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UNSTABLE INSTANTONS AND LOCALIZATION IN TWO-DIMENSIONAL GAUGE THEORIES

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Abstract

The success of localization in the non-perturbative evaluation of path integrals of gauge theories on compact manifolds has established the localization principle as an essential tool in quantum field theory. Despite this success, the conventional supersymmetric localization prescription fails to capture the necessary contribution of unstable instantons to the path integral of two-dimensional pure Yang-Mills (YM2) theories in the A-model, that is, topologically A-twisted $\mathcal{N} = (2, 2)$ supersymmetric gauge theory on a compact Riemannian manifold. At least two alternative approaches to localization remedy this failure, one of which invokes the Jeffrey-Kirwan residue theorem, the other of which draws from the original proof of non-abelian localization. This research aims to derive, from the failures and successes of different approaches to localization of YM2 theories in the A-model, more general insights about non-perturbative phenomena in gauge theories and the scope of supersymmetric localization.

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Chapter 1

Introduction

Quantum field theory (QFT) is a framework in which elementary particles are described as excitations of underlying quantum fields. In contrast to other scientific disciplines, QFT has no canonical definition but is instead a collection of physical and mathematical principles that unify quantum mechanics, special relativity, and classical field theory.

To date, a quantum field theoretic description has been provided for three of the four known fundamental interactions, or forces, occurring between elementary particles in nature – these are the electromagnetic force, the weak force, and the strong force. For instance, Quantum Electrodynamics (QED) describes the electromagnetic force, Electroweak (EW) theory describes both the electromagnetic and weak forces, and Quantum Chromodynamics (QCD) describes the strong force. Together, EW theory and QCD form the Standard Model of elementary particles (SM).

The experimental confirmation of the remarkably accurate predictions of the SM has established QFT as the most successful approach to understanding the physics of elementary particles and their interactions. This leaves the fourth fundamental force, gravitation, outside the scope of QFT. Although the classical description of gravity was established by General Relativity (GR), a consistent and experimentally verifiable quantum theory of gravity remains unknown. An overarching goal of contemporary physics is to formulate a quantum theory that describes all four fundamental interactions, which necessitates the unification of QFT with a quantum theory of gravity.

QED, EW theory, and QCD are all examples of particular types of QFTs called quantum gauge theories. A gauge theory is a field theory with a Lagrangian that is invariant under local transformations, called gauge transformations. Together, the gauge transformations form a Lie group, called the gauge group. The notion of gauge theories predates the development of QFTs, since the earliest example of a gauge theory is the classical theory of electromagnetism. In particular, the

Lagrangian of classical electromagnetism is invariant under gauge transformations. In this case, the gauge transformations form an abelian (commuting) gauge group. Not surprisingly, the quantum theory of electromagnetism, QED, is also an abelian gauge theory. QCD, on the other hand, is a non-abelian gauge theory. In this case, the gauge transformations form a non-abelian (non-commuting) gauge group.

Gauge theories serve as a bridge between physics and mathematics, as several objects and notions from mathematics have a formulation in terms of gauge theory and vice versa. From the physics perspective, the interest in gauge theories stems from the fact that QED, QCD, and the SM are examples of gauge theories occurring in nature. From the perspective of mathematics, gauge theories are of interest due to their utility in geometry and topology.

QCD is a physical example of a particular type of non-abelian gauge theory called four-dimensional Yang-Mills (YM4). Generally, Yang-Mills theories are theories whose action functional includes a kinetic term for the gauge fields called the Yang-Mills term. When a Yang-Mills theory excludes matter fields, it is called a pure Yang-Mills theory. Since QCD includes matter (e.g. quarks), it is not an example of pure YM4. Other YM theories have been studied in physical and non-physical settings and have played a crucial role in developing QFT. For instance, YM theories have been investigated on spacetime manifolds of different dimensionality and signature, with and without curvature, in cases with and without supersymmetry. The most important contemporary example of a non-physical YM theory appears in the AdS/CFT correspondence [1]. In its strongest form, the AdS/CFT correspondence relates $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory on $d = 3 + 1$ to type IIB superstring theory on $AdS_5 \times S^5$. By relating gauge theories to gravitational theories, the AdS/CFT correspondence constitutes significant progress in formulating a theory involving all four fundamental interactions.

Observations in physical QFTs typically involve complicated phenomena mediated by subtle underlying mechanisms. To gain insight into physical QFTs, it is often helpful to consider simpler toy models. In the case of YM4, a particularly useful toy model is two-dimensional Yang-Mills theory (YM2). On the one hand, YM4 and YM2 share several key features. On the other hand, YM2 is significantly simpler than YM4 and provides a more tractable and mathematically controlled setting to investigate shared features.

The most crucial distinguishing property is that YM4 has propagating degrees of freedom (gluons) while YM2 does not. Another difference between YM4 and YM2 is the metric dependence of the action. On a compact closed two-dimensional Riemannian manifold Σ with metric $g_{\mu\nu}$, the YM2 action only depends on the metric through the Riemannian measure $d\mu = \sqrt{\det g_{\mu\nu}} d^2x$. Here, x^μ are coordinates on Σ and $\mu = 1, 2$ are Euclidean spacetime indices. Remarkably, the partition function of pure YM2 on Σ depends only on the dimensionless combination $\varepsilon = A_\Sigma e^2$, where A_Σ is the area of Σ, and e^2 is the coupling constant. Moreover, the small ε limit of YM2 is a topological quantum field theory (TQFT) called BF-theory, whose partition function computes topological invariants. It is possible to formulate several variants of YM2 by introducing additional auxiliary fields and additional symmetry. For instance, the auxiliary fields may be bosons, fermions, or both, and the symmetry may be BRST symmetry or supersymmetry.

Despite the differences, both YM2 and YM4 exhibit instantons, i.e., finite-action extrema of the classical Euclidean action. Instantons are non-perturbative configurations appearing in a variety of theories and dimensions. For instance, YM4 permits instantons in the guise of self-dual connections saturating the BPS bound. In this case, the instanton configurations are topologically stable minima, and the instanton number is the second Chern class. In QCD, instantons appear in the U(1) problem. Specifically, the anomalous axial U(1) symmetry is broken by the presence of instanton sectors. Three-dimensional theories permit instantons in the guise of 't Hooft-Polyakov monopoles. In one-dimensional Quantum Mechanics, instanton configurations mediate tunneling phenomena.

The instanton configurations of YM2 on a compact manifold Σ are known, sophisticated, and classified by two types of G-connections $A = A_\mu dx^\mu$ that can intertwine. The first type is flat connections, and the second type is Yang-Mills connections, also called GNO connections. Flat connections are zero-action solutions of the classical Euclidean equations of motion, that may be regarded as pseudo-vacua. Equivalently, flat connections are solutions of the flatness equation $F = 0$, where F is the curvature of A. On the other hand, Yang-Mills connections are finite-action solutions of the classical Euclidean equations of motion. Equivalently, Yang-Mills connections are solutions of the Yang-Mills equation $D * F = 0$, where $*$ is the Hodge star, and D is the gauge-covariant derivative. If A is a Yang-Mills connection, its flux $\frac{1}{2\pi}\int_\Sigma F$ is GNO quantized, whereas if A is not a Yang-Mills connection, its flux is not GNO quantized. A GNO monopole is a saddle-point configuration specified by a particular flux and topological charge. GNO monopoles play the role of instantons in YM2 on Σ. GNO monopoles are unstable configurations, except when the fundamental group of the gauge group is non-trivial, in which case, GNO monopoles are stable configurations that generalize the Dirac monopole.

In the path integral formulation of QFT, observables are computed by evaluating path integrals. Path integration is typically non-trivial due to the lack of a mathematically precise framework. For instance, the domain of integration is the space of all field configurations, integration measures are infinite dimensional, the weight functions are oscillatory, and path integrals often suffer from ultraviolet (UV) and infrared (IR) divergences.

The most well-established path integration technique is perturbation theory. In perturbation theory, path integrals are evaluated approximately using power-series expansions about a small parameter in the QFT. When the QFT is weakly-coupled, the coupling constant of the theory is often a convenient choice for the small parameter about which to expand. For this reason, perturbation theory has been remarkably successful in studying weakly-coupled QFTs. However, when a QFT has no appropriately small parameter about which to expand, it falls outside the scope of perturbative analysis. This shortcoming of perturbation theory necessitated the development of alternative path integration techniques to study non-perturbative phenomena in QFTs.

Currently, there is a collection of non-perturbative path integration techniques, and the applicability of each technique depends on the particular QFT. For instance, non-perturbative techniques have been developed for QFTs with and without supersymmetry, on spacetime manifolds with and without curvature, in dimensions $d = 2, 3, 4, 5$. Examples of non-perturbative techniques include integrability, holography, lattice methods, and localization. Integrability utilizes a hidden enhancement of symmetry in the QFT. Holography leverages the AdS/CFT correspondence to map the QFT computation to a more straightforward gravitational setting. Lattice methods utilize a discrete spacetime to simplify the path integral. Localization utilizes equivariant integration to reduce path integrals of QFTs possessing a fermionic (Grassmann-odd) symmetry.

In particular, localization is a technique to reduce infinite-dimensional path integrals of QFTs exhibiting fermionic symmetries to lower-dimensional integrals. The idea is that, in favorable situations, the path integral only receives contributions from the fixed points of the fermionic symmetry $\mathcal{Q}$ due to the rules of fermionic (Berezin) integration. Localization reduces the path integration domain to a lower dimensional locus of $\mathcal{Q}$-invariant field configurations. To localize a path integral, the fermionic symmetry, the integration measure, and the action must all satisfy specific and well-established criteria. To illustrate, suppose that the situation is favorable and that we would like to compute a partition function by localizing its supersymmetric path integral. Note that the illustration here follows the exposition in [2]. The partition function may be computed by constructing a deformed partition function involving a deformed action

$$Z(t) = \int_{\mathcal{F}} DX e^{-S[X] - t\mathcal{Q}V_F[X]}. \tag{1.0.1}$$

Here, $Z(0)$ is the original partition function, $\mathcal{F}$ is the configuration space of fields, DX is the measure on $\mathcal{F}$, X are the fields, S is the action functional, t is a real positive parameter, $\mathcal{Q}$ is the localizing supercharge, V_F is a fermionic functional, and the bosonic functional $\mathcal{Q}V_F$ is called the localizing scheme, localizing term, or localizing action.

If the necessary criteria are satisfied, the partition function is independent of the deformation, and $\frac{d}{dt}Z(t) = 0$. Then, the original partition function is computed as

$$Z(0) = \lim_{t\to\infty} Z(t) = \int dX_0 Z_{\text{cl}}[X_0] Z_{\text{1-loop}}[X_0], \tag{1.0.2}$$

where X_0 are moduli, Z_{cl} is the classical contribution, and $Z_{\text{1-loop}}$ is the one-loop contribution, The moduli, or zero-modes, are bosonic coordinates on the localization locus of $\mathcal{Q}$-invariant field configurations. In the best-case scenarios, localization reduces path integrals to zero-dimensional integrals over matrix models, and path integration results in an exact analytical expression. In contrast, the result of perturbative path integration is always an approximate expression to some order in the power series expansion.

Localization has been a remarkably successful method to evaluate path integrals of supersymmetric gauge theories on compact manifolds. However, the localization principle was first described in the mathematical context of equivariant cohomology. The foundation of localization in the context of gauge theories was established in the seminal paper [3], in which Witten introduced non-abelian localization. In doing so, the localization principle of equivariant cohomology was incorporated into the methodology available to quantum field theory. In particular, non-abelian localization is the generalization of the Duistermaat-Heckman localization formula, by which finite-dimensional integrals are evaluated, to the case of infinite-dimensional functional integrals in gauge theories. The particular gauge theory whose path integral was localized was a cohomological variant of pure YM2 theory on a compact Riemannian manifold, which we will call cohomological YM2. Cohomological YM2 is a variant of conventional YM2 in a similar sense to how supersymmetric Yang-Mills theories are variants of conventional Yang-Mills.

A key result of [3] is the non-abelian localization of the cohomological YM2 path integral using two separate localization schemes. One of the localization schemes localizes the path integral to the moduli space of flat G-connections, and the other localization scheme localizes the path integral to the moduli space of Yang-Mills G-connections. The moduli space of Yang-Mills connections may also be referred to as the instanton moduli space. Another significant result is the map between cohomological YM2 and "physical" YM2. The map was established by recovering the physical YM2 partition function from the expectation value of a cohomological operator evaluated by localization.

The localization of cohomological YM2 builds on earlier results from [4], where the partition function of physical YM2 was first evaluated using lattice techniques, then reevaluated using techniques from topological quantum field theory (TQFT). The result of the lattice and TQFT approaches was an analytical expression, in which the partition function of physical YM2 is described as a sum over irreducible representations of the gauge group. On the other hand, the localization approach resulted in an analytical expression in which the partition function of physical YM2 is described as a sum over instantons. The Poisson summation formula relates the two descriptions of the physical YM2 partition function.

A key result of [4] was the evaluation of the symplectic volume of the moduli space of flat G-connections, which was extended in [3] to include general expressions for the intersection numbers of the moduli space of flat G-connections. Similar considerations date back to [5], in which the partition function of YM2 was evaluated using lattice techniques.

The Jeffrey-Kirwan (JK) residue theorem was introduced in [6] and used to give an alternative proof to Witten's non-abelian localization formula [3]. The JK residue technique was further developed in [7, 8] and evaluates integrals over meromorphic functions via a systematic choice of integration contour.

In [9], Witten studied a variant of cohomological YM2 in the A-model, which is the topologically A-twisted $\mathcal{N} = (2,2)$ supersymmetric gauged theory of vector and chiral multiplets. In particular, the standard multiplet of cohomological YM2 is related to the vector multiplet of the A-model by field redefinitions. Moreover, the standard cohomological multiplet is acted upon by an odd scalar BRST-like charge Q_{BRST}, the A-model vector multiplet is acted upon by an odd scalar supercharge Q_A, and the odd symmetries Q_A and Q_{BRST} exhibit similar properties. These considerations were extended in [10] to compute the instanton expansion of A-model correlators for the case in which the target space of a sigma model was a toric variety or Calabi-Yau hypersurface. This study constitutes a critical preliminary investigation of the instanton moduli space of the A-model.

The topological A-twist was introduced in [11] as a method to preserve half of the supercharges of the $\mathcal{N} = (2,2)$ supersymmetry algebra on two-dimensional manifolds with curvature. The A-twist permits the definition of curved space topological field theory actions preserving half of the $\mathcal{N} = (2,2)$ supercharges. Since the introduction of the A-twist, more sophisticated techniques have been developed to preserve supersymmetry on manifolds with curvature. For instance, the Ω-deformation generalizes the A-twist by permitting a U(1)-equivariant deformation of the A-twisted $\mathcal{N} = (2,2)$ algebra [12].

In [13], path integrals of $\mathcal{N} = (2,2)$ supersymmetric gauge theories were computed on S^2 using supersymmetric localization. Results include analytic expressions for partition functions and observables for $\mathcal{N} = (2,2)$ theories on S^2, but do not incorporate the A-model. These results constitute the first modern supersymmetric localization computation in two dimensions, following the foundational works of [14, 15].

Path integrals of $\mathcal{N} = 2$ supersymmetric gauge theories were computed on T^2 for rank-one gauge groups in [16], and considerations were extended to higher-rank gauge groups in [17]. Results include analytical expressions for partition functions and observables, but they do not incorporate the A-model. These results were achieved using a novel path integration technique that combines supersymmetric localization with the JK residue theorem. In particular, aspects of localization were utilized to reduce path integrals to contour integrals, which were then evaluated in terms of JK residues by choosing the integration contour according to the JK residue theorem. We will refer to this novel path integration technique as "JK-aided localization".

In [18], Benini and Zaffaroni computed path integrals of A-twisted and Ω-deformed $\mathcal{N} = (2, 2)$, $\mathcal{N} = 2$, $\mathcal{N} = 1$ supersymmetric gauge theories on S^2, $S^2 \times S^1$, $S^2 \times T^2$, respectively. In [19], the same authors extended the results to incorporate A-twisted and Ω-deformed theories on Σ_g,$\Sigma_g \times S^1$,$\Sigma_g \times T^2$, where Σ_g is a two-dimensional genus g Riemannian manifold. In [20], Closset, Cremonesi, and Park used similar techniques to evaluate path integrals of A-twisted and Ω-deformed $\mathcal{N} = (2, 2)$ supersymmetric gauge theories on S^2. In all cases, the A-twisted path integrals were evaluated using the JK-aided localization technique. The results include analytic expressions for partition functions and observables, as well as precision tests of dualities in two, three, and four dimensions. Importantly, all the cases involving A-twisted $\mathcal{N} = (2, 2)$ theories are examples of path integrals in the A-model being evaluated using JK-aided localization. These examples constitute a modern study of the instanton moduli space in the A-model.

The A-model presents several new challenges in the context of supersymmetric localization. The primary challenge is that the conventional (semi-canonical) prescription for localizing supersymmetric path integrals yields the incorrect result when applied to path integrals in the A-model. There are, however, at least two alternative approaches to A-model localization that remedy the shortcomings of the conventional localization prescription. Benini, Zaffaroni, and others developed one of these alternative approaches in [18, 19, 20], while Witten developed the other alternative approach in [3].

To describe the challenges encountered when attempting to localize path integrals in the A-model, we will compare the conventional prescription to the two alternative approaches. In doing so, we will refer to the conventional prescription for how to localize supersymmetric path integrals as the "follow-your-nose" (FYN) approach; we will refer to the alternative approach developed in [18, 19, 20] as the Benini-Zaffaroni (BZ) approach, and we will refer to the alternative approach developed in

[3] as the Witten approach.

Let us begin by comparing the FYN localization of a supersymmetric path integral to the saddle point approximation of a non-supersymmetric path integral. In the saddle point approximation, the path integral receives contributions from saddle point configurations, which are finite-action ($S \geq 0$) solutions of the equations of motion. In FYN localization, the supersymmetric path integral receives contributions from BPS configurations, which are zero-action ($S = 0$) solutions of the equations of motion. Evaluating a supersymmetric path integral using FYN localization is similar to a 1-loop exact saddle point approximation. The critical difference, however, is that the path integral contribution of non-BPS saddle point configurations is omitted in FYN localization and retained in the saddle point approximation. In other words, the saddle point approximation retains the contribution of all saddle points, while FYN localization retains BPS saddle points and omits non-BPS saddle points.

An alternative description of FYN localization goes as follows. Supersymmetric path integrals, evaluated using FYN localization, only receive contributions from the fixed-points of the supersymmetry. This description emphasizes that the BPS configurations form the fixed-point locus for the action of the localizing supercharge on the space of all field configurations. Regardless of the description, the result is the same: FYN localization reduces supersymmetric path integrals to lower-dimensional integrals (or sums) over BPS moduli. The BPS moduli are coordinates on the locus of BPS configurations. If a modulus is discrete, its "integral" is a sum, and if a modulus is continuous, its integral is simply an integral.

In the A-model, FYN localization fails because the assumption that the path integral only receives contributions from BPS configurations is incorrect. Evidence shows that A-model path integrals receive dramatic contributions from non-BPS saddle point configurations. For instance, the A-model Yang-Mills partition function has a description as a sum over a discrete non-BPS modulus (the GNO quantized gauge flux), which requires path integral contributions from non-BPS saddle point configurations (Yang-Mills connections). These non-BPS saddle point configurations play the role of unstable instantons of Yang-Mills in the A-model and are not accounted for by FYN localization. The alternative approaches to A-model localization remedy the failure of the FYN approach by capturing the contribution of unstable instantons to the path integral. Moreover, the path integral contribution of unstable instantons is included in the saddle point approximation but excluded in FYN localization.

Unstable instantons are not unique to Yang-Mills in the A-model but are known to contribute to the non-supersymmetric path integral of pure YM2, as noted previously. In particular, the fact that the A-model instantons are unstable configurations is a fact from pure YM2. Usually, instantons are stable configurations in the sense that, starting from the instanton configuration, it is not possible to use a continuous deformation to lower the action. For instance, YM4 permits instantons that are stable configurations. What is striking about pure YM2 is not that the saddle-point approximation captures the contribution of unstable instantons to the path integral but that the entire YM2 partition function has a description as a sum over specifically those unstable instantons.

To further illustrate the differences between the FYN, BZ, and Witten approaches to A-model localization, we will consider localizing the Euclidean path integral of the standard supersymmetric Yang-Mills Lagrangian L_{YM} for the A-model on a two-sphere. Before proceeding, let us record a few details. Observe that i) the A-model on S^2 is the topologically A-twisted $\mathcal{N} = (2,2)$ supersymmetric gauge theory of vector and chiral multiplets on S^2; ii) the fields in the A-model vector multiplet in Wess-Zumino (WZ) gauge are $\mathcal{V} = (A_\mu, \sigma, \tilde{\sigma}, D, \Lambda_\mu, \lambda, \tilde{\lambda})$ for $\mu = 1,2$ where A_μ is the gauge field, $\sigma, \tilde{\sigma}$ are complex bosonic scalars, D is the bosonic scalar auxiliary field, Λ_μ is the fermionic vector field, and $\lambda, \tilde{\lambda}$ are fermionic scalars; iii) the fields in $\mathcal{V}$ are all valued in the adjoint representation of the complexification of the Lie algebra of the gauge group, and are treated as generically complex and independent in Euclidean signature; iv) L_{YM} is a functional of the fields in $\mathcal{V}$; v) the localizing supercharge is the scalar A-model supercharge $Q_A = Q + \widetilde{Q}$; and vi) the action of L_{YM} is Q_A-exact: $S_{\text{YM}} = Q_A V_{\text{YM}}$.

We begin by describing the unsuccessful FYN approach to A-model localization. In this approach, one localizes to the locus of BPS configurations by setting to zero the real bosonic part of the localizing Lagrangian L_{YM}, and imposing the "real contour". The BPS configurations are the bosonic field configurations that set both the fermionic supersymmetry variations and the action to zero along the real contour. In particular, the BPS configurations are the $A_\mu, \sigma, \tilde{\sigma}, D$ solutions of the equations $0 = Q_A \Lambda_\mu = Q_A \lambda = Q_A \tilde{\lambda}$ constrained by the real contour $A_\mu^\dagger = A_\mu$, $\sigma^\dagger = \tilde{\sigma}$, $D^\dagger = D$, where $\dagger$ is hermitian conjugation. On the BPS locus, we have

$$0 = F_{12},\ 0 = D, \tag{1.0.3}$$

where F_{12} is the scalar gauge field strength appearing quadratically in L_{YM}. In this case, the gauge fields are localized to the locus of flat connections; that is, the locus of BPS configurations includes the locus of flat connections $\{A_\mu | F_{12} = 0\}$. Localizing to the locus of flat connections excludes the possibility of Yang-Mills connections $\{A_\mu | D_\nu F_{12} = 0\}$ as well as the GNO quantization of the gauge flux. Following the FYN approach through to the end, the contributions received by the path integral will be incorrect since the necessary Yang-Mills connections have already been omitted.

Next, we describe the BZ approach to A-model localization, which is known to remedy the shortcomings of the FYN approach successfully. In this approach, one localizes to the locus of "almost-BPS" configurations by setting to zero the real bosonic part of the localizing Lagrangian $L_{\rm YM}$, and imposing the real contour on all fields except the auxiliary field D. Specifically, the almost-BPS configurations are the $A_\mu, \sigma, \tilde{\sigma}, D$ solutions of the equations $0 = Q_A \Lambda_\mu = Q_A \lambda = Q_A \tilde{\lambda}$, for $A_\mu^\dagger = A_\mu$, $\sigma^\dagger = \tilde{\sigma}$ and generically complex D. On the BZ locus, we have

$$0 = iF_{12} - D, \tag{1.0.4}$$

for real F_{12} and complex D. In this case, the locus of almost-BPS configurations includes configurations $\{A_\mu | D = iF_{12}\}$ that permit Yang-Mills connections when the auxiliary field is covariantly constant. In particular, $D = iF_{12}$ together with $D_\nu D = 0$ permits Yang-Mills connections $\{A_\mu | D_\nu F_{12} = 0\}$, or equivalently, the GNO quantization of the gauge flux.

Following the BZ approach through to the end, the path integral is reduced to lower-dimensional integrals over two moduli. One of the moduli is the discrete GNO quantized gauge flux, and its "integral" is a sum. The other modulus is the vacuum expectation value (vev) of the complex bosonic scalar σ in $\mathcal{V}$. This modulus is continuous, complex, and its integral is a contour integral. To evaluate the contour integral correctly, the contour is chosen according to the Jeffrey-Kirwan residue theorem.

Before proceeding, let us make a few observations regarding the BZ approach. The almost-BPS locus is a superspace in the sense that it has both bosonic and fermionic directions, and the bosonic subspace of the locus is singular and non–compact. Together, the almost-BPS modes form an off-shell supersymmetry multiplet. The almost-BPS and BPS loci differ due to the complex auxiliary field D: the almost-BPS locus has $D \neq 0$ while the BPS locus has $D = 0$. Sending the value of D to zero in the almost BPS locus, one recovers the BPS locus. The nonzero complex auxiliary field plays at least three important roles. First, D is necessary to ensure the GNO quantization of the gauge flux. Second, D is necessary to close the off-shell supersymmetry multiplet of almost-BPS modes. Third, D is used to regulate bosonic singularities on the almost-BPS locus, and to reduce the non-compact integral over the vev of σ to a contour integral.

Now, we describe the Witten approach to A-model localization, which we expect to remedy the shortcomings of the FYN approach successfully. Although the Witten approach was originally outlined in the context of non-abelian localization of path integrals of two-dimensional cohomological gauge theories, it may be realized in the supersymmetric localization of A-model path integrals. More concretely, since the A-model vector multiplet $\mathcal{V}$ and the standard cohomological multiplet are related by field redefinitions, one may construct A-model localizing terms that are equivalent to localizing terms in the cohomological theory. Moreover, the established cohomological localization procedure is a benchmark for checking our A-model localization computation.

The Witten approach amounts to choosing the A-model localizing term to be the same as the particular "cohomological" localizing term that resulted in the correct contributions to cohomological path integrals. In particular, the A-model localizing term is a Q_A-exact linear combination of a non-standard Yang-Mills Lagrangian and a quadratic twisted chiral superpotential $L_{\text{loc}} = L_{\text{YM}}^{\text{ns}} + L_{\text{t.c.s.}}^{\text{quad}}$. In this case, one localizes to a "Witten" locus by setting to zero the real bosonic part of L_{loc} and imposing the real contour. On the Witten locus, we have

$$0 = F_{12} - it\tilde{\sigma},\ 0 = D_\mu\tilde{\sigma} \tag{1.0.5}$$

for real F_{12}, t and imaginary $\tilde{\sigma}$. Importantly, the Witten locus permits Yang-Mills connections $\{A_\mu | D_\nu F_{12} = 0\}$, since $F_{12} = it\tilde{\sigma}$ and $D_\mu\tilde{\sigma} = 0$. Following the Witten approach to the end, we expect the A-model path integral to receive contributions correctly and that the final result is lower-dimensional integrals over two moduli. One modulus should be the discrete GNO quantized gauge flux, and the other modulus should be the continuous vev of σ.

Let us summarize. Localization of path integrals in the A-model has been investigated in numerous studies. Although supersymmetric localization is always a subtle matter, the typical expectation was that the FYN approach succeeds in most situations and that it is possible to understand which path integral contributions to retain or omit after constructing the localizing term. In the case of the A-model, however, the FYN approach to localization is known to fail. In principle, the failure of the FYN approach is understood, and may be remedied by either the BZ or Witten approaches. Despite this, current explanations for why the FYN approach fails and how the BZ or Witten approaches rectify these shortcomings remain vague. Although the BZ and Witten approaches both successfully capture the correct contributions to A-model path integrals, the two approaches achieve this in significantly different ways. More concretely, we currently have no precise understanding of how the BZ and Witten approaches fix the FYN approach to A-model localization or why it is appropriate to retain the contribution of unstable instantons to A-model path integrals.

Classic investigations employed the Witten approach in the context of two-dimensional cohomological gauge theories, to prove the non-abelian localization theorem, to study the geometric properties of the moduli space of flat G-connections, and to map cohomological YM2 to physical YM2. Although these investigations outlined the Witten approach to localization, little emphasis was placed on explaining why the Witten localizing term is the correct choice and how this localizing term results in the correct contributions to the path integral.

These considerations present a problem when attempting to extend the Witten approach to other supersymmetric localization computations. In particular, the Witten localizing term may be constructed using the fields of the A-model vector multiplet in WZ gauge, but the construction of a Witten-type localizing term using other supersymmetry multiplets remains unclear. As a result, we are ill-equipped to employ the Witten approach in other scenarios where supersymmetric path integrals receive contributions from non-BPS saddle point configurations.

Another issue is the current explanation of how the Witten approach succeeds. Specifically, the approach succeeds because the localizing term localizes to a locus that permits Yang-Mills connections to “flow in from infinity” in field space. Although this explanation is undoubtedly true, it lacks detail.

Modern investigations employed the BZ approach to evaluate observables, test dualities, and derive general formulae for partition function in terms of integrals over moduli. However, these investigations focused on the results obtained using the BZ approach rather than explaining how and why the BZ approach succeeds or clarifying the scope of its applicability.

One issue is the current explanation of how the BZ approach succeeds and the necessary conditions for this success. At the level of the localization locus, the BZ approach succeeds because the almost-BPS configuration is on the complexified gauge orbit of the auxiliary zero-mode originating from the BPS configuration. The necessary conditions for this to happen are that the auxiliary field is permitted to remain generically complex and that the gauge flux is GNO quantized. If, for instance, the gauge flux is not GNO quantized, then the BZ approach fails because the gauge orbit of the auxiliary zero-mode originating from the BPS configuration never intersects the almost-BPS configuration. Moreover, the gauge flux is only GNO quantized when Yang-Mills connections exist. Even though the explanation of how the BZ approach succeeds appears valid, the motivation for the conditions on the gauge and auxiliary fields needs to be more specific.

Another problem is the scope of the BZ approach. It remains unclear when to employ the BZ approach instead of the FYN or Witten approaches, and when to expect the BZ approach to fail. Consequently, we are ill-equipped to generalize the BZ approach to cases other than the A-twisted theories already considered.

A third issue is the reconciliation of the BZ and Witten approaches to A-model localization. On the one hand, both the BZ and Witten approaches result in the path integral receiving contributions from non-BPS saddle point configurations correctly. On the other hand, the BZ and Witten approaches accomplish this in considerably different manners. For one thing, the localization loci in the two approaches differ significantly. The BZ approach involves specific choices at different stages in the localization computation. For instance, the localizing limit is a double scaling limit, there are two types of singularities in the moduli space that must be regulated differently, and the final contour integral must be evaluated according to the JK residue theorem. In contrast, the Witten approach involves a specific localizing term, and it is not clear how this compares to the BZ approach. For example, the Witten approach does not require the JK residue theorem. Interestingly, the JK residue theorem was used to give an alternative proof of the non-abelian localization theorem, and the first proof of non-abelian localization employed the Witten approach.

Given the lack of research in two-dimensional Yang-Mills theories, the A-model, and localization, this thesis aims to evaluate the FYN, BZ, and Witten approaches to supersymmetric localization of path integrals in the A-model.

The first objective is to confirm the map between physical YM2 and supersymmetric YM2 in the A-model. This will be achieved by recovering the physical YM2 partition function from the A-model partition function formula derived using the BZ approach to localization. This constitutes the A-model realization of the map between physical YM2 and cohomological YM2. The map was originally described by recovering the physical YM2 partition function from the expectation value of a cohomological operator, computed using the Witten approach to localization.

The second objective is to compare and contrast the FYN, BZ, and Witten approaches to A-model localization. This will be achieved by localizing the path integral A-model using each of the three approaches and comparing them at the level of the one-loop fluctuation determinant. In order to gain a microscopic understanding of each of the three approaches, the fluctuation determinants will be evaluated mode-by-mode. First, we aim to identify the problematic zero-modes in the singular FYN determinant, which cause the FYN approach to fail. Second, we aim to compare the mode-by-mode analysis of the FYN determinant to the same analyses of the BZ and Witten determinants. We expect the problematic zero-modes in the FYN determinant to be lifted in the non-singular Witten determinant. By appending coefficients to terms in a general localizing Lagrangian, we expect to be able to interpolate between the FYN, BZ, and Witten determinants.

The third objective is to use the mode-by-mode analysis of fluctuation determinants to develop criteria for when supersymmetric localization is expected to fail. The aim is to establish the criteria at the level of the target space or the supersymmetry algebra rather than at the level of the localizing term. Such criteria would contribute to supersymmetric localization by clarifying how to avoid bad localizing terms and how to identify dangerous field configurations.

Outline of thesis

Chapter 2 reviews the general features of modern supersymmetric localization and outlines the follow-your-nose (FYN) approach to localization. To begin, we introduce the two supersymmetric localization arguments. The first argument is the less common TQFT argument described in [21]. The second argument is the more common deformation invariance argument, described in, e.g., [2]. After that, we review the criteria for successful localization by deformation. By dropping one criterion, we describe how localization by deformation fails in the A-model. Following this, we describe the follow-your-nose approach to choosing the localization scheme and evaluating the localization locus. After describing the parametrization of fields on the localization locus, we describe the functional Taylor expansion of the deformed action and the decomposition of the path integral into classical and one-loop contributions. We compare localization to effective field theory and sketch the evaluation of the classical and one-loop contributions to the path integral. Finally, we state the final result of localization in terms of integrals over moduli.

Chapter 3 reviews two-dimensional Yang-Mills theories and non-abelian localization. In section 3.1, we introduce two-dimensional pure Yang-Mills theory in the mathematical context of G-connections on fiber bundles, then in the physics context of gauge fields. After describing the features of YM2, we explain how YM2 partition functions can differ by a renormalization ambiguity. Following this, we recount the procedure to obtain variants of YM2 by introducing auxiliary fields and a fermionic BRST-like charge Q and detail how to recover the original partition function defining YM2 from each variant. In particular, we describe the basic, ghost, projection, and standard Q-multiplets of cohomological YM2. Finally, we construct two localizing terms for non-abelian localization from the fields of the standard multiplet. Section 3.2 reviews the analytical evaluation of the YM2 partition function using the lattice gauge theory approach, following the exposition in [4].

Section 3.3 reviews the non-abelian localization of two-dimensional cohomological gauge theories, following the exposition in [3]. We outline the localization procedure, the two localizing terms, and their respective loci. One locus involves flat connections; the other locus involves Yang-Mills connections. In section 3.3.1, we review localizing to the locus of flat connections using the first localizing term. We describe reality conditions for fields and the evaluation of the moduli space. In section 3.3.2, we review localizing to the locus of Yang-Mills connections using the second localizing term. We describe reality conditions for fields and the evaluation of the moduli space. We explain the novelty of the second localizing term and its relationship to the first.

In section 3.3.3, we review the evaluation of the path integral after localizing to the locus of Yang-Mills connections using the second localizing term. We supplement the analysis of [3] by providing technical details and references for evaluating specific integrals. Section 3.4 reviews the relationship between physical YM2 and cohomological YM2. We describe how the expectation value of an operator in cohomological YM2, computed via localization, may be related to the partition function of physical YM2 using the Poisson summation formula.

Chapter 4 reviews the supersymmetric localization of A-twisted gauge theories on compact manifolds, following the exposition in [18, 19]. In section 4.1, we summarize the BZ approach to A-twisted localization. We collect results derived using the BZ approach, including general formulae for A-twisted partition functions and correlators.

Section 4.2 reviews the topological A-twist and the resulting A-twisted theories. Section 4.2.1 reviews the three-dimensional A-twisted $\mathcal{N} = 2$ supersymmetric theories. Section 4.2.2 reviews the two-dimensional A-twisted $\mathcal{N} = (2, 2)$ supersymmetric theories, i.e., the A-model, and details the vector multiplet.

Section 4.3 reviews the BZ approach to localizing the path integral of the A-model vector multiplet. Section 4.3.1 reviews localizing to the BPS locus following the FYN approach. Section 4.3.2 reviews localizing to the almost-BPS locus following the BZ approach, details the necessary conditions, and compares the BPS and almost-BPS loci. Section 4.3.3 reviews the almost-BPS zero modes and the off-shell supermultiplet of zero modes. Section 4.3.4 reviews the bosonic moduli space in the BZ approach and details the singular hyperplanes in two and three dimensions. Section 4.3.5 records the regulated one-loop determinants in the BZ approach for vector and chiral multiplets in two and three dimensions.

Section 4.4 reviews the asymptotic behavior of chiral one-loop determinants and provides details regarding regularization in two and three dimensions. Section 4.4.1 details the three-dimensional effective Chern-Simons coupling. Section 4.4.2 details the two-dimensional effective twisted chiral superpotential.

Section 4.5 reviews the evaluation of integrals over zero modes. Section 4.5.1 outlines the treatment of singularities in the bosonic moduli space. Section 4.5.2 details the evaluation of integrals over three-dimensional bosonic zero modes and the double-scaling localization limit. Section 4.5.3 reviews the evaluation of the integrals over fermionic zero modes for A-twisted theories on $S^2 \times S^1$, and details the role of the off-shell supersymmetry multiplet of zero modes. Section 4.5.4 reviews the evaluation of the integrals over fermionic zero modes for A-twisted theories on $\Sigma_g \times S^1$. Section 4.5.5 reviews the evaluation of the final contour integral over the continuous bosonic modulus in terms of Jeffrey-Kirwan residues.

In chapter 5, we recover the analytic expression for the physical YM2 partition function from the general formulae for A-model correlators derived using the BZ approach to supersymmetric localization in [18, 19]. This chapter constitutes an original contribution and confirms that the A-model correlator formula realizes the map between physical YM2 and cohomological YM2. The representation theory appendix B supplements this chapter.

Chapter 6 compares the FYN, BZ, and Witten approaches to A-model localization by evaluating the one-loop fluctuation determinant mode-by-mode using each approach. This chapter constitutes an original contribution.

Section 6.1 describes the A-model vector multiplet on S2 and details the supersymmetry algebra and supersymmetric actions. Section 6.2 describes the construction of our general localizing term. The general localizing term includes coefficients that, when turned on or off, result in the BZ, FYN, and Witten localizing terms. We record the localizing supercharge and the localizing limit.

Section 6.3 describes localizing to the FYN, BZ, and Witten loci. We detail the evaluation of moduli for each locus. Section 6.4 describes the parametrization of fields in terms of zero modes and fluctuating modes and details the locus expansion resulting in a localizing Lagrangian to quadratic order in fluctuations.

Section 6.5 describes the gauge-fixing procedure for the localizing Lagrangian. Section 6.6 describes the Cartan-Weyl basis decomposition of fluctuating modes and details how to express the Localizing Lagrangian in terms of matrices. Section 6.7 describes how to use monopole spherical harmonics to evaluate one-loop determinants from localizing Lagrangians expressed as matrices. We illustrate the procedure with two simple examples.

Section 6.8 describes integrating bosonic scalars using an integration contour novel to A-model localization. Instead of taking the bosonic scalars to be complex conjugates, we take one real and the other imaginary, resulting in a Lagrange multiplier.

Section 6.9 describes the mode-by-mode evaluation and analysis of 1-loop fluctuation determinants resulting from the FYN, BZ, and Witten approaches to A-model localization. We begin by detailing our conventions for monopole spherical harmonics.

Section 6.9.1 describes the evaluation of the fluctuation determinant in FYN and BZ approaches. This determinant results from choosing the localizing term to be the standard Yang-Mills Lagrangian. As a result, we identify two singular bosonic modes and one singular ghost mode.

Section 6.9.2 describes the evaluation of a fluctuation determinant intermediate between the Witten and FYN approaches. This determinant results from choosing the localizing term to be the standard Yang-Mills Lagrangian with quadratic twisted chiral superpotential. As a result, we obtain expressions for the determinant in terms of moduli and a coefficient tracing the twisted chiral superpotential.

Section 6.9.2.1 describes an extension to the case when the fermionic fluctuation operator is evaluated in terms of a Pfaffian instead of a determinant. Section 6.9.2.2 describes an extension of the fermionic Pfaffian in which the fermionic scalars in the A-model vector multiplet are redefined as the fermionic scalars in the standard cohomological multiplet.

Section 6.9.3 describes the evaluation of the fluctuation determinant in the Witten approach. This determinant results from choosing the localizing term to be a non-standard Yang-Mills Lagrangian with quadratic twisted chiral superpotential. As a result, we obtain expressions for the determinant in terms of moduli and a coefficient tracing the twisted chiral superpotential.

Section 6.9.4 describes the evaluation of a general fluctuation determinant intermediate between the BZ, FYN, and Witten approaches. This determinant results from choosing a general localizing term that retains coefficients that interpolate between BZ, FYN, and Witten localizing terms. As a result, we obtain expressions for the general determinant in terms of moduli and interpolating coefficients.

Section 6.9.5 describes the evaluation of an alternative fluctuation determinant that does not pertain to the BZ, FYN, and Witten approaches. This determinant results from choosing the localizing term to be an alternative non-standard Yang-Mills Lagrangian with quadratic twisted chiral superpotential. As a result, we obtain expressions for the alternative determinant in terms of moduli and a coefficient tracing the twisted chiral superpotential.

In chapter 7, we discuss the results, implications, and value of the results, suggest future directions and conclude the thesis.

Appendix A provides a general review of $\mathcal{N} = (2,2)$ supersymmetry and the A-model. In A.1, we provide conventions for spinors, coordinates, and gauge fields. In A.2, we review Lorentzian $\mathcal{N} = (2,2)$ supersymmetry following the exposition in [22], focusing on the vector multiplet in Wess-Zumino gauge. In A.3, we review Wick rotation and the Euclidean $\mathcal{N} = (2,2)$ supersymmetric vector multiplet. In A.4, we review the topological A-twist in $\mathcal{N} = (2,2)$ supersymmetry, and describe the A-model vector multiplet. In A.5, we detail the dictionary between the A-model vector multiplet and the standard cohomological multiplet. In A.6, we formulate our localizing term using the fields of the standard cohomological multiplet. This can be expressed in terms of the fields of the A-model vector multiplet using the dictionary. Specifically, we write the localizing term used by Benini and Zaffaroni in [18, 19] (the SYM action), and the two localizing terms used by Witten in [3] (the Donaldson action and the deformed Donaldson action).

In appendix B, we review the representation theory of Lie groups and Lie algebras. This section serves as the background for chapter 5, in which we recover the pure YM2 partition function from the Benini and Zaffaroni A-model partition function formula [18, 19]. In B.1, we review representations of Lie algebras and weights. In B.2, we review complexified Lie algebras and roots. In B.3, we review root systems, root diagrams, and the Weyl group. In B.4, we review the action of the Weyl group in terms of orbits, stabilizers, and fixed point sets. In B.5, we review covering groups, the fundamental group, and the relationship between Lie groups and Lie algebras. In B.6, we review lattices of Lie algebras and Lie groups and their relationship. In B.7, we prove some propositions concerning how elements of the character lattice behave under translations by the Weyl vector.

In appendix C, we collect details on A-model localization. This section serves as background material for chapter 6. In C.1, we provide the A-model vector multiplet we considered in the conventions of [20]. In C.2, we explain our conventions for monopole spherical harmonics. This appendix includes a table describing the existence of monopole harmonics, as well as the action of operators on monopole harmonics. We conclude the section by deriving a result of [20] using our conventions for monopole harmonics, thereby confirming their validity. In C.3, we provide details for the Cartan-Weyl basis decomposition procedure and give an example. In C.4, we provide a dictionary between the A-twisted vector multiplet in the conventions of [20] and the standard cohomological multiplet in the conventions of [3]. In C.5, we provide a partially completed dictionary between the A-twisted vector multiplet in the conventions of [18, 19] and the standard cohomological multiplet in the conventions of [3].

In appendix D, we provide some mathematical background for YM2. In particular, we recall the definitions of symplectic manifolds, symplectic forms, the Liouville form, Hamiltonian actions on symplectic manifolds, the moment map equation, Poisson brackets, and affine spaces. We record that the Riemann surface Σ is a symplectic manifold, that the space of G-connections $\mathcal{A}$ is both a symplectic manifold and an affine space, and that the action of the group of gauge transformations $\mathcal{G}$ on $\mathcal{A}$ is Hamiltonian. Finally, we recall that the Hodge star on the Riemann surface plays the role of an almost complex structure and is used to define the metric on the space of connections.

Original contributions

The primary original contribution of this thesis is the mode-by-mode evaluation of the 1-loop determinants of the bosonic, fermionic, and ghost fluctuation operators for three different localizing terms constructed from the two-dimensional A-model vector multiplet, using a novel contour of integration for the complex bosonic scalars in each case. The derivations and results are provided in chapter 6. The three localizing terms include the localizing term considered by Benini and Zaffaroni in [18, 19], and a localizing term resembling the deformed cohomological Donaldson action considered by Witten in [3]. In two cases, the 1-loop contribution of the fermionic fluctuation operator was evaluated in terms of a Pfaffian instead of a determinant. In the case of the FYN localizing term, the complete analysis and evaluation of the ratio of 1-loop determinants was performed and resulted in the Vandermonde determinant. This serves as a confirmation of known results and our novel choice of integration contour for the complex bosonic scalars. In order to compute 1-loop determinants mode-by-mode, it was necessary to understand and then use the monopole spherical harmonics provided in [20]. One of products of this research is presented in appendix C.2, where we have collected, and in some cases supplemented, the monopole spherical harmonics described in [20]. Moreover, it was necessary to write a dictionary to compare the localization terms in the A-model to those of cohomological gauge theory. As a result, we have provided two dictionaries between the standard cohomological multiplet in [3] and the A-model vector multiplet in the conventions of both [20] (appendix C.4) and [23] (appendix A.5).

A second original contribution is the recovery of the pure YM2 partition function, given as a sum over irreducible representations, from the A-model partition function formula of Benini and Zaffaroni, given as integrals over moduli. This serves as a confirmation of the A-model partition function formula. In doing so, it was necessary to prove some propositions about how translation by the Weyl vector affects different elements of the character lattice. The proofs are provided in Appendix B.7, and are valid for generic covering groups, including both the universal and adjoint covers.

Chapter 2

Supersymmetric localization

In this chapter, we review supersymmetric localization. In particular, this chapter focuses on the "follow-your-nose" (FYN) approach to supersymmetric localization, which is sometimes also called canonical supersymmetric localization or conventional supersymmetric localization. Note that the FYN approach to localization is a semi-canonical prescription that is expected to succeed in most settings. Modern reviews of supersymmetric localization include [24, 25, 2, 26]. A mathematically rigorous description of supersymmetric localization may be found in [27], and a clear description of equivariant localization is provided in [28].

Supersymmetric localization is a procedure by which infinite-dimensional path integrals exhibiting a fermionic, or Grassmann-odd, symmetry are reduced to lower dimensional integrals. In the best cases, the path integral is reduced to an integral over a matrix model.

There are two arguments for supersymmetric localization, which apply to supersymmetric path integrals of the form

$$\int_{\mathcal{F}} DXe^{-S[X]}. \tag{2.0.1}$$

Here, X denotes the fields of the theory, $\mathcal{F}$ is the configuration space of X, DX is the measure on $\mathcal{F}$, and $S[X]$ is a commuting action functional in Euclidean signature. X includes both bosons and fermions, and $\mathcal{F}$ is a superspace in the sense that it involves both bosonic and fermionic directions.

The first localization argument applies to the situation in which $\mathcal{F}$ is acted on by a global symmetry group G generated by a fermionic charge $\mathcal{Q}$. By introducing collective coordinates for the action of G on $\mathcal{F}$, then integrating, one obtains the volume of the symmetry group

$$\int_{\mathcal{F}} DXe^{-S[X]}. = \frac{1}{\text{vol}G}\int_{\mathcal{F}/G} DXe^{-S[X]}. \tag{2.0.2}$$

Since the collective coordinate associated to the fermionic charge $\mathcal{Q}$ is a Grassmann-odd quantity θ , the volume of the symmetry group is a vanishing Berezin integral $\mathrm{vol}G = \int \mathrm{d}\theta \cdot 1 = 0$. In this case, the only non-vanishing contributions to the path integral come from the locus of fixed points of the fermionic charge

$$\mathcal{F}_{\mathcal{Q}} = \{\ [X] \in \mathcal{F} \mid \text{fermions} = 0,\ \mathcal{Q}\,(\text{fermions}) = 0\ \} \tag{2.0.3}$$

where $[X]$ denotes equivalence classes of X. The path integral therefore reduces to

$$\frac{1}{\mathrm{vol}G}\int_{\mathcal{F}/G} DXe^{-S[X]} = \int_{\mathcal{F}_{\mathcal{Q}}} DXe^{-S[X]} + \underbrace{\int_{\mathcal{F}\setminus\mathcal{F}_{\mathcal{Q}}} DXe^{-S[X]}}_{=0}. \tag{2.0.4}$$

For details of the first localization argument, see the derivation in [29] or the reviews in [2, 22].

By the second localization argument, path integrals exhibiting a fermionic symmetry can be forced to localize to the locus of fixed points of the fermionic symmetry by a particular deformation of the action. This argument applies to a deformed path integral of the form

$$Z(t) = \int_{\mathcal{F}} DXe^{-S[X]-t\mathcal{Q}V_F[X]}. \tag{2.0.5}$$

Here, $V_F[X]$ is an anti-commuting functional, $\mathcal{Q}V_F[X]$ is a commuting functional, and $t \in \mathbb{R}_{\geq 0}$ is a parameter. $\mathcal{Q}V_F[X]$ is called the localization term or deformation term, and the choice of $\mathcal{Q}$ and V_F is referred to as a choice of localization scheme. For localization to occur, we require that:

1. The action is $\mathcal{Q}$-closed $\mathcal{Q}S = 0$. This implies that $\mathcal{Q}$ is nilpotent when acting on the action $\mathcal{Q}^2 S = 0$.

2. The measure DX is $\mathcal{Q}$-invariant. This implies that both $\mathcal{Q}$ and $\mathcal{Q}^2$ are non-anomalous.

3. The localizing term is $\mathcal{Q}$-closed $\mathcal{Q}^2 V_F = 0$. This ensures that both $\mathcal{Q}V_F$ and V_F are well-defined functionals, which is necessary for integration by parts to work correctly.

4. The real part of the localizing term is positive semi-definite $\mathrm{Re}\,(\mathcal{Q}V_F\,[X]) \geq 0$. This ensures the convergence of subsequent integrals.

When the requirements are satisfied, the path integral exhibits deformation invariance; that is, it is independent of the deformation. In particular, we have

$$\begin{aligned}\frac{d}{dt}Z(t) &= -\int_{\mathcal{F}} DX\,\mathcal{Q}\left(V_F[X]\right)e^{-S[X]-t\mathcal{Q}V_F[X]} \\ &= -\int_{\mathcal{F}} DX\,\mathcal{Q}\left(V_F[X]e^{-S[X]-t\mathcal{Q}V_F[X]}\right) \\ &= -\int_{\mathcal{F}} \mathcal{Q}\left(DXV_F[X]e^{-S[X]-t\mathcal{Q}V_F[X]}\right) \\ &= 0\end{aligned}$$

The first equality is the result of the derivative w.r.t. t. The second equality holds due to requirements that the action is $\mathcal{Q}$-closed $\mathcal{Q}S = 0$, and the localizing term is $\mathcal{Q}$-closed $\mathcal{Q}^2 V_F = 0$. The third equality holds due to the requirement that the measure DX is $\mathcal{Q}$-invariant. The fourth equality holds if $\int_{\mathcal{F}} \mathcal{Q}(\dots)$ is a total derivative in superspace.

In particular, the last step is the superspace analog of Stokes's theorem vanishing in the absence of boundary contributions

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega = 0. \tag{2.0.6}$$

Here, ω is a differential form on the manifold M with boundary ∂M, and d is the exterior derivative. For details regarding the $\mathcal{Q}$-invariance of superspace measures and the superspace analogs of Stokes's theorem, see e.g. [27]. If the partition function exhibits deformation invariance $\frac{d}{dt}Z(t) = 0$, the path integral localizes to the locus of $\mathcal{Q}$ fixed points in the localizing limit $t \to \infty$.

If the requirements for localization are unfulfilled, however, the localization argument fails, and the partition function does not exhibit deformation invariance $\frac{d}{dt}Z(t) \neq 0$. How and why the localization argument fails depends on the particular unfulfilled requirement. Consider, for instance, the situation in which only the second requirement is unfulfilled, and the superspace measure DX is not $\mathcal{Q}$-invariant. What can happen is that the superspace analog of Stokes's theorem does not vanish due to the presence of boundary contributions. Specifically, one has

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega \neq 0. \tag{2.0.7}$$

The supersymmetric localization procedure involves a conventional choice of localizing term called the canonical localization scheme. The canonical localization scheme is

$$L_{\mathrm{loc}}^{\mathrm{canon}} = \mathcal{Q}V_F^{\mathrm{canon}} = \mathcal{Q}\sum_{\psi_X}\left(\left(\mathcal{Q}\psi_X\right)^\dagger \psi_X + \psi_X\left(\mathcal{Q}\psi_X^\dagger\right)^\dagger\right). \tag{2.0.8}$$

Here, ψ_X denotes the fermions of the theory, and the sum runs over all ψ_X, The $\dagger$ denotes a conjugation operation, typically Hermitian conjugation. When evaluated, the bosonic part of the canonical localization scheme is a sum of squares

$$(L_{\text{loc}}^{\text{canon}})|_{\text{bos}} = \sum_{\psi_X} \left(|\mathcal{Q}\psi_X|^2 + \left|\mathcal{Q}\psi_X^\dagger\right|^2 \right). \tag{2.0.9}$$

Note that choosing the localizing term according to the canonical localizing scheme is conventional but not necessary.

Given a deformation invariant partition function, localization proceeds by identifying the bosonic moduli. The moduli are the coordinates on the locus of bosonic field configurations for which the real bosonic part of the localizing term vanishes

$$\left\{ X_0 \in \mathcal{F} \mid \text{Re}\left[\left(\mathcal{Q}V_F\left[X\right] \right)|_{\text{bos}} \right] = 0 \right\} \tag{2.0.10}$$

This is called the localization locus. Observe that this is the locus of $\mathcal{Q}$ fixed points, that is to say, bosonic configurations for which the variation of the fermions vanish $\mathcal{Q}\psi_X = \mathcal{Q}\psi_X^\dagger = 0$. The moduli X_0 are minima due to the fourth requirement $\text{Re}\left(\mathcal{Q}V_F\left[X\right]\right) \geq 0$. The moduli are also referred to as zero modes.

The next step in the localization procedure is to parametrize the fields near the localization locus as

$$X = X_0 + \frac{1}{\sqrt{t}} X_f. \tag{2.0.11}$$

In particular, this is a change of coordinates in which the fields X are expressed in terms of zero modes X_0, and fluctuating modes X_f. The point of this change of coordinates is that in the localizing limit $t \to \infty$, the functional Taylor-expansion of the deformed action is

$$S\left[X\right] + t\mathcal{Q}V\left[X\right] = S\left[X_0\right] + \frac{1}{2} \iint \left. \frac{\delta^2 S_{\text{loc}}}{\delta X^2} \right|_{X=X_0} \left(\delta X_f\right)^2 + \mathcal{O}\left(\frac{1}{\sqrt{t}}\right). \tag{2.0.12}$$

The zeroth-order term is the classical action, the first-order term vanishes since X_0 are minima, and the second-order term is quadratic in fluctuations.

It follows that, in the localizing limit, the deformed partition function reduces to

$$\lim_{t\to\infty} Z(t) = J \int DX_0 \overbrace{\Bigg(\underbrace{e^{-S[X_0]}}_{Z_{\text{class}}} \underbrace{\int DX_f e^{-\frac{1}{2}\iint \left.\frac{\delta^2 S_{\text{loc}}}{\delta X^2}\right|_{X=X_0} \left(\delta X_f\right)^2}}_{Z_{\text{1-loop}}} \Bigg)}^{Z_{\text{eff}}=\exp(-S_{\text{eff}})} \tag{2.0.13}$$

The resulting quantities are referred to as the classical contribution Z_{class}, the 1-

loop contribution $Z_{\text{1-loop}}$, and the effective action Z_{eff}. Moreover, J is the Jacobian due to the change of coordinates, which is typically trivial. Notice that there are many similarities between localization and effective field theory (EFT). The result of the localization argument is a type of 1-loop exact EFT, in which zero modes and fluctuating modes play the roles of light modes and heavy modes, respectively.

The evaluation of the classical contribution is typically straightforward – one evaluates the original action on the zero-mode configurations X_0. On the other hand, the evaluation of the 1-loop contribution is more subtle. In particular, it is necessary to fix a gauge and integrate out the fluctuating modes. The contours of integration for the fluctuating modes are chosen according to the reality conditions specified when evaluating the moduli. One way to evaluate the integrals over fluctuating modes is the mode-by-mode approach which utilizes the Cartan Weyl basis of the gauge algebra and spherical harmonics.

The result of integrating out the fluctuating modes is a superdeterminant

$$Z_{\text{1-loop}}\left[X_0\right] = \frac{1}{\text{Sdet}\left(S_{\text{loc}}^{\text{quad}}\right)}. \tag{2.0.14}$$

A superdeterminant, or Berezinian, is an algebraic operation on a supermatrix, and a supermatrix is a matrix in which certain entries are Grassmann even, and other entries are Grassmann odd. The evaluation of the Berezinian results in a ratio of standard determinants. For a reference on supermatrices and superdeterminants, see, e.g., [30].

The final result of localization is the partition function expressed in terms of integrals over moduli

$$Z = \sum_{X_0^{(d)}} \int dX_0^{(c)} e^{-S[X_0]} Z_{\text{1-loop}}\left[X_0\right]. \tag{2.0.15}$$

Here, $X_0^{(d)}$ denotes discrete moduli to be summed over, while $X_0^{(c)}$ denotes continuous moduli to be integrated over.

Chapter 3

Two-dimensional Yang-Mills theories & non-abelian localization

In this chapter, we review YM2, the exact solution of the pure YM2 partition function, non-abelian localization of cohomological YM2, and the relationship between pure YM2 and cohomological YM2. Witten's non-abelian localization computation will eventually be compared to the Benini-Zaffaroni supersymmetric localization computation.

3.1 Two-dimensional Yang-Mills

To define two-dimensional Yang-Mills theory, let Σ be a compact closed Riemann surface, let G be a simple compact connected Lie group with Lie algebra $\mathfrak{g} = \mathrm{Lie}G$, let $P \to \Sigma$ be a principle bundle with fiber G, and and let $\mathrm{ad}\,(P)$ be the adjoint bundle. Let $\mathcal{A}\,(P)$ be the space of G-connections A on P, and let $\mathcal{G}\,(P)$ be the group of gauge transformations i.e. bundle automorphisms $\mathcal{G}\,(P) = \mathrm{Aut}P$.

The curvature of the G-connection A is the adjoint-valued two-form

$$F_A = \mathrm{d}A + A \wedge A, \tag{3.1.1}$$

where d is the exterior derivative on Σ. The covariant derivative on $\mathrm{ad}\,(P)$ is

$$D_A = \mathrm{d} + A. \tag{3.1.2}$$

The exterior derivative is nilpotent $\mathrm{d}^2 = 0$ when acting on differential forms on Σ. The covariant derivative squares to the curvature of the G-connection

$$D_A^2 = F_A. \tag{3.1.3}$$

The two-dimensional Bianchi identity $D_A F_A = 0$ is trivial since Σ does not admit three forms and d is nilpotent.

The gauge transformations $g \in \mathcal{G}$ are maps $g : \Sigma \to G$. The gauge transformations act on connections by

$$(g, A) \to g^{-1} A g + g^{-1} \mathrm{d} g. \tag{3.1.4}$$

The infinitesimal gauge transformations are elements of the tangent space $u \in T\mathcal{G}$. The infinitesimal gauge transformations act on connections by

$$(u, A) \to -D_A u. \tag{3.1.5}$$

The YM2 action is

$$S[A] = \frac{1}{2e^2} \int_\Sigma \mathrm{Tr}\left(F_A \wedge *F_A\right) \tag{3.1.6}$$

where the gauge coupling e is a real constant of mass dimension $[e] = 1$, and $*$ is the Hodge star operator on Σ. The action is a $\mathcal{G}(P)$-invariant functional on the space of connections $\mathcal{A}(P)$.

To express the G-connection A in terms of the gauge field A_i, we choose local coordinates $\left\{x^i | i = 1, 2\right\}$ on a chart of Σ, a tangent space basis $\left\{\partial/\partial x^i\right\}$, a cotangent space basis $\left\{\mathrm{d}x^i\right\}$, and a basis of anti-hermitian generators $\{T_a | a = 1, \ldots, \mathrm{rank} G\}$ in the adjoint representation of $\mathfrak{g}$. The generators obey $[T_a, T_b] = f_{ab}^{\ \ c} T_c$, where $f_{ab}^{\ \ c}$ are structure constants. In components, the G-connection and its curvature are

$$A = A_i^a(x) T_a \mathrm{d}x^i \tag{3.1.7}$$

$$F_A = \frac{1}{2} F_{ij}^a(x) T_a \mathrm{d}x^i \wedge \mathrm{d}x^j. \tag{3.1.8}$$

Here, $A_i^a(x)$ is the gauge field and $F_{ij}^a(x)$ is the field strength. The field strength and covariant derivative are given by

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j], \quad D_i = \partial_i + A_i. \tag{3.1.9}$$

Together, these satisfy

$$F_{ij} = [D_i, D_j]. \tag{3.1.10}$$

The gauge transformations act on A_i and F_{ij} as

$$A_i \to A_i^g = g^{-1} A_i g + g^{-1} \mathrm{d} g, \tag{3.1.11}$$

$$F_{ij} \to F_{ij}^g = g^{-1} F_{ij} g. \tag{3.1.12}$$

In components, the action of YM2 is

$$S[A] = -\frac{1}{4e^2}\int_{\Sigma_g} \mathrm{d}\mu \mathrm{Tr} F_{ij}F^{ij} \tag{3.1.13}$$

Here, the trace is in the fundamental representation, and $\mathrm{d}\mu = *1 = \mathrm{d}^2x\sqrt{\det g_{ij}}$ is the Riemannian measure. This is referred to as the action of pure YM2.

The YM2 action exhibits invariance under three types of transformations which we now describe. The first invariance is under gauge transformations, which is straightforward to check:

$$\begin{aligned} S[A^g] &= -\frac{1}{4e^2}\int_{\Sigma_g} \mathrm{d}\mu \mathrm{Tr} g^{-1}F_{ij}gg^{-1}F^{ij}g & (3.1.14)\\ &= -\frac{1}{4e^2}\int_{\Sigma_g} \mathrm{d}\mu \mathrm{Tr} gg^{-1}F_{ij}F^{ij} & (3.1.15)\\ &= S[A]. & (3.1.16)\end{aligned}$$

The second invariance of the YM2 action is under area-preserving diffeomorphisms, that is to say, under general coordinate transformations that preserve the area of the Riemann surface $a_\Sigma = \int_{\Sigma_g} \mathrm{d}\mu$. To clarify, note first that the choice of metric g_{ij} on Σ determines an area form $\hat{\varepsilon}$. Using $\hat{\varepsilon}$, the curvature two-form F_A may be mapped to the adjoint-valued scalar $f = *F_A$ as

$$F_A = \hat{\varepsilon} f. \tag{3.1.17}$$

In components, this reads

$$F^a_{ij} = \sqrt{\det g_{ij}}\epsilon_{ij} f^a. \tag{3.1.18}$$

When expressed in terms of f, the action is

$$S[A] = -\frac{1}{2e^2}\int_{\Sigma_g} \mathrm{d}\mu \mathrm{Tr} f^2 \tag{3.1.19}$$

Since f is scalar, diffeomorphisms that keep the area a_Σ fixed are symmetries of the action.

The third invariance of the YM2 action is under simultaneous scalings of the coupling constant and the measure by a real constant:

$$e^2 \to te^2,\ \mathrm{d}\mu \to \mathrm{d}\mu/t,\ t \in \mathbb{R}, \tag{3.1.20}$$

from which it follows that $f \to tf$. Note that the measure is scaled by scaling the metric g_{ij}.

The partition function

$$Z\left(e, a_{\Sigma}; \Sigma_g, G\right) = \frac{1}{\mathrm{vol}\mathcal{G}} \int_{\mathcal{A}} DA e^{-S[A]} \tag{3.1.21}$$

defines YM2 on Σ_g. Here, the integration domain is the space of connections $\mathcal{A}$, DA is the path integral measure on $\mathcal{A}$, and vol$\mathcal{G}$ is the volume of the group of gauge transformations. Due to the invariance of the action under the scalings 3.1.20, the path integral is a function of only the invariant combination $\varepsilon \equiv e^2 a_{\Sigma}$. Since the mass dimensions of the squared coupling and area are $\left[e^2\right] = 2,\ \left[a_{\Sigma}\right] = -2$, the invariant combination is dimensionless $[\varepsilon] = 0$. The partition function is denoted more compactly by $Z(\varepsilon) \equiv Z\left(e, a_{\Sigma}; \Sigma_g, G\right)$.

Various definitions of YM2 may differ by local counter-terms depending on the topology and metric of the Riemann surface

$$\Delta S = u \int_{\Sigma_g} \mathrm{d}\mu + v \int_{\Sigma_g} \mathrm{d}\mu \frac{\mathcal{R}}{2\pi}, \tag{3.1.22}$$

Here, the first term depends on the area a_{Σ}, while in the second term, $\mathcal{R}$ is the Ricci scalar curvature and $\chi = \int_{\Sigma_g} \mathrm{d}\mu \frac{\mathcal{R}}{2\pi}$ is the Euler characteristic $\chi = 2 - 2g$. The counter-terms arise when using different regularization schemes to renormalize quadratic and logarithmic divergences occurring in one-loop amplitudes in an external gravitational field. Due to this renormalization ambiguity, partition functions computed in two different regularization schemes differ by

$$Z'(\varepsilon) = Z(\varepsilon) \exp\left(\Delta u \int_{\Sigma_g} \mathrm{d}\mu + \Delta v \int_{\Sigma_g} \mathrm{d}\mu \frac{R}{2\pi}\right), \tag{3.1.23}$$

Here, $\Delta u, \Delta v$ are constants that are understood as corrections to u, v. If the actions in the separate regularization schemes are both invariant under the scaling symmetry 3.1.20, the corrections reduce to

$$\Delta u = e^2 u_0, \quad \Delta v = v_0. \tag{3.1.24}$$

where u_0, v_0 are independent of e^2. For a reference, see [4].

Other variants of YM2 may be formulated by introducing additional auxiliary fields with quantum ghost numbers U. The ghost number is the charge under the $\mathbb{Z}$-grading of $\mathcal{G}$-invariant equivariant differential forms on $\mathcal{A}$. The first variant of YM2 we consider is constructed from the gauge field A_i together with an adjoint bosonic scalar field ϕ. In particular, $A_i = A_i^a(x)\, T_a$ is a commuting Lie algebra valued one form on Σ with ghost number $U = 0$, and $\phi = \phi^a(x)\, T_a$ is a commuting

Lie algebra-valued zero form on Σ with ghost number $U = 2$. The action is

$$S_1[A,\phi] = -\frac{e^2}{2}\int_\Sigma \mathrm{d}\mu \mathrm{Tr}\phi^2 - i\int_\Sigma \mathrm{Tr}\phi F_A, \tag{3.1.25}$$

The action is invariant under area-preserving diffeomorphisms, simultaneous scalings 3.1.20, and gauge transformations

$$A_i \rightarrow A_i^g = g^{-1}A_i g + g^{-1}\mathrm{d}g, \tag{3.1.26}$$

$$\phi \rightarrow \phi^g = g^{-1}\phi g. \tag{3.1.27}$$

The partition function

$$Z_1(\varepsilon;\Sigma_g,G) = \frac{1}{\mathrm{vol}\mathcal{G}}\int DAD\phi e^{-S_1[A,\phi]} \tag{3.1.28}$$

defines the theory, which is denoted more compactly by $Z_1(\varepsilon) \equiv Z_1(\varepsilon;\Sigma_g,G)$. The derivative of ϕ does not appear in the action, so it is an auxiliary field. Accordingly, ϕ may be eliminated from the theory by setting it to its on-shell value in the action. The on-shell value is the solution of the equation of motion for ϕ, which is of the form $\phi \propto f$. Eliminating ϕ in this manner is equivalent to performing the Gaussian integral over ϕ, which is peaked around $\phi \propto f$. In particular, ϕ may be integrated out as

$$\int_{\mathbb{R}} \frac{\mathrm{d}x}{\sqrt{2\pi}} e^{-\frac{e^2}{2}x^2 - \mathrm{i}xy} = e^{-\frac{1}{2e^2}y^2}. \tag{3.1.29}$$

By evaluating the ϕ integral in $Z_1(\varepsilon)$, one recovers the original partition function of pure YM2 3.1.21. In other words, we have

$$Z_1(\varepsilon;\Sigma_g,G) \simeq Z(\varepsilon;\Sigma_g,G), \tag{3.1.30}$$

such that the theory defined by $Z_1(\varepsilon)$ is equivalent to the pure YM2 theory defined by $Z(\varepsilon)$ up to local counter-terms.

An advantage of this alternative formulation is that it is straightforward to take the topological $\varepsilon \rightarrow 0$ limit

$$Z_{\mathrm{BF}}(\Sigma_g,G) = \lim_{\varepsilon\rightarrow 0} Z_1(\varepsilon;\Sigma_g,G) = \frac{1}{\mathrm{vol}\mathcal{G}}\int_{\mathcal{A}} DAD\phi e^{-S_{\mathrm{BF}}[A,\phi]} \tag{3.1.31}$$

where

$$S_{\mathrm{BF}}[A,\phi] = -i\int \mathrm{Tr}\phi F_A. \tag{3.1.32}$$

In this limit, YM2 becomes a Witten-type topological quantum field theory

(TQFT) called BF-theory. BF-theory is a TQFT in the sense that all dependence on the metric of Σ is lost. The partition function may be used to compute topological quantities. Specifically, we will later describe how this partition function is used to compute the symplectic volume $\mathrm{vol}\mathcal{M}_{\mathrm{flat}}$ where $\mathcal{M}_{\mathrm{flat}} = \{A \in \mathcal{A}(P) \,|F_A = 0\} /\mathcal{G}(P)$ is the moduli space of flat connections.

Further variants of YM2 may be formulated by introducing an auxiliary adjoint fermion ψ and a fermionic BRST-like charge Q. In particular, $\psi = \psi_i^a(x) T_a \mathrm{d}x^i$ is an anti-commuting adjoint valued one form on Σ with ghost number $U = 1$, and Q is an anti-commuting BRST-like operator which acts on fields as $\delta\Phi = -i\epsilon\{Q, \Phi\}$. Here, ϵ is an anti-commuting variational parameter, δ is a commuting variation, and Φ is any of the fields A_i, ϕ, ψ_i.

The second variant of YM2 we consider is constructed from A_i and ψ_i. The transformations under the BRST-like symmetry are

$$\delta A_i = i\epsilon\psi_i, \tag{3.1.33}$$

$$\delta\psi_i = -\epsilon D_i f, \tag{3.1.34}$$

and the square of the operator is

$$Q^2 A_i = \{Q, [Q, A_i]\} = -\{Q, \psi_i\} = iD_i f. \tag{3.1.35}$$

The action is

$$S_2[A, \psi] = -\frac{1}{2e^2}\int_\Sigma \mathrm{Tr}\left(F_A \wedge *F_A - \frac{i}{2}\psi \wedge \psi\right). \tag{3.1.36}$$

The fermionic terms in the action are non-vanishing due to ψ being an adjoint valued anti-commuting one-form. Moreover, the action is closed under the BRST-like symmetry $\delta S = 0$. The on-shell value of ψ_i is $\psi_i = 0$, while the on-shell value of A_i is $D_i f = 0$, i.e. the Yang-Mills equations. Observe that the algebra of Q only closes on-shell, since the operator is nilpotent $Q^2 = 0$ for the on-shell value $D_i f = 0$. The partition function

$$Z_2(\varepsilon; \Sigma_g, G) = \frac{1}{\mathrm{vol}\mathcal{G}}\int DAD\psi e^{-S_2[A,\psi]} \tag{3.1.37}$$

defines the theory. Since the derivative of ψ does not appear in the action, ψ is an auxiliary field which can be integrated out by setting it to its on-shell value in the action. By integrating out ψ, one recovers the partition function of pure YM2

$$Z_2(\varepsilon; \Sigma_g, G) \simeq Z(\varepsilon; \Sigma_g, G), \tag{3.1.38}$$

up to local counter-terms.

The third variant of YM2 we consider is constructed from A_i,ϕ, and ψ_i. Together, these fields transform under the BRST-like operator Q to form a multiplet, denoted (A_i, ψ_i, ϕ). This is called the basic multiplet of cohomological Yang-Mills. The transformations are

$$\delta A_i = i\epsilon\psi_i, \tag{3.1.39}$$

$$\delta\psi_i = -\epsilon D_i\phi = -\epsilon\left(\partial_i\phi + [A_i, \phi]\right), \tag{3.1.40}$$

$$\delta\phi = 0, \tag{3.1.41}$$

The square of the BRST-like operator is

$$Q^2 = -i\delta_\phi, \tag{3.1.42}$$

where δ_ϕ denotes an infinitesimal gauge transformation generated by the gauge parameter ϕ. Since Q is anti-commuting, Q^2 is commuting. This can be understood, for instance, by comparing a generic infinitesimal gauge transformation to the result obtained when squaring the BRST-like operator. A generic infinitesimal gauge transformation reads $A_i^a \to A_i^a + \delta_u A_i^a$, where $\delta_u A_i^a = -D_i u^a$ is the infinitesimal variation of A_i^a generated by a generic gauge parameter u^a. The square of the BRST operator, on the other hand, is evaluated by acting twice with Q on A_i^a according to 3.1.39-3.1.41. This reads

$$Q^2 A_i^a = \{Q, [Q, A_i^a]\} = -\{Q, \psi_i^a\} = iD_i\phi^a = -i\delta_\phi A_i^a. \tag{3.1.43}$$

where $\delta_\phi A_i^a = -D_i\phi^a$ is the infinitesimal variation of A_i^a generated by the gauge parameter ϕ^a.

Since the BRST-like operator squares to an infinitesimal gauge transformation, it is nilpotent when acting on gauge-invariant functionals of the fields in the basic multiplet. In particular, if V is a gauge-invariant functional obeying $\delta_\phi V = 0$, then by 3.1.42 we have nilpotency

$$Q^2 V = \delta_\phi V = 0. \tag{3.1.44}$$

Furthermore, any Q-exact functional, defined by acting with Q on a gauge-invariant functional, will be Q-closed. That is to say, if $W = QV$ and $\delta_\phi V = 0$, then by 3.1.42 we have closure

$$QW = Q^2 V = -i\delta_\phi V = 0. \tag{3.1.45}$$

The action is

$$S_3\left[A,\psi,\phi\right]=-i\int_{\Sigma}\mathrm{Tr}\left(\phi F_A-\frac{1}{2}\psi\wedge\psi\right)-\frac{e^2}{2}\int_{\Sigma}\mathrm{d}\mu\mathrm{Tr}\phi^2. \tag{3.1.46}$$

The action is invariant under gauge transformations, area-preserving diffeomorphisms, and simultaneous scalings of the coupling and measure. Both ϕ and ψ are auxiliary fields, with on-shell values $\phi\propto f$ and $\psi=0$, respectively. The on-shell value of A_i is again the Yang-Mills equations $D_i f=0$. Since the action is gauge-invariant, the BRST-like operator is nilpotent $Q^2S_3=0$. Notice that, in contrast with the previous case, the algebra of Q closes off-shell.

The partition function

$$Z_3\left(\varepsilon;\Sigma_g,G\right)=\frac{1}{\mathrm{vol}\mathcal{G}}\int DAD\phi D\psi e^{-S_3[A,\psi,\phi]} \tag{3.1.47}$$

defines the theory. The auxilliary fields ϕ and ψ may be integrated out by setting them to their on-shell values in the action. The result of integrating out ϕ and ψ is the partition function of pure YM2 3.1.21, up to local counter-terms.

Further variants of YM2 may be formulated by once again introducing auxiliary adjoint fields with ghost numbers transforming under Q. In particular, we introduce two additional multiplets, in a similar manner to the introduction of anti-ghost multiplets in the conventional Fadde'ev-Popov BRST procedure. The new multiplets consist of pairs of adjoint-valued auxiliary fields (v,w), one of which is commuting, the other of which is anti-commuting, with ghost numbers $(n,n+1)$ for some integer n. The transformations are

$$\delta v = i\epsilon w, \tag{3.1.48}$$

$$\delta w = \epsilon\left[\phi,v\right]. \tag{3.1.49}$$

The first additional multiplet is called the ghost multiplet (λ,η), where $\lambda^a\left(x\right)$ is a commuting adjoint scalar with ghost number $U=-2$, and $\eta^a\left(x\right)$ is an anti-commuting adjoint scalar with ghost number $U=-1$. The second additional multiplet is called the projection multiplet $(\chi,-iH)$, where $\chi^a\left(x\right)$ is an anti-commuting adjoint scalar with ghost number $U=-1$, and $H^a\left(x\right)$ is a commuting adjoint scalar with ghost number $U=0$. The transformations of the fields in the ghost multiplet are obtained by replacing v,w with λ,η in 3.1.48-3.1.49, and similarly for the projection multiplet fields.

Together, the basic, ghost, and projection multiplets form the standard multiplet of cohomological YM2. In particular, the standard multiplet consists of the adjoint-valued fields $(A_i,\psi_i,\phi,\lambda,\eta,\chi,H)$, with ghost numbers $(0,1,2,-2,-1,-1,0)$, and whose statistics are (B, F, B, B, F, F, B) where B is bosonic and F is fermionic. The transformations of the standard multiplet are

$$\delta A_i = i\epsilon\psi_i, \quad (3.1.50)$$

$$\delta\psi_i = -\epsilon D_i\phi, \quad (3.1.51)$$

$$\delta\phi = 0, \quad (3.1.52)$$

$$\delta\lambda = i\epsilon\eta, \quad (3.1.53)$$

$$\delta\eta = \epsilon\left[\phi,\lambda\right], \quad (3.1.54)$$

$$\delta\chi = \epsilon H, \quad (3.1.55)$$

$$\delta H = i\epsilon\left[\phi,\chi\right]. \quad (3.1.56)$$

For later convenience, we note that the transformation of the scalar field strength is

$$\delta f = i\star D\psi \quad (3.1.57)$$

where $f = \star F_A = \frac{1}{2}\epsilon^{ij}F_{ij} = F_{12}$, and $\star D\psi = \epsilon^{ij}D_i\psi_j = D_1\psi_2 - D_2\psi_1$.

The fourth variant of YM2 we consider is constructed from the fields of the standard multiplet. The theory is defined by the partition function

$$Z_4 = \frac{1}{\mathrm{vol}\mathcal{G}}\int DAD\psi D\phi D\lambda D\eta D\chi DHe^{i\{Q,V\}} \quad (3.1.58)$$

In this case, the Q-exact action $S_4 = -i\{Q,V\}$ is defined by the functional

$$V = \frac{1}{h^2}\int_\Sigma \mathrm{d}\mu\mathrm{Tr}\left(\frac{1}{2}\chi\left(H-2f\right) + g^{ij}D_i\lambda\psi_j\right), \quad (3.1.59)$$

where the coupling constant h is a real parameter. In particular, V is an anti-commuting functional that preserves ghost number, and results in an action $S_4 = -i\{Q,V\}$ with degenerate kinetic energy for fields.

To derive the explicit form of the action, one acts with the BRST-like symmetry on the terms in V according to the transformations of the standard multiplet 3.1.50-

3.1.56. This reads

$$\delta\left(\frac{1}{2}\chi\left(H-2f\right)+g^{ij}D_i\lambda\psi_j\right) \tag{3.1.60}$$
$$= \frac{1}{2}\delta\chi\left(H-2f\right)+\frac{1}{2}\chi\left(\delta H-2\delta f\right) \tag{3.1.61}$$
$$+g^{ij}\left(D_i\delta\lambda\psi_j+\left[\delta A_i,\lambda\right]\psi_j+D_i\lambda\delta\psi_j\right) \tag{3.1.62}$$
$$= \frac{1}{2}H\left(H-2f\right)+\frac{1}{2}\chi\left(i\left[\phi,\chi\right]-2i\star D\psi\right) \tag{3.1.63}$$
$$+g^{ij}\left(iD_i\eta\psi_j+i\left[\psi_i,\lambda\right]\psi_j-D_i\lambda D_j\phi\right) \tag{3.1.64}$$
$$= \frac{1}{2}\left(H-f\right)^2-\frac{1}{2}f^2+\frac{i}{2}\chi\left[\phi,\chi\right]-i\chi\star D\psi \tag{3.1.65}$$
$$+iD_i\eta\psi^i+i\left[\psi_i,\lambda\right]\psi^i-D_i\lambda D^i\phi \tag{3.1.66}$$

The first equality is due to commuting δ, while in the second equality, we used the transformations of the fields in the standard multiplet. In the third equality, the square was completed for H, and the indices were raised.

In view of this, the action reads

$$S_4 = -i\left\{Q,V\right\}$$
$$= \frac{1}{h^2}\int_\Sigma \mathrm{d}\mu\mathrm{Tr}\Bigg(\frac{1}{2}\left(H-f\right)^2-\frac{1}{2}f^2-D_i\lambda D^i\phi. \tag{3.1.67}$$
$$+\frac{i}{2}\chi\left[\phi,\chi\right]-i\chi\star D\psi+iD_i\eta\psi^i+i\left[\psi_i,\lambda\right]\psi^i\Bigg) \tag{3.1.68}$$

This is the standard action for the two-dimensional analog of four-dimensional Donaldson theory [3].

The fifth variant of YM2 we consider is again constructed from the fields of the standard multiplet. In this case, the action is a Q-exact deformation of the two-dimensional Donaldson action S_4. The theory is defined by the partition function

$$Z_5 = \frac{1}{\mathrm{vol}\mathcal{G}}\int DAD\psi D\phi D\lambda D\eta D\chi DHe^{i\left\{Q,V+tV'\right\}} \tag{3.1.69}$$

In this case, the action $S_5\left(t\right) = -i\left\{Q,V+tV'\right\}$ is a Q-exact deformation of the two-dimensional Donaldson action $S_4 = -i\left\{Q,V\right\}$, defined by functionals 3.1.59 and

$$V' = -\frac{1}{h^2}\int_\Sigma \mathrm{d}\mu\mathrm{Tr}\chi\lambda. \tag{3.1.70}$$

Here, V' is an anti-commuting functional that results in an action $S_5\left(t\right)$ with non-degenerate kinetic energy for all fields. In contrast with before, however, V' does not preserve ghost number. Notice that by setting the deformation parameter to zero, we recover the two-dimensional Donaldson action $S_5\left(t=0\right) = S_4$.

To derive the explicit form of the action, we act with the BRST-like symmetry on the terms in V'. This reads

$$\delta\left(\chi\lambda\right) = \left(\delta\chi\right)\lambda + \chi\left(\delta\lambda\right) = H\lambda + i\chi\eta. \tag{3.1.71}$$

In view of this, the action reads

$$\begin{aligned} S_5\left(t\right) &= -i\left\{Q, V + tV'\right\} && (3.1.72)\\ &= \frac{1}{h^2}\int_\Sigma \mathrm{d}\mu \mathrm{Tr}\bigg(\frac{1}{2}\left(H-f\right)^2 - \frac{1}{2}f^2 - D_i\lambda D^i\phi - tH\lambda && (3.1.73)\\ &\quad + \frac{i}{2}\chi\left[\phi,\chi\right] - i\chi\star D\psi + iD_i\eta\psi^i + i\left[\psi_i,\lambda\right]\psi^i - it\chi\eta\bigg) && (3.1.74)\end{aligned}$$

To express the action more conveniently, the squares are completed for H and f, which reads

$$\frac{1}{2}\left(H-f\right)^2 - \frac{1}{2}f^2 - tH\lambda = \frac{1}{2}\left(H-f-\lambda t\right)^2 - \frac{1}{2}\left(f+t\lambda\right)^2. \tag{3.1.75}$$

Accordingly, we have

$$\begin{aligned} S_5\left(t\right) &= \frac{1}{h^2}\int_\Sigma \mathrm{d}\mu \mathrm{Tr}\bigg(\frac{1}{2}\left(H-f-\lambda t\right)^2 - \frac{1}{2}\left(f+t\lambda\right)^2 - D_i\lambda D^i\phi && (3.1.76)\\ &\quad + \frac{i}{2}\chi\left[\phi,\chi\right] - i\chi\star D\psi + iD_i\eta\psi^i + i\left[\psi_i,\lambda\right]\psi^i - it\chi\eta\bigg) && (3.1.77)\end{aligned}$$

3.2 Exact solution of YM2 partition function

In this section we review the derivation of the exact solution of the partition function of YM2 using the lattice approach.

The partition function of YM2 is known to be

$$Z\left(e^2a, G, \Sigma_g\right) = e^{k_1(2-2g)}\sum_{R_\mu}\frac{1}{\left(\dim R_\mu\right)^{2g-2}}e^{-\frac{1}{2}e^2a\left(C_2(R_\mu)+k_2\right)} \tag{3.2.1}$$

Here, $C_2\left(R_\mu\right)$ is the quadratic Casimir of R_μ, e^2 is the Yang-Mills coupling constant, a is the area of Σ_g, while k_1 and k_2 are renormalization scheme dependent constants. This result can be derived using a combinatorial approach, involving techniques from lattice gauge theory (LGT), or a cut and paste approach, involving techniques from topological quantum field theory (TQFT). Here we describe the combinatorial approach. For references, see [3, 31, 4, 32].

In the combinatorial approach, the path integral of YM2

$$\int_{\mathcal{A}} DAe^{-\int_{\Sigma}\mathcal{L}_{\mathrm{YM}}} \tag{3.2.2}$$

is computed by formulating a discrete approximation of the continuum theory, then solving it using LGT techniques. There are two versions of the lattice approach to QFT. The first version is the Hamiltonian approach, in which space is discretized and time remains continuous, while the second version is the Euclidean approach, in which both space and time are discretized. Here, we consider the Euclidean approach, and the discrete approximation of spacetime is made by "triangulating" the Riemann surface Σ_g, i.e. covering Σ_g in polygons. The triangulation has a finite number of elements, consisting of vertices, and lines between vertices. The interiors of polygons are called plaquettes, and the edges of the polygons are the lines between vertices.

To discretize a scalar theory, instead of considering the value of the scalar field at every point in spacetime, one only considers the value of the scalar field at the vertices. The degrees of freedom of the discretized theory are then contained in the finite number of scalar fields defined at the vertices. To approximate the continuum theory, the triangulation is made finer and finer thereby increasing the number of vertices and degrees of freedom.

To discretize a gauge field theory, instead of thinking of the holonomy between every point in spacetime, one considers only the holonomy between vertices. In other words, the gauge field degrees of freedom in the continuum theory are assigned to the lines in the discrete theory. Recall that the holonomy is a map from curves on Σ to the structure group G

$$\mathrm{Hol}_C\left(A\right) = \mathcal{P}\exp\left(\int_C A\right), \tag{3.2.3}$$

where C is a smooth curve on Σ, $\int_C A$ is the integral of a one-form pulled back to C, and $\mathcal{P}$ is the path ordering symbol. The holonomy is necessary to compare points on fibers at different spacetime positions, in a similar manner to how parallel transport is necessary to compare tangent vectors in tangent vector spaces at different spacetime positions.

In the lattice gauge theory approximation of YM2, a gauge transformation is a map from the finite set of vertices to the structure group $g : S \to G$, i.e. each vertex $x \in S$ is assigned a group element $g\left(x\right) \in G$. To each line γ between vertices $x, y \in S$, one assigns a holonomy $U_\gamma \in G$, or Wilson line. The holonomy parametrizes the degrees of freedom of the gauge field in the continuum theory, and transforms under gauge transformations according to

$$U_\gamma \to g\left(y\right) U_\gamma g^{-1}\left(x\right). \qquad (3.2.4)$$

To proceed with the lattice approximation of the YM2 path integral, let w_i be a plaquette and let ρ_i be the area of w_i. The sum of plaquette areas $\sum_i \rho_i = \rho$ is the total area of Σ. In the lattice theory, the YM2 action is approximated as a sum of local contributions from each plaquette:

$$e^{-\int_\Sigma \mathcal{L}_{\rm YM}} = \prod_i e^{-\int_{w_i} \mathcal{L}_{\rm YM}}. \qquad (3.2.5)$$

In other words, the integral over the whole Riemann surface is decomposed into a sum of integrals over plaquettes. To write a gauge-invariant lattice approximation of the integrand, consider a generic plaquette w of area ρ_w. Let $x_1, \ldots, x_n$ label the edges of w, and let $U_1, U_2, \ldots, U_n$ be the group-valued holonomies assigned to each edge. The quantity

$$\mathcal{U} = U_1 U_2 \ldots U_n, \qquad (3.2.6)$$

is the holonomy on the whole plaquette as opposed to the holonomy on an specific edge.

The lattice approximation of the integrand of the YM2 path integral is taken to be

$$\prod_i \Gamma\left(\mathcal{U}_i, \rho_i\right) = \prod_i \sum_\alpha \dim \alpha \chi_\alpha\left(\mathcal{U}_i\right) \exp\left(-\rho_i c_2\left(\alpha\right)/2\right) \qquad (3.2.7)$$

where i labels plaquettes w_i. Here, α labels isomorphism classes of irreducible representations of G, $\chi_\alpha\left(\mathcal{U}_i\right)$ is the character in the α representation, and $C_2\left(\alpha\right)$ is the quadratic Casimir of the α representation.

Recall that for fixed representation π, the character is defined as the trace in the π-representation $\chi_\pi\left(u\right) = \mathrm{Tr}\left(\pi\left(u\right)\right)$ for $u \in G$. The function $\Gamma\left(\mathcal{U}, \rho\right)$ is a class function on the structure group G, that is to say, a function f on G obeying $f\left(vuv^{-1}\right) = f\left(u\right)$ for all $u, v \in G$. Since $\Gamma\left(\mathcal{U}, \rho\right)$ is a class function, it is invariant under lattice gauge transformations. Moreover, when the area of the plaquette is taken to zero $\rho \to 0$, the function $\Gamma\left(\mathcal{U}, \rho\right)$ obeys

$$\Gamma\left(\mathcal{U}, 0\right) = \sum_\alpha \dim \alpha \chi_\alpha\left(\mathcal{U}\right) = \delta\left(\mathcal{U} - 1\right) \qquad (3.2.8)$$

where $\delta\left(\mathcal{U} - 1\right)$ is the delta function. Note that the inclusion of the quadratic Casimir $C_2\left(\alpha\right)$ in the exponential is a choice that is motivated by the desire to obtain $\mathrm{Tr} f^2$ in the continuum theory.

The definition of $\Gamma\left(\mathcal{U}, \rho\right)$ relies on the Peter-Weyl theorem, which relates the representation space of the compact Lie group G to the space of functions on G as a manifold. For a reference, see e.g. [33]. More precisely, the theorem states that the matrix entries $\pi^{\alpha}_{m,n}$ of irreducible representations of G on a vector space V_{α} are functions on G that satisfy the Schur orthogonality relations, and form a complete set [34]. A simple but illustrative example of the Peter-Weyl theorem is $G = U(1) \simeq S^1$. In this case, the irreducible representations are one-dimensional $\pi^{\alpha}(e^{i\theta}) = e^{i\alpha\theta}$ for $\alpha \in \mathbb{Z}$, and the entries of the one-dimensional matrices are the functions $\pi^{\alpha}_{m,n}\left(e^{i\theta}\right) = e^{i\alpha\theta}$. The functions form an orthonormal basis for square integrable functions on S^1, which is equivalent to the statement of Fourier series.

The lattice approximation of the path integral is

$$Z_{\Sigma,X}(\rho) = \int \prod_{\gamma} dU_{\gamma} \prod_{i} \Gamma\left(\mathcal{U}_i, \rho_i\right) \tag{3.2.9}$$

where X denotes the triangulation, and dU is the normalized Haar measure. The evaluation of the integral is as follows. Consider a single plaquette, taken to be a square, for which the local factor is

$$\Gamma = \sum_{\alpha} \dim \alpha \chi_{\alpha}\left(U_1 U_2 U_3 U_4\right) e^{-\rho c_2(\alpha)/2}. \tag{3.2.10}$$

where ρ is the area of the square. On the other hand, if the square is divided into two triangles, the local factors are

$$\Gamma' \Gamma'' = \sum_{\alpha,\beta} \dim \alpha \dim \beta \chi_{\alpha}\left(U_1 U_2 V\right) \chi_{\beta}\left(V^{-1} U_3 U_4\right) e^{-\rho' c_2(\alpha)/2 - \rho'' c_2(\beta)/2}. \tag{3.2.11}$$

Here, V denotes the lattice connection along the edge of both triangles, and the area of each triangle sums to the area of the square $\rho = \rho' + \rho''$. To prove that the theory is invariant under subdivisions of the lattice, one must show that by integrating over the factors of the triangles, one obtains the factor of the square. Specifically, one must show that

$$\int \mathrm{d}V \Gamma' \Gamma'' = \Gamma. \tag{3.2.12}$$

The proof follows from the cutting and gluing formulae

$$\int \mathrm{d}V \chi_{\alpha}(UV) \chi_{\beta}(V^{-1}W) = \delta_{\alpha\beta} \frac{1}{\dim \alpha} \chi_{\alpha}(UW) \tag{3.2.13}$$

$$\int \mathrm{d}U \chi_{\alpha}(AUBU^{-1}) = \frac{1}{\dim \alpha} \chi_{\alpha}(A) \chi_{\alpha}(B). \tag{3.2.14}$$

To evaluate the Yang-Mills partition function in the higher genus case, one evaluates the integrals in

$$Z_{X,\Sigma}(\rho) = \sum_{\alpha} \dim \alpha e^{-\rho c_2(\alpha)/2} \int \mathrm{d}U_i dV_j \chi_\alpha \left(U_1 V_1 U_1^{-1} V_1^{-1} \cdots U_g V_g U_g^{-1} V_g^{-1} \right), \tag{3.2.15}$$

according to the procedure described above. By repeated application of the cutting and gluing formulae, one obtains factors of $1/\dim\alpha$, and for the final integral, one obtains $\chi_\alpha(1) = \dim\alpha$. The result of evaluating the integrals is

$$Z_g\left(\rho\right) = \sum_{\alpha} \frac{1}{(\dim\alpha)^{2g-2}} e^{-\frac{1}{2}\rho C_2(\alpha)}. \tag{3.2.16}$$

This may be related to other derivations of the YM2 partition function by taking into account the counter-terms of the renormalization ambiguity.

3.3 Cohomological localization

This section reviews two approaches to localizing path integrals of cohomological YM2 on a Riemann surface Σ, described originally in sections 3.1-3.2 of [3]. The two approaches differ by the choice of localizing term, as well as the resulting localization loci. To illustrate the different loci, let A_i be the gauge field of cohomological YM2, where $i = 1, 2$ are spacetime indices on Σ. The first localizing term reduces the space of gauge field configurations to the locus of solutions of the flatness equation $F_{12} = 0$, where F_{12} is the two-dimensional scalar field strength. Solutions A_i of $F_{12} = 0$ are referred to as flat connections. The second localizing term reduces the space of gauge field configurations to the locus of solutions A_i of the Yang-Mills equations $D_i F_{12} = 0$, where $D_i = \partial_i + A_i$ is the gauge-covariant derivative. Solutions A_i of $D_i F_{12} = 0$ are referred to as Yang-Mills connections.

The path integrals to be localized are schematically of the form

$$I \quad = \quad \int_{\mathcal{F}_\varphi} D\varphi \exp\left(-i\left\{Q, V\right\}\right). \tag{3.3.1}$$

Here, φ denotes the fields of standard cohomological multiplet of YM2, $\mathcal{F}_\varphi$ is the configuration space of φ, $D\varphi$ is the measure on $\mathcal{F}_\varphi$, Q is the anti-commuting BRST-like operator, V is an anti-commuting functional of φ, and $\{Q, V\}$ is a commuting functional of φ . The localizing charge is Q, and the localization scheme is $S = -i\{Q, V\}$. The path integral 3.3.1 defines a cohomological theory in a similar manner to how a path integral defines a conventional quantum field theory, and cohomological YM2 may be regarded as a particular variant of pure YM2 with additional auxiliary fields.

3.3.1 Moduli space of flat connections

The first localizing term is the action for the two-dimensional analog of four-dimensional Donaldson theory 3.1.67. As we will see, this action localizes to flat connections, i.e. solutions A_i of $f = 0$, where $f = F_{12}$. The action is

$$\begin{aligned} S_4 &= -i\{Q, V\} && (3.3.2)\\ &= \frac{1}{h^2}\int_\Sigma \mathrm{d}\mu \mathrm{Tr}\bigg(\frac{1}{2}(H-f)^2 - \frac{1}{2}f^2 - D_i\lambda D^i\phi. && (3.3.3)\\ &\qquad + \frac{i}{2}\chi[\phi,\chi] - i\chi \star D\psi + iD_i\eta\psi^i + i[\psi_i,\lambda]\psi^i\bigg) && (3.3.4)\end{aligned}$$

The path integral that defines the theory on Σ is

$$I = \int DAD\psi D\phi D\lambda D\eta D\chi DHe^{-S_4}$$

The path integral fulfills the criteria for localization to occur in the limit $h \to 0$, as outlined in chapter 2. In particular, the action is Q-closed, the measure is Q-invariant, both V and S_4 are well defined functionals, and reality conditions will be chosen such that the path integral converges.

The localization locus is the space of bosonic field configurations for which the real bosonic part of the action S_4 vanishes. To evaluate the localization locus, we must solve

$$0 = \mathrm{Re}\left[\mathrm{Tr}\left(\frac{1}{2}(H-f)^2 - \frac{1}{2}f^2 - D_i\lambda D^i\phi\right)\right] \qquad (3.3.5)$$

for the fields H, A_i, λ, and ϕ. Note that A_i sits in both $f = F_{12}$ and $D_i = \partial_i + A_i$.

Each of the fields is valued in the adjoint representation of the Lie algebra. The Lie algebra consists of anti-hermitian matrices for which the positive definite metric is $(a, b) = -\mathrm{Tr}ab$. When expressed in the basis of anti-hermitian generators T_a, the fields take the form $\Phi = \Phi^a T_a$ while the covariant derivatives take the form $D_i\Phi = (D_i\Phi)^a T_a$.

Currently, the fields are in Euclidean signature and each is assumed to be generically complex. To solve 3.3.5, reality conditions must be imposed on H^a, A_i^a, λ^a, and ϕ^a. For the gauge field A_i^a, the only reasonable reality condition is to take it to be real. This choice fixes the scalar field strength f^a to be real. Next, we take the auxiliary field H^a to be real.

For the complex scalars, on the other hand, there are two reasonable reality conditions. The first possibility is to take ϕ^a and λ^a to be complex conjugates of one another. For instance, taking ϕ^a to be complex, and its complex conjugate to be $\lambda^a = \overline{\phi}^a$, the scalar kinetic term reduces to $D_i\lambda D^i\phi = |D_i\phi|^2$. The second possibility is to take one to be real and the other to be purely imaginary. For example, taking real ϕ^a and purely imaginary λ^a, results in a purely imaginary scalar kinetic $(D_i\lambda)^a \left(D^i\phi\right)^b = i\,(D_i\lambda')^a \left(D^i\phi\right)^b$, where λ'^a is the imaginary part of $\lambda^a = i\lambda'^a$.

By first expanding in the basis of anti-hermitian generators T_a, then choosing real H^a, A_i^a, and complex conjugate ϕ^a and $\lambda^a = \overline{\phi}^a$, 3.3.5 reduces to

$$0 = H^a - f^a = f^a = (D_i\lambda)^a = (D_i\phi)^a\,. \tag{3.3.6}$$

The implications of these equations are as follows. To begin, the solution for the auxiliary field is

$$H = f. \tag{3.3.7}$$

Next, the complex scalar ϕ is constrained to be covariantly constant

$$0 = D_i\phi = \partial_i\phi + [A_i, \phi]\,, \tag{3.3.8}$$

and so too for its conjugate $\lambda = \overline{\phi}$. Lastly, the scalar field strength f is constrained by

$$0 = f \tag{3.3.9}$$

This is the flatness equation for the gauge fields A_i. The same conclusion is reached by choosing real ϕ^a and purely imaginary λ^a. In this case, $D_i\lambda = 0$ is a consequence of stationary phase.

The solutions of $0 = D_i\phi = f$ are configurations of A_i and ϕ, which together, form the space of solutions

$$\mathcal{U} = \{A_i, \phi, H \mid 0 = f = D_i\phi,\ H = f\}\,. \tag{3.3.10}$$

This is the localization locus. Note that the solutions are still subject to gauge transformations, and must be further constrained.

By looking at the transformations of the fermions in the standard cohomological multiplet 3.1.50, it is clear that the localization locus corresponds to the fixed-points of the BRST-like symmetry. In other words, $\mathcal{U}$ are the bosonic configurations for which the fermionic variations vanish. To see this, notice that the configurations $0 = D_i\phi = f$ imply

$$\delta\psi_i = -\epsilon D_i\phi = 0 \tag{3.3.11}$$

$$\delta\chi = \epsilon H = \epsilon f = 0 \tag{3.3.12}$$

For the last fermionic variation $\delta\eta = \epsilon\,[\phi, \lambda]$ to vanish, one must further constrain ϕ and λ to be commuting.

The coordinates on the localization locus are the moduli. The moduli fall into two cases depending on whether the G-connection $A = A_i \mathrm{d}x^i$ is irreducible or reducible. A is irreducible when its stabilizer coincides with the center of the structure group $\mathrm{Stab}_G(A) = Z(G)$, and irreducible when $\mathrm{Stab}_G(A) \neq Z(G)$. Beginning with the simpler of the two cases, if A is irreducible, then the solution of $D_i\phi = 0$ is $\phi = 0$. In this case, the space of solutions $\mathcal{U}$ coincides with the moduli space of flat connections, up to gauge transformations. The more subtle situation is when A is reducible, then $D_i\phi = 0$ has solutions $\phi \neq 0$. For a reference, see, e.g., [3] or page 1323 of [32].

3.3.2 Moduli space of Yang-Mills connections

The second localizing term is a Q-exact deformation of the first action, as described in 3.1.76. As we will see, this action localizes to Yang-Mills connections, i.e. solutions A_i of $D_i f = 0$, where $f = F_{12}$. The action is

$$\begin{aligned} S_5(t) &= -i\left\{Q, V + tV'\right\} && (3.3.13)\\ &= \frac{1}{h^2}\int_\Sigma \mathrm{d}\mu \mathrm{Tr}\bigg(\frac{1}{2}(H - f - \lambda t)^2 - \frac{1}{2}(f + t\lambda)^2 - D_i\lambda D^i\phi && (3.3.14)\\ &\quad + \frac{i}{2}\chi[\phi,\chi] - i\chi \star D\psi + iD_i\eta\psi^i + i[\psi_i,\lambda]\psi^i - it\chi\eta\bigg) && (3.3.15) \end{aligned}$$

The path integral that defines the theory on Σ is

$$I' = \int DAD\psi D\phi D\lambda D\eta D\chi DHe^{-S_5} \qquad (3.3.16)$$

The path integral fulfills the criteria for localization to occur in the limit $h \to 0$, as outlined in chapter 2. In particular, the action is Q-closed, the measure is Q-invariant, both V and S_4 are well defined functionals, and reality conditions will be chosen such that the path integral converges.

To evaluate the localization locus of S_5, we must solve

$$0 = \mathrm{Re}\left[\mathrm{Tr}\frac{1}{2}(H - f - \lambda t)^2 - \frac{1}{2}(f + t\lambda)^2 - D_i\lambda D^i\phi\right] \qquad (3.3.17)$$

for the fields H, A_i, λ, and ϕ. As before, A_i sits in both $f = F_{12}$ and $D_i = \partial_i + A_i$.

First expanding the fields as $\Phi = \Phi^a T_a$, and the covariant derivatives as $D_i\Phi = (D_i\Phi)^a T_a$, for anti-hermitian T_a, then choosing e.g. real H^a, A_i^a, ϕ^a, imaginary λ^a, and setting $t = -iu$, equation 3.3.17 reduces to

$$0 = H^a - f^a - u\lambda'^a = f^a + u\lambda'^a = i\left(D_i\lambda'\right)^a = (D_i\phi)^a\,. \qquad (3.3.18)$$

Here, $\lambda' = \mathrm{Im}(\lambda)$, or equivalently, $\lambda = i\lambda'$ for real λ'. The implications are as follows. To begin, the solution for the auxiliary field is

$$H = f + u\lambda'. \tag{3.3.19}$$

Next, the real scalar ϕ is constrained to be covariantly constant

$$0 = D_i\phi = \partial_i\phi + [A_i, \phi]\,. \tag{3.3.20}$$

The crucial point is that the two equations for λ' imply the Yang-Mills equations. On the one hand, λ' is constrained to be covariantly constant

$$iD_i\lambda' = 0, \tag{3.3.21}$$

while, on the other hand, we have

$$\lambda' = -\frac{f}{u}. \tag{3.3.22}$$

Importantly, this equation is *singular* for $u = it = 0$. Replacing λ' in 3.3.21 with its value from 3.3.22, one obtains

$$0 = iD_i\lambda' = -\frac{i}{u}D_i f. \tag{3.3.23}$$

In particular, the scalar field strength f is constrained to be covariantly constant

$$0 = D_i f, \tag{3.3.24}$$

which are precisely the classical Yang-Mills equations. Solutions of A_i, ϕ, λ', H of the combined equations 3.3.18 once again result in $0 = \delta\psi_i = \delta\chi$, but this time, $\delta\chi = \epsilon H$ vanishes due to $H = f + u\lambda' = f - f = 0$.

The upshot of the deformed $S\,(t \neq 0)$ theory is that it localizes to a locus permitting new components that are absent in the original $S\,(t = 0)$ theory, since the $\lambda \sim 1/t$ solutions are singular at $t = 0$. For this reason, the new components of the localization locus are said to flow in from infinity. The difference is most significant for the gauge field. In the case of the $S\,(t = 0)$ theory, the gauge field localizes to the moduli space of flat connections, up to gauge transformations, while in the case of the $S\,(t \neq 0)$ theory, the gauge field localizes to the moduli space of Yang-Mills connections, up to gauge transformations.

3.3.3 Localization computation details

In this section, we provide technical details on how to localize to the moduli space of Yang-Mills connections following the localization computation of [3]. Specifically, we review how to evaluate the necessary integrals. We rederive the locus of Yang-Mills connections, but do not perform the full localization computation.

The path integral to localize is the same as before

$$I' = \int DAD\psi D\phi D\lambda D\eta D\chi DH e^{-\frac{1}{h^2}\int_\Sigma \mathrm{d}\mu L(t)} \tag{3.3.25}$$

where the Lagrangian density is

$$\begin{aligned} L(t) &= \mathrm{Tr}\bigg(\frac{1}{2}(H-f-\lambda t)^2 - \frac{1}{2}(f+t\lambda)^2 + \frac{i}{2}\chi[\phi,\chi] \\ &\quad -i\chi\star D\psi + iD_i\eta\psi^i + i[\psi_i,\lambda]\psi^i - D_i\lambda D^i\phi - ti\chi\eta\bigg) \end{aligned}$$

Localization proceeds by first integrating out H, then taking the localizing limit $h\to 0$ of the path integral. Following this, we find a situation in which it is possible to evaluate the path integrals of λ, χ, η, and ϕ for $t \neq 0$, and localize to Yang-Mills connections.

To begin, observe that the integral over H in is a Gaussian integral peaked around $H = f + \lambda t$. Therefore, H may be integrated out according to

$$\int_{-\infty}^{\infty} \mathrm{d}x e^{-\frac{1}{2}(x-y)^2 - \frac{1}{2}y^2} = \sqrt{2\pi}e^{-\frac{1}{2}y^2}, \tag{3.3.26}$$

where x and y play the roles of H and $f+\lambda t$, respectively. That is to say, H is a conventional auxiliary field and is eliminated by setting it to its on-shell value $H = f + \lambda t$ in the Lagrangian $L(t)$.

Having eliminated H, the path integral to evaluate is

$$I_1' = \int DAD\psi D\phi D\lambda D\eta D\chi e^{-\frac{1}{h^2}\int_\Sigma \mathrm{d}\mu L_1(t)} \tag{3.3.27}$$

where

$$L_1(t) = \mathrm{Tr}\bigg(-\frac{1}{2}(f+t\lambda)^2 + \lambda\left(D_iD^i\phi - i\left[\psi_i,\psi^i\right]\right) \tag{3.3.28}$$

$$+\frac{i}{2}\chi[\phi,\chi] - i\chi\star D\psi + iD_i\eta\psi^i - ti\chi\eta.\bigg) \tag{3.3.29}$$

Here, integration by parts was used to express $-D_i\lambda D^i\phi$ as $\lambda D_iD^i\phi$, and the cyclicity of the trace was used to equate $\mathrm{Tr}\left([\psi_i,\lambda]\psi^i\right) = \mathrm{Tr}\left(-\lambda\left[\psi_i,\psi^i\right]\right)$, where the anti-commutativity of λ and ψ_i was taken into account.

In the $h \to 0$ limit, it is possible to evaluate the integral over the bosonic scalar λ for $t \neq 0$. In particular, λ may be integrated out by setting

$$\lambda = -\frac{f}{t}. \tag{3.3.30}$$

in $L_1(t)$. Notice that this equation is singular for $t = 0$. To illustrate, consider an analogous one-dimensional integral

$$I_x = \int_{-\infty}^{+\infty} \mathrm{d}x e^{\frac{1}{h^2}\left(-\frac{1}{2}(ixt+y)^2+ixb+c\right)}, \tag{3.3.31}$$

which corresponds to integrating out purely imaginary $\lambda \in i\mathbb{R}$. Here, ix and y assume the roles of λ and f, while b denotes $D_i D^i \phi - i\left[\psi_i, \psi^i\right]$, and c denotes the remaining terms. This integral may be evaluated in the limit of small real h and large imaginary t, while keeping the ratio $g = h/t$ fixed. Specifically, setting $t = -iu$ and $h = -igu$, the integral becomes

$$I_x = \int_{-\infty}^{+\infty} \mathrm{d}x e^{-\frac{1}{2g^2}\left(x+\frac{1}{u}y+\frac{i}{u^2}b\right)^2 - \frac{1}{2g^2u^4}\left(b^2-2iuby+2u^2c\right)} \tag{3.3.32}$$

This Gaussian integral peaked around $x = -\frac{1}{u}y - \frac{i}{u^2}b$. The integral evaluates to

$$I_x = \sqrt{2\pi} g e^{-\frac{1}{2g^2u^4}\left(b^2-2iuby+2u^2c\right)} \tag{3.3.33}$$

Here, the factor of g does not pose a problem as it is a fixed ratio.

After integrating out λ, the path integral is

$$I_2' = \int DAD\psi D\phi D\eta D\chi \frac{h}{t} e^{-S_2(t)} \tag{3.3.34}$$

for

$$\begin{aligned} S_2(t) &= \frac{1}{h^2}\int_\Sigma \mathrm{d}\mu \mathrm{Tr}\Big(-\frac{1}{t}\left(fD_iD^i\phi - if\left[\psi_i,\psi^i\right]\right) + \frac{1}{2t^2}\left(D_iD^i\phi - i\left[\psi_i,\psi^i\right]\right)^2 \\ &\quad + i\chi \star D\psi - i\eta\left(D_i\psi^i + t\chi\right) + \frac{i}{2}\chi\left[\chi,\phi\right]\Big) \end{aligned}$$

The integral over the fermionic scalars η and χ may be evaluated using the properties of Grassmann-odd δ-functions, which can be found in section 1.5.5 of [35]. The odd δ-function is

$$\int \mathrm{d}\xi e^{i\xi\theta} = i\delta(\theta), \tag{3.3.35}$$

for Grassmann-odd θ and ξ. To integrate out η, the necessary property of the odd

δ-function is

$$\beta\delta\left(\theta-\alpha\right)=\delta\left(\beta\theta-\beta\alpha\right) \tag{3.3.36}$$

for Grassmann-even α, β. To integrate out χ, on the other hand, the relevant property is

$$\int \mathrm{d}\theta\,\delta\left(\theta-\alpha\right)f\left(\theta\right)=f\left(\alpha\right). \tag{3.3.37}$$

Using property 3.3.35, the integral over η is

$$\frac{h}{t}\int D\eta\exp\left(\frac{1}{ih^2}\int_\Sigma \mathrm{d}\mu\mathrm{Tr}\left(\eta\left(D_i\psi^i+t\chi\right)\right)\right) \sim \frac{h}{it}\prod_{x\in\Sigma}\delta\left(D_i\psi^i+t\chi\right) \tag{3.3.38}$$

By property 3.3.36, this simplifies to

$$\frac{h}{it}\prod_{x\in\Sigma}\delta\left(D_i\psi^i+t\chi\right)=\prod_{x\in\Sigma}\delta\left(\frac{h}{it}D_i\psi^i-ih\chi\right) \tag{3.3.39}$$

After integrating out η, the path integral is

$$I_3'=\int DAD\psi D\phi D\chi\prod_{x\in\Sigma}\delta\left(-i\frac{h}{t}D_i\psi^i-ih\chi\right)e^{-S_3(t)} \tag{3.3.40}$$

for

$$S_3\left(t\right) = \frac{1}{h^2}\int_\Sigma \mathrm{d}\mu\mathrm{Tr}\Bigg(-\frac{1}{t}\left(fD_iD^i\phi-if\left[\psi_i,\psi^i\right]\right) \tag{3.3.41}$$

$$+\frac{1}{2t^2}\left(D_iD^i\phi-i\left[\psi_i,\psi^i\right]\right)^2+i\chi\star D\psi+\frac{i}{2}\chi\left[\chi,\phi\right]\Bigg) \tag{3.3.42}$$

By property 3.3.37, one finds that χ may be integrated out of the path integral by setting it to $\chi=-\frac{1}{t}D_i\psi^i$ in the action. The result of integrating out χ is

$$I_4'=\int DAD\psi D\phi e^{-S_4(t)} \tag{3.3.43}$$

for

$$S_4\left(t\right) = \frac{1}{h^2}\int_\Sigma \mathrm{d}\mu\mathrm{Tr}\Bigg(\frac{1}{t}\left(\phi D_iD^if+if\left[\psi_i,\psi^i\right]-iD_k\psi^k\epsilon^{ij}D_i\psi_j\right) \tag{3.3.44}$$

$$+\frac{1}{t^2}\left(\frac{i}{2}D_i\psi^i\left[D_i\psi^i,\phi\right]+\frac{1}{2}\left(-D_iD^i\phi+i\left[\psi_i,\psi^i\right]\right)^2\right)\Bigg) \tag{3.3.45}$$

Here, integration by parts was used to express $-fD_iD^i\phi$ as ϕD_iD^if.

The integral over the bosonic scalar ϕ may be evaluated in the limit of large imaginary t using Gaussian integration. In this case, ϕ acts as a Lagrange multiplier that imposes the Yang-Mills equations. Following the approach of [3], we set $t=-iu$, and discard terms of order $1/t^2$ since terms of order $1/t$ already yield non-degenerate kinetic energy. In this case, the integral over ϕ is

$$\int D\phi \exp\left(\frac{i}{h^2 u}\int_\Sigma \mathrm{d}\mu \mathrm{Tr}\phi D_i D^i f\right) \sim \prod_{x\in\Sigma} \delta\left(D_i D^i f\right). \tag{3.3.46}$$

To derive the Yang-Mills equations from the argument of the δ-function, integration by parts may be used to write

$$0 = \int_\Sigma \mathrm{d}\mu \mathrm{Tr} f D_i D^i f = -\int_\Sigma \mathrm{Tr}\left(D_i f\right)^2. \tag{3.3.47}$$

From this, it follows that the δ-function only has support on the Yang-Mills equations $D_i f = 0$.

3.4 Recovering physical YM2 from cohomological YM2

In this section, we briefly outline the recovery of the "physical" YM2 partition function from the result of localizing the cohomological gauge theory path integral following the exposition in sections 3 and 4 of [3]. To recover physical YM2 from cohomological YM2, the path integral to localize is

$$\frac{1}{\mathrm{vol}G}\int DAD\psi D\phi \cdot \beta e^{-S_{\mathrm{YM}}^{\mathrm{phys.}} - \frac{1}{u}\delta V_{\mathrm{loc}}} \tag{3.4.1}$$

where

$$S_{\mathrm{YM}}^{\mathrm{phys.}} = \frac{1}{4\pi^2}\int_\Sigma \mathrm{Tr}\left(-i\phi F - \frac{1}{2}\psi\wedge\psi\right) - \frac{\epsilon}{8\pi^2}\int_\Sigma \mathrm{d}\mu \mathrm{Tr}\phi^2 \tag{3.4.2}$$

$$\delta V_{\mathrm{loc}} = \frac{i}{h^2}\int_\Sigma \mathrm{d}\mu \mathrm{Tr}\left(D_i f D^i \phi + if\left[\psi_i, \psi^i\right] - iD_l \psi^l \epsilon^{ij} D_i \psi_j\right) \tag{3.4.3}$$

The relationship to operators described in [3] is $S_{\mathrm{YM}}^{\mathrm{phys.}} = -\omega - \epsilon\Theta$, and $\delta V_{\mathrm{loc}} = u \cdot L''(u)$. The localizing limit takes one from the cohomological theory $S_{\mathrm{YM}}^{\mathrm{coh.}} = S_{\mathrm{YM}}^{\mathrm{phys.}} + \frac{1}{u}\delta V_{\mathrm{loc}}$to the physical theory $S_{\mathrm{YM}}^{\mathrm{phys.}}$, that is

$$\lim_{u\to\infty} \frac{1}{\mathrm{vol}G}\int DAD\psi D\phi \cdot \beta e^{-S_{\mathrm{YM}}^{\mathrm{phys.}} - \frac{1}{u}\delta V_{\mathrm{loc}}} \tag{3.4.4}$$

$$= \frac{1}{\mathrm{vol}G}\int DAD\psi D\phi \cdot \beta e^{-S_{\mathrm{YM}}^{\mathrm{phys.}}} \tag{3.4.5}$$

The physical partition function, derived using TQFT techniques, reads

$$Z\left(\epsilon, G, \Sigma_g\right) = \frac{1}{(\mathrm{vol}G)^{2-2g}}\sum_\alpha \frac{1}{(\dim\alpha)^{2g-2}} e^{-\frac{1}{2}\epsilon\tilde{C}_2(\alpha)} \tag{3.4.6}$$

For $G = SU(2)$ we have $\mathrm{vol}SU(2) = 2^{5/2}\pi^2$, $\tilde{C}_2(\alpha_n) = \frac{1}{2}n^2$, $\dim SU(2) = 3$, and the physical partition function is expressed as

$$Z(\epsilon, SU(2), \Sigma_g) = \left(32\pi^4\right)^{g-1} \sum_{n=1}^{\infty} \frac{1}{n^{2g-2}} e^{-\frac{1}{4}\epsilon n^2} \tag{3.4.7}$$

To compare the physical partition function to the result of localization, the first step is to remove n^{2-2g} term from the summand. This is done by taking the $(g-1)$th derivative of Z with respect to ϵ, which brings down the appropriate factors from the exponential to cancel $n^{2(1-g)}$:

$$\frac{\partial^{g-1} Z}{\partial \epsilon^{g-1}} = \left(-8\pi^4\right)^{g-1} \sum_{n=1}^{\infty} e^{-\frac{1}{4}\epsilon n^2}. \tag{3.4.8}$$

Then the sum may be recast as

$$\sum_{n=1}^{\infty} e^{-\frac{1}{4}\epsilon n^2} = -\frac{1}{2} + \frac{1}{2} \sum_{n\in\mathbb{Z}} e^{-\frac{1}{4}\epsilon n^2} \tag{3.4.9}$$

in order to obtain the expression

$$\frac{\partial^{g-1} Z}{\partial \epsilon^{g-1}} = \frac{1}{2}\left(-8\pi^4\right)^{g-1} \left(-1 + \sum_{n\in\mathbb{Z}} e^{-\frac{1}{4}\epsilon n^2}\right). \tag{3.4.10}$$

The sum is then in the correct form to apply the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} e^{-\frac{1}{4}\epsilon n^2} = \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} dn e^{-\frac{1}{4}\epsilon n^2} e^{2\pi i n m} = \sqrt{\frac{4\pi}{\epsilon}} \sum_{m\in\mathbb{Z}} e^{-\frac{1}{\epsilon}(2\pi m)^2}, \tag{3.4.11}$$

in order to obtain the expression

$$\frac{\partial^{g-1} Z}{\partial \epsilon^{g-1}} = \frac{1}{2}\left(-8\pi^4\right)^{g-1} \left(-1 + \sqrt{\frac{4\pi}{\epsilon}} \sum_{m\in\mathbb{Z}} e^{-\frac{1}{\epsilon}(2\pi m)^2}\right). \tag{3.4.12}$$

In particular, this is the result obtained by localization, and we have gone from a sum over elements of the character lattice $n \in \mathbb{Z}$ which label irreducible representations, to a sum over elements of the cocharacter lattice $m \in \mathbb{Z}$ which labels instantons.

To identify the higher critical points of degree m, we consider the classical Yang-Mills action

$$S_{\mathrm{YM}}^{\mathrm{cl.}} = \frac{1}{2\epsilon}(\mu, \mu) = -\frac{1}{2\epsilon}\int_{\Sigma} \mathrm{d}\mu \mathrm{Tr}\left(F_A \wedge \star F_A\right) = -\frac{1}{2\epsilon}\int_{\Sigma} \mathrm{d}\mu \mathrm{Tr} f^2 \tag{3.4.13}$$

Given that the scalar field strength is

$$f = \star F_A = 2\pi i \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix} \tag{3.4.14}$$

and assuming that Σ has unit area $\int_\Sigma \mathrm{d}\mu = 1$, the classical Yang-Mills action is determined to be

$$S_{\mathrm{YM}}^{\mathrm{cl.}} = -\frac{1}{2\epsilon}\int_\Sigma \mathrm{d}\mu \mathrm{Tr} f^2 = \frac{1}{\epsilon}\left(2\pi m\right)^2. \tag{3.4.15}$$

The contribution of higher critical points to $\partial^{g-1}Z/\partial\epsilon^{g-1}$ is

$$\left.\frac{\partial^{g-1}Z}{\partial\epsilon^{g-1}}\right|_{\mathcal{M}_m} = \frac{|W|}{2}\left(-8\pi^4\right)^{g-1}\sqrt{\frac{4\pi}{\epsilon}}e^{-\frac{1}{\epsilon}(2\pi m)^2} \tag{3.4.16}$$

where $|W|$ is the order of the Weyl group, and $\mathcal{M}_m$ is the locus of higher critical points. Note$|W| = 2$, due to the symmetry between $\pm m$. For a reference, see sections 4.2-4.3 of [36].

Chapter 4

The Benini-Zaffaroni approach to A-twisted supersymmetric localization

In this chapter, we review the localization computation of Benini & Zaffaroni in [18, 19]. In particular, we review the Benini-Zaffaroni (BZ) approach to supersymmetric localization of A-twisted gauge theories on compact manifolds, in which localization is aided by the Jeffrey-Kirwan residue theorem. Several features of the BZ approach are reviewed in, e.g., [37, 38].

4.1 Summary and results

In [18], Benini & Zaffaroni studied topologically A-twisted supersymmetric gauge theories on compact manifolds, obtaining a general formula for the partition function of $\mathcal{N} = 2$ gauge theories on $S^2 \times S^1$. The critical step was to reduce the path integral, using aspects of supersymmetric localization, to a contour integral over a meromorphic function and thereafter evaluate the contour integral in terms of Jeffrey-Kirwan (JK) residues.

The three-dimensional partition function was generalized to the omega background, $\mathcal{N} = (2, 2)$ gauge theories on S^2, and $\mathcal{N} = 1$ gauge theories on $S^2 \times T^2$. The resulting partition functions were utilized to compute supersymmetric observables and confirm non-perturbative dualities in several examples in two, three, and four dimensions. Given their novel procedure in evaluating the path integral, Benini & Zaffaroni provided an alternative approach to computing two-dimensional partition functions and observables, including genus-zero A-model topological amplitudes and Gromov-Witten invariants.

In [19], The partition function formulae were generalized to $\mathcal{N} = (2, 2)$ gauge theories on Σ_g, $\mathcal{N}=2$ gauge theories on $\Sigma_g \times S^1$, and $\mathcal{N}=1$ gauge theories on $\Sigma_g \times T^2$, where Σ_g is an arbitrary Riemann surface of genus g. Consequently, they obtained new non-perturbative results, which included the computation of observables and the testing of dualities.

The path integrals localize to BPS configurations specified by the pair $(u, \mathfrak{m})$ which are moduli, or gauge parameters, and are valued as

$$\mathfrak{m} = \frac{1}{2\pi}\int_{\Sigma_g} F \in \Lambda^G_{\text{coch}}, \quad u \in \mathfrak{M}, \tag{4.1.1}$$

up to gauge transformations. In both two and three dimensions, the gauge flux $\mathfrak{m} \in \Gamma_{\mathfrak{h}}$ is quantized to the cocharacter (GNO) lattice $\Lambda^G_{\text{coch}} = \{y \in \mathfrak{h} \mid e^{2\pi i y} = 1_G\}$, where $\mathfrak{h}$ is the Cartan subalgebra, and 1_G is the identity element of G. In the three-dimensional theory, the continuous parameter is

$$u = A_t + i\beta\sigma, \quad \mathfrak{M} = H \times \mathfrak{h}, \tag{4.1.2}$$

where $A_t = \beta A_3$ is a flat connection along the S^1 whose radius is β, and σ is the real scalar in the three-dimensional vector multiplet, H is the maximal torus of G. It is also convenient to define $x = e^{iu}$. In the two-dimensional theory, the continuous parameter is

$$u = \sigma, \quad \mathfrak{M} \simeq \mathfrak{h}_{\mathbb{C}} \tag{4.1.3}$$

where σ is the complex scalar in the two-dimensional vector multiplet, and $\mathfrak{h}_{\mathbb{C}}$ is the complexified Cartan subalgebra.

In addition to gauge parameters, the theories are assumed to have a flavor symmetry group G_F, and flavor parameters $(v, \mathfrak{n})$, where

$$\mathfrak{n} = \frac{1}{2\pi}\int_{\Sigma_g} F^{\text{flav}} \in \Lambda^{G_F}_{\text{coch}}. \tag{4.1.4}$$

In both 3d and 2d, the flavor flux $\mathfrak{n}$ is GNO quantized, meaning that it is valued as $\mathfrak{n} \in \Lambda^{G_F}_{\text{coch}}$ where $\Lambda^{G_F}_{\text{coch}} = \{s \in \mathfrak{h}_F | e^{2\pi i s} = 1_{G_F}\}$ is the cocharacter (GNO) lattice of the flavor group G_F. Here, $\mathfrak{h}_F = \text{Lie}H_F$ is the Cartan subalgebra, H_F is a maximal torus of G_F, and 1_{G_F} is the identity element in G_F.

Using localization techniques, the partition function of each A-twisted theory is reduced to a contour integral which schematically reads

$$Z_{\Sigma_g \times T^n} = \frac{1}{|W|}\sum_{\mathfrak{m}\in\Gamma_{\mathfrak{h}}} \oint_{\mathcal{C}} Z_{\text{int}}\left(u, \mathfrak{m}; v, \mathfrak{n}\right)\ , \tag{4.1.5}$$

Here, T^n is an $n = 1, 2, 3$ dimensional torus, $|W|$ is the order of the Weyl group of G, the discrete modulus $\mathfrak{m}$ is summed over, while the continuous modulus u is integrated over. The integrand, which includes both the classical and one-loop contributions, is a function of both the gauge parameters $(u, \mathfrak{m})$ and the flavor parameters $(v, \mathfrak{n})$.

An important result of [18, 19] is the partition function formula

$$Z_{\Sigma_g \times T^n} = \sum_{u=u_{(\alpha)}} Z_{\text{cl},1\text{l}}\big|_{\mathfrak{m}=0} \left(\det_{ab} \frac{\partial B_a}{\partial u_b} \right)^{g-1} \tag{4.1.6}$$

where

$$iB_a = \frac{\partial \log Z_{\text{cl},1\text{l}}}{\partial \mathfrak{m}_a}, \tag{4.1.7}$$

and $u_{(\alpha)}$ are a set of solutions to the so-called equations Bethe-Ansatz equations (BAE)

$$e^{iB_a} = 1, \tag{4.1.8}$$

for which the Vandermonde determinant is non-zero. This form of the partition function is only valid when the roots of the BAE are simple. For non-simple BAE roots, the more general formula for the partition function is provided as a sum over JK-residues

$$Z_{\Sigma_g \times T^n} = \frac{1}{|W|} \sum_{\mathfrak{m} \in \Lambda^G_{\text{coch}}} \sum_{u_* \in \mathfrak{M}^*_{\text{sing}}} \text{JKRes}_{u=u_*} (\mathsf{Q}_{u_*}, \eta) \left(-\det_{ab} \frac{\partial^2 \log Z_{\text{cl},1\text{l}}}{\partial u_a \, \partial \mathfrak{m}_b} \right)^g Z_{\text{cl},1\text{l}} + \text{bound. contrib.} \tag{4.1.9}$$

Here, $u_* \in \mathfrak{M}^*_{\text{sing}}$ are points in the singular submanifold of the bosonic moduli space $\mathfrak{M}$, Q_{u_*} is a set of charges, η is an auxiliary parameter, a, b are adjoint indices, and the term with exponent is g is the contribution of higher-genus zero modes.

4.2 Topological A-twist

Supersymmetry is preserved on Σ_g by performing a topological A-twist. In particular, half of the flat space supercharges are preserved on Σ_g by solving the twisted Killing spinor equations for half of the supersymmetry parameters of the flat space theory. The A-twist is performed by turning on a connection V_μ on Σ_g coupled to a background $U(1)_R$ R-symmetry current:

$$V = -\frac{1}{2}\omega^{12}, \; W = \text{d}V = -\frac{\mathcal{R}}{4} e^1 \wedge e^2, \; \frac{1}{2\pi} \int_{\Sigma_g} W = g - 1 \tag{4.2.1}$$

where ω^{ab}_μ is the spin connection, and $\mathcal{R}$ is the Ricci scalar curvature. The supersymmetry parameters are commuting Dirac spinors $\epsilon, \tilde{\epsilon}$ of R-charge -1. Using the chiral projection operators $P_\pm = \frac{1}{2}(1 \pm \gamma_3)$, these may be expressed in terms of four

Weyl spinors $\epsilon_+, \epsilon_-, \tilde{\epsilon}_+, \tilde{\epsilon}_-$. The twisted Killing spinor equations

$$D_\mu \epsilon = D_\mu \tilde{\epsilon} = 0 \tag{4.2.2}$$

are solved by constant positive chirality spinors $\epsilon_+, \tilde{\epsilon}_+$ satisfying $0 = \partial_\mu \epsilon_+ = \partial_\mu \tilde{\epsilon}_+$, and vanishing negative chirality spinors $0 = \epsilon_- = \tilde{\epsilon}_-$. The solution follows from $D_\mu \epsilon = (\partial_\mu - i\omega_\mu^{12} P_-)\epsilon$, and similarly for $\tilde{\epsilon}$. Notice that this is slightly unconventional; usually the two-dimensional A-twist preserves supersymmetry parameters of opposite chirality and R-charge.

4.2.1 3d A-twisted theories

Here we provide details regarding the three dimensional topologically A-twisted theories considered in [18, 19]. The three-dimensional theories considered in the localization computation were constructed from the supermultiplets of topologically twisted $\mathcal{N} = 2$ supersymmetry on $\Sigma_g \times S^1$.

The supermultiplets are the chiral multiplet and the vector multiplet. The chiral multiplet is $\Phi^{(3d)} = (\phi, \psi, F)$, where all the component fields are in a representation $\mathfrak{R}$ of the gauge group, and the R-charges are $(q_R, q_R - 1, q_R - 2)$, respectively. Here, ϕ is a complex scalar, ψ is a Dirac spinor, and the auxiliary field F is a complex scalar. The anti-chiral multiplet is $\Phi^{\dagger(3d)} = \left(\phi^\dagger, \psi^\dagger, F^\dagger\right)$, where the component fields are in the conjugate representation $\overline{\mathfrak{R}}$, and the R-charges are $(-q_R, 1 - q_R, 2 - q_R)$, respectively. The vector multiplet is $\mathcal{V}^{(3d)} = \left(A_\mu, \sigma, \lambda, \lambda^\dagger, D\right)$ in Wess-Zumino (WZ) gauge, where all the component fields are in the adjoint representation of the gauge group, and the R-charges are respectively $(0, 0, -1, 1, 0)$. Here $A_{\mu=1,2,3}$ is the gauge field, σ is a scalar, $\lambda, \lambda^\dagger$ are independent complex Dirac spinors, while the auxiliary field D is a bosonic scalar.

The supersymmetry parameters of the theory are two positive chirality covariantly constant commuting spinors $\epsilon, \tilde{\epsilon}$ of R-charge -1, satisfying $D_\mu \epsilon = 0$, $\gamma_3 \epsilon = 0$, and similarly for $\tilde{\epsilon}$. The two anti-commuting complex supercharges are $Q, \widetilde{Q}$ of vanishing R-charge. The supersymmetry variation is

$$\delta = \delta_\epsilon + \delta_{\tilde{\epsilon}} = \epsilon^\alpha Q_\alpha + \bar{\epsilon}^\alpha \widetilde{Q}_\alpha, \quad Q = \epsilon^\alpha Q_\alpha, \quad \widetilde{Q} = \tilde{\epsilon}^{c\alpha} \widetilde{Q}_\alpha = -\left(\tilde{\epsilon}^\dagger C\right)^\alpha \widetilde{Q}_\alpha \tag{4.2.3}$$

where $\tilde{\epsilon} = -C\bar{\epsilon}^*$.

The three-dimensional algebra of the supercharges is

$$\{Q, \widetilde{Q}\} = -i\mathcal{L}_v^A - \delta_{\text{gauge}}(\sigma), \qquad Q^2 = \widetilde{Q}^2 = 0 \tag{4.2.4}$$

Here, $\mathcal{L}_v^A$ is a Lie derivative along the covariantly constant Killing vector field $v =$

$\beta^{-1}\tilde{\epsilon}^\dagger\epsilon\partial_t$, and includes both the gauge connection and the R-symmetry connection. The term $\delta_{\text{gauge}}(\sigma)$ denotes an infinitesimal gauge transformation in which the gauge parameter is the 3d real scalar σ.

The localizing supercharge is the linear combination

$$\mathcal{Q} = Q + \tilde{Q} \tag{4.2.5}$$

4.2.2 2d A-twisted theories

The two-dimensional theories considered in the localization computation were constructed from the supermultiplets of topologically twisted $\mathcal{N} = (2,2)$ supersymmetry on Σ_g .

The supermultiplets considered were the chiral multiplet $\Phi^{(2d)}$, the vector multiplet $\mathcal{V}^{(2d)}$. The chiral multiplet is $\Phi^{(2d)} = (\phi, \psi, F)$, where all the component fields are in a representation $\mathfrak{R}$ of the gauge group. Here, ϕ is a complex scalar, ψ is a complex Dirac spinor, and the auxiliary field F is a complex scalar. The anti-chiral multiplet $\Phi^{\dagger(2d)}$ is defined similarly to the three-dimensional case. The vector multiplet is $\mathcal{V}^{(2d)} = \left(A_i, \sigma, \overline{\sigma}, \lambda, \lambda^\dagger, D\right)$ in WZ gauge, where all the component fields are in the adjoint representation of the gauge group. Here, $A_{i=1,2}$ is the gauge field, $\sigma, \overline{\sigma}$ are independent complex scalars, $\lambda, \lambda^\dagger$ are independent complex Dirac spinors, while the auxiliary field D is a bosonic scalar. The adjoint-valued fields take the form $\Phi = \Phi^a T_a$ for hermitian T_a, where a is an adjoint index.

The two-dimensional results were primarily obtained by dimensional reduction from the three-dimensional results. We record here the dimensional reduction from the three-dimensional vector multiplet $\mathcal{V}^{(3d)}$ to the two dimensional vector multiplet $\mathcal{V}^{(2d)}$. The procedure for the dimensional reduction is to take $A_3 \to \sigma_1, \sigma \to \sigma_2$, then $\sigma_1 + i\sigma_2 \to -i\sigma$, $\sigma_1 - i\sigma_2 \to i\bar{\sigma}$, followed by $F_{\mu 3} \to D_\mu \sigma_1$, $D_3 \to -i\left[\sigma_1, \cdot\right]$, and $[\sigma_1, \sigma_2] \to \frac{i}{2}\left[\sigma, \bar{\sigma}\right]$.

The two-dimensional algebra of the supercharges is

$$\{Q, \widetilde{Q}\} = -\delta_{\text{gauge}}(\sigma), \qquad Q^2 = \widetilde{Q}^2 = 0. \tag{4.2.6}$$

Here, $\delta_{\text{gauge}}(\sigma)$ denotes an infinitesimal gauge transformation in which the gauge parameter is the 2d complex scalar σ. Notice that the 2d algebra no longer involves the Lie derivative $\mathcal{L}_v^A$ along the Killing vector field $v = \beta^{-1}\tilde{\epsilon}^\dagger\epsilon\partial_t$, since it is along S^1 which was shrunk to a point in the dimensional reduction.

The full supersymmetric Lagrangian of the two-dimensional theory is

$$L = L_{\text{YM}} + L_{\widetilde{W}+\overline{\widetilde{W}}} + L_{\text{mat}} + L_{W+\overline{W}} \tag{4.2.7}$$

where $L_{\rm YM}$ is the Yang-Mills Lagrangian, $L_{\widetilde{W}+\overline{\widetilde{W}}} = L_{\widetilde{W}} + L_{\overline{\widetilde{W}}}$ is the Lagrangian for the twisted chiral superpotential, $L_{\rm mat}$ is the matter kinetic Lagrangian, and $L_{W+\overline{W}} = L_W + L_{\overline{W}}$ is the Lagrangian for the chiral superpotential. The exact Lagrangians are

$$\tilde{\epsilon}^\dagger \epsilon L_{\rm YM} \simeq Q\widetilde{Q} V_{\rm YM}, \quad \tilde{\epsilon}^\dagger \epsilon L_{\rm mat} \simeq Q\widetilde{Q} V_{\rm mat}, \quad \tilde{\epsilon}^\dagger \epsilon L_W \simeq Q\widetilde{Q} V_W. \tag{4.2.8}$$

up to total derivatives. Here, $V_{\rm YM}, V_{\rm mat}, V_W$ are fermionic functionals. The localizing term considered was

$$L_{\rm loc} = \frac{1}{\mathsf{e}^2} L_{\rm YM} + \frac{1}{\mathsf{g}^2} L_{\rm mat}. \tag{4.2.9}$$

The actions that are functionals of the vector multiplet fields are the Yang-Mills action and the twisted superpotential action. The Yang-Mills action is

$$S_{\rm YM} = \int d^2x \sqrt{g} L_{\rm YM}. \tag{4.2.10}$$

$$L_{\rm YM} = \mathrm{Tr}\left(\frac{1}{2} F_{12}^2 + \frac{1}{2} D^2 + \frac{1}{2} D_\mu \overline{\sigma} D^\mu \sigma + \frac{1}{8}[\sigma, \overline{\sigma}]^2 + \text{fermions}\right), \tag{4.2.11}$$

The twisted superpotential action is

$$S_{\widetilde{W}} = \int d^2x \sqrt{g} \left(L_{\widetilde{W}} + L_{\overline{\widetilde{W}}} \right) \tag{4.2.12}$$

$$L_{\widetilde{W}} = i\mathrm{Tr}\frac{\partial \widetilde{W}}{\partial \sigma_a}(D + iF_{12})_a - \frac{i}{2}\mathrm{Tr}\frac{\partial \widetilde{W}}{\partial \sigma_a \partial \sigma_b}\lambda_a^\dagger (1 - \gamma_3)\lambda_b \tag{4.2.13}$$

$$L_{\overline{\widetilde{W}}} = -i\mathrm{Tr}\frac{\partial \overline{\widetilde{W}}}{\partial \overline{\sigma}_a}(D - iF_{12})_a + \frac{i}{2}\mathrm{Tr}\frac{\partial \overline{\widetilde{W}}}{\partial \overline{\sigma}_a \partial \overline{\sigma}_b}\lambda_a^\dagger (1 + \gamma_3)\lambda_b \tag{4.2.14}$$

Here, $\widetilde{W}(\sigma)$ is the twisted chiral superpotential, $\overline{\widetilde{W}}(\overline{\sigma})$ is the twisted anti-chiral superpotential, and a, b are indices of the adjoint representation. $\widetilde{W}(\sigma)$ and $\overline{\widetilde{W}}(\overline{\sigma})$ are independent gauge-invariant holomorphic functions of their arguments.

The fields of the vector multiplet transform as

$$Q A_i = \frac{i}{2}\lambda^\dagger \gamma_i \epsilon \tag{4.2.15}$$

$$\widetilde{Q} A_i = \frac{i}{2}\tilde{\epsilon}^\dagger \gamma_i \lambda \tag{4.2.16}$$

$$Q\sigma = 0 \tag{4.2.17}$$

$$\widetilde{Q}\sigma = 0 \tag{4.2.18}$$

$$Q\bar{\sigma} = \lambda^\dagger \epsilon \tag{4.2.19}$$

$$\widetilde{Q}\bar{\sigma} = \tilde{\epsilon}^\dagger \lambda \tag{4.2.20}$$

$$QD = -\frac{i}{2} D_i \lambda^\dagger \gamma^i \epsilon + \frac{i}{2}[\sigma, \lambda^\dagger \epsilon] \tag{4.2.21}$$

$$\widetilde{Q}D = +\frac{i}{2}\tilde{\epsilon}^\dagger \gamma^i D_i \lambda - \frac{i}{2}[\sigma, \tilde{\epsilon}^\dagger \lambda] \tag{4.2.22}$$

$$Q\lambda = \Big(iF_{12} - D + \frac{i}{2}[\sigma, \bar{\sigma}] - i\gamma^i D_i \sigma\Big)\epsilon \tag{4.2.23}$$

$$\widetilde{Q}\lambda = 0 \tag{4.2.24}$$

$$Q\lambda^\dagger = 0 \tag{4.2.25}$$

$$\widetilde{Q}\lambda^\dagger = \tilde{\epsilon}^\dagger \Big(- iF_{12} + D + \frac{i}{2}[\sigma, \bar{\sigma}] - i\gamma^i D_i \sigma\Big) \tag{4.2.26}$$

4.3 Localization

4.3.1 Follow-your-nose localization locus (BPS configurations)

In this section we describe how to find the bosonic zero modes after choosing the canonical localization scheme for the $\mathcal{N} = (2,2)$ A-twisted vector multiplet in WZ gauge. In this case, the localizing Lagrangian is the non-negative, $\mathcal{Q}$-exact, Yang-Mills Lagrangian $L_{\text{YM}} \simeq \mathcal{Q}V_{\text{YM}}$. Observe that this section departs from the BZ approach to localization and is intended to provide context.

To obtain the bosonic zero modes, the procedure is to find the field configurations which set the bosonic part of L_{YM} to zero along the real contour. Note that the reality conditions chosen for component fields when determining the bosonic zero modes fixes integration contours for the fluctuating modes when evaluating the fluctuation determinant. It is for this reason that the choice of reality conditions on component fields at this stage is referred to as a choice of contour.

The task is then to find the field configurations for which the real bosonic part of the localizing Lagrangian vanishes. That is to say, solve

$$0 = \text{Tr}\left((F_{12})^2 + (D_E)^2 + D_\mu \overline{\sigma} D^\mu \sigma + \frac{1}{4}\left[\sigma, \overline{\sigma}\right]^2 \right). \tag{4.3.1}$$

for A_μ, D_E, σ, and $\overline{\sigma}$. The component fields are valued in the adjoint representation of $\mathfrak{g}$, and each field takes the form $\Phi = \Phi^a T_a$, while covariant derivatives acting on adjoint fields take the form $D_\mu \Phi = (D_\mu \Phi)^a$. The Lie algebra consists of hermitian matrices and the metric $(a, b) = +\mathrm{Tr}\,(a, b)$ is positive definite.

The choice of reality conditions is then to take the gauge field A_μ to be real, the complex scalars σ and $\overline{\sigma}$ to be complex conjugates, and the auxiliary field D_E to be real. That is to say

$$A_\mu \in \mathbb{R},\ \sigma = \overline{\sigma}^\dagger \in \mathbb{C},\ D_E \in \mathbb{R} \tag{4.3.2}$$

where $\sigma = i(\sigma_1 + i\sigma_2)$, $\overline{\sigma} = -i\,(\sigma_1 - i\sigma_2)$, for $\sigma_1, \sigma_2 \in \mathbb{R}$. For this choice, (4.3.1) is an equation of the type $0 = \mathrm{Tr}(A^2 + B^2)$ where A, B are hermitian matrices whose elements are real $a_{ij}, b_{ij} \in \mathbb{R}$ for $i, j = 1, 2$. This reduces to a sum of squares of real matrix elements

$$0 = \mathrm{Tr}\left(A^2 + B^2\right) = \mathrm{Tr}A^\dagger A + \mathrm{Tr}B^\dagger B = \sum_{i,j} |a_{ij}|^2 + \sum_{i,j} |b_{ij}|^2\,, \tag{4.3.3}$$

with the solution $a_{ij} = b_{ij} = 0$ since $a_{ij}, b_{ij} \neq 0$ leads to a contradiction. Consequently, we have $A = B = 0$. Applying this argument to (4.3.1), we have the solution

$$0 = F_{12} = D_E = D_\mu \sigma = D_\mu \overline{\sigma} = [\sigma, \overline{\sigma}]\,, \tag{4.3.4}$$

modulo gauge transformations. One may worry that the contracted vector indices in the kinetic term $D_\mu \overline{\sigma} D^\mu \sigma$ spoil the argument, but this is not the case as the metric used for contracting the indices is positive definite.

To proceed, (4.3.4) is solved iteratively, for which we give some details. Let us start with $0 = D_\mu \sigma = \partial_\mu \sigma - i\,[A_\mu, \sigma]$, which is an equation for A_μ and σ. There are two general cases depending on whether the gauge connection is irreducible $\mathrm{Stab}_G(A) = Z(G)$ or reducible $\mathrm{Stab}_G(A) \neq Z(G)$, where $\mathrm{Stab}_G(A) = \left\{g \in G | A = gAg^{-1} + \mathrm{d}gg^{-1}\right\}$ is the stabilizer subgroup of G and $Z(G) = \{z \in G | zg = gz\}$ is the center of G. The first case is that A_μ is irreducible and $D_\mu \sigma = 0$, then the solution is $\sigma = 0$. This solution is not considered in [18, 19], or in what follows. The second case is that A_μ is reducible and $D_\mu \sigma = 0$, then $\sigma \neq 0$, and σ is covariantly constant. In particular, the solution is that the derivative and commutator acting on σ must vanish independently, i.e. σ is constant $\partial_\mu \sigma = 0$, and commutes with the gauge field $[A_\mu, \sigma] = 0$.

Next we have $[\sigma, \overline{\sigma}] = 0$, which states that the adjoint complex scalar $\sigma = \sigma^a T_a$ commutes with its complex conjugate $\overline{\sigma} = \overline{\sigma}^b T_b$. In this case, σ can be simultaneously diagonalized by an element of the gauge group $g \in G$. That is to say, σ can be conjugated into the Cartan subalgebra $\mathfrak{h}_\mathbb{C}$ of the complexification $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$ of the Lie algebra of the gauge group $\mathfrak{g} = \mathrm{Lie}G$. After diagonalization, we have $\sigma, \overline{\sigma} \in \mathfrak{h}_\mathbb{C}$.

The equation $D_E = 0$ is the straightforward statement that the auxiliary field vanishes. Typically the auxiliary field is simply integrated out by setting it to its on-shell value.

Finally, for the field strength, we have $F_{12} = 0$ which is an equation for A_μ. This is the statement that the gauge connection A_μ must set its curvature F_{12} to zero, which is referred to as a flat connection. In view of this, A_μ is valued in the space of flat connections $\{A_\mu \in \mathcal{A} \mid F_{12} = 0\}$.

These BPS configurations are the results obtained by choosing localization scheme according to the FYN approach to localization, for the case of the A-twisted $\mathcal{N} = (2,2)$ vector multiplet in the WZ gauge. Observe that the FYN approach to localization fails to evaluate the path integral of A-twisted $\mathcal{N} = (2,2)$ theories correctly, and the rest of this chapter proceeds with the BZ approach to localization.

4.3.2 Benini-Zaffaroni localization locus (almost BPS configurations)

In this section we describe how to evaluate the Benini-Zaffaroni localization locus of almost BPS configurations [18, 19] for the case of the $\mathcal{N} = (2,2)$ A-twisted vector multiplet in WZ gauge.

The locus of almost BPS configurations differs from the locus of BPS configurations in several important ways. Firstly, the locus of almost BPS configurations is a larger locus of configurations in field space than the locus of BPS configurations. Moreover, the almost BPS configurations result in non-vanishing fermionic variations $\mathcal{Q}\lambda \neq 0$, $\mathcal{Q}\lambda^\dagger \neq 0$, and non-vanishing Localizing Lagrangian. In particular, the almost-BPS configurations are non-supersymmetric in the sense that do not set to zero the Localizing Lagrangian $L_{\text{loc}} = \mathcal{Q}V$. The upshot of localizing to the locus of almost BPS configurations is that it permits the reduction of the path integral to a contour integral over a meromorphic function, which may be evaluated correctly in terms of JK residues.

The almost-BPS configurations are obtained as follows. Consider the bosonic configurations for which the fermionic variations vanish before any reality conditions are chosen for the component fields. In particular, one solves

$$0 = \mathcal{Q}\lambda \quad = \quad \Big(iF_{12} - D + \frac{i}{2}[\sigma, \bar{\sigma}] - i\gamma^i D_i \sigma\Big)\epsilon \tag{4.3.5}$$

$$0 = \mathcal{Q}\lambda^\dagger \quad = \quad \tilde{\epsilon}^\dagger \Big(- iF_{12} + D + \frac{i}{2}[\sigma, \bar{\sigma}] - i\gamma^i D_i \sigma\Big) \tag{4.3.6}$$

for $A_\mu, D_E, \sigma, \overline{\sigma}$. The equations reduce to

$$0 = D_E - iF_{12} = D_\mu \sigma = [\sigma, \overline{\sigma}] \,, \tag{4.3.7}$$

modulo gauge transformations. Choosing all the reality conditions to be the same as in the canonical case, except for the auxiliary field, which is now permitted to remain complex. That is to say, take

$$A_\mu \in \mathbb{R},\ \sigma = \overline{\sigma}^\dagger \in \mathbb{C},\ D_E \in \mathbb{C}. \tag{4.3.8}$$

The procedure is the same as before for the equations involving the complex scalars $\sigma, \overline{\sigma}$. Solving $D_\mu \sigma = 0$ with reducible A_μ results in a scalar σ which is constant $\partial_\mu \sigma = 0$, and which commutes with the gauge field $[A_\mu, \sigma] = 0$. Using the equation $[\sigma, \overline{\sigma}] = 0$, σ is simultaneously diagonalized to the complexified Cartan subalgebra $\sigma, \overline{\sigma} \in \mathfrak{h}_{\mathbb{C}}$.

The configuration involving the field strength, however, now permits solutions that differ significantly from those considered previously. Namely, for the configuration

$$0 = D_E - iF_{12} \tag{4.3.9}$$

it is now possible to have gauge connections that are *not* flat connections. Equation 4.3.9 concerns the complex, adjoint valued, scalar, auxiliary field $D_E = D_E^a T_a$, and the real, adjoint valued gauge connection $A_\mu = A_\mu^a T_a$. To begin with, the real F_{12} and complex D_E are scalar fields, both of which are functions of the coordinates x on the manifold: $F_{12} = F_{12}(x)$, $D_E = D_E(x)$. Moreover, both F_{12} and D_E are subject to gauge transformations. There are two cases depending on whether the field strength is covariantly constant $D_\mu F_{12} = 0$, or otherwise. Observe that the field strength being covariantly constant is indeed the Yang-Mills equations.

If the field strength F_{12} satisfies the Yang-Mills equations, then the flux $\mathfrak{m} = \frac{1}{2\pi}\int_{\Sigma_g} F_{12}$ is quantized to the cocharacter (GNO) lattice $\mathfrak{m} \in \Lambda^G_{\text{coch}}$ in $\mathfrak{h}$. Accordingly, the field strength can be expressed as $F_{12} = 2\pi\mathfrak{m}/\text{Vol}\Sigma_g$. In view of this, A_μ is valued in the space of Yang-Mills connections $\{A_\mu \in \mathcal{A} | D_\mu F_{12} = 0\}$, as opposed to the space of flat connections described in the previous section. Then, the BPS configuration $D_E(x) - 2\pi i\mathfrak{m}/\text{Vol}\Sigma_g = 0$ is in the complexified gauge orbit of the configuration

$$D_E(x) - \frac{2\pi i\mathfrak{m}}{\text{Vol}\Sigma_g} = D_0, \tag{4.3.10}$$

where D_0 denotes a constant, complex, zero-mode of the complex auxiliary field $D_E(x)$. In other words, the BPS configuration is gauge equivalent to the configuration including D_0. The utility of the zero-mode D_0 is that, by starting with $D_E(x) \in \mathbb{C}$, and taking $D_0 \in \mathbb{R} - 2\pi i\mathfrak{m}/\text{Vol}\Sigma_g$, one can reach the real line

$$D_E(x) = D_0 + \frac{2\pi i\mathfrak{m}}{\text{Vol}\Sigma_g} \in \mathbb{R}. \tag{4.3.11}$$

That is to say, permitting $D_E \in \mathbb{C}$, it is possible to reach to the original choice of reality condition for the auxiliary field $D_E \in \mathbb{R}$ described in (4.3.2).

If F_{12} does not satisfy the Yang-Mills equations, then the flux is not GNO quantized, and the configuration $0 = D_E - iF_{12}$ results in real $F_{12}(x) \in \mathbb{R}$ and purely imaginary $D_E(x) \in i\mathbb{R}$. The difference in this case is that, although one has an auxiliary zero-mode

$$D_E(x) - iF_{12}(x) = D_0, \tag{4.3.12}$$

the configuration is not gauge-equivalent to the BPS configuration where $D_E = 0 \in \mathbb{R}$. In this case, it is not possible to reach a configuration for which $D_E(x) \in \mathbb{R}$ using gauge transformations, because the complexified gauge orbit of the auxiliary zero mode D_0 spans

$$D_E(x) = D_0 + iF_{12}(x) \in \mathbb{C}. \tag{4.3.13}$$

4.3.3 Moduli space and zero modes

In this section, we describe the zero modes and the moduli space obtained in Benini-Zaffaroni approach to A-twisted localization. Taking A_μ to be a Yang-Mills connection, the fields (σ, F_{12}) are parametrized on the moduli space by the coordinates $(u, \mathfrak{m})$, valued as

$$u \in \mathfrak{h}_{\mathbb{C}}, \quad \mathfrak{m} = \frac{1}{2\pi}\int_{\Sigma_g} F_{12} \in \Lambda^G_{\text{coch}}. \tag{4.3.14}$$

Here, the complex modulus u is a continuous parameter and the real modulus $\mathfrak{m}$ is the discrete GNO quantized flux. The moduli space of bosonic zero modes is

$$\mathcal{M} = \frac{\mathfrak{M} \times \Lambda^G_{\text{coch}}}{W}, \quad \mathfrak{M} = \mathfrak{h} \times \mathfrak{h} = \mathfrak{h}_{\mathbb{C}} \tag{4.3.15}$$

where the moduli are valued as $u \in \mathfrak{M} = \mathfrak{h}_{\mathbb{C}}$, $\mathfrak{m} \in \Lambda^G_{\text{coch}} \subset \mathfrak{h}$, and the Weyl group W is what remains of the gauge transformations. The Weyl group is a subgroup of the isometry group of the root system and acts both $\mathfrak{h}$ and $\mathfrak{h}^*$ through Weyl reflections $w \in W$.

In addition to the bosonic zero modes $u, \overline{u}, \mathfrak{m}$, the localization locus involves fermionic zero modes. In the three-dimensional conventions, the additional zero modes are

$$\lambda_0^\dagger = \beta\, \tilde{\epsilon}^\dagger \lambda, \qquad \lambda_0^\dagger = \beta\, \lambda^\dagger \epsilon, \qquad D_0 = \beta\, \tilde{\epsilon}^\dagger \epsilon\, (D - iF_{12}) \tag{4.3.16}$$

where $\lambda_0^\dagger, \lambda_0$ are fermionic and D_0 is bosonic. The zero modes sit in a zero dimensional supersymmetry multiplet, and transform according to

$$\begin{array}{ccccc} Qu = 0, & Q\bar{u} = i\lambda_0^\dagger, & Q\lambda_0 = -D_0, & Q\lambda_0^\dagger = 0, & QD_0 = 0, \\ \tilde{Q}u = 0, & \tilde{Q}\bar{u} = i\lambda_0, & \tilde{Q}\lambda_0 = 0, & \tilde{Q}\lambda_0^\dagger = D_0, & \tilde{Q}D_0 = 0. \end{array} \tag{4.3.17}$$

4.3.4 Singular hyperplanes in moduli space

In this section, we record the singularities present in the localization locus obtained in Benini-Zaffaroni approach to A-twisted localization. One of the obstacles to integrating over the zero modes is that the bosonic moduli space $\mathfrak{M} \subset \mathcal{M}$ contains singularities in both the two- and three-dimensional cases. In three dimensions $\mathfrak{M}_{(3d)} = H \times \mathfrak{h}$ is moduli space of $u = \beta\,(A_3 + i\sigma)$ where σ is the real scalar in the three-dimensional vector multiplet. In two-dimensions, $\mathfrak{M}^{(2d)} = \mathfrak{h}_{\mathbb{C}}$ is the moduli space of $u = i\sigma$, where σ is the complex scalar in the two-dimensional vector multiplet.

There are two types of singularities occurring in $\mathfrak{M}^{(3d)}$. The first type of singularity occurs at a point $u = u_*$ in $\mathfrak{M}^{(3d)}$ for which the three-dimensional chiral multiplet $\Phi^{(3d)}$ becomes massless. These points take values in a singular submanifold $u_* \in \mathfrak{M}^{(3d)}_{\text{sing}} \subset \mathfrak{M}^{(3d)}$. As $u = u_*$ is a finite value in $\mathfrak{M}^{(3d)}$, this type of singularity occurs in the bulk of $\mathfrak{M}^{(3d)}$ as opposed to its boundary $\partial\mathfrak{M}^{(3d)}$. The second type of singularity occurs in the $\sigma \to \pm\infty$ limit of the real modulus $u \in \mathfrak{M}^{(3d)}, u = \beta\,(A_3 + i\sigma)$. As this is an infinite value in $\mathfrak{M}^{(3d)}$, this type of singularity occurs at the boundary $\partial\mathfrak{M}^{(3d)}$. The singular hyperplanes are

$$H_i = \left\{ u \in \mathfrak{M} \,\middle|\, e^{i\rho_i(u) + i\gamma_i(v)} = 1 \right\}. \tag{4.3.18}$$

$$H_\alpha = \left\{ u \in \mathfrak{M} \,\middle|\, e^{i\alpha(u)} = 1 \right\}. \tag{4.3.19}$$

There are two types of singularities occurring in $\mathfrak{M}^{(2d)}$. As before, the first type of singularity occurs at a point $u = u_*$ in $\mathfrak{M}^{(2d)}$ for which the two-dimensional chiral multiplet $\Phi^{(2d)}$ becomes massless. These points take values in a singular submanifold $u_* \in \mathfrak{M}^{(2d)}_{\text{sing}} \subset \mathfrak{M}^{(2d)}$ and occur in the bulk. The second type of singularity occurs in the $\sigma \to \pm\infty$ limit of the complex modulus $u \in \mathfrak{M}^{(2d)}, u = i\sigma$. This differs from the three dimensional case where the scalar was real, and requires an alternative treatment for the boundary of $\partial\mathfrak{M}^{(2d)}$. Observe that in the rank-one case, the boundary $\partial\mathfrak{M}^{(2d)} = \partial\mathfrak{h}_{\mathbb{C}}$ is the region at infinity in the complex σ-plane.

4.3.5 1-loop contributions

In this section, we record the 1-loop contributions resulting from the Benini-Zaffaroni approach to A-twisted localization. After regularization, the 1-loop determinants of the two- and three-dimensional chiral multiplets are

$$Z^{\text{chiral,3d}}_{\text{1-loop,reg}} = \prod_{\rho\in\mathfrak{R}} \Big(\frac{x^{\rho/2}y^{\gamma/2}}{1-x^\rho y^\gamma}\Big)^{\rho(\mathfrak{m})+\gamma(\mathfrak{n})+(g-1)(q_\rho-1)}, \tag{4.3.20}$$

$$Z^{\text{chiral,2d}}_{\text{1-loop,reg}} = \prod_{\rho\in\mathfrak{R}} \Big[\frac{1}{\rho(u)\,\gamma(v)}\Big]^{\rho(\mathfrak{m})+\gamma(\mathfrak{n})+(g-1)(q_\rho-1)}, \tag{4.3.21}$$

where $x = e^{iu}$ and $y = e^{iv}$ such that $x^\rho = e^{i\rho(u)}$ and $y^\gamma = e^{i\gamma(v)}$ for gauge weight ρ and flavor weight γ,

The 1-loop determinants of the vector multiplet are obtained from the regularized chiral 1-loop determinants, see section 2.4.4 of [18] for details. The 1-loop determinants of the two- and three-dimensional vector multiplets are

$$Z^{\text{vector,3d}}_{\text{1-loop,reg}} = \prod_{\alpha\in G}(1-x^\alpha)^{1-g}, \tag{4.3.22}$$

$$Z^{\text{vector,2d}}_{\text{1-loop,reg}} = (-1)^{\sum_{\alpha>0}\alpha(\mathfrak{m})}\prod_{\alpha\in G}\alpha(u), \tag{4.3.23}$$

where $x^\alpha = e^{i\alpha(u)}$ for roots α of G.

4.4 Asymptotic behavior

In this section, we review the Benini-Zaffaroni treatment of asymptotic chiral 1-loop determinants, the shift in the Chern-Simons (CS) coupling, and the shift in the twisted chiral superpotential.

Localizing to the Benini-Zaffaroni locus splits the path integral into integrals over moduli and integrals over fluctuations. In the 3d case, evaluating the integrals of fluctuating modes results in a singular 1-loop contribution and a shift in the bare CS coupling k. The shift of k is due to the chiral 1-loop contribution for large values of the real scalar σ in the 3d vector multiplet. Note that in 3d, $\sigma^{(3d)}$ is parametrized by $u^{(3d)} = A_t + i\beta\sigma^{(3d)}$ on $\mathfrak{M}^{(3d)} = H\times\mathfrak{h}$, and the asymptotic limit is $\left|\beta\sigma^{(3d)}\right| \gg 1$.

In the 2d case, the integrals of fluctuating modes evaluate to a singular 1-loop contribution and a shift in the twisted chiral superpotential $\widetilde{W}$. This can be stated as a shift in the complexified FI term ζ by choosing $\widetilde{W}$ to be linear in its argument. The shift of $\widetilde{W}$ (or ζ) is due to the chiral 1-loop contribution for large values of the complex scalar σ in the 2d vector multiplet. Note that in 2d, $\sigma^{(2d)}$ is parametrized by $u^{(2d)} = i\sigma^{(2d)}$ on $\mathfrak{M}^{(2d)} = \mathfrak{h}_{\mathbb{C}}$, and the asymptotic limit is $\left|\sigma^{(2d)}\right| \to \pm\infty$.

We proceed to explain this shift by studying the asymptotic behavior of the chiral 1-loop determinants, first in the 3d case, then in the 2d case. Observe that this section includes some verbatim statements from [18] where these derivations were originally described.

4.4.1 3d effective Chern-Simons coupling

After localizing to the Benini-Zaffaroni locus, the 1-loop contribution of a 3d chiral multiplet Φ of charge $Q = 1$ is

$$\frac{\det \mathcal{O}_\psi}{\det \mathcal{O}_\phi} = \prod_{k\in\mathbb{Z}} \frac{1}{\left(\frac{2\pi k - A_t}{i\beta} - \sigma\right)^{b+1}} \prod_{n\geq 0} \left(\frac{\frac{n(n+1+b)}{R^2} + \sigma^2 + \frac{(2\pi k - A_t)^2}{\beta^2}}{\frac{n(n+1+b)}{R^2} + \sigma^2 + \frac{(2\pi k - A_t)^2}{\beta^2} + iD_0} \right)^{2n+b+1} \tag{4.4.1}$$

Here, $\mathcal{O}_\psi$ and $\mathcal{O}_\phi$ are the fermionic and bosonic fluctuation operators, respectively, and $b \in \mathbb{Z}_{>1}$. For details regarding this expression, as well as the regularization of the first product, see section 2.2.5 in [18]. After regularization, the asymptotic contribution of the first product is

$$\prod_{k\in\mathbb{Z}} \frac{1}{\left(\frac{2\pi k - A_t}{i\beta} - \sigma\right)^{b+1}} \quad \underset{\text{reg.}}{\longrightarrow} \quad \prod_{\rho} \left(\frac{x^\rho}{1 - x^\rho} \right)^{\rho(\mathfrak{m}) - q_\rho + 1} \tag{4.4.2}$$

$$\underset{|\beta\sigma|\gg 1}{\longrightarrow} \quad x^{\frac{1}{2} Q_i^2 \operatorname{sign}(Q_i \sigma)\mathfrak{m}}. \tag{4.4.3}$$

where $x = e^{iu}$. The term on the right hand side of the first line is the final form of the regulated chiral 1-loop determinant. Note that we have written this for a general chiral multiplet Φ of charge Q_i.

To determine the asymptotic contribution of the term involving n, on the other hand, one begins by evaluating the convergent product over k. This reads

$$F = \prod_{k\in\mathbb{Z}} \prod_{n\geq 0} \left(\frac{\frac{n(n+1+b)}{R^2} + \sigma^2 + \frac{(2\pi k - A_t)^2}{\beta^2}}{\frac{n(n+1+b)}{R^2} + \sigma^2 + \frac{(2\pi k - A_t)^2}{\beta^2} + iD_0} \right)^{2n+b+1} \tag{4.4.4}$$

$$= \prod_{n\geq 0} f(n)^{2n+b+1} \tag{4.4.5}$$

where

$$f(n) = \frac{\cosh\left(\beta\sqrt{z}\right) - \cos A_t}{\cosh\left(\beta\sqrt{z + iD_0}\right) - \cos A_t}, \; z = R^{-2} n (n + 1 + b) + \sigma^2. \tag{4.4.6}$$

The product over n, on the other hand, is treated with ζ-function regularization.

For a reference, see e.g. [39]. This proceeds by writing

$$\log F = \log \prod_{n\geq 0} f(n)^{2n+b+1}. \tag{4.4.7}$$

$$= \sum_{n\geq 0} (2n+b+1) \log f(n). \tag{4.4.8}$$

In the asymptotic limit $|\beta\sigma| \gg 1$, we have

$$\log f = \log\left(\frac{\cosh(\beta\sqrt{z}) - \cos A_t}{\cosh\left(\beta\sqrt{z+iD_0}\right) - \cos A_t}\right) \tag{4.4.9}$$

$$= \beta\sqrt{z} - \beta\sqrt{z+iD_0} + \mathcal{O}\left(e^{-\beta\sqrt{z}}\right) \tag{4.4.10}$$

Inserting this back into the expression for $\log F$, we have

$$\log F = \sum_{n\geq 0} (2n+b+1)\left(\beta\sqrt{z} - \beta\sqrt{z+iD_0} + \mathcal{O}\left(e^{-\beta\sqrt{z}}\right)\right) \tag{4.4.11}$$

The linearly divergent term in $\log F$ is computed using ζ-function regularization:

$$-\sum_{n\geq 0} i\beta R D_0 = \frac{i}{2}\beta R D_0. \tag{4.4.12}$$

What remains of $\log F$ is a convergent sum over n, which is approximated by an integral:

$$\beta \int_0^\infty \mathrm{d}n\, (2n+b+1)\left(\beta\sqrt{z} - \beta\sqrt{z+iD_0} + iRD_0\right) \tag{4.4.13}$$

$$= i\beta R^2 |\sigma| D_0 - \frac{i}{2}(b+1)\beta R D_0 + \mathcal{O}\left(\frac{\beta R^2 D_0^2}{\sigma}\right) \tag{4.4.14}$$

From this, it follows that the contribution of the second term in the asymptotic limit is

$$F = \exp\left(i\beta R^2 |\sigma| D_0 - \frac{i}{2}(b+1)\beta R D_0 + \mathcal{O}\left(\frac{\beta R^2 D_0^2}{\sigma}\right)\right) \tag{4.4.15}$$

for a chiral multiplet of charge $Q = 1$. To obtain the corresponding expression for a generic charge, one places Q_i in front of σ and D_0. In particular, for a chiral multiplet of charge Q_i, the contribution in the asymptotic limit is

$$(F)\Big|_{\text{reg}} = \exp\left[i\beta R^2 \text{sign}(Q_i\sigma)\, Q_i^2 \sigma D_0 - \frac{i}{2}\beta R b Q_i D_0 + \mathcal{O}\left(\frac{\beta R^2 D_0^2}{\sigma}\right)\right]. \tag{4.4.16}$$

Together, the asymptotic contribution of the two terms is

$$\left.\frac{\det \mathcal{O}_\psi}{\det \mathcal{O}_\phi}\right|_{\text{reg}} \underset{|\beta\sigma|\gg 1}{\longrightarrow} x^{\frac{1}{2}Q_i^2 \text{sign}(Q_i\sigma)\mathfrak{m}} e^{i\beta R^2 \text{sign}(Q_i\sigma) Q_i^2 \sigma D_0+\ldots} \tag{4.4.17}$$

The shift in the CS coupling k is a result of the combination of the asymptotic contribution with the classical abelian Chern-Simons action

$$Z_{\text{cl}}^{\text{CS,Ab}} = e^{-\int \mathrm{d}^3x\sqrt{g}L_{\text{CS}}|_{\text{on-shell}}} = x^{k\mathfrak{m}} e^{2ik\beta R^2\sigma D_0}. \tag{4.4.18}$$

Together, these contributions conspire to produce a shift in k, which reads

$$\begin{aligned} & \left(x^{k\mathfrak{m}} e^{2ik\beta R^2\sigma D_0}\right)\left(x^{\frac{1}{2}Q_i^2 \text{sign}(Q_i\sigma)\mathfrak{m}} e^{iQ_i^2 \text{sign}(Q_i\sigma)\beta R^2\sigma D_0}\right) && (4.4.19)\\ = \; & e^{ik(u,\mathfrak{m})} e^{2ik\beta R^2\sigma D_0} e^{i\frac{1}{2}Q_i^2 \text{sign}(Q_i\sigma)(u,\mathfrak{m})} e^{iQ_i^2 \text{sign}(Q_i\sigma)\beta R^2\sigma D_0} && (4.4.20)\\ = \; & e^{ik(u,\mathfrak{m})+2ik\beta R^2\sigma D_0+\frac{i}{2}Q_i^2 \text{sign}(Q_i\sigma)(u,\mathfrak{m})+iQ_i^2 \text{sign}(Q_i\sigma)\beta R^2\sigma D_0} && (4.4.21)\\ = \; & e^{i\left(k+\frac{1}{2}Q_i^2 \text{sign}(Q_i\sigma)\right)(u,\mathfrak{m})+2i\left(k+\frac{1}{2}Q_i^2 \text{sign}(Q_i\sigma)\right)\beta R^2\sigma D_0} && (4.4.22)\\ = \; & x^{k_{\text{eff}}(\sigma)\mathfrak{m}} e^{2ik_{\text{eff}}(\sigma)\beta R^2\sigma D_0}. && (4.4.23) \end{aligned}$$

In the last line, we have written the effective CS coupling

$$k_{\text{eff}}(\sigma) = k + \frac{1}{2}\sum_i Q_i^2 \text{sign}(Q_i\sigma) \tag{4.4.24}$$

For a general theory, it is possible to have a mixed CS term, in which case the effective CS coupling is

$$k_{\text{eff}}^{ab}(\sigma) = k^{ab} + \frac{1}{2}\sum_{i,c} Q_i^a Q_i^b \text{sign}(Q_i^c\sigma_c) \tag{4.4.25}$$

where a, b, c run over the generators of the Abelian gauge group, i runs over the different matter fields, and Q_i^a are gauge charges.

4.4.2 2d effective twisted chiral superpotential

The shift of the 2d twisted chiral superpotential (or FI term) is derived in a similar manner to the shift of the 3d CS coupling, which we now describe. One localizes to the Benini-Zaffaroni locus, and studies the 1-loop contribution of a 2d chiral multiplet Φ, transforming as the weight ρ of a representation $\mathfrak{R}$ of the gauge group. The difference is that, instead of dealing with the real scalar $\sigma^{(3d)}$ of the 3d vector multiplet, one treats the complex scalar $\sigma^{(2d)}$ of the 2d vector multiplet. The novelty in the 2d case is the treatment of the boundary $\partial\mathfrak{M}^{(2d)} = \partial\mathfrak{h}_{\mathbb{C}}$, that is to say, the region at complex infinity in the $\sigma^{(2d)}$-plane.

In this case, the important term in the asymptotic limit of the chiral 1-loop determinant is

$$F=\prod_{n\geq 0}\left(\frac{\frac{n(n+1+b)}{R^2}+|\rho(\sigma)|^2}{\frac{n(n+1+b)}{R^2}+|\rho(\sigma)|^2+i\rho(D_0)}\right)^{2n+b+1} \tag{4.4.26}$$

where $b\in\mathbb{Z}_{\geq 0}$. To treat the product over n, we write

$$\begin{aligned}\log F &= \log\prod_{n\geq 0}\left(\frac{n(n+1+b)+R^2|\rho(\sigma)|^2}{n(n+1+b)+R^2|\rho(\sigma)|^2+iR^2\rho(D_0)}\right)^{2n+b+1} && (4.4.27)\\ &= \sum_{n\geq 0}(2n+b+1)\log\left(\frac{n(n+1+b)+a}{n(n+1+b)+a+iR^2\rho(D_0)}\right) && (4.4.28)\\ &= \sum_{n\geq 0}(2n+b+1)\log\left(\frac{1}{1+\frac{iR^2\rho(D_0)}{n(n+1+b)+a}}\right) && (4.4.29)\end{aligned}$$

where $a=R^2|\rho(\sigma)|^2$. To simplify this expression, one considers the convergence properties of a Taylor series of $\log(1+iy)^{-1}$ about point $y=0$. Specifically, the series

$$\log\left(\frac{1}{1+iy}\right)=-iy-\frac{1}{2}y^2+\mathcal{O}(y^3), \tag{4.4.30}$$

converges for $|y|<1$, or $y=1$, or $y=-1$. In the case at hand, we have that $y=\frac{R^2\rho(D_0)}{n(n+1+b)+a}$ goes to zero for $a\to\infty$. It follows that

$$\begin{aligned}\log F &= \sum_{n\geq 0}(2n+b+1)\log\left(\frac{1}{1+\frac{iR^2\rho(D_0)}{n(n+1+b)+a}}\right) && (4.4.31)\\ &= \sum_{n\geq 0}(2n+b+1)\left(-\frac{iR^2\rho(D_0)}{n(n+1+b)+a}\right) && (4.4.32)\\ &= -iR^2\rho(D_0)\sum_{n\geq 0}f(n)+\mathcal{O}\left(a^{-2}\right) && (4.4.33)\\ f(n) &= \frac{(2n+b+1)}{n(n+1+b)+a} && (4.4.34)\end{aligned}$$

Observe that $\sum_{n\geq 0}f(n)$ diverges as $\sum\frac{1}{n}$, and consequently, $\log F$ cannot be treated using ζ-function regularization. Instead, the resolution is to subtract $\frac{2}{n+1}$ from the summand. For a reference regarding similar considerations, see e.g. series representations 8.362 in [40]. Specifically, the resolution is

$$\left(\sum_{n\geq 0}f(n)\right)\Bigg|_{\text{reg}}=\sum_{n\geq 0}\left(f(n)-\frac{2}{n+1}\right). \tag{4.4.35}$$

Consequently, when regulated, $\log F$ can be written in terms of special functions

$$(\log F)|_{\text{reg}} = -iR^2\rho(D_0)\sum_{n\geq 0}\left(\frac{(2n+b+1)}{n(n+1+b)+a} - \frac{2}{n+1}\right) + \mathcal{O}\left(a^{-2}\right) \quad (4.4.36)$$

$$= -iR^2\rho(D_0)\left(-2\gamma - \sum_{\pm}\psi\left(\frac{1+b\pm\sqrt{-4a+(1+b)^2}}{2}\right)\right) \quad (4.4.37)$$

$$= -iR^2\rho(D_0)\left(-2\gamma - \log a + \mathcal{O}\left(a^{-1}\right)\right) \quad (4.4.38)$$

where γ is Euler's constant and $\psi(z) = \Gamma'(z)/\Gamma(z)$. Accordingly, the regulated contribution of F in the asymptotic limit $|\sigma| \to \pm\infty$ is

$$(F)\Big|_{\text{reg}} \simeq \exp\left(+iR^2\rho(D_0)2\gamma + iR^2\rho(D_0)\log a\right) \quad (4.4.39)$$

It follows that the chiral 1-loop contribution for large values of the complex scalar σ is

$$e^{2iR^2\log\left|\rho(\sigma)\right|\rho(D_0)}. \quad (4.4.40)$$

The shift in the twisted chiral superpotential $\widetilde{W}(\sigma)$ is a result of the combination of the asymptotic contribution with the classical twisted chiral superpotential action

$$Z_{\text{cl}}^{\widetilde{W}} = e^{-\int d^2x\sqrt{g}L_{\widetilde{W}}|_{\text{on-shell}}} = e^{4\pi\widetilde{W}'(\sigma)\mathfrak{m} - 8\pi iR^2\text{Re}\left(\widetilde{W}'(\sigma)\right)D_0}. \quad (4.4.41)$$

Together, these contributions conspire to produce a shift in $\widetilde{W}$, and the effective twisted superpotential is

$$\widetilde{W}_{\text{eff}} = -\frac{1}{4\pi}\rho(\sigma)\left(\log\rho(\sigma) - 1\right). \quad (4.4.42)$$

Observe that the role of the effective CS level k_{eff} in 3d is played by the effective twisted chiral superpotential $\widetilde{W}_{\text{eff}}$ in 2d.

4.5 Integration contours

4.5.1 The dangerous regions in the 3d bosonic moduli space

In this section, we outline the Benini & Zaffaroni treatment of singularities in the 3d bosonic moduli space. By expanding the deformed action around the almost-BPS configurations, then evaluating the classical and one-loop contributions, the partition function is eventually reduced to

$$Z = \frac{1}{|W|}\lim_{e,g\to 0}\sum_{\mathfrak{m}\in\Gamma_{\mathfrak{h}}}\int_{C_D} dD_0 \int_{\mathfrak{M}} d^2u \int d\lambda_0 d\lambda_0^\dagger \mathcal{Z}_{e,g}(u,\bar{u},\lambda_0,\lambda_0^\dagger,D_0;\mathfrak{m}) \quad (4.5.1)$$

Here, $\mathcal{Z}_{e,g}(u,\bar{u},\lambda_0,\lambda_0^\dagger,D_0;\mathfrak{m})$ is the effective action consisting of the classical and one-loop contributions, $\mathfrak{m}$ is a discrete bosonic zero-mode, $u,\bar{u},D_0$ are continuous bosonic zero modes, and $\lambda_0,\lambda_0^\dagger$ are fermionic zero modes. In particular, the integration domain of zero modes is a supermanifold in which there are both bosonic and fermionic directions. The integration domain $\mathfrak{M}\simeq\mathfrak{h}_{\mathbb{C}}\simeq(\mathbb{C}^*)^{\mathrm{rank}G}$ has dangerous regions in which the integrand becomes singular. In particular, $\mathcal{Z}_{e,g}(u,\bar{u},0,0,0;\mathfrak{m})$ becomes singular for various points $u\in\mathfrak{M}$.

The dangerous regions are dealt with by taking a double scaling limit, as we will now describe. For the dangerous integral

$$\lim_{e,g\to 0}\int_{\mathfrak{M}}\mathrm{d}^2u\mathcal{Z}_{e,g},\tag{4.5.2}$$

the procedure is to first take $g\to 0$, then take $e\to 0$. The $g\to 0$ limit is not a problem, and can be absorbed into a redefinition of the measure. We have then

$$\lim_{e,g\to 0}\int_{\mathfrak{M}}\mathrm{d}^2u\mathcal{Z}_{e,g}\sim\lim_{e\to 0}\int_{\mathfrak{M}}\mathrm{d}^2u\mathcal{Z}_{e,0}\tag{4.5.3}$$

The $e\to 0$ limit, on the other hand, is a problem. The resolution is to take a scaling limit in order to regulate singularities. This proceeds by splitting the integral as

$$\int_{\mathfrak{M}}\mathrm{d}^2u\mathcal{Z}_{e,0}=\int_{\mathfrak{M}\backslash\Delta_\varepsilon}\mathrm{d}^2u\mathcal{Z}_{e,0}+\int_{\Delta_\varepsilon}\mathrm{d}^2u\mathcal{Z}_{e,0}\tag{4.5.4}$$

where Δ_ε is an ε-neighborhood of singular hyperplanes. The contribution $\int_{\Delta_\varepsilon}$ bounded as long as e is kept finite, therefore it vanishes in the limit $\varepsilon\to 0$ faster than $e\to 0$.

4.5.2 Integrating out 3d bosonic zero modes

In this section, we review in detail the Benini & Zaffaroni approach to treating with the singularities in the 3d bosonic moduli space. Observe that the localizing term is $\mathcal{Q}V=\frac{1}{e^2}L_{\mathrm{YM}}+\frac{1}{g^2}L_{\mathrm{mat}}$, the localizing limit is $e,g\to 0$, the moduli space is $\mathcal{M}=(\mathfrak{M}\times\Lambda_{\mathrm{coch}}^G)/W$, and the singular submanifold is $\mathfrak{M}_{\mathrm{sing}}\subset\mathfrak{M}$. To deal with problematic integral associated to the modulus $u\in\mathfrak{M}$, we consider the partition function before taking the localizing limit $(e,g\neq 0)$, which reads

$$Z=\int_{\mathfrak{M}}\mathrm{d}^2uF_{e,g}\left(u,\overline{u}\right).\tag{4.5.5}$$

Here, $F_{e,g}$ denotes the incomplete result of integrating out fermionic zero modes and fluctuations. The measure is

$$d^2u = d(\mathrm{Re}u) \wedge d(\mathrm{Im}u) = \frac{i}{2} du \wedge d\overline{u}, \tag{4.5.6}$$

where $\overline{u}$ is the complex conjugate of u. For later convenience we record that u is

$$u = A_t + i\beta\sigma = \beta(A_3 + i\sigma), \quad \mathrm{Re}u = A_t, \quad \mathrm{Im}u = \beta\sigma, \tag{4.5.7}$$

where β is the radius of S^1, $A_t = \beta A_3$ is the gauge field along S^1, and σ is the real scalar in the 3d vector multiplet.

The dangerous region of the integral is at $u_* \in \mathfrak{M}_{\text{sing}} \subset \mathfrak{M}$, and taking the limit of $F_{e,g}(u, \overline{u})$ as $e, g \to 0$ at $u = u_*$ is problematic because extra zero modes appear for chiral scalars ϕ. This corresponds to the chiral multiplet Φ becoming massless at $u = u_*$. These are called bulk singularities because they occur at finite values of u. To resolve this, we consider the situation in which u is near $u_* \in \mathfrak{M}_{\text{sing}}$. Let $\phi_i, i = 1, \ldots, 2M$ denote the quasi 0-modes of the chiral scalars (i.e. the modes that become zero modes at $u = u_*$), let Q_i denote the charge of each ϕ_i, and assume that charges are projective (all Q_i have the same sign). The problematic integral is

$$I_0 = \int d^{2M}\phi \exp\left[\underbrace{-\frac{1}{g^2}\sum_i |Q_i(u-u_*)|^2 |\phi_i|^2}_{\#_A} \underbrace{-\frac{e^2}{2}\left(\zeta_{\text{eff}} - \sum_i \frac{Q_i}{g^2}|\phi_i|^2\right)^2}_{\#_B}\right] \tag{4.5.8}$$

where ζ_{eff} is the effective FI term at the point u. For convenience we have labeled the different terms in the integral $\#_A$ and $\#_B$.

The $g \to 0$ limit of I_0 is not a problem. This is because g appears as $\frac{1}{g^2}|\phi_i|^2$ and can therefore be absorbed into the measure by rescaling. The rescaling is

$$\phi \;\to\; \phi' = \sqrt{g^2}\phi \tag{4.5.9}$$

$$d^{2M}\phi \;\to\; d^{2M}\phi' = g^{2M} d^{2M}\phi \tag{4.5.10}$$

$$I_0 \;\to\; I_0' = g^{2M}\int d^{2M}\phi e^{-\sum_i |Q_i(u-u_*)|^2|\phi_i|^2 - \frac{e^2}{2}\left(\zeta_{\text{eff}} - \sum_i Q_i|\phi_i|^2\right)^2} \tag{4.5.11}$$

and we have $\lim_{g\to 0} I_0' = 0$.

The $e \to 0$ limit of I_0, on the other hand, is a problem. The term $\#_B$ in I_0 comes from the D-term potential, and ensures the convergence of integral (including at the point $u = u_*$). To begin, let us find an upper bound on $|I_0| = h(u)$ for small fixed e and u near u_*. The maximum of $|I_0| = h(u)$ is at $u = u_*$, that is to say, $h'(u_*) = 0$ and $h''(u_*) < 0$ where the primes denote derivatives. By rescaling

$\phi_i \to \phi_i' = \sqrt{\frac{g^2}{Q_i e}}\phi_i$, one obtains the pre-integration bound

$$|I_0| \leq \frac{g^{2M}}{e^M \prod_i |Q_i|} \int \mathrm{d}^{2M}\phi \exp\left[-\frac{1}{2}\left(\underbrace{e\zeta_{\text{eff}}\text{sign}\,(Q_i)}_{\#_C} - \underbrace{\sum_i |\phi_i|^2}_{\#_D}\right)^2\right]. \qquad (4.5.12)$$

For small e, one can neglect the term $\#_C$, and evaluate the contribution of $\#_D$. The result is

$$\int \mathrm{d}^{2M}\phi \exp\left[-\frac{1}{2}\left(\sum_i |\phi_i|^2\right)^2\right] = \frac{2^{\frac{M-2}{2}}\pi^M \Gamma(M/2)}{\Gamma(M)} \qquad (4.5.13)$$

Thereafter, one can obtain the post-integration bound

$$|I_0| \lesssim \frac{C_0}{e^M}, \quad C_0 = \frac{g^{2M}}{\prod_i |Q_i|} \frac{2^{\frac{M-2}{2}}\pi^M \Gamma(M/2)}{\Gamma(M)} \qquad (4.5.14)$$

Despite having obtained an upper bound, taking the limit of $F_{e,0}(u, \overline{u})$ as $e \to 0$ at $u = u_*$ remains problematic due to singularities and the removal of the quartic potential.

The resolution is to decompose the integral over $\mathfrak{M}$, then take the scaling limit. The decomposition is $\mathfrak{M} = (\mathfrak{M}\backslash\Delta_\varepsilon) \cup \Delta_\varepsilon$, where Δ_ε is an ε-neighborhood of $\mathfrak{M}_{\text{sing}}$ in $\mathfrak{M}$. In other words, Δ_ε is an ε-neighborhood around a (bulk) singular point $u_* \in \mathfrak{M}_{\text{sing}}$. To illustrate Δ_ε, let us oversimplify and consider $\mathfrak{M} = \mathbb{C}$. If this were the case, then $\Delta_\varepsilon \subset \mathfrak{M}$ could be an open disk of radius ε around $u_* \in \mathfrak{M}$

$$\Delta_\varepsilon = \left\{u \in \mathfrak{M} \;\middle|\; |u_* - u| < \varepsilon\right\}. \qquad (4.5.15)$$

In this case, the closed disk and boundary would respectively be

$$\overline{\Delta}_\varepsilon = \left\{u \in \mathfrak{M} \;\middle|\; |u_* - u| \leq \varepsilon\right\} \qquad (4.5.16)$$

$$\partial\Delta_\varepsilon = \left\{u \in \mathfrak{M} \;\middle|\; |u_* - u| = \varepsilon\right\} \qquad (4.5.17)$$

Observe that this is for the sake of illustration and not necessarily the case. We proceed with generic Δ_ε. The point of the decomposition of $\mathfrak{M}$ was that the partition function decomposes as

$$Z = \underbrace{\int_{\mathfrak{M}\backslash\Delta_\varepsilon} \mathrm{d}^2u F_{e,0}(u, \overline{u})}_{I_1} + \underbrace{\int_{\Delta_\varepsilon} \mathrm{d}^2u F_{e,0}(u, \overline{u})}_{I_2} \qquad (4.5.18)$$

The aim is then to evaluate $\lim_{e\to 0} Z$ in such a way that I_2 does not contribute.

The integral I_2 is bounded by

$$|I_2| \lesssim \frac{\varepsilon^2}{e^M} C_2 \tag{4.5.19}$$

where C_2 is constant We have then different options for how to take the limits. One option is to take $e \to 0$ before $\varepsilon \to 0$, which results in

$$\lim_{\varepsilon\to 0}\left(\lim_{e\to 0}\left(\frac{\varepsilon^2}{e^M}C_2\right)\right) = \underbrace{\lim_{\varepsilon\to 0}\left(\frac{\varepsilon^2}{0}C_2\right)}_{\text{singular}} \tag{4.5.20}$$

A second option is to take $\varepsilon \to 0$ before $e \to 0$, which results in

$$\lim_{e\to 0}\left(\lim_{\varepsilon\to 0}\left(\frac{\varepsilon^2}{e^M}C_2\right)\right) = \underbrace{\lim_{e\to 0}\left(\frac{0}{e^M}C_2\right)}_{\text{non-singular}} \tag{4.5.21}$$

Yet another option is to take a scaling limit $\varepsilon, e \to 0$ such that $\frac{\varepsilon^2}{e^M} \to 0$, which results in

$$\lim_{\frac{\varepsilon^2}{e^M}\to 0}\left(\frac{\varepsilon^2}{e^M}C_2\right) = \underbrace{0 \cdot C_2}_{\text{non-singular}} . \tag{4.5.22}$$

The takeaway is that the scaling limit ($\lim_{(\varepsilon^2)/(e^M)\to 0} I_2$) exists as long as one keeps $\varepsilon^2 < e^M$ while taking $(\varepsilon^2)/(e^M) \to 0$. In particular, one has

$$\lim_{(\varepsilon^2)/(e^M)\to 0} I_2 = \lim_{(\varepsilon^2)/(e^M)\to 0} \int_{\Delta_\varepsilon} \mathrm{d}^2u F_{e,0}\left(u, \overline{u}\right) = 0. \tag{4.5.23}$$

To keep notation clean, the scaling limit $\lim_{(\varepsilon^2)/(e^M)\to 0}$ is denoted $\lim_{\varepsilon,e\to 0}$. In the scaling limit, the partition function is

$$Z = \lim_{\varepsilon,e\to 0} \int_{\mathfrak{M}\setminus\Delta_\varepsilon} \mathrm{d}^2u F_{e,0}\left(u, \overline{u}\right) \tag{4.5.24}$$

This scaling limit is the resolution for the bulk singularities at $u = u_*$.

In addition to bulk singularities, $\mathfrak{M}$ exhibits boundary singularities, whose resolution we briefly describe. These have to do with the fact that $\mathfrak{M}$ is non-compact. For instance, in the rank 1 case, the space is a cylinder $\mathfrak{M} \simeq \mathbb{C}/2\pi$. The boundary singularities occur at large values of σ which appears in the integrand as $\mathrm{Im}u = \beta\sigma$, and in the integration measure as $\mathrm{d}\left(\mathrm{Im}u\right) = \beta \mathrm{d}\sigma$.

Before proceeding, observe that in the asymptotic $\sigma \to \pm\infty$ limit, all chiral multiplets Φ are massive, and their effect is to shift the bare CS level k. Put differently, the CS level that one sees at one-loop after integrating out matter fields is $k_{\text{eff}}(\sigma) = k + \frac{1}{2}\sum_i Q_i^2 \text{sign}\left(Q_i\sigma\right)$. Moreover, the effective CS level is used to assign charges to the boundaries. Specifically, the points $x = 0, \infty$ (where $x = e^{iu}, u =$

$A_t + i\beta\sigma$) are assigned charges proportional to the effective CS coupling at infinity on the Coulomb branch

$$k_\pm \equiv k_{\text{eff}}\left(\sigma = \pm\infty\right) = k + \frac{1}{2}\sum_i Q_i^2 \text{sign}\left(\pm Q_i \sigma\right), \quad Q_{x=0} = -k_+, \quad Q_{x=\infty} = k_- \tag{4.5.25}$$

That being said, the resolution for the boundary singularities proceeds by estimating the problematic integral in the $\sigma \to \pm\infty$ limit. After integrating out D, the problematic integral is

$$I_3 = \int_{\mathbb{R}} \mathrm{d}\sigma \exp\left[-\frac{e^2}{2}\left(k_\pm \sigma + \zeta\right)^2\right]. \tag{4.5.26}$$

To clarify the convergence and divergence of the integral, we write $I_3 = I_3\left(e, k_\pm\right)$. The integral is convergent for $I_3\left(e \neq 0, k_\pm \neq 0\right)$. The singularities come in two cases depending on whether $k_\pm$ is zero or non-zero. The first case is that

$$\lim_{e \to 0} I_3\left(e, k_\pm \neq 0\right) \tag{4.5.27}$$

is singular. The resolution is to decompose the integral I_3 as

$$\int_{\mathbb{R}} \mathrm{d}\sigma e^{-\frac{e^2}{2}\left(k_\pm \sigma + \zeta\right)^2} = \int_L^\infty \mathrm{d}\sigma e^{-\frac{e^2}{2}\left(k_\pm \sigma + \zeta\right)^2} + \int_{-\infty}^{-L} \mathrm{d}\sigma e^{-\frac{e^2}{2}\left(k_\pm \sigma + \zeta\right)^2} \tag{4.5.28}$$

where L is a cutoff, then take the scaling limit. The other case is that

$$\lim_{e \to 0} I_3\left(e, k_\pm = 0\right) \tag{4.5.29}$$

is singular. In this case, the resolution is to use a Lagrangian regulator κ_{reg}. For details see [18].

4.5.3 Integrating out fermionic zero modes on $S^2 \times S^1$

In this section, we describe how to integrate out the fermionic zero modes for the three-dimensional genus $g = 0$ case. The fermionic zero modes $\lambda_0, \lambda_0^\dagger$ are Grassmann odd scalars. Therefore, the integrals over $\lambda_0, \lambda_0^\dagger$ may be recast as Grassmann odd derivatives

$$\int \mathrm{d}\lambda_0\, \mathrm{d}\lambda_0^\dagger\, \mathcal{Z}(u, \bar{u}, \lambda_0, \lambda_0^\dagger, D_0; \mathfrak{m}) = \frac{\partial}{\partial \lambda_0} \frac{\partial}{\partial \lambda_0^\dagger}\, \mathcal{Z}\left(u, \bar{u}, \lambda_0, \lambda_0^\dagger, D_0; \mathfrak{m}\right)\Big|_{\lambda_0 = \lambda_0^\dagger = 0}. \tag{4.5.30}$$

The supersymmetric effective action $\mathcal{Z}(u, \bar{u}, \lambda_0, \lambda_0^\dagger, D_0; \mathfrak{m})$ is closed under both supercharges $Q\mathcal{Z} = \tilde{Q}\mathcal{Z} = 0$. Expressing the $\tilde{Q}$-variation of $\mathcal{Z}$ as a total derivative, then using the supersymmetry transformations of the zero modes 4.3.17, we have

$$0 = \tilde{Q}\mathcal{Z} \tag{4.5.31}$$

$$= \left(\tilde{Q}u\frac{\partial}{\partial u} + \tilde{Q}\overline{u}\frac{\partial}{\partial \overline{u}} + \tilde{Q}\lambda_0\frac{\partial}{\partial \lambda_0} + \tilde{Q}\lambda_0^\dagger\frac{\partial}{\partial \lambda_0^\dagger} + \tilde{Q}D_0\frac{\partial}{\partial D_0}\right)\mathcal{Z} \tag{4.5.32}$$

$$= \left(i\lambda_0\frac{\partial}{\partial \overline{u}} + D_0\frac{\partial}{\partial \lambda_0^\dagger}\right)\mathcal{Z} \tag{4.5.33}$$

Acting with $\partial/\partial\lambda_0$ on $0 = \tilde{Q}\mathcal{Z}$, the expression becomes

$$0 = \left(i\frac{\partial}{\partial \overline{u}} + D_0\frac{\partial}{\partial \lambda_0}\frac{\partial}{\partial \lambda_0^\dagger}\right)\mathcal{Z}, \tag{4.5.34}$$

or equivalently,

$$\left.\frac{\partial^2\mathcal{Z}}{\partial\lambda_0\partial\lambda_0^\dagger}\right|_{\lambda_0=\lambda_0^\dagger=0} = -\frac{i}{D_0}\left.\frac{\partial\mathcal{Z}}{\partial\overline{u}}\right|_{\lambda_0=\lambda_0^\dagger=0}. \tag{4.5.35}$$

The same result holds for $Q\mathcal{Z} = 0$, in which case it is necessary to first differentiate the equation with respect to $\lambda_0^\dagger$, then use the anti-commutativity of Grassmann odd derivatives. From these considerations, we see that the integrals over the fermionic zero modes evaluate to

$$\int \mathrm{d}\lambda_0\,\mathrm{d}\lambda_0^\dagger\;\mathcal{Z}(u,\bar{u},\lambda_0,\lambda_0^\dagger,D_0;\mathfrak{m}) = \left.\frac{\partial^2}{\partial\lambda_0\partial\lambda_0^\dagger}\mathcal{Z}(u,\bar{u},\lambda_0,\lambda_0^\dagger,D_0;\mathfrak{m})\right|_{\lambda_0=\lambda_0^\dagger=0} \tag{4.5.36}$$

$$= -\frac{i}{D_0}\frac{\partial}{\partial\overline{u}}\mathcal{Z}(u,\bar{u},D_0;\mathfrak{m}) \tag{4.5.37}$$

where $\mathcal{Z}(u,\bar{u},D_0;\mathfrak{m}) \equiv \mathcal{Z}(u,\bar{u},0,0,D_0;\mathfrak{m})$. After integrating out the fermionic zero modes, the summand reads

$$Z_{\mathfrak{m}} = \frac{1}{2\pi^2}\lim_{\mathbf{e},\varepsilon\to 0}\int_{\mathfrak{M}\setminus\Delta_\varepsilon}\mathrm{d}^2u\int_{\mathbb{R}+i\eta}\frac{\mathrm{d}D_0}{D_0}\frac{\partial}{\partial\overline{u}}\mathcal{Z}(u,\bar{u},D_0;\mathfrak{m}). \tag{4.5.38}$$

Note that the apparent pole at $D_0 = 0$ is cancelled by $\partial_{\overline{u}}\mathcal{Z} = 0$, and that the measure is $\mathrm{d}^2u = \mathrm{d}\,(\mathrm{Re}u)\wedge\mathrm{d}\,(\mathrm{Im}u) = \frac{i}{2}\mathrm{d}u\wedge\mathrm{d}\overline{u}$.

4.5.4 Integrating out fermionic zero modes on $\Sigma_g \times S^1$

In this section, we describe how to integrate out the fermionic zero modes for the three-dimensional genus $g \geq 1$ case. The integral over fermionic zero modes is

$$\int \mathrm{d}\lambda_0 \mathrm{d}\lambda_0^\dagger \left(\prod_{\alpha=1}^{g} d\eta_0^{(\alpha)} d\eta_0^{(\alpha)\dagger} \right) \mathcal{Z} = \left(\prod_{\alpha=1}^{g} \frac{\partial^2}{\partial \eta_0^{(\alpha)} \partial \eta_0^{(\alpha)\dagger}} \right) \frac{\partial^2}{\partial \lambda_0 \partial \lambda_0^\dagger} \mathcal{Z} \tag{4.5.39}$$

Note that while $\lambda_0, \lambda_0^\dagger$ are scalars, $\eta^\dagger = \eta_z^\dagger = \eta_i^\dagger \mathrm{d}z^i$ is a holomorphic one-form, and $\eta = \eta_{\bar{z}} = \eta_{\bar{j}} \mathrm{d}\bar{z}^{\bar{j}}$ is an anti-holomorphic one-form. For details regarding the fermionic zero modes $\eta_0, \eta_0^\dagger$, which appear for genus $g \geq 1$, see [19]. Since the effective action is topological, it can generically be expressed as

$$\mathcal{Z} = \mathcal{A} \exp \int_{\Sigma_g} \left(\mathcal{B} F + c \mathcal{G} \eta^\dagger \wedge \eta \right). \tag{4.5.40}$$

Here, the one-forms ($F = \mathrm{d}A$, $\eta, \eta^\dagger$) remain explicit, while $\mathcal{A}, \mathcal{B}, \mathcal{G}$ are functions of the scalars $u, \bar{u}, \lambda_0, \lambda_0^\dagger, D_0$. The partition function $\mathcal{Z}$ is closed under both Q and $\tilde{Q}$ (i.e. supersymmetric), and the same is true for the functions $\mathcal{A}, \mathcal{B}, \mathcal{G}$. Recall that Q and $\tilde{Q}$ act as differential operators. Acting with Q, then inserting the supersymmetry transformations, we obtain

$$0 = Q\mathcal{Z} = i\lambda_0^\dagger \frac{\partial \mathcal{Z}}{\partial \overline{u}} - D_0 \frac{\partial \mathcal{Z}}{\partial \lambda_0} + \mathcal{Z} Q \int_{\Sigma_g} \left(\mathcal{B} F + c \mathcal{G} \eta^\dagger \wedge \eta \right). \tag{4.5.41}$$

where the last term is due to exponential part of $\mathcal{Z}$. Specifically, the last term follows from the chain rule $\frac{d}{dx} e^{f(x)} = f'(x) e^{f(x)}$ where $f(x)$ is a differentiable function. To evaluate the action of Q on this integrand, note that

$$\begin{aligned} QF &= Q(\mathrm{d}A) && (4.5.42)\\ &= \mathrm{d}\eta^\dagger && (4.5.43)\\ Q\left(\eta_z^\dagger \wedge \eta_{\bar{z}}\right) &= Q\eta_z^\dagger \wedge \eta_{\bar{z}} + (-1)^{\deg \eta_z^\dagger} \eta_z^\dagger \wedge Q\eta_{\bar{z}} && (4.5.44)\\ &= -\eta_z^\dagger \wedge (i\mathrm{d}u) && (4.5.45)\\ &= -i\eta_z^\dagger \wedge \mathrm{d}u && (4.5.46) \end{aligned}$$

where $\mathrm{d}u = D_{\bar{z}} u = D_{\bar{j}} u \mathrm{d}\bar{z}^{\bar{j}}$ is anti-holomorphic. Accordingly, Q acts on the integrand as

$$Q \int_{\Sigma_g} \left(\mathcal{B} F + c \mathcal{G} \eta^\dagger \wedge \eta \right) = \int_{\Sigma_g} \left(\mathcal{B} \mathrm{d}\eta^\dagger - i \mathcal{G} \eta^\dagger \wedge \mathrm{d}u \right). \tag{4.5.47}$$

The functions $\mathcal{B}, \mathcal{G}$ are related using integration by parts, with the conclusion that

$$\mathcal{G} = \frac{\partial \mathcal{B}}{\partial i u}. \tag{4.5.48}$$

The necessary identity to obtain this result is $\overline{\partial}\alpha = \frac{\partial \alpha_i}{\partial \bar{z}^{\bar{j}}} \mathrm{d}\bar{z}^{\bar{j}} \wedge \mathrm{d}z^i$, where $\alpha = \alpha_i \mathrm{d}z^i$ (see e.g. [35]).

4.5.5 Evaluating the contour integral

Here, we describe the evaluation of the final contour integral resulting from the Benini-Zaffaroni approach. The integral to evaluate is

$$Z_{\mathfrak{m}} = \frac{i}{4\pi^2} \lim_{\mathbf{e},\varepsilon \to 0} \int_{\partial \Delta_\varepsilon} \mathrm{d}u \int_{\mathbb{R}+i\eta} \frac{\mathrm{d}D_0}{D_0} \mathcal{Z}(u, \bar{u}, D_0; \mathfrak{m}) \tag{4.5.49}$$

Let $\Gamma_\eta = \mathbb{R} + i\eta$, and let $\Gamma_+ = \Gamma_{\eta>0} \in$ UHP and $\Gamma_- = \Gamma_{\eta<0} \in$ LHP, where UHP (LHP) denotes the upper (lower) D_0-half-plane.

- If $\rho < 0$, the poles are in the LHP, then the $\varepsilon \to 0$ limit sends the poles to the origin from below (poles$\underset{\varepsilon \to 0}{\longrightarrow} D_0 = 0$ from below). Since the LHP-poles do not intersect the UHP contour Γ^+ as $\varepsilon \to 0$, taking the limit does not result in a singular configuration whose residue contributes.
- If $\rho > 0$, the poles are in the UHP, then the limit $\varepsilon \to 0$ sends the poles to the origin from above (poles $\underset{\varepsilon \to 0}{\longrightarrow} D_0 = 0$ from above) The poles intersect the UHP contour Γ^+ as $\varepsilon \to 0$, so taking the limit results in a non-trivial singularity whose residue contributes. In other words, the limit cannot be taken while holding the contour fixed.

The resolution is to decompose the contour as

$$\Gamma_+ = \Gamma_- + C_0 \tag{4.5.50}$$

where C_0 is a circle of radius smaller than r_{C_0} that goes around $D_0 = 0$ counter clockwise. After decomposing the contour, the integral reads

$$Z_{\mathfrak{m}} = \frac{i}{4\pi^2} \lim_{\mathbf{e},\varepsilon \to 0} \int_{\partial \Delta_\varepsilon} \mathrm{d}u \left(\int_{\Gamma_-} \frac{\mathrm{d}D_0}{D_0} \mathcal{Z}(u, \bar{u}, D_0; \mathfrak{m}) \right. \tag{4.5.51}$$

$$\left. + \int_{C_0} \frac{\mathrm{d}D_0}{D_0} \mathcal{Z}(u, \bar{u}, D_0; \mathfrak{m}) \right) \tag{4.5.52}$$

the first term in the parentheses term does not contribute. The second term is evaluated as

$$\begin{aligned}
& \frac{i}{4\pi^2} \lim_{\mathbf{e},\varepsilon\to 0} \int_{\partial\Delta_\varepsilon} \mathrm{d}u \int_{C_0} \frac{\mathrm{d}D_0}{D_0} \mathcal{Z}(u,\bar{u},D_0;\mathfrak{m}) && (4.5.53)\\
= \; & \frac{i}{4\pi^2} \lim_{\mathbf{e},\varepsilon\to 0} \int_{\partial\Delta_\varepsilon} \mathrm{d}u \, (2\pi i \mathcal{Z}(u,\bar{u},0;\mathfrak{m})) && (4.5.54)\\
= \; & -\frac{1}{2\pi} \lim_{\varepsilon\to 0} \int_{\partial\Delta_\varepsilon} \mathrm{d}u \mathcal{Z}(u,\bar{u},0;\mathfrak{m}) && (4.5.55)\\
= \; & -\frac{1}{2\pi} \left(2\pi i \operatorname*{Res}_{u=u_*} \mathcal{Z}(u,\bar{u},0;\mathfrak{m}) \right) && (4.5.56)\\
= \; & -i \operatorname*{Res}_{u=u_*} \mathcal{Z}(u,\bar{u},0;\mathfrak{m}) && (4.5.57)
\end{aligned}$$

The prescription is to take minus the residue, in which case $Z_{\mathfrak{m}}$ receives a contribution of

$$i \operatorname*{Res}_{u=u_*} \mathcal{Z}(u,\bar{u},0;\mathfrak{m}). \qquad (4.5.58)$$

In the higher rank-case, the prescription for the contour and the evaluation of contributing residues can be framed in terms of Jeffrey-Kirwan residues.

Chapter 5

Recovering pure YM2 from A-model YM2

In this section, we confirm the map between physical YM2 and supersymmetric YM2 in the A-model by recovering the physical YM2 partition function from the expectation value of an A-model operator, evaluated using the Benini-Zaffaroni (BZ) formula for A-model correlators [18, 19]. We use the same strategy that established the map between physical YM2 and cohomological YM2 [3]. The strategy was to recover the physical YM2 partition function from the expectation value of an operator, evaluated by localizing the path integral of a variant of YM2 in two-dimensional cohomological gauge theory. For details regarding the strategy, see sections 3.2 and 4.3 of [3], equations 2.81-2.90 in [41], or section 4.2 of [42].

Let us begin by describing the physical YM2 partition function, then the strategy that established the map between physical YM2 and cohomological YM2, followed by the BZ formula for A-model correlators.

In [29], Witten derived a description of the physical YM2 partition function as a sum over irreducible representations (irreps) of the gauge group, first using lattice techniques, then using TQFT techniques. In order to describe the physical YM2 partition function, let Σ_g be a closed oriented Riemannian manifold of genus g, equipped with a Euclidean metric, let $\mathrm{d}\mu = \sqrt{g}d^2x$ be the Riemannian measure, let $a = \int_\Sigma \mathrm{d}\mu$ be the area of Σ_g, let G be the gauge group, let R_μ be the irreps of G, and let e^2 be the YM coupling constant. The partition function of physical YM2 on Σ_g, with gauge group G, reads

$$Z\left(e^2 a, G, \Sigma_g\right) = e^{k_1(2-2g)} \sum_{R_\mu} \frac{1}{\left(\dim R_\mu\right)^{2g-2}} e^{-\frac{1}{2}e^2 a(C_2(\alpha)+k_2)} \tag{5.0.1}$$

Here, $C_2\left(R_\mu\right)$ is the quadratic Casimir of R_μ, and k_1 and k_2 are renormalization scheme dependent constants. In particular, various choices of renormalization of the quantum action can differ by factors of $\Delta S = k_1 \int \frac{R}{4\pi} + k_2 e^2 \int \mathrm{d}\mu$ where R is the Ricci scalar. For details, see chapter 3 or , e.g., section 3.1 of [31].

In [3], Witten established a map from cohomological YM2 to physical YM2 by recovering the irrep description of the physical YM2 partition function from the expectation value of an operator in a two-dimensional cohomological gauge theory, evaluated using non-abelian localization. For details regarding the operator expectation value, see equations 3.37-3.40 in [3]. The theory that was localized was a variant of YM2 defined using the standard multiplet of two-dimensional cohomological gauge theory. It is possible to realize Witten's procedure to recover the physical YM2 partition function from the BZ formula for A-model correlators because the standard cohomological multiplet and the A-twisted $\mathcal{N} = (2, 2)$ Euclidean vector multiplet in Wess-Zumino (WZ) gauge are related by field redefinitions. So, the BZ formula can be used to repeat Witten's procedure in the context of the A-model.

In [18, 19], Benini & Zaffaroni used supersymmetric localization to derive general formulae for partition functions and correlators in the A-model, that is, topologically A-twisted $\mathcal{N} = (2, 2)$ supersymmetric theories of vector and chiral multiplets defined on Σ_g with gauge group G. Recall that localization reduces supersymmetric path integrals to lower-dimensional integrals over moduli, the moduli are coordinates on the localization locus, and the localization locus is a subspace of the integration domain of the path integral i.e., the space of all field configurations. So, the BZ formulae are provided in terms of integrals over two moduli $\mathfrak{m}$ and u, and an integrand that consists of a classical contribution $Z_{\text{cl}}(u, \mathfrak{m})$, a one-loop contribution $Z_{11}(u, \mathfrak{m})$, and possible operator insertions $\mathcal{O}$. Here, $\mathfrak{m}$ is a discrete modulus and u is a continuous modulus .

In order to describe the BZ formula for A-model correlators, let us begin by providing some details regarding the A-model vector multiplet and the moduli. Before BZ localization, the fields in the A-model vector multiplet in WZ gauge are $\mathcal{V} = (A_\mu, \sigma, \tilde{\sigma}, D, \Lambda_\mu, \lambda, \tilde{\lambda})$ for $\mu = 1, 2$, where A_μ is the gauge field, $\sigma, \tilde{\sigma}$ are complex bosonic scalars, D is the bosonic scalar auxiliary field, Λ_μ is the fermionic vector field, and $\lambda, \tilde{\lambda}$ are fermionic scalars. Each field is valued in the adjoint representation of the complexification $\mathfrak{g}_{\mathbb{C}}$ of the Lie algebra $\mathfrak{g} = \text{Lie}G$, and is treated as generically complex and independent.

After BZ localization, the moduli u and $\mathfrak{m}$ parametrize σ and F_{12}, respectively, where F_{12} is the scalar gauge field strength. The continuous modulus u is the vacuum expectation value of σ, and the discrete modulus $\mathfrak{m}$ is the GNO quantized gauge flux.

Specifically, the moduli are

$$u \in \mathfrak{h}_{\mathbb{C}}, \qquad \mathfrak{m} = \frac{1}{2\pi} \int\limits_{\Sigma_g} F \in \Lambda_{\text{coch}}^{G}. \tag{5.0.2}$$

where $\mathfrak{h}_{\mathbb{C}}$ is the Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$, and $\Lambda_{\text{coch}}^{G}$ is the cocharacter (GNO) lattice. The cocharacter lattice is defined as the elements of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ whose image under the exponential map is the identity element of G, that is, $\Lambda_{\text{coch}}^{G} = \{\gamma \in \mathfrak{h} | \exp(2\pi i \gamma) = 1_G\}$. Note that the modulus u is complex-valued due to the BZ choice of reality conditions $\sigma^{\dagger} = \tilde{\sigma}$ during localization, and that this is not the only possible reality condition. Furthermore, the space $\mathfrak{h}_{\mathbb{C}}$ is non-compact, e.g., $\mathfrak{h}_{\mathbb{C}}$ is the complex u-plane for rank$G = 1$.

The BZ formula for correlators of the A-model vector multiplet is

$$Z(u, \mathfrak{m}, \Sigma_g, G) \quad = \quad \frac{(-1)^r}{|W|} \int \mathrm{d}u^r \sum_{\mathfrak{m} \in \Lambda_{\text{coch}}^{G}} Z_{\text{cl}}\, Z_{11}\, \mathcal{O} \tag{5.0.3}$$

where

$$Z_{\text{cl}} = e^{4\pi \widetilde{W}'(u) \cdot \mathfrak{m}}, \qquad Z_{11} = (-1)^{\sum_{\alpha \in \Delta_+} \alpha(\mathfrak{m})} \prod_{\alpha \in \Delta} \alpha(u)^{1-g} \tag{5.0.4}$$

Here, the rank is $r = \text{rank}G$, the order of the Weyl group is $|W|$, α are the roots of $\mathfrak{g}$, Δ is the set of roots, and Δ_+ are the positive roots. Furthermore, $\widetilde{W}(u)$ is the u-holomorphic twisted chiral superpotential and $\widetilde{W}'(u) = \frac{\partial}{\partial u} \widetilde{W}(u)$, $\alpha(u) = \langle \alpha, u \rangle$ denotes the pairing between dual vector spaces $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \otimes \mathfrak{h} \to \mathbb{R}$, and $\widetilde{W}'(u) \cdot \mathfrak{m} = \left(\widetilde{W}'(u), \mathfrak{m}\right)$ denotes the pairing with the scalar product $(\cdot, \cdot)$ on $\mathfrak{h}$. For the classical and one-loop contributions, see equations 5.5 and 5.9 in [18], or equations 3.3 and 3.6 in [19].

To recover the irrep description of the physical YM2 partition function from the BZ formula, we begin by making the following choices. In the integrand, we choose the holomorphic superpotential to be quadratic in its argument $\widetilde{W}(u) = \frac{i}{4} u^2$, and choose the operator insertion to be

$$\mathcal{O} = \exp\left(\frac{1}{2} \epsilon \text{Tr} u^2\right) \tag{5.0.5}$$

where $\epsilon > 0$. The operator $\mathcal{O}$ is gauge-invariant, closed under the localizing supercharge, and represents a characteristic class (roughly speaking). Furthermore, ϵ represents the combination $e^2 a$ where e^2 is the Yang-Mills coupling constant and a is the area of Σ_g. Together, the operator and quadratic superpotential correspond to an A-model action that is equivalent to the conventional pure YM2 action. For the cohomological analog of the operator insertion, see equation 3.38 in [3], or equation

3.2 in [43]. For the cohomological analog of the A-model action, see equation 3.42 in [3].

Before proceeding with the actual derivation, let us provide some details regarding our approach. There are two options to evaluate the integral over u and the sum over $\mathfrak{m}$: either sum $\mathfrak{m}$ then integrate u, or integrate u then sum $\mathfrak{m}$. For the order of evaluation, we choose to first sum $\mathfrak{m}$, then integrate u.

The sum over the elements of the cocharacter lattice $\mathfrak{m} \in \Lambda^G_{\text{coch}} \subset \mathfrak{h}$ is evaluated using the Poisson summation formula, resulting in a Dirac comb in which the sum is over character lattice elements $\mu \in \Lambda^G_{\text{ch}} \subset \mathfrak{h}^*$. Thereafter, u is integrated out along the real contour, so that the u-integral only receives contributions from points at which the Dirac comb has support. By integrating u along the real contour instead of the "Jeffery-Kirwan (JK) contour", we are departing from the BZ prescription for evaluation. For details regarding the JK contour, see, e.g., equation 2.46 in [19].

Observe that to evaluate the u-integral over the Dirac comb, it is necessary to choose the real contour for u. Whether u is real, purely imaginary, or complex depends on the reality conditions imposed on σ and $\tilde{\sigma}$ during localization. There are essentially two choices: one can either take complex conjugate $\tilde{\sigma} = \sigma^\dagger$, or take real σ and purely imaginary $\tilde{\sigma}$ as described on page 34 of [3]. Since u parametrizes σ, complex σ results in complex u, while real σ results in real u. In the BZ localization computation, the reality condition is $\tilde{\sigma} = \sigma^\dagger$, such that u is complex. In our derivation, we take real u, real σ, and purely imaginary $\tilde{\sigma}$, which is an acceptable reality condition that must yield the same final result. To be precise, we take u to be valued in the real part of $\mathfrak{h}_\mathbb{C}$, that is, $u \in \mathfrak{h}_\mathbb{R}$ for $\mathfrak{h}_\mathbb{R} = \{z \in \mathfrak{h}_\mathbb{C} | \overline{z} = z\}$ where $\overline{z}$ is the complex conjugate of z. Moreover, we show in chapter 6 that real σ and purely imaginary $\tilde{\sigma}$ yields the correct 1-loop determinant.

The result of summing $\mathfrak{m}$ then integrating out real u is an expression involving a sum over elements of the character lattice $\Lambda^G_{\text{ch}} \subset \mathfrak{h}^*$, and a summand described by the root system of G in $\mathfrak{h}^*$. The sum only receives contributions from the subset of regular elements in Λ^G_{ch}, and the summand is invariant under the action of the Weyl group W. Eliminating the W-invariance in the expression results in a sum over the subset of dominant integral elements of the character lattice $\Lambda^d_{\text{ch}} = \Lambda^G_{\text{ch}} \cap C^0$. By the theorem of highest weights, the dominant integral elements of the character lattice are in one-to-one correspondence with equivalence classes of irreducible representations, such that each $\mu \in \Lambda^d_{\text{ch}}$ is uniquely associated to an irreducible representation R_μ of G. The summand may therefore be recast in terms of the dimension and quadratic Casimir of R_μ using

$$\dim(R_\mu) = \frac{\prod_{\alpha>0}(\alpha, \mu+\varrho)}{\prod_{\alpha>0}(\alpha, \varrho)}, \qquad C_2(R_\mu) = (\mu+\varrho, \mu+\varrho) - (\varrho, \varrho). \tag{5.0.6}$$

where ϱ is the Weyl vector, at which point we have evaluated the irrep description of the physical YM2 partition function (in the renormalization scheme of BZ). Finally, we identify the renormalization scheme dependent constants (k_1 and k_2) so that our result may be related to physical YM2 partition functions evaluated in other renormalization schemes.

Now, let us proceed to the actual derivation. For quadratic superpotential $\widetilde{W}(u) = \frac{i}{4}u^2$, and operator insertion $\mathcal{O} = \exp(\frac{1}{2}\epsilon \mathrm{Tr} u^2)$, the BZ formula for correlators of the A-model vector multiplet 5.0.3 reads

$$Z(u, \mathfrak{m}, \Sigma_g, G) = \frac{(-1)^r}{|W|} \int \mathrm{d}u^r \sum_{\mathfrak{m} \in \Lambda^G_{\mathrm{coch}}} e^{2\pi i (u, \mathfrak{m}) - 2\pi i \langle \varrho, \mathfrak{m} \rangle - \frac{\epsilon}{2}(u,u)} \prod_{\alpha \in \Delta} \langle \alpha, u \rangle^{1-g} \tag{5.0.7}$$

Here, the phase factor in Z_{11} was recast as $(-1)^{\sum_{\alpha \in \Delta_+} \alpha(\mathfrak{m})} = e^{-2\pi i \langle \varrho, \mathfrak{m} \rangle}$ where $\varrho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$ is the Weyl vector.

By the generalized Poisson summation formula B.6.3, the sum over $\mathfrak{m} \in \Lambda^G_{\mathrm{coch}}$ in 5.0.7 evaluates to

$$\sum_{\mathfrak{m} \in \Lambda^G_{\mathrm{coch}}} e^{2\pi i (u, \mathfrak{m}) - 2\pi i \langle \varrho, \mathfrak{m} \rangle} f(u) = \frac{1}{\mathrm{covol}\left(\Lambda^G_{\mathrm{coch}}\right)} \sum_{\mu \in \Lambda^G_{\mathrm{ch}}} \delta^{(r)}(u - \mu - \varrho) f(u) \tag{5.0.8}$$

where

$$f(u) = \prod_{\alpha \in \Delta} \langle \alpha, u \rangle^{1-g} e^{-\frac{\epsilon}{2}(u,u)}. \tag{5.0.9}$$

Here, $\delta^{(r)}(\cdot)$ is the r-dimensional delta function, and the character lattice Λ^G_{ch} is the integral dual lattice of $\Lambda^G_{\mathrm{coch}}$, as described in section B.6.

After evaluating the sum, the full expression 5.0.7 reads

$$Z(u, \mu, \Sigma_g, G) = \frac{(-1)^r}{|W| \, \mathrm{covol}\left(\Lambda^G_{\mathrm{coch}}\right)} \int \mathrm{d}u^r \sum_{\mu \in \Lambda^G_{\mathrm{ch}}} \delta^{(r)}(u - \mu - \varrho) \prod_{\alpha \in \Delta} \langle \alpha, u \rangle^{1-g} e^{-\frac{\epsilon}{2}(u,u)}. \tag{5.0.10}$$

Since u is assumed to be real, its integral can be evaluated, and only receives contributions from points where the Dirac comb has support $u - \mu - \varrho = 0$. Integrating out real u, we have

$$\int \mathrm{d}u^r \sum_{\mu \in \Lambda^G_{\mathrm{ch}}} \delta^{(r)}(u - \mu - \varrho) \prod_{\alpha \in \Delta} \langle \alpha, u \rangle^{1-g} e^{-\frac{\epsilon}{2}(u,u)} \tag{5.0.11}$$

$$= \sum_{\mu \in \Lambda^G_{\mathrm{ch}}} \prod_{\alpha \in \Delta} (\alpha, \mu + \varrho)^{1-g} e^{-\frac{\epsilon}{2}(\mu + \varrho, \mu + \varrho)} \tag{5.0.12}$$

where in the second line, $(\cdot, \cdot)$ denotes the scalar product on $\mathfrak{h}^*$.

In terms of the full expression 5.0.10, we have

$$Z\left(\mu,\Sigma_g,G\right)=\frac{(-1)^r}{|W|\,\mathrm{covol}\left(\Lambda^G_{\mathrm{coch}}\right)}\sum_{\mu\in\Lambda^G_{\mathrm{ch}}}\prod_{\alpha\in\Delta}(\alpha,\mu+\varrho)^{1-g}\,e^{-\frac{\epsilon}{2}(\mu+\varrho,\mu+\varrho)},\tag{5.0.13}$$

in which both the sum and summand are described by the root system in $\mathfrak{h}^*$.

The root system is acted upon by the Weyl group W, which is the finite reflection group generated by reflections about the orthogonal hyperplanes π_α defined in section B.3. To eliminate the Weyl group invariance, 5.0.13 is expressed in terms of the fundamental Weyl chamber C^0, which serves as a fundamental domain for the action of W on $\mathfrak{h}^*$. Details concerning the action of the Weyl group on root systems can be found in sections B.3 and B.4.

We proceed by first absorbing the shift by the Weyl vector into the set being summed over, then observing that only regular elements contribute to the sum over the shifted set. Following this, the sum over the shifted set of regular elements is expressed as the W orbit through the set of strictly dominant elements, the resulting Weyl-invariant summand is simplified, and the product over all roots is recast in terms of positive roots. Finally, the sum over the ϱ-shifted set of strictly dominant elements is expressed as a sum over the dominant elements of the character lattice $\Lambda^d_{\mathrm{ch}}=\Lambda^G_{\mathrm{ch}}\cap C^0$, such that the dimension and quadratic Casimir formulae 5.0.6 can be applied.

That being said, the sum over $\mu\in\Lambda^G_{\mathrm{ch}}$ in 5.0.13 is recast as

$$\sum_{\mu\in\Lambda^G_{\mathrm{ch}}}\prod_{\alpha\in\Delta}(\alpha,\mu+\varrho)^{\chi/2}\,e^{-\frac{\epsilon}{2}(\mu+\varrho,\mu+\varrho)}=\sum_{q\in Q}\prod_{\alpha\in\Delta}(\alpha,q)^{\chi/2}\,e^{-\frac{\epsilon}{2}(q,q)}\tag{5.0.14}$$

where the new summation variable is $q=\mu+\varrho$ for all $\mu\in\Lambda^G_{\mathrm{ch}}$, the Euler characteristic is $\chi=2-2g$, and the ϱ-shifted set

$$Q=\left\{\mu+\varrho|\mu\in\Lambda^G_{\mathrm{ch}}\right\}\tag{5.0.15}$$

is an infinite set of integral elements in $\mathfrak{h}^*$. Observe that in the case when the gauge group is the universal cover $G=\tilde{G}$, both the character lattice $\Lambda^{\tilde{G}}_{\mathrm{ch}}$ and the set $Q=\left\{\mu+\varrho|\mu\in\Lambda^{\tilde{G}}_{\mathrm{ch}}\right\}$ coincide with the weight lattice, i.e., $\Lambda^{\tilde{G}}_{\mathrm{ch}}\simeq\Lambda^{\mathfrak{g}}_{\mathrm{wt}}$ and $Q\simeq\Lambda^{\mathfrak{g}}_{\mathrm{wt}}$. This is because the ϱ-shift maps from $\mu\in\Lambda^{\mathfrak{g}}_{\mathrm{wt}}$ to $\mu+\varrho\in\Lambda^{\mathfrak{g}}_{\mathrm{wt}}$. Details concerning the shifted set Q can be found in section B.7. We proceed without specifying any particular cover of G.

The sum over $q\in Q$ only receives contributions from regular elements, that is to say, elements valued in the interiors of Weyl chambers. Equivalently, the sum does not receive contributions from the subset of elements in Q valued on the union of hyperplanes $\Pi = \bigcup_{\alpha\in\Delta} \pi_\alpha$ where $\pi_\alpha = \{\gamma \subset \mathfrak{t}^* | (\alpha, \gamma) = 0\}$. This is due to the vanishing of the product over roots $\prod_{\alpha\in\Delta} (\alpha, q) = 0$ for all $q \in Q \cap \Pi$ in the summand. To see that the product vanishes, consider a particular hyperplane $\pi_\beta \subset \Pi$ orthogonal to the root $\beta \in \Delta$. If an element is valued on the hyperplane $q \in Q \cap \pi_\beta$, then its scalar product with the root is $(\beta, q) = 0$, that it to say, q and β are orthogonal. Since the product over roots includes $\beta \in \Delta$, we have $\prod_{\alpha\in\Delta} (\alpha, q) = (\alpha', q) \ldots (\beta, q) \ldots (\alpha'', q) = 0$ where $\alpha', \alpha'' \in \Delta$. In view of this, we decompose the set as $Q = (Q \cap \Pi) \cup (Q \backslash \Pi)$. Observe that since Π is equivalent to the union of all Weyl chamber boundaries, and its complement $\Pi^c = \mathfrak{t}^* \backslash \Pi$ is equivalent to the union of all Weyl chamber interiors, $Q \cap \Pi$ is the subset of elements valued on the union of Weyl chamber boundaries, while $Q\backslash\Pi$ is the subset of elements valued in the union of Weyl chamber interiors. As the sum does not receive contributions from the subset $Q \cap \Pi$ of Weyl chamber boundary valued elements, 5.0.14 reduces to a sum over the subset $Q\backslash\Pi$ of Weyl chamber interior valued elements. This reads

$$\sum_{q\in Q} \prod_{\alpha\in\Delta} (\alpha, q)^{\chi/2} e^{-\frac{\epsilon}{2}(q,q)} \tag{5.0.16}$$

$$= \sum_{q\in Q\backslash\Pi} \prod_{\alpha\in\Delta} (\alpha, q)^{\chi/2} e^{-\frac{\epsilon}{2}(q,q)} + \sum_{q\in Q\cap\Pi} \prod_{\alpha\in\Delta} (\alpha, q)^{\chi/2} e^{-\frac{\epsilon}{2}(q,q)} \tag{5.0.17}$$

$$= \sum_{q\in Q\backslash\Pi} \prod_{\alpha\in\Delta} (\alpha, q)^{\chi/2} e^{-\frac{\epsilon}{2}(q,q)}. \tag{5.0.18}$$

Next, we express the summation set $Q\backslash\Pi$ as the W-orbit through the set of strictly dominant elements $(Q\backslash\Pi) \cap C^0$. The Weyl group W acts on the elements of the set $Q\backslash\Pi$ through Weyl reflections $w \in W$. As $Q\backslash\Pi$ is an infinite set of integral elements valued in the union of Weyl chamber interiors, the action of the Weyl group on $Q\backslash\Pi$ is regular, that is to say, both transitive and free. In particular, for any two elements $q', q'' \in Q\backslash\Pi$, there is precisely one $w \in W$ for which $w \cdot q' = q''$. Since the W-action on $Q\backslash\Pi$ is regular, every element $q \in Q\backslash\Pi$ is valued in the orbit of W through a unique element in the interior of the fundamental Weyl chamber $q_0 \in Q \cap C^{0,i}$. Specifically, for each $q \in Q\backslash\Pi$ there is a unique $q_0 \in Q \cap C^{0,i}$ such that $q \in \mathrm{Orb}_W (q_0) = \{w \cdot q_0 | w \in W\}$. This is extended to express all elements $Q\backslash\Pi$ as the orbit of W through all elements of $Q \cap C^{0,i}$, which reads

$$Q\backslash\Pi = \mathrm{Orb}_W \left(Q \cap C^{0,i}\right) = \left\{w \cdot q_0 \mid w \in W, q_0 \in Q \cap C^{0,i}\right\}. \tag{5.0.19}$$

In view of this, the sum 5.0.18 simplifies to

$$\sum_{q\in Q\setminus\Pi}\prod_{\alpha\in\Delta}(\alpha,q)^{\chi/2}\,e^{-\frac{\epsilon}{2}(q,q)} \tag{5.0.20}$$

$$= \sum_{q\in \mathrm{Orb}_W(Q\cap C^{0,i})}\prod_{\alpha\in\Delta}(\alpha,q)^{\chi/2}\,e^{-\frac{\epsilon}{2}(q,q)} \tag{5.0.21}$$

$$= \sum_{w\in W}\sum_{q_0\in Q\cap C^{0,i}}\prod_{\alpha\in\Delta}(\alpha,w\cdot q_0)^{\chi/2}\,e^{-\frac{\epsilon}{2}(w\cdot q_0,w\cdot q_0)} \tag{5.0.22}$$

Here, the summand is invariant under the action of the Weyl group. This is due to two properties of Weyl reflections. The first property is that, when acting on elements in scalar products, Weyl reflections obey

$$(w\cdot q, w\cdot q) = (q,q)\,,\ (\alpha, w\cdot q) = \left(w^{-1}\cdot\alpha, q\right). \tag{5.0.23}$$

The second property, is that the set of roots Δ is closed under the W-action

$$W \ : \ \Delta\to\Delta, \tag{5.0.24}$$

$$\alpha\mapsto\alpha' = w\cdot\alpha. \tag{5.0.25}$$

In particular, Weyl reflections permute the elements in the set of roots Δ, in the sense that each root $\alpha\in\Delta$ is mapped so some other root $\alpha' = w\cdot\alpha\in\Delta$. By acting on both sides of $\alpha' = w\cdot\alpha$ with the inverse Weyl reflection w^{-1}, we have also $w^{-1}\alpha' = \alpha\in\Delta$. In view of these properties, the summand in 5.0.22 simplifies as

$$\prod_{\alpha\in\Delta}(\alpha,w\cdot q)^{\chi/2}\,e^{-\frac{\epsilon}{2}(w\cdot q,w\cdot q)} \tag{5.0.26}$$

$$= \prod_{\alpha\in\Delta}\left(w^{-1}\alpha,q\right)^{\chi/2}\,e^{-\frac{\epsilon}{2}(q,q)} \tag{5.0.27}$$

$$= \prod_{\alpha\in\Delta}(\alpha,q)^{\chi/2}\,e^{-\frac{\epsilon}{2}(q,q)} \tag{5.0.28}$$

In the first equality we used the first property. In the second equality, we used the second property, together with the fact that the order of the product over roots does not matter. Due to the invariance of the summand under the action of the Weyl group, the sum 5.0.22 reduces as

$$\sum_{w\in W}\sum_{q\in Q\cap C^{0,i}}\prod_{\alpha\in\Delta}(\alpha, w\cdot q)^{\chi/2}\, e^{-\frac{\epsilon}{2}(w\cdot q, w\cdot q)} \tag{5.0.29}$$

$$= \sum_{w\in W}\sum_{q\in Q\cap C^{0,i}}\prod_{\alpha\in\Delta}(\alpha, q)^{\chi/2}\, e^{-\frac{\epsilon}{2}(q,q)} \tag{5.0.30}$$

$$= |W| \sum_{q\in Q\cap C^{0,i}}\prod_{\alpha\in\Delta}(\alpha, q)^{\chi/2}\, e^{-\frac{\epsilon}{2}(q,q)} \tag{5.0.31}$$

The only effect of the sum over $w \in W$ is the overall factor $|W|$, i.e., the order of the Weyl group.

To recover the dimension of R_μ of G, it is necessary to express 5.0.31 as a product over the set of positive roots. The set of roots Δ is therefore decomposed into positive and negative subsets $\Delta = \Delta_+ \cup \Delta_-$. For each positive root $\alpha \in \Delta_+$, there is an associated negative root $-\alpha \in \Delta_-$. Under this decomposition, the product over roots simplifies as

$$\prod_{\alpha\in\Delta}(\alpha, q)^{\chi/2} = \left(\prod_{\alpha\in\Delta_+}(\alpha, q)\prod_{\alpha\in\Delta_-}(\alpha, q)\right)^{\chi/2} \tag{5.0.32}$$

$$= \left(\prod_{\alpha\in\Delta_+}(\alpha, q)\prod_{\alpha\in\Delta_+}(-\alpha, q)\right)^{\chi/2} \tag{5.0.33}$$

$$= \left((-1)^{|\Delta|/2}\left(\prod_{\alpha\in\Delta_+}(\alpha, q)\right)^2\right)^{\chi/2} \tag{5.0.34}$$

$$= \left((-1)^{|\Delta|/2}\right)^{\chi/2}\prod_{\alpha\in\Delta_+}(\alpha, q)^{\chi}\,, \tag{5.0.35}$$

the last equality is due to $(\alpha, q) > 0$ for $\alpha \in \Delta_+, q \in C^{0,i}$. Note that $|\Delta| = \dim\mathfrak{g} - \mathrm{rank}\mathfrak{g}$. Thus, expressing the sum 5.0.31 in terms of positive roots, we have

$$\sum_{q\in Q\cap C^{0,i}}\prod_{\alpha\in\Delta}(\alpha, q)^{\chi/2}\, e^{-\frac{\epsilon}{2}(q,q)} = \left((-1)^{|\Delta|/2}\right)^{\chi/2}\sum_{q\in Q\cap C^{0,i}}\prod_{\alpha\in\Delta_+}(\alpha, q)^{\chi}\, e^{-\frac{\epsilon}{2}(q,q)} \tag{5.0.36}$$

The final step is to shift back, that is to say, express the sum over the set Q in terms of the character lattice Λ^G_{ch}. Since the translation by ϱ is a bijective map between dominant integral elements and strictly dominant integral elements, each element of $Q \cap C^{0,i}$ is uniquely related to an element of $\Lambda^d_{\mathrm{ch}} = \Lambda^G_{\mathrm{ch}} \cap C^0$ by the the ϱ-shift. See section B.7 for details. When expressed in terms of the character lattice, the sum 5.0.36 reads

$$\sum_{q\in Q\cap C^{0,i}}\prod_{\alpha\in\Delta_+}(\alpha,q)^\chi\, e^{-\frac{\epsilon}{2}(q,q)} = \sum_{\mu\in\Lambda^d_{\text{ch}}}\prod_{\alpha\in\Delta_+}(\alpha,\mu+\varrho)^\chi\, e^{-\frac{\epsilon}{2}(\mu+\varrho,\mu+\varrho)}. \qquad (5.0.37)$$

Thus, the Weyl invariance has been eliminated from the sum. Collecting the factors, the sum in the original expression 5.0.13 simplifies as

$$Z(\mu,\Sigma_g,G) = \frac{(-1)^r}{|W|\,\text{covol}\left(\Lambda^G_{\text{coch}}\right)}\sum_{\mu\in\Lambda^G_{\text{ch}}}\prod_{\alpha\in\Delta}(\alpha,\mu+\varrho)^{\chi/2}\, e^{-\frac{\epsilon}{2}(\mu+\varrho,\mu+\varrho)} \qquad (5.0.38)$$

$$= \frac{(-1)^r}{|W|\,\text{covol}\left(\Lambda^G_{\text{coch}}\right)}\times \qquad (5.0.39)$$

$$\left(|W|\left((-1)^{|\Delta|/2}\right)^{\chi/2}\sum_{\mu\in\Lambda^d_{\text{ch}}}\prod_{\alpha\in\Delta_+}(\alpha,\mu+\varrho)^\chi\, e^{-\frac{\epsilon}{2}(\mu+\varrho,\mu+\varrho)}\right) \qquad (5.0.40)$$

$$= \frac{(-1)^r\left((-1)^{|\Delta|/2}\right)^{\chi/2}}{\text{covol}\left(\Lambda^G_{\text{coch}}\right)}\sum_{\mu\in\Lambda^d_{\text{ch}}}\prod_{\alpha\in\Delta_+}(\alpha,\mu+\varrho)^\chi\, e^{-\frac{\epsilon}{2}(\mu+\varrho,\mu+\varrho)} \qquad (5.0.41)$$

This is in the appropriate form to recover the dimension and quadratic Casimir of the irreducible representation R_μ of G. In particular, from the formulae 5.0.6 the terms in the above summand are identified as

$$\prod_{\alpha\in\Delta_+}(\alpha,\mu+\varrho) = \prod_{\alpha\in\Delta_+}(\alpha,\varrho)\dim(R_\mu), \qquad (\mu+\varrho,\mu+\varrho) = C_2(R_\mu)+(\varrho,\varrho). \qquad (5.0.42)$$

Thus, inserting 5.0.42 into 5.0.41 yields

$$Z(\mu,\Sigma_g,G) = \frac{(-1)^r\left((-1)^{|\Delta|/2}\right)^{\chi/2}}{\text{covol}\left(\Lambda^G_{\text{coch}}\right)}\times \qquad (5.0.43)$$

$$\sum_{\mu\in\Lambda^{\text{d}}_{\text{ch}}}\prod_{\alpha\in\Delta_+}\left(\dim(R_\mu)\prod_{\alpha>0}(\alpha,\varrho)\right)^\chi e^{-\frac{\epsilon}{2}(C_2(R_\mu)+(\varrho,\varrho))} \qquad (5.0.44)$$

$$= \frac{(-1)^r\left((-1)^{|\Delta|/2}\right)^{\chi/2} e^{-\frac{\epsilon}{2}(\varrho,\varrho)}\prod_{\alpha\in\Delta_+}(\alpha,\varrho)^\chi}{\text{covol}\left(\Lambda^G_{\text{coch}}\right)}\times \qquad (5.0.45)$$

$$\sum_{\mu\in\Lambda^d_{\text{ch}}}\dim(R_\mu)^\chi\, e^{-\frac{\epsilon}{2}C_2(R_\mu)} \qquad (5.0.46)$$

As a check, we consider the case $\Sigma_{g=0} = S^2$ for $G = SU(2)$. For $Z\left(\mu, S^2, SU(2)\right)$ we have $\chi = 2$, $\Delta = \{\pm 2\}$, $|\Delta| = 2$, $\varrho = 1$, $\Lambda^{\text{d}}_{\text{ch}} = \mathbb{Z}_{\geq 0}$, and $\text{covol}\left(\Lambda^G_{\text{coch}}\right)$ is not applicable. Setting $\epsilon = e^2 a$, yields

$$Z\left(\mu, S^2, SU(2)\right) = 4 \sum_{\mu \in \mathbb{Z}_{\geq 0}} \dim\left(R_\mu\right)^2 e^{-\frac{1}{2}e^2 a\left(C_2\left(R_\mu\right)+1\right)}. \tag{5.0.47}$$

When compared with the original irrep partition function 5.0.1, this results in the equations for the renormalization ambiguity constants k_1, k_2:

$$4 = e^{2k_1},\ e^{-\frac{1}{2}\epsilon\left(C_2\left(R_\mu\right)+1\right)} = e^{-\frac{1}{2}e^2 a\left(C_2(R)+k_2\right)}. \tag{5.0.48}$$

From this we identify $k_1 = i\pi n + \log 2, n \in \mathbb{Z}$ and $k_2 = 1$.

Chapter 6

A-model localization

In this chapter, we compare the follow-your-nose (FYN), Benini-Zaffaroni (BZ), and Witten approaches to supersymmetric localization of YM2 theories in the A-model. In particular, we localize path integrals according to each of the three approaches, then compare in each case the mode-by-mode evaluation of fluctuation determinants.

6.1 A-model vector multiplet on S^2

In this section we describe the A-model on the two-sphere, focusing on the vector multiplet in Wess-Zumino gauge. This proceeds by first outlining the supersymmetric background, then the vector multiplet, and finally, the supersymmetric actions constructed from fields in the vector multiplet.

The A-model on the two-sphere is defined by performing the topological A-twist of the $\mathcal{N}=(2,2)$ supersymmetric theory on $\mathbb{R}^2$. Note that this procedure may be used to define the A-model on any compact closed oriented Riemannian manifold Σ_g of genus g, not only on the S^2.

The line element on the S^2 is

$$\mathrm{d}s^2 = 2g_{z\overline{z}}\left(|z|^2\right)\mathrm{d}z\mathrm{d}\overline{z} = \sqrt{g}\mathrm{d}z\mathrm{d}\overline{z} = e^1 e^{\overline{1}}. \tag{6.1.1}$$

Here, $z = x^1 + ix^2$, $\overline{z} = x^1 - ix^2$ are complex coordinates, and $e^1 = e^1_z\mathrm{d}z = g^{1/4}\mathrm{d}z$, $e^{\overline{1}} = e^{\overline{1}}_{\overline{z}}d\overline{z} = g^{1/4}\mathrm{d}\overline{z}$ is the complex frame. In these conventions, $X_z = \frac{1}{2}\left(X_1 - iX_2\right)$ is a holomorphic vector, $X_{\overline{z}} = \frac{1}{2}\left(X_1 + iX_2\right)$ is an anti-holomorphic vector, and in the complex frame, these are $X_1 = e^z_1 X_z$ and $X_{\overline{1}} = e^{\overline{z}}_{\overline{1}}X_{\overline{z}}$.

The A-twist is a solution of the Killing spinor equations of S^2 that preserves half of the $\mathcal{N}=(2,2)$ supercharges of $\mathbb{R}^2$. The solution is

$$A^R_\mu = \frac{1}{2}\omega_\mu,\ \zeta = \begin{pmatrix} 0 \\ \zeta_+ \end{pmatrix},\ \tilde{\zeta} = \begin{pmatrix} \tilde{\zeta}_- \\ 0 \end{pmatrix},\ \mathcal{H} = \widetilde{\mathcal{H}} = 0 \tag{6.1.2}$$

for constant Killing spinors $\partial_\mu \zeta = \partial_\mu \tilde{\zeta} = 0$. Here, A_μ^R is the connection of the background vector-like $U(1)_R$ symmetry, ω_μ is the spin connection on S^2, and $\mathcal{H}, \widetilde{\mathcal{H}}$ are bosonic scalars from the supergravity multiplet that we do not consider.

The A-twist modifies the spin of the supercharges, the supersymmetry parameters, and the gaugini of the $\mathcal{N} = (2,2)$ theory on $\mathbb{R}^2$. In particular, the supercharges $Q_-, \widetilde{Q}_+$ are anti-commuting scalars, while the supersymmetry parameters $\zeta_+, \tilde{\zeta}_-$ are commuting scalars. The A-model scalar supercharge is the linear combination $Q_A = Q_- + \widetilde{Q}_+$.

The supersymmetry transformations of the A-model vector multiplet in Wess-Zumino gauge are

$$\delta_A a_1 = -i\Lambda_1 \tag{6.1.3}$$
$$\delta_A a_{\bar{1}} = +i\tilde{\Lambda}_{\bar{1}} \tag{6.1.4}$$
$$\delta_A \sigma = 0 \tag{6.1.5}$$
$$\delta_A \tilde{\sigma} = -2\left(\tilde{\lambda} + \lambda\right) \tag{6.1.6}$$
$$\delta_A \Lambda_1 = +2iD_1\sigma \tag{6.1.7}$$
$$\delta_A \tilde{\Lambda}_{\bar{1}} = -2iD_{\bar{1}}\sigma \tag{6.1.8}$$
$$\delta_A \lambda = +i\left(D - 2if_{1\bar{1}} - \frac{1}{2}[\sigma, \tilde{\sigma}]\right) \tag{6.1.9}$$
$$\delta_A \tilde{\lambda} = -i\left(D - 2if_{1\bar{1}} + \frac{1}{2}[\sigma, \tilde{\sigma}]\right) \tag{6.1.10}$$
$$\delta_A D = -2D_1\tilde{\Lambda}_{\bar{1}} - \left[\sigma, \tilde{\lambda}\right] - 2D_{\bar{1}}\Lambda_1 + [\sigma, \lambda] \tag{6.1.11}$$

Here, $\delta_A = \delta + \tilde{\delta}$ is the anti-commuting scalar supersymmetry variation with respect to Q_A. The A-twisted fermions have been defined as $\Lambda_1 = \tilde{\zeta}_-\lambda_-$, $\tilde{\Lambda}_{\bar{1}} = \zeta_+\tilde{\lambda}_+$ $\lambda = \tilde{\zeta}_-\lambda_+$, $\tilde{\lambda} = \zeta_+\tilde{\lambda}_-$. Note that $\pm$ does not indicate spin, but is instead meant to clarify how the A-twisted fermionic vectors and scalars on S^2 are related to the untwisted fermionic spinors from $\mathbb{R}^2$.

The algebra of the supercharges is

$$\delta^2\varphi = \tilde{\delta}^2\varphi = 0, \; . \tag{6.1.12}$$
$$\left\{\delta, \tilde{\delta}\right\}\varphi = 2i[\sigma, \varphi] \tag{6.1.13}$$
$$\left\{\delta, \tilde{\delta}\right\}a_\mu = 2D_\mu\sigma \tag{6.1.14}$$

where φ denotes any of the fields in the vector multiplet omitting a_μ. In particular, the supercharges square to an infinitesimal gauge transformation. Accordingly, gauge-invariant functionals of the fields are nilpotent under $\delta_A = \delta + \tilde{\delta}$.

A standard supersymmetric action in the A-model is that of the twisted chiral multiplet

$$S = \int d^2x\sqrt{g}\left(\frac{1}{h^2}L_{\text{YM}} + L_{\widetilde{W}} + L_{\overline{\widetilde{W}}}\right) \tag{6.1.15}$$

Here, L_{YM} is the standard Yang-Mills Lagrangian, while $L_{\widetilde{W}}$ is the twisted chiral superpotential Lagrangian and $L_{\overline{\widetilde{W}}}$ is its conjugate. In components, the Lagrangians are

$$L_{\text{YM}} = \text{Tr}\Bigg(\frac{1}{2}\left(-2if_{1\bar{1}}\right)^2 - \frac{1}{2}D^2 + \frac{1}{2}D_\mu\tilde{\sigma}D^\mu\sigma + \frac{1}{8}[\sigma,\tilde{\sigma}]^2 \tag{6.1.16}$$

$$+2i\tilde{\Lambda}_{\bar{1}}D_1\lambda - 2i\Lambda_1 D_{\bar{1}}\tilde{\lambda} - i\tilde{\Lambda}_{\bar{1}}[\tilde{\sigma},\Lambda_1] + i\tilde{\lambda}[\sigma,\lambda]\Bigg), \tag{6.1.17}$$

$$L_{\widetilde{W}} = \text{Tr}\Big(+ i\left(D + 2if_{1\bar{1}}\right)\widetilde{W}'(\sigma) - 2\Lambda_1\tilde{\Lambda}_{\bar{1}}\widetilde{W}''(\sigma)\Big), \tag{6.1.18}$$

$$L_{\overline{\widetilde{W}}} = \text{Tr}\left(-i\left(D - 2if_{1\bar{1}}\right)\overline{\widetilde{W}}'(\tilde{\sigma}) + 2\tilde{\lambda}\lambda\overline{\widetilde{W}}''(\tilde{\sigma})\right). \tag{6.1.19}$$

Here, $\widetilde{W}$ and $\overline{\widetilde{W}}$ are holomorphic and anti-holomorphic functionals, respectively, prime indicates derivatives, the gauge field strength is $f_{1\bar{1}} = \partial_1 a_{\bar{1}} - \partial_{\bar{1}}a_1 - i\left[a_1, a_{\bar{1}}\right]$, and $D_\mu D^\mu = 2\left\{D_1, D_{\bar{1}}\right\}$. All the Lagrangians are δ_A-closed (since the action is supersymmetric), while $L_{\text{YM}}, L_{\overline{\widetilde{W}}}$ are δ_A-exact up to total derivatives.

A less common supersymmetric action in A-model (noted in [20]) is defined by

$$L_{\Sigma\tilde{\Sigma}} = \text{Tr}\Bigg(\left(-2if_{1\bar{1}}\right)^2 - 2if_{1\bar{1}}D + \frac{1}{2}D_\mu\tilde{\sigma}D^\mu\sigma \tag{6.1.20}$$

$$+2i\tilde{\Lambda}_{\bar{1}}D_1\lambda - 2i\Lambda_1 D_{\bar{1}}\tilde{\lambda} - i\tilde{\Lambda}_{\bar{1}}\left[\tilde{\sigma},\Lambda_1\right]\Bigg). \tag{6.1.21}$$

This leads to an action that is both closed and exact under δ_A. Observe that $L_{\Sigma\tilde{\Sigma}}$ is a type of non-standard Yang-Mills Lagrangian that results in degenerate kinetic energy for fields, while L_{YM} is a standard Yang-Mills Lagrangian that results in non-degenerate kinetic energy. When written as $\delta, \tilde{\delta}$ exact variations, the two types of Yang-Mills Lagrangians read

$$L_{\text{YM}} = \delta\tilde{\delta}\text{Tr}\left(\tilde{\sigma}f_{1\bar{1}} - \frac{1}{2}\lambda\tilde{\lambda}\right) \tag{6.1.22}$$

$$L_{\Sigma\tilde{\Sigma}} = \delta\tilde{\delta}\text{Tr}\left(\tilde{\sigma}f_{1\bar{1}}\right) \tag{6.1.23}$$

A small curiosity is the relationship between $L_{\Sigma\tilde{\Sigma}}$ and the standard Donaldson Lagrangian in two-dimensional cohomological gauge theory L_{Don} (eq. 3.21 in [3]). Both $L_{\Sigma\tilde{\Sigma}}$ and L_{Don} are degenerate Yang-Mills type Lagrangians, and neither involve a bosonic scalar commutator term. It is not yet clear to the author whether $L_{\Sigma\tilde{\Sigma}}$ is the A-model analog of L_{Don}.

6.2 Localizing terms

In this section, we construct our localizing term. This proceeds by first sketching the setup for the localization computation, then constructing our localizing term from the available supersymmetric actions of the A-model vector multiplet, and finally, recording the path integral to be localized.

Schematically, the quantity we will evaluate is

$$\lim_{h\to 0}\int_{\mathcal{F}}\mathcal{D}\varphi e^{-\frac{1}{h^2}\{Q_A,V\}} \tag{6.2.1}$$

Here, the fields of the A-model vector multiplet are collectively denoted φ, and $\mathcal{D}\varphi$ is the path integral measure on configuration space $\mathcal{F}$ of φ. The term $S=\{Q_A,V\}$ is called the localization scheme, or localizing term where S is a commuting action functional of φ, and V is an anti-commuting functional of φ. For the localizing supercharge, we take the linear combination $Q_A=Q_-+\widetilde{Q}_+$, where $Q_-,\widetilde{Q}_+$ are the scalar supercharges in the A-model. Q_A is called the A-model supercharge. The localizing limit is $h\to 0$, where h is taken to be the Yang-Mills coupling constant.

Our localization scheme the Q_A-exact action functional

$$S(t,\tau)=\{Q_A,V(t,\tau)\}=\int \mathrm{d}^2x\sqrt{g}L(t,\tau) \tag{6.2.2}$$

where

$$\begin{aligned} L(t,\tau) &= L_{\mathrm{YM}}+tL^{\mathrm{quad}}_{\widetilde{\overline{W}}}-\tau L_{\mathrm{com}} && (6.2.3)\\ &= \left\{Q_A,V_{\mathrm{YM}}+tV^{\mathrm{quad}}_{\widetilde{\overline{W}}}-\tau V_{\mathrm{com}}\right\} && (6.2.4)\end{aligned}$$

is our general localizing Lagrangian. Here, t and τ are parameters that may be turned on or off, corresponding to different choices of localization scheme, and different localization computations.

The first term in $L(t,\tau)$ is the standard Yang-Mills Lagrangian, which has a Q_A-exact expression $L_{\mathrm{YM}}=\{Q_A,V_{\mathrm{YM}}\}$ for a fermionic functional V_{YM}.

The second term in $L(t,\tau)$ is the standard twisted anti-chiral superpotential Lagrangian $L_{\widetilde{\overline{W}}}$, for the quadratic choice of anti-holomorphic functional $\widetilde{\overline{W}}(\tilde{\sigma})=\frac{1}{2}\tilde{\sigma}^2$. This reads

$$\begin{aligned} L^{\mathrm{quad}}_{\widetilde{\overline{W}}} &= \mathrm{Tr}\left(-i(D-2if_{1\bar{1}})\widetilde{\overline{W}}'(\tilde{\sigma})+2\tilde{\lambda}\lambda\widetilde{\overline{W}}''(\tilde{\sigma})\right)\Big|_{\widetilde{\overline{W}}(\tilde{\sigma})=\frac{1}{2}\tilde{\sigma}^2} && (6.2.5)\\ &= \mathrm{Tr}\left(-i(D-2if_{1\bar{1}})\tilde{\sigma}+2\tilde{\lambda}\lambda\right). && (6.2.6)\end{aligned}$$

This Lagrangian also has a Q_A-exact expression $L^{\text{quad}}_{\widetilde{\widetilde{W}}} = \{Q_A, V^{\text{quad}}_{\widetilde{\widetilde{W}}}\}$ for a fermionic functional $V^{\text{quad}}_{\widetilde{\widetilde{W}}}$.

For $L\,(t=0,\tau)$ the $L^{\text{quad}}_{\widetilde{\widetilde{W}}}$ term is turned off. For $L\,(t\neq 0,\tau)$, the $L^{\text{quad}}_{\widetilde{\widetilde{W}}}$ term is turned on, and the parameter t can be used to track the behavior of the quadratic twisted chiral superpotential throughout the localization computation.

Note that one is free to set t to different values, and this corresponds to different choices of the functional $\widetilde{\widetilde{W}}\,(\tilde{\sigma})$. For instance, $t=\frac{1}{2}t'$ or $t=it''$ corresponds to $\widetilde{\widetilde{W}}\,(\tilde{\sigma})=\frac{1}{4}\tilde{\sigma}^2$ or $\widetilde{\widetilde{W}}\,(\tilde{\sigma})=\frac{i}{2}\tilde{\sigma}^2$, respectively.

The third term in $L\,(t,\tau)$ is the squared commutator of the complex bosonic scalars together with it's superpartner

$$\begin{aligned} L_{\text{com}} &= \delta_A V_{\text{com}} && (6.2.7)\\ &= \delta_A \text{Tr}\left(\frac{i}{8}\left(\lambda+\tilde{\lambda}\right)[\sigma,\tilde{\sigma}]\right) && (6.2.8)\\ &= \text{Tr}\left(\frac{1}{8}[\sigma,\tilde{\sigma}]^2+\frac{i}{2}\tilde{\lambda}\,[\sigma,\lambda]\right) && (6.2.9)\end{aligned}$$

To see the corresponding Q_A-exact expression, one sets $\tilde{\zeta}_- = \zeta_+ = 1$ in $\delta_A = \delta + \tilde{\delta}$ and $\lambda, \tilde{\lambda}$.

For $L\,(t,\tau=0)$, the L_{com} term is turned off. For $L\,(t,\tau=1)$, the squared bosonic scalar commutator terms in L_{com} and L_{YM} cancel, and one introduces a fermionic scalar commutator term. For $L\,(t,\tau\neq 0)$, the parameter τ is retained and may be used to track the behavior of the squared bosonic scalar commutator term in L_{YM} throughout the localization computation.

In components, the localizing Lagrangian is

$$\begin{aligned} L\,(t,\tau) &= \text{Tr}\Bigg(\frac{1}{2}\left(-2if_{1\bar{1}}-it\tilde{\sigma}\right)^2+\frac{1}{2}\left(D_E+t\tilde{\sigma}\right)^2 && (6.2.10)\\ &\quad +\frac{1}{2}D_\mu\tilde{\sigma}D^\mu\sigma+\frac{1-\tau}{8}[\sigma,\tilde{\sigma}]^2+2t\tilde{\lambda}\lambda-\frac{i\tau}{2}\tilde{\lambda}\,[\sigma,\lambda] && (6.2.11)\\ &\quad +2i\tilde{\Lambda}_{\bar{1}}D_1\lambda-2i\Lambda_1 D_{\bar{1}}\tilde{\lambda}-i\tilde{\Lambda}_{\bar{1}}[\tilde{\sigma},\Lambda_1]+i\tilde{\lambda}[\sigma,\lambda]\Bigg). && (6.2.12)\end{aligned}$$

Here, we have switched from the "Lorentzian" auxiliary field D to the "Euclidean" one $D_E=-iD$, and completed the square for the terms involving $f_{1\bar{1}}$ and D_E.

The A-model path integral to be evaluated using supersymmetric localization is

$$\lim_{h\to 0}\int \mathcal{D}a\mathcal{D}\sigma\mathcal{D}\tilde{\sigma}\mathcal{D}D_E\mathcal{D}\Lambda\mathcal{D}\tilde{\Lambda}\mathcal{D}\lambda\mathcal{D}\tilde{\lambda}e^{-\frac{1}{h^2}\int \mathrm{d}^2x\sqrt{g}L(t,\tau)} \qquad (6.2.13)$$

The choice of localization scheme is $L\,(t=0,\tau=0) = L_{\text{YM}}$ in the Benini-Zaffaroni approach, and $L\,(t\neq 0,\tau=1) = L_{\text{YM}}+tL^{\text{quad}}_{\widetilde{\widetilde{W}}}-L_{\text{com}}$ in the Witten approach.

We will consider other values of the parameters, for instance $t \neq 0$ and $\tau = 0$ or $\tau \neq 0$, to further understand the subtleties of the computation. Note, however, that these correspond to non-physical cases.

6.3 Localization loci

In this section, we describe localization loci in the A-model. First, we recall how localization loci are evaluated from localizing terms, and outline a few features of our general localizing term. Then, we consider the follow-your-nose approach to A-model localization, and evaluate the localization locus for the most conventional choice of localizing term, namely, the Yang-Mills Lagrangian. We see that the localizing term in the follow-your-nose approach localizes to a locus of flat gauge connections.

Following this, we consider the Witten approach to A-model localization, and evaluate the localization locus in the case where the localizing term is the Yang-Mills Lagrangian, without the bosonic commutator term, with a quadratic twisted chiral superpotential. We see that the localizing term in the Witten approach localizes to a locus of Yang-Mills connections, which implies the GNO quantization of the gauge flux.

Finally, we consider the Benini-Zaffaroni approach to A-model localization, and evaluate the locus of bosonic field configurations that are "almost-BPS", in the sense that they almost set the fermionic supersymmetry variations to zero. We recall the argument for the GNO quantization of the flux in the Benini-Zaffaroni approach.

Generally, the localization locus is the space of bosonic field configurations for which the real bosonic part of the localizing term vanishes. To evaluate the localization locus, one solves

$$
\begin{aligned}
0 &= L(t,\tau)|_{\text{bos}} && (6.3.1)\\
& \operatorname{Tr}\left(\frac{1}{2}(f_{12}-it\tilde{\sigma})^2+\frac{1}{2}(D_E+t\tilde{\sigma})^2+\frac{1}{2}D_\mu\tilde{\sigma}D^\mu\sigma+\frac{1-\tau}{8}[\sigma,\tilde{\sigma}]^2\right) && (6.3.2)
\end{aligned}
$$

for the fields $a_\mu, \sigma, \tilde{\sigma}$, and D_E. Note that a_μ sits in both $f_{12} = -2if_{1\bar{1}}$ and $D_\mu = \partial_\mu - i[a_\mu, \cdot]$. Different values of the parameters t and τ correspond to different localization loci.

Each field is valued in the adjoint representation of the complexified Lie algebra of the gauge group. The Lie algebra consists of hermitian matrices for which the positive definite metric is $(M, N) = \operatorname{Tr}(M, N)$. When expressed in the basis of hermitian generators T_a, the fields take the form $\Phi = \Phi^a T_a$ while the covariant derivatives take the form $D_\mu\Phi = (D_\mu\Phi)^a T_a$. As we are working in Euclidean signature, each of the fields is assumed to be generically complex, and reality conditions are chosen to ensure the convergence of the path integral.

Next, we consider the follow-your-nose approach to A-model localization, and evaluate the localization locus for the case in which the localizing term is the Yang-Mills Lagrangian. In doing so, we choose standard reality conditions for the component fields. In the follow-your-nose approach, we must solve

$$\begin{aligned} 0 &= L\left(t=0,\tau=0\right)\big|_{\text{bos}} && (6.3.3)\\ &= \operatorname{Tr}\left(\frac{1}{2}\left(f_{12}\right)^2+\frac{1}{2}\left(D_E\right)^2+\frac{1}{2}D_\mu\tilde{\sigma}D^\mu\sigma+\frac{1}{8}[\sigma,\tilde{\sigma}]^2\right) && (6.3.4)\end{aligned}$$

for $a_\mu, \sigma, \tilde{\sigma}, D_E$. Choosing real a_μ, D_E, and complex conjugate $\sigma, \tilde{\sigma}$, this reduces to the BPS equations

$$0 = f_{12} = D_E = D_\mu\sigma = D_\mu\tilde{\sigma} = [\sigma,\tilde{\sigma}] \qquad (6.3.5)$$

The solutions $a_\mu, D_E, \sigma, \tilde{\sigma}$ of the BPS equations are the BPS configurations (along the real contour), up to gauge transformations. So, for $a_\mu^\dagger = a_\mu$, $D_E^\dagger = D_E$, $\sigma^\dagger = \tilde{\sigma}$, the BPS configurations are

$$\mathcal{M}_{\text{BPS}} = \left\{a_\mu, \sigma, \tilde{\sigma}, D_E \mid 0 = f_{12} = D_E = D_\mu\sigma = D_\mu\tilde{\sigma} = [\sigma,\tilde{\sigma}]\right\}/\mathcal{G} \qquad (6.3.6)$$

where $\mathcal{G}$ is the group of gauge transformations.

Note that $\mathcal{M}_{\text{BPS}}$ is a space of bosonic field configurations for which the supersymmetry variation of the fermions in the vector multiplet vanish. Let us clarify this point. The fermionic supersymmetry variations are

$$\begin{aligned} \delta_A\Lambda_1 &= +2iD_1\sigma, && (6.3.7)\\ \delta_A\tilde{\Lambda}_{\bar{1}} &= -2iD_{\bar{1}}\sigma, && (6.3.8)\\ \delta_A\lambda &= +i\left(iD_E+f_{12}-\frac{1}{2}\left[\sigma,\tilde{\sigma}\right]\right), && (6.3.9)\\ \delta_A\tilde{\lambda} &= -i\left(iD_E+f_{12}+\frac{1}{2}\left[\sigma,\tilde{\sigma}\right]\right), && (6.3.10)\end{aligned}$$

Setting the fermionic supersymmetry variations to zero

$$0 = \delta_A\Lambda_1 = \delta_A\tilde{\Lambda}_{\bar{1}} = \delta_A\lambda = \delta_A\tilde{\lambda}, \qquad (6.3.11)$$

corresponds to the following bosonic equations

$$0 = iD_E + f_{12} = D_\mu\sigma = \left[\sigma,\tilde{\sigma}\right]. \qquad (6.3.12)$$

for generically complexified fields $a_\mu, D_E, \sigma, \tilde{\sigma}$. Clearly, these equations are solved by the configurations $\mathcal{M}_{\text{BPS}}$, which set $f_{12} = 0$, $D_E = 0$, $D_\mu \sigma = 0$, and $[\sigma, \tilde{\sigma}] = 0$. Moreover, $\mathcal{M}_{\text{BPS}}$ is a real space that is much smaller than the space of generically complexified fields that set the fermionic supersymmetry variations to zero.

We proceed by sketching the iterative solution of the BPS equations. From the equation $0 = f_{12}$, we have that a_μ is a flat connection. From the equation $0 = D_E$, we have that D_E must vanish. From the equation $0 = [\sigma, \tilde{\sigma}]$, we have that σ and $\tilde{\sigma}$ can be simultaneously diagonalized by an element of the gauge group G. That is to say, σ and $\tilde{\sigma}$ can be conjugated into the Cartan subalgebra of the complexification of the Lie algebra of G using gauge transformations.

The solutions of $0 = D_\mu \sigma = \partial_\mu \sigma - i\,[a_\mu, \sigma]$ fall into two cases depending on whether the connection $a = a_\mu dx^\mu$ is reducible or irreducible. Recall that a is reducible if $\text{Stab}_G(a) \neq Z(G)$, and irreducible if $\text{Stab}_G(a) = Z(G)$, where G is the gauge group, $\text{Stab}_G(a)$ is the stabilizer group of a, and $Z(G)$ is the center of G. If a is irreducible, then $D_\mu \sigma = 0$ is solved by $\sigma = 0$. If a is reducible, then $D_\mu \sigma = 0$ is solved by $\sigma \neq 0$ satisfying $0 = \partial_\mu \sigma = [a_\mu, \sigma]$, that is, constant configurations of σ that commute with a_μ.

To summarize, the follow-your-nose approach results in a localization locus in which the gauge fields a_μ are flat connections, the real auxiliary field D_E vanishes, and the complex conjugate bosonic scalars $\sigma, \tilde{\sigma}$ are constant, covariantly constant, and commuting. The localization locus is the space of BPS configurations along the real contour.

Now, we consider the Witten approach to A-model localization, and evaluate the localization locus in the case where the localizing term is the Yang-Mills Lagrangian, without the bosonic commutator term, with a quadratic twisted chiral superpotential. In the Witten approach, we must solve

$$\begin{aligned} 0 &= L\,(t \in \mathbb{R}_{>0}, \tau = 1)|_{\text{bos}} && (6.3.13) \\ &= \text{Tr}\left(\frac{1}{2}\,(f_{12} - it\tilde{\sigma})^2 + \frac{1}{2}\,(D_E + t\tilde{\sigma})^2 + \frac{1}{2} D_\mu \tilde{\sigma} D^\mu \sigma\right) && (6.3.14) \end{aligned}$$

for $a_\mu, \sigma, \tilde{\sigma}, D_E$. Choosing real $a_\mu, \sigma \in \mathbb{R}$, and purely imaginary $\tilde{\sigma}, D_E \in i\mathbb{R}$, we have the following situation.

First, to minimize the scalar kinetic energy, we require that the real scalar is covariantly constant $D_\mu \sigma = 0$. This requirement comes from stationary phase. It follows that the imaginary scalar is also covariantly constant $D_\mu \tilde{\sigma} = 0$. The matrix equation reduces to

$$0 = f_{12} - it\tilde{\sigma} = D_E + t\tilde{\sigma} = D_\mu \tilde{\sigma} = D_\mu \sigma \qquad (6.3.15)$$

Together, $f_{12} = it\tilde{\sigma}$ and $D_\mu \tilde{\sigma} = 0$ imply $D_\mu f_{12} = 0$, such that a_μ is a Yang-Mills connection. If a_μ is a Yang-Mills connection, the gauge flux is GNO quantized:

$$\mathfrak{m} = \frac{1}{2\pi}\int \mathrm{d}^2x\sqrt{g}\,(-2if_{1\bar{1}}) \in \Lambda^G_{\text{cochar}} \tag{6.3.16}$$

Here, $\Lambda^G_{\text{cochar}} = \{x \in \mathfrak{h} |\, \exp(2\pi \mathrm{i} x) = 1_G\}$ is the cocharacter lattice of the gauge group G, x is an element of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g} = \mathrm{Lie}G$, exp is the exponential map from $\mathfrak{g}$ to G, and 1_G is the identity element in G. The cocharacter lattice is defined as the kernel of the exponential map restricted to $\mathfrak{h} \subset \mathfrak{g}$, and is sometimes also called the GNO lattice. Finally, $\mathfrak{m} \in \Lambda^G_{\text{cochar}} \subset \mathfrak{h}$ is the GNO quantized gauge flux.

From the GNO quantization condition on the gauge flux, we have

$$f_{12} = \frac{2\pi}{V_{S^2}}\mathfrak{m} =, \quad \tilde{\sigma} = -\frac{2\pi i}{tV_{S^2}}\mathfrak{m}, \quad D_E = \frac{2\pi i}{V_{S^2}}\mathfrak{m}. \tag{6.3.17}$$

Here, $V_{S^2} = \mathrm{vol}S^2 = 4\pi R^2$ where R is the radius of S^2. So, $a_\mu, \tilde{\sigma}, D_E$ are parametrized by the discrete modulus $\mathfrak{m}$ on the localization locus.

From the equation $0 = D_\mu\sigma = \partial_\mu\sigma - i\,[a_\mu, \sigma]$ we have that the real scalar σ is covariantly constant, and it suffices to study its structure over a single point on S^2. For reducible connections, $0 = D_\mu\sigma$ permits non-zero, real, continuous solutions σ. So, σ is parametrized by $\sigma = 2\pi V_{S^2}^{-1/2} u$ on the localization locus, where u is a continuous real dimensionless modulus.

To summarize, the Witten approach results in a localization locus in which the gauge fields a_μ are Yang-Mills connections, or equivalently, the gauge flux is GNO quantized. On the localization locus, the purely imaginary scalars D_E and $\tilde{\sigma}$ are parametrized by the discrete GNO quantized flux $\mathfrak{m}$, and the real scalar σ is parametrized by the continuous modulus u.

Next, we consider the Benini-Zaffaroni approach to A-model localization, and briefly describe the locus of bosonic field configurations that are "almost-BPS". Following the Benini-Zaffaroni approach, we do not evaluate bosonic field configurations that set the bosonic part of $L\,(t = 0, \tau = 0) = L_{\text{YM}}$ to zero along the real contour, but instead, we study the bosonic field configurations for which the fermionic supersymmetry variations vanish. Once again, we set the fermionic supersymmetry variations to zero

$$0 = \delta_A\Lambda_1 = \delta_A\tilde{\Lambda}_{\bar{1}} = \delta_A\lambda = \delta_A\tilde{\lambda}, \tag{6.3.18}$$

and obtain the following bosonic equations

$$0 = iD_E + f_{12} = D_\mu\sigma = [\sigma, \tilde{\sigma}]\,, \tag{6.3.19}$$

for generically complexified fields $a_\mu, D_E, \sigma, \tilde{\sigma}$. Choosing real a_μ, complex conjugate

$\sigma, \tilde{\sigma}$, but permitting D_E to remain generically complex, we obtain the almost-BPS equations

$$0 = f_{12} + iD_E^{\mathbb{C}} = D_\mu \sigma = D_\mu \tilde{\sigma} = [\sigma, \tilde{\sigma}] \tag{6.3.20}$$

for real f_{12}, and complex $D_E^{\mathbb{C}}$. Note that, for $D_E^{\mathbb{C}} = 0$, one recovers the BPS equations.

So, for $a_\mu^\dagger = a_\mu$, $\sigma^\dagger = \tilde{\sigma}$, and generically complexified $D_E^{\mathbb{C}}$, the space of almost BPS configurations is

$$\mathcal{M}_{\text{aBPS}} = \left\{ a_\mu, \sigma, \tilde{\sigma}, D_E^{\mathbb{C}} \mid 0 = f_{12} + iD_E^{\mathbb{C}} = D_\mu \sigma = D_\mu \tilde{\sigma} = [\sigma, \tilde{\sigma}] \right\} / \mathcal{G} \tag{6.3.21}$$

where $\mathcal{G}$ is the group of gauge transformations. Note that the infinitesimal gauge transformations are $\delta_\sigma a_\mu = 2D_\mu \sigma$ for a_μ, and $\delta_\sigma \varphi = 2i\, [\sigma, \varphi]$ for $D_E^{\mathbb{C}}, \sigma, \tilde{\sigma}$.

Roughly speaking, the considerations regarding the GNO quantization of the flux and the configuration $0 = f_{12} + iD_E^{\mathbb{C}}$ are as follows. If the gauge flux is GNO quantized $\mathfrak{m} = \frac{1}{2\pi} \int_{S^2} f$, the Yang-Mills equations $D_\mu f_{12} = 0$ are satisfied, and $f_{12} = -iD_E^{\mathbb{C}}$ implies $D_\mu D_E^{\mathbb{C}} = 0$. In this case, the configurations

$$0 = D_E^{\mathbb{C}} - \frac{2\pi i}{V_{S^2}} \mathfrak{m}, \quad D_0 = D_E^{\mathbb{C}} - \frac{2\pi i}{V_{S^2}} \mathfrak{m} \tag{6.3.22}$$

are gauge equivalent, where $D_0 \in \mathbb{C}$ is a constant zero-mode of $D_E^{\mathbb{C}}$. Moreover, the gauge orbit of D_0 intersects the real contour of $D_E^{\mathbb{C}}$, that is

$$D_E^{\mathbb{C}} = D_0 + \frac{2\pi i}{V_{S^2}} \mathfrak{m} \in \mathbb{R}, \text{ for } D_0 \in \mathbb{R} - \frac{2\pi i}{V_{S^2}} \mathfrak{m}. \tag{6.3.23}$$

If the gauge flux is not GNO quantized, these considerations do not hold, and the gauge orbit of D_0 spans $D_E^{\mathbb{C}} = f_{12} + \mathbb{C}$.

To summarize, in the Benini-Zaffaroni approach to A-model localization, one considers a locus of almost-BPS field configurations by permitting the auxiliary field to remain generically complex $D_E^{\mathbb{C}}$. The locus of almost BPS configurations is non-compact, and singular. In this approach, the auxiliary field $D_E^{\mathbb{C}}$ plays at least three important roles. First, $D_E^{\mathbb{C}}$ plays a central role in the GNO quantization of the flux. Second, $D_E^{\mathbb{C}}$ is subsequently used to close the off-shell supersymmetry multiplet of zero modes. Third, $D_E^{\mathbb{C}}$ is used to regulate singularities in the bosonic moduli space.

Finally, let us record an interesting feature regarding the relationship of the A-model to the standard multiplet of two-dimensional cohomological gauge theory. The fields of the Euclidean A-model vector multiplet in WZ gauge are related, by field redefinitions, to the fields of the standard cohomological multiplet. Let Y denote the auxiliary field in the standard cohomological multiplet. The cohomological auxiliary field Y is related to the A-model auxiliary field $D = iD_E$ by the field redefinition

$$Y = -i\left(D - 2if_{1\bar{1}}\right). \tag{6.3.24}$$

Restricting the supersymmetry parameters to $\tilde{\zeta}_- = \zeta_+ = 1$, and expressing D in terms of Y, the supersymmetry transformations of the fermions and the auxiliary field are

$$\begin{aligned}
Q_A\Lambda_1 &= +2iD_1\sigma, && (6.3.25)\\
Q_A\tilde{\Lambda}_{\bar{1}} &= -2iD_{\bar{1}}\sigma, && (6.3.26)\\
Q_A\lambda &= -Y - \frac{i}{2}\left[\sigma, \tilde{\sigma}\right] && (6.3.27)\\
Q_A\tilde{\lambda} &= +Y - \frac{i}{2}\left[\sigma, \tilde{\sigma}\right] && (6.3.28)\\
Q_AY &= i\left[\sigma, \tilde{\lambda} - \lambda\right] && (6.3.29)
\end{aligned}$$

Setting the fermionic supersymmetry variations to zero

$$0 = Q_A\Lambda_1 = Q_A\tilde{\Lambda}_{\bar{1}} = Q_A\lambda = Q_A\tilde{\lambda}, \tag{6.3.30}$$

we obtain the following bosonic equations

$$0 = Y = D_\mu\sigma = \left[\sigma, \tilde{\sigma}\right]. \tag{6.3.31}$$

The interesting feature is that $0 = Y$ is the same equation as $0 = iD_E + f_{12}$.

6.4 Locus expansion

In this section, we describe the locus expansion of the localizing Lagrangian, resulting in an expression for the localizing Lagrangian up to quadratic order in fluctuations.

The bosonic fields $a_1, a_{\bar{1}}, \sigma, \tilde{\sigma}, D_E$ are expanded as

$$\varphi = \varphi_0 + h\varphi' \tag{6.4.1}$$

where φ_0 denotes zero modes determined by the localization locus, and φ' denotes fluctuating modes.

The fermionic fields $\Lambda_1, \tilde{\Lambda}_{\bar{1}}\lambda, \tilde{\lambda}$ are expanded as

$$\psi = h\psi' \tag{6.4.2}$$

where ψ' denotes fluctuating modes.

First, the fields in the localizing Lagrangian are recast in terms of zero modes and fluctuations, then we take the localizing limit $h \to 0$. To illustrate, we consider the locus expansion of the bosonic terms in

$$L(t,\tau) = \mathrm{Tr}\left(\frac{1}{2}\left(-2if_{1\bar{1}} - it\tilde{\sigma}\right)^2 + \frac{1}{2}\left(D_E + t\tilde{\sigma}\right)^2 + \frac{1}{2}D_\mu\tilde{\sigma}D^\mu\sigma + \frac{1-\tau}{8}[\sigma,\tilde{\sigma}]^2 + \dots\right) \tag{6.4.3}$$

For the gauge kinetic term, we have

$$\frac{1}{2h^2}\left(-2if_{1\bar{1}} - it\tilde{\sigma}\right)^2 \underset{\varphi\to\varphi_0+h\varphi'}{\longrightarrow} \frac{1}{2h^2}\left(-2if^0_{1\bar{1}} - it\tilde{\sigma}_0 + h\left(-2if'_{1\bar{1}} - it\tilde{\sigma}'\right)\right)^2 \tag{6.4.4}$$

$$\underset{h\to 0}{\longrightarrow} \frac{1}{2}\left(-2if'_{1\bar{1}} - it\tilde{\sigma}'\right)^2 \tag{6.4.5}$$

where $f'_{1\bar{1}} = D^0_1 a'_{\bar{1}} - D^0_{\bar{1}} a'_1$, and $-2if^0_{1\bar{1}} - it\tilde{\sigma}_0 = 0$ on the locus.

For the auxiliary field term, we have

$$\frac{1}{2h^2}\left(D_E + t\tilde{\sigma}\right)^2 \underset{\varphi\to\varphi_0+h\varphi'}{\longrightarrow} \frac{1}{2h^2}\left(D^0_E + t\tilde{\sigma}_0 + h\left(D'_E + t\tilde{\sigma}'\right)\right)^2 \tag{6.4.6}$$

$$\underset{h\to 0}{\longrightarrow} \frac{1}{2}\left(D'_E + t\tilde{\sigma}'\right)^2 \tag{6.4.7}$$

where $D^0_E + t\tilde{\sigma}_0 = 0$ on the locus.

For the bosonic scalar kinetic term, we have

$$\frac{1}{2h^2}D_\mu\tilde{\sigma}D^\mu\sigma \underset{\varphi\to\varphi_0+h\varphi'}{\longrightarrow} \frac{1}{h^2}\Big[\left(\partial_1 - i\left[a^0_1 + ha'_1, \tilde{\sigma}_0 + h\tilde{\sigma}'\right]\right) \tag{6.4.8}$$

$$\times\left(\partial_{\bar{1}} - i\left[a^0_{\bar{1}} + ha'_{\bar{1}}, \sigma_0 + h\sigma'\right]\right) \tag{6.4.9}$$

$$+\left(\partial_{\bar{1}} - i\left[a^0_{\bar{1}} + ha'_{\bar{1}}, \tilde{\sigma}_0 + h\tilde{\sigma}'\right]\right) \tag{6.4.10}$$

$$\times\left(\partial_1 - i\left[a^0_1 + ha'_1, \sigma_0 + h\sigma'\right]\right)\Big] \tag{6.4.11}$$

$$\underset{h\to 0}{\longrightarrow} \left(D^0_1\tilde{\sigma}' - i\left[a'_1, \tilde{\sigma}_0\right]\right)\left(D^0_{\bar{1}}\sigma' - i\left[a'_{\bar{1}}, \sigma_0\right]\right) \tag{6.4.12}$$

$$+\left(D^0_{\bar{1}}\tilde{\sigma}' - i\left[a'_{\bar{1}}, \tilde{\sigma}_0\right]\right)\left(D^0_1\sigma' - i\left[a'_1, \sigma_0\right]\right) \tag{6.4.13}$$

where we have used holomorphic coordinates $D_\mu\tilde{\sigma}D^\mu\sigma = 2\left(D_1\tilde{\sigma}D_{\bar{1}}\sigma + D_{\bar{1}}\tilde{\sigma}D_1\sigma\right)$. Note that if we first integrate by parts

$$\int_{S^2} \mathrm{d}^2x\sqrt{g}\,\mathrm{Tr}D_\mu\tilde{\sigma}D^\mu\sigma \tag{6.4.14}$$

$$= \int_{S^2} \mathrm{d}^2x\sqrt{g}\,\mathrm{Tr}\frac{1}{2}\left(D_\mu\tilde{\sigma}D^\mu\sigma + D_\mu\sigma D^\mu\tilde{\sigma}\right), \tag{6.4.15}$$

then expand in holomorphic coordinates, we obtain an equivalent (but larger) expression.

For the bosonic scalar commutator term, we have

$$\frac{1}{8h^2}\left[\sigma,\tilde{\sigma}\right]^2 \underset{\varphi\to\varphi_0+h\varphi}{\longrightarrow} \frac{1}{8h^2}\left[\sigma_0+h\sigma',\tilde{\sigma}_0+h\tilde{\sigma}'\right]^2 \tag{6.4.16}$$

$$\underset{h\to 0}{\longrightarrow} \frac{1}{8}\left(\left[\sigma',\tilde{\sigma}_0\right]+\left[\sigma_0,\tilde{\sigma}'\right]\right)^2 \tag{6.4.17}$$

Collecting the results, the localizing Lagrangian up to quadratic order in fluctuations is

$$\frac{1}{h^2}\tilde{L}\left(t,\tau\right) = \mathrm{Tr}\left[\frac{1}{2}\left(-2if'_{1\bar{1}}-it\tilde{\sigma}'\right)^2+\frac{1}{2}\left(D'_E+t\tilde{\sigma}'\right)^2\right. \tag{6.4.18}$$

$$+\frac{1-\tau}{8}\left(\left[\sigma',\tilde{\sigma}_0\right]+\left[\sigma_0,\tilde{\sigma}'\right]\right)^2 \tag{6.4.19}$$

$$+\left(D^0_1\tilde{\sigma}'-i\left[a'_1,\tilde{\sigma}_0\right]\right)\left(D^0_{\bar{1}}\sigma'-i\left[a'_{\bar{1}},\sigma_0\right]\right) \tag{6.4.20}$$

$$+\left(D^0_{\bar{1}}\tilde{\sigma}'-i\left[a'_{\bar{1}},\tilde{\sigma}_0\right]\right)\left(D^0_1\sigma'-i\left[a'_1,\sigma_0\right]\right) \tag{6.4.21}$$

$$+2i\tilde{\Lambda}'_{\bar{1}}D^0_1\lambda'-2i\Lambda'_1D^0_{\bar{1}}\tilde{\lambda}'-i\tilde{\Lambda}'_{\bar{1}}\left[\tilde{\sigma}_0,\Lambda'_1\right]+i\tilde{\lambda}'\left[\sigma_0,\lambda'\right] \tag{6.4.22}$$

$$\left.+2t\tilde{\lambda}'\lambda'-\frac{i\tau}{4}\lambda'\left[\sigma_0,\tilde{\lambda}'\right]-\frac{i\tau}{4}\tilde{\lambda}'\left[\sigma_0,\lambda'\right]\right]+\mathcal{O}\left(h\right) \tag{6.4.23}$$

where $f'_{1\bar{1}} = D^0_1 a'_{\bar{1}} - D^0_{\bar{1}} a'_1$, and $D^0_1, D^0_{\bar{1}}$ denote the covariant derivatives w.r.t. gauge fields $a^0_1, a^0_{\bar{1}}$.

6.5 Gauge-fixing

In this section, we describe the gauge fixing procedure for the localizing Lagrangian. To gauge fix, we add a gauge-fixing Lagrangian to the Localizing Lagrangian. The gauge-fixing Lagrangian is exact under both the BRST symmetry δ_{b} and the A-model supersymmetry $\delta_A = \delta + \tilde{\delta}$.

The BRST transformations are

$$\delta_{\mathrm{b}} a_\mu = D_\mu c \tag{6.5.1}$$

$$\delta_{\mathrm{b}}\varphi = i\left[c,\varphi\right] \tag{6.5.2}$$

$$\delta_{\mathrm{b}}\psi = i\left\{c,\psi\right\} \tag{6.5.3}$$

$$\delta_{\mathrm{b}} c = \frac{i}{2}\left\{c,c\right\} \tag{6.5.4}$$

$$\delta_{\mathrm{b}}\tilde{c} = -B \tag{6.5.5}$$

$$\delta_{\mathrm{b}} B = 0 \tag{6.5.6}$$

where B is the Nakanishi-Lautrap field, $c, \tilde{c}$ are Fadeev-Popov ghosts, φ denotes the bosons $\sigma, \tilde{\sigma}, D_E$, and ψ denotes the fermions $\Lambda_1, \tilde{\Lambda}_{\bar{1}}, \lambda, \tilde{\lambda}$.

The gauge-fixing Lagrangian is defined as

$$L_{\mathrm{gf}} = \frac{1}{2}\left(\delta' + \tilde{\delta}'\right)\left(\tilde{c}\left(G_{\mathrm{gf}} + \frac{\xi}{2}B\right)\right), \tag{6.5.7}$$

where $\delta' = \delta + \delta_{\mathrm{b}}, \tilde{\delta}' = \tilde{\delta} + \delta_{\mathrm{b}}$, G_{gf} is the gauge-fixing function for physical fields, and ξ is a parameter.

When evaluated for generic G_{gf}, the gauge-fixing Lagrangian reads

$$\begin{aligned} L_{\mathrm{gf}} &= \frac{1}{2}\left(\delta' + \tilde{\delta}'\right)\left(\tilde{c}\left(G_{\mathrm{gf}} + \frac{\xi}{2}B\right)\right) && (6.5.8)\\ &= \frac{\xi}{2}\delta_{\mathrm{b}}\left(\tilde{c}B\right) + \delta_{\mathrm{b}}\left(\tilde{c}G_{\mathrm{gf}}\right) + \frac{\xi}{4}\delta\left(\tilde{c}B\right) + \frac{1}{4}\xi\tilde{\delta}\left(\tilde{c}B\right) + \frac{1}{2}\delta\left(\tilde{c}G_{\mathrm{gf}}\right) + \frac{1}{2}\tilde{\delta}\left(\tilde{c}G_{\mathrm{gf}}\right) && (6.5.9)\\ &= -\frac{\xi}{2}B^2 - BG_{\mathrm{gf}} - \tilde{c}\delta_{\mathrm{b}}\left(G_{\mathrm{gf}}\right) - \frac{1}{2}\tilde{c}\delta\left(G_{\mathrm{gf}}\right) - \frac{1}{2}\tilde{c}\tilde{\delta}\left(G_{\mathrm{gf}}\right) && (6.5.10) \end{aligned}$$

Completing the square, then integrating B, the expression becomes

$$\begin{aligned} L_{\mathrm{gf}} &= \frac{1}{2\xi}G_{\mathrm{gf}}^2 - \frac{\xi}{2}\left(B + \frac{1}{\xi}G_{\mathrm{gf}}\right)^2 - \tilde{c}\delta_{\mathrm{b}}\left(G_{\mathrm{gf}}\right) + \ldots && (6.5.11)\\ &= \frac{1}{2\xi}G_{\mathrm{gf}}^2 - \tilde{c}\delta_{\mathrm{b}}\left(G_{\mathrm{gf}}\right) + \ldots && (6.5.12) \end{aligned}$$

For the gauge fixing function, we take

$$G_{\mathrm{gf}} = D_\mu a^\mu + \frac{i}{2}\xi\gamma\left[\sigma, \tilde{\sigma}\right] \tag{6.5.13}$$

where γ is a parameter that we introduce to retain or remove the bosonic scalar commutator term.

For this choice of G_{gf}, the gauge-fixing Lagrangian is

$$\begin{aligned} L_{\mathrm{gf}} &= \frac{1}{2\xi}\left(2D_1 a_{\bar{1}} + 2D_{\bar{1}}a_1 + \frac{i}{2}\xi\gamma\left[\sigma, \tilde{\sigma}\right]\right)^2 && (6.5.14)\\ &\quad -2\tilde{c}\delta_{\mathrm{b}}\left(D_1 a_{\bar{1}}\right) - 2\tilde{c}\delta_{\mathrm{b}}\left(D_{\bar{1}}a_1\right) - \tilde{c}\delta\left(D_1 a_{\bar{1}}\right) - \tilde{c}\delta\left(D_{\bar{1}}a_1\right) && (6.5.15)\\ &\quad -\tilde{c}\tilde{\delta}\left(D_1 a_{\bar{1}}\right) - \tilde{c}\tilde{\delta}\left(D_{\bar{1}}a_1\right) + \frac{i\gamma\xi}{2}\tilde{c}\delta_{\mathrm{b}}\left(\left[\sigma, \tilde{\sigma}\right]\right) && (6.5.16)\\ &\quad +\frac{i\gamma\xi}{4}\tilde{c}\delta\left(\left[\sigma, \tilde{\sigma}\right]\right) + \frac{i\gamma\xi}{4}\tilde{c}\tilde{\delta}\left(\left[\sigma, \tilde{\sigma}\right]\right) && (6.5.17)\\ &= \frac{1}{2\xi}\left(2D_1 a_{\bar{1}} + 2D_{\bar{1}}a_1 + \frac{i}{2}\xi\gamma\left[\sigma, \tilde{\sigma}\right]\right)^2 && (6.5.18)\\ &\quad -2\tilde{c}D_1 D_{\bar{1}}c - 2\tilde{c}D_{\bar{1}}D_1 c - i\tilde{c}\partial_1\tilde{\Lambda}_{\bar{1}} + i\tilde{c}\partial_{\bar{1}}\Lambda_1 && (6.5.19)\\ &\quad -\frac{i\gamma\xi}{2}\tilde{c}\left[\sigma, \lambda + \tilde{\lambda}\right] - \frac{\gamma\xi}{2}\left(\tilde{c}\left[\sigma, \left[c, \tilde{\sigma}\right]\right] + \tilde{c}\left[\left[c, \sigma\right], \tilde{\sigma}\right]\right) && (6.5.20) \end{aligned}$$

The part of the gauge-fixing Lagrangian that effects the 1-loop determinant is

$$L_{\rm gf} = \frac{1}{2\xi}\left(D_\mu a^\mu + \frac{i}{2}\xi\gamma\left[\sigma,\tilde{\sigma}\right]\right)^2 + D_\mu\tilde{c}D^\mu c + \dots \quad (6.5.21)$$

Setting $\xi = 1, \gamma = 1$ corresponds to the Feynman-like gauge used in appendix C.4 of [20], and setting $\xi = 1, \gamma = 0$ corresponds to the Landau-like gauge considered in appendix D.2 of [13].

For later convenience, we define

$$\begin{aligned} L_{\rm gf,gh}\left(\xi,\gamma\right) &= L_{\rm gf}\left(\xi,\gamma\right) + L_{\rm gh} && (6.5.22)\\ &= \frac{1}{2\xi}\left(D_\mu a^\mu + \frac{i\xi\gamma}{2}\left[\sigma,\tilde{\sigma}\right]\right)^2 + D_\mu\tilde{c}D^\mu c. && (6.5.23)\end{aligned}$$

The gauge-fixed localizing Lagrangian to quadratic order in fluctuations is

$$\begin{aligned} \tilde{L}\left(t,\tau,\xi,\gamma\right) &= \tilde{L}\left(t,\tau\right) + \tilde{L}_{\rm gf,gh}\left(\xi,\gamma\right) && (6.5.24)\\ &= {\rm Tr}\Big[\frac{1}{2}\left(-2if'_{1\bar{1}} - it\tilde{\sigma}'\right)^2 + \frac{1}{2}\left(D'_E + t\tilde{\sigma}'\right)^2 && (6.5.25)\\ &\quad + \frac{1-\tau}{8}\left(\left[\sigma',\tilde{\sigma}_0\right] + \left[\sigma_0,\tilde{\sigma}'\right]\right)^2 && (6.5.26)\\ &\quad + \left(D_1^0\tilde{\sigma}' - i\left[a'_1,\tilde{\sigma}_0\right]\right)\left(D_{\bar{1}}^0\sigma' - i\left[a'_{\bar{1}},\sigma_0\right]\right) && (6.5.27)\\ &\quad + \left(D_{\bar{1}}^0\tilde{\sigma}' - i\left[a'_{\bar{1}},\tilde{\sigma}_0\right]\right)\left(D_1^0\sigma' - i\left[a'_1,\sigma_0\right]\right) && (6.5.28)\\ &\quad + 2i\tilde{\Lambda}'_{\bar{1}}D_1^0\lambda' - 2i\Lambda'_1D_{\bar{1}}^0\tilde{\lambda}' - i\tilde{\Lambda}'_{\bar{1}}\left[\tilde{\sigma}_0,\Lambda'_1\right] + i\tilde{\lambda}'\left[\sigma_0,\lambda'\right] && (6.5.29)\\ &\quad + 2t\tilde{\lambda}'\lambda' - \frac{i\tau}{4}\lambda'\left[\sigma_0,\tilde{\lambda}'\right] - \frac{i\tau}{4}\tilde{\lambda}'\left[\sigma_0,\lambda'\right] && (6.5.30)\\ &\quad + \frac{1}{2\xi}\left(2D_1^0a'_{\bar{1}} + 2D_{\bar{1}}a'_1 + \frac{i\xi\gamma}{2}\left(\left[\sigma',\tilde{\sigma}_0\right] + \left[\sigma_0,\tilde{\sigma}'\right]\right)\right)^2 && (6.5.31)\\ &\quad - 2\tilde{c}'D_1^0D_{\bar{1}}^0c' - 2\tilde{c}'D_{\bar{1}}^0D_1^0c'\Big] && (6.5.32)\end{aligned}$$

where $f'_{1\bar{1}} = D_1^0a'_{\bar{1}} - D_{\bar{1}}^0a'_1$, and $D_1^0, D_{\bar{1}}^0$ contain gauge fields $a_1^0, a_{\bar{1}}^0$.

6.6 Fluctuation operators

In this section, we describe how to use the Cartan-Weyl basis to construct matrix expressions for localizing Lagrangians to quadratic order in fluctuations. This proceeds by first describing the Cartan Weyl basis, then the Cartan-Weyl decomposition of fields, and finally, how to express Lagrangians in terms of matrices. In the following section, we will discuss the monopole spherical harmonic decomposition of fluctuating modes, and outline the explicit evaluation of 1-loop determinants.

The Cartan-Weyl (CW) basis for the Lie algebra of the gauge group $\mathfrak{g} = {\rm Lie}G$

is $\{H_a, E_\alpha\}$, where H_a is a generator of the Cartan subalgebra, and E_α is a ladder operator. Here, a is an index for the Cartan subalgebra, and α is a non-vanishing root. The generators and ladder operators satisfy

$$[H_a, E_\alpha] = \alpha(H_a) E_\alpha, \quad [H_a, H_b] = 0, \quad [E_\alpha, E_{-\alpha}] = \frac{2}{|\alpha|^2}\alpha_a H_a \tag{6.6.1}$$

$$\mathrm{Tr}(E_\alpha E_\beta) = \delta_{\alpha+\beta,0}, \quad \mathrm{Tr}(H_a E_\beta) = 0, \quad E_\alpha^\dagger = E_{-\alpha}. \tag{6.6.2}$$

The Cartan-Weyl decomposition of the adjoint-valued bosonic fields $a_1, a_{\bar{1}}, \sigma, \tilde{\sigma}, D_E$ is

$$\varphi = \varphi_0 + h\varphi' \tag{6.6.3}$$

$$= \sum_a H_a \varphi_0^a + h\left(\sum_a H_a \varphi'^a + \sum_\alpha E_\alpha \varphi'^\alpha\right) \tag{6.6.4}$$

$$= \sum_a H_a \varphi_0^a + h\left(\sum_a H_a \varphi'^a + \sum_{\alpha>0}\left(E_\alpha \varphi'^\alpha + E_{-\alpha}\varphi'^{-\alpha}\right)\right) \tag{6.6.5}$$

where φ_0 denotes zero modes determined by the localization locus, and φ' denotes fluctuating modes.

The Cartan-Weyl decomposition of the adjoint-valued fermionic fields $\Lambda_1, \tilde{\Lambda}_{\bar{1}}, \lambda, \tilde{\lambda}$ is

$$\psi = h\psi' \tag{6.6.6}$$

$$= h\left(\sum_a H_a \psi'^a + \sum_\alpha E_\alpha \psi'^\alpha\right) \tag{6.6.7}$$

$$= h\left(\sum_a H_a \psi'^a + \sum_{\alpha>0}\left(E_\alpha \psi'^\alpha + E_{-\alpha}\psi'^{-\alpha}\right)\right) \tag{6.6.8}$$

where ψ' denotes fluctuating modes.

To illustrate, we sketch the procedure to obtain matrix expressions for a general localizing Lagrangian, first for the bosons, then for the fermions. Consider the bosonic part of the localizing Lagrangian together with the gauge-fixing Lagrangian, in the form

$$L(t,\tau,\xi,\gamma)|_{\mathrm{bos}} = L(t,\tau)|_{\mathrm{bos}} + L_{\mathrm{gf,gh}}(\xi,\gamma) \tag{6.6.9}$$

$$= \frac{1}{h^2}\mathrm{Tr}\bigg(\frac{1}{2}\left(-2if_{1\bar{1}} - it\tilde{\sigma}\right)^2 + \frac{1}{2}\left(D_E + t\tilde{\sigma}\right)^2 \tag{6.6.10}$$

$$+\frac{1}{4}\left(D_\mu\tilde{\sigma}D^\mu\sigma + D_\mu\sigma D^\mu\tilde{\sigma}\right) + \frac{1-\tau}{8}[\sigma,\tilde{\sigma}]^2 \tag{6.6.11}$$

$$\frac{1}{2\xi}\left(D_\mu a^\mu + \frac{i\xi\gamma}{2}[\sigma,\tilde{\sigma}]\right)^2 + D_\mu\tilde{c}D^\mu c\bigg) \tag{6.6.12}$$

We do not consider the auxiliary field D_E since it can be integrated out by setting it to its on-shell value.

First, we perform the locus expansion and obtain an expression for the localizing Lagrangian to quadratic order in fluctuations. Then we expand in the fields in the Cartan-Weyl basis. The result is an action to quadratic order in fluctuations, which we denote

$$\tilde{S}_{\rm bos} = \int_{S^2} \mathrm{d}^2x\sqrt{g}\, \tilde{L}\left(t,\tau,\xi,\gamma\right)\Big|_{\rm bos} \tag{6.6.13}$$

This action may be expressed in terms of a 4×4 matrix for physical modes, and a 1×1 matrix for ghost modes, which reads

$$\tilde{S}_{\rm bos} = \int_{S^2} \mathrm{d}^2x\sqrt{g}\sum_\alpha \mathrm{Tr}\Bigg(\left(a_{\bar{1}}^{-\alpha}, a_1^{-\alpha}, \sigma^{-\alpha}, \tilde{\sigma}^{-\alpha}\right)\Delta_{\rm bos}^{(\alpha)}\begin{pmatrix} a_1^\alpha \\ a_{\bar{1}}^\alpha \\ \sigma^\alpha \\ \tilde{\sigma}^\alpha\end{pmatrix} + \tilde{c}^{-\alpha}\Delta_{\rm gho}^{(\alpha)}c^\alpha\Bigg) \tag{6.6.14}$$

where the bosonic fluctuation matrix $\Delta_{\rm bos}^{(\alpha)}$ is

$$\begin{pmatrix} \frac{-2(\xi+1)D_1^0 D_{\bar{1}}^0 - \xi\alpha(\sigma_0)\alpha(\tilde{\sigma}_0)}{\xi} & \frac{2(\xi-1)D_1^0 D_1^0}{\xi} & \frac{i(\gamma+1)\alpha(\tilde{\sigma}_0)D_1^0}{2} & \frac{(2t-i(\gamma-1)\alpha(\sigma_0))D_{\bar{1}}^0}{2} \\ \frac{2(\xi-1)D_{\bar{1}}^0 D_{\bar{1}}^0}{\xi} & \frac{-2(\xi+1)D_{\bar{1}}^0 D_1^0 - \xi\alpha(\sigma_0)\alpha(\tilde{\sigma}_0)}{\xi} & \frac{i(\gamma+1)\alpha(\tilde{\sigma}_0)D_{\bar{1}}^0}{2} & \frac{(-2t-i(\gamma-1)\alpha(\sigma_0))D_{\bar{1}}^0}{2} \\ \frac{-i(\gamma+1)\alpha(\tilde{\sigma}_0)D_{\bar{1}}^0}{2} & \frac{-i(\gamma+1)\alpha(\tilde{\sigma}_0)D_1^0}{2} & -\frac{\left(\gamma^2\xi+\tau-1\right)\alpha(\tilde{\sigma}_0)^2}{8} & \frac{\left(\gamma^2\xi+\tau-1\right)\alpha(\sigma_0)\alpha(\tilde{\sigma}_0)-D^2}{8} \\ \frac{(2t+i(\gamma-1)\alpha(\sigma_0))D_{\bar{1}}^0}{2} & \frac{(-2t+i(\gamma-1)\alpha(\sigma_0))D_1^0}{2} & \frac{\left(\gamma^2\xi+\tau-1\right)\alpha(\sigma_0)\alpha(\tilde{\sigma}_0)-2D^2}{8} & \frac{-\left(\gamma^2\xi+\tau-1\right)\alpha(\sigma_0)^2-4t^2}{8} \end{pmatrix} \tag{6.6.15}$$

the ghost fluctuation matrix is

$$\Delta_{\rm gho}^{(\alpha)} = -D^2 \tag{6.6.16}$$

and $D^2 = 2\{D_1^0, D_{\bar{1}}^0\}$. Note that we have assumed complex conjugate $a_{\bar{1}}'^\dagger = a_1'$, $a_1'^\dagger = a_{\bar{1}}'$, and independent σ', $\tilde{\sigma}'$. In this case, a_μ, σ would be integrated out along the real contour, and $\tilde{\sigma}$ would be integrated out along the purely imaginary contour.

Next, we consider the fermionic part of the localizing Lagrangian

$$L\left(t,\tau\right)\big|_{\rm fer} = \mathrm{Tr}\Big(2i\tilde{\Lambda}_{\bar{1}}D_1\lambda - 2i\Lambda_1 D_{\bar{1}}\tilde{\lambda} - i\tilde{\Lambda}_{\bar{1}}[\tilde{\sigma},\Lambda_1] \tag{6.6.17}$$

$$+ i\tilde{\lambda}[\sigma,\lambda] + 2t\tilde{\lambda}\lambda - \frac{i\tau}{2}\tilde{\lambda}\left[\sigma,\lambda\right]\Big) \tag{6.6.18}$$

Following the same procedure, we obtain an expression to quadratic order in fluctuations in the Cartan-Weyl basis, in the localizing limit. In this case, there are two possible matrix expressions for $\tilde{L}\left(t,\tau\right)\Big|_{\rm fer}$ depending on the reality conditions of the fluctuating modes $\Lambda_1', \tilde{\Lambda}_{\bar{1}}', \lambda', \tilde{\lambda}'$.

If we take the modes to be complex conjugates $\Lambda_{\bar{1}}'^\dagger = \Lambda_1'$, $\tilde{\lambda}'^\dagger = \lambda'$, we have an expression in terms of a 2×2 matrix. This reads

$$\tilde{S}_{\text{fer}} = \int_{S^2} \mathrm{d}^2x\sqrt{g}\sum_{\alpha}\mathrm{Tr}\Bigg(\frac{1}{2}\left(2\tilde{\Lambda}_{\bar{1}}^{(-\alpha)},\tilde{\lambda}^{(-\alpha)}\right)\Delta_{\text{fer}}^{(\alpha)}\begin{pmatrix}2\Lambda_1^{(\alpha)}\\ \lambda^{(\alpha)}\end{pmatrix}\Bigg) \quad (6.6.19)$$

$$\Delta_{\text{fer}}^{(\alpha)} = -2i\begin{pmatrix}\frac{1}{4}\alpha\left(\tilde{\sigma}_0\right) & -D_1^0\\ D_{\bar{1}}^0 & \frac{(\tau-2)}{2}\alpha\left(\sigma_0\right)+2it\end{pmatrix} \quad (6.6.20)$$

This matrix has a 1-loop contribution in terms of a determinant $\det \Delta_{\text{fer}}^{(\alpha)}$. To evaluate this, however, we must first decompose the fluctuations in the basis of monopole spherical harmonics.

Alternatively, we can expand the fermionic Lagrangian, repeat the procedure, and take the fermionic modes to be independent. In this case, one obtains a 4×4 matrix whose 1-loop contribution is a Pfaffian. The expanded Lagrangian is

$$L\left(t,\tau\right)\big|_{\text{fer}} = \frac{1}{2}\mathrm{Tr}\bigg(2i\left(\lambda D_1\tilde{\Lambda}_{\bar{1}}+\tilde{\Lambda}_{\bar{1}}D_1\lambda\right)-2i\left(\tilde{\lambda}D_{\bar{1}}\Lambda_1+\Lambda_1D_{\bar{1}}\tilde{\lambda}\right) \quad (6.6.21)$$

$$-i\left(\Lambda_1\left[\tilde{\sigma},\tilde{\Lambda}_{\bar{1}}\right]+\tilde{\Lambda}_{\bar{1}}\left[\tilde{\sigma},\Lambda_1\right]\right)+i\left(\lambda\left[\sigma,\tilde{\lambda}\right]+\tilde{\lambda}\left[\sigma,\lambda\right]\right) \quad (6.6.22)$$

$$+2t\left(\tilde{\lambda}\lambda-\lambda\tilde{\lambda}\right)-\frac{i\tau}{2}\left(\lambda\left[\sigma,\tilde{\lambda}\right]+\tilde{\lambda}\left[\sigma,\lambda\right]\right)\bigg) \quad (6.6.23)$$

Here we have used integration by parts and the cyclicity of the trace for anti-commuting $\Lambda_1, \tilde{\Lambda}_{\bar{1}}, \lambda, \tilde{\lambda}$.

Repeating the procedure to obtain a quadratic order Lagrangian in the Cartan Weyl basis, and taking the modes to be independent $\Lambda_{\bar{1}}^{\prime\dagger} \neq \Lambda_1^{\prime}, \tilde{\lambda}^{\prime\dagger} \neq \lambda^{\prime}$, we have

$$\tilde{S}_{\text{fer}} = \int_{S^2} \mathrm{d}^2x\sqrt{g}\sum_{\alpha}\mathrm{Tr}\left(\left(\Lambda_1^{(-\alpha)},\tilde{\Lambda}_{\bar{1}}^{(-\alpha)},\lambda^{(-\alpha)},\tilde{\lambda}^{(-\alpha)}\right)\Delta_{\text{fer}}^{(\alpha)}\begin{pmatrix}\Lambda_1^{(\alpha)}\\ \tilde{\Lambda}_{\bar{1}}^{(\alpha)}\\ \lambda^{(\alpha)}\\ \tilde{\lambda}^{(\alpha)}\end{pmatrix}\right) \quad (6.6.24)$$

$$\Delta_{\text{fer}}^{(\alpha)} = \begin{pmatrix} 0 & -\frac{i\alpha(\tilde{\sigma}_0)}{2} & 0 & -iD_{\bar{1}}^0\\ -\frac{i\alpha(\tilde{\sigma}_0)}{2} & 0 & iD_1^0 & 0\\ 0 & iD_1^0 & 0 & -\frac{i(\tau-2)\alpha(\sigma_0)}{4}-t\\ -iD_{\bar{1}}^0 & 0 & -\frac{i(\tau-2)\alpha(\sigma_0)}{4}+t & 0\end{pmatrix} \quad (6.6.25)$$

This matrix has a 1-loop contribution in terms of a Pfaffian $\mathrm{Pf}\Delta_{\text{fer}}^{(\alpha)}$.

6.7 Monopole spherical harmonics

In this section, we will outline how to use monopole spherical harmonics to evaluate the 1-loop determinant of a localizing Lagrangian at quadratic order in fluctuations. We illustrate the procedure using two simple examples. In the first example, we evaluate a bosonic 1-loop determinant. In the second example, we evaluate a fermionic 1-loop determinant. A similar exposition may be found in appendix C of [20], and we follow the same conventions. In both cases, we consider a monopole background of charge $\mathbf{r} = -c_1$ on the A-twisted S^2 of unit radius, where c_1 is the first Chern class. The charge may be written as $\mathbf{r} = r - Q(\mathfrak{m})$ for R-charge r, gauge charge Q, and GNO quantized flux $\mathfrak{m}$.

Consider the quadratic bosonic action

$$\tilde{S}_{\text{bos}} = \int_{S^2} \mathrm{d}^2 x \sqrt{g} \sum_{\alpha} \operatorname{Tr}\left(2 a_{\bar{1}}^{(-\alpha)} \Delta_{\text{bos}}^{(\alpha)} a_1^{(\alpha)}\right) \tag{6.7.1}$$

$$\Delta_{\text{bos}}^{(\alpha)} = -4 D_1 D_{\bar{1}} + \alpha\left(\tilde{\sigma}_0\right) \alpha\left(\sigma_0\right) \tag{6.7.2}$$

for complex conjugate $a_{\bar{1}}^{(-\alpha)}, a_1^{(\alpha)}$.

In the basis of monopole harmonics, the fluctuating modes are expressed as

$$a_1^{(\alpha)} = \sum_{j \geq j_0(\mathbf{r})+1} \sum_{m=-j}^{j} A_{j,m} Y_{\frac{\mathbf{r}}{2},j,m} \tag{6.7.3}$$

$$a_{\bar{1}}^{(-\alpha)} = \sum_{j' \geq j(\mathbf{r})+1}^{\infty} \sum_{m'=-j'}^{j'} \overline{A}_{j',m'} Y^{\dagger}_{\frac{\mathbf{r}}{2},j',m'} \tag{6.7.4}$$

where $Y_{\frac{\mathbf{r}}{2},j,m}$ is the top vector harmonic, and $j_0\left(\mathbf{r}\right) = \frac{|\mathbf{r}-1|}{2} - \frac{1}{2}$.

Expressing the fluctuating modes in $\tilde{S}_{\text{bos}}$ in terms of harmonics, we have

$$\int_{S^2} \mathrm{d}^2 x \sqrt{g} \sum_{\alpha} \sum_{j',j,m',m} \left(2 \overline{A}_{j',m'} Y^{\dagger}_{\frac{\mathbf{r}}{2},j',m'} \left(-4 D_1 D_{\bar{1}} + \alpha\left(\tilde{\sigma}_0\right) \alpha\left(\sigma_0\right)\right) A_{j,m} Y_{\frac{\mathbf{r}}{2},j,m}\right) \tag{6.7.5}$$

The operator acting on the harmonic satisfies the eigenvalue equation

$$-4 D_1 D_{\bar{1}} Y_{\frac{\mathbf{r}}{2},j,m} = \left(j\left(j+1\right) - \frac{\mathbf{r}}{2}\left(\frac{\mathbf{r}}{2} - 1\right)\right) Y_{\frac{\mathbf{r}}{2},j,m}. \tag{6.7.6}$$

Replacing the operator with its eigenvalue, $\tilde{S}_{\text{bos}}$ reads

$$\int_{S^2} \mathrm{d}^2 x \sqrt{g} \sum_{\alpha} \sum_{j',j,m',m} \left(2 \overline{A}_{j',m'} Y^{\dagger}_{\frac{\mathbf{r}}{2},j',m'} \left(j\left(j+1\right) - \frac{\mathbf{r}}{2}\left(\frac{\mathbf{r}}{2} - 1\right) + \alpha\left(\tilde{\sigma}_0\right) \alpha\left(\sigma_0\right)\right) A_{j,m} Y_{\frac{\mathbf{r}}{2},j,m}\right) \tag{6.7.7}$$

The harmonics form an orthonormal basis. For scalar harmonics the orthonormal condition is

$$\int_{S^2} d^2x\sqrt{g}\, Y^{\dagger}_{\frac{\mathbf{r-2}}{2},j',m'} Y_{\frac{\mathbf{r-2}}{2},j,m} = \delta_{j,j'}\delta_{m,m'}. \tag{6.7.8}$$

Similar considerations hold for vector harmonics, except that there is an additional delta function.

For this reason, the integral over the two-sphere in $\tilde{S}_{\rm bos}$ evaluates to

$$2\sum_{\alpha}\sum_{j\geq j_0(\mathbf{r})+1}\sum_{m=-j}^{j}\overline{A}_{j,m}\left(j\,(j+1)-\frac{\mathbf{r}}{2}\left(\frac{\mathbf{r}}{2}-1\right)+\alpha\left(\tilde{\sigma}_0\right)\alpha\left(\sigma_0\right)\right)A_{j,m}. \tag{6.7.9}$$

It follows that the bosonic 1-loop determinant, obtained by integrating out complex conjugate fluctuations $a^{(-\alpha)}_{\bar{1}}$ and $a^{(\alpha)}_{1}$, is

$$\frac{1}{\sqrt{\det\Delta_{\rm bos}}} \tag{6.7.10}$$

where

$$\det\Delta_{\rm bos} = \prod_{\alpha}\prod_{m=-\frac{\mathbf{r}}{2}+1}^{\frac{\mathbf{r}}{2}-1}\left(\alpha\left(\tilde{\sigma}_0\right)\alpha\left(\sigma_0\right)\right) \tag{6.7.11}$$

$$\prod_{j=j_0(\mathbf{r})+1}^{\infty}\prod_{m=-j}^{j}\left(j\,(j+1)-\frac{\mathbf{r}}{2}\left(\frac{\mathbf{r}}{2}-1\right)+\alpha\left(\tilde{\sigma}_0\right)\alpha\left(\sigma_0\right)\right) \tag{6.7.12}$$

Next, we evaluate the 1-loop determinant of the fermionic Lagrangian

$$L_{\rm fer} = L\left(t,\tau=0\right)\big|_{\rm fer} \tag{6.7.13}$$

$$= {\rm Tr}\left(2i\tilde{\Lambda}_{\bar{1}}D_1\lambda-2i\Lambda_1D_{\bar{1}}\tilde{\lambda}-i\tilde{\Lambda}_{\bar{1}}[\tilde{\sigma},\Lambda_1]+i\tilde{\lambda}[\sigma,\lambda]+2t\tilde{\lambda}\lambda\right) \tag{6.7.14}$$

To quadratic order in fluctuations, the action reads

$$\tilde{S}_{\rm fer} = \int_{S^2}{\rm d}^2x\sqrt{g}\sum_{\alpha}{\rm Tr}\left(\frac{1}{2}\left(2\tilde{\Lambda}^{(-\alpha)}_{\bar{1}},\tilde{\lambda}^{(-\alpha)}\right)\Delta^{(\alpha)}_{\rm fer}\begin{pmatrix}2\Lambda^{(\alpha)}_1\\ \lambda^{(\alpha)}\end{pmatrix}\right) \tag{6.7.15}$$

$$\Delta^{(\alpha)}_{\rm fer} = -2i\begin{pmatrix}\frac{1}{4}\alpha\left(\tilde{\sigma}_0\right) & -D_1\\ D_{\bar{1}} & \alpha\left(\sigma_0\right)+2it\end{pmatrix} \tag{6.7.16}$$

for complex conjugate pairs $\tilde{\Lambda}^{(-\alpha)}_{\bar{1}}$,$\Lambda^{(\alpha)}_1$ and $\tilde{\lambda}^{(-\alpha)}$, $\lambda^{(\alpha)}$.

In the basis of monopole harmonics, the fluctuating modes are expressed as

$$\Lambda_1^{(\alpha)} = \sum_{j\geq j_0(\mathbf{r})+1} \sum_{m=-j}^{j} L_{jm} Y_{\frac{\mathbf{r}}{2},j,m} \tag{6.7.17}$$

$$\tilde{\Lambda}_{\bar{1}}^{(-\alpha)} = \sum_{j'\geq j(\mathbf{r})+1}^{\infty} \sum_{m'=-j'}^{j'} \tilde{L}_{j'm'} Y^{\dagger}_{\frac{\mathbf{r}}{2},j',m'} \tag{6.7.18}$$

$$\lambda^{(\alpha)} = \sum_{j\geq j_0(\mathbf{r})+1} \sum_{m=-j}^{j} \ell_{jm} Y_{\frac{\mathbf{r}-2}{2},j,m} \tag{6.7.19}$$

$$\tilde{\lambda}^{(-\alpha)} = \sum_{j'\geq j(\mathbf{r})+1}^{\infty} \sum_{m'=-j'}^{j'} \tilde{\ell}_{j'm'} Y^{\dagger}_{\frac{\mathbf{r}-2}{2},j',m'} \tag{6.7.20}$$

where $Y_{\frac{\mathbf{r}}{2},j,m}$ is the top vector harmonic, $Y_{\frac{\mathbf{r}-2}{2},j,m}$ is the scalar harmonic, and $j_0(\mathbf{r}) = \frac{|\mathbf{r}-1|}{2} - \frac{1}{2}$.

The operators acting on harmonics in $\tilde{S}_{\text{fer}}$ satisfy the eigenvalue equations

$$(-2iD_{\bar{1}})\,\Lambda_1^{(\alpha)} = \sum_{j,m} L_{jm}\,(-2iD_{\bar{1}})\,Y_{\frac{\mathbf{r}}{2},j,m} \tag{6.7.21}$$

$$= \sum_{j,m} L_{jm}\left(-\lambda_{j,m}^{(-)(\mathbf{r})}\right) Y_{\frac{\mathbf{r}-2}{2},j,m} \tag{6.7.22}$$

$$(2iD_1)\,\lambda^{(\alpha)} = \sum_{j,m} \ell_{jm}\,(2iD_1)\,Y_{\frac{\mathbf{r}-2}{2},j,m} \tag{6.7.23}$$

$$= \sum_{j,m} \ell_{jm}\left(-\lambda_{j,m}^{(+)(\mathbf{r}-2)}\right) Y_{\frac{\mathbf{r}}{2},j,m} \tag{6.7.24}$$

where the eigenvalues are

$$\lambda_{jm}^{(\pm)(\mathbf{r})} = \sqrt{\left(j+\frac{1}{2}\right)^2 - \frac{(\mathbf{r}\pm 1)^2}{4}}. \tag{6.7.25}$$

Note that the eigenvalues obey $\lambda_{jm}^{(\pm)(\mathbf{r})} = \lambda_{jm}^{(\mp)(\mathbf{r}\pm 2)}$, that is to say, the eigenvalues coincide when $\mathbf{r}$ is shifted by two.

Using the fact that both the vector harmonics $Y^{\dagger}_{\frac{\mathbf{r}}{2},j',m'}$,$Y_{\frac{\mathbf{r}}{2},j,m}$, and the scalar harmonics$Y^{\dagger}_{\frac{\mathbf{r}-2}{2},j',m'}$,$Y_{\frac{\mathbf{r}-2}{2},j,m}$, form an orthonormal basis, we integrate over the two-sphere in $\tilde{S}_{\text{fer}}$ and obtain the expression

$$\sum_{\alpha} \sum_{j\geq j_0(\mathbf{r})+1} \sum_{m=-j}^{j} \mathrm{Tr}\Big(\tilde{L}_{jm}\,(-i\alpha\,(\tilde{\sigma}_0))\,L_{jm} + \tilde{L}_{jm}\left(-\lambda_{j,m}^{(+)(\mathbf{r}-2)}\right)\ell_{jm} \tag{6.7.26}$$

$$+\tilde{\ell}_{jm}\left(-\lambda_{j,m}^{(-)(\mathbf{r})}\right) L_{jm} + \tilde{\ell}_{jm}\,(-2i\alpha\,(\sigma_0)+4t)\,\ell_{jm}\Big) \tag{6.7.27}$$

It follows that the fermionic 1-loop determinant, obtained by integrating out complex conjugate pairs $\tilde{\Lambda}_{\bar{1}}^{(-\alpha)}$,$\Lambda_{1}^{(\alpha)}$ and $\tilde{\lambda}^{(-\alpha)}$, $\lambda^{(\alpha)}$, is

$$\det \Delta_{\text{fer}}^{(\alpha)} = \prod_{\alpha} \left(-2i\alpha\left(\sigma_0\right) + 4t\right)^{\alpha(\mathfrak{m})-1} \prod_{\alpha} \left(-i\alpha\left(\tilde{\sigma}_0\right)\right)^{1-\alpha(\mathfrak{m})} \tag{6.7.28}$$

$$\times \prod_{\alpha} \prod_{j=j_0(\alpha)+1}^{\infty} \left[j\left(j+1\right) - \frac{\alpha(\mathfrak{m})}{2}\left(\frac{\alpha(\mathfrak{m})}{2} - 1\right) - 2\left(\alpha\left(\sigma_0\right) - 2it\right)\alpha\left(\tilde{\sigma}_0\right)\right]^{2j+1} \tag{6.7.29}$$

where $j_0(\alpha) = \frac{|1-\alpha(\mathfrak{m})|}{2} - \frac{1}{2}$ was translated from $j_0\left(\mathbf{r}\right) = \frac{|\mathbf{r}-1|}{2} - \frac{1}{2}$ using $\mathbf{r} = 2 - \alpha(\mathfrak{m})$.

6.8 Integrating bosonic scalars

Here, we discuss the integration contours for the bosonic scalars $\sigma, \tilde{\sigma}$. Consider the bosonic gauge fixed Localizing Lagrangian

$$\begin{aligned} L\left(t,\tau,\xi,\gamma\right)|_{\text{bos}} &= L\left(t,\tau\right)|_{\text{bos}} + L_{\text{gf,gh}}\left(\xi,\gamma\right) && (6.8.1)\\ &= \frac{1}{h^2}\text{Tr}\bigg(\frac{1}{2}\left(f_{12} - it\tilde{\sigma}\right)^2 + \frac{1}{2}\left(D_E + t\tilde{\sigma}\right)^2 && (6.8.2)\\ &\quad + \frac{1}{2}D_\mu\sigma D^\mu\tilde{\sigma} + \frac{1-\tau}{8}[\sigma,\tilde{\sigma}]^2 && (6.8.3)\\ &\quad \frac{1}{2\xi}\left(D_\mu a^\mu + \frac{i\xi\gamma}{2}\left[\sigma,\tilde{\sigma}\right]\right)^2 + D_\mu\tilde{c}D^\mu c\bigg) && (6.8.4)\end{aligned}$$

When the superpotential is turned on ($t \neq 0$), the integration contour for $\tilde{\sigma}$ is constrained by the quadratic term involving $f_{12} - it\tilde{\sigma}$. Since the only reasonable integration contour for the gauge field is $a_\mu \in \mathbb{R}$, we always have $f_{12} \in \mathbb{R}$, as well as $it\tilde{\sigma} \in \mathbb{R}$. So, the integration contour for $\tilde{\sigma}$ then depends on whether t is a real or purely imaginary parameter. Specifically, to satisfy $it\tilde{\sigma} \in \mathbb{R}$ we must have either $t \in \mathbb{R}$ and $\tilde{\sigma} \in i\mathbb{R}$ or $t \in i\mathbb{R}$ and $\tilde{\sigma} \in \mathbb{R}$.

Furthermore, the integration contour of σ is constrained by that of $\tilde{\sigma}$. Typically, one takes the bosonic scalars to be complex conjugates, but if one is real, the other must be purely imaginary. Thus, for $t \in \mathbb{R}$ we have $\tilde{\sigma} \in i\mathbb{R}$ and $\sigma \in \mathbb{R}$, while for $t \in i\mathbb{R}$ we have $\tilde{\sigma} \in \mathbb{R}$ and $\sigma \in i\mathbb{R}$.

When the superpotential is turned off ($t = 0$), $\tilde{\sigma}$ no longer appears in the quadratic term involving f_{12}, and it's integration contour is not constrained as before. In this case, the standard choice is to take σ and $\tilde{\sigma}$ to be complex conjugates, but it remains possible to take one real and the other purely imaginary.

Let us describe how to evaluate the integral for real σ and purely imaginary $\tilde{\sigma}$, at quadratic order in fluctuations. To do so, we set $\tilde{\sigma} =: i\tilde{\sigma}_{\text{R}}$ for $\tilde{\sigma}_{\text{R}} \in \mathbb{R}$. The bosonic scalar kinetic term is

$$\sigma^{-\alpha}\left(-\frac{1}{2}D^2\right)\tilde{\sigma}^{\alpha} \longrightarrow i\sigma^{\mathrm{T}}\left(-\frac{1}{2}D^2\right)\tilde{\sigma}_{\mathrm{R}} \tag{6.8.5}$$

where $D^2 = D_\mu D^\mu$ is the Laplacian. Integrating over $\sigma \in \mathbb{R}$ yields the delta-function constraint $\delta\left(-D^2\tilde{\sigma}_{\mathrm{R}}\right)$. In this case, the measure can be recast as

$$\mathcal{D}\tilde{\sigma}_{\mathrm{R}}\delta\left(-D^2\tilde{\sigma}_{\mathrm{R}}\right) = \mathcal{D}\tilde{\sigma}_{\mathrm{R}}\delta\left(\tilde{\sigma}_{\mathrm{R}}\right)\det\left(-D^2_{\tilde{\sigma}}\right)^{-1/2} \tag{6.8.6}$$

where $D^2_{\tilde{\sigma}}$ is the contribution of the Laplacian acting on $\tilde{\sigma}_{\mathrm{R}}$. Integrating over $\tilde{\sigma}_{\mathrm{R}} \in \mathbb{R}$ then sets to zero all $\tilde{\sigma}_{\mathrm{R}}$ fluctuations in the Lagrangian due to the delta function $\delta\left(\tilde{\sigma}_{\mathrm{R}}\right)$.

Moreover, the contribution $\det\left(-D^2_{\tilde{\sigma}}\right)^{-1/2}$ partially cancels the contribution from integrating out the ghosts $c, \tilde{c}$. In particular, integrating out the ghosts contributes a factor of $\det\left(-D^2_c\right)$ and the cancellation that occurs is

$$\frac{\det\left(-D^2_c\right)}{\sqrt{\det\left(-D^2_{\tilde{\sigma}}\right)}} = \sqrt{\det\left(-D^2\right)} \tag{6.8.7}$$

6.9 1-loop determinants

In this section, we use monopole spherical harmonics to evaluate 1-loop determinants mode-by-mode, for several different localizing Lagrangians in the A-model. In doing so, we hope to understand and compare the three approaches to A-model localization, namely, the follow-your-nose approach, the Witten approach, and the Benini-Zaffaroni approach. The three approaches correspond to different localizing Lagrangians, different localization loci, and different 1-loop determinants.

Let us begin by outlining the procedure to evaluate 1-loop contributions mode-by-mode. First, we specify our choice of localizing Lagrangian $L(t, \tau, \xi, \gamma)$, as well as parameters t, τ, ξ, γ. Then we construct a matrix expression for the localizing Lagrangian at quadratic order in fluctuations in the Cartan Weyl basis, first for the bosons, then for the fermions. If the matrix is not in the appropriate form to be evaluated, it is brought into the correct form. For instance, the bosonic matrices must be hermitian. Following this, the fluctuating modes are decomposed in the basis of monopole spherical harmonics.

Our monopole harmonic conventions are precisely those of [20], for a monopole background of charge $\mathbf{r} = -c_1 = 2 - \alpha(\mathfrak{m})$. Here, c_1 is the first Chern class, 2 is the vector like R-symmetry charge, the gauge charge α is a root (weight of the adjoint representation), and $\mathfrak{m}$ is the GNO quantized flux. This $\mathbf{r}$ is appropriate for the A-model vector multiplet $\mathcal{V}$, which transforms like a chiral multiplet Φ of R-charge

2 in the adjoint representation. For this choice, the monopole harmonics are

$$Y^{r-1}_{jm}\ \forall j \geq |r-1|\,,\ Y^{r}_{jm}\ \forall j \geq |r|\,,\ Y^{r+1}_{jm}\ \forall j \geq |r+1|\,,\ \text{for } r=-\frac{\alpha(\mathfrak{m})}{2} \tag{6.9.1}$$

where

$$j \geq j_0\left(\alpha\right)+1,\ j_0(\alpha)=\frac{|1-\alpha(\mathfrak{m})|}{2}-\frac{1}{2},\ m=-j,-j+1,\cdots,j. \tag{6.9.2}$$

Here, $Y^{r}_{j,m}$ is the scalar harmonic, $Y^{r+1}_{j,m}$ is the top vector harmonic, $Y^{r-1}_{j,m}$ is the bottom vector harmonic, and the statement $Y^{r}_{jm}\ \forall j \geq |r|$ should be understood as: the harmonic Y^{r}_{jm} that exists for all $j \geq |r|$ where $r=-\frac{\alpha(\mathfrak{m})}{2}$ and $\alpha(\mathfrak{m}) \in \mathbb{Z}$.

The fluctuating modes, in the quadratic order Lagrangian, are expressed in terms of harmonics according to whether they are scalars or vectors. The scalar fluctuations $\sigma^{(\alpha)}, \tilde{\sigma}^{(\alpha)}, D^{(\alpha)}_{E}, \lambda^{(\alpha)}, \tilde{\lambda}^{(\alpha)}$ are expressed in terms of the scalar harmonic $Y^{r}_{j,m}$, the holomorphic vector fluctuations $a^{(\alpha)}_{1}, \Lambda^{(\alpha)}_{1}$ are expressed in terms of the top vector harmonic $Y^{r+1}_{j,m}$, and the anti-holomorphic vector fluctuations $a^{(\alpha)}_{\bar{1}}, \Lambda^{(\alpha)}_{\bar{1}}$ are expressed in terms of the bottom vector harmonic $Y^{r-1}_{j,m}$.

As a result, the quadratic order Lagrangian now involves various combinations of D_1 and $D_{\bar{1}}$ acting on harmonics. The operators act on harmonics according to the eigenvalue equations

$$\begin{aligned}
D_1 Y^{r}_{j,m} &= \tfrac{r_+}{2} Y^{r+1}_{j,m}, & D_{\bar{1}} Y^{r+1}_{j,m} &= -\tfrac{r_+}{2} Y^{r}_{j,m},\\
D_{\bar{1}} Y^{r}_{j,m} &= -\tfrac{r_-}{2} Y^{r-1}_{j,m}, & D_1 Y^{r-1}_{j,m} &= \tfrac{r_-}{2} Y^{r}_{j,m}.\\
D_1 D_{\bar{1}} Y^{r+1}_{j,m} &= -\tfrac{r_+^2}{4} Y^{r+1}_{j,m}, & D_1 D_1 Y^{r-1}_{j,m} &= \tfrac{r_- r_+}{4} Y^{r+1}_{j,m},\\
D_{\bar{1}} D_{\bar{1}} Y^{r+1}_{j,m} &= \tfrac{r_+ r_-}{4} Y^{r-1}_{j,m}, & D_{\bar{1}} D_1 Y^{r-1}_{j,m} &= -\tfrac{r_-^2}{4} Y^{r-1}_{j,m},\\
D_1 D_{\bar{1}} Y^{r}_{j,m} &= -\tfrac{r_-^2}{4} Y^{r}_{j,m}, & D_{\bar{1}} D_1 Y^{r}_{j,m} &= -\tfrac{r_+^2}{4} Y^{r}_{j,m}.
\end{aligned} \tag{6.9.3}$$

where we have denoted the eigenvalues as

$$r_{\pm}=\sqrt{j\left(j+1\right)-r\left(r\pm 1\right)}\ \text{for } r=-\frac{\alpha(\mathfrak{m})}{2} \tag{6.9.4}$$

Observe, however, that the harmonics only exist for specific values of j, e.g. Y^{r+1}_{jm} exists for $j \geq |r+1|$, and the eigenvalues that contribute depend on which harmonics exist. We proceed by first specifying values for $j=j_0\left(\alpha\right)+1$ and $\alpha\left(\mathfrak{m}\right) \in \mathbb{Z}$, then determining which harmonics exist and which eigenvalues contribute, then evaluating the corresponding 1-loop contribution. This is implemented iteratively for all possible values of j. For later convenience, the existence of harmonics is summarized in Table C.1.

For each value of j and $\alpha\left(\mathfrak{m}\right)$ we obtain a ratio of determinants that is the collective contribution of integrating out the bosonic, fermionic, and ghost fluctuation

	$\exists Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}+1}$ $\forall\, j\geq\left\|-\frac{\alpha(\mathfrak{m})}{2}+1\right\|$	$\exists Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}}$ $\forall\, j\geq\left\|-\frac{\alpha(\mathfrak{m})}{2}\right\|$	$\exists Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}-1}$ $\forall j\geq\left\|-\frac{\alpha(\mathfrak{m})}{2}-1\right\|$	$d_G=2j+1$
$j=\frac{\alpha(\mathfrak{m})}{2}-1$	$\alpha(\mathfrak{m})\geq 2$	-	-	$\alpha(\mathfrak{m})-1$
$j=-\frac{\alpha(\mathfrak{m})}{2}$	-	$\alpha(\mathfrak{m})\leq 0$	$\alpha(\mathfrak{m})\leq -1$	$1-\alpha(\mathfrak{m})$
$j=-\frac{\alpha(\mathfrak{m})}{2}-1$	-	-	$\alpha(\mathfrak{m})\leq -2$	$-\alpha(\mathfrak{m})-1$
$j=\frac{\alpha(\mathfrak{m})}{2}$	$\alpha(\mathfrak{m})\geq 1$	$\alpha(\mathfrak{m})\geq 0$	-	$\alpha(\mathfrak{m})+1$

Table 6.1: **Harmonic Existence** This table describes the existence of harmonics for various values of j and $\alpha(\mathfrak{m})$, where d_G is degeneracy. Note that the harmonics $Y_{jm}^{r+1}, Y_{jm}^{r}, Y_{jm}^{r-1}$ are respectively denoted $Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}+1}$, $Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}}$, $Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}-1}$ using using $r=-\frac{\alpha(\mathfrak{m})}{2}$. For example, for $j=-\frac{\alpha(\mathfrak{m})}{2}$, $Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}}$ exists when $\alpha(\mathfrak{m})\leq 0$, $Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}-1}$ exists when $\alpha(\mathfrak{m})\leq -1$, and the degeneracy is $d_G=1-\alpha(\mathfrak{m})$. For $j\geq\frac{|\alpha(\mathfrak{m})|}{2}+1$ all harmonics exist with degeneracy $|\alpha(\mathfrak{m})|+3$. The choices $j=\frac{\alpha(\mathfrak{m})}{2}-1$ or $j=-\frac{\alpha(\mathfrak{m})}{2}$ give $r_+=0$, while $j=-\frac{\alpha(\mathfrak{m})}{2}-1$ or $j=\frac{\alpha(\mathfrak{m})}{2}$ give $r_-=0$, for $r_\pm=\sqrt{j\,(j+1)-\alpha(\mathfrak{m})/2\,(\alpha(\mathfrak{m})/2\mp 1)}$.

operators. The ratio of determinants is then simplified by identifying cancellations that occur between different contributions.

6.9.1 Case 1: Yang-Mills Lagrangian

Here we compute the 1-loop determinant mode-by-mode, for the case in which the localizing term is the Yang-Mills Lagrangian. This is the localizing Lagrangian for the follow-your-nose approach to localization in the A-model.

The Localizing Lagrangian together with the gauge-fixing Lagrangian is

$$
\begin{aligned}
& L\,(t=0,\tau=0,\xi=1,\gamma=0) && (6.9.5)\\
=\;& L_{\mathrm{YM}}+L_{\mathrm{gf,gh}}\,(\xi=1,\gamma=0) && (6.9.6)\\
=\;& \frac{1}{h^2}\mathrm{Tr}\bigg(\frac{1}{2}\,(-2if_{1\bar{1}})^2+\frac{1}{2}D_E^2+\frac{1}{2}D_\mu\tilde{\sigma}D^\mu\sigma+\frac{1}{8}\,[\sigma,\tilde{\sigma}]^2 && (6.9.7)\\
& +2i\tilde{\Lambda}_{\bar{1}}D_1\lambda-2i\Lambda_1D_{\bar{1}}\tilde{\lambda}-i\tilde{\Lambda}_{\bar{1}}[\tilde{\sigma},\Lambda_1]+i\tilde{\lambda}[\sigma,\lambda] && (6.9.8)\\
& +\frac{1}{2}\,(D_\mu a^\mu)^2+D_\mu\tilde{c}D^\mu c\bigg) && (6.9.9)
\end{aligned}
$$

Note that we immediately integrate out the auxiliary field D_E by setting it to its on-shell value in the action.

The bosonic action to quadratic order in fluctuations is

$$\tilde{S}_{\rm bos} = \int_{S^2} \mathrm{d}^2 x \sqrt{g} \tilde{L}_{\rm bos} \tag{6.9.10}$$

$$\tilde{L}_{\rm bos} = \sum_\alpha \mathrm{Tr}\Bigg(\left(a_{\bar{1}}^{-\alpha}, a_1^{-\alpha}, \sigma^{-\alpha}, \tilde{\sigma}^{-\alpha}\right) \Delta_{\rm bos,0} \begin{pmatrix} a_1^\alpha \\ a_{\bar{1}}^\alpha \\ \sigma^\alpha \\ \tilde{\sigma}^\alpha \end{pmatrix} + \tilde{c}^{-\alpha} \left(\Delta_{\rm gho}\right) c^\alpha \Bigg) \tag{6.9.11}$$

$$\Delta_{\rm bos,0} = \begin{pmatrix} -4D_1 D_{\bar{1}} + \alpha(\tilde{\sigma}_0)\alpha(\sigma_0) & 0 & 0 & -i\alpha(\sigma_0) D_1 \\ 0 & -4D_{\bar{1}} D_1 + \alpha(\tilde{\sigma}_0)\alpha(\sigma_0) & 0 & -i\alpha(\sigma_0) D_{\bar{1}} \\ -i\alpha(\tilde{\sigma}_0) D_{\bar{1}} & -i\alpha(\tilde{\sigma}_0) D_1 & -\frac{1}{8}\alpha(\tilde{\sigma}_0)^2 & -\frac{1}{2}D^2 + \frac{\alpha(\tilde{\sigma}_0)\alpha(\tilde{\sigma}_0)}{8} \\ 0 & 0 & \frac{\alpha(\tilde{\sigma}_0)\alpha(\sigma_0)}{8} & -\frac{\alpha(\sigma_0)^2}{8} \end{pmatrix} \tag{6.9.12}$$

$$\Delta_{\rm gho} = -D^2 \tag{6.9.13}$$

where $D^2 = 2\{D_1, D_{\bar{1}}\}$. Note that we have chosen the complex conjugate pairs for the gauge modes ($a_{\bar{1}}^\dagger = a_1$) and the ghost modes ($\tilde{c}^\dagger = c$), but not the bosonic scalar modes ($\sigma^\dagger \neq \tilde{\sigma}$).

The matrix $\Delta_{\rm bos,0}$ is not hermitian, and therefore not in the appropriate form. To extract the hermitian part of $\Delta_{\rm bos,0}$, we recall that a matrix M can be decomposed into a hermitian part $A = A^\dagger$ and an anti-hermitian part $B = -B^\dagger$. The decomposition of the matrix is $M = A + B$ for $A = \frac{1}{2}\left(M + M^\dagger\right)$ and $B = \frac{1}{2}\left(M - M^\dagger\right)$. Using $A = M - B$ for $M = \Delta_{\rm bos,0}$, we obtain the hermitian matrix $A = \Delta_{\rm bos}$.

In terms of the appropriate hermitian matrix $\Delta_{\rm bos}$, the quadratic Lagrangian is

$$\tilde{L}_{\rm bos} = \sum_\alpha \mathrm{Tr}\Bigg(\left(a_{\bar{1}}^{-\alpha}, a_1^{-\alpha}, \sigma^{-\alpha}, \tilde{\sigma}^{-\alpha}\right) (2\Delta_{\rm bos}) \begin{pmatrix} a_1^\alpha \\ a_{\bar{1}}^\alpha \\ \sigma^\alpha \\ \tilde{\sigma}^\alpha \end{pmatrix} + \tilde{c}^{-\alpha} \left(-D^2\right) c^\alpha \Bigg) \tag{6.9.14}$$

$$\Delta_{\rm bos} = \begin{pmatrix} -4D_1 D_{\bar{1}} + \alpha(\tilde{\sigma}_0)\alpha(\sigma_0) & 0 & \frac{i\alpha(\tilde{\sigma}_0)}{2} D_1 & -\frac{i\alpha(\sigma_0)}{2} D_1 \\ 0 & -4D_{\bar{1}} D_1 + \alpha(\tilde{\sigma}_0)\alpha(\sigma_0) & \frac{i\alpha(\tilde{\sigma}_0)}{2} D_{\bar{1}} & -\frac{i\alpha(\sigma_0)}{2} D_{\bar{1}} \\ -\frac{i\alpha(\tilde{\sigma}_0)}{2} D_{\bar{1}} & -\frac{i\alpha(\tilde{\sigma}_0)}{2} D_1 & -\frac{\alpha(\tilde{\sigma}_0)^2}{8} & -\frac{D^2}{4} + \frac{\alpha(\tilde{\sigma}_0)\alpha(\tilde{\sigma}_0)}{8} \\ \frac{i\alpha(\sigma_0)}{2} D_{\bar{1}} & \frac{i\alpha(\sigma_0)}{2} D_1 & \frac{D^2}{4} + \frac{\alpha(\tilde{\sigma}_0)\alpha(\sigma_0)}{8} & -\frac{\alpha(\sigma_0)^2}{8} \end{pmatrix} \tag{6.9.15}$$

The fermionic action to quadratic order in fluctuations is

$$\tilde{S}_{\text{fer}} = \int_{S^2} \mathrm{d}^2x\sqrt{g}\tilde{L}_{\text{fer}} \tag{6.9.16}$$

$$\tilde{L}_{\text{fer}} = \sum_{\alpha} \mathrm{Tr}\left(\frac{1}{2}\left(2\tilde{\Lambda}_{\bar{1}}^{(-\alpha)}, \tilde{\lambda}^{(-\alpha)}\right)\Delta_{\text{fer}}^{(\alpha)}\begin{pmatrix} 2\Lambda_1^{(\alpha)} \\ \lambda^{(\alpha)} \end{pmatrix}\right) \tag{6.9.17}$$

$$\Delta_{\text{fer}} = -2i\begin{pmatrix} \frac{\alpha(\tilde{\sigma}_0)}{4} & -D_1 \\ D_{\bar{1}} & -\alpha(\sigma_0) \end{pmatrix} \tag{6.9.18}$$

Note that we have chosen the complex conjugate pairs for both the fermionic vector modes ($\Lambda_{\bar{1}}^{\dagger} = \Lambda_1$) and the fermionic scalar modes ($\tilde{\lambda}^{\dagger} = \lambda$).

In the basis of monopole harmonics, the fluctuating modes are expressed as

$$a_1^{(\alpha)} = \sum_{j\geq j_0(\alpha)+1}\sum_{m=-j}^{j} A_{jm}Y_{j,m}^{r+1}, \tag{6.9.19}$$

$$a_1^{(-\alpha)} = \sum_{j'\geq j(\alpha)+1}^{\infty}\sum_{m'=-j'}^{j'} \overline{A}_{j'm'}Y_{j',m'}^{r-1,\dagger}, \tag{6.9.20}$$

$$a_{\bar{1}}^{(\alpha)} = \sum_{j\geq j_0(\alpha)+1}\sum_{m=-j}^{j} \tilde{A}_{jm}Y_{j,m}^{r-1}, \tag{6.9.21}$$

$$a_{\bar{1}}^{(-\alpha)} = \sum_{j'\geq j(\alpha)+1}^{\infty}\sum_{m'=-j'}^{j'} \overline{\tilde{A}}_{j'm'}Y_{j',m'}^{r+1,\dagger}, \tag{6.9.22}$$

$$\sigma^{(\alpha)} = \sum_{j\geq j(\alpha)+1}\sum_{m=-j}^{j} S_{jm}Y_{j,m}^{r}, \tag{6.9.23}$$

$$\sigma^{(-\alpha)} = \sum_{j'\geq j(\alpha)+1}^{\infty}\sum_{m'=-j'}^{j'} \overline{S}_{jm}Y_{j',m'}^{r,\dagger}, \tag{6.9.24}$$

$$\tilde{\sigma}^{(\alpha)} = \sum_{j\geq j(\alpha)+1}\sum_{m=-j}^{j} \tilde{S}_{jm}Y_{j,m}^{r}, \tag{6.9.25}$$

$$\tilde{\sigma}^{(-\alpha)} = \sum_{j'\geq j(\alpha)+1}^{\infty}\sum_{m'=-j'}^{j'} \overline{\tilde{S}}_{j'm'}Y_{j',m'}^{r,\dagger}, \tag{6.9.26}$$

$$c^{(\alpha)} = \sum_{j\geq j(\alpha)+1}\sum_{m=-j}^{j} C_{jm}Y^{r}_{j,m}, \tag{6.9.27}$$

$$\tilde{c}^{(-\alpha)} = \sum_{j'\geq j(\alpha)+1}^{\infty}\sum_{m'=-j'}^{j'} \overline{\tilde{C}}_{j'm'}Y^{r,\dagger}_{j',m'}, \tag{6.9.28}$$

$$\Lambda_1^{(\alpha)} = \sum_{j\geq j_0(\alpha)+1}\sum_{m=-j}^{j} L_{jm}Y^{r+1}_{j,m}, \tag{6.9.29}$$

$$\tilde{\Lambda}_{\bar{1}}^{(-\alpha)} = \sum_{j'\geq j(\alpha)+1}\sum_{m'=-j'}^{j'} \tilde{L}_{j'm'}Y^{r+1,\dagger}_{j',m'}, \tag{6.9.30}$$

$$\lambda^{(\alpha)} = \sum_{j\geq j_0(\alpha)+1}\sum_{m=-j}^{j} \ell_{jm}Y^{r}_{j,m}, \tag{6.9.31}$$

$$\tilde{\lambda}^{(-\alpha)} = \sum_{j'\geq j(\alpha)+1}\sum_{m'=-j'}^{j'} \tilde{\ell}_{j'm'}Y^{r,\dagger}_{j',m'}. \tag{6.9.32}$$

for $j_0(\alpha) = \frac{|1-\alpha(\mathfrak{m})|}{2} - \frac{1}{2}$ and $r = -\frac{\alpha(\mathfrak{m})}{2}$.

The order of evaluation is as follows. We choose a specific value of j and $\alpha(\mathfrak{m})$ and determine which harmonics exist. If a harmonic does not exist, we remove it by deleting its associated row and column from the matrices Δ_{bos},Δ_{fer} and Δ_{gho}. Then, the operators in the matrices are replaced by their appropriate eigenvalues, e.g. $D_1 Y^r_{j,m} = \frac{r_+}{2} Y^{r+1}_{j,m}$. Following this, we integrate over the two-sphere using the fact that the harmonics form an orthonormal basis, e.g. $\int_{S^2} d^2x\sqrt{g}\, Y^{r,\dagger}_{j',m'}Y^r_{j,m} = \delta_{j,j'}\delta_{m,m'}$. Next, we evaluate the determinant of each matrix, and obtain an expression involving an infinite product over j, which corresponds to evaluating the integrals over fluctuating modes. We construct the ratio of determinants, identify the cancellations occurring between infinite products, then simplify the expression. By repeating this process for all values of j and $\alpha(\mathfrak{m})$, we obtain the collective 1-loop contribution.

To be clear, the harmonic existence criteria is

$$Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}-1}\ \forall j \geq \left|-\frac{\alpha(\mathfrak{m})}{2}-1\right|,\ Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}}\ \forall j \geq \left|-\frac{\alpha(\mathfrak{m})}{2}\right|,\ Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}+1}\ \forall j \geq \left|-\frac{\alpha(\mathfrak{m})}{2}+1\right| \tag{6.9.33}$$

and the cases for specific j that we now iterate through are as follows.

1. For $j \geq \frac{|\alpha(\mathfrak{m})|}{2}+1$, $\alpha(\mathfrak{m}) \in \mathbb{Z}$, the existing harmonics are $Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}-1}, Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}}, Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}+1}$ with degeneracy $2j+1$.

2. For $j = \frac{\alpha(\mathfrak{m})}{2} - 1$, $\alpha(\mathfrak{m}) \geq 2$, the existing harmonic is $Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}+1}$ with degeneracy $\alpha(\mathfrak{m}) - 1$.

3. For $j = -\frac{\alpha(\mathfrak{m})}{2}$, $\alpha(\mathfrak{m}) \leq -1$, the existing harmonics are $Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}-1}, Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}}$ with degeneracy $1 - \alpha(\mathfrak{m})$.

   (a) For $j = 0$, $\alpha(\mathfrak{m}) = 0$, the existing harmonic is $Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}}$ with degeneracy -1

4. For $j = -\frac{\alpha(\mathfrak{m})}{2} - 1$, $\alpha(\mathfrak{m}) \leq -2$, the existing harmonic is $Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}-1}$ with degeneracy $-\alpha(\mathfrak{m}) - 1$.

5. For $j = \frac{\alpha(\mathfrak{m})}{2}$, $\alpha(\mathfrak{m}) \geq 1$, the existing harmonics are $Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}}, Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}+1}$ with degeneracy $\alpha(\mathfrak{m}) + 1$.

   (a) For $j = 0$, $\alpha(\mathfrak{m}) = 0$, the existing harmonic is $Y_{jm}^{\frac{\alpha(\mathfrak{m})}{2}}$ with degeneracy $+1$

After iterating through these cases, we collect the results, and evaluate the total 1-loop contribution. Now, we proceed with the iterative evaluation of the 1-loop contribution for all cases of j.

1. For $j \geq \frac{|\alpha(\mathfrak{m})|}{2} + 1$, all harmonics exist and the determinants are

$$\det \Delta_{\text{bos}} = \prod_{\alpha} \prod_{j=\frac{|\alpha(\mathfrak{m})|}{2}+1}^{\infty} \prod_{m=-j}^{j} \left(j^2 + j - \frac{\alpha(\mathfrak{m})^2}{4}\right)^2 \tag{6.9.34}$$

$$\times \left(\left(j + \frac{\alpha(\mathfrak{m})}{2}\right)\left(j - \frac{\alpha(\mathfrak{m})}{2} + 1\right) + \alpha(\tilde{\sigma}_0)\alpha(\sigma_0)\right) \tag{6.9.35}$$

$$\times \left(\left(j + \frac{\alpha(\mathfrak{m})}{2} + 1\right)\left(j - \frac{\alpha(\mathfrak{m})}{2}\right) + \alpha(\tilde{\sigma}_0)\alpha(\sigma_0)\right) \tag{6.9.36}$$

$$\det \Delta_{\text{gh}} = \prod_{\alpha} \prod_{j=\frac{|\alpha(\mathfrak{m})|}{2}+1}^{\infty} \prod_{m=-j}^{j} \left(j^2 + j - \frac{\alpha(\mathfrak{m})^2}{4}\right) \tag{6.9.37}$$

$$\det \Delta_{\text{fer}} = \prod_{\alpha} \prod_{j=\frac{|\alpha(\mathfrak{m})|}{2}+1}^{\infty} \prod_{m=-j}^{j} \left(j + \frac{\alpha(\mathfrak{m})}{2}\right)\left(j - \frac{\alpha(\mathfrak{m})}{2} + 1\right) + \alpha(\tilde{\sigma}_0)\alpha(\sigma_0) \tag{6.9.38}$$

The ghosts cancel part of the bosonic operator

$$\frac{\det \Delta_{\text{gh}}}{\sqrt{\det \Delta_{\text{bos}}}} = \frac{1}{\left(\begin{array}{c}\prod_{\alpha} \prod_{j=\frac{|\alpha(\mathfrak{m})|}{2}+1}^{\infty} \prod_{m=-j}^{j} \left(\left(j+\frac{\alpha(\mathfrak{m})}{2}\right)\left(j-\frac{\alpha(\mathfrak{m})}{2}+1\right)+\alpha(\tilde{\sigma}_0)\alpha(\sigma_0)\right) \\ \times\left(\left(j+\frac{\alpha(\mathfrak{m})}{2}+1\right)\left(j-\frac{\alpha(\mathfrak{m})}{2}\right)+\alpha(\tilde{\sigma}_0)\alpha(\sigma_0)\right)\end{array}\right)^{1/2}} \tag{6.9.39}$$

Since the roots come in $\pm\alpha$ pairs, the product can be simplified as

$$\prod_{\alpha} f(\alpha) = \prod_{\alpha>0} f(\alpha) \prod_{\alpha<0} f(\alpha) = \prod_{\alpha>0} f(\alpha) f(-\alpha). \tag{6.9.40}$$

The ratio of determinants becomes

$$\frac{\det\Delta_{\text{gh}}\det\Delta_{\text{fer}}}{\sqrt{\det\Delta_{\text{bos}}}} = \frac{\begin{array}{c}\prod_{\alpha>0}\prod_{j=\frac{|\alpha(\mathfrak{m})|}{2}+1}^{\infty}\prod_{m=-j}^{j}\left(\left(j+\frac{\alpha(\mathfrak{m})}{2}\right)\left(j-\frac{\alpha(\mathfrak{m})}{2}+1\right)+\alpha(\tilde{\sigma}_0)\alpha(\sigma_0)\right)\\ \times\left(\left(j+\frac{\alpha(\mathfrak{m})}{2}+1\right)\left(j-\frac{\alpha(\mathfrak{m})}{2}\right)+\alpha(\tilde{\sigma}_0)\alpha(\sigma_0)\right)\end{array}}{\left(\begin{array}{c}\prod_{\alpha}\prod_{j=\frac{|\alpha(\mathfrak{m})|}{2}+1}^{\infty}\prod_{m=-j}^{j}\left(\left(j+\frac{\alpha(\mathfrak{m})}{2}\right)\left(j-\frac{\alpha(\mathfrak{m})}{2}+1\right)+\alpha(\tilde{\sigma}_0)\alpha(\sigma_0)\right)\\ \times\left(\left(j+\frac{\alpha(\mathfrak{m})}{2}+1\right)\left(j-\frac{\alpha(\mathfrak{m})}{2}\right)+\alpha(\tilde{\sigma}_0)\alpha(\sigma_0)\right)\end{array}\right)^{1/2}} \tag{6.9.41}$$

$$= 1 \tag{6.9.42}$$

2. For $j = \frac{\alpha(\mathfrak{m})}{2} - 1$ with $\alpha(\mathfrak{m}) \geq 2$, only the $Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}+1}$ harmonic exists. The degeneracy is $\alpha(\mathfrak{m}) - 1$.

$$\det\Delta_{\text{bos}} = \prod_{\alpha}\left(2\alpha(\tilde{\sigma}_0)\alpha(\sigma_0)\right)^{\alpha(\mathfrak{m})-1} \tag{6.9.43}$$

$$\det\Delta_{\text{gh}} = \emptyset \tag{6.9.44}$$

$$\det\Delta_{\text{fer}} = \prod_{\alpha}\left(-\frac{i}{2}\alpha(\tilde{\sigma}_0)\right)^{\alpha(\mathfrak{m})-1} \tag{6.9.45}$$

Together, these are

$$\frac{\det\Delta_{\text{gh}}\det\Delta_{\text{fer}}}{\sqrt{\det\Delta_{\text{bos}}}} = \prod_{\alpha,\alpha(\mathfrak{m})\geq 2}\frac{2^{1-\alpha(\mathfrak{m})}\left(-i\alpha(\tilde{\sigma}_0)\right)^{\alpha(\mathfrak{m})-1}}{\sqrt{2^{\alpha(\mathfrak{m})-1}\left(\alpha(\tilde{\sigma}_0)\alpha(\sigma_0)\right)^{\alpha(\mathfrak{m})-1}}} \tag{6.9.46}$$

(a) When α is a root (i.e. weight of the adjoint representation), with $\alpha(\tilde{\sigma}_0) \neq 0$, $\alpha(\sigma_0) \neq 0$, and there are no generic zero modes.

(b) When α is an element of the Cartan, both the fermionic and bosonic determinants are zero because $0 = \alpha(\tilde{\sigma}_0) = \alpha(\sigma_0)$, and $\sigma_0, \tilde{\sigma}_0 \in \mathfrak{h}$ by the moduli space equations.

3. For $j = -\frac{\alpha(\mathfrak{m})}{2}$ with $\alpha(\mathfrak{m}) \leq -1$, both the $Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}}$ and $Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}-1}$ harmonics

exists. The degeneracy is $1-\alpha(\mathfrak{m})$. The contributions are

$$\det \Delta_{\text{bos}} = \prod_\alpha \left(\frac{\alpha(\mathfrak{m})^2}{8} \left(\alpha(\tilde{\sigma}_0)\alpha(\sigma_0) - \alpha(\mathfrak{m})\right) \right)^{1-\alpha(\mathfrak{m})} \tag{6.9.47}$$

$$\det \Delta_{\text{gh}} = \prod_\alpha \left(-\frac{\alpha(\mathfrak{m})}{2} \right)^{1-\alpha(\mathfrak{m})} \tag{6.9.48}$$

$$\det \Delta_{\text{fer}} = \prod_\alpha \left(2i\alpha(\sigma_0)\right)^{1-\alpha(\mathfrak{m})} \tag{6.9.49}$$

Together, these are

$$\frac{\det \Delta_{\text{gh}} \det \Delta_{\text{fer}}}{\sqrt{\det \Delta_{\text{bos}}}} = \prod_{\alpha, \alpha(\mathfrak{m}) \leq -1} \frac{\left(-i\alpha(\mathfrak{m})\alpha(\sigma_0)\right)^{1-\alpha(\mathfrak{m})}}{\sqrt{8^{\alpha(\mathfrak{m})-1} \left(\alpha(\mathfrak{m})^2 \left(\alpha(\tilde{\sigma}_0)\alpha(\sigma_0) - \alpha(\mathfrak{m})\right)\right)^{1-\alpha(\mathfrak{m})}}} \tag{6.9.50}$$

(a) For $\alpha(\mathfrak{m}) = 0$, only the $Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}}$ harmonic exists. To study this case, we proceed by evaluating the determinants for $\alpha(\mathfrak{m}) \neq 0$, then taking $\alpha(\mathfrak{m}) \to 0$. The determinants are

$$\det \Delta_{\text{bos}} = \prod_\alpha \left(\frac{\alpha(\mathfrak{m})^2}{16} \right)^{1-\alpha(\mathfrak{m})} \underset{\alpha(\mathfrak{m}) \to 0}{\longrightarrow} \prod_\alpha \left(\frac{0}{4} \cdot \frac{0}{4} \right) \tag{6.9.51}$$

$$\det \Delta_{\text{gh}} = \prod_\alpha \left(-\frac{\alpha(\mathfrak{m})}{2} \right)^{1-\alpha(\mathfrak{m})} \underset{\alpha(\mathfrak{m}) \to 0}{\longrightarrow} \prod_\alpha \left(-\frac{0}{2} \right) \tag{6.9.52}$$

$$\det \Delta_{\text{fer}} = \prod_\alpha \left(2i\alpha(\sigma_0)\right)^{1-\alpha(\mathfrak{m})} \underset{\alpha(\mathfrak{m}) \to 0}{\longrightarrow} \prod_\alpha \left(2i\alpha(\sigma_0)\right) \tag{6.9.53}$$

$$\frac{\det \Delta_{\text{gh}} \det \Delta_{\text{fer}}}{\sqrt{\det \Delta_{\text{bos}}}} = \frac{\prod_\alpha \left(-\frac{\alpha(\mathfrak{m})}{2}\right)^{1-\alpha(\mathfrak{m})} \prod_\alpha \left(2i\alpha(\sigma_0)\right)^{1-\alpha(\mathfrak{m})}}{\sqrt{\prod_\alpha \left(\frac{\alpha(\mathfrak{m})^2}{16}\right)^{1-\alpha(\mathfrak{m})}}} \tag{6.9.54}$$

$$\underset{\alpha(\mathfrak{m}) \to 0}{\longrightarrow} \prod_\alpha \frac{0}{\sqrt{0^2}} \left(-4i\alpha(\sigma_0)\right) \tag{6.9.55}$$

Here there is one ghost zero mode, and two bosonic zero modes (one for each $\alpha(\mathfrak{m})^2$) coming from the harmonic expansion of the scalars $\sigma, \tilde{\sigma}$.

i. When α is a root, $\alpha(\tilde{\sigma}_0) \neq 0$, $\alpha(\sigma_0) \neq 0$, and $\alpha(\mathfrak{m}) = 0$, there are two generic bosonic zero modes from $\alpha(\mathfrak{m})^2$, and one ghost zero mode. These bosonic zero modes come from the harmonic expansion of the scalars $\sigma, \tilde{\sigma}$, each of which contributes a factor of $\alpha(\mathfrak{m})$.

ii. When α is an element of the Cartan, all eigenvalues are zero because $0 = \alpha(\tilde{\sigma}_0) = \alpha(\sigma_0) = \alpha(\mathfrak{m})$, and $\sigma_0, \tilde{\sigma}_0, \mathfrak{m} \in \mathfrak{h}$ by the moduli space constraint.

4. For $j = -\frac{\alpha(\mathfrak{m})}{2} - 1$ with $\alpha(\mathfrak{m}) \leq -2$, only the $Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}-1}$ harmonic exists. The degeneracy is $-\alpha(\mathfrak{m}) - 1$. The contributions are

$$\det \Delta_{\text{bos}} = \prod_{\alpha} (2\alpha(\tilde{\sigma}_0)\alpha(\sigma_0))^{-\alpha(\mathfrak{m})-1} \tag{6.9.56}$$

$$\det \Delta_{\text{gh}} = \emptyset \tag{6.9.57}$$

$$\det \Delta_{\text{fer}} = \emptyset \tag{6.9.58}$$

Together, these are

$$\frac{\det \Delta_{\text{gh}} \det \Delta_{\text{fer}}}{\sqrt{\det \Delta_{\text{bos}}}} = \prod_{\alpha, \alpha(\mathfrak{m}) \leq -2} \frac{1}{\sqrt{(2\alpha(\tilde{\sigma}_0)\alpha(\sigma_0))^{-\alpha(\mathfrak{m})-1}}} \tag{6.9.59}$$

(a) When α is a root, with $\alpha(\tilde{\sigma}_0) \neq 0 \neq \alpha(\sigma_0)$, and there are no generic zero modes.

(b) When α is an element of the Cartan, both the fermionic and bosonic determinants are zero because $\alpha(\tilde{\sigma}_0) = 0 = \alpha(\sigma_0)$ because $\sigma, \tilde{\sigma} \in \mathfrak{t}$ by the moduli space equations.

5. For $j = \frac{\alpha(\mathfrak{m})}{2}$ with $\alpha(\mathfrak{m}) \geq 1$, both the $Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}+1}$ and $Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}}$ harmonics exists.. The degeneracy is $\alpha(\mathfrak{m}) + 1$.

$$\det \Delta_{\text{bos}} = \prod_{\alpha} \left(\frac{\alpha(\mathfrak{m})^2}{8} (\alpha(\tilde{\sigma}_0)\alpha(\sigma_0) + \alpha(\mathfrak{m})) \right)^{\alpha(\mathfrak{m})+1} \tag{6.9.60}$$

$$\det \Delta_{\text{gh}} = \prod_{\alpha} \left(\frac{\alpha(\mathfrak{m})}{2} \right)^{\alpha(\mathfrak{m})+1} \tag{6.9.61}$$

$$\det \Delta_{\text{fer}} = \prod_{\alpha} (\alpha(\mathfrak{m}) + \alpha(\tilde{\sigma}_0)\alpha(\sigma_0))^{\alpha(\mathfrak{m})+1} \tag{6.9.62}$$

Together, these are

$$\frac{\det \Delta_{\text{gh}} \det \Delta_{\text{fer}}}{\sqrt{\det \Delta_{\text{bos}}}} = \prod_{\alpha \in \mathfrak{g}, \alpha(\mathfrak{m}) \geq 1} 2^{\frac{1}{2}(\alpha(\mathfrak{m})+1)} \sqrt{(\alpha(\mathfrak{m}) + \alpha(\tilde{\sigma}_0)\alpha(\sigma_0))^{\alpha(\mathfrak{m})+1}} \tag{6.9.63}$$

(a) For $\alpha(\mathfrak{m}) = 0$, only the $Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}}$ harmonic exists. To study this case, we proceed by evaluating the determinants for $\alpha(\mathfrak{m}) \neq 0$, then taking

$\alpha(\mathfrak{m}) \to 0$. The determinants are

$$\det \Delta_{\text{bos}} = \prod_\alpha \left(\frac{\alpha(\mathfrak{m})^2}{16}\right)^{\alpha(\mathfrak{m})+1} \underset{\alpha(\mathfrak{m})\to 0}{\longrightarrow} \prod_\alpha \left(\frac{0}{4}\cdot\frac{0}{4}\right) \tag{6.9.64}$$

$$\det \Delta_{\text{gh}} = \prod_\alpha \left(\frac{\alpha(\mathfrak{m})}{2}\right)^{\alpha(\mathfrak{m})+1} \underset{\alpha(\mathfrak{m})\to 0}{\longrightarrow} \prod_\alpha \left(\frac{0}{2}\right) \tag{6.9.65}$$

$$\det \Delta_{\text{fer}} = \prod_\alpha (2i\alpha(\sigma_0))^{\alpha(\mathfrak{m})+1} \underset{\alpha(\mathfrak{m})\to 0}{\longrightarrow} \prod_\alpha (2i\alpha(\sigma_0)) \tag{6.9.66}$$

$$\frac{\det \Delta_{\text{gh}} \det \Delta_{\text{fer}}}{\sqrt{\det \Delta_{\text{bos}}}} = \frac{\prod_\alpha \left(\frac{\alpha(\mathfrak{m})}{2}\right)^{\alpha(\mathfrak{m})+1} \prod_\alpha (2i\alpha(\sigma_0))^{\alpha(\mathfrak{m})+1}}{\sqrt{\prod_\alpha \left(\frac{\alpha(\mathfrak{m})^2}{16}\right)^{\alpha(\mathfrak{m})+1}}} \tag{6.9.67}$$

$$\underset{\alpha(\mathfrak{m})\to 0}{\longrightarrow} \prod_\alpha \frac{0}{\sqrt{0^2}} (4i\alpha(\sigma_0)) \tag{6.9.68}$$

i. When α is a root, $\alpha(\tilde\sigma_0) \neq 0 \neq \alpha(\sigma_0)$, and $\alpha(\mathfrak{m}) = 0$, there are two generic bosonic zero modes from $\alpha(\mathfrak{m})^2$, and one ghost zero mode. These bosonic zero modes come from the harmonic expansion of the scalars $\sigma, \tilde\sigma$, each of which contributes a factor of $\alpha(\mathfrak{m})$.

ii. When α is an element of the Cartan, all eigenvalues are zero because $\alpha(\tilde\sigma_0) = 0 = \alpha(\sigma_0)$ as well as $\alpha(\mathfrak{m}) = 0$ because $\sigma, \tilde\sigma, \mathfrak{m} \in \mathfrak{h}$ by the moduli space constraint.

To collect the results, we denote the ratio of determinants, evaluated for a particular value of j, by

$$Z^j_{1\text{L}} = \left.\left(\frac{\det \Delta_{\text{gh}} \det \Delta_{\text{fer}}}{\sqrt{\det \Delta_{\text{bos}}}}\right)\right|_j . \tag{6.9.69}$$

Thus, for $\alpha(\mathfrak{m}) \neq 0$, the various contributions are

$$Z_{1\text{L}}^{j=\frac{\alpha(\mathfrak{m})}{2}-1} = \prod_{\alpha,\alpha(\mathfrak{m})\geq 2} \frac{\left(-\frac{i}{2}\alpha(\tilde\sigma_0)\right)^{\alpha(\mathfrak{m})-1}}{\sqrt{(2\alpha(\tilde\sigma_0)\alpha(\sigma_0))^{\alpha(\mathfrak{m})-1}}} \tag{6.9.70}$$

$$Z_{1\text{L}}^{j=-\frac{\alpha(\mathfrak{m})}{2}} = \prod_{\alpha,\alpha(\mathfrak{m})\leq -1} \frac{\left(-\frac{\alpha(\mathfrak{m})}{2}\right)^{1-\alpha(\mathfrak{m})} (2i\alpha(\sigma_0))^{1-\alpha(\mathfrak{m})}}{\sqrt{\left(\frac{\alpha(\mathfrak{m})^2}{8}(\alpha(\tilde\sigma_0)\alpha(\sigma_0) - \alpha(\mathfrak{m}))\right)^{1-\alpha(\mathfrak{m})}}} \tag{6.9.71}$$

$$Z_{1\text{L}}^{j=-\frac{\alpha(\mathfrak{m})}{2}-1} = \prod_{\alpha,\alpha(\mathfrak{m})\leq -2} \frac{1}{\sqrt{(2\alpha(\tilde\sigma_0)\alpha(\sigma_0))^{-\alpha(\mathfrak{m})-1}}} \tag{6.9.72}$$

$$Z_{1\text{L}}^{j=\frac{\alpha(\mathfrak{m})}{2}} = \prod_{\alpha,\alpha(\mathfrak{m})\geq 1} \frac{\left(\frac{\alpha(\mathfrak{m})}{2}\right)^{\alpha(\mathfrak{m})+1} (\alpha(\mathfrak{m}) + \alpha(\tilde\sigma_0)\alpha(\sigma_0))^{\alpha(\mathfrak{m})+1}}{\sqrt{\left(\frac{\alpha(\mathfrak{m})^2}{8}(\alpha(\tilde\sigma_0)\alpha(\sigma_0) + \alpha(\mathfrak{m}))\right)^{\alpha(\mathfrak{m})+1}}} \tag{6.9.73}$$

The contribution from the terms with $|\alpha(\mathfrak{m})| \geq 2$ is

$$\begin{aligned} Z_{1L}^{|\alpha(\mathfrak{m})|\geq 2} &= Z_{1L}^{j=\frac{\alpha(\mathfrak{m})}{2}-1} Z_{1L}^{j=-\frac{\alpha(\mathfrak{m})}{2}-1} \\ &= \prod_{\alpha,\alpha(\mathfrak{m})\leq -2} \frac{1}{\sqrt{(2\alpha(\tilde{\sigma}_0)\alpha(\sigma_0))^{-\alpha(\mathfrak{m})-1}}} \\ &\times \prod_{\alpha,\alpha(\mathfrak{m})\geq 2} \frac{\left(-\frac{i}{2}\alpha(\tilde{\sigma}_0)\right)^{\alpha(\mathfrak{m})-1}}{\sqrt{(2\alpha(\tilde{\sigma}_0)\alpha(\sigma_0))^{\alpha(\mathfrak{m})-1}}} \end{aligned}$$

Switching the sign of $\alpha(\mathfrak{m})$ in the second product, we combine the two expressions to get

$$Z_{1L}^{|\alpha(\mathfrak{m})|\geq 2} = \prod_{\alpha,\alpha(\mathfrak{m})\leq -2} \frac{\left(-\frac{i}{2}\alpha(\tilde{\sigma}_0)\right)^{-\alpha(\mathfrak{m})-1}}{(2\alpha(\tilde{\sigma}_0)\alpha(\sigma_0))^{-\alpha(\mathfrak{m})-1}} \tag{6.9.74}$$

$$= \prod_{\alpha,\alpha(\mathfrak{m})\leq -2} \frac{1}{(4i\alpha(\sigma_0))^{-\alpha(\mathfrak{m})-1}} \tag{6.9.75}$$

Switching to positive roots using $\prod_\alpha f(\alpha) = \prod_{\alpha>0} f(\alpha) f(-\alpha)$, we get for the $\alpha(\mathfrak{m}) \leq -2$ contribution

$$Z_{1L}^{\alpha(\mathfrak{m})\leq -2} = \prod_{\alpha>0} 16\,(-1)^{\alpha(\mathfrak{m})}\,\alpha(\sigma_0)^2 \tag{6.9.76}$$

and similarly for $\alpha(\mathfrak{m}) \geq 2$. So, the contribution for $|\alpha(\mathfrak{m})| \geq 2$ is

$$Z_{1L}^{|\alpha(\mathfrak{m})|\geq 2} = \prod_{\alpha>0} 16\,(-1)^{\alpha(\mathfrak{m})}\,\alpha(\sigma_0)^2. \tag{6.9.77}$$

The contribution from the $|\alpha(\mathfrak{m})| \geq 1$ terms are

$$Z_{1L}^{|\alpha(\mathfrak{m})|\geq 1} = Z_{1L}^{j=\frac{\alpha(\mathfrak{m})}{2}} Z_{1L}^{j=-\frac{\alpha(\mathfrak{m})}{2}} \tag{6.9.78}$$

$$= \prod_{\alpha,\alpha(\mathfrak{m})\geq 1} \frac{\left(\frac{\alpha(\mathfrak{m})}{2}\right)^{\alpha(\mathfrak{m})+1} (\alpha(\mathfrak{m}) + \alpha(\tilde{\sigma}_0)\alpha(\sigma_0))^{\alpha(\mathfrak{m})+1}}{\sqrt{\left(\frac{\alpha(\mathfrak{m})^2}{8}(\alpha(\tilde{\sigma}_0)\alpha(\sigma_0) + \alpha(\mathfrak{m}))\right)^{\alpha(\mathfrak{m})+1}}} \tag{6.9.79}$$

$$\times \prod_{\alpha,\alpha(\mathfrak{m})\leq -1} \frac{\left(-\frac{\alpha(\mathfrak{m})}{2}\right)^{1-\alpha(\mathfrak{m})} (2i\alpha(\sigma_0))^{1-\alpha(\mathfrak{m})}}{\sqrt{\left(\frac{\alpha(\mathfrak{m})^2}{8}(\alpha(\tilde{\sigma}_0)\alpha(\sigma_0) - \alpha(\mathfrak{m}))\right)^{1-\alpha(\mathfrak{m})}}} \tag{6.9.80}$$

Switching the sign of $\alpha(\mathfrak{m})$ in the second product, we combine the above expressions to get

$$Z_{1\text{L}}^{|\alpha(\mathfrak{m})|\geq 1} = \prod_{\alpha,\alpha(\mathfrak{m})\geq 1} \frac{\left(\frac{\alpha(\mathfrak{m})}{2}\right)^{2(\alpha(\mathfrak{m})+1)} (\alpha(\mathfrak{m})+\alpha(\tilde{\sigma}_0)\alpha(\sigma_0))^{\alpha(\mathfrak{m})+1} (2i\alpha(\sigma_0))^{1+\alpha(\mathfrak{m})}}{\left(\frac{\alpha(\mathfrak{m})^2}{8}(\alpha(\tilde{\sigma}_0)\alpha(\sigma_0)+\alpha(\mathfrak{m}))\right)^{\alpha(\mathfrak{m})+1}} \tag{6.9.81}$$

$$= \prod_{\alpha,\alpha(\mathfrak{m})\geq 1} (4i\alpha(\sigma_0))^{1+\alpha(\mathfrak{m})} \tag{6.9.82}$$

Switching to positive roots using $\prod_\alpha f(\alpha) = \prod_{\alpha>0} f(\alpha) f(-\alpha)$, we have that the $\alpha(\mathfrak{m}) \geq 1$ contribution is

$$Z_{1\text{L}}^{\alpha(\mathfrak{m})\geq 1} = \prod_{\alpha>0} 16\,(-1)^{\alpha(\mathfrak{m})}\,\alpha(\sigma_0)^2, \tag{6.9.83}$$

and similarly for $\alpha(\mathfrak{m}) \leq -1$. So, the contribution for $|\alpha(\mathfrak{m})| \geq 1$ is

$$Z_{1\text{L}}^{|\alpha(\mathfrak{m})|\geq 1} = \prod_{\alpha>0} 16\,(-1)^{\alpha(\mathfrak{m})}\,\alpha(\sigma_0)^2. \tag{6.9.84}$$

Thus, the ratio of determinants for $\alpha(\mathfrak{m}) \neq 0$ is

$$Z_{1\text{-loop}}^{\alpha(\mathfrak{m})\neq 0} = \prod_{\alpha>0} 16\,(-1)^{\alpha(\mathfrak{m})}\,\alpha(\sigma_0)^2. \tag{6.9.85}$$

6.9.2 Case 2: Yang-Mills with quadratic potential

Here we compute the 1-loop determinant mode-by-mode, for the case in which the localizing Lagrangian is the Yang-Mills Lagrangian with the quadratic twisted chiral superpotential Lagrangian.

The Localizing Lagrangian together with the gauge-fixing Lagrangian is

$$L\,(t \in \mathbb{R}_{>0}, \tau = 0, \xi = 1, \gamma = 0) \tag{6.9.86}$$

$$= L_{\text{YM}} + tL_{\widetilde{\widetilde{W}}} + L_{\text{gf,gh}}\,(\xi = 1, \gamma = 0) \tag{6.9.87}$$

$$= \frac{1}{h^2}\text{Tr}\bigg(\frac{1}{2}\,(-2if_{1\bar{1}} - it\tilde{\sigma})^2 + \frac{1}{2}\,(D_E + t\tilde{\sigma})^2 \tag{6.9.88}$$

$$+\frac{1}{2}D_\mu\tilde{\sigma}D^\mu\sigma + \frac{1}{8}\,[\sigma,\tilde{\sigma}]^2 + 2i\tilde{\Lambda}_{\bar{1}}D_1\lambda - 2i\Lambda_1 D_{\bar{1}}\tilde{\lambda} \tag{6.9.89}$$

$$-i\tilde{\Lambda}_{\bar{1}}\,[\tilde{\sigma},\Lambda_1] + i\tilde{\lambda}[\sigma,\lambda] + 2t\tilde{\lambda}\lambda \tag{6.9.90}$$

$$+\frac{1}{2}\,(D_\mu a^\mu)^2 + D_\mu\tilde{c}D^\mu c\bigg) \tag{6.9.91}$$

Note that we immediately integrate out the auxiliary field D_E by setting it to its on-shell value in the action.

The bosonic Lagrangian to quadratic order in fluctuations is

$$L = \sum_\alpha \mathrm{Tr}\Bigg(\left(a_{\bar{1}}^{-\alpha}, a_1^{-\alpha}, \sigma^{-\alpha}, \tilde{\sigma}^{-\alpha}\right) \mathcal{M} \begin{pmatrix} a_1^\alpha \\ a_{\bar{1}}^\alpha \\ \sigma^\alpha \\ \tilde{\sigma}^\alpha \end{pmatrix} + \tilde{c}^{-\alpha} \left(-D_\mu D^\mu\right) c^\alpha \Bigg) \tag{6.9.92}$$

$$\mathcal{M} = \begin{pmatrix} -4D_1 D_{\bar{1}} + \alpha(\tilde{\sigma}_0)\alpha(\sigma_0) & 0 & 0 & \left(-i\alpha(\sigma_0) + t\right) D_1 \\ 0 & -4D_{\bar{1}} D_1 + \alpha(\tilde{\sigma}_0)\alpha(\sigma_0) & 0 & \left(-i\alpha(\sigma_0) - t\right) D_{\bar{1}} \\ -i\alpha(\tilde{\sigma}_0) D_{\bar{1}} & -i\alpha(\tilde{\sigma}_0) D_1 & -\frac{1}{8}\alpha(\tilde{\sigma}_0)^2 & -\left\{D_1, D_{\bar{1}}\right\} + \frac{1}{8}\alpha(\tilde{\sigma}_0)\alpha(\sigma_0) \\ tD_{\bar{1}} & -tD_1 & \frac{1}{8}\alpha(\tilde{\sigma}_0)\alpha(\sigma_0) & -\frac{1}{8}\alpha(\sigma_0)^2 + \frac{t^2}{2} \end{pmatrix} \tag{6.9.93}$$

Decomposing $\mathcal{M} = \mathcal{A} + \mathcal{B}$ where $\mathcal{A} = \frac{1}{2}\left(\mathcal{M} + \mathcal{M}^\dagger\right)$ is Hermitian and $\mathcal{B} = \frac{1}{2}\left(\mathcal{M} - \mathcal{M}^\dagger\right)$ is anti-Hermitian, we have for $\mathcal{A} = \mathcal{M} - \mathcal{B}$ the matrix

$$\mathcal{A} = \begin{pmatrix} -4D_1 D_{\bar{1}} + \alpha(\tilde{\sigma}_0)\alpha(\sigma_0) & 0 & \frac{1}{2}i\alpha(\tilde{\sigma}_0) D_1 & \left(-\frac{1}{2}i\alpha(\sigma_0) + t\right) D_1 \\ 0 & -4D_{\bar{1}} D_1 + \alpha(\tilde{\sigma}_0)\alpha(\sigma_0) & \frac{1}{2}i\alpha(\tilde{\sigma}_0) D_{\bar{1}} & \left(-\frac{1}{2}i\alpha(\sigma_0) - t\right) D_{\bar{1}} \\ -\frac{1}{2}i\alpha(\tilde{\sigma}_0) D_{\bar{1}} & -\frac{1}{2}i\alpha(\tilde{\sigma}_0) D_1 & -\frac{\alpha(\tilde{\sigma}_0)^2}{8} & -\frac{\left\{D_1, D_{\bar{1}}\right\}}{2} + \frac{\alpha(\tilde{\sigma}_0)\alpha(\sigma_0)}{8} \\ \left(\frac{1}{2}i\alpha(\sigma_0) + t\right) D_{\bar{1}} & \left(\frac{1}{2}i\alpha(\sigma_0) - t\right) D_1 & \frac{\left\{D_1, D_{\bar{1}}\right\}}{2} + \frac{\alpha(\tilde{\sigma}_0)\alpha(\sigma_0)}{8} & -\frac{\alpha(\sigma_0)^2}{8} + \frac{t^2}{2} \end{pmatrix} \tag{6.9.94}$$

This is half of $\mathcal{M}$, and the operator to be evaluated is $\Delta_{\text{bos}}^{(\alpha)} = 2\mathcal{A}$. The harmonics are

$$Y_{jm}^{r-1}\ \forall j \geq |r-1|\,, \qquad Y_{jm}^{r}\ \forall j \geq |r|\,, \qquad Y_{jm}^{r+1}\ \forall j \geq |r+1| \tag{6.9.95}$$

where $r = -\frac{\alpha(\mathfrak{m})}{2}$. The fields $\left(a_1^\alpha, a_{\bar{1}}^\alpha, \sigma^\alpha, \tilde{\sigma}^\alpha\right)$ are expanded in $\left(Y_{jm}^{r+1}, Y_{jm}^{r-1}, Y_{jm}^{r}, Y_{jm}^{r}\right)$, while the ghosts are Grassmann odd scalars $(c, \tilde{c})$ expanded in Y_{jm}^r reading

$$\left(-2D_1 D_{\bar{1}} - 2D_{\bar{1}} D_1\right) Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}} \tag{6.9.96}$$

The fermionic part of the quadratic Lagrangian is

$$L_{\text{fer}} = \sum_\alpha \mathrm{Tr}\left(\frac{1}{2} \left(2\tilde{\Lambda}_{\bar{1}}^{(-\alpha)}, \tilde{\lambda}^{(-\alpha)}\right) \Delta_{\text{fer}}^{(\alpha)} \begin{pmatrix} 2\Lambda_1^{(\alpha)} \\ \lambda^{(\alpha)} \end{pmatrix} \right) \tag{6.9.97}$$

$$\Delta_{\text{fer}}^{(\alpha)} = -2i \begin{pmatrix} \frac{\alpha(\tilde{\sigma}_0)}{4} & -D_1 \\ D_{\bar{1}} & -\alpha\left(\sigma_0\right) + it \end{pmatrix} \tag{6.9.98}$$

The fermionic fields $\left(2\Lambda_1^{(\alpha)}, \lambda^{(\alpha)}\right)$ are expanded in $\left(Y_{jm}^{r+1}, Y_{jm}^{r}\right)$. The matrix reads

$$\begin{pmatrix} -\frac{i\alpha(\tilde{\sigma}_0)}{2} & 2iD_1^{(-\alpha(\mathfrak{m})/2)} \\ -2iD_{\bar{1}}^{(-\alpha(\mathfrak{m})/2+1)} & 2\left(i\alpha(\sigma_0) + t\right) \end{pmatrix} \begin{pmatrix} Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}+1} \\ Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}} \end{pmatrix}. \tag{6.9.99}$$

The contributions are evaluated using the table for harmonic existence C.1.

For $j \geq \frac{|\alpha(\mathfrak{m})|}{2} + 1$ all harmonics exist, and the determinants are

$$\det \Delta_{\text{bos}} = \prod_{\alpha} \prod_{j=\frac{|\alpha(\mathfrak{m})|}{2}+1}^{\infty} \prod_{m=-j}^{j} \frac{1}{4}\Big(2it\alpha(\tilde{\sigma}_0)^2\alpha(\sigma_0)\left(r_-^4 - r_+^4\right) \tag{6.9.100}$$

$$-4t^2\alpha(\tilde{\sigma}_0)^3\alpha(\sigma_0)\left(\alpha(\tilde{\sigma}_0)\alpha(\sigma_0) + r_-^2 + r_+^2\right) \tag{6.9.101}$$

$$+\left(r_-^2 + r_+^2\right)^2\left(\alpha(\tilde{\sigma}_0)\alpha(\sigma_0) + r_-^2\right)\left(\alpha(\tilde{\sigma}_0)\alpha(\sigma_0) + r_+^2\right)\Big) \tag{6.9.102}$$

$$\det \Delta_{\text{gh}} = \prod_{\alpha} \prod_{j=\frac{|\alpha(\mathfrak{m})|}{2}+1}^{\infty} \prod_{m=-j}^{j} \left(j^2 + j - \frac{\alpha(\mathfrak{m})^2}{4}\right) \tag{6.9.103}$$

$$\det \Delta_{\text{fer}} = \prod_{\alpha} \prod_{j=\frac{|\alpha(\mathfrak{m})|}{2}+1}^{\infty} \prod_{m=-j}^{j} \left(j + \frac{\alpha(\mathfrak{m})}{2}\right)\left(j - \frac{\alpha(\mathfrak{m})}{2} + 1\right) + \alpha(\tilde{\sigma}_0)\left(\alpha(\sigma_0) - it\right) \tag{6.9.104}$$

where $r_\pm = \sqrt{j\,(j+1) - r\,(r \pm 1)}$ for $r = -\frac{\alpha(\mathfrak{m})}{2}$.

Denoting the ratio of determinants, evaluated for a particular value of j, by

$$Z_{\text{1L}}^{j} = \left.\left(\frac{\det \Delta_{\text{gh}} \det \Delta_{\text{fer}}}{\sqrt{\det \Delta_{\text{bos}}}}\right)\right|_{j}, \tag{6.9.105}$$

the repetition of the mode-by-mode analysis of the previous section results in

$$Z_{\text{1L},t\neq 0}^{j=\frac{\alpha(\mathfrak{m})}{2}-1} = \prod_{\alpha,\alpha(\mathfrak{m})\geq 2} \left(\frac{-\frac{i}{2}\alpha(\tilde{\sigma}_0)}{\sqrt{2\alpha(\tilde{\sigma}_0)\alpha(\sigma_0)}}\right)^{\alpha(\mathfrak{m})-1} \tag{6.9.106}$$

$$Z_{\text{1L},t\neq 0}^{j=-\frac{\alpha(\mathfrak{m})}{2}} = \prod_{\alpha,\alpha(\mathfrak{m})\leq -1} \left(-\frac{2\alpha(\mathfrak{m})\left(t + i\alpha(\sigma_0)\right)}{\sqrt{\frac{1}{2}\alpha(\mathfrak{m})^2(\alpha(\tilde{\sigma}_0)\alpha(\sigma_0)-\alpha(\mathfrak{m}))+it\alpha(\mathfrak{m})^2\alpha(\tilde{\sigma}_0) +2t^2\alpha(\tilde{\sigma}_0)^2(\alpha(\mathfrak{m})-\alpha(\tilde{\sigma}_0)\alpha(\sigma_0))}}\right)^{1-\alpha(\mathfrak{m})} \tag{6.9.107}$$

$$Z_{\text{1L},t\neq 0}^{j=-\frac{\alpha(\mathfrak{m})}{2}-1} = \prod_{\alpha,\alpha(\mathfrak{m})\leq -2} \left(\frac{1}{\sqrt{2\alpha(\tilde{\sigma}_0)\alpha(\sigma_0)}}\right)^{-\alpha(\mathfrak{m})-1} \tag{6.9.108}$$

$$Z_{\text{1L},t\neq 0}^{j=\frac{\alpha(\mathfrak{m})}{2}} = \prod_{\alpha,\alpha(\mathfrak{m})\geq 1} \left(\frac{\alpha(\mathfrak{m})\left(\alpha(\mathfrak{m}) + \alpha(\tilde{\sigma}_0)\left(\alpha(\sigma_0) - it\right)\right)}{\sqrt{\frac{1}{2}\alpha(\mathfrak{m})^2(\alpha(\tilde{\sigma}_0)\alpha(\sigma_0)+\alpha(\mathfrak{m}))-it\alpha(\mathfrak{m})^2\alpha(\tilde{\sigma}_0) -2t^2\alpha(\tilde{\sigma}_0)^2(\alpha(\mathfrak{m})+\alpha(\tilde{\sigma}_0)\alpha(\sigma_0))}}\right)^{\alpha(\mathfrak{m})+1} \tag{6.9.109}$$

6.9.2.1 Case 2a: Pfaffian

Here, we compute fermionic 1-loop contributions in terms of a Pfaffian as opposed to a determinant. We consider first the fermionic part of the localizing term

$$L_{\text{loc}}^{\text{fer}} = \frac{1}{h^2}\text{Tr}\left(2i\tilde{\Lambda}_{\bar{1}}D_1\lambda - 2i\Lambda_1 D_{\bar{1}}\tilde{\lambda} - i\tilde{\Lambda}_{\bar{1}}\left[\tilde{\sigma}, \Lambda_1\right] + i\tilde{\lambda}\left[\sigma, \lambda\right] + 2t\tilde{\lambda}\lambda\right) \tag{6.9.110}$$

The Grassmann-odd fermionic fields are contracted with the supersymmetry parameters as

$$\left(\Lambda_1^{(\alpha)},\tilde{\Lambda}_{\bar{1}}^{(\alpha)},\lambda^{(\alpha)},\tilde{\lambda}^{(\alpha)}\right)=\left(\left(\tilde{\zeta}_-\lambda_-\right)_1^{(\alpha)},\left(\zeta_+\tilde{\lambda}_+\right)_{\bar{1}}^{(\alpha)},\left(\tilde{\zeta}_-\lambda_+\right)^{(\alpha)},\left(\zeta_+\tilde{\lambda}_-\right)^{(\alpha)}\right), \tag{6.9.111}$$

which are expanded in the monopole harmonics as $(Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}+1},Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}-1},Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}},Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}})$, respectively.

To compute the determinant from the Pfaffian, the fermionic fluctuation operator needs to be recast as a 4×4 matrix. After integrating by parts, one may express the fermionic part of the localizing term as

$$L_{\rm loc}^{\rm fer} = \frac{1}{h^2}{\rm Tr}\left(2i\tilde{\Lambda}_{\bar{1}}D_1\lambda-2i\Lambda_1D_{\bar{1}}\tilde{\lambda}-i\tilde{\Lambda}_{\bar{1}}\left[\tilde{\sigma},\Lambda_1\right]+i\tilde{\lambda}\left[\sigma,\lambda\right]+2t\tilde{\lambda}\lambda\right) \tag{6.9.112}$$

$$= \frac{1}{h^2}{\rm Tr}\Big(i\tilde{\Lambda}_{\bar{1}}D_1\lambda+i\lambda D_1\tilde{\Lambda}_{\bar{1}}-i\Lambda_1D_{\bar{1}}\tilde{\lambda}-i\tilde{\lambda}D_{\bar{1}}\Lambda_1 \tag{6.9.113}$$

$$-\frac{i}{2}\tilde{\Lambda}_{\bar{1}}\left[\tilde{\sigma},\Lambda_1\right]-\frac{i}{2}\Lambda_1\left[\tilde{\sigma},\tilde{\Lambda}_{\bar{1}}\right]+\frac{i}{2}\tilde{\lambda}\left[\sigma,\lambda\right] \tag{6.9.114}$$

$$+\frac{i}{2}\lambda\left[\sigma,\tilde{\lambda}\right]+t\tilde{\lambda}\lambda-t\lambda\tilde{\lambda}\Big) \tag{6.9.115}$$

In terms of a matrix, this reads

$$L_{\rm fer}^2 = \sum_\alpha {\rm Tr}\left(\Lambda_1^{(-\alpha)},\tilde{\Lambda}_{\bar{1}}^{(-\alpha)},\lambda^{(-\alpha)},\tilde{\lambda}^{(-\alpha)}\right)\Delta_{\rm fer}^{(2)}\begin{pmatrix}\Lambda_1^{(\alpha)}\\ \tilde{\Lambda}_{\bar{1}}^{(\alpha)}\\ \lambda^{(\alpha)}\\ \tilde{\lambda}^{(\alpha)}\end{pmatrix} \tag{6.9.116}$$

$$\Delta_{\rm fer}^{(2)}=\begin{pmatrix}0 & -\frac{i}{2}\alpha(\tilde{\sigma}_0) & 0 & -iD_{\bar{1}}\\ -\frac{i}{2}\alpha(\tilde{\sigma}_0) & 0 & iD_1 & 0\\ 0 & iD_1 & 0 & \frac{i}{2}\alpha(\sigma_0)-t\\ -iD_{\bar{1}} & 0 & \frac{i}{2}\alpha(\sigma_0)+t & 0\end{pmatrix} \tag{6.9.117}$$

Computing the Pfaffian and using $({\rm Pf}A)^2=\det A$, the determinant is

$$\left({\rm Pf}\Delta_{\rm fer}^{(2)}\right)^2 \tag{6.9.118}$$

$$=\prod_\alpha\prod_{j=\frac{|\alpha(\mathfrak{m})|}{2}+1}^{\infty}\prod_{m=-j}^{j}\frac{1}{16}\left(\left(j+\frac{\alpha(\mathfrak{m})}{2}\right)\left(j-\frac{\alpha(\mathfrak{m})}{2}+1\right)+\alpha(\tilde{\sigma}_0)\left(\alpha(\sigma_0)-2it\right)\right) \tag{6.9.119}$$

$$\left(\left(j+\frac{\alpha(\mathfrak{m})}{2}+1\right)\left(j-\frac{\alpha(\mathfrak{m})}{2}\right)+\alpha(\tilde{\sigma}_0)\left(\alpha(\sigma_0)+2it\right)\right) \tag{6.9.120}$$

The case excluding t is obtained by setting $t=0$.

6.9.2.2 Case 2b: Pfaffian (alternative)

Here, we provide a very preliminary sketch for how to compute the fermionic 1-loop contribution as a Pfaffian, in the case where the A-model fermionic scalars $\lambda, \tilde{\lambda}$ are redefined as fermionic scalars η, χ of the standard cohomological multiplet. We redefine the fermionic scalars in $L^{\text{fer}}_{\text{loc}}$ as

$$\eta = -2\left(\lambda + \tilde{\lambda}\right), \quad \chi = \frac{1}{2}\left(\lambda - \tilde{\lambda}\right), \quad \lambda = -\frac{\eta}{4} + \chi, \quad \tilde{\lambda} = -\frac{\eta}{4} - \chi. \tag{6.9.121}$$

The fermionic part of the localizing term then reads

$$\begin{aligned}
L^{\text{fer}}_{\text{loc}} &= \frac{1}{h^2}\text{Tr}\Big(i\tilde{\Lambda}_{\bar{1}}D_1\lambda + i\lambda D_1\tilde{\Lambda}_{\bar{1}} - i\Lambda_1 D_{\bar{1}}\tilde{\lambda} - i\tilde{\lambda}D_{\bar{1}}\Lambda_1 && (6.9.122)\\
&\quad -\frac{i}{2}\tilde{\Lambda}_{\bar{1}}\left[\tilde{\sigma}, \Lambda_1\right] - \frac{i}{2}\Lambda_1\left[\tilde{\sigma}, \tilde{\Lambda}_{\bar{1}}\right] + \frac{i}{2}\tilde{\lambda}\left[\sigma, \lambda\right] + \frac{i}{2}\lambda\left[\sigma, \tilde{\lambda}\right] && (6.9.123)\\
&\quad +t\tilde{\lambda}\lambda - t\lambda\tilde{\lambda}\Big) && (6.9.124)\\
&= \frac{1}{h^2}\text{Tr}\Big(-\frac{i}{4}\eta D_1\Lambda_{\bar{1}} - \frac{i}{4}\Lambda_{\bar{1}}D_1\eta + i\chi D_1\Lambda_{\bar{1}} + i\Lambda_{\bar{1}}D_1\chi && (6.9.125)\\
&\quad +\frac{i}{4}\eta D_{\bar{1}}\Lambda_1 + \frac{i}{4}\Lambda_1 D_{\bar{1}}\eta + i\chi D_{\bar{1}}\Lambda_1 + i\Lambda_1 D_{\bar{1}}\chi && (6.9.126)\\
&\quad -\frac{i}{2}\tilde{\Lambda}_{\bar{1}}\left[\tilde{\sigma}, \Lambda_1\right] - \frac{i}{2}\Lambda_1\left[\tilde{\sigma}, \tilde{\Lambda}_{\bar{1}}\right] + \frac{i}{16}\eta\left[\sigma, \eta\right] - i\chi\left[\sigma, \chi\right] && (6.9.127)\\
&\quad -\frac{t}{2}\eta\chi + \frac{t}{2}\chi\eta\Big) && (6.9.128)
\end{aligned}$$

In terms of a matrix, this reads

$$L^3_{\text{fer}} = \sum_{\alpha}\text{Tr}\left(\Lambda_1^{(-\alpha)}, \tilde{\Lambda}_{\bar{1}}^{(-\alpha)}, \chi^{(-\alpha)}, \eta^{(-\alpha)}\right)\Delta^{(3)}_{\text{fer}}\begin{pmatrix}\Lambda_1^{(\alpha)}\\ \tilde{\Lambda}_{\bar{1}}^{(\alpha)}\\ \chi^{(\alpha)}\\ \eta^{(\alpha)}\end{pmatrix} \tag{6.9.129}$$

$$\Delta^{(3)}_{\text{fer}} = \begin{pmatrix} 0 & -\frac{i}{2}\alpha(\tilde{\sigma}_0) & iD_{\bar{1}} & \frac{1}{4}iD_{\bar{1}}\\ -\frac{i}{2}\alpha(\tilde{\sigma}_0) & 0 & iD_1 & -\frac{1}{4}iD_1\\ iD_{\bar{1}} & iD_1 & -i\alpha(\sigma_0) & \frac{t}{2}\\ \frac{1}{4}iD_{\bar{1}} & -\frac{1}{4}iD_1 & -\frac{t}{2} & \frac{1}{16}i\alpha(\sigma_0)\end{pmatrix} \tag{6.9.130}$$

Making this expression anti-symmetric, expanding in harmonics, then computing the Pfaffian and using $(\mathrm{Pf}A)^2 = \det A$, the determinant is

$$\left(\mathrm{Pf}\Delta_{\mathrm{fer}}^{(3)}\right)^2 = \begin{array}{r} \prod_\alpha \prod_{j=\frac{|\alpha(\mathfrak{m})|}{2}+1}^{\infty} \prod_{m=-j}^{j} \frac{1}{64}\left(\left(j+\frac{\alpha(\mathfrak{m})}{2}\right)\left(j-\frac{\alpha(\mathfrak{m})}{2}+1\right)+\alpha(\tilde\sigma_0)\left(\alpha(\sigma_0)-2it\right)\right) \\ \times\left(\left(j+\frac{\alpha(\mathfrak{m})}{2}+1\right)\left(j-\frac{\alpha(\mathfrak{m})}{2}\right)+\alpha(\tilde\sigma_0)\left(\alpha(\sigma_0)+2it\right)\right) \end{array} \tag{6.9.131}$$

6.9.3 Case 3: Yang-Mills without commutator with quadratic potential

Here, we compute the 1-loop determinant mode-by-mode, for the case in which the localizing Lagrangian is the Yang-Mills Lagrangian with the quadratic twisted chiral superpotential Lagrangian, but without the commutator term for the bosonic scalars σ and $\tilde\sigma$. This is the localizing Lagrangian for the Witten approach to A-model localization.

The Localizing Lagrangian together with the gauge-fixing Lagrangian is

$$\begin{aligned} & L\left(t\in\mathbb{R}_{>0},\tau=1,\xi=1,\gamma=0\right) && (6.9.132)\\ = \; & L_{\mathrm{YM}} + tL_{\widetilde{\widetilde{W}}}^{\mathrm{quad}} - L_{\mathrm{com}} + L_{\mathrm{gf,gh}}\left(\xi=1,\gamma=0\right) && (6.9.133)\\ = \; & \frac{1}{h^2}\mathrm{Tr}\bigg(\frac{1}{2}\left(-2if_{1\bar 1}-it\tilde\sigma\right)^2+\frac{1}{2}\left(D_E+t\tilde\sigma\right)^2 && (6.9.134)\\ & +\frac{1}{2}D_\mu\tilde\sigma D^\mu\sigma + 2i\tilde\Lambda_{\bar 1}D_1\lambda - 2i\Lambda_1 D_{\bar 1}\tilde\lambda && (6.9.135)\\ & -i\tilde\Lambda_{\bar 1}\left[\tilde\sigma,\Lambda_1\right]+\frac{i}{2}\tilde\lambda[\sigma,\lambda]+2t\tilde\lambda\lambda && (6.9.136)\\ & +\frac{1}{2}\left(D_\mu a^\mu\right)^2 + D_\mu\tilde c D^\mu c\bigg) && (6.9.137) \end{aligned}$$

Note that we immediately integrate out the auxiliary field D_E by setting it to it's on-shell value in the action.

The bosonic Lagrangian to quadratic order in fluctuations is

$$L = \sum_\alpha \mathrm{Tr}\left(\left(a_{\bar 1}^{-\alpha}, a_1^{-\alpha}, \sigma^{-\alpha}, \tilde\sigma^{-\alpha}\right)\mathcal{M}\begin{pmatrix} a_1^\alpha \\ a_{\bar 1}^\alpha \\ \sigma^\alpha \\ \tilde\sigma^\alpha \end{pmatrix} + \tilde c^{-\alpha}\left(-D_\mu D^\mu\right)c^\alpha\right) \tag{6.9.138}$$

$$\mathcal{M} = \begin{pmatrix} -4D_1D_{\bar 1}+\alpha(\tilde\sigma_0)\alpha(\sigma_0) & 0 & 0 & \left(-i\alpha(\sigma_0)+t\right)D_1 \\ 0 & -4D_{\bar 1}D_1+\alpha(\tilde\sigma_0)\alpha(\sigma_0) & 0 & \left(-i\alpha(\sigma_0)-t\right)D_{\bar 1} \\ -i\alpha(\tilde\sigma_0)D_{\bar 1} & -i\alpha(\tilde\sigma_0)D_1 & 0 & -\left\{D_1,D_{\bar 1}\right\} \\ tD_{\bar 1} & -tD_1 & 0 & -\frac{t^2}{2} \end{pmatrix} \tag{6.9.139}$$

Decomposing $\mathcal{M} = \mathcal{A} + \mathcal{B}$ where $\mathcal{A} = \frac{1}{2}\left(\mathcal{M} + \mathcal{M}^\dagger\right)$ is Hermitian and $\mathcal{B} = \frac{1}{2}\left(\mathcal{M} - \mathcal{M}^\dagger\right)$ is anti-Hermitian, we have for $\mathcal{A} = \mathcal{M} - \mathcal{B}$ the matrix

$$\mathcal{A} = \begin{pmatrix} -4D_1 D_{\bar{1}} + \alpha(\tilde{\sigma}_0)\alpha(\sigma_0) & 0 & \frac{1}{2}i\alpha(\tilde{\sigma}_0)D_1 & \left(-\frac{1}{2}i\alpha(\sigma_0) + t\right) D_1 \\ 0 & -4D_{\bar{1}}D_1 + \alpha(\tilde{\sigma}_0)\alpha(\sigma_0) & \frac{1}{2}i\alpha(\tilde{\sigma}_0)D_{\bar{1}} & \left(-\frac{1}{2}i\alpha(\sigma_0) - t\right) D_{\bar{1}} \\ -\frac{1}{2}i\alpha(\tilde{\sigma}_0)D_{\bar{1}} & -\frac{1}{2}i\alpha(\tilde{\sigma}_0)D_1 & 0 & -\frac{1}{2}\left\{D_1, D_{\bar{1}}\right\} \\ \left(\frac{1}{2}i\alpha(\sigma_0) + t\right) D_{\bar{1}} & \left(\frac{1}{2}i\alpha(\sigma_0) - t\right) D_1 & \frac{1}{2}\left\{D_1, D_{\bar{1}}\right\} & -\frac{1}{2}t^2 \end{pmatrix} \tag{6.9.140}$$

This is half of $\mathcal{M}$, and the operator to be evaluated is $\Delta^{(\alpha)}_{\text{bos}} = 2\mathcal{A}$. The fermionic part of the quadratic Lagrangian is

$$L_{\text{fer}} = \sum_\alpha \text{Tr}\left(\frac{1}{2}\left(2\tilde{\Lambda}^{(-\alpha)}_{\bar{1}}, \tilde{\lambda}^{(-\alpha)}\right)\Delta^{(\alpha)}_{\text{fer}}\begin{pmatrix} 2\Lambda^{(\alpha)}_1 \\ \lambda^{(\alpha)} \end{pmatrix}\right) \tag{6.9.141}$$

$$\Delta^{(\alpha)}_{\text{fer}} = -2i\begin{pmatrix} \frac{\alpha(\tilde{\sigma}_0)}{4} & -D_1 \\ D_{\bar{1}} & -\frac{1}{2}\alpha(\sigma_0) + it \end{pmatrix} \tag{6.9.142}$$

The fermionic fields $\left(2\Lambda^{(\alpha)}_1, \lambda^{(\alpha)}\right)$ are expanded in $\left(Y^{r+1}_{jm}, Y^r_{jm}\right)$. The matrix reads

$$\begin{pmatrix} -\frac{i}{2}\alpha(\tilde{\sigma}_0) & 2iD_1^{(-\alpha(\mathfrak{m})/2)} \\ -2iD_{\bar{1}}^{(-\alpha(\mathfrak{m})/2+1)} & i\alpha(\sigma_0) + 2t \end{pmatrix}\begin{pmatrix} Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}+1} \\ Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}} \end{pmatrix}. \tag{6.9.143}$$

The contributions are evaluated using the table for harmonic existence C.1.

For $j \geq \frac{|\alpha(\mathfrak{m})|}{2} + 1$ all harmonics exist, and the determinants are

$$\det \Delta_{\text{bos}} = \prod_\alpha \prod_{j=\frac{|\alpha(\mathfrak{m})|}{2}+1}^{\infty} \prod_{m=-j}^{j} \left(j^2 + j - \frac{\alpha(\mathfrak{m})^2}{4}\right)^2 \tag{6.9.144}$$

$$\times\left(j^2 + j - \frac{1}{4}\left(\alpha(\mathfrak{m}) - 2\right)\alpha(\mathfrak{m}) + \alpha(\tilde{\sigma}_0)\alpha(\sigma_0)\right) \tag{6.9.145}$$

$$\times\Bigg(\left(j^2 + j - \frac{1}{4}\alpha(\mathfrak{m})\left(\alpha(\mathfrak{m}) + 2\right) + \alpha(\tilde{\sigma}_0)\alpha(\sigma_0)\right) \tag{6.9.146}$$

$$-it\alpha(\mathfrak{m})\alpha(\tilde{\sigma}_0)^2\alpha(\sigma_0) - t^2\alpha(\tilde{\sigma}_0)^3\alpha(\sigma_0)\Bigg) \tag{6.9.147}$$

$$\det \Delta_{\text{gh}} = \prod_\alpha \prod_{j=\frac{|\alpha(\mathfrak{m})|}{2}+1}^{\infty} \prod_{m=-j}^{j} \left(j^2 + j - \frac{\alpha(\mathfrak{m})^2}{4}\right) \tag{6.9.148}$$

$$\det \Delta_{\text{fer}} = \prod_\alpha \prod_{j=\frac{|\alpha(\mathfrak{m})|}{2}+1}^{\infty} \prod_{m=-j}^{j} \left(j + \frac{\alpha(\mathfrak{m})}{2}\right)\left(j - \frac{\alpha(\mathfrak{m})}{2} + 1\right) + \alpha(\tilde{\sigma}_0)\left(\frac{1}{2}\alpha(\sigma_0) - it\right) \tag{6.9.149}$$

Denoting the ratio of determinants, evaluated for a particular value of j, by

$$Z^j_{1\mathrm{L}} = \left(\frac{\det\Delta_{\mathrm{gh}}\det\Delta_{\mathrm{fer}}}{\sqrt{\det\Delta_{\mathrm{bos}}}}\right)\Bigg|_j, \tag{6.9.150}$$

the repetition of the mode-by-mode analysis of the previous sections results in

$$Z^{j=\frac{\alpha(\mathfrak{m})}{2}-1}_{1\mathrm{L},t\neq0} = \prod_{\alpha,\alpha(\mathfrak{m})\geq2}\left(\frac{-\frac{i}{2}\alpha(\tilde\sigma_0)}{\sqrt{2\alpha(\tilde\sigma_0)\alpha(\sigma_0)}}\right)^{\alpha(\mathfrak{m})-1} \tag{6.9.151}$$

$$Z^{j=-\frac{\alpha(\mathfrak{m})}{2}}_{1\mathrm{L},t\neq0} = \prod_{\alpha,\alpha(\mathfrak{m})\leq-1}\left(-\frac{\sqrt{2}\alpha(\mathfrak{m})\left(2t+i\alpha(\sigma_0)\right)}{\sqrt{\alpha(\mathfrak{m})\left(\substack{-\alpha(\mathfrak{m})^2+\alpha(\mathfrak{m})\alpha(\tilde\sigma_0)(\alpha(\sigma_0)+2it)\\ +2t^2\alpha(\tilde\sigma_0)^2}\right)}}\right)^{1-\alpha(\mathfrak{m})} \tag{6.9.152}$$

$$Z^{j=-\frac{\alpha(\mathfrak{m})}{2}-1}_{1\mathrm{L},t\neq0} = \prod_{\alpha,\alpha(\mathfrak{m})\leq-2}\left(\frac{1}{\sqrt{2\alpha(\tilde\sigma_0)\alpha(\sigma_0)}}\right)^{-\alpha(\mathfrak{m})-1} \tag{6.9.153}$$

$$Z^{j=\frac{\alpha(\mathfrak{m})}{2}}_{1\mathrm{L},t\neq0} = \prod_{\alpha,\alpha(\mathfrak{m})\geq1}\left(\frac{\sqrt{2}\alpha(\mathfrak{m})\left(\alpha(\mathfrak{m})+\frac{1}{2}\alpha(\tilde\sigma_0)\left(\alpha(\sigma_0)-2it\right)\right)}{\sqrt{\alpha(\mathfrak{m})\left(\substack{\alpha(\mathfrak{m})^2+\alpha(\mathfrak{m})\alpha(\tilde\sigma_0)(\alpha(\sigma_0)-2it)\\ -2t^2\alpha(\tilde\sigma_0)^2}\right)}}\right)^{\alpha(\mathfrak{m})+1} \tag{6.9.154}$$

6.9.4 Case 4: Yang-Mills with coefficient on commutator and quadratic potential

Here we compute the 1-loop determinant mode-by-mode, for the case in which the localizing Lagrangian is the Yang-Mills Lagrangian with the quadratic twisted chiral superpotential Lagrangian, and with a coefficient on the commutator term for the bosonic scalars σ and $\tilde\sigma$. This localizing Lagrangian is intermediate between the Witten and follow-your-nose approaches to A-model localization.

The Localizing Lagrangian together with the gauge-fixing Lagrangian is

$$\begin{aligned}
& L\left(t\in\mathbb{R}_{>0},\tau=1-\hat\tau,\xi=1,\gamma=0\right) && (6.9.155)\\
= \;& L_{\mathrm{YM}}+tL^{\mathrm{quad}}_{\widetilde{W}}-(1-\hat\tau)L_{\mathrm{com}}+L_{\mathrm{gf,gh}}\left(\xi=1,\gamma=0\right) && (6.9.156)\\
= \;& \frac{1}{h^2}\mathrm{Tr}\Bigg(\frac{1}{2}\left(-2if_{1\bar1}-it\tilde\sigma\right)^2+\frac{1}{2}\left(D_E+t\tilde\sigma\right)^2 && (6.9.157)\\
& +\frac{1}{2}D_\mu\tilde\sigma D^\mu\sigma+\frac{\hat\tau}{8}\left[\sigma,\tilde\sigma\right]^2 && (6.9.158)\\
& +2i\tilde\Lambda_{\bar1}D_1\lambda-2i\Lambda_1D_{\bar1}\tilde\lambda-i\tilde\Lambda_{\bar1}\left[\tilde\sigma,\Lambda_1\right] && (6.9.159)\\
& +\frac{i}{2}\left(1+\hat\tau\right)\tilde\lambda[\sigma,\lambda]+2t\tilde\lambda\lambda && (6.9.160)\\
& +\frac{1}{2}\left(D_\mu a^\mu\right)^2+D_\mu\tilde cD^\mu c\Bigg) && (6.9.161)
\end{aligned}$$

Here, the commutator $[\sigma,\tilde{\sigma}]^2$ and its superpartner $\frac{i}{2}\tilde{\lambda}[\sigma,\lambda]$ can be turned on or off by setting the coefficient to $\hat{\tau}=1,0$, respectively. Note that $\hat{\tau}=0$ is case 3 and $\hat{\tau}=1$ is case 2. For convenience, we record that the $\delta_A=\delta+\tilde{\delta}$ exact term involving the commutator is $\frac{1}{8}\delta_A\left(i\left(\lambda+\tilde{\lambda}\right)[\sigma,\tilde{\sigma}]\right)=\frac{1}{8}[\sigma,\tilde{\sigma}]^2+\frac{i}{2}\tilde{\lambda}[\sigma,\lambda]$. Note that we immediately integrate out the auxiliary field D_E by setting it to it's on-shell value in the action.

The bosonic Lagrangian to quadratic order in fluctuations is

$$L = \sum_\alpha \mathrm{Tr}\left(\left(a_{\bar{1}}^{-\alpha},a_1^{-\alpha},\sigma^{-\alpha},\tilde{\sigma}^{-\alpha}\right)\mathcal{M}\begin{pmatrix} a_1^\alpha \\ a_{\bar{1}}^\alpha \\ \sigma^\alpha \\ \tilde{\sigma}^\alpha\end{pmatrix}+\tilde{c}^{-\alpha}\left(-D_\mu D^\mu\right)c^\alpha\right) \quad (6.9.162)$$

$$\mathcal{M}=\begin{pmatrix} -4D_1D_{\bar{1}}+\alpha(\tilde{\sigma}_0)\alpha(\sigma_0) & 0 & 0 & \left(-i\alpha(\sigma_0)+t\right)D_1 \\ 0 & -4D_{\bar{1}}D_1+\alpha(\tilde{\sigma}_0)\alpha(\sigma_0) & 0 & \left(-i\alpha(\sigma_0)-t\right)D_{\bar{1}} \\ -i\alpha(\tilde{\sigma}_0)D_{\bar{1}} & -i\alpha(\tilde{\sigma}_0)D_1 & -\frac{\hat{\tau}\alpha(\tilde{\sigma}_0)^2}{8} & -\{D_1,D_{\bar{1}}\}+\frac{\hat{\tau}\alpha(\tilde{\sigma}_0)\alpha(\sigma_0)}{8} \\ tD_{\bar{1}} & -tD_1 & \frac{\hat{\tau}\alpha(\tilde{\sigma}_0)\alpha(\sigma_0)}{8} & -\frac{\hat{\tau}\alpha(\sigma_0)^2}{8}-\frac{t^2}{2}\end{pmatrix} \quad (6.9.163)$$

Decomposing $\mathcal{M}=\mathcal{A}+\mathcal{B}$ where $\mathcal{A}=\frac{1}{2}\left(\mathcal{M}+\mathcal{M}^\dagger\right)$ is Hermitian and $\mathcal{B}=\frac{1}{2}\left(\mathcal{M}-\mathcal{M}^\dagger\right)$ is anti-Hermitian, we have for $\mathcal{A}=\mathcal{M}-\mathcal{B}$ the matrix

$$\mathcal{A}=\begin{pmatrix} -4D_1D_{\bar{1}}+\alpha(\tilde{\sigma}_0)\alpha(\sigma_0) & 0 & \frac{1}{2}i\alpha(\tilde{\sigma}_0)D_1 & \left(-\frac{1}{2}i\alpha(\sigma_0)+t\right)D_1 \\ 0 & -4D_{\bar{1}}D_1+\alpha(\tilde{\sigma}_0)\alpha(\sigma_0) & \frac{1}{2}i\alpha(\tilde{\sigma}_0)D_{\bar{1}} & \left(-\frac{1}{2}i\alpha(\sigma_0)-t\right)D_{\bar{1}} \\ -\frac{1}{2}i\alpha(\tilde{\sigma}_0)D_{\bar{1}} & -\frac{1}{2}i\alpha(\tilde{\sigma}_0)D_1 & -\frac{\hat{\tau}}{8}\alpha(\tilde{\sigma}_0)^2 & -\frac{1}{2}\{D_1,D_{\bar{1}}\}+\frac{\hat{\tau}\alpha(\tilde{\sigma}_0)\alpha(\sigma_0)}{8} \\ \left(\frac{1}{2}i\alpha(\sigma_0)+t\right)D_{\bar{1}} & \left(\frac{1}{2}i\alpha(\sigma_0)-t\right)D_1 & \frac{1}{2}\{D_1,D_{\bar{1}}\}+\frac{\hat{\tau}\alpha(\tilde{\sigma}_0)\alpha(\sigma_0)}{8} & -\frac{\hat{\tau}\alpha(\sigma_0)^2}{8}+\frac{t^2}{2}\end{pmatrix} \quad (6.9.164)$$

which is half of $\mathcal{M}$, and the operator to be evaluated is $\Delta_{\mathrm{bos}}^{(\alpha)}=2\mathcal{A}$. The fermionic part of the quadratic Lagrangian is

$$L_{\mathrm{fer}} = \sum_\alpha \mathrm{Tr}\left(\frac{1}{2}\left(2\tilde{\Lambda}_{\bar{1}}^{(-\alpha)},\tilde{\lambda}^{(-\alpha)}\right)\Delta_{\mathrm{fer}}^{(\alpha)}\begin{pmatrix} 2\Lambda_1^{(\alpha)} \\ \lambda^{(\alpha)}\end{pmatrix}\right) \quad (6.9.165)$$

$$\Delta_{\mathrm{fer}}^{(\alpha)} = -2i\begin{pmatrix} \frac{\alpha(\tilde{\sigma}_0)}{4} & -D_1 \\ D_{\bar{1}} & -\frac{1}{2}\left(1+\hat{\tau}\right)\alpha(\sigma_0)+it\end{pmatrix} \quad (6.9.166)$$

The fermionic fields $\left(2\Lambda_1^{(\alpha)},\lambda^{(\alpha)}\right)$ are expanded in $\left(Y_{jm}^{r+1},Y_{jm}^r\right)$. The matrix reads

$$\begin{pmatrix} -\frac{i}{2}\alpha(\tilde{\sigma}_0) & 2iD_1^{(-\alpha(\mathfrak{m})/2)} \\ -2iD_{\bar{1}}^{(-\alpha(\mathfrak{m})/2+1)} & i\left(1+\hat{\tau}\right)\alpha(\sigma_0)+2t\end{pmatrix}\begin{pmatrix} Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}+1} \\ Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}}\end{pmatrix}. \quad (6.9.167)$$

The contributions are evaluated using the table for harmonic existence C.1.

For $j \geq \frac{|\alpha(\mathfrak{m})|}{2}+1$ all harmonics exist, and the determinants are

$$\det\Delta_{\text{bos}} = \prod_{\alpha}\prod_{j=\frac{|\alpha(\mathfrak{m})|}{2}+1}^{\infty}\prod_{m=-j}^{j}\Bigg[\tag{6.9.168}$$

$$t^2\alpha(\tilde\sigma_0)^3\alpha(\sigma_0)\left((\hat\tau-1)\left(j^2+j-\frac{\alpha(\mathfrak{m})^2}{4}\right)+\hat\tau\alpha(\tilde\sigma_0)\alpha(\sigma_0)\right) \tag{6.9.169}$$

$$+\frac{1}{4}it\alpha(\mathfrak{m})\alpha(\tilde\sigma_0)^2\alpha(\sigma_0)\left(\alpha(\mathfrak{m})^2-4j(j+1)\right)+\frac{1}{16}\left(j^2+j-\frac{\alpha(\mathfrak{m})^2}{4}\right)^2 \tag{6.9.170}$$

$$\times\left(\left(\alpha(\mathfrak{m})-2\right)\alpha(\mathfrak{m})-4\left(j^2+j+\alpha(\tilde\sigma_0)\alpha(\sigma_0)\right)\right) \tag{6.9.171}$$

$$\times\left(\alpha(\mathfrak{m})\left(\alpha(\mathfrak{m})+2\right)-4\left(j^2+j+\alpha(\tilde\sigma_0)\alpha(\sigma_0)\right)\right)\Bigg] \tag{6.9.172}$$

$$\det\Delta_{\text{gh}} = \prod_{\alpha}\prod_{j=\frac{|\alpha(\mathfrak{m})|}{2}+1}^{\infty}\prod_{m=-j}^{j}\left(j^2+j-\frac{\alpha(\mathfrak{m})^2}{4}\right) \tag{6.9.173}$$

$$\det\Delta_{\text{fer}} = \prod_{\alpha}\prod_{j=\frac{|\alpha(\mathfrak{m})|}{2}+1}^{\infty}\prod_{m=-j}^{j}\left(j+\frac{\alpha(\mathfrak{m})}{2}\right)\left(j-\frac{\alpha(\mathfrak{m})}{2}+1\right) \tag{6.9.174}$$

$$-\frac{i}{2}\alpha(\tilde\sigma_0)\left(i\alpha(\sigma_0)\left(1+\hat\tau\right)+2t\right) \tag{6.9.175}$$

Denoting the ratio of determinants, evaluated for a particular value of j, by

$$Z^{j}_{1\text{L}} = \left(\frac{\det\Delta_{\text{gh}}\det\Delta_{\text{fer}}}{\sqrt{\det\Delta_{\text{bos}}}}\right)\Bigg|_{j}, \tag{6.9.176}$$

the repetition of the mode-by-mode analysis of the previous sections results in

$$Z^{j=\frac{\alpha(\mathfrak{m})}{2}-1}_{1\text{L},t\neq0,\hat\tau\neq0} = \prod_{\alpha,\alpha(\mathfrak{m})\geq2}\left(\frac{-\frac{i}{2}\alpha(\tilde\sigma_0)}{\sqrt{2\alpha(\tilde\sigma_0)\alpha(\sigma_0)}}\right)^{\alpha(\mathfrak{m})-1} \tag{6.9.177}$$

$$Z^{j=-\frac{\alpha(\mathfrak{m})}{2}}_{1\text{L},t\neq0,\hat\tau\neq0} = \prod_{\alpha,\alpha(\mathfrak{m})\leq-1}\left(\frac{\left(-\frac{\alpha(\mathfrak{m})}{2}\right)\left(i\alpha(\sigma_0)\left(1+\hat\tau\right)+2t\right)}{\sqrt{\frac{1}{8}\left(\begin{array}{c}-2(\hat\tau-1)t^2\alpha(\mathfrak{m})\alpha(\tilde\sigma_0)^2+4\hat\tau t^2\alpha(\tilde\sigma_0)^3\alpha(\sigma_0)\\ -\alpha(\mathfrak{m})^3+\alpha(\mathfrak{m})^2\alpha(\tilde\sigma_0)(\alpha(\sigma_0)+2it)\end{array}\right)}}\right)^{1-\alpha(\mathfrak{m})} \tag{6.9.178}$$

$$Z^{j=-\frac{\alpha(\mathfrak{m})}{2}-1}_{1\text{L},t\neq0,\hat\tau\neq0} = \prod_{\alpha,\alpha(\mathfrak{m})\leq-2}\left(\frac{1}{\sqrt{2\alpha(\tilde\sigma_0)\alpha(\sigma_0)}}\right)^{-\alpha(\mathfrak{m})-1} \tag{6.9.179}$$

$$Z^{j=\frac{\alpha(\mathfrak{m})}{2}}_{1\text{L},t\neq0,\hat\tau\neq0} = \prod_{\alpha,\alpha(\mathfrak{m})\geq1}\left(\frac{\left(\frac{\alpha(\mathfrak{m})}{2}\right)\left(\alpha(\mathfrak{m})-\frac{i}{2}\alpha(\tilde\sigma_0)\left(i\alpha(\sigma_0)\left(1+\hat\tau\right)+2t\right)\right)}{\sqrt{\frac{1}{8}\left(\begin{array}{c}2(\hat\tau-1)t^2\alpha(\mathfrak{m})\alpha(\tilde\sigma_0)^2+4\hat\tau t^2\alpha(\tilde\sigma_0)^3\alpha(\sigma_0)\\ +\alpha(\mathfrak{m})^3+\alpha(\mathfrak{m})^2\alpha(\tilde\sigma_0)(\alpha(\sigma_0)-2it)\end{array}\right)}}\right)^{\alpha(\mathfrak{m})+1} \tag{6.9.180}$$

We now consider the case in which the commutator of the bosonic scalars and its superpartner are removed by setting $\hat{\tau}=0$. Moreover, we would like to express the generic zero-mode $\tilde{\sigma}_0$ in terms of the GNO quantized flux $\mathfrak{m}=\frac{1}{2\pi}\int_{S^2}\mathrm{d}^2x\sqrt{g}\,(-2if_{1\bar{1}})\in\Lambda_{\text{char}}$ using the localization locus equation $-2if^0_{1\bar{1}}-it\tilde{\sigma}_0=0$. As a preliminary step, we make the replacement $\alpha(\tilde{\sigma}_0)=-\frac{i}{t}\alpha(\mathfrak{m})$. The ratios of determinants become

$$Z^{j=\frac{\alpha(\mathfrak{m})}{2}-1}_{1\text{L},t\neq0,\hat{\tau}=0} = \prod_{\alpha,\alpha(\mathfrak{m})\geq2}\left(\frac{-\frac{1}{2t}\alpha(\mathfrak{m})}{\sqrt{-\frac{2i}{t}\alpha(\mathfrak{m})\alpha(\sigma_0)}}\right)^{\alpha(\mathfrak{m})-1} \tag{6.9.181}$$

$$Z^{j=-\frac{\alpha(\mathfrak{m})}{2}}_{1\text{L},t\neq0\hat{\tau}=0} = \prod_{\alpha,\alpha(\mathfrak{m})\leq-1}\left(\frac{\left(-\frac{\alpha(\mathfrak{m})}{2}\right)(i\alpha(\sigma_0)+2t)}{\sqrt{\frac{\alpha(\mathfrak{m})^2}{8}\left(-\frac{i}{t}\alpha(\mathfrak{m})\alpha(\sigma_0)-\alpha(\mathfrak{m})\right)}}\right)^{1-\alpha(\mathfrak{m})} \tag{6.9.182}$$

$$Z^{j=-\frac{\alpha(\mathfrak{m})}{2}-1}_{1\text{L},t\neq0,\hat{\tau}=0} = \prod_{\alpha,\alpha(\mathfrak{m})\leq-2}\left(\frac{1}{\sqrt{-\frac{2i}{t}\alpha(\mathfrak{m})\alpha(\sigma_0)}}\right)^{-\alpha(\mathfrak{m})-1} \tag{6.9.183}$$

$$Z^{j=\frac{\alpha(\mathfrak{m})}{2}}_{1\text{L},t\neq0,\hat{\tau}=0} = \prod_{\alpha,\alpha(\mathfrak{m})\geq1}\left(\frac{\left(\frac{\alpha(\mathfrak{m})}{2}\right)\left(-\frac{i}{2t}\alpha(\mathfrak{m})\alpha(\sigma_0)\right)}{\sqrt{\frac{\alpha(\mathfrak{m})^2}{8}\left(-\frac{i}{t}\alpha(\mathfrak{m})\alpha(\sigma_0)+\alpha(\mathfrak{m})\right)}}\right)^{\alpha(\mathfrak{m})+1} \tag{6.9.184}$$

6.9.5 Case 5: Degenerate Yang-Mills with quadratic potential

Here, we provide a very preliminary sketch for how to compute 1-loop determinant mode-by-mode, for the case in which the localizing Lagrangian is a degenerate Yang-Mills Lagrangian $L_{\Sigma\tilde{\Sigma}}$ with the quadratic twisted chiral superpotential Lagrangian.

The Localizing Lagrangian together with the gauge-fixing Lagrangian is

$$\begin{aligned}
& L\,(t\in\mathbb{R}_{>0},\xi=1,\gamma=0) && (6.9.185)\\
= \;& L_{\tilde{\Sigma}\Sigma}+tL^{\text{quad}}_{\widetilde{W}}+L_{\text{gf,gh}}\,(\xi=1,\gamma=0) && (6.9.186)\\
= \;& \frac{1}{h^2}\mathrm{Tr}\bigg(+\frac{1}{2}D_\mu\tilde{\sigma}D^\mu\sigma+(f_{12})^2+f_{12}D && (6.9.187)\\
& -it\,(D+f_{12})\,\tilde{\sigma} && (6.9.188)\\
& +2i\tilde{\Lambda}_{\bar{1}}D_1\lambda-2i\Lambda_1D_{\bar{1}}\tilde{\lambda}-i\tilde{\Lambda}_{\bar{1}}\,[\tilde{\sigma},\Lambda_1]+2t\tilde{\lambda}\lambda && (6.9.189)\\
& +\frac{1}{2}\,(D_\mu a^\mu)^2+D_\mu\tilde{c}D^\mu\tilde{c}\bigg) && (6.9.190)
\end{aligned}$$

Completing the square for $f_{12}=-2if_{1\bar{1}}$ and setting $D=iD_E$ we have

$$(f_{12})^2+f_{12}\,(iD_E)-it\,(iD_E+f_{12})\,\tilde{\sigma}=\left(f_{12}+\frac{1}{2}\,(iD_E-it\tilde{\sigma})\right)^2+\frac{1}{4}\,(D_E+t\sigma)^2. \tag{6.9.191}$$

The localizing term then reads

$$L_{\text{loc}} = \frac{1}{h^2}\text{Tr}\bigg(\frac{1}{2}D_\mu\tilde{\sigma}D^\mu\sigma + \Big(f_{12} + \frac{1}{2}\left(iD_E - it\tilde{\sigma}\right)\Big)^2 + \frac{1}{4}\left(D_E + t\tilde{\sigma}\right)^2 \tag{6.9.192}$$

$$+2i\tilde{\Lambda}_{\bar{1}}D_1\lambda - 2i\Lambda_1 D_{\bar{1}}\tilde{\lambda} - i\tilde{\Lambda}_{\bar{1}}\left[\tilde{\sigma}, \Lambda_1\right] + 2t\tilde{\lambda}\lambda\bigg). \tag{6.9.193}$$

The auxiliary field can be integrated out by setting $D_E + t\tilde{\sigma} = 0$, after which the localizing term reads

$$L_{\text{loc}} = \frac{1}{h^2}\text{Tr}\bigg(\frac{1}{2}D_\mu\tilde{\sigma}D^\mu\sigma + \left(f_{12} - it\tilde{\sigma}\right)^2 \tag{6.9.194}$$

$$+2i\tilde{\Lambda}_{\bar{1}}D_1\lambda - 2i\Lambda_1 D_{\bar{1}}\tilde{\lambda} - i\tilde{\Lambda}_{\bar{1}}\left[\tilde{\sigma}, \Lambda_1\right] + 2t\tilde{\lambda}\lambda\bigg) \tag{6.9.195}$$

Taking the $h \to 0$ limit while keeping t fixed the expansion of the term involving f_{12} reads

$$\frac{1}{h^2}\left(f_{12} - it\tilde{\sigma}\right)^2 \tag{6.9.196}$$

$$\underset{\varphi\to\varphi_0+h\varphi}{\longrightarrow} \frac{1}{h^2}\bigg(\partial_1\left(a^0_{\bar{1}} + ha_{\bar{1}}\right) - \partial_{\bar{1}}\left(a^0_1 + ha_1\right) \tag{6.9.197}$$

$$-i\left[a^0_1 + ha_1, a^0_{\bar{1}} + ha_{\bar{1}}\right] - it\left(\tilde{\sigma}_0 + h\tilde{\sigma}\right)\bigg)^2 \tag{6.9.198}$$

$$\underset{h\to 0}{\longrightarrow} -4\left(D^0_1 a_{\bar{1}} - D^0_{\bar{1}}a_1\right)^2 - 2t\left(D^0_1 a_{\bar{1}} - D^0_{\bar{1}}a_1\right)\tilde{\sigma} \tag{6.9.199}$$

$$-2t\tilde{\sigma}\left(D^0_1 a_{\bar{1}} - D^0_{\bar{1}}a_1\right) - t^2\tilde{\sigma}^2 \tag{6.9.200}$$

This term cancels with the gauge fixing Lagrangian

$$L_{\text{gf}} = \left(D_\mu a^\mu\right)^2 = 4\left(D^0_1 a_{\bar{1}} + D^0_{\bar{1}}a_1\right)^2 \tag{6.9.201}$$

Noting that

$$4\left(D^0_1 a_{\bar{1}} + D^0_{\bar{1}}a_1\right)^2 = 4\left(D^0_1 a_{\bar{1}}D^0_1 a_{\bar{1}} + D^0_1 a_{\bar{1}}D^0_{\bar{1}}a_1 + D^0_{\bar{1}}a_1 D^0_1 a_{\bar{1}} + D^0_{\bar{1}}a_1 D^0_{\bar{1}}a_1\right) \tag{6.9.202}$$

The cancellation occurring in $L_{\text{loc}} + L_{\text{gf}}$ is

$$-4\left(D^0_1 a_{\bar{1}} - D^0_{\bar{1}}a_1\right)^2 + 2\left(D^0_1 a_{\bar{1}} + D^0_{\bar{1}}a_1\right)^2 \tag{6.9.203}$$

$$= -\cancel{4D^0_1 a_{\bar{1}}D^0_1 a_{\bar{1}}} + 4D^0_1 a_{\bar{1}}D^0_{\bar{1}}a_1 + 4D^0_{\bar{1}}a_1 D^0_1 a_{\bar{1}} - \cancel{4D^0_{\bar{1}}a_1 D^0_{\bar{1}}a_1} \tag{6.9.204}$$

$$+\cancel{4D^0_1 a_{\bar{1}}D^0_1 a_{\bar{1}}} + 4D^0_1 a_{\bar{1}}D^0_{\bar{1}}a_1 + 4D^0_{\bar{1}}a_1 D^0_1 a_{\bar{1}} + \cancel{4D^0_{\bar{1}}a_1 D^0_{\bar{1}}a_1} \tag{6.9.205}$$

$$= 8D^0_1 a_{\bar{1}}D^0_{\bar{1}}a_1 + 8D^0_{\bar{1}}a_1 D^0_1 a_{\bar{1}} \tag{6.9.206}$$

In particular, the result is

$$\frac{1}{h^2}\left(f_{12} - it\tilde{\sigma}\right)^2 + \left(D_\mu a^\mu\right)^2 \tag{6.9.207}$$

$$\longrightarrow \quad 8D_1^0 a_{\bar{1}} D_{\bar{1}}^0 a_1 + 8D_{\bar{1}}^0 a_1 D_1^0 a_{\bar{1}} - 2t\left(D_1^0 a_{\bar{1}} - D_{\bar{1}}^0 a_1\right)\tilde{\sigma} \tag{6.9.208}$$

$$-2t\tilde{\sigma}\left(D_1^0 a_{\bar{1}} - D_{\bar{1}}^0 a_1\right) - t^2\tilde{\sigma}^2 \tag{6.9.209}$$

The kinetic term goes as

$$\frac{1}{2h^2} D_\mu \tilde{\sigma} D^\mu \sigma \underset{\varphi\to\varphi_0+h\varphi}{\longrightarrow} \frac{1}{2h^2}\Big(h\left(D_\mu^0\tilde{\sigma} - i\left[a_\mu, \tilde{\sigma}_0\right]\right) - ih^2\left[a_\mu, \tilde{\sigma}\right]\Big) \tag{6.9.210}$$

$$\times\Big(h\left(D_0^\mu\sigma - i\left[a^\mu, \sigma_0\right]\right) - ih^2\left[a^\mu, \sigma\right]\Big)\Bigg) \tag{6.9.211}$$

$$\underset{h\to 0}{\longrightarrow} \left(D_1^0\tilde{\sigma} - i\left[a_1, \tilde{\sigma}_0\right]\right)\left(D_{\bar{1}}^0\sigma - i\left[a_{\bar{1}}, \sigma_0\right]\right) \tag{6.9.212}$$

$$+\left(D_{\bar{1}}^0\tilde{\sigma} - i\left[a_{\bar{1}}, \tilde{\sigma}_0\right]\right)\left(D_1^0\sigma - i\left[a_1, \sigma_0\right]\right) \tag{6.9.213}$$

In view of these calculations, the bosonic terms at quadratic order read

$$L_{\text{loc,gf}}^{(2)} = 8D_1^0 a_{\bar{1}} D_{\bar{1}}^0 a_1 + 8D_{\bar{1}}^0 a_1 D_1^0 a_{\bar{1}} - 2t\left(D_1^0 a_{\bar{1}} - D_{\bar{1}}^0 a_1\right)\tilde{\sigma} \tag{6.9.214}$$

$$-2t\tilde{\sigma}\left(D_1^0 a_{\bar{1}} - D_{\bar{1}}^0 a_1\right) - t^2\tilde{\sigma}^2 \tag{6.9.215}$$

$$\left(D_1^0\tilde{\sigma} - i\left[a_1, \tilde{\sigma}_0\right]\right)\left(D_{\bar{1}}^0\sigma - i\left[a_{\bar{1}}, \sigma_0\right]\right) \tag{6.9.216}$$

$$+\left(D_{\bar{1}}^0\tilde{\sigma} - i\left[a_{\bar{1}}, \tilde{\sigma}_0\right]\right)\left(D_1^0\sigma - i\left[a_1, \sigma_0\right]\right) \tag{6.9.217}$$

$$= a_{\bar{1}}\left(-8D_1^0 D_{\bar{1}}^0\right)a_1 + a_1\left(-8D_{\bar{1}}^0 D_1^0\right)a_{\bar{1}} \tag{6.9.218}$$

$$+2t\left(a_{\bar{1}} D_1^0\tilde{\sigma} - a_1 D_{\bar{1}}^0\tilde{\sigma}\right) - 2t\tilde{\sigma}\left(D_1^0 a_{\bar{1}} - D_{\bar{1}}^0 a_1\right) - t^2\tilde{\sigma}^2 \tag{6.9.219}$$

$$\left(D_1^0\tilde{\sigma} - i\left[a_1, \tilde{\sigma}_0\right]\right)\left(D_{\bar{1}}^0\sigma - i\left[a_{\bar{1}}, \sigma_0\right]\right) \tag{6.9.220}$$

$$+\left(D_{\bar{1}}^0\tilde{\sigma} - i\left[a_{\bar{1}}, \tilde{\sigma}_0\right]\right)\left(D_1^0\sigma - i\left[a_1, \sigma_0\right]\right) \tag{6.9.221}$$

The corresponding operator is

$$L_{\text{loc,gf}}^{(2)} = \sum_\alpha \operatorname{Tr}\left(a_{\bar{1}}^{-\alpha}, a_1^{-\alpha}, \sigma^{-\alpha}, \tilde{\sigma}^{-\alpha}\right)\Delta_{\text{bos}}\begin{pmatrix} a_1^\alpha \\ a_{\bar{1}}^\alpha \\ \sigma^\alpha \\ \tilde{\sigma}^\alpha \end{pmatrix} \tag{6.9.222}$$

where

$$\Delta_{\text{bos}} = \begin{pmatrix} -8D_1 D_{\bar{1}} + \alpha(\tilde{\sigma}_0)\alpha(\sigma_0) & 0 & \left(-\frac{1}{2}i\alpha(\sigma_0) + 2t\right)D_1 & \frac{1}{2}i\alpha(\tilde{\sigma}_0)D_1 \\ 0 & -8D_{\bar{1}}D_1 + \alpha(\tilde{\sigma}_0)\alpha(\sigma_0) & \left(-\frac{1}{2}i\alpha(\sigma_0) - 2t\right)D_{\bar{1}} & \frac{1}{2}i\alpha(\tilde{\sigma}_0)D_{\bar{1}} \\ \left(\frac{1}{2}i\alpha(\sigma_0) + 2t\right)D_{\bar{1}} & \left(\frac{1}{2}i\alpha(\sigma_0) - 2t\right)D_1 & -t^2 & -\frac{\{D_1, D_{\bar{1}}\}}{2} \\ -\frac{1}{2}i\alpha(\tilde{\sigma}_0)D_{\bar{1}} & -\frac{1}{2}i\alpha(\tilde{\sigma}_0)D_1 & \frac{\{D_1, D_{\bar{1}}\}}{2} & 0 \end{pmatrix} \tag{6.9.223}$$

If we instead choose to have imaginary $\tilde{\sigma} \to i\eta$, the bosonic terms at quadratic order are

$$L'^{(2)}_{\text{loc,gf}} = 8D^0_1 a_{\bar{1}} D^0_{\bar{1}} a_1 + 8D^0_{\bar{1}} a_1 D^0_1 a_{\bar{1}} \tag{6.9.224}$$
$$-2it\left(D^0_1 a_{\bar{1}} - D^0_{\bar{1}} a_1\right)\eta - 2it\eta\left(D^0_1 a_{\bar{1}} - D^0_{\bar{1}} a_1\right) + t^2\eta^2 \tag{6.9.225}$$
$$i\left(D^0_1\eta - i\left[a_1, \eta_0\right]\right)\left(D^0_{\bar{1}}\sigma - i\left[a_{\bar{1}}, \sigma_0\right]\right) \tag{6.9.226}$$
$$+i\left(D^0_{\bar{1}}\eta - i\left[a_{\bar{1}}, \eta_0\right]\right)\left(D^0_1\sigma - i\left[a_1, \sigma_0\right]\right) \tag{6.9.227}$$

The corresponding operator is

$$L'^{(2)}_{\text{loc,gf}} = \sum_\alpha \text{Tr}\left(a^{-\alpha}_{\bar{1}}, a^{-\alpha}_1, \sigma^{-\alpha}, \tilde{\sigma}^{-\alpha}\right)\Delta'_{\text{bos}}\begin{pmatrix} a^\alpha_1 \\ a^\alpha_{\bar{1}} \\ \sigma^\alpha \\ \tilde{\sigma}^\alpha \end{pmatrix} \tag{6.9.228}$$

where

$$\Delta'_{\text{bos}} = \begin{pmatrix} -8D_1D_{\bar{1}} + i\alpha(\tilde{\sigma}_0)\alpha(\sigma_0) & 0 & \left(\frac{1}{2}\alpha(\sigma_0) + 2it\right)D_1 & -\frac{1}{2}\alpha(\tilde{\sigma}_0)D_1 \\ 0 & -8D_{\bar{1}}D_1 + i\alpha(\tilde{\sigma}_0)\alpha(\sigma_0) & \left(\frac{1}{2}\alpha(\sigma_0) - 2it\right)D_{\bar{1}} & -\frac{1}{2}\alpha(\tilde{\sigma}_0)D_{\bar{1}} \\ \left(-\frac{1}{2}\alpha(\sigma_0) + 2it\right)D_{\bar{1}} & \left(-\frac{1}{2}\alpha(\sigma_0) - 2it\right)D_1 & t^2 & -\frac{i}{2}\left\{D_1, D_{\bar{1}}\right\} \\ \frac{1}{2}\alpha(\tilde{\sigma}_0)D_{\bar{1}} & \frac{1}{2}\alpha(\tilde{\sigma}_0)D_1 & \frac{i}{2}\left\{D_1, D_{\bar{1}}\right\} & 0 \end{pmatrix} \tag{6.9.229}$$

For both cases the fermionic operator is

$$\Delta_{\text{fer}} = \frac{1}{2}\begin{pmatrix} -\frac{i}{2}\alpha(\tilde{\sigma}_0) & 2iD_1^{(-\alpha(\mathfrak{m})/2)} \\ -2iD_{\bar{1}}^{(-\alpha(\mathfrak{m})/2+1)} & 2t \end{pmatrix}\begin{pmatrix} Y^{-\frac{\alpha(\mathfrak{m})}{2}+1}_{jm} \\ Y^{-\frac{\alpha(\mathfrak{m})}{2}}_{jm} \end{pmatrix}. \tag{6.9.230}$$

Denoting the ratio of determinants, evaluated for a particular value of j, by

$$Z^j_{1\text{L}} = \left(\frac{\det\Delta_{\text{gh}}\det\Delta_{\text{fer}}}{\sqrt{\det\Delta_{\text{bos}}}}\right)\Bigg|_j, \tag{6.9.231}$$

the repetition of the mode-by-mode analysis of the previous sections results in

$$Z_{1L}^{j=\frac{\alpha(\mathfrak{m})}{2}-1} = \prod_{\alpha,\alpha(\mathfrak{m})\geq 2} \left(\frac{-\frac{i}{4}\alpha(\tilde{\sigma}_0)}{\sqrt{\alpha(\tilde{\sigma}_0)\alpha(\sigma_0)}}\right)^{\alpha(\mathfrak{m})-1} \tag{6.9.232}$$

$$Z_{1L}^{j=-\frac{\alpha(\mathfrak{m})}{2}} = \prod_{\alpha,\alpha(\mathfrak{m})\leq -1} \left(\frac{(-\alpha(\mathfrak{m}))\,(t)}{\sqrt{\frac{\alpha(\mathfrak{m})}{64}\begin{pmatrix}-2\alpha(\mathfrak{m})^2+\alpha(\mathfrak{m})\alpha(\tilde{\sigma}_0)(\alpha(\sigma_0)-4it)\\+4t^2\alpha(\tilde{\sigma}_0)^2\end{pmatrix}}}\right)^{1-\alpha(\mathfrak{m})} \tag{6.9.233}$$

$$Z_{1L}^{j=-\frac{\alpha(\mathfrak{m})}{2}-1} = \prod_{\alpha,\alpha(\mathfrak{m})\leq -2} \left(\frac{1}{\sqrt{\alpha(\tilde{\sigma}_0)\alpha(\sigma_0)}}\right)^{-\alpha(\mathfrak{m})-1} \tag{6.9.234}$$

$$Z_{1L}^{j=\frac{\alpha(\mathfrak{m})}{2}} = \prod_{\alpha,\alpha(\mathfrak{m})\geq 1} \left(\frac{(\alpha(\mathfrak{m}))\left(\frac{1}{4}\left(\alpha(\mathfrak{m})-it\alpha(\tilde{\sigma}_0)\right)\right)}{\sqrt{\frac{\alpha(\mathfrak{m})}{64}\begin{pmatrix}2\alpha(\mathfrak{m})^2+\alpha(\mathfrak{m})\alpha(\tilde{\sigma}_0)(\alpha(\sigma_0)+4it)\\-4t^2\alpha(\tilde{\sigma}_0)^2\end{pmatrix}}}\right)^{\alpha(\mathfrak{m})+1} \tag{6.9.235}$$

Using the fact that on the locus we have $(f_{12}-it\tilde{\sigma})=0$, we replace $\alpha(\tilde{\sigma}_0)=\frac{i}{t}\alpha(\mathfrak{m})$. Consequently, the 1-loop contributions are

$$Z_{1L}^{j=\frac{\alpha(\mathfrak{m})}{2}-1} = \prod_{\alpha,\alpha(\mathfrak{m})\geq 2} \left(\frac{-\frac{i}{4}\alpha(\tilde{\sigma}_0)}{\sqrt{\alpha(\tilde{\sigma}_0)\alpha(\sigma_0)}}\right)^{\alpha(\mathfrak{m})-1} \tag{6.9.236}$$

$$Z_{1L}^{j=-\frac{\alpha(\mathfrak{m})}{2}} = \prod_{\alpha,\alpha(\mathfrak{m})\leq -1} \left(\frac{(-\alpha(\mathfrak{m}))\,(t)}{\sqrt{\frac{i\alpha(\mathfrak{m})^3(\alpha(\sigma_0)+2it)}{64t}}}\right)^{1-\alpha(\mathfrak{m})} \tag{6.9.237}$$

$$Z_{1L}^{j=-\frac{\alpha(\mathfrak{m})}{2}-1} = \prod_{\alpha,\alpha(\mathfrak{m})\leq -2} \left(\frac{1}{\sqrt{\alpha(\tilde{\sigma}_0)\alpha(\sigma_0)}}\right)^{-\alpha(\mathfrak{m})-1} \tag{6.9.238}$$

$$Z_{1L}^{j=\frac{\alpha(\mathfrak{m})}{2}} = \prod_{\alpha,\alpha(\mathfrak{m})\geq 1} \left(\frac{(\alpha(\mathfrak{m}))\left(\frac{1}{4}\left(\alpha(\mathfrak{m})-it\alpha(\tilde{\sigma}_0)\right)\right)}{\sqrt{\frac{\alpha(\mathfrak{m})^3(2t+i\alpha(\sigma_0))}{64t}}}\right)^{\alpha(\mathfrak{m})+1} \tag{6.9.239}$$

Chapter 7

Discussion & Conclusions

This chapter will discuss and conclude the thesis by summarizing our findings, their relation to the original aims, as well as their significance. It will also review the limitations of our approach and opportunities for future extensions and generalizations.

We studied the failure of supersymmetric localization in the simple context of two-dimensional Yang-Mills (YM2) theories in the A-model (A-twisted $\mathcal{N} = (2, 2)$ supersymmetry on a compact Riemannian manifold Σ_g of genus g), where the conventional follow-your-nose (FYN) approach to localization is known to fail, the Benini-Zaffaroni (BZ) approach to localization is known to succeed, and the Witten approach to localization was expected to succeed.

More generally, the FYN localization is known to fail for A-twisted supersymmetric gauge theories in dimensions $d = 2, 3, 4$. There are at least a few vague explanations for why this happens, including i) FYN localization fails to capture path integral contributions from infinity, ii) FYN localization fails due to the non-compactness of the resulting moduli space; and iii) FYN localization fails because the deformed partition function depends on the deformation term such that the superspace analogue of Stokes's theorem results in non-zero boundary contributions. There are at least two alternative approaches to localization of A-twisted theories that remedy the failure of the FYN approach – these are the BZ and Witten approaches.

The overarching goal was to use the established failure of FYN localization in the case of the A-model to develop criterion that identify the possible failure of FYN localization in other settings. Given a supersymmetric localization computation, the hope was that the criteria would apriori identify the possible failure of FYN localization due to missing path integral contributions from non-BPS saddle point configurations.

The approach to establishing criterion that identify the possible failure of FYN localization, was to begin by studying the specific non-BPS saddle point configurations that spoil FYN localization in the case of YM2 in the A-model on Σ_g. These non-BPS saddle point configurations are the unstable instantons of YM2. In particular, the aim was to understand the role of unstable instantons in the FYN, BZ, and Witten approaches to A-model localization. The unstable instantons are points in the space of field configurations whose path integral contribution is omitted in FYN localization, even though these points are solutions of the Yang-Mills equations. In FYN localization, the unstable instantons are non-supersymmetric in the sense that they do not set the FYN localization scheme, and the fermionic supersymmetry variations of the theory, to zero. We placed a particular emphasis on understanding the features of the unstable instantons at the level of the localization locus and the fluctuation determinant obtained in the FYN, BZ, and Witten approaches to A-model localization. We hoped that, near the instanton configuration, the localization locus and fluctuation determinant would exhibit features that could be used to identify the instanton as a dangerous field configuration that might spoil FYN localization.

In the FYN approach, the localizing term is the standard non-degenerate YM2 action in the A-model. In the Witten approach, the localizing term is a non-standard degenerate YM2 action deformed by a Q_A-exact term. The BZ approach uses the standard non-degenerate YM2 action, but it does not play the role of a conventional localizing term. In the FYN and Witten approaches, the localization locus is evaluated by identifying the bosonic field configurations that set the localizing term to zero along the real contour. The FYN locus excludes instanton configurations, while the Witten locus includes instanton configurations. In the BZ approach, the localization locus is evaluated by identifying the bosonic field configurations that almost set the fermionic supersymmetry variations to zero along the real contour, except for the auxiliary field, which is permitted to remain generically complex. When the auxiliary field is covariantly constant, the BZ locus configuration for the auxiliary field and gauge field strength is related to the instanton configuration by gauge transformations. Simply put, the FYN locus excludes instanton configurations in a straightforward manner, the Witten locus includes instanton configurations in a straightforward manner, and the BZ locus includes instanton configurations in a subtle manner.

The instanton configurations are absolute minima of the Witten localizing term but not the FYN localizing term or the action used in the BZ approach. In the Witten approach, one expects the properties of the instanton configurations to be rather simple – at least for the theory on S^2. On the Witten locus, the fluctuations around the instanton configuration are all massive, and the locus should not exhibit any interesting modes. Moreover, the Witten fluctuation determinant should not involve any “zero-divided-by-zero” cancellations or blow-ups. In the BZ approach, on the other hand, the expectation is that exciting things happen near the instanton configurations, both on the localization locus, and in the fluctuation determinant.

We wrote a general localizing term that includes parameters that interpolate between the localizing terms in the FYN and Witten approaches. Specifically, the parameters interpolate between the standard YM2 action in the A-model and a non-standard YM2 action with a Q_A-exact deformation. The localization locus evaluated from the general localizing term then included parameters that interpolate between the loci of the different approaches. For instance, by varying the parameters one may interpolate between the FYN locus excluding instantons and the Witten locus including instantons. A particular value of the interpolating parameters corresponds to the transition point between including or excluding instanton configurations and between successful or unsuccessful localization. Since the interpolating parameters may be retained in the fluctuation determinant, it is possible to interpolate between the determinants obtained in the different approaches. By interpolating between different approaches to A-model localization, we aimed to identify characteristic features of good and bad localizing terms, localization loci, and fluctuation determinants. Thereafter, the hope was to extend the notion of good and bad approaches to A-model localization to incorporate the supersymmetry algebra and target space.

More concretely, we formulated three objectives in the introduction. The first objective was to confirm the map between physical YM2 and supersymmetric YM2 in the A-model by combining the results of Benini and Zaffaroni (BZ) in [18, 19] with the procedure established by Witten in [3]. In particular, the aim was to recover the physical YM2 partition function from the expectation value of an A-model operator, using the BZ formula for A-model correlators. The second objective was to compare and contrast the FYN, BZ, and Witten approaches to A-model localization. The aim was to localize A-model path integrals according to each of the three approaches, then compare the mode-by-mode evaluation of the one-loop fluctuation determinants. The third objective was to use the comparison of the FYN, BZ, and Witten fluctuation determinants to develop criteria for when supersymmetric localization is expected to fail. In particular, the hope was to establish a criteria, at the level of the supersymmetry algebra or target space, that would identify dangerous localizing terms and field configurations for which the FYN, BZ, and Witten approaches to localization fail.

To realize these objectives, we began by reviewing the necessary background. In chapter 2, we reviewed supersymmetric localization. In chapter 3, we reviewed physical YM2 theories, cohomological YM2 theories, and the non-Abelian localization of path integrals of cohomological gauge theories on compact manifolds in two dimensions. In chapter 4, we reviewed the BZ approach to supersymmetric localization of path integrals of A-twisted supersymmetric gauge theories on compact manifolds in two and three dimensions.

In chapter 5, we succeeded with our first objective by confirming the map between physical YM2 and supersymmetric YM2 in the A-model on Σ_g. Specifically, we recovered the physical YM2 partition function from the expectation value of an A-model operator by evaluating the BZ formula for correlators of the A-twisted $\mathcal{N} = (2,2)$ supersymmetric vector multiplet in Wess-Zumino (WZ) gauge. To provide context, let us recall the results of BZ and the procedure of Witten.

On the one hand, BZ used supersymmetric localization to derive general formulae for partition functions and correlators of A-twisted $\mathcal{N} = (2,2)$ supersymmetric gauge theories on Σ_g. The BZ formula is provided in terms of integrals over two moduli $\mathfrak{m}$ and u, and an integrand that consists of a classical contribution, a one-loop contribution, and possible operator insertions. Here, the discrete modulus $\mathfrak{m}$ is the GNO quantized gauge flux, and the continuous modulus u is the vev of the bosonic scalar in the vector multiplet. The moduli are valued as $\mathfrak{m} \in \Lambda^G_{\text{coch}}$ and $u \in \mathfrak{h}_{\mathbb{C}}$, where Λ^G_{coch} is the cocharacter lattice in $\mathfrak{h}$, and $\mathfrak{h}_{\mathbb{C}}$ is the Cartan subalgebra of the complexification of the Lie algebra $\mathfrak{g}$ of the gauge group G. On the other hand, Witten used non-abelian localization to establish a map between physical YM2 and cohomological YM2 on Σ_g. The map was established by recovering the physical YM2 partition function from the expectation value of a cohomological operator, evaluated by localizing the path integral over the fields of the standard multiplet of two-dimensional cohomological gauge theory.

We confirmed the map between physical YM2 and A-twisted YM2 using the same strategy that established the map between physical YM2 and cohomological YM2. The reason that it was possible to realize Witten's procedure using the BZ formula for A-model correlators is because the A-twisted $\mathcal{N} = (2,2)$ vector multiplet in WZ gauge is related, by field redefinitions, to the standard cohomological multiplet. Recovering the physical YM2 partition function from the expectation value of this A-twisted operator, evaluated using the BZ formula, constitutes the extension of the map between physical YM2 and cohomological YM2 to a map between physical YM2 and A-twisted YM2.

To recover the YM2 partition function from the BZ formula for A-twisted $\mathcal{N} = (2,2)$ correlators, we began by making the following choices. In the integrand, we chose the holomorphic superpotential to be quadratic in its argument $W(u) \propto u^2$, and included the operator insertion $\exp(\frac{1}{2}\varepsilon \text{Tr} u^2)$ for $\varepsilon > 0$. Together, the operator and quadratic superpotential correspond to an A-model action that is equivalent to the conventional pure YM2 action. For the cohomological analogue of this A-model action, see equation 3.42 in [3].

There were two possibilities to evaluate the integrals over moduli: either sum $\mathfrak{m}$ then integrate u, or integrate u then sum $\mathfrak{m}$. For the order of evaluation, we chose to first sum $\mathfrak{m}$, then integrate u along the real contour. Observe that we departed from the BZ prescription by integrating u along the real contour. Specifically, the BZ prescription to evaluate the integral over u is to choose the contour of integration to be the 'Jeffery-Kirwan (JK) contour'. For details regarding the JK contour, see e.g. equation 2.46 in [19].

The sum over $\mathfrak{m} \in \Lambda^G_{\text{coch}}$ was evaluated using the Poisson summation formula, resulting in a periodic δ-function in which the summation variable is $\mu \in \Lambda^G_{\text{ch}}$, where Λ^G_{ch} is the character lattice in $\mathfrak{h}^*$, and the argument of the δ-function includes both u and μ. Then, the integrand of u was a δ-function multiplying a u-dependent function, and the non-zero contributions to the integral over u came from points at which the delta function was supported. Note that integrating u along the real contour was necessary to collapse the δ-function. After summing $\mathfrak{m}$ then integrating u, the expectation value of the A-model operator was an expression involving a sum over $\mu \in \Lambda^G_{\text{ch}}$ and a summand described by the root system of G in $\mathfrak{h}^*$.

By eliminating the Weyl group invariance, the expectation value of the A-model operator was expressed as a sum over dominant elements of the character lattice $\mu \in \Lambda^G_{\text{ch}} \cap C^0$, where C^0 is the fundamental Weyl chamber. Thereafter, the dominant elements of the character lattice were related to the irreducible representations R_μ of G, and the result was the physical YM2 partition function expressed as a sum over $\mu \in \Lambda^G_{\text{ch}} \cap C^0$ and a summand involving the dimension and quadratic Casimir of R_μ. Finally, we matched our result to other YM2 partition functions by identifying the constants by which YM2 partition functions may differ when evaluated using different renormalization schemes.

Our derivation appears to have an implication for the final BZ formula for partition functions (eq. 2.53 in [19]). In particular, the sum in the BZ final formula is over the solutions of the Bethe ansatz equations (BAE), and the prescription is to retain (discard) the subset of BAE solutions for which the Vandermonde determinant is non-zero (zero). In our derivation, after summing $\mathfrak{m}$ then integrating u, the YM2 partition function is expressed as a sum over $\mu \in \Lambda^G_{\text{ch}}$ and a summand involving the Vandermonde determinant. When the gauge group is the universal covering group $G = \tilde{G}$, the sum over $\mu \in \Lambda^{G=\tilde{G}}_{\text{ch}}$ does not receive contributions from the subset of character lattice elements that intersect the union of Weyl chamber boundaries $\Lambda^{G=\tilde{G}}_{\text{ch}} \cap \Pi$. The reason that these elements do not contribute is the Vandermonde determinant in the summand is zero for all $\mu \in \Lambda^{G=\tilde{G}}_{\text{ch}} \cap \Pi$. When the gauge group is some other covering group $G \neq \tilde{G}$, the summation variable $\mu \in \Lambda^{G\neq\tilde{G}}_{\text{ch}}$ is shifted by the Weyl vector ϱ in the summand, and we defined the ϱ-shifted summation variable $q \in Q$ where $Q = \{\mu + \varrho | \mu \in \Lambda^{G\neq\tilde{G}}_{\text{ch}}\}$. In this case, the sum over $q \in Q$ does not receive contributions from the subset of elements $Q \cap \Pi$, because they set the Vandermonde determinant in the summand to zero. The implication for the final BZ partition function formula is that this appears to motivate why some of the BAE solutions should be discarded, and others retained. In particular, our derivation suggests that the subset of BAE solutions for which the Vandermonde determinant is zero (non-zero) should be discarded (retained) because they correspond to the set $\Lambda_{\rm ch}^{G=\tilde{G}} \cap \Pi$ ($\Lambda_{\rm ch}^{G=\tilde{G}} \backslash \Pi$) when $G = \tilde{G}$, or the set $Q \cap \Pi$ ($Q \backslash \Pi$) when $G \neq \tilde{G}$.

In chapter 6, we were partially successful with our second objective of comparing and contrasting the FYN, BZ, and Witten approaches to A-model localization. We successfully evaluated the localization loci for all three approaches, but the comparison of the one-loop fluctuation determinants remains incomplete. We saw that the FYN locus includes flat connections, while the BZ and Witten loci include Yang-Mills connections. The Yang-Mills connections are non-BPS saddle point configurations, whose contribution to the path integral is missed by the FYN approach, but captured by the BZ and Witten approaches. Already at the level of the localization locus, it is clear the FYN approach fails. This is because the final partition function must be a sum over GNO quantized gauge fluxes, and the flux of a Yang-Mills connection is GNO quantized, while the flux of a flat connection is not.

The BZ and Witten approaches capture the contribution of Yang-Mills connections in significantly different manners. The BZ locus is evaluated by setting the fermionic supersymmetry variations to zero, then imposing the real contour on all fields except the auxiliary field D, which is permitted to remain generically complex. When D is covariantly constant, the gauge flux is GNO quantized, and the configuration for the gauge and auxiliary fields on the BZ locus is related to the BPS configuration by gauge transformations. The BZ locus is a non-compact singular superspace, including Yang-Mills connections. In particular, the bosonic part of the BZ locus is non-compact and singular, and the fermionic part of the BZ locus is due to the presence of fermionic scalar zero-modes.

The Witten locus is evaluated by setting the bosonic part of the Witten localizing term to zero and imposing the real contour. The Witten localizing term includes a mixing term for the gauge field strength and a bosonic scalar and a mass term for the fermionic scalars. The Witten locus is a non-singular compact space including Yang-Mills connections. The presence of the Yang-Mills connections is due to the mixing term together with the kinetic term for the bosonic scalars, and the fermionic scalar mass term lifts the fermionic scalar zero-modes.

We provide the complete analysis of the mode-by-mode evaluation of the FYN fluctuation determinant, where we identified two bosonic zero modes and one ghost zero mode. The bosonic zero modes have zero action to quadratic order and finite action to non-quadratic order. Importantly, the bosonic zero-modes cause singularities and their presence constitutes a catastrophe for FYN localization.

The Witten fluctuation determinant was partially evaluated, resulting in ratios of infinite products. To complete the analysis of the Witten determinant, the ratio of infinite products must be reduced, and the precise relationship between the quadratic superpotential parameter t and the singular bosonic zero modes in the FYN determinant must be described. We expect that the catastrophic bosonic zero modes in the FYN determinant are lifted by the parameter t in the Witten determinant.

The analysis of the FYN determinant incorporates aspects the BZ determinant, since both FYN and BZ localization use the standard non-degenerate YM2 action. The critical difference between the FYN and BZ approaches, at the level of the one-loop determinant, is the auxiliary field D. In the analysis of the FYN determinant, we integrated out D at the beginning. In a more complete analysis of the BZ determinant, D would be retained instead of integrated out at the beginning.

The immediate continuation of this research is to complete the mode-by-mode evaluation of the fluctuation determinant including parameters that interpolate between the Witten and FYN approaches. We expect that the factor of t, which tracks the quadratic twisted chiral superpotential in the Witten localizing term, lifts the catastrophic bosonic zero-modes identified in the FYN fluctuation determinant. Thereafter, the objective is to identify characteristic features of good and bad approaches to A-model localization at the level of localizing terms, localization loci, and fluctuation determinants. Following this, the hope is to describe good and bad approaches to A-model localization and dangerous field configurations at the level of the supersymmetry algebra and target space.

A description of the failure of FYN localization in the A-model would lay the foundation for the development of criterion that identify the possible failure of FYN localization in other settings. The first extension would be to understand the simplest setting in which the FYN approach to supersymmetric localization might fail to capture the path integral contribution of non-BPS saddle point configurations. Presumably, the simplest setting to consider the failure of FYN localization would be supersymmetric quantum mechanics (SQM). SQM is a one-dimensional quantum field theory in which the fields are maps from spacetime to a target space. It is relatively straightforward to sketch a setup in SQM that exhibits similar features to YM2, namely, exact minima and saddle point configurations.

For instance, consider SQM for fields that are maps from a spacetime S^1 to a target space S^2. In this case, the FYN localization locus is the space of trivial (contractible) loops on S^2, where the loops are the images of S^1 in S^2. The trivial loops are exact minima of the action. If however, one loop contracts to a point at the north pole, and another loop contracts to a point at the south pole, then there is a path between two minima of the action. If one has a smooth function along a path, and the path interpolates between two minima, then somewhere along the path there exists a saddle point of the function. The saddle point between minima of the action at the north and south poles is the loop that wraps the equator of the S^2. Like a rubber band sliding down a ball, the loop grows as it goes from the north pole to the equator, then shrinks as it goes from the equator to the south pole. As the loop grows, it costs more action, and as it shrinks, it costs less action. At the equator, the loop is a saddle point of the action.

The objective would be to identify a setting in which unstable saddle point configurations contribute to SQM localization in a similar manner to how unstable instantons contribute to localization of YM2 theories in the A-model. The appearance of these contributions to SQM localization could be motivated by describing the features of the fluctuations around the unstable saddle point configurations. Although the general expectation in SQM is that unstable saddle point configurations do not contribute to localization, the possibility of such contributions is not omitted. It would be interesting to understand how this relates to the SQM localization results of [44].

Having identified and characterized unstable saddle point contributions in SQM localization, the next step would be to extend the results to higher dimensional supersymmetric indicies. Since supersymmetric indices are partition functions of theories defined on compact manifolds of the form $S^1 \times (\dots)$, they exhibit similar features to SQM. In particular, supersymmetric indies may be regarded as partition functions of theories that are variants of SQM with incredibly complicated target spaces in which the time coordinate wraps the S^1. In particular, the hope would be to take a simple example of unstable saddle point contributions in SQM localization and extend it to the much broader context of unstable saddle point contributions to supersymmetric indices evaluated by localization.

A separate possibility for future research would be to study the role and necessity of Jeffrey-Kirwan residues in the BZ approach to localization [18, 19, 20, 16, 17, 45, 44]. In this case, one would extend our comparison of the Witten and BZ approaches to A-model localization with an emphasis on understanding the features of JK-aided localization. Since the JK contour is not understood in general [38], an important objective would be to understand precisely how the JK contour captures the correct contributions to the A-model path integrals.

A direct extension of chapter 5 would be to repeat the derivation with the integrals over moduli evaluated in the opposite order. Specifically, one would repeat the recovery of the physical YM2 partition function, using the BZ formula for A-twisted $\mathcal{N} = (2, 2)$ correlators, by first integrating the continuous modulus u along the JK contour then summing the discrete modulus $\mathfrak{m}$. Thereafter, the derivation would be compared to the derivation in chapter 5, where we chose to first sum $\mathfrak{m}$ using the Poisson summation formula, then integrate u along the real contour. This would provide a simple and well-established setting in which the JK contour could be compared to a detailed derivation. In this case, it would be interesting to understand the precise relationship between the elements of Λ_{ch} and the solutions of the BAE.

A direct extension of chapter 6 would be to make a full comparison of the fluctuation determinants in the BZ and Witten approaches to A-model localization. It would be particularly interesting to elaborate on the BZ argument for the GNO quantization of the flux, and understand precisely how the covariantly constant generically complex auxiliary field captures unstable instanton configurations. The BZ approach to localization involves several specific choices, as well as the introduction of an additional JK parameter $\hat{\eta} \in \mathfrak{h}^*$. To correctly evaluate JK residues, it is necessary to specify a value for $\hat{\eta}$, which amounts to an arbitrary choice of positive direction in $\mathfrak{h}^*$. The JK residue is used in [18, 19, 20] under the assumption that the charges are projective, that is, the charges all lie in one half-plane of $\mathfrak{h}^*$. It was suggested in [45], that if the charges are not projective, the JK residue may be evaluated using two JK parameters $\hat{\eta}', \hat{\eta}'' \in \mathfrak{h}^*$ pointing into opposite half-planes of $\mathfrak{h}^*$. It would be interesting to understand whether the JK residues in the A-model may be evaluated for non-projective charges using two JK parameters.

Another possibility for future research would be to generalize the Witten approach to supersymmetric localization. We studied the Witten approach to supersymmetric localization by constructing a specific Witten localizing term from the fields of the A-twisted $\mathcal{N} = (2, 2)$ vector multiplet in WZ gauge on Σ_g. In particular, the Witten localizing term is the A-twisted $\mathcal{N} = (2.2)$ supersymmetric analog of the deformed Donaldson action in two-dimensional cohomological gauge theory (eq. 3.27 in [3]). The construction of Witten-type localizing terms from the fields of other supermultiplets remains unclear. It would be interesting to understand the extent to which the Witten approach to localization generalizes to other supersymmetric gauge theories. Specifically, when is the Witten approach to localization appropriate, and how can a Witten-type localizing term be constructed from the fields of supermultiplets other than the A-twisted $\mathcal{N} = (2.2)$ vector multiplet in WZ gauge?

A natural starting point to address these questions would be to contrast Witten localization with the current collection of examples of BZ localization, that is, JK-residue aided supersymmetric localization. The collection includes examples of BZ localization for A-twisted $\mathcal{N} = (2, 2)$, $\mathcal{N} = 2$, $\mathcal{N} = 1$ supersymmetric gauge theories on Σ_g, $\Sigma_g \times S^1$, $\Sigma_g \times T^2$ [18, 19], and Ω-deformed $\mathcal{N} = (2, 2)$ supersymmetric gauge theories on S^2 [20]. In the case of A-twisted $\mathcal{N} = (2, 2)$ theories on Σ_g, there are at least two unanswered questions.

The first question is whether it is possible to realize an analog of the Witten localizing term without imposing the WZ gauge condition on the A-twisted $\mathcal{N} = (2.2)$ vector multiplet. More generally, what is the role of the WZ gauge choice in the failure of FYN localization and the success of both BZ and Witten localization?

To address this question, one could begin by A-twisting $\mathcal{N} = (2,2)$ supersymmetry without imposing WZ gauge on the vector multiplet. In this case, the lower components in the θ-expansion of the vector superfield $V_{(2,2)}$ would be retained instead of eliminated by choosing WZ gauge. To proceed, one would define the twisted chiral superfields $\Sigma, \overline{\Sigma}$ in the standard way, that is, by acting on the vector superfield $V_{(2,2)}$ with the superspace derivatives $D_\pm, \overline{D}_\pm$. Then, the BZ and FYN localizing terms would correspond to the standard D-term Lagrangian $L_{\overline{\Sigma}\Sigma} = \int \mathrm{d}^4\theta \overline{\Sigma}\Sigma$. The Witten localizing term, on the other hand, is not as obvious. A first attempt at defining the Witten localizing term would be along the lines of $L_{\mathrm{Wtn}} = L_{\overline{\Sigma}\Sigma} + L^{\mathrm{quad}}_{\mathrm{tw.ch.sp}} + L_{\mathrm{cm}}$. Here, the F-term $L^{\mathrm{quad}}_{\mathrm{tw.ch.sp}}$ denotes the generalization of the quadratic twisted chiral superpotential, and L_{cm} denotes the generalization of the Q_A-exact term that, in WZ gauge, removed the bosonic scalar commutator term from $L_{\overline{\Sigma}\Sigma}$. Another subtlety to address would be the treatment of the additional vector multiplet component fields that were not present in WZ gauge.

The second question is whether the Witten approach may be realized when localizing A-twisted $\mathcal{N} = (2.2)$ supersymmetric gauge theories of vector and chiral multiplets. For instance, one may consider a localizing term $L = L_{\mathrm{Wtn}} + L_{\overline{\Phi}\Phi} + L_{\mathrm{ch.sp}}$, where $L_{\overline{\Phi}\Phi}$ and $L_{\mathrm{ch.sp}}$ are respectively the D- and F-term Lagrangians for the chiral superfields $\Phi, \overline{\Phi}$, and $L_{\mathrm{Wtn}} = L_{\overline{\Sigma}\Sigma} + L^{\mathrm{quad}}_{\mathrm{tw.ch.sp}} + L_{\mathrm{cm}}$ is the Witten localizing term described above. It is not immediately obvious how the inclusion of the chiral multiplet affects the Witten approach to localization. Presumably, the localization locus would include singular points at which the chiral multiplets become massless, and the singular chiral 1-loop determinant would require regulation. The BZ approach treats these chiral singularities by retaining a generic complex auxiliary field D, but the Witten approach appears to omit this option. It would be interesting to understand whether an analog of the Witten localizing term can be constructed from the fields of the chiral multiplet.

Another question is whether an analog of the Witten approach can be realized for the Ω-deformed $\mathcal{N} = (2.2)$ supersymmetric vector multiplet on S^2. The Ω-deformed vector multiplet is a one-parameter deformation of the A-twisted vector multiplet, and it is unclear whether the Ω-deformation parameter ϵ_Ω spoils the Witten approach to localization. If an Ω-deformed analog of the Witten localizing term can be defined, the Witten approach could be compared to several examples of BZ localization of Ω-deformed theories [18, 19, 20].

Similar questions extend to three-dimensional A-twisted $\mathcal{N} = 2$ supersymmetric gauge theories on $\Sigma_g \times S^1$, and four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theories on $\Sigma_g \times T^2$. For instance, what are the three- and four-dimensional analogs of the two-dimensional Witten localizing term L_{Wtn}? To answer this, one might attempt to invert the dimensional reduction map, provided in appendix B of [18], that relates the A-twisted vector multiplets in two and three dimensions. One could attempt to identify the four-dimensional analog of the Witten localizing term in a similar manner. It is not obvious that such dimensional uplifts would correspond to higher dimensional analogues of the two-dimensional Witten localizing term. Presumably, the Witten approach to localization of three- and four-dimensional A-twisted theories of vector multiplets will not work in precisely the same way as in the two-dimensional case.

Further research directions include incorporating Higgs branch localization into our description of A-model localization, or using localization to evaluate supersymmetric observables. For instance, it would be interesting to understand how Higgs branch localization fits into our comparison of the FYN, BZ, and Witten approaches to localization of YM2 theories in the A-model. In [20], Higgs branch localization was compared to BZ localization for Ω-deformed $\mathcal{N} = (2,2)$ supersymmetric gauge theories on S^2. Higgs branch localization differs significantly from the BZ approach to localization, but some features appear to overlap with the Witten approach to localization. As a first step, one could repeat Higgs branch localization for the case of the A-twisted vector multiplet on S^2 by turning off the Ω-deformation parameter ϵ_Ω. Along the way, one could attempt to recover the vortex equations. Then it would be possible to compare the Higgs branch, FYN, BZ, and Witten approaches to A-model localization, and generalizations thereof.

A standard question in supersymmetric localization is whether there are exciting observables to compute. A typically interesting quantity to evaluate using localization is the expectation value of Wilson loops (WLs). For the result of localization to be non-trivial, the WL operator insertion should be $\mathcal{Q}$-closed, but not $\mathcal{Q}$-exact, where $\mathcal{Q}$ is the localizing supercharge. To understand whether the A-twisted $\mathcal{N} = (2,2)$ vector multiplet permits any interesting WL operator insertions, we investigated the fate of the untwisted $\mathcal{N} = (2,2)$ WL operators of [46] after A-twisting. We concluded that the A-twisted $\mathcal{N} = (2,2)$ vector multiplet does not permit the $\mathcal{N} = (2,2)$ WL operators of [46], because it is not possible to formulate a counterterm to cancel the Q_A-variation of the gauge field A_μ in the A-twisted theory. However, we realized that at least one of the $\mathcal{N} = (2,2)$ WL operators of [46] appears to survive under the A/2-twist of $\mathcal{N} = (2,2)$ supersymmetry. In view of this, it would be interesting to identify this A/2-twisted WL operator, investigate its properties, and hopefully provide a geometric interpretation of its expectation value. Another extension of [46] is to understand the fate of the $\mathcal{N} = (2,2)$ WL operators when relaxing the WZ gauge condition imposed on the $\mathcal{N} = (2,2)$ vector multiplet. In this case, the hope would be to define an $\mathcal{N} = (2,2)$ supersymmetric WL operator outside of WZ gauge.

Although there remain many possibilities beyond the future directions mentioned, let us restrict the discussion to three final points of interest. It would be interesting to understand if our study can be extended to incorporate higher-form symmetries, test dualities of A-twisted $\mathcal{N} = (2, 2)$ theories, or evaluate interesting exact results on the A-twisted $\mathcal{N} = (2, 2)$ squashed two-sphere, or hemisphere.

Appendix A

$\mathcal{N} = (2, 2)$ supersymmetry

In this chapter, we review 2,2 supersymmetry. This begins with conventions for spinors and coordinates. Then, we describe the Lorentzian 2,2 superalgebra , superspace, and the vector multiplet in WZ gauge. Following this, we describe the Euclidean vector multiplet and its actions. Thereafter, we describe the topological A-twist of the vector multiplet. The A-twisted vector multiplet is then related to the standard multiplet of two-dimensional cohomological Yang-mills using field redefinitions. Finally, we summarize the different localizing terms used in [18, 19, 3] in terms of the standard multiplet.

A.1 Conventions

The conventions are similar to [9, 47, 48, 23, 22]. Spinor indices are denoted using Greek letters α, β, taking values $+, -$. Here, $\pm$ indicates chirality under Lorentz transformations. The values, "$+$" and "$-$" may be referred to as right- and left-moving, respectively. Weyl spinors are decomposed as

$$\psi_\alpha = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \tag{A.1.1}$$

Spinor indices are raised and lowered as

$$\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta,\ \psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta \tag{A.1.2}$$

where ε denotes the antisymmetric tensor

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.1.3}$$

Accordingly, we have

$$\psi_+ = \psi^-,\ \psi_- = -\psi^+,\ \psi^\alpha\chi_\alpha = +\psi_+\chi_- - \psi_-\chi_+ \tag{A.1.4}$$

In two-dimensional Lorentzian flat space the coordinates are $x^\mu, \mu = 0, 1$, with metric $\eta_{\mu\nu} = \text{diag}\,(-1, 1)$. Spacetime derivatives are $\partial_\mu = \partial/\partial x^\mu$, and the integration measure is $\mathrm{d}^2x_{(L)} = \mathrm{d}x^0\mathrm{d}x^1$. Light cone coordinates are $x^\pm = x^0 \pm x^1$, while derivatives are $\partial_\pm = \partial_0 \pm \partial_1$. An arbitrary vector field v_μ can therefore be expressed in light-cone coordinates as $v_\pm = v_0 \pm v_1$.

Euclidean flat space is obtained by Wick rotation of the Lorentzian time coordinate $x^0 \to -ix^2$, such that the derivative with respect to time changes as $\partial_0 \to i\partial_2$, and similarly for a vector field $v_0 \to iv_2$. The Euclidean metric is $\delta_{\mu\nu} = \text{diag}\,(1, 1)$. The Euclidean integration measure is obtained by Wick rotation of the Lorentzian measure $\mathrm{d}^2x_{(L)} = \mathrm{d}x^0\mathrm{d}x^1 \to -i\mathrm{d}x^1\mathrm{d}x^2 = \mathrm{d}^2x_{(E)}$. In Euclidean space, we have holomorphic coordinates

$$z = x^1 + ix^2,\ \overline{z} = x^1 - ix^2, \tag{A.1.5}$$

as well as their derivatives

$$\partial_z = \frac{1}{2}\left(\partial_1 - i\partial_2\right),\ \partial_{\overline{z}} = \frac{1}{2}\left(\partial_1 + i\partial_2\right). \tag{A.1.6}$$

An arbitrary Euclidean vector field v_μ can be expressed in holomorphic coordinates as

$$v_z = \frac{1}{2}\left(v_1 - iv_2\right),\ v_{\overline{z}} = \frac{1}{2}\left(v_1 + iv_2\right). \tag{A.1.7}$$

By Wick rotating expressions in light-cone coordinates we can obtain expressions holomorphic coordinates $x^+ \to \overline{z},\ x^- \to z$, while for vectors we have

$$\partial_0 + \partial_1 \to 2\partial_{\overline{z}},\ \partial_0 - \partial_1 \to -2\partial_z. \tag{A.1.8}$$

The gauge conventions are as follows. For the gauge field A_μ, the field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\left[A_\mu, A_\nu\right] \tag{A.1.9}$$

and the gauge covariant derivative is

$$D_\mu = \partial_\mu - iA_\mu. \tag{A.1.10}$$

Together, these satisfy

$$F_{\mu\nu} = i\left[D_\mu, D_\nu\right]. \tag{A.1.11}$$

In particular, these are the physics conventions, in which the Lie algebra of the gauge group is taken to consist of Hermitian matrices. In the physics conventions, the bilinear form on the Lie algebra is positive semi-definite $\mathrm{Tr}\,(a,b) \geq 0$.

When switching between holomorphic and Euclidean coordinates the covariant derivatives and field strengths obey

$$[D_z, D_{\overline{z}}] = iF_{z\overline{z}}, \quad D_\mu D^\mu = 2\{D_z, D_{\overline{z}}\}, \quad F_{12} = -2iF_{z\overline{z}} \tag{A.1.12}$$

$$D_z D_{\overline{z}} = \frac{1}{2}\{D_z, D_{\overline{z}}\} + \frac{1}{2}[D_z, D_{\overline{z}}] = \frac{1}{4}(D_\mu D^\mu + F_{12}) \tag{A.1.13}$$

$$D_{\overline{z}} D_z = \frac{1}{2}\{D_{\overline{z}}, D_z\} + \frac{1}{2}[D_{\overline{z}}, D_z] = \frac{1}{4}(D_\mu D^\mu - F_{12}) \tag{A.1.14}$$

We also use the math conventions, in which the Lie algebra of the gauge group is taken to consist of anti-Hermitian matrices. To go from the physics conventions to the math conventions, one replaces $A = iA'$ and $F = iF'$, where A', F' are represented by anti-Hermitian matrices. In the math conventions, the field strength is

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu + \left[A'_\mu, A'_\nu\right], \tag{A.1.15}$$

and the gauge covariant derivative is

$$D_\mu = \partial_\mu + A'_\mu. \tag{A.1.16}$$

Together, these satisfy,

$$F'_{\mu\nu} = [D_\mu, D_\nu]. \tag{A.1.17}$$

In the math convention, the bilinear form on the Lie algebra of the gauge group is negative semi-definite $\mathrm{Tr}\,(a', b') \leq 0$.

A.2 Lorentzian signature

In $d = 1{+}1$, an $\mathcal{N} = (2,2)$ supersymmetric theory is a QFT with a $\mathbb{Z}_2$ graded Hilbert space $\mathcal{H} = \mathcal{H}^B \oplus \mathcal{H}^F$. The even operators are the Hamiltonian H, momentum operator P, and the generator of Lorentz transformations M. Together, H and P generate spacetime translations. The generators of spacetime translations are Hermitian $H = H^\dagger, P = P^\dagger$, while the generator of Lorentz transformations is anti-hermitian $M = M^\dagger$. The even operators act independently on the bosonic and fermionic Hilbert spaces $\mathcal{H}^B \circlearrowleft$ and $\mathcal{H}^F \circlearrowleft$.

The odd operators are the supercharges $Q_+, Q_-, \overline{Q}_+, \overline{Q}_-$, where $\pm$ denotes chirality under Lorentz transformations. The supercharges are hermitian $\overline{Q}_\pm = Q_\pm^\dagger$. The odd operators act between the bosonic and fermionic Hilbert spaces $\mathcal{H}^B \leftrightarrows \mathcal{H}^F$.

There are two optional even operators. The first is the generator of the vector-like $U(1)_V$ R-symmetry, denoted F_V, the second is the generator of the axial $U(1)_A$ R-symmetries, denoted F_A. These have the property $(-1)^{F_V} = \pm(-1)^{F_A} = \pm 1$.

The operator algebra is

$$\begin{gathered} Q_+^2 = Q_-^2 = \bar{Q}_+^2 = \bar{Q}_-^2 = 0 \\ \{Q_- \bar{Q}_\pm\} = H \pm P \\ \{\bar{Q}_+, \bar{Q}_-\} = Z, \qquad \{Q_+, Q_-\} = Z^* \\ \{Q_-, \bar{Q}_+\} = \tilde{Z} \qquad \{Q_+, \bar{Q}_-\} = \tilde{Z}^* \\ [iM, Q_\pm] = \mp Q_\pm, \qquad [iM, \bar{Q}_\pm] = \mp \bar{Q}_+ \\ [iF_V, Q_\pm] = -iQ_\pm, \qquad [iF_V, \bar{Q}_\pm] = i\bar{Q}_\pm \\ [iF_A, Q_\pm] = \mp iQ_\pm, \qquad [iF_A, \bar{Q}_+] = \pm i\bar{Q}_\pm \end{gathered} \tag{A.2.1}$$

Here, $Z, \tilde{Z}$ are central charges, and $Z^*, \tilde{Z}^*$ are their complex conjugates. The central charges will be set to zero in what follows. The linear combination

$$Q_A := \overline{Q}_+ + Q_-, \tag{A.2.2}$$

is called the A-model supercharge. The pair of generators (Q_A, F_A) then obey the subalgebra

$$Q_A^2 = 0, \qquad \left\{Q_A, Q_A^\dagger\right\} = H, \qquad [F_A, Q_A] = Q_A. \tag{A.2.3}$$

The A-model supercharge forms the twisted chiral ring

$$\mathcal{R}_A = \{Q_A\text{-cohomology class of fields}\}. \tag{A.2.4}$$

In particular, an operator $\mathcal{O}(x)$ is called twisted chiral if it obeys

$$[Q_A, \mathcal{O}(x)]' = Q_A \mathcal{O}(x) - (-1)^{|\mathcal{O}|} \mathcal{O}(x) Q_A = 0. \tag{A.2.5}$$

The prime denotes two separate cases depending on whether $\mathcal{O}$ is commuting or anti-commuting, If $\mathcal{O}$ is commuting, then $|\mathcal{O}| = 0$ and brackets are a commutator. If $\mathcal{O}$ is anti-commuting, then $|\mathcal{O}| = 1$ and the brackets denote an anti-commutator.

$\mathcal{N} = (2,2)$ superspace is obtained by extending the commuting x^μ coordinates to include complex anti-commuting coordinates

$$\theta^+,\ \theta^-,\ \overline{\theta}^+,\ \overline{\theta}^- \tag{A.2.6}$$

where $\pm$ denotes chirality under Lorentz transformations. The anti-commuting coordinates are Grassmann-odd, obeying $\theta^2 = \overline{\theta}^2 = 0$. In Lorentzian signature, the odd coordinates are related by conjugation $(\theta^\pm)^\dagger = \overline{\theta}^\pm$, while in Euclidean signature

	Twisted Lorentz Group	Scalar super-charges	"BRST" super-charge	Other super-charges	Super-charge rading
$\mathcal{N}$=(2,2) A-model	$U(1)_E{\times}U(1)_V$	$Q_-,\widetilde{Q}_+$	$Q_A{=}Q_-{+}\widetilde{Q}_+$	$\widetilde{Q}_-dz$, $Q_+d\overline{z}$	$[F_A,Q_A]{=}Q_A$
$\mathcal{N}$=(2,2) B-model	$U(1)_E{\times}U(1)_A$	$\widetilde{Q}_-,\widetilde{Q}_+$	$Q_B{=}\widetilde{Q}_-{+}\widetilde{Q}_+$	Q_-dz, $Q_+d\overline{z}$	$[F_V,Q_B]{=}Q_B$
$\mathcal{N}$=(2,0) A/2-model	$U(1)_E{\times}U(1)_L$	$\widetilde{Q}_+$	$Q_{A/2}{=}\widetilde{Q}_+$	Q_+dz	
$\mathcal{N}$=(0,2) B/2-model	$U(1)_E{\times}U(1)_R$	Q_-	$Q_{B/2}{=}Q_-$	$\widetilde{Q}_-d\overline{z}$	

Table A.1: **Twisted $\mathcal{N}=(2,2)$ supersymmetry**

they understood to be independent $(\theta^\pm)^\dagger \neq \overline{\theta}^\pm$. The vector $U(1)_V$ and axial $U(1)_A$ R-symmetry charges of $\left(\theta^+,\theta^-,\overline{\theta}^+,\overline{\theta}^-\right)$ are assigned to be $(-1,-1,+1,+1)$ and $(-1,+1,+1,-1)$ respectively.

$\mathcal{N}=(2,2)$ superspace is the space with coordinates $\left(x^\mu,\theta^\pm,\overline{\theta}^\pm\right)$, while a function of the superspace coordinates $f\left(x^\mu,\theta^\pm,\overline{\theta}^\pm\right)$ is a superfield. Odd coordinates $\theta^\pm,\overline{\theta}^\pm$ are also referred to as fermionic coordinates, while commuting coordinates x^μ are referred to as bosonic coordinates.

The fermionic coordinates obey the rules of derivation and integration of Grassmann variables. The derivative of a Grassmann variable θ is

$$\frac{\partial}{\partial\theta^\beta}\theta^\alpha = \delta^\alpha_\beta \tag{A.2.7}$$

while integrals obey Berezin integration

$$\int \mathrm{d}\theta\ \theta = \frac{\partial}{\partial\theta}\theta = 1,\ \int \mathrm{d}\theta\ 1 = \frac{\partial}{\partial\theta}1 = 0. \tag{A.2.8}$$

Using the fermionic coordinates, one defines fermionic integration measures on superspace which are necessary for the superspace formulation of supersymmetric Lagrangians. The odd integration measures on superspace are

$$\mathrm{d}^4\theta = \frac{1}{4}\mathrm{d}\theta^+\mathrm{d}\theta^-\mathrm{d}\overline{\theta}^+\mathrm{d}\overline{\theta}^-, \tag{A.2.9}$$

$$\mathrm{d}^2\theta = \frac{1}{2}\mathrm{d}\theta^-\mathrm{d}\theta^+,\ \mathrm{d}^2\overline{\theta} = \frac{1}{2}\mathrm{d}\overline{\theta}^+\mathrm{d}\overline{\theta}^-, \tag{A.2.10}$$

$$\mathrm{d}^2\tilde{\theta} = \frac{1}{2}\mathrm{d}\overline{\theta}^-\mathrm{d}\theta^+,\ \mathrm{d}^2\overline{\tilde{\theta}} = \frac{1}{2}\mathrm{d}\overline{\theta}^+\mathrm{d}\theta^-, \tag{A.2.11}$$

with the convention that the integrand should be in the same order as the measure

$\int d\theta^+ d\overline{\theta}^- \left(\theta^+\overline{\theta}^-\right) = 1.$

The differential operators acting on superspace are

$$\mathcal{Q}_\pm = \frac{\partial}{\partial\theta^\pm} + i\overline{\theta}^\pm\partial_\pm \tag{A.2.12}$$

$$\overline{\mathcal{Q}}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} - i\theta^\pm\partial_\pm \tag{A.2.13}$$

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - i\overline{\theta}^\pm\partial_\pm \tag{A.2.14}$$

$$\overline{D}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} + i\theta^\pm\partial_\pm \tag{A.2.15}$$

The algebra of the differential operators is

$$\mathcal{Q}_\pm^2 = \overline{\mathcal{Q}}_\pm^2 = 0,\ \left\{\mathcal{Q}_\pm, \overline{\mathcal{Q}}_\pm\right\} = -2i\partial_\pm \tag{A.2.16}$$

$$D_\pm^2 = \overline{D}_\pm^2 = 0,\ \left\{D_\pm, \overline{D}_\pm\right\} = 2i\partial_\pm \tag{A.2.17}$$

The supermultiplets of $\mathcal{N} = (2,2)$ supersymmetry are the chiral multiplet Φ and its conjugate $\overline{\Phi}$, the twisted chiral multiplets $\tilde{\Phi}$ and its conjugate $\overline{\tilde{\Phi}}$, the vector multiplet V, and the field strength multiplet Σ and its conjugate $\tilde{\Sigma}$. In what follows we focus primarily on the vector multiplet V and the field strength multiplet $\Sigma = \overline{D}_+ D_- V$, both in Wess-Zumino gauge. For details concerning other multiplets, see [22].

The vector multiplet is $V = \left(v_\mu, \sigma, \overline{\sigma}, \lambda_\pm, \overline{\lambda}_\pm, D\right)$ where v_μ is the gauge field, $\sigma = \sigma_1 - i\sigma_2$, $\overline{\sigma} = \sigma_1 + i\sigma_2$ are complex bosonic scalars, $\lambda_\pm, \overline{\lambda}_\pm$ is a Dirac spinor, and D is an auxiliary field. The component fields of the vector multiplet are all valued in the adjoint representation of $\mathfrak{g} = \mathrm{Lie}G$. In Lorentzian signature, the fermions are related by $\lambda_\pm^\dagger = \overline{\lambda}_\pm$, and similarly for the bosonic scalars $\sigma^\dagger = \overline{\sigma}$.

The component fields of the vector multiplet transform under the commuting supersymmetry variation

$$\delta = i\epsilon^\alpha Q_\alpha + i\bar{\epsilon}_\alpha \overline{Q}^\alpha = i\epsilon_+ Q_- - i\epsilon_- Q_+ - i\bar{\epsilon}_+ \overline{Q}_- + i\bar{\epsilon}_- \overline{Q}_+. \tag{A.2.18}$$

Here, the supersymmetry parameters $\epsilon^\pm$ and $\overline{\epsilon}^\pm$ are anti-commuting spinors with $U(1)_V$ charges $+1$ and -1 respectively, obeying $\epsilon^\mp = \pm\epsilon_\pm$. The transformations of the vector multiplet are

$$\delta v_\pm = i\bar{\epsilon}_\pm\lambda_\pm + i\epsilon_\pm\bar{\lambda}_\pm \quad \text{(A.2.19)}$$

$$\delta\sigma = -i\bar{\epsilon}_+\lambda_- - i\epsilon_-\bar{\lambda}_+ \quad \text{(A.2.20)}$$

$$\delta\bar{\sigma} = -i\epsilon_+\bar{\lambda}_- - i\bar{\epsilon}_-\lambda_+ \quad \text{(A.2.21)}$$

$$\delta D = \frac{1}{2}\big(-\bar{\epsilon}_+D_-\lambda_+ - \bar{\epsilon}_-D_+\lambda_- + \epsilon_+D_-\bar{\lambda}_+ + \epsilon_-D_+\bar{\lambda}_- \quad \text{(A.2.22)}$$

$$+i\epsilon_+\left[\sigma,\bar{\lambda}_-\right] + i\epsilon_-\left[\bar{\sigma},\bar{\lambda}_+\right] - i\bar{\epsilon}_-\left[\sigma,\lambda_+\right] - i\bar{\epsilon}_+\left[\bar{\sigma},\lambda_-\right]\big), \quad \text{(A.2.23)}$$

$$\delta\lambda_+ = i\epsilon_+\left(D + iF_{01} + \frac{1}{2}[\sigma,\bar{\sigma}]\right) + \epsilon_-D_+\bar{\sigma} \quad \text{(A.2.24)}$$

$$\delta\lambda_- = i\epsilon_-\left(D - iF_{01} - \frac{1}{2}[\sigma,\bar{\sigma}]\right) + \epsilon_+D_-\sigma \quad \text{(A.2.25)}$$

$$\delta\bar{\lambda}_+ = -i\bar{\epsilon}_+\left(D - iF_{01} + \frac{1}{2}[\sigma,\bar{\sigma}]\right) + \bar{\epsilon}_-D_+\sigma \quad \text{(A.2.26)}$$

$$\delta\bar{\lambda}_- = -i\bar{\epsilon}_-\left(D + iF_{01} - \frac{1}{2}[\sigma,\bar{\sigma}]\right) + \bar{\epsilon}_+D_-\bar{\sigma}, \quad \text{(A.2.27)}$$

where $v_\pm = v_0 \pm v_1, v_\pm = v_0 \pm v_1$.

A.3 Euclidean signature

The transformations of the vector multiplet in Euclidean signature are obtained by Wick rotation $x^0 \to -ix^2$. The effect of Wick rotation on the Lorentz Group is $SO(1,1) \to SO(2)_E \simeq U(1)_E$, and the generator of Lorentz transformations is modified as $M \to iM_E$. Accordingly, the vectors change as $D_+ \to 2D_{\overline{z}}$, $D_- \to -2D_z$, $v_+ \to 2v_{\overline{z}}$, $v_- \to -2v_z$, and $F_{01} \to -iF_{12}$. The hermiticity conditions from Minkowski signature no longer hold for the Euclidean fermions $\lambda_\pm^\dagger \neq \overline{\lambda}_\pm$ or the scalars $\sigma^\dagger \neq \overline{\sigma}$. Instead, these fields are understood to be independent.

In Euclidean signature, the transformations of the vector multiplet are

$$\delta v_z = -\frac{i}{2}\left(\epsilon_-\bar{\lambda}_- + \bar{\epsilon}_-\lambda_-\right), \tag{A.3.1}$$

$$\delta v_{\bar{z}} = \frac{i}{2}\left(\epsilon_+\bar{\lambda}_+ + \bar{\epsilon}_+\lambda_+\right), \tag{A.3.2}$$

$$\delta\sigma = -i\left(\epsilon_-\bar{\lambda}_+ + \bar{\epsilon}_+\lambda_-\right), \tag{A.3.3}$$

$$\delta\bar{\sigma} = -i\left(\bar{\epsilon}_-\lambda_+ + \epsilon_+\bar{\lambda}_-\right), \tag{A.3.4}$$

$$\delta D = \frac{i}{2}\left(\epsilon_-\left[\bar{\sigma},\bar{\lambda}_+\right] - \bar{\epsilon}_-\left[\sigma,\lambda_+\right] + \epsilon_+\left[\sigma,\bar{\lambda}_-\right] - \bar{\epsilon}_+\left[\bar{\sigma},\lambda_-\right]\right) \tag{A.3.5}$$

$$+\epsilon_- D_{\bar{z}}\bar{\lambda}_- - \bar{\epsilon}_- D_{\bar{z}}\lambda_- - \epsilon_+ D_z\bar{\lambda}_+ + \bar{\epsilon}_+ D_z\lambda_+, \tag{A.3.6}$$

$$\delta\lambda_+ = +i\epsilon_+\left(D + F_{12} + \frac{1}{2}\left[\sigma,\bar{\sigma}\right]\right) + 2\epsilon_- D_{\bar{z}}\bar{\sigma}, \tag{A.3.7}$$

$$\delta\lambda_- = +i\epsilon_-\left(D - F_{12} - \frac{1}{2}\left[\sigma,\bar{\sigma}\right]\right) - 2\epsilon_+ D_z\sigma, \tag{A.3.8}$$

$$\delta\bar{\lambda}_+ = -i\bar{\epsilon}_+\left(D - F_{12} + \frac{1}{2}\left[\sigma,\bar{\sigma}\right]\right) + 2\bar{\epsilon}_- D_{\bar{z}}\sigma, \tag{A.3.9}$$

$$\delta\bar{\lambda}_- = -i\bar{\epsilon}_-\left(D + F_{12} - \frac{1}{2}\left[\sigma,\bar{\sigma}\right]\right) - 2\bar{\epsilon}_+ D_z\bar{\sigma}, \tag{A.3.10}$$

The D-term of the Euclidean vector multiplet is the action of supersymmetric YM2

$$S_{\mathrm{SYM}} = \int d^2x\sqrt{g}\,\mathrm{Tr}\left[\frac{1}{2}\left(F_{12}\right)^2 - \frac{1}{2}\left(-D\right)^2 + \frac{1}{8}\left[\sigma,\overline{\sigma}\right]^2\right. \tag{A.3.11}$$

$$+D_{\bar{z}}\overline{\sigma}D_z\sigma + D_z\overline{\sigma}D_{\bar{z}}\sigma + i\overline{\lambda}_+ D_z\lambda_+ - i\overline{\lambda}_- D_{\bar{z}}\lambda_- \tag{A.3.12}$$

$$\left.+\frac{1}{2}\overline{\lambda}_-\left[\sigma,\lambda_+\right] + \frac{1}{2}\overline{\lambda}_+\left[\overline{\sigma},\lambda_-\right]\right]. \tag{A.3.13}$$

The F-term is the actions of the twisted chiral superpotential

$$S_{\widetilde{W}} = -\int d^2x\sqrt{g}\Big(\left(D - F_{12}\right)\partial_j\widetilde{W}\left(\sigma\right) + \lambda_-^i\overline{\lambda}_+^j\partial_i\partial_j\widetilde{W}\left(\sigma\right)\Big). \tag{A.3.14}$$

$$S_{\overline{\widetilde{W}}} = -\int d^2x\sqrt{g}\Big(\left(D + F_{12}\right)^j\partial_j\overline{\widetilde{W}}\left(\overline{\sigma}\right) - \overline{\lambda}_-^i\lambda_+^j\partial_i\partial_j\overline{\widetilde{W}}\left(\overline{\sigma}\right)\Big). \tag{A.3.15}$$

A.4 Topological A-twist

Given the Euclidean theory, we proceed to perform the topological A-twist. Topological twisting is a procedure by which a topological quantum field theory is obtained from a supersymmetric quantum field theory. The key result of twisting is that a subset of the original supercharges have spin zero under Lorentz transformations defined with respect to the twisted stress-energy tensor. Given Lorentz scalar supercharges, one is free to define supersymmetric actions on manifolds with curvature.

The A-twisted theory is obtained by replacing the Lorentz group $SO(2)_E \simeq U(1)_E$ of the Euclidean theory by the A-twisted Lorentz group $U'(1)_E$, defined as the diagonal subgroup of $U(1)_E \times U(1)_V$. After A-twisting, the energy momentum tensor is $T^{\text{twisted}}_{\mu\nu} = T_{\mu\nu} + \frac{1}{4}\left(\epsilon_\mu^{\ \lambda}\partial_\lambda J^V_\nu + \epsilon_\nu^{\ \lambda}\partial_\lambda J^V_\mu\right)$ where J^V_ν is the $U(1)_V$ current, while the generator of Lorentz transformations of the twisted theory is $M'_E = M_E + F_V$. The effect of the A-twist on the supercharges $Q_\pm, \overline{Q}_\pm$, variational parameters $\epsilon_\pm, \bar{\epsilon}_\pm$, and fermions $\lambda_\pm, \overline{\lambda}_\pm$ is summarized in the table.

	$U(1)_V$	$U(1)_A$	$U(1)_E$	$U(1)'_E$
Q_-	-1	+1	+1	0
Q_+	-1	-1	-1	-2
$\overline{Q}_-$	+1	-1	+1	+2
$\overline{Q}_+$	+1	+1	-1	0
ϵ_-	+1	+1	+1	+2
ϵ_+	+1	-1	-1	0
$\bar{\epsilon}_-$	-1	-1	+1	0
$\bar{\epsilon}_+$	-1	+1	-1	-2
λ_-	+1	+1	+1	+2
λ_+	+1	-1	-1	0
$\overline{\lambda}_-$	-1	-1	+1	0
$\overline{\lambda}_+$	-1	+1	-1	-2

Table A.2: Topological A-twist of the Euclidean vector multiplet. Furthest to the left are the supercharges $Q_\pm, \overline{Q}_\pm$, variational parameters $\epsilon_\pm, \bar{\epsilon}_\pm$, and fermions $\lambda_\pm, \overline{\lambda}_\pm$ being A-twisted. Observe that the supersymmetry parameters and the fermions have the same charge. In the middle are the charges under the vector R-symmetry $U(1)_V$ generated by F_V, the axial R-symmetry $U(1)_A$ generated by F_A, and the Euclidean Lorentz group $U(1)_E \simeq SO(2)_E$ generated by M_E. Furthest to the right is the A-twisted Lorentz group $U'(1)_E$ generated by $M'_E = M_E + F_V$. Under $U(1)'_E$, scalars have charge zero, holomorphic vectors have charge +2, while anti-holomorphic vectors have charge -2. The scalars of the twisted theory are $Q_-, \overline{Q}_+, \epsilon_+, \bar{\epsilon}_-, \lambda_+, \overline{\lambda}_-$, while $\lambda_- = \lambda_{-,z}$ and is a holomorphic vector, and $\overline{\lambda}_+ = \overline{\lambda}_{+,\bar{z}}$ is an anti-holomorphic vector. The A-model supercharge $Q_A = \overline{Q}_+ + Q_-$is a scalar of the twisted theory.

The A-twisted supersymmetry transformations of the vector multiplet are obtained from the Euclidean transformations by declaring the fermions $\lambda_\pm, \overline{\lambda}_\pm$ to have new spins, and restricting to the supercharges and variational parameters that are Lorentz scalars $Q_-, \overline{Q}_+, \epsilon_+, \bar{\epsilon}_-$. The component fields of the A-twisted theory transform under commuting scalar supersymmetry variation

$$\delta = i\epsilon_+ Q_- + i\bar{\epsilon}_- \bar{Q}_+. \tag{A.4.1}$$

The transformations of the A-twisted vector multiplet are

$$\delta v_z = -\frac{i}{2}\bar{\epsilon}_-\lambda_-, \tag{A.4.2}$$

$$\delta v_{\bar{z}} = +\frac{i}{2}\epsilon_+\bar{\lambda}_+, \tag{A.4.3}$$

$$\delta\sigma = 0, \tag{A.4.4}$$

$$\delta\bar{\sigma} = -i\left(\bar{\epsilon}_-\lambda_+ + \epsilon_+\bar{\lambda}_-\right), \tag{A.4.5}$$

$$\delta D = -\frac{i}{2}\left[\sigma, \bar{\epsilon}_-\lambda_+ - \epsilon_+\bar{\lambda}_-\right] - \bar{\epsilon}_- D_{\bar{z}}\lambda_- - \epsilon_+ D_z\bar{\lambda}_+, \tag{A.4.6}$$

$$\delta\lambda_+ = +i\epsilon_+\left(D + F_{12} + \frac{1}{2}[\sigma,\bar{\sigma}]\right), \tag{A.4.7}$$

$$\delta\lambda_- = -2\epsilon_+ D_z\sigma, \tag{A.4.8}$$

$$\delta\bar{\lambda}_+ = +2\bar{\epsilon}_- D_{\bar{z}}\sigma, \tag{A.4.9}$$

$$\delta\bar{\lambda}_- = -i\bar{\epsilon}_-\left(D + F_{12} - \frac{1}{2}[\sigma,\bar{\sigma}]\right), \tag{A.4.10}$$

Here, $\lambda_+, \overline{\lambda}_-$ are scalars, while $\lambda_- = \lambda_{-,z}$ is a holomorphic 1-form, and $\overline{\lambda}_+ = \overline{\lambda}_{+,\bar{z}}$ is an anti-holomorphic 1-form. For completeness, we record that

$$\delta F_{12} = \epsilon_+ D_z\bar{\lambda}_+ + \bar{\epsilon}_- D_{\bar{z}}\lambda_-. \tag{A.4.11}$$

To obtain the variation under the A-model supercharge $Q_A = Q_- + \bar{Q}_+$, one sets $\epsilon_+ = \bar{\epsilon}_- = \epsilon$.

A.5 Standard cohomological multiplet

Given the transformations of the A-twisted vector multiplet, we proceed to identify the standard cohomological multiplet. This is achieved by redefining the fields. After setting $\epsilon_+ = \bar{\epsilon}_- = \epsilon$, the first field redefinition is

$$Y = i\left(D + F_{12}\right). \tag{A.5.1}$$

The transformations of A-twisted vector multiplet are then

$$\delta v_z = -\frac{i}{2}\epsilon\lambda_-, \tag{A.5.2}$$
$$\delta v_{\bar{z}} = +\frac{i}{2}\epsilon\bar{\lambda}_+, \tag{A.5.3}$$
$$\delta\sigma = 0, \tag{A.5.4}$$
$$\delta\bar{\sigma} = -i\epsilon\left(\lambda_+ + \bar{\lambda}_-\right), \tag{A.5.5}$$
$$\delta Y = +\frac{1}{2}\epsilon\left[\sigma, \lambda_+ - \bar{\lambda}_-\right], \tag{A.5.6}$$
$$\delta\lambda_+ = +\epsilon Y + \frac{i}{2}\epsilon\left[\sigma, \bar{\sigma}\right], \tag{A.5.7}$$
$$\delta\lambda_- = -2\epsilon D_z\sigma, \tag{A.5.8}$$
$$\delta\bar{\lambda}_+ = +2\epsilon D_{\bar{z}}\sigma, \tag{A.5.9}$$
$$\delta\bar{\lambda}_- = -\epsilon Y + \frac{i}{2}\epsilon\left[\sigma, \bar{\sigma}\right] \tag{A.5.10}$$

Next, we go from the physics conventions, in which the Lie algebra consists of hermitian matrices, to the math conventions, in which the Lie algebra consists of anti-hermitian matrices. This is achieved by the second field redefinition $\varphi \to i\varphi$ where φ denotes all component fields. The only effect is a change in the factors of i in front of the commutators. The result is

$$\delta v_z = -\frac{i}{2}\epsilon\lambda_-, \tag{A.5.11}$$
$$\delta v_{\bar{z}} = +\frac{i}{2}\epsilon\bar{\lambda}_+, \tag{A.5.12}$$
$$\delta\sigma = 0, \tag{A.5.13}$$
$$\delta\bar{\sigma} = -i\epsilon\left(\lambda_+ + \bar{\lambda}_-\right), \tag{A.5.14}$$
$$\delta Y = \frac{i}{2}\epsilon\left[\sigma, \lambda_+ - \bar{\lambda}_-\right], \tag{A.5.15}$$
$$\delta\lambda_+ = +\epsilon Y - \frac{1}{2}\epsilon\left[\sigma, \bar{\sigma}\right], \tag{A.5.16}$$
$$\delta\lambda_- = -2\epsilon D_z\sigma, \tag{A.5.17}$$
$$\delta\bar{\lambda}_+ = +2\epsilon D_{\bar{z}}\sigma, \tag{A.5.18}$$
$$\delta\bar{\lambda}_- = -\epsilon Y - \frac{1}{2}\epsilon\left[\sigma, \bar{\sigma}\right] \tag{A.5.19}$$

The third step is to redefine the fields as

$$v_{\bar{z}} \rightarrow -A_{\bar{z}} \quad \text{(A.5.20)}$$
$$v_z \rightarrow -A_z \quad \text{(A.5.21)}$$
$$\lambda_- \rightarrow +2\psi_z, \quad \text{(A.5.22)}$$
$$\bar{\lambda}_+ \rightarrow -2\psi_{\bar{z}}, \quad \text{(A.5.23)}$$
$$\sigma \rightarrow \phi, \quad \text{(A.5.24)}$$
$$\bar{\sigma} \rightarrow \lambda, \quad \text{(A.5.25)}$$
$$\lambda_+ \rightarrow +\chi - \frac{\eta}{2}, \quad \text{(A.5.26)}$$
$$\bar{\lambda}_- \rightarrow -\chi - \frac{\eta}{2}, \quad \text{(A.5.27)}$$
$$G \rightarrow H \quad \text{(A.5.28)}$$

The result of the third field redefinition is

$$\delta A_z = i\epsilon\psi_z, \quad \text{(A.5.29)}$$
$$\delta A_{\bar{z}} = i\epsilon\psi_{\bar{z}}, \quad \text{(A.5.30)}$$
$$\delta\psi_z = -\epsilon D_z\phi, \quad \text{(A.5.31)}$$
$$\delta\psi_{\bar{z}} = -\epsilon D_{\bar{z}}\phi, \quad \text{(A.5.32)}$$
$$\delta\phi = 0, \quad \text{(A.5.33)}$$
$$\delta\lambda = i\epsilon\eta, \quad \text{(A.5.34)}$$
$$\delta\eta = \epsilon\left[\phi, \lambda\right], \quad \text{(A.5.35)}$$
$$\delta\chi = \epsilon H, \quad \text{(A.5.36)}$$
$$\delta H = i\epsilon\left[\phi, \chi\right]. \quad \text{(A.5.37)}$$

The final step is to change from holomorphic coordinates to Euclidean coordinates

$$A_z \rightarrow \frac{1}{2}\left(A_1 - iA_2\right), \; A_{\bar{z}} \rightarrow \frac{1}{2}\left(A_1 + iA_2\right), \; \psi_z \rightarrow \frac{1}{2}\left(\psi_1 - i\psi_2\right), \psi_{\bar{z}} \rightarrow \frac{1}{2}\left(\psi_1 + i\psi_2\right). \quad \text{(A.5.38)}$$

The result of changing coordinates is the standard multiplet of cohomological YM2

$$\delta A_\mu = i\epsilon\psi_\mu, \quad \text{(A.5.39)}$$
$$\delta\psi_\mu = -\epsilon D_\mu\phi = -\epsilon\left(\partial_\mu\phi + [A_\mu, \phi]\right), \quad \text{(A.5.40)}$$
$$\delta\phi = 0, \quad \text{(A.5.41)}$$
$$\delta\lambda = i\epsilon\eta, \quad \text{(A.5.42)}$$
$$\delta\eta = \epsilon c\,[\phi, \lambda], \quad \text{(A.5.43)}$$
$$\delta\chi = \epsilon H, \quad \text{(A.5.44)}$$
$$\delta H = i\epsilon\,[\phi, \chi]. \quad \text{(A.5.45)}$$

A.6 Summary of localizing terms

Here we compare the localizing terms of used by Benini-Zaffaroni in [18, 19] and Witten in [3]. We present the localizing terms using the standard cohomological multiplet. The localizing terms, as well as the localization computations, can be compared using the dictionary between the standard cohomological multiplet and the vector multiplet of A-twisted (2,2) supersymmetry. Consider the following fermionic functionals of the fields of the standard multiplet

$$V_0 = \frac{1}{h^2}\mathrm{Tr}\left(\frac{1}{2}\chi\left(H - 2f\right) + g^{ij}D_i\lambda\psi_j\right) \quad \text{(A.6.1)}$$
$$V_1 = -\frac{1}{h^2}\mathrm{Tr}\chi\lambda \quad \text{(A.6.2)}$$
$$V_2 = \frac{1}{h^2}\mathrm{Tr}\left(\frac{1}{8}\eta\,[\phi, \lambda]\right) \quad \text{(A.6.3)}$$

Acting on these terms with the BRST-like variation, one obtains the action

$$S\left(c_0, c_1, c_2\right) = \delta\left(\int d^2x\sqrt{g}\left(c_0V_0 + c_1V_1 + c_2V_2\right)\right), \quad \text{(A.6.4)}$$

where c_0, c_1c_2 are real coefficients. The three different localizing terms in [18, 19, 3] are

$$S_{\mathrm{BZ}} = S\left(1, 0, 1\right) = \delta\left(\int d^2x\sqrt{g}\left(V_0 + V_2\right)\right) \quad \text{(A.6.5)}$$
$$S^{\mathrm{flat}}_{\mathrm{WTN}} = S\left(1, 0, 0\right) = \delta\left(\int d^2x\sqrt{g}\left(V_0\right)\right) \quad \text{(A.6.6)}$$
$$S^{\mathrm{YM}}_{\mathrm{WTN}} = S\left(1, t, 0\right) = \delta\left(\int d^2x\sqrt{g}\left(V_0 + tV_1\right)\right) \quad \text{(A.6.7)}$$

Here, S_{BZ} is the localizing term used in [18, 19], $S_{\mathrm{WTN}}^{\mathrm{flat}}$ is the localizing term used to localize to the moduli space of flat connections in [3], and $S_{\mathrm{WTN}}^{\mathrm{YM}}$ is the localizing term used to localize to the moduli space of Yang-Mills connections in [3].

Appendix B

Representation theory of Lie algebras and Lie groups

This appendix provides background for derivation in section 5 by reviewing the necessary material from the representation theory of Lie algebra and Lie groups. Standard textbook references on representation theory include [49, 50, 33, 51, 52]. Other useful resources are the lecture notes [53, 34, 54], appendix A of [55], appendix A of [56], and appendix A of [57].

B.1 Representations of Lie algebras

In this section, we describe representations of Lie algebras, weight spaces, the adjoint representation, and the Killing form. For a reference, see e.g. [51, 34]. A concise account is provided in appendix A of [55].

Let $\mathfrak{g}$ be a Lie algebra, and let $\mathfrak{t}$ be the Cartan subalgebra of $\mathfrak{g}$. A Lie algebra representation $(\rho, \mathfrak{g})$ is a linear map

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V_\rho) \tag{B.1.1}$$

that preserves the Lie bracket $\rho\left([X,Y]\right) = \rho\left(X\right)\rho\left(Y\right) - \rho\left(Y\right)\rho\left(X\right)$ for all $X, Y \in \mathfrak{g}$. Here, $\mathfrak{gl}(V_\rho)$ is the space of linear endomorphisms of the vector space V_ρ, that is to say, linear maps of V_ρ to itself. The vector space V_ρ is called the representation space.

The decomposition of the representation space can be viewed as a generalized Eigen-decomposition. In particular, V_ρ decomposes into a direct sum of Eigen-spaces $V_\rho[\lambda]$ called weight spaces

$$V_\rho = \bigoplus_{\lambda \in \mathfrak{t}^*} V_\rho[\lambda]. \tag{B.1.2}$$

The non-zero elements of these spaces $e_\lambda \in V_\rho[\lambda]$ perform the function of Eigenvectors. The Eigenvalue equation is obtained by acting on e_λ with ρ-representatives of $\mathfrak{t}$

$$\rho(H)e_\lambda = \lambda(H)e_\lambda, \qquad H \in \mathfrak{t}, \tag{B.1.3}$$

where λ is a linear functional called the weight of the representation ρ, see e.g. [55]

The adjoint representation $(\mathrm{ad}, \mathfrak{g})$ is the case when the representation space is the Lie algebra itself

$$\mathrm{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}), \tag{B.1.4}$$

$$X \mapsto \mathrm{ad}(X). \tag{B.1.5}$$

Representatives of the adjoint act on the Lie algebra by $\mathrm{ad}(X)(Y) = [X, Y]$ where $X, Y \in \mathfrak{g}$.

From the adjoint representation, it is possible to specify Killing forms, and subsequently, determine dual elements. The Killing form on $\mathfrak{g}$ is defined by $K_\mathfrak{g}(X, Y) \equiv \mathrm{Tr}\,(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$ where $X, Y \in \mathfrak{g}$, and $\circ$ denotes composition of $\mathfrak{g}$-valued linear operators. See e.g. Chapter VII.9 in [51] for explicit Killing forms for $SU(n)$ and $SO(2n+1)$. Choosing a particular $K_\mathfrak{g}(\cdot,\cdot)$, then restricting $\mathfrak{g}$ to $\mathfrak{t}$, determines the Killing forms $K_\mathfrak{t}(\cdot,\cdot)$ on $\mathfrak{t}$ and $K_{\mathfrak{t}^*}(\cdot,\cdot)$ on $\mathfrak{t}^*$. The dual element of $H \in \mathfrak{t}$, denoted $H^* \in \mathfrak{t}^*$, is then defined by

$$\langle H^*, X\rangle = K_\mathfrak{t}\,(H, X) \text{ for all } X \in \mathfrak{t}. \tag{B.1.6}$$

where $\langle\cdot,\cdot\rangle : \mathfrak{t}^* \otimes \mathfrak{t} \rightarrow \mathbb{R}$ is the inner product between dual vector spaces.

B.2 Complexification of Lie algebras

In this section, we review the complexification of Lie algebras, the adjoint representation space, and roots. For a reference, see e.g. chapter 8 of [51], or appendix A of [55].

The representations of the real Lie algebra $\mathfrak{g}$ are studied through its complexification $\mathfrak{g}_\mathbb{C}$, which can be regarded as the representation space of the adjoint representation. $\mathfrak{g}_\mathbb{C}$ is a complex Lie algebra, defined by permitting complex linear combinations of elements of $\mathfrak{g}$ with imaginary coefficients

$$\mathfrak{g}_\mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}. \tag{B.2.1}$$

For example, the complexification of the real algebra $\mathfrak{su}(2)$ is the complex algebra $\mathfrak{su}(2)_\mathbb{C} = \mathfrak{sl}(2, \mathbb{C})$. The complexified Cartan subalgebra $\mathfrak{t}_\mathbb{C}$ is determined by restricting $\mathfrak{g}$ to $\mathfrak{t}$, then complexifying. Just as in the real case, $\mathfrak{t}_{\mathbb{C}}$ is the maximal abelian subalgebra of $\mathfrak{g}_{\mathbb{C}}$. The subset of real elements of the complexification $\mathfrak{g}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{C}}$ is

$$\mathfrak{g}_{\mathbb{R}} = \left\{ \, Z \in \mathfrak{g}_{\mathbb{C}} \mid \overline{Z} = Z \, \right\} = i\mathfrak{g}, \tag{B.2.2}$$

where the bar denotes complex conjugation. Similarly, $\mathfrak{t}_{\mathbb{R}}$ is the real part of $\mathfrak{t}_{\mathbb{C}}$.

The decomposition of the adjoint representation space is captured by the decomposition of the complexification $\mathfrak{g}_{\mathbb{C}}$. In particular, $\mathfrak{g}_{\mathbb{C}}$ decomposes into a complexified Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$ and a direct sum of root spaces V_α

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} V_\alpha. \tag{B.2.3}$$

where α are roots, while Δ is the set of roots. The roots are the weights of the adjoint representation.

The complexified subalgebra $\mathfrak{t}_{\mathbb{C}}$ consists of elements of $\mathfrak{g}_{\mathbb{C}}$ that commute with elements of the real subalgebra $\mathfrak{t}$, that is to say

$$\mathfrak{t}_{\mathbb{C}} = \left\{ Y \in \mathfrak{g}_{\mathbb{C}} \mid \mathrm{ad}\,(X)\, Y = 0 \text{ for all } X \in \mathfrak{t} \right\}. \tag{B.2.4}$$

The roots and the root spaces emerge from the generalized Eigenvalue equation, obtained by acting with adjoint representatives of $\mathfrak{t}_{\mathbb{R}}$ on non-zero elements of the complexification $\mathfrak{g}_{\mathbb{C}}$. This is given by

$$\mathrm{ad}(X)(Y) = \alpha(X)Y, \quad \text{for all} \quad X \in \mathfrak{t}_{\mathbb{R}}, \tag{B.2.5}$$

where $Y \in \mathfrak{g}_{\mathbb{C}}$. The roots α are real, non-zero, linear functionals on $\mathfrak{t}_{\mathbb{R}}$. The subspace of $Y \in \mathfrak{g}_{\mathbb{C}}$ satisfying B.2.5 is the root space V_α. That is to say, the root spaces $V_\alpha \subset \mathfrak{g}_{\mathbb{C}}$ are

$$V_\alpha = \left\{ Y \in \mathfrak{g}_{\mathbb{C}} \mid \mathrm{ad}\,(X)\, Y = \alpha\,(X)\, Y \text{ for all } X \in \mathfrak{t}_{\mathbb{R}} \right\}, \tag{B.2.6}$$

with the property that they are one-dimensional $\dim V_\alpha = 1$. This is Eigen-decomposition in the sense that the roles of Eigen-spaces, Eigenvectors, and Eigenvalues are assumed by V_α, Y, and $\alpha(X)$, respectively.

The roots have several important properties. Roots are vectors in $\mathfrak{t}^*$, belonging to the set $\alpha \in \Delta$. Roots come in pairs with opposite sign, in the sense that that for every $\alpha \in \Delta$, there is an $-\alpha \in \Delta$. Each root in $\alpha \in \mathfrak{t}^*$ is associated to a dual root $\alpha^* \in \mathfrak{t}$, and a coroot $H_\alpha \in \mathfrak{t}$. The coroot can be expressed in terms of the dual root

$H_\alpha = \frac{2\alpha^*}{K(\alpha,\alpha)}$. Consequently, we have

$$\alpha(H_\beta) = \langle \alpha, H_\beta \rangle = \frac{2K(\alpha,\beta)}{K(\beta,\beta)} \tag{B.2.7}$$

as well as $\alpha(H_\alpha) = 2$.

B.3 Root systems

In this section, we review root systems of Lie algebras, fundamental systems, and the Weyl group. For a reference, see [49, 51], or appendix A of [56].

In general, a root system are defined as follows. Let E be a Euclidean space with scalar product $(\ ,\ )$. A finite nonzero set of vectors Φ in E is referred to as a root system if i) Φ spans E, ii) for each $\beta \in \Phi$ the only other multiple in the set is $-\beta \in \Phi$, iii) for $\beta, \gamma \in \Phi$, the reflected element $\sigma_\beta(\gamma) = \gamma - 2(\beta,\gamma)/(\beta,\beta)\,\beta$, is also in the set $\sigma_\beta(\gamma) \in \Phi$, and iv) for $\beta, \gamma \in \Phi$, the quantity $\langle \beta, \gamma \rangle =: 2(\beta,\gamma)/(\beta,\beta)$ is integer $\langle \beta, \gamma \rangle \in \mathbb{Z}$.

The set of roots Δ, of the Lie algebra $\mathfrak{g}$, is a root system in $\mathfrak{t}^*$. This is referred to as the root system Δ of G. In particular, Δ is a finite non-zero set of vectors in the fixed Euclidean vector space $\mathfrak{t}^*$, satisfying requirements i) - iv) of being a root system. In a similar fashion, the coroots H_α are valued in a set $\tilde{\Delta}$ that is a root system in the dual vector space $\mathfrak{t}^*$.

The Weyl group W of the root system is the finite reflection group of $\mathfrak{t}^*$, generated by reflections about hyperplanes which are orthogonal to the roots $\alpha \in \Delta$ and pass through the origin. As a finite reflection group, the Weyl group is a subgroup of the isometry group of $\mathfrak{t}^*$. The Weyl group is also the finite reflection group of the dual root system $\tilde{\Delta}$ in $\mathfrak{t}$, generated by reflecting about hyperplanes $\tilde{\pi}_\alpha$ orthogonal to coroots H_α.

A reflection hyperplane, orthogonal to a root $\alpha \in \Delta$, is defined using the scalar product on $\mathfrak{t}^*$:

$$\pi_\alpha = \{\gamma \subset \mathfrak{t}^* \,|\, (\alpha,\gamma) = 0\}\,. \tag{B.3.1}$$

Each hyperplane breaks $\mathfrak{t}^*$ into two disconnected components, or regions. The complement $\pi_\alpha \backslash \mathfrak{t}^*$ consists of two disconnected components, where each connected component of the complement shares π_α as a boundary. Similarly, the union of the hyperplanes

$$\Pi = \bigcup_{\alpha \in \Delta} \pi_\alpha \tag{B.3.2}$$

splits $\mathfrak{t}^*$ into several disconnected components. The complement of the union of hyperplanes $\Pi \backslash \mathfrak{t}^*$ consists of several disconnected components, where the connected components of the complement are referred to as Weyl chambers. Each Weyl chamber, denoted C, is a convex cone. The union of all Weyl chambers is the union of convex cones in the complement $\Pi\backslash\mathfrak{t}^*$, while a particular Weyl chamber is a particular convex cone. The same arguments apply to the dual space $\mathfrak{t}$, where the chambers are defined using hyperplanes $\tilde{\pi}_\alpha$ orthogonal to coroots H_α.

The elements of the Weyl group $w \in W$, referred to as Weyl reflections, are reflections about hyperplanes π_α in $\mathfrak{t}^*$ or $\tilde{\pi}_\alpha$ in $\mathfrak{t}$. The Weyl group acts on $\mathfrak{t}^*$ and $\mathfrak{t}$ separately, sending $W : \mathfrak{t}^* \to \mathfrak{t}^*, X \mapsto w \cdot X$ and $W : \mathfrak{t} \to \mathfrak{t}, Y \mapsto w \cdot Y$. These are reflections in the standard sense: linear maps from a Euclidean space itself, which preserve the reflection hyperplane. In what follows, we denote an arbitrary Weyl reflection by w, and a particular Weyl reflection about the hyperplane π_α or $\tilde{\pi}_\alpha$, by s_α.

The Weyl reflection of an element in $\mathfrak{t}^*$ about the hyperplane π_α is

$$s_\alpha (X) = X - X (H_\alpha) \alpha, \quad X \in \mathfrak{t}^*. \tag{B.3.3}$$

The Weyl reflection of an element in $\mathfrak{t}$ about the hyperplane $\tilde{\pi}_\alpha$ is

$$s_\alpha (Y) = Y - \alpha (Y) H_\alpha, \quad Y \in \mathfrak{t}. \tag{B.3.4}$$

It is useful to note that, with respect to the scalar product on $\mathfrak{t}^*$or $\mathfrak{t}$, Weyl reflections obey $(w \cdot X, Y) = \left(X, w^{-1} \cdot Y\right)$. This, in turn, implies the identity $(w \cdot X, w \cdot Y) = (X, Y)$.

As an example, consider $\mathfrak{g} = \mathfrak{su}(2)$, where $\mathfrak{t}^* = \mathbb{R}$, and the set of roots is $\Delta = \{+2, -2\}$. Both roots specify the same hyperplane $\pi_\alpha = 0$. The Weyl chambers are the convex cones $C = \mathbb{R}_{\geq 0}$ and $C' = \mathbb{R}_{\leq 0}$. Weyl reflections map between $\alpha = +2$ and $\alpha' = -2$, such that the set Δ is closed under the action of the Weyl group.

At this point, $\mathfrak{t}^*$ includes the set of vectors Δ, the reflection hyperplanes π_α, the chambers C, and the group W generated by reflections about π_α. To define a basis of vectors, it is necessary to choose a positive direction in $\mathfrak{t}^*$.

Choosing a positive direction in $\mathfrak{t}^*$ is referred to as the choice of fundamental system for the root system. This choice can be equivalently regarded as choosing which one of the Weyl chambers will serve as the fundamental Weyl chamber, or choosing which of the roots will serve as simple roots. The choice of fundamental system is arbitrary, but necessary, in the sense that one can pick any Weyl chamber to be the fundamental one, but a particular fundamental chamber must be specified to define a basis.

Choosing a fundamental system breaks $\mathfrak{t}^*$ into two regions, one of which is regarded as the positive half space, the other of which is regarded as the negative half space.

Half of the set of roots Δ will be valued in the positive space, and are called positive roots Δ_+, while the other half of Δ will be valued in the negative half space, and are called negative roots Δ_-. In particular, the set of roots decomposes as $\Delta = \Delta_+ \cup \Delta_-$.

The positive roots fix an element, unique to each root system, called the Weyl vector. By definition, the Weyl vector is half the sum of the positive roots $\varrho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$.

Furthermore, the positive roots fix the simple roots, which serve as a basis for $\mathfrak{t}^*$. By definition, a simple root is any positive root that cannot be expressed as the sum of two positive roots. That is to say, any $\alpha \in \Delta_+$ that cannot be expressed as $\beta + \gamma$ for $\beta, \gamma \in \Delta_+$. The set of simple roots

$$\{\alpha_I | I = 1, \ldots, r = \mathrm{rank}\mathfrak{g}\} \tag{B.3.5}$$

serve as a basis for $\mathfrak{t}^*$. The full set of roots Δ can then be expressed as the set of simple roots acted upon by all Weyl reflections $\Delta = \{w \cdot \alpha_I | I = 1, \ldots, r, w \in W\}$. The simple roots fix the fundamental Weyl chamber and the simple Weyl reflections, which we now describe.

The fundamental Weyl chamber, denoted C^0, is the convex cone in $\mathfrak{t}^*$, defined as the closure of the hyperplanes π_{α_I} orthogonal to simple roots α_I. In particular, the fundamental Weyl chamber is the region

$$C^0 = \{\gamma \subset \mathfrak{t}^* | \, (\gamma, \alpha_I) \geq 0, \; I = 1, \ldots, r\}, \tag{B.3.6}$$

where $(\ ,\ )$ is the scalar product on $\mathfrak{t}^*$. The interior of C^0 is then $C^{0,i} = \{\gamma | \, (\gamma, \alpha_I) > 0\}$, while the boundaries of C^0 are $C^{0,b} = \{\gamma | \, (\gamma, \alpha_I) = 0\}$. The fundamental Weyl chamber is a fundamental domain for the action of the Weyl group on $\mathfrak{t}^*$, in the sense that every Weyl chamber is an image, under the W-action, of the fundamental chamber. This rational applies to other elements in $\mathfrak{t}^*$ including vectors and scalars: an element in any Weyl chamber is necessarily a image, under Weyl reflection, of a unique element in the fundamental Weyl chamber.

The simple Weyl reflections $s_{\alpha_I} =: s_I$, are Weyl reflections about hyperplanes π_{α_I} orthogonal to simple roots. As the simple roots are a subset of the set of all roots $\{\alpha_I\} \subset \Delta$, the simple Weyl reflections are a subset of the set of all Weyl reflections $\{s_I\} \subset \{s_\alpha | \alpha \in \Delta\}$. An arbitrary Weyl reflection $w \in W$ can always be expressed as either a simple Weyl reflection $w = s_I$ or the composition of repeated simple Weyl reflections $w = s_I s_J \cdots s_K$.

The fundamental Weyl chamber is a fundamental domain for the action of the

Weyl group on $\mathfrak{t}^*$, in the sense that every Weyl chamber is an image, under the W-action, of the fundamental chamber. In view of this, the Weyl chamber associated to a particular reflection, denoted C^w, can be expressed as

$$C^w = \{\gamma \subset \mathfrak{t}^* | \, (w \cdot \gamma, \alpha_I) \geq 0, \; I = 1, \ldots, r\}, \; w \in W. \tag{B.3.7}$$

The interior and boundaries of all Weyl chambers can be expressed in a similar fashion. This rational applies to other elements in $\mathfrak{t}^*$ including vectors and scalars: an element in any Weyl chamber is necessarily an image, under Weyl reflection, of a unique element in the fundamental Weyl chamber.

As an example, consider again $\mathfrak{g} = \mathfrak{su}(2)$ with $\mathfrak{t}^* = \mathbb{R}$. One can choose either $\mathbb{R}_{\geq 0}$ or $\mathbb{R}_{\leq 0}$ to be the positive direction. Choosing $\mathbb{R}_{\geq 0}$ to be the positive direction, the set of roots $\Delta = \{+2, -2\}$ decomposes as $\Delta_+ = \{+2\}$ and $\Delta_- = \{-2\}$. In this case, the positive root $\alpha = 2$ serves as the simple root, and the Weyl vector is $\varrho = 1$. There are two Weyl reflections, both of which are reflections about the hyperplane $\pi_\alpha = 0$. The simple Weyl reflection is the one that is associated to the simple root. The fundamental Weyl chamber is $C^0 = \mathbb{R}_{\geq 0}$, its interior is $C^{0,d} = \mathbb{R}_{>0}$, while its boundary is $C^{0,b} = 0$ is equivalent to the hyperplane $\pi_\alpha = 0$. Finally, the negative root is the Weyl reflected image of the unique simple root in the fundamental chamber.

In addition to the basis of simple roots $\{\alpha_I\}$ in $\mathfrak{t}^*$, there are three more bases, one of which is in $\mathfrak{t}^*$, and two of which are in $\mathfrak{t}$. Since each root in $\mathfrak{t}^*$ is associated to a coroot in $\mathfrak{t}$, each simple root α_I is associated to a simple coroot H_{α_I}. The set of simple coroots

$$\{H_{\alpha_I} | I = 1, \ldots, r = \mathrm{rank}\mathfrak{g}\}. \tag{B.3.8}$$

serve as a basis for $\mathfrak{t}$.

The remaining two bases obtained by the definition of dual elements. Dual to the basis of simple roots, we have the coweight basis

$$\left\{h^J \mid \left\langle \alpha_I, h^J \right\rangle = \delta_I{}^J \right\}, \tag{B.3.9}$$

which serves as a second basis for $\mathfrak{t}$. Note that coweight basis is sometimes referred to as magnetic weight basis. Dual to the basis of simple coroots, we have the weight basis

$$\left\{\lambda^I \mid \left\langle \lambda^I, H_J \right\rangle = \delta^I{}_J \right\}. \tag{B.3.10}$$

which serves as a second basis for $\mathfrak{t}^*$, where $H_J =: H_{\alpha_J}$. Note that the weight basis is sometimes referred to as the basis of fundamental weights. The weight basis provides an alternative expression for the Weyl vector $\varrho = \frac{1}{2}\sum_{\alpha \in \Delta_+} \alpha = \sum_I \lambda^I$.

In summary, the two bases of $\mathfrak{t}^*$ are then $\{\alpha_I\}$ and $\{\lambda^I\}$, while the two bases of $\mathfrak{t}$ are $\{H_I\}$ and $\{h^I\}$. Each of the four bases will be used to define a lattice, which will be studied in detail in what follows. Here, it suffices to note that given a basis $\{v_k\}$ of a vector space V, a lattice $L \subset V$ is the set of all integer linear combinations of the basis $L = \{\sum a_k v_k | a_k \in \mathbb{Z}\}$. In view of this, the weight lattice is $\Lambda_{\text{wt}} = \{\sum n_I \lambda^I | n_I \in \mathbb{Z}\} \subset \mathfrak{t}^*$.

Elements in $\mathfrak{t}^*$ are referred to as integral, dominant, strictly dominant depending on where they are valued. The integrality of an element in $\mathfrak{t}^*$ is a statement concerning how it is valued with respect to the weight basis. If an element $X \in \mathfrak{t}^*$ is integral, it can be expressed as an integer linear combination in the weight basis $X = \sum m_I \lambda^I$ for $m_I \in \mathbb{Z}$. Equivalently, if an element $X \in \mathfrak{t}^*$ is valued in the weight lattice $X \in \Lambda_{\text{wt}}$, it is an integral element. For this reason, the weight lattice is sometimes referred to as the lattice of integral forms. Since the Weyl vector is the sum of the weight basis vectors $\varrho = \sum_I \lambda^I$, it is sometimes referred to as the lowest integral form.

The dominance and strict dominance of an element in $\mathfrak{t}^*$ is a statement concerning how the element is valued with respect to the fundamental Weyl chamber. If $X \in \mathfrak{t}^*$ is dominant, it is valued in the closure of the fundamental Weyl chamber $X \in C^0$. If $X \in \mathfrak{t}^*$ is strictly dominant, it is valued in the interior of the fundamental Weyl chamber $X \in C^{0,d}$.

If, for example, an element $X \in \mathfrak{t}^*$ is both integral and strictly dominant, then we have $X \in \Lambda_{\text{wt}} \cap C^{d,0}$. The same nomenclature applies to sets of elements S in $\mathfrak{t}^*$. If S is a set of integral elements then it takes values in the weight lattice $S \cap \Lambda_{\text{wt}} = S$ and nowhere else $S \backslash \Lambda_{\text{wt}} = \emptyset$. The dominant elements of a set are those in closure of the fundamental Weyl chamber $S^d = S \cap C^0$, while the non-dominant elements are those in the complement $S^{nd} = S \backslash C^0$. The strictly dominant elements of a set are those in interior of the fundamental Weyl chamber $S^{sd} = S \cap C^{0,i}$, while the non-strictly dominant elements are those in the complement $S^{nsd} = S \backslash C^{0,i}$

B.4 Action of the Weyl group

In this section, we review the action of the Weyl group on root systems, focusing on the orbit, stabilizer, and fixed-point set. For a reference, see e.g. the textbook [33], or the lecture notes [53].

Let G be a group acting on a set X. The orbit of G through a point $x \in X$, is the set of points $y \in X$ which can be reached by the action of G

$$\text{orb}_G(x) = \{y = g \cdot x | g \in G\} \tag{B.4.1}$$

In other words, the orbit is the set attainable by acting on $x \in X$ with all $g \in G$. Given a point $x \in X$, the stabilizer group of x is the set of group elements

$$\text{stab}_G(x) = \{g \in G | g \cdot x = x\} \tag{B.4.2}$$

This is the set of group elements which leave invariant a particular $x \in X$. The stabilizer group is also referred to as the isotropy group at x. Given a group element $g \in G$, the fixed point set of G is the set

$$\text{fix}_X(g) = \{x \in X | g \cdot x = x\} \tag{B.4.3}$$

This is the set of elements left invariant when acting with a particular $g \in G$. The order of the stabilizer is equal to the order of the group divided by the number of distinct orbits

$$|\text{stab}_G(x)| = \frac{|G|}{|\text{orb}_G(x)|}. \tag{B.4.4}$$

This equality is a consequence of the orbit-stabilizer theorem in combination with Lagrange's theorem. Furthermore, the order of the stabilizer and the number of elements in the fixed point set are related by summing over their respective arguments

$$\sum_{g\in G} |\text{fix}_X(g)| = \sum_{x\in X} |\text{stab}_G(x)|. \tag{B.4.5}$$

Replacing the right hand side of 2nd with right hand side of 1st, we have that the number of distinct orbits is equal to the average number of fixed points.

$$|\text{orb}_G(x)| = \frac{1}{|G|}\sum_{g\in G} |\text{fix}_X(g)| \tag{B.4.6}$$

This is referred to as the lemma that is not Burnside's in [53].

Next, we proceed to put these results into the context of the action of the Weyl group on the root system. In what follows we focus on the W-action on the root system in $\mathfrak{t}^*$, but note that the above arguments are just as applicable to the dual root system in $\mathfrak{t}$.

The conventions are as follows. Let C,C^i, C^b denote the closure, interior, and boundary of an arbitrary Weyl chamber, and let $C^0, C^{0,i}, C^{0,b}$ denote the same quantities for the fundamental Weyl chamber. Let $\Pi = \bigcup_{\alpha\in\Delta} \pi_\alpha$ denote the union of all reflection hyperplanes, let $\Pi^c = \mathfrak{t}^* \backslash \Pi$ denote the complement of Π, and let $\Pi_0 = \bigcap_{\alpha\in\Delta} \pi_\alpha$ denote the intersection of all hyperplanes. Let X^b denote a generic boundary element valued on an arbitrary chamber boundary C^b. Let X^i denote a generic interior element valued on an arbitrary chamber interior C^i. Let $X^{(0)}$ denote an element valued at the intersection of all hyperplanes $\Pi^{(0)}$, which is equivalent to being valued on *all* chamber boundaries, or sitting at the origin in $\mathfrak{t}^*$. Observe that each chamber boundary is a subspace of the union of hyperplanes $C^b \subset \Pi$, and each chamber interior is a subspace of the complement of all hyperplanes $C^i \subset \Pi^c$.

We now proceed to describe the orbit, stabilizer, and fixed point set of Weyl group action. The orbit of W through an element $X \in \mathfrak{t}^*$ is the set

$$\mathrm{Orb}_W(X) = \{w \cdot X | w \in W\}, \tag{B.4.7}$$

consisting of all distinct images of X attainable by Weyl reflection. The orbit of a boundary element X^b is a set of boundary elements $\mathrm{Orb}_W(X^b) = \left\{X^b, X'^b, X''^b, \dots\right\}$. The number of distinct images in the orbit is fewer than the number of reflections $\left|\mathrm{Orb}_W(X^b)\right| < |W|$. The orbit of an interior element X^i is a set of interior elements $\mathrm{Orb}_W(X^i) = \left\{X^i, X'^i, X''^i, \dots\right\}$. The number of distinct images in the orbit is equal to the number of reflections $\left|\mathrm{Orb}_W(X^i)\right| = |W|$. The orbit of an element valued on all boundaries $X^{(0)}$ is trivial $\mathrm{Orb}_W(X^{(0)}) = \left\{X^{(0)}\right\}$.

Given an element $X \in \mathfrak{t}^*$, the stabilizer group of X is the set of Weyl reflections

$$\mathrm{Stab}_W(X) = \{w \in W | w \cdot X = X\}. \tag{B.4.8}$$

The stabilizer of a boundary element is a subgroup of the Weyl group $\mathrm{Stab}_W(X^b) \subset W$. If, for example, a particular $X^b = Y$ lies on the hyperplane $Y \in \pi_\alpha$, it is invariant under the Weyl reflection $s_\alpha(Y) = Y$, and therefore has stabilizer $\mathrm{Stab}_W(Y) = \{1_W, s_\alpha\}$. The stabilizer of an interior element is trivial $\mathrm{Stab}_W(X_i) = \{1_W\}$. The stabilizer of an element valued on all boundaries $X^{(0)} \in \Pi^{(0)}$ is the whole Weyl group $\mathrm{Stab}_W(X^{(0)}) = W$.

Given a Weyl reflection $w \in W$, the fixed point set of W is the set

$$\mathrm{fix}_{\mathfrak{t}^*}(w) = \{X \in \mathfrak{t}^* | w \cdot X = X\}. \tag{B.4.9}$$

To illustrate, consider the particular Weyl reflection s_α about the hyperplane π_α. Then, $\mathrm{Fix}_{\mathfrak{t}^*}(s_\alpha) = \{Y \in \mathfrak{t}^* | s_\alpha Y = Y\}$ is the set of all elements valued on the hyperplane $Y \in \pi_\alpha \cap \mathfrak{t}^*$. A fixed point set of a Weyl reflection cannot cannot contain interior elements X^i, but can contain boundary elements X^b, or an element valued on all boundaries $X^{(0)}$.

B.5 Covering groups

In this section we review the relationship between covers of Lie groups and Lie algebras. This includes a description of based homotopy, the fundamental group, and the classification of covering groups. For a reference, see the textbooks [35, 51], appendix A of [56] , or appendix A of [55].

The relationship between semi-simple compact Lie groups G and Lie algebras $\mathfrak{g}$ is not one-to-one, but quantifiable. In particular, a finite number Lie groups $G_1, G_2, G_3, \ldots$ that might differ globally, can correspond to the same Lie algebra $\mathfrak{g}$. For instance, $SU(2)$ and $SO(3)$ map to the same Lie algebra $\mathfrak{su}(2)$, but are not the same group. The collection of groups $G_1, G_2, G_3, \ldots$ corresponding to the same algebra $\mathfrak{g}$ are classified according to their fundamental group $\pi_1(G)$ and the center of the group $Z(G)$.

To describe the fundamental group $\pi_1(X)$ of a topological space X, we first recall based homotopy and concatenation of loops. For details see e.g. section 4.1.4 of [35]. A loop γ, based at a point $p_0 \in X$, is a continuous map $\gamma : [0,1] \to X$ which begins and ends at the base-point $\gamma(0) = \gamma(1) = p_0$. Two loops $\alpha, \beta : [0,1] \to X$, based at $p_0 \in X$, are called homotopic if there exists a homotopy, i.e. a continuous map $F : [0,1] \times [0,1] \to X$ satisfying i) $F(0,t) = p_0$, ii) $F(1,t) = p_0$, iii) $F(s,0) = \alpha(s)$, and iv) $F(s,1) = \beta(s)$ for all $t, s \in [0,1]$. Homotopy is an equivalence relation, and homotopic loops form an equivalence class $[\gamma]$. The concatenation, or product, of two based loops $\alpha, \beta : [0,1] \to X$, is a third loop $\delta = \alpha \circ \beta$ defined as

$$\delta(t) = (\alpha \circ \beta)(t) = \begin{cases} \alpha(2t) & 0 \leq t \leq 1/2, \\ \beta(2t-1) & 1/2 \leq t \leq 1. \end{cases} \tag{B.5.1}$$

Concatenation of loops generalizes to concatenation of distinct equivalence classes of homotopic loops on X. In particular, two distinct homotopy classes $[\gamma_1]$ and $[\gamma_2]$ may be concatenated to produce a third $[\gamma_1] \circ [\gamma_2] = [\gamma_1 \circ \gamma_2] = [\gamma_3]$.

The fundamental group $\pi_1(X)$ is then the group in which the elements are the closed set of distinct homotopy classes of loops in X, and the group operation is concatenation. The identity element $e \in \pi_1$ is the homotopy class $[\gamma_0]$ where $\gamma_0 = \gamma_0(t)$ is a constant loop based at $p_0 \in X$ for all t, and the inverse of an element $[\gamma] \in \pi_1$ is $\left[\gamma^{-1}\right]$ where $\gamma^{-1}(t) = \gamma(1-t)$.

The fundamental group of a Lie group $\pi_1(G)$ is the case where the topological space X is taken to be G as a manifold. When the topological space is also a Lie group, π_1 is Abelian. If the fundamental group consists of only the identity element $\pi_1(G) = \{e\}$, it is said to be trivial. In this case, the manifold is simply-connected as a topological space, such that all loops can be continuously deformed to a point.

The center of the Lie group $Z(G)$, on the other hand, is the subgroup of G containing all elements that commute with all other elements

$$Z(G) = \{z \in G \mid zg = gz \;\forall\; g \in G\}. \tag{B.5.2}$$

To have a trivial center $Z(G) = \{e\}$ means that there are no elements of G other than the identity that commute with all $g \in G$.

For an arbitrary G' belonging to the collection $G_1, G_2, G_3, \ldots$, there exist two other groups denoted $\tilde{G}$ and H' (theorem VII.6.4 [51]). The group $\tilde{G}$ is called universal covering group of G', with the property that it is a simply-connected group with trivial fundamental $\pi_1(\tilde{G}) = \{e\}$. The group H', on the other hand, is a discrete normal subgroup of the universal cover $H' \subset \tilde{G}$, characterized by $H' \simeq \pi_1(G')$. H' is an Abelian group, since $\pi_1(G')$ is Abelian when G' is a Lie group. In addition to being a subgroup of the universal cover, H' is also a subgroup of its center $H' \subseteq Z(\tilde{G})$.

The arbitrary G' in $G_1, G_2, G_3, \ldots$ is the quotient group of the universal cover and the subgroup $G' = \tilde{G}/H'$. There are as many of these quotient groups as there are discrete Abelian subgroups of $Z(\tilde{G})$. For instance, if $Z(\tilde{G})$ can be decomposed into two subgroups H', H'', there are two quotient groups $G' = \tilde{G}/H'$, $G'' = \tilde{G}/H''$.

The final quotient group is called the adjoint group G_{ad}, which is obtained by quotienting the universal cover by its entire center $G_{ad} = \tilde{G}/Z(\tilde{G})$. The adjoint group is characterized by trivial center $Z(G_{ad}) = \{e\}$.

It is then possible to sort the $G_1, G_2, G_3, \ldots$ sharing the same $\mathfrak{g}$ in a list according to $\pi_1(G)$ and $Z(G)$. On the one end of the list is the universal cover $\tilde{G}$ with trivial $\pi_1(\tilde{G})$, in between are all the quotient groups G',G'', ... obtained from subgroups $Z(\tilde{G})$, and on the other end is the adjoint group $G_{ad} = \tilde{G}/Z(\tilde{G})$ with trivial $Z(G_{ad})$. The list is then

$$\tilde{G}, \ldots, G' = \frac{\tilde{G}}{H'},\ G'' = \frac{\tilde{G}}{H''}, \ldots, G_{ad} = \frac{\tilde{G}}{Z(\tilde{G})}. \tag{B.5.3}$$

where $\tilde{G}$ is the group in $G_1, G_2, G_3, \ldots$ with trivial $\pi_1(\tilde{G})$, G_{ad} is the group with trivial $Z(G_{ad})$, and the quotient groups $G', G'', \ldots$ are obtained from all discrete Abelian subgroups $H', H'', \ldots$ of $Z(\tilde{G})$.

B.6 Lattices

In this section, we review lattices, lattices of Lie algebras, then lattices of Lie groups. For a reference see [51, 53], or the lecture notes [54].

Let V be a finite dimensional real Euclidean vector space of dimension $m = \dim V$, and let $\{v_k | k = 1, \ldots, m\}$ be a basis of V. A lattice

$$L = \left\{ \sum a_k v_k \mid a_k \in \mathbb{Z} \right\}, \tag{B.6.1}$$

is a subgroup of V, with addition as the group operation.

The dual lattice L^* of L in the dual vector space V^* of V is

$$L^* = \{\ y \in V^* \mid \langle x, y\rangle \in \mathbb{Z} \text{ for all } x \in L\ \} \tag{B.6.2}$$

where $\langle\cdot,\cdot\rangle : V \otimes V^* \to \mathbb{R}$ is the inner product between dual vector spaces. The dual lattice is the set of all elements dual to the original lattice.

Given two lattices L_1, L_2 in the same vector space V, L_1 is a called a sublattice of L_2 if L_1 is a subset of L_2. If L_1 is a sublattice of L_2, it is denoted by the inclusion $L_1 \subset L_2$. Then, L_1 is the coarser lattice, while L_2 is the finer lattice.

As lattices are discrete groups, the notion of quotient groups extends to lattices. The quotient of two lattices in the same vector space is L_1/L_2. Duality acts on quotients as $L_1/L_2 \simeq L_2^*/L_1^*$. For several inclusions $L_1 \supset L_2 \supset L_3$, the quotient obeys $L_1/L_2 = (L_1/L_3)/(L_2/L_3)$.

In the context of lattices, the Poisson summation formula has a neat generalization. For the m-dimensional lattice $L \subset \mathbb{R}^m$, the generalized Poisson summation formula is

$$\sum_{x\in L} f(x) = \frac{1}{\operatorname{covol}(L)} \sum_{y\in L^*} \hat{f}(y). \tag{B.6.3}$$

Here, L^* is the dual lattice of L, the Fourier transform of $f(x)$ is

$$\hat{f}(y) = \int_{\mathbb{R}^m} e^{-2\pi i\langle x,y\rangle} f(x)\, \mathrm{d}x, \tag{B.6.4}$$

and $\operatorname{covol}(L) = \operatorname{vol}(\mathbb{R}^n/L)$ is the volume of a fundamental domain of L, i.e. the unit plaquette.

Since conventions for the Fourier transform often differ, we note here that a general Fourier transform may be written

$$\hat{f}(\xi) = \mathcal{F}_x^{(a,b)}[f(x)](\xi) = \sqrt{\frac{|b|}{(2\pi)^{1-a}}} \int_{-\infty}^{+\infty} f(x)\, e^{ibx\xi} \mathrm{d}x, \tag{B.6.5}$$

where a, b are convention-dependent coefficients. A common choice for (a, b) is $(0, 1)$, while in B.6.4 the choice is $(1, -2\pi)$.

Recall from the discussion of root systems that there are four vector space bases associated to the algebra $\mathfrak{g} = \mathrm{Lie}G$, two of which are in $\mathfrak{t}^*$, the other two of which are in $\mathfrak{t}$. In $\mathfrak{t}^*$ we have the basis of simple roots $\{\alpha_I\}$ and the basis of weights $\{\lambda^I\}$, while in $\mathfrak{t}$ we have the basis of simple coroots $\{H_I\}$, and the basis of coweights $\{h^I\}$. In what follows, we use each basis to define lattice. These four lattices are associated to the algebra $\mathfrak{g} = \mathrm{Lie}G$, as opposed to being associated to the group G, in the sense that they are defined using algebraic quantities such as roots, weights, coroots, and coweights.

Following this, two group associated lattices will be defined using the exponential map from the algebra to the group. As a reminder of this association, the lattices of the algebra will be denoted $L^{\mathfrak{g}}$, while the lattices of the group will be denoted L^G. Since a Lie group is a manifold, and a Lie algebras is tangent spaces of that manifold, group-valued quantities pertain to global properties of the manifold, while algebra valued quantities pertain to local properties of the manifold.

The root lattice in $\mathfrak{t}^*$ is the set of all integer linear combinations of the basis of simple roots

$$\Lambda^{\mathfrak{g}}_{\text{rt}} = \left\{ \sum n_I \alpha_I \mid n_I \in \mathbb{Z} \right\}, \ \Lambda^{\mathfrak{g}}_{\text{rt}} \subset \mathfrak{t}^*. \tag{B.6.6}$$

Similarly, the coroot lattice in $\mathfrak{t}$ is the set of all integer linear combinations of the simple coroots

$$\Lambda^{\mathfrak{g}}_{\text{cort}} = \left\{ \sum m_I H_I \mid m_I \in \mathbb{Z} \right\}, \ \Lambda^{\mathfrak{g}}_{\text{cort}} \subset \mathfrak{t} \tag{B.6.7}$$

The weight lattice in $\mathfrak{t}^*$ is the dual lattice of the coroot lattice

$$\Lambda^{\mathfrak{g}}_{\text{wt}} = \{ \lambda \in \mathfrak{t}^* \mid \langle \lambda, H \rangle \in \mathbb{Z}, \ \text{for all } H \in \Lambda_{\text{cr}} \}, \ \Lambda^{\mathfrak{g}}_{\text{wt}} \subset \mathfrak{t}^*. \tag{B.6.8}$$

Similarly, the coweight lattice in $\mathfrak{t}$ is the dual lattice of the root lattice

$$\Lambda^{\mathfrak{g}}_{\text{cowt}} = \{ h \in \mathfrak{t} \mid \langle \alpha, h \rangle \in \mathbb{Z}, \ \text{for all } \alpha \in \Lambda_{\text{rt}} \}, \ \Lambda^{\mathfrak{g}}_{\text{cowt}} \subset \mathfrak{t}. \tag{B.6.9}$$

The root lattice is a sublattice of the weight lattice

$$\Lambda^{\mathfrak{g}}_{\text{rt}} \subset \Lambda^{\mathfrak{g}}_{\text{wt}}, \tag{B.6.10}$$

while the coroot lattice is a sublattice of the coweight lattice

$$\Lambda^{\mathfrak{g}}_{\text{cort}} \subset \Lambda^{\mathfrak{g}}_{\text{cowt}}. \tag{B.6.11}$$

To define the group associated lattices, recall that the exponential map is a map from elements of the algebra $\mathfrak{g}$ to elements of the group G. When restricted to the Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, the exponential map is

$$\exp \ : \ \mathfrak{t} \to G, \tag{B.6.12}$$

$$X \mapsto e^{2\pi i X}. \tag{B.6.13}$$

The kernel of the exponential map, ker (exp), is the elements in the Cartan subalgebra $X \in \mathfrak{t}$ whose image is the identity element 1_G of G. The kernel of the exponential map defines a lattice in $\mathfrak{t}$, referred to as the cocharacter lattice, which is associated to the group G, as opposed to the algebra $\mathfrak{g}$.

The cocharacter lattice in $\mathfrak{t}$, defined as the kernel of the exponential map restricted to the Cartan subalgebra, is

$$\Lambda_{\text{coch}}^{G} = \{\gamma \in \mathfrak{t} | \exp(2\pi \mathrm{i}\gamma) = 1_G\}, \ \Lambda_{\text{coch}}^{G} \subset \mathfrak{t} \tag{B.6.14}$$

The cocharacter lattice satisfies the inclusions

$$\Lambda_{\text{cort}}^{\mathfrak{g}} \subseteq \Lambda_{\text{coch}}^{G} \subseteq \Lambda_{\text{cowt}}^{\mathfrak{g}}. \tag{B.6.15}$$

It is not possible for the three lattices in $\mathfrak{t}$ to coincide, since the coroot lattice is a sublattice of the coweight lattice $\Lambda_{\text{cort}}^{\mathfrak{g}} \subset \Lambda_{\text{cowt}}^{\mathfrak{g}}$. The character lattice in $\mathfrak{t}^*$ is the dual lattice of the cocharacter lattice

$$\Lambda_{\text{ch}}^{G} = \left\{ \mu \in \mathfrak{t}^* \mid \langle \mu, \gamma \rangle \in \mathbb{Z}, \text{ for all } \gamma \in \Lambda_{\text{coch}}^{G} \right\}, \ \Lambda_{\text{ch}}^{G} \subset \mathfrak{t}^*. \tag{B.6.16}$$

The character lattice satisfies the inclusions

$$\Lambda_{\text{rt}}^{\mathfrak{g}} \subseteq \Lambda_{\text{ch}}^{G} \subseteq \Lambda_{\text{wt}}^{\mathfrak{g}}. \tag{B.6.17}$$

It is not possible for the three lattices in $\mathfrak{t}^*$ to coincide, since the root lattice is a sublattice of the weight lattice $\Lambda_{\text{rt}}^{\mathfrak{g}} \subset \Lambda_{\text{wt}}^{\mathfrak{g}}$. Due to the inclusions, the elements of all three lattices in $\mathfrak{t}^*$ are integral. This is because being an integral element is equivalent to being an element of $\Lambda_{\text{wt}}^{\mathfrak{g}}$.

The values taken by the cocharacter and character lattices depends on the cover of G. In particular, the lattices depend on what value G takes in the list of possible covers $\tilde{G}, \ldots, G' = \tilde{G}/H', G'' = \tilde{G}/H'', \ldots, G_{ad} = \tilde{G}/Z(\tilde{G})$ described in the discussion on covering groups. The example to keep in mind is that for $\mathfrak{g} = \mathfrak{su}(2)$, the universal cover is $\tilde{G} = SU(2)$, while the adjoint cover is $G_{\text{ad}} = SO(3)$, where $SO(3) \simeq PSU(2) = SU(2)/\mathbb{Z}_2$. That being said, the three general cases to consider are the universal cover $G = \tilde{G}$, the adjoint cover $G = G_{\text{ad}}$, or an intermediate cover $G = G'$.

When the group is the universal cover $G = \tilde{G}$, the cocharacter lattice coincides with the coweight lattice, and the character lattice coincides with the weight lattice

$$\Lambda_{\text{cort}}^{\mathfrak{g}} \subset \Lambda_{\text{coch}}^{G=\tilde{G}} \simeq \Lambda_{\text{cowt}}^{\mathfrak{g}}, \quad \Lambda_{\text{rt}}^{\mathfrak{g}} \subset \Lambda_{\text{ch}}^{G=\tilde{G}} \simeq \Lambda_{\text{wt}}^{\mathfrak{g}}. \tag{B.6.18}$$

When the group is the adjoint cover $G = G_{\text{ad}}$, the cocharacter lattice coincides with the coroot lattice and the character lattice coincides with the root lattice

$$\Lambda^{\mathfrak{g}}_{\text{cort}} \simeq \Lambda^{G=G_{\text{ad}}}_{\text{coch}} \subset \Lambda^{\mathfrak{g}}_{\text{cowt}}, \quad \Lambda^{\mathfrak{g}}_{\text{rt}} \simeq \Lambda^{G=G_{\text{ad}}}_{\text{ch}} \subset \Lambda^{\mathfrak{g}}_{\text{wt}}. \tag{B.6.19}$$

When the group is some other cover $G = G'$, intermediate between the universal cover and the adjoint cover, we have the sublattices

$$\Lambda^{\mathfrak{g}}_{\text{cort}} \subset \Lambda^{G=G'}_{\text{coch}} \subset \Lambda^{\mathfrak{g}}_{\text{cowt}}, \quad \Lambda^{\mathfrak{g}}_{\text{rt}} \subset \Lambda^{G=G'}_{\text{ch}} \subset \Lambda^{\mathfrak{g}}_{\text{wt}}. \tag{B.6.20}$$

By considering lattice quotients, it is possible to determine the center $Z(G)$, as well as the fundamental group $\pi_1(G)$, for arbitrary G. The group isomorphisms obtained from the quotient lattices in $\mathfrak{t}$ are

$$Z(G) = \frac{\Lambda^{\mathfrak{g}}_{\text{cowt}}}{\Lambda^{G}_{\text{coch}}}, \quad \pi_1(G) = \frac{\Lambda^{G}_{\text{coch}}}{\Lambda^{\mathfrak{g}}_{\text{cort}}}. \tag{B.6.21}$$

Similarly, the group isomorphisms obtained from the quotient lattices in $\mathfrak{t}^*$ are

$$Z(G) = \frac{\Lambda^{G}_{\text{ch}}}{\Lambda^{\mathfrak{g}}_{\text{rt}}}, \quad \pi_1(G) = \frac{\Lambda^{\mathfrak{g}}_{\text{wt}}}{\Lambda^{G}_{\text{ch}}}. \tag{B.6.22}$$

B.7 Translations by the Weyl vector

In this section, we describe in detail how to shift elements of integral lattices by the Weyl vector. In doing so, we make and prove some propositions similar to proposition IX.2.5 in [51].

Let T_ϱ denote the shift $T_\varrho\,(X) = X + \varrho$ of an element $X \in \mathfrak{t}^*$ by the Weyl vector ϱ. The Weyl vector always points into the fundamental Weyl chamber, which is the positive direction chosen for $\mathfrak{t}^*$. In view of this, T_ϱ is the translation of an element X, by a distance ϱ, in the positive direction of $\mathfrak{t}^*$, that is to say, toward the interior of the fundamental Weyl chamber $C^{0,i}$. More generally, translation by the Weyl vector is a map between elements of $\mathfrak{t}^*$

$$\begin{aligned} T_\varrho \;\;:\;\; & \mathfrak{t}^* \to \mathfrak{t}^*, \\ & X \mapsto X' = X + \varrho. \end{aligned} \tag{B.7.1}$$

The first property of T_ϱ is that it maps integral elements to other integral elements. This can be verified by restricting T_ϱ to the set of all integral elements of $\mathfrak{t}^*$, or equivalently, the elements of the weight lattice $\Lambda^{\mathfrak{g}}_{\text{wt}}$ in $\mathfrak{t}^*$. The restriction of T_ϱ to the weight lattice is a map from the weight lattice to itself

$$\begin{aligned} T_\varrho \quad &: \quad \Lambda_{\mathrm{wt}}^{\mathfrak{g}} \to \Lambda_{\mathrm{wt}}^{\mathfrak{g}}, \\ &X \mapsto X' = X + \varrho. \end{aligned} \tag{B.7.2}$$

To show this, use the weight basis $\{\lambda^I\}$ to express the lattice as

$$\Lambda_{\mathrm{wt}} = \left\{ \sum_{I=1}^{r} n_I \lambda^I \;\middle|\; n_I \in \mathbb{Z} \right\}, \tag{B.7.3}$$

the lattice element as $X = \sum_{I=1}^{r} n_I \lambda^I$, and the Weyl vector as $\varrho = \sum_{I=1}^{r} \lambda^I$. The shift then reads

$$\sum_{I=1}^{r} n_I \lambda^I \mapsto X' = \sum_{I=1}^{r} n_I \lambda^I + \sum_{I=1}^{r} \lambda^I = \sum_{I=1}^{r} (n_I + 1)\, \lambda^I = \sum_{I=1}^{r} n'_I \lambda^I \tag{B.7.4}$$

Due to $n_I \in \mathbb{Z}$, the coefficient in the last expression $n'_I = n_I + 1$ is also an integer $n'_I \in \mathbb{Z}$. Since the shifted element X' can be expressed in the weight basis with integer coefficient $X' = \sum_{I=1}^{r} n'_I \lambda^I, n'_I \in \mathbb{Z}$, it is valued in the weight lattice $X' \in \Lambda_{\mathrm{wt}}$. This verifies that T_ϱ maps integral elements to other integral elements.

Next, we verify two properties of how T_ϱ acts on integral elements with respect to the fundamental Weyl chamber. The notation is as follows. Let the closure, interior, and boundary of the fundamental Weyl chamber be denoted by $C^0, C^{0,i}$, and $C^{0,b}$ respectively. Let the subsets of dominant, strictly-dominant, non-dominant, non-strictly dominant, and boundary elements of a lattice Λ be denoted respectively by the superscripts

$$\Lambda^{\mathrm{d}} = \Lambda \cap C^0,\ \Lambda^{\mathrm{sd}} = \Lambda \cap C^{0,i},\ \Lambda^{\mathrm{nd}} = \Lambda \backslash C^0,\ \Lambda^{\mathrm{nsd}} = \Lambda \backslash C^{0,i},\ \Lambda^{\mathrm{b}} = \Lambda \cap C^b \tag{B.7.5}$$

The first property to verify is that T_ϱ maps dominant integral elements $X \in \Lambda_{\mathrm{wt}}^{\mathrm{d}} = \Lambda_{\mathrm{wt}}^{\mathfrak{g}} \cap C^0$ to strictly dominant integral elements $X' \in \Lambda_{\mathrm{wt}}^{\mathrm{sd}} = \Lambda_{\mathrm{wt}}^{\mathfrak{g}} \cap C^{0,i}$, and that this is a bijection, in the sense that the inverse map exists T_ϱ^{-1}, and elements X and X' are in one-to-one correspondence. The second property to verify is that T_ϱ maps non-dominant integral elements $X \in \Lambda_{\mathrm{wt}}^{\mathrm{nd}} = \Lambda_{\mathrm{wt}}^{\mathfrak{g}} \backslash C^0$ to non-strictly dominant integral elements $X' \in \Lambda_{\mathrm{wt}}^{\mathrm{nsd}} = \Lambda_{\mathrm{wt}}^{\mathfrak{g}} \backslash C^{0,i}$.
Beginning with the first property, we want to show that the restriction of T_ϱ to dominant integral elements $\Lambda_{\mathrm{wt}}^{\mathrm{d}}$ is a map to unique strictly dominant integral elements

$$\begin{aligned} T_\varrho \quad &: \quad \Lambda_{\mathrm{wt}}^{\mathrm{d}} \to \Lambda_{\mathrm{wt}}^{\mathrm{sd}}, \\ &X \mapsto X' = X + \varrho \end{aligned} \tag{B.7.6}$$

To do so, we use the weight basis $\{\lambda^I\}$ to express the subsets as

$$\Lambda^{\mathrm{d}}_{\mathrm{wt}} = \left\{ \sum_{I=1}^{r} n_I \lambda^I \,\middle|\, n_I \geq 0, n_I \in \mathbb{Z} \right\}, \tag{B.7.7}$$

$$\Lambda^{\mathrm{sd}}_{\mathrm{wt}} = \left\{ \sum_{I=1}^{r} m_I \lambda^I \,\middle|\, m_I \geq 1, m_I \in \mathbb{Z} \right\}. \tag{B.7.8}$$

The shift then reads

$$\sum_{I=1}^{r} n_I \lambda^I \mapsto X' = \sum_{I=1}^{r} n_I \lambda^I + \sum_{I=1}^{r} \lambda^I = \sum_{I=1}^{r} (n_I + 1) \lambda^I = \sum_{I=1}^{r} n'_I \lambda^I \tag{B.7.9}$$

Due to $n_I \geq 0, n_I \in \mathbb{Z}$, the coefficient in the last expression $n'_I = n_I + 1$ is a positive integer $n'_I \geq 1, n'_I \in \mathbb{Z}$. Since the shifted element X' can be expressed in the weight basis with a positive integer coefficient, we have $n'_I = m_I$ for $m_I \geq 1, m_I \in \mathbb{Z}$, and consequently, X' is integral and strictly dominant $X' = \sum_{I=1}^{r} m_I \lambda^I \in \Lambda^{\mathrm{sd}}_{\mathrm{wt}}$. This verifies that T_ϱ maps each dominant integral element to a unique strictly dominant integral element.
Next, we want to show that T_ϱ^{-1} uniquely maps strictly dominant integral elements to dominant integral elements. The inverse map reads

$$\begin{aligned} T_\varrho^{-1} \;:\; & \Lambda^{\mathrm{sd}}_{\mathrm{wt}} \to \Lambda^{\mathrm{d}}_{\mathrm{wt}}, \\ & X' \mapsto X'' = X' - \varrho. \end{aligned} \tag{B.7.10}$$

Expressing the shift in weight basis, we have

$$\sum_{I=1}^{r} m_I \lambda^I \mapsto X'' = \sum_{I=1}^{r} m_I \lambda^I - \sum_{I=1}^{r} \lambda^I = \sum_{I=1}^{r} (m_I - 1) \lambda^I = \sum_{I=1}^{r} m''_I \lambda^I. \tag{B.7.11}$$

Due to $m_I \geq 1, m_I \in \mathbb{Z}$, the coefficient in the last expression $m''_I = m_I - 1$ is a non-negative integer $m''_I \geq 0, m''_I \in \mathbb{Z}$. Since X'' can be expressed in the weight basis with non-negative integer coefficient, we have $m''_I = n_I$ for $n_I \geq 0, n_I \in \mathbb{Z}$, and consequently, X'' is integral and dominant $X'' = \sum_{I=1}^{r} n_I \lambda^I \in \Lambda^{\mathrm{d}}_{\mathrm{wt}}$. This verifies that T_ϱ^{-1} maps each strictly dominant integral element to a unique dominant integral element. This also verifies the bijection, since the elements of the sets $\Lambda^{\mathrm{d}}_{\mathrm{wt}}$ and $\Lambda^{\mathrm{sd}}_{\mathrm{wt}}$ are uniquely related by T_ϱ and T_ϱ^{-1}.

Proceeding with the verification of the second property, we want to show that the restriction of T_ϱ to non-dominant integral elements is a map to non-strictly dominant integral elements

$$T_\varrho \quad : \quad \Lambda^{\text{nd}}_{\text{wt}} \to \Lambda^{\text{nsd}}_{\text{wt}}, \qquad Y \mapsto Y' = Y + \varrho \tag{B.7.12}$$

In the weight basis, the sets are

$$\Lambda^{\text{nd}}_{\text{wt}} = \left\{ \sum_{I=1}^{r} j_I \lambda^I \,\middle|\, j_I \leq -1, j_I \in \mathbb{Z} \right\} \tag{B.7.13}$$

$$\Lambda^{\text{nsd}}_{\text{wt}} = \left\{ \sum_{I=1}^{r} k_I \lambda^I \,\middle|\, k_I \leq 0, k_I \in \mathbb{Z} \right\} \tag{B.7.14}$$

The shift then reads

$$\sum_{I=1}^{r} j_I \lambda^I \mapsto Y' = \sum_{I=1}^{r} j_I \lambda^I + \sum_{I=1}^{r} \lambda^I = \sum_{I=1}^{r} (j_I + 1)\, \lambda^I = \sum_{I=1}^{r} j'_I \lambda^I. \tag{B.7.15}$$

Due to $j_I \leq -1, j_I \in \mathbb{Z}$, the coefficient in the last expression $j'_I = j_I + 1$ is a non-positive integer $j'_I \leq 0, j'_I \in \mathbb{Z}$. Since Y' can be can be expressed in the weight basis with non-positive integer coefficient, we have $j'_I = k_I$ for $k_I \leq 0, k_I \in \mathbb{Z}$, and consequently, Y' is a non-strictly dominant integral element $Y' = \sum_{I=1}^{r} k_I \lambda^I \in \Lambda^{\text{nsd}}_{\text{wt}}$.

Next, we turn our attention to how T_ϱ acts on different parts of the character lattice Λ^G_{ch}. To begin with, recall that the elements of Λ^G_{ch} are integral due to the inclusions $\Lambda^G_{\text{ch}} \subseteq \Lambda^{\mathfrak{g}}_{\text{wt}}$. As a consequence of the inclusions, we have the subsets $\Lambda^{\text{d}}_{\text{ch}} \subseteq \Lambda^{\text{d}}_{\text{wt}}$, $\Lambda^{\text{sd}}_{\text{ch}} \subseteq \Lambda^{\text{sd}}_{\text{wt}}$, $\Lambda^{\text{nd}}_{\text{ch}} \subseteq \Lambda^{\text{nd}}_{\text{wt}}$, $\Lambda^{\text{nsd}}_{\text{ch}} \subseteq \Lambda^{\text{nsd}}_{\text{wt}}$, and $\Lambda^{\text{b}}_{\text{ch}} \subseteq \Lambda^{\text{b}}_{\text{wt}}$. Furthermore, recall that Λ^G_{ch} takes values according to the cover of G. In particular, specifying the cover fixes the value the character lattice, which in turn fixes the properties its image under T_ϱ. There are two important cases for the cover when considering the properties of S. The first case is when the group is the universal cover $G = \tilde{G}$, and the character lattice coincides with the weight lattice $\Lambda^{\mathfrak{g}}_{\text{rt}} \subset \Lambda^{G=\tilde{G}}_{\text{ch}} \simeq \Lambda^{\mathfrak{g}}_{\text{wt}}$, while the second case is when the group is any other cover $G \neq \tilde{G}$, and the character lattice is a sublattice of the weight lattice $\Lambda^{\mathfrak{g}}_{\text{rt}} \subseteq \Lambda^{G\neq\tilde{G}}_{\text{ch}} \subset \Lambda^{\mathfrak{g}}_{\text{wt}}$. Observe that the second case includes both the adjoint cover $\Lambda^{\mathfrak{g}}_{\text{rt}} \simeq \Lambda^{G=G_{\text{ad}}}_{\text{ch}} \subset \Lambda^{\mathfrak{g}}_{\text{wt}}$, as well as any intermediate cover $\Lambda^{\mathfrak{g}}_{\text{rt}} \subset \Lambda^{G=G'}_{\text{ch}} \subset \Lambda^{\mathfrak{g}}_{\text{wt}}$.

The restriction of T_ϱ to a generic character lattice $\Lambda^G_{\text{ch}} \subseteq \Lambda^{\mathfrak{g}}_{\text{wt}}$, with unspecified cover of G, maps to a set

$$T_\varrho \quad : \quad \Lambda^G_{\text{ch}} \to S, \qquad X \mapsto X' = X + \varrho. \tag{B.7.16}$$

The properties of the image

$$S = \left\{ X + \varrho \,\middle|\, X \in \Lambda_{\text{ch}}^{G} \right\} \tag{B.7.17}$$

at this stage are that i) S is an infinite set since the lattice Λ_{ch}^{G} contains infinitely many elements; and ii) the elements of S are integral due to Λ_{ch}^{G} being a lattice of integral elements, and T_ϱ mapping between integral elements. In the case of the universal cover, the image of T_ϱ is

$$S = \left\{ X + \varrho \,\middle|\, X \in \Lambda_{\text{ch}}^{G=\tilde{G}} \simeq \Lambda_{\text{wt}}^{\mathfrak{g}} \right\}, \tag{B.7.18}$$

and the set is the weight lattice $S \simeq \Lambda_{\text{wt}}^{\mathfrak{g}}$. This case is described previously in (B.7.2), in which T_ϱ maps the weight lattice to itself. In the case of any other cover, the image of T_ϱ is

$$S = \left\{ X + \varrho \,\middle|\, X \in \Lambda_{\text{ch}}^{G\neq\tilde{G}} \subset \Lambda_{\text{wt}}^{\mathfrak{g}} \right\}, \tag{B.7.19}$$

and instead of a lattice we have an infinite set of integral elements $S \subset \Lambda_{\text{wt}}^{\mathfrak{g}}$. This is a new case, that applies to both the adjoint cover $G = G_{\text{ad}}$, as well as any intermediate cover $G = G'$.

Now, we consider the further restriction of T_ϱ, first to the subset of dominant character lattice elements $\Lambda_{\text{ch}}^{\text{d}} \subseteq \Lambda_{\text{wt}}^{\text{d}}$, then to the subset of non-dominant character lattice elements $\Lambda_{\text{ch}}^{\text{nd}} \subseteq \Lambda_{\text{wt}}^{\text{nd}}$.

The restriction of T_ϱ to the subset of dominant character lattice elements $\Lambda_{\text{ch}}^{\text{d}} = \Lambda_{\text{ch}}^{G} \cap C^0$ maps to a set

$$\begin{aligned} T_\varrho \quad : \quad & \Lambda_{\text{ch}}^{\text{d}} \to S', \\ & X \mapsto X' = X + \varrho. \end{aligned} \tag{B.7.20}$$

The properties of the image $S' = \left\{X + \varrho | X \in \Lambda_{\text{ch}}^{\text{d}}\right\}$ are i) S' is an infinite set because $\Lambda_{\text{ch}}^{\text{d}}$ is an infinite set; ii) the elements of S' are integral because T_ϱ maps between integral elements; and iii) the elements of S are strictly dominant because T_ϱ maps from dominant integral elements to strictly dominant integral elements. Therefore, S is an infinite set of strictly dominant integral elements $S' \subseteq \Lambda_{\text{wt}}^{\text{sd}}$.

Due to the inclusion $\Lambda_{\text{ch}}^{\text{d}} \subseteq \Lambda_{\text{wt}}^{\text{d}}$, the map (B.7.20) is a restriction of the bijective map (B.7.7) between $\Lambda_{\text{wt}}^{\text{d}}$ and $\Lambda_{\text{wt}}^{\text{sd}}$. Since the restriction of a bijection is itself a bijection, each element of $\Lambda_{\text{ch}}^{\text{d}}$ is uniquely related to an element of S' through T_ϱ and its inverse T_ϱ^{-1}. The inverse map is

$$\begin{aligned} T_\varrho^{-1} \quad : \quad & S' \to \Lambda_{\text{ch}}^{\text{d}}, \\ & X' \mapsto X'' = X' - \varrho. \end{aligned} \tag{B.7.21}$$

In the case of the universal cover, the image of T_ϱ is

$$S' = \left\{ X + \varrho \;\middle|\; X \in \Lambda_{\text{ch}}^{\text{d},G=\tilde{G}} \simeq \Lambda_{\text{wt}}^{\text{d}} \right\}, \tag{B.7.22}$$

and we have the set of all strictly dominant integral elements $S' \simeq \Lambda_{\text{wt}}^{\text{sd}}$. This case is described previously in (B.7.6).

In the case of any other cover, the image of T_ϱ is

$$S' = \left\{ X + \varrho \;\middle|\; X \in \Lambda_{\text{ch}}^{\text{d},G\neq\tilde{G}} \subset \Lambda_{\text{wt}}^{\text{d}} \right\}, \tag{B.7.23}$$

and we have a subset of strictly dominant integral elements $S' \subset \Lambda_{\text{wt}}^{\text{sd}}$. This is a new case, that applies to both the adjoint cover $G = G_{\text{ad}}$, as well as any intermediate cover $G = G'$.

The restriction of T_ϱ to the subset of non-dominant character lattice elements $\Lambda_{\text{ch}}^{\text{nd}} = \Lambda_{\text{ch}}^{G} \backslash C^0$ is the map

$$\begin{aligned} T_\varrho \quad : \quad & \Lambda_{\text{ch}}^{\text{nd}} \to S'', \\ & Y \mapsto Y' = Y + \varrho. \end{aligned} \tag{B.7.24}$$

The properties of the image $S'' = \left\{Y + \varrho | Y \in \Lambda_{\text{ch}}^{\text{nd}}\right\}$ are i) S'' is an infinite set because $\Lambda_{\text{ch}}^{\text{nd}}$ is an infinite set; ii) the elements of S'' are integral because T_ϱ maps between integral elements; and iii) the elements of S are non-strictly dominant because T_ϱ maps non-dominant integral elements to non-strictly dominant integral elements. Therefore, S'' is an infinite set of non-strictly dominant integral elements $S'' \subseteq \Lambda_{\text{wt}}^{\text{nsd}}$. In the case of the universal cover, the image of T_ϱ is

$$S'' = \left\{ Y + \varrho \;\middle|\; Y \in \Lambda_{\text{ch}}^{\text{nd},G=\tilde{G}} \simeq \Lambda_{\text{wt}}^{\text{nd}} \right\}, \tag{B.7.25}$$

and we have the set of all non-strictly dominant integral elements $S'' \simeq \Lambda_{\text{wt}}^{\text{nsd}}$. This case was described previously in (B.7.12). In the case of any other cover, the image of T_ϱ is

$$S'' = \left\{ Y + \varrho \;\middle|\; Y \in \Lambda_{\text{ch}}^{\text{nd},G\neq\tilde{G}} \subset \Lambda_{\text{wt}}^{\text{nd}} \right\}, \tag{B.7.26}$$

and we have a subset of non-strictly dominant integral elements $S'' \subset \Lambda_{\text{wt}}^{\text{nsd}}$. This is a new case, that applies to both the adjoint cover $G = G_{\mathrm{ad}}$, as well as any intermediate cover $G = G'$.

In view of (B.7.16), (B.7.20), and (B.7.24), we have the following properties when shifting elements of the character lattice by the Weyl vector. T_ϱ maps the dominant elements of the character lattice $\Lambda_{\mathrm{ch}}^{\mathrm{d}} = \Lambda_{\mathrm{ch}}^{G} \cap C^0$ to an infinite set $S' = \left\{X + \varrho \mid X \in \Lambda_{\mathrm{ch}}^{\mathrm{d}}\right\}$ of strictly dominant integral elements $S' \subseteq \Lambda_{\mathrm{wt}}^{\mathrm{sd}} = \Lambda_{\mathrm{wt}}^{\mathfrak{g}} \cap C^{0,i}$. The map is bijective, that is to say, the elements of the sets $\Lambda_{\mathrm{ch}}^{\mathrm{d}}$ and S' are in one-to-one correspondence under T_ϱ and its inverse T_ϱ^{-1}. In particular, T_ϱ shifts the elements of $\Lambda_{\mathrm{ch}}^{G}$ valued in the closure of the fundamental Weyl chamber in the positive direction chosen for $\mathfrak{t}^*$, i.e. toward the interior of the fundamental Weyl chamber. The set S' cannot have elements valued on the boundary of the fundamental Weyl chamber $S' \cap C^{0,b} = \emptyset$. T_ϱ maps the non-dominant elements of the character lattice $\Lambda_{\mathrm{ch}}^{\mathrm{nd}} = \Lambda_{\mathrm{ch}}^{G} \backslash C^0$ to an infinite set $S'' = \left\{X + \varrho \mid X \in \Lambda_{\mathrm{ch}}^{\mathrm{nd}}\right\}$ of non-strictly dominant integral elements $S'' \subseteq \Lambda_{\mathrm{wt}}^{\mathrm{nsd}} = \Lambda_{\mathrm{wt}}^{\mathfrak{g}} \backslash C^{0,i}$. In particular, the set S'' cannot have elements valued in the interior of the fundamental Weyl chamber $S'' \cap C^{0,i} = \emptyset$, but can have elements valued on the boundary of the fundamental Weyl chamber $S'' \cap C^{0,b} \neq \emptyset$.

Appendix C

Appendix: A-model localization

This chapter serves as an appendix to the chapter on A-model localization.

C.1 Supersymmetry conventions

The line element on S^2 is

$$ds^2 = \sqrt{g} dz d\overline{z} = 2g_{z\overline{z}} \left(|z|^2\right) dz d\overline{z} = e^1 e^{\overline{1}}. \tag{C.1.1}$$

The topological A-twist is

$$A_\mu^R = \frac{1}{2}\omega_\mu, \quad \zeta = \begin{pmatrix} 0 \\ \zeta_+ \end{pmatrix}, \quad \tilde{\zeta} = \begin{pmatrix} \tilde{\zeta}_- \\ 0 \end{pmatrix}, \quad \mathcal{H} = \widetilde{\mathcal{H}} = 0 \tag{C.1.2}$$

for constant generalized Killing spinors $\partial_\mu \zeta = \partial_\mu \tilde{\zeta} = 0$. Here, A_μ^R is the connection of the background vector-like $U(1)$ R-symmetry, ω_μ is the spin connection, and $\mathcal{H}, \widetilde{\mathcal{H}}$ are gravi-photon dual field strengths.

The A-twisted supersymmetry algebra is

$$\delta^2 = \tilde{\delta}^2 = 0, \qquad \left\{\delta, \tilde{\delta}\right\} = 0. \tag{C.1.3}$$

The transformations of the A-twisted vector multiplet in WZ gauge are

$$\begin{array}{cc} \delta a_1 = 0 & \tilde{\delta} a_1 = -i\Lambda_1 \\ \delta a_{\bar{1}} = i\tilde{\Lambda}_{\bar{1}} & \tilde{\delta} a_{\bar{1}} = 0 \\ \delta\sigma = 0 & \tilde{\delta}\sigma = 0 \\ \delta\tilde{\sigma} = -2\tilde{\lambda} & \tilde{\delta}\tilde{\sigma} = -2\lambda \\ \delta\Lambda_1 = +2iD_1\sigma & \tilde{\delta}\Lambda_1 = 0 \\ \delta\tilde{\Lambda}_{\bar{1}} = 0 & \tilde{\delta}\tilde{\Lambda}_{\bar{1}} = -2iD_{\bar{1}}\sigma \\ \delta\lambda = i\left(D - 2if_{1\bar{1}} - \frac{1}{2}\left[\sigma,\tilde{\sigma}\right]\right) & \tilde{\delta}\lambda = 0 \\ \delta\tilde{\lambda} = 0 & \tilde{\delta}\tilde{\lambda} = -i\left(D - 2if_{1\bar{1}} + \frac{1}{2}\left[\sigma,\tilde{\sigma}\right]\right) \\ \delta D = -2D_1\tilde{\Lambda}_{\bar{1}} - \left[\sigma,\tilde{\lambda}\right] & \tilde{\delta} D = -2D_{\bar{1}}\Lambda_1 + \left[\sigma,\lambda\right] \end{array} \tag{C.1.4}$$

Under linear combination $\delta_A = \delta + \tilde{\delta}$ the transformations read

$$\delta_A a_1 = -i\Lambda_1 \tag{C.1.5}$$

$$\delta_A a_{\bar{1}} = i\tilde{\Lambda}_{\bar{1}} \tag{C.1.6}$$

$$\delta_A \sigma = 0 \tag{C.1.7}$$

$$\delta_A \tilde{\sigma} = -2\tilde{\lambda} - 2\lambda \tag{C.1.8}$$

$$\delta_A \Lambda_1 = +2iD_1\sigma \tag{C.1.9}$$

$$\delta_A \tilde{\Lambda}_{\bar{1}} = -2iD_{\bar{1}}\sigma \tag{C.1.10}$$

$$\delta_A \lambda = i\left(D - 2if_{1\bar{1}} - \frac{1}{2}\left[\sigma,\tilde{\sigma}\right]\right) \tag{C.1.11}$$

$$\delta_A \tilde{\lambda} = -i\left(D - 2if_{1\bar{1}} + \frac{1}{2}\left[\sigma,\tilde{\sigma}\right]\right) \tag{C.1.12}$$

$$\delta_A D = -2D_1\tilde{\Lambda}_{\bar{1}} - \left[\sigma,\tilde{\lambda}\right] - 2D_{\bar{1}}\Lambda_1 + \left[\sigma,\lambda\right] \tag{C.1.13}$$

It is useful to note that

$$\delta f_{1\bar{1}} = iD_1\tilde{\Lambda}_{\bar{1}} \qquad \tilde{\delta} f_{1\bar{1}} = iD_{\bar{1}}\Lambda_1 \tag{C.1.14}$$

$$\delta\left(-2if_{1\bar{1}}\right) = \delta\left(*f\right) = 2D_1\tilde{\Lambda}_{\bar{1}} \qquad \tilde{\delta}\left(-2if_{1\bar{1}}\right) = \tilde{\delta}\left(*f\right) = 2D_{\bar{1}}\Lambda_1 \tag{C.1.15}$$

$$\delta\left[\sigma,\tilde{\sigma}\right] = -2\left[\sigma,\tilde{\lambda}\right] \qquad \tilde{\delta}\left[\sigma,\tilde{\sigma}\right] = -2\left[\sigma,\lambda\right] \tag{C.1.16}$$

$$\tilde{\delta}\delta\left[\sigma,\tilde{\sigma}\right] = -2\left[\sigma,\tilde{\delta}\tilde{\lambda}\right] \qquad \delta\tilde{\delta}\left[\sigma,\tilde{\sigma}\right] = -2\left[\sigma,\delta\lambda\right] \tag{C.1.17}$$

$$\delta\left(\tilde{\delta}\tilde{\lambda}\right) = 2i\left[\sigma,\tilde{\lambda}\right] \qquad \tilde{\delta}\left(\delta\lambda\right) = 2i\left[\sigma,\lambda\right] \tag{C.1.18}$$

$$2i\left(\delta\lambda + \tilde{\delta}\tilde{\lambda}\right) = 2\left[\sigma,\tilde{\sigma}\right] \tag{C.1.19}$$

$$-\frac{i}{2}\left(\delta\lambda - \tilde{\delta}\tilde{\lambda}\right) = D + *f \tag{C.1.20}$$

One of the vector multiplet actions considered in CCP15 is

$$L_{\widetilde{\Sigma}\Sigma} = \frac{1}{2}\left(\delta\tilde{\delta}-\tilde{\delta}\delta\right)\mathrm{tr}\left(\tilde{\sigma}f_{1\bar{1}}\right) \tag{C.1.21}$$

$$= \mathrm{tr}\Bigg(\frac{1}{2}D_\mu\tilde{\sigma}D^\mu\sigma+\left(-2if_{1\bar{1}}\right)^2-2if_{1\bar{1}}D \tag{C.1.22}$$

$$+2i\lambda D_1\tilde{\Lambda}_{\bar{1}}-2i\tilde{\lambda}D_{\bar{1}}\Lambda_1+i\tilde{\sigma}\left[\tilde{\Lambda}_{\bar{1}},\Lambda_1\right]\Bigg) \tag{C.1.23}$$

This follows from

$$\delta\tilde{\delta}\mathrm{tr}\left(\tilde{\sigma}f_{1\bar{1}}\right) = \mathrm{tr}\Bigg(D\left(-2if_{1\bar{1}}\right)+\left(-2if_{1\bar{1}}\right)^2 \tag{C.1.24}$$

$$-\frac{1}{2}\left[\sigma,\tilde{\sigma}\right]\left(-2if_{1\bar{1}}\right)-2\tilde{\sigma}D_{\bar{1}}D_1\sigma \tag{C.1.25}$$

$$+2i\lambda D_1\tilde{\Lambda}_{\bar{1}}-2i\tilde{\lambda}D_{\bar{1}}\Lambda_1+i\tilde{\sigma}\left[\tilde{\Lambda}_{\bar{1}},\Lambda_1\right] \tag{C.1.26}$$

$$\tilde{\delta}\delta\mathrm{tr}\left(\tilde{\sigma}f_{1\bar{1}}\right) = \mathrm{tr}\Bigg(-D\left(-2if_{1\bar{1}}\right)-\left(-2if_{1\bar{1}}\right)^2 \tag{C.1.27}$$

$$-\frac{1}{2}\left[\sigma,\tilde{\sigma}\right]\left(-2if_{1\bar{1}}\right)+2\tilde{\sigma}D_1D_{\bar{1}}\sigma \tag{C.1.28}$$

$$+2i\tilde{\lambda}D_{\bar{1}}\Lambda_1-2i\tilde{\lambda}D_1\tilde{\Lambda}_{\bar{1}}-i\tilde{\sigma}\left[\Lambda_1,\tilde{\Lambda}_{\bar{1}}\right] \tag{C.1.29}$$

The Yang-Mills action is

$$L_{YM} = \frac{1}{2}\left(\delta\tilde{\delta}-\tilde{\delta}\delta\right)\mathrm{tr}\left(\tilde{\sigma}f_{1\bar{1}}-\frac{1}{2}\tilde{\lambda}\lambda\right) \tag{C.1.30}$$

$$= \mathrm{tr}\Bigg(\frac{1}{2}\left(*f\right)^2-\frac{1}{2}D^2+\frac{1}{2}D_\mu\tilde{\sigma}D^\mu\sigma+\frac{1}{8}\left[\sigma,\tilde{\sigma}\right]^2 \tag{C.1.31}$$

$$+2i\tilde{\Lambda}_{\bar{1}}D_1\lambda-2i\Lambda_1D_{\bar{1}}\tilde{\lambda}-i\tilde{\Lambda}_{\bar{1}}\left[\tilde{\sigma},\Lambda_1\right]+i\tilde{\lambda}\left[\sigma,\lambda\right] \tag{C.1.32}$$

This follows from

$$\delta\tilde{\delta}\mathrm{tr}\left(\tilde{\lambda}\lambda\right) = \mathrm{tr}\Bigg(2i\left[\sigma,\tilde{\lambda}\right]\lambda \tag{C.1.33}$$

$$+\left(D+*f+\frac{1}{2}\left[\sigma,\tilde{\sigma}\right]\right)\left(D+*f-\frac{1}{2}\left[\sigma,\tilde{\sigma}\right]\right)\Bigg) \tag{C.1.34}$$

$$\tilde{\delta}\delta\mathrm{tr}\left(\tilde{\lambda}\lambda\right) = \mathrm{tr}\Bigg(-\left(D+*f+\frac{1}{2}\left[\sigma,\tilde{\sigma}\right]\right)\left(D+*f-\frac{1}{2}\left[\sigma,\tilde{\sigma}\right]\right) \tag{C.1.35}$$

$$+2i\tilde{\lambda}\left[\sigma,\lambda\right]\Bigg) \tag{C.1.36}$$

C.1.1 Euclidean vector multiplet in WZ gauge

Here, we describe the $\mathcal{N} = (2,2)$ supersymmetric vector multiplet in Wess-Zumino (WZ) gauge, on flat spacetime in Euclidean signature $\mathbb{R}^2$. The gauge group is G, the gauge algebra is $\mathfrak{g} = \mathrm{Lie}G$ with Cartan subalgebra $\mathfrak{h}$. The Euclidean flat space metric is $\delta_{\mu\nu}$ where $\mu, \nu = 1, 2$. The Levi-Civita symbol $\epsilon^{\mu\nu}$ satisfies $\epsilon^{12} = 1$. In holomorphic coordinates $z = x^1 + ix^2$, $\overline{z} = x^1 - ix^2$, vectors read $X_z = \frac{1}{2}(X_1 - iX_2)$ and $X_{\overline{z}} = \frac{1}{2}(X_1 + iX_2)$, the metric is $\delta_{z\overline{z}} = \delta_{\overline{z}z} = \frac{1}{2}, \delta_{zz} = \delta_{\overline{z}\overline{z}} = 0$, and $\epsilon^{z\overline{z}} = -2i$.

To define the vector multiplet in WZ gauge, we begin by recalling the role of the vector superfield. To do so, observing that the chiral and anti-chiral superfields are subject to phase transformations

$$\Phi \to e^{i\mathcal{Y}}\Phi, \ \overline{\Phi} \to \overline{\Phi}e^{-i\tilde{\mathcal{Y}}} \tag{C.1.37}$$

Here, $\mathcal{Y}$ and $\tilde{\mathcal{Y}}$ are respectively $\mathfrak{g}$-valued chiral and anti-chiral superfields of vanishing R- and $Z, \tilde{Z}$-charge. One cannot use the combination $\overline{\Phi}\Phi$ to write a phase transformation invariant D-term for the chiral fields, because

$$\overline{\Phi}\Phi \to \overline{\Phi}e^{i\mathcal{Y}-i\tilde{\mathcal{Y}}}\Phi \neq \overline{\Phi}\Phi. \tag{C.1.38}$$

Instead, one gauges the supersymmetric system by introducing a vector superfield $\mathcal{V}$ transforming as

$$-2\mathcal{V} \to -2\mathcal{V} - i\mathcal{Y} + i\tilde{\mathcal{Y}}. \tag{C.1.39}$$

Here, $\mathcal{V}$ is valued in the adjoint representation of $\mathfrak{g}$. Then $\int \mathrm{d}^4\theta\overline{\Phi}e^{-2\mathcal{V}}\Phi$ is a phase transformation invariant D-term, because

$$\overline{\Phi}e^{-2\mathcal{V}}\Phi \quad \to \quad \overline{\Phi}e^{-i\tilde{\mathcal{Y}}}e^{i\tilde{\mathcal{Y}}}e^{-2\mathcal{V}}e^{-i\mathcal{Y}}e^{i\mathcal{Y}}\Phi = \overline{\Phi}e^{-2\mathcal{V}}\Phi. \tag{C.1.40}$$

The lower components in the superspace expansion of $\mathcal{V}$ are eliminated by gauge-fixing to WZ gauge. Specifically, $\mathcal{Y}$ and $\tilde{\mathcal{Y}}$ are chosen such that the components of $\mathcal{V}$ are reduced to $(a_\mu, \sigma, \tilde{\sigma}, \lambda_\pm, \tilde{\lambda}_\pm, D)$. At first order in the gauge parameters, the transformation of the vector superfield is

$$\delta_{\mathcal{Y}}\mathcal{V} = \frac{i}{2}(\mathcal{Y} - \tilde{\mathcal{Y}}) + \frac{i}{2}[\mathcal{Y} + \tilde{\mathcal{Y}}, \mathcal{V}], \tag{C.1.41}$$

The vector superfield in WZ gauge remains subject to residual gauge transformations $\mathcal{Y} = \tilde{\mathcal{Y}} = (\omega, 0, 0, 0)$. In terms of component fields, the residual gauge transformations are

$$\delta_\omega a_\mu = \partial_\mu \omega + i[\omega, a_\mu] \tag{C.1.42}$$
$$\delta_\omega \sigma = i[\omega, \sigma] \ , \tag{C.1.43}$$
$$\delta_\omega \tilde{\sigma} = i[\omega, \tilde{\sigma}] \ , \tag{C.1.44}$$
$$\delta_\omega \lambda_\pm = i[\omega, \lambda_\pm] \tag{C.1.45}$$
$$\delta_\omega \tilde{\lambda}_\pm = i[\omega, \tilde{\lambda}_\pm] \tag{C.1.46}$$
$$\delta_\omega D = i[\omega, D] \tag{C.1.47}$$

The supersymmetry transformations of the vector multiplet in WZ gauge are

$$\delta a_z = -i\zeta_- \tilde{\lambda}_- - i\tilde{\zeta}_- \lambda_- \ , \tag{C.1.48}$$
$$\delta a_{\bar{z}} = i\zeta_+ \tilde{\lambda}_+ + i\tilde{\zeta}_+ \lambda_+ \ , \tag{C.1.49}$$
$$\delta \sigma = 2\zeta_- \tilde{\lambda}_+ + 2\tilde{\zeta}_+ \lambda_- \ , \tag{C.1.50}$$
$$\delta \tilde{\sigma} = -2\tilde{\zeta}_- \lambda_+ - 2\zeta_+ \tilde{\lambda}_- \ , \tag{C.1.51}$$
$$\delta \lambda_- = i\zeta_- \left(D + 2if_{z\bar{z}} + \frac{1}{2}[\sigma, \tilde{\sigma}] \right) + 2i\zeta_+ D_z \sigma \ , \tag{C.1.52}$$
$$\delta \lambda_+ = i\zeta_+ \left(D - 2if_{z\bar{z}} - \frac{1}{2}[\sigma, \tilde{\sigma}] \right) + 2i\zeta_- D_{\bar{z}} \tilde{\sigma} \ , \tag{C.1.53}$$
$$\delta \tilde{\lambda}_- = -i\tilde{\zeta}_- \left(D - 2if_{z\bar{z}} + \frac{1}{2}[\sigma, \tilde{\sigma}] \right) - 2i\tilde{\zeta}_+ D_z \tilde{\sigma} \ , \tag{C.1.54}$$
$$\delta \tilde{\lambda}_+ = -i\tilde{\zeta}_+ \left(D + 2if_{z\bar{z}} - \frac{1}{2}[\sigma, \tilde{\sigma}] \right) - 2i\tilde{\zeta}_- D_{\bar{z}} \sigma \ , \tag{C.1.55}$$
$$\delta D = -2D_z \left(\zeta_+ \tilde{\lambda}_+ - \tilde{\zeta}_+ \lambda_+ \right) + 2D_{\bar{z}} \left(\zeta_- \tilde{\lambda}_- - \tilde{\zeta}_- \lambda_- \right) \tag{C.1.56}$$
$$-[\sigma, \zeta_+ \tilde{\lambda}_- - \tilde{\zeta}_- \lambda_+] - [\tilde{\sigma}, \tilde{\zeta}_+ \lambda_- - \zeta_- \tilde{\lambda}_+] \tag{C.1.57}$$

Here, The field strength is

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - i[a_\mu, a_\nu] \ . \tag{C.1.58}$$

C.1.2 A-twisted vector multiplet

The A-twist can be described as coupling the $\mathcal{N} = (2,2)$ supersymmetric theory on $\mathbb{R}^2$ to an off-shell rigid supergravity background

$$\left\{ \Sigma_{\boldsymbol{g}}; g_{\mu\nu}, \omega_\mu, A_\mu^{(V)}, C_\mu, \widetilde{C}_\mu \right\} \tag{C.1.59}$$

where $\Sigma_{\boldsymbol{g}}$ is Riemann surface of genus $\boldsymbol{g}$, $g_{\mu\nu}$ is a metric for $\Sigma_{\boldsymbol{g}}$, ω_μ is a spin connection on $\Sigma_{\boldsymbol{g}}$, $A_\mu^{(V)}$ is a connection coupling to the $U(1)_V$-symmetry current, and $C_\mu, \widetilde{C}_\mu$ is a complex graviphoton coupling to the conserved current for central charges of the superalgebra. In the supergravity description, A-twisting is a particular solution to the generalized Killing spinor equations, by which half of the $\mathcal{N} = (2,2)$ supercharges on $\mathbb{R}^2$ may be preserved on a Riemann surface $\Sigma_{\boldsymbol{g}}$.

In what follows, we consider the genus zero case in which the Riemann surface is a two-sphere S^2. Moreover, we focus on the case of vanishing central charges, such that the graviphotons don't come into the picture. For details on other cases, see e.g. [12, 20]. That being said, we proceed to describe the A-twist necessary to define A-model action functionals on S^2. The line element on S^2 is

$$ds^2 = 2g_{z\overline{z}}\left(z,\overline{z}\right)dzd\overline{z} = \underbrace{\overbrace{\sqrt{2g_{z\overline{z}}}}^{e^1_z}dz}_{e^1}\underbrace{\overbrace{\sqrt{2g_{z\overline{z}}}}^{e^{\overline{1}}_{\overline{z}}}d\overline{z}}_{e^{\overline{1}}} = e^1 e^{\overline{1}} \tag{C.1.60}$$

where the complex zweibein is $e^1 = e^1_z dz,\ e^{\bar{1}} = e^{\bar{1}}_{\bar{z}} d\bar{z}$. For details on these curved space conventions, see appendix A.2 in [12]. In the complex zweibein, vectors read $X_1 = e^z_1 X_z$ and $X_{\overline{1}} = e^{\overline{z}}_{\overline{1}} X_{\overline{z}}$. The metric covariant derivative acts on an object $\varphi_{(q_S)}$ of $\mathrm{Spin}\,(2)_E$-charge q_S as

$$\nabla_\mu \varphi_{(q_S)} = \left(\partial_\mu - i q_S \omega_\mu\right)\varphi_{(q_S)}. \tag{C.1.61}$$

where ω_μ is the spin connection in the complex zweibein. In particular, $\omega_\mu \equiv -2i\omega_{\mu,1\overline{1}}$ where $\omega_{\mu ab}$ denotes the spin connection in a generic frame $\{e^a\}$. The metric covariant derivative acts on left- (right-) handed spinors ψ_- (ψ_+) of charge $q_S = \frac{1}{2}$ ($q_S = -\frac{1}{2}$) as

$$\nabla_\mu \psi_- = \left(\partial_\mu - \frac{i}{2}\omega_\mu\right)\psi_-,\ \nabla_\mu \psi_+ = \left(\partial_\mu + \frac{i}{2}\omega_\mu\right)\psi_+. \tag{C.1.62}$$

The generalized Killing spinor equations for the supersymmetry parameters $\zeta_\pm, \tilde{\zeta}_\pm$are

$$(\nabla_z - iA_z^{(V)})\zeta_- = 0 \tag{C.1.63}$$
$$(\nabla_{\bar{z}} - iA_{\bar{z}}^{(V)})\zeta_- = \frac{1}{2}\mathcal{H}\, e_{\bar{z}}^{\bar{1}}\,\zeta_+ \tag{C.1.64}$$
$$(\nabla_z - iA_z^{(V)})\zeta_+ = \frac{1}{2}\tilde{\mathcal{H}}\, e_z^{1}\,\zeta_- \tag{C.1.65}$$
$$(\nabla_{\bar{z}} - iA_{\bar{z}}^{(V)})\zeta_+ = 0 \tag{C.1.66}$$
$$(\nabla_z + iA_z^{(V)})\tilde{\zeta}_- = 0 \tag{C.1.67}$$
$$(\nabla_{\bar{z}} + iA_{\bar{z}}^{(V)})\tilde{\zeta}_- = \frac{1}{2}\tilde{\mathcal{H}}\, e_{\bar{z}}^{\bar{1}}\,\tilde{\zeta}_+ \tag{C.1.68}$$
$$(\nabla_z + iA_z^{(V)})\tilde{\zeta}_+ = \frac{1}{2}\mathcal{H}\, e_z^{1}\,\tilde{\zeta}_- \tag{C.1.69}$$
$$(\nabla_{\bar{z}} + iA_{\bar{z}}^{(V)})\tilde{\zeta}_+ = 0 \tag{C.1.70}$$

Here, the metric-covariant derivative ∇_μ has been expressed in holomorphic coordinates using $X_z = \frac{1}{2}(X_1 - iX_2)$ and $X_{\overline{z}} = \frac{1}{2}(X_1 + iX_2)$, and $\widetilde{\mathcal{H}}, \mathcal{H}$ are scalar graviphoton dual field strengths that vanish in the absence of central charges. The generalized Killing spinor equations are solved by the topological A-twist, which reads

$$A_\mu^{(V)} = \frac{1}{2}\omega_\mu, \quad \zeta = \begin{pmatrix} 0 \\ \zeta_+ \end{pmatrix}, \quad \tilde{\zeta} = \begin{pmatrix} \tilde{\zeta}_- \\ 0 \end{pmatrix}, \quad \mathcal{H} = \widetilde{\mathcal{H}} = 0, \tag{C.1.71}$$

for $\partial_\mu\zeta_+ = \partial_\mu\tilde{\zeta}_- = 0$.

C.2 Monopole spherical harmonics

In the conventions of [20], the raising and lowering operators act on states in Hilbert space as

$$-2iD_1 : \mathcal{H}_{\frac{\mathbf{r}-2}{2}} \longrightarrow \mathcal{H}_{\frac{\mathbf{r}}{2}} \tag{C.2.1}$$
$$+2iD_{\overline{1}} : \mathcal{H}_{\frac{\mathbf{r}}{2}} \longrightarrow \mathcal{H}_{\frac{\mathbf{r}-2}{2}} \tag{C.2.2}$$
$$-4D_1 D_{\overline{1}} : \mathcal{H}_{\frac{\mathbf{r}}{2}} \longrightarrow \mathcal{H}_{\frac{\mathbf{r}}{2}} \tag{C.2.3}$$
$$-4D_{\overline{1}} D_1 : \mathcal{H}_{\frac{\mathbf{r}-2}{2}} \longrightarrow \mathcal{H}_{\frac{\mathbf{r}-2}{2}} \tag{C.2.4}$$

Here, $\mathcal{H}_{\frac{\mathbf{r}}{2}}$ and $\mathcal{H}_{\frac{\mathbf{r}-2}{2}}$ denote the Hilbert spaces for fields of spin $\frac{\mathbf{r}}{2}$ and $\frac{\mathbf{r}-2}{2}$, where the first Chern class is $c_1 = -\mathbf{r}$. One may also write $\mathbf{r} = q_R - Q\mathfrak{m}$ where q_R is the R-charge, Q is the gauge charge, and k is the GNO quantized flux

$$\mathfrak{m} = \frac{1}{2\pi}\int_{S^2} f \in \Lambda_{\text{cochar}}. \tag{C.2.5}$$

	$\exists Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}+1}$ $\forall\, j\geq\left\|-\frac{\alpha(\mathfrak{m})}{2}+1\right\|$	$\exists Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}}$ $\forall\, j\geq\left\|-\frac{\alpha(\mathfrak{m})}{2}\right\|$	$\exists Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}-1}$ $\forall j\geq\left\|-\frac{\alpha(\mathfrak{m})}{2}-1\right\|$	$d_G = 2j+1$
$j=\frac{\alpha(\mathfrak{m})}{2}-1$	$\alpha(\mathfrak{m})\geq 2$	-	-	$\alpha(\mathfrak{m})-1$
$j=-\frac{\alpha(\mathfrak{m})}{2}$	-	$\alpha(\mathfrak{m})\leq 0$	$\alpha(\mathfrak{m})\leq -1$	$1-\alpha(\mathfrak{m})$
$j=-\frac{\alpha(\mathfrak{m})}{2}-1$	-	-	$\alpha(\mathfrak{m})\leq -2$	$-\alpha(\mathfrak{m})-1$
$j=\frac{\alpha(\mathfrak{m})}{2}$	$\alpha(\mathfrak{m})\geq 1$	$\alpha(\mathfrak{m})\geq 0$	-	$\alpha(\mathfrak{m})+1$

Table C.1: **Harmonic Existence** This table describes the existence of harmonics for various values of j and $\alpha(\mathfrak{m})$, where d_G is degeneracy. Note that the harmonics $Y_{jm}^{r+1}, Y_{jm}^{r}, Y_{jm}^{r-1}$ are respectively denoted $Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}+1}, Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}}, Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}-1}$ using $r = -\frac{\alpha(\mathfrak{m})}{2}$. For example, for $j=-\frac{\alpha(\mathfrak{m})}{2}$, $Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}}$ exists when $\alpha(\mathfrak{m})\leq 0$, $Y_{jm}^{-\frac{\alpha(\mathfrak{m})}{2}-1}$ exists when $\alpha(\mathfrak{m})\leq -1$, and the degeneracy is $d_G = 1-\alpha(\mathfrak{m})$. For $j\geq \frac{|\alpha(\mathfrak{m})|}{2}+1$ all harmonics exist with degeneracy $|\alpha(\mathfrak{m})|+3$. The choices $j=\frac{\alpha(\mathfrak{m})}{2}-1$ or $j=-\frac{\alpha(\mathfrak{m})}{2}$ give $r_+ = 0$, while $j=-\frac{\alpha(\mathfrak{m})}{2}-1$ or $j=\frac{\alpha(\mathfrak{m})}{2}$ give $r_-=0$, for $r_\pm = \sqrt{j\,(j+1)-\alpha(\mathfrak{m})/2\,(\alpha(\mathfrak{m})/2\mp 1)}$.

For our purposes, we are interested in $\mathbf{r} = 2-\alpha\,(\mathfrak{m})$ where α is a root (i.e. weight of the adjoint representation). It is convenient to rewrite the monopole harmonic conventions of [20] in terms of $r=-\frac{\alpha(\mathfrak{m})}{2}$ where $\alpha(\mathfrak{m}) = \alpha\,(\mathfrak{m})$. The raising and lowering operators acting on monopole harmonics may be described as

$$\begin{aligned}
D_1 Y_{j,m}^{r} &= \tfrac{r_+}{2} Y_{j,m}^{r+1}, & D_{\bar{1}} Y_{j,m}^{r+1} &= -\tfrac{r_+}{2} Y_{j,m}^{r},\\
D_{\bar{1}} Y_{j,m}^{r} &= -\tfrac{r_-}{2} Y_{j,m}^{r-1}, & D_1 Y_{j,m}^{r-1} &= \tfrac{r_-}{2} Y_{j,m}^{r}.\\
D_1 D_{\bar{1}} Y_{j,m}^{r+1} &= -\tfrac{r_+^2}{4} Y_{j,m}^{r+1}, & D_1 D_1 Y_{j,m}^{r-1} &= \tfrac{r_- r_+}{4} Y_{j,m}^{r+1},\\
D_{\bar{1}} D_{\bar{1}} Y_{j,m}^{r+1} &= \tfrac{r_+ r_-}{4} Y_{j,m}^{r-1}, & D_{\bar{1}} D_1 Y_{j,m}^{r-1} &= -\tfrac{r_-^2}{4} Y_{j,m}^{r-1},\\
D_1 D_{\bar{1}} Y_{j,m}^{r} &= -\tfrac{r_-^2}{4} Y_{j,m}^{r}, & D_{\bar{1}} D_1 Y_{j,m}^{r} &= -\tfrac{r_+^2}{4} Y_{j,m}^{r}.
\end{aligned} \tag{C.2.6}$$

Here, $Y_{j,m}^{r}$ are scalar harmonics, $Y_{j,m}^{r\pm 1}$ are vector harmonics. and

$$r_\pm = \sqrt{j\,(j+1) - r\,(r\pm 1)}. \tag{C.2.7}$$

As a check, we compare these conventions to those of [20]. For the Dirac operator on the two-sphere in the fermionic 1-loop determinant we have

$$-i\not{\nabla}_{S^2}\begin{pmatrix} 2\Lambda_1^\alpha \\ \lambda^\alpha \end{pmatrix} = -i\begin{pmatrix} 0 & -2D_1 \\ 2D_{\bar{1}} & 0 \end{pmatrix}\begin{pmatrix} 2\Lambda_1^\alpha \\ \lambda^\alpha \end{pmatrix} = \begin{pmatrix} 0 & 2iD_1 \\ -2iD_{\bar{1}} & 0 \end{pmatrix}\begin{pmatrix} 2\Lambda_1^\alpha \\ \lambda^\alpha \end{pmatrix} \tag{C.2.8}$$

we have $-2iD_{\bar{1}}\Lambda^{\alpha}_{\bar{1}} \to -2iD_{\bar{1}}Y^{r+1}$ and $2iD_1\lambda^{\alpha} \to 2iD_1Y^r$ with eigenvalues

$$\begin{aligned} -2iD_{\bar{1}}Y^{r+1} &= -2i\left(\frac{r_+}{2}\right)Y^r, && \text{(C.2.9)}\\ 2iD_1Y^r &= 2i\left(-\frac{r_+}{2}\right)Y^{r+1}. && \text{(C.2.10)} \end{aligned}$$

The formal determinant results in

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = AD-BC = AD-\left(2i\left(-\frac{r_+}{2}\right)\right)\left(-2i\left(\frac{r_+}{2}\right)\right) = AD+r_+^2 \quad \text{(C.2.11)}$$

replacing $r \to -\frac{\alpha(\mathfrak{m})}{2}$ in $r_+ = \sqrt{j\,(j+1) - r\,(r+1)}$ gives

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = AD + j\,(j+1) - \frac{\alpha(\mathfrak{m})^2}{4} + \frac{\alpha(\mathfrak{m})}{2} \quad \text{(C.2.12)}$$

as expected from [20].

C.3 Cartan-Weyl basis decomposition

The Cartan-Weyl (CW) basis for the Lie algebra of the gauge group $\mathfrak{g} = \text{Lie}G$ is $\{H_a, E_\alpha\}$, where H_a is a generator of the Cartan subalgebra, and E_α is a ladder operator. Here, a is an index for the Cartan subalgebra, and α is a non-vanishing root. In the conventions of [20], The generators and ladder operators satisfy

$$[H_a, E_\alpha] = \alpha\,(H_a)\,E_\alpha, \quad [H_a, H_b] = 0, \quad [E_\alpha, E_{-\alpha}] = \frac{2}{|\alpha|^2}\alpha_a H_a \quad \text{(C.3.1)}$$

$$\text{Tr}\,(E_\alpha E_\beta) = \delta_{\alpha+\beta,0}, \quad \text{Tr}\,(H_a E_\beta) = 0, \quad E_\alpha^\dagger = E_{-\alpha}. \quad \text{(C.3.2)}$$

When expressed in the CW basis, adjoint-valued zero modes read

$$\varphi_0 = \sum_a H_a \varphi_0^a, \quad \text{(C.3.3)}$$

while adjoint-valued fluctuating modes read

$$\varphi_f = \sum_a H_a \varphi^a + \sum_\alpha E_\alpha \varphi^\alpha. \quad \text{(C.3.4)}$$

The decomposition procedure is illustrated with two examples from a possible complex scalar kinetic $D_\mu \sigma D^\mu \tilde{\sigma}$ in $L^{quad.}_{loc,fix}$. At quadratic order, $D_\mu \sigma D^\mu \tilde{\sigma}$ reads

$$\begin{aligned} L^{quad.}_{loc,fix} \supset \ & \mathrm{Tr}\Big(D^0_1 \sigma D^0_{\bar{1}} \tilde{\sigma} - i D^0_1 \sigma \left[a_{\bar{1}}, \tilde{\sigma}_0\right] - i \left[a_1, \sigma_0\right] D^0_{\bar{1}} \tilde{\sigma} - \left[a_1, \sigma_0\right]\left[a_{\bar{1}}, \tilde{\sigma}_0\right] \\ & + D^0_{\bar{1}} \sigma D^0_1 \tilde{\sigma} - i D^0_{\bar{1}} \sigma \left[a_1, \tilde{\sigma}_0\right] - i \left[a_{\bar{1}}, \sigma_0\right] D^0_1 \tilde{\sigma} - \left[a_{\bar{1}}, \sigma_0\right]\left[a_1, \tilde{\sigma}_0\right] \Big) \end{aligned}$$

Considering first the CW decomposition of the third term $\mathrm{Tr}\left(-i\left[a_1, \sigma_0\right] D_{\bar{1}} \tilde{\sigma}\right) = \mathrm{Tr}\left(i\left[\sigma_0, a_1\right] D_{\bar{1}} \tilde{\sigma}\right)$ we have

$$\sum_{abcd} \sum_{\alpha\beta} \mathrm{Tr}\left(i\left[H_a \sigma^a_0, H_b a^b_1 + E_\alpha a^\alpha_1\right] \left(\partial_{\bar{1}} \left(H_d \tilde{\sigma}^d + E_\beta \tilde{\sigma}^\beta \right) - i\left[H_c a^{0,c}_{\bar{1}}, H_d \tilde{\sigma}^d + E_\beta \tilde{\sigma}^\beta\right]\right)\right) \tag{C.3.5}$$

Due to commuting generators $[H_a, H_b] = 0$, this simplifies as

$$\sum_{abcd} \sum_{\alpha\beta} \mathrm{Tr}\left(i\left[H_a \sigma^a_0, E_\alpha a^\alpha_1\right] \left(\partial_{\bar{1}} \left(H_d \tilde{\sigma}^d + E_\beta \tilde{\sigma}^\beta \right) - i\left[H_c a^{0,c}_{\bar{1}}, E_\beta \tilde{\sigma}^\beta\right]\right)\right). \tag{C.3.6}$$

The eigenvalue equation $[H_a, E_\alpha] = \alpha(H_a) E_\alpha$ and $\mathrm{Tr}(H_a E_\beta) = 0$ results in

$$\sum_{\alpha\beta} \mathrm{Tr}\left(i\alpha(\sigma_0) E_\alpha a^\alpha_1 D_{\bar{1}} E_\beta \tilde{\sigma}^\beta \right), \tag{C.3.7}$$

and from the trace $\mathrm{Tr}(E_\alpha E_\beta) = \delta_{\alpha+\beta}$, this reduces to

$$\sum_{\alpha\beta} i\alpha(\sigma_0) a^\alpha_1 D_{\bar{1}} \left(\tilde{\sigma}^\beta\right) \delta_{\alpha+\beta}. \tag{C.3.8}$$

Since this vanishes except when $\alpha + \beta = 0$, we have

$$\sum_{\alpha} a^{-\alpha}_1 \left(-i\alpha(\sigma_0) D_{\bar{1}}\right) \tilde{\sigma}^\alpha. \tag{C.3.9}$$

Considering the fourth term $\mathrm{Tr}\left(-\left[a_1, \sigma_0\right]\left[a_{\bar{1}}, \tilde{\sigma}_0\right]\right) = -\mathrm{Tr}\left(\left[\sigma_0, a_1\right]\left[\tilde{\sigma}_0, a_{\bar{1}}\right]\right)$ we have

$$-\sum_{abcd} \sum_{\alpha\beta} \mathrm{Tr}\left(\left[H_a \sigma^a_0, H_c a^c_1 + E_\alpha a^\alpha_1\right] \left[H_b \tilde{\sigma}^b_0, H_d a^d_{\bar{1}} + E_\beta a^\beta_{\bar{1}}\right]\right) \tag{C.3.10}$$

$$= -\sum_{ab} \sum_{\alpha\beta} \mathrm{Tr}\left(\left[H_a \sigma^a_0, E_\alpha a^\alpha_1\right] \left[H_b \tilde{\sigma}^b_0, E_\beta a^\beta_{\bar{1}}\right]\right) \tag{C.3.11}$$

due to $[H_a, H_b] = 0$. The eigenvalue equation results in

$$-\sum_{\alpha\beta} \mathrm{Tr}\left(\alpha\left(\sigma^a_0\right) E_\alpha a^\alpha_1 \beta\left(\tilde{\sigma}^b_0\right) E_\beta a^\beta_{\bar{1}} \right), \tag{C.3.12}$$

which due to $\mathrm{Tr}\,(E_\alpha E_\beta) = \delta_{\alpha+\beta}$ simplifies as

$$-\sum_{\alpha\beta} \alpha\left(\sigma_0^a\right) a_1^\alpha \beta\left(\tilde{\sigma}_0^b\right) a_{\bar{1}}^\beta \delta_{\alpha+\beta} \tag{C.3.13}$$

evaluating again $\delta_{\alpha+\beta}$, and choosing $\alpha = -\beta$, yields

$$\sum_\alpha a_1^{-\alpha}\left(\alpha\left(\tilde{\sigma}_0\right)\alpha\left(\sigma_0\right)\right) a_{\bar{1}}^\alpha. \tag{C.3.14}$$

Decomposing in this manner, the kinetic term $\frac{1}{2}D_\mu\sigma D^\mu\tilde{\sigma}$ results in

$$\sum_\alpha \Big(\sigma^{-\alpha}\left(-D_1 D_{\bar{1}}\right)\tilde{\sigma}^\alpha + \sigma^{-\alpha}\left(-i\alpha\left(\tilde{\sigma}_0\right)D_1\right)a_{\bar{1}}^\alpha \tag{C.3.15}$$

$$+a_1^{-\alpha}\left(-i\alpha\left(\sigma_0\right)D_{\bar{1}}\right)\tilde{\sigma}^\alpha + a_1^{-\alpha}\left(\alpha\left(\sigma_0\right)\alpha\left(\tilde{\sigma}_0\right)\right)a_{\bar{1}}^\alpha \tag{C.3.16}$$

$$+\sigma^{-\alpha}\left(-D_{\bar{1}}D_1\right)\tilde{\sigma}^\alpha + \sigma^{-\alpha}\left(-i\alpha\left(\tilde{\sigma}_0\right)D_{\bar{1}}\right)a_1^\alpha \tag{C.3.17}$$

$$+a_{\bar{1}}^{-\alpha}\left(-i\alpha\left(\sigma_0\right)D_1\right)\tilde{\sigma}^\alpha + a_{\bar{1}}^{-\alpha}\left(\alpha\left(\sigma_0\right)\alpha\left(\tilde{\sigma}_0\right)\right)a_1^\alpha\Big). \tag{C.3.18}$$

C.4 Dictionary: A-model vector multiplet (CCP15) to standard multiplet (W92)

In the conventions of [20], the transformations of the A-twisted vector multiplet in Wess-Zumino gauge are

$$\begin{aligned}
\delta a_1 &= -i\Lambda_1\\
\delta a_{\bar{1}} &= i\Lambda_{\bar{1}}\\
\delta\sigma &= 0\\
\delta\tilde{\sigma} &= -2(\lambda+\tilde{\lambda})\\
\delta\Lambda_1 &= +2iD_1\sigma\\
\delta\Lambda_{\bar{1}} &= -2iD_{\bar{1}}\sigma\\
\delta\lambda &= i\left(+D - 2if_{1\bar{1}} - \tfrac{1}{2}\left[\sigma,\tilde{\sigma}\right]\right)\\
\delta\tilde{\lambda} &= -i\left(+D - 2if_{1\bar{1}} + \tfrac{1}{2}\left[\sigma,\tilde{\sigma}\right]\right)\\
\delta D &= -2D_{\bar{1}}\Lambda_1 - 2D_1\Lambda_{\bar{1}} + \left[\sigma,\lambda-\tilde{\lambda}\right]
\end{aligned} \tag{C.4.1}$$

These are obtained from equations 2.12 and 2.13 in [20], by i) setting the omega-deformation parameter to zero $\epsilon_\Omega = 0$, and ii) considering the linear combination $\delta = \delta_{\mathrm{CCP15}} + \tilde{\delta}_{\mathrm{CCP15}}$. Here, $1, \bar{1}$ are indices for the holomorphic and anti-holomorphic frame, respectively, as described in appendix A.2 of CC14.

The A-twisted fields are defined as

$$\Lambda_z = \tilde{\zeta}_-\lambda_-,\ \Lambda_{\bar{z}} = \zeta_+\tilde{\lambda}_+,\ \lambda = \tilde{\zeta}_-\lambda_+,\ \tilde{\lambda} = \zeta_+\tilde{\lambda}_-. \tag{C.4.2}$$

where the right hand side of each equality is in the conventions of [12]. Note that the A-twisted fields are defined modulo powers of $\tilde{\zeta}_-\zeta_+ = 1$. See equation A.7 in [20], for the definition of A-twisted fields in the vector multiplet, and see equation 6.42 in [12] for the precise flat space transformations of the vector multiplet in Wess-Zumino gauge.

Expressing the A-twisted fields in terms of the standard component field of the vector multiplet, the transformations read

$$\delta a_1 = -i\tilde{\zeta}_-\lambda_- \ , \tag{C.4.3}$$

$$\delta a_{\bar{1}} = +i\zeta_+\tilde{\lambda}_+ \ , \tag{C.4.4}$$

$$\delta\sigma = 0 \ , \tag{C.4.5}$$

$$\delta\tilde{\sigma} = -2\tilde{\zeta}_-\lambda_+ - 2\zeta_+\tilde{\lambda}_- \ , \tag{C.4.6}$$

$$\delta\lambda_- = +2i\zeta_+ D_1\sigma \ , \tag{C.4.7}$$

$$\delta\lambda_+ = +i\zeta_+\left(D - 2if_{1\bar{1}} - \frac{1}{2}[\sigma,\tilde{\sigma}]\right) \ , \tag{C.4.8}$$

$$\delta\tilde{\lambda}_- = -i\tilde{\zeta}_-\left(D - 2if_{1\bar{1}} + \frac{1}{2}[\sigma,\tilde{\sigma}]\right) \ , \tag{C.4.9}$$

$$\delta\tilde{\lambda}_+ = -2i\tilde{\zeta}_- D_{\bar{1}}\sigma \ , \tag{C.4.10}$$

$$\delta D = -2\zeta_+ D_1\tilde{\lambda}_+ - 2\tilde{\zeta}_- D_{\bar{1}}\lambda_- - \left[\sigma, \zeta_+\tilde{\lambda}_- - \tilde{\zeta}_-\lambda_+\right] \ . \tag{C.4.11}$$

Note that the transformation of the field strength is

$$\delta f_{1\bar{1}} = i\left(\zeta_+ D_1\tilde{\lambda}_+ + \tilde{\zeta}_- D_{\bar{1}}\lambda_-\right) . \tag{C.4.12}$$

where

$$f_{1\bar{1}} = \partial_1 a_{\bar{1}} - \partial_{\bar{1}} a_1 - i\,[a_1, a_{\bar{1}}] \,. \tag{C.4.13}$$

Next, we relate the transformations A-twisted vector multiplet to the standard cohomological multiplet described in [3]. This will be achieved by redefining the fields, and switching the generators of the Lie algebra from the current physics conventions to the math conventions. The physics conventions are those in which the Lie algebra of the gauge group is taken to consist of hermitian matrices, while the math conventions are those in which the Lie algebra of the gauge group is taken to consist of anti-hermitian matrices. The first field redefinition is

$$Y = -i\,(D - 2if_{1\bar{1}}) \,. \tag{C.4.14}$$

Restricting to $\tilde{\zeta}_- = \zeta_+ = \zeta$, and expressing D in terms of Y, the transformations reduce to

$$\begin{aligned}
\delta a_1 &= -i\zeta\lambda_- \\
\delta a_{\bar{1}} &= i\zeta\tilde{\lambda}_+ \\
\delta\sigma &= 0 \\
\delta\tilde{\sigma} &= -2\zeta\left(\lambda_+ + \tilde{\lambda}_-\right) \\
\delta\lambda_- &= +2i\zeta D_1\sigma \\
\delta\tilde{\lambda}_+ &= -2i\zeta D_{\bar{1}}\sigma \\
\delta\lambda_+ &= -\zeta Y - \tfrac{i}{2}\zeta\left[\sigma,\tilde{\sigma}\right] \\
\delta\tilde{\lambda}_- &= +\zeta Y - \tfrac{i}{2}\zeta\left[\sigma,\tilde{\sigma}\right] \\
\delta Y &= i\zeta\left[\sigma,\tilde{\lambda}_- - \lambda_+\right]
\end{aligned} \tag{C.4.15}$$

The hermitian generators of the Lie algebra are expressed in terms of anti-hermitian generators by redefining each component field as $\varphi \to i\varphi$. The only effect is to modify the factors of i in front of the commutators. Thereafter, the component fields are redefined as

$$\begin{aligned}
A_1 &= a_1 \\
A_{\bar{1}} &= a_{\bar{1}} \\
\psi_1 &= -\lambda_- \\
\psi_{\bar{1}} &= +\tilde{\lambda}_+ \\
\phi &= +2i\sigma \\
\lambda &= -2i\tilde{\sigma} \\
\eta &= 4\left(\tilde{\lambda}_- + \lambda_+\right) \\
\chi &= \frac{1}{2}\left(\tilde{\lambda}_- - \lambda_+\right) \\
H &= Y
\end{aligned}$$

The field redefinitions result in the transformations of the standard cohomological multiplet

$$\begin{aligned}
\delta A_1 &= i\zeta\psi_1 \\
\delta A_{\bar{1}} &= i\zeta\psi_{\bar{1}} \\
\delta\psi_1 &= -\zeta D_1\phi \\
\delta\psi_{\bar{1}} &= -\zeta D_{\bar{1}}\phi \\
\delta\phi &= 0 \\
\delta\lambda &= i\zeta\eta \\
\delta\eta &= \zeta\,[\phi,\lambda] \\
\delta\chi &= \zeta H \\
\delta H &= i\zeta\,[\phi,\chi]
\end{aligned} \tag{C.4.16}$$

C.5 Dictionary: A-model vector multiplet (BZ16) to standard multiplet (W92)

Here, we begin to recover the standard cohomological multiplet from the transformations of the A-twisted vector multiplet in [18, 19]. The fields of the A-twisted vector multiplet transform as

$$QA_i = \frac{i}{2}\lambda^\dagger\gamma_i\epsilon \tag{C.5.1}$$

$$\widetilde{Q}A_i = \frac{i}{2}\tilde{\epsilon}^\dagger\gamma_i\lambda \tag{C.5.2}$$

$$Q\sigma = 0 \tag{C.5.3}$$

$$\widetilde{Q}\sigma = 0 \tag{C.5.4}$$

$$Q\bar{\sigma} = \lambda^\dagger\epsilon \tag{C.5.5}$$

$$\widetilde{Q}\bar{\sigma} = \tilde{\epsilon}^\dagger\lambda \tag{C.5.6}$$

$$QD = -\frac{i}{2}D_i\lambda^\dagger\gamma^i\epsilon + \frac{i}{2}[\sigma,\lambda^\dagger\epsilon] \tag{C.5.7}$$

$$\widetilde{Q}D = +\frac{i}{2}\tilde{\epsilon}^\dagger\gamma^i D_i\lambda - \frac{i}{2}[\sigma,\tilde{\epsilon}^\dagger\lambda] \tag{C.5.8}$$

$$Q\lambda = \Big(iF_{12} - D + \frac{i}{2}[\sigma,\bar{\sigma}] - i\gamma^i D_i\sigma\Big)\epsilon \tag{C.5.9}$$

$$\widetilde{Q}\lambda = 0 \tag{C.5.10}$$

$$Q\lambda^\dagger = 0 \tag{C.5.11}$$

$$\widetilde{Q}\lambda^\dagger = \tilde{\epsilon}^\dagger\Big(-iF_{12} + D + \frac{i}{2}[\sigma,\bar{\sigma}] - i\gamma^i D_i\sigma\Big) \tag{C.5.12}$$

First let us define

$$\eta_\mu = \frac{i}{2}\tilde{\epsilon}^\dagger\gamma_\mu\lambda, \quad \eta_\mu^\dagger = \frac{i}{2}\lambda^\dagger\gamma_\mu\epsilon, \quad \zeta = \tilde{\epsilon}^\dagger\lambda, \quad \zeta^\dagger = \lambda^\dagger\epsilon, \tag{C.5.13}$$

as described in [19]. Then in holomorphic vielbein $e^z = e^1 + ie^2$, the transformations under $\mathcal{Q} = Q + \tilde{Q}$ are

$$\mathcal{Q}A_z = \eta_z^\dagger \tag{C.5.14}$$
$$\mathcal{Q}A_{\overline{z}} = \eta_{\overline{z}} \tag{C.5.15}$$
$$\mathcal{Q}\eta_z^\dagger = \tilde{\epsilon}^\dagger \epsilon D_z \sigma \tag{C.5.16}$$
$$\mathcal{Q}\eta_{\overline{z}} = \tilde{\epsilon}^\dagger \epsilon D_{\overline{z}} \sigma \tag{C.5.17}$$
$$\mathcal{Q}\sigma = 0 \tag{C.5.18}$$
$$\mathcal{Q}\bar{\sigma} = \zeta^\dagger + \zeta \tag{C.5.19}$$
$$\mathcal{Q}D = 2\left(D_z \eta_{\overline{z}} - D_{\overline{z}} \eta_z^\dagger\right) + \frac{i}{2}[\sigma, \zeta^\dagger - \zeta] \tag{C.5.20}$$
$$\mathcal{Q}\zeta = -\tilde{\epsilon}^\dagger \epsilon \Big(D - iF_{12}\Big) + \frac{i}{2}\tilde{\epsilon}^\dagger \epsilon [\sigma, \bar{\sigma}] \tag{C.5.21}$$
$$\mathcal{Q}\zeta^\dagger = +\tilde{\epsilon}^\dagger \epsilon \Big(D - iF_{12}\Big) + \frac{i}{2}\tilde{\epsilon}^\dagger \epsilon [\sigma, \bar{\sigma}] \tag{C.5.22}$$

If we further define the linear combinations $\xi_\pm = \frac{1}{2}\left(\zeta^\dagger \pm \zeta\right)$ the fermionic part becomes

$$\mathcal{Q}\xi_+ = \frac{i}{2}\tilde{\epsilon}^\dagger \epsilon [\sigma, \bar{\sigma}] \tag{C.5.23}$$
$$\mathcal{Q}\xi_- = \tilde{\epsilon}^\dagger \epsilon \Big(D - iF_{12}\Big) \tag{C.5.24}$$

By noting that

$$\mathcal{Q}F_{12} = \mathcal{Q}\left(-2iF_{z\overline{z}}\right) = -2i\left(D_z \eta_{\overline{z}} - D_{\overline{z}} \eta_z^\dagger\right) \tag{C.5.25}$$

We have

$$Y = \Big(D - iF_{12}\Big) \tag{C.5.26}$$
$$\mathcal{Q}Y = \frac{i}{2}[\sigma, \xi_-] \tag{C.5.27}$$

Together, this reads

$$\mathcal{Q}A_z = \eta_z^\dagger \tag{C.5.28}$$

$$\mathcal{Q}A_{\overline{z}} = \eta_{\overline{z}} \tag{C.5.29}$$

$$\mathcal{Q}\eta_z^\dagger = \tilde{\epsilon}^\dagger \epsilon D_z \sigma \tag{C.5.30}$$

$$\mathcal{Q}\eta_{\overline{z}} = \tilde{\epsilon}^\dagger \epsilon D_{\overline{z}} \sigma \tag{C.5.31}$$

$$\mathcal{Q}\sigma = 0 \tag{C.5.32}$$

$$\mathcal{Q}\bar{\sigma} = 2\xi_+ \tag{C.5.33}$$

$$\mathcal{Q}Y = \frac{i}{2}[\sigma, \xi_-] \tag{C.5.34}$$

$$\mathcal{Q}\xi_+ = \frac{i}{2}\tilde{\epsilon}^\dagger \epsilon [\sigma, \bar{\sigma}] \tag{C.5.35}$$

$$\mathcal{Q}\xi_- = \tilde{\epsilon}^\dagger \epsilon Y \tag{C.5.36}$$

Appendix D

Appendix: YM2 & non-abelian localization

This section serves as an appendix for the chapter on YM2 and non-abelian localization.

D.1 Symplectic manifolds

Here, we record facts about symplectic manifolds. Let (X, ω) be a $2n$ dimensional symplectic manifold with symplectic two-form $\omega \in \Omega^2(X)$. The symplectic form ω is i) closed under the exterior derivative $\mathrm{d}\omega = 0$, ii) non-degenerate.

The tangent space at a point $p \in X$ is a vector space T_pX. When the tangent space is equipped with a symplectic form, we have a symplectic vector space (T_pX, ω). At the point p, the symplectic form is $\omega_p \in \wedge^2 T_pX$. In this context, non-degeneracy can be stated.

The Liouville volume form is $\frac{\omega^n}{n!} = \exp\omega$. If X is compact, then the Liouville volume form integrates to the symplectic volume $\int_X \frac{\omega^n}{n!} = \mathrm{vol}(X, \omega)$, and defines a canonical orientation for X. Non-degeneracy implies $\frac{\omega_p^n}{n!} \neq 0$ for all $p \in X$.

Upon choosing a local coordinates $\{x^i | i = 1, \ldots, 2n\}$, a basis $\{\partial/\partial x^i\}$ for the tangent space TX, and a basis $\{\mathrm{d}x^i\}$ for the cotangent space T^*X, the symplectic form reads

$$\omega = \frac{1}{2}\omega_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j. \tag{D.1.1}$$

The non-degeneracy condition in local coordinates is $\det\omega_{ij} \neq 0$. The symplectic form is antisymmetric $\omega_{ij} = -\omega_{ji}$. Non-degeneracy implies an inverse. When viewed as isomorphisms, we have

$$\begin{aligned} \omega &: TX \to T^*X \\ \omega^{-1} &: T^*X \to TX \end{aligned}$$

in components this is $\left(\omega^{-1}\right)^{ki}\omega_{ij} = \delta^k_j$.

D.2 Hamiltonian actions & moment maps

Here, we record some facts about Hamiltonian actions on symplectic manifolds. Let H be a connected Lie group with Lie algebra $\mathfrak{h}$ with dual $\mathfrak{h}^*$. If the action of H on the symplectic manifold (X, ω) is Hamiltonian, then

The moment map is

$$\mu : X \to \mathfrak{h}^*. \tag{D.2.1}$$

μ maps elements $h \in \mathfrak{h}$ to functions $\langle \mu, h \rangle \in X$ where $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \otimes \mathfrak{h} \to \mathbb{R}$ is the canonical dual pairing between vector spaces.

Hamiltonian vector fields V_f satisfy

$$\mathrm{d}f = \iota_{V_f}\omega. \tag{D.2.2}$$

The Poisson brackets are a bilinear function $\{\cdot,\cdot\} : \Omega^0(X) \otimes \Omega^0(X) \to \Omega^0(X)$ are determined by ω. They are defined, for a symplectic form ω and functions $f, g \in \Omega^0(X)$, by

$$\{f, g\} = \omega^{-1}(\mathrm{d}f, \mathrm{d}g) \tag{D.2.3}$$

In terms of components we have $\{f, g\} = \left(\omega^{-1}\right)^{ij}\partial_i f \partial_j g$, while in terms of Hamiltonian vector fields we have $\{f, g\} = \omega_{ij} V^i_f V^j_g$.

The moment map equation is

$$\mathrm{d}\langle \mu, h \rangle = \iota_{V(h)}\omega \tag{D.2.4}$$

where $h \in \mathfrak{h}$.

D.3 The space of connections

Here, we record feature $\mathcal{A}(P)$ as an affine space. Given a base point $b \in \mathcal{A}(P)$, there is a tangent space $T_b\mathcal{A} = \Omega^1_\Sigma \otimes \mathrm{ad}(P)$. Elements of the tangent space $u \in \Omega^1_\Sigma \otimes \mathrm{ad}(P)$ are Lie algebra valued one-forms on the Riemannian manifold Σ_g. An arbitrary connection $A \in \mathcal{A}(P)$ is written w.r.t. the base point $b \in \mathcal{A}(P)$ & tangent vector

$u \in \Omega^1_\Sigma \otimes \mathrm{ad}(P)$,

$$A = b + u. \tag{D.3.1}$$

Two connections $A, A' \in \mathcal{A}(P)$ may be subtracted but not added. For instance, for $u, v \in \Omega^1_\Sigma \otimes \mathrm{ad}(P)$ we have

$$A - A' \quad = \quad b + u - b - v = u - v$$

Here, we record features of $\mathcal{A}(P)$ as a symplectic manifold. The action of $\mathcal{G}(P)$ on $\mathcal{A}(P)$ is Hamiltonian. The choice of metric $g_\Sigma(\cdot,\cdot)$ on Σ_g gives a Hodge star $\star_\Sigma$. On Σ_g, the top form $\star_\Sigma 1$ is both the Riemannian measure $\mathrm{d}\mu$, and the symplectic form ω_Σ. The symplectic form on $\mathcal{A}(P)$ is for given by

$$\omega_{\mathcal{A}}(u, v) = -\int_\Sigma \mathrm{Tr}\,(u \wedge v) \tag{D.3.2}$$

where $u, v \in \Omega^1_\Sigma \otimes \mathrm{ad}(P)$ are tangent vectors. The metric on $\mathcal{A}(P)$ is given by

$$g_{\mathcal{A}}(\cdot,\cdot) = \omega_{\mathcal{A}}(\cdot, \star\cdot) \tag{D.3.3}$$

or equivalently

$$g_{\mathcal{A}}(u, v) = -\int_\Sigma \mathrm{Tr}\,(u \wedge \star v) \tag{D.3.4}$$

The Hodge star $\star_\Sigma$ gives a metric $g_{\mathcal{A}}(\cdot,\cdot)$ on $\mathcal{A}(P)$. This is because the Hodge star plays the role of an almost complex structure. An almost complex structure J on a symplectic manifold (X, ω) is $J : TX \to TX$ such that $J^2 = -1$. If the symplectic form ω is "compatible" with the almost complex structure J, the metric can be written $g_X(\cdot,\cdot) = \omega_X(\cdot, J\cdot)$.

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