-$homogeneous with respect to the p variable, since

$$\begin{aligned}
\mathrm{div}\nabla_p\phi &= \mathrm{div}_\tau\nabla_p\phi + \sum_i \nu_i\left(\partial_{x_i}\nabla_p\phi\right)[\nu] \\
&= \mathrm{div}_\tau\nabla_p\phi + \sum_i \nu_i\nabla_p(\partial_{x_i}\phi)\cdot\nu + \nu\cdot\left(\nabla_p^2\phi D\nu\right)[\nu] \\
&= \mathrm{div}_\tau\nabla_p\phi + \nabla_x\phi\cdot\nu.
\end{aligned}$$

Therefore, we define the first variation of a C^2-regular set E, induced by the vector field X, as

$$\delta P_\phi(E)[X\cdot\nu] := \int_{\partial E} (X(x)\cdot\nu(x))\,\mathrm{div}\nabla_p\phi(x,\nu(x))\,\mathrm{d}\mathcal{H}^{N-1}(x) \qquad (2.13)$$

and the $\phi-$curvature of the set E as

$$\mathrm{H}_E^\phi(x) := \mathrm{div}\nabla_p\phi(x,\nu(x)). \qquad (2.14)$$

If we now consider equation (2.12), we develop the tangential gradient to find

$$\nabla_p\phi\cdot(-\nabla_\tau(X\cdot\nu) + D\nu[X]) = \nabla_p\phi\cdot(-\nabla_\tau X[\nu] - D\nu[X] + D\nu[X]) = 0.$$

This shows that for any set E of class C^2 it holds

$$\delta P_\phi(E)[X\cdot\nu] := \int_{\partial E} \left(\nabla_x\phi\cdot X + \phi\,\mathrm{div}_\tau X\right)\,\mathrm{d}\mathcal{H}^{N-1},$$

where we dropped the evaluation of ϕ at $(x, \nu_E(x))$. We remark that the expression on the right hand side makes sense even if the set E is just of finite perimeter. Defining the ϕ−divergence operator div_ϕ as

$$\mathrm{div}_\phi X := \nabla_x \phi \cdot X + \phi \,\mathrm{div}_\tau X,$$

we are led to define the distributional ϕ−curvature of a set E of finite perimeter as an operator $\mathrm{H}^\phi_E \in L^1(\partial E)$ (if it exists) such that the following representation formula holds

$$\int_{\partial E} \mathrm{div}_\phi X \,\mathrm{d}\mathcal{H}^{N-1} = \int_{\partial E} \mathrm{H}^\phi_E \,\nu_E \cdot X \,\mathrm{d}\mathcal{H}^{N-1}, \qquad \forall X \in C^\infty_c(\mathbb{R}^N; \mathbb{R}^N). \tag{2.15}$$

The previous computations allow to say that the distributional ϕ−curvature can be expressed as (2.14) if the set is of class C^2. Finally, since ϕ is a regular elliptic integrand, one can prove the following monotonicity result.

Lemma 2.10. *Let E, F be two C^2 sets of finite ϕ−perimeter with $E \subseteq F$, and assume that $x \in \partial F \cap \partial E$: then $\mathrm{H}^\phi_F(x) \leq \mathrm{H}^\phi_E(x)$.*

Proof. Since the anisotropy is smooth, we can expand the curvature formula (2.14) as

$$\mathrm{H}^\phi = \mathrm{tr}\left(\nabla_x \nabla_p \phi(x,\nu) + \nabla^2_p \phi(x,\nu) D\nu\right) \tag{2.16}$$

and compare H^ϕ_E with H^ϕ_F. We consider separately the two terms appearing in (2.16). The first one depends on ν just by the value it has at the point x. Therefore, since $\nu_E(x) = \nu_F(x)$ we have the equality. The second one falls in the classical framework of smooth anisotropies that do not depend on the space variable. Since $D\nu_F \leq D\nu_E$ (as matrices) one concludes the proof. □

3 The minimizing movements approach

In this section we follow [155] (see also [8, 144]) to prove the existence for the mean curvature flow via the *minimizing movements* approach. We recall that in the whole chapter we will assume the hypothesis (H0).

3.1 The discrete scheme

In this subsection we will define the discrete scheme approximating the weak solution of the mean curvature flow, and we shall study some of its properties.

We define the following iterative scheme. Given $h > 0, f \in L^\infty(\mathbb{R}^N \times [0, \infty))$ and $t \geq h$, and given a bounded set of finite perimeter F, we minimize the energy functional

$$\mathscr{F}^F_{h,t}(E) = P_\phi(E) + \frac{1}{h}\int_E \mathrm{sd}^\psi_F(x)\,\mathrm{d}x - \int_E F_h(x,t)\,\mathrm{d}x \tag{2.17}$$

in the class of all measurable sets $E \subseteq \mathbb{R}^N$, and where we have set

$$F_h(x,t) := \fint_t^{t+h} f(x,s)\,\mathrm{d}s.$$

Equivalently, we could define the energy functional as

$$\mathscr{F}^F_{h,t}(E) = P_\phi(E) + \frac{1}{h}\int_{E \triangle F} |\mathrm{sd}^\psi_F| - \int_E F_h(x,t)\,\mathrm{d}x,$$

which agrees with (2.17) up to a constant. Then, we denote

$$T_{h,t}F = E \in \mathrm{argmin}\, \mathscr{F}^F_{h,t}.$$

We will refer to this minimizing procedure as the *incremental problem*. It is well-known (compare (2.13) and [145, Proposition 17.8]) that a minimimum of (2.17) of class C^2 satisfies the Euler-Lagrange equation

$$\int_{\partial E} \mathrm{H}^{\phi}_{E} X \cdot \nu_E \, \mathrm{d}\mathcal{H}^{N-1} = - \int_{\partial E} \left(\frac{1}{h} \mathrm{sd}^{\psi}_{F}(x) - F_h(x,t) \right) X(x) \cdot \nu_E(x) \, \mathrm{d}\mathcal{H}^{N-1}(x) \tag{2.18}$$

for all $X \in C^{\infty}_{c}(\mathbb{R}^N;\mathbb{R}^N)$. We can then define the *discrete flow*, which can be seen as a discrete-in-time approximation of the mean curvature flow starting from the initial set E_0. We define iteratively the *discrete flow* by setting $E^{(h)}_t = E_0$ for $t \in [0,h)$ and

$$E^{(h)}_t = T_{h,t-h} E^{(h)}_{t-h} = T_{h,([\frac{t}{h}]-1)h} E^{(h)}_{t-h}, \qquad t \in [h,+\infty), \tag{2.19}$$

where $[\cdot]$ denotes the integer part of a real number. This section is devoted to recall and prove some estimates on the discrete flow. The first one is a well-known existence result.

Lemma 2.11. *For any measurable function* $g : \mathbb{R}^N \to \mathbb{R}$ *such that* $\min\{g,0\} \in L^1_{loc}$, *the problem*

$$\min \left\{ P(E) + \int_E g \ : \ E \textit{ is of finite perimeter} \right\}$$

admits a solution.

Consider now F as a bounded set of finite perimeter. Then, the function $g = \mathrm{sd}^{\psi}_{F}/h - F_h$ is coercive, thus $\min\{g,0\} \in L^1$. Therefore, by the previous result and by classical arguments see [56, Proposition 6.1] for a proof, one can prove the following result.

Lemma 2.12. *For any given set F of finite perimeter, the problem* (2.17) *admits a solution E, which satisfies the discrete dissipation inequality*

$$P_{\phi}(E) + \frac{1}{h} \int_{E \triangle F} |\mathrm{sd}^{\psi}_{F}| \leq P_{\phi}(F) + \int_{E \setminus F} F_h(x,t) \, \mathrm{d}x - \int_{F \setminus E} F_h(x,t) \, \mathrm{d}x.$$

Moreover, the problem (2.17) *admits a minimal and a maximal solution.*

We define $T^{+}_{h,t}F$ (respectively $T^{-}_{h,t}F$) as the maximal (respectively minimal) solution to (2.17) having as initial datum F. In the following, whenever no confusion is possible, we shall write $T_{h,t}$ instead of $T^{-}_{h,t}$.

A comparison result holds. We will consider just bounded sets as datum for the problem (2.17), but the same result holds in general for unbounded sets (see also Section 4.1 for the case of unbounded sets with bounded boundary). The proof of this result is classical (see e.g. [56]) and it is based on the submodularity of the perimeter (2.10). We will omit it.

Lemma 2.13 (Weak comparison principle)**.** *Assume that F_1, F_2 are bounded sets with $F_1 \subset\subset F_2$ and consider $g_1, g_2 \in L^{\infty}$ with $g_1 \geq g_2$. Then, for any two solutions E_i, $i = 1,2$ of the problems*

$$\min \left\{ P_{\phi}(E) + \int_E \frac{\mathrm{sd}^{\psi}_{F_i}}{h} + g_i : E \text{ is of finite perimeter} \right\},$$

we have $E_1 \subseteq E_2$. If, instead, $F_1 \subseteq F_2$, then we have that the minimal (respectively maximal) solution to (2.17) *for $i = 1$ is contained in the minimal (respectively maximal) solution to* (2.17) *for $i = 2$.*

We now prove the volume-density estimates for minimizers of problem (2.17). This result is based on the minimality properties of almost-minimizers for perimeters induced by regular elliptic integrands (see [72, Remark 1.9] for further results). These estimates have the disadvantage that the smallness condition on the radius depends on the parameter h. Subsequently, we will recall a finer result in the spirit of [144], where we can drop this dependence by making some restrictions on the balls considered.

Lemma 2.14. *Let $g \in L^\infty$ and assume E minimizes the functional*

$$\mathscr{F}(F) = P_\phi(F) + \int_F g$$

among all measurable subsets of $\mathbb{R}^N$. Then the density estimate

$$\begin{aligned} \sigma\rho^N &\le |B_\rho(x) \cap E| \le (1-\sigma)\rho^N \\ \sigma\rho^{N-1} &\le P_\phi(E; B_\rho(x)) \le (1-\sigma)\rho^{N-1} \end{aligned} \tag{2.20}$$

holds for all $x \in \partial^* E$, $0 < \rho < (2\lambda\|g\|_\infty)^{-1} := \rho_0$, for a suitable $\sigma = \sigma(N, c_\psi, \lambda)$.

Proof. By minimality,

$$P_\phi(E) \le P_\phi(F) + \|g\|_\infty |E \triangle F| \qquad \forall F \subseteq \mathbb{R}^N,$$

thus [72, Lemma 2.8] implies the thesis. □

Remark 2.15. We remark that the previous result allows us to choose the minimal solution to (2.17) to be an open set, and the maximal one to be a closed set. This follows from the fact that the density estimates imply that the boundary of any minimizer has zero measure.

We now recall [52, Lemma 3.7], which is an anisotropic version of [144, Remark 1.4]. It provides volume-density estimates for minimizers of (2.17) starting from E, uniform in ψ and h, holding in the exterior of E. We remark that, even if in the reference the anisotropy ϕ considered did not depend on x, all the arguments hold with minor modifications also in our case. We recall the proof of this result, as similar techniques will be used later on.

Lemma 2.16. *Let E be a bounded, closed set, $h > 0$, and $g \in L^\infty(\mathbb{R}^N)$. Let E' be a minimizer of*

$$P_\phi(F) + \int_F \frac{\mathrm{sd}_E^\psi}{h} + g.$$

Then, there exists $\sigma > 0$, depending on λ, and $r_0 \in (0,1)$, depending only on $N, \lambda, G := \|g\|_{L^\infty(F)}$, with the following property: if $\bar{x}$ is such that $|E' \cap B_s(\bar{x})| > 0$ for all $s > 0$ and $B_r(\bar{x}) \cap E = \emptyset$ with $r \le r_0$, then

$$|E' \cap B_r(\bar{x})| \ge \sigma r^N. \tag{2.21}$$

Analogously, if $\bar{x}$ is such that $|B_s(\bar{x}) \setminus E'| > 0$ for all $s > 0$ and $B_r(\bar{x}) \subseteq E$ with $r \le r_0$, then

$$|B_r(\bar{x}) \setminus E'| \ge \sigma r^N.$$

Proof. For all $s \in (0, r)$, set $E'(s) := E' \setminus B_s(\bar{x})$. Note that, for a.e. s we have

$$P_\phi(E'(s)) = P_\phi(E') - P_\phi(E' \cap B_s(\bar{x})) + \int_{E' \cap \partial B_s(\bar{x})} (\phi(x, \nu(x)) + \phi(x, -\nu(x)))\, \mathrm{d}\mathcal{H}^{N-1}(x),$$

where ν denotes the outer normal vector of the set $E' \cap \partial B_s(\bar{x})$. Since $E' \cap B_s(\bar{x}) \subset E^c$ and $\mathrm{sd}_E^\psi \ge 0$ in E^c, one has $\int_{E' \cap B_s(\bar{x})} \mathrm{sd}_E^\psi \ge 0$, and therefore the minimality of E' implies

$$P_\phi(E' \cap B_s(\bar{x})) + \int_{E' \cap B_s(\bar{x})} g \le \int_{E' \cap \partial B_s(\bar{x})} (\phi(x, \nu(x)) + \phi(x, -\nu(x)))\, \mathrm{d}\mathcal{H}^{N-1}(x).$$

By the bound on the $\phi-$perimeter and using the classical isoperimetric inequality (whose constant is denoted C_N) we obtain

$$\begin{aligned} 2\lambda\mathcal{H}^{N-1}(E' \cap \partial B_s(\bar{x})) &\ge \frac{1}{\lambda} P(E' \cap B_s(\bar{x})) + \int_{E' \cap B_s(\bar{x})} g \\ &\ge \frac{1}{\lambda} C_N |E' \cap B_s(\bar{x})|^{\frac{N-1}{N}} - \|g\|_\infty |E' \cap B_s(\bar{x})| \ge \frac{C_N}{2\lambda} |E' \cap B_s(\bar{x})|^{\frac{N-1}{N}}, \end{aligned}$$

provided $|E' \cap B_s(\bar{x})|^{1/N} \leq C_N/(2\lambda\|g\|_\infty)$, which is true if r_0 is small enough. Since the *rhs* is positive for every s, we conclude

$$\frac{\mathrm{d}}{\mathrm{d}s}|E' \cap B_s(\bar{x})|^{\frac{1}{N}} \geq \frac{C_N}{4\lambda^2 N} \quad \text{for a.e. } s \in (0,r). \tag{2.22}$$

The thesis follows by integrating the above differential inequality. The other case is analogous. □

Remark 2.17. Requiring that the anisotropy ψ is bounded uniformly from above and below ensures that the results of the previous Lemmas 2.14 and 2.16 can be read in terms of the $\psi-$balls. For example, for any $r \geq 0$ and $x \in \mathbb{R}^N$, equation (2.21) could be read as $|E' \cap B_r^\psi(\bar{x})| \geq \sigma c_\psi^{-N} r^N$, provided $\bar{x}$ is such that $|E' \cap B_s^\psi(\bar{x})| > 0$ for all $s > 0$ and $B_r^\psi(\bar{x}) \cap E = \emptyset$, and holds for all $r \leq r_0/c_\psi$. Here, σ is as in Lemma 2.16 and depends only on λ. Analogous statements holds for Lemma 2.20.

We now provide some estimates on the evolution of balls under the discrete flow. We start by a simple remark concerning the boundedness of the evolving sets.

Remark 2.18. A simple estimate on the energies implies that the minimizers of (2.17) are bounded whenever F is bounded. Indeed, assume $F \subseteq B_R$ and consider $B_\rho(x) \cap (E \setminus B_R) \neq \emptyset$: testing the minimality of E against F we easily deduce

$$\frac{R}{2h}|B_\rho(x) \cap E| \leq \int_{E \cap B_\rho(x)} \frac{\mathrm{sd}_F^\psi}{h} \leq P_\phi(F) + \|F_h(\cdot,t)\|_\infty |E \triangle F| \leq P_\phi(F) + \|f\|_\infty(|F| + |E|).$$

Employing the density estimates of Lemma 2.16 and sending $R \to \infty$, we get a contradiction, as the isoperimetric inequality implies that $|E|$ is bounded since $\mathscr{F}_{h,t}^F(F) < \infty$.

We now want to prove finer estimates on the speed of evolution of balls. These estimates are classically a crucial step in order to prove existence of the flow. In the case under study, the main difficulties come from the inhomogeneity of the functionals considered, as in the homogeneous case convexity arguments easily yield the boundedness result, for example. We will use a "variational" approach in the spirit of [56] (but see also [155, Lemma 3.8] for a different proof relying more on the smoothness of the evolving set).

Lemma 2.19. *For every $R_0 > 0$ there exist $h_0(R_0) > 0$ and $C(R_0, \phi, \psi, f) > 0$ with the following property: For all $R \geq R_0$, $h \in (0, h_0)$, $t > 0$ and $x \in \mathbb{R}^N$ one has*

$$T_{h,t}(B_R(x)) \supset B_{R-Ch}(x). \tag{2.23}$$

Proof. We divide the proof into three steps. In the following, the constants σ, r_0 are those of Lemma 2.16. We will assume $x = 0$ for simplicity. We fix $R \geq R_0$ and denote $E := T_{h,t}B_R$.
Step 1. We prove that, given $a \in (0,\sigma), \varepsilon \in (0,1)$, we can ensure $|B_{R(1-\varepsilon)} \setminus E| < a\,R^N(1-\varepsilon)^N$ for h small enough. Indeed, assume by contradiction $|B_{R(1-\varepsilon)} \setminus E| \geq a\,R^N(1-\varepsilon)^N$. Testing the minimality of E against B_R, we obtain

$$\int_{(B_{R(1-\varepsilon)} \setminus E) \cup (E \setminus B_R)} \frac{|\mathrm{sd}_{B_R}^\psi|}{h} \leq \frac{1}{h}\int_{B_R \triangle E} |\mathrm{sd}_{B_R}^\psi| \leq P_\phi(B_R) - \int_{B_R \setminus E} F_h + \int_{E \setminus B_R} F_h,$$

and estimating $|\mathrm{sd}_{B_R}^\psi| \geq R\varepsilon/c_\psi$ on $B_{R(1-\varepsilon)} \setminus E$, we get

$$\frac{R\varepsilon}{hc_\psi}|B_{R(1-\varepsilon)} \setminus E| \leq P_\phi(B_R) + \|f\|_\infty\left(\omega_N R^N + |B_{R(1+\varepsilon)} \setminus B_R|\right) + \int_{E \setminus B_{R(1+\varepsilon)}} \left(F_h - \frac{|\mathrm{sd}_{B_R}^\psi|}{h}\right).$$

Taking $h \leq \varepsilon/(c_\psi\|f\|_\infty)$, the last term on the *rhs* is negative, thus

$$\frac{R\varepsilon}{hc_\psi}|B_{R(1-\varepsilon)} \setminus E| \leq P_\phi(B_R) + \|f\|_\infty R^N(\omega_N + 2^{N+1}\varepsilon).$$

We employ the hypothesis to obtain

$$\frac{a}{hc_\psi}\,\varepsilon(1-\varepsilon)^N R^{N+1} \le c_\psi N\omega_N R^{N-1} + cR^N,$$

a contradiction for $h \le ca\,\varepsilon\,(1-\varepsilon)^N \min\{1, R^2\}$, where c is a constant depending on $N, \phi, \psi, \|f\|_\infty$.
Step 2. Using Step 1, we prove that $B_{R/2} \subset E$ for h small. Assume that $R \le r_0$: by following the second part of the proof of Lemma 2.16 we obtain equation (2.22), which reads

$$\frac{\mathrm{d}}{\mathrm{d}s}|B_s \setminus E|^{1/N} \ge \frac{C_N}{4\lambda^2 N} = \sigma^{1/N} \quad \text{for a.e } s \in (0, R).$$

Applying the previous step with $\varepsilon = 1/4, a = \sigma/3^N$, it holds $|B_{3R/4} \setminus E| \le \sigma R^N/4^N$ for all $h \le c(N, \phi, \psi, f)R$. Therefore, one deduces the existence of a positive extinction radius

$$R^* = \frac{3R}{4} - \frac{|B_{3R/4} \setminus E|^{1/N}}{\sigma^{1/N}} \ge \frac{R}{2}$$

such that $|B_{R^*} \setminus E| = 0$, which proves the claim. Clearly, taking $h \le cR_0$ the smallness assumption on h is uniform for $R \ge R_0$.

If $R \ge r_0$ one simply uses a covering argument. For any $x \in B_{R-r_0}$, applying the previous result to the ball $B_{r_0}(x)$ and using the comparison principle of Lemma 2.13, we conclude that $\forall h \le c\,r_0$ it holds

$$\bigcup_{x\in B_{R-r_0}} B_{r_0/2}(x) \subset\subset E.$$

Step 3. We conclude the proof. By the previous two steps and Remark 2.18, taking h small enough, we see that

$$\rho := \sup\{r > 0 \ :\ |B_r \setminus E| = 0\} \in (R/2, +\infty).$$

We can assume $\rho \le R$, otherwise the result of the lemma is trivial. Consider the vector field $\nabla_p\phi\left(x, \frac{x}{|x|}\right) \in C^1(\mathbb{R}^N, \mathbb{R}^N)$. Then, recalling (2.11), we get $P_\phi(G) \ge -\int_{\mathbb{R}^N} D\chi_G \cdot \nabla_p\phi(x, x/|x|)$ for all G set of finite perimeter and

$$P_\phi((1+\varepsilon)B_\rho) = \int_{\mathbb{R}^N} D\chi_{(1+\varepsilon)B_\rho} \cdot \left(-\nabla_p\phi\left(x, \frac{x}{|x|}\right)\right).$$

Setting $W_\varepsilon = (1+\varepsilon)B_\rho \setminus E$, by submodularity on $(1+\varepsilon)B_\rho, E$ and exploiting the minimality of E, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} \nabla_p\phi\left(x, \frac{x}{|x|}\right) \cdot D\chi_{W_\varepsilon} &= \int_{\mathbb{R}^N} \nabla_p\phi\left(x, \frac{x}{|x|}\right) \cdot \left(D\chi_{(1+\varepsilon)B_\rho} - D\chi_{(1+\varepsilon)B_\rho\cap E}\right)\\
&\le P_\phi((1+\varepsilon)B_\rho \cap E) - P_\phi((1+\varepsilon)B_\rho)\\
&\le P_\phi(E) - P_\phi((1+\varepsilon)B_\rho \cup E)\\
&\le \frac{1}{h}\int_{W^\varepsilon} \mathrm{sd}^\psi_{B_R} - \int_{W_\varepsilon} F_h(x,t)\,\mathrm{d}x.
\end{aligned}$$

We conclude, using the divergence theorem ,

$$\int_{W^\varepsilon} -\mathrm{div}\nabla_p\phi\left(x, \frac{x}{|x|}\right) \le \frac{1}{h}\int_{W^\varepsilon} \mathrm{sd}^\psi_{B_R} + \|f\|_\infty |W_\varepsilon|.$$

Dividing by $|W^\varepsilon|$ and sending $\varepsilon \to 0$ we obtain

$$\fint_{\partial B_\rho \cap E} -\mathrm{div}\nabla_p\phi\left(x, \frac{x}{|x|}\right)\,\mathrm{d}\mathcal{H}^{N-1} \le \frac{1}{c_\psi}\frac{\rho - R}{h} + \|f\|_\infty.$$

Exploiting the regularity assumptions on ϕ, we remark that

$$|\mathrm{div}\nabla_p\phi| = |\mathrm{tr}\left(\nabla_x\nabla_p\phi + \nabla_p^2\phi\nabla(x/|x|)\right)| \leq C\left(1+\frac{1}{|x|}\right).$$

Thus, we obtain

$$-C\left(1+\frac{1}{\rho}\right) \leq \frac{\rho - R}{h},$$

which implies that $\rho \in (0,\rho_1)\cup(\rho_2,R)$ for $\rho_{1,2} = \left(R - Ch \mp \sqrt{(R-Ch)^2 - 4Ch}\right)/2$, as long as $h \leq R_0^2/(4C)$. Since the choice $\rho \leq \rho_1 < R/2$ is not admissible, we conclude the proof by estimating

$$\rho_2 = R - Ch + \frac{R-Ch}{2}\left(\sqrt{1-\frac{4Ch}{(R-Ch)^2}} - 1\right) \geq R - Ch - \frac{Ch}{R-Ch},$$

from which the thesis follows. □

The proof of the previous result can be employed to prove an estimate from above of the evolution speed of the flow, as the following result shows. Since the proof follows the same lines and is easier in this case, we only sketch it.

Lemma 2.20. *Fix $T > 0$ and $R_0 > 0$. Then, there exist positive constants $C = C(\phi,\psi,f,R_0)$ and $h_0 = h_0(R_0)$ such that, for every $R \geq R_0$ and $h \leq h_0$, if $E_0 \subseteq B_R$, then $E_t^{(h)} \subseteq B_{R+CT}$ for all $t \in (0,T)$.*

Proof. Choose h small as in the previous result and set

$$\rho = \inf\{r > 0 \ : \ |E \setminus B_r| = 0\} \in (R/2, +\infty).$$

We can assume $\rho \geq R$, otherwise the result is trivial. Defining $W_\varepsilon = E \setminus (1-\varepsilon)B_\rho$ and reasoning as before we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} \nabla_p\phi\left(x,\frac{x}{|x|}\right)\cdot D\chi_{W^\varepsilon} &= \int_{\mathbb{R}^N}\nabla_p\phi\left(x,\frac{x}{|x|}\right)\cdot\left(D\chi_{(1-\varepsilon)B_\rho\cup E} - D\chi_{(1-\varepsilon)B_\rho}\right)\\
&\geq -P_\phi((1-\varepsilon)B_\rho\cup E) + P_\phi((1-\varepsilon)B_\rho)\\
&\geq -P_\phi(E) + P_\phi((1-\varepsilon)B_\rho\cap E)\\
&\geq \frac{1}{h}\int_{W^\varepsilon}\mathrm{sd}^\psi_{B_R} - \int_{W_\varepsilon} F_h(x,t)\,\mathrm{d}x.
\end{aligned}$$

As in the previous proof, we arrive at

$$\frac{\rho - R}{h} \leq C\left(1+\frac{1}{\rho}\right),$$

which implies that $\rho \leq \rho_2 = \left(R + Ch + \sqrt{(R+Ch)^2 + 4Ch}\right)/2 \leq R + Ch$, up to changing C. □

3.2 Existence of flat flows

In the following, we will prove that the discrete flow (defined in (2.19)) defines a discrete-in-time approximation of a weak solution to the mean curvature flow, which is usually known as a "flat" flow (because the approximating surfaces $\partial^* E_t^{(h)}$ converge in the "flat" distance of Whitney to the limit $\partial^* E_t$, see [8]).

We start by proving uniform bounds on the distance between two consecutive sets of the discrete flow and on the symmetric difference between them. We introduce the time-discrete normal velocity: for all $t \geq 0$ and $x \in \mathbb{R}^N$, we set

$$v_h(x,t) := \begin{cases} \frac{1}{h}\mathrm{sd}^{\psi}_{E^{(h)}_{t-h}}(x) & \text{for } t \in [h,+\infty) \\ 0 & \text{for } t \in [0,h). \end{cases}$$

The following result provides a bound on the $L^\infty-$norm of the discrete velocity. Since the proof is essentially the same of [144, Lemma 2.1], we will omit it. The only difference is that we use the upper and lower bounds of (2.9) to work with Euclidean balls.

Lemma 2.21. *There exists a positive constant c_∞ depending only on N, ψ with the following property. Let E_0 be a bounded set of finite perimeter and let $\{E^{(h)}_t\}_{t\in(0,T)}$ be a discrete flow starting from E_0. Then,*

$$\sup_{E^{(h)}_t \triangle E^{(h)}_{t-h}} |v_h(\cdot,t)| \leq c_\infty h^{-1/2}$$

for all h sufficiently small.

The following result can be found in [155, Proposition 3.4] (see also [92, Lemma 2.2]): it provides an estimate on the volume of the symmetric difference of two consecutive sets of the discrete flow. The proof is analogous to the one in the reference.

Lemma 2.22. *There exists a constant C such that for every $t \geq h$ the discrete flow $E^{(h)}_t$ satisfies for all h sufficiently small*

$$|E^{(h)}_{t+h} \triangle E^{(h)}_t| \leq C\left(l P_\phi(E^{(h)}_t) + \frac{1}{l}\int_{E^{(h)}_t \triangle E^{(h)}_{t+h}} |\mathrm{sd}^{\psi}_{E^{(h)}_t}| \right) \quad \forall l \leq c\sqrt{h}, \tag{2.24}$$

where c is a positive constant depending on N, ψ.

We are now able to prove an uniform bound on the perimeter of the evolving sets. The proof follows [92, Proposition 2.3].

Lemma 2.23. *For any initial bounded set E_0 of finite $\phi-$perimeter and h small enough, the discrete flow $\{E^{(h)}_t\}$ satisfies*

$$P_\phi(E^{(h)}_t) \leq C_T \quad \forall t \in (0,T),$$

for a suitable constant $C_T = C_T(T, E_0, f, \phi, \psi)$.

Proof. By testing the minimality of $E^{(h)}_t$ against $E^{(h)}_{t-h}$ we obtain $\forall t \in [h,T)$

$$P_\phi(E^{(h)}_t) + \frac{1}{h}\int_{E^{(h)}_t \triangle E^{(h)}_{t-h}} |\mathrm{sd}^{\psi}_{E^{(h)}_{t-h}}| \leq P_\phi(E^{(h)}_{t-h}) + \|f\|_\infty |E^{(h)}_t \triangle E^{(h)}_{t-h}|. \tag{2.25}$$

Combining this estimate with (2.24) for $l = 2Ch\|f\|_\infty \ll \sqrt{h}$, where C is the constant appearing in equation (2.24), we obtain for h sufficiently small

$$P_\phi(E^{(h)}_t) + \frac{1}{2h}\int_{E^{(h)}_t \triangle E^{(h)}_{t-h}} |\mathrm{sd}^{\psi}_{E^{(h)}_{t-h}}| \leq \left(1 + 2C^2 h\|f\|^2_\infty\right) P_\phi(E^{(h)}_{t-h}) \tag{2.26}$$

Iterating the previous estimate, we find

$$P_\phi(E^{(h)}_t) \leq (1 + 2C^2\|f\|_\infty h)^{\left[\frac{t}{h}\right]-1} P_\phi(E^{(h)}_h).$$

In order to estimate $P_\phi(E^{(h)}_h)$ we start by observing that Remark 2.18, for $h = h(E_0)$ small enough, implies $E^{(h)}_h \subseteq B_{2r}$, where $E_0 \subseteq B_r$. Therefore, by (2.25) for $t = h$ we obtain $P_\phi(E^{(h)}_h) \leq P_\phi(E_0) + c$ and we conclude $P_\phi(E^{(h)}_t) \leq C_T(P_\phi(E_0) + 1)$. □

We then present a sketch of the proof of the local Hölder continuity in time of the discrete flow, uniformly in h, which can be deduced as in [92, Proposition 2.3]. We highlight the main differences.

Proposition 2.24. *Let E_0 be an initial bounded set of finite $\phi-$perimeter and $T > 0$. Then, for h small enough, for a discrete flow $\{E_t^{(h)}\}$ starting from E_0 it holds*

$$|E_t^{(h)} \triangle E_s^{(h)}| \leq C_T|t-s|^{1/2} \quad \forall h \leq t \leq s < T,$$

for a suitable constant $C_T = C_T(T, E_0, f, \phi, \psi)$.

Proof. Following the previous proof, employing again (2.26) we find

$$\begin{aligned} P_\phi(E_{2h}^{(h)}) + \frac{1}{2}\int_{E_{2h}^{(h)} \triangle E_h^{(h)}} |v_h(\cdot, 2h)| + \frac{1}{2}\int_{E_h^{(h)} \triangle E_0^{(h)}} |v_h(\cdot, h)| \\ \leq (1+ch)P_\phi(E_h^{(h)}) + \frac{1}{2}\int_{E_h^{(h)} \triangle E_0^{(h)}} |v_h(\cdot, h)| \\ \leq (1+ch)\left(P_\phi(E_h^{(h)}) + \int_{E_h^{(h)} \triangle E_0^{(h)}} |v_h|(\cdot, h)\right) \leq (1+ch)^2 P_\phi(E_0). \end{aligned}$$

Iterating, we conclude as before

$$\sum_{k=1}^{[T/h]} \int_{E_{kh}^{(h)} \triangle E_{(k-1)h}^{(h)}} |v_h(\cdot, kh)| \leq C_T(P_\phi(E_0) + 1). \tag{2.27}$$

Therefore, combining the previous results and applying (2.24) with $l = h \ll \sqrt{h}$, we obtain

$$\int_h^T |E_t^{(h)} \triangle E_{t-h}^{(h)}| \leq c\sum_{k=1}^{[T/h]} \left(hP_\phi(E_{kh}^{(h)}) + \int_{E_{kh}^{(h)} \triangle E_{(k-1)h}^{(h)}} |v_h(\cdot, kh)|\right) \leq C_T\left(P_\phi(E_0) + 1\right). \tag{2.28}$$

The proof then follows the one of [92, Proposition 2.3], from equation (2.5) onward. □

We finally prove the main result of this section, the existence of flat flows.

Proof of Theorem 2.1. The proof is classical and we only sketch it. By the uniform equicontinuity of the approximating sequence of Proposition 2.24 and compactness of sets of finite perimeter (by Lemma 2.20 and 2.23) we can use the Ascoli-Arzelà theorem to prove that the sequence $(E_t^{(h_k)})_{k\in\mathbb{N}}$ converges in L^1 to sets E_t for all times $t \geq 0$ and that the family $\{E_t\}_{t\geq 0}$ satisfies the $1/2-$Hölder continuity property, locally uniformly in time. The other property is then easily deduced. □

3.3 Existence of distributional solutions

From Theorem 2.1 we deduce the existence of a subsequence $(h_k)_{k\geq 0}$ such that

$$D\chi_{E_t^{(h_k)}} \overset{*}{\rightharpoonup} D\chi_{E_t} \qquad \forall t \geq 0. \tag{2.29}$$

We will also assume (2.2), remarking that it implies

$$\lim_{k\to\infty} P_\phi(E_t^{(h_k)}) = P_\phi(E_t) \qquad \text{for a.e. } t \in [0, +\infty). \tag{2.30}$$

Our aim is to derive (2.3) and (2.4) from the Euler-Lagrange equation (2.18) and passing to the limit $h \to 0$. To achieve this, we will prove that the discrete velocity is a good approximation (up to multiplicative factors) of the discrete evolution speed of the sets. Notice that (2.3) is a weak formulation of (2.1), while (2.4) establishes the link between v and the velocity of the boundaries of E_t. Indeed, law (2.1) can be interpreted as looking for a family $\{E_t\}_{t\geq 0}$ of sets, whose normal vector ν_{E_t} and $\phi-$curvature $\mathrm{H}_{E_t}^\phi$ are well-defined objects and a function $v : [0,\infty) \times \mathbb{R}^N \to \mathbb{R}$

such that for every $t \in [0,+\infty)$ and $x \in \partial E_t$

$$\begin{cases} v(x,t) &= -\mathrm{H}^{\phi}_{E_t}(x) + f(x,t) \\ V(x,t) &= \psi(x,\nu_{E_t}(x))v(x,t), \end{cases}$$

where V represents the normal velocity of evolution, obtained as the limit as $h \to 0$ (in a suitable sense) of the ratio

$$\frac{\chi_{E_t} - \chi_{E_{t-h}}}{h}.$$

In this whole section we will assume that hypothesis (2.11) holds. In particular, the sets defining the discrete flow are smooth hypersurfaces in $\mathbb{R}^N$. Moreover, we require hypotheses (H1) to hold.

We start by estimating in time the L^2-norm of the discrete velocity. The proof is the same as the one presented in [155, Lemma 3.6], up to using the density estimates on the $\phi-$perimeter of Lemma 2.14 and considering the $\psi-$balls instead of the Euclidean one.

Proposition 2.25. *Let $\{E^{(h)}_t\}_{t\geq 0}$ be a discrete flow starting from an initial bounded set E_0 of finite $\phi-$perimeter. Then, for any $T>0$ and for h small enough, it holds*

$$\int_0^T \int_{\partial E^{(h)}_t} v_h^2 \,\mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t \leq C_T,$$

for a suitable constant $C_T = C_T(T, E_0, \phi, \psi, f)$.

Recalling now the Euler-Lagrange equation (2.18) and Lemma 2.23 we conclude

$$\int_0^T \int_{\partial E^{(h)}_t} \left(\mathrm{H}^{\phi}_{E^{(h)}_t}\right)^2 = \int_0^T \int_{\partial E^{(h)}_t} (v_h - F_h)^2 \leq C_T, \tag{2.31}$$

We now prove an estimate on the error between the discrete velocity $\psi(\cdot,\nu_{E_t})v_h(\cdot,t)$ and the discrete time derivative of χ_h. The proof of this result is based on a double blow-up argument, and the smoothness of sets (locally) minimizing the $\phi-$perimeter is essential. We will split the proof in various lemmas: the first one concerns the composition of blow-ups, and is a well-known result to the experts. We present a simple proof since we could not find a reference.

Lemma 2.26 (Composition of blow-ups)**.** *Consider $0<\beta<\beta'<1$. Assume that $(A_h)_{h\in[0,1]}$ is a family of measurable sets such that the following blow-ups converge as $h \to 0$*

$$\frac{A_h - x_h}{h^\beta} \to A_1 \quad \text{in } L^1_{loc}$$

$$h^{-(\beta'-\beta)}A_1 \to A_2 \quad \text{in } L^1_{loc},$$

where $x_h \in \partial A_h$ for all $h \in [0,1]$. Then, if the composition of the blow-ups $h^{-\beta'}(A_h - x_h)$ converges in L^1_{loc}, the limit coincides with A_2.

Proof. We can assume *wlog* $x_h = 0$. Denote with $A_3 = L^1_{loc} - \lim_{h\to 0} h^{-\beta'}A_h$. We fix a ball B_M and $\varepsilon>0$. There exists h^* such that $\forall h \leq h^*$ it holds

$$|((h^{-\beta'}A_h)\triangle A_3)\cap B_M| \leq \varepsilon, \quad |((h^{-\beta'+\beta}A_1)\triangle A_2)\cap B_M| \leq \varepsilon.$$

We fix h and *wlog* assume $Mh^{\beta'-\beta}\leq 1$. Taking $\tilde h < h$ suitably small (depending on h,ε), we can ensure

$$|((\tilde h^{-\beta}A_h)\triangle A_1)\cap B_1| \leq \varepsilon h^{N(\beta'-\beta)}.$$

Since $\tilde h^{-\beta}h^{-(\beta'-\beta)} > h^{-\beta'}$, there exists $\bar h < h$ such that $\bar h^{-\beta'} = \tilde h^{-\beta}h^{-(\beta'-\beta)}$. We can then estimate

$$\begin{aligned}|(A_3\triangle A_2)\cap B_M| &\le |(A_3\triangle \bar{h}^{-\beta'}A_h)\cap B_M| + |((h^{-\beta'+\beta})A_1\triangle(\bar{h}^{-\beta'}A_h))\cap B_M| \\ &\quad + |((h^{-\beta'+\beta}A_1)\triangle A_2)\cap B_M| \\ &\le 2\varepsilon + h^{-N(\beta'-\beta)}|(A_1\triangle(\tilde{h}^{-\beta}A_h))\cap B_{Mh^{\beta'-\beta}}| \\ &\le 2\varepsilon + h^{-N(\beta'-\beta)}|(A_1\triangle(\tilde{h}^{-\beta}A_h))\cap B_1| \le 3\varepsilon.\end{aligned}$$

□

We now compute some estimates on the normal vector on the boundary of the evolving sets, following the proof of [155, Lemma 4.2] (see also [144, Proposition 2.2]). We fix c_∞ as the constant appearing in Lemma 2.21.

In the sequel, we will denote by $\omega(h)$ a modulus of continuity, that is a continuous increasing function $\omega : [0,1] \to \mathbb{R}$ with $\omega(0) = 0$, which can possibly change from statement to statement and line to line to absorb constants independent of h.

Lemma 2.27. *Assume* (H0) *and* (H1). *For given constants $1/2 < \beta' < \alpha < 1$ and $T > 2$, there exists a modulus of continuity ω with the following property. Consider $t \in [2h, T]$ and $x_h \in \partial E_t^{(h)}$ such that*

$$|v_h(t,y)| \le h^{\alpha-1} \quad \forall y \in B_{c_\infty\sqrt{h}}(x_h)\cap(E_t^{(h)}\triangle E_{t-h}^{(h)}). \tag{2.32}$$

Then, there exists $\nu \in S^{N-1}$ such that

$$\begin{aligned}&|\nu_{E_t^{(h)}}(\cdot)-\nu| \le \omega(h) \quad \text{in } B_{h^{\beta'}}(x_h)\cap\partial E_t^{(h)} \\ &|\nu_{E_{t-h}^{(h)}}(\cdot)-\nu| \le \omega(h) \quad \text{in } B_{h^{\beta'}}(x_h)\cap\partial E_{t-h}^{(h)}.\end{aligned} \tag{2.33}$$

Proof. We fix $\frac{1}{2} < \beta < \beta' < \alpha$ and $0 < R < h^{\frac{1}{2}-\beta}/c_\psi$. Testing the minimality of $E_s^{(h)}$, $s = t, t-h$, we find

$$P_\phi(E_s^{(h)}, B_{Rh^\beta}(x_h)) \le P_\phi(G, B_{Rh^\beta}(x_h)) + \frac{1}{h}\int_{G\triangle E_s^{(h)}} |\mathrm{sd}^\psi_{E_{s-h}^{(h)}}| + \int_{G\triangle E_s^{(h)}} |F_h|, \tag{2.34}$$

for any set G of finite perimeter such that $G\triangle E_s^{(h)} \subset\subset B_{Rh^\beta}(x_h)$. Using Lemma 2.21, the 1−Lipschitz regularity of sd^ψ and (2.32), we deduce $|v_h(s,y)| \le c_\psi Rh^{\beta-1} + c_\infty h^{-1/2} \le (1+c_\infty)h^{-1/2}$ for any $y \in B_{Rh^\beta}(x_h)\cap(E_s^{(h)}\triangle F)$. Plugging this inequality in (2.34), we find

$$P_\phi(E_s^{(h)}, B_{Rh^\beta}(x_h)) \le P_\phi(G, B_{Rh^\beta}(x_h)) + \frac{1+c}{\sqrt{h}}|F\triangle E_s^{(h)}| + \|f\|_\infty|G\triangle E_s^{(h)}|. \tag{2.35}$$

We then introduce the blown-up sets for $s = t, t-h$, defined as

$$E_s^{(h),\beta} := h^{-\beta}\left(E_s^{(h)} - x_h\right).$$

Rescaling equation (2.35), we easily find that $E_s^{(h),\beta}$ is a (Λ_h, r_h)−minimizer of the $\phi(x_h + h^\beta\cdot,\cdot)$−perimeter, with $\Lambda_h = (1+c)h^{\beta-1/2}$, $r_h = h^{1/2-\beta}$. Moreover, scaling the density estimates (2.20) we have a uniform bound on the perimeters of the sets $E_s^{(h),\beta}$ in each ball B_R. By compactness, there exist two sets E_1^β, E_2^β such that

$$E_t^{(h),\beta} \to E_1^\beta,\ E_{t-h}^{(h),\beta} \to E_2^\beta \qquad \text{in } L^1_{loc}.$$

Then, by scaling and (2.32) we find

$$|\mathrm{sd}^\psi_{E_{t-h}^{(h),\beta}}(\cdot)| \le c_\infty h^{\alpha-\beta} \qquad \text{on } B_{h^{1/2-\beta}}(0)\cap(E_t^{(h),\beta}\triangle E_{t-h}^{(h),\beta}),$$

thus we easily conclude that $E^\beta := E_1^\beta = E_2^\beta$. By Lemma 2.20 we can assume that $x_h \to x_0$ as

$h \to 0$, up to subsequences. Moreover, by closeness of Λ_h-minimizers under $L^1_{loc}-$convergence (see e.g. [72, Theorem 2.9]), one can see that E^β is a 0-minimizer for the $\phi(x_0,\cdot)-$perimeter. Thus, by complete regularity, it is a smooth C^2 set. We can then employ the classic blow-up theorem to deduce that, for a fixed $\beta' \in (\beta,\alpha)$, the blow-up $h^{-(\beta'-\beta)}E^\beta$ converges to a half-space $\mathbb{H} = \{x \cdot \nu \le 0\}$ as $h \to 0$. Finally, the blow-ups

$$E_s^{(h),\beta'} := \frac{E_s^{(h)} - x_h}{h^{\beta'}}$$

admit a converging subsequence by compactness of sets of finite perimeter and by rescaling equation (2.35). Thus, the previous Lemma 2.26 implies

$$E_s^{(h),\beta'} \to \mathbb{H} \quad \text{in } L^1_{loc}$$

as $h \to 0$. To conclude, the $\varepsilon-$regularity Theorem for $\Lambda-$minimizers (see e.g. [72, Theorem 3.1]) ensures that $E_s^{(h),\beta'}$ are uniformly $C^{1,\frac{1}{2}}$ sets in $B_1(0)$ for $s = t, t-h$ as $h \to 0$. □

We recall here an approximation result proved in [144] (see also [155] for a more detailed proof). We remark that the proof of this result is purely geometric and does not rely on the variational problem satisfied by the sets $E_t^{(h)}, E_{t-h}^{(h)}$.

Corollary 2.28 (Corollary 4.3 in [155]). *Under the hypotheses of Lemma 2.27, fix $0 < \beta < \alpha$ and let $\boldsymbol{C}_{h^\beta}$ be the open cylinder defined as*

$$\boldsymbol{C}_{h^\beta}(x_h,\nu) := \left\{ x \in \mathbb{R}^N : |(x - x_h)\cdot\nu| < \frac{h^\beta}{2},\ \Big|(x-x_h) - ((x-x_h)\cdot\nu)\,\nu\Big| < \frac{h^\beta}{2} \right\}.$$

Then, it holds

$$\Big|\int_{\boldsymbol{C}_{h^\beta/2}(x_h,\nu)} (\chi_{E_t^{(h)}} - \chi_{E_{t-h}^{(h)}})\,\mathrm{d}x - \int_{\partial E_t^{(h)} \cap \boldsymbol{C}_{h^\beta/2}(x_h,\nu)} \mathrm{sd}_{E_{t-h}^{(h)}}\,\mathrm{d}\mathcal{H}^{N-1}\Big| \le \omega(h)\int_{\boldsymbol{C}_{h^\beta/2}(x_h,\nu)} |\chi_{E_t^{(h)}} - \chi_{E_{t-h}^{(h)}}|.$$

Carefully inspecting the proof, one indeed proves that there exists a geometric constant C such that for any $y \in B^{N-1}_{h^\beta/2}(x_h)$

$$|\mathrm{sd}_{E_{t-h}^{(h)}}(y, f_t^{(h)}(y))\sqrt{1+|\nabla f_t^{(h)}(y)|^2} - \left(f_t^{(h)}(y) - f_{t-h}^{(h)}(y)\right)| \le \omega(h)|f_t^{(h)}(y) - f_{t-h}^{(h)}(y)|, \quad (2.36)$$

where we set

$$\partial E_s^{(h)} \cap \mathbf{C} = \{(y, f_s^{(h)}(y)) \in \mathbb{R}^{N-1}\times\mathbb{R},\ |y| \le h^\beta/2\},$$

for $s = t, t-h$.

We briefly recall some classical results. Consider an anisotropy ψ, independent of the position. It is well-known that, for any closed set $G \subseteq \mathbb{R}^N$, setting sd_G^ψ as the distance from G induced by ψ°, then the gradient of sd_G^ψ exists almost everywhere and satisfies the eikonal equation (for a proof see for instance [44, Remark 2.2])

$$\psi(\nabla \mathrm{sd}_G^\psi) = 1 \qquad (2.37)$$

almost everywhere. Moreover, in this particular case, in the definition of dist^ψ we can consider just straight lines as follows from a simple application of Jensen's inequality: for any curve γ as in the definition of dist^ψ, we have

$$\int_0^1 \psi^\circ(\dot\gamma(t))\,\mathrm{d}t \ge \psi^\circ\left(\int_0^1 \dot\gamma\right) = \psi^\circ(y-x).$$

Proposition 2.29 (Estimate on almost flat sets)**.** *Under the hypotheses of Lemma 2.27 and with the same notation, fix $\beta \in (0, \alpha)$ and let $\boldsymbol{C}_{h^\beta}$ be the open cylinder defined as*

$$\boldsymbol{C}_{h^\beta}(x_h, \nu) := \left\{ x \in \mathbb{R}^N : |(x - x_h) \cdot \nu| < \frac{h^\beta}{2}, \ \left|(x - x_h) - ((x - x_h) \cdot \nu)\, \nu\right| < \frac{h^\beta}{2} \right\}.$$

Then, it holds

$$\Big| \int_{\boldsymbol{C}_{h^\beta/2}(x_h,\nu)} (\chi_{E_t^{(h)}} - \chi_{E_{t-h}^{(h)}})\, \mathrm{d}x - \int_{\partial E_t^{(h)} \cap \boldsymbol{C}_{h^\beta/2}(x_h,\nu)} \psi(x, \nu_{E_t^{(h)}})\, \mathrm{sd}^{\psi}_{E_{t-h}^{(h)}}\, \mathrm{d}\mathcal{H}^{N-1} \Big| \\ \leq \omega(h) \int_{\boldsymbol{C}_{h^\beta/2}(x_h,\nu)} |\chi_{E_t^{(h)}} - \chi_{E_{t-h}^{(h)}}|.$$

Proof. We recall that the modulus of continuity ω may change from line to line to absorb constants independent of h.

From the previous Lemma 2.27 we know that, for h suitably small, both $\partial E_t^{(h)}$ and $\partial E_{t-h}^{(h)}$ in $\mathbf{C}_{h^\beta/2}(x_h, \nu)$ can be written as graphs of functions of class $C^{1,\frac{1}{2}}$. Up to a change of coordinates, we can assume *wlog* that $x_h = 0, \nu = e_N$. For simplicity, we set $\mathbf{C} = \mathbf{C}_{h^\beta/2}(0, e_N)$. We thus find

$$\partial E_s^{(h)} \cap \mathbf{C} = \{(y, f_s^{(h)}(y)) \in \mathbb{R}^{N-1} \times \mathbb{R}, \ |y| \leq h^\beta/2\}$$

for $s = t, t-h$, where $f_s^{(h)} : B^{N-1}_{h^\beta/2} \to \mathbb{R}$ are $C^{1,\frac{1}{2}}$ functions with

$$\|\nabla f_s^{(h)}\|_{L^\infty(B_{h^\beta/2})} \leq \omega(h).$$

We want to prove the following slightly stronger pointwise inequality: namely, that for any point $x = (y, f_t^{(h)}(y)) \in \partial E_t^{(h)} \cap \mathbf{C}$, it holds

$$\left| \mathrm{sd}^{\psi}_{E_{t-h}^{(h)}}(x)\, \psi(x, \nu_{E_t^{(h)}}(x)) \sqrt{1 + |\nabla f_t^{(h)}(y)|} - \Big(f_t^{(h)}(y) - f_{t-h}^{(h)}(y) \Big) \right| \leq \omega(h) |f_t^{(h)}(y) - f_{t-h}^{(h)}(y)|. \tag{2.38}$$

Integrating the previous inequality over $\mathbf{C}$ yields the thesis. Clearly, it is enough to prove (2.38) at each point x such that $|\mathrm{sd}^{\psi}_{E_{t-h}^{(h)}}(x)| > 0$. We thus fix $x = (y, f_t^{(h)}(y)) \in \partial E_t^{(h)} \cap \mathbf{C}$ and denote by $x' := (y, f_{t-h}^{(h)}(y))$. We remark that these points depend on h, but we drop the subscript to ease notation. It can be assumed without loss of generality that $x \notin E_{t-h}^{(h)}$, as the other case is analogous.

Step 1 We now prove that, with the notation previously introduced, it holds

$$|\mathrm{sd}'_{E_{t-h}^{(h)}}(x) - \mathrm{sd}^{\psi}_{E_{t-h}^{(h)}}(x)| \leq \omega(h) |f_t^{(h)}(y) - f_{t-h}^{(h)}(y)|, \tag{2.39}$$

where sd' denotes the signed distance function induced by the anisotropy $\psi(x', \cdot)$. Let γ be a smooth curve, with $\gamma(0) = x, \gamma(1) \in \partial E_{t-h}^{(h)}$ to be used in the definition of the geodesic distance $\mathrm{sd}^{\psi}_{E_{t-h}^{(h)}}$. Firstly, we remark that one could assume

$$\gamma([0,1]) \subseteq B(x, 2c_\psi^2 |f_t^{(h)}(y) - f_{t-h}^{(h)}(y)|) \tag{2.40}$$

Indeed, if it were not the case, the lower bounds contained in (2.9) and (2.36) allow us to estimate

$$\int_0^1 \psi^\circ(\gamma, \dot\gamma)\, \mathrm{d}t \geq \frac{1}{c_\psi} \int_0^1 |\dot\gamma|\, \mathrm{d}t \geq 2c_\psi |f_t^{(h)}(y) - f_{t-h}^{(h)}(y)| \geq 2c_\psi\, \mathrm{sd}_{E_{t-h}^{(h)}}(x) \geq 2\, \mathrm{sd}^{\psi}_{E_{t-h}^{(h)}}(x), \tag{2.41}$$

a contradiction for h small. We can reason analogously for $\mathrm{sd}'_{E_{t-h}^{(h)}}$. In particular, we can consider just curves having length $\int_0^1|\dot\gamma|\le c|f_t^{(h)}(y)-f_{t-h}^{(h)}(y)|$. Therefore, we obtain (by homogeneity)

$$\begin{aligned}\mathrm{sd}^{\psi}_{E^{(h)}_{t-h}}(x) &\le \int_0^1 \psi^\circ(\gamma,\dot\gamma)\,\mathrm{d}t \le \int_0^1 \psi^\circ(x',\dot\gamma)\,\mathrm{d}t + \sup_{\nu\in S^{N-1},\, t\in[0,1]} |\psi(\gamma(t),\nu)-\psi(x',\nu)| \int_0^1 |\dot\gamma| \\ &\le \int_0^1 \psi^\circ(x',\dot\gamma)\,\mathrm{d}t + c\,\omega(h)|f_t^{(h)}(y)-f_{t-h}^{(h)}(h)|,\end{aligned}$$

and, taking the $\inf_\gamma$, we obtain $\mathrm{sd}^{\psi}_{E^{(h)}_{t-h}}(x)\le \mathrm{sd}'_{E^{(h)}_{t-h}}(x)+\omega(h)|f_t^{(h)}(y)-f_{t-h}^{(h)}(y)|$. The converse inequality can be proved analogously, yielding (2.39).

Therefore, in what follows we will consider always the anisotropy frozen in x', and use sd' instead of sd^ψ. Finally, let $p\in\partial E^{(h)}_{t-h}$ a minimizer for the definition of $\mathrm{sd}'_{E^{(h)}_{t-h}}(x)$. In the following, with $\Pi^v_{\mathbb{H}}z$, $\Pi_{\mathbb{H}}z$ we denote respectively the projection on the hyperplane $\mathbb{H}$ of z along the direction v and the orthogonal projection of z on $\mathbb{H}$.

Step 2. In this step we assume that $E^{(h)}_{t-h}\cap\mathbf{C}$ coincides with the halfspace $\mathbb{H}=p+\{z\cdot\nu\le 0\}$ intersected with the same cylinder and prove claim (2.38).
To this aim, we start noticing that by translation we may assume $p=0$ and that $\mathrm{sd}'_{\mathbb{H}}(z+\xi)=\mathrm{sd}'_{\mathbb{H}}(z)$ for all $z\in\mathbb{R}^N$ and for all ξ orthogonal to ν. Hence, in fact,

$$\mathrm{sd}'_{\mathbb{H}}(z)=\mathrm{sd}'_{\mathbb{H}}((z\cdot\nu)\nu)=(z\cdot\nu)\mathrm{sd}'_{\mathbb{H}}(\nu)\,. \tag{2.42}$$

Therefore, $\mathrm{sd}'_{\mathbb{H}}$ is differentiable everywhere, with $\nabla\mathrm{sd}'_{\mathbb{H}}=\mathrm{sd}'_{\mathbb{H}}(\nu)\nu$. Recalling the eikonal equation (2.37), it must hold $\mathrm{sd}'_{\mathbb{H}}(\nu)=1/\psi(x',\nu)$ and in turn, from (2.42), and choosing $z=x$, we have

$$\mathrm{sd}'_{\mathbb{H}}(x)\psi(x',\nu)=x\cdot\nu=\mathrm{sd}_{\mathbb{H}}(x). \tag{2.43}$$

We remark that $\mathrm{sd}'_{\mathbb{H}}(x)=\mathrm{sd}'_{E^{(h)}_{t-h}}(x)$ by (2.32), thus we conclude (2.38) by combining (2.43) with (2.36).

Step 3. We now conclude in the general case. With the notation introduced at the end of Step 1, set $\nu=\nu_{E^{(h)}_{t-h}}(p)$, and consider the half-space $\mathcal{H}^1=p+\{z\cdot\nu\le 0\}$ and $w:=x'-\Pi_{\mathcal{H}^1}(x')$ as depicted in Figure 2.1. We shall prove that

$$|w|\le\omega(h)|f_t^{(h)}(y)-f_{t-h}^{(h)}(y)|.$$

To see this, we start by remarking that (2.33) implies

$$|e_N-e_N(e_N\cdot\nu_{E^{(h)}_{t-h}})|\le\omega(h)\qquad\text{in }\partial E^{(h)}_{t-h}\cap\mathbf{C},$$

implying $e_N\cdot\nu_{E^{(h)}_{t-h}}\ge 1-\omega(h)$, and thus, for any versor v tangent to $\partial E^{(h)}_{t-h}\cap\mathbf{C}$ one has $|v\cdot e_N|\le\omega(h)$. Therefore, we have $(x'-p)\cdot e_N\le\omega(h)|x'-p|$ and also

$$\begin{aligned}\frac{x'-p}{|x'-p|}\cdot\nu &= \frac{x'-p}{|x'-p|}\cdot(e_N(\nu\cdot e_N)+\nu-e_N(\nu\cdot e_N)) \\ &\le\omega(h)+|\nu-e_N(\nu\cdot e_N)|=\omega(h)+\left(1-|\nu\cdot e_N|^2\right)^{1/2} \\ &\le 3\sqrt{\omega(h)},\end{aligned}$$

by choosing h small. Up to defining $\sqrt{\omega}$ as ω, using the previous estimate and the bounds (2.40) we see that

$$|w|=|x'-p|\left(\frac{x'-p}{|x'-p|}\cdot\nu\right)\le\omega(h)|x'-p|\le\omega(h)|f_t^{(h)}(y)-f_{t-h}^{(h)}(y)|. \tag{2.44}$$

We now remark that $\mathrm{sd}'_{E^{(h)}_{t-h}}(x)=\mathrm{sd}'_{\mathcal{H}^1}(x)$ (by convexity of the anisotropy $\psi(x',\cdot)$) and so, applying

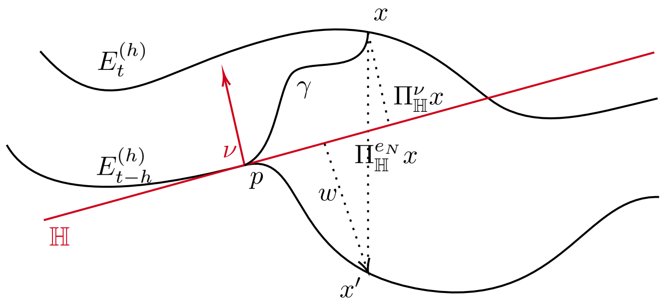

Figure 2.1: The situation in the proof of the lemma.

the previous step to $\mathcal{H}^1$ and using also (2.39), we get

$$\left| \mathrm{sd}^{\psi}_{E^{(h)}_{t-h}}(x)\, \psi(x, \nu_{E^{(h)}_t}(x)) \sqrt{1 + |\nabla f^{(h)}_t(y)|} - |x - \Pi^{e_N}_{\mathbb{H}} x| \right| \le \omega(h) |x - \Pi^{e_N}_{\mathbb{H}} x|.$$

We conclude (2.38) by estimating

$$\big| |x - \Pi^{e_N}_{\mathbb{H}} x| - |x - x'| \big| \le |x' - \Pi^{e_N}_{\mathbb{H}} x| = |w|/|\nu \cdot e_N| \le \frac{\omega(h)}{1 - \omega(h)} |f^{(h)}_t(y) - f^{(h)}_{t-h}(y)|,$$

where we used (2.44). We conclude the proof by a simple change of coordinates and using (2.38) to find

$$\begin{aligned}
&\left| \int_{\partial E^{(h)}_t \cap \mathbf{C}} \psi(x, \nu_{E^{(h)}_t}(x))\, \mathrm{sd}^{\psi}_{E^{(h)}_{t-h}}(x)\, \mathrm{d}\mathcal{H}^{N-1} - \int_{B_{h^\beta/2}} f^{(h)}_t(y) - f^{(h)}_{t-h}(y)\, \mathrm{d}y \right| \\
&= \left| \int_{B_{h^\beta/2}} \psi((y, f^{(h)}_t(y)), \nu_{E^{(h)}_t}(y, f^{(h)}_t(y)))\, \mathrm{sd}^{\psi}_{E^{(h)}_{t-h}}(y, f^{(h)}_t(y)) \sqrt{1 + |\nabla f^{(h)}_t(y)|^2} - (f^{(h)}_t(y) - f^{(h)}_{t-h}(y))\, \mathrm{d}y \right| \\
&\le \omega(h) \int_{B_{h^\beta/2}} |f^{(h)}_t - f^{(h)}_{t-h}|\, \mathrm{d}y.
\end{aligned}$$

□

Finally, we are able to prove that the error generated by approximating the discrete velocity with v_h goes to zero as $h \to 0$. We follow the lines of [144, Proposition 2.2].

Proposition 2.30 (Error estimate)**.** *Under the hypothesis of Lemma 2.27, the error in the discrete curvature equation vanishes in the limit $h \to 0$, namely*

$$\lim_{h \to 0} \left| \frac{1}{h} \int_0^T \left(\int_{E^{(h)}_t} \eta\, \mathrm{d}x - \int_{E^{(h)}_{t-h}} \eta\, \mathrm{d}x \right) \mathrm{d}t - \int_0^T \int_{\partial E^{(h)}_t} \psi(x, \nu_{E^{(h)}_t}) v_h \eta\, \mathrm{d}\mathcal{H}^{N-1}(x)\, \mathrm{d}t \right| = 0 \quad (2.45)$$

for all $\eta \in C^1_c(\mathbb{R}^N \times [0, T))$.

Proof. We fix $t \in [2h, \infty)$ and $\alpha \in (\frac{1}{2}, \frac{N+2}{2N+2})$. For any point $x_h \in \partial E^{(h)}_t$ we define the open set A_{x_h} defined as follows:

(i) if (2.32) holds, we set $A_{x_h} = \mathbf{C}_{h^\beta/2}(x_h, \nu)$, with the notations of Corollary 2.29;

(ii) otherwise we set $A_{x_h} = B(x_h, c_\infty \sqrt{h})$, where c_∞ is the constant of Lemma 2.21.

By Lemma 2.21, the family $\{A_{x_h} : x_h \in \partial E^{(h)}_t\}$ is a covering of $E^{(h)}_t \triangle E^{(h)}_{t-h}$. By a simple application of Besicovitch's theorem (see e.g. [145]), we find a finite collection of points $I \subseteq \partial E^{(h)}_t$ such that $\{A_{x_h}\}_{x_h \in I}$ is a covering of $E^{(h)}_t \triangle E^{(h)}_{t-h}$ with the finite intersection property. We proceed to estimate (2.45) on each A_{x_h} belonging to this family.

Estimate in case (i) We use Proposition 2.29 to deduce

$$\begin{aligned}
&\left|\int_{A_{x_h}} (\chi_{E_t^{(h)}} - \chi_{E_{t-h}^{(h)}})\eta \,\mathrm{d}x - \int_{\partial E_t^{(h)}\cap A_{x_h}} \psi(x,\nu_{E_t^{(h)}})\mathrm{sd}^{\psi}_{E_{t-h}^{(h)}}\,\eta\,\mathrm{d}\mathcal{H}^{N-1}\right| \\
&\le |\eta(x_h,t)|\left|\int_{A_{x_h}} (\chi_{E_t^{(h)}} - \chi_{E_{t-h}^{(h)}}) - \int_{\partial E_t^{(h)}\cap A_{x_h}} \psi(x,\nu_{E_t^{(h)}})\mathrm{sd}^{\psi}_{E_{t-h}^{(h)}}\,\mathrm{d}\mathcal{H}^{N-1}\right| \\
&+ \left|\int_{A_{x_h}} (\chi_{E_t^{(h)}} - \chi_{E_{t-h}^{(h)}})(\eta-\eta(x_h,t)) - \int_{\partial E_t^{(h)}\cap A_{x_h}} (\eta-\eta(x_h,t))\,\psi(x,\nu_{E_t^{(h)}})\,\mathrm{sd}^{\psi}_{E_{t-h}^{(h)}}\,\mathrm{d}\mathcal{H}^{N-1}\right| \\
&\le C(\omega(h)\|\eta\|_\infty + h^\beta\|\nabla\eta\|_\infty)\int_{A_{x_h}} |\chi_{E_t^{(h)}} - \chi_{E_{t-h}^{(h)}}|\,\mathrm{d}\mathcal{H}^{N-1} + ch^\beta\|\nabla\eta\|_\infty P(E_t^{(h)},A_{x_h}). \qquad (2.46)
\end{aligned}$$

Estimate in case (ii) By assumption $\exists y\in B_{c_\infty\sqrt{h}}(x_h)\cap(E_t^{(h)}\triangle E_{t-h}^{(h)})$ such that $|v_h(t,y)|>h^{\alpha-1}$. We can assume *wlog* $y\in E_t^{(h)}$. We then have $B(y,h^\alpha/(2c_\psi))\subseteq\mathbb{R}^N\setminus E_{t-h}^{(h)}$ and $\mathrm{sd}^{\psi}_{E_{t-h}^{(h)}}>h^\alpha/(2c_\psi^2)$ on $B(y,h^\alpha/(2c_\psi))$. Since $h^\alpha<<h^{1/2}$, we can use the density estimates of Lemma 2.14 to deduce

$$ch^{(N+1)\alpha-1}\le\int_{B(y,h^\alpha/(2c_\psi))\cap(E_t^{(h)}\triangle E_{t-h}^{(h)})}|v_h|\,\mathrm{d}x.$$

Analogously, recalling also Lemma 2.21, we deduce

$$\int_{B(x_h,c_\infty\sqrt{h})\cap\partial E_t^{(h)}}|\psi(x,\nu_{E_{t-h}^{(h)}})\,\mathrm{sd}^{\psi}_{E_{t-h}^{(h)}}|\,\mathrm{d}\mathcal{H}^{N-1}(x)\le ch^{\frac{N}{2}}.$$

Combining the two previous equations and $B(y,h^\alpha/(2c_\psi))\subseteq B(y,c\sqrt{h})$, we infer

$$\begin{aligned}
&\int_{A_{x_h}}|\chi_{E_t^{(h)}}-\chi_{E_{t-h}^{(h)}}|+\int_{A_{x_h}\cap\partial E_t^{(h)}}|\psi(x,\nu_{E_{t-h}^{(h)}})\,\mathrm{sd}^{\psi}_{E_{t-h}^{(h)}}|\,\mathrm{d}\mathcal{H}^{N-1}\\
&\qquad\le ch^{\frac{N}{2}-(N+1)\alpha+1}\int_{A_{x_h}\cap(E_t^{(h)}\triangle E_{t-h}^{(h)})}|\psi(x,\nu_{E_{t-h}^{(h)}})v_h|. \qquad (2.47)
\end{aligned}$$

Summing over $x_h\in I$ both (2.46) and (2.47), and using the local finiteness of the covering, we get

$$\begin{aligned}
&\left|\int(\chi_{E_t^{(h)}}-\chi_{E_{t-h}^{(h)}})\eta\,\mathrm{d}x-\int_{\partial E_t^{(h)}}\psi(x,\nu_{E_t^{(h)}})\mathrm{sd}^{\psi}_{E_{t-h}^{(h)}}\,\eta\,\mathrm{d}\mathcal{H}^{N-1}\right|\\
&\le\sum_{x_h\in I}\left|\int_{A_{x_h}}(\chi_{E_t^{(h)}}-\chi_{E_{t-h}^{(h)}})\eta\,\mathrm{d}x-\int_{\partial E_t^{(h)}\cap A_{x_h}}\psi(x,\nu_{E_t^{(h)}})\mathrm{sd}^{\psi}_{E_{t-h}^{(h)}}\,\eta\,\mathrm{d}\mathcal{H}^{N-1}\right|\\
&\le c\left(\omega(h)\|\eta\|_\infty+h^\beta\|\nabla\eta\|_\infty+h^{\frac{N}{2}-(n+1)\alpha+1}\|\eta\|_\infty\right)\cdot\\
&\quad\cdot\left(P(E_t^{(h)})+|E_t^{(h)}\triangle E_{t-h}^{(h)}|+\int_{E_t^{(h)}\triangle E_{t-h}^{(h)}}|v_h|\right)
\end{aligned}$$

where the last constant c depends on N,ψ. We then use Lemma 2.23, (2.27) and (2.28) to conclude

$$\begin{aligned}
&\left|\int_{2h}^T\frac{1}{h}\left(\int_{E_t^{(h)}}\eta\,\mathrm{d}x-\int_{E_{t-h}^{(h)}}\eta\,\mathrm{d}x\right)-\int_h^T\int_{\partial E_t^{(h)}}\psi(x,\nu_{E_t^{(h)}})v_h\eta\,\mathrm{d}\mathcal{H}^{N-1}\right|\\
&\qquad\le c\left(\omega(h)\|\eta\|_\infty+h^\beta\|\nabla\eta\|_\infty+h^{\frac{N}{2}-(n+1)\alpha+1}\|\eta\|_\infty\right),
\end{aligned}$$

where $c=c(E_0,f,T,\psi)$ and T is chosen such that $\operatorname{spt}\eta\subset\subset\mathbb{R}^N\times[0,T]$. The conclusion follows using the definition of α and taking the limit $h\to0$.

□

The proof of Theorem 2.2 is now a consequence of the previous results. In particular, hypothesis

(2.29) and (2.30) imply that the discrete flow converges to the flat flow in the sense of varifolds and this allows to prove (2.3), while (2.4) is a consequence of Proposition 2.30. In order to prove the convergence of the approximations in time of the forcing term, we need to require additionally that (H2) holds.

Proof of Theorem 2.2. Firstly, combining [124, Theorem 4.4.2] with the bounds contained in (2.31) and in Proposition 2.25, we conclude the existence of functions $v, \mathrm{H}^\phi, \tilde{f} : \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$ satisfying

$$\int_0^T \int_{\partial E_t} |v|^2 + |\mathrm{H}^\phi|^2 + |\tilde{f}|^2 \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}t \leq C_T$$

and the following properties

$$\begin{aligned} \lim_k \int_0^T \int_{\partial E_t^{(h_k)}} v_{h_k} \eta \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}t &= \int_0^T \int_{\partial E_t} \eta v \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}t \\ \lim_k \int_0^T \int_{\partial E_t^{(h_k)}} F_{h_k}(x,t) \eta \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}t &= \int_0^T \int_{\partial E_t} \eta \tilde{f} \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}t \\ \lim_k \int_0^T \int_{\partial E_t^{(h_k)}} \mathrm{H}^\phi_{E_t^{(h_k)}} \eta \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}t &= \int_0^T \int_{\partial E_t} \eta \mathrm{H}^\phi \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}t, \end{aligned} \tag{2.48}$$

for any $\eta \in C_c^0(\mathbb{R}^N \times [0,T))$. We now employ an approximation procedure to prove that $\mathrm{H}^\phi(\cdot,t)$ is the $\phi-$mean curvature of E_t for a.e. $t \in [0,\infty)$, following the lines of [144, 155]. Fixed $t \in [0,+\infty)$ and $\varepsilon > 0$, set ν_ε a continuous function such that $\int_{\partial E_t} (\nu_{E_t} - \nu_\varepsilon)^2 \, \mathrm{d}\mathcal{H}^{N-1} < \varepsilon$. Then, by (2.29) one could prove that $\lim_{k\to\infty} \int_{\partial E_t^{(h_k)}} (\nu_{E_t^{(h_k)}} - \nu_\varepsilon)^2 \, \mathrm{d}\mathcal{H}^{N-1} < \varepsilon$. Considering test functions in (2.48) of the form $\eta(x,t) = a(t)g(x)$, one has for a.e. $t \in [0,+\infty)$

$$\lim_k \int_{\partial E_t^{(h_k)}} \mathrm{H}^\phi_{E_t^{(h_k)}} g \, \mathrm{d}\mathcal{H}^{N-1} = \int_{\partial E_t} \mathrm{H}^\phi g \, \mathrm{d}\mathcal{H}^{N-1}.$$

Thus, for a.e. $t \in [0,+\infty)$ and for any $X \in C_c^0(\mathbb{R}^N;\mathbb{R}^N)$ it holds

$$\lim_k \int_{\partial E_t^{(h_k)}} \mathrm{H}^\phi_{E_t^{(h_k)}} \nu_{E_t^{(h_k)}} \cdot X \, \mathrm{d}\mathcal{H}^{N-1} = \int_{\partial E_t} \mathrm{H}^\phi \nu_{E_t} \cdot X \, \mathrm{d}\mathcal{H}^{N-1}$$

by approximating the normal vectors of $E_t^{(h_k)}$ with ν_ε. Furthermore, by the convergence (2.29) and the hypothesis (2.30) we can use the Reshetnyak's continuity theorem (see e.g. [13, Theorem 2.39]), ensuring

$$\int_{\partial E_t^{(h_k)}} L(x, \nu_{E_t^{(h_k)}}) \, \mathrm{d}\mathcal{H}^{N-1} \to \int_{E_t} L(x, \nu_{E_t}) \, \mathrm{d}\mathcal{H}^{N-1}$$

as $k \to \infty$, for any $L \in C_c^0(\mathbb{R}^N \times \mathbb{R}^N)$. We choose $L(x,\nu) = \mathrm{div}_\phi X$ for some $X \in C_c^1(\mathbb{R}^N;\mathbb{R}^N)$ to obtain

$$\begin{aligned} \int_{\partial E_t} \mathrm{div}_\phi X \, \mathrm{d}\mathcal{H}^{N-1} &= \lim_k \int_{\partial E_t^{(h_k)}} \mathrm{div}_\phi X \, \mathrm{d}\mathcal{H}^{N-1} \\ &= \lim_k \int_{\partial E_t^{(h_k)}} X \cdot \nu_{E_t^{(h_k)}} \mathrm{H}^\phi_{E_t^{(h_k)}} \, \mathrm{d}\mathcal{H}^{N-1} \\ &= \int_{\partial E_t} X \cdot \nu_{E_t} \mathrm{H}^\phi \, \mathrm{d}\mathcal{H}^{N-1}, \end{aligned}$$

which shows that $\mathrm{H}^\phi(\cdot,t)$ is the $\phi-$mean curvature of the set E_t for a.e. $t \in [0,+\infty)$. Moreover, we remark that $F_{h_k}(x,t) \to f(x,t)$ for every (x,t), thus for any test function $\eta \in C_c^0(\mathbb{R}^N \times [0,+\infty))$ and $t \in [0,+\infty)$ we have

$$\left|\int_{\partial E_t^{(h)}} F_{h_k}(x,t)\eta(x,t)\,\mathrm{d}\mathcal{H}^{N-1}_x - \int_{\partial E} f\eta\,\mathrm{d}\mathcal{H}^{N-1}_x\right| \le \left|\int_{\partial E_t^{(h)}} F_{h_k}\eta - \int_{\partial E_t} F_{h_k}\eta\right| + \int_{\partial E_t} |F_{h_k} - f|\eta$$
$$\le \|f\|_\infty \|\eta\|_\infty \left(P(E_t^{(h)}) - P(E_t)\right) + \int_{\partial E_t} |F_{h_k} - f|\eta \to 0$$

applying the dominated convergence theorem and recalling Lemma 2.20. Thus, $\tilde{f} = f$. We then prove (2.3) by passing to the limit in the Euler-Lagrange equation (2.18).

To prove (2.4) we employ Proposition 2.30: for every $\eta \in C^0_c(\mathbb{R}^N \times [0,T))$, by a change of variables we have that

$$\int_h^T \left[\int_{E_t^{(h)}} \eta\,\mathrm{d}x - \int_{E_{t-h}^{(h)}} \eta\,\mathrm{d}x\right] \mathrm{d}t = \int_h^T \int_{E_t^{(h)}} (\eta(x,t) - \eta(x,t-h))\,\mathrm{d}x\,\mathrm{d}t - h\int_{E_0} \eta\,\mathrm{d}x$$

where we have used that $E_t^{(h)} = E_0$ for $t \in [0,h)$. Therefore, a simple convergence argument yields

$$\lim_{h\to 0} \frac{1}{h}\int_h^T \left[\int_{E_t^{(h)}} \eta\,\mathrm{d}x - \int_{E_{t-h}^{(h)}} \eta\,\mathrm{d}x\right] \mathrm{d}t = -\int_h^T \partial_t \eta(x,t)\,\mathrm{d}x\,\mathrm{d}t - \int_{E_0} \eta.$$

Combining the previous estimate with Proposition 2.30 and passing to the limit, we obtain (2.4). □

4 Viscosity solutions

In this section we will prove the existence of another weak notion of solution for the mean curvature flow starting from a compact set. We will follow the so-called level set approach based on the theory of viscosity solution. We recall that in the first part we work with the standing assumptions of the chapter (H0). Additionally, we require (H1).

4.1 The discrete scheme for unbounded sets

In this short subsection we will define the discrete evolution scheme for unbounded sets having compact boundary. The idea would be to define this evolution simply as the complement of the evolution of the complementary set, but since the anisotropies we are considering are not symmetric, we need additional care.

We recall that, given an anisotropy ϕ, we define $\tilde{\phi}(x,\nu) := \phi(x,-\nu)$. This anisotropy has all the properties of the original one, concerning regularity and bounds. We start remarking the following simple fact. One can see that $\mathrm{dist}^\psi(x,y) = \mathrm{dist}^{\tilde{\psi}}(y,x)$, since for any curve $\gamma \in W^{1,1}([0,1];\mathbb{R}^N), \gamma(0) = x, \gamma(1) = y$, a simple change of variable yields

$$\int_0^1 \psi^\circ(\gamma(t),\dot{\gamma}(t))\,\mathrm{d}t = \int_0^1 \psi^\circ\left(\gamma(1-t), -\frac{\mathrm{d}}{\mathrm{d}t}(\gamma(1-t))\right)\mathrm{d}t = \int_0^1 \widetilde{(\psi^\circ)}(\eta(t),\dot{\eta}(t))\,\mathrm{d}t,$$

for $\eta(t) = \gamma(1-t)$, once one sees that

$$\widetilde{(\psi^\circ)}(\cdot,\nu) = \sup_{\psi(\cdot,\xi)\le 1} \xi\cdot(-\nu) = \sup_{\tilde{\psi}(\cdot,-\xi)\le 1} (-\xi)\cdot\nu = (\tilde{\psi})^\circ(\cdot,\nu).$$

Therefore, by definition of signed distance we have

$$\mathrm{sd}^\psi_E(x) = -\mathrm{sd}^{\tilde{\psi}}_{E^c}(x). \tag{2.49}$$

For every compact set F and $h > 0, t \ge 0$, we will denote by $\tilde{T}^{\pm}_{h,t}F$ the maximal and the minimal solution to problem (2.17), according to Lemma 2.12 with P_ϕ and sd^ψ, respectively, replaced by

$P_{\tilde{\phi}}$ and $\mathrm{sd}^{\tilde{\psi}}$. Finally, for every set E with compact boundary we define

$$T^{\pm}_{h,t}E := \left(\tilde{T}^{\mp}_{h,t}E^c\right)^c. \tag{2.50}$$

As in the case for compact sets, we set $T_{h,t}E := T^{-}_{h,t}E$. Given an open, unbounded set E_0 having compact boundary, we can then define the discrete flow $\{E^{(h)}_t\}_{t\geq 0}$ as follows: $E^{(h)}_t := E_0$ for $t \in [0, h)$ and

$$E^{(h)}_t = T_{h,t}E^{(h)}_{t-h}, \quad \forall t \in [h, +\infty).$$

One easily checks that analogous results to Lemmas 2.13, 2.20 and 2.19 hold also for this problem. We state the corresponding results.

Lemma 2.31. *Let $F_1 \subseteq F_2$ be open, unbounded sets with compact boundary and fix $h > 0, t \geq 0$. Then, $T_{h,t}F_1 \subseteq T_{h,t}F_2$.*

Lemma 2.32. *For any $T > 0$ there exists a constant $C_T(\phi, \psi, f, T)$ such that for every $R > 0$ the following holds. If the initial open set $E \supset B^c_R$, then $E^{(h)}_t \supset B^c_{C_T R}$ for all $t \in [0, T]$.*

Lemma 2.33. *For every $R_0 > 0$ there exist $h_0(R_0) > 0$ and $C(R_0, \phi, \psi, f) > 0$ with the following property: For all $R \geq R_0$, $h \in (0, h_0)$, $t > 0$ and $x \in \mathbb{R}^N$ one has*

$$T_{h,t}((B_R(x))^c) \subseteq (B_{R-Ch}(x))^c.$$

We now state a comparison principle between bounded and unbounded sets, following the line of [56, Lemma 6.10].

Lemma 2.34. *Let E_1 be a compact set and let E_2 be an open, unbounded set, with compact boundary, and such that $E_1 \subseteq E_2$. Then, for every $h \in (0, 1), t \geq 0$ it holds $T^{\pm}_{h,t}E_1 \subseteq T^{\pm}_{h,t}E_2$.*

Proof. We fix $h \in (0, 1), t \in [0, T]$ for $T > 0$. Set $R > 0$ such that $E_1, E^c_2 \subseteq B_R$ and note that by Lemmas 2.13 and 2.20 (applied to $P_{\tilde{\phi}}$ instead of P_ϕ) we get

$$\left(T^{+}_{h,t}E_2\right)^c \subseteq \tilde{T}^{-}_{h,t}E^c_2 \subseteq T^{-}_{h,t}B_R \subseteq B_{C_T R}, \tag{2.51}$$

for some $C_T(\phi, \psi, f, T)$. Since $\tilde{T}^{-}_{h,t}E^c_2$ is the minimal solution of

$$\min\left\{P_{\tilde{\phi}}(E) + \frac{1}{h}\int_E \mathrm{sd}^{\tilde{\psi}}_{E^c_2}(x)\,\mathrm{d}x - \int_E F_h(x,t)\,\mathrm{d}x\right\},$$

considering the change of variables $\tilde{E} = E^c$ and using (2.49), we easily conclude that $T^{+}_{h,t}E_2 = \left(\tilde{T}^{-}_{h,t}E^c_2\right)^c$ is the maximal solution of

$$\min\left\{P_\phi(\tilde{E}) + \frac{1}{h}\int_{B_{C_T R}} \mathrm{sd}^{\psi}_{E_2} - \frac{1}{h}\int_{\tilde{E}^c} \mathrm{sd}^{\psi}_{E_2} - \int_{\tilde{E}^c} F_h(x,t)\,\mathrm{d}x\right\} - \frac{1}{h}\int_{B_{C_T R}} \mathrm{sd}^{\psi}_{E_2}.$$

we then note that

$$\int_{B_{C_T R}} \mathrm{sd}^{\psi}_{E_2} = \int_{\tilde{E}} \mathrm{sd}^{\psi}_{E_2}\chi_{B_{C_T R}} + \int_{\tilde{E}^c} \mathrm{sd}^{\psi}_{E_2},$$

for every $\tilde{E}$ such that $\tilde{E}^c \subseteq B_{C_T R}$. By (2.51), we conclude that $T^{+}_{h,t}E_2$ is the maximal solution of

$$\min\left\{P_\phi(\tilde{E}) + \frac{1}{h}\int_{\tilde{E}} \mathrm{sd}^{\psi}_{E_2}\chi_{B_{C_T R}} - \int_{\tilde{E}^c} F_h(x,t)\,\mathrm{d}x \ : \ \tilde{E}^c \subseteq B_{C_T R}\right\}. \tag{2.52}$$

Analogously, one proves that $T^{-}_{h,t}E_2$ is the minimal solution of (2.52). Finally, we remark that $\mathrm{sd}^{\psi}_{E_s}\chi_{B_{C_T R}} \leq \mathrm{sd}^{\psi}_{E_1}$ and that $T^{\pm}_{h,t}E_1 \cup T^{\pm}_{h,t}E_2, T^{\pm}_{h,t}E_1 \cap T^{\pm}_{h,t}E_2$ are both admissible competitors for (2.52), one argues exactly as in the proof of Lemma 2.13 to conclude $T^{\pm}_{h,t}E_1 \subseteq T^{\pm}_{h,t}E_2$. ∎

4.2 The level set approach

We recall that in this section we assume (H0), (H1). Consider a function $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ whose spatial superlevel sets $\{u(\cdot, t) \geq s\}$ evolve according to the mean curvature equation

$$V(x,t) = -\psi(x, \nu_{\{u(\cdot,t)\geq s\}}) \left(\mathrm{H}^{\phi}_{\{u(\cdot,t)\geq s\}}(x) - f(x,t) \right) \quad \text{for } x \in \partial\{u(\cdot,t) \geq s\}.$$

The function u then satisfies (recalling that $-\nabla u/|\nabla u|$ is the *outer normal vector* to the superlevel set $\{u(\cdot, t) \geq u(x, t)\}$) the equation

$$\begin{aligned}
\partial_t u = |\nabla u| V(x) &= -\psi(x, -\nabla u) \left(\mathrm{H}^{\phi}_{\{u(\cdot,t)\geq u(x,t)\}}(x) - f(x,t) \right) \\
&= -\psi(x, -\nabla u) \left(\mathrm{div} \nabla_p \phi(x, -\nabla u) - f(x,t) \right) \\
&= -\psi(x, -\nabla u) \left(\sum_i \partial_{x_i} \partial_p \phi(x, -\nabla u) - \nabla_p^2 \phi(x, -\nabla u) \,:\, \nabla^2 u - f(x,t) \right) \\
:&= -\psi(x, -\nabla u) \left(H(x, \nabla u, \nabla^2 u) - f(x,t) \right),
\end{aligned}$$

where we defined the Hamiltonian $H \,:\, \mathbb{R}^N \times \mathbb{R}^N \setminus \{0\} \times Sym_N \to \mathbb{R}$ as

$$H(x, p, X) := \sum_i \partial_{x_i} \partial_{p_i} \phi(x, -p) - \nabla_p^2 \phi(x, -p) \,:\, X. \tag{2.53}$$

We therefore focus on solving the parabolic Cauchy problem

$$\begin{cases} \partial_t u + \psi(x, -\nabla u) \left(H(x, \nabla u, \nabla^2 u) - f(x,t) \right) = 0 \\ u(\cdot, t) = u_0. \end{cases} \tag{2.54}$$

The appropriate setting for this type of geometric evolution equations is the one of viscosity solutions, in the framework of [102, 128] (see also [56]). We will focus on the evolution of sets with compact boundary on compact time intervals of the form $[0, T]$. We now define the notion of admissible test function. In the following, with a small abuse of language, we will say that a function $u : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ is constant outside a compact set if there exists a compact set $K \subset \mathbb{R}^N$ such that $u(\cdot, t)$ is constant in $\mathbb{R}^N \setminus K$ for every $t \in [0, T]$ (with the constant possibly depending on t).

Definition 2.35. Let $\hat{z} = (\hat{x}, \hat{t}) \in \mathbb{R}^N \times (0, T)$ and let $A \subseteq (0, T)$ be any open interval containing $\hat{t}$. We will say that $\eta \in C^0(\mathbb{R}^N \times \bar{A})$ is admissible at the point $\hat{z}$ if it is of class C^2 in a neighborhood of $\hat{z}$, if it is constant out of a compact set, and, in case $\nabla \eta(\hat{z}) = 0$, the following holds: for all $(x, t) \in \mathbb{R}^N \times A$, and there exist $b \in [0, +\infty)$ and $\Psi : [0, +\infty) \to [0, +\infty)$ such that

$$|\eta(x,t) - \eta(\hat{z}) - \eta_t(\hat{z})(t - \hat{t})| \leq \Psi(|x - \hat{x}|) + b|t - \hat{t}|^2,$$

where the function Ψ satisfies $\Psi(r) = o(r^2)$ as $r \to 0$.

We then recall one of the equivalent definitions of viscosity solutions.

Definition 2.36. An upper semicontinuous function $u : \mathbb{R}^N \times [0, T] \to \mathbb{R}$ (in short, $u \in usc(\mathbb{R}^N \times [0, T])$), constant outside a compact set, is a viscosity subsolution of the Cauchy problem (2.54) if $u(\cdot, 0) \leq u_0$ and for all $z := (x, t) \in \mathbb{R}^N \times (0, T)$ and all $C^\infty-$test functions η such that η is admissible at z and $u - \eta$ has a maximum at z (in the domain of definition of η) the following holds:

i) If $\nabla \eta(z) = 0$, then it holds
$$\eta_t(z) \leq 0 \tag{2.55}$$

ii) If $\nabla \eta(z) \neq 0$, then
$$\partial_t \eta(z) + \psi(z, -\nabla \eta(z)) \left(H(z, \nabla \eta(z), \nabla^2 \eta(z)) - f(z,t) \right) \leq 0. \tag{2.56}$$

A lower semicontinuous function $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ (in short, $u \in lsc(\mathbb{R}^N \times [0,T])$), constant outside a compact set, is a viscosity supersolution of the Cauchy problem (2.54) if $u(\cdot,0) \geq u_0$ and for all $z := (x,t) \in \mathbb{R}^N \times [0,T]$ and all $C^\infty-$test functions η such that η is admissible at z and $u - \eta$ has a minimum at z (in the domain of definition of η) the following holds:

i) If $\nabla\eta(z) = 0$, then $\eta_t(z) \geq 0$;

ii) If $\nabla\eta \neq 0$ then

$$\partial_t \eta(z) + \psi(z, -\nabla\eta(z)) \big(H(z, \nabla\eta(z), \nabla^2\eta(z)) - f(z,t) \big) \leq 0.$$

Finally, a function u is a viscosity solution for the Cauchy problem (2.54) if it is both a subsolution and a supersolution of (2.54).

Remark 2.37. By classical arguments, one could assume that the maximum of $u - \eta$ is strict in the definition of subsolution above (an analogous remark holds for supersolutions).

Remark 2.38. We remark that, if $-u$ is a subsolution to (2.54) with initial datum $-u_0$, then u is a supersolution for (2.54) for the initial datum u_0 and where ϕ, ψ are replaced by $\tilde{\phi}, \tilde{\psi}$ respectively, as defined in Section 4.1.

We will first prove existence for viscosity solutions of (2.54) via an approximation-in-time technique, and then prove uniqueness of solutions to (2.54) to link the approximate solution to the mean curvature flow equation. We would like to proceed with the classical construction of e.g. [45, 56, 85], but in our case the lack of continuity of the evolving functions forces us to be particularly careful with the procedure.

We use the shorthand notation of *lsc* for lower semicontinuous and *usc* for upper semicontinuous. Given a bounded, *usc* function v which is constant outside a compact set, we define the transformation

$$T^+_{h,t} v(x) = \sup \Big\{ s \;:\; x \in T^+_{h,t}\{v \geq s\} \Big\}. \tag{2.57}$$

Firstly, we see that $T^+_{h,t} v(x) \in \mathbb{R}$, as v is bounded. Moreover, it turns out that the function $T^+_{h,t} v$ is *usc*, bounded and constant outside a compact set. Indeed, definition (2.57) is equivalent to

$$T^+_{h,t} v(x) = \inf \Big\{ s \;:\; x \notin T^+_{h,t}\{v \geq s\} \Big\} = \inf_{s\in\mathbb{R}} \Big(s + \mathbb{1}_{(T^+_{h,t}\{v\geq s\})^c}(x) \Big),$$

where $\mathbb{1}_A(x)$ is the indicatrix function of a set A, being 0 on the set and $+\infty$ outside. By definition, $\mathbb{1}_A$ is an *usc* function for any open set A. Thus, recalling Remark 2.15, in the equation above we are taking the infimum of a family of *usc* functions, which is then a *usc* function. The other two properties follows from the previous study of the discrete evolution. Analogously, given a bounded *lsc* function g, we define

$$T^-_{h,t} g(x) = \sup \Big\{ s \;:\; x \in T^-_{h,t}\{g > s\} \Big\} = \sup_{s\in\mathbb{R}} \Big(s - \mathbb{1}_{T^-_{h,t}\{g>s\}} \Big),$$

which is now a bounded *lsc* function (as sup of *lsc* functions), constant outside a compact set.

We are now ready to give the definition of the discrete-in-time approximations of sub - and super solution to (2.54). Given an initial compact set E_0, set u_0 as a (uniformly) continuous function, spatially constant outside a compact set, such that $\{u_0 \geq 0\} = E_0$. We remark that for every $s \in \mathbb{R}$, the superlevel set $\{u_0 \geq s\}$ is either compact or it is unbounded with compact boundary. Then, for $h > 0$ we introduce the following family of maps as $u^\pm_h(\cdot,t) = u_0$ for $t \in [0,h)$ and

$$u^\pm_h(\cdot,t) := T^\pm_{h,t-h} u^\pm_h(\cdot, t-h) \quad \text{for } t \geq h. \tag{2.58}$$

We easily see that the maps above are functions (as implied by the comparison principle contained in Lemmas 2.13, 2.31 and 2.34) piecewise constant in time (as $T^\pm_{h,t} = T^\pm_{h,[t/h]h}$). Moreover, by the previous remarks, we have that $u^+_h(\cdot,t)$ is an *usc* function, while $u^-_h(\cdot,t)$ is a *lsc* function, for every $t \in [0,+\infty)$. Some further properties of the approximating scheme are listed below.

Lemma 2.39. *For any $h > 0$, $t \geq 0$ we have the following. It holds*

$$u_h^-(\cdot, t) \leq u_h^+(\cdot, t). \tag{2.59}$$

Furthermore, given any $\lambda \in \mathbb{R}$ and $t \geq h$ it holds

$$\begin{aligned} \{u_h^+(\cdot, t) > \lambda\} &\subseteq T^+_{h,t-h}\{u_h^+(\cdot, t-h) \geq \lambda\} \subseteq \{u_h^+(\cdot, t) \geq \lambda\} \\ \{u_h^-(\cdot, t) > \lambda\} &\subseteq T^-_{h,t-h}\{u_h^-(\cdot, t-h) > \lambda\} \subseteq \{u_h^-(\cdot, t) \geq \lambda\}. \end{aligned} \tag{2.60}$$

Proof. Fix $x \in \mathbb{R}^N$, $t \in [0, h)$. For any given $\sigma < u_h^-(x, h)$ we have that there exists a sequence $(s_n) \nearrow \sigma$ so that $x \in T^-_{h,t-h}\{u_0 > s_n\} \subseteq T^+_{h,t-h}\{u_0 \geq s_n\}$. Thus, $u_h^+(x, t) \geq \sigma$. We then conclude by induction. Then, (2.60) follows easily by the definition (2.58). □

We then prove that the half-relaxed limits (in the spirit of [19], see also the references therein) of the families of functions $u_h^\pm$

$$\begin{aligned} u^+(x, t) &:= \sup_{(x_h, t_h) \to (x, t)} \limsup_{h \to 0} u_h^+(x_h, t_h) \\ u^-(x, t) &:= \inf_{(x_h, t_h) \to (x, t)} \liminf_{h \to 0} u_h^-(x_h, t_h), \end{aligned} \tag{2.61}$$

are (respectively) sub - and supersolutions in the viscosity sense of (2.54), see Theorem 2.3 (note that, by definition, u^+ is *usc*, while u^- is *lsc*). The proof of this result is the subject of the following section and we recall that the hypothesis required are (H0), (H1) and $f \in C^0(\mathbb{R}^N \times [0, \infty))$ only. Once the existence of sub - and super-solutions to the equation is settled, we need to properly define the notion of level-set solution to the mean curvature flow. To do so, we first prove uniqueness for (2.54) via a comparison principle and under additional hypothesis. Then, we show that the evolution of the zero superlevel set of the solution does not depend on the choice of the initial function u_0.

We start with a comparison result between u^+, u^- and u_0 at the initial time: it will ensure that the classical hypothesis for the comparison principle are satisfied. We first prove an estimate for the speed of decay of the level sets of the evolving functions. While it will only be needed in the following section, in the proof of the forthcoming Lemma 2.41 we will use similar techniques, so we preferred to state it here.

Lemma 2.40. *Let $u^+(x, t)$ be the function defined in (2.61), let $\sigma \in \mathbb{R}$. Assume that, for a suitable x_0 and $R > 0$, it holds $B(x_0, R) \subseteq \{u^+(\cdot, t_0) \geq \sigma\}$. Then, there exists $C = C(R, \phi, \psi, f)$ such that $B(x_0, R - C(t - t_0)) \subseteq \{u^+(\cdot, t) \geq \sigma\}$ for every $t \leq t_0 + R/(2C)$. An analogous statement holds for u^- by considering its open sublevel sets.*

Proof. We focus on the case $\{u^+(\cdot, t_0) \geq \sigma\}$ bounded, the other case being analogous. By assumption, for any $R_0 < R$, if h is small enough, we have $B(x_0, R_0) \subseteq \{u_h^+(\cdot, t_0) \geq \sigma\}$. Set $C = C(R_0/2, \phi, \psi, f)$ as the constant of Lemma 2.19. Let R_n be defined recursively following law (2.23), that is $R_{n+1} = R_n - Ch$, as long as $R_n \geq R_0/2$. By simple iteration we find that $R_n = R_0 - nCh$, as long as $R_n \geq R_0/2$, which can be ensured enforcing $h\, n \leq R_0/(2C)$. Therefore, for any $t \geq t_0$ such that $t - t_0 \leq R_0/(2C)$, we set $n = [(t - t_0)/h]$ and send $h \to 0$ to deduce (recalling also Lemma 2.13)

$$\{u^+(\cdot, t) \geq \sigma\} \supset B(x_0, R_0 - C(t - t_0)).$$

Since the choice of R_0 is arbitrary, we conclude. □

We are now ready to prove a comparison result for the functions $u^\pm$ and a continuity estimate at the initial time $t = 0$.

Lemma 2.41. *For any $(x, t) \in \mathbb{R}^N \times [0, +\infty)$ it holds*

$$u^-(x, t) \leq u^+(x, t).$$

Moreover $u^-(\cdot,0) = u^+(\cdot,0) = u_0$, *so that there exists a modulus of continuity* ω *such that* $\forall x, y \in \mathbb{R}^N$

$$u^+(x,0) - u^-(y,0) \leq \omega(|x-y|).$$

Proof. The proof of the first inequality essentially follows from (2.59) and the definition of $u^\pm$. To prove the equality at the initial time $t = 0$, we start by remarking that $u^+(\cdot,0) \geq u_0$ as can be seen taking sequences of the form $(x_h, 0)$ in (2.61). Then, consider ω as a continuous, strictly increasing modulus of continuity for u_0. We can also see that $\forall \varepsilon > 0$ $\{u_0 \leq u_0(x) + \varepsilon\} \supseteq B(x, \omega^{-1}(\varepsilon))$ by uniform continuity. Thus, reasoning iteratively as in Lemma 2.40 and using (2.60), we obtain that there exists $h_0(\varepsilon)$ such that $\forall h \leq h_0$ it holds

$$\{u_h^+(\cdot,t) \leq u_h^+(x,0)+\varepsilon\} \supseteq \left(T^+_{h,t-h}\{u_0 > u_0(x)+\varepsilon\}\right)^c = T^-_{h,t-h}\{u_0 \leq u_0(x)+\varepsilon\} \supseteq B(x, \omega^{-1}(\varepsilon/2)),$$

as long as $t \leq (\omega^{-1}(\varepsilon) - \omega^{-1}(\varepsilon/2))/(2C) =: t_\varepsilon$, and where we recalled that $u_h^\pm(\cdot,0) = u_0$. Now, fix $\sigma > 0, x \in \mathbb{R}^N$ such that $u(x,0) > \sigma$ and a sequence $(x_{h_k}, t_{h_k}) \to (x,0)$ such that $\lim_k u^+_{h_k}(x_{h_k}, t_{h_k}) > \sigma$. Then, for k large enough $(x_{h_k}, t_{h_k}) \in B(x, \omega^{-1}(\varepsilon/2)) \times [0, t_\varepsilon)$ and so we conclude

$$\sigma < \lim_k u_h^+(x_{h_k}, t_{h_k}) \leq u_0(x,0) + \varepsilon.$$

Letting $\varepsilon \to 0$ we conclude $u(\cdot,0)^+ \leq u_0$. The proof for u^- is essentially the same. The last claim follows from the previous one, recalling that ω is a modulus of uniform continuity for u_0. □

In order to prove a comparison principle for (2.54), we will need to assume (H3). Under these additional hypotheses, we are able to prove uniqueness for the parabolic Cauchy problem (2.54). The proof of this result follows from [102, Theorem 4.2]: we will just show in detail that the assumption of the aforementioned theorem hold in our case, following [23, Proposition 6.1] and [102, pag. 463].

Proof of Theorem 2.4. The proof of this result essentially follows from [102, Theorem 4.2], combined with the existence result of Theorem 2.3. Referring to the notation of [102], we firstly remark that in our case $\Omega = \mathbb{R}^N$, thus the parabolic boundary of $U = \Omega \times [0,T]$ is simply $\partial_p U = \mathbb{R}^N \times \{0\}$. Therefore, the initial conditions $(A1) - (A3)$ are all verified by Lemma 2.41. We then define the continuous Hamiltonian $F : [0,T] \times \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}) \times M^{N\times N} \to \mathbb{R}$ as follows

$$F(t,x,p,X) := \psi(x,-p)\left(-\sum_i \partial_{x_i}\partial_{p_i}\phi(x,-p) + \nabla_p^2\phi(x,-p) : X + f(x,t)\right), \tag{2.62}$$

and focus on the conditions $(F1), (F3) - (F5), (F6'), (F7), (F9), (F10)$ that F must satisfy. The assumptions $(F1), (F3)-(F5), (F9)$ are easily checked. $(F6')$ follows from the Lipschitz regularity of ϕ and ψ, as $\forall t \in [0,T], x \in \mathbb{R}^N, |p| \geq \rho, |q| + |X| \leq R$ one has

$$\begin{aligned} &|F(t,x,p,X) - F(t,x,q,X)| \leq c_\psi |p-q| \left| -\sum_i \partial_{x_i}\partial_{p_i}\phi(x,-p) + \nabla_p^2\phi(x,-p) : X \right| \\ &+ \psi(x,-q)\left| -\sum_i (\partial_{x_i}\partial_{p_i}\phi(x,-p) - \partial_{x_i}\partial_{p_i}\phi(x,-q)) + \left(\nabla_p^2\phi(x,-p) - \nabla_p^2\phi(x,-q)\right) : X \right| \\ &\leq c_R|p-q|\left(1 + \frac{1}{|p|}\right) + c_R|p-q| \leq c_{R,\rho}|p-q|. \end{aligned}$$

For $(F7)$, we remark that the first term in the parenthesis in (2.62) is $0-$homogeneous in p, while the second one is $(-1)-$homogeneous in p but $1-$homogeneous in X. Lastly, we sketch how to prove $(F10)$. Since it concerns the X-terms, we focus simply on

$$\nabla_p^2\phi(x,-p) : X = \mathrm{tr}\left(\nabla_p^2\phi(x-,p)\, X^T\right).$$

Multiplying by $\phi(x,-p)$, we rewrite $\phi(x,-p)\mathrm{tr}\left(\nabla_p^2\phi(x-,p)\, X^T\right) = \mathrm{tr}(A(x,-p)X^T)$, where $A = B - (\nabla_p\phi \otimes \nabla_p\phi)$, with B being the uniformly elliptic operator $\frac{1}{2}\nabla_p^2\phi^2$. We can then factorize

$B = \tilde{L}\tilde{L}^T$, with $\tilde{L}$ being a nondegenerate, lower triangular matrix. Then, following the proof of [23, Proposition 6.1] and [102, pg. 463], we obtain $(F10)$. □

Once uniqueness is settled, one can finally define the notion of *level set* solution to the mean curvature flow as follows.

Definition 2.42. Let E_0 be a compact initial set. Define a uniformly continuous, bounded function $u_0 : \mathbb{R}^N \to \mathbb{R}$ such that $\{u_0 \geq 0\} = E_0$. Then, let $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ be the unique continuous viscosity solution to (2.54) given by Theorem 2.4. Then, the family $E_t := \{u^+(\cdot, t) \geq 0\}_{t\geq 0}$ will be called the level set solution to the mean curvature flow.

This definition is well posed since the Hamiltonian defined in (2.53) satisfies the so-called *geometricity condition*. Namely, one can easily check that for any $\lambda \neq 0, p \in \mathbb{R}^N \setminus 0, q \in \mathbb{R}^N$ and any symmetric $N \times N$ matrix X one has

$$H(x, \lambda p, \lambda X + p \otimes q + q \otimes p) = \frac{\lambda}{|\lambda|} H(x, p, X).$$

Thus, one can prove by classical arguments (see e.g. [56, Remark 3.9]) the following result.

Lemma 2.43. *Let $u_0, \tilde{u}_0$ two initial data for* (2.54) *such that $\{u_0 \geq 0\} = \{\tilde{u}_0 \geq 0\}$. Then, denoting by $u, \tilde{u}$ the corresponding solutions to* (2.54)*, one has*

$$\{u(\cdot, t) \geq 0\} = \{\tilde{u}(\cdot, t) \geq 0\} \quad \textit{for all } t \in [0, T],$$

and the same identity holds for the open superlevel sets.

4.3 Proof of Theorem 2.3

In this section we will prove that the limiting functions $u^\pm$ are respectively a viscosity sub- and supersolutions to (2.54). We recall the standing assumptions (H0), (H1) and $f \in C^0(\mathbb{R}^N \times [0, +\infty))$. We will be following the structure of the proof of [56, Theorem 6.16], but taking into account the weaker definition of u^+ holding in our case. We will be using the O, o notations with respect to $h \to 0$ and focus on proving that u^+ is a subsolution. The proof for u^- is analogous.

Proof of Theorem 2.3. Consider u^+ as defined in (2.6): we need to prove that it is a subsolution. In the following, we will denote $u := u^+$ and $u_h := u_h^+$. Let $\eta(x, t)$ be an admissible test function in $\bar{z} := (\bar{x}, \bar{t}) \in \mathbb{R}^N \times (0, T)$ and assume that $(\bar{x}, \bar{t})$ is a strict maximum point for $u - \eta$. Assume furthermore that $u - \eta = 0$ in such point. We need to show that either (2.55) or (2.56) holds at $\bar{z}$.
Case 1. Let us first assume that $\nabla\eta(\bar{z}) \neq 0$. By classical arguments, we can assume that $\bar{z}$ is a strict maximum point and that η is smooth. By the definition of u, there exists a sequence $\tilde{z}_k := (\tilde{x}_{h_k}, \tilde{t}_{h_k}) \to \bar{z}$ such that $\lim_k u_{h_k}(\tilde{z}_k) = u(\bar{z})$. We remark that we can substitute the functions u_{h_k} for $t > 0$ with their *usc* envelope in time, without changing the value of u. Indeed, the *usc* envelope of u_{h_k} is the function at all discrete times lh_k is given by

$$\max\{u_{h_k}(\cdot, (l-1)h_k), u_{h_k}(\cdot, lh_k)\}$$

and coincides with u_{h_k} elsewhere. Since now u_{h_k} is *usc* in time and space, by standard arguments (compare e.g. [18, Lemma 6.1]), there exists a radius $\rho > 0$ such that all functions $u_{h_k} - \eta$ achieve a local maximum in $B_\rho(\bar{z})$ at points $z_k = (x_k, t_k)$. Then, passing to a further subsequence we can ensure that $z_k \to w \in B_\rho(\bar{z})$, and we use the definition of u to obtain

$$(u - \eta)(w) \geq \limsup_k (u_{h_k} - \eta)(z_k) \geq \limsup_k (u_{h_k} - \eta)(\tilde{z}_k) = (u - \eta)(\bar{z}).$$

Therefore, $w = \bar{z}$ by maximality. Thus we can assume that each function $u_{h_k} - \eta$ achieves a local maximum in $B_\rho(\bar{z})$ at a point $z_{h_k} =: (x_k, t_k)$ and that $u_{h_k}(z_{h_k}) \to u(\bar{z})$ as $k \to \infty$. Finally, we can assume also that $\nabla\eta(x_k, t_k) \neq 0$ for k large enough.

Step 1. We start defining an appropriate set which is then used as a competitor for the minimality of the level sets of the functions u_h. From the previous computations, one has in particular that

$$u_h(x,t) \le \eta(x,t) + c_k \tag{2.63}$$

where $c_k := u_{h_k}(x_k,t_k) - \eta(x_k,t_k)$, with equality if $(x,t)=(x_k,t_k)$. Let $\sigma>0$ and set

$$\eta^\sigma_{h_k}(x) := \eta(x,t_k) + c_k + \frac{\sigma}{2}|x-x_k|^2.$$

Then, for all $x\in\mathbb{R}^N$,

$$u_{h_k}(x,t_k) \le \eta^\sigma_{h_k}(x)$$

with equality if and only if $x=x_k$. We set $l_k = u_{h_k}(x_k,t_k) = \eta^\sigma_{h_k}(x_k)$. We fix $\varepsilon>0$, to be chosen later, and write $E_{\varepsilon,k} := \{u_{h_k}(\cdot,t_k-h_k)\ge l_k-\varepsilon\}$. We define[1]

$$W_\varepsilon := \left(T^+_{h,t_k-h_k}E_{\varepsilon,k}\right) \setminus \left\{\eta^\sigma_{h_k}(\cdot) > l_k+\varepsilon\right\}.$$

We immediately see that $W_\varepsilon \to \{x_k\}$ in the Kuratowski sense as $\varepsilon\to 0$ since by (2.60)

$$\{u_{h_k}(\cdot,t_k) > l_k-\varepsilon\} \setminus \left\{\eta^\sigma_{h_k}(\cdot) > l_k+\varepsilon\right\} \subseteq W_\varepsilon \subseteq \{u_{h_k}(\cdot,t_k)\ge l_k-\varepsilon\}\setminus\left\{\eta^\sigma_{h_k}(\cdot)>l_k+\varepsilon\right\}, \tag{2.64}$$

see also (2.70) below. Then, we check that $|W_\varepsilon|>0$ for all ε small enough. By the continuity of η^σ and $|\nabla\eta(\bar z)|\ne 0$, for any ε there exist a radius r_ε such that $W_\varepsilon \supseteq B(x_k,r_\varepsilon)\cap T^+_{h,t_k-h_k}E_{\varepsilon,k}$. Furthermore, for any $\varepsilon>0$, using (2.60) again yields $x_k\in T^+_{h_k,t_k-h_k}\{u_{h_k}(\cdot,t_k-h_k)\ge l_k-\varepsilon\}$, and the latter set coincides with the closure of its points of density 1 by Lemma 2.14. Thus, x_k satisfies lower density estimates and so we conclude that $|W_\varepsilon|>0$. Now, assume $E_{\varepsilon,k}$ is bounded. By minimality we have

$$\begin{aligned} &P_\phi(T^+_{h,t_k-h_k}E_{\varepsilon,k}) + \frac{1}{h_k}\int_{T^+_{h,t_k-h_k}E_{\varepsilon,k}} \mathrm{sd}^\psi_{E_{\varepsilon,k}}(x)\,\mathrm{d}x + \int_{W_\varepsilon} F_{h_k}(x,t_k-h_k)\,\mathrm{d}x \\ &\le P_\phi\left(\left(T^+_{h,t_k-h_k}E_{\varepsilon,k}\right)\cap\{\eta^\sigma_{h_k}>l_k+\varepsilon\}\right) + \frac{1}{h_k}\int_{\left(T^+_{h,t_k-h_k}E_{\varepsilon,k}\right)\cap\{\eta^\sigma_{h_k}>l_k\}} \mathrm{sd}^\psi_{E_{\varepsilon,k}}. \end{aligned} \tag{2.65}$$

Adding to both sides the term $P_\phi\left(\{\eta^\sigma_{h_k}>l_k+\varepsilon\}\cup T^+_{h,t_k-h_k}E_{\varepsilon,k}\right)$ and using the submodularity (2.10), we obtain

$$\begin{aligned} P_\phi(\{\eta^\sigma_{h_k}>l_k+\varepsilon\}\cup W_\varepsilon) - P_\phi(\{\eta^\sigma_{h_k}>l_k+\varepsilon\}) + \frac{1}{h_k}\int_{W_\varepsilon}\mathrm{sd}^\psi_{E_{\varepsilon,k}}(x)\,\mathrm{d}x \\ + \int_{W_\varepsilon} F_{h_k}(x,t_k-h_k)\,\mathrm{d}x \le 0. \end{aligned}$$

By (2.63), $\{u_{h_k}(\cdot,t_k-h_k)\ge l_k-\varepsilon\}\subseteq\{\eta(\cdot,t_k-h_k)\ge l_k-c_k-\varepsilon\}$, therefore it holds

$$\begin{aligned} P_\phi(\{\eta^\sigma_{h_k}>l_k+\varepsilon\}\cup W_\varepsilon) - P_\phi(\{\eta^\sigma_{h_k}>l_k+\varepsilon\}) + \frac{1}{h_k}\int_{W_\varepsilon}\mathrm{sd}^\psi_{\{\eta(\cdot,t_k-h_k)\ge l_k-c_k-\varepsilon\}}(x)\,\mathrm{d}x \\ + \int_{W_\varepsilon} F_{h_k}(x,t_k-h_k)\,\mathrm{d}x \le 0. \end{aligned} \tag{2.66}$$

[1] We need to define the sets W_ε in this way (compare the different definition in [56]) since firstly, we can not rule out that the inclusions in (2.64) are strict, and secondly it is not clear if otherwise $|W_\varepsilon|>0$.

If instead $E_{\varepsilon,k}$ is an unbounded set with compact boundary, we replace inequality (2.65) by

$$P_\phi(T_{h,t_k-h_k}E_{\varepsilon,k}) + \frac{1}{h_k}\int_{\left(T^+_{h,t_k-h_k}E_{\varepsilon,k}\right)\cap B_R} \mathrm{sd}^\psi_{E_{\varepsilon,k}}(x)\,\mathrm{d}x + \int_{W_\varepsilon} F_{h_k}(x,t_k-h_k)\,\mathrm{d}x$$
$$\le P_\phi(\left(T^+_{h,t_k-h_k}E_{\varepsilon,k}\right)\cap\{\eta^\sigma_{h_k} > l_k+\varepsilon\}) + \frac{1}{h_k}\int_{\left(T^+_{h,t_k-h_k}E_{\varepsilon,k}\right)\cap\{\eta^\sigma_{h_k}>l_k+\varepsilon\}\cap B_R} \mathrm{sd}^\psi_{E_{\varepsilon,k}},$$

for $R>0$ sufficiently large, see (2.52). Then, one can argue as before to obtain (2.66).
Step 2. We estimate the first two terms in (2.66). The quantity $P_\phi(\{\eta^\sigma_{h_k} > l_k+\varepsilon\}\cup W_\varepsilon) - P_\phi(\{\eta^\sigma_{h_k} > l_k+\varepsilon\})$ can be estimated as done in Lemma 2.19. Indeed, we consider the vector field $v = \nabla_p\phi(x,\nabla\eta^\sigma_{h_k})$ in (2.11) and we use the divergence theorem to get

$$\begin{aligned} P_\phi(\{\eta^\sigma_{h_k} > l_k+\varepsilon\}\cup W_\varepsilon) - P_\phi(\{\eta^\sigma_{h_k} \ge l_k+\varepsilon\}) &\ge \int_{\partial(\{\eta^\sigma_{h_k}>l_k+\varepsilon\}\cup W_\varepsilon)} v\cdot\nu - \int_{\partial\{\eta^\sigma_{h_k}>l_k+\varepsilon\}} v\cdot\nu \\ &= |W_\varepsilon| \fint_{W_\varepsilon} \operatorname{div} v, \end{aligned} \tag{2.67}$$

where ν denotes the unit outer vector to the set we are integrating on. We then remark that $\fint_{W_\varepsilon}\operatorname{div} v \to \mathrm{H}^\phi_{\{\eta^\sigma_{h_k}>l_k\}}(x_k)$ and $\fint_{W_\varepsilon} F_{h_k}(x,t_k-h_k)\,\mathrm{d}x \to F_{h_k}(x_k,t_k-h_k)$ as $\varepsilon\to 0$ by continuity.
Step 3. We bound the distance term in (2.66) by showing that

$$\frac{1}{h_k}\mathrm{sd}^\psi_{\{\eta(\cdot,t_k-h_k)=l_k-c_k-\varepsilon\}}(z) \ge \frac{\partial_t\eta(z,t_k) - O(h_k)}{\psi(y,-\nabla\eta(y,t_k-h_k)) + O(h_k)}. \tag{2.68}$$

For any $z\in W_\varepsilon$, we have

$$\eta(z,t_k) + c_k + \frac{\sigma}{2}|z-x_k|^2 \le l_k+\varepsilon. \tag{2.69}$$

Since, in turn, $\eta(z,t_k)+c_k > l_k-\varepsilon$ it follows that $\sigma|z-x_k|^2 < 4\varepsilon$ and thus, for ε small enough,

$$W_\varepsilon \subseteq B_{c\sqrt{\varepsilon}}(x_k). \tag{2.70}$$

By a Taylor expansion, for every $z\in W_\varepsilon$ we have

$$\eta(z,t_k-h_k) = \eta(z,t_k) - h_k\partial_t\eta(z,t_k) + h_k^2\int_0^1(1-s)\partial^2_{tt}\eta(z,t_k-sh_k)\,\mathrm{d}s. \tag{2.71}$$

Then, we consider $y, y_e \in \{\eta(\cdot,t_k-h_k)(y) = l_k-c_k-\varepsilon\}$ being respectively, a point of minimal $\psi-$distance and Euclidean distance from z.
Claim: We claim that it holds

$$|z-y| = O(h_k). \tag{2.72}$$

In order to prove this result, we start remarking that for $k\to\infty$ and choosing $\varepsilon \ll h_k$, one has $\mathrm{sd}^\psi_{\{\eta(\cdot,t_k-h_k)\ge l_k-c_k-\varepsilon\}}(z) \to 0$ (as $z\to x_k$ for $\varepsilon\to 0$ and $x_k\in\{\eta(\cdot,t_k)\ge l_k-c_k\}$). In particular, recalling the bounds (2.9) one has

$$|z-y_e| \le c_\psi^2|z-y| \le c_\psi^3|\mathrm{sd}^\psi_{\{\eta(\cdot,t_k-h_k)\ge l_k-c_k-\varepsilon\}}(z)| \to 0$$

as $k\to\infty$. By (2.69) we deduce in particular $\eta(z,t_k)+c_k < l_k+\varepsilon$, that is,

$$0 \le \eta(z,t_k) - \eta(y,t_k-h_k) \le 2\varepsilon, \tag{2.73}$$

and the same inequality substituting y_e to y. Thus, one has

$$\eta(z,t_k) - \eta(y_e,t_k-h_k) = \nabla\eta(y,t_k-h_k)\cdot(z-y_e) - h_k\partial_t\eta(y,t_k-h_k) + O(|z-y_e|^2+h_k^2)$$

which we combine with $\nabla\eta(y,t_k-h_k)\cdot(z-y_e) = \pm|\nabla\eta(y,t_k-h_k)|\,|z-y_e|$ (see [56] for details) and (2.73) to get

$$|z - y_e|\,|\nabla\eta(y, t_k - h_k)| \le 2\varepsilon + O(h_k) + O(|z - y_e|^2).$$

Recalling that $|\nabla\eta(y, t_k - h_k)| \ge c > 0$ for h_k small enough, we divide by $|\nabla\eta(y, t_k - h_k)|$ to conclude $|z - y_e| = O(h_k)$ as $\varepsilon \ll h_k$. Finally, employing again (2.9), we prove the claimed (2.72).

Then, we consider a geodesic curve for the definition of $\mathrm{sd}^{\psi}_{\{\eta(\cdot,t_k-h_k)\ge l_k-c_k-\varepsilon\}}(z)$: if this distance is positive, we choose $\gamma : [0,1] \to \mathbb{R}^N$ with $\gamma(0) = z, \gamma(1) = y$, with y as before, otherwise we take γ such that $\gamma(0) = y, \gamma(1) = z$. In the following, we will assume $\mathrm{sd}^{\psi}_{\{\eta(\cdot,t_k-h_k)\ge l_k-c_k-\varepsilon\}}(z) > 0$, the other case being analogous. Recalling (2.7), we have

$$\begin{aligned}
\eta(z, t_k - h_k) &= \eta(y, t_k - h_k) + \int_0^1 \nabla\eta(\gamma, t_k - h_k)\cdot\dot\gamma\,\mathrm{d}t \\
&\ge \eta(y, t_k - h_k) - \int_0^1 \psi(\gamma, -\nabla\eta(\gamma, t_k - h_k))\psi^\circ(\gamma,\dot\gamma)\,\mathrm{d}t \\
&\ge \eta(y, t_k - h_k) - \psi(y, -\nabla\eta(y, t_k - h_k))\,\mathrm{sd}^{\psi}_{\{\eta(\cdot,t_k-h_k)=l_k-c_k-\varepsilon\}}(z) \\
&\quad - \int_0^1 \left(\psi(\gamma, -\nabla\eta(\gamma, t_k - h_k)) - \psi(y, -\nabla\eta(y, t_k - h_k))\right)\psi^\circ(\gamma,\dot\gamma)\,\mathrm{d}t \\
&\ge \eta(y, t_k - h_k) - \big(\psi(y, -\nabla\eta(y, t_k - h_k)) + c|z - y|\big)\,\mathrm{sd}^{\psi}_{\{\eta(\cdot,t_k-h_k)=l_k-c_k-\varepsilon\}}(z),
\end{aligned}$$

where in the last line we reasoned as in (2.41) to obtain the bound $\sup_t |\gamma(t) - y| \le c|z - y|$. Recalling (2.72) one has

$$\eta(z, t_k - h_k) \ge \eta(y, t_k - h_k) - \psi(y, -\nabla\eta(y, t_k - h_k))\,\mathrm{sd}^{\psi}_{\{\eta(\cdot,t_k-h_k)=l_k-c_k-\varepsilon\}}(z) + o(h_k). \tag{2.74}$$

Combining (2.71) with (2.74) and using (2.73), we deduce

$$\begin{aligned}
&\mathrm{sd}^{\psi}{}_{\{\eta(\cdot,t_k-h_k)=l_k-c_k-\varepsilon\}}(z)\,\psi(y, -\nabla\eta(y, t_k - h_k)) + o(h_k) \\
&\qquad \ge -2\varepsilon + h_k\partial_t\eta(z, t_k) - h_k^2\int_0^1 (1 - s)\partial^2_{tt}\eta(z, t_k - sh_k)\,\mathrm{d}s.
\end{aligned}$$

Note that, in view of (2.69) and (2.9), $|\eta(z, t_k) - \eta(y, t_k)| \le c\varepsilon + ch_k = O(h_k)$, provided $\varepsilon \ll h_k$ and small enough. We then conlude (2.68) by combining the previous inequality with (2.70),(2.72) as

$$\begin{aligned}
\frac{1}{h_k}\mathrm{sd}^{\psi}_{\{\eta(\cdot,t_k-h_k)=l_k-c_k-\varepsilon\}}(z) &\ge \frac{\partial_t\eta(z, t_k) - \frac{2\varepsilon}{h_k} - O(h_k) - O_{h_k}(1)}{\psi(y, -\nabla\eta(y, t_k - h_k))} \\
&= \frac{\partial_t\eta(x_k, t_k) + O(\sqrt{\varepsilon}) - \frac{2\varepsilon}{h_k} - O(h_k) - O_{h_k}(1)}{\psi(x_k, -\nabla\eta(x_k, t_k - h_k)) + O(\sqrt{\varepsilon}) + O(h_k)}.
\end{aligned}$$

Step 4. We conclude the proof by employing (2.66), (2.67) and (2.68), dividing by $|W_\varepsilon|$ and sending $\varepsilon \to 0$ to obtain

$$\frac{\partial_t\eta(x_k, t_k) - O_{h_k}(1)}{\psi(x_k, -\nabla\eta(x_k, t_k)) + O(h_k)} + \mathrm{H}^{\phi}_{\{\eta^\sigma_{h_k} \ge \eta^\sigma_{h_k}(x_k)\}}(x_k) - F_{h_k}(x_k, t_k - h_k) \le 0.$$

Letting simultaneously $\sigma \to 0$ and $k \to \infty$, recalling the continuity properties of H^{ϕ}, we deduce (2.56). Indeed the sets $\{\eta^\sigma_{h_k} > \eta^\sigma_{h_k}(x_k)\}$ are converging in C^2 to the set $\{\eta > \eta(x)\}$, $x_k \to x$ and thus

$$\mathrm{H}^{\phi}_{\{\eta^\sigma_{h_k} > \eta^\sigma_{h_k}(x_k)\}}(x_k) \to \mathrm{H}^{\phi}_{\{\eta>\eta(x)\}}(x),$$

and we conclude the proof of this step.

Case 2. Now we consider the case $\nabla\eta(\bar x, \bar t) = 0$ and we show that $\partial_t\eta(\bar x, \bar t) \le 0$. The proof follows the line of the one in [56], we just highlight the differences.

Since $\nabla\eta(\bar z)=0$, there exist b,Ψ as in Definition 2.36 such that

$$|\eta(x,t)-\eta(\bar z)-\partial_t\eta(\bar z)(t-\bar t)|\le\Psi(|x-\bar x|)+b|t-\bar t|^2,$$

thus, we can define

$$\tilde\eta(x,t)=\partial_t\eta(\bar z)(t-\bar t)+2\Psi(|x-\bar x|)+2b|t-\bar t|^2$$
$$\tilde\eta_k(x,t)=\tilde\eta(x,t)+\frac{1}{k(\bar t-t)}.$$

We remark that $u-\tilde\eta$ achieves a strict maximum in $\bar z$ and the local maxima of $u-\tilde\eta_k$ in $\mathbb{R}^N\times[0,\bar t]$ are in points $(x_k,t_k)\to\bar z$ as $k\to\infty$, with $t_n\le\bar t$. From now on, the only difference from [56] is in the case $x_k=\bar x$ for an (unrelabeled) subsequence. We assume $x_k=\bar x$ $\forall k>0$ and define $\tau_k=\bar t-t_k>0$ and the radii

$$r_k:=2\sqrt{C\tau_k},$$

where C is the constant of Lemma 2.40. Taking k large enough, by Lemma 2.40 the balls $B(\cdot,r_k)$ have an extinction time greater than $2(\bar t-t_k)$. We then have

$$\begin{aligned}B(\bar x,r_k)&\subseteq\{\tilde\eta_k(\cdot,t_k)\le\tilde\eta_k(\bar x,t_k)+2\Psi(r_k)\}\\&\subseteq\{u(\cdot,t_k)\le u(\bar x,t_k)+2\Psi(r_k)\},\end{aligned}$$

by definition of $\tilde\eta_k$ and the maximality of $u-\tilde\eta_k$ at z_k. Since the balls $B(\cdot,r_k)$ are not vanishing, we conclude

$$\bar x\in\{u(\cdot,\bar t)\le u(\bar x,t_k)+2\Psi(r_k)\}.$$

Finally, we use again the maximality of $u-\eta$ at $\bar z$ and the choice of r_k to obtain

$$\frac{\eta(\bar x,t_k)-\eta(\bar z)}{t_k-\bar t}=\frac{\eta(\bar x,t_k)-\eta(\bar z)}{-\tau_k}\le\frac{u(\bar x,t_k)-u(\bar x,\bar t)}{-\tau_k}\le\frac{-2\Psi(r_k)}{-\tau_k}.$$

Recalling that $\Psi(r)=o(r^2)$ as $r\to0$, we can pass to the limit $k\to\infty$ and conclude $\partial_t\eta(\bar z)\le0$. □

We conclude with two remarks concerning some possible generalizations of the results presented.

Remark 2.44. The results presented in this chapter can be immediately extended to unbounded initial open sets E_0, whose boundary is compact. Indeed, defining the discrete flow as $E_t^{(h)}=E_0$ if $t\in[0,h)$, otherwise by induction $E_t^{(h)}=T^-_{h,t}E^{(h)}_{t-h}$, where the operator $T^-_{h,n}$ is the one defined in (2.50), this evolution is uniquely characterized by the one of the complement. Thus, all the results presented in this chapter can be extended to this particular unbounded case.

Remark 2.45. Following the lines of [23] (in the spirit of [8]) one can see that the results of this chapter may be extended to prove existence of flat flows and level set solutions to the mean curvature flow on $\mathbb{R}^N$ endowed with the geometric structure induced by a Finsler metric ϕ°. For example, the perimeter functional in this setting is defined as follows. Given a set E of finite perimeter, its (intrinsic) perimeter is

$$\mathcal{P}_{\phi^\circ}(E)=\int_{\partial^*E}\phi(x,\nu_E(x))\,\mathrm{d}\mathcal{H}^{N-1}_{\phi^\circ}(x),$$

where the Hausdorff measure $\mathcal{H}^{N-1}_{\phi^\circ}$ is the one induced by the metric ϕ°. In particular, one can compute $\mathrm{d}\mathcal{H}^{N-1}_{\phi^\circ}(x)=\omega_N|B^{\phi^\circ}(x)|^{-1}\,\mathrm{d}\mathcal{H}^{N-1}(x)$ (see [23]), thus this approach is equivalent to consider in our framework a slightly different (but still regular) anisotropy, namely $\phi^*(x,\nu):=\omega_N|B^{\phi^\circ}(x)|^{-1}\phi(x,\nu)$. In particular, this approach leads to considering the evolution of hypersurfaces E_t moving according to the evolution law

$$V_{\phi^\circ}(x,t)=-\mathcal{H}_{E_t}(x)+f(x,t)\quad x\in\partial E_t,\ t\in(0,T)$$

where now V_{ϕ° represents the speed of evolution along the anisotropic normal outer vector $n_{\phi^\circ}(x) = \nabla_p \phi(x, \nu_E(x))$ and $\mathcal{H}$ is the "intrinsic" mean curvature, thus the first variation of the perimeter $\mathcal{P}_{\phi^\circ}$. Recalling that $n_{\phi^\circ}(x) \cdot \nu_E(x) = \phi(x, \nu_E(x))$, we see that the hypersurfaces are evolving with a normal (in the Euclidean sense) velocity given by the law

$$V(x,t) = \phi(x, \nu_{E_t}(x)) \left(-H^{\phi^*}_{E_t}(x) + f(x,t) \right).$$

After this transformation, we can apply the results previously proved.

Chapter 3

Minimizing Movements for Nonlinear Mean Curvature Flows

Contents

1 Introduction

In this chapter we study a nonlinear version of the anisotropic mean curvature flow (MCF in short) with forcing and mobility. In particular, given a continuous non decreasing function $G : \mathbb{R} \to \mathbb{R}$, we consider the flow of sets $t \mapsto E_t$ formally governed by the evolution law

$$V(x,t) = \psi(\nu_{E_t}(x))G\left(-\kappa^{\phi}_{E_t}(x) + \mathtt{f}(t)\right), \qquad \text{for all } x \in \partial^* E_t,\ t \geq 0, \tag{3.1}$$

where ψ, ϕ are two anisotropies (with ψ usually called the mobility), $\kappa^{\phi}_{E_t}$ denotes the ϕ-curvature of the set of finite perimeter E_t, ν_{E_t} denotes the outer normal vector and $\mathtt{f}$ is a forcing term constant in space. We are interested in showing that the mimizing movement approximation scheme produces discrete-in-time solutions that converge to the unique viscosity solution to (3.1) as the time-step parameter tends to 0.

The evolution law (3.1) is relevant from a numerical point of view, as suggested e.g. in [57, Remark 3.5]. For example, a truncation of the evolution speed is usually encoded in algorithms for the MCF, which would correspond to choosing $G(s) = (-M) \vee s \wedge M$ in (3.1). Another interesting choice could be $G(s) = -s^-$, which amounts to consider a purely shrinking evolution. Moreover, evolution by powers of the mean curvature have been previously studied in the smooth or convex setting [60, 169, 15] and have been used to prove isoperimetric inequalities [170], or considered in the setting of image processing algorithms [11, 167]. In particular, in [11, Section 4.5] it is remarked that the evolution law (3.1) with $G(s) = s^{\frac{1}{3}}$ and $\phi = \psi = |\cdot|$ is particularly interesting as it is invariant under affine motions (isometries and rescalings). We refer also to [79] for interesting links between a time-fractional Allen-Cahn equation and motion by powers of the mean curvature.

In the present nonlinear setting, only two concept of solutions are currently available: smooth solutions (starting from smooth sets, in general existing only in a finite time span) and viscosity/level-set solutions, which are weak solutions of (3.1) defined globally in time and starting from any initial compact set. On the other hand, it is not clear whether a notion of "BV-solutions" in the spirit of [8, 144] (and studied in the previous chapter) can be properly defined.

Inspired by the techniques developed in [56], and the recent study [50] (contained in Chapter 2), we show that the mimizing movement scheme à la Almgren-Taylor-Wang or Luckhaus-Sturzenhecker [8, 144] provides existence by approximation of a level set solution to the nonlinear MCF, under suitable smoothness assumptions on the quantities involved. This is an extension and an improvement of the unpublished (and unfinished) preprint [46], and is essentially based on techniques introduced in [56]. In [56], the authors prove the convergence of the minimizing movements scheme to viscosity solutions to a very general class of curvature flows of the form $V = -\kappa$, with κ being a "variational" curvature. We will also use some refinements of the techniques of [56] that we developed in Chapter 2, where we focused on similar evolutions driven by inhomogeneous curvatures (i.e. non translationally invariant). We want to point out that our result is more general than those of [46], as the authors work in the isotropic setting without mobility ($\psi = \phi = |\cdot|$), require further regularity on the function G and assume that

$$\lim_{s \to \pm\infty} G(s) = \pm\infty,$$

which simplifies many arguments. From a technical point of view, the main difficulties arise in the case where G is bounded from above or below, as some tools heavily employed in the linear setting are no longer available (see e.g. the commonly used reformulation (3.10)). Anyhow, by an approximation approach we can recover all the necessary results, which we then pair with the variational approach of [56] in order to prove our main result.

To conclude, it would be interesting to study the much more challenging case where ϕ is non smooth, i.e. the so-called crystalline case. In this setting the availability of the viscosity solutions of [104, 105] and the development of distribution solutions of [55, 53, 52] may suggest the premarkibility of a future investigation in this direction.

2 The minimizing movements scheme

2.1 Preliminaries

For the notations and some preliminary results, we refer to Section 2 in Chapter 2.

We focus on a nonlinear evolution by anisotropic mean curvature with mobility of a family of sets $\{E_t\}_{t\geq 0}$ starting from a set $E_0 \subseteq \mathbb{R}^N$ which is either bounded or has bounded complement. The evolution law is

$$V(x,t) = \psi(\nu_{E_t}(x))G\left(-\kappa^{\phi}_{E_t}(x) + \mathtt{f}(t)\right) \quad x \in \partial^* E_t,\, t \geq 0, \tag{3.2}$$

where $\nu_{E_t}(x)$ is the outer normal vector to E_t at x and ϕ, ψ are two anisotropies (as defined in Definition 2.5 in Chapter 2) that are homogeneous in space, and under the following hypotheses on the functions involved:

- $G : \mathbb{R} \to \mathbb{R}$ is a continuous, non-decreasing function, with $G(0) = 0$;
- $\mathtt{f} \in C^0_b(\mathbb{R})$;
- $\phi \in C^3$ and it is strictly convex.

We then set

$$\lim_{s\to-\infty} G(s) = -a \in [-\infty, 0], \qquad \lim_{s\to+\infty} G(s) = b \in [0, +\infty].$$

Consider a function $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ whose superlevel sets $E_s := \{u(\cdot, t) \geq s\}$ evolve according to the nonlinear mean curvature equation (3.2). By classical computations (see e.g. [101]), the function u satisfies the equation

$$\partial_t u = |\nabla u| V(x,t) = \psi(\nabla u)G\left(-\nabla^2\phi(\nabla u) : \nabla^2 u + \mathtt{f}\right) =: -H(t, \nabla u, \nabla^2 u)$$

where we defined the Hamiltonian $H \;:\; [0, +\infty) \times \mathbb{R}^N \setminus \{0\} \times S^N \to \mathbb{R}$ as

$$H(t, p, X) := -\psi\left(p\right) G(-\kappa^{\phi}(p, X) + \mathtt{f}(t)), \tag{3.3}$$

and $\kappa^{\phi}(p, X)$ is defined as $\kappa^{\phi}(p, X) = \nabla^2\phi(p) : X$. Therefore, one is led to solve the parabolic Cauchy problem in bounded time intervals $[0, T]$, for $T > 0$, given by

$$\begin{cases} \partial_t u + H(t, \nabla u, \nabla^2 u) = 0 & \text{on } \mathbb{R}^N \times [0, T] \\ u(\cdot, 0) = u_0 & \text{on } \mathbb{R}^N. \end{cases} \tag{3.4}$$

Existence and uniqueness to (3.4) can be proved in the framework of viscosity solutions as done in [128]. Let us recall the notion of viscosity solution used in [128], starting by a family of auxiliary functions.

Definition 3.1. The family $\mathcal{F}$ is composed of smooth functions $\ell \in C^\infty_c([0, +\infty))$ satisfying $\ell(0) = \ell'(0) = \ell''(0) = 0$, $\ell''(r) > 0$ in a neighborhood of 0, ℓ constant in $(0, M)^c$ for some $M > 0$ (depending on ℓ), and such that

$$\lim_{p\to 0} \frac{\ell'(|p|)}{|p|} H(t, p, \pm I) = 0$$

holds uniformly in time.

Remark 3.2. We note that $\mathcal{F} \neq \emptyset$. Since for all $t \in [0, +\infty), p \in \mathbb{R}^N$ it holds

$$G(-c/|p| - \|\mathtt{f}\|_\infty) \leq H(t, p, I) \leq H(t, p, -I) \leq G(c/|p| + \|\mathtt{f}\|_\infty),$$

for a suitable positive constant $c = c(\phi)$, one can repeat the construction used in [128, page 229] to show that $\mathcal{F} \neq \emptyset$.

With a slight abuse of notation, in the following we will say that a function is spatially constant outside a compact set even if the value of such constant is time-dependent.

Definition 3.3. Let $\hat{z} = (\hat{x}, \hat{t}) \in \mathbb{R}^N \times (0,T)$ and let $A \subseteq (0,T)$ be any open interval containing $\hat{t}$. We will say that $\eta \in C^0(\mathbb{R}^N \times \bar{A})$ is admissible at the point $\hat{z}$ if it is of class C^2 in a neighborhood of $\hat{z}$, it is spatially constant outside a compact set, and, in case $\nabla\eta(\hat{z}) = 0$, the following holds: there exists $\ell \in \mathcal{F}$ and $\omega \in C^\infty([0,\infty))$ with $\omega'(0) = 0, \omega(r) > 0$ for $r \neq 0$ and such that

$$|\eta(x,t) - \eta(\hat{z}) - \eta_t(\hat{z})(t-\hat{t})| \leq \ell(|x-\hat{x}|) + \omega(|t-\hat{t}|),$$

for all $(x,t) \in \mathbb{R}^N \times A$.

The notion of viscosity solution used in [128] is the following one.

Definition 3.4. An upper semicontinuous function $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$, constant outside a compact set, is a viscosity subsolution of the Cauchy problem (3.4) if $u(\cdot,0) \leq u_0$ and, for all $z := (x,t) \in \mathbb{R}^N \times [0,T]$ and all C^∞-test functions η such that η is admissible at z and $u - \eta$ has a maximum at z, the following holds:

i) If $\nabla\eta(z) = 0$, then
$$\eta_t(z) \leq 0 \tag{3.5}$$

ii) If $\nabla\eta(z) \neq 0$, then
$$\partial_t\eta(z) + H(\nabla\eta(z), \nabla^2\eta(z)) \leq 0. \tag{3.6}$$

A lower semicontinuous function $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$, constant outside a compact set, is a viscosity supersolution of the Cauchy problem (3.4) if $-u$ is a viscosity supersolution to (3.4), with $-u_0$ replacing u_0. Finally, a function u is a viscosity solution for the Cauchy problem (3.4) if it is both a subsolution and a supersolution of (3.4).

Remark 3.5. By classical arguments, one could assume that the maximum of $u - \eta$ is strict in the definition of subsolution above (an analogous remark holds for supersolutions).

We thus recall the existence and uniqueness result proved in [128].

Theorem 3.6. *Given an initial datum u_0 uniformly continuous and constant outside a compact set, the Cauchy problem* (3.4) *admits a unique viscosity solution. Moreover, if u, v are, respectively, a super- and subsolution to* (3.4) *satisfying $u(\cdot,0) \geq v(\cdot,0)$, then $u(\cdot,t) \geq v(\cdot,t)$ for every $t \in [0,T]$.*

2.2 The minimizing movements scheme

In order to give the definition of the discrete scheme we consider in this chapter, we introduce some notations. We set g as a selection of the set-valued inverse of G, that is $g(x) \in G^{-1}(x)$ for every $x \in (-a,b)$ and extend it setting $g = -\infty$ for every $x \leq -a$, $g = +\infty$ for every $x \geq b$. Here, we extended G to $[-\infty,+\infty]$ setting $G(\pm\infty) = \lim_{x\to\pm\infty} G(x)$. We assume also that $g(0) = 0$. Note that these definitions imply $G \circ g = id$ in $[-a,b]$. Moreover, g is strictly increasing. In the following we will denote for $k \in \mathbb{N}, h > 0$

$$f(kh) = \fint_{kh}^{(k+1)h} \mathtt{f}(s)\,\mathrm{d}s.$$

Given a bounded set of finite perimeter E and $h > 0, t \in (0,+\infty)$ we define a functional on the measurable sets as

$$\mathscr{F}^E_{h,t}(F) = P_\phi(F) + \int_{E\triangle F} \left| g\left(\frac{\mathrm{sd}_E}{h}\right) \right| - f([t/h]h)|F|, \tag{3.7}$$

where $[\cdot]$ denotes the integer part.

Lemma 3.7. *Let E be a bounded set of finite perimeter and $h>0, t\in[0,+\infty)$. Then, there exist minimizers of $\mathscr{F}^E_{h,t}$ and, denoting E' one such minimizer, it has the following properties: it is a bounded set of finite ϕ-perimeter such that (up to negligible sets)*

$$E_{-ah}\subseteq E'\subseteq E_{bh}.$$

Moreover, there exist a maximal and a minimal minimizer (with respect to inclusion) of $\mathscr{F}^E_{h,t}$.

Proof. Note that $\mathscr{F}^E_{h,t}(E)<+\infty$ and that

$$\mathscr{F}^E_{h,t}(F)\geq P_\phi(F)+\int_F |g\left(\mathrm{sd}_E/h\right)|\,\chi_{E^c}-\|f\|_\infty.$$

It is easy to see that the functional on the *rhs* admits a minimizer of finite energy by the coercivity of $g(\mathrm{sd}_E/h)$. Thus, by standard methods, one proves the existence of minimizers to $\mathscr{F}^E_{h,t}$. Since it has finite energy, it is straightforward to check that $\mathrm{sd}_E\in[-ah,bh]$ a.e. on $E'\triangle E$. If $b<+\infty$ this clearly implies that E' is bounded; if $b=+\infty$ a classic contradiction argument (essentially recalled in Lemma 3.11 below) yields the same result. Finally, by classical arguments one shows that, if E_1', E_2' are minimizers of $\mathscr{F}^E_{h,t}$, then so are $E_1'\cap E_2', E_1'\cup E_2'$, implying the existence of a minimal and a maximal solution (see e.g. [56, Proposition 6.1]). □

For a given bounded set E and $t\in(0,+\infty)$, we thus denote

$$T^-_{h,t}E=\min\operatorname{argmin}\,\mathscr{F}^E_{h,t},\qquad T^+_{h,t}E=\max\operatorname{argmin}\,\mathscr{F}^E_{h,t},\tag{3.8}$$

where the minimum and maximum above are made with respect to inclusion. We will often denote $T_{h,t}:=T^-_{h,t}$. We now prove some classical results following the lines of [144].

Lemma 3.8 (Weak comparison principle)**.** *Fix $h>0, t\in(0,+\infty)$ and assume that F_1,F_2 are bounded sets with $F_1\subset\subset F_2$. Then, for any two minimizers E_i of $\mathscr{F}^{F_i}_{h,t}$ for $i=1,2$, we have $E_1\subseteq E_2$. If, instead, $F_1\subseteq F_2$, then we have that the minimal (respectively maximal) minimizer of $\mathscr{F}^{F_1}_{h,t}$ is contained in the minimal (respectively maximal) minimizer of $\mathscr{F}^{F_2}_{h,t}$.*

Proof. Firstly, we assume $F_1\subset\subset F_2$, Testing the minimality of E_1,E_2 with their intersection and union, respectively, we obtain

$$P_\phi(E_1)+\int_{(E_1\setminus E_2)\setminus F_1} g\left(\frac{\mathrm{sd}_{F_1}}{h}\right)+\int_{(E_1\setminus E_2)\cap F_1} g\left(\frac{\mathrm{sd}_{F_1}}{h}\right)\leq P_\phi(E_1\cap E_2)+f([t/h]h)|E_1\setminus E_2|$$
$$P_\phi(E_2)\leq P_\phi(E_1\cup E_2)+\int_{(E_1\setminus E_2)\setminus F_2} g\left(\frac{\mathrm{sd}_{F_2}}{h}\right)+\int_{(E_1\setminus E_2)\cap F_2} g\left(\frac{\mathrm{sd}_{F_2}}{h}\right)-f([t/h]h)|E_1\setminus E_2|.$$

Summing the two inequalities above and using the submodularity of the perimeter we get

$$\int_{(E_1\setminus E_2)\setminus F_1} g\left(\frac{\mathrm{sd}_{F_1}}{h}\right)+\int_{(E_1\setminus E_2)\cap F_1} g\left(\frac{\mathrm{sd}_{F_1}}{h}\right)\leq\int_{(E_1\setminus E_2)\cap F_2} g\left(\frac{\mathrm{sd}_{F_2}}{h}\right)+\int_{(E_1\setminus E_2)\setminus F_2} g\left(\frac{\mathrm{sd}_{F_2}}{h}\right).\tag{3.9}$$

Assume by contradiction that $|E_1\setminus E_2|>0$. Since $\mathrm{sd}_{F_2}<\mathrm{sd}_{F_1}$ and by the strict monotonicity of g, we estimate the *rhs* of (3.9) by

$$\int_{(E_1\setminus E_2)\setminus F_2} g\left(\frac{\mathrm{sd}_{F_2}}{h}\right)+\int_{(E_1\setminus E_2)\cap F_2} g\left(\frac{\mathrm{sd}_{F_2}}{h}\right)<\int_{(E_1\setminus E_2)\setminus F_2} g\left(\frac{\mathrm{sd}_{F_1}}{h}\right)+\int_{(E_1\setminus E_2)\cap F_1} g\left(\frac{\mathrm{sd}_{F_1}}{h}\right)$$

and plug it in (3.9) to reach the desired contradiction. The other cases follow analogously, reasoning by approximation if $F_1\subseteq F_2$. □

Lemma 3.9. *Let $c\in\mathbb{R}$. Consider E a bounded set of finite perimeter and non-decreasing functions $g_1,g_2:\mathbb{R}\to\mathbb{R}$ such that $g_1<g_2$ in $\mathbb{R}\setminus\{0\}$ and $g_1(0)=g_2(0)=0$. Then, if E_i solves*

$$\min_F\left\{P_\phi(F)+\int_{E\triangle F}|g_i(\mathrm{sd}_E(x))|\,\mathrm{d}x+c|F|\right\}$$

for $i = 1, 2$, we have that $E_2 \subseteq E_1$. If $g_1 \leq g_2$ instead, an analogous statement holds for the maximal and minimal solutions.

Proof. Denote $g_i = g_i \circ \mathrm{sd}_E$ for $i = 1, 2$ and assume by contradiction that $|E_2 \setminus E_1| > 0$. Reasoning as in Lemma 3.8, one gets

$$\int_{E_1 \triangle E} |g_1| + \int_{E_2 \triangle E} |g_2| \leq \int_{(E_1 \cup E_2) \triangle E} |g_1| + \int_{(E_1 \cap E_2) \triangle E} |g_2|.$$

Simplifying[1] the above expression and recalling that $g_i \geq 0$ on E^c, $g_i \leq 0$ on E, we reach

$$0 \leq \int_{(E_2 \setminus E_1) \setminus E} (g_1 - g_2) + \int_{(E_2 \setminus E_1) \cap E} (g_1 - g_2) = \int_{E_2 \setminus E_1} (g_1 - g_2),$$

which implies the contradiction. The case $g_1 \leq g_2$ follows by approximation. □

In the linear case ($g = id$), minimizers of $\mathscr{F}^E_{h,t}$ minimize also the functional

$$F \mapsto P_\phi(F) + \int_F \mathrm{sd}_E / h - f([t/h]h)|F|. \tag{3.10}$$

In the present setting, since $\int_E g(\mathrm{sd}_E)$ may be infinite in the case $a < +\infty$, we can not draw this conclusion straightforwardly. We can nonetheless recover the minimal and the maximal solution to (3.8) by means of a sequence of minimizers of a functional similar to (3.10).

Corollary 3.10. *Let E be a bounded set of finite perimeter and $t \in (0, +\infty), h > 0$. Then, there exists a sequence of uniformly bounded sets $(E_n)_{n \in \mathbb{N}}$ such that $E_n \nearrow T^-_{h,t}E$ and for any $n \in \mathbb{N}$, E_n is a minimizer of*

$$F \mapsto P_\phi(F) + \int_F g\left(\frac{\mathrm{sd}_E}{h}\right) \vee (-n) - f([t/h]h)|F| =: \mathscr{F}^{E,n}_{h,t}(F). \tag{3.11}$$

Analogously, there exists a sequence of uniformly bounded sets $(E_n)_{n \in \mathbb{N}}$ such that $E_n \searrow T^+_{h,t}E$ in L^1 and for any $n \in \mathbb{N}$, E_n is a solution to

$$\min \left\{ P_\phi(F) + \int_{B_R \setminus F} g\left(\frac{\mathrm{sd}_E}{h}\right) \wedge n - f([t/h]h)|F| : F \subseteq B_R \right\}, \tag{3.12}$$

where $T^\pm_{h,t}E \subseteq B_R$.

Proof. We prove the statement for $T^-_{h,t}E$, the other case being analogous. Assume $a < +\infty$ (otherwise the result follows by the boundedness of $T^-_{h,t}E$). We set $c = f([t/h]h)$, $g_n := g(\mathrm{sd}_E/h) \vee (-n)$, and $E' = T^-_{h,t}E$. Consider the sequence of sets $(E_n)_{n \in \mathbb{N}}$, each being the minimal minimizer of $\mathscr{F}^{E,n}_{h,t}$. By the same arguments recalled above, note that that there exists a constant $R > 0$ such that $E_n \subseteq B_R$ for all $n \in \mathbb{N}$. By Lemma 3.9, the sequence E_n is increasing as $g_n \geq g_{n+1}$ and moreover $E' \supseteq E_n$ as $g \leq g_n$. Therefore, one has that $E_n \nearrow \tilde{E} := \bigcup_n E_n \subseteq E'$ and also $\chi_{E_n \triangle E'} = |\chi_{E_n} - \chi_{E'}| \to \chi_{\tilde{E} \triangle E'}$ a.e. as $n \to \infty$. By lower semicontinuity of the perimeter and

[1] Noting that

$$E_1 \triangle E = ((E_1 \setminus E_2) \setminus E) \cup ((E_1 \cap E_2) \setminus E) \cup ((E \setminus E_1) \setminus E_2) \cup ((E \cap E_2) \setminus E_1)$$
$$(E_1 \cup E_2) \triangle E = (E_2 \setminus E_1 \setminus E) \cup ((E_1 \cap E_2) \setminus E) \cup ((E_1 \setminus E_2) \setminus E) \cup ((E \setminus E_1) \setminus E_2)$$
$$(E_1 \cap E_2) \triangle E = ((E_2 \cap E_1) \setminus E) \cup ((E \setminus E_1) \setminus E_2) \cup ((E \cap E_1) \setminus E_2) \cup ((E \cap E_2) \setminus E_1).$$

Fatou's lemma we get

$$\begin{aligned}\mathscr{F}^E_{h,t}(\tilde E) &= P_\phi(\tilde E) - c|\tilde E| + \int_{\tilde E \triangle E'} |g(\mathrm{sd}_E/h)| = P_\phi(\tilde E) - c|\tilde E| + \int_{\mathbb{R}^N} \liminf_{n\to\infty} (|g_n|\chi_{E_n\triangle E}) \\ &\le \liminf_{n\to\infty}\left(P_\phi(E_n) - c|E_n| + \int_{E_n\triangle E} |g_n|\right).\end{aligned}$$

Since E_n minimizes $\mathscr{F}^{E,n}_{h,t}$ we get

$$\mathscr{F}^E_{h,t}(\tilde E) \le \liminf_n \left(P_\phi(E') + \int_{E'\triangle E} |g_n| - c|E'|\right) \le \mathscr{F}^E_{h,t}(E'), \tag{3.13}$$

where in the last inequality we used that $|g_n| \le |g|$. Since E' is the minimal minimizer of $\mathscr{F}^E_{h,t}$ we conclude $\tilde E = E'$. The functional (3.11) is obtained from (3.7) adding $\int_E g_n(\mathrm{sd}_E/h)$. Finally, the functional in (3.12) is obtained from functional (3.7) adding the (finite) term $-\int_{B_R\setminus E} g(\mathrm{sd}_E/h)\wedge n$ and restricting the family of competitors. □

We define the discrete flow starting from the initial set E_0 by setting $E^{(h)}_t = E_0$ for $t \in [0, h)$ and iteratively

$$E^{(h)}_t = T_{h,t-h} E^{(h)}_{t-h}, \qquad t \in [h, +\infty). \tag{3.14}$$

We now provide an estimate on the evolution speed of balls. It is interesting to note that, in the isotropic setting ($\psi = \phi = |\cdot|$) and under the hypothesis of strict monotonicity of G, an explicit evolution law for the radii of evolving balls can be obtained. In our more general case we need to employ the variational proofs of [56]. By Lemma 3.7, the relevant case is when $b = +\infty$.

Lemma 3.11. *Assume $b = +\infty$. For every $R > 0$ and every $t \in (0, +\infty), h > 0$ it holds*

$$T^\pm_{h,t} B_R \subseteq B_{R+\frac{h}{c_\psi}G(\|f\|_\infty)}.$$

Proof. We fix $h > 0$ and set $c := f([t/h]h)$ and $E' = T^\pm_{h,t}B_R$. Let $\varepsilon > 0$ and set $H \subseteq \mathbb{R}^N$ as an half-space containing the ball centered at 0 of radius $R + \frac{h}{c_\psi}G(c+\varepsilon)$. By the minimality of E' we get

$$\int_{E'\triangle B_R} |g(\mathrm{sd}_{B_R}/h)| - \int_{(E'\cap H)\triangle B_R} |g(\mathrm{sd}_{B_R}/h)| \le P_\phi(E'\cap H) - P_\phi(E') + c|E'\setminus H|.$$

By a simple computation, since $B_R \subset H$ we find

$$\int_{E'\setminus H} g(\mathrm{sd}_{B_R}/h) \le P_\phi(E'\cap H) - P_\phi(E') + c|E'\setminus H| \le c|E'\setminus H| \tag{3.15}$$

where in the last inequality we used that cutting sets of finite perimeter by half-spaces decreases P_ϕ. Therefore, since $\mathrm{sd}_{B_R} \ge hG(c+\varepsilon)$ on $E'\setminus H$, one concludes $|E'\setminus H| = 0$. Thus the result follows sending $\varepsilon \to 0$. □

We then provide an upper bound on the evolution speed of balls in the spirit of [56]. We remark that the significant case is $a = +\infty$ as otherwise Lemma 3.7 yields

$$T^\pm_{h,t} B_R \supseteq B_{R-ah}.$$

Lemma 3.12. *Let $R_0 > 0$ and $\sigma > 1$ be fixed. Assume $a = +\infty$. Then, there exist a positive constant c such that, if $h > 0$ is small enough, for all $R \ge R_0$ and $t \in (0, +\infty)$ it holds*

$$T^\pm_{h,t} B_R \supseteq B_{R+\frac{h}{c_\psi}G(-\sigma\frac{c}{R}-\|f\|_\infty)}. \tag{3.16}$$

Proof. We prove the result for $E := T_{h,t}B_R$. Take h small enough so that $T_{h,t}B_{\frac{1}{4}R_0} \ne \emptyset$. By

Lemma 3.11, translation invariance and taking h small, one can see that[2] $B_{\frac{R}{4}} \subseteq E$. We set

$$\bar\rho = \sup\{r \in [0,R] : |B_r \setminus E| = 0\} \in \left[\frac{R}{4}, R + \frac{h}{c_\psi} G(\|f\|_\infty)\right]. \tag{3.17}$$

Assume *wlog* $\bar\rho < R$. Let $\bar x \in \partial B_{\bar\rho}$ be such that $|B(\bar x, \varepsilon) \setminus E| > 0$ for any $\varepsilon > 0$. Set $\rho \in (0, \bar\rho)$ and $\tau = (1 - \rho/\bar\rho)\bar x$ such that $\partial B(\tau, \rho) \cap \partial B_{\bar\rho} = \{\bar x\}$. Setting $B^\varepsilon := ((1+\varepsilon)\tau, \rho)$, consider the sets

$$W^\varepsilon := B^\varepsilon \setminus E.$$

Notice that by construction, for ε small, W^ε has positive measure and it converges to $\{x\}$ as $\varepsilon \to 0$. By (2.11) with $v = \nabla\phi(x/|x|)$ and by submodularity, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} \nabla\phi\left(\frac{x}{|x|}\right) \cdot D\chi_{W^\varepsilon} = \int_{\mathbb{R}^N} \nabla\phi\left(\frac{x}{|x|}\right) \cdot \left(D\chi_{B^\varepsilon} - D\chi_{B^\varepsilon \cap T^\pm_{h,t} B_R}\right) \\ &\le P_\phi(B^\varepsilon \cap T^\pm_{h,t} B_R) - P_\phi(B^\varepsilon) \le P_\phi(T^\pm_{h,t} B_R) - P_\phi(B^\varepsilon \cup T^\pm_{h,t} B_R). \end{aligned} \tag{3.18}$$

Since E minimizes (3.10) (as $a = +\infty$), we use its minimality on the *rhs* of (3.18) and the divergence theorem on the *lhs* of (3.18) to arrive at

$$-\int_{W^\varepsilon} \operatorname{div} \nabla\phi\left(\frac{x}{|x|}\right) \le f([t/h]h)|W_\varepsilon| + \int_{W^\varepsilon} g\left(\frac{\mathrm{sd}_{B_R}}{h}\right). \tag{3.19}$$

By the regularity assumptions on ϕ we remark that it holds

$$|\mathrm{div}\nabla\phi(p)| = |\mathrm{tr}(\nabla^2\phi(p))| \le \frac{c}{|p|}.$$

We plug the estimate above in (3.19), divide by $|W^\varepsilon|$ and send $\varepsilon \to 0$ to conclude

$$-\frac{c}{\rho} - \|f\|_\infty \le \limsup_{s \to c_\psi(\bar\rho - R)/h} g(s).$$

Applying G to both sides and letting $\rho \to \bar\rho$, we conclude

$$\bar\rho \ge R + \frac{h}{c_\psi} G\left(-\frac{c}{\bar\rho} - \|f\|_\infty\right) \ge R + \frac{h}{c_\psi} G\left(-\frac{4c}{R} - \|f\|_\infty\right), \tag{3.20}$$

where in the last inequality we recalled that $\bar\rho \ge R/4$. Using again the previous analysis with the bound (3.20), we show (3.16) by taking h small enough. □

2.3 The scheme for unbounded sets

We now define the discrete evolution scheme for unbounded sets having compact boundary. For every compact set K and $h > 0, t \ge 0$, we will denote by $\tilde T^\pm_{h,t} K$ the maximal and the minimal minimizer of $\tilde{\mathscr{F}}^K_{h,t}$, which corresponds to (3.7) with $\tilde g(s) := -g(-s)$ instead of $g(s)$ and $-f$ instead of f. By changing variable $\tilde F := F^c$ in (3.7), we see that $(\tilde T^-_{h,t} K)^c$ is the maximal solution to

$$\min\left\{P_\phi(\tilde F) + \int_{\tilde F \triangle K^c} |g\left(\mathrm{sd}_{K^c}/h\right)| + f([t/h]h)|\tilde F^c|\right\}. \tag{3.21}$$

[2]Indeed, by translation invariance and Lemma 3.11 it holds

$$T_{h,t} B_{\frac{R}{4}} + B_{\frac{3}{4}R} \subseteq T_{h,t} B_R \subseteq B_{R+ch}.$$

Therefore, for every unbounded set E with compact boundary we define[3]

$$T^{\pm}_{h,t}E := \left(\tilde{T}^{\mp}_{h,t}E^c\right)^c. \tag{3.22}$$

As in the case of compact sets, we set $T_{h,t}E := T^-_{h,t}E$. Given an unbounded set E_0 having compact boundary, we define the discrete flow $\{E^{(h)}_t\}_{t\geq 0}$ as follows: $E^{(h)}_t := E_0$ for $t \in [0,h)$ and

$$E^{(h)}_t = T_{h,t-h}E^{(h)}_{t-h}, \quad \forall t \in [h,+\infty).$$

Since $\tilde{g}$ has the same properties of g, one easily checks that analogous results to Lemmas 3.11, 3.8 and 3.12 hold also for (3.22).

Lemma 3.13. *Let $t,h>0$. The following statements hold.*

- *Let $F_1 \subseteq F_2$ be unbounded sets with compact boundary. Then, $T_{h,t}F_1 \subseteq T_{h,t}F_2$.*
- *There exists $c>0$ such that for every $R>0, h>0$ it holds $T^{\pm}_{h,t}B^c_R \supseteq B^c_{R+ch}$.*
- *Let $R_0>0$ and $\sigma>1$ be fixed. Then, if $a=+\infty$ there exist $c>0$ such that for $h>0$ small enough and for all $R\geq R_0$, it holds*

$$T^{\pm}_{h,t}B^c_R \subseteq B^c_{R+\frac{h}{c_\psi}G(-\sigma\frac{c}{R}-\|f\|_\infty)}. \tag{3.23}$$

If instead $a<+\infty$ it holds

$$T^{\pm}_{h,t}B^c_R \subseteq B^c_{R-ah}. \tag{3.24}$$

Furthermore, Corollary 3.10 implies straightforwardly the following approximation result.

Corollary 3.14. *Set $t,h>0$ and let E be an unbounded set of finite perimeter with bounded complement. Then, there exists two sequences of sets $(E_n)_{n\in\mathbb{N}}, (E'_n)_{n\in\mathbb{N}}$ with uniformly bounded complement with the following property. Each $(E_n)^c$ is a minimizer of (3.21) with $g\vee(-n)$ substituting g, and $(E'_n)^c$ is a minimizer of (3.21) with $g\wedge n$ substituting g. Moreover $E_n \nearrow T^-_{h,t}E$ and $E'_n \searrow T^+_{h,t}E$.*

We now deduce an equivalent version of (3.21), which will be used in the final proof, following [56]. Let us consider E such that $E^c\subseteq B_R$ and assume $a=+\infty$. Recall that $T^{\pm}_{h,t}E\supseteq B^c_{R+ch}$ for some $c>0$ by Lemma 3.13. Adding to the functional in (3.21) the term $\int_{B_{R+ch}\setminus(T^-_{h,t}E)^c} g(\mathrm{sd}_E/h)$ and restricting the family of competitors, we note that $T^-_{h,t}E$ is the minimal solution to

$$\min\left\{P_\phi(\tilde{F}) + \int_{\tilde{F}\cap B_{R+ch}} g\left(\mathrm{sd}_E/h\right) + f([t/h]h)|\tilde{F}^c| : \tilde{F}^c\subseteq B_{R+ch}\right\}. \tag{3.25}$$

The case $a<+\infty$ needs to be treated by approximation using Corollary 3.14. Lastly, we state a comparison principle between bounded and unbounded sets. Its proof follows the one of [56, Lemma 6.10], up to employing Corollary 3.14.

Lemma 3.15. *Let E_1 be a compact set and let E_2 be an open, unbounded set with compact boundary, and such that $E_1\subseteq E_2$. Then, for every $h\in(0,1), t\geq 0$ it holds $T^{\pm}_{h,t}E_1 \subseteq T^{\pm}_{h,t}E_2$.*

3 Main result

We now describe the discrete-in-time approximation of the viscosity solution based on the operators $T^{\pm}_{h,t}$ previously defined. In this section we essentially follow [56], as done in Chapter 2.

[3]To justify this, one can check that if a set E is moving according to (3.2), its complement moves according to

$$V(x,t) = -\psi(\nu_{E^c}(x))G(\kappa_{E^c}(x)+\mathtt{f}) \quad \text{in the direction } \nu_{E^c},$$

from which the incremental problem follows.

Given a continuous function $v : \mathbb{R}^N \to \mathbb{R}$ which is constant outside a compact set, we define the transformation

$$T_{h,t}v(x) = \sup\{s \in \mathbb{R} \ : \ x \in T_{h,t}\{v \geq s\}\}, \tag{3.26}$$

which defines a new function on $\mathbb{R}^N \times [0, +\infty)$ by setting $v_h(x,t) = v(x)$ for $t \in [0,h)$ and

$$v_h(x,t) := (T_{h,t-h}v_h(\cdot, t-h))(x). \tag{3.27}$$

By lemmas 3.8 and 3.13, one can see that the operator $T_{h,t}$ maps functions into functions. Moreover, the following holds.

Lemma 3.16. *Given $t, h > 0$, the operator $T_{h,t}$ defined in* (3.26) *satisfies the following properties:*

- *$T_{h,t}$ is monotone, meaning that $u_0 \leq v_0$ implies $T_{h,t}u_0 \leq T_{h,t}v_0$;*
- *$T_{h,t}$ is translation invariant, as for any $z \in \mathbb{R}^N$, setting $\tau_z u_0(x) := u_0(x - z)$, it holds $T_{h,t}(\tau_z u_0) = \tau_z(T_{h,t}u_0)$;*
- *$T_{h,t}$ commutes with constants, meaning $T_{h,t}(u + c) = (T_{h,t}u) + c$ for every $c \in \mathbb{R}$.*

Proof. The first assertion follows from Lemma 3.8 and 3.13. The second one follows easily employing the definition (3.26), recalling the fact that the functional defined in (3.7) is invariant under translations and that $\{\tau_z u_0 \geq \lambda\} = \{u_0 \geq \lambda\} + z$ for all $\lambda \in \mathbb{R}$. The last result follows analogously. □

The previous properties satisfied by the operator, in turn, preserve the continuity in space of the initial function. Indeed, assume u_0 is uniformly continuous and let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing, continuous modulus of continuity for u_0. Then, for any $s > s'$ we have

$$\{u > s\} + B_{\omega^{-1}(s-s')} \subseteq \{u > s'\},$$

thus, by translation invariance we deduce

$$T_{h,t}\{u > s\} + B_{\omega^{-1}(s-s')} \subseteq T_{h,t}\{u > s'\}.$$

This inclusion implies that the function $T_{h,t}u_0$ is uniformly continuous in space, with the same modulus of continuity ω of u_0. The following lemma provides an estimate on the continuity in time of u_h. Here, equality between sets must be understood up to negligible sets.

Lemma 3.17. *Fix $t, h > 0$ and u_0 a uniformly continuous function. For all $\lambda \in \mathbb{R}$ it holds*

$$T_{h,t}\{u_h(\cdot,t) > \lambda\} = \{u_h(\cdot, t+h) > \lambda\}, \quad T^+_{h,t}\{u_h(\cdot,t) \geq \lambda\} = \{u_h(\cdot, t+h) \geq \lambda\}.$$

Proof. Given $\varepsilon > 0$, by definition it is easy to see that

$$\{T_{h,0}u_0 > \lambda + \varepsilon\} \subseteq T^{\pm}_{h,0}\{u_0 > \lambda\} \subseteq \{T_{h,0}u_0 > \lambda - \varepsilon\}.$$

Passing to the limit $\varepsilon \to 0$, we deduce

$$\{u_h(\cdot,h) \geq \lambda\} \subseteq T^{\pm}_{h,0}\{u_0 > \lambda\} \subseteq \{u_h(\cdot,h) \geq \lambda\}.$$

Finally, since $u_h(\cdot,h)$ is a continuous function, the equalities $\{u_h(\cdot,h) > \lambda\} = \mathrm{int}\{u_h(\cdot,h) \geq \lambda\}$ and $\{u_h(\cdot,h) \geq \lambda\} = \overline{\{u_h(\cdot,h) \geq \lambda\}}$ holds and we prove the result for $t = h$. The other cases follow by iteration. □

With the previous results and reasoning exactly as in [56, Lemma 6.13], we can prove that the functions u_h are uniformly continuous in time.

Lemma 3.18. *For any $\varepsilon > 0$, there exists $\tau > 0$ and $h_0 = h_0(\varepsilon) > 0$ such that for all $|t - t'| \leq \tau$ and $h \leq h_0$ we have $|u_h(\cdot,t) - u_h(\cdot,t')| \leq \varepsilon$.*

Thus, the family $\{u_h\}_{h>0}$ is equicontinuous and uniformly bounded as implied by Lemma 3.11. By the Ascoli-Arzelà theorem we can pass to the limit $h \to 0$ (up to subsequences) to conclude that $u_h \to u$ uniformly in any compact in time subset of $\mathbb{R}^N \times [0, +\infty)$, with u being a uniformly continuous function. Moreover, the function u is bounded and constant outside a compact set.

Proposition 3.19. *Let $T > 0$. Up to a subsequence, the family $\{u_h\}_{h>0}$ converges uniformly on $\mathbb{R}^N \times [0, T]$ to a uniformly continuous function u, which is bounded and constant out of a compact set.*

We can thus state the main result of the present chapter.

Theorem 3.20. *The function u defined in Proposition 3.19 coincides with the unique continuous viscosity solution of the Cauchy problem* (3.4).

We finally recall the notion of a level-set solution to the evolution equation (3.2) (cp. e.g. [101]).

Definition 3.21. Given an initial bounded set E_0 (or unbounded set with bounded complement) define an uniformly continuous function $u_0 : \mathbb{R}^N \to \mathbb{R}$ such that $\{u_0 > 0\} = E_0$. Then, setting u as the solution to (3.4) with initial datum u_0 given by Theorem 3.20, we define the level-set solution to the nonlinear mean curvature evolution (3.2) of E_0 as

$$E_t := \{u(\cdot, t) > 0\}.$$

Our result, Theorem 3.20 amounts thus in showing that the discrete flow converges to the unique level set solution to equation (3.1).

3.1 Proof of the main result

We start by an estimate on the evolution speed. For every $r > 0$, using the notation of Lemma 3.12, we set

$$\hat{\kappa}(r) = \min\left\{-1, \frac{1}{c_\psi} G\left(-\frac{c}{r} - \|f\|_\infty\right)\right\}$$

and, given $r_0 > 0$, we set $r(t)$ as the unique solution to

$$\begin{cases} \dot{r}(t) = \hat{\kappa}(r(t)) \\ r(0) = r_0. \end{cases} \tag{3.28}$$

Note that, in general, the solution $r(t)$ will exist in a finite time interval $[0, T^*(r_0)]$, where $T^*(r_0)$ denotes the extinction time of the solution starting from r_0 i.e. the first time t such that $r(t) = 0$.

Lemma 3.22. *Let u be the function given by Proposition 3.19 and assume that there exists $\lambda \in \mathbb{R}$ such that $B(x_0, r_0) \subseteq \{u(\cdot, t_0) > \lambda\}$. Then, if $a = +\infty$, it holds*

$$B(x_0, r(t - t_0)) \subseteq \{u(\cdot, t) > \lambda\}$$

for every $t \leq T^*(r_0) + t_0$, where $r(t)$ is the solution to (3.28) *with extinction time $T^*(r_0)$. If instead $a < +\infty$ it holds*

$$B(x_0, r_0 - a(t - t_0)) \subseteq \{u(\cdot, t) > \lambda\}$$

for all t such that $r_0 - a(t - t_0) \geq 0$. The same result holds for sublevels substituting superlevel sets.

Proof. The result in the case $a < +\infty$ follows directly by Lemma 3.7, so we assume $a = +\infty$. We consider *wlog* $\{u(\cdot, t_0) > \lambda\}$ bounded, as the other case is analogous. For a fixed $R_0 < r_0$, taking $h(R_0)$ small enough, we can ensure that $B(x_0, R_0) \subseteq \{u_h(\cdot, t_0) > \lambda\}$. We then fix $\sigma > 1$ and define recursively the radii R_n by

$$R_{n+1} = R_n + \frac{h}{c_\psi} G\left(-\sigma \frac{C_\phi}{R_n} - \|f\|_\infty\right).$$

By Lemmas 3.8, 3.12 and 3.17, we see that $B(x_0, R_{[(t-t_0)/h]+1}) \subseteq \{u(\cdot,t) > \lambda\}$ for every $t \geq t_0$ such that $R_{[(t-t_0)/h]+1} > 0$. Let then r_σ be the unique solution to the ODE

$$\begin{cases} \dot{r}_\sigma(t) = \hat{\kappa}(r_\sigma(t)/\sigma) \\ r_\sigma(0) = R_0. \end{cases} \tag{3.29}$$

Employing the monotonicity of $\hat{\kappa}$, if $r_\sigma(t) \leq R_n$, then

$$\begin{aligned} r_\sigma((n+1)h) &\leq R_n + \int_{nh}^{(n-1)h} \hat{\mathrm{H}}\left(\frac{r_\sigma(s)}{\sigma}\right) \mathrm{d}s \leq R_n + \int_{nh}^{(n-1)h} \hat{\mathrm{H}}\left(\frac{R_n}{\sigma}\right) \mathrm{d}s \\ &\leq R_n + \int_{nh}^{(n-1)h} \frac{1}{c_\psi} G\left(-\sigma\frac{C_\phi}{R_n} - \|f\|_\infty\right) \mathrm{d}s = R_{n+1}. \end{aligned}$$

Therefore, $B(x_0, r_\sigma(h[(t-t_0)/h] + h) \subseteq \{u_h(\cdot,t) > \lambda\}$ for $t \geq t_0$ as long as the radius is positive. We conclude sending $h \to 0$, then $R_0 \to r_0$ and $\sigma \to 1$. □

We are now in the position to prove our main result, reasoning as in [56] (and Chapter 2).

Proof of Theorem 3.20. Consider u as defined in (3.19): we show that u is a subsolution, as proving that it is a supersolution is analogous. Let $\eta(x,t)$ be an admissible test function in $\bar{z} := (\bar{x}, \bar{t})$ and assume that $(\bar{x}, \bar{t})$ is a strict maximum point for $u - \eta$. Assume furthermore that $u - \eta = 0$ in such point.

Case 1: We assume that $\nabla\eta(\bar{z}) \neq 0$. Firstly, in the case $a < +\infty$ we remark that if $\partial_t\eta/\psi(\nabla\eta(\hat{z})) \leq -a$, then (3.6) is trivially satisfied, thus we can assume *wlog* that

$$\frac{\partial_t\eta(\bar{z})}{\psi(\nabla\eta(\hat{z}))} > -a. \tag{3.30}$$

By classical arguments (recalled in Chapter 2) we can assume that each function $u_{h_k} - \eta$ assumes a local supremum in $B_\rho(\bar{z})$ at a point $z_{h_k} =: (x_k, t_k)$ and that $u_{h_k}(z_{h_k}) \to u(\bar{z})$ as $k \to \infty$. Moreover, we can assume that $\nabla\eta(z_k) \neq 0$ for k large enough.

Step 1: We define a suitable competitor for the minimality of the level sets of u_h. By the previous remarks we have that

$$u_h(x,t) \leq \eta(x,t) + c_k \tag{3.31}$$

where $c_k := u_{h_k}(x_k, t_k) - \eta(x_k, t_k)$, with equality if $(x,t) = (x_k, t_k)$. Let $\sigma > 0$ and set

$$\eta^\sigma_{h_k}(x) := \eta(x, t_k) + c_k + \frac{\sigma}{2}|x - x_k|^2.$$

Then, for all $x \in \mathbb{R}^N$,

$$u_{h_k}(x, t_k) \leq \eta^\sigma_{h_k}(x)$$

with equality if and only if $x = x_k$. We set $l_k = u_{h_k}(x_k, t_k) = \eta^\sigma_{h_k}(x_k)$. We fix $\varepsilon > 0$, to be chosen later, and define $E^k_\varepsilon := \{u_{h_k}(\cdot, t_k) > l_k - \varepsilon\} = T_{h_k, t_k - h_k}\{u_{h_k}(\cdot, t_k - h_k) > l_k - \varepsilon\}$[4] and

$$W^k_\varepsilon := E^k_\varepsilon \setminus \left\{\eta^\sigma_{h_k}(\cdot) > l_k + \varepsilon\right\}. \tag{3.32}$$

Assume that E^k_ε is bounded and let us define $E^k_{\varepsilon,n}$ as the sets constructed by Corollary 3.10 where $\{u_{h_k}(\cdot, t_k - h_k)\ > l_k - \varepsilon\}, E^k_\varepsilon$ substitute $E, T^-_{h,t}E$ respectively. We thus have that $E^k_{\varepsilon,n} \nearrow E^k_\varepsilon$ as $n \to \infty$ and that each $E^k_{\varepsilon,n}$ is the minimal minimizer of a problem in the form (3.10). We define

$$W^k_{\varepsilon,n} := E^k_{\varepsilon,n} \setminus \left\{\eta^\sigma_{h_k}(\cdot) > l_k + \varepsilon,\right\}. \tag{3.33}$$

It is easy to see that, along any subsequence $n(\varepsilon) \to \infty$ as $\varepsilon \to 0$, it holds $W^k_{\varepsilon,n(\varepsilon)} \to \{x\}$ as $\varepsilon \to 0$. Furthermore, we check that for every $\varepsilon, k > 0$ there exists $n(\varepsilon, k)$ large enough such that

[4] The choice of working with the open superlevel sets is motivated by our need to employ (3.11)

$|W^k_{\varepsilon,n}| > 0$ for all $n \geq n(\varepsilon, k)$. Indeed, by the continuity of η^σ and since $|\nabla\eta(\bar{z})| \neq 0$ there exists a positive radius r such that

$$(B(x_k, r) \cap E^k_\varepsilon) \subseteq W^k_\varepsilon.$$

Since $x_k \in E^k_\varepsilon$ and it is an open set, it holds $|W^k_\varepsilon| > 0$. Recalling that $E^k_{\varepsilon,n} \to E^k_\varepsilon$ in L^1, we conclude that $|W^k_{\varepsilon,n}| > 0$ for all $n = n(\varepsilon, k)$ large enough. Note also that, for every fixed k, $n(\varepsilon, k) \to \infty$ as $\varepsilon \to 0$.

By minimality of $E^k_{\varepsilon,n}$ we have

$$\begin{aligned} &P_\phi(E^k_{\varepsilon,n}) + \int_{E^k_{\varepsilon,n}} g\left(\mathrm{sd}_{\{u_{h_k}(\cdot,t_k-h_k)>l_k-\varepsilon\}}(x)/h_k\right) \vee (-n)\,\mathrm{d}x - f\left(\left[\frac{t}{h_k}\right] h_k\right) |W^k_{\varepsilon,n}| \\ &\leq P_\phi\left(E^k_{\varepsilon,n} \cap \{\eta^\sigma_{h_k} > l_k\}\right) + \int_{E^k_{\varepsilon,n}\cap\{\eta^\sigma_{h_k}>l_k\}} g\left(\mathrm{sd}_{\{u_{h_k}(\cdot,t_k-h_k)>l_k-\varepsilon\}}(x)/h_k\right) \vee (-n)\,\mathrm{d}x. \end{aligned} \tag{3.34}$$

Adding to both sides $P_\phi\left(\{\eta^\sigma_{h_k} > l_k\} \cup E^k_{\varepsilon,n}\right)$ and using the submodularity of the perimeter, we obtain

$$\begin{aligned} P_\phi(\{\eta^\sigma_{h_k} > l_k + \varepsilon\} \cup W^k_{\varepsilon,n}) - P_\phi(\{\eta^\sigma_{h_k} > l_k + \varepsilon\}) - f\left(\left[\frac{t}{h_k}\right] h_k\right) |W^k_{\varepsilon,n}| \\ + \int_{W^k_{\varepsilon,n}} g\left(\mathrm{sd}_{\{u_{h_k}(\cdot,t_k-h_k)>l_k-\varepsilon\}}(x)/h_k\right) \vee (-n)\,\mathrm{d}x \leq 0. \end{aligned}$$

Equation (3.31) implies $\{u_{h_k}(\cdot, t_k - h_k) > l_k - \varepsilon\} \subseteq \{\eta(\cdot, t_k - h_k) > l_k - c_k - \varepsilon\}$, therefore by monotonicity we get

$$\begin{aligned} P_\phi(\{\eta^\sigma_{h_k} > l_k + \varepsilon\} \cup W^k_{\varepsilon,n}) - P_\phi(\{\eta^\sigma_{h_k} > l_k + \varepsilon\}) - f\left(\left[\frac{t}{h_k}\right] h_k\right) |W^k_{\varepsilon,n}| \\ + \int_{W^k_{\varepsilon,n}} g\left(\mathrm{sd}_{\{\eta(\cdot,t_k-h_k)>l_k-c_k-\varepsilon\}}(x)/h_k\right) \vee (-n)\,\mathrm{d}x \leq 0. \end{aligned} \tag{3.35}$$

If instead E^k_ε is an unbounded set with compact boundary, we employ (3.25) instead of (3.34) to obtain (3.35) in the computations above. See [56] and Chapter 2 for details.

Step 2: We now estimate the terms appearing in (3.35). We start with the first two perimeter terms $P_\phi(\{\eta^\sigma_{h_k} > l_k + \varepsilon\} \cup W^k_{\varepsilon,n}) - P_\phi(\{\eta^\sigma_{h_k} > l_k + \varepsilon\})$. Reasoning as in Lemma 3.12, we use the divergence theorem and (2.11) with the vector field $v := \nabla\phi(\nabla\eta^\sigma/|\nabla\eta^\sigma|)$ to obtain

$$\begin{aligned} &P_\phi(\{\eta^\sigma_{h_k} > l_k + \varepsilon\} \cup W^k_{\varepsilon,n}) - P_\phi(\{\eta^\sigma_{h_k} \geq l_k + \varepsilon\}) \\ &\qquad \geq \int_{\partial(\{\eta^\sigma_{h_k}>l_k+\varepsilon\}\cup W^k_{\varepsilon,n})} v \cdot \nu - \int_{\partial\{\eta^\sigma_{h_k}>l_k+\varepsilon\}} v \cdot \nu = \int_{W^k_{\varepsilon,n}} \operatorname{div} v, \end{aligned} \tag{3.36}$$

where ν denotes the unit outer vector to the set we are integrating on.

The last term in (3.35) can be treated as follows. For any $z \in W_\varepsilon$, we have

$$\eta(z, t_k) + c_k + \frac{\sigma}{2}|z - x_k|^2 \leq l_k + \varepsilon. \tag{3.37}$$

Since, in turn, $\eta(z, t_k) + c_k > l_k - \varepsilon$ it follows that $\sigma|z - x_k|^2 < 4\varepsilon$ and thus, for ε small enough,

$$W^k_\varepsilon \subseteq B_{c\sqrt{\varepsilon}}(x_k). \tag{3.38}$$

Therefore, by Hausdorff convergence it holds that for every $\varepsilon, k > 0$ there exists $n = n(\varepsilon, k)$ large enough such that

$$W^k_{\varepsilon,n} \subseteq B_{2c\sqrt{\varepsilon}}(x_k). \tag{3.39}$$

On the other hand, by a Taylor expansion, for every $z \in W^k_{\varepsilon,n}$ we have

$$\eta(z, t_k - h_k) = \eta(z, t_k) - h_k \partial_t \eta(z, t_k) + h_k^2 \int_0^1 (1-s)\partial^2_{tt}\eta(z, t_k - sh_k)\,\mathrm{d}s. \tag{3.40}$$

Then, we consider $y \in \{\eta(\cdot, t_k - h_k)(y) = l_k - c_k - \varepsilon\}$ being a point of minimal ψ-distance from z, that is, $\psi(z-y) = |\mathrm{sd}_{\{\eta(\cdot,t_k-h_k)(y)>l_k-c_k-\varepsilon\}}(z)|$. One can prove (see Chapter 2 for details) that

$$|z - y| = O(h_k). \tag{3.41}$$

Moreover, it holds (see [56, eq (6.26)] for details)

$$(z-y)\cdot \frac{\nabla\eta(y, t_k - h_k)}{|\nabla\eta(y, t_k - h_k)|} = \pm\psi\left(\frac{\nabla\eta(y, t_k - h_k)}{|\nabla\eta(y, t_k - h_k)|}\right) \mathrm{dist}^{\psi}_{\{\eta(\cdot,t_k-h_k)(y)=l_k-c_k-\varepsilon\}}(z),$$

with a "+" if $z \in \{\eta(\cdot, t_k - h_k)(y) \le l_k - c_k - \varepsilon\}$ and a "-" otherwise. We get

$$\begin{aligned}
\eta(z, t_k - h_k) &= \eta(y, t_k - h_k) + (z-y)\cdot\nabla\eta(y, t_k - h_k) \\
&\quad + \int_0^1 (1-s)\left(\nabla^2\eta(y + s(z-y), t_k - h_k)(z-y)\right)\cdot(z-y)\,\mathrm{d}s \\
&= l_k - c_k - \varepsilon - \mathrm{sd}_{\{\eta(\cdot,t_k-h_k)(y)=l_k-c_k-\varepsilon\}}(z)\psi(\nabla\eta(y, t_k - h_k)) \\
&\quad + \int_0^1 (1-s)\left(\nabla^2\eta(y + s(z-y), t_k - h_k)(z-y)\right)\cdot(z-y)\,\mathrm{d}s.
\end{aligned} \tag{3.42}$$

Note that, in view of (3.37) it holds $|\eta(z, t_k) - \eta(y, t_k)| \le c\varepsilon + ch_k = O(h_k)$, provided $\varepsilon \ll h_k$ and small enough. Thus, using also (3.39),(3.41) we deduce

$$\begin{aligned}
\frac{1}{h_k}\mathrm{sd}_{\{\eta(\cdot,t_k-h_k)>l_k-c_k-\varepsilon\}}(z) &\ge \frac{\partial_t\eta(z, t_k) - \frac{2\varepsilon}{h_k} - O(h_k) - O_{h_k}(1)}{\psi(\nabla\eta(y, t_k - h_k))} \\
&= \frac{\partial_t\eta(x_k, t_k) + O(\sqrt{\varepsilon}) - \frac{2\varepsilon}{h_k} - O(h_k) - O_{h_k}(1)}{\psi(\nabla\eta(x_k, t_k - h_k)) + O(\sqrt{\varepsilon}) + O(h_k)},
\end{aligned}$$

and we apply g to both sides to conclude

$$g\left(\mathrm{sd}_{\{\eta(\cdot,t_k-h_k)>l_k-c_k-\varepsilon\}}(z)/h_k\right) \ge g\left(\frac{\partial_t\eta(x_k, t_k) - O_{h_k}(1)}{\psi(\nabla\eta(x_k, t_k - h_k)) + O(h_k)}\right) \tag{3.43}$$

Step 4: We conclude the proof. Combining (3.35), (3.36) and (3.43), we arrive at

$$0 \ge \int_{W^k_{\varepsilon,n}} \mathrm{div}\, v + |W^k_{\varepsilon,n}|\left(-f\left(\left[\frac{t}{h_k}\right]h_k\right) + g\left(\frac{\partial_t\eta(x_k, t_k) - O_{h_k}(1)}{\psi(\nabla\eta(x_k, t_k - h_k)) + O(h_k)}\right) \vee (-n)\right). \tag{3.44}$$

Choosing $n = n(\varepsilon, k)$, we can divide by $|W^k_{\varepsilon,n(\varepsilon,k)}| > 0$ and apply G to both sides to get

$$G\left(-\fint_{W^k_{\varepsilon,n(\varepsilon,k)}} \mathrm{div}\, v + f\left(\left[\frac{t}{h_k}\right]h_k\right)\right) \ge G\left(g\left(\frac{\partial_t\eta(x_k, t_k) - O_{h_k}(1)}{\psi(\nabla\eta(x_k, t_k - h_k)) + O(h_k)}\right) \vee (-n(\varepsilon, k))\right).$$

Let us fix $k > 0$ and send $\varepsilon \to 0$ (thus also $n(\varepsilon, k) \to 0$). Thanks to the continuity of G and recalling also that $W^k_{\varepsilon,n(\varepsilon,k)} \to \{x\}$ as $\varepsilon \to 0$, we arrive at

$$G\left(-\kappa^{\phi}_{\{\eta^{\sigma}_{h_k} \ge \eta^{\sigma}_{h_k}(x_k)\}}(x_k) + f\left(\left[\frac{t}{h_k}\right]h_k\right)\right) \ge \frac{\partial_t\eta(x_k, t_k) - O_{h_k}(1)}{\psi(\nabla\eta(x_k, t_k)) + O(h_k)},$$

which finally implies the thesis by letting simultaneously $\sigma \to 0$ and $k \to +\infty$.

Case 2: We assume $\nabla\eta(\bar{x}, \bar{t}) = 0$ and prove that $\partial_t\eta(\bar{x}, \bar{t}) \le 0$. The proof follows the line of the one in [56]. We focus on the case $a = +\infty$, the other being simpler.

Since $\nabla\eta(\bar z) = 0$, there exist $\ell \in \mathcal{F}$ and $\omega \in C^\infty(\mathbb{R})$ with $\omega'(0) = 0$ such that

$$|\eta(x,t) - \eta(\bar z) - \partial_t\eta(\bar z)(t - \bar t)| \le \ell(|x - \bar x|) + \omega(|t - \bar t|)$$

thus, we can define

$$\tilde\eta(x,t) = \partial_t\eta(\bar z)(t - \bar t) + 2\ell(|x - \bar x|) + 2\omega(|t - \bar t|)$$
$$\tilde\eta_k(x,t) = \tilde\eta(x,t) + \frac{1}{k(\bar t - t)}.$$

We remark that $u - \tilde\eta$ achieves a strict maximum in $\bar z$ and the local maxima of $u - \tilde\eta_k$ in $\mathbb{R}^N \times [0, \bar t]$ are in points $(x_k, t_k) \to \bar z$ as $k \to \infty$, with $t_n \le \bar t$. From now on, the only difference from [56] is in the case $x_k = \bar x$ for an (unrelabeled) subsequence. We thus assume $x_k = \bar x$ for all $k > 0$ and define $b_k = \bar t - t_k > 0$ and the radii

$$r_k := \ell^{-1}(a_k b_k),$$

where $a_k \to 0$ must be chosen such that the extinction time for the solution of (3.28) satisfies $T^*(r_k) \ge \bar t - t_k$, for k large enough. To show that such a choice for a_k is premarkible, we set

$$\beta(t) = \sup_{0 \le s \le t} \hat{\mathrm{H}}(\ell^{-1}(s))\ell'(\ell^{-1}(s)), \tag{3.45}$$

where $\hat\kappa$ is as in (3.28). Note that by Definition 3.1 it holds $\beta(t) \le \hat{\mathrm{H}}(t)$ for t small, β is non decreasing in t and $g(t) \to 0$ as $t \to 0$. We then have

$$\begin{aligned}\frac{T^*(r_k)}{b_k} &\ge \frac{1}{b_k}\int_{r_k/2}^{r_k} \frac{1}{\hat\kappa(s)}\,\mathrm{d}s = \frac{1}{b_k}\int_{\ell^{-1}(a_k b_k/2)}^{\ell^{-1}(a_k b_k)} \frac{1}{\hat\kappa(s)}\,\mathrm{d}s \\ &= \frac{a_k}{2} \fint_{a_k b_k/2}^{a_k b_k} \frac{1}{\hat\kappa(\ell^{-1}(r))\ell'(\ell^{-1}(r))}\,\mathrm{d}r \ge \frac{a_k}{2}\frac{1}{\beta(b_k)} = 2,\end{aligned} \tag{3.46}$$

where in the last equality we chose $a_k := 4\beta(b_k)$ which tends to 0 as $k \to \infty$.

By definition of $\tilde\eta_k$ it holds

$$\begin{aligned}B(\bar x, r_k) &\subseteq \{\tilde\eta_k(\cdot, t_k) \le \tilde\eta_k(\bar x, t_k) + 2\ell(r_k)\} \\ &\subseteq \{u(\cdot, t_k) \le u(\bar x, t_k) + 2\ell(r_k)\},\end{aligned}$$

by maximality of $u - \tilde\eta_k$ at z_k and since $u(z_k) = \tilde\eta_k(z_k)$. Since the balls $B(\cdot, r_k)$ are not vanishing, by Lemma 3.22 we have

$$\bar x \in \{u(\cdot, \bar t) \le u(\bar x, t_k) + 2\ell(r_k)\}. \tag{3.47}$$

Finally, using again the maximality of $u - \eta$ at $\bar z$, the choice of r_k and (3.47), we obtain

$$\frac{\eta(\bar z) - \eta(\bar x, t_k)}{\bar t - t_k} = \frac{\eta(\bar z) - \eta(\bar x, t_k)}{b_k} \le \frac{u(\bar z) - u(\bar x, t_k)}{b_k} \le \frac{2\ell(r_k)}{b_k} = 2a_k.$$

Passing to the limit $k \to \infty$, we conclude that $\partial_t\eta(\bar z) \le 0$. □

Chapter 4

Discrete-to-continuous crystalline curvature flows

Contents

1 Introduction

In this chapter we analyse a space- and time-discrete approximation of crystalline mean curvature flows of the form

$$V(x,t) = -\phi(\nu_{E(t)}(x))\mathrm{H}^{\phi}_{E(t)}(x), \qquad x \in \partial E(t), \quad t \geq 0, \tag{4.1}$$

for a class of crystalline norms ϕ. We recall that an anisotropy ϕ is said to be crystalline if and only if $\{\phi \leq 1\}$ is a polytope (or equivalently, ϕ is the support function of a polytope). Moreover, in this chapter we restrict ourselves to the case where $\{\phi \leq 1\}$ is a zonotope with rational generators [150, 26]. The evolution law (4.1) has been considered to describe some phenomena in materials science and crystal growth; see e.g. [115, 175]. The main result of this chapter is a convergence result of the discrete approximation to the continuous evolution, as the time and space steps go to zero, even in the somewhat surprising case where the space-step is greater or equal to the time-step.

From the mathematical point of view, the lack of regularity of the differential operator involved in the definition of the crystalline curvature (see [22, 23]) is the main reason why the well-posedness of the crystalline mean curvature flow in every dimension has been a long-standing open problem. After some partial results (see for instance [9, 16, 21, 41, 99, 100, 103]), important breakthroughs have been obtained simultaneously in [104, 105, 107], where a suitable crystalline theory of viscosity solutions was developed, and with a different approach in [55, 53, 52], where a new notion of distributional solutions was proposed.

Let us focus on the definition of distributional solutions, referring to the nice review [106] for further information on viscosity solutions to (4.1) (we just note that the two notions are equivalent in the present setting [52, Remark 6.1]). The exact definition of distributional solutions will be recalled in Definition 4.1, but when ϕ is smooth it can be motivated as follows: It is known (see for instance [173] for the isotropic case) that $E(t)$ evolves according to (4.1) if and only if the signed distance function $d(\cdot,t) := \mathrm{sd}^{\phi^\circ}_{E(t)}$ to $\partial E(t)$ induced by the polar norm[1] ϕ°, satisfies

$$\partial_t d \geq \operatorname{div}(\nabla\phi(\nabla d)) \quad \text{in } \{d > 0\}, \tag{4.2}$$

$$\partial_t d \leq \operatorname{div}(\nabla\phi(\nabla d)) \quad \text{in } \{d < 0\} \tag{4.3}$$

in the viscosity sense. The idea of the new definition introduced in [55] is to reinterpret the equations above in the distributional sense. In particular, note that replacing $\nabla\phi(\nabla u)$ by a vector field $z \in L^\infty(\{d > 0\}; \mathbb{R}^N)$ such that $z(x) \in \partial\phi(\nabla d)$ for a.e. x, where $\partial\phi$ denotes the subdifferential of ϕ, the equations (4.2), (4.3) make sense even when ϕ is crystalline. The corresponding notion of super- and sub-solutions bears a comparison principle, which yields uniqueness of the motion up to fattening. Existence is obtained either by a variant of the minimizing movements scheme of [8, 144] in the spirit of [45], which consists in building a discrete-in-time evolution obtained by a recursive minimization procedure (see [55, 52]), or by approximation with smooth anisotropies [53]. We observe that the convergence of such time discrete approaches to a motion characterized by (4.2)-(4.3) in the *viscosity sense* was shown in [129], including in the 2D crystalline setting, while convergence in a distributional sense was established in [41] in the convex case only. Briefly, given a time-step $h > 0$ and an initial closed set $E_0 =: E^{h,0}$, one defines $E^{h,k+1} = \{u^{h,k+1} \leq 0\}$, where $u^{h,k+1}$ is defined as the minimizer of a so-called "Rudin-Osher-Fatemi" [166] problem:

$$u^{h,k+1} \in \operatorname{argmin}\left\{\int_{\mathbb{R}^N} \phi(Du) + \frac{1}{2h}\int_{\mathbb{R}^N} |u - \mathrm{sd}^{\phi^\circ}_{E^{h,k}}|^2\right\}. \tag{4.4}$$

In this chapter we combine this discretization in time with a simultaneous discretization in space for the particular class of purely crystalline anisotropies ϕ of the following form

$$\phi(v) = \sum_{i\in\mathcal{E}} \beta(i)|i \cdot v|, \tag{4.5}$$

where $\beta(i) > 0$ and $\mathcal{E} \subseteq \mathbb{Z}^N \setminus \{0\}$ is a finite set of generators such that Span $\mathcal{E} = \mathbb{R}^N$.

[1] defined by $\phi^\circ(x) = \sup_{\phi(\nu)\leq 1} \nu \cdot x$ and which satisfies $\phi(x) = \sup_{\phi^\circ(x)\leq 1} \nu \cdot x$.

We now specify the discrete setting we are interested in, referring the reader to [30] for a more thorough introduction to related topics. We consider an ε-spaced square lattice $\varepsilon\mathbb{Z}^N$ and discrete functions $u : \varepsilon\mathbb{Z}^N \to \mathbb{R}$, and denote $u_i := u(i)$. We observe that we could also consider a general finite-dimensional Bravais lattice, at the expense of more tedious notation. A natural discrete version of total variation-like energies are those appearing in Ising systems, namely energies of the form

$$TV^\varepsilon_\beta(v) := \varepsilon^{N-1} \sum_{i,j\in\varepsilon\mathbb{Z}^N} \beta(i/\varepsilon - j/\varepsilon)|v_i - v_j|, \tag{4.6}$$

where β is as in (4.5) and extended to 0 in $\mathbb{Z}^N \setminus \mathcal{E}$. Under the hypotheses above on β, the functionals TV^ε_β are shown to Γ-converge[2] as $\varepsilon \to 0$ to the total variation functional

$$TV_\phi(v) = \int_{\mathbb{R}^N} \phi(Dv)$$

where ϕ is as in (4.5), see e.g. [51]. It is thus natural to define a minimizing movements scheme based on TV^ε_β which is the discrete counterpart of the minimizing procedure (4.4), as follows: given $E_0 \subseteq \mathbb{R}^N$, we define $E^0_{\varepsilon,h} = \{i \in \varepsilon\mathbb{Z}^N : (i + [0,\varepsilon)^N) \cap E_0 \neq \emptyset\}$ and for every $k \in \mathbb{N}$ we let $u^{k+1}_{\varepsilon,h}$ be such that

$$u^{k+1}_{\varepsilon,h} \in \operatorname{argmin}\left\{ TV^\varepsilon_\beta(v) + \frac{1}{2h}\sum_{i\in\varepsilon\mathbb{Z}^N} |v_i - (\mathrm{sd}^k_{\varepsilon,h})_i|^2 \ : \ v : \varepsilon\mathbb{Z}^N \to \mathbb{R} \right\}, \tag{4.7}$$

where $\mathrm{sd}^k_{\varepsilon,h}$ denotes a suitable signed ϕ°-distance function to $E^k_{\varepsilon,h}$ defined on $\varepsilon\mathbb{Z}^N$. (Actually, the energy in (4.7) is infinite and we rather consider the Euler-Lagrange equation of the problem.) Then, one sets $E^{k+1}_{\varepsilon,h} := \{u^{k+1}_{\varepsilon,h} \leq 0\}$.

The idea is to study the asymptotic behaviour of the discrete evolutions $E^k_{\varepsilon,h}$ as both $\varepsilon, h \to 0$. Notice that a similar analysis has been performed in [28] in the planar case, for $\phi = \|\cdot\|_1$ and $\mathrm{sd}^k_{\varepsilon,h}$ the continuous signed distance function from the discrete sets $E^k_{\varepsilon,h}$ restricted to the lattice $\varepsilon\mathbb{Z}^N$, see also [27, 29, 31, 146, 171] for further related results. With this choice, if $\varepsilon \gg h$ it is easy to see that the dissipation-like term in (4.7)

$$\frac{1}{2h}\sum_{i\in\varepsilon\mathbb{Z}^N} |v_i - (\mathrm{sd}^{k+1}_{\varepsilon,h})_i|^2$$

forces the functions $u^k_{\varepsilon,h}$ to be constant as k varies, therefore producing *pinning* on the moving interfaces. Moreover, when the two scales ε, h are going to zero at the same speed it is shown in [28] that a direct implementation of the standard scheme with the choice above for the distance, introduces a systematic error of order $\varepsilon = h$ at each step, which accumulates and produces a drift in the limiting evolution. As a result, low curvature shapes remain pinned, while sets with higher curvature evolve with a law which is a nonlinear modification of the crystalline curvature flow (4.1). Thus, the evolution law (4.1) can be approximated with the scheme of [28] only if $\varepsilon \ll h$.

In the main result of this chapter, Theorem 4.24, we show that with a new appropriate definition of the distance $\mathrm{sd}^k_{\varepsilon,h}$, we can recover in the limit $\varepsilon, h \to 0$ the actual distributional solution to (4.1) for every initial set $E_0 \subseteq \mathbb{R}^N$, for every purely crystalline anisotropy ϕ of the form (4.5) with rational coefficients, in any dimension and irrespective of relative size of the space- and time-steps. In fact, the assumption of the rational character of β can be removed in the regime $\varepsilon \leq O(h)$. This is the first general rigorous convergence result for a fully discrete scheme without restrictions on the dimension, on the initial sets and in which the spatial mesh is allowed to be of the same order or even coarser than the time step.

Let us further comment on the analysis carried out in [28] in the planar case (see also [30] for many more references on the topic). One important change between these older results and

[2] Note that we do not need to assume that the lattice generated by $\{e_k\}_{k=1,\dots,m}$ is $\mathbb{Z}^N$, which is necessary to ensure the equi-coercivity of the discrete functionals.

those presented in this chapter is that we consider distributional solutions to the crystalline mean curvature flow (4.1), instead of relying on the characterization of the motion via ODEs, which dates back to [9, 16]. The latter notion of solution is indeed suited only for planar evolutions, thus the limitation $N = 2$ in the past works. With the ODE definition and for $\phi = \|\cdot\|_1$, the authors of [28] precisely prove the following results. If $\varepsilon \ll h$ then the limiting motion is consistent with (4.1), while if $h \ll \varepsilon$ pinning happens for any nonempty initial data. As already mentioned, in the critical case $\varepsilon = h$, the limit planar motion is not driven by (4.1), but instead by a slightly modified nonlinear crystalline mean curvature flow, and pinning may happen for some particular (low curvature) initial data. This striking difference with our result may be (vaguely) justified by the following remark. While in [28], the focus is on discrete sets, we rather evolve, in accordance with the definition of distributional solutions, the *signed distance functions* to the boundaries. In this way we can effectively achieve a sub-pixel precision in our approximation, as $u_{\varepsilon,h}$ and the signed distance function carry more information than the evolving level set $\{u_{\varepsilon,h}(t) \le 0\}$. Our new definition of the interpolated signed distance is detailed in Section 4.

The consistency result in this chapter validates the numerical experiments which we carry on in Section 6 to illustrate our results. These experiments are derived from previous experiments in [47], which however were using a different redistancing operation for which no consistency was proven. Numerical schemes based on the variational approach [8, 144] have been introduced for crystal growth [10]. Since then, there have been many attempts to implement implicit schemes based on this approach for isotropic and anisotropic curvature flows in various settings [45, 85, 161, 164, 84], all relying on the consistency of the spatial discretization with respect to the time-discrete scheme (hence assuming $\varepsilon \ll h$).

Many other techniques have been considered to simulate crystalline flows after [176, 177], see e.g. [110, 111, 80] for the evolution of planar curves and [160, 163] for higher-dimensional algorithms.

Let us conclude this introduction with two comments. The first one concerns the hypothesis that ϕ is purely crystalline. It seems quite technical, as it implies that the associated interaction function β (in the sense of (4.5)) has finite range. While this is not necessary to carry our the existence part for the discrete minimizing movements scheme, it is essential for building a calibration which yields a bound on the speed of Wulff shapes, see Appendix 4.A.1. In practice, being the Wulff shape $\mathcal{W} := \{\phi^\circ \le 1\}$ a finite Minkowski sum of (rational) segments (which is called a *zonotope*), we can effectively handcraft a calibration along the directions identified by these segments. It is a remarkable difference between this discrete setting and the continuous one, where instead the vector field $x/\phi^\circ(x)$ in $\mathbb{R}^N$ is the right calibration *for any* anisotropy ϕ.

The second one is on possible generalizations of the present analysis to more general evolution laws than (4.1). The more general evolution law which is shown to admit a unique distributional solution is

$$V(x,t) = \psi(\nu_{E(t)}(x)) \left(-\mathrm{H}^\phi_{E(t)}(x) + f(x,t) \right), \qquad x \in \partial E(t), t \ge 0, \tag{4.8}$$

where ψ is a norm (usually referred to as the *mobility*), and f is a forcing term, see [55, 52] . We expect most of the present analysis to be valid even if $\psi \ne \phi$, under suitable compatibility assumptions on ψ (see [55, 52] for details), and it should not be difficult to consider a driving force f as long as it is Lipschitz in space and globally bounded, see [52] again.

2 Distributional crystalline curvature flows

We recall the distributional formulation for the crystalline mean curvature motion of sets evolving with normal velocity (4.1) introduced in [55] (see also [52]). For the notations concerning anisotropies, we refer to Section 2 in Chapter 2. We recall that a sequence of closed sets $(E_k)_{k\ge 1}$ in $\mathbb{R}^N$ converges to a closed set E in the *Kuratowski sense*: if the following conditions are satisfied

(i) if $x_k \in E_k$ for each k, any limit point of $\{x_k\}$ belongs to E;

(ii) for all $x \in E$ there exists a sequence $\{x_k\}$ such that $x_k \in E_k$ for each k and $x_k \to x$.

We will write in this case:

$$E_k \xrightarrow{\mathcal{K}} E\,.$$

One can easily verify that $E_k \xrightarrow{\mathcal{K}} E$ if and only if (for any norm ψ) $\mathrm{dist}^\psi(\cdot, E_k) \to \mathrm{dist}^\psi(\cdot, E)$ locally uniformly in $\mathbb{R}^N$. Hence, by Ascoli-Arzelà Theorem we have that any sequence of closed sets admits a converging subsequence in the Kuratowski sense (possibly to $\emptyset$, when $\mathrm{dist}^\psi(\cdot, E_k) \to +\infty$).

Definition 4.1. Let $E_0 \subseteq \mathbb{R}^N$ be a closed set. Let E be a closed set in $\mathbb{R}^N \times [0, +\infty)$ and for each $t \geq 0$ denote $E(t) := \{x \in \mathbb{R}^N \,:\, (x,t) \in E\}$. We say that E is a *superflow* for (4.1) with initial datum E_0 if

(a) $E(0) \subseteq E_0$;

(b) $E(s) \xrightarrow{\mathcal{K}} E(t)$ as $s \nearrow t$ for all $t > 0$;

(c) If $E(t) = \emptyset$ for some $t \geq 0$, then $E(s) = \emptyset$ for all $s > t$.

(d) Set $T^* := \inf\{t > 0 \,:\, E(s) = \emptyset \text{ for } s \geq t\}$, and

$$d(x,t) := \mathrm{dist}^{\phi^\circ}(x, E(t)) \qquad \text{for all } (x,t) \in \mathbb{R}^N \times (0, T^*) \setminus E.$$

Then,

$$\partial_t d \geq \mathrm{div} z \tag{4.9}$$

holds in the distributional sense in $\mathbb{R}^N \times (0,T^*) \setminus E$ for a suitable $z \in L^\infty(\mathbb{R}^N \times (0,T^*))$ such that $z \in \partial\phi(\nabla d)$ a.e., $\mathrm{div} z$ is a Radon measure in $\mathbb{R}^N \times (0,T^*) \setminus E$, and $(\mathrm{div} z)^+ \in L^\infty(\{(x,t) \in \mathbb{R}^N \times (0,T^*) : d(x,t) \geq \delta\})$ for every $\delta \in (0,1)$.

We say that A, open set in $\mathbb{R}^N \times [0, +\infty)$, is a *subflow* for (4.1) with initial datum E_0 if $\mathbb{R}^N \times [0,+\infty) \setminus A$ is a superflow for (4.1) with initial datum $\mathbb{R}^N \setminus \mathrm{int}(E_0)$.

Finally, we say that E, closed set in $\mathbb{R}^N \times [0, +\infty)$, is a *weak flow* for (4.1) with initial datum E_0 if it is a superflow and if $\mathrm{int}(E)$[3] is a subflow, both with initial datum E_0.

In [55] the following crucial inclusion principle between sub- and superflows is proven.

Theorem 4.2. *Let E be a superflow with initial datum E_0 and F be a subflow with initial datum F_0 in the sense of Definition 4.1. Assume that $\mathrm{dist}^{\phi^\circ}(E^0, \mathbb{R}^N \setminus F^0) =: \Delta > 0$. Then,*

$$\mathrm{dist}^{\phi^\circ}\big(E(t), \mathbb{R}^N \setminus F(t)\big) \geq \Delta \qquad \textit{for all } t \geq 0$$

(with the convention that $\mathrm{dist}^{\phi^\circ}(G, \emptyset) = \mathrm{dist}^{\phi^\circ}(\emptyset, G) = +\infty$ for any G).

We also recall the corresponding notion of sub- and supersolution to the level set flow associated with (4.1). In what follows $\mathrm{UC}(\mathbb{R}^N)$ stands for the space of uniformly continuous functions on $\mathbb{R}^N$.

Definition 4.3 (Level set subsolutions and supersolutions)**.** Let $u_0 \in \mathrm{UC}(\mathbb{R}^N)$. A lower semicontinuous function $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ is called a *level set superflow* for (4.1), with initial datum u_0, if $u(\cdot, 0) \geq u_0$ and if for a.e. $\lambda \in \mathbb{R}$ the closed sublevel set $\{u(\cdot, t) \leq \lambda\}$ is a superflow for (4.1) in the sense of Definition 4.1, with initial datum $\{u_0 \leq \lambda\}$.

An upper-semicontinuous function $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ is called a *level set subflow* for (4.1), with initial datum u_0, if $-u$ is level set superflow in the previous sense, with initial datum $-u_0$.

Finally, a continuous function $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ is called a *level set flow* for (4.1) if it is both a level set sub- and superflow.

Using Theorem 4.2, it is not difficult to deduce the following parabolic comparison principle between level set sub- and superflows, which yields in particular the uniqueness of level set flows (in the sense of Definition 4.3), see [52].

Theorem 4.4. *Let u_0, $v_0 \in \mathrm{UC}(\mathbb{R}^N)$ and let u, v be respectively a level set subflow starting from u_0 and a level set superflow starting from v_0. If $u_0 \leq v_0$, then $u \leq v$.*

[3] Here we are taking the interior with respect to $\mathbb{R}^N \times [0, +\infty)$

We finally recall that in [55] (see also [52]) the existence of level set flows is established by implementing a level-by-level minimizing movements scheme. This in turn yields existence and uniqueness (up to fattening) for weak flows. This is made precise in the following statement, see [55, Corollary 4.6] and [52, Theorem 4.8].

Theorem 4.5. *Let $u_0 \in \mathrm{UC}(\mathbb{R}^N)$. Then the following holds:*

(i) *There exists a unique level set flow u in the sense of Definition 4.3 starting u_0.*

(ii) *For all $\lambda \in \mathbb{R}$ the sets $\{(x,t) \,:\, u(x,t) \leq \lambda\}$ and $\{(x,t) \,:\, u(x,t) < \lambda\}$ are respectively the maximal superflow and minimal sublow with initial datum $\{u_0 \leq \lambda\}$.*

(iii) *For all but countably many $\lambda \in \mathbb{R}$, the fattening phenomenon does not occur; that is,*

$$\begin{aligned} \{(x,t) \,:\, u(x,t) < \lambda\} &= \mathrm{int}\left(\{(x,t) \,:\, u(x,t) \leq \lambda\}\right), \\ \mathrm{cl}\left(\{(x,t) \,:\, u(x,t) < \lambda\}\right) &= \{(x,t) \,:\, u(x,t) \leq \lambda\}, \end{aligned} \tag{4.10}$$

where interior and closure are relative to space-time.

For all such λ, $\{(x,t) \,:\, u(x,t) \leq \lambda\}$ is the unique weak flow in the sense of Definition 4.1, starting from $\{u_0 \leq \lambda\}$.

The aim of this chapter is to show that the convergence to the continuum level set flow holds true also when the Euler implicit time discretisation is combined with a suitable spatial discretisation procedure.

3 The discrete "Rudin-Osher-Fatemi" problem

In this part, we describe our discrete setting, and then introduce and analyse the discrete variant (4.7) of Problem (4.4).

3.1 Discrete functions spaces and operators

For $\varepsilon > 0$, we define the function spaces $X_\varepsilon = \mathbb{R}^{\varepsilon\mathbb{Z}^N}$ and $Y_\varepsilon = \mathbb{R}^{\varepsilon\mathbb{Z}^N \times \varepsilon\mathbb{Z}^N}$. Given a function $u \in X_\varepsilon$ and a discrete "vector field" $z \in Y_\varepsilon$, with a slight abuse of notation we will denote $u_i = u(i)$ and $z_{ij} = z(i,j)$, $i, j \in \varepsilon\mathbb{Z}^N$. The discrete gradient $D_\varepsilon : X_\varepsilon \to Y_\varepsilon$ is defined, for $u \in X_\varepsilon$ as

$$(D_\varepsilon u)_{ij} = \frac{u_i - u_j}{\varepsilon}.$$

We denote its adjoint operator by $D_\varepsilon^* : Y_\varepsilon \to X_\varepsilon$, namely the operator such that, for $\eta \in X_\varepsilon$ compactly supported and for $z \in Y_\varepsilon$, is defined as

$$\sum_i (D_\varepsilon^* z)_i \eta_i := \sum_{ij} z_{ij} (D_\varepsilon \eta)_{ij} = \sum_{ij} z_{ij} (\eta_i - \eta_j),$$

where the indexes, here and throughout the chapter, range over $\varepsilon\mathbb{Z}^N$ if not otherwise stated. In particular, taking $\eta = \chi_{\{i\}}$, one finds that

$$(D_\varepsilon^* z)_i = \sum_j \frac{z_{ij} - z_{ji}}{\varepsilon}, \tag{4.11}$$

which can be seen as a discrete divergence operator.

3.2 Discrete ROF problem

In this section we consider the discrete anisotropic ROF problem associated with the discrete total variation functional. Without loss of generality, we consider $\varepsilon = 1$ in this section, and denote $X := X_1$, $Y := Y_1$ and $D := D_1$. Given a nonnegative $\beta \in X$, which will be called the *interaction function*, satisfying

$$\sum_{i \in \mathbb{Z}^N} \beta(i) =: c_\beta < +\infty, \tag{4.12}$$

we set $\alpha_{ij} = \beta(i-j)$ and, for any $u \in X$ we define

$$TV(u) = \sum_{i,j\in\mathbb{Z}^N} \alpha_{ij}|u_i - u_j| = \sum_{i,j} \alpha_{ij}|(Du)_{i,j}|. \tag{4.13}$$

We also consider the discrete perimeter P_ϕ defined for every $E \subseteq \mathbb{Z}^N$ as

$$P_\phi(E) := TV(\chi^E) = \sum_{i,j\in\mathbb{Z}^N} \alpha_{ij}|\chi_i^E - \chi_j^E|.$$

We also consider a suitable localization of the perimeter: namely, for any set $A \subseteq \mathbb{R}^N$ we define

$$P_\phi(E;A) = \sum_{i\in A\cap\mathbb{Z}^N \text{ or } j\in A\cap\mathbb{Z}^N} \alpha_{ij}|\chi_i^E - \chi_j^E|.$$

Note that the quantities above may well be infinite.

Then, given $g \in X$, we consider the following problem: find a pair $(u,z) \in X \times Y$ such that

$$\begin{cases} D^*z + u = g \\ z_{ij}(u_i - u_j) = \alpha_{ij}|u_i - u_j|, \quad |z_{ij}| \le \alpha_{ij} \quad \forall i,j \in \mathbb{Z}^N. \end{cases} \tag{4.14}$$

Note that the equation above is the Euler-Lagrange equation of the ROF discrete functional

$$ROF_g(v) = TV(v) + \frac{1}{2}\sum_{i\in\mathbb{Z}^N} (v_i - g_i)^2. \tag{4.15}$$

However, (4.14) makes sense also for those g such that $ROF_g \equiv +\infty$.

We will also consider the following geometric minimization problem. Given $g \in X$, find

$$\min_{F\subseteq\mathbb{Z}^N} P_\phi(F) + \sum_{i\in\mathbb{Z}^N} \chi_i^F g_i. \tag{4.16}$$

In order to deal with unbounded sets, possibly with infinite perimeter, we will consider the following notion of global minimality with respect to compactly supported perturbations.

Definition 4.6. A set $E \subseteq \mathbb{Z}^N$ is a global minimizer for the problem (4.16) if for every $R > 0$

$$P_\phi(E;B_R) + \sum_{|i|<R} \chi_i^E g_i \le P_\phi(F;B_R) + \sum_{|i|<R} \chi_i^F g_i \tag{4.17}$$

for every $F \subseteq \mathbb{Z}^N$ such that $F \triangle E \subseteq B_R$. Here $B_R = \{x \in \mathbb{R}^N : |x| < R\}$ is the open ball of radius R centered in the origin.

Proposition 4.7. *Let $g, g \in X$ such that $g' - g \ge \delta > 0$. Let E, E' be two global minimizers of problem* (4.17)*, in the sense of Definition 4.6, corresponding to g, g' respectively. Then, $E' \subseteq E$.*

Proof. Let us denote in the following $\chi := \chi^{E_s}, \chi' := \chi^{E'_s}$. For a given $R > 0$ we define the competitor sets $F = (E_s \setminus B_R) \cup ((E'_s \cup E_s) \cap B_R)$ and $F' = (E'_s \setminus B_R) \cup ((E'_s \cap E_s) \cap B_R)$. By minimality of E_s, E'_s in B_R one has

$$\sum_{|i|<R \text{ or } |j|<R} \alpha_{ij}|\chi'_i - \chi'_j| + \sum_{|i|<R} g'_i(\chi'_i - \chi'_i \wedge \chi_i) \le \sum_{\substack{|i|<R\\|j|<R}} \alpha_{ij}|\chi'_i \wedge \chi_i - \chi'_j \wedge \chi_j| + \sum_{\substack{|i|<R\\|j|\ge R}} (\alpha_{ij}+\alpha_{ji})|\chi'_i \wedge \chi_i - \chi'_j| \tag{4.18}$$

$$\sum_{|i|<R \text{ or } |j|<R} \alpha_{ij}|\chi_i - \chi_j| + \sum_{|i|<R} g_i(\chi_i - \chi'_i \vee \chi_i) \le \sum_{\substack{|i|<R\\|j|<R}} \alpha_{ij}|\chi'_i \vee \chi_i - \chi'_j \vee \chi_j| + \sum_{\substack{|i|<R\\|j|\ge R}} (\alpha_{ij}+\alpha_{ji})|\chi'_i \vee \chi_i - \chi_j|. \tag{4.19}$$

Using the inequality[4] $|a\wedge b - c\wedge d| + |a\vee b - c\vee d| \le |a-c| + |b-d|$ and summing together (4.18) and (4.19) we obtain

$$\begin{aligned}&\sum_{\substack{|i|<R\\|j|\ge R}} (\alpha_{ij}+\alpha_{ji})\left(|\chi_i - \chi_j| + |\chi'_i - \chi'_j|\right) + 2\sum_{|i|<R}(g'_i - g_i)(\chi'_i - \chi_i)^+ \\ &\le \sum_{\substack{|i|<R\\|j|\ge R}} (\alpha_{ij}+\alpha_{ji})\left(|\chi'_i\wedge\chi_i - \chi'_j| + |\chi'_i \vee \chi_i - \chi_j|\right).\end{aligned} \tag{4.20}$$

We then remark that $|\chi'_i\wedge\chi_i - \chi'_j| \le |\chi'_i\wedge\chi_i - \chi'_i| + |\chi'_i - \chi'_j| = (\chi'_i - \chi_i)^+ + |\chi'_i - \chi'_j|$ and analogously $|\chi'_i\vee\chi_i - \chi_j| \le (\chi'_i - \chi_i)^+ + |\chi_i - \chi_j|$. Therefore, (4.20) entails

$$\sum_{|i|<R}(g'_i - g_i)(\chi'_i - \chi_i)^+ \le \sum_{|i|<R}(\chi'_i - \chi_i)^+ \sum_{|j|\ge R}(\alpha_{ij}+\alpha_{ji}). \tag{4.21}$$

Fix now $R_\delta > 0$ such that

$$\sum_{|k|\ge R_\delta} \beta(k) \le \frac{\delta}{4}$$

and define $V_R := \sum_{|i|<R}(\chi'_i - \chi_i)^+$. Assuming $R > R_\delta$, for every $\ell < R$ we use (4.21) and $g+\delta \le g'$ to get

$$\begin{aligned}\delta V_R &\le \sum_{|i|<\ell}(\chi'_i - \chi_i)^+ \sum_{|j|\ge R}(\alpha_{ij}+\alpha_{ji}) + 2c_\beta \sum_{\ell\le|i|<R}(\chi'_i - \chi_i)^+ \\ &\le 2\sum_{|i|<\ell}(\chi'_i - \chi_i)^+ \sum_{|k|\ge R-\ell}\beta(k) + 2c_\beta(V_R - V_\ell).\end{aligned} \tag{4.22}$$

Therefore, choosing $\ell = R - R_\delta$ in (4.22) we obtain

$$\frac{\delta}{2} V_R \le 2c_\beta (V_R - V_{R-R_\delta}), \tag{4.23}$$

which implies that for every $k, \ell \in \mathbb{N}$ it holds

$$V_{kR_\delta} \le \left(1 - \frac{\delta}{4c_\beta}\right)^\ell V_{(k+\ell)R_\delta}. \tag{4.24}$$

[4]Indeed, if $a \ge b$ and $c \ge d$, this is an equality, while if $a > b$ and $c < d$, one deduces that $b-d < a-d < a-c$, $b-d < b-c < a-c$ so that there exists $t\in(0,1)$ with $a-d = t(b-d)+(1-t)(a-c)$, $b-c = (1-t)(b-d)+t(a-c)$: the conclusion follow by convexity of $|\cdot|$.

Letting $\ell \to +\infty$, since $V_{(k+\ell)R_\delta} = O(\ell^N)$, we infer that $V_{kR_\delta} = 0$ for every $k \in \mathbb{N}$. In particular, this implies that $(\chi' - \chi)^+ = 0$ i.e. $\chi' \leq \chi$. □

We now prove the following theorem.

Theorem 4.8. *Given $g \in X$ there exists a unique function $u^g \in X$ and there exists a discrete vector field $z \in Y$ such that (u^g, z) is a solution to* (4.14)*. Moreover, the following comparison principle holds: if $g \leq g'$ then $u^g \leq u^{g'}$. Finally, for any $R > 0$ and $s \in \mathbb{R}$ the sublevel set $E_s := \{i \in \mathbb{Z}^N : u_i^g \leq s\}$ is a global minimizer (in the sense of Definition 4.6) for* (4.16) *with g replaced by $g - s$.*

Proof. **Step 1.** (Existence) For every $n \in \mathbb{N}$ set $g^n := g\chi^{B_n}$ and note that $g^n \in \ell^2(\mathbb{Z}^N)$. Therefore, by standard methods and by strict convexity the functional (4.15), with g replaced by g^n admits a unique minimizer u^n and, as previously observed, the optimality condition is the existence of a discrete field z^n such that (u^n, z^n) solves (4.14) (with g^n in place of g). Note that, for any $k \in \mathbb{Z}^N$, by equation (4.14) it holds

$$|u_k^n| \leq |g_k^n| + |(D^*z)_k| \leq |g_k| + c_\beta \quad \text{for every } n \in \mathbb{N}, \tag{4.25}$$

where the last inequality follows from the definition (4.11) and from $|z_{ij}| \leq \alpha_{ij}$ and $|g^n| \leq |g|$. Now, it is clear that we can extract a subsequence n_k and find (u, z) such that $u_i^{n_k} \to u_i$ and $z_{ij}^{n_k} \to z_{ij}$ as $k \to +\infty$. Clearly we have that $|z_{ij}| \leq \alpha_{ij}$ and $z_{ij}(u_i - u_j) = \alpha_{ij}|u_i - u_j|$ and it is immediate to check that (u, z) satisfies equation (4.14).

Step 2. (Minimality of the sublevelsets) Let $R > 0, s \in \mathbb{R}$ and let $F \subseteq \mathbb{Z}^N$ such that $E_s \triangle F \subseteq\subseteq B_R$. We first remark that $\alpha_{ij}|\chi_i^{E_s} - \chi_j^{E_s}| = -z_{ij}(\chi_i^{E_s} - \chi_j^{E_s})$, which follows easily from the definition of E_s and $z_{ij}(u_i - u_j) = \alpha_{ij}|u_i - u_j|$.

We set $I_R := \{(i,j) \in \mathbb{Z}^N \times \mathbb{Z}^N : |i| < R \text{ or } |j| < R\}$ and compute

$$\begin{aligned}
P_\phi(F; B_R) - P_\phi(E_s; B_R) &= \sum_{(i,j)\in I_R} \alpha_{ij}|\chi_i^F - \chi_j^F| - \sum_{(i,j)\in I_R} \alpha_{ij}|\chi_i^{E_s} - \chi_j^{E_s}| \\
&\geq - \sum_{(i,j)\in I_R} z_{ij}(\chi_i^F - \chi_j^F) + \sum_{(i,j)\in I_R} z_{ij}(\chi_i^{E_s} - \chi_j^{E_s}) \\
&= \sum_{(i,j)\in I_R} z_{ij}(\chi_i^{E_s} - \chi_i^F - (\chi_j^{E_s} - \chi_j^F)) \\
&= \sum_{ij} z_{ij}(\chi_i^{E_s} - \chi_i^F - (\chi_j^{E_s} - \chi_j^F)),
\end{aligned} \tag{4.26}$$

where in the last equality we used the fact that $\chi_i^{E_s} = \chi_i^F$ if $|i| \geq R$. Noting that the function $\chi^{E_s} - \chi^F$ is compactly supported, we may use it as a test function for (4.14). Therefore, from (4.26) we deduce

$$\begin{aligned}
P_\phi(F; B_R) - P_\phi(E_s; B_R) &\geq \sum_{ij} z_{ij}(\chi_i^{E_s} - \chi_i^F - (\chi_j^{E_s} - \chi_j^F)) \\
&= \sum_i (\chi_i^{E_s} - \chi_i^F)(g_i - u_i) \geq \sum_{i\in E_s\setminus F} (g_i - s) - \sum_{i \in F\setminus E_s} (g_i - s),
\end{aligned}$$

which shows the minimality of E_s.

Step 3. (Comparison and uniqueness for (4.14)) Assume $g \leq g'$ and let $(u, z), (u', z')$ two corresponding solutions for (4.14). Let $s > s'$ and recall that by Step 2 $\{u' \leq s'\}$ and $\{u \leq s\}$ are global minimizers for (4.16) according to Definition 4.6, with g replaced by $g' - s'$ and $g - s$ respectively. Since $g' - s' - (g - s) \geq s - s' > 0$, from Proposition 4.7 we obtain $\{u' \leq s'\} \subseteq \{u \leq s\}$. By the arbitrariness of s, s' we conclude that $u \leq u'$.

□

Remark 4.9. We remark that, given $g \in X$ it clearly holds that $u^{-g} = -u^g$.

4 The minimizing movements scheme

In this section we provide a combined spatial and time discretisation of the flow (4.1) for a particular class of norms ϕ and show the convergence of the scheme to the continuum flow. In what follows, we consider $\{e_1, \dots, e_m\} \subseteq \mathbb{Z}^N$ a finite number of integer vectors spanning the whole $\mathbb{R}^N$, and set $\mathcal{E} = \{\pm e_k\}_{k=1}^m$. We let $\beta \in X$ be a non-negative function such that

$$\beta(-i) = \beta(i) \quad \text{and} \quad \beta(i) > 0 \text{ if and only if } i \in \mathcal{E}.$$

One can naturally associate an anisotropy ϕ with the function β setting

$$\phi(v) = \sum_{i \in \mathcal{E}} \beta(i)|i \cdot v| = \sum_{k=1}^{m} 2\beta(e_k)|v \cdot e_k|. \tag{4.27}$$

Note that, in particular, it holds

$$\#\{k \in \mathbb{Z}^N : \beta(k) \neq 0\} < +\infty. \tag{4.28}$$

We recall that the ϕ-perimeter associated with (4.27)

$$P_\phi(E) = \int_{\partial^* E} \phi(\nu_E)\, d\mathcal{H}^{N-1}$$

(defined for every $E \subseteq \mathbb{R}^N$ of finite perimeter) is the Γ-limit (in a suitable sense) as $\varepsilon \to 0$ of the following scaled discrete perimeters

$$P_\phi^\varepsilon(E) := \varepsilon^{N-1} \sum_{i,j \in \varepsilon\mathbb{Z}^N} \alpha_{ij}^\varepsilon |\chi_i^E - \chi_j^E| = \varepsilon^N \sum_{i,j \in \varepsilon\mathbb{Z}^N} \alpha_{i,j}^\varepsilon |(D_\varepsilon \chi^E)_{i,j}|$$

defined for all $E \subseteq \varepsilon\mathbb{Z}^N$, see for instance [26]. Here we have set

$$\alpha_{ij}^\varepsilon := \beta\left(\frac{i}{\varepsilon} - \frac{j}{\varepsilon}\right). \tag{4.29}$$

4.1 The discrete scheme

In this section we describe our minimizing movements scheme, discretized in both time and space. Given ϕ a norm on $\mathbb{R}^N$ and a closed set $E \notin \{\emptyset, \mathbb{R}^N\}$, let us recall that we denote with $\mathrm{sd}_E^{\phi^\circ}$ the signed $\phi^\circ-$distance function from E, which is defined as

$$\mathrm{sd}_E^{\phi^\circ}(x) := \min_{y \in E} \phi^\circ(x-y) - \min_{y \notin E} \phi^\circ(x-y).$$

We also set $\mathrm{sd}_\emptyset^{\phi^\circ} \equiv +\infty$ and $\mathrm{sd}_{\mathbb{R}^N}^{\phi^\circ} \equiv -\infty$. We denote

$$C_\phi = \min_{i \in \mathbb{Z}^N \setminus \{0\}} \phi^\circ(i) > 0 \tag{4.30}$$

and define the ϕ-Wulff shape $\mathcal{W}_R(x)$ of radius $R > 0$ and center $x \in \mathbb{R}^N$ as $\mathcal{W}_R(x) = \{y \in \mathbb{R}^N : \phi^\circ(x-y) < R\}$.

Recalling (4.29), we rescale equation (4.14) on the lattice $\varepsilon\mathbb{Z}^N$ in the following way. We recall that $X_\varepsilon = \mathbb{R}^{\varepsilon\mathbb{Z}^N}$ and $Y_\varepsilon = \mathbb{R}^{\varepsilon\mathbb{Z}^N \times \varepsilon\mathbb{Z}^N}$. Given $g \in X_\varepsilon$ the problem (4.14) now becomes to find $(u,z) \in X_\varepsilon \times Y_\varepsilon$ satisfying

$$\begin{cases} hD_\varepsilon^* z + u = g & \text{on } \varepsilon\mathbb{Z}^N \\ z_{ij}(u_i - u_j) = \alpha_{ij}^\varepsilon |u_i - u_j|, & |z_{ij}| \leq \alpha_{ij}^\varepsilon, \end{cases} \tag{4.31}$$

where $D_\varepsilon^* z$ is defined in (4.11).

Given $u \in X_\varepsilon$ we define the operators $d_\pm^{\varepsilon,\phi^\circ}, \mathrm{sd}_\pm^{\varepsilon,\phi^\circ}, \mathrm{sd}^{\varepsilon,\phi^\circ} : X_\varepsilon \to X_\varepsilon$ in the following way: letting $E = \{i \in \varepsilon\mathbb{Z}^N : u_i \le 0\}$, we first define

$$\begin{aligned}
(d_-^{\varepsilon,\phi^\circ}(u))_i &= \sup_{j\in\{u\ge 0\}} \{u_j - \phi^\circ(i-j)\}\,,\\
(\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u))_i &= \inf_{j\in\{u\le 0\}} \Big\{(d_-^{\varepsilon,\phi^\circ}(u))_j + \phi^\circ(i-j)\Big\}\,,\\
(d_+^{\varepsilon,\phi^\circ}(u))_i &= \inf_{j\in\{u\le 0\}} \{u_j + \phi^\circ(i-j)\}\,,\\
(\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u))_i &= \sup_{j\in\{u\ge 0\}} \Big\{(d_+^{\varepsilon,\phi^\circ}(u))_j - \phi^\circ(i-j)\Big\}\,,\\
(\mathrm{sd}^{\varepsilon,\phi^\circ}(u))_i &= \frac{1}{2}(\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u))_i + \frac{1}{2}(\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u))_i.
\end{aligned} \tag{4.32}$$

Note that $d_+^{\varepsilon,\phi^\circ}(u) = -d_-^{\varepsilon,\phi^\circ}(-u)$ and $\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) = -\mathrm{sd}_-^{\varepsilon,\phi^\circ}(-u)$.

We will say that $f \in X_\varepsilon$ is (L,ϕ°)-Lipschitz if for all $i, j \in \varepsilon\mathbb{Z}^N$ it holds $|f_i - f_j| \le L\phi^\circ(i-j)$.

Remark 4.10. We assume in what follows that u is $(1,\phi^\circ)$-Lipschitz. Then, concerning $d_-^{\varepsilon,\phi^\circ}$, $\mathrm{sd}_-^{\varepsilon,\phi^\circ}$, we remark that

$$d_-^{\varepsilon,\phi^\circ}(u) = \min\{f \in X_\varepsilon : f \ge u \text{ in } \{u \ge 0\},\ f \text{ is } (1,\phi^\circ)\text{-Lipschitz}\}\,, \tag{4.33}$$

and analogously

$$\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u) = \max\Big\{f \in X_\varepsilon : f \le d_-^{\varepsilon,\phi^\circ}(u) \text{ in } \{u \le 0\},\ f \text{ is } (1,\phi^\circ)\text{-Lipschitz}\Big\}\,. \tag{4.34}$$

Correspondingly it holds

$$\begin{aligned}
d_+^{\varepsilon,\phi^\circ}(u) &= \max\{f \in X_\varepsilon : f \le u \text{ in } \{u \le 0\},\ f \text{ is } (1,\phi^\circ)\text{-Lipschitz}\}\,,\\
\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) &= \min\Big\{f \in X_\varepsilon : f \ge d_+^{\varepsilon,\phi^\circ}(u) \text{ in } \{u \ge 0\},\ f \text{ is } (1,\phi^\circ)\text{-Lipschitz}\Big\}\,,
\end{aligned} \tag{4.35}$$

In particular, the functions $d_\pm^{\varepsilon,\phi^\circ}(u), \mathrm{sd}_\pm^{\varepsilon,\phi^\circ}(u), \mathrm{sd}^{\varepsilon,\phi^\circ}(u)$ are also $(1,\phi^\circ)$-Lipschitz. Let us show (4.33) the other identities being analogous. To this aim, denote by $\hat{d}$ the function defined by the right-hand side of (4.33). Since $d_-^{\varepsilon,\phi^\circ}(u)$ is the pointwise supremum of $(1,\phi^\circ)$-Lipschitz functions, we clearly have that $d_-^{\varepsilon,\phi^\circ}(u)$ is itself $(1,\phi^\circ)$-Lipschitz. Moreover, testing with $j = i$ in the definition of $d_-^{\varepsilon,\phi^\circ}(u)$, we get $d_-^{\varepsilon,\phi^\circ}(u) \ge u$ in $\{u \ge 0\}$. Thus, we infer $\hat{d} \le d_-^{\varepsilon,\phi^\circ}(u)$. For the opposite inequality, let f be any functions as in the minimisation problem on the right-hand side of (4.33). Then for any $i \in \varepsilon\mathbb{Z}^N$ and $j \in \{u \ge 0\}$ we have

$$f_i \ge f_j - \phi^\circ(i-j) \ge u_j - \phi^\circ(i-j)\,.$$

By maximising with respect to $j \in \{u \ge 0\}$, we get $f \ge d_-^{\varepsilon,\phi^\circ}(u)$ and in turn, by the arbitrariness of f, $\hat{d} \ge d_-^{\varepsilon,\phi^\circ}(u)$, which concludes the proof of (4.33)

Since the functions $d_\pm^{\varepsilon,\phi^\circ}(u), \mathrm{sd}_\pm^{\varepsilon,\phi^\circ}(u), \mathrm{sd}^{\varepsilon,\phi^\circ}(u)$ are $(1,\phi^\circ)$-Lipschitz, from (4.33) it follows that

$$d_-^{\varepsilon,\phi^\circ}(u) \le u \text{ in } \varepsilon\mathbb{Z}^N, \quad d_-^{\varepsilon,\phi^\circ}(u) = u \text{ in } \{u \ge 0\}, \tag{4.36}$$

while (4.34) implies that

$$\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u) \ge d_-^{\varepsilon,\phi^\circ}(u) \text{ in } \varepsilon\mathbb{Z}^N, \quad \mathrm{sd}_-^{\varepsilon,\phi^\circ}(u) = d_-^{\varepsilon,\phi^\circ}(u) \text{ in } \{u \le 0\}. \tag{4.37}$$

Reasoning in the same way, we see that

$$\begin{aligned} d_+^{\varepsilon,\phi^\circ}(u) \geq u \text{ in } \varepsilon\mathbb{Z}^N, \quad & d_+^{\varepsilon,\phi^\circ}(u) = u \text{ in } \{u \leq 0\}, \\ \mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) \leq d_+^{\varepsilon,\phi^\circ}(u) \text{ in } \varepsilon\mathbb{Z}^N, \quad & \mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) = d_+^{\varepsilon,\phi^\circ}(u) \text{ in } \{u \geq 0\}. \end{aligned} \tag{4.38}$$

In particular we conclude

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(u) \geq u \text{ in } \{u \geq 0\}, \quad \mathrm{sd}^{\varepsilon,\phi^\circ}(u) \leq u \text{ in } \{u \leq 0\}. \tag{4.39}$$

We remark that, always for u a $(1,\phi^\circ)$-Lipschitz function, $\{u \leq 0\} = \{\mathrm{sd}_\pm^{\varepsilon,\phi^\circ}(u) \leq 0\}$ and $\{u \geq 0\} = \{\mathrm{sd}_\pm^{\varepsilon,\phi^\circ}(u) \geq 0\}$. In particular, if the level set 0 of u is "fat", then this is preserved by these discrete "signed distance functions". Further properties of these discrete signed distance functions are presented in Lemma 4.14 below and in Remark 4.18

Moreover, it follows directly from the definition of $d_\pm^{\varepsilon,\phi^\circ}(u), \mathrm{sd}_\pm^{\varepsilon,\phi^\circ}(u)$ that the function $\mathrm{sd}^{\varepsilon,\phi^\circ}(u)$ is invariant under integer translations, meaning that for any $i, \tau \in \varepsilon\mathbb{Z}^N$ it follows

$$\left(\mathrm{sd}^{\varepsilon,\phi^\circ}(u(\cdot + \tau))\right)_i = \left(\mathrm{sd}^{\varepsilon,\phi^\circ}(u)\right)_{i+\tau}. \tag{4.40}$$

Given a set $E \subseteq \varepsilon\mathbb{Z}^N$, we will denote with $\widehat{E} \subseteq \mathbb{R}^N$ the closed set defined by

$$\widehat{E} := E + [0,\varepsilon]^N.$$

We now define the discrete evolution scheme. For ease of notation we assume $\varepsilon = \varepsilon(h)$, with $\varepsilon \to 0$ as $h \to 0$ and we will specify the dependence on h only.

Let $E_0 \subseteq \mathbb{R}^N$ be a closed set. We define $E^{h,0} := \{i \in \varepsilon\mathbb{Z}^N : (i + [0,\varepsilon)^N) \cap E_0 \neq \emptyset\}$. We note that

$$\widehat{E}^{h,0} \to E_0, \quad E^{h,0} \to E_0 \tag{4.41}$$

as $h \to 0$ in the Kuratowski sense, where with a slight abuse of notation we write $\widehat{E}^{h,0}$ to denote the set $\widehat{E^{h,0}} = E^{h,0} + [0,\varepsilon]^N$.

Given a closed set $E_0 \subseteq \mathbb{R}^N$ with $E_0 \notin \{\emptyset, \mathbb{R}^N\}$, we consider $u^{h,0}$ a $(1,\phi^\circ)$-Lipschitz function on $\varepsilon\mathbb{Z}^N$ which is negative inside $E^{h,0}$ and positive outside. For instance, we set

$$u^{h,0} := \frac{1}{2}C_\phi\varepsilon(1 - \chi_{E^{h,0}}) - \frac{1}{2}C_\phi\varepsilon\chi_{E^{h,0}},$$

where C_ϕ is defined in (4.30), so that $u^{h,0}$ is $(1,\phi^\circ)$-Lipschitz. Let us set $(z^{h,0})_{ij} = 0$ for all $i,j \in \varepsilon\mathbb{Z}^N$. Then, as long as $E^{h,k} \notin \{\emptyset, \mathbb{R}^N\}$, we can iteratively define $u^{h,k+1}, z^{h,k+1}$ for $k \in \mathbb{N}$ by solving (4.31) with $g = \mathrm{sd}^{\varepsilon,\phi^\circ}(u^{h,k})$; i.e.,

$$\begin{cases} hD_\varepsilon^* z^{h,k+1} + u^{h,k+1} = \mathrm{sd}^{\varepsilon,\phi^\circ}(u^{h,k}) & \text{on } \varepsilon\mathbb{Z}^N \\ z_{ij}^{h,k+1}(u_i^{h,k+1} - u_j^{h,k+1}) = \alpha_{ij}^\varepsilon |u_i^{h,k+1} - u_j^{h,k+1}|, & |z_{ij}^{h,k+1}| \leq \alpha_{ij}^\varepsilon. \end{cases} \tag{4.42}$$

We then set

$$E^{h,k+1} = \{i \in \varepsilon\mathbb{Z}^N : u_i^{h,k+1} \leq 0\}.$$

If either $E^{h,k} = \emptyset$ or $E^{h,k} = \mathbb{R}^N$, we define $E^{h,k+1} = E^{h,k}$. We denote by T_h^* the first discrete time hk such that $E^{h,k} = \emptyset$, if any; otherwise we let $T_h^* = +\infty$. Analogously, we set ${T'}_h^*$ first discrete time hk such that $E^{h,k} = \mathbb{R}^N$, if any; otherwise we let $T_h'^* = +\infty$.

For ease of notation we will set

$$
\begin{aligned}
&E^h(t) := E^{h,[t/h]} \subseteq \varepsilon\mathbb{Z}^N \\
&d^h(t) := \mathrm{sd}^{\varepsilon,\phi^\circ}(u^{h,[t/h]}) \in X_\varepsilon \\
&u^h(t) := u^{h,[t/h]} \in X_\varepsilon \\
&z^h(t) := z^{h,[t/h]} \in Y_\varepsilon \\
&\widehat{d}^h(\cdot,t) := \mathrm{sd}^{\phi^\circ}_{\widehat{E}^h(t)} \in \mathrm{Lip}(\mathbb{R}^N),
\end{aligned}
\tag{4.43}
$$

where again, with a slight abuse of notation, $\widehat{E}^h(t)$ stands for $\widehat{E^h(t)}$. Note that in the definition of $\widehat{d}^h(\cdot,t)$ we are possibly using the convention $\mathrm{sd}^{\phi^\circ}_{\emptyset} \equiv +\infty$ and $\mathrm{sd}^{\phi^\circ}_{\mathbb{R}^N} \equiv -\infty$. Note also that $z^h(t)$ is well defined only for $0 \le t < \min\{T^*_h, T'^*_h\}$; however, if needed, we can set $z^h(t) = 0$ for $t \ge \min\{T^*_h, T'^*_h\}$.

Remark 4.11. If u is the solution to (4.31) with datum (L,ϕ°)-Lipschitz datum g, by standard arguments, based on the comparison principle and translation invariance, one can show that u satisfies the same Lipschitz bound of g. Indeed, given $j \in \varepsilon\mathbb{Z}^N$, the function $u(\cdot - j) \pm L\phi^\circ(j)$ solves (4.31) with datum $g(\cdot - j) \pm L\phi^\circ(j)$. By comparison one concludes as $g(\cdot - j) - L\phi^\circ(j) \le g(\cdot) \le g(\cdot - j) + L\phi^\circ(j)$.

Lemma 4.12. *Let u^h, E^h, d^h be defined as in* (4.43)*. Then, d^h is $(1,\phi^\circ)$-Lipschitz and satisfies for every $t \ge 0$*

$$
\begin{cases}
u^h(t) \le d^h(t) & \text{in } \varepsilon\mathbb{Z}^N \setminus E^h(t) \\
u^h(t) \ge d^h(t) & \text{in } E^h(t).
\end{cases}
\tag{4.44}
$$

Proof. It follows from Remarks 4.10 and 4.11. □

Lemma 4.13. *Given a $(1,\phi^\circ)$-Lipschitz function $u \in X_\varepsilon$, one has that*

$$
\sup_{\varepsilon\mathbb{Z}^N\setminus E} |\mathrm{sd}^{\varepsilon,\phi^\circ}_{\pm}(u) - \mathrm{sd}^{\phi^\circ}_{\widehat{E}}| \le c_\phi \varepsilon,
\tag{4.45}
$$

for a suitable positive constant c_ϕ, where $E = \{i \in \varepsilon\mathbb{Z}^N : u_i \le 0\}$. Moreover,

$$
\mathrm{sd}^{\varepsilon,\phi^\circ}_{\pm}(u) \ge \mathrm{sd}^{\phi^\circ}_{\widehat{E}} - c_\phi\varepsilon \quad \text{in } \varepsilon\mathbb{Z}^N.
\tag{4.46}
$$

Proof. In this proof we let c_ϕ denote a positive constant which depends on ϕ and that may change from line to line and also within the same line.

We start introducing a slightly modified definition of the discrete signed distance $\mathrm{sd}^{\varepsilon,\phi^\circ}(u)$. Namely, setting

$$
\begin{aligned}
&\partial^+_\varepsilon E := \{i \in \varepsilon\mathbb{Z}^N \setminus E \;:\; \exists j \in E \text{ with } \|i-j\|_\infty = \varepsilon\} \\
&\partial^-_\varepsilon E := \{i \in E \;:\; \exists j \in \varepsilon\mathbb{Z}^N \setminus E \text{ with } \|i-j\|_\infty = \varepsilon\}
\end{aligned},
\tag{4.47}
$$

we define

$$
\tilde{d}_i = \begin{cases}
\inf\{u_j + \phi^\circ(i-j) \;:\; j \in \partial^-_\varepsilon E\}, & \text{for } i \in \varepsilon\mathbb{Z}^N \setminus E \\
\sup\{u_j - \phi^\circ(i-j) \;:\; j \in \partial^+_\varepsilon E\} & \text{for } i \in E
\end{cases}.
\tag{4.48}
$$

We start by showing that

$$
\begin{aligned}
&\mathrm{sd}^{\varepsilon,\phi^\circ}_{\pm}(u) \ge \tilde{d} \quad \text{in } E, \\
&\mathrm{sd}^{\varepsilon,\phi^\circ}_{\pm}(u) \le \tilde{d} \quad \text{in } \varepsilon\mathbb{Z}^N \setminus E.
\end{aligned}
\tag{4.49}
$$

Indeed, we note that for every $i \in E$ we have

$$
(\mathrm{sd}^{\varepsilon,\phi^\circ}_{-}(u))_i = (d^{\varepsilon,\phi^\circ}_{-}(u))_i = \sup_{j\in\{u\ge 0\}} \{u_j - \phi^\circ(i-j)\} \ge \sup_{j\in\partial^+_\varepsilon E} \{u_j - \phi^\circ(i-j)\} = \tilde{d}_i.
$$

On the other hand, recalling that $d_-^{\varepsilon,\phi^\circ}(u) \le u$ in E, for every $i \in \varepsilon\mathbb{Z}^N \setminus E$ we see

$$(\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u))_i = \inf_{j\in\{u\le 0\}}\{(d_-^{\varepsilon,\phi^\circ}(u))_j + \phi^\circ(i-j)\} \le \inf_{j\in\partial_\varepsilon^- E}\{u_j + \phi^\circ(i-j)\} = \tilde{d}_i.$$

Reasoning analogously we show the same inequalities between $\mathrm{sd}_+^{\varepsilon,\phi^\circ}$ and $\tilde{d}$ and thus prove (4.49).

Next, we prove

$$\sup_{\varepsilon\mathbb{Z}^N} |\tilde{d} - \mathrm{sd}_{\widehat{E}}^{\phi^\circ}| \le c_\phi \varepsilon. \tag{4.50}$$

Recall that by definition (4.47), since $u \le 0$ in E and $u > 0$ in $\varepsilon\mathbb{Z}^N \setminus E$ and since u is $(1,\phi^\circ)$-Lipschitz, it holds

$$|u_j| \le c_\phi \varepsilon \quad \text{for } j \in \partial_\varepsilon^\pm E.$$

Then, for every $i \in \varepsilon\mathbb{Z}^N \setminus E$ we have

$$\tilde{d}_i = \inf_{j\in\partial_\varepsilon^- E}\{u_j + \phi^\circ(i-j)\} \ge \inf_{j\in\partial_\varepsilon^- E} \phi^\circ(i-j) - c_\phi\varepsilon \ge \mathrm{sd}_{\widehat{E}}^{\phi^\circ}(i) - c_\phi\varepsilon. \tag{4.51}$$

On the other hand, by definition of $\mathrm{sd}_{\widehat{E}}^{\phi^\circ}$ there exists $x \in \partial\widehat{E}$ such that $\mathrm{sd}_{\widehat{E}}^{\phi^\circ}(i) = \phi^\circ(i-x)$. Let $k \in \varepsilon\mathbb{Z}^N$ be the closest point from x in $\partial_\varepsilon^- E$. We have

$$\begin{aligned} \mathrm{sd}_{\widehat{E}}^{\phi^\circ}(i) &= \phi^\circ(i-x) \ge \phi^\circ(i-k) - c_\phi\varepsilon \\ &\ge \phi^\circ(i-k) + u_k - c_\phi\varepsilon \ge \tilde{d}_i - c_\phi\varepsilon. \end{aligned} \tag{4.52}$$

Finally, equation (4.51) and (4.52) imply (4.50) outside E. The other case is analogous.

We now finally prove (4.45) outside E. From (4.49) and (4.50) it holds

$$d_-^{\varepsilon,\phi^\circ}(u) = \mathrm{sd}_-^{\varepsilon,\phi^\circ}(u) \ge \tilde{d} \ge \mathrm{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon \quad \text{in } E.$$

In particular, $\mathrm{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon$ is an admissible competitor in (4.34), thus $\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u) \ge \mathrm{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon$ in $\varepsilon\mathbb{Z}^N$. On the other hand, in $\varepsilon\mathbb{Z}^N \setminus E$ it holds (4.49), thus we conclude (4.45) for $\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u)$. Concerning $\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u)$, we note that by Remark 4.10 and the equation above it holds

$$u \ge \mathrm{sd}_-^{\varepsilon,\phi^\circ}(u) \ge \mathrm{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon \quad \text{in } E.$$

The function $\mathrm{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon$ is therefore admissible in (4.35), thus by maximality

$$d_+^{\varepsilon,\phi^\circ}(u) \ge \mathrm{sd}_{\widehat{E}}^{\phi^\circ} - c_\phi\varepsilon.$$

Since $\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) = d_+^{\varepsilon,\phi^\circ}(u)$ in $\varepsilon\mathbb{Z}^N \setminus E$ we conclude (4.45), taking also into account again (4.49) and (4.50). Finally, (4.46) follows by combining (4.45), (4.49) and (4.50). □

Lemma 4.14. *Given $u \in X_\varepsilon$ and $(1,\phi^\circ)$-Lipschitz, it holds*

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(-u) = -\mathrm{sd}^{\varepsilon,\phi^\circ}(u). \tag{4.53}$$

Furthermore, if $u_1, u_2 \in X_\varepsilon$ are $(1,\phi^\circ)$-Lipschitz and $u_1 \le u_2$ then

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(u_1) \le \mathrm{sd}^{\varepsilon,\phi^\circ}(u_2). \tag{4.54}$$

Finally, for any $s > 0$ and $u \in X_\varepsilon$ and $(1,\phi^\circ)$-Lipschitz, it holds

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(u-s) \le \mathrm{sd}^{\varepsilon,\phi^\circ}(u) - s. \tag{4.55}$$

Proof. For every $i \in \varepsilon\mathbb{Z}^N$ it holds

$$(d_-^{\varepsilon,\phi^\circ}(-u))_i = \max_{j\in\{(-u)\geq 0\}} \{-u_j - \phi^\circ(i-j)\} = -\min_{j\in\{u\leq 0\}} \{u_j + \phi^\circ(i-j)\} = -(d_+^{\varepsilon,\phi^\circ}(u))_i.$$

In turn,

$$\begin{aligned}(\mathrm{sd}_-^{\varepsilon,\phi^\circ}(-u))_i &= \min_{j\in\{(-u)\leq 0\}} \Big\{(d_-^{\varepsilon,\phi^\circ}(-u))_j + \phi^\circ(i-j)\Big\} \\ &= -\max_{j\in\{u\geq 0\}} \Big\{(d_+^{\varepsilon,\phi^\circ}(u))_j - \phi^\circ(i-j)\Big\} = -(\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u))_i.\end{aligned}$$

Reasoning in the same way for $d_+^{\varepsilon,\phi^\circ}, \mathrm{sd}_+^{\varepsilon,\phi^\circ}$ we arrive at

$$\mathrm{sd}_\pm^{\varepsilon,\phi^\circ}(-u) = -\mathrm{sd}_\mp^{\varepsilon,\phi^\circ}(u) \tag{4.56}$$

and thus $\mathrm{sd}^{\varepsilon,\phi^\circ}(-u) = -\mathrm{sd}^{\varepsilon,\phi^\circ}(u)$. The monotonicity property (4.54) follows easily from Definition (4.32). The proofs of the other results also follow from Definition (4.32), we present only the one concerning (4.55). Fix $s > 0$ and $u \in X_\varepsilon$ be a $(1, \phi^\circ)$-Lipschitz function. By definition of $d_-^{\varepsilon,\phi^\circ}(u)$ we have

$$(d_-^{\varepsilon,\phi^\circ}(u))_i = \sup_{j\in\{u\geq 0\}} \{u_j - \phi^\circ(i-j)\} \geq s + \sup_{j\in\{u\geq s\}} \{(u_j - s) - \phi^\circ(i-j)\} = (d_-^{\varepsilon,\phi^\circ}(u-s))_i + s.$$

Analogously

$$\begin{aligned}(\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u))_i &= \inf_{j\in\{u\leq 0\}} \Big\{(d_-^{\varepsilon,\phi^\circ}(u))_j + \phi^\circ(i-j)\Big\} \\ &\geq s + \inf_{j\in\{u\leq s\}} \Big\{(d_-^{\varepsilon,\phi^\circ}(u-s))_j + \phi^\circ(i-j)\Big\} = s + (\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u-s))_i.\end{aligned}$$

Since the proofs for $d_+^{\varepsilon,\phi^\circ}(u), \mathrm{sd}_+^{\varepsilon,\phi^\circ}(u)$ are analogous, we conclude. □

Remark 4.15. (Evolution of the complement) Let $E^h(t), u^h(t)$ be as in (4.43). We note that, if $F_0 \subseteq \mathbb{R}^N$ is a closed set such that $F^{h,0} = \varepsilon\mathbb{Z}^N \setminus E^{h,0}$, then the discrete evolution starting from F_0 coincides with $\{u^h(t) \geq 0\}$ for every $t \geq 0$. Indeed, denoting v^h the discrete evolution starting from F_0, it holds by definition $v^{h,0} = -u^{h,0}$, thus recalling (4.53) we have

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(v^{h,0}) = -\mathrm{sd}^{\varepsilon,\phi^\circ}(u^{h,0})$$

and, by uniqueness for (4.31) it follows that $v^h(h) = -u^h(h)$. Then we can iterate to conclude.

Remark 4.16 (Comparison principle)**.** Let E_0, F_0 be closed sets in $\mathbb{R}^N$ such that $E^{h,0} \subseteq F^{h,0}$ (note that this condition is satisfied if $E_0 \subseteq F_0$). Let $E^h(t), F^h(t)$ be the corresponding discrete evolutions and let $u^h(t), v^h(t)$ be the associated functions as in (4.43). Then, for every $t \geq 0$ it holds $E^h(t) \subseteq F^h(t)$. This follows easily by iteration from the monotonicity property (4.54) and from the comparison principle for (4.31). One in fact could also consider the "open" discrete evolution given by $\mathring{E}^h(t) := \{u^h(t) < 0\}$ and $\mathring{F}^h(t) := \{v^h(t) < 0\}$. Then, by the same argument one also have that $\mathring{E}^h(t) \subseteq \mathring{F}^h(t)$.

Remark 4.17 (Avoidance principle)**.** Let $E_0, F_0 \subseteq \mathbb{R}^N$ be closed sets such that $E^{h,0} \cap F^{h,0} = \emptyset$ (which is, for example, implied by $\mathrm{dist}(E_0, F_0) > c_\phi \varepsilon$ for a suitable $c_\phi > 0$). Let E^h, u^h and $\mathring{F}^h(t), v^h$ be the closed and open discrete evolutions starting from E_0, F_0 respectively (where the open discrete evolution has been defined in Remark 4.16). Then,

$$\mathring{F}^h(t) \subseteq \varepsilon\mathbb{Z}^N \setminus E^h(t).$$

Indeed, $F^{h,0} \subseteq \varepsilon\mathbb{Z}^N \setminus E^{h,0}$ implies that $-u^{h,0} \leq v^{h,0}$ and thus by (4.53) and (4.54)

$$-\mathrm{sd}^{\varepsilon,\phi^\circ}(u^{h,0}) = \mathrm{sd}^{\varepsilon,\phi^\circ}(-u^{h,0}) \leq \mathrm{sd}^{\varepsilon,\phi^\circ}(v^{h,0}).$$

By the comparison principle for (4.31) and iterating one sees that $-u^h(t) \le v^h(t)$ for all $t \ge 0$, which implies

$$\mathring{F}^h(t) = \{v^h(t) < 0\} \subseteq \{u^h(t) > 0\} = \varepsilon\mathbb{Z}^N \setminus E^h(t).$$

Remark 4.18. We conclude this section by observing that we could have made different choices of the distance function, without affecting the final convergence result. In definition (4.32) we could have set

$$\begin{aligned}(d^<(u))_i &= \inf_{j\in\{u<0\}} \{u_j + \phi^\circ(i-j)\}\,,\\ (\mathrm{sd}^<(u))_i &= \sup_{j\in\{u\ge 0\}} \left\{(d^<(u))_j - \phi^\circ(i-j)\right\},\\ (d^\le(u))_i &= \inf_{j\in\{u\le 0\}} \{u_j + \phi^\circ(i-j)\}\,,\\ (\mathrm{sd}^\le(u))_i &= \sup_{j\in\{u>0\}} \left\{(d^<(u))_j - \phi^\circ(i-j)\right\}.\end{aligned} \tag{4.57}$$

One can see that $\mathrm{sd}^\le(u)$ mimics the signed distance function to the boundary of $\{u \le 0\}$ while $\mathrm{sd}^<(u)$ mimics the signed distance function to the boundary of $\{u < 0\}$. Defining the algorithm as in (4.42) but with $\mathrm{sd}^<, \mathrm{sd}^\le$ replacing $\mathrm{sd}^{\varepsilon,\phi^\circ}$, adapting our proof one can conclude the same convergence result. Let us further comment on the relation between $\mathrm{sd}^{\varepsilon,\phi^\circ}, \mathrm{sd}^\le, \mathrm{sd}^<$. One can prove that for any $(1,\phi^\circ)$-Lipschitz function $u \in X_\varepsilon$, then

$$\mathrm{sd}^\le(u) \le \mathrm{sd}_-^{\varepsilon,\phi^\circ}(u) \le \mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) \le \mathrm{sd}^<(u). \tag{4.58}$$

Thus, between the many possible choices we could have performed in (4.32), it turns out that $\mathrm{sd}^<$ is the "maximal" one, while $\mathrm{sd}^\le$ is the "minimal". Indeed, let us show that $\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u) \le \mathrm{sd}_+^{\varepsilon,\phi^\circ}(u)$. By definition (4.32) and (4.36), (4.38) for every $i \in \{u \ge 0\}$ it holds

$$(\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u))_i = \inf_{j\in\{u\le 0\}} \left\{(d_-^{\varepsilon,\phi^\circ}(u))_j + \phi^\circ(i-j)\right\} = \inf_{j\in\{u\le 0\}} \{u_j + \phi^\circ(i-j)\} = (\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u))_i.$$

Reasoning analogously, for every $i \in \{u \le 0\}$ it holds

$$(\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u))_i = \sup_{j\in\{u\ge 0\}} \left\{(d_+^{\varepsilon,\phi^\circ}(u))_j - \phi^\circ(i-j)\right\} = \sup_{j\in\{u\ge 0\}} \{u_j - \phi^\circ(i-j)\} = (\mathrm{sd}_-^{\varepsilon,\phi^\circ}(u))_i.$$

Furthermore, for any two $(1,\phi^\circ)$-Lipschitz functions $u, u' \in X_\varepsilon$, if $u \le u' - s$ for $s > 0$ then

$$\mathrm{sd}^<(u) \le \mathrm{sd}^\le(u') - s.$$

In particular, this implies that for any $(1,\phi^\circ)$-Lipschitz function $u \in X_\varepsilon$ and $s' > s$ then

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(u-s) \le \mathrm{sd}^{\varepsilon,\phi^\circ}(u-s') + s' - s.$$

Fix $u_0 \in X_\varepsilon$ is a $(1,\phi^\circ)$-Lipschitz function. Using the properties above and standard arguments, one can see that for all but countably many $s \in \mathbb{R}$ the discrete evolutions starting from $\{u_0 \le s\}$ and corresponding to the three possible choices of distances in (4.58) coincide.

4.2 Discrete evolution of Wulff shapes

In this section we provide some control on the evolution speed of discrete Wulff shapes. The first result estimates the solution to (4.31) for the distance to the Wulff shape.

Lemma 4.19. *There exists a constant $C = C(\phi) > 0$ with the following property. If u is the solution to* (4.31) *with $g = \phi^\circ$, then $u \le \phi^h$, where $\phi^h \in X_\varepsilon$ is defined as*

$$\phi_i^h := \begin{cases} \phi^\circ(i) + \frac{Ch}{\phi^\circ(i)} & \text{if } \phi^\circ(i) \ge C(\sqrt{h} \vee \varepsilon) \\ C(\sqrt{h} \vee \varepsilon) + \frac{Ch}{\sqrt{h}\vee\varepsilon} & \text{otherwise.} \end{cases} \tag{4.59}$$

The proof of Lemma 4.19, based on the construction of a calibration, is postponed to Appendix 4.A.1. We now prove a useful lemma used to estimate the redistancing step in our algorithm for functions of the form of (4.59).

Lemma 4.20. *Let $R \geq \delta > 0$ and set*

$$u := (\phi^\circ - R) \vee (\delta/2 - R).$$

Then, for ε, h small enough depending on δ it holds

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(u) \leq \phi^\circ - R + \hat{c}\varepsilon \quad \text{in } \varepsilon\mathbb{Z}^N, \tag{4.60}$$

for a suitable positive constant $\hat{c}$, depending on ϕ. Furthermore, if we assume (4.118)*, it holds*

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(u) \leq \phi^\circ - R \quad \text{in } \varepsilon\mathbb{Z}^N. \tag{4.61}$$

Proof. By (4.58), it is sufficient to prove the claim for $\mathrm{sd}_+^{\varepsilon,\phi^\circ}$. We start showing that $d_+^{\varepsilon,\phi^\circ}(u) = u$, noting that by (4.38) it suffices to prove $d_+^{\varepsilon,\phi^\circ}(u) \leq u$ in $\{u \geq 0\} = \{\phi^\circ \geq R\}$. Assuming (4.118), given $i \in \{u \geq 0\}$ we note that $\phi^\circ(i) \geq R$ thus by Lemma 4.29 there exists $j \in \mathcal{W}_R \setminus \mathcal{W}_{R-2\varepsilon\ell_1}$ satisfying

$$\phi^\circ(j) + \phi^\circ(i-j) = \phi^\circ(i).$$

Taking $\varepsilon = \varepsilon(\delta)$ we can ensure that $R - 2\varepsilon\ell_1 \geq \delta/2$, so that $j \in (\mathcal{W}_R \setminus \mathcal{W}_{\delta/2}) \cap \varepsilon\mathbb{Z}^N$. By definition (4.57) and the equation above we conclude that

$$d_+^{\varepsilon,\phi^\circ}(u) \leq u_j + \phi^\circ(i-j) = \phi^\circ(j) - R + \phi^\circ(i-j) = \phi^\circ(i) - R,$$

hence we have shown that $d_+^{\varepsilon,\phi^\circ}(u) = u$. Finally, from the definition (4.32) and since $d_+^{\varepsilon,\phi^\circ}(u) = u = \phi^\circ - R$ on $\{u \geq 0\}$, we conclude by the triangular inequality that $\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) \leq \phi^\circ - R$. All in all, we have obtained (4.61).

If instead (4.118) does not hold, using the first part of Lemma 4.29 and reasoning as above, one concludes that

$$\mathrm{sd}_+^{\varepsilon,\phi^\circ}(u) \leq \phi^\circ - R + \hat{c}\varepsilon,$$

for a positive constant $\hat{c}$, and then the conclusion follows. □

Combining the two results above we can provide a bound on the evolution speed of Wulff shapes in the algorithm (4.42).

Proposition 4.21. *Assume either $\varepsilon \leq O(h)$ or that* (4.118) *holds. For every $\delta > 0$ there exist ε_0, h_0, c_0 positive constants depending on δ with the following property. If $R \geq \delta$, $\varepsilon \leq \varepsilon_0$ and $h \leq h_0$, then the discrete evolution of $\mathcal{W}_R$ defined in* (4.42)*, denoted $\mathcal{W}^h(t)$, satisfies*

$$\mathcal{W}^h(t) \supseteq (\mathcal{W}_{R-c_0(t+\varepsilon)}) \cap \varepsilon\mathbb{Z}^N), \tag{4.62}$$

as long as $R - c_0(t+\varepsilon) \geq \delta/2$.

Proof. Let $\mathring{\mathcal{W}}^h(t)$ be the open discrete evolution (see Remark 4.16) starting from the closure of $\mathcal{W}_R$, for some $R > 0$ and let $v^h(t)$ be the associated function as in (4.43). Using the definition of $v^{h,0}$, (4.37) and the first definition in (4.32), it is easy to see that

$$(\mathrm{sd}_-^{\varepsilon,\phi^\circ}(v^{h,0}))_0 = (d_-^{\varepsilon,\phi^\circ}(v^{h,0}))_0 \leq -R + c_\phi\varepsilon. \tag{4.63}$$

On the other hand, consider $i \in \{v^{h,0} \geq 0\}$ and let $x' \in \partial\mathcal{W}_R$ be such that

$$\phi^\circ(i - x') = \phi^\circ(i) - \phi^\circ(x') = \phi^\circ(i) - R.$$

Since there exists $j' \in \{v^{h,0} \leq 0\}$ such that $\phi^\circ(j' - x') \leq c_\phi\varepsilon$, then by triangular inequality

$$\phi^\circ(i - j') \leq \phi^\circ(i) - R + c_\phi\varepsilon.$$

Thus, using again definition (4.32), we get

$$(d_+^{\varepsilon,\phi^\circ}(v^{h,0}))_i \leq \inf_{j\in\{v^{h,0}\leq 0\}} \phi^\circ(i-j) \leq \phi^\circ(i) - R + c_\phi\varepsilon,$$

which implies

$$(\mathrm{sd}_+^{\varepsilon,\phi^\circ}(v^{h,0}))_0 \leq \sup_{j\in\{v^{h,0}\geq 0\}} (d_+^{h,0}(v^{h,0}))_j - \phi^\circ(j) \leq -R + c_\phi\varepsilon. \tag{4.64}$$

Therefore, since $\mathrm{sd}^{\varepsilon,\phi^\circ}(v^{h,0})$ is a $(1,\phi^\circ)$-Lipschitz function, from (4.63), (4.64) we get that

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(v^{h,0}) \leq \phi^\circ - R + c_\phi\varepsilon \quad \text{in } \varepsilon\mathbb{Z}^N.$$

By comparison and Lemma 4.19 we obtain

$$v^h(h) \leq \phi^h - R + c_\phi\varepsilon, \tag{4.65}$$

where $\phi^h \in X_\varepsilon$ is defined in (4.59). Considering $R \geq \delta$ and $h = h(\delta), \varepsilon = \varepsilon(\delta)$ small enough, the equation above implies that

$$v^h(h) \leq (\phi^\circ - R + c_0 h + c_\phi\varepsilon) \vee \left(\frac{\delta}{2} - R\right) \tag{4.66}$$

where $c_0 = 4C/\delta$, with C the same as in (4.59). Assume first (4.118). From Lemma 4.20, with R replaced by $R - c_0 h - c_\phi\varepsilon$, we get

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(v^h(h)) \leq \phi^\circ - R + c_0 h + c_\phi\varepsilon, \tag{4.67}$$

therefore by comparison and Lemma 4.19 we get

$$v^h(2h) \leq \phi^h - R + c_0 h + c_\phi\varepsilon,$$

which, reasoning as above, implies for $\varepsilon(\delta), h(\delta)$ small

$$v^h(2h) \leq (\phi^\circ - R + 2c_0 h + c_\phi\varepsilon) \vee \left(\frac{\delta}{2} - R\right).$$

Hence, we can iterate the argument to conclude that

$$v^h(t) \leq (\phi^\circ - R + c_0 t + c_\phi\varepsilon) \vee \left(\frac{\delta}{2} - R\right), \tag{4.68}$$

as long as $R - c_0 t - c_\phi\varepsilon \geq \delta/2$ and ε, h are sufficiently small. In particular, this implies (4.62) (possibly changing the value of c_0).

If instead (4.118) does not hold and $\varepsilon \leq O(h)$, we obtain (4.65), (4.66) in the same way. Then, using the first part of Lemma 4.20 we get

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(v^h(h)) \leq \phi^\circ - R + c_0 h + \hat{c}\varepsilon + c_\phi\varepsilon, \tag{4.69}$$

then iterating we get

$$v^h(kh) \leq (\phi^\circ - R + kc_0 h + k\hat{c}\varepsilon + c_\phi\varepsilon) \vee \left(\frac{\delta}{2} - R\right),$$

hence, recalling that $\varepsilon \leq O(h)$ we conclude (4.68) and (4.62), as long as $R - c_0 t - c_\phi\varepsilon \geq \delta/2$, with ε, h sufficiently small and possibly changing the value of c_0. □

As a corollary of the previous result, we deduce an estimate of the evolution of the distance function $\widehat{d^h}$ at distance from the evolving boundary, which we show next.

Corollary 4.22. *Let $E_0 \subseteq \mathbb{R}^N$ be a closed set and consider the discrete evolution defined in (4.43). Assume either that $\varepsilon \le O(h)$ or that (4.118) holds. Then, for every $\delta > 0$ there exist $c_0 = c_0(\delta) > 0$, $h_0 = h_0(\delta) > 0$ and $\varepsilon_0 = \varepsilon_0(\delta)$ such that the following holds. If $\widehat{d}^h(x,t) \ge \delta$, then for $s \ge t$,*

$$\widehat{d}^h(x,s) \ge \widehat{d}^h(x,t) - c_0(s - t + \varepsilon + h) \tag{4.70}$$

provided $0 < h \le h_0$, $0 < \varepsilon < \varepsilon_0$ and as long as $\widehat{d}^h(x,t) - c_0(s - t + \varepsilon + h) \ge \delta/2$. Similarly, if $\widehat{d}^h(x,t) \le -\delta$, then for $s \ge t$,

$$\widehat{d}^h(x,s) \le \widehat{d}^h(x,t) + c_0(s - t + \varepsilon + h) \tag{4.71}$$

provided $0 < h \le h_0$ and as long as $\widehat{d}^h(x,t) + c_0(s - t + \varepsilon + h) \le -\delta/2$.

Proof. As usual, in this proof we denote by c_ϕ a positive constant depending on ϕ whose value may change from line to line and also within the same line.

Assume $\widehat{d}^h(x,t) \ge \delta$. Without loss of generality we may assume $t \in [0, T_h^*)$ so that $\widehat{d}^h(x,t)$ is finite. Denote by $x_\varepsilon \in \varepsilon\mathbb{Z}^N$ such that $x \in x_\varepsilon + [0,\varepsilon)^N$. Note that there exists a constant $c_\phi > 0$ such that, setting $R := \widehat{d}^h(x,t) - c_\phi\varepsilon$, one has $(\mathcal{W}_R(x_\varepsilon))^{h,0} \cap E^h(t) = \emptyset$ and $R > \delta/2$ (if ε, h are sufficiently small, depending on δ). By the avoidance principle stated in Remark 4.17, we deduce that the open discrete evolution of $\mathcal{W}_R(x_\varepsilon)$, which we denote by $F(\tau)$, lies outside $E^h([\frac{t}{h}]h + \tau)$ for all $\tau \ge 0$. By Proposition 4.21 we deduce

$$F(\tau) \supseteq \mathcal{W}_{R - c_0(\tau+\varepsilon)}(x_\varepsilon) \cap \varepsilon\mathbb{Z}^N, \tag{4.72}$$

provided that $R - c_0(\tau + \varepsilon) \ge \delta/2$. Note that in particular

$$(\mathcal{W}_{R - c_0(\tau+h+\varepsilon)}(x_h) \cap \varepsilon\mathbb{Z}^N) \subseteq (\varepsilon\mathbb{Z}^N \setminus E^h(t+\tau)),$$

as long as $R - c_0(\tau + h + \varepsilon) \ge \delta/2$. In turn, we get

$$\widehat{d}^h(x_\varepsilon, t+\tau) \ge R - c_0(\tau + h + \varepsilon), \tag{4.73}$$

provided $R - c_0(\tau + h + \varepsilon) \ge \delta/2$ (for a possibly larger value of c_0). Recalling the definition of R and x_ε and possibly increasing the value of c_0, we infer

$$\widehat{d}^h(x, t+\tau) \ge \widehat{d}^h(x,t) - c_0(\tau + h + \varepsilon) \tag{4.74}$$

as long as $\widehat{d}^h(x,t) - c_0(\tau + h + \varepsilon) \ge \delta$. The case $\widehat{d}^h(x,t) \le -\delta$ is analogous. □

5 Convergence of the scheme

We now are ready to study the convergence of the scheme as $\varepsilon \to 0, h \to 0$. Recall that we assumed that $\varepsilon = \varepsilon(h)$ goes to 0 as $h \to 0$. In this section we assume that either $\varepsilon \le O(h)$ or that (4.118) holds. Let $E^h(\cdot)$ be the discrete evolution defined in (4.43) and recall that $\widehat{E}^h(\cdot) = E^h(\cdot) + [0,\varepsilon]^N$. We introduce the closed space-time tubes

$$\overline{E}^h := \mathrm{cl}\big(\{(x,t) \in \mathbb{R}^N \times [0,+\infty) : x \in \widehat{E}^h(t)\}\big) \tag{4.75}$$

where the closure is in space-time. Then, there exist A, E open and closed (respectively) subsets of $\mathbb{R}^N \times [0,+\infty)$, with $A \subseteq E$, and a subsequence $h_k \to 0$ such that

$$\overline{E}^{h_k} \xrightarrow{\mathcal{K}} E \qquad \text{and} \qquad \mathbb{R}^N \times [0,+\infty) \setminus \mathrm{int}\big(\overline{E}^{h_k}\big) \xrightarrow{\mathcal{K}} \mathbb{R}^N \times [0,+\infty) \setminus A,$$

where interior, and Kuratowski convergence are meant in space-time. Let $E(t)$ and $A(t)$ be the t-time slice of E and A, respectively..

Note that if $E(t) = \emptyset$ for some $t \ge 0$, then (4.70) implies $E(s) = \emptyset$ for all $s \ge t$ so that we can define, as in Definition 4.1, the extinction time T^* of E. In the same fashion one can define the extinction time T'^* of $\mathbb{R}^N \times [0,+\infty) \setminus A$ (notice that at least one between T^* and T'^* is $+\infty$). Possibly extracting a further (not relabelled) subsequence and arguing exactly as in [55, Proof of Proposition 4.4] (and relying on the bounds (4.70) and (4.71)), one can in fact show the following result.

Proposition 4.23. *There exists a countable set $\mathcal{N} \subseteq (0,+\infty)$ such that $\widehat{d}^{h_k}(\cdot,t)^+ \to \mathrm{dist}^{\phi^\circ}(\cdot,E(t))$ and $\widehat{d}^{h_k}(\cdot,t)^- \to \mathrm{dist}^{\phi^\circ}(\cdot,\mathbb{R}^N \setminus A(t))$ locally uniformly for all $t \in (0,+\infty) \setminus \mathcal{N}$. Moreover, E and $\mathbb{R}^N \times [0,+\infty) \setminus A$ satisfy the continuity properties (b) and (c) of Definition 4.1. In addition, if $T^* > 0$, then $\{\widehat{d}^{h_k}\}$ is locally uniformly bounded in $\mathbb{R}^N \times (0,T^*) \setminus E$ and analogously $\{\widehat{d}^{h_k}\}$ is locally uniformly bounded in $\mathbb{R}^N \times (0,T'^*) \cap A$ if $T'^* > 0$. Finally, $E(0) = E_0$ and $A(0) = \mathrm{int}(E_0)$.*

The main result of the chapter is the following one.

Theorem 4.24. *The set E is a superflow in the sense of Definition 4.1 with initial datum E_0, while A is a subflow with initial datum E_0.*

The proof of this result follows the main lines of the proof of [55, Theorem 4.5]. One important difference with respect to the local, continuous setting is that the variable z^{h_k} is defined on the *edges* (i,j) between the vertices $i \in \varepsilon\mathbb{Z}^N$ and it is therefore unclear how to pass to the limit in this variable to obtain the limiting vector field $z(x,t)$. In order to do so, we associate with the discrete vector field $z^h_{ij}(t) \in Y_\varepsilon$ a vector field $\mathbf{z}^h(\cdot,t)$ in $\mathbb{R}^N$ defined as follows:

$$\mathbf{z}^h(x,t) := \frac{1}{\varepsilon} \sum_{j \in \varepsilon\mathbb{Z}^N} z^h_{ij}(t)(i-j), \tag{4.76}$$

where $i \in \varepsilon\mathbb{Z}^N$ is such that $x \in i + [0,\varepsilon)^N$. Recall that we can take $z^h_{ij}(t)$ and thus $\mathbf{z}^h(\cdot,t)$ identically zero for $t \geq \min\{T^*_h, T'^*_h\}$. First, we show the following:

Lemma 4.25. *The vector field $\mathbf{z}^h$ satisfies*

$$\phi^\circ(\mathbf{z}^h) \leq 1. \tag{4.77}$$

Proof. Take $v \neq 0$ in $\mathbb{R}^N$. Recalling that $\phi(v) = \sum_{\ell \in \mathbb{Z}^N} \beta(\ell)|v \cdot \ell|$, one has for any $x \in \mathbb{R}^N$ and $i \in \varepsilon\mathbb{Z}^N$ such that $x \in i + [0,\varepsilon)^N$

$$\mathbf{z}^h(x,t) \cdot v = \frac{1}{\varepsilon} \sum_{j \in \varepsilon\mathbb{Z}^N} z^h_{ij}(t)(i-j) \cdot v = \sum_{\ell \in \mathbb{Z}^N} z^h_{i,i+\varepsilon\ell}(t)\ell \cdot v \leq \phi(v), \tag{4.78}$$

where we used that $|z^h_{i,i+\varepsilon\ell}(t)| \leq \beta(\ell)$. □

Hence, being globally bounded, this vector field is weakly-$*$ compact in $L^\infty(\mathbb{R}^N \times (0,T);\mathbb{R}^N)$ for any $T > 0$. The following lemma establishes a relationship between the divergence of its limits and the limits of the discrete divergences of z^h.

Lemma 4.26. *Assume that $\mathbf{z}^{h_k} \overset{*}{\rightharpoonup} z$ in $L^\infty(\mathbb{R}^N \times (0,T);\mathbb{R}^N)$ along a subsequence $h_k \to 0$. Then, for every $\varphi \in C^\infty(\mathbb{R}^N \times (0,T))$ and $\eta \in C^\infty_c(\mathbb{R}^N \times (0,T))$ it holds*

$$\lim_{k\to\infty} \left(\varepsilon_k^N \int \sum_{i,j \in \varepsilon_k\mathbb{Z}^N} z^{h_k}_{ij}(t)\eta(i,t)\, \frac{\varphi(i,t)-\varphi(j,t)}{\varepsilon_k}\, \mathrm{d}t \right) = \iint \eta\, z \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t.$$

Proof. Let $\varphi \in C^\infty(\mathbb{R}^N)$ and $\eta \in C^\infty_c(\mathbb{R}^N)$ and denote $S(t) = \mathrm{supp}(\eta(t))$ and $Q_k := [0,\varepsilon_k)^N$. We have

$$\varepsilon_k^N \sum_{i,j \in \varepsilon_k\mathbb{Z}^N} z^{h_k}_{ij}(t)\eta(i,t)\, \frac{\varphi(i,t)-\varphi(j,t)}{\varepsilon_k} = \varepsilon_k^N \sum_{i,j \in \varepsilon_k\mathbb{Z}^N} \frac{z^{h_k}_{ij}(t)}{\varepsilon_k}\eta(i,t)\,\nabla\varphi(x_{ij}) \cdot (i-j), \tag{4.79}$$

where x_{ij} belongs to the segment between i and j. Furthermore we have

$$\begin{aligned}
&\left|\varepsilon_k^N \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)\,\frac{\varphi(i,t)-\varphi(j,t)}{\varepsilon_k} - \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} \frac{z_{ij}^{h_k}(t)}{\varepsilon_k}\int_{i+Q_k}\eta\nabla\varphi\cdot(i-j)\,\mathrm{d}x\right| \\
&\le \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} \frac{\alpha_{ij}^{\varepsilon_k}}{\varepsilon_k}|\eta(i,t)|\int_{i+Q_k}|(\nabla\varphi(x_{ij},t)-\nabla\varphi(x,t))\cdot(i-j)|\,\mathrm{d}x + O(\varepsilon_k^N) \qquad (4.80)\\
&\le 2\|\eta\|_\infty \sum_{i\in S(t)\cap\varepsilon_k\mathbb{Z}^N}\sum_{j\in\varepsilon_k\mathbb{Z}^N} \frac{\alpha_{ij}^{\varepsilon_k}}{\varepsilon_k}\int_{i+Q_k}|(\nabla\varphi(x_{ij},t)-\nabla\varphi(x,t))\cdot(i-j)|\,\mathrm{d}x + O(\varepsilon_k^N)\\
&\le c\varepsilon_k^N \sum_{i\in S\cap\varepsilon_k\mathbb{Z}^N}\sum_{j\in\varepsilon_k\mathbb{Z}^N}\frac{\alpha_{ij}^{\varepsilon_k}+\alpha_{ji}^{\varepsilon_k}}{\varepsilon_k}|i-j|^2 + O(\varepsilon_k^N) \qquad (4.81)\\
&= c\varepsilon_k^{N+1}\sum_{\substack{i\in\mathbb{Z}^N\\ \varepsilon_k i\in S(t)}}\sum_{j\in\mathbb{Z}^N}\alpha_{ij}|i-j|^2 + O(\varepsilon_k^N)\\
&\le c\varepsilon_k^{N+1}\left(\sum_{\ell\in\mathbb{Z}^N}\beta(\ell)|\ell|^2\right)(\#S(t)\cap\varepsilon_k\mathbb{Z}^N) + O(\varepsilon_k^N)\\
&\le c\varepsilon_k\sum_{\ell\in\mathbb{Z}^N}\beta(\ell)|\ell|^2 + O(\varepsilon_k^N) \qquad (4.82)
\end{aligned}$$

where in (4.80) we used the Lipschitz property of η and (4.28), while in (4.81) we used the Lipschitz property of $\nabla\varphi$ and $|x_{ij}-x| \le (1+\sqrt{N})|i-j|$ for $i \ne j$ and $x \in i+Q_k$, and finally in (4.82) we used that $\#(S(t)\cap\varepsilon\mathbb{Z}^N) = O(\varepsilon_k^{-N})$, which holds locally uniformly in time. Moreover, note that the the estimate provided above is uniform as t varies in compact subsets of $(0,T)$. Recalling (4.28), we conclude integrating in time and sending $k\to\infty$. ☐

At this point, we may proceed with the proof of Theorem 4.24.

Proof of Theorem 4.24. As usual, in this proof we denote by c_ϕ a positive constant depending on ϕ whose value may change from line to line and also within the same line.

We only show that E is a superflow, as the subflow property of A can be proven analogously. Points (a), (b) and (c) of Definition 4.1 follow from Proposition 4.23. We are left with showing (d). Without loss of generality we may assume $T^*>0$ (which follows from Corollary 4.22 if the initial set is not trivial). Note also that by Proposition 4.23 we have $\liminf_k T^*_{h_k} \ge T^*$.

Step 1: (Proof of (4.9)). For $(x,t)\in\mathbb{R}^N\times(0,T^*)\setminus E$ we set $d(x,t) := \mathrm{dist}^{\phi^\circ}(\cdot,E(t))$. By Lemma 4.13 and Proposition 4.23 we have

$$\sup_{\varepsilon_k\mathbb{Z}^N\cap K}|d^{h_k}(t)-d(\cdot,t)| \to 0 \text{ as } k\to\infty \text{ for } t\in(0,T^*)\setminus\mathcal{N} \text{ and for any compact } K\subseteq\mathbb{R}^N\setminus E(t). \qquad (4.83)$$

Moreover, d^{h_k} and d are locally uniformly bounded in $\mathbb{R}^N\times(0,T^*)\setminus E$. Set $\mathbf{z}^{h_k}(\cdot,t) := 0$ for $t > T^*_{h_k}$ if $T^*_{h_k} < T^*$. Extracting a further subsequence, if needed, and recalling Lemma 4.25, we may assume that $\mathbf{z}^{h_k}$ converges weakly-$*$ in $L^\infty(\mathbb{R}^N\times(0,T^*);\mathbb{R}^N)$ to some vector-field z satisfying

$$\phi^\circ(z) \le 1 \qquad (4.84)$$

almost everywhere. Recall that by (4.44) we have $u^{h_k}(t) \le d^{h_k}(t)$ in $\varepsilon_k\mathbb{Z}^N\setminus E^{h_k}(t)$; i.e., in the region where $d^{h_k}(t)$ is nonnegative. Combining with (4.42) (and recalling (4.43)) we infer that for $t < T^*_{h_k}$ it holds

$$-D^*_{\varepsilon_k} z^{h_k}(t+h_k) \le \frac{d^{h_k}(t+h_k)-d^{h_k}(t)}{h_k} \qquad \text{in } \varepsilon_k\mathbb{Z}^N\setminus E^{h_k}(t). \qquad (4.85)$$

Consider a nonnegative test function $\varphi\in C_c^\infty((\mathbb{R}^N\times(0,T^*))\setminus E)$. If k is large enough, then the distance of the support of φ from $\overline{E}^{h_k}$ is bounded away from zero. In particular, d^{h_k} is finite and positive on supp φ. We deduce from (4.85) that

$$\begin{aligned}
&\varepsilon_k^N \int \sum_{i\in\varepsilon_k\mathbb{Z}^N} \varphi(i,t)\left(\frac{d_i^{h_k}(t+h_k)-d_i^{h_k}(t)}{h_k} + (D^*_{\varepsilon_k} z^{h_k}(t+h_k))_i\right)\mathrm{d}t \\
&= -\varepsilon_k^N \int \sum_{i\in\varepsilon_k\mathbb{Z}^N} \frac{\varphi(i,t)-\varphi(i,t-h_k)}{h_k} d_i^{h_k}(t)\,\mathrm{d}t + \varepsilon_k^N \int \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} \frac{z_{ij}^{h_k}(t+h_k)-z_{ji}^{h_k}(t+h_k)}{h_k}\varphi(i,t)\,\mathrm{d}t \\
&= -\varepsilon_k^N \int \sum_{i\in\varepsilon_k\mathbb{Z}^N} \frac{\varphi(i,t)-\varphi(i,t-h_k)}{h_k} d_i^{h_k}(t)\,\mathrm{d}t + \varepsilon_k^N \int \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t+h_k)\frac{\varphi(i,t)-\varphi(j,t)}{h_k}\,\mathrm{d}t \geq 0.
\end{aligned} \tag{4.86}$$

It is easy to check that the first integral in (4.86) converges to $-\iint d\,\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t$ as $k\to\infty$ thanks to (4.83) and since d^{h_k}, d are uniformly bounded. Recalling that $\mathbf{z}^{h_k}$ converges weakly-$*$ in $L^\infty(\mathbb{R}^N\times(0,T^*))$ to z, we use Lemma 4.26 to conclude that the second integral in (4.86) converges to $\iint z\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t$. We thus conclude (4.9).

Step 2: (Convergence of u^{h_k} to d). Firstly, we establish an upper bound for $-D^*_{\varepsilon_k} z_{h_k}$ away from E^{h_k}. We start by noting that definition (4.32) implies

$$\mathrm{sd}^{\varepsilon,\phi^\circ}(u) \leq \frac{1}{2}\left((d_-^{\varepsilon,\phi^\circ}(u))_j + u_\ell + \phi^\circ(\cdot - j) + \phi^\circ(\cdot-\ell)\right) \quad \text{in } \varepsilon\mathbb{Z}^N\setminus\{u\leq 0\}, \tag{4.87}$$

for every $(1,\phi^\circ)$-Lipschitz function $u\in X_\varepsilon$ and $j,\ell\in\{u\leq 0\}$. Therefore, specifying the inequality above for $u^{h_k}(t)$, by the comparison principle and Lemma 4.19 we conclude

$$u_i^{h_k}(t+h_k) \leq \frac{1}{2}\left(\phi^{h_k}_{i-j} + \phi^{h_k}_{i-\ell} + (d_-^{\varepsilon,\phi^\circ}(u^{h_k}(t))_j + u_\ell^{h_k}(t)\right), \quad \forall i\in\varepsilon_k\mathbb{Z}^N\setminus E^{h_k}(t), \tag{4.88}$$

where $j,\ell\in E^{h_k}(t)$. If $\widehat{d}^{h_k}(i,t)\geq R>0$, recalling the definition of ϕ^h, we get

$$u_i^{h_k}(t+h_k) \leq \frac{1}{2}\left(\phi^\circ(i-j) + \phi^\circ(i-\ell) + (d_-^{\varepsilon,\phi^\circ}(u^{h_k}(t))_j + u_\ell^{h_k}(t)\right) + \frac{Ch_k}{R-c_\phi\varepsilon}, \tag{4.89}$$

for all $i\in\varepsilon_k\mathbb{Z}^N\setminus E^{h_k}(t)$. Infimizing in j,ℓ over $E^{h_k}(t)$ in (4.89) and using again (4.32) and (4.38), we conclude

$$u_i^{h_k}(t+h_k) \leq d_i^{h_k}(t) + h_k\frac{C}{R-c_\phi\varepsilon_k} \leq d_i^{h_k}(t) + h_k\frac{C}{R}. \tag{4.90}$$

provided h_k,ε_k are small enough depending on R, and for a possibly larger value of C. As a consequence of (4.90), we obtain

$$-D^*_{\varepsilon_k} z^{h_k}(t+h_k) \leq \frac{C}{R} \qquad \text{in } \{\widehat{d}^{h_k}(\cdot,t)\geq R\}\cap\varepsilon_k\mathbb{Z}^N. \tag{4.91}$$

Using again Lemma 4.26 and the convergences of E_{h_k} and d_{h_k} it follows that

$$\mathrm{div} z \leq \frac{C}{R} \qquad \text{in } \{(x,t)\in\mathbb{R}^N\times(0,T^*):\, d(x,t)>R\}$$

in the sense of distributions. Hence div$z$ is a Radon measure in $\mathbb{R}^N\times(0,T^*)\setminus E$, and $(\mathrm{div} z)^+\in L^\infty(\{(x,t)\in\mathbb{R}^N\times(0,T^*):\, d(x,t)\geq\delta\})$ for every $\delta>0$.

On the other hand, note that for every $i\in\varepsilon_k\mathbb{Z}^N$ it holds

$$d^{h_k}(t) \geq d_i^{h_k}(t) - \phi^\circ(\cdot - i).$$

Thus, by Lemma 4.19 and by comparison as before, we get

$$u_i^{h_k}(t+h_k) \geq d_i^{h_k}(t) - \phi_0^{h_k} = d_i^{h_k}(t) - (C+1)\sqrt{h_k}.$$

Combining the above inequality with (4.90), we deduce for all $t \in (0,T^*) \setminus \mathcal{N}$ and any $\delta > 0$ that

$$\sup_{\{\widehat{d}_{h_k}(\cdot,t)\geq\delta\}\cap\varepsilon_k\mathbb{Z}^N} |u^{h_k}(t+h_k) - d^{h_k}(t)| \leq \sqrt{h_k}(C+2),$$

provided that k is large enough. In particular, recalling also (4.83), we deduce that

$$\sup_{\varepsilon_k\mathbb{Z}^N\cap K} |u^{h_k}(t) - d(\cdot,t)| \to 0 \text{ as } k\to\infty \text{ for } t\in(0,T^*)\setminus\mathcal{N} \text{ and for any compact } K\subseteq\mathbb{R}^N\setminus E(t), \tag{4.92}$$

also with the sequence $\{u^{h_k}\}$ locally (in space and time) uniformly bounded.
Step 3: (The subdifferential inclusion). It remains to show that

$$z \in \partial\phi(\nabla d) \quad \text{a.e. in } \mathbb{R}^N \times (0,T^*) \setminus E. \tag{4.93}$$

Recall that $\xi \in \partial\phi(\eta)$ if and only if $\xi \in \{v : \phi^\circ(v) \leq 1,\ v\cdot\eta \geq \phi(\eta)\}$. Since one inequality has been proved in (4.84), we show the other one. Consider a test function $\eta \geq 0$, $\eta \in C_c^\infty((\mathbb{R}^N \times (0,T^*)) \setminus E)$. Let $\sigma > 0$ and set $d_\sigma \in C^\infty(\mathbb{R}^N \times (0,T^*))$ as $d_\sigma = d * \rho_\sigma$, where ρ_σ are space-time mollifiers. Obviously

$$\begin{aligned}\sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)(u_i^{h_k}(t) - u_j^{h_k}(t)) &= \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t))(d_\sigma(i,t) - d_\sigma(j,t)) \\ &+ \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)\left(u_i^{h_k}(t) - d_\sigma(i,t) - u_j^{h_k}(t) + d_\sigma(j,t)\right).\end{aligned} \tag{4.94}$$

In turn, Lemma 4.26 implies that

$$\lim_{k\to\infty} \varepsilon_k^N \int \left(\sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)\frac{d_\sigma(i,t) - d_\sigma(j,t)}{\varepsilon_k} \right) \mathrm{d}t = \iint z\cdot\nabla d_\sigma\,\eta\,\mathrm{d}x\,\mathrm{d}t. \tag{4.95}$$

Let us thus show that

$$\lim_{\sigma\to0}\lim_{k\to\infty} \varepsilon_k^N \int \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} \left(z_{ij}^{h_k}(t)\eta(i,t)\frac{u_i^{h_k}(t) - d_\sigma(i,t) - u_j^{h_k}(t) - d_\sigma(j,t)}{\varepsilon_k} \right) \mathrm{d}t = 0, \tag{4.96}$$

We set for every $t \in (0,T_h^*)$ and $\sigma > 0$

$$\begin{aligned} m_{k,\sigma}(t) &:= \min_{i\in\mathrm{supp}(\eta)\cap\varepsilon_k\mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t)), \\ M_{k,\sigma}(t) &:= \max_{i\in\mathrm{supp}(\eta)\cap\varepsilon_k\mathbb{Z}^N} (u_i^{h_k}(t) - d_\sigma(i,t)). \end{aligned}$$

The convergence (4.92) implies that these quantities are uniformly bounded and

$$\lim_{\sigma\to0}\lim_{k\to+\infty} m_{k,\sigma}(t) = 0, \qquad \lim_{\sigma\to0}\lim_{k\to+\infty} M_{k,\sigma}(t) = 0, \tag{4.97}$$

uniformly for all $t \notin \mathcal{N}$. For all times $t \in (0,T^*) \setminus \mathcal{N}$ it holds

$$\begin{aligned}
&\varepsilon_k^N \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)\frac{u_i^{h_k}(t)-d_\sigma(i,t)-u_j^{h_k}(t)+d_\sigma(j,t)}{\varepsilon_k}\\
&=\varepsilon_k^N \sum_{i,j\in\varepsilon_k\mathbb{Z}^N} z_{ij}^{h_k}(t)\eta(i,t)\frac{(u_i^{h_k}(t)-d_\sigma(i,t)-m_{k,\sigma}(t))-(u_j^{h_k}(t)-d_\sigma(j,t)-m_{k,\sigma}(t))}{\varepsilon_k}\\
&=\varepsilon_k^N \sum_{i\in\varepsilon_k\mathbb{Z}^N}(u_i^{h_k}(t)-d_\sigma(i,t)-m_{k,\sigma}(t))\sum_{j\in\varepsilon_k\mathbb{Z}^N}\left(\frac{z_{ij}^{h_k}(t)-z_{ji}^{h_k}(t)}{\varepsilon_k}\eta(i,t)+z_{ji}^{h_k}(t)\frac{\eta(i,t)-\eta(j,t)}{\varepsilon_k}\right).
\end{aligned} \tag{4.98}$$

For k large enough, since the support of η is at positive distance from E, by the bound (4.91) one has $D^*_{\varepsilon_k} z^{h_k}(t) \geq -c(\delta)$ on the support for h_k small enough. Thus it holds

$$\begin{aligned}
\varepsilon_k^N \sum_{i\in\varepsilon_k\mathbb{Z}^N}(u_i^{h_k}(t)-d_\sigma(i,t)-m_{k,\sigma}(t))\eta(i,t)\sum_{j\in\varepsilon_k\mathbb{Z}^N}\frac{z_{ij}^{h_k}(t)-z_{ji}^{h_k}(t)}{\varepsilon_k}\\
\geq -c(\delta)\varepsilon_k^N \sum_{i\in\varepsilon_k\mathbb{Z}^N}(u_i^{h_k}(t)-d_\sigma(i,t)-m_{k,\sigma}(t))\eta(i,t).
\end{aligned}$$

Recalling that $\#(\operatorname{supp}(\eta)\cap\varepsilon_k\mathbb{Z}^N) = O(h_k^{-N})$ uniformly in time, by uniform convergence and (4.92) we conclude that

$$\lim_{\sigma\to 0}\liminf_{k\to\infty}\varepsilon_k^N\int\sum_{i\in\varepsilon_k\mathbb{Z}^N}(u_i^{h_k}(t)-d_\sigma(i,t)-m_{\varepsilon,k}(t))\eta(i,t)\sum_{j\in\varepsilon_k\mathbb{Z}^N}\frac{z_{ij}^{h_k}(t)-z_{ji}^{h_k}(t)}{\varepsilon_k}\,\mathrm{d}t\geq 0. \tag{4.99}$$

The other term in (4.98) can be estimated using the Lipschitz constant of η:

$$\begin{aligned}
&\left|\int\sum_{i,j\in\varepsilon_k\mathbb{Z}^N}\varepsilon_k^N(u_i^{h_k}(t)-d_\sigma(i,t)-m_{\varepsilon,k}(t))z_{ji}^{h_k}(t)\frac{\eta(i,t)-\eta(j,t)}{\varepsilon_k}\,\mathrm{d}t\right|\\
&\leq\|\nabla\eta\|_\infty\varepsilon_k^N\int\sum_{i,j\in\varepsilon_k\mathbb{Z}^N}(u_i^{h_k}(t)-d_\sigma(i,t)-m_{\varepsilon,k}(t))\alpha_{ji}^{h_k}\frac{|i-j|}{\varepsilon_k}\,\mathrm{d}t\to 0
\end{aligned}$$

letting first $k\to+\infty$ and then $\sigma\to 0$, thanks to (4.92) and (4.97). Note now that adding and subtracting $M_{\varepsilon,k}(t)$ to (4.96) instead of $m_{\varepsilon,k}(t)$ and reasoning as above, one proves that

$$\begin{aligned}
&\lim_{\sigma\to 0}\limsup_{k\to\infty}\varepsilon_k^N\int\left(\sum_{i\in\varepsilon_k\mathbb{Z}^N}((u_i^{h_k}(t)-d_\sigma(i,t)-M_{\varepsilon,k}(t))\eta(i,t)\sum_{j\in\varepsilon_k\mathbb{Z}^N}\frac{z_{ij}^{h_k}(t)-z_{ji}^{h_k}(t)}{\varepsilon_k}\right)\mathrm{d}t\leq 0,\\
&\lim_{\varepsilon\to 0}\lim_{k\to\infty}\varepsilon_k^N\int\left|\sum_{i,j\in\varepsilon_k\mathbb{Z}^N}((u_i^{h_k}(t)-d_\sigma(i,t)-M_{\varepsilon,k}(t))z_{ji}^{h_k}(t)\frac{\eta(i,t)-\eta(j,t)}{\varepsilon_k}\right|\mathrm{d}t=0.
\end{aligned} \tag{4.100}$$

Combining (4.98), (4.99) and (4.100), we conclude (4.96).

Integrating in time (4.94) and combining (4.95) and (4.96), since $\nabla d_\sigma = \rho_\sigma * \nabla d \to \nabla d$ pointwise a.e. and are uniformly bounded in $L^\infty(\mathbb{R}^N\times(0,T^*);\mathbb{R}^N)$, it holds

$$\lim_{k\to\infty}\varepsilon_k^N\int\left(\sum_{i,j\in\varepsilon_k\mathbb{Z}^N}\eta(i,t)\,z_{ij}^{h_k}(t)\frac{u_i^{h_k}(t)-u_j^{h_k}(t)}{\varepsilon_k}\right)\mathrm{d}t=\iint z\cdot\nabla d\,\eta\,\mathrm{d}x\,\mathrm{d}t.$$

The convergence above can be paired with the lower semicontinuity of the Γ-convergence of the discrete total variations (which follows from an adaptation of classical arguments, see e.g. [51]) and $z_{ij}^{(h)}(u_i^{(h)} - u_j^{(h)}) = \alpha_{ij}^{(h)}|u_i^{(h)} - u_j^{(h)}|$ to obtain

$$\iint \phi(\nabla d)\eta \leq \liminf_{k\to\infty} \varepsilon_k^N \int \left(\sum_{i,j\in\varepsilon_k\mathbb{Z}^N} \eta(i,t)\,\alpha_{ij}^{h_k} \frac{|u_i^{h_k}(t) - u_j^{h_k}(t)|}{\varepsilon_k} \right) \mathrm{d}t$$

$$= \liminf_{k\to\infty} \varepsilon_k^N \int \left(\sum_{i,j\in\varepsilon_k\mathbb{Z}^N} \eta(i,t)\,z_{ij}^{h_k}(t) \frac{u_i^{h_k}(t) - u_j^{h_k}(t)}{\varepsilon_k} \right) \mathrm{d}t = \iint z\cdot\nabla d\,\eta,$$

which shows that $\phi(\nabla d) = z\cdot\nabla d$ a.e. on the support of η, from which we deduce (4.93). □

We conclude this section by observing that the discrete scheme converges to the unique weak flow (in the sense of Definition 4.1) starting from E_0 for "generic" initial data E_0, i.e. whenever fattening does not occur. More precisely, we have the following Corollary.

Corollary 4.27. *Let $u_0 \in \mathrm{UC}(\mathbb{R}^N)$ and for every $\lambda \in \mathbb{R}$ let $\overline{E}_\lambda^h$ be the closed space-time tube of the h-discrete evolution starting from $\{u_0 \leq \lambda\}$; i.e., as in (4.75) with $E_0 = \{u_0 \leq \lambda\}$. Then, there exists a countable set $\mathcal{N}$ such that for all $\lambda \in \mathbb{R}^N \setminus \mathcal{N}$*

$$\overline{E}_\lambda^h \xrightarrow{\mathcal{K}} E_\lambda \quad \text{in } \mathbb{R}^N \times [0,+\infty)$$

as $h \to 0$, where E_λ is the unique weak flow in the sense of Definition 4.1 starting from $\{u_0 \leq \lambda\}$.

Proof. It follows by combining Theorems 4.24 and 4.5. □

6 Numerical experiments

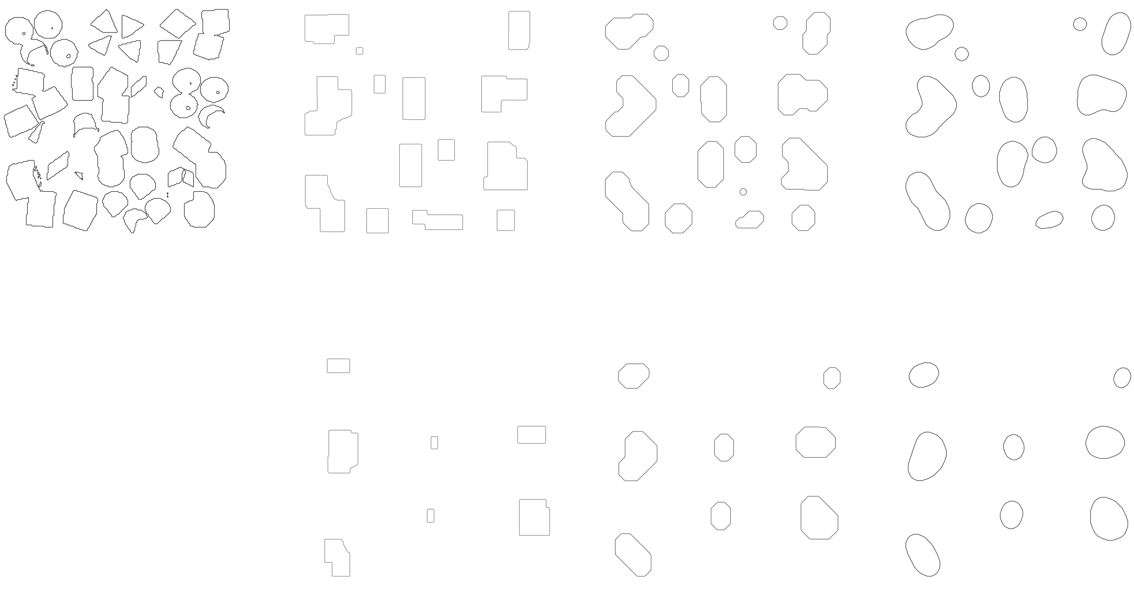

Figure 4.1: An initial datum and evolutions for square, octagonal and "almost isotropic" anisotropies, at two different times.

We show some numerical experiments to illustrate our results, in dimension 2. We follow the implementation described in [47] (see also [48]), except that now the distance is properly computed using using the inf/sup-convolution formulas (4.32). The (exact) numerical resolution of the discrete ROF functional is computed using Hochbaum's parametric maximum flow algorithm [119, 120], implemented upon the maxflow/mincut implementation of Boykov and Kolmogorov [25].

Other implementations of the algorithm yielding approximate minimizers have been considered for instance in [45, 161], of course they work in practice and allow to address more (an)isotropies than those considered here, yet the joint convergence as $\varepsilon = h \to 0$ is not clear in these contexts. For numerical speedup, the infimum and supremum of definition (4.32) are computed only in a neighborhood of fixed size and not on the whole grid. Similarly, the ROF minimization is only performed in a neighborhood of the boundary. We observe that Corollary 4.30 in Appendix 4.A.2 justifies this restriction in some particular case, notably the case $\phi = \|\cdot\|_{\ell^1}, \phi^\circ = \|\cdot\|_{\ell^\infty}$ (where $\ell_1 = 1$ can be chosen in Lemma 4.29), which is particularly relevant. The code is available at `https://plmlab.math.cnrs.fr/chambolle/chapters/fig/discretecrystals/` (implemented in `C/C++` and running on GNU/linux with `gcc`).

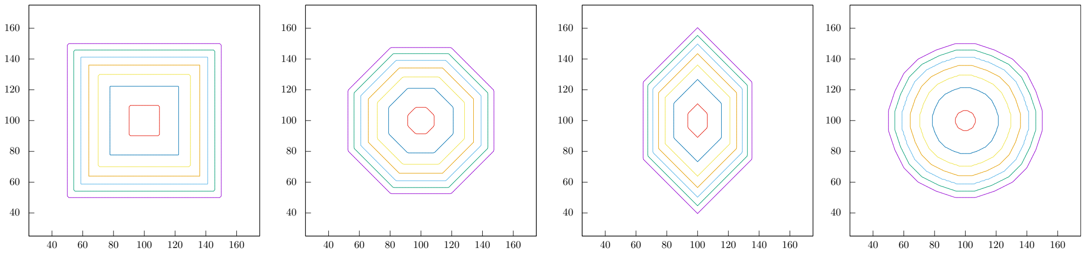

Figure 4.2: Wulff shapes of initial radius $R_0 = 50$ evolved at times $t = 0, 200, 400, \dots, 1200$ for four different anisotropies (square, octagon, diamond and "almost isotropic").

Figure 4.1 shows three examples of flows from the same starting set, composed of random shapes. The anisotropies are square (nearest neighbours interactions), octagonal (next nearest neighbours, weighted so that the corresponding Wulff shape is a regular octagon), and "almost isotropic", which is generated by the interactions in the directions $(0, \pm 1)$, $(\pm 1, 0)$, $(\pm 1, \pm 2)$, $(\pm 1, \pm 3)$ weighted so that the Wulff shape is a polygon with 24 facets of equal lengths.

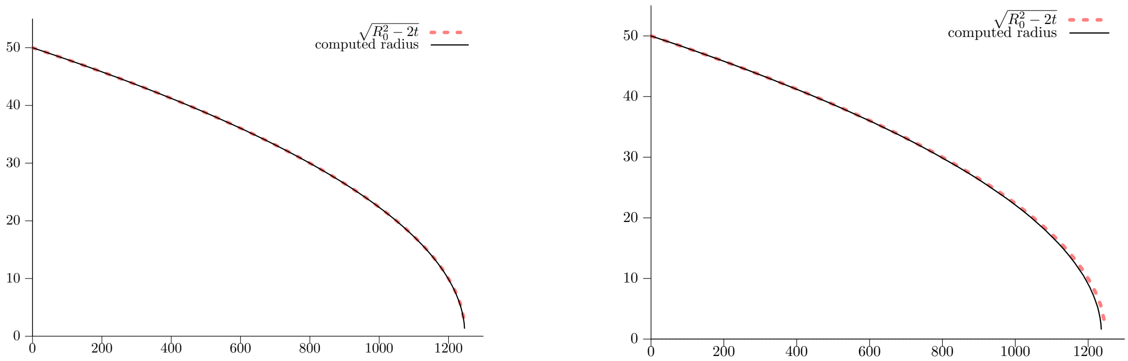

Figure 4.3: Evolution of the radius for the square (left) and octogonal (right) anisotropies.

Then, we estimate the decay of the radius of an initial Wulff shape $\mathcal{W}_{R_0} = \{\phi \le R_0\}$ along the evolution, up to extinction. In our experiment, $R_0 = 50$. It is well known that the solution is the Wulff shape of radius $R(t) = \sqrt{R_0^2 - 2(N-1)t}$ (where here $N = 2$). The evolutions are depicted in Figure 4.2. We use the same anisotropies as in figure 4.1, with additionally a "diamond" Wulff shape generated by the directions $(0, \pm 1)$, $(\pm 1, \pm 2)$ and with sides of equal lengths. In all cases, the weights have been calibrated so that the perimeters of the Wulff shapes are $6.28 \approx 2\pi$.

The plots in Figure 4.4 show that the decay of the radii is remarkably close to the theoretical prediction, even if this is less precise when more directions of interactions are involved, near extinction. This might be due in part to the fact that the computation of the distance through truncated variants of (4.32) become less precise.

Finally, we perform the same experiment with varying ε and h. We observe that the results look remarkably close even if, at low resolution, the error becomes huge when the size of the Wulff shape is of the order of the discretization. Figure 4.5 shows the shapes. Observe that the shape at time $t = 49$ is only computed for $\varepsilon = 0.1$ and $h = 0.1$ (the shape vanishes before for the two

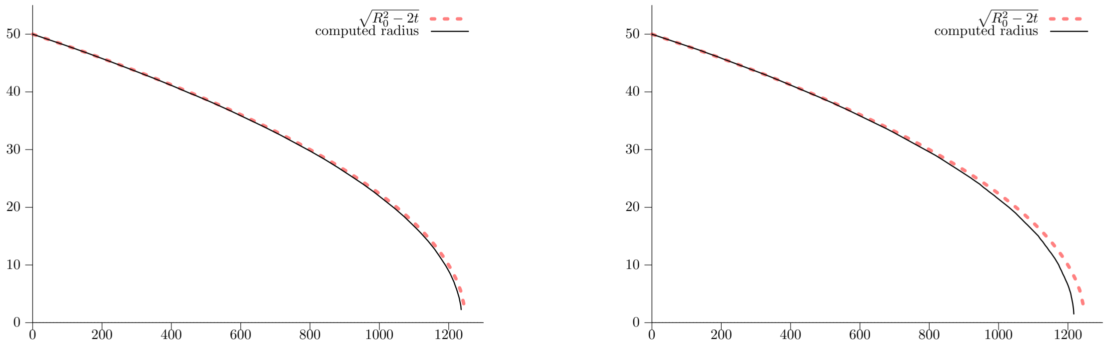

Figure 4.4: Evolution of the radius for the diamond (left) and "almost isotropic" (right) anisotropies.

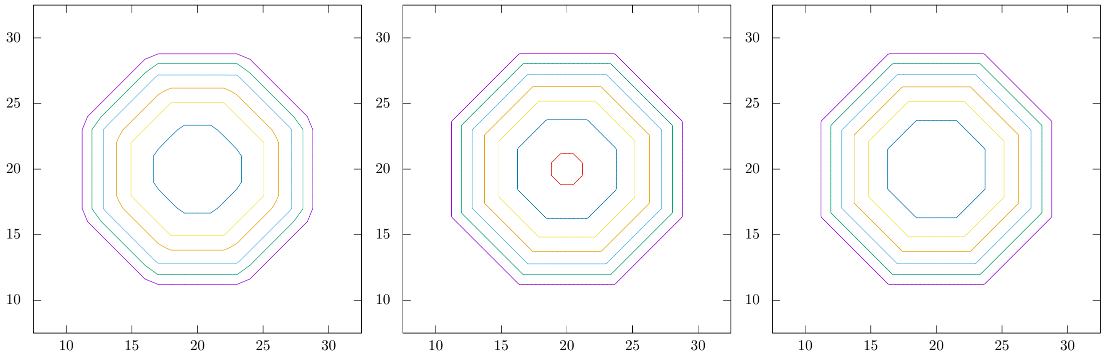

Figure 4.5: Evolution of an initial octagon with $R_0 = 10$ at times $0, 7, 14, \ldots$. Left: $\varepsilon = 1$, $h = 0.1$, middle: $\varepsilon = 0.1$, $h = 0.1$, right: $\varepsilon = 0.1$, $h = 0.5$.

other experiments). On the other hand, this computation took more than one hour, while the case $\varepsilon = 1$ took less than a minute and the case $\varepsilon = 0.1$, $h = 0.5$ a bit less than an hour. Figure 4.6 shows the decay of the radii, which should be $\sqrt{R_0^2 - 2t}$ for $R_0 = 10$ and $t \in [0, 50]$.

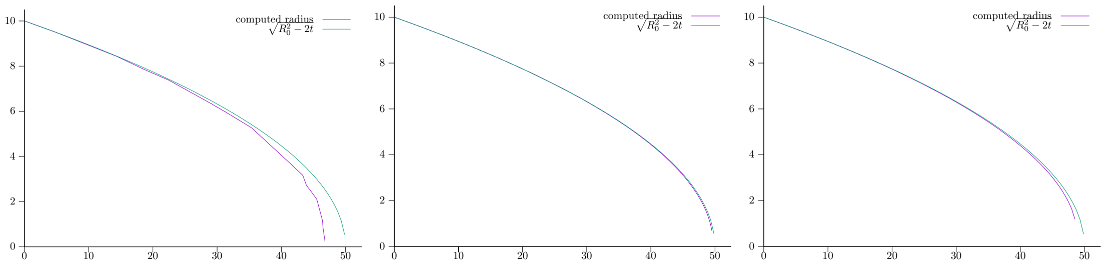

Figure 4.6: Evolution of the radius for an initial octagon with $R_0 = 10$ until the vanishing time $t = 50$. Left: $\varepsilon = 1$, $h = 0.1$, middle: $\varepsilon = 0.1$, $h = 0.1$, right: $\varepsilon = 0.1$, $h = 0.5$.

Appendix 4.A Proof of technical lemmas

4.A.1 Proof of Lemma 4.19

We build here a supersolution to Problem (4.31) when $g = \phi^\circ$. Let us first recall some notation and results concerning zonotopes (see e.g. [150]). Recall that $\mathcal{E} = \{\pm e_k\}_{k=1}^m \subseteq \mathbb{Z}^N$ where, without loss of generality, the vectors $e_1, \ldots, e_m$ span the whole $\mathbb{R}^N$. Given a non-negative interaction function $\beta \in X$, we assume that $\beta = 0$ on $\mathbb{Z}^N \setminus \mathcal{E}$ and that $\beta(-i) = \beta(i)$ for every $i \in \mathbb{Z}^N$. The anisotropy ϕ associated to β, as defined in (4.5), is such that its 1-Wulff shape $\mathcal{W}_1 \subseteq \mathbb{R}^N$ is a zonotope, which can be expressed as the Minkowski sum

$$\mathcal{W}_1 = \sum_{e\in\mathcal{E}} \beta(e)(-e,e) = \sum_{k=1}^{m} 2\beta(e_k)(-e_k,e_k).$$

Alternatively, one can define the zonotope $\mathcal{W}_1$ as the image of a cube under an affine map. Indeed, it holds

$$\mathcal{W}_1 = V(Q^{(m)}) \tag{4.101}$$

where $V = (2\beta(e_1)e_1, \dots, 2\beta(e_m)e_m) \in \mathbb{R}^{N\times m}$ and $Q^{(m)} = (-1,1)^m$. Since the set $\mathcal{E}$ is uniquely defined up to sign changes, the matrix V is also uniquely detemined up to permutations of columns or sign changes.

Note that by definition of zonotope any element $x \in \overline{\mathcal{W}_\ell}$ for $\ell > 0$ can be written as

$$x = \ell \sum_{k=1}^{m} 2\beta(e_k)\lambda_k e_k,$$

for suitable coefficients $|\lambda_k| \leq 1$. We note that (the closure of) a facet F (of non-zero dimension) of the zonotope $\mathcal{W}_\ell$ can be described in the following form:

$$F = \ell \sum_{j=1}^{r} 2\beta(e_{\sigma(j)})[-e_{\sigma(j)}, e_{\sigma(j)}] + \ell \sum_{j=r+1}^{m} 2\beta(e_{\sigma(j)})\varepsilon_{\sigma(j)} e_{\sigma(j)}, \tag{4.102}$$

where σ is a permutation of $\{1,\dots,m\}$, $1 \leq r \leq m$ and $|\varepsilon_j| = 1$. Moreover (see [150, page 206] for details) the vectors $e_{\sigma(1)}, \dots, e_{\sigma(r)}$ uniquely identify

$$\{e \in \mathcal{E} : e \parallel F\},$$

and r is uniquely defined as the number of vectors in the family $\mathcal{E}$ which are parallel to the facet F. Analogously, any vertex v of the zonotope $\mathcal{W}_\ell$ is of the form

$$v = \ell \sum_{j=1}^{m} 2\beta(e_{\sigma(j)})\varepsilon_{\sigma(j)} e_{\sigma(j)}, \tag{4.103}$$

where $\varepsilon_j \in \{\pm 1\}$ for every $j = 1,\dots,m$ and σ is a permutation of $\{1,\dots,m\}$. Note however that not every point of this form is a vertex of the zonotope.

Lemma 4.28. *There exists $\ell_0 > 0$ such that for every $\varepsilon > 0$ and every $\ell \geq \ell_0$, if $i \in \varepsilon\mathbb{Z}^N$ belongs to $\partial\mathcal{W}_{\varepsilon\ell}$, then for each $k \in \{1,\dots,m\}$ either one of the following holds:*

i) neither $i+\varepsilon e_k$ nor $i-\varepsilon e_k$ belong to $\partial\mathcal{W}_{\varepsilon\ell}$. In this case it holds either $\phi^\circ(i+\varepsilon e_k) > \phi^\circ(i) > \phi^\circ(i-\varepsilon e_k)$ or $\phi^\circ(i-\varepsilon e_k) > \phi^\circ(i) > \phi^\circ(i+\varepsilon e_k)$;

ii) one between $i \pm \varepsilon e_k$ belongs to $\partial\mathcal{W}_{\varepsilon\ell}$. In this case $\phi^\circ(i\pm\varepsilon e_k) \geq \ell$ and it holds

$$\#((i+\varepsilon\mathbb{Z}e_k) \cap \partial\mathcal{W}_{\varepsilon\ell}) \geq 2[\ell/\ell_0]. \tag{4.104}$$

Proof. By scaling, it suffices to prove the result in the case $\varepsilon = 1$. We take ℓ_0 such that

$$\ell_0 \geq \max_{k=1,\dots,m} \frac{1}{2\beta(e_k)} \tag{4.105}$$

and remark that $\ell_0 \in (0,+\infty)$. Note that the choice (4.105) implies for every $j = 1,\dots,m$ that

$$|(-2\ell\beta(e_j)e_j, 2\ell\beta(e_j)e_j)| = 4\ell\beta(e_j)|e_j| \geq 2\frac{\ell}{\ell_0}|e_j|.$$

We then fix $i \in \partial\mathcal{W}_\ell \cap \mathbb{Z}^N$ and $e_k \in \mathcal{E}$. We have to distinguish two cases.

Case 1. There exists a facet $F \ni i$ of $\mathcal{W}_\ell$ such that $e_k \parallel F$. By (4.102) we then see that

$$i \in 2\ell\beta(e_k)[-e_k, e_k] + j,$$

where $j \in F$. This implies in particular that $\{n \in \mathbb{Z} : i + ne_k \in F\}$ is an interval of $\mathbb{Z}$ containing 0. Furthermore, by the assumption (4.105), it contains at least $[2\ell|e_k|/\ell_0]$ points and we conclude (4.104). Since i and one between $i \pm e_k$ belong to $\partial\mathcal{W}_\ell$, then $\phi^\circ(i \pm e_k) \geq \ell$ by convexity.
Case 2. For every facet $F \ni i$ of $\mathcal{W}_\ell$ it holds $e_k \nparallel F$. Let us fix a facet $F \ni i$ and note that by (4.102) and up to relabelling the indexes, it holds

$$i \in \ell \sum_{j=1}^{r} 2\beta(e_j)[-e_j, e_j] + \ell \sum_{j=r+1}^{m} 2\beta(e_j)\varepsilon_j e_j,$$

with $k > r$ and $|\varepsilon_j| = 1$ for $j = r+1, \ldots, m$. Recalling (4.101), we see that

$$i - \varepsilon_k e_k = \ell V(y - \frac{\varepsilon_k}{\ell\beta(e_k)}\tilde{e}_k),$$

where $\tilde{e}_1, \ldots \tilde{e}_m$ denotes the canonical base of $\mathbb{R}^m$ and $y \in \sum_{j=1}^r [-\tilde{e}_j, \tilde{e}_j] + \sum_{j=r+1}^m \varepsilon_j \tilde{e}_j \subseteq \partial Q^{(m)}$. By the choice (4.105) and since $k > r$, one deduces that $y - \frac{\varepsilon_k}{\ell\beta(e_k)}\tilde{e}_k \in Q^{(m)}$, thus $i - \varepsilon_k e_k \in \overline{\mathcal{W}_\ell}$. Since then $e_k \nparallel F$ for any facet containing i, it must hold $\phi^\circ(i - \varepsilon_k e_k) < \ell$. By convexity one easily concludes that $\phi^\circ(i + \varepsilon_k e_k) > \ell$, which shows i). □

We now define a calibration z_{ij} for every $(i,j) \in \left(\{\phi^\circ > \varepsilon\ell_0\} \cap \varepsilon\mathbb{Z}^N\right) \times \varepsilon\mathbb{Z}^N$. Fix $i \in \varepsilon\mathbb{Z}^N$ with $\phi^\circ(i) > \varepsilon\ell_0$. In the following we write $i \sim j$ if $\frac{i-j}{\varepsilon} \in \mathcal{E}$. We start defining

$$z_{ij} = \begin{cases} 0 & \text{if } j \not\sim i \\ -\beta(e_k) & \text{if } j = i \pm \varepsilon e_k \text{ and } \phi^\circ(j) > \phi^\circ(i) \\ \beta(e_k) & \text{if } j = i \pm \varepsilon e_k \text{ and } \phi^\circ(j) < \phi^\circ(i). \end{cases} \tag{4.106}$$

In particular, this definition covers case i) in Lemma 4.28. Assume then that there exists $j \sim i$ with $\phi^\circ(j) = \phi^\circ(i)$ and $\frac{j-i}{\varepsilon} = e_k \in \mathcal{E}$. Since $i \in \varepsilon\mathbb{Z}^N$ and $e_k \in \mathcal{E}$ fall in case ii) of Lemma 4.28, there exists an interval $[-\underline{n}, \bar{n}] \cap \mathbb{Z}$ for $\underline{n}, \bar{n} \in \mathbb{N}$ such that

$$(i + \varepsilon\mathbb{Z}e_k) \cap \partial W^{\phi^\circ}_{\phi^\circ(i)} = i + ([-\underline{n}, \bar{n}] \cap \mathbb{Z})\varepsilon e_k$$

and moreover

$$\#([-\underline{n}, \bar{n}] \cap \mathbb{Z}) \geq 2[\phi^\circ(i)/(\varepsilon\ell_0)]. \tag{4.107}$$

Thus, we define z_{ij} as a linear interpolation of the values assumed at the extremal points of $i + [-\underline{n}, \bar{n}]\varepsilon e_k$ as

$$\begin{aligned} z_{i+t\varepsilon e_k, i+(t+1)\varepsilon e_k} &:= \beta(e_k)\left(1 - 2\frac{t + \underline{n} + 1}{\underline{n} + \bar{n} + 1}\right) \quad \forall t \in [-\underline{n} - 1, \bar{n}] \cap \mathbb{Z}, \\ z_{i+t\varepsilon e_k, i+(t-1)\varepsilon e_k} &:= \beta(e_k)\left(1 - 2\frac{-t + \underline{n} + 1}{\underline{n} + \bar{n} + 1}\right) \quad \forall t \in [-\underline{n}, \bar{n} + 1] \cap \mathbb{Z}. \end{aligned} \tag{4.108}$$

By definition one easily sees that

$$|z_{ij}| \leq \alpha^\varepsilon_{ij}, \qquad z_{ij}(\phi^\circ(i) - \phi^\circ(j)) = \alpha^\varepsilon_{ij}|\phi^\circ(i) - \phi^\circ(j)|. \tag{4.109}$$

We now show how to bound the divergence $(D^*_\varepsilon z)_i$. Assume that $\phi^\circ(i + \varepsilon e_k) = \phi^\circ(i)$ or that $\phi^\circ(i - \varepsilon e_k) = \phi^\circ(i)$. Then by definition (4.108) and by (4.107) one deduces

$$z_{i,i+\varepsilon e_k} + z_{i,i-\varepsilon e_k} - z_{i+\varepsilon e_k,i} - z_{i-\varepsilon e_k,i} = -\frac{4\beta(e_k)}{\underline{n} + \bar{n} + 1} \geq -\frac{2\beta(e_k)}{[\phi^\circ(i)/(\varepsilon\ell_0)]} \geq -\frac{C\varepsilon}{\phi^\circ(i)}, \tag{4.110}$$

and similarly if $\phi^\circ(i-\varepsilon e_k)=\phi^\circ(i)$. If instead $\phi^\circ(i\pm\varepsilon e_k)\neq\phi^\circ(i)$ and $\phi^\circ(i\pm\varepsilon e_k)\geq\varepsilon\ell_0$, one sees that

$$z_{i,i+\varepsilon e_k}+z_{i,i-\varepsilon e_k}=0 \text{ and } z_{i+\varepsilon e_k,i}+z_{i-\varepsilon e_k,i}=0 \tag{4.111}$$

Combining (4.110) and (4.111) and recalling (4.28) we conclude that if $\phi^\circ(i)\geq\ell_1\varepsilon$ then

$$h(D^*_\varepsilon z)_i\geq-\frac{c_\phi h}{\phi^\circ(i)} \tag{4.112}$$

for a suitable positive constant c_ϕ depending on ϕ.

We now illustrate a procedure that allows to extend the calibration above to $\varepsilon\mathbb{Z}^N\times\varepsilon\mathbb{Z}^N$. We set $C>1$ a sufficiently big constant and define a function $v\in X_\varepsilon$ setting

$$v:=\begin{cases}\phi^\circ+\dfrac{Ch}{\phi^\circ} & \text{on } \{\phi^\circ\geq C(\sqrt{h}\vee\varepsilon)\}\cap\varepsilon\mathbb{Z}^N\\ C(\sqrt{h}\vee\varepsilon)+\frac{h}{\sqrt{h}\vee\varepsilon} & \text{on } \{\phi^\circ< C(\sqrt{h}\vee\varepsilon)\}\cap\varepsilon\mathbb{Z}^N\end{cases}. \tag{4.113}$$

A calibration $w\in Y_\varepsilon$ can be defined setting for $i,j\in\varepsilon\mathbb{Z}^N$

$$w_{ij}:=\begin{cases}z_{ij} & \text{if } \phi^\circ(i)\geq 2\sqrt{C}(\sqrt{h}\vee\varepsilon)\\ -\alpha^\varepsilon_{ij} & \text{if } \phi^\circ(i)< 2\sqrt{C}(\sqrt{h}\vee\varepsilon)\end{cases}. \tag{4.114}$$

Since $x\mapsto x+Chx^{-1}$ is strictly monotone in the region $\{x\geq\sqrt{Ch}\}$, we can employ (4.109) to prove that, for every $i,j\in\varepsilon\mathbb{Z}^N$ with $\phi^\circ(i)\geq C(\sqrt{h}\vee\varepsilon)$, it holds

$$w_{ij}(v_i-v_j)=\alpha^\varepsilon_{ij}|v_i-v_j|,\quad |w_{ij}|\leq\alpha^\varepsilon_{ij}. \tag{4.115}$$

Moreover, taking C large enough ensures that whenever $j\sim i$, then

$$\begin{aligned}\phi^\circ(i)\leq 2\sqrt{C}(\sqrt{h}\vee\varepsilon) &\implies \phi^\circ(j)\leq C(\sqrt{h}\vee\varepsilon)\\ \phi^\circ(i)\geq 2\sqrt{C}(\sqrt{h}\vee\varepsilon) &\implies \phi^\circ(j)\geq \sqrt{C}(\sqrt{h}\vee\varepsilon)\end{aligned}. \tag{4.116}$$

Thus, equation (4.115) can be directly checked in the case $\phi^\circ(i)\leq 2\sqrt{C}(\sqrt{h}\vee\varepsilon)$ using the definition (4.114).

Note now that definition (4.114) implies $D^*_\varepsilon w=0$ in the region $\{\phi^\circ<2\sqrt{C}(\sqrt{h}\vee\varepsilon)\}$ thus we assume $\phi^\circ(i)\geq 2\sqrt{C}(\sqrt{h}\vee\varepsilon)$ and estimate $(D^*_\varepsilon w)_i$. If $\phi^\circ(i-\varepsilon e_k)<2\sqrt{C}(\sqrt{h}\vee\varepsilon)$ by convexity $\phi^\circ(i+\varepsilon e_k)>2\sqrt{C}(\sqrt{h}\vee\varepsilon)$, thus by definition (4.114) we get

$$z_{i,i+\varepsilon e_k}-z_{i+\varepsilon e_k,i}+z_{i,i-\varepsilon e_k}-z_{i-\varepsilon e_k,i}=-\beta(e_k)-\beta(e_k)+\beta(e_k)-(-\beta(e_k))=0.$$

The symmetric case is analogous. On the other hand, if every $j\sim i$ is in $\{\phi^\circ\geq 2\sqrt{C}(\sqrt{h}\vee\varepsilon)\}$ equation (4.112) holds. Therefore, we have shown

$$hD^*_\varepsilon w\geq-\frac{c_\phi h}{\phi^\circ}\chi_{\{\phi^\circ\geq\sqrt{C}(\sqrt{h}\vee\varepsilon)\}}. \tag{4.117}$$

By a direct computation, using (4.117) and assuming the $C>c_\phi$, we see that the pair (v,w) defined above satisfies

$$\begin{cases}hD^*_\varepsilon w+v\geq\phi^\circ\\ w_{ij}(v_i-v_j)=\alpha^\varepsilon_{ij}|v_i-v_j|,\quad |w_{ij}|\leq\alpha^\varepsilon_{ij}.\end{cases}$$

Recalling the comparison result in Theorem 4.8, we conclude that the solution u to (4.14) satisfies $u\leq v$ in $\varepsilon\mathbb{Z}^N$.

4.A.2 A remark on the inf/sup-convolution formulas (4.32)

In this section we show that in some particular cases, the $\inf, \sup$ in the definition (4.32) can be replaced by $\min, \max$ and that this minimization/maximization procedure can be made in a fixed neighborhood of the point considered. Yet, our proof also shows that this neighborhood can become very large, depending on the weights of the interaction, and it seems that we cannot expect in general cases that the $\min, \max$ are actually reached.

We assume here that ϕ satisfies the following assumption. There exists $\ell_\phi > 0$ such that for every $\varepsilon_k \in \{0, \pm 1\}$ for $k = 1, \ldots, m$, there exists $\ell \leq \ell_\phi$ such that

$$\ell \sum_{k=1}^{m} 2\beta(e_k)\varepsilon_k e_k \in \mathbb{Z}^N. \tag{4.118}$$

Note that this condition is satisfied if and only if $\beta(e_k) \in \mathbb{Q}$ for all $k = 1, \ldots, m$.

Lemma 4.29. *There exists $\ell_1 > 0$ with the following property. For any $i \in \varepsilon\mathbb{Z}^N$ with $\phi^\circ(i) \geq \varepsilon\ell_1$ there exists $j \in \varepsilon\mathbb{Z}^N \setminus \{0\}$ with $\phi^\circ(j) < \phi^\circ(i)$ and satisfying*

$$\phi^\circ(i) \geq \phi^\circ(j) + \phi^\circ(i-j) - c_\phi\varepsilon. \tag{4.119}$$

If (4.118) *holds, for any $i \in \varepsilon\mathbb{Z}^N$ with $\phi^\circ(i) \geq 2\varepsilon\ell_1$ there exists $j \in (\mathcal{W}_{\varepsilon\ell_1} \setminus \{0\}) \cap \varepsilon\mathbb{Z}^N$ such that*

$$\phi^\circ(i) = \phi^\circ(j) + \phi^\circ(i-j). \tag{4.120}$$

Moreover, for every $R \in (2\varepsilon\ell_1, \phi^\circ(i))$ there exists $j \in \mathcal{W}_R \setminus \mathcal{W}_{R-2\varepsilon\ell_1}$ such that (4.120) *holds.*

Proof. By scaling we prove the result in the case $\varepsilon = 1$. Given $i \in \mathbb{Z}^N \setminus \{0\}$, inequality (4.119) follows easily choosing $\ell_1 \geq 2$, considering $\sigma i \in \mathbb{R}^N \setminus \{0\}$ for an appropriate $\sigma \in (0,1)$ and $j \in \mathbb{Z}^N$ so that $\sigma i \in (j + [0,1]^N)$.

We now assume (4.118) and denote by ℓ_ϕ the radius associated to ϕ. We then choose $\ell_1 = \ell_\phi$. Let us fix $i \in \mathbb{Z}^N$ with $\phi^\circ(i) = \ell \geq 2\ell_1$. By (4.102) there exist $r > 0$, ε_k, λ_k with $|\varepsilon_k| = 1$ and $|\lambda_k| < 1$ such that

$$i = \ell \left(\sum_{k=1}^{r} 2\beta(e_k)\varepsilon_k e_k + \sum_{k=r+1}^{m} \lambda_k 2\beta(e_k)e_k \right).$$

Let us denote the point

$$v = \sum_{k=1}^{r} 2\beta(e_k)\varepsilon_k e_k \in \partial\mathcal{W}_1,$$

and define the function sign by $\mathrm{sign}(x) = x/|x|$ if $x \neq 0$ and 0 otherwise. For any $\ell' \leq \ell_\phi$ we rewrite i as follows

$$\begin{aligned} i &= \ell' \left(v + \sum_{k=r+1}^{m} 2\beta(e_k)\mathrm{sign}(\lambda_k)e_k \right) + (\ell - \ell') \left(v + \sum_{k=r+1}^{m} 2\beta(e_k) \left(\frac{\ell}{\ell - \ell'}\lambda_k - \frac{\ell'}{\ell - \ell'}\mathrm{sign}(\lambda_k) \right) e_k \right) \\ &=: \ell' w + (\ell - \ell') \left(v + \sum_{k=r+1}^{m} 2\beta(e_k)\lambda'_k e_k \right). \end{aligned}$$

Notice that, since $\ell \geq 2\ell'$ and $|\lambda_k| \leq 1$ it holds $|\lambda'_k| \leq 1$, thus by formula (4.102) we get

$$v + \sum_{k=r+1}^{m} 2\beta(e_k)\lambda'_k e_k \in \partial\mathcal{W}_1$$

and therefore $\phi^\circ(i - \ell' w) = \ell - \ell'$. We conclude noting that by the hypothesis (4.118) we can choose $\ell' \leq \ell_1$ so that $\ell' w \in \mathbb{Z}^N$, which implies (4.120) since $\phi^\circ(\ell' w) = \ell'$.

We now prove the last assertion. Since $\phi^\circ(i) \geq 2\ell_1$, by the previous result there exists $j_0 \in (\mathcal{W}_{\ell_1} \setminus \{0\})$ so that $\phi^\circ(i) = \phi^\circ(j_0) + \phi^\circ(i - j_0)$. Now, if $R - 2\ell_1 \leq \phi^\circ(j_0)$ we conclude. If not, then

$\phi^\circ(i-j_0) \geq 2\ell_1$ by (4.120), and thus we can find $k_0 \in (\mathcal{W}_{\ell_1} \setminus \{0\})$ so that

$$\phi^\circ(i-j_0) = \phi^\circ(k_0) + \phi^\circ(i-j_0-k_0). \tag{4.121}$$

Denoting $j_1 = j_0 + k_0$, on one hand (4.121) implies

$$\phi^\circ(i) = \phi^\circ(j_0) + \phi^\circ(j_1-j_0) + \phi^\circ(i-j_1) \geq \phi^\circ(j_1) + \phi^\circ(i-j_1) \tag{4.122}$$

thus equality holds instead. If $\phi^\circ(j_1) \geq R - 2\ell_1$ we conclude, if not (4.122) yields $\phi^\circ(i-j_1) \geq 2\ell_1$ and we can iterate. Recalling that $\phi^\circ \geq c_\phi > 0$ on $\varepsilon\mathbb{Z}^N \setminus \{0\}$, it is clear that after a finite number of iterations the process stops, and one can check that the required properties are satisfied. □

By the previous lemma it is easy to prove the following result.

Corollary 4.30. *Let $u \in X$ be a $(1,\phi)$-Lipschitz function and ℓ_1 as in Lemma 4.29. Then, for all $i \in \varepsilon\mathbb{Z}^N$ it holds*

$$\sup_{j\in\{u\geq 0\}} \{u_j - \phi^\circ(i-j)\} = \max_{j\in\{u\geq 0\}} \{u_j - \phi^\circ(i-j)\}.$$

In addition, if $i \in \{u \leq 0\}$, the maximum is reached in a point in $(\{u \leq 0\} + \mathcal{W}_{2\varepsilon\ell_1}) \cap \varepsilon\mathbb{Z}^N$.

Proof. It is enough to consider $i \in \{u < 0\} \cap \varepsilon\mathbb{Z}^N$. Let us denote $F = (\{u \leq 0\} + \mathcal{W}_{2\varepsilon\ell_1}) \cap \{u > 0\}$. Firstly, by a variant of the argument by iteration employed in the proof of Lemma 4.29, one can prove that

$$\sup_{j\in\{u\geq 0\}} \{u_j - \phi^\circ(i-j)\} = \sup_{j\in F} \{u_j - \phi^\circ(i-j)\}. \tag{4.123}$$

On the other hand, take a point $j_0 \in \{u > 0\}$. If $j \in F$ satisfies $u_j - \phi^\circ(i-j) \geq u_{j_0} - \phi^\circ(i-j_0)$, since $u \leq 2\varepsilon\ell_1$ in F (as u is $(1,\phi^\circ)$-Lipschitz) we obtain

$$2\varepsilon\ell_1 + \phi^\circ(i-j_0) \geq \phi^\circ(i-j),$$

which implies that the sup in (4.123) is indeed a max. □

Appendix 4.B Extension to the isotropic case

In this appendix we show a modification of the algorithm proposed in this chapter tailored to the isotropic case.

4.B.1 Definition of the Algorithm and Main Result.

Given $u \in X_\varepsilon$, we consider the discrete Laplacian operator

$$(\Delta_\varepsilon u)_i := \frac{1}{\varepsilon^2}\sum_{n=1}^{N}(u_{i+\varepsilon e_n} - 2u_i + u_{i-\varepsilon e_n}) = \frac{1}{\varepsilon^2}\sum_{n=1}^{N}(u_{i+\varepsilon e_n} + u_{i-\varepsilon e_n}) - \frac{2N}{\varepsilon^2}u_i.$$

We also consider the redistancing operator $\mathrm{sd}^{\varepsilon,\phi^\circ}$ defined in (4.32), for the particular choice $\phi = \phi^\circ = |\cdot|$ In the following, we write $\mathrm{sd}^\varepsilon(u) := \mathrm{sd}^{\varepsilon,|\cdot|}(u)$, and for ease of notation we drop the argument whenever clear from the context.
We modify the algorithm proposed in (4.43)in the following way. Given a set $E^0 \subseteq \mathbb{R}^N$, we let $u^{0,\varepsilon} = \mathrm{sd}^{0,\varepsilon} = d^{0,\varepsilon}$ be a 1-Lipschitz function on $\varepsilon\mathbb{Z}^N$, such that $\mathrm{sd}^{0,\varepsilon}_i \leq 0$ for any $i \in E^0 \cap \varepsilon\mathbb{Z}^N$, and it is positive elsewhere. Then, we define for all k:

$$\begin{aligned} u_i^{k+1,\varepsilon} &= d_i^{k,\varepsilon} + h(\Delta_\varepsilon d^{k,\varepsilon})_i, \\ d^{k+1,\varepsilon} &= \mathrm{sd}^\varepsilon(u^{k+1,\varepsilon}). \end{aligned} \tag{4.124}$$

We set $E^\varepsilon(t) = \{u_i^{[\frac{t}{h}],\varepsilon} < 0\}$. Note that if we choose $h \in (0, \frac{\varepsilon^2}{2N}]$ and let $\theta = \frac{2Nh}{\varepsilon^2} \leq 1$, (4.124) becomes:

$$u_i^{k+1} = (1-\theta)d_i^{k,\varepsilon} + \frac{\theta}{2N}\sum_{n=1}^{N}\left(d_{i+\varepsilon e_n}^{k,\varepsilon} + d_{i-\varepsilon e_n}^{k,\varepsilon}\right),$$

where $(e_n)_{n=1}^N$ is the canonical basis of $\mathbb{R}^N$ ($\mathbb{Z}^N$). In particular, this ensures that $u_i^{k+1,\varepsilon}$ is 1-Lipschitz as a convex combination of 1-Lipschitz functions.

Moreover, from Lemma 4.44 we see that where $u^{k+1,\varepsilon} \geq 0$ (that is, where $d^{k+1,\varepsilon} \geq 0$), then

$$\frac{d_i^{k+1,\varepsilon} - d_i^{k,\varepsilon}}{h} \geq \frac{1}{\varepsilon^2}(\Delta_\varepsilon d^{k,\varepsilon})_i, \tag{4.125}$$

while where $u_i^{k+1,\varepsilon} < 0$ (i.e. $d_i^{k+1,\varepsilon} < 0$), it holds

$$\frac{d_i^{k+1,\varepsilon} - d_i^{k,\varepsilon}}{h} \leq \frac{1}{\varepsilon^2}(\Delta_\varepsilon d^{k,\varepsilon})_i. \tag{4.126}$$

It is thus reasonable to expect that this scheme approximates weak solutions to the (isotropic) mean curvature flow as defined in Definition 4.1. Indeed, we can show the following.

Theorem 4.31. *As $\varepsilon \to 0$, the function $(d^\varepsilon(t)_i)$, defined for $t \geq 0$ and $i \in \varepsilon\mathbb{Z}^N$ by $d^\varepsilon(t)_i = d_i^{[t/h],\varepsilon}$, converge up to subsequences, for almost all time and locally uniformly in space to a function $d(x,t)$ such that $d^+ = \max\{d, 0\}$ is the distance function to a supersolution to the (generalized) mean curvature flow starting from E^0, and d^- is the distance function to a supersolution starting from $\varepsilon\mathbb{Z}^N \setminus E^0$. In particular, if the mean curvature flow $E(t)$ starting from E^0 is unique, then $d^\varepsilon(t)$ converges to the signed distance function to $E(t)$, up to extinction.*

Here, by generalized solution, we mean a solution in the viscosity sense [64], as defined in [87, 20, 19, 173] (see also Chapter 2), or, equivalently, in the distributional sense as in Definition 4.1. We make this precise in the next section, before proving Theorem 4.31. Note that our results requires $h = \varepsilon^2\theta/2N$.

The setting is essentially the same one presented in this chapter. We set for each $t \geq 0$, $d^\varepsilon(t) = \mathrm{sd}^{[t/h],\varepsilon}$ and then let, for $t > 0$, $E_\varepsilon(t) = \{i \in \varepsilon\mathbb{Z}^N : d^\varepsilon(t)_i < 0\}$. Then we let $E_\varepsilon = \{(i,t) \in \varepsilon\mathbb{Z}^N \times [0,+\infty) : i \in E_\varepsilon(t)\}$ and $F_\varepsilon = \{(i,t) \in \varepsilon\mathbb{Z}^N \times [0,+\infty) : i \notin E_\varepsilon(t)\}$. We find a subsequence such that both $E_{\varepsilon_k} \to E$ and $F_{\varepsilon_k} \to \varepsilon\mathbb{Z}^N \setminus A$ in the Kuratowski sense in $\mathbb{R}^N \times [0,+\infty)$, where A is an open set and E a closed set. Observe that $A \subset E$. We let $T^* = \inf\{t > 0 : E \cap (\mathbb{R}^N \times (t,+\infty)) = \emptyset \text{ or } \varepsilon\mathbb{Z}^N \setminus A \cap (\mathbb{R}^N \times (t,+\infty)) = \emptyset\} \in [0,+\infty]$. Note that it may (in general will) happen that after some time, $u^{k+1,\varepsilon}$ defined by (4.124) becomes positive (or negative) everywhere, in which case $d^{k+1,\varepsilon}$ will be $+\infty$ (respectively, $-\infty$) and the corresponding sets $E_\varepsilon(t)$ (or $\varepsilon\mathbb{Z}^N \setminus E_\varepsilon(t)$) will be empty: in the limit, this corresponds to times which are past the extinction time T^* of E or $\varepsilon\mathbb{Z}^N \setminus A$.

4.B.2 Estimate on Balls and Consequences

A crucial point for proving the convergence of the method is to control the behaviour of the algorithm when $d_i^{k,\varepsilon}$ represents the distance to a ball of radius $R > 0$. For this, given $\theta \in (0,1]$, we let $u_i := |i| - R$, $i \in \varepsilon\mathbb{Z}^N$,

$$v_i = (1-\theta)u_i + \frac{\theta}{2N}\sum_{n=1}^{N}(u_{i+\varepsilon e_n} + u_{i-\varepsilon e_n})$$

and let $d = \mathrm{sd}^\varepsilon(v)$. Then, we show the following estimate.

Lemma 4.32. *There exist $C \geq 1$ such that if ε/R is small enough (depending only on the dimension N), then it holds*

$$d_i \leq |i| - R + \frac{C}{R}\varepsilon^2 = |i| - R + \frac{2NC}{\theta R}h.$$

Here as before h, ε, θ must satisfy $\theta = 2Nh/\varepsilon^2$.

Proof. Without loss of generality we assume $\theta = 1$. Also, since $\mathrm{sd}_-^\varepsilon \le \mathrm{sd}_+^\varepsilon$ (see Remark 4.18), we can assume $d = \mathrm{sd}_+^\varepsilon(v)$. We first observe that

$$v_i = -R + \frac{1}{2N}\sum_{n=1}^{N} |i + \varepsilon e_n| + |i - \varepsilon e_n|.$$

We remark that if $|i| \ge 2\varepsilon$, $e \in \{e_n, -e_n : n = 1, \dots, N\}$,

$$|i + \varepsilon e| = |i| + \varepsilon \frac{i \cdot e}{|i|} + \int_0^1 (1-t)\frac{1}{|i + t\varepsilon e|}\left(I - \frac{(i + t\varepsilon e)\otimes(i + t\varepsilon e)}{|i + t\varepsilon e|^2}\right)\varepsilon e \cdot \varepsilon e \, dt$$
$$\le |i| + \varepsilon\frac{i \cdot e}{|i|} + \frac{1}{|i| - \varepsilon}\frac{\varepsilon^2}{2} \quad (4.127)$$

so that

$$v_i \le |i| - R + \frac{1}{|i| - \varepsilon}\frac{\varepsilon^2}{2}.$$

If we assume that $\varepsilon \le \min\{1, R/2\}$, then for $|i| \le R - \varepsilon$, $v_i < 0$. Hence, we may estimate, for i with $v_i \ge 0$ (hence with $|i| \ge R - \varepsilon$):

$$(d_+^\varepsilon(v))_i := \inf_{j: v_j < 0} v_j + |j - i| \le \inf_{j: \frac{R}{2} + \varepsilon \le |j| \le R - \varepsilon} v_j + |j - i| \le \inf_{j: \frac{R}{2} + \varepsilon \le |j| \le R - \varepsilon} |j| - R + \frac{\varepsilon^2}{R} + |j - i|,$$

assuming $\varepsilon \le R/8$ so that the set of j's is not empty. Consider j with $\frac{R}{2} + \varepsilon \le |j| \le R - \varepsilon$, close to the segment $[0, i]$: if $\tilde{\jmath}$ is the projection of j onto $[0, i]$, one has

$$|j| + |j - i| = \sqrt{|j - \tilde{\jmath}|^2 + |\tilde{\jmath}|^2} + \sqrt{|\tilde{\jmath} - i|^2 + |j - \tilde{\jmath}|^2}$$
$$\le |\tilde{\jmath}| + \frac{|j - \tilde{\jmath}|^2}{2|\tilde{\jmath}|} + |\tilde{\jmath} - i| + \frac{|j - \tilde{\jmath}|^2}{2|\tilde{\jmath} - i|} = |i| + \left(\frac{1}{2|\tilde{\jmath}|} + \frac{1}{2|\tilde{\jmath} - i|}\right)|j - \tilde{\jmath}|^2$$

If ε/R is small enough (depending only on N), we can find $j, \tilde{\jmath}$ such that $|j - \tilde{\jmath}|^2 \le N\varepsilon^2$ and $R/2 \le |\tilde{\jmath}| \le 3R/4$, so that $|\tilde{\jmath} - i| \ge R/4$. We obtain for such a choice:

$$|j| + |j - i| \le |i| + \frac{3N}{R}\varepsilon^2$$

This shows that where $(d_+^\varepsilon)_i$ is non-negative, it is less than $|i| - R + \frac{3N+1}{R}\varepsilon^2$ as soon as ε/R is small enough. As $\mathrm{sd}_+^\varepsilon$ is the smallest 1-Lipschitz function larger than d_+^ε where it is non-negative (Remark 4.10), this achieves the proof. □

We now continue the study of the properties of d^ε. We note that if for some k, $d_i^{k,\varepsilon} \ge R > 0$ at $i \in \varepsilon\mathbb{Z}^N$, then $d_j^{k,\varepsilon} \ge R - |j - i|$ (as it is 1-Lipschitz) and iterations of Lemma 4.32 show that for some $C \ge 1$ depending only on the dimension, if ε is small enough,

$$d_i^{\ell,\varepsilon} \ge R - \frac{C}{R}(\ell - k)h$$

for $\ell \ge k$ and as long as the right-hand side is larger than $R/2$ (that is, $(\ell - k)h \le R^2/2C$). This may also be written:

$$d^\varepsilon(s)_i \ge d^\varepsilon(t)_i - C/R(s - t + h) \quad (4.128)$$

for $d^\varepsilon(t)_i \ge R > 0$, $0 \le t \le s \le CR^2$ for some constant C depending only on the dimension. A similar, symmetric statement holds if $d^\varepsilon(t)_i \le -R < 0$. In particular, by our choice of E^0, one has $T^* > 0$.

The estimate (4.128) allows to reproduce the proof of [55, Proposition 4.4] and find that except for a countable set of times the function

$$\sum_{i\in\varepsilon_k\mathbb{Z}^N} d_i^{\varepsilon_k}(t)\chi_{i+[0,\varepsilon)^N}$$

converges locally uniformly to some function $d(x,t)$ which is locally finite for $t < T^*$ and such that its positive and negative parts respectively satisfy:

$$d^+(\cdot,t) = \mathrm{dist}(\cdot, E(t)) \quad \text{and} \quad d^-(\cdot,t) = \mathrm{dist}(\cdot, \varepsilon\mathbb{Z}^N \setminus A(t)), \tag{4.129}$$

where $E(t) = \{x \in \mathbb{R}^N : (x,t) \in E\}$ and $A(t) = \{x \in \mathbb{R}^N : (x,t) \in A\}$, for $t < T^*$. Moreover, for every $x \in \mathbb{R}^N$ the functions $\mathrm{dist}(x, E(\cdot))$ and $\mathrm{dist}(x, \varepsilon\mathbb{Z}^N \setminus A(\cdot))$ are left-continuous and right-lower-semicontinuous. Equivalently, the maps $E(\cdot)$ and $\varepsilon\mathbb{Z}^N \setminus A(\cdot)$ are left-continuous and right-upper-semicontinuous with respect to the Kuratowski convergence. Finally, $E(0) = E^0$ and $A(0) = \mathring{E}^0$. In addition, $d(\cdot,t) \equiv +\infty$ or $-\infty$ for all $t > T^*$.

From (4.129) one has in particular that $\varepsilon\mathbb{Z}^N \setminus E \cap (\mathbb{R}^N \times (0,T^*)) = \{(x,t) \in \mathbb{R}^N \times (0,T^*) : d(x,t) > 0\}$ while $A = \{(x,t) \in \mathbb{R}^N \times (0,T^*) : d(x,t) < 0\}$, and

$$d^+(x,t) = \inf\left\{\liminf_{k\to+\infty} \max\{0, d^\varepsilon_{i_k}(t_k)\} : \varepsilon_k\mathbb{Z}^N \times \mathbb{R}_+ \ni (i_k,t_k) \overset{k\to\infty}{\longrightarrow} (x,t)\right\}$$
$$-d^-(x,t) = \sup\left\{\limsup_{k\to+\infty} \min\{0, d^\varepsilon_{i_k}(t_k)\} : \varepsilon_k\mathbb{Z}^N \times \mathbb{R}_+ \ni (i_k,t_k) \overset{k\to\infty}{\longrightarrow} (x,t)\right\}$$

are the classical relaxed half-limits (see for instance [19, 20]).

4.B.3 Consistency of the algorithm

As before, in this section we fix $\theta \in (0,1]$ and the parameters ε, h are linked through $h = \theta\varepsilon^2/2N$. We investigate the limit of the scheme as $\varepsilon, h \to 0$.

At this point there are two elementary directions to prove the convergence of the algorithm. One can establish the consistency with the viscosity approach of [173], showing that d is a viscosity super-solution to the heat equation in $\{d > 0\}$ (and a subsolution in $\{d < 0\}$), or equivalently the consistency with respect to the distributional Definition 4.1.

For the viscosity point of view, let us note that d^+ is lower semicontinuous, as explained before. Then, let us consider a smooth test function $\eta(x,t)$ with $\eta \le d$, $\eta(\bar{x},\bar{t}) = d(\bar{x},\bar{t}) > 0$, and assume without loss of generality that the contact point is unique [64]. Then, it is standard that for small ε_k, there is $i_k \to \bar{x}$, $i_k \in \varepsilon_k\mathbb{Z}^N$, and $t_k \to \bar{t}$ such that for all $t > 0$ and $i \in \varepsilon_k\mathbb{Z}^N$:

$$\eta_k(i,t) = \eta(i,t) + (d^{\varepsilon_k}(t_k)_{i_k} - \eta(i_k,t_k)) \le d^{\varepsilon_k}(t)_i, \quad \eta_k(i_k,t_k) = d^{\varepsilon_k}(t_k)_{i_k} > 0.$$

We have:

$$\eta_k(i_k,t_k) = d^{\varepsilon_k}(t_k)_{i_k} \ge (1-\theta)d^{\varepsilon_k}_{i_k}(t_k - h_k) + \frac{\theta}{2N}\sum_{n=1}^N \left(d^{\varepsilon_k}_{i_k+\varepsilon_k e_n}(t_k - h_k) + d^{\varepsilon_k}_{i_k-\varepsilon_k e_n}(t_k - h_k)\right)$$
$$\ge (1-\theta)\eta_k(i_k, t_k - h_k) + \frac{\theta}{2N}\sum_{n=1}^N \left(\eta_k(i_k + \varepsilon_k e_n, t_k - h_k) + \eta_k(i_k - \varepsilon_k e_n, t_k - h_k)\right),$$

so that:

$$\frac{\eta_k(i_k,t_k) - \eta_k(i_k, t_k - h_k)}{h_k} \ge \left(\Delta_{\varepsilon_k}\eta_k(\cdot, t_k - h_k)\right)_{i_k}.$$

Using that η is smooth and passing to the limit, we recover:

$$\frac{\partial\eta}{\partial t}(\bar{x},\bar{t}) \ge \Delta\eta(\bar{x},\bar{t}),$$

showing that d^+ is a viscosity supersolution to the heat equation in $\{d > 0\}$.

On the other hand, the variational point of view is tackled as follows, considering rather a test function $\eta \in C_c^\infty((\varepsilon\mathbb{Z}^N \setminus E) \cap (\mathbb{R}^N \times (0,T^*)); \mathbb{R}_+)$. The support U_η is at distance from E, hence for ε_k small enough it is also at positive distance from E_{ε_k} so that d^{ε_k} is bounded from below by a positive number on $\overline{U}_\eta \cap (\varepsilon\mathbb{Z}^N \times [0,+\infty))$. Thanks to (4.125), it follows that

$$\frac{d^{\varepsilon_k}(t)_i - d^{\varepsilon_k}(t-h_k)_i}{h_k} \geq (\Delta_{\varepsilon_k} d^{\varepsilon_k}(t-h_k))_i$$

for $(i,t) \in \overline{U}_\eta$, hence

$$\int_0^{T^*} \varepsilon^N \sum_{i \in \varepsilon\mathbb{Z}^N} \left(\frac{d^{\varepsilon_k}(t)_i - d^{\varepsilon_k}(t-h_k)_i}{h_k} - (\Delta_{\varepsilon_k} d^{\varepsilon_k}(t-h_k))_i \right) \eta(i,t)\,\mathrm{d}t \geq 0.$$

Rearranging the sums, this reads

$$\int_0^{T^*} \varepsilon^N \sum_{i \in \varepsilon\mathbb{Z}^N} \left(\frac{\eta(i,t) - \eta(i,t+h_k)}{h_k} - (\Delta_{\varepsilon_k}(\eta(t+h_k,\cdot))_i \right) d^{\varepsilon_k}(t)_i\,\mathrm{d}t \geq 0.$$

In the limit (since η is smooth, and d^{ε_k} converges uniformly for almost every time), we obtain

$$\int_0^{T^*} \int_{\mathbb{R}^N} (-\partial_t \eta - \Delta\eta) d\,\mathrm{d}x\,\mathrm{d}t \geq 0$$

so that

$$\frac{\partial d}{\partial t} \geq \Delta d \quad \text{in} \quad \mathcal{D}'(\{(x,t) \in \mathbb{R}^N \times [0,T^*) : d(x,t) > 0\}), \tag{4.130}$$

that is in the sense of distributions (or measures). In the same way, we have:

$$\frac{\partial d}{\partial t} \leq \Delta d \quad \text{in} \quad \mathcal{D}'(\{(x,t) \in \mathbb{R}^N \times [0,T^*) : d(x,t) < 0\}). \tag{4.131}$$

Lastly, one still needs to prove that Δd is bounded above in $\{d \geq R\}$ for any $R > 0$ (*cf.* Definition 4.1, point (d)); observe however that together with (4.128), (4.130)–(4.131) imply that $\partial d^\pm / \partial t$ and $\Delta d^\pm$ are Radon measures where they are positive, and the proof in [55, Appendix] then shows that (4.130)–(4.131) also hold in the viscosity sense.

To prove the L^∞ bound, one observes that if for some $t \in (h,T^*)$ it holds $d_i^\varepsilon(t) = d_i^{k,\varepsilon} \geq R > 0$ ($k = [t/h]$) for some $i \in \varepsilon\mathbb{Z}^N$, by definition of $\mathrm{sd}_\pm^\varepsilon$ in (4.32) there is $x \in \mathbb{R}$ with $x \leq 0$ and $j \in \varepsilon\mathbb{Z}^N$ such that $x + |j-i| - \varepsilon^2 \leq d_i^{k,\varepsilon} \leq x + |j-i|$ (and in particular $|j-i| \geq R$). Using (4.127), one sees that

$$(\Delta_\varepsilon d^{k,\varepsilon})_i \leq \frac{2N}{\varepsilon^2}(x + |j-i| - d_i^{k,\varepsilon}) + \frac{N}{|i-j| - \varepsilon} \leq 2N + \frac{2N}{R},$$

where the last inequality holds for $\varepsilon \leq \frac{R}{2}$. Hence considering now a smooth, non-negative test function η with support in $\{d > R\}$, we can reproduce the previous arguments to show that in the limit,

$$\int_0^{T^*} \int_{\mathbb{R}^N} \Delta\eta\, d\,\mathrm{d}x\,\mathrm{d}t \leq 2N\frac{R+1}{R} \int_0^{T^*} \int_{\mathbb{R}^N} \eta\,\mathrm{d}x\,\mathrm{d}t,$$

showing that $(\Delta d)^+ \in L^\infty(\{d > R\})$, as needed.

Part II

Stability of some Volume-Preserving Curvature Flows

Chapter 5

Long Time Behaviour of the Discrete Volume Preserving Mean Curvature Flow in the Flat Torus

Contents

1 Introduction

In this chapter we consider the geometric evolution of sets called *the volume preserving mean curvature flow*. This evolution is a modification of the *classical mean curvature flow* defined as a flow of sets $(E_t)_{0\leq t\leq T}$ in $\mathbb{R}^N$ following the motion law

$$v_t = \overline{\mathrm{H}}_{E_t} - H_{E_t} \quad \text{on} \quad \partial E_t \tag{5.1}$$

for all $t \in [0,T]$, where $\overline{\mathrm{H}}_{E_t}$ denotes the average of H_{E_t} over ∂E_t. One can observe that the volume of the evolving sets is indeed preserved during the evolution and that the perimeters of the sets E_t are non-increasing.

One of the main mathematical difficulties of the volume preserving mean curvature flow is the non-local nature of the functional given by the constraint. Moreover, the generated flow may present singularities of different kinds, even in a finite time-span and even if the initial data is smooth. For example, we can see merging or collision of near sets, pinch-offs or shrinking of connected components to points. There exist examples of singular solutions even in the two dimensional case, see [148, 149]. After the onset of singularities, the classical or smooth formulation of the flow (5.1) ceases to hold and needs to be replaced by a weaker one. Due to the lack of a comparison principle, a natural approach is the minimizing movement approach proposed independently by Almgren, Taylor and Wang in [8] and by Luckhaus and Sturzenhecker in [144] for the unconstrained case and adapted to the volume-preserving setting in [155].

We briefly recall the scheme in the volume contrained setting. First of all we define a discrete-in-time approximation of the flow that will be called the *discrete (volume-preserving) flow*. Given any initial set E_0 and a time-step $h>0$ we define iteratively $E_h^0 := E_0$ and for all $n \geq 0$

$$E_h^{n+1} \in \operatorname{argmin}\left\{ P(F) + \frac{1}{h}\int_{F\triangle E_h^n} \mathrm{dist}_{\partial E_h^n}(x)\,\mathrm{d}x \ :\ F\subset \mathbb{T}^N,\, |F| = |E_0| \right\},$$

where $\mathrm{dist}_{\partial E_h^n}$ is the distance function from the set ∂E_h^n. We can define for every $t \geq 0$, the approximate flow by $E_h(t) := E_h^{[t/h]}$. It can be proved (see [154, Proposition 2.2]) that the discrete flow is well defined. Any limit point of this flow as the time-step h converges to zero will be called a *flat flow*. As for the classical mean curvature flow, this approach produces global-in-time solutions as shown in [155]. The existence of such global solutions then allows to analyse the equilibrium configurations reached in the long time asymptotics.

The long time behaviour of the volume preserving mean curvature flow has been previously studied only in some particular cases, when the existence of global smooth solutions could be ensured by choosing suitably regular initial sets. For example one can consider uniformly convex and nearly spherical initial sets (see [82, 122]), or $C^\infty-$regular initial sets that are H^3-close to strictly stable critical sets in the three and four dimensional flat torus (see [159]). For more general initial data, the long time behaviour in the context of flat flows of convex and star-shaped sets (see [21, 136]) has been characterized only up to (possibly diverging in the case of [21]) translations. In [154] the authors characterized the long-time limits of the discrete-in-time approximate flows constructed by the Euler implicit scheme introduced in [8, 144] under the volume constraint in arbitrary space dimension. They proved that the discrete flow starting from an arbitrary bounded initial set converges exponentially fast to a finite union of disjoint balls with equal radii. The same authors and collaborators were also able to send the discretization parameter h to 0 in [133], in the case $N=2$. Indeed, an explicit penalization is used in order to enforce the volume constraint.

In this chapter the long-time convergence analysis is developed in the flat torus $\mathbb{T}^N$ for the discrete flow. In such framework the class of possible long-time limits is much richer as it includes not only union of balls with equal radii but also different type of critical sets for the perimeter. The notion of *strictly stable critical set* is crucial to our result; for the precise definition we refer to Section 2 of this chapter, but it can be summarized as a regular, critical set for the perimeter (i.e. with a constant mean curvature boundary) with strictly positive (volume-constrained) second variation. The main result of this chapter is the theorem below. It provides a complete characterization of the long-time behaviour of the discrete mean curvature flow in the flat torus starting near a strictly stable critical set. Moreover, an estimate on the convergence speed is provided.

Theorem 5.1. *Let E be a strictly stable critical set in the flat torus. Then there exist $\delta^* = \delta^*(E) > 0$ and $h^* = h^*(E) > 0$ with the following property: if $h < h^*$ and $E_0 \subset \mathbb{T}^N$ is a set of finite perimeter satisfying*

$$|E_0| = |E|, \qquad \overline{E}_0 \subset (E)_{\delta^*},$$

then every discrete volume preserving mean curvature flow $(E_h^n)_{n\in\mathbb{N}}$ starting from E_0 converges to a translate of E in C^k for every $k \in \mathbb{N}$ and the convergence is exponentially fast.

We would like to give some details to highlight the major differences between the results presented in this chapter and the analysis carried out in [159]. In the aforementioned work, the author studied the flat flow, albeit in low dimension ($N \leq 4$). In the article, it was assumed the initial set to be a $C^\infty-$deformation of a strictly stable critical set, close in the H^3-sense to the latter set. Under these assumptions, it was proved the exponential convergence of the flat flow to a translated of the strictly stable critical set. We remark that our result addresses the long time behaviour of the discrete flow but holds in much weaker hypotheses: we only assume the initial set to be of finite perimeter and close in the Hausdorff sense to a strictly stable critical set. Moreover, our result holds in every dimension and we are also able to provide the complete characterization of the long-time behaviour starting from any initial set in dimension $N = 2$. In order to state the precise result in the two-dimensional case we first introduce the following notation.

We will call *lamella* any connected set in $\mathbb{T}^2$ whose $1-$periodic extension in $\mathbb{R}^2$ is a stripe bounded by two parallel lines. Our final result in two dimension is the following theorem.

Theorem 5.2. *Fix h, $m > 0$ and an initial set $E_0 \subset \mathbb{T}^2$ with finite perimeter and such that $|E_0| = m$. Let $(E_h^n)_{n\in\mathbb{N}}$ be a discrete flow starting from E_0 and let P_∞ be the limit of the non-increasing sequence $P(E_h^n)$. Then either one of the following holds:*

i) $(E_h^n)_{n\in\mathbb{N}}$ converges to a disjoint union of l discs of equal radii and total area m, where $l = \pi^{-1}(4m)^{-1}P_\infty^2 \in \mathbb{N}$;

ii) $((E_h^n)^c)_{n\in\mathbb{N}}$ converges to a disjoint union of l discs of equal radii and total area $1-m$, where $l = \pi^{-1}(4-4m)^{-1}P_\infty^2 \in \mathbb{N}$;

iii) $(E_h^n)_{n\in\mathbb{N}}$ converges to a disjoint union of l lamellae of total area m, with the same slope and $l \leq P_\infty/2$. Moreover, the equality $l = P_\infty/2 \in \mathbb{N}$ holds if and only if the limit is given by vertical or horizontal lamellae.

In all cases the convergence is exponentially fast in C^k for every $k \in N$.

1.1 Comments about the proof of Theorem 5.1

The first step towards proving our main result Theorem 5.1 is Proposition 5.30. More precisely, we prove the convergence up to translations of any discrete flow, starting Hausdorff-close to a strictly stable critical set E, to the latter set. Such a convergence holds in the C^k-norm for every $k \in \mathbb{N}$. Since at this point we can not rule out that different subsequences of the discrete flow may converge to different translates of E, the subsequent step consists in proving the convergence of the whole flow to a unique translate of the set E (with exponential rate).

In order to prove Proposition 5.30, in a first step we show that every long-time limit of the flow is a critical point of the perimeter. When the ambient space is $\mathbb{R}^N$, this implies that the limit points can only be balls or finite union of balls with the same radii. However, in the periodic setting, we may end up with different critical points of the perimeter. Indeed, already in the three dimensional torus $\mathbb{T}^3$ we find a wealth of different critical points in addition to balls: for example, lamellae, cylinders and gyroids (see Figure 5.1) . We then exploit the strict stability of E (Proposition 5.27) to ensure that the flow remains L^1-close up to translations to the set E. To conclude, a regularity argument shows that the convergence in L^1 of the flow to a regular stable set implies the convergence in C^k for every $k \in \mathbb{N}$, thus proving Proposition 5.30.

The proof of Proposition 5.27 is based on the following idea: from a stability result in [4], one can estimate the L^1-distance (up to translations) of a set F from a strictly stable critical set E in terms of the differences of the perimeters, provided that the L^1-distance between E and F

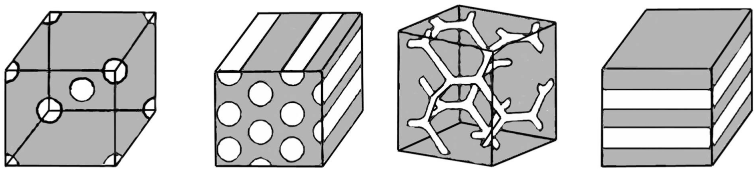

Figure 5.1: The critical points in $\mathbb{T}^3$. Balls, cylinders, gyroids and $3-$dimensional lamellae.

remains below a certain threshold. Moreover, a counterexample shows that the Hausdorff-closeness assumption can not be weakened to L^1-closeness, as we will discuss in details in Subsection 4.3.

In order to establish the uniqueness of the limit and, therefore, prove Theorem 5.1, Section 5.2 is devoted to proving the convergence of the barycenters of the evolving sets. A crucial intermediate result consists in generalizing the Alexandrov-type estimate [154, Theorem 1.3] (see also [138]) to the flat torus. This result provides a stability inequality for C^1-normal deformations of strictly stable critical sets in the periodic setting. It could also be seen as an higher-order Łojasiewicz-Simon inequality for the perimeter functional. We briefly give some definitions to present some further details. Given a set E of class C^1 and a function $f : \partial E \to \mathbb{R}$ such that $\|f\|_{L^\infty(\partial E)}$ is sufficiently small, the *normal deformation* E_f of the set E is defined as

$$\partial E_f := \{x + f(x)\nu_E(x) \ : \ x \in \partial E\},$$

where ν_E is the normal outer vector of E. A normal deformation E_f is said to be of class C^k if E is of class C^k and $f \in C^k(\partial E)$. The result proved in [154] is the following.

Theorem 5.3. *There exist $\delta \in (0, 1/2)$ and $C > 0$ with the following property: for any $f \in C^1(\partial B) \cap H^2(\partial B)$ such that $\|f\|_{C^1(\partial B)} \leq \delta$, $|E_f| = \omega_N$ and $\mathrm{bar}(E_f) = 0$, we have*

$$\|f\|_{H^1(\partial B)} \leq C\|H_{E_f} - \overline{H}_{E_f}\|_{L^2(\partial B)}.$$

We are able to show that in the periodic setting the above quantitative estimate holds with B replaced by any strictly stable critical set. More precisely, we have the following:

Theorem 5.4. *Let $E \subset \mathbb{T}^N$ be a strictly stable critical set. There exist $\delta \in (0, 1/2)$ and $C > 0$ with the following property: for any $f \in C^1(\partial E) \cap H^2(\partial E)$ such that $\|f\|_{C^1(\partial E)} \leq \delta$ and satisfying*

$$\left|\int_{\partial E} f \,\mathrm{d}\mathcal{H}^{N-1}\right| \leq \delta\|f\|_{L^2(\partial E)}, \qquad \left|\int_{\partial E} f\nu_E \,\mathrm{d}\mathcal{H}^{N-1}\right| \leq \delta\|f\|_{L^2(\partial E)}, \tag{5.2}$$

we have

$$\|f\|_{H^1(\partial E)} \leq C\|H_{E_f} - \overline{H}_{E_f}\|_{L^2(\partial E)}. \tag{5.3}$$

We will prove in details in Section 3 that the conditions (5.2) have a geometric explanation. Indeed, the first one ensures that $|E_f| \approx |E|$, up to higher-order error terms, and the second one, for some choices of E, is implied by imposing $\mathrm{bar}(E_f) \approx \mathrm{bar}(E)$. We finally remark that the estimate (5.3) is optimal for what concerns the power of the norms, see [154, Remark 1.5].

The last section of the chapter is devoted to the two-dimensional case. This particular choice of the dimension is purely technical and it is motivated by the availability of a complete characterization of the critical points of the perimeter in the two-dimensional flat torus. In this setting we are able to prove the exponential convergence of the flow starting from any initial set to either a finite union of balls or a finite union of lamellae or the complement of these configurations.

2 Preliminaries

Let $\mathbb{T}^N := \mathbb{R}^N/\mathbb{Z}^N$ be the $N-$dimensional torus, that is the quotient space $\mathbb{R}^N/\sim$ where $\sim$ is the equivalence relation given by $x \sim y$ if and only if $x - y \in \mathbb{Z}^N$. We can define the distance between two points $x, y \in \mathbb{T}^N$ simply by

$$\operatorname{dist}(x,y) = \min_{z \in \mathbb{Z}^N} |(x+z) - y|.$$

The function spaces $C^k(\mathbb{T}^N)$ and $W^{k,p}(\mathbb{T}^N)$, for $k \in \mathbb{N}$ and $p \in [1,\infty]$, are defined as the restriction of $C^k(\mathbb{R}^N)$ and $W^{k,p}_{loc}(\mathbb{R}^N)$, respectively, to the functions that are one-periodic. When we need to be specific about functions on the torus, it is often convenient to give coordinates to $\mathbb{T}^N$ via the unit cube $Q = [0,1)^N$. With $B_r(x)$ we denote the ball in $\mathbb{R}^N$ of center x and radius r, while B_r will be a short-hand notation for $B_r(0)$. Given $x \in \mathbb{R}^N$, we will write $x = (x', x_N)$ where $x' \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$. Similarly, we denote by $B'_r(x') \subset \mathbb{R}^{N-1}$ the ball in $\mathbb{R}^{N-1}$ with radius $r > 0$ and center $x' \in \mathbb{R}^{N-1}$. Let $F \subset \mathbb{T}^N$ we denote with $\operatorname{dist}_F(\cdot)$ the distance from the set F and with $C^{1,1}(\partial F)$ the set of functions continuously differentiable with derivative Lipschitz continuous on ∂F. Moreover, we denote by c, C some constants, which could be changing from line to line and always depend on the dimension N, and by $\frac{\partial}{\partial t}$ (or equivalently ∂_t) the partial derivative with respect to the variable t.

Let us recall the definition of functions of bounded variation in the periodic setting. We say that a function $u \in L^1(\mathbb{T}^N)$ is of bounded variation if its total variation is finite, that is

$$|Du| = \sup\left\{ \int_{\mathbb{T}^N} u \operatorname{div}\varphi \,\mathrm{d}x : \varphi \in C^1(\mathbb{T}^N;\mathbb{R}^N),\ |\varphi| \le 1 \right\} < +\infty.$$

We denote the space of such functions by $BV(\mathbb{T}^N)$. We say that a measurable set $E \subset \mathbb{T}^N$ is of finite perimeter in $\mathbb{T}^N$ if its characteristic function $\chi_E \in BV(\mathbb{T}^N)$. The perimeter $P(E)$ of E in $\mathbb{T}^N$ is nothing but the total variation $|D\chi_E|(\mathbb{T}^N)$. We refer to Maggi's book [145] for a more complete reference about sets of finite perimeter and their properties.

We introduce the following notation.

Definition 5.5. Let E be a set of class C^1. Given a function $f : \partial E \to \mathbb{R}$ such that $\|f\|_{L^\infty(\partial E)}$ is sufficiently small, we set

$$\partial E_f := \{x + f(x)\nu_E(x) : x \in \partial E\} \tag{5.4}$$

and we call E_f the *normal deformation* of E induced by f.

With a slight abuse of notation, we give the following definition.

Definition 5.6. Let E be a set of class C^1. Let $X(\partial E)$ denote a functional space that can either be $L^p(\partial E)$, $W^{k,p}(\partial E)$, $C^{k,\alpha}(\partial E)$ for some $k \in \mathbb{N}$, $p \ge 1$ and $\alpha \in [0,1]$. For any $F = E_f$ with $f \in X(\partial E)$, we set

$$\operatorname{dist}_X(F,E) = \|f\|_{X(\partial E)}.$$

We recall the classical definition of $C^{1,\alpha}-$convergence of sets.

Definition 5.7. Given $\alpha \in [0,1]$, a sequence $(E_n)_{n\in\mathbb{N}}$ of $C^{1,\alpha}-$regular sets is said to converge in $C^{1,\alpha}$ to a set E if:

- for any $x \in \partial E$, up to rotations and relabelling the coordinates, we can find a cylinder $C = B' \times (-1,1)$, where $B' \subset \mathbb{R}^{N-1}$ is the unit ball centred at the origin, and functions $f, f_n \in C^{1,\beta}(B';(-1,1))$ such that for n large enough, it holds

$$(E - x) \cap C = \{(x', x_N) \in B' \times (-1,1) \ : \ x_N \le f(x')\}$$
$$(E_n - x) \cap C = \{(x', x_N) \in B' \times (-1,1) \ : \ x_N \le f_n(x')\};$$

- it holds

$$f_n \to f \quad \text{in } C^{1,\alpha}(B').$$

The following is a simple rephrasing of a classical result concerning the $C^{1,\alpha}-$convergence of $\Lambda-$minimizers of the perimeter (see e.g. [4, Theorem 4.2]).

Theorem 5.8. *Let $\Lambda > 0$ and let E be a set of class C^2. Then for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, E) > 0$ with the following property: for every $\Lambda-$minimizer F such that $|E \triangle F| \leq \delta$, then F is of class $C^{1,1/2}$ and*

$$\operatorname{dist}_{C^{1,\beta}}(E, F) \leq \varepsilon \quad \text{for} \quad \beta \in (0, 1/2).$$

We now recall some preliminary results from [4] regarding the second variation of the perimeter in the flat torus. Let $E \subset \mathbb{T}^N$ be a set of class C^2 and let ν_E be its outer normal. Throughout the section, when no confusion is possible, we shall omit the subscript E and write ν instead of ν_E. Given a vector X, its tangential part on ∂E is defined as $X_\tau = X - (X \cdot \nu)\nu$. In particular, we will denote by D_τ the tangential gradient operator given by $D_\tau \varphi = (D\varphi)_\tau$. We also recall that the second fundamental form B_E of ∂E is given by $D_\tau \nu$, its eigenvalues are called principal curvatures and its trace is called mean curvature, and we denote it by H_E.

Let $X : \mathbb{T}^N \to \mathbb{R}^N$ be a vector field of class C^2. Consider the associated flow $\Phi : \mathbb{T}^N \times (-1, 1) \to \mathbb{T}^N$ defined by $\frac{\partial \Phi}{\partial t} = X(\Phi)$, $\Phi(\cdot, 0) = Id$. We define the *first and second variation of the perimeter at E* with respect to the flow Φ to be respectively the values

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P(E_t), \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} P(E_t)$$

where $E_t = \Phi(\cdot, t)(E)$. It is a classical result of the theory of sets of finite perimeter (see [145]) that the the first variation of the perimeter has the following expression

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P(E_t) = \int_{\partial^* E} \mathrm{H}_E \nu_E \cdot X \, \mathrm{d}\mathcal{H}^{N-1},$$

where $\mathrm{H}_E \in L^2(\partial^* E)$ is the (weak) scalar mean curvature of E. The following equation for the second variation of the perimeter holds.

Theorem 5.9 (Theorem 3.1 in [4])**.** *If E, X and ν are as above, we have*

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} P(E_t) = &\int_{\partial E} \left(|D_\tau(X \cdot \nu)|^2 - |B_E|^2 (X \cdot \nu)^2\right) \mathrm{d}\mathcal{H}^{N-1} - \int_{\partial E} \mathrm{H}_E \operatorname{div}_\tau (X_\tau (X \cdot \nu)) \, \mathrm{d}\mathcal{H}^{N-1} \\ &+ \int_{\partial E} \mathrm{H}_E (\operatorname{div} X)(X \cdot \nu) \, \mathrm{d}\mathcal{H}^{N-1}.\end{aligned}$$

Remark 5.10. We remark that the last two integral in the above expression vanish when E is a critical set for the perimeter and if $|\Phi(\cdot, t)(E)| = |E|$ for all $t \in [0, 1]$. Indeed, if E is a regular critical set for the perimeter then its curvature is constant, therefore the second integral vanishes. Moreover, if the flow Φ is volume-preserving then it can be shown (see equation (2.30) in [59]) that

$$0 = \frac{\mathrm{d}^2 |E_t|}{\mathrm{d}t^2} = \int_{\partial E} (\operatorname{div} X)(X \cdot \nu) \, \mathrm{d}\mathcal{H}^{N-1}.$$

Hence, if Φ is a volume-preserving variation of a regular critical set E we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} P(E_t) = \int_{\partial E} \left(|D_\tau(X \cdot \nu)|^2 - |B_E|^2 (X \cdot \nu)^2\right) \mathrm{d}\mathcal{H}^{N-1} =: \delta^2 P(E)[X \cdot \nu_E].$$

We remark that due to the translation invariance of the perimeter functional, the second variation degenerates along flows of the form $\Phi(x, t) = x + t\eta$, where $\eta \in \mathbb{R}^N$. In view of this it is convenient to introduce the subspace $T(\partial E)$ of $\tilde{H}^1(\partial E) := \left\{\varphi \in H^1(\partial E) : \int_{\partial E} \varphi \, \mathrm{d}\mathcal{H}^{N-1} = 0\right\}$ generated by the functions ν_i, $i = 1, \dots, N$. Its orthogonal subspace, in the L^2-sense, will be denoted by $T^\perp(\partial E)$ and is given by

$$T^\perp(\partial E) = \left\{\varphi \in \tilde{H}^1(\partial E) : \int_{\partial E} \varphi \nu_i \, \mathrm{d}\mathcal{H}^{N-1} = 0, \ i = 1, \dots, N\right\}.$$

Definition 5.11. We say that a regular critical set E is *a strictly stable set* if it has positive second variation of the perimeter, in the sense that

$$\delta^2 P(E)[\varphi] > 0, \qquad \forall \varphi \in T^{\perp}(\partial E) \setminus \{0\}.$$

The following result ensures that the second variation of a strictly stable set E is coercive on the subspace $T^{\perp}(\partial E)$.

Lemma 5.12 (Lemma 3.6 in [4]). *Assume that E is a strictly stable set, then*

$$m_0 := \inf\{\delta^2 P(E)[\varphi] : \varphi \in T^{\perp}(\partial E),\, \|\varphi\|_{H^1(\partial E)} = 1\} > 0$$

and

$$\delta^2 P(E)[\varphi] \geq m_0 \|\varphi\|^2_{H^1(\partial E)} \quad \forall \varphi \in T^{\perp}(\partial E).$$

Moreover, from the Step 1 in the proof of [4, Theorem 3.9] we obtain also the following result.

Lemma 5.13. *Assume that E is a strictly stable set, then*

$$\inf\left\{\delta^2 P(E)[\varphi] : \varphi \in \tilde{H}^1,\ \|\varphi\|_{H^1(\partial E)} = 1,\ \left|\int_{\partial E} \varphi \nu_E \, \mathrm{d}\mathcal{H}^{N-1}\right| \leq \delta\right\} \geq \frac{m_0}{2},$$

where the constant m_0 is the one in Lemma 5.12.

In the proof of Theorem 5.1 we will also need the following key lemma.

Lemma 5.14 (Lemma 3.8 in [4]). *Let $E \subset \mathbb{T}^N$ be of class C^3 and let $p > N-1$. For every $\delta > 0$ there exist $C > 0$ and $\eta_0 > 0$ such that if $F \subset \mathbb{T}^N$ satisfies $\partial F = \{x + \psi(x)\nu_E(x) : x \in \partial E\}$ for some $\psi \in C^2(\partial E)$ with $\|\psi\|_{W^{2,p}(\partial E)} \leq \eta_0$, then there exist $\sigma \in \mathbb{T}^N$ and $\varphi \in W^{2,p}(\partial E)$ with the properties that*

$$|\sigma| \leq C\|\psi\|_{W^{2,p}(\partial E)}, \quad \|\varphi\|_{W^{2,p}(\partial E)} \leq C\|\psi\|_{W^{2,p}(\partial E)}$$

and

$$\partial F + \sigma = \{x + \varphi(x)\nu_E(x) : x \in \partial E\}, \quad \left|\int_{\partial E} \varphi \nu_E \, \mathrm{d}\mathcal{H}^{N-1}\right| \leq \delta \|\varphi\|_{L^2(\partial E)}.$$

We also recall the definition of inner and outer ball condition.

Definition 5.15. We say that a open set $E \subset \mathbb{T}^N$ satisfies a *uniform inner (respectively outer) ball condition* with radius r if there exists $r > 0$ such that for every $x \in \partial E$ there exists a ball $B_r(y) \subset E$ (resp. $B_r(y) \subset E^c$) with $x \in \partial B_r(y)$.

Note that all sets $E \subset \mathbb{T}^N$ of class $C^{1,1}$ satisfy a uniform inner and outer ball condition (see e.g. [65]). Arguing as in the proof of [4, Lemma 3.8], we can prove the following result.

Lemma 5.16. *Let $E \subset \mathbb{T}^N$ be of class C^∞ and let $m > 0$. There exists $\eta = \eta(m, E) > 0$ such that, for every $k \in \mathbb{N}$, $u \in C^k(\partial E)$ with $\|u\|_{C^k(\partial E)} \leq m$, $\|u\|_{C^0(\partial E)} \leq \eta$ and for every $\sigma \in \mathbb{T}^N$ with $|\sigma| \leq \eta$, then $E_u + \sigma$ can be written as a normal deformation of E induced by a function $v : \partial E \to \partial E$ such that*

$$\|v\|_{C^0(\partial E)} \leq 2\eta, \quad \|v\|_{C^k(\partial E)} \leq C(\|u\|_{C^k(\partial E)} + |\sigma|),$$

where $C = C(E) > 0$.

Proof. Being the set E smooth, it satisfy the uniform inner and outer ball condition, hence there exists a positive radius $r > 0$ such that the signed distance sd_E from the set E, defined by

$$\mathrm{sd}_E(x) = \begin{cases} \mathrm{dist}_{\partial E}(x) & \text{if } x \in E^c \\ -\mathrm{dist}_{\partial E}(x) & \text{if } x \in E, \end{cases}$$

is a function of class C^∞ (from the regularity of ∂E) in the r-tubular neighborhood $(\partial E)_r$, that is $(\partial E)_r := \{x\colon \mathrm{dist}_{\partial E}(x) < r\}$ (for further properties of the distance function see [109, section

14.6]). Since, for some $k \geq 2$, u has C^k-norm bounded by m, we also have $\|u\|_{C^{1,1}(\partial E)} \leq m$. Then, there exists a radius $\rho = \rho(m, E)$ such that ∂E_u satisfies a uniform inner and outer ball condition of radius ρ. We can assume without loss of generality that $\rho < r$.

We now let $\eta \leq \rho/2$ to be chosen later, take any $|\sigma| < \eta$ and set $F = E_u + \sigma$. Clearly, F still satisfies a uniform inner and outer ball condition of radius ρ. Then, for every $y \in \partial F$ there exists $x \in \partial E_u$ such that $y = x + \sigma$, hence we have

$$\text{dist}_{\partial E}(y) \leq |\sigma| + \text{dist}_{\partial E}(x) < \eta + \|u\|_{C^0(\partial E)} \leq 2\eta,$$

and in particular $\partial F \subset (\partial E)_{2\eta} \subset (\partial E)_r$. We now define the map $T_u : \partial E \to \partial E$ as

$$T_u(x) \coloneqq \pi_E(x + u(x)\nu_E(x) + \sigma) = y - \text{sd}_E(y)\nabla \text{sd}_E(y), \tag{5.5}$$

where π_E is the projection map on ∂E and $y = x + u(x)\nu_E(x) + \sigma \in \partial F$. By choosing η smaller, by interpolation, it holds $\|u\|_{C^1(\partial E)} + |\sigma| < \frac{1}{2}$, which implies that the function $x \mapsto x + u(x)\nu_E(x) + \sigma$ is a diffeomorphism (since it is a small perturbation of the identity). Moreover, since E is of class C^∞ (and possibly for η smaller), $\pi_E\big|_{\partial F} : \partial F \to \partial E$ is a diffeomorphism of class C^k, C^k-close to the identity. Therefore, $T_u \in C^k(\partial E)$ and, by (5.5), we get

$$\|T_u - I\|_{C^k(\partial E)} \leq C(\|u\|_{C^k(\partial E)} + |\sigma|). \tag{5.6}$$

Moreover, using again (5.5) and the invertibility of the map $x \mapsto x + u(x)\nu_E(x) + \sigma$, we obtain

$$\|T_u^{-1} - I\|_{C^k(\partial E)} \leq C(\|u\|_{C^k(\partial E)} + |\sigma|). \tag{5.7}$$

Using the fact that T_u is a diffeomorphism and (5.5), we can find a function $v : \partial E \to \mathbb{R}$ such that F is the normal deformation of E induced by v, more precisely for every $x \in \partial E$ it holds

$$x + u(x)\nu_E(x) + \sigma = T_u(x) + v(T_u(x))\nu_E(T_u(x)).$$

Finally, using the above expression and the bounds in (5.6) and (5.7), we conclude that

$$\|v\|_{C^k(\partial E)} \leq \|T_u^{-1}\|_{C^k(\partial E)}(\|u\|_{C^k(\partial E)} + |\sigma| + \|T_u - I\|_{C^k(\partial E)}) \leq C(\|u\|_{C^k(\partial E)} + |\sigma|),$$

for some constant $C = C(E) > 0$. □

Let $E, F \subset \mathbb{T}^N$ be measurable sets. We define

$$\alpha(E, F) := \min_{x \in \mathbb{T}^N} |E \triangle (F + x)|.$$

In one of the main results of [4] the authors proved the following quantitative isoperimetric inequality in the periodic setting.

Theorem 5.17 (Corollary 1.2 in [4]). *Let $E \subset \mathbb{T}^N$ be a strictly stable set. Then, there exist $\sigma = \sigma(E)$, $C = C(E) > 0$ such that*

$$C\alpha^2(E, F) \leq P(F) - P(E)$$

for all $F \subset \mathbb{T}^N$ with $|F| = |E|$ and $\alpha(E, F) < \sigma$.

3 A quantitative generalized Alexandrov Theorem

In this section, we will prove that local minimizers of the perimeter in the flat torus satisfy a quantitative Alexandrov-type estimate. We reproduce some arguments similar to the ones used in the proof of Theorem 1.3 in [154]. In the follwoing, we denote by $E \subset \mathbb{T}^N$ a strictly stable set. Thanks to some classical results for sets of finite perimeter (see for example [145, Theorem 27.4]), the previous hypothesis implies that E is of class C^∞.

First of all, we compute the $(N-1)-$Jacobian of the map

$$\Phi : \partial E \to \partial E_f \subset \mathbb{R}^N, \qquad x \mapsto x + f(x)\nu_E(x).$$

Given $x \in \partial E$, we choose an orthonormal basis

$$\mathcal{B}' := \{v_1(x), \dots, v_{N-1}(x)\}$$

of T_xE such that in this basis the second fundamental form of E, $B_E(x) : T_xE \to T_xE \subset \mathbb{R}^N$, has the following expression

$$B_E(x) = \begin{pmatrix} \kappa_1(x) & & \\ & \ddots & \\ & & \kappa_{N-1}(x) \\ 0 & \dots & 0 \end{pmatrix},$$

where $\kappa_1(x), \dots, \kappa_{N-1}(x)$ are the principal curvatures of E in x. We then complete $\mathcal{B}'$ to a basis $\mathcal{B}$ of the whole $\mathbb{R}^N$ with the normal vector $v_N(x) := \nu_E(x)$. In the following, to simplify the notation, we will drop the dependence on x. The tangential differential of Φ with respect to the basis $\mathcal{B}$ is given by

$$D\Phi = I + \nu_E \otimes \nabla f + f D\nu_E,$$

where I is the immersion $T_xE \hookrightarrow \mathbb{R}^N$, ∇f is the tangential gradient of f and $D\nu_E$ is the tangential differential of ν_E. Given the regularity of ∂E, we recall that $D\nu_E$ is equal to B_E. Moreover, by definition of $\mathcal{B}$, we have that

$$(\nu_E \otimes \nabla f)(v_i, v_j) = \delta_{N,i} \, \nabla f \cdot v_j, \quad i = 1, \dots, N, \; j = 1, \dots, N-1.$$

Thanks to the previous observations we find the following expression

$$D\Phi = \begin{pmatrix} 1+\kappa_1 f & & \\ & \ddots & \\ & & 1+\kappa_{N-1} f \\ \partial_{v_1} f & \dots & \partial_{v_{N-1}} f \end{pmatrix}. \tag{5.8}$$

By Binet formula, the Jacobian $J\Phi$ can be explicitly computed as

$$\begin{aligned} J\Phi &= \left(\prod_{i=1}^{N-1} (1+\kappa_i f)^2 + \sum_{j=1}^{N-1} (\partial_{v_j} f)^2 \prod_{i \neq j} (1+\kappa_i f)^2 \right)^{1/2} \\ &= \prod_{i=1}^{N-1} (1+\kappa_i f) \left(1 + \sum_{j=1}^{N-1} \frac{(\partial_{v_j} f)^2}{(1+\kappa_j f)^2} \right)^{1/2}. \end{aligned} \tag{5.9}$$

To show the previous formula, we characterize the minors of $D\Phi$. If we omit the $N-$th row of $D\Phi$, we obtain the minor

$$M_N = \begin{pmatrix} 1+\kappa_1 f & & \\ & \ddots & \\ & & 1+\kappa_{N-1} f \end{pmatrix},$$

if we omit the $i-$th row of $D\Phi$ for $1 \le i \le N-1$, we obtain the minor

$$M_i = \begin{pmatrix} 1+\kappa_1 f & & & & & \\ & \ddots & & & & \\ & & 1+\kappa_{i-1} f & & & \\ & & & 1+\kappa_{i+1} f & & \\ & & & & \ddots & \\ & & & & & 1+\kappa_{N-1} f \\ \partial_{v_1} f & \dots & \partial_{v_{i-1}} f & \partial_{v_{i+1}} f & \dots & \partial_{v_{N-1}} f \end{pmatrix}.$$

We then deduce (5.9) by explicitly computing

$$\det(M_N)^2 = \prod_{i=1}^{N-1} (1+\kappa_i f)^2, \quad \det(M_i)^2 = (\partial_{v_i} f)^2 \prod_{j \neq i} (1+\kappa_j f)^2.$$

Note that, if $\|f\|_{C^1}$ is small enough, the map Φ is a diffeomorphism from ∂E to $\Phi(\partial E) = \partial E_f$, and thus the tangential differential $D\Phi : T_x E \to T_{\Phi(x)} E_f$ is a surjective map. In particular, this allows us to calculate the normal vector ν_{E_f} in $\Phi(x)$. We remark that a vector v orthogonal to every column of (5.8) is a normal vector to the whole tangent space $T_{\Phi(x)} E_f$, therefore a possible v is given by

$$v = -\sum_{i=1}^{N-1} \frac{\partial_{v_i} f}{1+\kappa_i f} v_i + \nu_E,$$

where the sign of the component along ν_E is taken positive so that the case $f = 0$ is consistent with the orientation of ν_E. Since $|v| \ge 1$, by normalizing v we obtain the normal vector

$$\nu_{E_f} = \left(\nu_E - \sum_{i=1}^{N-1} \frac{\partial_{v_i} f}{1+\kappa_i f} v_i \right) \left(1 + \sum_{j=1}^{N-1} \frac{(\partial_{v_j} f)^2}{(1+\kappa_j f)^2} \right)^{-1/2}, \tag{5.10}$$

moreover, we remark that

$$\nu_E \cdot \nu_{E_f} = \left(1 + \sum_{j=1}^{N-1} \frac{(\partial_{v_j} f)^2}{(1+\kappa_j f)^2} \right)^{-1/2}. \tag{5.11}$$

We can now compute explicitly the formula for the first variation of the perimeter.

Lemma 5.18. *Setting* $Q := \left(1 + \sum_{j=1}^{N-1} \frac{(\partial_{v_j} f)^2}{(1+\kappa_j f)^2} \right)^{1/2}$*, the following formulas hold true:*

1. *If* $f \in L^\infty(\partial E) \cap H^1(\partial E)$ *with* $\|f\|_{L^\infty}$ *sufficiently small, then*

$$P(E_f) = \int_{\partial E} Q \prod_{i=1}^{N-1} (1+\kappa_i f) \, \mathrm{d}\mathcal{H}^{N-1}.$$

2. *If* $f \in L^\infty(\partial E) \cap H^1(\partial E)$ *with* $\|f\|_{L^\infty}$ *sufficiently small, then the first variation* $\delta P(E_f)[\varphi]$

exists for all $\varphi \in C^1(\partial E)$ *and is given by*

$$\delta P(E_f)[\varphi] = \int_{\partial E} \varphi\, Q \sum_{i=1}^{N-1} \kappa_i \prod_{j\neq i} (1+\kappa_j f)\, \mathrm{d}\mathcal{H}^{N-1} + \int_{\partial E} \frac{1}{Q} \prod_{i=1}^{N-1} (1+\kappa_i f) \left(\sum_{j=1}^{N-1} \frac{\partial_{v_j}\varphi\, \partial_{v_j} f}{(1+k_j f)^2} - \varphi \sum_{j=1}^{N-1} \frac{k_j\, (\partial_{v_j} f)^2}{(1+k_j f)^3} \right) \mathrm{d}\mathcal{H}^{N-1}. \tag{5.12}$$

Proof. The first formula is a straightforward consequence of the area formula

$$P(E_f) = \int_{\partial E_f} \mathrm{d}\mathcal{H}^{N-1} = \int_{\partial E} J\Phi\, \mathrm{d}\mathcal{H}^{N-1}$$

and of the expression of the Jacobian $J\Phi$ in (5.9). Now, (5.12) easily follows by taking the derivatives

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} P(E_{f+\varepsilon\varphi})$$

in the first formula. □

Remark 5.19. We observe that, if $\|f\|_{L^\infty(\partial E)}$ is small enough and $|E_f| = |E|$, then there exists a constant $C > 0$, only depending on E, such that

$$\left| \int_{\partial E} f(x)\, \mathrm{d}\mathcal{H}^{N-1}(x) \right| \le C \int_{\partial E} f(x)^2\, \mathrm{d}\mathcal{H}^{N-1}(x). \tag{5.13}$$

Firstly, since ∂E is regular, for every $\varepsilon > 0$ sufficiently small there exists a tubular neighborhood $\mathcal{N}$ of ∂E such that $\mathcal{N}$ is diffeomorphic to $\partial E \times (-\varepsilon, \varepsilon)$ via the diffeomeorphism $\Psi(x,t) = x + \nu_E(x)t$. The Jacobian of Ψ is given by

$$J\Psi(x,t) = \prod_{i=1}^{N-1} (1+\kappa_i(x)t). \tag{5.14}$$

Secondly, if $\|f\|_{L^\infty(\partial E)}$ is small enough, we remark that the condition $|E_f| = |E|$ is equivalent to

$$0 = |E_f| - |E| = \int_{\partial E} \int_0^{f(x)} J\Psi(x,t)\, \mathrm{d}t\, \mathrm{d}\mathcal{H}^{N-1}(x).$$

Then, we can conclude that

$$\begin{aligned} 0 &= \int_{\partial E} \int_0^{f(x)} J\Psi(x,t)\, \mathrm{d}t\, \mathrm{d}\mathcal{H}^{N-1}(x) \\ &= \int_{\partial E} f(x)\, \mathrm{d}\mathcal{H}^{N-1}(x) + \int_{\partial E} \int_0^{f(x)} (J\Psi(x,t) - 1)\, \mathrm{d}t\, \mathrm{d}\mathcal{H}^{N-1}(x) \\ &= \int_{\partial E} f(x)\, \mathrm{d}\mathcal{H}^{N-1}(x) + \int_{\partial E} \int_0^{f(x)} (H_E(x)\, t + o(t))\, \mathrm{d}t\, \mathrm{d}\mathcal{H}^{N-1}(x), \end{aligned}$$

that implies (5.13) for a constant depending only on N and the principal curvatures of E.

We are now able to prove the following stability result; it ensures that the second variation of the perimeter remains strictly positive for small normal deformations of a strictly stable set E.

Lemma 5.20. *Fix* $N \ge 2$. *There exists* $\delta = \delta(E) > 0$ *small such that, if* $f \in L^\infty(\partial E) \cap H^1(\partial E)$ *with* $\|f\|_{L^\infty(\partial E)} \le \delta$,

$$\left| \int_{\partial E} f(x)\, \mathrm{d}\mathcal{H}^{N-1}(x) \right| \le \delta \|f\|_{L^2(\partial E)} \quad \text{and} \quad \left| \int_{\partial E} f(x)\nu_E(x)\, \mathrm{d}\mathcal{H}^{N-1}(x) \right| \le \delta \|f\|_{L^2(\partial E)}, \tag{5.15}$$

then we have

$$\delta^2 P(E)[f] = \int_{\partial E} (|\nabla f(x)|^2 - |B_E(x)|^2 f(x)^2)\, \mathrm{d}\mathcal{H}^{N-1}(x) \geq \frac{m_0}{8} \|f\|^2_{H^1(\partial E)},$$

where m_0 is the constant given by Lemma 5.12.

Proof. Set $g = f - \bar{f}$, where $\bar{f} = \fint_{\partial E} f \, \mathrm{d}\mathcal{H}^{N-1}$, then g has zero average and, by the first inequality in (5.15), we have

$$\bar{f}^2 = \frac{1}{P(E)^2} \left(\int_{\partial E} f \, \mathrm{d}\mathcal{H}^{N-1} \right)^2 \leq C\delta^2 \|f\|^2_{L^2(\partial E)}. \tag{5.16}$$

If δ is sufficiently small, from (5.16) we obtain

$$\|g\|^2_{L^2(\partial E)} = \|f - \bar{f}\|^2_{L^2(\partial E)} = \|f\|^2_{L^2(\partial E)} - \bar{f}^2 P(E) \geq \|f\|^2_{L^2(\partial E)} \left(1 - C\delta^2\right) \geq \frac{1}{2} \|f\|^2_{L^2(\partial E)}.$$

Using the previous inequality, (5.16) again and the second inequality in (5.15) we infer that the function g satisfies

$$\left| \int_{\partial E} g\nu_E \, \mathrm{d}\mathcal{H}^{N-1} \right| \leq \left| \int_{\partial E} f\nu_E \, \mathrm{d}\mathcal{H}^{N-1} \right| + \left| \int_{\partial E} \bar{f}\nu_E \, \mathrm{d}\mathcal{H}^{N-1} \right| \leq C\delta \|g\|_{L^2(\partial E)}.$$

Then, we can apply Lemma 5.13 to obtain

$$\delta^2 P(E)[g] \geq \frac{m_0}{2} \|g\|^2_{H^1(\partial E)},$$

provided δ small enough. We conclude

$$\begin{aligned}
\delta^2 P(E)[f] &= \delta^2 P(E)[g] - \delta^2 P(E)[g] + \delta^2 P(E)[f] \\
&= \delta^2 P(E)[g] - 2\bar{f} \int_{\partial E} |B_E(x)|^2 f(x) \, \mathrm{d}\mathcal{H}^{N-1}(x) + \bar{f}^2 \int_{\partial E} |B_E(x)|^2 \, \mathrm{d}\mathcal{H}^{N-1}(x) \\
&\geq \frac{m_0}{2} \|g\|^2_{H^1(\partial E)} - C|\bar{f}| \|f\|_{L^2(\partial E)} \geq \frac{m_0}{2} (\|g\|^2_{L^2(\partial E)} + \|\nabla g\|^2_{L^2(\partial E)}) - C\delta \|f\|^2_{L^2(\partial E)} \\
&\geq \frac{m_0}{4} (\|f\|^2_{L^2(\partial E)} + \|\nabla f\|^2_{L^2(\partial E)}) - C\delta \|f\|^2_{L^2(\partial E)} \geq \frac{m_0}{8} \|f\|^2_{H^1(\partial E)},
\end{aligned}$$

up to taking δ smaller if needed, and where the constant $C > 0$ only depends on E. $\square$

Remark 5.21. Remark 5.19 ensures that the conclusion of the previous lemma also holds if we replace the hypothesis $|\fint_{\partial E} f \, \mathrm{d}\mathcal{H}^{N-1}| \leq \delta \|f\|_{L^2(\partial E)}$ with $\|f\|_{L^\infty(\partial E)}$ small enough and $|E_f| = |E|$.

We are now able to prove the generalized version of the quantitative Alexandrov's inequality in the periodic setting, Theorem 5.4.

Proof of Theorem 5.4. First of all we notice that, if we take the constant C in (5.3) to be bigger than $\sqrt{P(E)/2}$, then it is enough to consider only the case $\|H_{E_f} - \overline{H}_{E_f}\|_{L^2(\partial E)} \leq 1$.

Set $p = x + f(x)\nu_E(x)$ and let $\varphi \in C^1(\partial E)$, by the definition of scalar mean curvature H_{E_f} and a change of coordinates we obtain

$$\delta P(E_f)[\varphi] = \int_{\partial E} (H_{E_f} \nu_{E_f})(p) \cdot \nu_E \, \varphi \, J\Phi \, \mathrm{d}\mathcal{H}^{N-1}. \tag{5.17}$$

Combining (5.17), (5.9) and (5.11) we obtain

$$\delta P(E_f)[\varphi] = \int_{\partial E} H_{E_f} \varphi \, J\Phi \left(1 + \sum_{j=1}^{N-1} \frac{(\partial_{v_j} f)^2}{(1 + \kappa_j f)^2} \right)^{-1/2} \mathrm{d}\mathcal{H}^{N-1} = \int_{\partial E} H_{E_f} \varphi \prod_{i=1}^{N-1} (1 + \kappa_i f) \, \mathrm{d}\mathcal{H}^{N-1}.$$

In the following, with a slight abuse of notation, with the symbol $O(g)$ we will mean any function

h of the form $h(x) = r(x)g(x)$, where $|r(x)| \leq C$ for all $x \in \partial E$ and C is a constant depending only on N and E.

By a simple Taylor expansion we have

$$\delta P(E_f)[\varphi] = \int_{\partial E} H_{E_f}\varphi \left(1 + H_E f + O(f^2)\right) \mathrm{d}\mathcal{H}^{N-1}. \tag{5.18}$$

From (5.12) and again by Taylor expansion, we obtain

$$\begin{aligned}\delta P(E_f)[\varphi] &= \int_{\partial E} \left(H_E + f \sum_{i=1}^{N-1} \kappa_i \sum_{s\neq i} \kappa_s + O(f^2) + O(|\nabla f|^2) \right) \varphi \, \mathrm{d}\mathcal{H}^{N-1} \\ &\quad + \int_{\partial E} (\nabla f + h) \cdot \nabla\varphi \, \mathrm{d}\mathcal{H}^{N-1} \\ &= \int_{\partial E} \left(H_E + f H_E^2 - |B_E|^2 f + O(f^2) + O(|\nabla f|^2) \right) \varphi \, \mathrm{d}\mathcal{H}^{N-1} \\ &\quad + \int_{\partial E} (\nabla f + h) \cdot \nabla\varphi \, \mathrm{d}\mathcal{H}^{N-1} \end{aligned} \tag{5.19}$$

where $\nabla f, \nabla\varphi$ are respectively the tangent gradient of f, φ on ∂E and h is a vector field satisfying $|h| \leq C(|f| + |\nabla f|^2))|\nabla f|$. Set $R = O(f^2) + O(|\nabla f|^2)$, by comparing (5.18) and (5.19) we infer that

$$\begin{aligned}\int_{\partial E} (\nabla f \cdot \nabla\varphi - |B_E|^2 f\varphi) \, \mathrm{d}\mathcal{H}^{N-1} &= \int_{\partial E} (H_{E_f} - H_E)\,(1 + H_E f + R)\, \varphi \, \mathrm{d}\mathcal{H}^{N-1} \\ &\quad - \int_{\partial E} (h \cdot \nabla\varphi + (O(f^2) + O(|\nabla f|^2))\varphi) \, \mathrm{d}\mathcal{H}^{N-1}. \end{aligned} \tag{5.20}$$

Testing (5.20) with $\varphi = 1$, we get

$$\int_{\partial E} (H_{E_f} - H_E)\,(1 + H_E f + R) \, \mathrm{d}\mathcal{H}^{N-1} = \int_{\partial E} (O(|f|) + O(|\nabla f|^2)) \, \mathrm{d}\mathcal{H}^{N-1},$$

then, for δ sufficiently small, using Hölder inequality we obtain

$$\begin{aligned} \left|\overline{H}_{E_f} - H_E\right| &= \left| -\fint_{\partial E} (H_{E_f} - H_E)(H_E f + R) \, \mathrm{d}\mathcal{H}^{N-1} + \fint_{\partial E} (O(|f|) + O(|\nabla f|^2)) \, \mathrm{d}\mathcal{H}^{N-1} \right| \\ &\leq \left| \fint_{\partial E} (H_{E_f} - \overline{H}_{E_f})(H_E f + R) \, \mathrm{d}\mathcal{H}^{N-1} \right| + \left| \fint_{\partial E} (\overline{H}_{E_f} - H_E)(H_E f + R) \, \mathrm{d}\mathcal{H}^{N-1} \right| \\ &\quad + \int_{\partial E} (O(|f|) + O(|\nabla f|^2)) \, \mathrm{d}\mathcal{H}^{N-1} \\ &\leq \delta \, \frac{|H_E| + C\delta}{P(E)} \|H_{E_f} - \overline{H}_{E_f}\|_{L^2} + \delta \, (|H_E| + C\delta) \, |\overline{H}_{E_f} - H_E| \\ &\quad + \int_{\partial E} (O(|f|) + O(|\nabla f|^2)) \, \mathrm{d}\mathcal{H}^{N-1}, \end{aligned}$$

with $C = C(N, E)$ since $\delta \leq 1$. For δ small enough, recalling that $\|H_{E_f} - \overline{H}_{E_f}\|_{L^2} \leq 1$, the previous inequality implies

$$\frac{1}{2}|\overline{H}_{E_f} - H_E| \leq C\delta \|H_{E_f} - \overline{H}_{E_f}\|_{L^2} + \int_{\partial E} (O(|f|) + O(|\nabla f|^2)) \, \mathrm{d}\mathcal{H}^{N-1} \leq C\delta. \tag{5.21}$$

Using the bound $\|f\|_{C^1} \leq \delta$ and the definition of h we easily see that

$$h \cdot \nabla f = \delta \, O(|\nabla f|^2).$$

Testing (5.20) with $\varphi = f$, using Hölder's inequality and by the previous remark, we get

$$\begin{aligned}
\int_{\partial E} (|\nabla f|^2 - |B_E|^2 f^2)\, \mathrm{d}\mathcal{H}^{N-1} &= \int_{\partial E} \left(H_{E_f} - H_E\right)(1 + H_E f + R) f \, \mathrm{d}\mathcal{H}^{N-1} \\
&\quad + \delta \int_{\partial E} (O(f^2) + O(|\nabla f|^2))\, \mathrm{d}\mathcal{H}^{N-1} \\
&= \int (H_{E_f} - \overline{H}_{E_f})(1 + H_E f + R) f \, \mathrm{d}\mathcal{H}^{N-1} + \int (\overline{H}_{E_f} - H_E)(1 + H_E f + R) f \, \mathrm{d}\mathcal{H}^{N-1} \\
&\quad + \delta \int_{\partial E} (O(f^2) + O(|\nabla f|^2))\, \mathrm{d}\mathcal{H}^{N-1} \\
&\le C \|H_{E_f} - \overline{H}_{E_f}\|_{L^2} \|f\|_{L^2} + |\overline{H}_{E_f} - H_E| \int (1 + H_E f + R) f \, \mathrm{d}\mathcal{H}^{N-1} \\
&\quad + \delta \int_{\partial E} (O(f^2) + O(|\nabla f|^2))\, \mathrm{d}\mathcal{H}^{N-1} \\
&= C \|H_{E_f} - \overline{H}_{E_f}\|_{L^2} \|f\|_{L^2} + |\overline{H}_{E_f} - H_E| \int (f + O(f^2) + f O(|\nabla f|^2))\, \mathrm{d}\mathcal{H}^{N-1} \\
&\quad + \delta \int_{\partial E} (O(f^2) + O(|\nabla f|^2))\, \mathrm{d}\mathcal{H}^{N-1}. \qquad (5.22)
\end{aligned}$$

By (5.13), (5.21) and by Hölder inequality, we obtain

$$|\overline{H}_{E_f} - H_E| \int (f + O(f^2) + f O(|\nabla f|^2))\, \mathrm{d}\mathcal{H}^{N-1} \le \delta \int_{\partial E} (O(f^2) + O(|\nabla f|^2)).$$

Finally, by the above inequality, (5.13) again and by combining (5.22) with (5.21) we deduce that, for any $\eta > 0$, it holds

$$\begin{aligned}
\int_{\partial E} (|\nabla f|^2 - |B_E|^2 f^2)\, \mathrm{d}\mathcal{H}^{N-1} &\le C \|H_{E_f} - \overline{H}_{E_f}\|_{L^2} \|f\|_{H^1} + \delta \int_{\partial E} (O(f^2) + O(|\nabla f|^2))\, \mathrm{d}\mathcal{H}^{N-1} \\
&\le \frac{1}{\eta} C^2 \|H_{E_f} - \overline{H}_{E_f}\|_{L^2}^2 + \eta \|f\|_{H^1}^2 + C\delta \|f\|_{H^1}^2. \qquad (5.23)
\end{aligned}$$

The conclusion then follows combining (5.23) with Lemma 5.20 and taking δ and η sufficiently small. □

4 Uniform L^1–estimate on the discrete flow

In this section we give the precise definition of the discrete volume preserving flow in the flat torus and we study some of its properties.

4.1 Discrete volume preserving mean-curvature flow

Let $E \neq \emptyset$ be a measurable subset of $\mathbb{T}^N$. In the following we will always assume that E coincides with its Lebesgue representative. Fixed $h > 0$, $m \in (0, 1)$, we consider the minimum problem

$$\min \left\{ P(F) + \frac{1}{h} \int_F \mathrm{sd}_E(x)\, \mathrm{d}x : F \subset \mathbb{T}^N,\, |F| = m \right\}, \qquad (5.24)$$

where $\mathrm{sd}_E(x) := \mathrm{dist}_E(x) - \mathrm{dist}_{\mathbb{T}^N \setminus E}(x)$ is the signed distance from the set E. Observe that the minimum problem (5.24) is equivalent to the problem

$$\min \left\{ P(F) + \frac{1}{h} \int_{F \triangle E} \mathrm{dist}_{\partial E}(x)\, \mathrm{d}x : F \subset \mathbb{T}^N,\, |F| = m \right\}.$$

For every $F \subset \mathbb{T}^N$, we set

$$J_h^E(F) := P(F) + \frac{1}{h}\int_{F\triangle E} \mathrm{dist}_{\partial E}(x)\,\mathrm{d}x =: P(F) + \frac{1}{h}\mathcal{D}(F,E), \tag{5.25}$$

with a little abuse of notation we will sometimes denote by J_h^E also the functional

$$F \mapsto P(F) + \frac{1}{h}\int_F \mathrm{sd}_E(x)\,\mathrm{d}x$$

and, when no ambiguity arises, we will write J_h instead of J_h^E.

By induction we can now define the *discrete-in-time, volume preserving mean curvature flow* $(E_h^n)_{n\in\mathbb{N}}$ and we will refer to it as the *discrete flow.* Let $E_0 \subset \mathbb{T}^N$ be a measurable set such that $|E_0| = m$. We set $E_h^0 = E_0$ and iteratively define for $n \geq 1$

$$E_h^n \in \operatorname{argmin}\left\{P(F) + \frac{1}{h}\int_F \mathrm{sd}_{E_h^{n-1}}(x)\,\mathrm{d}x : F \subset \mathbb{T}^N,\ |F| = m\right\}.$$

Remark 5.22. We start by remarking that the sequence of the perimeters along the discrete flow is non-increasing. Indeed, from the minimality of E_h^n and considering E_h^{n-1} as a competitor we obtain

$$P(E_h^n) \leq P(E_h^n) + \frac{1}{h}\int_{E_h^{n-1}\triangle E_h^n} \mathrm{dist}_{\partial E_h^{n-1}}(x)\,\mathrm{d}x \leq P(E_h^{n-1}).$$

From this simple remark we observe that, even if the starting set of the flow E_0 is not of finite perimeter, the perimeters of the sets E_h^n are uniformly bounded by a constant that only depends on the dimension N, the fixed volume m and h. Given any set E_0 of volume m, consider the cube Q_m of the same volume. From the minimality of E_h^1 and using Q_m as a competitor we obtain

$$\begin{aligned} P(E_h^1) &\leq P(Q_m) + \frac{1}{h}\int_{E_0\triangle Q_m} \mathrm{dist}_{\partial E_0}(x)\,\mathrm{d}x - \frac{1}{h}\int_{E_0\triangle E_1^h} \mathrm{dist}_{\partial E_0}(x)\,\mathrm{d}x \\ &\leq P(Q_m) + \frac{1}{h}\int_{\mathbb{T}^N} \sqrt{N} = C(N,m,h), \end{aligned}$$

where we estimated $\mathrm{dist}_{\partial E_0} \leq \mathrm{diam}(\mathbb{T}^N) = \sqrt{N}$.

We recall some preliminary results that can be found in [154]. If not otherwise stated, their original proofs can be easily adapted to the periodic case, the major difference being that in our case we work in the flat torus, which is compact, thus simplifying some arguments. First of all, we observe that that the problem (5.24) admits solutions via the standard method of the calculus of variations.

The regularity properties of the discrete flow are investigated in the following proposition. Some of the results are classical, others follow from [154, Proposition 2.3].

Proposition 5.23. *Let h, m, $M > 0$ and let $E \subset \mathbb{T}^N$ be a set with $|E| = m$ and $P(E) \leq M$. Then, any solution $F \subset \mathbb{T}^N$ to* (5.24) *satisfies the following regularity properties:*

i) There exist $c_0 = c_0(N) > 0$ and a radius $r_0 = r_0(m,h,N,M) > 0$ such that for every $x \in \partial^* F$ and $r \in (0, r_0]$ we have

$$|B_r(x) \cap F| \geq c_0 r^N \quad \text{and} \quad |B_r(x) \setminus F| \geq c_0 r^N.$$

In particular, F admits an open representative whose topological boundary coincides with the closure of its reduced boundary, i.e. $\partial F = \overline{\partial^* F}$.

ii) There exists $\Lambda = \Lambda(m,h,N,M) > 0$ such that F is a Λ-minimizer of the perimeter, that is

$$P(F) \leq P(F') + \Lambda|F\triangle F'|$$

for all measurable set $F' \subset \mathbb{T}^N$.

iii) The following Euler-Lagrange equation holds: there exists $\lambda \in \mathbb{R}$ *such that for all* $X \in C^1_c(\mathbb{T}^N, \mathbb{T}^N)$ *we have*

$$\int_{\partial^* F} \frac{\mathrm{sd}_E}{h} X \cdot \nu_F \, \mathrm{d}\mathcal{H}^{N-1} + \int_{\partial^* F} \mathrm{div}_\tau X \, \mathrm{d}\mathcal{H}^{N-1} = \lambda \int_{\partial^* F} X \cdot \nu_F \, \mathrm{d}\mathcal{H}^{N-1}. \tag{5.26}$$

iv) There exists a closed set Σ*, whose Hausdorff dimension is less than or equal to* $N-8$*, such that* $\partial^* F = \partial F \setminus \Sigma$ *is an* $(N-1)$*-submanifold of class* $C^{2,\alpha}$ *for all* $\alpha \in (0,1)$ *with*

$$|H_F(x)| \leq \Lambda, \quad \text{for all } x \in \partial F \setminus \Sigma.$$

v) There exists $k_0 = k_0(m, h, N, M) \in \mathbb{N}$ *and* $s_0 = s_0(m, h, N, M) > 0$ *such that* F *is made up of at most* k_0 *connected components having mutual Hausdorff distance at least* s_0*.*

The following result characterizes the stationary sets of the discrete scheme. The last assertion of the proposition is a technical result that will be employed in the proof of Lemma 5.25.

Proposition 5.24. *Every stationary set* E *for the discrete flow is a critical set of the perimeter. Viceversa, if* E *is a regular critical set of the perimeter, then there exists* $h^* = h^*(E) > 0$ *such that, for every* $h < h^*$*, the volume preserving discrete flow starting from* E *is unique and given by* $E^n_h = E$*. Moreover, if* E *is a strictly stable set then it is also the unique volume-constrained minimizer of the functional*

$$\tilde{J}_h(F) := P(F) + \frac{1}{h} \int_F \mathrm{dist}_E(x) \, \mathrm{d}x.$$

Proof. The first statement is an immediate consequence of (5.26). Since E is a stationary point for the discrete flow, it satisfies

$$\int_{\partial^* E} \mathrm{div}_\tau X \, \mathrm{d}\mathcal{H}^{N-1} = \lambda \int_{\partial^* E} X \cdot \nu_E \, \mathrm{d}\mathcal{H}^{N-1}$$

for all $X \in C^1_c(\mathbb{T}^N, \mathbb{T}^N)$, i.e. E is a critical point for the perimeter.

The second part follows using the same argument of the proof of [154, Proposition 3.2]. Indeed, recall that the second variation has the following expression

$$\partial^2 J_h(E)[\varphi] = \int_{\partial E} |\nabla \varphi|^2 + \left(\frac{1}{h} - |B_E|^2\right) \varphi^2 \, \mathrm{d}\mathcal{H}^{N-1},$$

which is positive if h is small enough. Then we procede as in the proof of [154, Proposition 3.2].

Analogously, we prove that E is the unique volume-constrained minimizer of $\tilde{J}_h$. Firstly, observe that, by Theorem 5.17, E is a strict local L^1-minimizer of the perimeter and it is a global minimizer of the second term in $\tilde{J}_h$. Therefore, there exists $\varepsilon > 0$ such that

$$\tilde{J}_h(E) < \tilde{J}_h(F)$$

for all measurable set F such that $|F| = |E|$ and $|E \triangle F| \leq \varepsilon$, i.e. E is an isolated local minimizer for $\tilde{J}_h$ in L^1 with the volume constraint, with minimality neighbourhood uniform with respect to h. Now, given any sequence $(h_n)_{n \in \mathbb{N}}$ going to zero, let F_n be a volume constrained minimizer of J_{h_n}; we then easily deduce that $|E \triangle F_n| \to 0$ as $n \to \infty$, and therefore, for n large enough, $|E \triangle F_n| \leq \varepsilon$. The strict minimality of E therefore implies $F_n = E$.

□

4.2 Uniform L^1 estimate

In this subsection we prove a uniform L^1-estimate on the discrete flow starting from an initial set E_0 sufficiently "close" to a strictly stable set of the perimeter. We will devote the next subsection to a discussion upon the hypotheses of the estimate. Before we recall the definition of

Hausdorff distance and some of its properties, for a complete reference see e.g. [14, Section 4.4], [153, Section 10.1].

Given a set $C \subset \mathbb{T}^N$, we denote by $(C)_\delta$ the δ fattened of C, that is the set

$$\{x \in \mathbb{T}^N : \operatorname{dist}_C(x) \le \delta\}.$$

Let C_1, $C_2 \subset \mathbb{T}^N$ be closed sets, we define the *Hausdorff distance* between C_1 and C_2 as

$$\mathrm{d}_H(C_1, C_2) := \inf\{\rho > 0 : C_1 \subset (C_2)_\rho,\, C_2 \subset (C_1)_\rho\}.$$

Given C_n, C closed sets in $\mathbb{T}^N$, we say that $(C_n)_{n\in\mathbb{N}}$ converges to C in the Hausdorff distance and we write $C_n \stackrel{H}{\to} C$, if $\mathrm{d}_H(C_n, C) \to 0$ as $n \to \infty$. We recall that the space of closed subsets of a compact set equipped with the Hausdorff metric is compact (see e.g [14, Theorem 4.4.15] or [153, Proposition 10.1]) and also that the convergence in the Hausdorff distance is equivalent to the uniform convergence of the respective distance functions, i.e.

$$C_n \stackrel{H}{\to} C \qquad \Longleftrightarrow \qquad \operatorname{dist}_{C_n} \to \operatorname{dist}_C \quad \text{uniformly}.$$

In the following, given two open smooth sets E_1, E_2, we will denote by $\mathrm{d}_H(E_1, E_2)$ the Hausdorff distance between their closures.

Lemma 5.25. *Let $E \subset \mathbb{T}^N$ be a strictly stable set and let $\varepsilon > 0$. Then, there exist $\delta = \delta(\varepsilon, E) > 0$ and $h^* = h^*(E) > 0$ such that, for every $h < h^*$ and for every set E_0 satisfying*

$$|E_0| = |E|, \qquad \mathrm{d}_H(\overline{E}_0, \overline{E}) \le \delta,$$

we have

$$|E \triangle F| \le \varepsilon,$$

where F is a solution of (5.24) with E_0 replacing E.

Proof. Let $h^* = h^*(E)$ be the constant given by Proposition 5.24 so that, for every $h < h^*$, E is the unique volume-constrained global minimizer of the functional

$$\tilde{J}_h(G) := P(G) + \frac{1}{h}\int_G \operatorname{dist}_E(x)\,\mathrm{d}x. \tag{5.27}$$

Fix $h < h^*$ and let $(E_n)_{n\in\mathbb{N}}$ be a sequence of sets satisfying

$$|E_n| = |E|, \qquad \overline{E}_n \stackrel{H}{\to} \overline{E}. \tag{5.28}$$

Consider F_n a solution of (5.24) with E_n replacing E. We claim that

$$F_n \stackrel{L^1}{\to} E.$$

If we prove the claim, the conclusion easily follows.

First, Remark 5.22 ensures that $(F_n)_{n\in\mathbb{N}}$ is a sequence of sets with uniformly bounded perimeters, with the bound depending only on N, m, h. Therefore, there exist F a set of finite perimeter such that $|F| = m$ and a (unrelabelled) subsequence of $(F_n)_{n\in\mathbb{N}}$ such that

$$F_n \stackrel{L^1}{\to} F.$$

Now, let K be a compact subset of $\mathbb{T}^N$ such that, up to a subsequence, we have

$$\overline{E_n^c} \stackrel{H}{\to} K.$$

From the second property in (5.28) we easily deduce that $(\overline{E})^c \subset K$, and therefore $K^c \subset \overline{E}$. In particular, this inclusion implies that

$$\int_{K^c} \mathrm{dist}_K(x)\,\mathrm{d}x = \int_E \mathrm{dist}_K(x)\,\mathrm{d}x \geq \int_G \mathrm{dist}_K(x)\,\mathrm{d}x$$

for every $G \subset \mathbb{T}^N$. Setting

$$\bar{J}_h(G) := P(G) + \frac{1}{h}\int_G (\mathrm{dist}_E(x) - \mathrm{dist}_K(x))\,\mathrm{d}x,$$

from the previous remark and from the fact that E is the unique minimizer of (5.27), we have

$$\begin{aligned}\bar{J}_h(G) &= \tilde{J}_h(G) - \frac{1}{h}\int_G \mathrm{dist}_K(x)\,\mathrm{d}x \\ &> \tilde{J}_h(E) - \frac{1}{h}\int_G \mathrm{dist}_K(x)\,\mathrm{d}x \\ &\geq \tilde{J}_h(E) - \frac{1}{h}\int_E \mathrm{dist}_K(x)\,\mathrm{d}x = \bar{J}_h(E),\end{aligned}$$

for any measurable set $G \subset \mathbb{T}^N$ with $|G| = |E|$. Finally, we obtain

$$\begin{aligned}\bar{J}_h(F) &= P(F) + \frac{1}{h}\int_F (\mathrm{dist}_E(x) - \mathrm{dist}_K(x))\,\mathrm{d}x \\ &\leq \liminf_{n\to\infty} P(F_n) + \frac{1}{h}\int_F (\mathrm{dist}_E(x) - \mathrm{dist}_K(x))\,\mathrm{d}x \\ &= \liminf_{n\to\infty}\left(P(F_n) + \frac{1}{h}\int_{F_n}\big(\mathrm{dist}_{E_n}(x) - \mathrm{dist}_{E_n^c}(x)\big)\,\mathrm{d}x\right) \\ &\leq \liminf_{n\to\infty}\left(P(E) + \frac{1}{h}\int_{E}\big(\mathrm{dist}_{E_n}(x) - \mathrm{dist}_{E_n^c}(x)\big)\,\mathrm{d}x\right) \\ &= P(E) - \frac{1}{h}\int_E \mathrm{dist}_K(x)\,\mathrm{d}x = \bar{J}_h(E)\end{aligned}$$

where we exploited the lower-semicontinuity of the perimeter and the minimality of F_n. Since E is the unique volume-constrained minimizer of $\bar{J}_h$, the set F must coincide with E and this concludes the proof. □

Remark 5.26. We remark that under the hypotheses of Lemma 5.25 we could have just assumed the one-sided inclusion

$$\overline{E}_0 \subset (E)_{\delta^*}$$

instead of

$$\mathrm{d}_H(\overline{E}_0, \overline{E}) \leq \delta$$

for a suitable $\delta^* \leq \delta$. Indeed, let $\delta_n \to 0$ and $E_n \subset (E)_{\delta_n}$ such that $|E_n| = |E|$. We prove that $\overline{E}_n$ converges to $\overline{E}$ in the sense of Kuratowski (and thus with respect to Hausdorff). Let $(x_n)_{n\in\mathbb{N}}$ be a sequence such that $x_n \in \overline{E}_n$ and $x_n \to y$. For every $n \in \mathbb{N}$, there exists $y_n \in E$ such that $|x_n - y_n| \leq \delta_n$. Therefore, for any $\varepsilon > 0$ there exists n_0 such that, for $n \geq n_0$, we have

$$|y_n - y| \leq |y_n - x_n| + |x_n - y| \leq \delta_n + \varepsilon,$$

that is $y_n \to y$. Since $(y_n)_{n\in\mathbb{N}} \subset E$, we have $y \in \overline{E}$.

Fix now $y \in \overline{E}$. Assume by contradiction that there exists $\delta > 0$ such that $\mathrm{dist}_{E_n}(y) > \delta$, i.e. it doesn't exist a sequence of elements in $\overline{E}_n$ converging to y. From this (and up to subsequences) it follows

$$E_n \subset (E)_{\delta_n} \setminus B_\delta(y) \qquad \forall n \in \mathbb{N}.$$

Thus we have

$$\begin{aligned} m = \lim_{n\to\infty} |E_n| &\leq \lim_{n\to\infty} |(E)_{\delta_n} \setminus B_\delta(y)| \\ &\leq \lim_{n\to\infty} |(E)_{\delta_n} \setminus (B_\delta(y)\cap E)\,| \\ &= \lim_{n\to\infty} |(E)_{\delta_n}| - |B_\delta(y)\cap E| = m - |B_\delta(y)\cap E| \end{aligned}$$

which is a contradiction.

We are now able to prove the main estimate that will be used in the proof of Proposition 5.30.

Proposition 5.27 (Uniform L^1-estimate)**.** *Let $E\subset\mathbb{T}^N$ be a strictly stable set. Then, for every $\varepsilon>0$ there exist $\delta^*=\delta^*(\varepsilon,E)>0$ and $h^*=h^*(E)>0$ with the following property: for every $h<h^*$, if E_0 is a measurable set such that*

$$|E_0| = |E|, \qquad \overline{E}_0 \subset (E)_{\delta^*},$$

then the discrete flow $(E_h^n)_{n\in\mathbb{N}}$ starting from E_0 satisfies

$$\alpha(E, E_h^n) \leq \varepsilon$$

for every $n\in\mathbb{N}$.

Proof. Fix $h<h^*$, where $h^*=h^*(E)$ is the constant given by Lemma 5.25 and let $\sigma=\sigma(E)$, $C=C(E)$ be the constants of Theorem 5.17. Moreover, let $\delta:=\delta(\sigma,E)$ be the constant given by Lemma 5.25 with σ replacing ε. Set $\delta^*\leq\delta$ to be chosen later and consider E_0 such that

$$|E_0| = |E|, \qquad \overline{E}_0 \subset (E)_{\delta^*}.$$

Recall that, from Remark 5.26 and from the hypothesis $\overline{E}_0\subset(E)_{\delta^*}$, without loss of generality, we can assume $\mathrm{d}_H(\overline{E}_0,\overline{E})\leq\delta^*$. Moreover, by the regularity of E, we can also suppose $\alpha(E_0,E)\leq\tilde{C}\delta^*$, for a suitable constant $\tilde{C}>0$ that only depends on E. From Lemma 5.25 we have that

$$|E_h^1 \triangle E| \leq \sigma. \tag{5.29}$$

Let x_0 be such that $\alpha(E_0,E)=|E_0\triangle(E+x_0)|$. By choosing $E+x_0$ as a competitor for the minimality of E_h^1 and estimating $\mathrm{dist}_{\partial E_0}\leq\mathrm{diam}(\mathbb{T}^N)=\sqrt{N}$, we find

$$P(E_h^1) - P(E) \leq \frac{1}{h}\int_{E_0\triangle(E+x_0)} \mathrm{dist}_{\partial E_0}(x)\,\mathrm{d}x \leq \frac{\sqrt{N}}{h}\alpha(E_0,E) \leq \frac{\sqrt{N}}{h}\tilde{C}\delta^*.$$

By (5.29), we can apply Theorem 5.17 and the previous estimate to obtain

$$\alpha(E_h^1, E) \leq \frac{1}{\sqrt{C}}\sqrt{P(E_h^1)-P(E)} \leq \frac{1}{\sqrt{C}}\sqrt{\frac{\sqrt{N}}{h}\alpha(E,E_0)} \leq \frac{1}{\sqrt{C}}\sqrt{\frac{\sqrt{N}}{h}\tilde{C}\delta^*} \leq \min\{\sigma,\delta,\varepsilon\},$$

where we have chosen δ^* such that $\delta^*\leq Ch\left(\min\{\sigma,\delta,\varepsilon\}\right)^2/(\tilde{C}\sqrt{N})$. Since E_h^1 is a $\Lambda-$minimizer and E is regular, up to taking δ^* smaller, the classical regularity theory for $\Lambda-$minimizers (see Theorem 5.8) implies

$$\mathrm{d}_H(\partial E_h^1, \partial E + x_1) \leq \delta,$$

where x_1 is such that $\alpha(E_h^1,E)=|E_h^1\triangle(E+x_1)|$.

Now we iterate the procedure: by induction, suppose that

$$\alpha(E_h^{n-1}, E) \leq \min\{\sigma,\delta,\varepsilon\}, \qquad \mathrm{d}_H(\partial E_h^{n-1}, \partial E + x_{n-1}) \leq \delta \tag{5.30}$$

where x_{n-1} is such that $|E_h^{n-1}\triangle(E+x_{n-1})|=\alpha(E_h^{n-1},E)$. Observe that the second inequality in (5.30) implies that $\mathrm{d}_H(\overline{E}_h^{n-1},\overline{E}+x_{n-1})\leq\delta$, therefore E_h^{n-1} and $E+x_{n-1}$ satisfy the hypotheses of Lemma 5.25 and thus

$$|E_h^n \triangle (E + x_{n-1})| \leq \sigma.$$

Observe that by definition $\alpha(E_h^n, E + x_{n-1}) = \alpha(E_h^n, E)$. Now, by Theorem 5.17 and the monotonicity of the perimeters along the discrete flow we obtain

$$\begin{aligned}\alpha(E_h^n, E) &\leq \frac{1}{\sqrt{C}}\sqrt{P(E_h^n) - P(E)} \\ &\leq \frac{1}{\sqrt{C}}\sqrt{P(E_h^1) - P(E)} \\ &\leq \frac{1}{\sqrt{C}}\sqrt{\frac{\sqrt{N}}{h}\tilde{C}\delta^*} \leq \min\{\sigma, \delta, \varepsilon\}.\end{aligned}$$

Again, thanks to the choice of δ^*, the hypotheses of Theorem 5.8 are satisfied and thus

$$\mathrm{d}_H(\partial E_h^n, \partial E + x_n) \leq \delta,$$

where x_n is such that $\alpha(E_h^n, E) = |E_h^n \triangle (E + x_n)|$. This concludes the proof. □

4.3 Some remarks on the hypothesis of the L^1−estimate

In this subsection we show that Proposition 5.27 does not hold if we weaken the hypothesis of closeness in the Hausdorff distance between the starting set E_0 and the strictly stable set E. In particular, we prove that the sole hypothesis of closeness in L^1 and in perimeter is not enough. We remark that a modification of this example yields the same result in $\mathbb{R}^N$.

Fix $h > 0$ and $G \subset \mathbb{T}^N$. Recall that, for any set $F \subset \mathbb{T}^N$ such that $|F| = |G|$, we have set

$$J_h^G(F) := P(F) + \frac{1}{h}\int_{F \triangle G} \mathrm{dist}_{\partial G}(x)\,\mathrm{d}x. \tag{5.31}$$

Proposition 5.28. *There exist $m > 0$ and a sequence $(E_n)_{n\in\mathbb{N}} \subset \mathbb{T}^N$ with the following properties: $|E_n| = m$ for every $n \in \mathbb{N}$, $P(E_n)$ is uniformly bounded and, letting F_n be any volume-constrained minimizer of* (5.31) *with E_n instead of G, we have*

$$E_n \overset{L^1}{\to} E, \quad P(E_n) \to P(E) \quad \textit{but} \quad F_n \overset{L^1}{\to} F,$$

where E is a lamella and F is such that $|E \triangle F| > 0$.

Proof. Let $m > 0$ such that the ball of volume m has perimeter strictly less than the one of the lamella of the same volume; we remark that for every smaller volume $m' \leq m$ the same property holds. Let E be a lamella of measure m, recall that E is a strictly stable set of the perimeter in $\mathbb{T}^N$. From the assumption on m it follows that E is only a local minimizer of the perimeter and not a global one.

Step 1. Firstly, we construct a sequence $(E_n)_{n\in\mathbb{N}}$ such that $E_n \to E$ in L^1 and $\partial E_n \to \mathbb{T}^N$ in the Hausdorff distance. We define E_n by adding to E some balls contained in $\mathbb{T}^N \setminus E$ and of overall small volume, and by subtracting to E balls contained in E with the same overall volume.

Recall that $\mathbb{T}^N = [0,1]^N/\mathbb{Z}^N$. In the following, with a little abuse of notation, we will identify $\mathbb{T}^N$ and $[0,1)^N$. We define

$$\begin{aligned} I_n &:= \left\{\underline{k} = (k_1, \dots, k_N) \in \mathbb{Z}^N : 0 \leq k_i \leq 2^n - 1 \quad \forall i = 1, \dots N\right\}, \\ \mathcal{P}_n &:= \left\{Q_{n,\underline{k}} := \left[0, \frac{1}{2^n}\right)^N + \frac{\underline{k}}{2^n} : \underline{k} \in I_n\right\}, \end{aligned}$$

for every $n \in \mathbb{N}$. Up to choosing m smaller, we can assume that $m = 1/2^s$ for some $s \in \mathbb{N}$. Moreover, we can suppose, up to translations, that $E = [0,1)^{N-1} \times (0, 1/2^s)$, thus for $n \geq s$ we have

$$E = \mathrm{Int}\left(\bigcup_{\underline{k}\in I_n,\ 0\le k_N\le 2^{n-s}-1} Q_{n,\underline{k}}\right),$$

where $\mathrm{Int}(\cdot)$ denotes the interior of a set in $\mathbb{T}^N$. For every $n \ge s$ and $\underline{k} \in I_n$, we consider the balls $B_{n,\underline{k}} \subset Q_{n,\underline{k}}$ centered in the center of the cube $Q_{n,\underline{k}}$ and of radius $r_{n,\underline{k}}$ chosen in such a way that

$$\left|\bigcup_{\underline{k}\in I_n,\ 0\le k_N\le 2^{n-s}-1} B_{n,\underline{k}}\right| = \left|\bigcup_{\underline{k}\in I_n,\ 2^{n-s}\le k_N\le 2^{n}-1} B_{n,\underline{k}}\right|. \tag{5.32}$$

Moreover, we can also take the radii $r_{n,\underline{k}}$ sufficiently small so that

$$\lim_{n\to\infty}\left|\bigcup_{\underline{k}\in I_n} B_{n,\underline{k}}\right| = 0, \qquad \lim_{n\to\infty} P\left(\bigcup_{\underline{k}\in I_n} B_{n,\underline{k}}\right) = 0. \tag{5.33}$$

Set now

$$\begin{aligned} A_n &:= \bigcup_{\underline{k}\in I_n,\ 0\le k_N\le 2^{n-s}-1} B_{n,\underline{k}} \subset \mathrm{Int}\left(\bigcup_{\underline{k}\in I_n,\ 0\le k_N\le 2^{n-s}-1} Q_{n,\underline{k}}\right) = E,\\ C_n &:= \bigcup_{\underline{k}\in I_n,\ 2^{n-s}\le k_N\le 2^{n}-1} B_{n,\underline{k}} \subset \bigcup_{\underline{k}\in I_n,\ 2^{n-s}\le k_N\le 2^{n}-1} Q_{n,\underline{k}} \subset \mathbb{T}^N\setminus E. \end{aligned}$$

Define $E_n = (E \cup C_n) \setminus A_n$ and observe that, by (5.32), we have $|E_n| = |E|$. Now, by (5.33), we also obtain

$$E_n \stackrel{L^1}{\to} E \quad \text{ and } \quad P(E_n) \to P(E).$$

Observe that, from the definition of A_n and C_n, we have that

$$(\partial A_n)_{\sqrt{N}/2^n} \cup (\partial C_n)_{\sqrt{N}/2^n} = \mathbb{T}^N$$

and therefore the set $\partial E_n = \partial E \cup \partial C_n \cup \partial A_n$ converges in the Hausdorff metric to the whole $\mathbb{T}^N$ as $n \to +\infty$. Therefore we have constructed a sequence $(E_n)_{n\in\mathbb{N}}$ that satisfies

$$E_n \stackrel{L^1}{\to} E, \qquad P(E_n) \to P(E), \qquad \partial E_n \stackrel{H}{\to} \mathbb{T}^N. \tag{5.34}$$

Step 2. Let E_n be the sets previously defined. We consider the space $X = \{F \subset \mathbb{T}^N : F \text{ is measurable}\}$ endowed with the L^1−distance, i.e. $\mathrm{dist}_{L^1}(F,G) = |F\triangle G|$ for every $F, G \in X$. We extend our functional in the following way

$$\tilde{J}_h^E(F) := \begin{cases} J_h^E(F) & \text{if } P(F) < \infty,\ |F| = m,\\ +\infty & \text{otherwise} \end{cases}$$

and we set $J_n := \tilde{J}_h^{E_n}$. We then prove the Γ−convergence of the functionals J_n to the perimeter functional in X, that is

$$\Gamma(X) - \lim_{n\to\infty} J_n = P. \tag{5.35}$$

We can clearly restrict ourselves to consider sets of finite perimeter and volume m, otherwise the result is trivial. For any given set F of measure m and finite perimeter we choose the sequence constantly equal to F as a recovery sequence for F. Indeed, by (5.34) we have

$$J_n(F) = P(F) + \frac{1}{h}\int_{F\triangle E_n} \mathrm{dist}_{\partial E_n} \to P(F).$$

We now prove the lim inf inequality. Given a sequence $F_n \to F$ in L^1, by lower semicontinuity

$$P(F) \le \liminf_{n\to\infty} P(F_n) \le \liminf_{n\to\infty} \left(P(F_n) + \frac{1}{h}\int_{F_n \triangle E_n} \mathrm{dist}_{\partial E_n} \right)$$

and thus (5.35) is proved. Therefore, thanks to the equi-coercivity of the functionals J_n, any sequence of volume-constrained global minimizers of J_n converges in L^1, up to a subsequence, to a volume-constrained global minimizer of the perimeter in the torus. Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of global minimizers of the functional J_n and let F be such that $F_n \to F$ in L^1. We know that F is a global minimizer of the perimeter and that by the choice of m the lamella is not a global minimizer. Therefore it must hold $|E\triangle F| > 0$. □

5 Convergence of the flow

In this section, we will prove the main result of the chapter concerning the convergence of the discrete flow that mainly relies on Proposition 5.27.

5.1 Convergence of the flow up to translations

We start by recalling [154, Lemma 3.6]: it will be used in the proof of the following proposition.

Lemma 5.29. *Let $(E_h^n)_{n\in\mathbb{N}}$ be a volume preserving discrete flow starting from E_0 and let $E_h^{k_n}$ be a subsequence such that $E_h^{k_n} + \tau_n \to F$ in $L^1(\mathbb{T}^N)$ for some set F and a suitable sequence $(\tau_n)_{n\in\mathbb{N}} \subset \mathbb{T}^N$. Then $\mathrm{dist}_{\partial E_h^{k_n - 1}}(\cdot + \tau_n) \to \mathrm{dist}_{\partial F}$ uniformly.*

In the following proposition we characterize the long-time behaviour up to translations of the discrete mean curvature flow in the flat torus starting near a regular strictly stable set.

Proposition 5.30. *Let $E \subset \mathbb{T}^N$ be a strictly stable set. Then there exist $\delta^* = \delta^*(E) > 0$ and $h^* = h^*(E) > 0$ with the following property: if $h < h^*$ and $E_0 \subset \mathbb{T}^N$ is a set of finite perimeter satisfying*

$$|E_0| = |E|, \qquad \overline{E}_0 \subset (E)_{\delta^*},$$

then, for every discrete flow $(E_h^n)_{n\in\mathbb{N}}$ starting from E_0, there exists a sequence of translations $\tau_n \in \mathbb{T}^N$ such that

$$E_h^n + \tau_n \to E \quad in \quad C^k, \quad \forall k \in \mathbb{N}.$$

Proof. Let $\varepsilon > 0$ be sufficiently small and let $\delta^* = \delta^*(\varepsilon, E)$, $h^* = h^*(E)$ be the constants given by Proposition 5.27. Fix E_0 an initial set satisfying $|E| = |E_0|$ and $\overline{E}_0 \subset (E)_{\delta^*}$. It is enough to show that any (unrelabelled) subsequence of the discrete flow starting from E_0 admits a further subsequence converging in C^k and up to translations to E. We divide the proof into three steps.
Step 1. (Existence and regularity of a limit point) From Proposition 5.23 we remark that, for $n \ge 1$, the sets E_h^n are uniform $\Lambda-$minimizers with uniformly bounded, non-increasing perimeters. Therefore, by the compactness of (uniform) $\Lambda-$minimizers, we can conclude that there exists a subsequence $(E_h^{k_n})_{n\in\mathbb{N}}$ and a $\Lambda-$minimizer E_h^∞ such that

$$E_h^{k_n} \xrightarrow{L^1} E_h^\infty, \quad P(E_h^{k_n}) \to P(E_h^\infty), \quad \mathrm{sd}_{E_h^{k_n - 1}} \to \mathrm{sd}_{E_h^\infty} \text{ uniformly.}$$

Let G be a set of finite perimeter such that $|G| = m$. By the minimality of $E_h^{k_n}$ we have

$$P(E_h^{k_n}) + \frac{1}{h}\int_{E_h^{k_n}} \mathrm{sd}_{E_h^{k_n-1}}(x)\,\mathrm{d}x \le P(G) + \frac{1}{h}\int_G \mathrm{sd}_{E_h^{k_n-1}}(x)\,\mathrm{d}x$$

and, taking the limit as $n \to \infty$, we obtain

$$P(E_h^\infty) + \frac{1}{h}\int_{E_h^\infty} \mathrm{sd}_{E_h^\infty}(x)\,\mathrm{d}x \le P(G) + \frac{1}{h}\int_G \mathrm{sd}_{E_h^\infty}(x)\,\mathrm{d}x.$$

We have thus proved that E_h^∞ is a fixed point for the discrete flow and thus, by Proposition 5.24, it is a critical point for the perimeter.

Let $\tau_\infty \in \operatorname{argmin}_x |(E_h^\infty + x)\triangle E|$. By Proposition 5.27 we have $\alpha(E, E_h^{k_n}) \le \varepsilon$ for every $n \in \mathbb{N}$. Now, up to taking ε smaller, Theorem 5.8 and the smoothness of E, yields both the $C^{1,\beta}$-closeness between $E_h^\infty + \tau_\infty$ and E, and the $C^{1,\beta}$ regularity of $E_h^\infty + \tau_\infty$ (and thus of E_h^∞), for every $\beta \in (0,1)$. From Proposition 5.23 *(iv)* it follows that E_h^∞ is of class $C^{2,\beta}$, therefore we conclude that E_h^∞ has constant classical mean curvature and thus it is of class C^∞. To conclude, the smoothness of E_h^∞ allow us to use Theorem 5.8 to improve the convergence of the subsequence to

$$E_h^{k_n} \to E_h^\infty \quad \text{in} \quad C^{1,\beta} \tag{5.36}$$

and to ensure that the sets $E_h^{k_n}$ are of class $C^{1,\beta}$ for n large enough.

Step 2. (Convergence in $C^{2,\beta}$ of the flow and $C^{2,\beta}-$closeness to E) In this step we we will prove that E_h^∞ is $C^{2,\beta}-$close to E and that the convergence of $E_h^{k_n}$ to E_h^∞ is in $C^{2,\beta}$. Without loss of generality, we assume that $\alpha(E, E_h^\infty) = |E \triangle E_h^\infty|$ so that the translation introduced by the previous step does not appear.

First of all we remark that, owing to the compactness of ∂E_h^∞, it suffices to show that the result holds locally. By a compactness argument and the definition of convergence of sets in $C^{1,\beta}$ (Definition 5.7), up to rotations and relabelling the coordinates, we can find a cylinder $C = B' \times (-L, L)$, where $B' \subset \mathbb{R}^{N-1}$ is a ball centred at the origin, and functions $f_\infty, f_n \in C^{1,\beta}(B'; (-L, L))$ describing locally $\partial E_h^\infty \cap C$ and $\partial E_h^{k_n} \cap C$ respectively. We remark that the convergence (5.36) now reads as

$$f_{k_n} \to f_\infty \quad \text{in} \quad C^{1,\beta}(B'). \tag{5.37}$$

We now prove that the curvatures $H_{E_h^{k_n}}$ of the sequence $E_h^{k_n}$ are converging in $C^{0,\beta}$ to the curvature of E_h^∞ in the following sense

$$H_{E_h^{k_n}}(\cdot, f_{k_n}(\cdot)) \to H_{E_h^\infty}(\cdot, f_\infty(\cdot)) \quad \text{in} \quad C^{0,\beta}(B'). \tag{5.38}$$

We will follow an argument used in Step 3 of the proof of [4, Theorem 4.3].

Since we described $\partial E_h^{k_n} \cap C$ as a graph, the following formula for the curvature of $\partial E_h^{k_n}$ holds

$$\operatorname{div}\left(\frac{\nabla f_{k_n}(\cdot)}{\sqrt{1 + |\nabla f_{k_n}(\cdot)|^2}}\right) = H_{E_h^{k_n}}(\cdot, f_{k_n}(\cdot)) \quad \text{on} \quad B' \tag{5.39}$$

and an analogous formula holds for ∂E_h^∞. From (5.39) and the Euler-Lagrange equation (5.26), by integrating on B', we then obtain

$$\begin{aligned} \lambda_{k_n}\mathcal{H}^{N-1}(B') - \frac{1}{h}\int_{B'} sd_{E_h^{k_n-1}}(x', f_{k_n}(x'))\,\mathrm{d}\mathcal{H}^{N-1}(x') \\ = \int_{B'} H_{E_h^{k_n}}(x', f_{k_n}(x'))\,\mathrm{d}\mathcal{H}^{N-1}(x') \\ = \int_{B'} \operatorname{div}\left(\frac{\nabla f_{k_n}(x')}{\sqrt{1+|\nabla f_{k_n}(x')|^2}}\right)\mathrm{d}\mathcal{H}^{N-1}(x') \\ = \int_{\partial B'} \frac{\nabla f_{k_n}(y)}{\sqrt{1+|\nabla f_{k_n}(y)|^2}} \cdot y \,\mathrm{d}\mathcal{H}^{N-2}(y), \end{aligned} \tag{5.40}$$

where we set $y = x'/|x'|$ and integrated by parts in the last line. We can then exploit the convergence (5.37) and the formula (5.39) for the curvature of E_h^∞ to prove

$$\begin{aligned}\int_{\partial B'} \frac{\nabla f_{k_n}(y)}{\sqrt{1+|\nabla f_{k_n}(y)|^2}} \cdot y \,\mathrm{d}\mathcal{H}^{N-2}(y) \to \int_{\partial B'} \frac{\nabla f_\infty(y)}{\sqrt{1+|\nabla f_\infty|^2(y)}} \cdot y \,\mathrm{d}\mathcal{H}^{N-2}(y)\\
= \int_{B'} \operatorname{div}\left(\frac{\nabla f_\infty(x')}{\sqrt{1+|\nabla f_\infty(x')|^2}}\right) \mathrm{d}\mathcal{H}^{N-1}(x')\\
= H_{E_h^\infty}\,\mathcal{H}^{N-1}(B').\end{aligned}$$

Now, Lemma 5.29 ensures that $sd_{E_h^{k_n-1}} \to sd_{E_h^\infty}$ uniformly and we can use the convergence (5.37) to obtain

$$sd_{E_h^{k_n-1}}((\cdot, f_{k_n}(\cdot))) \to sd_{E_h^\infty}((\cdot, f_\infty(\cdot))) = 0 \quad \text{uniformly on } B',$$

since $\partial E_h^\infty \cap C = \{(x', f_\infty(x')) : x' \in B'\}$ by definition. Therefore we find

$$\int_{B'} sd_{E_h^{k_n-1}}((x', f_{k_n}(x')))\,\mathrm{d}\mathcal{H}^{N-1}(x') \to \int_{B'} sd_{E_h^\infty}((x', f_\infty(x')))\,\mathrm{d}\mathcal{H}^{N-1}(x') = 0.$$

We then conclude that (5.40) converges to $H_{E_h^\infty}\mathcal{H}^{N-1}(B')$ and thus it must hold

$$\lambda_{k_n} \to H_{E_h^\infty}.$$

From (5.26), the previous result and the fact that the signed distance functions are all equi-lipschitz, we conclude that for any $\beta \in (0,1)$, the sequence $(H_{E_h^{k_n}}(\cdot, f_{k_n}(\cdot)))$ is bounded in $C^{0,\beta}(B')$ and thus it converges uniformly to $H_{E_h^\infty}(\cdot, f_\infty(\cdot))$. This proves the convergence (5.38).

We remark that the previous result also hold if we describe the sets of the flow $E_h^{k_n}$ as normal deformations of E_h^∞, that is there exist functions $\varphi_{k_n} : \partial E_h^\infty \to \mathbb{R}$ such that $E_h^{k_n} = (E_h^\infty)_{\varphi_{k_n}}$. In this case the convergence (5.36) reads as

$$\varphi_{k_n} \to 0 \quad \text{in} \quad C^{1,\beta}(\partial E_h^\infty),$$

and this and Lemma 5.29 ensure that

$$sd_{E_h^{k_n-1}}(\cdot + \varphi_{k_n}(\cdot)\nu_{E_h^\infty}(\cdot)) \to sd_{E_h^\infty}(\cdot) = 0 \quad \text{uniformly on } \partial E_h^\infty.$$

Now, the convergence of the curvatures reads as

$$H_{E_h^{k_n}}(\cdot + \varphi_{k_n}(\cdot)\nu_{E_h^\infty}(\cdot)) \to H_{E_h^\infty}(\cdot) \quad \text{in} \quad C^{0,\beta}(\partial E_h^\infty).$$

We can then apply directly [4, Lemma 7.2] to obtain that the subsequence $E_h^{k_n}$ is converging to E_h^∞ in $C^{2,\beta}$.

To prove the $C^{2,\beta}-$closeness of the limit point we argue by contradiction. Assume that a sequence of limit points $(E_h^{\infty,l})_{l\in\mathbb{N}}$ is converging in $C^{1,\beta}$ to E but there exists $\sigma > 0$ such that

$$\operatorname{dist}_{C^{2,\beta}}(E, E_h^{\infty,l}) > \sigma$$

for every l large enough. Again, we describe locally $\partial E_h^{\infty,l}$ and ∂E as graphs of suitable functions $f_{\infty,l}, f : B' \to (-L, L)$ and we can repeat the same argument previously employed to prove that

$$H_{E_h^{\infty,l}}((\cdot, f_{\infty,l}(\cdot))) \to H_E((\cdot, f(\cdot))) \qquad \text{in } C^{0,\beta}(B').$$

This time the argument is simpler, since the limit points are stationary sets for the perimeter and thus their Euler-Lagrange equation is

$$H_{E_h^{\infty,l}} = \lambda_{E_h^{\infty,l}} \in \mathbb{R} \quad \text{on} \quad \partial E_h^{\infty,l}.$$

Again, Lemma 7.2 in [4] yields the desired contradiction.
Step 3. (Uniqueness up to translations and C^k convergence) By the previous step we can find a suitable function $\varphi_\infty \in C^{2,\beta}(\partial E)$ such that $E_h^\infty = E_{\varphi_\infty}$. Up to introducing a further translation given by Lemma 5.14, the hypotheses of Theorem 5.4 are satisfied and thus

$$\|\varphi_\infty\|_{H^1(\partial E)} \leq C\|H_{E_h^\infty} - \overline{H}_{E_h^\infty}\|_{L^2(\partial E)} = 0,$$

since the set E_h^∞ is a stationary set for the perimeter. Therefore E_h^∞ is a translated of the set E.

A standard bootstrap method based on the elliptic regularity theory combined with the Euler-Lagrange equation (5.26) yields the convergence in C^k for every $k \in \mathbb{N}$. □

5.2 Exponential convergence of the whole flow

In this subsection we will prove that the translations introduced in Proposition 5.30 decay to zero exponentially fast. In order to prove this result we will estimate the decay of the dissipations via a dissipation-dissipation inequality, which in turn relies on the quantitative Alexandrov type estimate established in Theorem 5.4. We start by recalling some preliminary results from [154].

Lemma 5.31. *Let $\eta > 0$ and let $E \subset \mathbb{T}^N$ be a strictly stable set. There exists $\delta > 0$ with the following property: if $f_1, f_2 \in C^1(\partial E)$ with $\|f_i\|_{C^1(\partial E)} \leq \delta$ and $|E_{f_i}| = |E|$ for $i = 1, 2$ we have*

$$C_1(1-\eta)\|f_1 - f_2\|_{L^2}^2 \leq \mathcal{D}(E_{f_1}, E_{f_2}) \leq C_1(1+\eta)\|f_1 - f_2\|_{L^2}^2 \tag{5.41}$$

$$\frac{1-\eta}{2}\int_{\partial E_{f_1}} \mathrm{sd}_{E_{f_2}}^2 \,\mathrm{d}\mathcal{H}^{N-1} \leq \mathcal{D}(E_{f_1}, E_{f_2}) \leq \frac{1+\eta}{2}\int_{\partial E_{f_1}} \mathrm{sd}_{E_{f_2}}^2 \,\mathrm{d}\mathcal{H}^{N-1} \tag{5.42}$$

$$|\mathrm{bar}(E_{f_1}) - \mathrm{bar}(E_{f_2})|^2 \leq C_2\|f_1 - f_2\|_{L^2}^2 \leq \frac{C_2}{C_1(1-\eta)}\mathcal{D}(E_{f_1}, E_{f_2}) \tag{5.43}$$

for suitable constants C_1, $C_2 > 0$.

The following lemma proves the crucial dissipation-dissipation inequality (5.45) (see [154, Lemma 3.9]). This result will play a central role in the proof of Theorem 5.1. Its proof is based on the Alexandrov-type estimate contained in Theorem 5.4.

Lemma 5.32. *Let $h > 0$ and let $E \subset \mathbb{T}^N$ be a strictly stable set. There exist constants C, $\delta > 0$ with the following property: for any pair of normal deformations E_{f_1}, E_{f_2} with $f_i \in C^2(\partial E)$, $\|f_i\|_{C^1(\partial E)} \leq \delta$, and such that $|E_{f_2}| = |E|$, $|\int_{\partial E} \nu_E f_2 \,\mathrm{d}\mathcal{H}^{N-1}| \leq \delta\|f_2\|_{L^2(\partial E)}$ and*

$$H_{E_{f_2}} + \frac{\mathrm{sd}_{E_{f_1}}}{h} = \lambda \quad on \quad \partial E_{f_2} \tag{5.44}$$

for some $\lambda \in \mathbb{R}$, we have

$$\mathcal{D}(E, E_{f_2}) \leq C\mathcal{D}(E_{f_2}, E_{f_1}). \tag{5.45}$$

Proof. By Theorem 5.4, for δ sufficiently small, we get

$$\begin{aligned}\|f_2\|_{L^2(\partial E)}^2 &\leq C\|H_{E_{f_2}} - \overline{H}_{E_{f_2}}\|_{L^2(\partial E)}^2 \leq C\|H_{E_{f_2}} - \lambda\|_{L^2(\partial E)}^2 \\ &\leq 2C\|H_{E_{f_2}} - \lambda\|_{L^2(\partial E_{f_2})}^2 = \frac{2C}{h^2}\int_{\partial E_{f_2}} \mathrm{sd}_{E_{f_1}}^2 \,\mathrm{d}\mathcal{H}^{N-1},\end{aligned}$$

where the third inequality follows by bounding the Jacobian of the change of variables by 2 (see (5.9)). By combining the previous inequalities with (5.41) and (5.42), we obtain the thesis. □

We are now able to prove the main result of the chapter, Theorem 5.1.

Proof of Theorem 5.1. Let $h^* > 0$, $\delta^* > 0$ and $(\tau_n)_{n\in\mathbb{N}}$ be given by Proposition 5.30. Fix $h < h^*$ and set $E_n := E_h^n$. We split the proof in three steps.
Step 1. (Exponential convergence of dissipations) Testing the minimality of E_n with E_{n-1} we obtain

$$P(E_n) + \frac{1}{h}\mathcal{D}(E_n, E_{n-1}) \leq P(E_{n-1}).$$

Recalling that $P(E_n) \to P(E)$ and summing the previous inequality from $n+1$ to $+\infty$ we get

$$\sum_{k=n+1}^{+\infty} \frac{1}{h}\mathcal{D}(E_k, E_{k-1}) \leq P(E_n) - P(E). \tag{5.46}$$

We will now construct a suitable competitor to estimate the dissipation at the step $n-1$ with the difference of perimeters. Since, by Proposition 5.30, we have

$$E_n + \tau_n \to E \quad \text{in} \quad C^k \quad \forall k \in \mathbb{N}, \tag{5.47}$$

the translated sets of the flow, for n large enough, can be written as normal deformations of the set E, that is there exists $g_n : \partial E \to \mathbb{R}$ such that

$$E_n + \tau_n = E_{g_n},$$

where E_{g_n} was defined in (5.4). The convergence (5.47) then reads as $g_n \to 0$ in C^k as $n \to \infty$. Let σ_n be the translations introduced by Lemma 5.14 with $E_n + \tau_n$ instead of F. From the convergence in C^k of $E_n + \tau_n$ to E, we deduce that $\sigma_n \to 0$ as $n \to \infty$. Therefore, setting

$$F_n := E_n + \tau_n + \sigma_n,$$

we have that $F_n \to E$ in C^k and $F_n = E_{f_n}$ with $f_n : \partial E \to \mathbb{R}$ satisfying

$$\left|\int_{\partial E} f_n \nu_E \,\mathrm{d}\mathcal{H}^{N-1}\right| \leq \delta\|f_n\|_{L^2(\partial E)} \quad \text{and} \quad \|f_n\|_{W^{2,p}(\partial E)} \leq C\|g_n\|_{W^{2,p}(\partial E)}$$

for $p > N-1$. Consider now the competitor

$$\mathscr{E}_n := E - \tau_{n-1} - \sigma_{n-1}.$$

From the minimality of E_n we easily deduce

$$P(E_n) + \frac{1}{h}\mathcal{D}(E_n, E_{n-1}) \leq P(\mathscr{E}_n) + \frac{1}{h}\mathcal{D}(\mathscr{E}_n, E_{n-1}) = P(E) + \frac{1}{h}\mathcal{D}(E, E_{n-1} + \tau_{n-1} + \sigma_{n-1}) \tag{5.48}$$

where we used the translational invariance of the dissipations. From Lemma 5.29 we obtain that the sequence $E_{n-2} + \tau_{n-1} + \sigma_{n-1}$ converges in C^k to the same limit of $E_{n-1} + \tau_{n-1} + \sigma_{n-1}$, that is to E. In particular, for n large enough we can write $E_{n-2} + \tau_{n-1} + \sigma_{n-1} = E_\psi$ for a suitable function $\psi : \partial E \to \mathbb{R}$ (depending on n) and with $\|\psi\|_{C^1(\partial E)}$ small. From Lemma 5.32 we can then estimate the right hand side of (5.48) with

$$\begin{aligned}\mathcal{D}(E, E_{n-1} + \tau_{n-1} + \sigma_{n-1}) =& \mathcal{D}(E, F_{n-1}) = \mathcal{D}(E, E_{f_{n-1}}) \leq C\mathcal{D}(E_{f_{n-1}}, E_\psi)\\ =& C\mathcal{D}(E_{n-1} + \tau_{n-1} + \sigma_{n-1}, E_{n-2} + \tau_{n-1} + \sigma_{n-1})\\ =& C\mathcal{D}(E_{n-1}, E_{n-2}).\end{aligned}$$

From the previous inequality and (5.48) we obtain

$$P(E_n) - P(E) = P(E_n) - P(\mathscr{E}_n) \leq \frac{C}{h}\mathcal{D}(E_{n-1}, E_{n-2}). \tag{5.49}$$

Now, (5.46) and (5.49) yield

$$\begin{aligned}\sum_{k=n-1}^{\infty}\frac{1}{h}\mathcal{D}(E_k,E_{k-1}) &= \sum_{k=n+1}^{\infty}\frac{1}{h}\mathcal{D}(E_k,E_{k-1})+\frac{1}{h}\mathcal{D}(E_n,E_{n-1})+\frac{1}{h}\mathcal{D}(E_{n-1},E_{n-2})\\ &\le \frac{C+1}{h}\mathcal{D}(E_{n-1},E_{n-2})+\frac{1}{h}\mathcal{D}(E_n,E_{n-1})\\ &\le \frac{C+1}{h}\left(\mathcal{D}(E_{n-1},E_{n-2})+\mathcal{D}(E_n,E_{n-1})\right).\end{aligned}$$

We then apply Lemma 5.33 below (with $l=2$) to conclude

$$\mathcal{D}(E_n,E_{n-1}) \le \left(1-\frac{1}{C+1}\right)^{n/2}(P(E_0)-P(E)). \tag{5.50}$$

Step 2. (Exponential convergence of barycenters) Set

$$b=\left(1-\frac{1}{C+1}\right)^{\frac{1}{4}} \in (0,1). \tag{5.51}$$

From (5.47) and Lemma 5.29 both the sequences $(E_n+\tau_n)_{n\in\mathbb{N}}$ and $(E_{n-1}+\tau_n)_{n\in\mathbb{N}}$ converge in C^k to E. Therefore, for n large enough, there exist some functions $f_{1,n},\ f_{2,n} \in C^k(\partial E)$ such that

$$E_n+\tau_n = E_{f_{1,n}}, \quad E_{n-1}+\tau_n = E_{f_{2,n}}$$

and $\|f_{i,n}\|_{C^k(\partial E)} \to 0$ as $n\to\infty$ for $i=1,2$. From (5.43) and (5.50) we can estimate for n sufficiently large

$$\begin{aligned}|\mathrm{bar}(E_n)-\mathrm{bar}(E_{n-1})| &= |\mathrm{bar}(E_n+\tau_n)-\mathrm{bar}(E_{n-1}+\tau_n)|\\ &= |\mathrm{bar}(E_{f_{1,n}})-\mathrm{bar}(E_{f_{2,n}})|\\ &\le C\sqrt{\mathcal{D}(E_{f_{1,n}},E_{f_{2,n}})} = \sqrt{\mathcal{D}(E_n,E_{n-1})}\\ &\le C\,(P(E_0)-P(E))^{1/2}\,b^n.\end{aligned}$$

In turn, the above estimate implies that $(\mathrm{bar}(E_n))_{n\in\mathbb{N}}$ satisfies the Cauchy condition, thus the whole sequence admits a limit $\bar{\xi}\in\mathbb{T}^N$. Moreover, the convergence is exponentially fast in the sense that

$$|\mathrm{bar}(E_{f_{1,n}})-\bar{\xi}| \le \sum_{k=n+1}^{\infty}|\mathrm{bar}(E_{f_{1,n}})-\mathrm{bar}(E_{f_{2,n}})| \le C\,(P(E_0)-P(E))^{1/2}\,\frac{b^n}{1-b}$$

for n large enough. Recalling (5.47) we thus conclude that there exists a suitable translation $\xi\in\mathbb{T}^N$ such that for every $k\in\mathbb{N}$

$$E_n \to E-\xi \quad \text{in } C^k \quad \text{as } n\to\infty.$$

Step 3. (Exponential convergence of the sets) By the previous step we can write, for n large, the boundaries of the evolving sets as radial graphs of the limit set $E-\xi$. Precisely, for n large enough there exist functions f_n such that

$$E_n+\xi = E_{f_n} \quad \text{and} \quad \|f_n\|_{C^k(\partial E)} \to 0 \quad \text{as } n\to\infty. \tag{5.52}$$

From (5.41) and for n large enough we have $\|f_n-f_{n-1}\|_{L^2(\partial E)} \le 2\sqrt{\mathcal{D}(E_n,E_{n-1})}$ and thus, recalling (5.50) and arguing as in Step 2, we get

$$\|f_n\|_{L^2(\partial E)} \le \sum_{k=n+1}^{\infty}\|f_n-f_{n-1}\|_{L^2(\partial E)} \le (P(E_0)-P(E))^{1/2}\,\frac{b^n}{1-b} \tag{5.53}$$

where b is as in (5.51). The above estimate yields the exponential decay of the L^2-norms of the radial graphs. We recall the well-known Gagliardo-Nieremberg inequality: for every $j \in \mathbb{N}$ there exists $C > 0$ such that, if g is smooth enough on the boundary of a smooth set E, then

$$\|D^k g\|_{L^2(\partial E)} \leq C \|D^{2k} g\|^{1/2}_{L^2(\partial E)} \|g\|^{1/2}_{L^2(\partial E)} \tag{5.54}$$

where D^k stands for the collection of all the $k-$th order derivatives of g, see e.g. [17, Theorem 3.70]. Now, by (5.52) for every k there exists n_k such that $\sup_{n \geq n_k} \|D^{2k} f_n\|_{L^2(\partial E)} \leq 1$, therefore we may apply (5.54) to f_n to deduce from (5.53) that also $\|D^k f_n\|_{L^2(\partial E)}$ decays exponentially fast for all $k \in \mathbb{N}$. The Sobolev immersion Theorem then yields the exponential decay in C^k for every k thus completing the proof of the result. □

Lemma 5.33. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of non-negative numbers. Assume furthermore that there exist $c > 1$, $l \in \mathbb{N}$ such that $\sum_{n=k}^{\infty} a_n \leq c \sum_{j=k}^{k+l-1} a_j$ for every $k \in \mathbb{N}$. Then*

$$a_k \leq \left(1 + \frac{1}{c}\right)^{\frac{k}{l}} S$$

for every $k \in \mathbb{N}$, where $S = \sum_{n=1}^{\infty} a_n$.

The proof of the previous lemma can be found in [154, Lemma 3.11].

6 Two-dimensional case

In this section, we completely characterize the long-time behaviour of the discrete flow in dimension two. This particular choice for the dimension is purely technical and can be justified as follows. In the two-dimensional flat torus we have a complete characterization of the critical points of the perimeter: they consist in unions of disjoint discs (having the same area) or in unions of disjoint lamellae (possibly having different areas), or their complements. It turns out that these sets are all strictly stable. This allows us to conclude that either the connected components of any limit point of the discrete flow or the ones of their complements are strictly stable sets. We remark that in higher dimension this could not be true anymore.

Fix h, $m > 0$ and let $(E_h^n)_{n\in\mathbb{N}}$ be a flow with initial set $E_0 \subset \mathbb{T}^2$ such that $|E_0| = m$. We recall that, by Proposition 5.23, there exists $s_0 > 0$ such that the distance between the connected components of the set E_h^n is at least s_0. Moreover, the proposition also provides a bound from below on the diameter of the connected components. Set

$$P_\infty := \lim_n P(E_h^n)$$

as the limit of the monotone sequence of the perimeters along the discrete flow. Let F be any possible limit point of the sequence $(E_n^h)_{n\in\mathbb{N}}$. We observe that if F is a union of discs then its number of connected components must be $\pi^{-1} P_\infty^2/(4m)$ and therefore the form of the limit point is uniquely determinated up to translations. Analogously, if F is the complement of a union of discs, F^c is made of $\pi^{-1} P_\infty^2/(4 - 4m)$ connected components and thus it is uniquely determinated up to translations of its complement. In the case when F is a union of lamellae the number of connected components is, in general, less than or equal to $P_\infty/2$, and we have no information on the area of the single components.

Since we will consider h as a fixed parameter, from now on we will denote by E_n the set E_h^n.

Remark 5.34 (Remarks on the uniform $C^{1,\alpha}-$closeness to limit points)**.** We remark that for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that for every $n \geq n_0$ it holds

$$|E_n \triangle \bigcup_{i=1}^{l_n} F_{i,n}| \leq \varepsilon \quad \text{or} \quad |E_n^c \triangle \bigcup_{i=1}^{L_n} F_{i,n}| \leq \varepsilon, \tag{5.55}$$

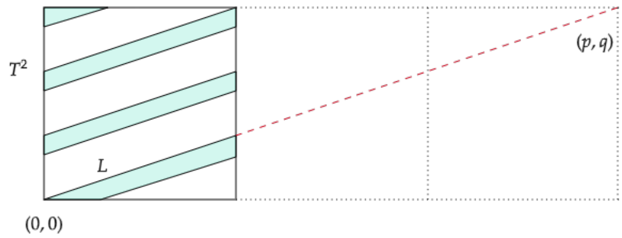

Figure 5.2: The lamella L in light blue, the line a dashed in red.

where, in the first case, $\bigcup_{i=1}^{l_n} F_{i,n}$ is a union of disjoint lamellae or a union of disjoint discs, with $F_{i,n}$ having the same mass of the $i-$th connected component of E_n; l_n is either less than or equal to $P_\infty/2$ if $F_{i,n}$, $i = 1,\dots,l_n$, are lamellae or $l_n = \pi^{-1}P_\infty^2/(4m)$ if they are discs; in the second case, $\bigcup_{i=1}^{L_n} F_{i,n}$ is a union of disjoint discs, with $F_{i,n}$ having the same mass of the $i-$th connected component of E_n^c and $L_n = \pi^{-1}P_\infty^2/(4-4m)$. This can be easily proved recalling that any subsequence of the flow admits a further subsequence converging in L^1 to a set of the aforementioned form.

Moreover, the classical regularity theory of $\Lambda-$minimizers implies that the previous result can be improved. Consider, for the sake of simplicity, that E_n satisfies the first inequality in (5.55)(the other case is analogous). Then one can prove that for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that for every $n \geq n_0$ it holds

$$E_n = \bigcup_{i=1}^{l_n} (F_{i,n})_{f_{i,n}} \quad \text{where} \quad f_{i,n} \in C^{1,\alpha}(\partial F_{i,n}),\ \|f_{i,n}\|_{C^{1,\alpha}(\partial F_{i,n})} \leq \varepsilon. \tag{5.56}$$

Remark 5.35. In this remark, we identify $\mathbb{T}^2$ with the unit square $[0,1)^2$. We prove that for a fixed $M > 0$ there exists a finite number of slopes such that, for any lamella L having one of those slopes, we have $P(L) \leq M$.

Fix a lamella L. Let $a \subset \mathbb{T}^2$ be one of the two components of the boundary of L, and suppose that $(0,0) \in a$. Since a is a closed curve in $\mathbb{T}^2$, by periodicity, the line in $\mathbb{R}^2$ passing through the origin and with the same slope of a must also pass through a point of the form $(p,q) \in \mathbb{N}\times\mathbb{N}$ with p, q coprime or equal to $(0,1)$ or $(1,0)$. We then remark that the length in $\mathbb{T}^2$ of a is equal to the one of the segment between the origin and (p,q), that is $\text{length}(a) = |(p,q)|$. Since $P(L) = 2\,\text{length}(a)$, in order to have $P(L) \leq M$, the point (p,q) must be contained in the disc of radius $M/2$. Our claim follows since in the disc of radius $M/2$ there is a finite number of points belonging to $\mathbb{N}\times\mathbb{N}$.

In the following lemma we characterize the geometric form of any limit point of the discrete flow.

Lemma 5.36 (Uniqueness of the form of the limit)**.** *Fix h, $m > 0$ and an initial set $E_0 \subset \mathbb{T}^2$ with mass m. Let $(E_n)_{n\in\mathbb{N}}$ be a discrete flow starting from E_0. Then either one of the following holds:*

i) the limit points of the flow are disjoint unions of l discs of total area m, where $l = \pi^{-1}(4m)^{-1}P_\infty^2$ belongs to $\mathbb{N}$,

ii) the limit points of the flow are the complement of disjoint unions of l discs of total area $1-m$, where $l = \pi^{-1}(4-4m)^{-1}P_\infty^2$ belongs to $\mathbb{N}$.

iii) the limit points of the flow are disjoint unions of l lamellae of total area m, with the same slope and $l \leq P_\infty/2$. Moreover, the equality $l = P_\infty/2 \in \mathbb{N}$ holds if and only if the limit is given by vertical or horizontal lamellae.

Proof. We first employ a compactness argument and then use Lemma 5.29 to conclude. We start by fixing some notation. We denote by

$$\mathscr{E}_B := \bigcup_{i=1}^{l_B} B_i \tag{5.57}$$

any disjoint union of $l_B = 4^{-1}\pi m^{-1} P_\infty^2$ discs each having radius $2m/P_\infty$; we denote by

$$\mathscr{E}_{B^c} := \left(\bigcup_{i=1}^{l_{B^c}} B_i \right)^c \tag{5.58}$$

the complement of any disjoint union of $l_{B^c} = 4^{-1}\pi(1-m)^{-1}P_\infty^2$ discs, each of radius $2(1-m)/P_\infty$; we denote by

$$\mathscr{E}_L := \bigcup_{i=1}^{l_L} L_i \tag{5.59}$$

any disjoint union of $l_L \leq P_\infty/2$ lamellae having the same slope (and possibly having different masses). We remark that, for every fixed P_∞ and m, the following holds

$$i := \inf\{\, \mathrm{d}_H(\mathscr{E}_B, \mathscr{E}_L) \wedge \mathrm{d}_H(\mathscr{E}_{B^c}, \mathscr{E}_L) \wedge \mathrm{d}_H(\mathscr{E}_{B^c}, \mathscr{E}_B) \ : \ \mathscr{E}_L, \mathscr{E}_B, \mathscr{E}_{B^c} \text{ as above}\} > 0, \tag{5.60}$$

This is clear if we compare the families $\mathscr{E}_B, \mathscr{E}_{B^c}$ and a union of lamellae having the same slope. Since, by Remark 5.35, there is a finite number of possible slopes for the lamellae, we conclude (5.60). From Remark 5.34 the discrete flow is eventually C^1-close to a limit point of the form $\mathscr{E}_L, \mathscr{E}_B$ or $\mathscr{E}_{B^c}$. Assume now by contradiction that the flow does not converge to a fixed configuration. Then, without loss of generality, we can assume that for every $0 < \varepsilon < i/3$ there exist infinitely many indexes such that

$$\mathrm{d}_H(E_{n-1}, \mathscr{E}_B) \leq \varepsilon \quad \text{and} \quad \mathrm{d}_H(E_n, \mathscr{E}_L) \leq \varepsilon.$$

Therefore we get

$$\mathrm{d}_H(\mathscr{E}_B, \mathscr{E}_L) \leq \mathrm{d}_H(\mathscr{E}_B, E_{n-1}) + \mathrm{d}_H(\mathscr{E}_L, E_n) + \mathrm{d}_H(E_n, E_{n-1}) \leq 2\varepsilon + \mathrm{d}_H(E_n, E_{n-1}).$$

To reach the contradiction (compare (5.60)), it is enough to show that for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that for every $n \geq n_0$ it holds

$$\mathrm{d}_H(E_{n-1}, E_n) \leq \varepsilon. \tag{5.61}$$

Assume by contradiction the existence of a subsequence n_k along which the flow satisfies

$$\mathrm{d}_H(E_{n_k-1}, E_{n_k}) > \varepsilon.$$

Up to a further subsequence, $E_{n_k} \to F$, with F being a set of the form $\mathscr{E}_B, \mathscr{E}_L$ or $\mathscr{E}_{B^c}$. But then Lemma 5.29 implies $\mathrm{sd}_{E_{n_k-1}} \to \mathrm{sd}_F$ uniformly, which is clearly a contradiction.

Finally, we observe that in case *iii)* the number of connected component is given by $\frac{P_\infty}{2|(p,q)|}$, where we used the same notation of Remark 5.35. Thus, $l = P_\infty/2$ if and only if (p, q) is equal to $(0, 1)$ or to $(1, 0)$ that means that the lamella is either vertical or horizontal. □

Thanks to the previous lemma we can then conclude the proof of Theorem 5.2, the main result of this section. While the proofs of assertions *i)* and *ii)* of Theorem 5.2 are similar to the one of [154, Theorem 3.4], the third one is slightly different, the main issue being that we can not fix the mass of the connected components of the limiting configuration. We will prove nonetheless the exponential convergence of the dissipations that, in turn, yields the convergence of the mass of the connected components of the flow. We start by a simple remark.

Remark 5.37 ($C^{1,\alpha}$-closeness to lamellae)**.** Let $\varepsilon > 0$. Consider two lamellae L_1, L_2 having the same slope, possibly having different area and two $C^{1,\alpha}-$deformations E_1, E_2, respectively, of L_1

and L_2. Suppose also that

$$\mathrm{dist}_{C^{1,\alpha}}(E_i, L_i) \leq \varepsilon, \quad i = 1, 2.$$

Then the closeness in L^∞ of E_1 and E_2 implies that E_2 and L_1 are close in $C^{1,\alpha}$. Indeed, we first remark that

$$\mathrm{dist}_{C^{1,\alpha}}(L_2, L_1) = \mathrm{dist}_{L^\infty}(L_2, L_1)$$

since the components of the boundaries of L_1 and L_2 differ only by a translation. Moreover, the hypothesis $\mathrm{dist}_{L^\infty}(E_1, E_2) \leq \varepsilon$ implies $\mathrm{dist}_{L^\infty}(L_2, L_1) \leq 2\varepsilon$. Now, let f_2 be a suitable function such that $E_2 = (L_2)_{f_2}$, then $\|f_2\|_{C^{1,\alpha}(\partial L_2)} \leq \varepsilon$ and there exists a constant $|c| \leq \mathrm{dist}_{L^\infty}(L_1, L_2) \leq 2\varepsilon$ such that $E_2 = (L_1)_{f_2+c}$. Therefore we obtain

$$\mathrm{dist}_{C^{1,\alpha}}(E_2, L_1) = \|f_2 + c\|_{C^{1,\alpha}(\partial L_1)} \leq \|f_2\|_{C^{1,\alpha}(\partial L_2)} + |c| \leq \varepsilon + 2\varepsilon = 3\varepsilon.$$

Proof of Theorem 5.2. By Lemma 5.36, we can assume that all the limit points of the flow are sets either of the form $\mathscr{E}_B$, $\mathscr{E}_{B^c}$ or $\mathscr{E}_L$ (see (5.57), (5.58), (5.59)). To conclude we need to prove that the whole sequence converges in C^k and exponentially fast to a unique configuration.

In the case when the limit points are of the form $\mathscr{E}_B$, the proof follows the same spirit of [154, Theorem 3.4], but it is easier since we work in a compact space. The case when the limit points are of the form $\mathscr{E}_{B^c}$ is at all analogous: we simply remark that, if F is a minimizer of 5.25, then its complement is a minimizer of the same problem with E^c instead of E and with $1 - m$ instead of m. By studying the evolution of the complement of the discrete flow, we can conclude as before.

Now, suppose that the limit points are of the form $\mathscr{E}_L$. We begin by observing that any subsequence of the flow admits a further subsequence converging in L^1 to a union of disjoint lamellae. Firstly, we prove the exponential decay of the dissipations. Testing the minimality of E_n with E_{n-1} we obtain

$$P(E_n) + \frac{1}{h}\mathcal{D}(E_n, E_{n-1}) \leq P(E_{n-1}).$$

Summing for $s \geq n+1$ we have

$$\sum_{s=n+1}^{+\infty} \frac{1}{h}\mathcal{D}(E_s, E_{s-1}) \leq P(E_n) - P_\infty. \tag{5.62}$$

With the notation previously introduced, for every ε we can choose n large enough such that (5.56) holds. Let $F_{i,n}$ be the sets given by (5.56): by Lemma 5.36, we know that $F_{i,n}$, $i = 1, \ldots, l_n$, are eventually lamellae and $l_n = l \geq P_\infty/2$.

We will now construct a suitable competitor to estimate the dissipation at the step $n-1$ with the difference of the perimeters. For n large enough consider the competitor

$$\mathscr{L}_n = \bigcup_{i=1}^{l} F_{i,n-1}.$$

We remark that, by definition and for n large enough, this competitor has perimeter $P(\mathscr{L}_n) = P_\infty$. By Proposition 5.23, there exists $s_0 = s_0(m, h, N, E_0) > 0$ such that the connected components $E_{i,n}$ of E_n satisfy

$$\mathrm{dist}\,(E_{i,n}, E_{j,n}) \geq s_0$$

for every $i \neq j$, moreover Remark 5.34 ensures that

$$\mathrm{dist}\,(F_{i,n-1}, F_{j,n-1}) \geq \frac{s_0}{2}$$

holds for n large enough and $i \neq j$. Thus, we can localize the dissipations

$$\mathcal{D}(E_n, E_{n-1}) = \sum_{i=1}^{l} \mathcal{D}(E_{i,n}, E_{i,n-1}), \tag{5.63}$$
$$\mathcal{D}(\mathscr{L}_n, E_{n-1}) = \sum_{i=1}^{l} \mathcal{D}(F_{i,n-1}, E_{i,n-1}).$$

Testing the minimality of E_n with $\mathscr{L}_n$ and using the previous equality we have

$$P(E_n) + \frac{1}{h}\mathcal{D}(E_n, E_{n-1}) \leq P(\mathscr{L}_n) + \frac{1}{h}\sum_{i=1}^{l} \mathcal{D}(F_{i,n-1}, E_{i,n-1}). \tag{5.64}$$

Recalling Remark 5.37 and equations (5.56) and (5.61), we then obtain that the connected components of both E_{n-1} and E_{n-2} are small normal $C^{1,\alpha}-$deformations of the connected components of $\mathscr{L}_{n-1}$. Thus we can assume that both $E_{i,n-1}$ and $E_{i,n-2}$ can be described as normal deformation of $F_{i,n-1}$ for $i = 1, \ldots, k$. Let $f_{i,n-1}$ and $f_{i,n-2}$ be the functions (having small $C^{1,\alpha}-$norms) that describe respectively these deformations. Now, recalling Lemma 5.32, we can estimate

$$\begin{aligned}\mathcal{D}(F_{i,n-1}, E_{i,n-1}) =& \mathcal{D}(F_{i,n-1}, (F_{i,n-1})_{f_{i,n-1}}) \leq C\mathcal{D}((F_{i,n-1})_{f_{i,n-1}}, (F_{i,n-1})_{f_{i,n-2}}) \\ =& C\mathcal{D}(E_{i,n-1}, E_{i,n-2}).\end{aligned}$$

Thus, from equations (5.63) and (5.64) we get

$$P(E_n) - P_\infty = P(E_n) - P(\mathscr{L}_n) \leq \frac{C}{h}\sum_{i=1}^{l} \mathcal{D}(E_{i,n-1}, E_{i,n-2}) = \frac{C}{h}\mathcal{D}(E_{n-1}, E_{n-2})$$

and then (5.62) clearly yields

$$\begin{aligned}\sum_{s=n-1}^{\infty} \frac{1}{h}\mathcal{D}(E_s, E_{s-1}) =& \sum_{s=n+1}^{\infty} \frac{1}{h}\mathcal{D}(E_s, E_{s-1}) + \frac{1}{h}\mathcal{D}(E_{n-1}, E_{n-2}) + \frac{1}{h}\mathcal{D}(E_n, E_{n-1}) \\ &\leq P(E_n) - P_\infty + \frac{1}{h}\mathcal{D}(E_{n-1}, E_{n-2}) + \frac{1}{h}\mathcal{D}(E_n, E_{n-1}) \\ &\leq \frac{C+1}{h}\mathcal{D}(E_{n-1}, E_{n-2}) + \frac{1}{h}\mathcal{D}(E_n, E_{n-1}) \\ &\leq \left(\frac{C+1}{h}\mathcal{D}(E_{n-1}, E_{n-2}) + \frac{1}{h}\mathcal{D}(E_n, E_{n-1})\right).\end{aligned}$$

We can then conclude using the same arguments of [154, Theorem 3.4]. □

Chapter 6

Asymptotic of the Discrete Volume-Preserving Fractional Mean Curvature Flow via a Nonlocal Quantitative Alexandrov Theorem

Contents

1 Introduction

In this chapter we consider the geometric evolution of sets called *the volume preserving fractional mean curvature flow.* It is the fractional counterpart of the classical *volume preserving mean curvature flow*, which is defined as the flow of sets $(E_t)_{0\le t\le T}$ in $\mathbb{R}^N$ following the motion law

$$v_t = \bar{H}_{E_t} - H_{E_t} \quad \text{on} \quad \partial E_t$$

for all $t \in [0,T]$, where $\bar{H}_{E_t}$ denotes the average of H_{E_t} over ∂E_t. In the fractional setting, the velocity of the flow is related to the *fractional mean curvature*, a geometric quantity introduced by Caffarelli, Roquejoffre and Savin in [36] and defined as the first variation of the fractional perimeter functional. The latter functional is defined on a measurable set $E \subset \mathbb{R}^N$ as

$$P^s(E) = \int_E \int_{E^c} \frac{1}{|x-y|^{N+s}} \, \mathrm{d}x \, \mathrm{d}y.$$

One can then compute the fractional curvature of a smooth enough set E as in [36], and find the expression

$$H^s_E(x) = \int_{\mathbb{R}^N} \frac{\chi_E(y) - \chi_{E^c}(y)}{|x-y|^{N+s}} \, \mathrm{d}y, \quad x \in \partial E.$$

In both the previous formulae, the integrals are intended in the principal value sense. In analogy with the classical case, the evolution law for the *volume preserving fractional mean curvature flow* is given by

$$v_t = \bar{H}^s_{E_t} - H^s_{E_t} \quad \text{on} \quad \partial E_t, \tag{6.1}$$

with the notations previously introduced.

Up to now, a satisfactory study of this type of evolution is still missing. While the evolution without the volume constraint is well-understood (see e.g. [56, 127]), the lack of a comparison principle in our case makes the study much harder. Moreover, the generated flow may present singularities of different kinds, as happens for the classical mean curvature flow: see [62] for some explicit examples of pinch-like singularities. In [131] short-time existence is proved for the smooth flow (6.1), while existence of the smooth flow starting from convex sets (under suitable assumptions) is provided in [61]. I this chapter we focus on a discrete-in-time approximation of the flow, obtained via the minimizing movements scheme in the spirit of [8, 144]. First of all we define the *discrete flow.* Given any initial set E_0, with $|E_0| = m$, and a time-step $h > 0$ we define $E^{(h)}_0 := E_0$ and, iteratively, for $n \ge 0$ we set

$$E^{(h)}_{n+1} \in \operatorname{argmin} \left\{ P^s(F) + \frac{1}{h} \int_F \mathrm{sd}_{E^{(h)}_n}(x) \, \mathrm{d}x + \frac{1}{h^{\frac{s}{1+s}}} ||F| - m| : F \subset \mathbb{R}^N \text{ measurable} \right\},$$

where $\mathrm{sd}_{E^{(h)}_n}$ is the signed distance function from the set $E^{(h)}_n$. This variational problem is the fractional counterpart of the one studied in [155]. We define for every $t \ge 0$, the discrete flow by $E^{(h)}(t) := E^{(h)}_{[t/h]}$, and we will prove that such a flow is well-defined. Any L^1_{loc}-limit point of this flow as the time-step h converges to zero will be called a *flat flow.* For the classical mean curvature flow, under the hypothesis of convergence of the perimeters, this approach produces global-in-time distributional solutions of the evolution law (6.1), as shown in [155]. In the fractional case, we fall short of this result, since we lack some regularity results needed to characterize the evolution law of a flat flow. Moreover, from a technical point of view, proving the uniform boundness of the discrete flow in the fractional setting is nontrivial.

In the recent years, the study of the long time behaviour of the volume preserving mean curvature flow has attracted more and more attention. In the local case, after some classical studies [82, 122], in a recent paper [154] the authors proved the asymptotic behaviour of the classical discrete flow by showing its convergence to unions of equal balls. Then, they improved their results in [133], proving uniform estimates with respects to the time parameter h in dimension $N = 2$, thus obtaining the same result for the flat flow. In the fractional setting some recent results have been proved. For example, in [61] the authors prove that the smooth flow starting from a convex set converges to a ball, up to translations possibly depending on time and under the hypothesis of equiboundedness for the fractional curvatures along the flow.

In this chapter the long-time convergence analysis for the discrete flow is developed in the fractional setting. The main result of the chapter is Theorem 6.1. It provides a complete characterization of the long-time behaviour of the discrete fractional mean curvature flow starting from any bounded set of finite fractional perimeter, providing also an estimate on the convergence speed. We will assume that the dimension N is such that any $\Lambda-$minimizer of the fractional perimeter is a smooth set. Namely, we will assume that either:

- $N = 2$;
- $N \leq 7$ and $s \in (s_0, 1)$, where s_0 is the constant of Proposition 6.13, item *ii)*.

This is a technical hypothesis that could be dropped if, for example, we knew that the evolving sets were smooth. In particular, it is essential to characterize the possible long-time limit points for the discrete flow. In the local case such characterization has been proved in [73].

Theorem 6.1. *Let m, $M > 0$ and let E_0 be an initial bounded set with $P^s(E_0) \leq M$, $|E_0| = m$. Then, for $h = h(s, M, m) > 0$ small enough the following holds: for any discrete flow $E_n^{(h)}$ starting from E_0, there exists $\xi \in \mathbb{R}^N$ such that*

$$E_n^{(h)} - \xi \to B^{(m)} \quad \text{as } n \to \infty \text{ in } C^k$$

for all $k \in \mathbb{N}$, where $B^{(m)}$ denotes the ball centered at the origin with volume equal to m. Moreover, the convergence is exponentially fast, meaning that there exist functions $f_n \in C^\infty(B^{(m)})$ such that $E_n^{(h)} - \xi = B_{f_n}^{(m)}$ and $\|f_n\|_{C^k(\partial B^{(m)})} \leq c_k e^{-c_k n}$, for some constants c_k depending on k, m and M.

We stress the difference between this result and the one holding in the classic setting, where the limit points of the discrete flow are in general unions of disjointed balls having the same radius and not necessarily only a single ball. This is a peculiar feature of the nonlocal perimeter considered, that penalizes non-connected components.

A crucial intermediate result consists in generalizing the Alexandrov-type estimate [154, Theorem 1.3] and Theorem 5.4 in Chapter 5 (see also [138]) to the fractional setting. This result provides a stability inequality for normal deformations of balls which can be seen as a sharp Łojasiewicz-Simon inequality.

Theorem 6.2. *There exist $\delta = \delta(N) > 0$ with the following property: for any $f \in C^2(\partial B)$ such that $\|f\|_{C^1(\partial B)} \leq \delta$, $|B_f| = \omega_N$ and $\mathrm{bar}(B_f) = \fint_{B_f} x \,\mathrm{d}x = 0$, and for any $s \in (0, 1)$, there exists $C = C(N, s) > 0$ such that*

$$\|f\|_{H^{\frac{1+s}{2}}(\partial B)} \leq C \|H_{B_f}^s - \bar{H}_{B_f}^s\|_{L^2(\partial B)},$$

where we have set $\bar{H}_{B_f}^s := \fint_{\partial B} H_{B_f}^s(x + f(x)x) \,\mathrm{d}\mathcal{H}^{N-1}(x)$. Furthermore, there exists $s^* > 1$ such that for every $s \in (s^*, 1)$ it holds

$$(1 - s)\|f\|_{H^{\frac{1+s}{2}}(\partial B)}^2 \leq C(1 - s)^2 \|H_{B_f}^s - \bar{H}_{B_f}^s\|_{L^2(\partial B)}^2, \tag{6.2}$$

for a dimensional constant C.

The proof of the previous theorem follows closely the proof of the quantitative Alexandrov type estimate contained in Chapter 5. In particular, the approach is based on some Taylor approximations of the factor $\bar{H}_{B_f}^s - H_{B_f}^s(x)$ combined with the coercivity of the second variation of the fractional perimeter, proved in [88]. The additional regularity assumption $f \in C^2$ is technical and needed to properly define $H_{B_f}^s$.

2 Preliminaries

We work in the Euclidian space $\mathbb{R}^N$, with $N \geq 2$. We denote with $|\cdot|$ the standard Lebesgue measure in $\mathbb{R}^N$, $\mathrm{M}(\mathbb{R}^N)$ is the family of measurable set of $\mathbb{R}^N$. We denote with E^c the complement of a set $E \subset \mathbb{R}^N$. We denote by $\mathcal{H}^{N-1}$ the Hausdorff measure, and sometimes we denote $\mathrm{d}\mathcal{H}^{N-1}_x := \mathrm{d}\mathcal{H}^{N-1}(x)$. If E is a set with C^1 boundary the outer normal to E at a point x in ∂E is denoted by $\nu = \nu_E(x)$. We denote the ball of radius r and center x both as $B(x,r)$ and $B_r(x)$, and we set $B = B(0,1)$. Also, with $B^{(m)}$ we denote the ball centered at zero and having volume $|B^{(m)}| = m$. Let f be a real valued function, with $O(f)$ we will denote the family of all function g such that $|g| \le C|f|$. Finally, we denote by $C(*, \cdots, *)$ a constant that depends on $*, \cdots, *$; such a constant may change from line to line.

Let $s \in (0,1)$ we define the s-fractional perimeter as the following function

$$P^s : \mathrm{M}(\mathbb{R}^N) \to [0,+\infty], \quad P^s(E) := \int_E \int_{E^c} \frac{1}{|x-y|^{N+s}} \,\mathrm{d}x \,\mathrm{d}y = \frac{1}{2}[\chi_E]^2_{H^{\frac{s}{2}}}.$$

More in general, for every $E, F \in \mathrm{M}(\mathbb{R}^N)$ we set

$$\mathcal{L}_s(E,F) := \int_E \int_F \frac{1}{|x-y|^{N+s}} \,\mathrm{d}x \,\mathrm{d}y$$

and, for any bounded set Ω, we define the fractional perimeter of E relative to Ω as

$$P^s(E;\Omega) := \mathcal{L}_s(E\cap\Omega, E^c\cap\Omega) + \mathcal{L}_s(E\cap\Omega, E^c\setminus\Omega) + \mathcal{L}_s(E\setminus\Omega, E^c\cap\Omega).$$

Let $E \in \mathrm{M}(\mathbb{R}^N)$ be a set of class C^2. Given a vector field $X \in C^1_c(\mathbb{R}^N;\mathbb{R}^N)$, let

$$\Phi : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N, \quad \Phi(t,x) = x + tX(x).$$

We recall that the first variation of the s-fractional perimeter of E in the direction of X is given by

$$\partial P^s(E)[X] := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P^s(\Phi(t,E)) = \int_{\partial E} H^s_E(x) X(x)\cdot\nu_E(x)\,\mathrm{d}\mathcal{H}^{N-1}_x,$$

where $H^s_E(x)$ is the s-fractional mean curvature of E evaluated at $x \in \partial E$, that is

$$H^s_E(x) := \int_{\mathbb{R}^N} \frac{\chi_E(y) - \chi_{E^c}(y)}{|x-y|^{N+s}}\,\mathrm{d}y,$$

where the integral has to be intended in the principal value sense. Applying the divergence theorem in the above formula, with $\mathrm{div}(-\frac{1}{s}\frac{\xi}{|\xi|^{N+s}}) = \frac{1}{|\xi|^{N+s}}$, the fractional curvature can be written as

$$H^s_E(p) = \frac{1}{s}\int_{\partial E} \frac{(x-p)\cdot\nu_E(x)}{|x-p|^{N+s}}\,\mathrm{d}\mathcal{H}^{N-1}(x) \quad \forall\, p \in \partial E.$$

We recall some useful results concerning sets of finite fractional perimeter. The proofs of the following results can be found, respectively, in [36, Proposition 3.1], [78, Theorem 7.1] and [77, Lemma 2.5].

Proposition 6.3 (Lower semi-continuity). *Let $\{E_n\}_{n\in\mathbb{N}} \subset \mathrm{M}(\mathbb{R}^N)$ such that $\chi_{E_n} \to \chi_E$ in L^1_{loc}, as $n \to +\infty$, for some $E \in \mathrm{M}(\mathbb{R}^N)$. Then, for all $s \in (0,1)$, we have*

$$P^s(E) \le \liminf_{n\to+\infty} P^s(E_n).$$

Theorem 6.4 (Compactness). *If $R > 0$ and $\{E_n\}_{n\in\mathbb{N}} \subset \mathrm{M}(\mathbb{R}^N)$, with*

$$E_n \subset B(0,R) \quad \forall n \in \mathbb{N} \quad \text{and} \quad \sup_{n\in\mathbb{N}} P^s(E_n) < +\infty,$$

then, up to a subsequence, $E_n \to E$ in $L^1(\mathbb{R}^N)$, where $E \subset B(0,R)$ and $P^s(E) < +\infty$.

Theorem 6.5. *(Relative isoperimetric inequality) Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz continuous boundary and let $E \in \mathrm{M}(\mathbb{R}^N)$. Then there exists a constant $C = C(s,N,\Omega) >$*

0 *such that*

$$P^s(E,\Omega) \geq \mathcal{L}_s(E\cap\Omega, E^c\cap\Omega) \geq C \min\left\{|E\cap\Omega|^{\frac{N-s}{N}},\, |E\setminus\Omega|^{\frac{N-s}{N}}\right\}.$$

We recall the following convergence theorems. The first one concerns the convergence of the fractional perimeter to the classical one and its proof can be found in [37, Theorem 1].

Theorem 6.6. *Let E be a bounded set of class $C^{1,\alpha}$ for $\alpha \in (0,1)$. Then,*

$$\lim_{s\to 1^-} (1-s)P^s(E) = \omega_{N-1}P(E).$$

The second one relates to the convergence of the fractional curvatures. It was proved in a more general setting in [1, 37, 43].

Theorem 6.7. *Let E be a bounded set of class C^2. Then,*

$$\lim_{s\to 1^-} (1-s)H_E^s = \omega_{N-1}H_E$$

uniformly on ∂E.

Finally, we recall the pointwise convergence of the fractional Gagliardo seminorms to the Sobolev one. The classical proof is contained in [24, Corollary 2], see also [116, Proposition 3.7] for the same result in a more general setting. Here and in the following with ∇ we denote the tangential gradient on a hypersurface.

Theorem 6.8. *Assume $f \in H^s(\partial B)$. Then*

$$\lim_{s\to 1^-} (1-s)[f]^2_{H^{\frac{1+s}{2}}(\partial B)} = C\|\nabla f\|^2_{L^2(\partial B)},$$

where $C > 0$ is a constant that depends only on N.

3 A fractional quantitative Alexandrov type estimate

In this section, we are going to prove the quantitative Alexandrov inequality Theorem 6.2 in the nonlocal setting of the fractional perimeter. From now on we set

$$[f]^2_{\frac{1+s}{2}} \coloneqq [f]^2_{H^{\frac{1+s}{2}}(\partial B)} = \int_{\partial B}\int_{\partial B} \frac{|f(x)-f(y)|^2}{|x-y|^{N+s}}\, \mathrm{d}\mathcal{H}^{N-1}_x\, \mathrm{d}\mathcal{H}^{N-1}_y.$$

We start by recalling representation formulas for the s-fractional perimeter and its first variation on smooth sets. As usual, given a vector field $X \in C^1_c(\mathbb{R}^N;\mathbb{R}^N)$, we define the first variation of the fractional perimeter of a C^2 set E with respect to X as

$$\partial P^s(E)[X] \coloneqq \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P^s(\Phi(t,E)),$$

where $\Phi : \mathbb{R}^N \times (-1,1) \to \mathbb{R}^N$ is the flow defined by $\Phi(x,t) \coloneqq x + tX$. Analogously, we define the second variation of the fractional perimeter of a C^2 set E with respect to X as

$$\partial^2 P^s(E)[X] \coloneqq \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} P^s(\Phi(t,E)).$$

For a normal deformation B_f of B induced by a function $f \in C^1(\partial B)$, and for every function $\psi \in C^1(\partial B)$, with a slight abuse of notation, we set

$$\partial P^s(B_f)[\psi] \coloneqq \partial P^s(B_f)[X] = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P^s(B_{f+t\psi}),$$

where the field X is defined by

$$X(x) = \psi\left(\frac{x}{|x|}\right)\frac{x}{|x|}, \qquad x \in \mathbb{R}^N \setminus \{0\}.$$

Lemma 6.9. *The following equalities hold true:*

1. *If $f \in C^2(\partial B)$ with $\|f\|_\infty$ sufficiently small, then*

$$\begin{aligned} P^s(B_f) = \frac{P^s(B)}{P(B)} \int_{\partial B} (1+f)^{N-s}\, \mathrm{d}\mathcal{H}^{N-1} + \\ + \frac{1}{2}\int_{\partial B}\int_{\partial B}\int_{1+f(y)}^{1+f(x)}\int_{1+f(y)}^{1+f(x)} F_{|x-y|}(r,\rho)\,\mathrm{d}r\,\mathrm{d}\rho\,\mathrm{d}\mathcal{H}_x^{N-1}\,\mathrm{d}\mathcal{H}_y^{N-1}, \end{aligned} \tag{6.3}$$

where, for every $\theta, r, \rho \in (0,+\infty)$, we have set

$$F_\theta(r,\rho) := \frac{r^{N-1}\rho^{N-1}}{((r-\rho)^2 + r\rho\theta^2)^{\frac{N+s}{2}}}.$$

2. *If $f \in C^2(\partial B)$ with $\|f\|_\infty$ sufficiently small, then, for every $\psi \in C^1(\partial B)$, we have*

$$\begin{aligned} \partial P^s(B_f)[\psi] = (N-s)\frac{P^s(B)}{P(B)}\int_{\partial B}(1+f)^{N-s-1}\psi\,\mathrm{d}\mathcal{H}^{N-1} \\ + \int_{\partial B}\int_{\partial B}\int_{f(y)}^{f(x)} \left(\psi(x)F_{|x-y|}(1+f(x),1+\rho) - \psi(y)F_{|x-y|}(1+f(y),1+\rho)\right). \end{aligned} \tag{6.4}$$

Proof. By explicit computations one can obtain equation (6.3), see for example the calculations in the proof of [88, Theorem 2.1]. To prove (6.4), we take the derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P^s(B_{f+t\psi})$$

in formula (6.3) and, recalling that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\alpha(t)}^{\beta(t)}\int_{\alpha(t)}^{\beta(t)} f(r,\rho)\,\mathrm{d}\rho\,\mathrm{d}r\right] = \int_{\alpha(t)}^{\beta(t)} (f(\beta(t),\rho)\beta'(t) - f(\alpha(t),\rho)\alpha'(t))\,\mathrm{d}\rho \\ + \int_{\alpha(t)}^{\beta(t)} (f(r,\beta(t))\beta'(t) - f(r,\alpha(t))\alpha'(t))\,\mathrm{d}r \end{aligned}$$

for every function α, $\beta : \mathbb{R} \to \mathbb{R}$ of class C^1 and $f \in L^1_{loc}(\mathbb{R}\times\mathbb{R})$, we conclude

$$\begin{aligned} \partial P^s(B_f)[\psi] = \int_{\partial B}\int_{\partial B}\int_{1+f(y)}^{1+f(x)} \left(\psi(x)F_{|x-y|}(1+f(x),\rho) - \psi(y)F_{|x-y|}(1+f(y),\rho)\right)\,\mathrm{d}\rho \\ + (N-s)\frac{P^s(B)}{P(B)}\int_{\partial B}(1+f)^{N-s-1}\psi\,\mathrm{d}\mathcal{H}^{N-1}. \end{aligned}$$

A simple change of coordinates then yields the thesis. □

Lemma 6.10. *If $f \in C^2(\partial B)$ with $\|f\|_{C^1(\partial B)} \le \delta$ sufficiently small, then we have*

$$\partial P^s(B_f)[1] = (N-s)\frac{P^s(B)}{P(B)}\int_{\partial B}\left(1 + (N-s-1)f + O(f^2)\right)\,\mathrm{d}\mathcal{H}^{N-1} + O([f]^2_{\frac{1+s}{2}}), \tag{6.5}$$

$$\begin{aligned}\partial P^s(B_f)[f] =&(N-s)\frac{P^s(B)}{P(B)}\int_{\partial B}\big(1+(N-s-1)f+O(f^2)\big)\,f\,\mathrm{d}\mathcal{H}^{N-1}\\&+\int_{\partial B}\int_{\partial B}\frac{(f(x)-f(y))^2}{|x-y|^{N+s}}+O([f]^2_{\frac{1+s}{2}})\|f\|_{C^1}.\end{aligned}\tag{6.6}$$

Proof. Let $\psi\in C^1(\partial B)$, we remark that, by expanding the first term in (6.4), we obtain

$$\begin{aligned}\partial P^s(B_f)[\psi] =&(N-s)\frac{P^s(B)}{P(B)}\int_{\partial B}(1+(N-s-1)f+O(f^2))\psi\,\mathrm{d}\mathcal{H}^{N-1}\\&+\int_{\partial B}\int_{\partial B}\int_{f(y)}^{f(x)}\big(\psi(x)F_{|x-y|}(1+f(x),1+\rho)-\psi(y)F_{|x-y|}(1+f(y),1+\rho)\big)\,\mathrm{d}\rho.\end{aligned}$$

By symmetry, using a change of variables in the formula above, we get

$$\begin{aligned}\partial P^s(B_f)[\psi] =&(N-s)\frac{P^s(B)}{P(B)}\int_{\partial B}(1+(N-s-1)f+O(f^2))\psi\,\mathrm{d}\mathcal{H}^{N-1}\\&+2\int_{\partial B}\int_{\partial B}\int_{f(y)}^{f(x)}\psi(x)F_{|x-y|}(1+f(x),1+\rho)\,\mathrm{d}\rho\,\mathrm{d}\mathcal{H}^{N-1}_x\,\mathrm{d}\mathcal{H}^{N-1}_y.\end{aligned}\tag{6.7}$$

We remark that, fixed $x,y\in\partial B$ and $x\neq y$, if $\|f\|_{C^1}\leq\delta$ is sufficiently small, and if ρ varies between the values $f(y)$ and $f(x)$, then we have $|f(x)-\rho|\leq\|\nabla f\|_\infty|x-y|\leq\delta|x-y|$. From this observation we can expand the denominator of $F_{|x-y|}(1+f(x),1+\rho)$ and get

$$\begin{aligned}&|(f(x)-\rho)^2+(1+f(x))(1+\rho)|x-y|^2|^{-\frac{N+s}{2}}\\&=\frac{1}{|x-y|^{N+s}}((f(x)-\rho)^2/|x-y|^2+f(x)+\rho+f(x)\rho+1)^{-\frac{N+s}{2}}\\&=\frac{1}{|x-y|^{N+s}}\left(1+O(\|f\|_{C^1})\right).\end{aligned}\tag{6.8}$$

Plugging formula (6.8) into the second addend of (6.7) and by symmetry again, we obtain

$$\begin{aligned}&2\int_{\partial B}\int_{\partial B}\int_{f(y)}^{f(x)}\psi(x)F_{|x-y|}(1+f(x),1+\rho)\,\mathrm{d}\rho\,\mathrm{d}\mathcal{H}^{N-1}_x\,\mathrm{d}\mathcal{H}^{N-1}_y\\&=2\int_{\partial B\times\partial B}\frac{\psi(x)}{N}\frac{(1+f(x))^{N-1}}{|x-y|^{N+s}}\left((1+f(x))^N-(1+f(y))^N\right)(1+O(\|f\|_{C^1}))\,\mathrm{d}\mathcal{H}^{N-1}_x\,\mathrm{d}\mathcal{H}^{N-1}_y\\&=\int\frac{\left(\psi(x)(1+f(x))^{N-1}-\psi(y)(1+f(y))^{N-1}\right)\left((1+f(x))^N-(1+f(y))^N\right)}{N|x-y|^{N+s}}(1+O(\|f\|_{C^1})).\end{aligned}$$

Now, if $\psi=1$ by a simple Taylor expansion we conclude

$$2\int_{\partial B\times\partial B}\int_{f(y)}^{f(x)}F_{|x-y|}(1+f(x),1+\rho)=(N-1)\int_{\partial B\times\partial B}\frac{(f(x)-f(y))^2}{|x-y|^{N+s}}(1+O(\|f\|_{C^1}))=O([f]^2_{\frac{1+s}{2}}),$$

while the choice $\psi=f$ yields

$$2\int_{\partial B\times\partial B}\int_{f(y)}^{f(x)}f(x)F_{|x-y|}(1+f(x),1+\rho)=\int_{\partial B\times\partial B}\frac{(f(x)-f(y))^2}{|x-y|^{N+s}}(1+O(\|f\|_{C^1})).$$

□

In order to prove Theorem 6.2, we need the following lemma, which states the coercivity of the second variation of the fractional perimeter of a ball with respect to normal deformations. Its proof is contained in [88, Theorem 8.1]. We start by defining

$$\lambda_1^s:=s(N-s)\frac{P^s(B)}{P(B)}.\tag{6.9}$$

Lemma 6.11. *There exists $\delta > 0$ small such that, if $f \in C^2(\partial B)$ with $\|f\|_{C^1(\partial B)} \le \delta$, $|B_f| = \omega_N$ and $\mathrm{bar}(B_f) = 0$, then we have*

$$\begin{aligned}\partial^2 P^s(B)[f] &= \int_{\partial B}\int_{\partial B} \frac{(f(x)-f(y))^2}{|x-y|^{N+s}}\,\mathrm{d}\mathcal{H}^{N-1}_x\,\mathrm{d}\mathcal{H}^{N-1}_y - \lambda_1^s \int_{\partial B} |f|^2\,\mathrm{d}\mathcal{H}^{N-1}\\ &\ge \frac{1}{4}\Big([f]^2_{\frac{1+s}{2}} + \lambda_1^s \|f\|^2_{L^2(\partial B)}\Big).\end{aligned}$$

We are now in position to prove Theorem 6.2.

Proof of Theorem 6.2. Without loss of generality, we assume that $\|H^s_{B_f} - \bar{H}^s_{B_f}\|_{L^2} \le 1$. Let $\Phi : \partial B \to \partial B_f \subset \mathbb{R}^N$ be the map defined by $\Phi(x) = (1 + f(x))x$, by direct computations one can prove that

$$J\Phi(x) = (1+f(x))^{N-1}(1+(1+f(x))^{-2}|\nabla f(x)|^{1/2}.$$

For every $\psi \in C^1(\partial B)$, let

$$X : \mathbb{R}^N \to \mathbb{R}^N, \quad X(x) := \frac{x}{|x|}\psi\left(\frac{x}{|x|}\right).$$

Employing the area formula we get

$$\begin{aligned}\partial P^s(B_f)[\psi] &= \int_{\partial B_f} H^s_{B_f}\nu_{B_f}\cdot X\,\mathrm{d}\mathcal{H}^{N-1}\\ &= \int_{\partial B} H^s_{B_f}(p)\nu_{B_f}(p)\cdot x\psi(x)J\Phi(x)\,\mathrm{d}\mathcal{H}^{N-1}_x\\ &= \int_{\partial B} H^s_{B_f}(p)\,\psi(x)\,(1+f(x))^{N-1}\,\mathrm{d}\mathcal{H}^{N-1}_x,\end{aligned}$$

where we have set $p = (1 + f(x))x$ (for more details see [154, Section 1] and Chapter 5). Now, by a simple Taylor expansion we obtain

$$\partial P^s(B_f)[\psi] = \int_{\partial B} H^s_{B_f}(p)\,\psi(x)\,(1+(N-1)f(x)+O(f^2))\,\mathrm{d}\mathcal{H}^{N-1}_x. \tag{6.10}$$

We recall that

$$H^s_B(x) = (N-s)\frac{P^s(B)}{P(B)} \quad \text{for all } x \in \partial B.$$

If $\psi = 1$, by combining formulas (6.10) and (6.5), we infer

$$\int_{\partial B}(H^s_{B_f}(p) - H^s_B)(1+(N-1)f(x)+O(f^2))\,\mathrm{d}\mathcal{H}^{N-1}_x = \int_{\partial B} O(f)\,\mathrm{d}\mathcal{H}^{N-1} + O([f]^2_{\frac{1+s}{2}}) \tag{6.11}$$

and if $\psi = f$, by combining equations (6.10) and (6.6), we get

$$\begin{aligned}&\int_{\partial B}\int_{\partial B} \frac{(f(x)-f(y))^2}{|x-y|^{N+s}}\,\mathrm{d}\mathcal{H}^{N-1}_x\,\mathrm{d}\mathcal{H}^{N-1}_y - s(N-s)\frac{P^s(B)}{P(B)}\int_{\partial B} f^2\,\mathrm{d}\mathcal{H}^{N-1}\\ &= \int_{\partial B}\Big(H^s_{B_f}(p) - H^s_B\Big)(1+(N-1)f(x)+O(f^2))f(x)\,\mathrm{d}\mathcal{H}^{N-1}_x\\ &\quad + O([f]^2_{\frac{1+s}{2}})\|f\|_{C^1}.\end{aligned} \tag{6.12}$$

Using the same arguments of the proof of Theorem 5.4 in Chapter 5 (see also [154, Theorem 1.3]) we can conclude.

For the interested reader we present a sketch of the proof. By (6.11), for δ sufficiently small, using Hölder's inequality we obtain

$$
\begin{aligned}
\left|\bar{H}^s_{B_f} - H^s_B\right| &\leq \left|-\!\!\!\!\!\!\int_{\partial B} (H^s_{B_f} - H^s_B)((N-1)f + O(f^2))\,\mathrm{d}\mathcal{H}^{N-1}\right| \\
&\quad + \int_{\partial B} O(|f|)\,\mathrm{d}\mathcal{H}^{N-1} + O([f]^2_{\frac{1+s}{2}}) \\
&\leq \left|-\!\!\!\!\!\!\int_{\partial B} (H^s_{B_f} - \bar{H}^s_{B_f})((N-1)f + O(f^2))\,\mathrm{d}\mathcal{H}^{N-1}\right| \\
&\quad + \left|-\!\!\!\!\!\!\int_{\partial B} (\bar{H}^s_{B_f} - H^s_B)((N-1)f + O(f^2))\,\mathrm{d}\mathcal{H}^{N-1}\right| \\
&\quad + \int_{\partial B} O(|f|)\,\mathrm{d}\mathcal{H}^{N-1} + O([f]^2_{\frac{1+s}{2}}) \\
&\leq \delta\,\frac{N-1+C\delta}{P(B)^{1/2}}\|H^s_{B_f} - \bar{H}^s_{B_f}\|_{L^2} + \delta\,(N-1+C\delta)\,|\bar{H}^s_{B_f} - H^s_B| \\
&\quad + \int_{\partial B} O(|f|)\,\mathrm{d}\mathcal{H}^{N-1} + O([f]^2_{\frac{1+s}{2}}),
\end{aligned}
$$

with $C = C(N)$. For δ small enough, recalling that $\|H_{B_f} - \bar{H}_{B_f}\|_{L^2} \leq 1$, the previous inequality implies

$$
\frac{1}{2}|\bar{H}^s_{B_f} - H^s_B| \leq C\delta\|H^s_{B_f} - \bar{H}^s_{B_f}\|_{L^2} + \int_{\partial B} O(|f|)\,\mathrm{d}\mathcal{H}^{N-1} + O([f]^2_{\frac{1+s}{2}}) \leq C\delta. \tag{6.13}
$$

By (6.12), using again Hölder's inequality and by the previous remark, we get

$$
\begin{aligned}
&\int_{\partial B}\int_{\partial B} \frac{(f(x)-f(y))^2}{|x-y|^{N+s}}\,\mathrm{d}\mathcal{H}^{N-1}_x\,\mathrm{d}\mathcal{H}^{N-1}_y - s(N-s)\frac{P^s(B)}{P(B)}\int_{\partial B} f^2\,\mathrm{d}\mathcal{H}^{N-1} \\
&= \int_{\partial B}\left(H^s_{B_f}(p) - H^s_B\right)(1 + (N-1)f + O(f^2))f\,\mathrm{d}\mathcal{H}^{N-1} \\
&\quad + O([f]^2_{\frac{1+s}{2}})\|f\|_{C^1} \\
&= \int_{\partial B}(H^s_{B_f}(p) - \bar{H}^s_{B_f})(1 + (N-1)f + O(f^2))f\,\mathrm{d}\mathcal{H}^{N-1} \\
&\quad + \int_{\partial B}(\bar{H}^s_{B_f} - H^s_B)(1 + (N-1)f + O(f^2))f\,\mathrm{d}\mathcal{H}^{N-1} \\
&\quad + O([f]^2_{\frac{1+s}{2}})\|f\|_{C^1} \\
&\leq C\|H^s_{B_f} - \bar{H}^s_{B_f}\|_{L^2}\|f\|_{L^2} + |\bar{H}^s_{B_f} - H^s_B|\int_{\partial B}(1 + (N-1)f + O(f^2))f\,\mathrm{d}\mathcal{H}^{N-1} \\
&\quad + O([f]^2_{\frac{1+s}{2}})\|f\|_{C^1}.
\end{aligned} \tag{6.14}
$$

Since $|B_f| = \omega_N$, we have

$$
\left|\int_{\partial B} f\,\mathrm{d}\mathcal{H}^{N-1}\right| = \int_{\partial B} O(f^2)\,\mathrm{d}\mathcal{H}^{N-1}. \tag{6.15}
$$

By (6.15) and (6.13), we obtain

$$
|\bar{H}^s_{B_f} - H^s_B|\int_{\partial B}(f + O(f^2))\,\mathrm{d}\mathcal{H}^{N-1} \leq \delta\int_{\partial B} O(f^2).
$$

Finally, by the above inequality, (6.15) again and by combining (6.14) with (6.13) we deduce that, for any $\eta > 0$, it holds

$$\begin{aligned}
&\int_{\partial B}\int_{\partial B}\frac{(f(x)-f(y))^2}{|x-y|^{N+s}}\,\mathrm{d}\mathcal{H}^{N-1}_x\,\mathrm{d}\mathcal{H}^{N-1}_y - s(N-s)\frac{P^s(B)}{P(B)}\int_{\partial B} f^2\,\mathrm{d}\mathcal{H}^{N-1}\\
&\quad\le C\|H^s_{B_f}-\bar{H}^s_{B_f}\|_{L^2}\|f\|_{L^2}+C\delta(\|f\|^2_{L^2}+[f]^2_{\frac{1+s}{2}}) \qquad (6.16)\\
&\quad\le \frac{1}{\eta}C^2\|H^s_{B_f}-\bar{H}^s_{B_f}\|^2_{L^2}+\eta\|f\|^2_{L^2}+C\delta(\|f\|^2_{L^2}+[f]^2_{\frac{1+s}{2}}). \qquad (6.17)
\end{aligned}$$

The conclusion then follows combining (6.17) with Lemma 6.11 and taking δ and η sufficiently small. □

Remark 6.12. By slightly changing the last step in the previous proof we can prove the quantitative Alexandrov result in the classical case [154, Theorem 1.1]. First, we remark that (6.16) reads

$$\begin{aligned}
&(1-s)\left(\int_{\partial B}\int_{\partial B}\frac{(f(x)-f(y))^2}{|x-y|^{N+s}}\,\mathrm{d}\mathcal{H}^{N-1}_x\,\mathrm{d}\mathcal{H}^{N-1}_y - s(N-s)\frac{P^s(B)}{P(B)}\int_{\partial B} f^2\,\mathrm{d}\mathcal{H}^{N-1}\right)\\
&\quad\le C\left\|(1-s)\left(H^s_{B_f}-\bar{H}^s_{B_f}\right)\right\|_{L^2}\|f\|_{L^2}+C\delta(1-s)(\|f\|^2_{L^2}+[f]^2_{\frac{1+s}{2}}),
\end{aligned}$$

from which we obtain

$$\begin{aligned}
\frac{1-s}{4}\left(\lambda^s_1\|f\|^2_{L^2}+[f]^2_{\frac{1+s}{2}}\right) \le &\frac{C^2}{\eta}\left\|(1-s)\left(H^s_{B_f}-\bar{H}^s_{B_f}\right)\right\|^2_{L^2}+\eta\|f\|^2_{L^2}\\
&+C\delta(1-s)(\|f\|^2_{L^2}+[f]^2_{\frac{1+s}{2}}).
\end{aligned} \qquad (6.18)$$

By recalling the definition of λ^s_1 (see (6.9)), and by Theorem 6.6 we obtain

$$\lim_{s\to 1}(1-s)\lambda^s_1=(N-1)\omega_{N-1}.$$

Finally, using Theorems 6.7, 6.8, we can take the limit as $s \to 1^-$ in the inequality 6.18 and get

$$\frac{1}{4}\left((N-1)\omega_{N-1}\|f\|^2_{L^2}+C\|\nabla f\|^2_{L^2}\right)\le\frac{C^2}{\eta}\left\|\omega_{N-1}\left(H_{B_f}-\bar{H}_{B_f}\right)\right\|^2_{L^2}+\eta\|f\|^2_{L^2}+C\delta\|\nabla f\|^2_{L^2},$$

where $C = C(N)$ and we also used that, by uniform convergence, $(1-s)\bar{H}^s_{B_f} \to \omega_{N-1}\bar{H}_{B_f}$. We then conclude by taking η and δ sufficiently small. Finally, the hypothesis $f \in C^2(\partial B)$ can be weakened to $f \in C^1(\partial B)\cap H^2(\partial B)$ by approximation.

4 Asymptotic Behaviour

We start this section by introducing the incremental minimum problem which defines the discrete-in-time approximation of the volume preserving fractional mean curvature flow.

Let $E \neq \emptyset$ be a bounded, measurable subset of $\mathbb{R}^N$. In the following we will always assume that E coincides with its Lebesgue representative. Fixed $h > 0$, $m > 0$, we consider the minimum problem

$$\min\left\{P^s(F)+\frac{1}{h}\int_F \mathrm{sd}_E(x)\,\mathrm{d}x+\frac{1}{h^{\frac{s}{s+1}}}||F|-m| : F\subset\mathbb{R}^N\right\}, \qquad (6.19)$$

where $\mathrm{sd}_E(x) := \mathrm{dist}_E(x) - \mathrm{dist}_{E^c}(x)$ is the signed distance from the set E. Observe that the minimum problem (6.19) is equivalent to the problem

$$\min\left\{P^s(F)+\frac{1}{h}\int_{F\triangle E}\mathrm{dist}_{\partial E}(x)\,\mathrm{d}x+\frac{1}{h^{\frac{s}{s+1}}}||F|-m| : F\subset\mathbb{R}^N\right\}.$$

We set $\mathcal{F}_h(\cdot,\, E)$ as the functional

$$\mathcal{F}_h(F,\, E) = P^s(F) + \frac{1}{h}\int_F \mathrm{sd}_E(x)\,\mathrm{d}x + \frac{1}{h^{\frac{s}{s+1}}}||F| - m|.$$

Let $E,\, F$ be measurable sets, we define

$$\mathcal{D}(E,\, F) := \int_{E\Delta F} \mathrm{dist}_{\partial E}(x)\,\mathrm{d}x.$$

The following proposition recalls some properties of minimizers of problem (6.19).

Proposition 6.13. *Let $M > 0, h > 0, s \in (0,1)$ and $m > 0$. Let $E \subset \mathbb{R}^N$ be a bounded, measurable set such that $P^s(E) \le M$ and $|E| \le M$. Then, there exists a minimizer F of (6.19). Moreover, it is bounded and satisfies the following properties:*

i) There exists $\Lambda = \Lambda(h, N, s) > 0$ such that F is a Λ-minimizer of the fractional perimeter, namely

$$P^s(F) \le P^s(F') + \Lambda|F \triangle F'|$$

for all measurable set $F' \subset \mathbb{R}^N$ such that $\mathrm{diam}(F \triangle F') \le 1$.

ii) The boundary ∂F is of class $C^{2,\alpha}$ for any $\alpha \in (0, s)$ outside of a closed set Σ of Hausdorff dimension at most $N - 3$. Moreover, there exists $s_0 \in (0,1)$ such that, if $s \in (s_0, 1)$, then ∂F is of class $C^{1,\alpha}$ for any $\alpha \in (0,1)$ outside a closed set Σ of Hausdorff dimension at most $N - 8$.

iii) There exist $c_0 = c_0(N, s) > 0$ and a radius $r_0 = r_0(h, N, s) > 0$ such that for every $x \in \partial F \backslash \Sigma$ and $r \in (0, r_0]$ we have

$$|B_r(x) \cap F| \ge c_0 r^N \quad \text{and} \quad |B_r(x) \setminus F| \ge c_0 r^N.$$

iv) The following Euler-Lagrange equation holds: for all $X \in C^1_c(\mathbb{R}^N, \mathbb{R}^N)$ we have

$$\int_{\partial F} \frac{\mathrm{sd}_E}{h} X \cdot \nu_F \,\mathrm{d}\mathcal{H}^{N-1} + \int_{\partial F} H^s_F X \cdot \nu_F \,\mathrm{d}\mathcal{H}^{N-1} = \lambda \int_{\partial F} X \cdot \nu_F \,\mathrm{d}\mathcal{H}^{N-1}, \tag{6.20}$$

where $\lambda = \fint (H^s_F + \frac{1}{h}\mathrm{sd}_E)$ and, if $|F| \ne m$, it also holds $\lambda = \mathrm{sgn}(m - |F|)h^{-\frac{s}{1+s}}$.

v) There exist $k_0 = k_0(h, N, s, M, m) \in \mathbb{N}$ and $d_0 = d_0(h, N, s, M, m) > 0$ such that F is made up of at most k_0 connected components having diameter larger than d_0.

Proof. For the existence of minimizers of (6.19) see for example [42, Theorem 1.1]. The $\Lambda-$minimality property is easily deduced, for instance we can choose $\Lambda = 2(h^{-1} + h^{-\frac{s}{1+s}})$. Concerning property *ii*), it follows from [39, 168] and [37, Theorem 5]. The density esimates can be found in [36, Theorem 4.1]. Item iv) can be proved as in the local case (see [155, Lemma 3.7]). The bound on the number of connected components and on the diameter of the components follows from a covering argument as in [154, Proposition 2.3]. □

By induction we can now define the discrete-in-time, volume preserving fractional mean curvature flow.

Definition 6.14. Fixed $h > 0$ and $m > 0$, let $E_0 \subset \mathbb{R}^N$ be a measurable set such that $|E_0| = m$. Let $E^{(h)}_1$ be a solution of the problem (6.19) with E_0 instead of E. Assume that $E^{(h)}_k$ is defined for $1 \le k \le n-1$, let $E^{(h)}_n$ be a solution of (6.19) with E replaced by $E^{(h)}_{n-1}$. The sequence $\{E^{(h)}_n\}_{n\in\mathbb{N}}$ will be called a *discrete flow.*

We recall the density estimate holding for one-sided minimizers of the fractional perimeter, which can be found in [36, Theorem 4.1] .

Proposition 6.15. *There exists a constant $C = C(N, s) > 0$ with the following property: given $E \subset \mathbb{R}^N$, R, $\mu > 0$ and $x_0 \in \partial E$ such that*

$$P^s(E) \leq P^s(E \setminus B_r(x_0)) + \mu |E \cap B_r(x_0)| \quad \forall 0 < r < R,$$

then

$$Cr^N \leq |E \cap B_r(x_0)| \quad \forall 0 < r < \min\{R, \mu^{-1/s}\}.$$

We employ the density estimates above to bound the distance function between two consecutive sets of the discrete flow. The proof follows the line of [155, Proposition 3.2] where it is proved in the of local case, see also [144].

Proposition 6.16. *There exists a constant $\gamma = \gamma(N, s) > 0$ with the following property. Let $F \subset \mathbb{R}^N$ be a bounded set of finite fractional perimeter and let E be a minimizer of $\mathcal{F}_h(\cdot, F)$, then*

$$\sup_{E \triangle F} \mathrm{dist}_{\partial F} \leq \gamma h^{1/1+s}.$$

Proof. Let $\gamma = \max\{3, 2^{s+1/s} P^s(B)^{1/s} C^{-1/s}\}$, where $C = C(N, s)$ is the constant given by the Proposition 6.15. Let $c > \gamma$ and $x_0 \in E \triangle F$. Suppose by contradiction that $\mathrm{dist}_{\partial F}(x_0) > ch^{1/1+s}$. Since the other case is analogous, we assume $x_0 \in E \setminus F$. We then have

$$\mathrm{sd}_F(x_0) > ch^{1/1+s} \tag{6.21}$$

and thus any ball $B_r(x_0)$ of radius $r \leq ch^{1/1+s}/2$ is contained in F^c. By the minimality of E, we have $\mathcal{F}_h(E, F) \leq \mathcal{F}_h(E \setminus B_r(x_0), F)$, therefore

$$P^s(E) \leq P^s(E \setminus B_r(x_0)) - \frac{1}{h} \int_{E \cap B_r(x_0)} \mathrm{sd}_F \, \mathrm{d}x + \frac{1}{h^{s/1+s}} |E \cap B_r(x_0)|.$$

We use (6.21) and $r \leq ch^{1/1+s}/2$ to infer that

$$-\frac{1}{h} \int_{E \cap B_r(x_0)} \mathrm{sd}_F \, \mathrm{d}x < -\frac{c}{2h^{s/1+s}} |E \cap B_r(x_0)|.$$

Then we have

$$P^s(E) \leq P^s(E \setminus B_r(x_0)) - \frac{1}{h^{s/1+s}} \left(\frac{c}{2} - 1 \right) |E \cap B_r(x_0)|. \tag{6.22}$$

By assumption $c > 3$ and we can apply Proposition 6.15 with $\mu = 0$ and obtain

$$Cr^N \leq |E \cap B_r(x_0)| \qquad \forall 0 < r < \frac{c}{2} h^{1/1+s}. \tag{6.23}$$

On the other hand, from (6.22) we deduce, for every $0 < r < ch^{1/1+s}/2$, that

$$\frac{1}{h^{s/1+s}} \left(\frac{c}{2} - 1 \right) |E \cap B_r(x_0)| \leq P^s(E \setminus B_r(x_0)) - P^s(E) \leq P^s(B_r^c) = P^s(B) r^{N-s} \tag{6.24}$$

(where the last inequality follows from the subadditivity of the perimeter on E and B_r^c). Combining (6.23) and (6.24), we get that

$$Cr^N \leq |E \cap B_r(x_0)| \leq P^s(B) \left(\frac{c}{2} - 1 \right)^{-1} h^{s/1+s} r^{N-s} \leq 2P^s(B) h^{s/1+s} r^{N-s}$$

for all $0 < r < ch^{1/1+s}/2$, which gives the desired contradiction to the choice of c as soon as $r \to ch^{1/1+s}/2$. □

As a corollary of the previous result we obtain the following density estimates, their proof is an adaptation of the one of [155, Corollary 3.3].

Corollary 6.17. *Let $E \subset \mathbb{R}^N$ be a bounded set of finite fractional perimeter and let F be a* *minimizer of* $\mathcal{F}_h(\cdot, E)$*. Then for every* $r \in (0, \gamma h^{1/1+s})$ *and for every* $x_0 \in \partial^* F$*, it holds*

$$\min\{|B_r(x_0) \setminus F|, |F \cap B_r(x_0)|\} \geq cr^N \tag{6.25}$$

$$cr^{N-s} \leq P^s(F, B_r(x_0)) \leq Cr^{N-s}, \tag{6.26}$$

where γ *is the constant given by Proposition 6.16 and the constants* c*,* C *only depend on* N *and* s*.*

Proof. Since F is a minimizer of $\mathcal{F}_h(\cdot, E)$, for any $x_0 \in \partial F$, it holds that $\mathcal{F}_h(F, E) \leq \mathcal{F}_h(F \cup B_r(x_0), E)$, which implies

$$\begin{aligned} P^s(F) &\leq P^s(F \cup B_r(x_0)) + \frac{1}{h} \int_{B_r(x_0) \setminus F} \mathrm{sd}_E \, \mathrm{d}x + \frac{1}{h^{s/1+s}} |B_r(x_0) \setminus F| \\ &\leq P^s(F \cup B_r(x_0)) + \frac{C}{h^{s/1+s}} |B_r(x_0) \setminus F|, \end{aligned}$$

where we bounded $\mathrm{sd}_E \leq \gamma h^{1/1+s}$ by Proposition 6.16. Analogously, one can show that

$$\begin{aligned} P^s(F) &\leq P^s(F \setminus B_r(x_0)) + \frac{C}{h^{s/1+s}} |F \cap B_r(x_0)| \qquad (6.27) \\ &= \mathcal{L}_s(F \setminus B_r(x_0), F^c \setminus B_r(x_0)) + \mathcal{L}_s(F \setminus B_r(x_0), B_r(x_0)) + \frac{C}{h^{s/1+s}} |F \cap B_r(x_0)| \end{aligned}$$

Therefore, by Proposition 6.15, we deduce

$$\min\{|F \cap B_r(x_0)|, |B_r(x_0) \setminus F|\} \geq cr^N \qquad \forall 0 < r < \gamma h^{1/1+s}.$$

The first inequality in (6.26) is now an immediate consequence of the relative isoperimetric inequality. To prove the second inequality, by (6.27) we get

$$\begin{aligned} P^s(F, B_r(x_0)) &= \mathcal{L}_s(F \cap B_r(x_0), F^c) + \mathcal{L}_s(F \setminus B_r(x_0), F^c \cap B_r(x_0)) \\ &= P^s(F) - \mathcal{L}_s(F \setminus B_r(x_0), F^c \setminus B_r(x_0)) \\ &\leq \mathcal{L}_s(F \setminus B_r(x_0), B_r(x_0)) + \frac{C}{h^{s/1+s}} |B_r(x_0) \setminus F| \\ &\leq P^s(B_r(x_0)) + \frac{C\gamma^s}{r^s} \omega_N r^N \leq C(N,s) r^{N-s}, \end{aligned}$$

where we used that $r \leq \gamma h^{1/1+s}$. □

Remark 6.18. From the monotonicity of the energy $P^s(\cdot) + h^{-\frac{s}{1+s}} || \cdot | - m|$ along the discrete flow starting from E_0 with $|E_0| = m$, $P^s(E_0) \leq M$, one can observe that $|E_n^{(h)}| \in (m/2, 3m/2)$ for all $n \in \mathbb{N}$ and for $h = h(m, M)$ small,.

We now characterize the stationary sets E for the discrete flow. We say that E is a *stationary set* for the discrete flow if it is a fixed set for the functional (6.19), that is,

$$E = E_n^{(h)} \quad \forall n \in \mathbb{N}.$$

In the following, we will always assume that either:

- $N = 2$;
- $N \leq 7$ and $s \in (s_0, 1)$, where s_0 is the constant of Proposition 6.13, item *ii)*.

This hypothesis is essential for the proof of the following result.

Proposition 6.19. *Every stationary set* E *for the discrete flow is a critical set of the* $s-$*perimeter, that is, a single ball.*

Proof. It is an immediate consequence of the Euler-Lagrange equation (6.20). Since E is a stationary point for the discrete flow, it satisfies

$$\int_{\partial E} H^s_E \, \mathrm{d}\mathcal{H}^{N-1} = \lambda \int_{\partial E} X \cdot \nu_E \, \mathrm{d}\mathcal{H}^{N-1}$$

for all $X \in C^1_c(\mathbb{R}^N, \mathbb{R}^N)$, i.e. E is a critical point for the $s-$perimeter. By [35, Theorem 1.1] and [63, Theorem 1.1], we conclude that E is a single ball having constant fractional mean curvature $H^s_E = \lambda$. □

Before proving the convergence of the flow up to translations, we recall [154, Lemma 3.5] that will be used in the proof of the next proposition. The proof in the fractional setting is analogous and will be omitted.

Lemma 6.20. *Let $\{E^{(h)}_n\}_{n\in\mathbb{N}}$ be a discrete flow starting from E_0 and let $E^{(h)}_{k_n}$ be a subsequence such that $E^{(h)}_{k_n} + \tau_n \to F$ in L^1 for some set F and a suitable sequence $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$. Then $\mathrm{dist}_{\partial E^{(h)}_{k_n-1}}(\cdot + \tau_n) \to \mathrm{dist}_{\partial F}$ uniformly.*

The following result proves the convergence of the discrete flow to a union of disjointed balls, all having the same radius. The proof follows closely the one of [154, Proposition 3.6]. Moreover, we prove that the flow eventually has fixed volume. A this point, we can not rule out that the flow is converging to different balls (each at infinite distance from the others) and that the translations introduced are different along different subsequences.

Proposition 6.21. *Let m, $M > 0$ and E_0 be an initial bounded set with $P^s(E_0) \leq M$, $|E_0| = m$. Then there exists $h^* = h^*(s, M, m) > 0$ such that, for any $h < h^*$ and for any discrete flow $E^{(h)}_n$ starting from E_0, the following hold:*

i) for n sufficiently large $|E^{(h)}_n| = m$;

ii) there exists
$$P^s_\infty = \lim_{n\to\infty} P^s(E^{(h)}_n);$$

iii) $E^{(h)}_n$ is made of $K = (P^s_\infty/\omega^s_N)^{\frac{N}{s}}(\omega_N/m)^{\frac{N}{s}-1}$ distinct connected components $E^{(h)}_{n,i}$, and $E^{(h)}_{n,i} - \mathrm{bar}(E^{(h)}_{n,i})$ converges in C^k, for every $k \in \mathbb{N}$, to the ball centered at the origin and having mass m/K.

Proof. Let $\{E^{(h)}_{k_n}\}_{n\in\mathbb{N}}$ be any given subsequence of $\{E^{(h)}_n\}_{n\in\mathbb{N}}$. By Proposition 6.13, each set $E^{(h)}_{k_n}$ is made up of $l_n \leq k_0$ connected components having diameter uniformly bounded by d_0. Therefore, there exist l_n balls $\{B_{d_0}(\xi^i_n)\}$, each containing a different component of $E^{(h)}_{k_n}$ and such that $E^{(h)}_{k_n} \subset \cup_{i=1}^{l_n} B_{d_0}(\xi^i_n)$. Up to subsequences, we can assume that $l_n = \tilde{l}$, and for all $1 \leq i < j \leq \tilde{l}$ the following limits exist
$$\limsup_{n\to\infty} |\xi^i_n - \xi^j_n| =: d^{i,j} \in [0, +\infty].$$

Now we define the following equivalence classes: we say that $i \equiv j$ if and only if $d^{i,j} < +\infty$. Denote by $l \leq \tilde{l}$ the number of such equivalence classes, let $j(i)$ be a representative for each class $i \in \{1, \dots, l\}$, and set $\sigma^i_n := \xi^{j(i)}$ for $i = 1, \dots, l$. We have constructed a subsequence $E^{(h)}_{k_n}$ satisfying $E^{(h)}_{k_n} \subset \cup_{i=1}^{l} B_R(\sigma^i_n)$, where $R = d_0 + \max\{d^{i,j} : d^{i,j} < +\infty\} + 1$, and for all $i \not\equiv j$ it holds $|\sigma^i_n - \sigma^j_n| \to +\infty$ as $n \to +\infty$.

Now, fix $1 \leq i \leq l$, and set

$$F^i_n := E^{(h)}_{k_n} - \sigma^i_n, \quad \tilde{F}^i_n := (E^{(h)}_{k_n} - \sigma^i_n) \cap B_R, \quad m^i_n := |\tilde{F}^i_n|.$$

Up to a subsequence, we have $m^i_n \to m^i > 0$. Moreover, by Lemma 6.20 and by the compactness of sets of equi-bounded fractional perimeters, there exist $\tilde{F}^i \Subset B_R$ such that, up to a subsequence,

$$\tilde{F}^i_n \to \tilde{F}^i \text{ in } L^1, \quad \mathrm{sd}_{E^{(h)}_{k_n-1}}(\cdot + \sigma^i_n) \to \mathrm{sd}_{\tilde{F}^i}(\cdot) \text{ locally uniformly.} \tag{6.28}$$

Let $\tilde{G}^i$ be any bounded set with $|\tilde{G}^i| = m_n^i$ and let $\tilde{G}_n^i := \left(\frac{m_n^i}{m^i}\right)^{\frac{1}{N}} \tilde{G}^i$. We set now $G_n^i := (F_n^i \setminus \tilde{F}_n^i) \cup \tilde{G}_n^i$ so that, for n sufficiently large, $|F_n^i| = |G_n^i|$. By the minimality of $E_{k_n}^{(h)}$ we have

$$P^s(F_n^i) + \frac{1}{h}\int_{F_n^i} \mathrm{sd}_{E_{k_n-1}^{(h)}}(x+\sigma_n^i)\,\mathrm{d}x \le P^s(G_n^i) + \frac{1}{h}\int_{G_n^i} \mathrm{sd}_{E_{k_n-1}^{(h)}}(x+\sigma_n^i)\,\mathrm{d}x.$$

For n sufficiently large, we obtain

$$\begin{aligned} &P^s(\tilde{F}_n^i) + \int_{\tilde{F}_n^i}\int_{F_n^i\setminus\tilde{F}_n^i} \frac{1}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{h}\int_{\tilde{F}_n^i} \mathrm{sd}_{E_{k_n-1}^{(h)}}(x+\sigma_n^i)\,\mathrm{d}x \\ &\le P^s(\tilde{G}_n^i) + \int_{\tilde{G}_n^i}\int_{F_n^i\setminus\tilde{F}_n^i} \frac{1}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{h}\int_{\tilde{G}_n^i} \mathrm{sd}_{E_{k_n-1}^{(h)}}(x+\sigma_n^i)\,\mathrm{d}x. \end{aligned}$$

Passing to the limit as $n \to \infty$, using (6.28) and the uniform boundedness of $\tilde{F}_n^i$ and $\tilde{G}_n^i$, we deduce that

$$P^s(\tilde{F}^i) + \frac{1}{h}\int_{\tilde{F}^i} \mathrm{sd}_{\tilde{F}^i}(x)\,\mathrm{d}x \le P^s(G^i) + \frac{1}{h}\int_{G^i} \mathrm{sd}_{\tilde{F}^i}(x)\,\mathrm{d}x.$$

This minimality property extends by density to all competitors G^i with finite perimeter and volume m^i, so that we deduce that $\tilde{F}^i$ is a fixed point for the discrete scheme with prescribed volume m^i, and, whence by Proposition 6.19, it is a ball. Moreover, since $\tilde{F}^i$ are uniform $\Lambda-$minimizer by Proposition 6.13, we also deduce that $\tilde{F}_n^i$ converge to $\tilde{F}^i$ in $C^{1,\alpha}$ for every $\alpha \in (0,1)$. In particular, for n large enough, $\tilde{F}_n^i$ has only one connected component.

We have shown that, for n large enough, $E_{k_n}^{(h)}$ is made up by a fixed number K of connected components $E_{k_n}^{(h),i}$, $i = 1,\dots,K$ and $E_{k_n}^{(h),i} - \mathrm{bar}(E_{k_n}^{(h),i}) \to B_{R_i}$ where $|B_{R_i}| = m_i$. Now, we show that all the radii R_i are equal to R. To this aim, we consider the Euler-Lagrange equation (6.20)

$$\frac{1}{h}\mathrm{sd}_{E_{k_n-1}^{(h)}} + H^s_{E_{k_n}^{(h)}} = \lambda_n \quad \text{on } \partial E_{k_n}^{(h)}.$$

By Proposition 6.16, we deduce that

$$|\lambda_n| \le h^{-1}\|\mathrm{sd}_{E_{k_n-1}^{(h)}}\|_{L^\infty(\partial E_{k_n}^{(h)})} + \|H^s_{E_{k_n}^{(h)}}\|_{L^\infty(\partial E_{k_n}^{(h)})} \le c + \|H^s_{E_{k_n}^{(h)}}\|_{L^\infty(\partial E_{k_n}^{(h)})}.$$

To bound the right hand side, we use the $\Lambda-$minimality of $E_{k_n}^{(h)}$ to obtain

$$\|H^s_{E_{k_n}^{(h)}}\|_{L^\infty(\partial E_{k_n}^{(h)})} \le \Lambda.$$

Therefore, by passing to a further subsequence, we can assume $\lambda_n \to \lambda \in \mathbb{R}$. Arguing as before, we can localize the Euler-Lagrange equation to each single F_n^i and obtain

$$\frac{1}{h}\mathrm{sd}_{E_{k_n-1}^{(h)}}(x+\sigma_n^i) + H^s_{F_n^i}(x) = \lambda_n \quad x \in \partial F_n^i.$$

We can pass to the limit as $n \to \infty$ thanks to Lemma 6.20 and the continuity property of the fractional mean curvature (see e.g. [63, Lemma 2.1]). Thus, taking into account that $\tilde{F}^i$ is a fixed set for (6.19), we deduce that

$$H^s_{\tilde{F}^i} = \lambda \quad \text{on } \partial\tilde{F}^i.$$

In particular, this shows that $R_i = c\lambda^{-s}$, for a suitable constant c depending only on s and N. In order to prove that, eventually, $|E_n^{(h)}| = m$, we proceed as follows. Set $|B_{R_i}| = c_1\lambda^{-sN}$ and $P^s(B_{R_i}) = c_2\lambda^{-s(N-s)}$, for suitable constants c_1, c_2 depending on N, s. From Remark 6.18, we take $h = h(s,M)$ small enough such that

$$|E_{k_n}^{(h)}| \in \left[\frac{m}{2}, \frac{3m}{2}\right], \quad P^s(E_{k_n}^{(h)}) \le P^s(E_0) \le M$$

and, for n large enough, this implies

$$\sum_{i=1}^{K} m_n^i \in \left[\frac{m}{2}, \frac{3m}{2}\right], \quad \sum_{i=1}^{K} P^s(\tilde{F}_n^i) \leq M.$$

Passing to the limit as $n \to \infty$ we obtain

$$Kc_1\lambda^{-sN} \in \left[\frac{m}{2}, \frac{3m}{2}\right], \quad Kc_2\lambda^{-s(N-s)} \leq M,$$

which implies

$$\lambda^{s^2} \leq \frac{2c_1\, M}{m\, c_2}. \tag{6.29}$$

If we suppose that $|E_{k_n}^{(h)}| \neq m$ for infinitely many indexes, then $\lambda = \mathrm{sgn}(m - |E_{k_n}^{(h)}|)h^{-\frac{s}{1+s}}$ which is a contradiction to (6.29) if h is sufficiently small. We have thus proved item *i)*. Since, for n large enough, $|E_n^{(h)}| = m$, the sequence $\{P^s(E_n^{(h)})\}_{n\in\mathbb{N}}$ is eventually non-increasing, from which item *ii)* follows. Knowing the exact values of the volume and $s-$perimeter of any limit point, we are able to compute K and obtain the convergence in L^1 of the whole sequence. Moreover, arguing as in [42] we conclude the convergence in C^k for every $k \in \mathbb{N}$ via a bootstrap method. □

We then recall some results of [154].

Lemma 6.22. *Let $\eta > 0$. There exists $\delta > 0$ with the following property: if $f_1, f_2 \in C^1(\partial B)$ with $\|f_i\|_{C^1(\partial B)} \leq \delta$ and $|B_{f_i}| = |B|$ for $i = 1, 2$ we have*

$$C_1(1-\eta)\|f_1 - f_2\|^2_{L^2(B)} \leq \mathcal{D}(B_{f_1}, B_{f_2}) \leq C_1(1+\eta)\|f_1 - f_2\|^2_{L^2(B)} \tag{6.30}$$

$$\frac{1-\eta}{2}\int_{\partial B_{f-1}} \mathrm{sd}^2_{B_{f_2}}\, \mathrm{d}\mathcal{H}^{N-1} \leq \mathcal{D}(B_{f_1}, B_{f_2}) \leq \frac{1+\eta}{2}\int_{\partial B_{f-1}} \mathrm{sd}^2_{B_{f_2}}\, \mathrm{d}\mathcal{H}^{N-1} \tag{6.31}$$

$$|\mathrm{bar}(B_{f_1}) - \mathrm{bar}(B_{f_2})|^2 \leq C_2\|f_1 - f_2\|^2_{L^2(B)} \leq \frac{C_2}{C_1(1-\eta)}\mathcal{D}(B_{f_1}, B_{f_2})$$

for suitable constants C_1, $C_2 > 0$.

The following crucial lemma is based on the Alexandrov-type estimate contained in Theorem 6.2. Its proof is the same of the one presented in Chapter 5.

Lemma 6.23. *Let $h > 0$. There exist constants $C(h, m, s)$, $\delta > 0$ with the following property: given two normal deformations $B_{f_1}^{(m)}$, $B_{f_2}^{(m)}$ of $B^{(m)}$ with $f_i \in C^2(\partial B^{(m)})$, $\|f_i\|_{C^1(\partial B^{(m)})} \leq \delta$, and such that $|B_{f_2}^{(m)}| = m$, $\mathrm{bar}(B_{f_2}^{(m)}) = 0$ and*

$$H^s_{B_{f_2}^{(m)}} + \frac{\mathrm{sd}_{B_{f_1}^{(m)}}}{h} = \lambda \quad \text{on} \quad \partial B_{f_2}^{(m)} \tag{6.32}$$

for some $\lambda \in \mathbb{R}$, we have

$$\mathcal{D}(B^{(m)}, B_{f_2}^{(m)}) \leq C\mathcal{D}(B_{f_2}^{(m)}, B_{f_1}^{(m)}).$$

Proof. By Theorem 6.2, for δ sufficiently small, we get by using (6.32)

$$\begin{aligned}\|f_2\|^2_{L^2(\partial B^{(m)})} &\leq C\|H^s_{B_{f_2}^{(m)}} - \overline{H}^s_{B_{f_2}^{(m)}}\|^2_{L^2(\partial B^{(m)})} \leq C\|H^s_{B_{f_2}^{(m)}} - \lambda\|^2_{L^2(\partial B^{(m)})} \\ &\leq C\|H^s_{B_{f_2}^{(m)}} - \lambda\|^2_{L^2(\partial B_{f_2}^{(m)})} = \frac{C}{h^2}\int_{\partial B_{f_2}^{(m)}} \mathrm{sd}^2_{B_{f_1}^{(m)}}\, \mathrm{d}\mathcal{H}^{N-1},\end{aligned}$$

where the third inequality follows by bounding the Jacobian of the change of variables by 1 (up to taking δ sufficiently small). By combining the previous inequalities with (6.30) and (6.31), we obtain the thesis. □

We now prove Theorem 6.1. We will follow closely the proofs of [154, Theorem 3.3] and Theorem 5.4 in Chapter 5. The main difference is that we use the fractional perimeter framework previously studied instead of the classical one. We present a sketch of the proof.

Proof. We start by sketching the proof of the exponential decay of the dissipations following Step 1 in [154, Theorem 3.3].

From Proposition 6.21 we know that any limit point of the discrete flow is given by the union of K disjoint balls, all having volume m/K. We then use two competitors to obtain a discrete Gronwall-type inequality. Firstly, testing the minimality of $E_k^{(h)}$ with $E_{k-1}^{(h)}$ and summing from $n+1$ to infinity, we obtain

$$\sum_{k\geq n+1} \mathcal{D}(E_k^{(h)}, E_{k-1}^{(h)}) \leq P^s(E_n^{(h)}) - P_\infty^s = P^s(E_n^{(h)}) - KP^s(B^{(m/K)}).$$

On the other hand, recalling Proposition 6.21, the sets $(E_n^{(h)})^i - \mathrm{bar}((E_n^{(h)})^i) =: (E_n^{(h)})^i - \xi_n^i$ are eventually $C^{1,\alpha}-$deformations of $B^{(m/K)}$, having volume $|(E_n^{(h)})^i| = m_n^i$. We consider the admissible competitor for $E_n^{(h)}$ given by

$$\mathcal{B}_n = \bigcup_{i=1}^{K} \left(B^{(m_{n-1}^i)} + \xi_{n-1}^i \right).$$

Testing the minimality of $E_n^{(h)}$ against $\mathcal{B}_n$, one can obtain, by employing Lemma 6.23, that

$$P^s(E_n^{(h)}) - P^s(\mathcal{B}_n) \leq C\mathcal{D}(E_{n-1}^{(h)}, E_{n-2}^{(h)}).$$

Recalling that, if a measurable set F has L disjointed connected components F^i, $i = 1, \dots, L$, then

$$P^s(F) = \sum_{i=1}^{L} P^s(F^i) - 2\sum_{i<j} \int_{F^i}\int_{F^j} \frac{1}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y,$$

by concavity, we estimate

$$P^s(\mathcal{B}_n) \leq \sum_{i=1}^{K} P^s(B^{(m_{n-1}^i)}) \leq KP^s(B^{(m/K)}).$$

Thus, combining the previous two estimates, we obtain the discrete Gronwall-type estimate

$$\sum_{k\geq n+1} \mathcal{D}(E_k^{(h)}, E_{k-1}^{(h)}) \leq C\mathcal{D}(E_{n-1}^{(h)}, E_{n-2}^{(h)}).$$

Finally, employing [154, Lemma 3.10] we conclude the exponential convergence of the dissipations

$$\mathcal{D}(E_n^{(h)}, E_{n-1}^{(h)}) \leq \left(1 - \frac{1}{C+1}\right)^{\frac{n}{2}} (P^s(E_0) - KP^s(B^{(m/K)})).$$

From now on, one can follow directly the proof of [154, Theorem 3.3] employing Lemma 6.22 to conclude that the discrete flow $E_n^{(h)}$ is eventually contained in a compact set and converges in C^k to a union of K disjoint balls. Now, from Proposition 6.19 we deduce that the limit point is indeed a single ball, having volume equal to m, thus reaching the conclusion of the proof. □

Chapter 7

Stability of the surface diffusion flow and volume-preserving mean curvature flow in the flat torus

Contents

Introduction

In this chapter we establish global in time existence and convergence towards equilibrium of two physically relevant volume-preserving geometric motions, namely the volume-preserving mean curvature flow and the surface diffusion flow.

On the one hand, the first one is the volume-preserving counterpart of the well-known mean curvature flow, and it is defined as a smooth evolution of sets E_t governed by the law

$$V_t = -\mathrm{H}_{E_t} + \bar{\mathrm{H}}_{E_t} \quad \text{on } \partial E_t, \tag{7.1}$$

where V_t and H_{E_t} are the outer normal velocity and the mean curvature of ∂E_t, respectively, while $\bar{\mathrm{H}}_{E_t} = \fint_{\partial E_t} \mathrm{H}_{E_t}$.

On the other hand, the surface diffusion flow is a smooth flow of sets E_t evolving according to the law

$$V_t = \Delta_{E_t} \mathrm{H}_{E_t} \quad \text{on } \partial E_t, \tag{7.2}$$

where Δ_{E_t} denotes the Laplace-Beltrami operator on ∂E_t. Similar to the mean curvature flow, the surface diffusion flow has important applications in material science, especially in physical systems with multiple phases. It has been proposed in the physical literature by Mullins [157] to model surface dynamics for phase interfaces when the evolution is governed by mass diffusion in the interface.

The volume preserving mean curvature flow can be seen as a simplified, second-order version of the surface diffusion flow as both flows share several common properties. Indeed, from the evolution laws (7.1) and (7.2) it follows that the volume of the evolving sets is preserved along the two flows, as can be easily seen from the following computation

$$\frac{\mathrm{d}}{\mathrm{d}t}|E_t| = \int_{\partial E_t} V_t \, \mathrm{d}\mathcal{H}^{N-1} = 0,$$

the perimeter is decreasing, since the evolution (7.1) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} P(E_t) = \int_{\partial E_t} V_t \mathrm{H}_{E_t} \, \mathrm{d}\mathcal{H}^{N-1} = \int_{\partial E_t} \left(\mathrm{H}_{E_t} - \bar{\mathrm{H}}_{E_t}\right)^2 \, \mathrm{d}\mathcal{H}^{N-1} \le 0,$$

and an integration by parts shows for (7.2) that

$$\frac{\mathrm{d}}{\mathrm{d}t} P(E_t) = \int_{\partial E_t} V_t \mathrm{H}_{E_t} \, \mathrm{d}\mathcal{H}^{N-1} = -\int_{\partial E_t} |\nabla \mathrm{H}_{E_t}|^2 \, \mathrm{d}\mathcal{H}^{N-1} \le 0.$$

Moreover, these two evolutions can be regarded (at least formally) as gradient flows of the perimeter according to suitable metrics. In particular, the mean curvature flow can be considered as (a volume preserving modification of) the L^2-gradient flow of the perimeter, while the surface diffusion can be interpreted as its H^{-1}-gradient flow.

In both cases, singularities may appear in a finite time even for initial smooth sets (see [149]), therefore in general only short-time existence results are available, see for instance [82, 122] for the mean curvature flow and [81] for the surface diffusion flow (see also [97] for the case of triple junction clusters). Because of the (formal) gradient flow structure of the two flows, it is reasonable to expect that if the initial set is sufficiently close to a stable point (or a local minimizer) E of the perimeter, then the flow exists for all times and asymptotically converges to E. We refer to this property as *dynamical stability.* The notion of strict stability can be summarized as follows: stable sets are sets whose boundary has constant mean curvature and positive definite second variation of the perimeter (i.e., they are “stable” for the perimeter functional). In this chapter (as in Chapter 5), we will focus on the flat torus $\mathbb{T}^N$, which is particularly interesting due to the variety of possible limit points of the flows, namely periodic constant mean curvature hypersurfaces. In the Euclidean space only unions of balls have constant mean curvature, whereas the flat torus admits a much broader range of such surfaces. However, a full characterization of constant mean curvature hypersurfaces in $\mathbb{T}^N$ is not available in any dimension. In dimension $N = 2$, the only sets with constant mean curvature are (unions of) discs and stripes (also called lamellae), or their complement. On the other hand, for $N \geq 3$ there exist many nontrivial examples, as cylinders and triply periodic surfaces known as gyroids.

The aforementioned approach of studying the dynamical stability of stable sets has been used in many instances in the literature. Concerning the surface diffusion, this method was employed in [3, 93, 94], where the authors considered the surface diffusion (also with an extra elastic term) and the Mullins-Sekerka flows in the 2, 3-dimensional flat torus (see also the survey [74]) and proved the dynamical stability of stable sets. It should be noted that the flows considered in these works include nonlocal terms, but their results also apply to the evolution driven solely by the perimeter energy. In the Euclidan setting, other results for the surface diffusion deal with the stability of balls [81, 141, 178], infinite cylinders [142], two-dimensional triple junctions [98], as well double bubbles [2, 96] (see also [141] for similar results in different settings).
Regarding the volume preserving mean curvature flow, recent progresses have been made in proving the dynamical stability of strictly stable sets in the 3-dimensional flat torus [159], while older results mainly concern convex sets, balls, or the 2-dimensional setting. The dynamical stability of balls has been proven in the Euclidean setting under various hypoteses on the dimension or on the initial set in [82, 95, 122, 143]. We refer also to [165], where global existence and convergence results for a large class of geometric evolution laws have been considered, relying on the concept of L^p-maximal regularity for quasilinear parabolic equations.

In the present chapter we show in any dimensions the dynamical stability of strictly stable sets in the flat torus both for the surface diffusion flow and the volume preserving mean curvature flow. By assuming the initial set to be close in the $C^{1,1}$-topology to a strictly stable set, we obtain global existence and asymptotic convergence of both the flows to (a translated of) the underlying stable set. This is quite surprising for the surface diffusion flow, which is a fourth-order flow. Our main result of the chapter is the following.

Theorem 7.1. *Let $E \subset \mathbb{T}^N$ be a strictly stable set and let $E_0 = E_{u_0} \subset \mathbb{T}^N$ be the normal deformation of E induced by $u_0 \in C^{1,1}(\partial E)$ with $|E_0| = |E|$. There exists $\delta = \delta(E) > 0$ such that if $\|u_0\|_{C^{1,1}(\partial E)} \leq \delta$, then*

(i) the volume-preserving mean curvature flow E_t starting from E_0 (defined in (7.3)) exists smooth for all times $t \geq 0$, and $E_t \to E + \tau$ as $t \to \infty$, for some $\tau \in \mathbb{T}^N$, in C^k for every $k \in \mathbb{N}$ exponentially fast;

(ii) the surface diffusion flow E_t starting from E_0 (defined in (7.10)) exists smooth for all times $t \geq 0$, and $E_t \to E + \tau$ as $t \to \infty$, for some $\tau \in \mathbb{T}^N$, in C^k for every $k \in \mathbb{N}$ exponentially fast.

Where with exponentially fast we mean that the sets E_t can be written as normal deformations of $E + \tau$ induced by functions $u(\cdot, t) \in C^\infty(\partial E + \tau)$ such that

$$\|u(\cdot, t)\|_{C^k(\partial E+\tau)} \leq C_k e^{-C_k t} \quad \text{for} \quad t > 0.$$

The main technical novelty of our argument is the use a quantitative Alexandrov-type inequality, Theorem 5.4 in Chapter 5, that is applied for the first time to a continuous-in-time setting. This technique allows us to treat in a unified fashion both the geometric flows considered. However, it seems to be quite general, in the sense that it can be adapted to other gradient flows of the perimeter functional. For instance, we are confident that the Mullins-Sekerka flow or, more in general, fractional gradient flows of the perimeter could be treated analogously, provided one has sufficient control on the Schauder estimates for the linearized system governing the evolutions. Moreover, since this stability inequality can be seen as a Łojasiewicz-Simon inequality with sharp exponents, one is able to derive the optimal decay of the dissipation along the flow, immediately yielding the exponential convergence in any norm of the flow to the subjacent strictly stable set. In particular, our line of proof works in any dimension without the need of deriving energy estimates for the high derivatives of the curvature, which was one the main bottleneck of the previous methods developed in [3, 93, 94]. Lastly, the Schauder-type estimates we provide following the lines of [117] seems to be new in this setting.

We now outline the strategy of the proof, which is based on the gradient flow structure of the evolution. Firstly, applying the Alexandrov-type inequality Theorem 5.4 in Chapter 5, combined with the quantitative isoperimetric inequality of [4], we are able to to bound the velocity in terms of the displacement. By iterating this procedure for the whole time of existence and using higher order estimates, we can extend the flow for all times. In order to do so, we need to show that the short-time existence and regularity results depend only on the bounds of the initial datum. This is not *a priori* clear from previous existence results [81, 82]. More precisely, we rely on Schauder estimates on the linearized problem solved by the flows, which is a quasilinear perturbation of the heat equation for the mean curvature flow and a quasilinear perturbation of the biharmonic heat equation for the surface diffusion flow. While Schauder-type estimates for general quasilinear parabolic PDEs of the second order are well known (see for instance [90]), we couldn't find a precise reference for the fourth-order equation. Although an approach by scaling (in the spirit of [137]) could be feasible by working in local coordinates, we preferred to rely on the estimates provided in [117], where time-weighted Hölder norms are employed. After establishing the global existence of both flows, we obtain the exponential convergence up to translations via a Gronwall-type inequality. This is where it comes into play the optimality of the exponent in the aforementioned Alexandrov theorem, which yields the exponential rate of convergence. Finally, we prove the convergence of these translations by exploiting the decay of geometric quantities along the flow, as in [3].

1 Preliminary results

For the notations used in this Chapter, and some preliminary results, we refer to Section 2 in Chapter 5.

1.1 Short-time existence for the mean curvature flow

Given $T > 0$ and $E_0 \subset \mathbb{T}^N$ an open smooth set, the *volume-preserving mean curvature flow* in $[0,T)$ starting from E_0 is the family of sets $(E_t)_{0\leq t<T}$ whose outer normal velocity is given by

$$V_t(x) = -\mathrm{H}_{E_t}(x) + \bar{\mathrm{H}}_{E_t}, \quad x \in \partial E_t,\ t \in (0,T). \tag{7.3}$$

We remark that this equation should be intended as follows: there exist a smooth open set $E \subset \mathbb{T}^N$ and a 1-parameter family of smooth diffeomorphism $\Phi_t : \partial E \to \mathbb{T}^N$ given by $\Phi_t(x) = x + u(x,t)\nu_E(x)$, such that $\Phi_0(\partial E) = \partial E_0$, $\Phi_t(\partial E) = \partial E_t$, and

$$\partial_t u(x,t)\nu(x) \cdot \nu_{E_t}(\Phi_t(x)) = -\mathrm{H}_{E_t}(\Phi_t(x)) + \bar{\mathrm{H}}_{E_t}, \quad x \in \partial E,\ t \in (0,T).$$

Assuming that the flow starting from E_0 exists, following classical computations (see for instance [147]) one can deduce that the evolution equation satisfied by u is

$$\partial_t u = \Delta_E u + \langle A(x,u,\nabla u), \nabla^2 u\rangle + J(x,u,\nabla u) + \mathrm{H}_E,$$

where Δ_E is the Laplace-Beltrami operator on ∂E, A is a smooth tensor such that $A(\cdot,0,0) = 0$, and J is a smooth function.

In order to prove the stability of such flow, we need the following short-time existence result.

Theorem 7.2. *Let $\varepsilon > 0$, let $\beta \in (0,1)$ and let $E \subset \mathbb{T}^N$ be a smooth open set. There exists $\delta = \delta(\varepsilon, E, \beta) > 0$ with the following property: if E_0 is the normal deformation of E induced by $u_0 \in C^{1,1}(\partial E)$, $\|u_0\|_{C^{1,1}(\partial E)} \leq \delta$, and $|E_0| = |E|$, then there exists $T > 0$, which only depends on E, β and the bound on $\|u_0\|_{C^{1,1}(\partial E)}$, such that the volume preserving mean curvature flow E_t starting from E_0 exists in $[0,T)$, the sets E_t are normal deformation of E induced by $u(\cdot,t) \in C^\infty(\partial E)$ for all $t \in (0,T)$, and*

$$\sup_{t\in(0,T)} \|u(\cdot,t)\|_{C^{1,\beta}(\partial E)} \leq \varepsilon. \tag{7.4}$$

Moreover, for every $k \in \mathbb{N}$, there exist two constants $c_k = c_k(N) > 0$ and $C_k = C_k(E) > 0$ such *that*

$$\sup_{t\in(0,T)} t^{c_k}\|\nabla^{k+2}u(\cdot,t)\|_{C^0(\partial E)} \le C_k(\|u_0\|_{C^{1,1}(\partial E)}+1). \tag{7.5}$$

We remark that the proof of this result is classical and can be derived from the Schauder estimates for quasi-linear parabolic equations, as u solves a lower-order, nonlinear perturbation of the heat equation. In the following section we will provide a brief outline of the proof for an analogous short-time existence result for the surface diffusion flow (see Theorem 7.12). Similar and simplified arguments would prove the previous result for the mean curvature flow, which is a second order flow.

For the sake of completeness, we provide here an alternative proof of Theorem 7.2 which follows from some results found in the literature. Even if these results are shown in the ambient space $\mathbb{R}^N$, the same arguments can be repeated in the flat torus. The first part of the Theorem is the short-time existence result of [82].

Theorem 7.3 ([82, Main Theorem])**.** *Let $E \subset \mathbb{T}^N$ be a smooth open set and $\beta \in (0,1)$. There exists $\delta = \delta(E,\beta) > 0$ with the following property: if E_0 is the normal deformation of E induced by $u_0 \in C^{1,1}(\partial E)$, $\|u_0\|_{C^{1,1}(\partial E)} \le \delta$, and $|E_0| = |E|$, then there exists $T > 0$, only depending on E, β and the bound on $\|u_0\|_{C^{1,1}(\partial E)}$, such that the volume-preserving mean curvature flow E_t starting from E_0 exists in $[0,T)$, and the sets E_t are normal deformations induced by $u(\cdot,t) \in C^\infty(\partial E)$ for all $t \in (0,T)$. Furthermore, the mapping $(t,E_0) \mapsto E_t$ is a local smooth semiflow on $C^{1,\beta}(E)$.*

We remark that the local smooth semiflow property in particular implies that $\|u(\cdot)\|_{C^{1,\beta}}$ depends continuously on $\|u_0\|_{C^{1,\beta}}$ (see for instance [12, pag. 66]). In particular, for every $\varepsilon > 0$ there exists $\delta(E,\varepsilon,\beta) > 0$ and $T(E,\varepsilon,\beta) > 0$ such that if $\|u_0\|_{C^{1,\beta}} \le \delta$ then

$$\|u(\cdot,t)\|_{C^{1,\beta}} \le \varepsilon \quad \text{for every } t \in (0,T). \tag{7.6}$$

In order to obtain the higher-order regularity inequalities, we apply some curvature estimates obtained recently in [134].

Theorem 7.4 ([134, Theorem 1.1])**.** *Assume that $E_0 \subset \mathbb{R}^N$ is an open bounded set satisfying a uniform inner and outer ball condition with radius r. Then, there exists a time $T = T(r,N) > 0$ such that the volume preserving mean curvature flow E_t starting from E_0 exists in $[0,T)$ and it satisfies a uniform inner and outer ball condition of radius $r/2$. Moreover, it is smooth in $(0,T)$ and satisfies for every $k \in \mathbb{N}$*

$$\sup_{t\in(0,T)} \left(t^k\|\mathrm{H}_{E_t}\|^2_{H^k(\partial E_t)}\right) \le C_k, \tag{7.7}$$

where C_k depends on $k, |E_0|, r$.

Before proving the short time existence result, we remark a classical result concerning the uniform ball condition.

Remark 7.5. Let E be a smooth set satisfying a uniform ball condition of radius r_E. Then every small $C^{1,1}$-normal deformations of E satisfy a uniform ball condition of radius $r \approx r_E$. Indeed, it is easy to see that if E_f is the normal deformation of E induced by $f \in C^{1,1}(\partial E)$, then the Hausdorff distance between E and E_f is bounded by $\|f\|_{C^0(\partial E)}$. Furthermore, since $\nabla \mathrm{sd}_{E_f} = \nu_{E_f}$ and ν_{E_f} can be written as

$$\nu_{E_f} = \left(\nu_E - \sum_{i=1}^{N-1}\frac{\nabla f\cdot v_i}{1+\kappa_i f}v_i\right)\left(1+\sum_{i=1}^{N-1}\frac{(\nabla f\cdot v_i)^2}{(1+\kappa_i f)^2}\right)^{-1/2}, \tag{7.8}$$

where the family $\{v_i\}_{i=1,\dots,N-1}$ denotes an orthonormal frame of the tangent space on ∂E (see Chapter 5), by differentiating (7.8) one can see that

$$\|\mathrm{sd}_{E_f} - \mathrm{sd}_E\|_{C^{1,1}(\partial E)} \le C_E\|f\|_{C^{1,1}(\partial E)},$$

which then implies that $E_f \to E$ in $C^{1,1}$ if $\|f\|_{C^{1,1}} \to 0$. Therefore, by [65, Theorem 2.6] and [65, Remark 2.7] one infers that the radius r of the uniform ball condition of the set E_f depends continuously on $\|f\|_{C^{1,1}}$ when it is small enough. In particular, for every $\varepsilon > 0$ there exists $\delta(r_E, \varepsilon) > 0$ such that, if $\|f\|_{C^{1,1}} \leq \delta$ then

$$|r_E - r| \leq \varepsilon. \tag{7.9}$$

Proof of Theorem 7.2. By Theorem 7.3 there exist a time $T' > 0$ and a family of evolving functions $u(\cdot, t)$, which are smooth in $(0, T')$ and satisfy the inequality (7.4). The second bound follows from classic elliptic regularity arguments that we now sketch.
Fix $t \in (0, T')$, from the bound on $\sup_{t\in(0,T')} \|u\|_{C^{1,\beta}(\partial E)}$ and (up to rotations) for any given point $x = (x', x_N) \in \partial E$ we can parametrize in a cylinder $C = B'_r(x) \times (-L, L)$ both ∂E and ∂E_t as graphs of smooth functions g, g_t. From Theorem 7.4 there exists a time T'' (depending on E, δ by Remark(7.5)) such that the evolving sets E_t satisfy a uniform inner and outer ball condition of radius $r/2$ for any $t \in (0, T'')$. Let us set $T = \min\{T', T''\}$. From estimate (7.7) we get that

$$\mathrm{H}_{E_t} = \operatorname{div}\left(\frac{\nabla g_t}{\sqrt{1 + |\nabla g_t|^2}}\right) = \frac{1}{\sqrt{1 + |\nabla g_t|}}\left(I - \frac{\nabla g_t \otimes \nabla g_t}{1 + |\nabla g_t|^2}\right) : \nabla^2 g_t$$

is bounded in $L^2(B'_r(x'))$ by a constant which depends on $|E_0|, T, r$. Then, by uniform geometric Calderon-Zygmund inequality (see [75, Section 3] or [4, Lemma 7.2]) we deduce that, for some $\rho < r$, in the ball $B'_\rho(x')$ the function g_t is bounded in $H^2(B'_\rho(x'))$ by a constant, depending only on the L^2-bound on H_{E_t}, the norm of the coefficients of the elliptic operator, which are in turn bounded by $\|u_0\|_{C^{1,1}}$ thanks to the previous step. Iterating this procedure, we bound the higher norms $H^k(B'_\rho(x'))$ of g_t, for every $k \in \mathbb{N}$. Then, we conclude by means of Sobolev embeddings and by a covering argument. □

1.2 Short-time existence for the surface diffusion flow

We now consider the evolution called *surface diffusion flow*, defined by

$$V_t(x) = \Delta_{E_t}\mathrm{H}_{E_t}(x), \quad x \in \partial E_t,\ t \in (0, T). \tag{7.10}$$

As for the mean curvature flow, the equation above means that there exist a smooth open set $E \subset \mathbb{T}^N$ and a 1-parameter family of smooth diffeomorphism $\Phi_t : E \to \mathbb{T}^N$ such that $\Phi_t(x) = x + u(x,t)\nu_E(x)$, $\Phi_t(\partial E) = \partial E_t$ and

$$\partial_t u(x,t)\nu_E(x) \cdot \nu_{E_t}(\Phi_t(x)) = \Delta_{E_t}\mathrm{H}_{E_t}(\Phi_t(x)).$$

Assuming that the diffeomorphisms above exist, arguing as in [147, pag. 21], one can deduce that the evolution equation satisfied by u is

$$\begin{aligned}\partial_t u &= -\Delta^2_{E_t} u - \frac{1}{\nu_E \cdot \nu_{E_t}}\Delta_{E_t}(\nu_E \cdot \nu_{E_t})\Delta_{E_t} u + \frac{1}{\nu_E \cdot \nu_{E_t}}\Delta_{E_t} P(x, u, \nabla u)\\ &= -\Delta^2_{E_t} u + \tilde{J}(x, u, \nabla u, \nabla^2 u, \nabla^3 u),\end{aligned} \tag{7.11}$$

where P is a smooth function (assuming that u and ∇u are small), the function $\tilde{J}$ can be written as

$$\tilde{J}(x, u, \nabla u, \nabla^2 u, \nabla^3 u) = \langle \tilde{B}_1, \nabla^2 u\rangle + \langle \tilde{B}_2, \nabla^2 u \otimes \nabla^2 u\rangle + \langle \tilde{B}_3, \nabla^3 u\rangle + \tilde{b}_4$$

and $\tilde{B}_1, \tilde{B}_2, \tilde{B}_3$ and $\tilde{b}_4$ are tensor-valued, respectively scalar-valued functions depending on $(x, u, \nabla u)$ and smooth if their arguments are small enough. Here, with a little abuse of notation, ∇ denotes the covariant derivative on ∂E.

On the other hand, linearizing the Laplace-Beltrami operator yields the evolution equation (compare with [94, Section 3.1])

$$\partial_t u = -\Delta^2_E u + \langle A(x, u, \nabla u), \nabla^4 u\rangle + J(x, u, \nabla u, \nabla^2 u, \nabla^3 u), \tag{7.12}$$

where A is a smooth 4th-order tensor, vanishing when both h and ∇h vanish, and J is given by

$$\begin{aligned} J =& \langle B_1, \nabla^3 u \otimes \nabla^2 u\rangle + \langle B_2, \nabla^3 u\rangle + \langle B_3, \nabla^2 u \otimes \nabla^2 u \otimes \nabla^2 u\rangle \\ &+ \langle B_4, \nabla^2 u \otimes \nabla^2 u\rangle + \langle B_5, \nabla^2 u\rangle + b_6, \end{aligned} \tag{7.13}$$

where B_i, $i = 1, \dots 5$ and b_6 are smooth tensor-valued, respectively scalar-valued functions depending on $(x, u, \nabla u)$.

In this subsection we want to prove a short-time existence result for the surface diffusion flow, in particular we will obtain a priori estimates that will be used to prove the stability of the flow. We will follow the classical approach of linearization and fixed point to solve the nonlinear evolution problem, and then employ Shauder-type estimates to show higher order regularity of the flow. We will follow closely what has been done in [94], combining it with the results of [117].

To start we recall some classical results concerning the Cauchy problem for the biharmonic heat equation on a smooth Riemannian manifold Σ with metric g, which is the solution to the following problem

$$\begin{cases} \partial_t u = -\Delta_\Sigma^2 u + f(x,t) & \text{on } \Sigma \times [0,\infty) \\ u(\cdot, 0) = u_0 & \text{on } \Sigma, \end{cases} \tag{7.14}$$

once the functions f, u_0 are assigned.

Theorem 7.6 (p. 251, [90, Theorem 2])**.** *Given (Σ, g) a smooth Riemannian manifold, there exists a unique biharmonic heat kernel with respect to g denoted as $b_g \in C^\infty\big(\Sigma \times \Sigma \times (0,\infty)\big)$. Moreover let $T > 0$, for any integers $k, p, q \geq 0$ and for any $(x, y, t) \in \Sigma \times \Sigma \times (0, T)$ we have*

$$|\partial_t^k \nabla_x^p \nabla_y^q b_g(x,y,t)|_g \leq C t^{-\frac{n+4k+p+q}{4}} \exp\{-\delta\big(t^{-\frac14} d_g(x,y)\big)^{\frac43}\}, \tag{7.15}$$

where $|\cdot|_g = \sqrt{g(\cdot,\cdot)}$, ∇_x and ∇_y are covariant derivatives with respect to g, and the constants $C, \delta > 0$ depend on T, g and $p + q + 4k$.

Given the biharmonic heat kernel $b_g \in C^\infty\big(\Sigma\times\Sigma\times(0,\infty)\big)$ on (Σ, g) and a function $u_0 \in C^0(\Sigma)$, we define for $(x,t) \in \Sigma \times (0,\infty)$

$$S u_0(x,t) = \int_\Sigma b_g(x,y,t) u_0(y) \mathrm{d}V_g(y) \tag{7.16}$$

where V_g is the Riemannian volume form. Hence, as usual, Su_0 is the solution to the homogeneous Cauchy problem

$$\begin{cases} \partial_t v + \Delta_\Sigma^2 v = 0 & \text{on } \Sigma \times (0,+\infty) \\ v(\cdot,0) = u_0(\cdot) & \text{on } \Sigma. \end{cases} \tag{7.17}$$

Moreover, since the biharmonic heat kernel is smooth for every $t > 0$, we get $Su_0 \in C^\infty\big(\Sigma \times (0,+\infty)\big)$. We now collect some results, which are shown in [117], about the solution of (7.14). The following Schauder-type estimates on the solution of the homogeneous problem (7.17) can then be proved, see [117, Theorem 3.8]. In particular, we modify slightly the formulation of the result, to fit our purposes. One can inspect the proof of [117, Theorem 3.8] (see pag. 7487,7489 in particular) to check the result.

Theorem 7.7. *Suppose $u_0 \in C^{1,1}(\Sigma)$ and fix $T > 0$. Then there exists $C_1(\Sigma, T) > 0$ such that*

$$\sup_{t\in(0,T)} \||Su_0|_g\|_{C^{1,1}(\Sigma)} \leq C_1 \|u_0\|_{C^{1,1}(\Sigma)}, \tag{7.18}$$

Furthermore, for any $l, k \in \mathbb{N}$, we have

$$\sup_{t\in(0,T)} t^{l+\frac{k}{4}} \left\| \left| (\partial_t)^l \nabla_g^{k+2} S u_0(t) \right|_g \right\|_{C^0(\Sigma)} \leq C_{l,k} \|u_0\|_{C^{1,1}(\Sigma)}, \tag{7.19}$$

for some constants $C_{l,k} > 0$ depending on l, k, Σ and T.

In order to study the evolution problem (7.12) we introduce the following two Banach spaces. Fix $0 < T < \infty$ and $0 < \beta < 1$. We define

$$Y_T \coloneqq \left\{ u \in C^0\big(\Sigma \times (0,T)\big) : \|u\|_{Y_T} < \infty \right\}, \tag{7.20}$$

where

$$\begin{aligned} \|u\|_{Y_T} \coloneqq & \sup_{t\in(0,T)} \Big(t^{\frac{1}{2}} \|u(\cdot,t)\|_{C^0(\Sigma)} + t^{\frac{1}{2}+\frac{\beta}{4}} [u(\cdot,t)]_{C^\beta(\Sigma)} \Big) \\ & + \sup_{(x,t)\in\Sigma\times(0,T)} \sup_{0<h<T-t} t^{\frac{1}{2}+\frac{\beta}{4}} \frac{|u(x,t+h)-u(x,t)|}{|h|^{\frac{\beta}{4}}} \end{aligned} \tag{7.21}$$

and $[\cdot]_{C^\beta}$ is the usual Hölder seminorm. Similarly, we introduce the space

$$X_T \coloneqq \left\{ u \in C^0(\Sigma \times (0,T)) : u(\cdot,t) \in C^4(\Sigma),\ \|u\|_{X_T} < \infty \right\}, \tag{7.22}$$

where

$$\begin{aligned} \|u\|_{X_T} \coloneqq & \sup_{t\in(0,T)} \Big(\sum_{k=0}^{4} t^{-\frac{1}{2}+\frac{k}{4}} \|\nabla^k u(\cdot,t)\|_{C^0(\Sigma)} + t^{\frac{1}{2}+\frac{\beta}{4}} [\nabla^4 u(\cdot,t)]_{C^\beta(\Sigma)} \\ & + t^{\frac{1}{2}} \|\partial_t u(\cdot,t)\|_{C^0(\Sigma)} + t^{\frac{1}{2}+\frac{\beta}{4}} [\partial_t u(\cdot,t)]_{C^\beta(\Sigma)} \Big) \\ & + \sup_{(x,t)\in\Sigma\times(0,T)} \sup_{0<h<T-t} t^{\frac{1}{2}+\frac{\beta}{4}} \frac{|\nabla^4 u(x,t+h)-\nabla^4 u(x,t)|_g}{|h|^{\frac{\beta}{4}}} \\ & + \sup_{(x,t)\in\Sigma\times(0,T)} \sup_{0<h<T-t} t^{\frac{1}{2}+\frac{\beta}{4}} \frac{|\partial_t u(x,t+h)-\partial_t u(x,t)|}{|h|^{\frac{\beta}{4}}}. \end{aligned} \tag{7.23}$$

Proposition 7.8. *The spaces $(Y_T, \|\cdot\|_{Y_T})$ and $(X_T, \|\cdot\|_{X_T})$ are Banach spaces.*

The proof of the completeness of the spaces Y_T and X_T is standard, indeed one can prove directly that all Cauchy sequence converge to a function in the space and the candidate limit is obtained using a diagonal argument.

Remark 7.9. Since the norm $\sum_{k=0}^4 \|\nabla^k u\|_{C^0}$ is equivalent to the norm $\|u\|_{C^0} + \|\nabla^4 u\|_{C^0}$ for $C^4(\Sigma)$, we have that the norm $\|\cdot\|_{X_T}$ defined in (7.23) is equivalent to the following norm

$$\|u\|_{X_T}' \coloneqq \|u\|_{X_T} + \sum_{k=0}^{3} \sup_{(x,t)\in\Sigma\times(0,T)} \sup_{0<h<T-t} t^{-\frac{1}{2}+\frac{k}{4}+\frac{\beta}{4}} \frac{|\nabla^k u(x,t+h)-\nabla^k u(x,t)|_g}{|h|^{\frac{\beta}{4}}}.$$

Now we study the nonhomogeneous initial value problem

$$\begin{cases} \partial_t u + \Delta_\Sigma^2 u = f & \text{on } \Sigma\times(0,T) \\ u(\cdot,0) = 0 & \text{on } \Sigma, \end{cases} \tag{7.24}$$

where f is a function on $\Sigma\times(0,T)$. Given the biharmonic heat kernel $b_g \in C^\infty\big(\Sigma\times\Sigma\times(0,T)\big)$ on (Σ, g), the solution (if it exists) to the nonhomogeneous problem (7.24) should be given by Duhamel's principle

$$Vf(x,t) \coloneqq \int_0^t \int_\Sigma b_g(x,y,t-s) f(y,s) \mathrm{d}V_g(y)\mathrm{d}s, \tag{7.25}$$

and, for every $\lambda > 0$, $Vf \in C^\infty(\Sigma\times(\frac{\lambda}{2},\lambda))$.

We then recall the following fundamental Schauder-type estimates proved in [117] on solutions of (7.24) (see [117, Remark 3.12] for the final comments on the constant C).

Theorem 7.10 ([117, Theorem 3.10])**.** *Fix $0 < T < \infty$, if $f \in Y_T$, then $Vf \in X_T$ and there exists a constant $C > 0$ depending on Σ, T such that*

$$\|Vf\|_{X_T} \le C\|f\|_{Y_T}. \tag{7.26}$$

Moreover, equation $(\partial_t + \Delta^2_\Sigma)Vf = f$ *holds in the classical sense on* $\Sigma \times (0,T)$ *and thus* $Vf \in C^\infty(\Sigma \times (0,T))$.

We now turn our attention to the evolution equation (7.12), and use the results above for the particular choice $\Sigma = \partial E$ with the Riemaniann metric induced by the Euclidean one. We consider the map

$$f[u](x) \coloneqq \langle A(x,u,\nabla u), \nabla^4 u\rangle + J(x,u,\nabla u,\nabla^2 u,\nabla^3 u), \tag{7.27}$$

where A, J are the operators defined in (7.12). We now provide the fundamentals estimates on $f[u]$, which represents the nonlinear error generated linearizing (7.12).

Lemma 7.11. *For any* ε, $m > 0$ *there exist* T, $\delta > 0$ *depending on* E, ε *with the following properties. For every* $u_0 \in C^{1,1}(\Sigma)$ *and* $\psi \in X_T$ *satisfying* $\|\psi\|_{X_T} \le m$ *it holds*

$$f[\psi + Su_0] \in Y_T. \tag{7.28}$$

Moreover, if $\|u_0\|_{C^{1,1}(\Sigma)} \le \delta$ *it holds*

$$\|f[Su_0]\|_{Y_T} \le \varepsilon(\|u_0\|_{C^{1,1}(\Sigma)} + 1). \tag{7.29}$$

Finally, $\psi_1, \psi_2 \in X_T$ *satisfying* $\|\psi_i\|_{X_T} \le m$*, it holds*

$$\|f[\psi_1 + Su_0] - f[\psi_2 + Su_0]\|_{Y_T} \le \varepsilon\|\psi_1 - \psi_2\|_{X_T}. \tag{7.30}$$

Proof. Let $T < 1$ to be chosen later and fic $\varepsilon, m > 0$. We prove only equation (7.29), giving a sketch of the proof for (7.30) and (7.28) as they are analogous; we also drop the dependence on the set E in the norms. For clarity of exposition, we prove the results for the simplified error term

$$\tilde f[u](x,t) \coloneqq \langle A(x,u(x,t),\nabla u(x,t)), \nabla^4 u(x,t)\rangle + \langle B, \nabla^3 u(x,t) \otimes \nabla^2 u(x,t)\rangle, \tag{7.31}$$

where B is a (constant) tensor of the same dimension of $\nabla^3 u \otimes \nabla^2 u$ with $\|B\| < 1$. The general case is explained in the appendix, but follows by analogous computations. We will also write $A(x,t)$ and assume implicitly the dependence on $u, \nabla u$.

Firstly, we prove (7.29). In what follows we use the short-hand notation $u = Su_0$. From the definition of $\tilde f[\cdot]$ we have

$$\begin{aligned} \|\tilde f[u]\|_{C^0} &\le \|A\|_{C^0}\|\nabla^4 u\|_{C^0} + \|\nabla^3 u\|_{C^0}\|\nabla^2 u\|_{C^0}, \\ [\tilde f[u]]_{C^\beta} &\le \|\nabla^4 u\|_{C^0} \sup_{\tau\in\mathbb{T}^N}\big(|\tau|^{-\beta}|A(x+\tau,t) - A(x,t)|\big) + \|A\|_{C^0}[\nabla^4 u]_{C^\beta} \\ &\quad + [\nabla^3 u]_{C^\beta}\|\nabla^2 u\|_{C^0} + \|\nabla^3 u\|_{C^0}[\nabla^2 u]_{C^\beta}. \end{aligned} \tag{7.32}$$

Then, we multiply by $t^{\frac12}$ the first equation in (7.32) to get

$$t^{\frac12}\|\tilde f[u]\|_{C^0} \le \|A\|_{C^0} t^{\frac12}\|\nabla^4 u\|_{C^0} + t^{\frac14}t^{\frac14}\|\nabla^3 u\|_{C^0}\|\nabla^2 u\|_{C^0}.$$

By (7.19), with the choice of $l = 0$, $k = 0,1,2$, we have that all the terms $t^{\frac12}\|\nabla^4 u\|_{C^0}$, $t^{\frac14}\|\nabla^3 u\|_{C^0}$ and $\|\nabla^2 u\|_{C^0}$ are bounded by $\|u\|_{C^{1,1}}$ (times a constant that depends on E which we can suppose equal to one for simplicity). We now fix $\delta > 0$ sufficiently small, depending on ε and E, so that $\|A\|_{C^0}$ is bounded by ε, which can be done since A is a smooth tensor and $A(\cdot,0,0) = 0$. Finally, taking T small enough, depending on ε and E, we conclude

$$\sup_{t\in(0,T)} t^{\frac12}\|\tilde f[u]\|_{C^0} \le \varepsilon\|u_0\|_{C^{1,1}}.$$

Therefore, taking into account the full expression for the error term $f[u]$ given by (7.27), one can show that

$$\sup_{t\in(0,T)} t^{\frac12}\|f[u]\|_{C^0} \le C\varepsilon\left(\|u_0\|_{C^{1,1}} + 1\right),$$

where the last constant comes from the term b_6.

Concerning the Hölder seminorm in space, we first remark that

$$\sup_{\tau\in\mathbb{T}^N}\frac{|A(x+\tau,t)-A(x,t)|}{|\tau|^\beta}\le[A(\cdot,u,\nabla u)]_{C^\beta}+\|\partial_2A\|_{C^0}[u]_{C^\beta}+\|\partial_3A\|_{C^0}[\nabla u]_{C^\beta},$$

where ∂_2A and ∂_3A denote the derivative of $A(x,y,z)$ with respect to the second and third components. Therefore, employing again the bounds in (7.18) and (7.19) we can bound

$$t^{\frac12}\|\nabla^4u\|_{C^0}\sup_\tau\frac{|A(x+\tau,t)-A(x,t)|}{|\tau|^\beta}\le\varepsilon\|u_0\|_{C^{1,1}}, \tag{7.33}$$

where we took $\delta>0$ sufficiently small, depending on ε and E, such that

$$[A(\cdot,u,\nabla u)]_{C^\beta}+\|\partial_2A\|_{C^0}[u]_{C^\beta}+\|\partial_3A\|_{C^0}[\nabla u]_{C^\beta}\le\varepsilon,$$

which is possible since A is smooth and $A(\cdot,0,0)=0$. Thus, multiplying by $t^{\frac12+\frac\beta4}$ the second equation in (7.32) we obtain

$$\begin{aligned}t^{\frac12+\frac\beta4}[\tilde f[u]]_{C^\beta}&\le t^{\frac\beta4}\varepsilon\|u_0\|_{C^{1,1}}+\|A\|_{C^0}t^{\frac12+\frac\beta4}[\nabla^4u]_{C^\beta}\\&\quad+t^{\frac14}t^{\frac14+\frac\beta4}\|\nabla^3u\|_{C^\beta}\|\nabla^2u\|_{C^0}+t^{\frac14}t^{\frac14}\|\nabla^3u\|_{C^0}t^{\frac\beta4}\|\nabla^2u\|_{C^\beta}.\end{aligned} \tag{7.34}$$

Then, all the terms in (7.34) with the norms of u can be bounded employing (7.18) and (7.19), thus we can make the right-hand side above as small as needed taking T,δ small enough. Analogous calculations show a similar inequality for the complete error term $f[u]$.

Finally, we show how to bound the Hölder seminorm in time appearing in $\|\tilde f[u]\|_{Y_T}$. We fix $t\in(0,T), h\in(0,T-t)$. To ease notation, we omit to write the evaluation at x in the following. We have by the very definition of $\tilde f[u](t)$ that

$$\begin{aligned}&|\tilde f[u](t+h)-\tilde f[u](t)|\\&\quad\le|\langle A(u(t+h),\nabla u(t+h)),\nabla^4u(t+h)\rangle-\langle A(u(t),\nabla u(t)),\nabla^4u(t)\rangle|\\&\quad+|\langle B,\big(\nabla^3u(t+h)\otimes\nabla^2u(t+h)\big)\rangle-\langle B,\big(\nabla^3u(t)\otimes\nabla^2u(t)\big)\rangle|.\end{aligned}$$

Now by the triangular inequality we obtain

$$\begin{aligned}&|\langle A(u(t+h),\nabla u(t+h)),\nabla^4u(t+h)\rangle-\langle A(u(t),\nabla u(t)),\nabla^4u(t)\rangle|\\&\le\|A\|_{C^0}|\nabla^4u(t+h)-\nabla^4u(t)|+\|\partial_3A\|_{C^0}|\nabla u(t+h)-\nabla u(t)|\|\nabla^4u(t)\|_{C^0}\\&+\|\partial_2A\|_{C^0}|u(t+h)-u(t)|\|\nabla^4u\|_{C^0},\end{aligned} \tag{7.35}$$

and analogously

$$\begin{aligned}&|\langle B,\big(\nabla^3u(t+h)\otimes\nabla^2u(t+h)\big)\rangle-\langle B,\big(\nabla^3u(x,t)\otimes\nabla^2u(x,t)\big)\rangle|\\&\le|\nabla^3u(t+h)-\nabla^3u(t)|\|\nabla^2u\|_{C^0}+\|\nabla^3u\|_{C^0}|\nabla^2u(t+h)-\nabla^2u(t)|.\end{aligned} \tag{7.36}$$

Therefore from formulas (7.35) and (7.36), we obtain

$$\begin{aligned}&|\tilde f[u](t+h)-\tilde f[u](t)|\\&\qquad\le\left(\|\partial_2A\|_{C^0}|u(t+h)-u(t)|+\|\partial_3A\|_{C^0}|\nabla u(t+h)-\nabla u(t)|\right)\|\nabla^4u(t)\|_{C^0}\\&\qquad+\|A\|_{C^0}|\nabla^4u(t+h)-\nabla^4u(t)|+|\nabla^3u(t+h)-\nabla^3u(t)|\|\nabla^2u\|_{C^0}\\&\qquad+\|\nabla^3u\|_{C^0}|\nabla^2u(t+h)-\nabla^2u(t)|.\end{aligned}$$

Applying again (7.18), (7.19), and using the smallness of $\|A\|_{C^0}$, we conclude (7.29) by taking $T,\ \delta$ small enough.

Following the computations above one can easily prove that if $u_0\in C^{1,1}(\Sigma)$ and $\|\psi\|_{X_T}\le m$, it holds

$$f[\psi + Su_0] \in Y_T.$$

The only difference is that, in addition to (7.18), (7.19) one can directly exploit the definition of $\|\cdot\|_{X_T}$ to obtain the required bounds. Also the proof for (7.30) is essentially the same, only much more tedious to write. We show the computations only for the term $\sup_{t\in(0,T)} t^{1/2}\|\cdot\|_{C^0}$ appearing in the norm of Y_T and for the simplified error term (7.31). For $u_i := \psi_i + Su_0$ we can write

$$\begin{aligned}
&|\tilde f[u_1] - \tilde f[u_2]| \\
&= \big|\langle A(x,u_1,\nabla u_1),\nabla^4 u_1\rangle - \langle A(x,u_2,\nabla u_2),\nabla^4 u_2\rangle + \langle B,(\nabla^3 u_1\otimes\nabla^2 u_1 - \nabla^3 u_2\otimes\nabla^2 u_2)\rangle\big| \\
&\le \|\nabla^4 u_1\|_{C^0}\left(\|\partial_1 A\|_{C^0}|\psi_1-\psi_2| + \|\partial_2 A\|_{C^0}|\nabla\psi_1 - \nabla\psi_2|\right) + \|A\|_{C^0}|\nabla^2\psi_1 - \nabla^2\psi_2| \\
&\quad + \|\nabla^3 u_1\|_{C^0}|\nabla^2\psi_1-\nabla^2\psi_2| + \|\nabla^2 u_2\|_{C^0}|\nabla^3\psi_1 - \nabla^3\psi_2|.
\end{aligned}$$

Multiplying the inequality above by $t^{\frac12}$ we have

$$\begin{aligned}
&t^{\frac12}|\tilde f[u_1]-\tilde f[u_2]| \\
&\le \Big(\|\nabla^4 u_1\|_{C^0}\Big(t\|\partial_1 A\|_{C^0} + t^{\frac34}\|\partial_2 A\|_{C^0}\Big) + t^{\frac12}\left(\|A\|_{C^0} + \|\nabla^3 u_1\|_{C^0}\right) \\
&\quad + t^{\frac14}\|\nabla^2 u_2\|_{C^0}\Big)\|\psi_1-\psi_2\|_{X_T} \\
&\le t^{\frac14}\Big(t^{\frac12}\|\nabla^4 u_1\|_{C^0}\|A\|_{C^1} + \|A\|_{C^0} + t^{\frac14}\|\nabla^3 u_1\|_{C^0} + |\nabla^2 u_2\|_{C^0}\Big)\|\psi_1-\psi_2\|_{X_T}.
\end{aligned}$$

Again, by definition of $\|\cdot\|_{X_T}$ and by (7.18),(7.19) we conclude taking T, δ small enough. □

We are now able to prove a short-time existence result for the surface diffusion evolution. Thanks to the previous lemmas, we provide also higher order regularity estimates depending on the $C^{1,1}-$bound on the initial datum only. The proof follows closely the corresponding one in [117, 94].

Theorem 7.12. *Let $\varepsilon > 0$ and let $E \subset \mathbb{T}^N$ be a smooth open set. There exist $\delta = \delta(\varepsilon, E)$, $T = T(\varepsilon, E) > 0$ with the following property: if E_0 is the normal deformation of E induced by $u_0 \in C^{1,1}(\partial E)$, $\|u_0\|_{C^{1,1}(\partial E)} \le \delta$, and $|E_0| = |E|$, then the surface diffusion flow E_t starting from E_0 exists in $[0,T)$, the sets E_t are normal deformations of E induced by $u(\cdot,t) \in C^\infty(\partial E)$ for all $t \in (0,T)$, and*

$$\sup_{t\in(0,T)} \|u\|_{C^2(\partial E)} \le \varepsilon. \tag{7.37}$$

Moreover, for every $k \in \mathbb{N}\setminus\{0\}$, there exist constants $C_k = C_k(\varepsilon, E) > 0$ such that

$$\sup_{t\in[\frac{T}{2},T)} \|\nabla^{k+2}u\|_{C^0(\partial E)} \le C_k(\|u_0\|_{C^{1,1}(\partial E)} + 1). \tag{7.38}$$

Proof. In this proof we denote by $C > 0$ a constant that depends on N and E and may change from line to line. Fix $\varepsilon > 0$.

Step 1: We show existence for (7.12) via a fixed point argument. Let $T < 1$, $\delta < 1$ to be chosen later, and let $u_1 \in C^\infty((0,T); C^\infty(\partial E))$ be the solution of

$$\begin{cases} \partial_t u_1 = -\Delta_E^2 u_1 & \text{on } \partial E\times[0,T), \\ u_1(\cdot,0) = u_0 & \text{on } \partial E, \end{cases}$$

where $u_0 \in C^{1,1}(\partial E)$ is such that $\|u_0\|_{C^{1,1}(\partial E)} \le \delta$. The solution exists and it is given by (7.16), that is $u_1 = 0 + Su_0 =: \psi_1 + Su_0$. Moreover (7.37) and (7.38) are satisfied by u_1 thanks to

Theorem 7.7, for δ small enough depending on ε. Let now u_2 be the solution of

$$\begin{cases} \partial_t u_2 = -\Delta_E^2 u_2 + f[u_1] & \text{on } \partial E \times [0,T), \\ u_2(\cdot,0) = u_0 & \text{on } \partial E, \end{cases}$$

where $f[u]$ is defined as in (7.27). By (7.16) and (7.25), the unique solution is given by $u_2 = Vf[u_1] + Su_0 = Vf[Su_0] + Su_0 =: \psi_2 + Su_0$. Moreover, by Theorem 7.10 and (7.29) we have the estimate

$$\|\psi_2\|_{X_T} \le C\|f[Su_0]\|_{Y_T} \le C\varepsilon(\|u_0\|_{C^{1,1}(\partial E)} + 1) \le m,$$

for m sufficiently large. We are then led to define an iterative scheme. We set u_1, u_2 as above and for $n \ge 3$ we let u_n be the solution to

$$\begin{cases} \partial_t u_n = -\Delta_E^2 u_n + f[u_{n-1}] & \text{on } \partial E \times [0,T), \\ u_n(\cdot,0) = u_0 & \text{on } \partial E, \end{cases} \tag{7.39}$$

and we split it as $u_n = Su_0 + Vf[u_{n-1}] =: \psi_n + Su_0$. We will show that the sequence ψ_n is converging in X_T. To do so, assume that $\psi_j \in X_T$ for $j = 1, \dots, n-1$ with

$$\|\psi_j\|_{X_T} \le m.$$

Then, by Theorem 7.10 and Lemma 7.11 we get $\psi_n \in X_T$ and

$$\begin{aligned} \|\psi_n\|_{X_T} &= \|Vf[u_{n-1}]\|_{X_T} \le C\|f[u_{n-1}]\|_{Y_T} = C\|f[\psi_{n-1} + Su_0]\|_{Y_T} \\ &\le C\sum_{j=2}^{n-1} \|f[\psi_j + Su_0] - f[\psi_{j-1} + Su_0]\|_{Y_T} + C\|f[Su_0]\|_{Y_T} \\ &\le C\Big(\sum_{j=1}^{n-1} \varepsilon^j\Big)(\|u_0\|_{C^{1,1}(\partial E)} + 1) \\ &\le C\varepsilon\Big(1 + \sum_{j=1}^{+\infty} \varepsilon^j\Big)(\|u_0\|_{C^{1,1}(\partial E)} + 1) \\ &\le C\varepsilon(\|u_0\|_{C^{1,1}(\partial E)} + 1) \le m. \end{aligned} \tag{7.40}$$

Moreover, Lemma 7.11 implies that, for $\delta(\varepsilon, E)$, $T(\varepsilon, E)$ small enough, it holds for all $n \ge 3$

$$\|\psi_{n+1} - \psi_n\|_{X_T} \le \varepsilon\|\psi_n - \psi_{n-1}\|_{X_T},$$

therefore ψ_n is a Cauchy sequence and admits a limit point ψ satisfying

$$\|\psi\|_{X_T} \le C\varepsilon(\|u_0\|_{C^{1,1}(\partial E)} + 1). \tag{7.41}$$

We thus showed the existence of a fixed point $u = \psi + Su_0$ for the problem (7.39). Finally, by (7.18) and (7.41) it holds

$$\|u\|_{C^2(\partial E)} = \|\psi + Su_0\|_{C^2(\partial E)} \le \|\psi\|_{X_T} + \|Su_0\|_{C^2(\partial E)} \le C\varepsilon(\|u_0\|_{C^{1,1}(\partial E)} + 1). \tag{7.42}$$

Step 2: By (7.42) we get straightforwardly that (7.38) holds for $k = 0, 1, 2$. In order to prove (7.38) for $k \ge 3$, we consider $x \in \partial E$ and we work under local coordinate, $B'_r \cong U \subset \partial E$ such that the metric $(g^{ij})_{i,j=1,\dots,N-1}$ of ∂E satisfies $\frac{1}{2}\delta_{ij} \le g_E^{ij} \le 2\delta_{ij}$. Note in particular that the operator $-\Delta_E^2$ is uniformly elliptic in U. In the following we identify B'_r and $U \subset \partial E$. We also set g_t as the metric on ∂E_t (see [147, pag. 20] for details). Observe that u restricted to $B'_r \times [\frac{T}{2}, T)$ is of class C^∞ by the previous step. Recalling that $u = \psi + Su_0$, we have that the function ψ satisfies

$$\partial_t \psi = -\Delta^2_{g_t}\psi + (\partial_t + \Delta^2_{g_t})(Su_0) + f' =: -\Delta^2_{g_t}\psi + \tilde{f}. \tag{7.43}$$

Taking ∇_g in (7.43) shows that the function $\nabla_g\psi$ satisfies the equation

$$\begin{aligned}\partial_t \nabla_g \psi &= -\Delta^2_{g_t}\nabla_g\psi - (\nabla_g g_t^{ij})g_t^{kl}(\psi)_{ijkl} - g_t^{ij}(\nabla_g g_t^{kl})(\psi)_{ijkl} + \nabla_g \tilde{f} \\ &=: -\Delta^2_{g_t}\nabla_g\psi + F,\end{aligned} \tag{7.44}$$

where the error term F contains the derivative of ψ up to order four. To estimate $\|F\|_{C^{\beta/4}([\frac{T}{2},T];C^\beta(B'_r))}$ we first observe that, by (7.19), it follows

$$\|\nabla_g\left((\partial_t + \Delta^2_{g_t})(Su_0)\right)\|_{C^{\beta/4}([\frac{T}{2},T);C^\beta(B'_1))} \le C\varepsilon(\|u_0\|_{C^{1,1}(\partial E)} + 1).$$

Secondly, we remark that the other terms of F can be bounded analogously, recalling that they contain derivatives of ψ up to order four and using (7.41), to show that

$$\|F\|_{C^{\beta/4}([\frac{T}{2},T);C^\beta(B'_r))} \le C\varepsilon(\|u_0\|_{C^{1,1}(\partial E)} + 1). \tag{7.45}$$

Note now that $\partial_t + \Delta^2_{g_t}$ is a uniformly parabolic operator, since the coefficients of $\Delta^2_{g_t}$ are close to the ones of Δ^2_E depending on $\|u(\cdot,t)\|_{C^{1,1}(\partial E)}$ as $g^{ij}_{E_u} - g^{ij}_E = B(x,u,\nabla u)$ and B is a smooth function with $B(x,0,0) = 0$, see again [147, pag. 20]. Since $\nabla_g\psi$ solves (7.44), by the standard interior Schauder estimates and the bound (7.45), there exists $C > 0$, which depends on T and thus on ε and E, such that

$$\begin{aligned}\|\nabla_g\psi\|_{C^{1,\beta/4}([\frac{T}{2},T);C^{4,\beta}(B'_{r/2}))} &\le C\left(\|F\|_{C^{\beta/4}([\frac{T}{4},T);C^\beta(B'_r))} + \|\nabla_g\psi\|_{C^0(B'_r\times[\frac{T}{4},T))}\right) \\ &\le C\varepsilon(\|u_0\|_{C^{1,1}(\partial E)} + 1),\end{aligned}$$

where we noted that $\|\psi\|_{C^1((B'_r\times[\frac{T}{4},T)))} \le \|\psi\|_{X_T}$ and employed again (7.41). Finally, we conclude

$$\sup_{t\in[\frac{T}{2},T)} \|\nabla^5 u\|_{C^0(\partial E)} \le C(\|u_0\|_{C^{1,1}(\partial E)} + 1).$$

By induction, one can prove (7.38) for every $k \in \mathbb{N}$. □

2 Stability

2.1 Stability of the volume preserving mean curvature flow

In this subsection, we study the evolution by mean curvature (7.3) of normal deformations of a strictly stable set. Suppose that E is a strictly stable set and that $E_0 = E_{u_0}$ is a smooth normal deformation of E. By Theorem 7.2, the volume preserving mean curvature flow starting from E_0 exists in a short time interval, and the evolving sets E_t can be parametrized as normal deformations of the set E induced by functions $u(\cdot,t)$ satisfying

$$\begin{cases} u_t(x,t)\nu_{E_t}(p)\cdot\nu_E(x) = -\left(\mathrm{H}_{E_t}(p) - \bar{\mathrm{H}}_{E_t}\right) & x \in \partial E, \\ u(\cdot,0) = u_0 \end{cases}$$

where $p = x + u(x,t)\nu_E(x)$ and $\bar{\mathrm{H}}_{E_t} = \fint_{\partial E_t}\mathrm{H}_{E_t}$. The scalar product above can be written as

$$\nu_{E_t}(p)\cdot\nu_E(x) = \left(1 + \sum_{j=1}^{N-1}\frac{(\partial_{\tau_j}u(x,t))^2}{(1+\kappa_j(x)u(x,t))^2}\right)^{-1/2},$$

where $\kappa_j(x)$ and $\tau_j(x)$ are, respectively, the principal curvatures and the principal directions of E at x. In particular, we remark that $\nu_{E_t}(p)\cdot\nu_E(x) = 1 + O(\|u(\cdot,t)\|_{H^1})$. We can then prove the first part of the main result, that is Theorem 7.1, concerning the long time behaviour of the volume preserving mean curvature flow.

Proof of (i) Theorem 7.1. Let ε, $\delta(\varepsilon)\in(0,1)$ to be chosen later. In the following, if not otherwise stated, the constants depends on N, E and may change from line to line. Fix for instance $\beta = 1/2$ and suppose that δ is smaller than the constant given by Theorem 7.2. We also use the short-hand notation $\pi_f := (\pi_E|_{E_f})^{-1}$.

Step 1. We start by proving that $P(E_t) - P(E) \le Ce^{-ct}$ as long as the flow exists.

Let $u_0 \in C^{1,1}(\partial E)$ with $\|u_0\|_{C^{1,1}} \le \delta < 1$. By Theorem 7.2 there exist a time $T > 0$, which depends on E and the bound on $\|u_0\|_{C^{1,1}} < 1$, and a smooth flow E_t starting from E_0 for $t\in[0,T)$. Moreover, $E_t = E_{u(\cdot,t)}$ and $u(\cdot,t)$ satisfies (7.4) and (7.5). Without loss of generality we can assume $T<\infty$. We also note that the value of T does not change taking ε,δ smaller.

We recall the following well-known identities, holding along the smooth flow:

$$\frac{\mathrm{d}}{\mathrm{d}t}|E_t| = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}P(E_t) = -\|\mathrm{H}_{E_t} - \bar{\mathrm{H}}_{E_t}\|^2_{L^2(\partial E_t)}. \tag{7.46}$$

Let δ^* be the constant given by Theorem 5.4 in Chapter 5, $p > N-1$ and $\eta = \eta(\delta^*, p)$ given by Lemma 5.14 in Chapter 5. By estimates (7.4), (7.5) and by interpolation we have that $\|u(\cdot,t)\|_{W^{2,p}(\partial E)} \le \eta$ for every $t\in[T/2,T)$, up to taking ε smaller and therefore δ smaller. Thus for any $t\in[T/2,T)$ we can apply Lemma 5.14 of Chapter 5 to find $\sigma_t\in\mathbb{T}^N$ and a function $\tilde u(\cdot,t)$ such that $E_t+\sigma_t = E_{\tilde u(\cdot,t)}$ and

$$|\sigma_t| \le C\|u(\cdot,t)\|_{W^{2,p}(\partial E)},\ \|\tilde u(\cdot,t)\|_{W^{2,p}(\partial E)} \le C\|u(\cdot,t)\|_{W^{2,p}(\partial E)},$$
$$\left|\int_{\partial E_t}\tilde u(\cdot,t)\nu_{E_t}\right| \le \delta^*\|\tilde u(\cdot,t)\|_{L^2(\partial E)}.$$

Furthermore, Lemma 5.16 (taking δ smaller if needed) implies that $\|\tilde u(\cdot,t)\|_{C^1(\partial E)} \le \delta^*$. We then apply Theorem 5.4 of Chapter 5 to the set $E_t+\sigma_t$ to obtain

$$\|\tilde u(\cdot,t)\|_{H^1(\partial E)} \le C\|\mathscr{H}_{E_t+\sigma_t} - \lambda\|_{L^2(\partial E)} \tag{7.47}$$

for any $\lambda\in\mathbb{R}$, where we recall $\mathscr{H}_{E_t+\sigma_t}(x) = \mathrm{H}_{E_t}(x+\tilde u(x)\nu_E(x))$. From the previous equation, first by the change of variable $y = x+\tilde u(x,t)\nu_E(x)$ (estimating the Jacobian with the bounds on $\tilde u$ and Lemma 5.16), and then by translation invariance, we arrive at

$$\|\tilde u(\cdot,t)\|_{H^1(\partial E)} \le C\|\mathrm{H}_{E_t+\sigma_t} - \lambda\|_{L^2(\partial E_t+\sigma_t)} = C\|\mathrm{H}_{E_t} - \lambda\|_{L^2(\partial E_t)}. \tag{7.48}$$

We now claim that

$$P(E_t+\sigma_t) - P(E) = P(E_{\tilde u(\cdot,t)}) - P(E) \le C\|\tilde u(\cdot,t)\|^2_{H^1(\partial E)}, \tag{7.49}$$

which is a classical result but we provide a proof for the sake of completeness.

Let us define, for every $x\in\partial E$, the function

$$Q(x) := \left(1 + \sum_{j=1}^{N-1}\frac{(\partial_{\tau_j}\tilde u(x,t))^2}{(1+\kappa_j(x)\tilde u(x,t))^2}\right)^{\frac12}$$

where $\tau_1(x),\dots,\tau_{N-1}(x)$ and $\kappa_1(x),\dots,\kappa_{N-1}(x)$ are, respectively, the principal directions and curvatures of ∂E at x. Then, we have

$$\begin{aligned} P(E_t+\sigma_t) = P(E_{\tilde{u}(\cdot,t)}) &= \int_{\partial E} Q(x) \prod_{i=1}^{N-1} \left(1+\kappa_i(x)\tilde{u}(t,\,x)\right)\, d\mathcal{H}^{N-1}(x) \\ &= P(E) + \int_{\partial E} \left(\mathrm{H}_E \tilde{u}(\cdot,t) + O(\tilde{u}(\cdot,t)^2) + O(|D\tilde{u}(\cdot,t)|^2)\right) d\mathcal{H}^{N-1} \\ &\le P(E) + \mathrm{H}_E \int_{\partial E} \tilde{u}(\cdot,t)\, d\mathcal{H}^{N-1} + C\int_{\partial E} \left(\tilde{u}(\cdot,t)^2 + |D\tilde{u}(\cdot,t)|^2\right) d\mathcal{H}^{N-1} \\ &\le P(E) + C\|\tilde{u}(\cdot,t)\|^2_{H^1(\partial E)}, \end{aligned}$$

where we have used that $\mathrm{H}_E = \sum_{i=1}^{N-1} \kappa_i$ and the inequality

$$\left| \int_{\partial E} \tilde{u}(\cdot,t)\, d\mathcal{H}^{N-1} \right| \le C \int_{\partial E} \tilde{u}(\cdot,t)^2\, d\mathcal{H}^{N-1},$$

which follows from the fact that $|E_t| = |E_0|$. Hence, we prove the claim in (7.49).

We now define the Lyapunov functional $\mathscr{E}(t) = P(E_t) - P(E)$, which is non increasing by (7.46). Moreover, by translation invariance, from (7.48), (7.49) and for any $\lambda \in \mathbb{R}$ we have

$$P(E_t) - P(E) = P(E_t+\sigma_t) - P(E) \le C\|\mathrm{H}_{E_t} - \lambda\|^2_{L^2(\partial E_t)}. \tag{7.50}$$

Since for any $t \in (0,T)$ equation (7.50) for the particular choice of $\lambda = \bar{\mathrm{H}}_{E_t}$ implies

$$\mathscr{E}'(t) = -\|\mathrm{H}_{E_t} - \bar{\mathrm{H}}_{E_t}\|^2_{L^2(\partial E_t)} \le -C\mathscr{E}(t),$$

by Gronwall's inequality we conclude (recalling $\mathscr{E}(0) \ge \mathscr{E}(T/2)$)

$$\mathscr{E}(t) \le \mathscr{E}(0) e^{-C(t-T/2)}, \qquad \forall t \in [T/2, T). \tag{7.51}$$

Step 2. We now show that the flow exists for every $t \ge 0$ and it converges exponentially fast to E up to translations.

Up to taking δ smaller, we can use the quantitative isoperimetric inequality in Theorem 5.17 in Chapter 5 to find the existence of translations τ_t such that

$$C|E \triangle (E_t + \tau_t)|^2 \le P(E_t) - P(E) \le P(E_0) - P(E).$$

Furthermore, since all the evolving sets $\{E_t\}_{t\in[T/2,T)}$ satisfy a uniform inner and outer ball condition by Remark 7.5, by classical convergence results (see e.g. [65, Theorem 3.2]) we have that $E_t+\tau_t$ is C^1−close to E. In particular, there exist smooth (by the implicit map theorem) functions $v(\cdot,t) : \partial E \to \mathbb{R}$ such that $E_t + \tau_t = E_{v(\cdot,t)}$ and

$$|\tau_t| \le \max_{x\in\partial E_t+\sigma_t} \mathrm{dist}_{\partial E_t}(x) \le \|u(\cdot,t)\|_{C^0(\partial E)} + \|v(\cdot,t)\|_{C^0(\partial E)} \le 2\varepsilon,$$

up to taking δ smaller. Therefore, recalling (7.51), we have

$$\|v(\cdot,t)\|^2_{L^1(\partial E)} \le C(P(E_0) - P(E)) e^{-C(t-T/2)}. \tag{7.52}$$

By Lemma 5.16, we also have $\|v(\cdot,t)\|_{C^k(\partial E)} \le C(\|u(\cdot,t)\|_{C^k(\partial E)} + |\tau_t|)$ for every $k \ge 2$. For every $t \in [T/2, T)$, by combining the previous estimate with (7.5), (7.52) and interpolation inequalities, for any $l \in \mathbb{N}$ there exist $k(l) \in \mathbb{N}, \theta(l) \in (0,1)$ and $C = C(E,l) > 0$ such that

$$\|\nabla^l v(\cdot,t)\|_{C^0} \le C\|v(\cdot,t)\|^{\theta}_{L^1} \|v(\cdot,t)\|^{1-\theta}_{C^k} \le CT^{-\frac{k}{4}(1-\theta)} (P(E_0) - P(E))^{\frac{\theta}{2}} e^{-C(t-T/2)}. \tag{7.53}$$

Choosing $\mathscr{E}(0) = P(E_0) - P(E)$ small (hence choosing δ small) we can then apply again Theorem 7.2 with the new initial set $E_{v(\cdot,T/2)} = E_{T/2} + \tau_{T/2}$ to get existence of the translated flow up to the time $3T/2$. We remark that, by uniqueness, the flow above is well defined since it coincides in $[T/2, T)$ with the flow E_t translated by τ_t and estimate (7.51) now holds for all $t \in [T/2, 3T/2)$. Since now the bound (7.53) is uniform along the flow, choosing at every step the times $t = nT/2$, we can iterate the procedure above to prove that the flow exists for all times $t \in [0, \infty)$. Moreover, for every $t \in (0, \infty)$ there exists a translation τ_t such that $E_t + \tau_t = E_{v(\cdot,t)}$ with v satisfying (7.53). In particular, we have that $v \to 0$ exponentially in C^k for any k, as $t \to \infty$ and thus $E_t + \tau_t \to E$ in C^k for every k. This also implies (reasoning as in (7.48)) that $\|\mathrm{H}_{E_t} - \bar{\mathrm{H}}_{E_t}\|_{L^2(\partial E)} \to 0$ exponentially fast.

Step 3. We conclude by showing the convergence of the whole flow to a translate of E.

Let us prove the convergence of the translations $\{\tau_t\}_{t\geq 0}$. By compactness we can find a sequence $t_n \to \infty$ such that $\tau_{t_n} \to \tau$. Defining

$$\mathcal{D}(F, G) := \int_{F \triangle G} \mathrm{dist}_{\partial G}(x)\, \mathrm{d}x, \tag{7.54}$$

following the computations of [3, pag. 21] we see

$$\begin{aligned} \left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}(E_t, E - \tau)\right| &= \left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{E_t \triangle (E-\tau)} \mathrm{dist}_{\partial E \tau_t}(x)\, \mathrm{d}x\right| \\ &= \left|\int_{E_t} \mathrm{div}(\mathrm{sd}_{E-\tau}(x) V_t(x) \nu_{E_t}(x))\, \mathrm{d}x\right| \\ &= \left|-\int_{\partial E_t} \mathrm{sd}_{E-\tau}(x)(\mathrm{H}_{E_t}(x) - \bar{\mathrm{H}}_{E_t}(x))\, \mathrm{d}\mathcal{H}^{N-1}(x)\right| \\ &\leq P(E_0)\|\mathrm{H}_{E_t} - \bar{\mathrm{H}}_{E_t}\|_{L^2(\partial E)} \left(\sup_{x \in \partial E_t} \mathrm{dist}_{\partial E - \tau}(x)\right) \\ &\leq Ce^{-Ct}\left(\sup_{x \in \mathbb{T}^N} \mathrm{dist}_{\partial E - \tau}(x)\right) \leq Ce^{-Ct}, \end{aligned} \tag{7.55}$$

where we recall that V_t is the velocity of the flow in the normal direction (see (7.3)). Clearly, condition (7.55) implies that $\mathcal{D}(E_t, E - \tau)$ admits a limit as $t \to +\infty$. By the previous step and since $\tau_{t_n} \to \tau$, we deduce that

$$\mathcal{D}(E_t, E - \tau) \to 0 \quad \text{as } t \to +\infty.$$

Assume now that $\sigma \in \mathbb{T}^N$ is the limit of τ_{s_n} along a subsequence $s_n \to \infty$ as $n \to +\infty$. By the previous step, $E_{s_n} \to E - \sigma$, therefore

$$0 = \lim_{n \to +\infty} \mathcal{D}(E_{s_n}, E - \tau) = \mathcal{D}(E - \sigma, E - \tau),$$

which implies $\sigma = \tau$ by definition (7.54). This concludes the proof as the exponential convergence follows from Step 2. □

2.2 Stability of the surface diffusion flow

We now focus on surface diffusion flow (7.10). As in the previous subsection, we consider E a strictly stable set and $E_0 = E_{u_0}$ a smooth normal deformation of E. By Theorem 7.12, the surface diffusion flow starting from E_0 exists smooth in an interval $[0, T)$, moreover the evolving sets E_t can be written as normal deformations of E induced by functions $u(\cdot, t)$ satisfying

$$\begin{cases} u_t(x,t)\nu_{E_t}(p) \cdot \nu_E(x) = \Delta_{E_t} \mathrm{H}_{E_t}(p) & \forall x \in \partial E, \\ u(x,0) = u_0(x) \end{cases}$$

where $p = x + u(x,t)\nu_E(x)$.

Now, we aim to show the stability result (ii) of Theorem 7.1 for the surface diffusion flow. Due to the similarity of the arguments needed with those employed to prove item (i) of Theorem 7.1, we will only highlight the main differences between the two.

Proof of (ii) Theorem 7.1. Firstly, Theorem 7.12 ensures the existence of a smooth flow E_t for $t \in (0,T)$ of normal deformations of E induced by functions $u(\cdot,t) \in C^\infty(\partial E)$ and satisfying (7.37) and (7.38). We recall the following identities, holding along the flow E_t as long as it exists smooth,

$$\frac{\mathrm{d}}{\mathrm{d}t}|E_t| = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}P(E_t) = \int_{\partial E} \mathrm{H}_{E_t}(x)\Delta_{E_t}\mathrm{H}_{E_t}(x)\,dx = -\|\nabla \mathrm{H}_{E_t}\|^2_{L^2(\partial E_t)} \leq 0. \tag{7.56}$$

Denoting by C_{E_t} the constant in the Poincaré inequality, we get

$$\|\mathrm{H}_{E_t} - \bar{\mathrm{H}}_{E_t}\|_{L^2(\partial E_t)} \leq C_{E_t}\|\nabla \mathrm{H}_{E_t}\|_{L^2(\partial E_t)}.$$

Combining the previous inequality with (7.56), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}P(E_t) \leq -C_{E_t}\|\mathrm{H}_{E_t} - \bar{\mathrm{H}}_{E_t}\|^2_{L^2(\partial E_t)}.$$

Since $\|u(\cdot,t)\|_{C^{1,1}(\partial E)} \leq c$ for every $t \in (0,T)$, the Poincaré constants C_{E_t} are uniformly bounded in the same time interval and the bound depends on $E, \|u\|_{C^{1,1}(\partial E)}$ (see e.g. the results in [75]). Thus, we obtain the estimate $\frac{\mathrm{d}}{\mathrm{d}t}P(E_t) \leq -C\|\mathrm{H}_{E_t} - \bar{\mathrm{H}}_{E_t}\|^2_{L^2(\partial E_t)}$ uniformly in $(0,T)$. We then conclude by following the same arguments of part (i). $\square$

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Crystalline and Anisotropic, Nonlinear or Nonlocal Curvature Flows

Abstract

This thesis is devoted to the study of geometric flows, with particular focus on the mean curvature flow. It is divided in two thematic parts. The first part, Part I, contains Chapters 2, 3 and 4, and concerns convergence results for the minimizing movements scheme, which is a variational procedure extending Euler's implicit scheme to evolutions having a gradient flow-like structure. We implement this scheme for anisotropic or crystalline, nonlocal or inhomogeneous curvature flows, in linear and nonlinear instances, and study its convergence towards weak solutions to the flows. In Chapter 4 we also pair this study with a discrete-to-continuum limit. The second part, Part II, is devoted to the study of asymptotic behaviour of volume-preserving curvature flows both in the discrete- and continuus-in-time instances. The main technical tool employed is a new Łojasiewicz-Simon inequality suited to the study of these kind of evolutions.

Keywords: Geometric Evolution Equations, Mean Curvature Flows, Crystalline Curvature Flows, Minimizing Movements

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Flot de la Courbure Cristalline et Anisotrope, Non Linéaire ou Non Locale

Résumé

Cette thèse est consacrée à l'étude de flots géométriques, avec un accent particulier sur le flot de la courbure moyenne. La thèse est divisée en deux parties thématiques. La première partie, Partie I, contient les Chapitres 2, 3 et 4, et concerne des résultats de convergence pour le schéma des mouvements minimisants, qui est une procédure variationnelle étendant le schéma implicite d'Euler aux évolutions ayant une structure de type flot gradient. Nous mettons en œuvre ce schéma pour des flots, linéaires ou non linéaires, de la courbure anisotrope ou cristalline, non locale ou inhomogène, et nous étudions sa convergence vers des solutions faibles. Au Chapitre 4, nous associons également cette étude à une limite discrète-continue. La deuxième partie, Partie II, est consacrée à l'étude du comportement asymptotique des flots de la courbure avec une contrainte de volume, à la fois en temps discret et en temps continu. Le principal outil technique utilisé est une nouvelle inégalité de Łojasiewicz-Simon adaptée à l'étude de ce type d'évolutions.

Mots clés : Equations d'évolution géométrique, Flot de la Courbure Moyenne, Mouvements Minimisants

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MOTS CLÉS

Equations d'Évolution Géométrique, Flot de la Corbure Moyenne, Mouvements Minimisants

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KEYWORDS

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