



# UNIVERSITÀ DI PARMA

PHD IN INDUSTRIAL ENGINEERING  
CYCLE XXXVIII

## FRACTIONAL VISCOELASTIC MODELLING OF LAMINATED GLASS

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# Abstract

This doctoral research investigates the time-dependent mechanical behavior of laminated glass, focusing on the viscoelastic response of polymeric interlayers. These materials, which play a key role in the structural performance and durability of laminated glass systems, exhibit complex relaxation phenomena that require advanced constitutive modeling beyond classical approaches.

In the first part of the thesis, a comprehensive review of viscoelastic models is presented, highlighting the limitations of the traditional Prony series formulation in capturing wide-spectrum relaxation behaviors. To overcome these constraints, the research proposes an alternative interpolation framework based on fractional viscoelastic theory and power-law relaxation functions. This approach, rooted in the concept of fractional derivatives, allows a continuous and physically meaningful description of the stress-strain relationship with a reduced number of parameters and enhanced fitting accuracy.

A detailed experimental campaign was conducted to characterize the relaxation behavior of different polymeric interlayers across a broad temperature and time range. The experimental data were analyzed and fitted using both Prony series and power-law formulations, providing a quantitative comparison of their predictive capabilities. The results demonstrated that the power-law approach, expressed as a piecewise fractional model, offers superior agreement with experimental observations and significantly simplifies the parameter identification process.

The final part of the thesis focuses on the implementation of the proposed fractional model into finite element (FE) simulations of laminated glass structures. The developed numerical framework enables accurate prediction of structural response under realistic loading and boundary conditions, including short- and long-term behaviors. Validation against experimental bending and creep tests confirmed the reliability and robustness of the model, opening the way to efficient simulation-based design of laminated glass components.

Overall, this work contributes to the advancement of viscoelastic modeling in glass engineering by integrating experimental evidence, mathematical innovation, and numerical implementation. The fractional viscoelastic approach proposed herein not only improves the predictive capability of current models but also provides a versatile and computationally efficient tool for industrial applications and further academic research in the mechanics of time-dependent materials.



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# Introduction

Laminated glass is a key material in modern engineering applications, ranging from building façades and safety glazing to the automotive and aerospace industries [1]. Laminated glass is a composite material produced by permanently bonding multiple glass plies with thin polymeric interlayers through an autoclave process involving elevated temperature and pressure [2], as illustrated in 1. The present research focuses specifically on the pre-fracture viscoelastic behavior of laminated glass under different strain rates and thermal conditions. Although the post-fracture structural integrity of laminated glass is of undeniable importance [3], its investigation lies beyond the scope of this work and is treated as a separate research endeavor. Laminated glass’s appeal lies in the ability to combine the transparency and durability of glass with the ductility and energy-absorbing properties of polymeric interlayers. The resulting composite structure is able to provide enhanced mechanical resistance, improved acoustic insulation, and increased safety in the event of failure [4], also depending to the the type of glass [5]. For these reasons, laminated glass has become an indispensable component of structural design, especially where both functional and safety requirements must be satisfied simultaneously [6–8].

A landmark example of laminated glass architecture is the Kunstmuseum Stuttgart, located at Kleiner Schloßplatz 1 in Stuttgart, Germany. Designed by Hascher Jehle Architektur and completed in 2005 as illustrated in Figure 2, the museum is enclosed within a striking transparent glass cube, approximately 25 meters high, 25 meters wide, and 25 meters deep. The façade consists of large laminated glass panels, supported by minimal steel fittings and structural silicone joints, creating the appearance of a completely self-supporting glass volume.

Each panel is composed of multiple glass plies bonded by a polymer interlayer, which ensures both load transfer and post-fracture integrity. The interlayer’s viscoelastic properties play a fundamental role in distributing stresses and providing redundancy under service and ultimate limit states [9, 10]. This project exemplifies the evolution of glass from a purely enclosing material to a structural element, capable of carrying its own weight and withstanding environmental loads while maintaining full transparency.

In the context of this thesis, the Kunstmuseum Stuttgart serves as a reference for architectural-scale laminated glass systems, where accurate modeling of viscoelastic interlayers is essential for predicting long-term deflection, creep behavior, and residual load-bearing capacity. The methodologies developed herein — based on fractional viscoelastic constitutive laws and power-law representations of the relaxation function —

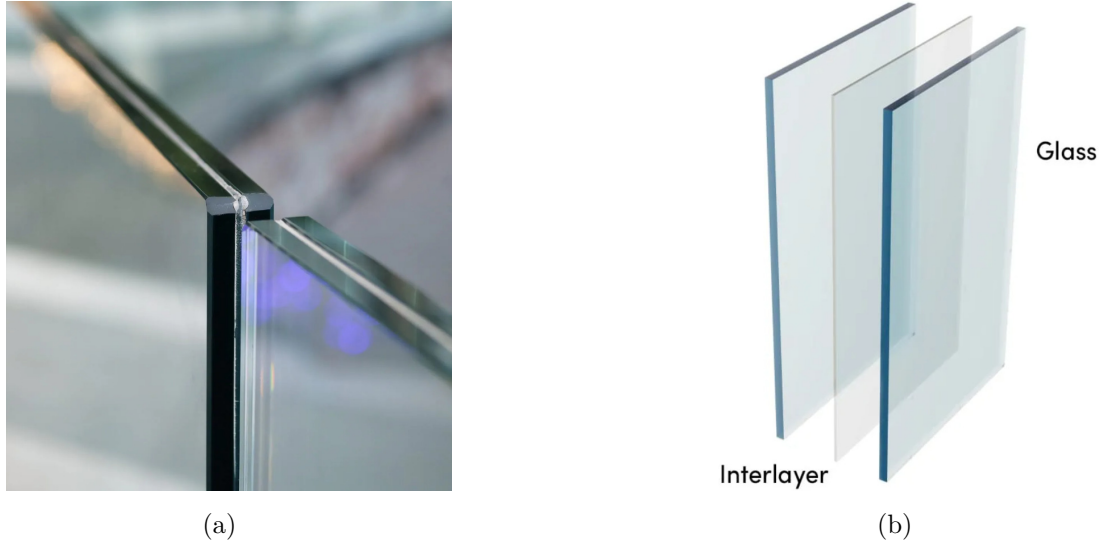


Figure 1: (a) Edge view of a laminated glass panel showing the bonding of multiple glass plies through a transparent polymer interlayer. (b) Schematic representation of the laminated glass structure, consisting of two (or more) glass sheets permanently bonded by a viscoelastic interlayer that ensures load transfer, impact resistance, and post-fracture integrity.

provide an advanced framework to simulate such time-dependent effects with improved physical fidelity and computational efficiency.

Recent developments in the automotive industry have demonstrated how laminated glass technologies—traditionally used in architecture—are being increasingly adopted for structural and safety-critical components, as illustrated in Figure 3. A prominent example is found in the latest Tesla models, which employ large-area laminated glazing systems for windshields, panoramic roofs, and side windows. These components integrate multi-layer laminated glass panels with polymer interlayers that combine high optical quality, impact resistance, and acoustic damping [11, 12]. In particular, the panoramic roof systems of Tesla’s Model 3 and Model Y represent some of the largest structural glass applications in passenger vehicles, designed to withstand aerodynamic loads, thermal gradients, and potential impact events while maintaining stringent optical and noise-vibration-harshness (NVH) requirements.

These applications exemplify how advances in viscoelastic modeling and materials characterization directly influence automotive design, where lightweight transparent structures are subject to both short-term dynamic and long-term environmental loadings. The polymer interlayer’s time- and temperature-dependent behavior plays a decisive role in the performance and durability of such systems. Accordingly, the methodologies developed in this thesis—based on fractional viscoelastic laws and power-law relaxation functions—are equally relevant for predicting and optimizing the mechanical response of

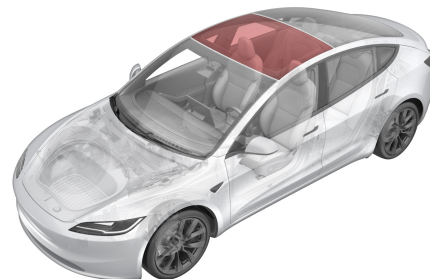


Figure 2: Kunstmuseum Stuttgart, Kleiner Schloßplatz 1, Stuttgart, Baden-Wuerttemberg, Germany.

automotive laminated glass, particularly under coupled thermal, mechanical, and time-dependent effects.



(a)



(b)

Figure 3: (a) Interior view of the Tesla Model X, featuring extensive use of laminated glazing for the panoramic windshield and roof. (b) Schematic rendering of the Tesla Model X body structure, where laminated glazing is used for both the windscreen and roof sections, demonstrating the growing use of laminated glass as a multifunctional structural and safety component in modern electric vehicles.

An emblematic example of the architectural potential of cold-bent laminated glass is the Bridge of Peace in Tbilisi, Georgia shown in Figure 4. Designed by architect Michele De Lucchi and completed in 2010, the bridge features a curved glass canopy composed of numerous cold-bent laminated glass panels supported by a steel space frame. The panels

were fabricated as flat laminates and subsequently elastically bent on site to achieve the desired double-curved geometry.

This construction technique allows for the realization of complex free-form surfaces while avoiding costly and energy-intensive glass tempering or hot-forming processes. By exploiting the elastic deformability of glass and the viscoelastic compliance of the polymer interlayer, the cold-bending process induces controlled pre-stress states that remain within safe design limits [13]. As such, the Bridge of Peace exemplifies how advanced material understanding and accurate numerical modeling can enable innovative architectural expressions without compromising safety or durability.

In the context of this thesis, the bridge serves as a reference case illustrating the increasing importance of viscoelastic modeling in the design and analysis of modern glass structures. The long-term structural performance of cold-bent laminated glass is governed by the time-dependent behavior of the interlayer, whose relaxation and creep characteristics influence the distribution of stresses and the preservation of the intended geometry over time [14]. The modeling approaches developed in this work—based on power-law relaxation functions and fractional viscoelastic formulations—aim to provide a more accurate and efficient framework to capture these effects, thereby supporting the design of safer and more reliable cold-bent glass systems.



Figure 4: The bridge of peace, Tbilisi (Georgia).

In the aerospace sector, laminated glass and polymeric composites play a crucial role in the design of transparency systems, such as cockpit canopies, windshields, and observation windows. These components must combine optical clarity, impact resistance, lightweight design, and resistance to extreme environmental conditions including large temperature gradients, pressure fluctuations, and ultraviolet radiation exposure [15].

A representative example is the F-35 Lightning II canopy in Figure 5, which consists of a multi-layer laminated transparency made of stretched acrylic and polycarbonate layers bonded with viscoelastic interlayers. This design ensures high ballistic resistance, bird-strike protection, and aerodynamic continuity, while maintaining the optical preci-

sion required for pilot visibility and head-up display systems. Similar laminated assemblies are also employed in commercial aviation, such as the Boeing 787 Dreamliner and Airbus A350 cockpit windshields, which utilize chemically strengthened glass–polymer laminates for their durability and damage tolerance [16]. The importance of artificial ageing of glass is also underlined in [17].

These aerospace applications clearly demonstrate the importance of accurate viscoelastic characterization of polymer interlayers under multi-axial and time-dependent loading. Temperature variations during high-altitude flight, coupled with cyclic pressurization, lead to long-term mechanical and rheological effects that directly influence stress redistribution, optical alignment, and fatigue life. The methodologies developed in this thesis — based on power-law approximations of the relaxation function and fractional viscoelastic constitutive laws — provide a versatile framework for modeling these phenomena with high accuracy across the multiple time scales typical of aerospace service conditions.



Figure 5: Pilot seated within the cockpit of a modern fighter aircraft featuring a laminated transparency canopy. The canopy consists of multiple polymer–glass laminate layers designed to provide high optical clarity, aerodynamic smoothness, and resistance to impact and thermal gradients. Such aerospace glazing systems exemplify advanced laminated materials where the viscoelastic properties of polymer interlayers play a critical role in ensuring safety, durability, and optical performance under extreme service conditions.

The mechanical performance of laminated glass is largely governed by the rheological behavior of the polymer interlayer. Unlike glass, which behaves elastically and whose response is relatively insensitive to environmental conditions, polymers exhibit a strongly time- and temperature-dependent viscoelastic response [18]. This means that the structural performance of laminated glass elements cannot be considered constant but evolves depending on the load duration, the loading history, and the environmental conditions to which the system is exposed. As a result, reliable design requires accurate constitutive models capable of capturing these complex dependencies [19]. It is well established that the load-bearing capacity of laminated glass in the pre-fracture phase is primarily governed by the shear interaction between the glass plies, facilitated by the polymeric

interlayers.

Central to the description of viscoelasticity is the concept of the relaxation function, which quantifies the stress decay in a material subjected to a constant strain [20]. The relaxation function provides the link between stress and strain through hereditary integrals, embodying the memory effect that is characteristic of viscoelastic materials. Its accurate determination and representation are therefore essential to predict the structural response of laminated glass under service and ultimate loading conditions. However, this task is far from trivial [21]. The relaxation behavior of polymers spans several orders of magnitude in time, covering both short-term and long-term phenomena, and is strongly influenced by temperature.

Although these interlayers exhibit viscoelastic behavior, most design approaches adopt quasi-elastic approximations, wherein the polymer is treated as a linear elastic material with an effective shear modulus adjusted based on the duration of loading and the ambient temperature. This method, widely accepted and codified in design standards, is generally suitable for a broad range of conditions [22]. However, in scenarios involving impulsive loads—such as impacts or blast waves—and especially under non-monotonic loading, the time-dependent and memory-dependent nature of viscoelasticity, which is ignored in the quasi-elastic approach, can become critical [23]. In such cases, a comprehensive viscoelastic analysis becomes necessary. The rheological behavior of laminated glass is governed by the viscoelastic properties of its polymeric interlayer [24].

The theoretical modeling is based on Boltzmann’s superposition principle and requires, as input data, the relaxation function of the polymer, which describes the temporal decay of the secant elastic modulus in a homogeneously strained specimen [25]. Experimental studies on most commercial polymers have shown that the relaxation function can be accurately approximated by segments of power-law functions of time [26]. When represented in a bi-logarithmic plot, this corresponds to a piecewise linear function (polyline), where each segment is fully defined by two parameters: the slope and the intercept with the vertical axis [27,28]. As will be demonstrated later, three such segments typically suffice to capture the long-term behavior of the material, meaning that only six parameters are needed for a comprehensive description. This representation is not only efficient but also offers intuitive geometric interpretation, facilitating parameter calibration [29]. In contrast, the conventional approach based on the classical Wiechert model approximates the relaxation function using a Prony series—i.e., a sum of exponential terms, each characterized by its own relaxation time. Achieving reliable predictions over a wide time range with this method requires 20~30 parameters, whose identification from experimental data is considerably more complex and less transparent [27].

Traditionally, the relaxation function has been represented using Prony series, that is, a sum of exponential terms determined by curve-fitting experimental data [27,25]. This approach has the advantage of being well established and easy to implement in numerical tools such as finite element solvers. Nevertheless, it presents notable drawbacks. Accurately representing a relaxation function over wide time scales requires a large number of exponential terms, which makes parameter identification computationally demanding and may lead to numerical inefficiencies [30]. Moreover, the physical interpretation of the

fitted parameters is often limited, and extrapolation beyond the range of experimental data is generally unreliable [31].

An alternative and potentially more powerful representation relies on power-law functions. Polymers often exhibit relaxation behavior that naturally follows power-law trends, particularly when the relaxation modulus is observed over extended time scales [25, 32]. Approximating the relaxation function with continuously connected power-law branches offers a more faithful and parsimonious description of the underlying material behavior. From a modeling perspective, this formulation naturally connects with the mathematical framework of fractional calculus, where stress-strain relations involve fractional derivatives in time [33–36]. Such models have attracted growing interest in recent years due to their ability to represent viscoelastic phenomena with fewer parameters and greater accuracy across multiple time scales [31]. The fractional description has been used in many problems in mechanics, and its applicability to laminated glass was first recognized in [37].

The present thesis explores new approaches to modeling viscoelastic polymers for laminated glass applications, with particular emphasis on power-law-based formulations and their fractional calculus counterparts. The research addresses both the theoretical foundations and the numerical implementation of these models, aiming to develop tools that are accurate, computationally efficient, and physically meaningful for the analysis and design of laminated glass structures.

Building upon established viscoelastic theory [25, 27] and recent developments in fractional modeling [33, 36], the proposed methods provide an alternative to the traditional Prony-series-based approaches commonly used in finite element implementations [38]. Experimental investigations and numerical validations demonstrate that the new formulations can capture the time-dependent behavior of polymer interlayers with fewer parameters and enhanced predictive robustness.

By advancing the understanding and modeling of viscoelastic interlayers, this work contributes to the development of more reliable and optimized design strategies for laminated glass elements [18]. Moreover, it establishes a general methodological framework that not only advances laminated glass modeling but also offers a transferable approach to the broader class of polymer-based composites and rate-dependent materials used in modern structural applications.

## Outline

This work is divided in six chapters. To guide the reader, the structure of this thesis is outlined below.

### Chapter 1 - Linear Viscoelasticity

This chapter introduces the fundamental concepts of linear viscoelasticity, which provide the theoretical basis for modeling time- and frequency-dependent behavior in polymeric materials. After recalling the essential notions of stress, strain, and constitutive relationships, the discussion focuses on the linear superposition principle and its implications

for viscoelastic systems. The main formulations in both the time and frequency domains are presented, including relaxation and creep functions, convolution integrals, and their Laplace representations. Classical rheological models are reviewed to highlight their role in approximating material response and their limitations in capturing complex behaviors. The chapter concludes by establishing the theoretical framework that underpins the advanced fractional viscoelastic models developed in the subsequent chapters.

## **Chapter 2 - Application of the fractional calculus in viscoelasticity**

This chapter addresses the numerical strategies adopted for approximating fractional derivatives in the modeling of viscoelastic materials. First, experimental data obtained through Dynamic Thermo-Mechanical Analysis (DTMA) are presented as the basis for characterizing the relaxation behavior of polymeric interlayers. Two main interpolation schemes for the relaxation curve are introduced and compared: the classical Prony series representation and an alternative formulation based on power laws. Building on the latter, the transition from empirical power-law descriptions to fractional constitutive equations is established, with emphasis on their relevance for linear viscoelastic materials. Numerical approximation methods are then discussed in detail, focusing on the Grünwald–Letnikov scheme and its limitations due to the requirement of constant time steps. To overcome these drawbacks, the L1 formula with variable time-step is introduced, which offers improved efficiency and accuracy in capturing both short- and long-term responses. The chapter concludes with a numerical comparison of the methods, highlighting the advantages of fractional formulations with adaptive time discretization for practical applications in laminated glass analysis.

## **Chapter 3 - Numerical framework of the inflexed laminated Beam**

Chapter 3 presents the theoretical and numerical formulation of the inflexed laminated beam model with a viscoelastic interlayer, incorporating a full viscoelastic description of the material behavior. The proposed model introduces an innovative approach to characterize the interlayer viscoelasticity through power-law approximations of the relaxation function, leading to a fractional constitutive law that directly relates stress and strain via fractional derivatives. The analysis initially focuses on the simply supported configuration, for which the governing equations are derived and solved using two fractional-derivative discretization schemes: the L1 approximation with variable time step and the Grünwald–Letnikov (GL) method. Finally, the chapter develops a variational formulation for laminated beams with viscoelastic interlayers, providing an energy-based framework suitable for finite element implementation and for extension to more complex geometries and loading conditions.

## **Chapter 4 - Numerical Solution for the semi-analytical problem**

This chapter investigates the viscoelastic response of laminated glass structures by employing a fractional calculus framework for modeling polymer interlayers presented in

Chapter 3. Such a description is particularly effective when the relaxation function of the interlayer can be represented by continuously connected branches of power laws, as is typically the case for commercial laminating polymers. Numerical solutions are obtained by approximating fractional derivatives with the L1 algorithm, which allows for variable time-stepping and efficiently captures both short-term loading effects and long-term creep behavior. This approach overcomes the limitations of the classical Grünwald–Letnikov scheme, which requires constant time-steps and may lose accuracy over extended observation times. The methodology is applied to laminated glass beams and to single-curvature cold-bent laminates, enabling a comprehensive study of their rheological and structural response. Parametric analyses highlight the influence of polymer type, interlayer thickness, temperature, and deformation history, while comparisons with quasi-elastic approximations clarify the role of viscoelastic memory in stress evolution. The results demonstrate that the proposed fractional framework provides accurate predictions across different time scales and structural configurations, supporting both the reliable characterization of laminated glass behavior and the optimization of innovative design processes.

## **Chapter 5 - Fractional-Viscoelastic-Finite-Element Model**

This chapter develops a one-dimensional viscoelastic finite element model for laminated glass, where fractional calculus is employed to capture the shear-coupling behavior of polymeric interlayers. The relaxation function of the polymer is represented by a piecewise power-law formulation, enabling a consistent link between Boltzmann’s superposition principle and the Caputo definition of fractional derivatives. The weak form of the governing equations is derived from Hamilton’s principle, and temporal discretization is carried out with the L1 scheme, which allows for variable time-stepping on a logarithmic scale to efficiently cover both short- and long-term responses. The predictive capability of the model is assessed through comparison with experimental data, demonstrating its effectiveness across a broad range of loading conditions and operating temperatures. This establishes the proposed approach as a robust tool for the structural analysis and design of laminated glass elements.

## **Chapter 6 - Multi-Material Polymeric Interlayers**

This chapter investigates the mechanical behavior of multi-material polymeric interlayers (MMPIs), where different polymers are stacked between glass plies to provide hybrid acoustic and structural properties. The time- and temperature-dependent rheology of such interlayers makes their modeling particularly complex, as the relaxation function of the laminate must account for the viscoelastic response of each constituent layer. A fractional calculus approach is proposed to capture this behavior, with relaxation functions expressed as piecewise power laws and fractional derivatives approximated numerically using the L1 scheme with variable time-stepping. This formulation simplifies parameter identification, improves computational efficiency, and enhances accuracy compared

to conventional Prony-series-based models. The methodology is validated by combining the relaxation functions of different polymers within a laminated glass package and comparing predictions with experimental data. Finally, results are contrasted with those from the quasi-elastic approach to highlight the relevance of viscoelastic memory in the design and optimization of MMPI systems.

# Chapter 1

## Linear Viscoelasticity

Solids are characterized by having their own shape. In particular, elastic solids deform when subjected to external loads, but once the action of these forces ceases, they return to their original configuration. Liquids, on the other hand, do not possess a definite shape; as a consequence, their internal stresses do not depend on deformation (which, in fact, cannot be properly defined). Specifically, if the liquids are Newtonian, the stresses vary linearly with the rate at which the deformation is applied.

Viscoelastic materials exhibit characteristics of both purely elastic solids and purely viscous fluids, and their behavior can be described through mechanical analogues that combine these two responses. Such behavior is typical of polymers, human bones, various mortars and resins employed in construction, certain families of rocks, and many other materials. A viscoelastic material is therefore characterized by exhibiting two asymptotic behaviors: that of an elastic solid and that of a viscous fluid.

In civil engineering, the viscoelastic properties of materials are of primary importance, as they govern the long-term performance of structural elements. Their characterization requires the explicit introduction of time, leading to the concepts of stress and strain histories. Within the framework of linear viscoelasticity, the material response at any given instant depends on the entire sequence of preceding load events, in accordance with the principle of superposition [39].

All materials exhibit time-dependent phenomena, manifested either as stress relaxation (under sustained deformation) or as creep (under sustained loading). The distinction lies in the characteristic timescales of these processes, which depend not only on intrinsic material properties (such as elastic modulus and viscosity) but also on environmental conditions (temperature, humidity) and the magnitude of applied stresses [7, 40].

This chapter outlines the fundamental concepts of viscoelasticity, focusing first on the classical approach based on discrete-element models (springs and dashpots) and the integral formulation (Boltzmann's superposition principle), and subsequently on fractional viscoelasticity, where the Spring-Pot model provides accurate numerical predictions in agreement with experimental observations.

### 1.0.1 The Elastic Model (Hooke)

The elastic model used to describe the behavior of solids under prescribed stress (or prescribed strain) is expressed as:

$$\sigma = E \varepsilon \quad (1.1)$$

Equation (1.1), known as *Hooke's law*, represents the constitutive relation of purely elastic solids. This model is commonly schematized as a spring with stiffness  $E$ , as illustrated in Figure 1.1(a).

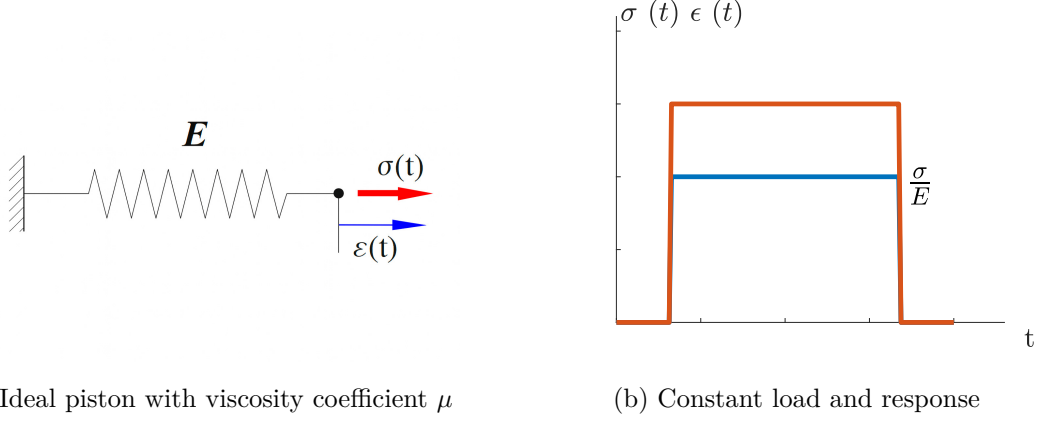


Figure 1.1: Newton–Petroff model, the stress  $\sigma(t)$  showed with red line, and the strain response  $\varepsilon(t)$  with the blue line.

In Equation (1.1),  $\sigma(t)$  denotes the stress (in *Pascal*),  $\varepsilon(t)$  the corresponding strain, and  $E$  the elastic modulus (in *Pascal*). This modulus is characteristic of each material, yet it also varies with external factors such as temperature.

Equation (1.1) expresses that, for a prescribed stress level, the material element undergoes a deformation that remains constant over time. Furthermore, if after a load increment that brings the element to a given stress level the load is subsequently removed, restoring the element to its initial stress state, the element returns to its original configuration. This implies that the work performed during the loading phase is entirely recovered during unloading, at the expense of the potential energy stored in the element during loading.

### 1.0.2 The Viscous Model (Newton–Petroff)

The viscous model, which describes the behavior of fluids, is expressed by the following law:

$$\sigma(t) = \mu \dot{\varepsilon}(t) \quad (1.2)$$

where  $\sigma(t)$  is the stress history,  $\dot{\varepsilon}(t)$  the strain rate (in  $\text{sec}^{-1}$ ), and  $\mu$  a constant depending on the fluid viscosity (with  $1 \text{ Pa} \cdot \text{s} = 10 \text{ Poise}$ ). Viscosity is a material property that generally decreases as temperature increases.

This model is typically represented as a piston immersed in an oil bath, as illustrated in Figure 1.2(a).

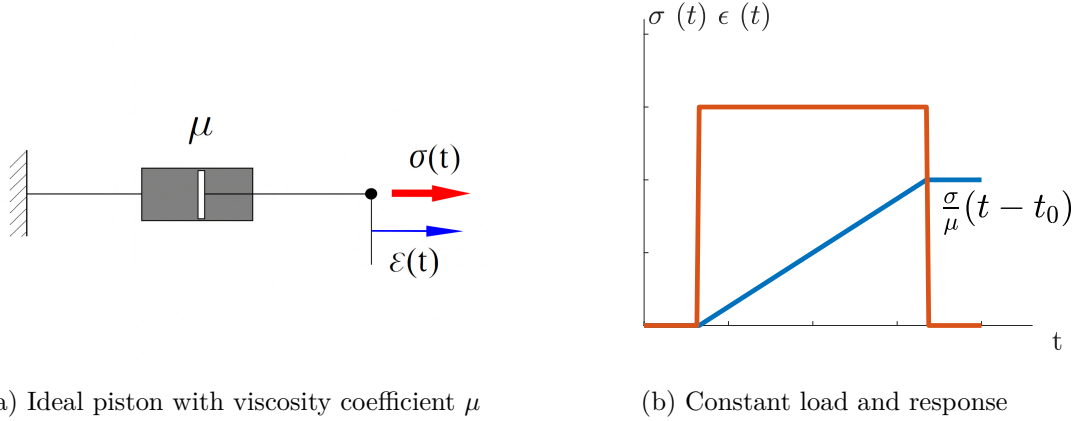


Figure 1.2: Newton–Petroff model, the stress  $\sigma(t)$  showed with red line, and the strain response  $\varepsilon(t)$  with the blue line.

Equation (1.2), known as *Newton–Petroff’s law*, shows that  $\sigma(t)$  does not depend directly on the strain  $\varepsilon(t)$  but rather on its time derivative. The energy expended is not stored within the fluid (since the system does not return to its initial configuration) and is instead fully dissipated as heat. In other words, a viscous fluid deforms irreversibly under the action of an external load.

## 1.1 Viscoelastic Models

Viscoelastic materials exhibit an intermediate behavior between that of a purely elastic solid and a purely viscous fluid. In fact, all materials are viscoelastic to some extent. For example, if a constant load is applied to a solid specimen, the deformation gradually increases over time (creep). Conversely, if a constant deformation is imposed, the stress progressively decreases (relaxation). These phenomena are particularly pronounced in certain classes of materials, such as natural rubbers and polymers, and less evident in steels.

Historically, this intermediate behavior has been modeled through combinations of springs and dashpots arranged in various configurations.

Before proceeding to illustrate the material’s model with viscoelastic behaviours it is important to clarify the properties of such materials.

### 1.1.1 Creep and Relaxation Functions

In materials science, stress relaxation refers to the progressive decrease in stress that occurs when a material is maintained at a constant strain. This phenomenon arises

because a sustained deformation over time promotes time-dependent mechanisms such as viscoelastic or plastic strain. Stress relaxation must be clearly distinguished from creep, which instead describes the gradual accumulation of strain under constant stress. From a structural perspective, stress relaxation reduces the internal stress state and, consequently, the reaction forces transmitted to supports or connected components. In this sense, its effect is analogous to that of cold springing, though it develops gradually rather than instantaneously. The degree of relaxation is governed by parameters such as time, temperature, and stress level. While its exact influence on a system cannot usually be predicted with certainty, it can be bounded within practical limits.

The *creep function* is the strain response to a unit stress. Such a function, hereafter denoted as  $J(t)$ , is a monotonically increasing function and, within the framework of linear viscoelasticity, since the strain increases proportionally with stress, it can be written as:

$$\varepsilon(t) = J(t)\sigma_0 \quad (1.3)$$

where  $\sigma_0$  is the applied stress level. The function  $J(t)$  is therefore the response to a stress history  $\sigma(t) = H(t)$  and can thus be defined as the unit-step creep function. Figure 1.3 shows the strain history and the corresponding creep function  $J(t)$ . The evolution of  $J(t)$  depends on the material under consideration.

The *relaxation function*, hereafter denoted as  $R(t)$ , is the counterpart of the creep function and represents the stress response to a unit strain imposed as  $\varepsilon(t) = H(t)$ . In linear viscoelasticity it can be expressed as:

$$\sigma(t) = R(t)\varepsilon_0 \quad (1.4)$$

where  $\varepsilon_0$  is the applied strain level.  $R(t)$  represents the response to a unit-step strain  $\varepsilon(t) = H(t)$ .

In Figure 1.4 the strain history and the corresponding relaxation function  $R(t)$  are reported. This function is monotonically decreasing.

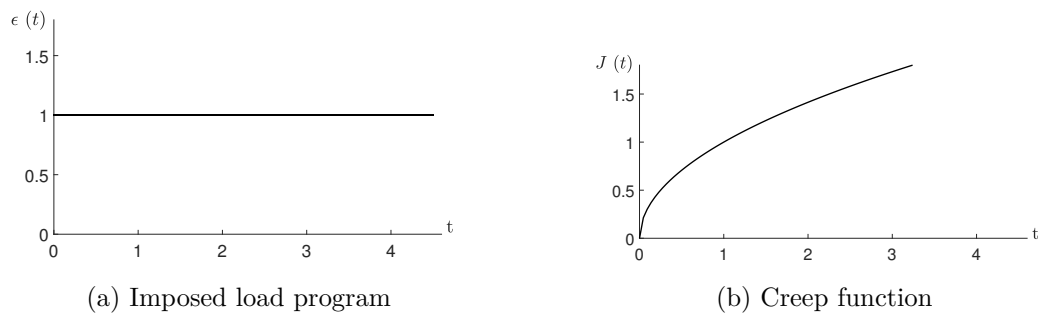


Figure 1.3: Generic trend of a creep function  $J(t)$  over time, indicating the monotonic crescent trend of the curve .

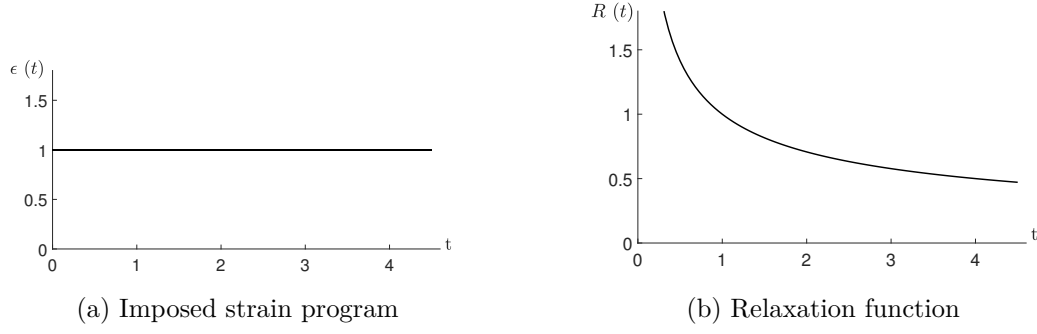


Figure 1.4: Relaxation function.

The two functions  $R(t)$  and  $J(t)$  contain all the information regarding the viscoelastic behavior of any given material. It is worth noting that  $R(t)$  and  $J(t)$  are positive functions for all  $t \geq 0$ , while they are identically zero for  $t < 0$ .

### 1.1.2 Boltzmann's Superposition Principle

Let us consider the creep experiment, imposing a load program such as the one depicted in Figure 1.5(a).

At time  $t = t_1$  we have  $\sigma(t) = \sigma_1 H(t - t_1)$ , while for every  $t > t_2$  the total stress is  $\sigma_1 + \sigma_2$ , and thus the overall loading history can be written as:

$$\sigma(t) = \sigma_1 H(t - t_1) + \sigma_2 H(t - t_2) \quad (1.5)$$

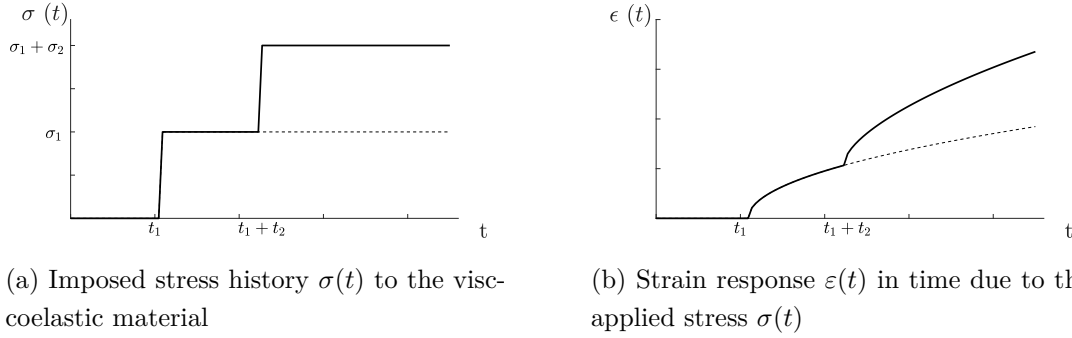


Figure 1.5: Loading program and corresponding strain response of a generic viscoelastic material.

Since  $J(t)$  is defined only for  $t \geq 0$ , the response of (1.5) to the load program described in Figure 1.5(a), for every  $t : t_1 < t < t_2$ , is given by:

$$\varepsilon(t) = \sigma_1 J(t - t_1) \quad (1.6)$$

while for every  $t > t_2$  the response is the superposition of two contributions:

$$\varepsilon(t) = \sigma_1 J(t - t_1) + \sigma_2 C(t - t_2) \quad (1.7)$$

Equation (1.7) is valid under the assumption of *linearity* of the system, and since  $J(t) \neq 0$  for  $t \geq 0$ , we have  $J(t - t_i)$  with  $t_i = t_1, t_2$  different from zero only for  $t \geq t_i$ .

Finally, if the loading program consists of  $n$  steps at instants  $t_1, t_2, \dots, t_{n-1}$  with amplitudes  $\sigma_1, \sigma_2, \dots, \sigma_n$ , then equation (1.7) can be written in the form:

$$\varepsilon(t) = \sum_{j=1}^n \sigma_j J(t - t_j) \quad (1.8)$$

If the loading program is a continuous function, it can be discretized into intervals of amplitude  $\Delta t$ , so that the stress increments  $\sigma_j$  correspond to  $\Delta\sigma$ , and it follows that, in the limit as  $\Delta t \rightarrow 0$  and  $\Delta\sigma \rightarrow d\sigma(t)$ , equation (1.8) becomes the *Boltzmann integral*:

$$\varepsilon(t) = \int_0^t d\sigma(\tau) J(t - \tau) = \int_0^t \dot{\sigma}(\tau) J(t - \tau) d\tau. \quad (1.9)$$

If, at the initial time  $t = 0$ , the applied stress assumes a nonzero value  $\sigma(0) = \sigma_0$ , then the overall response is given by:

$$\varepsilon(t) = \int_0^t \dot{\sigma}(\tau) J(t - \tau) d\tau + \sigma_0 J(t). \quad (1.10)$$

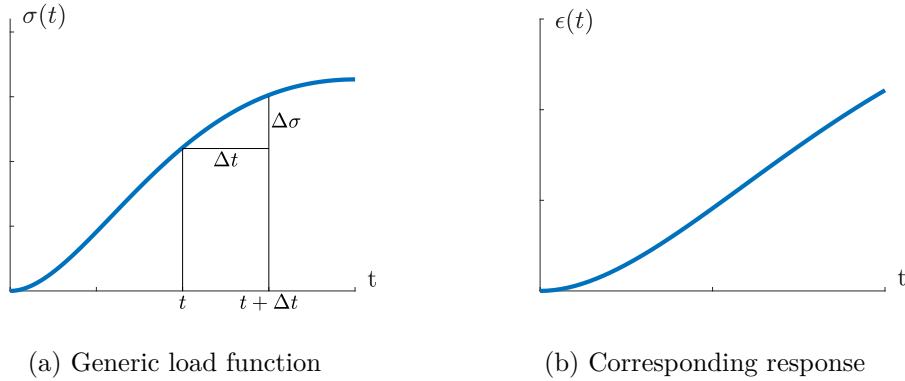


Figure 1.6: Generic load program and corresponding response.

Equation (1.10) is a convolution integral (or *faltung*) and expresses the fact that the present deformation depends on the entire loading history starting from an initial instant. In practice, Boltzmann's idea is that the material retains the memory of all past events that have affected it, so that the mechanical response at any given time is a function of the entire preceding history.

An analogous reasoning can be made starting from the relaxation function, thus obtaining the stress history as a function of the strain history:

$$\sigma(t) = \int_0^t \dot{\epsilon}(\tau) R(t - \tau) d\tau + \epsilon_0 R(t) \quad (1.11)$$

where  $\epsilon(0) = \epsilon_0$  is the imposed strain value at  $t = 0$ .

On the other hand,  $\Phi(t)$  and  $\Psi(t)$  must necessarily be related. To highlight this relation, consider a loading history with null initial conditions for both stress and strain. It is possible to assume an arbitrary stress history  $\sigma(t)$  (with  $\sigma_0 = 0$ ), to which a corresponding strain history  $\epsilon(t)$  is associated according to equation (1.9).

If one now considers the derivative  $\dot{\epsilon}(t)$  corresponding to  $\sigma(t)$  and substitutes it into equation (1.11), the result will be precisely  $\sigma(t)$ , i.e. the stress history that generated  $\epsilon(t)$  in the first place.

Consequently, by applying the Laplace transform to equations (1.10) and (1.11), with  $\sigma_0 = 0$  and  $\epsilon_0 = 0$ , we obtain:

$$J_{\mathcal{L}}(s) R_{\mathcal{L}}(s) = \frac{1}{s^2} \quad (1.12)$$

where  $J_{\mathcal{L}}(s)$  and  $R_{\mathcal{L}}(s)$  denote the Laplace transforms of the creep and relaxation functions, respectively.

This fundamental relationship expresses the mutual correspondence between  $R(t)$  and  $J(t)$  as functions connected to one another in the Laplace domain. Therefore, if either  $J(t)$  or  $R(t)$  is experimentally determined, the other function can be consequently derived, and vice versa.

### 1.1.3 The Maxwell Model

The Maxwell model, illustrated in Figure 1.7(a), consists of a spring with stiffness  $E$  in series with a dashpot characterized by a viscosity coefficient  $\mu$ , immersed in an oil bath.

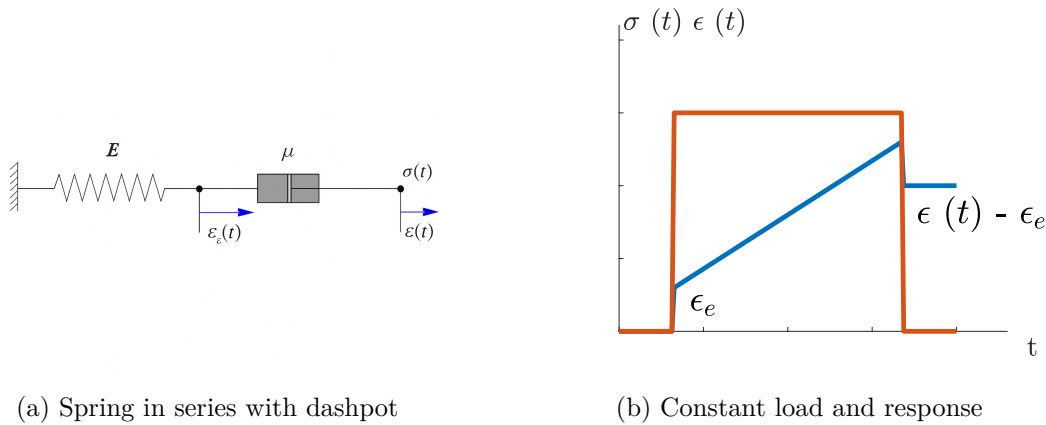


Figure 1.7: Maxwell model.

Let  $\varepsilon_e(t)$  denote the elongation of the spring and  $\varepsilon(t)$  the total elongation. The rheological behavior is then governed by the following pair of equations:

$$\sigma(t) = E\varepsilon_e(t), \quad \sigma(t) = \mu[\dot{\varepsilon}(t) - \dot{\varepsilon}_e(t)] \quad (1.13)$$

By combining the two equations, one obtains the constitutive relation:

$$\mu\dot{\sigma}(t) + E\sigma(t) = E\mu\dot{\varepsilon}(t) \quad (1.14)$$

which, when rewritten in canonical form, yields:

$$\dot{\sigma}(t) + \nu\sigma(t) = E\dot{\varepsilon}(t) \quad (1.15)$$

where

$$\nu = \frac{E}{\mu} \quad [\text{sec}^{-1}]s. \quad (1.16)$$

That equation is supplemented by the initial stress condition at time  $t = 0$  ( $\sigma(0) = \sigma_0$ ). The combination of the initial condition and (1.15) makes it possible to determine the stress history from the prescribed strain history  $\varepsilon(t)$ .

The general solution of (1.15) is the sum of the homogeneous solution  $\sigma_{om}(t)$  and a particular solution  $\sigma_p(t)$ , the latter depending on the strain-rate history  $\dot{\varepsilon}(t)$ . Thus, we obtain:

$$\sigma(t) = Be^{-\nu t} + \sigma_p(t) \quad (1.17)$$

where  $B$  is an integration constant determined by the initial condition  $\sigma_0$  and  $\sigma_p(t)|_{t=0}$ .

In the case where  $\dot{\varepsilon}(t) = 0$ ,  $\forall t > 0$ , i.e. when a zero strain rate is imposed with the initial condition  $\sigma(0) = \sigma_0$ , we have:

$$\sigma(t) = \sigma_0 e^{-\mu t} \quad (1.18)$$

which shows that, as time evolves, the stress decreases—precisely as observed experimentally.

The physical interpretation of this rheological model is as follows: if a constant deformation is applied to the specimen, at the initial time  $t = 0$  the spring is deformed with  $\varepsilon_e(0) = \frac{\sigma_0}{E}$ , so that the piston behaves like an incompressible fluid element (no volume variation). As time progresses, the fluid inside the piston flows from one chamber to the other, causing the deformation of the spring to decrease. Consequently, the elastic deformation  $\varepsilon_e(t)$  decreases until it reaches zero, thus cancelling the initial stress at  $t \rightarrow \infty$ .

In practical terms, if a constant strain is suddenly imposed at  $t = 0$ , the system will immediately sustain a stress equal to the elastic response  $E\varepsilon_0$ , and subsequently exhibit a relaxation process until reaching zero stress after sufficient time, with the imposed strain being entirely accommodated by the dashpot.

Assuming a strain  $\varepsilon(t) = H(t)$ , and therefore  $\dot{\varepsilon}(t) = \delta(t)$ , in the Maxwell equation (1.15) the corresponding stress history is obtained as  $\sigma(t) = R(t)$ , i.e. the relaxation function. This yields the following differential equation:

$$\dot{R}(t) + \nu R(t) = E\delta(t) \quad (1.19)$$

together with the initial condition

$$R(0) = \sigma_0 = E.$$

Hence, the relaxation function is given by

$$R(t) = Ee^{-\nu t}. \quad (1.20)$$

The creep law can be obtained from equation (1.12), which yields:

$$J(t) = \frac{\nu}{E}t + \frac{1}{E}. \quad (1.21)$$

Therefore, the Boltzmann integrals for the Maxwell model take the form:

$$\sigma(t) = E \int_0^t e^{-\nu(t-\tau)} \dot{\varepsilon}(\tau) d\tau \quad (1.22)$$

$$\varepsilon(t) = \frac{\nu}{E} \int_0^t (t-\tau) \dot{\sigma}(\tau) d\tau + \frac{1}{\nu} \int_0^t \dot{\sigma}(\tau) d\tau. \quad (1.23)$$

In Figure 1.8 the laws  $R(t)$  and  $J(t)$  for the Maxwell model are reported. From this figure it is evident that the relaxation function decays exponentially, which qualitatively agrees with the behavior presented in the subsection 1.1.1. However, the creep response is completely inconsistent with the experimental behavior of materials under constant load. Therefore, it follows that the Maxwell model is inadequate to describe creep.

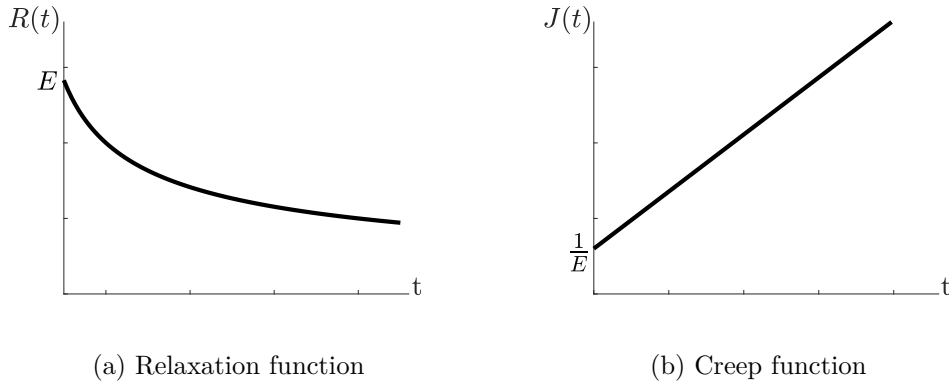


Figure 1.8: Relaxation and creep functions for the Maxwell model.

It should also be noted that equation (1.31) is a convolution integral with an exponential kernel. This form of kernel is common to all ordinary differential equations. In other words, for ordinary differential equations, the constitutive stress–strain relations  $\sigma - \varepsilon$  always correspond to kernels of exponential type.

#### 1.1.4 The Kelvin–Voigt Model

Another model commonly employed to describe viscoelastic behavior is the *Kelvin–Voigt* model, obtained by arranging in parallel a spring of stiffness  $E$  and a dashpot of viscosity  $\mu$ , as represented in Figure 1.9.

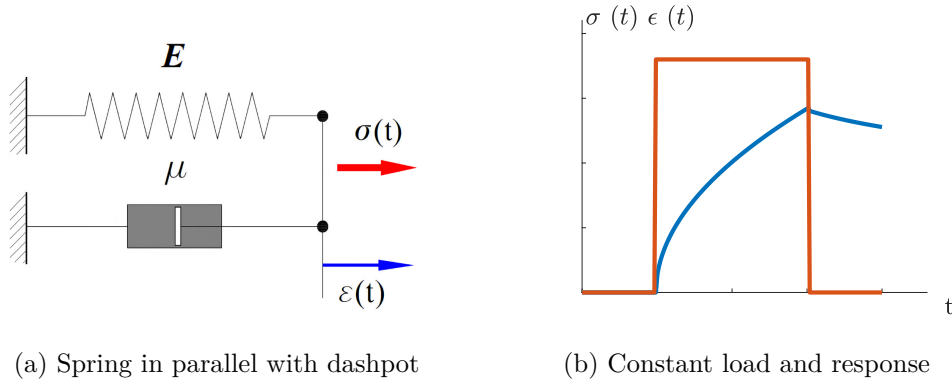


Figure 1.9: Kelvin–Voigt model.

In this case, the stress in the spring and that in the dashpot differ from one another, while the strain of the two elements is identical. The stress in the spring  $\sigma_e(t)$  and that in the dashpot  $\sigma_\mu(t)$  are given by:

$$\sigma_e(t) = E\varepsilon(t), \quad \sigma_\mu(t) = \mu\dot{\varepsilon}(t) \quad (1.24)$$

On the other hand, equilibrium requires that

$$\sigma(t) = \sigma_e(t) + \sigma_\mu(t), \quad (1.25)$$

from which one obtains

$$\mu\dot{\varepsilon}(t) + E\varepsilon(t) = \sigma(t) \quad (1.26)$$

which, rewritten in canonical form, yields:

$$\dot{\varepsilon}(t) + \nu\varepsilon(t) = \frac{\sigma(t)}{\mu} \quad (1.27)$$

where

$$\nu = \frac{E}{\mu}.$$

The initial condition is  $\varepsilon(0) = \varepsilon_0$ . If it is assumed that at  $t = 0$  the response to a prescribed deformation is applied, the solution of (3.10) yields  $\varepsilon(t) = \varepsilon_0 e^{-\nu t}$ , i.e. the deformation is no longer sustained in the absence of increasing external load and tends to vanish as  $t \rightarrow \infty$ .

Proceeding analogously to the previous case, let us now assume a stress  $\sigma(t) = H(t)$  and perform a creep test. Recalling equation (1.27) and taking into account that in this case the strain history represents the creep function  $\varepsilon(t) = J(t)$ , one obtains the following differential equation:

$$\dot{J}(t) + \nu J(t) = \frac{H(t)}{\mu} \quad (1.28)$$

with the initial condition

$$J(0) = \varepsilon(0) = 0.$$

The solution is therefore:

$$J(t) = \frac{1}{E} \left( 1 - e^{-\nu t} \right), \quad t \geq 0. \quad (1.29)$$

Recalling the constitutive relation between the two functions in the Laplace domain (equation (1.12)), the relaxation function is obtained as:

$$R(t) = E \left[ 1 + \frac{\delta(t)}{\nu} \right]. \quad (1.30)$$

Hence, the Boltzmann integrals for the Kelvin–Voigt model take the following form:

$$\sigma(t) = E \int_0^t \left[ 1 + \frac{1}{\nu} \delta(t - \tau) \right] \dot{\varepsilon}(\tau) d\tau \quad (1.31)$$

$$\varepsilon(t) = \frac{1}{E} \int_0^t \left[ 1 - e^{-\nu(t-\tau)} \right] \dot{\sigma}(\tau) d\tau. \quad (1.32)$$

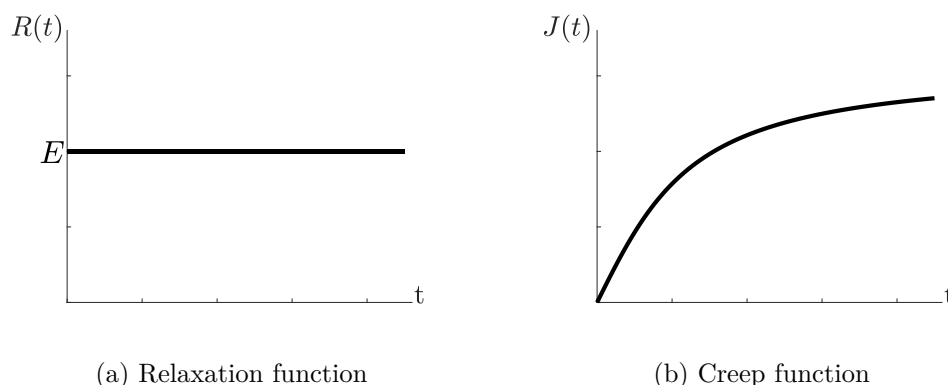


Figure 1.10: Relaxation and creep functions for the Kelvin–Voigt model.

In this case, the creep function grows according to an exponential law, showing qualitatively acceptable agreement with experimental observations. On the other hand, the relaxation function remains constant, which is not experimentally observed. It follows that the Kelvin–Voigt model, similarly to the Maxwell model, is inconsistent when used in isolation.

### 1.1.5 Other Classical Models

An extension of the elementary models, such as the Kelvin–Voigt model, can be obtained by introducing a larger number of elastic and viscous elements. This approach allows for a more accurate description of the viscoelastic behavior of materials, capturing both creep and relaxation responses. Among the most widely used models in this family is the *Standard Linear Solid* (SLS), also known as the *Zener model*. These configurations represent a significant improvement over simpler models, as they are capable of simulating more complex time-dependent phenomena typical of polymeric materials.

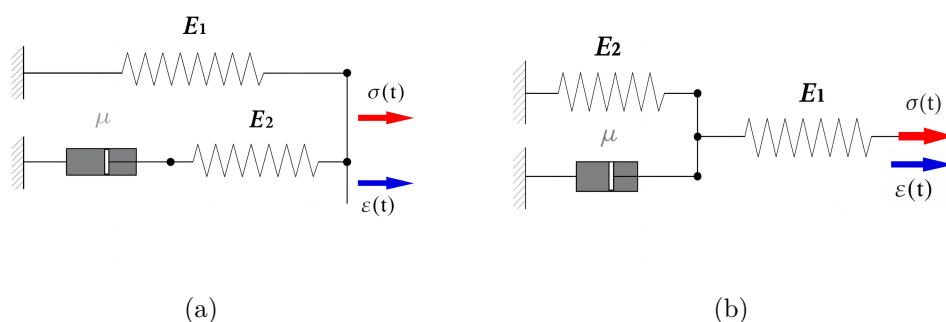


Figure 1.11: Standard Linear Solid (SLS) or Zener models.

The model shown in Figure 1.11(a), consisting of a Hookean spring in parallel with

a Maxwell element, is described by the following differential equation:

$$\dot{\sigma}(t) + \beta\sigma(t) = \alpha\mu\dot{\varepsilon}(t) + \beta E_1\varepsilon(t), \quad (1.33)$$

where the coefficients are defined as:

$$\alpha = \frac{E_1 + E_2}{\mu}, \quad \beta = \frac{E_2}{\mu}.$$

Similarly, the model shown in Figure 1.11(b), obtained by connecting a Hookean spring in series with a Kelvin–Voigt element, is governed by the following differential equation:

$$\dot{\sigma}(t) + \alpha\sigma(t) = E_1\dot{\varepsilon}(t) + \beta\varepsilon(t), \quad (1.34)$$

where  $\alpha$  and  $\beta$  are defined as in the previous case.

These two models prove to be qualitatively satisfactory, since with only a few parameters to calibrate they can be optimized to fit experimental data with good accuracy (best-fitting). However, this comes at the expense of higher computational cost.

More recent viscoelastic models have been developed by combining different Maxwell and Kelvin–Voigt elements with Hookean springs, leading to constitutive equations that include several parameters to be identified. The increase in the number of parameters naturally raises the complexity of the model. In general, the choice of a specific configuration depends on the balance between model accuracy and computational efficiency.

The constitutive relation of these *multi-element models* can be expressed in the following general form:

$$\sum_{k=0}^n a_k \frac{d^k}{dt^k} \sigma(t) = \sum_{k=0}^m b_k \frac{d^k}{dt^k} \varepsilon(t), \quad (1.35)$$

where the number of parameters  $m$  and  $n$  depends on the complexity of the model. It is worth noting that Equations (3.5), (3.10), (3.11), and (3.12) are all special cases of this general expression.

However, increasing the number of elementary elements in a model, although generally improving accuracy, also complicates its resolution and often leads to a loss of physical meaning. In fact, it may occur that some elements exhibit negative stiffness or viscosity, which is clearly unacceptable from a physical standpoint.

Moreover, the response equations derived from such models are ordinary differential equations, which necessarily take the form of integral equations with exponential kernels. This formulation, however, does not align with experimental data, which typically reveal constitutive relationships between stress and strain (or stress and time) that follow a power-law behaviour rather than an exponential one.

### 1.1.6 Maxwell–Wiechert model

The *generalized Maxwell model*, often referred to as the Maxwell–Wiechert model (after J. C. Maxwell and E. Wiechert), represents one of the most comprehensive linear formulations for viscoelasticity. In this model, several Maxwell elements are connected in parallel, allowing the relaxation process to be described over a spectrum of characteristic times rather than a single relaxation time. This feature reflects the molecular nature of polymers, where chains or segments of different lengths contribute to the mechanical response at different time scales: shorter chains relax quickly and have a limited impact, whereas longer chains relax more slowly and dominate the long-term behaviour. The Wiechert model reproduces this distribution of relaxation times by incorporating a sufficient number of spring–dashpot elements to match the experimental response.

Because of its ability to capture complex time-dependent phenomena, the generalized Maxwell model has become a standard tool for describing the mechanical behaviour of viscoelastic materials. It assumes linear viscoelasticity and is therefore applicable to problems involving small deformations, making it particularly suitable for polymers, biological tissues, and other soft viscoelastic solids. From a mathematical standpoint, the constitutive relation can be expressed either in the time domain, through the relaxation function, or in the frequency domain, via the complex modulus. This dual formulation makes the model versatile and well-suited for both experimental characterisation and numerical simulations. In engineering practice, its implementation is commonly achieved through a *Prony series* expansion, which provides an efficient representation of the relaxation spectrum and facilitates its use within finite element analysis of viscoelastic structures. The model scheme is presented in Figure 1.12, where, a series of Maxwell units  $i$  composed by a spring of stiffness  $E_i$  and a dashpot of viscosity  $\mu_i$  in series are puts in parallel with the adjunct of a single spring unit  $E_0$  that represent the material stiffness when time tends to infinity.

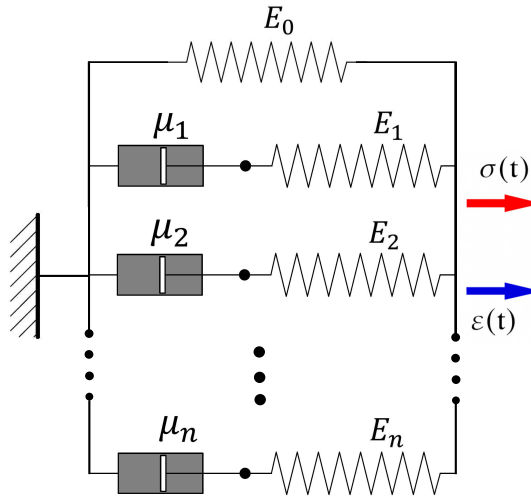


Figure 1.12: Scheme of the Maxwell-Wiechert model.

A common mathematical formulation of the generalized Maxwell model is based on the concept of a *discrete relaxation spectrum*, where each Maxwell element contributes a term to the overall stress relaxation response. This naturally leads to a Prony series representation of the relaxation modulus, expressed as

$$E(t) = E_0 + \sum_{i=1}^n E_i \exp\left(-\frac{t}{\tau_i}\right),$$

where  $E_i$  denotes the modulus and  $\tau_i$  the relaxation time of the  $i$ -th Maxwell element. This representation is particularly convenient when the relaxation times of the material can be either estimated experimentally or assumed based on prior knowledge. In practice, a common guideline is to include approximately one relaxation mode for each decade of time or frequency in order to capture the behaviour accurately. More advanced statistical methods can also be employed to identify the optimal number of modes that provide a good fit while avoiding over-parameterization and ensuring that the model remains physically meaningful.

The Prony series formulation is widely used in engineering applications, since it provides an efficient numerical tool for finite element implementations. Nevertheless, its accuracy strongly depends on the selection of the relaxation times and weights, which may require laborious calibration procedures, especially when long-term responses are of interest.

## 1.2 Fractional-Order Models

In the previous sections, it has been observed that classical models are not able to fully capture the viscoelastic behavior of materials. In fact, the constant-stress test, which allows the creep phenomenon to be studied, is well represented by the Kelvin–Voigt model, which however proves ineffective in simulating relaxation. Conversely, the Maxwell model reproduces relaxation correctly but fails to describe creep.

This situation implies that, unless complicated multi-element models are adopted, a double modeling approach would be required to describe the full behavior of the same material (though subjected to different tests). Such an approach is clearly unacceptable, as it would imply that the same material behaves differently under different experimental conditions.

Moreover, if one uses the creep law derived from the Maxwell model together with the relaxation law obtained from the Kelvin–Voigt model, the fundamental link between the two functions in the Laplace domain is violated.

### 1.2.1 The Riemann–Liouville Formulation

The concept of repeated integration was first introduced by Cauchy. His definition is given by:

$${}_a I_t^n f(t) = \frac{d^{-n} f(t)}{d(t-a)^{-n}} = \int_a^t \int_a^{\tau_{n-1}} \cdots \int_a^{\tau_1} f(\tau) d\tau d\tau_1 \cdots d\tau_{n-1} = \frac{1}{(n-1)!} \int_a^t (t-\tau)^{n-1} f(\tau) d\tau, \quad (1.36)$$

which expresses a multiple integral in the form of a convolution integral, with kernel  $(t-\tau)^{n-1}$ .

This formulation can be naturally extended to non-integer orders, leading to the definition of the *Riemann–Liouville fractional integral*. To this end, the factorial function is replaced by the Gamma function, thereby generalizing the order of integration from integer values  $n$  to any real or even complex value  $\alpha$ :

$${}_a I_t^\alpha f(t) = \frac{d^{-\alpha} f(t)}{d(t-a)^{-\alpha}} = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad \Re(\alpha) > 0. \quad (1.37)$$

Equation (1.37) is commonly referred to in the literature as the *Riemann–Liouville fractional integral*. It represents the so-called *left-sided integral*, since the lower limit of integration is fixed at  $a$ .

Alternatively, by introducing a finite upper limit  $b$  and integrating backward from  $t$ , one obtains the *right-sided integral*:

$${}_t I_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau) d\tau. \quad (1.38)$$

The fractional derivative of Riemann–Liouville (R–L) can be introduced by observing that an integer-order derivative of order  $n$  may be regarded as the  $(n+m)$ -th derivative of the  $m$ -th primitive of the function. Generalizing this concept leads to the definition:

$${}_a D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad (n-1) < \Re(\alpha) < n, \quad (1.39)$$

which is referred to as the *left-sided derivative*, since it is defined over the interval  $(a, t)$ .

Analogously, by setting a finite upper limit  $b > t$ , one obtains the *right-sided derivative*:

$${}_t D_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left( -\frac{d}{dt} \right)^n \int_t^b \frac{f(\tau)}{(\tau-t)^{\alpha-n+1}} d\tau. \quad (1.40)$$

A peculiarity of the R–L derivative is that the derivative of a constant is not equal to zero. In fact:

$${}_0 D_t^\alpha C = \frac{C t^{-\alpha}}{\Gamma(1-\alpha)}. \quad (1.41)$$

This feature highlights one of the main limitations of the R–L formulation when applied to physical problems, as it contradicts the intuitive notion that the derivative of a constant should vanish.

Finally, if forward differences  $(t+\tau)$  are employed instead of backward differences  $(t-\tau)$ , an analogous definition is obtained, commonly referred to as the *Weil differintegral*.

### 1.2.2 The Caputo Approach

A definition of fractional derivative was provided by Michele Caputo and is particularly suitable for applications to concrete problems.

The Riemann–Liouville definition represents a refined mathematical tool, but it often proves inadequate for the modeling of physical phenomena. Indeed, in order to solve differential equations involving the Riemann–Liouville derivative, it is necessary to specify initial conditions in terms of fractional derivatives; such derivatives, however, generally lack a clear physical meaning and are therefore not widely employed in practical problems.

Caputo’s approach overcomes these limitations, since it allows the fractional derivative and/or integral of a function  $f(t)$  to be defined by means of initial conditions expressed through integer-order derivatives. In modern rheology, where fractional derivatives are used to model the behavior of many materials, it is often necessary to solve viscoelasticity problems starting from initial conditions expressed as integer-order derivatives (for instance, the strain rate, which corresponds to the first derivative of the strain function with respect to time). Thanks to Caputo’s formulation, it is possible to solve differential equations with fractional derivatives while specifying initial conditions in terms of integer-order derivatives.

Around 1967, Caputo introduced the following definition of fractional operator:

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \quad n-1 < \alpha < n, \quad (1.42)$$

which is called the *Caputo differintegral*, valid for  $n-1 < \alpha < n$ .

This expression results from an interpolation between integer-order derivatives: when  $\alpha \rightarrow n$  it reduces to the  $n$ -th order derivative of the function  $f(t)$ . Unlike the Riemann–Liouville derivative, the Caputo derivative of a constant is zero.

Thus, in those cases where the function under analysis has prescribed characteristics at  $t \rightarrow 0$ , and the initial conditions are given, Caputo’s derivative is more convenient than the Riemann–Liouville one.

### 1.2.3 Nutting’s Experiment

The process that led to the modeling of viscoelastic phenomena through fractional calculus has been long and complex, involving decades of study and numerous scientists. Probably, the first innovative step in this direction was taken by P. G. Nutting, whose experiments [41] are certainly well-known and frequently cited.

Around 1921, Nutting focused his attention on the study of the viscoelastic behavior of materials. He conducted several experiments and pointed out that the two classical equations used to describe perfectly elastic solids and perfectly viscous fluids, although apparently very different, were in reality special cases of a more general law. Moreover, from the best fitting of his experimental data, he observed that the strain–time relationship could not be described by laws where the time dependence was exponential (as

obtained with classical models). Instead, the curve that interpolated the data points had to necessarily follow a *power law*.

Nutting's experiments produced two types of curves: one relating the displacement  $u$  to time  $t$ , and another relating the displacement to the applied force  $F$  (at constant time).

The displacement–time curves  $u - t$  showed a proportionality between the displacement and the  $n$ -th power of the time variable  $t^n$ . This implied a linear relationship between the logarithm of the displacement and the logarithm of time, namely:

$$\log u \propto \log t$$

Furthermore, the exponent  $n$  turned out to be independent of the force magnitude. From the displacement–force curves  $u - F$ , a similar relationship was obtained:

$$\log u \propto \log F$$

Thus, the displacement was proportional to the  $m$ -th power of the applied force, i.e.,  $u \propto F^m$ .

On the basis of these observations, Nutting proposed a *power-law relationship* capable of expressing the link between the three variables under consideration, namely  $u$ ,  $F$ , and  $t$ .

The empirical law, with a simple mathematical structure and general validity, was expressed as:

$$u = at^n F^m \quad (1.43)$$

This relation describes the displacement as a function of time and the applied force. The parameters  $n$  and  $m$ , characteristic of each material, are independent of  $u$ ,  $t$ ,  $F$ , and of the specimen geometry, but vary with temperature. The constant  $a$ , instead, was found to be independent of  $u$ ,  $t$ , and  $F$ , and dependent only on the type of test.

It is worth noting that Hooke's law, expressed in equation (1.1), is a particular case of equation (1.43) with  $n = 0$  and  $m = 1$ , while the governing equation of viscous fluids according to Newton–Petroff, equation (1.2), is obtained by setting  $n = m = 1$ . From his experimental data, Nutting further observed that the values of  $n$  ranged between 0.2 and 0.91, while those of  $m$  ranged between 0.75 and 3.5. Values of  $n$  close to zero were characteristic of materials with solid-like behavior, while values close to unity denoted a behavior more similar to liquids.

Nutting's experiment and his observations highlighted that classical models, even in their more elaborate forms (such as series of Maxwell or Kelvin–Voigt elements), in which the response is always tied to an exponential law with time, could not adequately reproduce the experimentally observed behavior of materials. This led to the necessity of seeking new constitutive laws for creep and relaxation functions, characterized by a different mathematical structure and capable of simultaneously describing stress and strain histories without violating the Laplace-domain relation introduced in equation (1.12).

Later, between 1936 and 1938, A. N. Gemant, based on further experimental data, suggested the adoption of *fractional derivatives* for the accurate description of the viscoelastic behavior of certain materials [42, 43].

### 1.2.4 The Spring–Pot

Around the 1950s, Scott Blair, G. W. and Caffyn, J. E. [44], in seeking a mathematical law that could justify Nutting’s results, introduced a stress–strain relation based on fractional derivatives.

The difficulty with discrete models, introduced in the previous sections, lies in the fact that in order to achieve an acceptable level of approximation, they require very complex structures that are difficult to handle. Moreover, their solutions always lead to an exponential form of the response function, in contrast to Nutting’s experimental evidence, which instead suggested a power-law model.

The *fractional differential model* (FDM) offers greater flexibility with fewer concentrated parameters to be calibrated, resulting in a significant simplification. The flexibility of these models stems from the fact that the order of the fractional derivative can vary continuously, thus providing a constitutive law adapted to the material in question.

In elastic solids, stress is proportional to the zero-order derivative of strain, whereas in viscous fluids stress is proportional to the first derivative of strain. Nutting’s observation naturally supports the hypothesis that for viscoelastic materials stress is proportional to a derivative of fractional order, lying between 0 and 1, of the imposed strain over time.

This hypothesis leads to the conception of a mathematical model known as the *Spring–Pot*, whose schematic representation is shown in Figure 1.13.

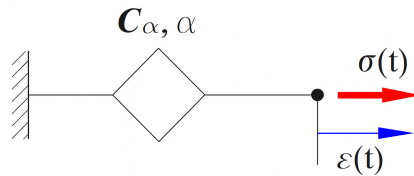


Figure 1.13: The Spring–Pot.

The governing equation of the Spring–Pot, proposed by Scott Blair [44], is:

$$\sigma(t) = E {}_0^C D_t^\alpha \varepsilon(t), \quad 0 \leq \alpha \leq 1 \quad (1.44)$$

where  $\alpha$  and  $E$  are material-dependent constants. The symbol  ${}_0^C D_t^\alpha$  is used referring to the Caputo fractional derivative before introduced in Equation 1.42, and it will be adopted in the following section of the manuscript.

Subsequently, A. N. Gerasimov suggested an analogous expression to generalize Hooke's law of deformation, making use of the Caputo fractional derivative:

$$\sigma(t) = k {}_0^C D_t^\alpha \varepsilon(t), \quad 0 \leq \alpha \leq 1 \quad (1.45)$$

where  $k$  is a material constant that can be interpreted as a *generalized viscosity*.

It should be noted that the expression of Scott Blair (1.44) and that of Gerasimov (1.45) coincide if one assumes zero initial conditions, namely  $\varepsilon(t) = 0$  for all  $t \leq 0$ .

In general, one can thus write:

$$\sigma(t) = C_\alpha {}_0^C D_t^\alpha \varepsilon(t), \quad 0 \leq \alpha \leq 1, \quad (1.46)$$

and, conversely,

$$\varepsilon(t) = \frac{1}{C_\alpha} {}_0^C I_t^\alpha \sigma(t), \quad 0 \leq \alpha \leq 1, \quad (1.47)$$

where  $C_\alpha$  is a material constant that can be obtained by *best-fitting* experimental data.

Although the Spring-Pot provides an analytical model capable of simulating viscoelastic behavior, it still lacks a complete physical interpretation. In classical models, it is possible to distinguish the contribution of the solid phase from that of the fluid phase; in the Spring-Pot such distinction is not straightforward.

Indeed, the coefficient  $C_\alpha$ , which represents the proportionality between stress and the fractional derivative of strain, lacks a clear physical definition: it does not correspond to either an elastic modulus or a viscosity. The only statement that can be made is that this coefficient must satisfy the relation:

$$C_\alpha = E \eta^\alpha \quad (1.48)$$

where  $E$  is the elastic modulus (in Pascal) and  $\eta$  is a characteristic time of the material (in seconds).

Several studies aimed at investigating the physical meaning of the fractional model were carried out by R. L. Bagley and P. J. Torvik [45, 33, 46].

### 1.2.5 Integral Formulation of the Fractional Model

In this section it will be shown how, starting from Boltzmann's superposition principle, one can arrive at the spring-pot model described previously. To this end, let us assume that the relaxation function, which constitutes the kernel of Boltzmann's integral, follows a power-law behavior that can adequately approximate the experimental data. Thus, we write:

$$R(t) \propto t^{-\alpha}, \quad 0 \leq \alpha \leq 1, \quad (1.49)$$

or equivalently:

$$R(t) = \nu_\alpha t^{-\alpha}, \quad 0 \leq \alpha \leq 1, \quad (1.50)$$

where the coefficient  $\nu_\alpha$  must be chosen such that the formulation is consistent with the constitutive law (1.46). Recalling the definition of the Caputo fractional derivative introduced in equation (1.42), the relaxation function can be expressed as

$$\nu_\alpha = \frac{C_\alpha}{\Gamma(1-\alpha)}, \quad (1.51)$$

which leads to

$$R(t) = \frac{C_\alpha}{\Gamma(1-\alpha)} t^{-\alpha} = \frac{E}{\Gamma(1-\alpha)} \left(\frac{t}{\eta}\right)^{-\alpha}. \quad (1.52)$$

By substituting expression (1.52) into Boltzmann's integral (equation (1.11)), one obtains

$$\sigma(t) = \frac{C_\alpha}{\Gamma(1-\alpha)} \int_0^t \dot{\varepsilon}(\tau) (t-\tau)^{-\alpha} d\tau = C_\alpha {}^C D_t^\alpha \varepsilon(t), \quad (1.53)$$

which reproduces precisely the expression (1.46).

Thus, by assuming in Boltzmann's integral a kernel of power-law type, the stress-strain relation is directly obtained as a convolution integral involving the Caputo fractional derivative, in agreement with the constitutive relation proposed by Gerasimov and subsequently formalized by Scott Blair.

Once the relaxation function is assigned, the creep function can be obtained via relation (1.12). To this end, applying the Laplace transform to  $R(t)$  given in equation (1.52), one obtains:

$$R_{\mathcal{L}}(s) = C_\alpha s^{\alpha-1}, \quad (1.54)$$

Recalling equation (1.12), the creep function in the Laplace domain is then:

$$J_{\mathcal{L}}(s) = \frac{1}{C_\alpha s^{\alpha+1}}, \quad (1.55)$$

Performing the inverse Laplace transform yields

$$J(t) = \mathcal{L}^{-1}\{C_{\mathcal{L}}(s); t\} = \frac{1}{C_\alpha \Gamma(1+\alpha)} t^\alpha = \frac{1}{E \Gamma(1+\alpha)} \left(\frac{t}{\eta}\right)^\alpha. \quad (1.56)$$

A deeper explanation of the Laplace transformation applied to fractional operators can be found in the Appendix section 8.1.3.

Substituting expression (1.56) into the Boltzmann integral (1.9), one obtains

$$\varepsilon(t) = \frac{1}{C_\alpha \Gamma(1+\alpha)} \int_0^t \dot{\sigma}(\tau) (t-\tau)^\alpha d\tau. \quad (1.57)$$

By integrating by parts and recalling Euler's gamma function property ( $\Gamma(z+1) = z\Gamma(z)$ ), this can be rewritten as:

$$\varepsilon(t) = \frac{1}{C_\alpha \Gamma(\alpha)} \int_0^t \sigma(\tau) (t - \tau)^{\alpha-1} d\tau = \frac{1}{C_\alpha} {}^C I_t^\alpha \sigma(t), \quad (1.58)$$

which coincides with the formulation proposed by Slonimsky [47]. A more complete treatment of the Euler Gamma function can be found in the appendix in section 8.2.

It is worth noting that the adoption of a power-law form for the relaxation function makes it possible to simulate, with the aid of a single model characterized by only two parameters,  $C_\alpha$  and  $\alpha$ , not only the relaxation test but also the creep test.

This represents a significant advantage of fractional models compared with classical models, in which only integer-order derivatives appear.

### Final Remarks

In this chapter the classical viscoelastic modelling has been investigated, starting from the well known Maxwell and Kelvin-Voight models underlining their limitations. In particular, the Maxwell model becomes less reliable when it is used to interpret the Creep function of a polymeric material. That is described with a crescent linear function which does not fit with the qualitative experimental behaviour. Also the Kelvin-Voight model is not reliable when it is used to represent the material behaviour under constant deformation. Other ways to interpret the viscoelastic behaviour of materials has been proposed, in particular by employing the fractional derivative proposed by Caputo. That follows the intuition from Nutting, that the relaxation function well fits with power laws, instead all the classical models and their combinations are based on exponential fitting (Prony series). Using power laws to fit the relaxation functions the constitutive equation of the material naturally brings to a link between stress and strain with a derivative of order  $\alpha$  intermediate between 0 and 1, called fractional derivative. The mathematical tool that has this kind of particular constitutive law is called Spring-Pot. The advantages of this approach and its application are the core of the thesis and they will be analysed in the next paragraphs.

## Chapter 2

# Application of the fractional calculus in viscoelasticity

In engineering applications, the viscoelastic properties of materials play a crucial role, particularly because of their influence on the long-term performance of structural elements. Consequently, in order to characterize viscoelastic materials, it is necessary to explicitly introduce the time variable. For this reason, one must consider both the stress history and the strain history. Moreover, within the framework of linear viscoelasticity, the response of a material (in terms of stress or strain) at any given time depends on the entire sequence of previous events that have affected it, in accordance with the principle of superposition of effects.

Two numerical methods are compared for the solution of the fractional differential equation under consideration. The first is the classical Grünwald–Letnikov (GL) scheme, which employs a constant subdivision of the observation time interval. The second is the L1 scheme, which instead allows for a variable time-step discretization, enabling greater flexibility in capturing the time-dependent behavior of the solution.

### 2.1 Tests on viscoelastic materials

The viscoelastic behavior of the PVB interlayer is characterized via Dynamic Thermomechanical Analysis (DTMA), employing a three-stage experimental protocol that combined temperature ramp tests, frequency sweep measurements, and stress relaxation experiments. All tests were conducted using an *Anton Paar MCR 302 Rheometer* in Figure 2.1, operated in plate–plate configuration under ambient air conditions. The instrument was equipped with an environmental chamber and a liquid nitrogen-based cooling system to enable precise thermal control across a wide temperature range.

During DMA testing, a sinusoidal stress is applied to a polymer specimen, and the resulting strain response is recorded. The phase angle  $\delta$  between stress and strain—ranging from  $0^\circ$  for purely elastic materials to  $90^\circ$  for ideal viscous behavior—is a fundamental



Figure 2.1: Anton Paar MCR 302 Rheometer.

metric of viscoelasticity, encapsulating the material's time- and temperature-dependent mechanical response.

The rheometer system integrates four key subsystems to ensure accurate measurement and environmental control:

1. A displacement transducer based on a Linear Variable Differential Transformer (LVDT), which resolves probe position via magnetic-core-induced voltage modulation;
2. A PID-controlled thermal chamber, capable of maintaining thermal stability within  $\pm 0.5^\circ\text{C}$ ;
3. A servo-driven linear actuator with a guided drive shaft to apply highly accurate, programmable force inputs;
4. Modular specimen grips designed to accommodate a range of sample geometries, enabling broad experimental flexibility.

Polymeric specimens were prepared in prismatic geometry with nominal dimensions of  $8.0 \pm 0.2 \text{ mm} \times 8.0 \pm 0.2 \text{ mm} \times 0.76 \pm 0.02 \text{ mm}$ . Prior to testing, samples were conditioned at  $23 \pm 1^\circ\text{C}$  and pre-dried for 48 h in a desiccator maintained at  $5 \pm 0.5$  relative humidity. This pre-conditioning protocol was adopted to minimize moisture absorption, which is known to affect PVB's rheological response, and to prevent bubble formation during thermal loading.

A custom shear sandwich fixture was utilized to replicate the constrained shear deformation typically encountered in laminated glass configurations. Daily calibrations of both force and displacement sensors were performed using NIST-traceable reference standards to ensure high measurement accuracy and repeatability.

The temperature-dependent viscoelastic response was quantified via temperature sweep tests ranging from 10°C to 70°C, performed at a controlled heating rate of 3 K min<sup>-1</sup>. Mechanical testing was conducted over a frequency range centered at 1 Hz, in alignment with the experimental conditions adopted for the laminated glass study. Specifically, while high-frequency regimes (10 Hz to 100 Hz) are representative of dynamic, impact-like loading, lower frequency ranges (0.1 Hz to 1 Hz) capture slow viscoelastic phenomena such as creep and stress relaxation. The secant shear modulus,  $G(t)$ , was evaluated using the IRIS RHEO-HUB software, following the methodology proposed by Baumgärtel and Winter [48]. The temperature dependence of the shift factors was modeled with the Williams–Landel–Ferry (WLF) equation [49], using the software *Origin* for curve fitting and parameter estimation.

The machine used for the four-point bending experiments is shown in Figure 2.2. Manufactured by MTS (MTS Criterion® Series 40 Electromechanical Universal Test System), it features an integrated oven that allows testing of the laminated glass plate at the desired temperature. The vertical displacement of the beam is measured using an LVDT positioned at the center of the beam’s intradox, as illustrated in Figure 2.2a, where the maximum deflection occurs. Load is applied by vertically displacing the crosshead, which pushes the two rollers onto the laminated glass beam, inducing flexural stress. The machine can operate in either displacement-controlled or force-controlled mode.

The experimental setup also addresses potential challenges arising from system imperfections, such as support friction, partial fixity, and misalignments. Roller supports are employed to minimize frictional effects, and careful alignment is maintained throughout the testing procedure to ensure accurate and reliable measurements.

All dimensional measurements were recorded with an accuracy of  $\pm 0.1$  mm. The apparatus incorporates roller supports to eliminate frictional effects, ensuring free longitudinal movement. Particular care was taken to maintain perfect alignment throughout the testing procedure to prevent any measurement artifacts.

In this section, the results of an experimental campaign conducted to assess the response of layered glass beams are presented, laminated with PVB Extra Stiff PRO (B231) interlayer produced by Kuraray Europe GmbH, at various operating temperatures: 20°C, 30°C, 40°C and 50°C. Figure 2.3 presents the relaxation curves<sup>1</sup> for this material across these temperatures over a time range from one second to fifty years, obtained via Dynamic Mechanical Thermal Analysis (DMTA), following the same procedure described in [50].

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<sup>1</sup>Kuraray’s technical portal (<https://www.trosifol.com>) offers complete product specifications across their entire range, including elastic property datasheets and standardized classifications. For this study, we examined their commercially available product PVB Extra Stiff PRO (Kuraray internal reference B231). While the website provides general material data, the complete relaxation curves required for our analysis were not publicly available and were specifically characterized for this research.

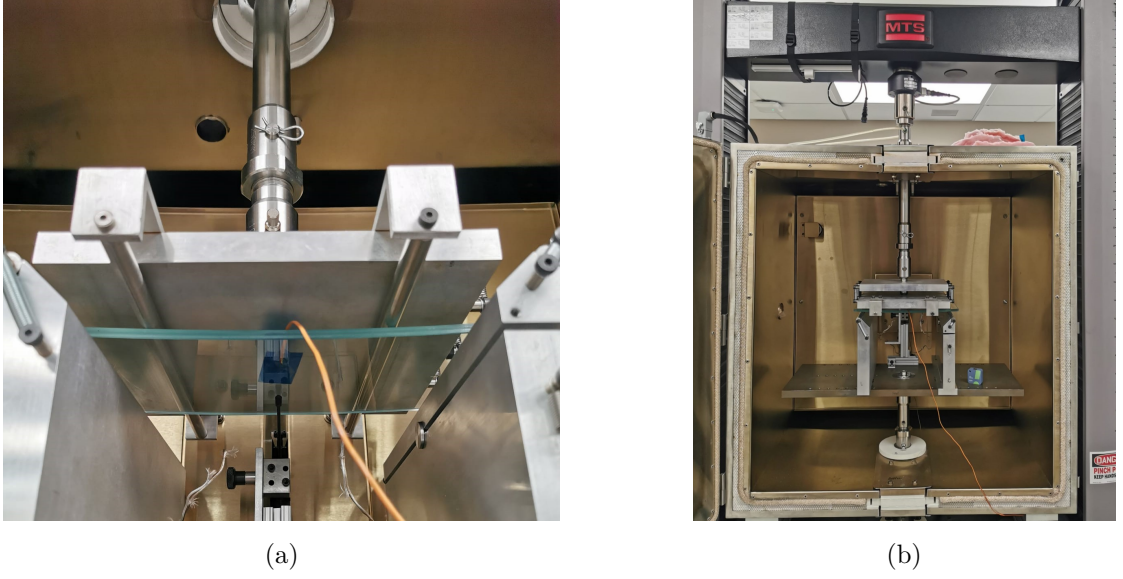


Figure 2.2: MTS Four point bending machine used for the experimental test.

An important application of *Dynamic Thermomechanical Analysis (DTMA)* is the identification of the glass transition temperature,  $T_g$ . In amorphous polymers, this temperature marks a fundamental shift in behavior: above  $T_g$ , the material transitions from a rigid, glassy state to a flexible, rubbery one, accompanied by a notable decrease in both stiffness and viscosity. This transformation is typically indicated by a pronounced peak in the loss tangent ( $\tan \delta$ ), which reflects the phase lag between applied stress and the resulting strain due to viscoelastic effects.

Given its central role in our data processing workflow, the WLF model deserves specific attention. The time-temperature superposition principle is implemented using the temperature-dependent logarithmic shift factor,  $\log a_T$ , which shifts relaxation curves horizontally along the logarithmic time axis. Williams, Landel & Ferry proposed the following expression:

$$\log a_T = \frac{-C_1(T - T_{\text{ref}})}{C_2 + (T - T_{\text{ref}})}, \quad (2.1)$$

where  $C_1$  and  $C_2$  are material-specific coefficients, and  $T_{\text{ref}}$  is the reference temperature. The polymer master curve is constructed through successive application of the time-temperature superposition principle. Each isothermal relaxation curve is horizontally shifted along the logarithmic time axis to align with the reference curve. The resulting master curve enables the prediction of long-term viscoelastic behavior based on short-term experimental data.

The coefficients  $C_1$  and  $C_2$  in equation (2.1) were calibrated by analyzing relaxation curves obtained via DTMA at multiple temperatures. For each pair of curves measured at different temperatures, the shift factor  $a_T$  was determined from the time ratio needed

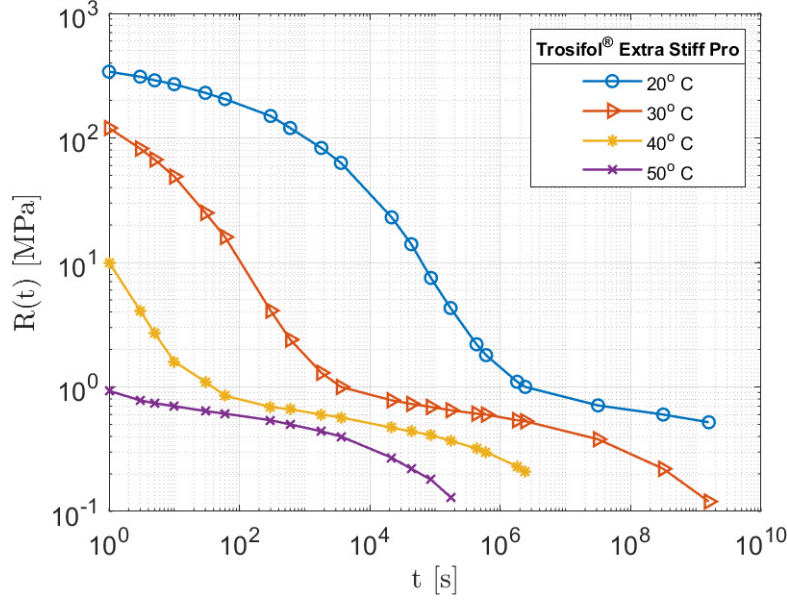


Figure 2.3: Experimental relaxation curves of PVB Extra Stiff PRO (B231), obtained via DTMA, at the environmental temperatures of 20°C, 30°C, 40°C and 50°C.

to superpose their shear modulus values. Using a fixed reference temperature of  $T_{\text{ref}} = 298 \text{ K}$  (25°C), two distinct test temperatures  $T$ , and their corresponding shift factors, a system of two equations was formulated to solve for the unknowns  $C_1$  and  $C_2$ . The validation procedure involved two independent applications of this method, each based on a different pair of temperatures.

Figure 2.4 shows the experimentally obtained values of  $\log a_T$  (blue triangles), derived from direct comparison of the shear relaxation curves at various temperatures with the reference curve at  $T_{\text{ref}} = 298 \text{ K}$  (25°C). The same figure includes the theoretical predictions of the WLF equation (2.1), with parameters calibrated using two temperature pairs: (i)  $T = 283 \text{ K}$  (10°C) and  $T = 303 \text{ K}$  (30°C) (solid red curve), and (ii)  $T = 293 \text{ K}$  (20°C) and  $T = 303 \text{ K}$  (30°C) (dashed yellow curve). The significantly better agreement of curve (ii) with experimental data highlights a well-known limitation of the WLF model: its predictive accuracy deteriorates when applied to temperatures far from  $T_g$ . Therefore, reliable determination of the coefficients requires selecting temperature points sufficiently close to the glass transition.

The WLF coefficients for Trosifol Extra Stiff were determined as  $C_1 = 21.82$  and  $C_2 = 78.88 \text{ K}$ , with  $T_g = 323 \text{ K}$  (50°C). The WLF equation was applied for temperatures in the range 20 – 50°C, an interval of applicability corroborated by experimental data, as per Figure 2.4.

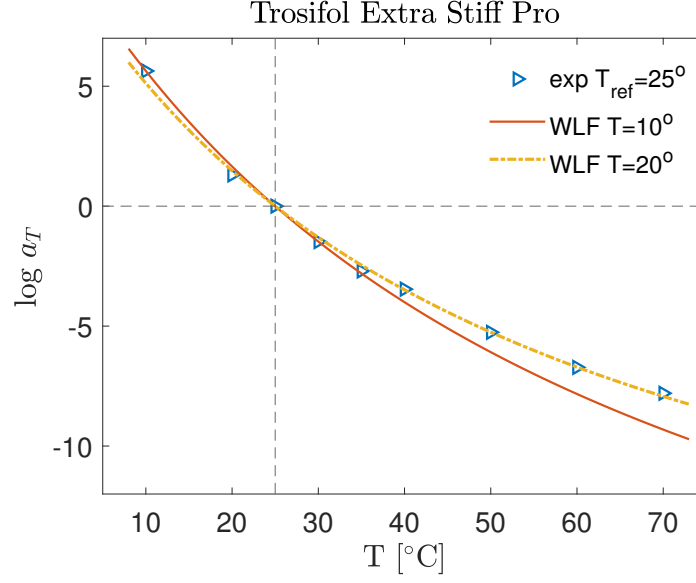


Figure 2.4: Logarithmic shift factors as a function of the temperatures [°C]. Experimental results (blue triangles), measured from comparisons at  $T_{\text{ref}} = 25^\circ\text{C}$ , and theoretical predictions from the WLF equation, with coefficients calibrated for two temperature pairs: (i)  $T = 10^\circ\text{C}$  and  $T = 30^\circ\text{C}$  (solid red curve), or (ii)  $T = 20^\circ\text{C}$  and  $T = 30^\circ\text{C}$  (dashed yellow curve).

## 2.2 Interpolation methods for the Relaxation Curve

Accurate numerical interpolation of the relaxation curve of a polymeric material is essential, as this curve serves as a fundamental input for constitutive models used in stress-strain analysis. In the case of laminated glass, a simplified approach often assumes the polymeric interlayer to behave as a linear elastic material, characterized by a single elastic modulus. While this assumption enables the derivation of a static stress solution, it neglects both the time-dependent stiffness evolution and the memory effects inherent to viscoelastic materials.

Such a simplification may be sufficient for many practical design applications where a high degree of precision is not required. However, when detailed material characterization is necessary—for example, in advanced structural analysis or safety-critical applications—a more comprehensive viscoelastic representation must be adopted. This requires the full relaxation function of the polymeric material to be taken into account, ensuring that the constitutive model captures its time-dependent mechanical response.

### 2.2.1 Prony Series

Rheological modelling of laminated glass typically relies on linear viscoelasticity theory. The viscoelastic constitutive behavior is defined by the relaxation curve of the polymer, which is commonly interpreted using the Maxwell–Wiechert model, composed of arrays of spring–dashpot units [51]. This framework suggests expanding the relaxation function into a series of exponential terms (Prony series), each corresponding to an individual rheological unit. The coefficients of the series can be calibrated through creep or relaxation tests [4], or alternatively via Dynamic Mechanical Analysis (DMA) [52]. The time–temperature superposition principle [49] establishes a relationship between the characteristic time scales of viscoelastic phenomena and the operating temperature.

However, experimental investigations on a range of commercial polymers [4, 53–57] have shown that the relaxation curve, when plotted on bi-logarithmic axes (shear modulus versus time), typically exhibits a shape that is well-approximated by a polyline. Since a power-law function appears as a straight line in such a plot, it is natural to model the relaxation response using piecewise power-law functions [58]. Typically, 3–4 branches (i.e., 6–8 coefficients) are sufficient to achieve an accurate fit, a process further facilitated by the geometric interpretation of the parameters [59]. By contrast, a conventional Prony series representation generally requires 20–30 terms to attain comparable accuracy [51].

A typical relaxation curve over a long observation time, for a polymer commonly used to laminate glass, is represented in Figure 2.5. The graph shows, in a bi-logarithmic plot, the secant shear modulus [MPa] measured in a relaxation test at 20° C on a Stiff PVB, as a function of time [s]. The classical way to interpolate the trend is through a Prony series of exponentials terms of the form  $R_i e^{-t/\theta_i}$ , where  $R_i$  is the  $i$ -th relaxation shear modulus, and  $\theta_i$  the corresponding relaxation time. Consequently, the relaxation function reads

$$R(t) = R_0 + \sum_{i=1}^N R_i e^{-t/\theta_i}. \quad (2.2)$$

This series interprets the Wiechert model of viscoelasticity, which consists of an array of Maxwell units in parallel with a spring of stiffness  $R_0$ , which represents the residual stiffness of the viscoelastic material when time tends to infinity. Figure 2.5 represents the resulting approximation of the curve: 9 Maxwell units, in addition to the spring, are necessary to obtain a good fit. This implies that 19 parameters need to be calibrated. Practical methods have been proposed to calibrate the coefficients; examples are the Domain of Influence method [60], but its application is not straightforward.

### 2.2.2 Power Laws

It has been observed that classical models are unable to fully capture the viscoelastic behavior of materials. For example, the constant-stress test, which allows for the study of the creep phenomenon, is well represented by the Kelvin–Voigt model, which, however, fails to reproduce relaxation. Conversely, the Maxwell model provides a good description

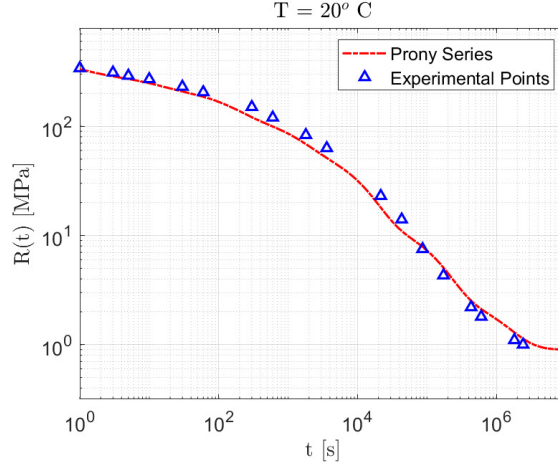


Figure 2.5: Bi-logarithmic plot of the relaxation curve for an Acoustic Monolayer PVB tested at the temperature of  $20^\circ\text{C}$  and its approximations. Experimentally measured tangent shear modulus  $R(t)$  [MPa] as a function of time  $t$  [s] indicated as red triangles; approximation with a Prony series, red dashed line, with 9 terms and 19 parameters to calibrate.

of relaxation but proves inadequate in simulating creep. As a result, unless one resorts to complex multi-element models, a dual modeling approach is required in order to describe the behavior of the same material when subjected to different tests.

Such a situation is unsatisfactory, since it is not reasonable to assume that a material exhibits fundamentally different behaviors under the two testing conditions. Moreover, if one adopts the creep law derived from the Maxwell model and the relaxation law derived from the Kelvin–Voigt model, the intrinsic consistency of the formulation is lost, as the fundamental relationship in the Laplace domain is violated.

The distinct shape of the relaxation function suggests that an alternative interpolation can be achieved using continuously connected branches of lines in the bilogarithmic plot, as illustrated in Figure 2.6, which correspond to power laws of the form  $C_\alpha t^{-\alpha}$ , with  $0 < \alpha < 1$ . Each branch is defined by two parameters: the slope  $\alpha$  of the line and  $C_\alpha$  [MPa s $^\alpha$ ], representing the value at  $t = 1$  s. Typically, the relaxation functions of polymeric interlayers can be interpolated with no more than four branches of power laws over an observation window in the range ( $10^{-4}$  s,  $10^{12}$  s). For phenomena observed over shorter time intervals, fewer branches are sufficient. Polymers such as SentryGlas<sup>®</sup>, in a relatively wide time-temperature range, may only require a single power law to achieve an accurate fit.

The four lines that interpolate the curve in the bilogarithmic graph represent characteristic phases of the polymer response. The first line, where the modulus is high, corresponds to the “glassy region”, where the interlayer provides the higher coupling between the glass pliers. The second line indicates the “glassy transition”, with a faster

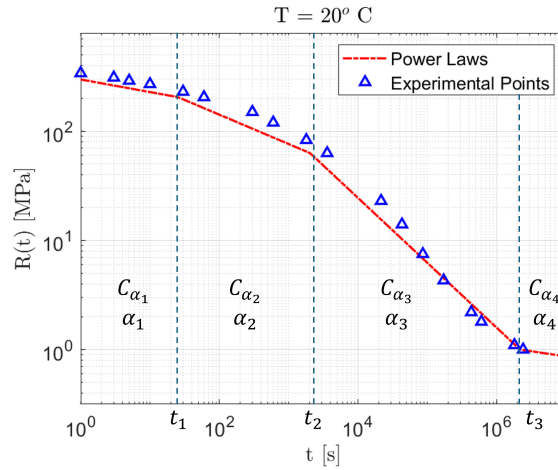


Figure 2.6: Bilogarithmic representation of the relaxation curve for Trosifol<sup>®</sup> Extra Stiff (PVB) at 20° C. Interpolation of experimental points (dots) with connected branches of power laws (dashed line).

descent in time. The third branch is the “rubbery plateau”: the relaxation function settles to a lower value and does not decrease rapidly. The fourth last branch is associated with the “viscous flow”, where the negative slope of the line increases in absolute value, so that the coupling between the glass plies through the material becomes very mild.

The use of power laws is advantageous with respect to Prony series because, on the one hand, less parameters need to be calibrated and, on the other hand, such parameters have an intuitive geometric interpretation. An even more important aspect is that there is a mathematical description of the equations of viscoelasticity [61], founded on fractional calculus, which naturally provides power laws of time as fundamental solutions. This is because Boltzmann’s convolution integral coincides with the Caputo fractional derivative of order  $\alpha$  when the relaxation function is expressed by a power law. However, in order to establish such a precise correspondence, it is necessary to formally write the power-law terms by defining the coefficients in terms of Euler’s Gamma function  $\Gamma$ , which is the generalization of the factorial  $n!$  to non-integer or complex values of  $n$ . In conclusions, the power-law approximation defined in Figure 2.6 takes the analytical form

$$R(t) = \begin{cases} \frac{C_{\alpha_1}}{\Gamma(1-\alpha_1)} t^{-\alpha_1}, & t \leq t_A, \\ \frac{C_{\alpha_2}}{\Gamma(1-\alpha_2)} t^{-\alpha_2}, & t_A \leq t \leq t_B, \\ \frac{C_{\alpha_3}}{\Gamma(1-\alpha_3)} t^{-\alpha_3}, & t_B \leq t \leq t_C, \\ \frac{C_{\alpha_4}}{\Gamma(1-\alpha_4)} t^{-\alpha_4}, & t \geq t_C. \end{cases} \quad (2.3)$$

The relaxation curves correspond to the complete time trend of the elastic shear

modulus of the polymeric material at a fixed temperature, but can also be obtained by varying the temperature in a short observation period. The Time-Temperature Superposition principle (TTS) establishes that any variation of the test temperature is associated with a variation of the time scale for the effects of viscosity [4]. This property is defined by models such as that proposed by the Williams Landel & Ferry [49], which establishes a correlation between the time scale of the rheological phenomenon and the operating temperature.

Table 2.1 collects the parameters that describe the branches of power-law approximating the relaxation curves at various temperatures of Figure 2.3. These will be used in the numerical simulations.

Table 2.1: Fractional parameters  $\alpha_i$ ,  $C_{\alpha_i}$  [MPa s $^\alpha$ ],  $\tau_i$  [s] of Trosifol Extra Stif PRO at different temperatures 20°C, 30°C, 40°C and 50°C.

| Temp. (°C) | $i$ | $\alpha_i$ | $C_{\alpha_i}$ (MPa s $^\alpha$ ) | $\tau_i$ (s) |
|------------|-----|------------|-----------------------------------|--------------|
| 20         | 1   | 0.1149     | 340.8                             | 61.2         |
|            | 2   | 0.2930     | 709.4                             | 3608.9       |
|            | 3   | 0.6055     | 9173.3                            | –            |
| 30         | 1   | 0.1335     | 148.5003                          | 0.7          |
|            | 2   | 0.4955     | 128.3280                          | 64.6         |
|            | 3   | 0.8607     | 588.0325                          | 593.9        |
|            | 4   | 0.4761     | 50.4192                           | 4773.4       |
|            | 5   | 0.0800     | 1.7600                            | –            |
| 40         | 1   | 0.1649     | 55.2290                           | 0.0196       |
|            | 2   | 0.5982     | 10.0398                           | 1.0222       |
|            | 3   | 0.7841     | 10.0809                           | 18.6076      |
|            | 4   | 0.1009     | 1.3680                            | –            |
| 50         | 1   | 0.0825     | 0.8631                            | 294.1        |
|            | 2   | 0.1201     | 1.0689                            | 3576.2       |
|            | 3   | 0.2557     | 3.2415                            | –            |

## 2.3 From power law to fractional derivatives

The study of viscoelastic behavior in polymeric materials represents a cornerstone of materials science and engineering, owing to its profound implications for both fundamental understanding and practical applications. Polymers, due to their complex molecular architectures and time-dependent mechanical responses, often exhibit behaviors that can-

not be adequately described by classical integer-order models of elasticity and viscosity. Traditional constitutive models, while successful in capturing simple stress-strain relationships, often fail to represent the rich spectrum of relaxation and creep phenomena observed experimentally in polymeric systems.

In recent decades, fractional calculus has emerged as a powerful and versatile mathematical framework for modeling complex viscoelastic behavior. Fractional derivatives, which generalize classical differentiation to non-integer orders, offer a unique capability to capture memory and hereditary effects inherent in polymeric materials. Unlike conventional models that require multiple spring-dashpot elements to approximate the same phenomena, fractional-order models can often achieve high accuracy with fewer parameters, providing both computational efficiency and deeper physical insight.

The application of fractional derivatives in viscoelastic modeling bridges the gap between theoretical formulations and experimental observations, enabling a more precise characterization of time-dependent mechanical responses. These models have demonstrated superior performance in describing phenomena such as stress relaxation, creep compliance, and dynamic mechanical behavior over wide frequency and temperature ranges, which are critical for the design and optimization of polymer-based materials and devices.

This thesis focuses on the development, analysis, and application of fractional-order models to study viscoelasticity in polymeric materials. By integrating theoretical formulations, numerical simulations, and experimental validation, this work aims to advance the understanding of the underlying mechanisms governing polymer viscoelasticity and provide robust predictive tools for material design. The study also explores the potential of fractional calculus to unify and extend classical viscoelastic theories, offering a coherent framework that can adapt to the intrinsic complexity of real-world polymeric systems.

### 2.3.1 Constitutive equation for viscoelastic materials

In viscoelasticity, the constitutive response of a material is governed not only by the instantaneous value of the applied stress or strain, but also by the entire history of loading. This hereditary nature is commonly described by the Boltzmann superposition principle, which establishes that the total strain at a given time  $t$  can be obtained as the linear superposition of the infinitesimal contributions associated with all past stress increments. Conversely, the stress at time  $t$  can be expressed as the superposition of the infinitesimal effects of past strain increments. In mathematical terms, the principle leads to a convolution integral between the relaxation function and the strain rate (in the stress-controlled formulation), or between the creep function and the stress rate (in the strain-controlled formulation). This formulation naturally incorporates memory effects and provides the fundamental basis for the constitutive equations of polymeric materials, whose time-dependent behavior cannot be described by a single elastic modulus.

If the applied load is a continuous function, the Boltzmann integral is

$$\tau_{yz}(z, t) = \gamma_{yz}(z, 0) R(t) + \int_0^t \frac{\partial \gamma_{yz}(z, \bar{t})}{\partial \bar{t}} R(t - \bar{t}) d\bar{t}. \quad (2.4)$$

where  $R(\cdot)$  is the relaxation function of the polymeric material subjected to a generic shear strain  $\gamma_{yz}(z, t)$  that consequently generate a shear stress  $\tau_{yz}(z, t)$ . In general, we will suppose that the structure is undistorted at  $t = 0$ , so that  $\gamma_{yz}(z, 0)$ .

Equation 2.4 is a convolution integral (or *Faltung*) and expresses the fact that the current deformation depends on the entire loading history, starting from a virgin material. In essence, Boltzmann's idea is that the material possesses memory of the events it has undergone, so that the mechanical response at a given time is a function of the entire past history. This theory, known as the integral formulation of linear viscoelasticity, was later taken up at the beginning of the twentieth century by V. Volterra and further developed around 1965 by Benvenuti.

If the relaxation function is a simple power-law of the form  $R(t) = \frac{C_\alpha}{\Gamma(1-\alpha)} t^{-\alpha}$ , since  $\gamma_{yz}(z, 0) = 0$  as an initial condition, the constitutive equation (2.4) takes the form

$$\tau_{yz}(z, t) = \int_0^t \frac{\partial \gamma_{yz}(z, \bar{t})}{\partial \bar{t}} \frac{C_\alpha}{\Gamma(1-\alpha)} (t - \bar{t})^{-\alpha} d\bar{t} = C_\alpha {}^C_0 D_t^\alpha [\gamma_{yz}(z, \cdot)](t), \quad (2.5)$$

because of the definition (1.42) of Caputo fractional derivative. This is the model that has been considered in [62, 59].

In a more general setting, consider a relaxation function piecewise defined by power laws as in Figure 2.6 and of the type (2.3). Now the shear stress  $\tau_{y,z}(z, t)$  is related to the strain  $\gamma_{y,z}(z, t)$  by

$$\tau_{yz}(z, t) = \begin{cases} \int_0^t \frac{\gamma_{yz}(z, \bar{t})}{\bar{t}} \frac{C_{\alpha_1}}{\Gamma(1 - \alpha_1)} (t - \bar{t})^{-\alpha_1} d\bar{t}, \\ t \leq t_A, \\ \\ \int_0^{t-t_A} \frac{\partial \gamma_{yz}(z, \bar{t})}{\partial \bar{t}} \frac{C_{\alpha_2}}{\Gamma(1 - \alpha_2)} (t - \bar{t})^{-\alpha_2} d\bar{t} \\ + \int_{t-t_A}^t \frac{\partial \gamma_{yz}(z, \bar{t})}{\partial \bar{t}} \frac{C_{\alpha_1}}{\Gamma(1 - \alpha_1)} (t - \bar{t})^{-\alpha_1} d\bar{t}, \\ t_A \leq t \leq t_B, \\ \\ \int_0^{t-t_B} \frac{\partial \gamma_{yz}(z, \bar{t})}{\partial \bar{t}} \frac{C_{\alpha_3}}{\Gamma(1 - \alpha_3)} (t - \bar{t})^{-\alpha_3} d\bar{t} \\ + \int_{t-t_B}^{t-t_A} \frac{\partial \gamma_{yz}(z, \bar{t})}{\partial \bar{t}} \frac{C_{\alpha_2}}{\Gamma(1 - \alpha_2)} (t - \bar{t})^{-\alpha_2} d\bar{t} \\ + \int_{t-t_A}^t \frac{\partial \gamma_{yz}(z, \bar{t})}{\partial \bar{t}} \frac{C_{\alpha_1}}{\Gamma(1 - \alpha_1)} (t - \bar{t})^{-\alpha_1} d\bar{t}, \\ t_B \leq t \leq t_C, \\ \\ \int_0^{t-t_C} \frac{\partial \gamma_{yz}(z, \bar{t})}{\partial \bar{t}} \frac{C_{\alpha_4}}{\Gamma(1 - \alpha_4)} (t - \bar{t})^{-\alpha_4} d\bar{t} \\ + \int_{t-t_C}^{t-t_B} \frac{\partial \gamma_{yz}(z, \bar{t})}{\partial \bar{t}} \frac{C_{\alpha_3}}{\Gamma(1 - \alpha_3)} (t - \bar{t})^{-\alpha_3} d\bar{t} \\ + \int_{t-t_B}^{t-t_A} \frac{\partial \gamma_{yz}(z, \bar{t})}{\partial \bar{t}} \frac{C_{\alpha_2}}{\Gamma(1 - \alpha_2)} (t - \bar{t})^{-\alpha_2} d\bar{t} \\ + \int_{t-t_A}^t \frac{\partial \gamma_{yz}(z, \bar{t})}{\partial \bar{t}} \frac{C_{\alpha_1}}{\Gamma(1 - \alpha_1)} (t - \bar{t})^{-\alpha_1} d\bar{t}, \\ t \geq t_C. \end{cases} \quad (2.6)$$

Using Caputo's definition (1.42) of fractional derivative, it is possible to rewrite Equation 2.6 in the fractional form:

$$\tau_{yz}(z, t) = \begin{cases} C_{\alpha_1} {}^C D_t^{\alpha_1} [\gamma_{yz}(z, \cdot)](t), \\ t \leq t_A, \\ \\ C_{\alpha_2} {}^C D_{t-t_A}^{\alpha_2} [\gamma_{yz}(z, \cdot)](t) \\ + C_{\alpha_1} {}^C D_{t-t_A}^{\alpha_1} [\gamma_{yz}(z, \cdot)](t), \\ t_A \leq t \leq t_B, \\ \\ C_{\alpha_3} {}^C D_{t-t_B}^{\alpha_1} [\gamma_{yz}(z, \cdot)](t) \\ + C_{\alpha_2} {}^C D_{t-t_B}^{\alpha_2} [\gamma_{yz}(z, \cdot)](t) \\ + C_{\alpha_1} {}^C D_{t-t_A}^{\alpha_1} [\gamma_{yz}(z, \cdot)](t), \\ t_B \leq t \leq t_C, \\ \\ C_{\alpha_4} {}^C D_{t-t_C}^{\alpha_4} [\gamma_{yz}(z, \cdot)](t) \\ + C_{\alpha_3} {}^C D_{t-t_C}^{\alpha_3} [\gamma_{yz}(z, \cdot)](t) \\ + C_{\alpha_2} {}^C D_{t-t_B}^{\alpha_2} [\gamma_{yz}(z, \cdot)](t) \\ + C_{\alpha_1} {}^C D_{t-t_A}^{\alpha_1} [\gamma_{yz}(z, \cdot)](t), \\ t \geq t_C. \end{cases} \quad (2.7)$$

This is a piece-wise defined fractional derivative that, in a more general framework, the relaxation function can be expressed in a piecewise form using power laws, as described in Equation (2.3). Consequently, in Equation (2.5), the parameters  $\alpha$  and  $C_\alpha$  are no longer constant, but instead vary with time. These quantities can therefore be represented as vectors, denoted by  $\tilde{\alpha}$  and  $\tilde{C}_{\tilde{\alpha}}$ , following the procedure outlined in [63]. Equation (2.5) can thus be reformulated as:

$$\tau_{yz}(z, t) = C_{\tilde{\alpha}} {}^C D_t^{\tilde{\alpha}} [\gamma_{yz}(z, \cdot)](t). \quad (2.8)$$

The formulation of the constitutive equation for the polymeric interlayer can be extended to an arbitrary number of branches used to approximate the relaxation function. In the example discussed above, the relaxation curve reported in Figure 2.6, composed of four branches, was considered. However, selecting a different curve with a larger or smaller number of branches does not complicate the formulation, since the piecewise definition of the relaxation function can be adapted to any chosen level of discretization.

## 2.4 Numerical approximation with Grünwald-Letnikov (GL) method

Among the various approaches to fractional differentiation, the Grünwald–Letnikov (GL) method is one of the most widely used due to its simplicity and direct connection with

the definition of a fractional derivative as a limit of fractional differences. The method relies on discretizing the time domain with a constant step size and approximating the convolution integral of the fractional derivative by a weighted sum of past values of the function. This approach has the advantage of being straightforward to implement, as it only requires the computation of binomial coefficients of fractional order. However, since every new evaluation of the derivative involves the entire history of the function, the GL scheme can become computationally demanding for long time horizons. Despite this limitation, the method is commonly used as a reference scheme and provides a useful benchmark for assessing more advanced numerical algorithms.

Subdivide the observation time in  $s$  subintervals with constant amplitude  $\Delta t = t_i - t_{i-1}$ , with  $i = 1, 2 \dots s$ . The Caputo's fractional derivative of order  $\alpha$ , with  $0 \leq \alpha \leq 1$  of the generic function  $f(t)$ , defined by (1.42), is approximated by the Grünwald-Letnikov (GL) operators defined as

$${}_0^C D_t^\alpha [f(\cdot)](t_s) \simeq \Delta t^{-\alpha} \sum_{r=0}^{s-1} \frac{\Gamma(r-\alpha)}{\Gamma(r)\Gamma(-\alpha)} f(t_s - r\Delta t), \quad (2.9)$$

where, recall,  $\Delta t$  is the constant time step. This can be expressed in compact matrix form as

$$\nabla^\alpha \mathbf{f}_s = \mathbf{A}_s^{(\alpha)} \mathbf{f}_s, \quad (2.10)$$

where  $\mathbf{f}_s$  is the vector collecting the values of  $f(t)$  at the nodal points and  $\mathbf{A}_s^{(\alpha)}$  is a lower  $s \times s$  lower band strip matrix. In particular

$$\mathbf{A}_s^{(\alpha)} = \frac{1}{\Delta t^\alpha} \begin{bmatrix} \omega_1(\alpha) & \cdots & 0 \\ \omega_2(\alpha) & \ddots & \vdots \\ \vdots & & \\ \omega_s(\alpha) & \cdots & \omega_2(\alpha) & \omega_1(\alpha) \end{bmatrix}, \quad \mathbf{f}_s = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_s \end{bmatrix}, \quad (2.11)$$

with  $\omega_1 = 1$ ,  $\omega_2 = -\alpha$ ,  $\omega_{s+1} = \frac{s-\alpha-1}{s} \omega_s(\alpha)$ .

When the relaxation function is composed of branches of power-laws, one can divide the observation time in such a way that each branch is defined by a certain number of time steps. With reference to the form (2.3), suppose that the first interval  $[0, t_A]$  takes the first  $p$  points, the second interval  $[t_A, t_B]$  takes  $u$  points, the third one  $[t_B, t_C]$  further  $d$  points and the last interval  $[t_C, t]$  the remaining  $g = s - d - u - p$  points, forming the partition

$$\begin{aligned} & t_1, t_2, \dots, t_p; t_{p+1}, t_{p+2} \dots t_{p+u}; \\ & t_{p+u+1}, t_{p+u+2} \dots t_{p+u+d}; \\ & t_{p+u+d+1}, t_{p+u+d+2} \dots t_{p+u+d+g}. \end{aligned} \quad (2.12)$$

The first column of the triangular strip matrix is built by multiplying each times group, as previously defined, for the correspondent coefficient  $\frac{C_{\alpha_i}}{\Delta t^{\alpha_i}}$ , for  $i = 1 \dots 4$ . Schematically, one constructs the vector

$$\begin{aligned}
\left(\mathbf{\Omega}_s^{(\tilde{\alpha})}\right)^T &= \left[ \begin{aligned} &\frac{C_{\alpha_1}}{\Delta t^{\alpha_1}}(\omega_1, \dots, \omega_p), \\ &\frac{C_{\alpha_2}}{\Delta t^{\alpha_2}}(\omega_{p+1}, \dots, \omega_{p+u}), \\ &\frac{C_{\alpha_3}}{\Delta t^{\alpha_3}}(\omega_{p+u+1}, \dots, \omega_{p+u+d}), \\ &\frac{C_{\alpha_4}}{\Delta t^{\alpha_4}}(\omega_{p+u+d+1}, \dots, \omega_{p+u+d+g}) \end{aligned} \right] \\
&= \left[ \left(\mathbf{\Omega}_p^{(\alpha_1)}\right)^T, \left(\mathbf{\Omega}_u^{(\alpha_2)}\right)^T, \left(\mathbf{\Omega}_d^{(\alpha_3)}\right)^T, \left(\mathbf{\Omega}_g^{(\alpha_4)}\right)^T \right].
\end{aligned} \tag{2.13}$$

The Grünwald-Letnikov matrix  $\mathbf{A}_s^{(\tilde{\alpha})}$  becomes an  $s \times s$  lower triangular Toeplitz square matrix, whose first column is  $\mathbf{\Omega}_s^{(\tilde{\alpha})}$  and such that  $[\mathbf{A}_s^{(\tilde{\alpha})}]_{i,j} = 0$  if  $i < j$  and  $[\mathbf{A}_s^{(\tilde{\alpha})}]_{i+1,j+1} = [\mathbf{A}_s^{(\tilde{\alpha})}]_{i,j}$  if  $i \geq j$ . Schematically indicated as

$$\mathbf{A}_s^{\tilde{\alpha}} = \begin{bmatrix} \mathbf{\Omega}_p^{(\alpha_1)} & 0 & 0 & 0 \\ \mathbf{\Omega}_u^{(\alpha_2)} & \ddots & 0 & 0 \\ \mathbf{\Omega}_d^{(\alpha_3)} & \ddots & \ddots & 0 \\ \mathbf{\Omega}_g^{(\alpha_4)} & \ddots & \ddots & \ddots \end{bmatrix}, \tag{2.14}$$

it is constructed in such a way that each descending diagonal from left to right is constant. It is possible to verify that when  $\alpha_i$  is a natural number, the GL matrix provides the finite difference scheme for ordinary derivatives (the triangular matrix becomes a banded matrix). For example,  $\mathbf{A}_s^{(2)}$  approximates the second derivative of the function.

For each term  $v_n(t)$  and  $q_n(t)$  of the series expansion (3.11), which appear in the governing equation (3.13), the discretized version of the vertical displacement and the applied load is defined by the arrays

$$\mathbf{v}_n = \begin{bmatrix} v_n(t_1) \\ v_n(t_2) \\ \vdots \\ v_n(t_s) \end{bmatrix}, \quad \mathbf{q}_n = \begin{bmatrix} q_n(t_1) \\ q_n(t_2) \\ \vdots \\ q_n(t_s) \end{bmatrix}. \tag{2.15}$$

This is a really classical approach to numerically interpret the fractional derivative of a function due to its easy and fast application, but with some important limitations, for example it requires a constant step time, that makes it not always suitable for every application.

## 2.5 L1 Formula With Variable Time-Step

To follow the long-term response, it is convenient to use a variable time step, which shall be small at the beginning, when the derivative of the relaxation curve is high, and large in the long run, when the relaxation curve tends to zero. Observing that each branch of power-law is a straight segment in the bi-log graph, it is natural to consider a discretized time vector such that  $\text{Log}(t_{i+1}) - \text{Log}(t_i) = c = \text{const}$ , i.e.,  $t_{i+1} = 10^c t_i$ , which corresponds to a constant interval in the bi-log graph. This partition, following a geometric progression, is shown in figure 2.7.



Figure 2.7: Example of time mesh following a geometric progression. Graphs in (a) logarithmic scale and (b) linear scale.

With reference to a typical relaxation function interpolated by four branches of power law (PVB stiff polymer at 20° C), Figure 2.6 evidences the time mesh in geometric progression for an observation time of  $10^7$  s. Observe that each branch is homogeneously represented, in the sense that the variation of the relaxation curve, in the bi-log graphs, is equally taken into account by the proposed partition.

As an alternative to the Grünwald-Letnikov approach, it is possible to discretize the Caputo's derivative of a generic function  $v(t)$  via the expression [64–66]

$$\begin{aligned} {}^C_0 D_t^\alpha [v(\cdot)](t_s) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_s} (t_s - \tau)^{-\alpha} \frac{dv}{d\tau}(\tau) d\tau \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^s \frac{v(t_k) - v(t_{k-1})}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} (t_s - \tau)^{-\alpha} d\tau + R_n. \end{aligned} \quad (2.16)$$

This is the L1 formula, where  $R_n$  is the local truncation error.

The L1 formula can be extended to a piecewise relaxation function composed by four temporal branches represented by power-laws, as per Figure 4.9, where the coefficients  $\alpha$  and  $C_\alpha$  vary in time. The observation time is divided in  $s^*$  intervals: the number of intervals is  $p^*$  in the first interval  $[0, t_A]$ ,  $u^*$  in  $[t_A, t_B]$ ,  $d^*$  in  $[t_B, t_C]$  and  $g^* = s^* - p^* - u^* - d^*$  in last interval  $[t_C, t]$ . Within of each temporal branch  $C_\alpha$  and  $\alpha$  assume different values, for example for the firsts  $p^*$  elements one has  $C_{\alpha_1}$  and  $\alpha_1$ , as showed in figure 4.9, and so on.

With reference to the governing equation (3.13), let us consider the components  $v_n(t)$  in the series expansion (3.11a). The fractional operator  ${}^C_0 D_t^\alpha [v_n(\cdot)](t)$ , defined in (2.8), can be schematically approximated as

$${}_0^C D_t^{\tilde{\alpha}} [v_n(\cdot)](t_j) = \frac{C_{\alpha_i}}{\Gamma(1-\alpha_i)} \sum_{k=1}^j \frac{v_n(t_k) - v_n(t_{k-1})}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} (t_j - \tau)^{-\alpha_i} d\tau, \quad (2.17)$$

where the index  $i$  can vary between one and the number of branches used to subdivide the relaxation function. In this example the curve has been subdivided in four different branches each one with different constant values  $\alpha_i$  and  $C_{\alpha_i}$ . Each branch stands for a specific number of time steps counted with the index  $j$ , so the scheme used to assign the correct number of  $i$  according to the actual generic time step  $j$  is the following

$$i = \begin{cases} 1 & \text{for } j = 1, \dots, p^*, \\ 2 & \text{for } j = p^* + 1, \dots, p^* + u^*, \\ 3 & \text{for } j = p^* + u^* + 1, \dots, p^* + u^* + d^*, \\ 4 & \text{for } j = p^* + u^* + d^* + 1, \dots, s^*, \end{cases} \quad (2.18)$$

In this example, the first branch contains  $p^*$  time steps, the second  $u^*$ , the third  $d^*$  and the last one  $g^* = s^* - (p^* + u^* + d^*)$ . It is also possible to rewrite the defined integral  $\int_{t_{k-1}}^{t_k} (t_j - \tau)^{-\alpha_i} d\tau$  as

$$\frac{1}{\Gamma(1-\alpha_i)} \int_{t_{k-1}}^{t_k} (t_j - \tau)^{-\alpha_i} d\tau = \frac{1}{\Gamma(2-\alpha_i)} [(t_j - t_{k-1})^{1-\alpha_i} - (t_j - t_k)^{1-\alpha_i}]. \quad (2.19)$$

It is possible to define the temporal fractional step  $T_{j,k}^{\alpha_i}$  by grouping the temporal quantities in the finite difference scheme

$$T_{j,k}^{\alpha_i} = \frac{(t_j - t_{k-1})^{1-\alpha_i} - (t_j - t_k)^{1-\alpha_i}}{t_k - t_{k-1}}. \quad (2.20)$$

The fractional derivative  ${}_0^C D_t^{\tilde{\alpha}} [v(\cdot)](t_j)$  at  $t_j$  can be now rewritten using the L1 formula, that consist in the finite difference scheme as follows

$${}_0^C D_t^{\tilde{\alpha}} [v(\cdot)](t_j) = \frac{C_{\alpha_i}}{\Gamma(2-\alpha_i)} \sum_{k=1}^j (v(t_k) - v(t_{k-1})) T_{j,k}^{\alpha_i}. \quad (2.21)$$

An analogous expression can be derived for the approximation of  ${}_0^{L1} D_t^{2+\tilde{\alpha}} [v_n(\cdot)](t)$ , for which it is sufficient to substitute in (2.21) the quantity  $2+\alpha_i$  in place of  $\alpha_i$ ,  $i = 1 \dots 4$ .

## 2.6 Numerical Comparison

The two numerical approximations of the fractional differential problem can be solved via the Grünwald-Letnikov and the L1 formula, are implemented with MATLAB. Here, the advantages and disadvantages of the two approaches are discussed in representative

examples. A key aspect in the study of viscoelastic materials is the numerical approximation of fractional derivatives of order  $\alpha$ , with  $0 < \alpha < 1$ , for a generic function. As a paradigmatic example, consider the function  $f(t) = \arctan t$  for  $t \geq 0$  which asymptotically approaches  $\pi/2$  as  $t \rightarrow +\infty$ . This function exhibits the same qualitative behavior as the creep displacement at a representative point of a viscoelastic structure, namely a gradual increase over time towards a limiting value.

The Grünwald–Letnikov scheme for the fractional derivative of order  $\alpha$  is defined in (2.11), and relies on a constant time step  $\Delta t$ . In contrast, the L1 method employs the formulation given in (2.19), where the time discretization is constructed on a geometric progression.

Figures 2.8 report the numerical results for the fractional derivatives of order  $\alpha = 0.4$ , respectively. In both cases, the derivatives are computed using the two approaches over the interval  $0 \leq t \leq 100$ , s, discretized into 1000 time steps.

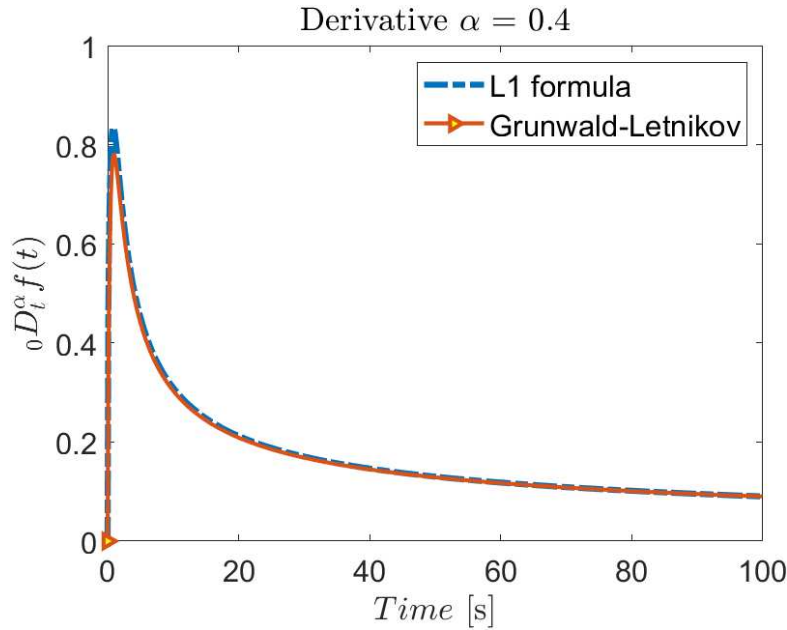


Figure 2.8: Fractional derivative of order  $\alpha = 0.4$  of the arctangent function using two different numerical methods: in blue dashed line the L1 formula and in red the Grünwald Letnikov method.

The difference between the two numerical methods is quantified using the MAPE (Mean Absolute Percentage Error), also known as mean absolute percentage deviation (MAPD), is a measure of prediction accuracy of a forecasting method in statistics. It usually expresses the accuracy as a ratio defined by the expression

$$MAPE = 100 \frac{1}{n} \sum_{i=1}^n \left| \frac{{}^{GL}_0 D_t^\alpha - {}^{L1}_0 D_t^\alpha}{{}^{GL}_0 D_t^\alpha} \right|. \quad (2.22)$$

Let  ${}^G_0D_t^\alpha$  denote the Caputo fractional derivative approximated through the Grünwald–Letnikov method and  ${}^L_0D_t^\alpha$  the corresponding forecast value before introduced. Their difference, normalized by the actual value  $A_t$ , is used to compute the error at each time step. The absolute values of these normalized errors are then summed over all forecast points and divided by the number of fitted points  $n$ , yielding the Mean Absolute Percentage Error (MAPE).

It should be emphasized that MAPE must be interpreted with caution in forecasting applications, since very small actual values can lead to disproportionately large error percentages, thereby inflating the metric. For this reason, the Weighted Mean Absolute Percentage Error (wMAPE) is generally preferred, as it mitigates this sensitivity to small denominators.

The difference between the results obtained with the two methods is smaller intermediate values of  $\alpha$  as shown in Figure 2.9. For many engineering applications the values of  $\alpha$  obtained from the power law fitting of the relaxation functions are within 0.2 and 0.8. In that range, the MAPE between the two numerical methods is always below 2%, and that makes the L1 formula a suitable method for this kind of application. In general, one would expect the error associated with the L1 formula to increase with time, since the time step grows progressively larger. However, when the time derivative of the function becomes very small—as in the present example—the contribution to the convolution integral becomes negligible. Consequently, enlarging the time step does not introduce a significant error.

It is worth recalling that, as shown in [64], the truncation error  $R_s$  tends to zero as the time step decreases. In that work, the proposed scheme relied on progressive time-step refinement, in contrast with the present approach, which enlarges the time step. The rationale for the present scheme lies in the observation that, at large times, the function under consideration tends to flatten, while the relaxation function (see Figure 4.3) also decreases significantly. For these reasons, the proposed increasing time-step strategy is justified. In fact, it can be interpreted as a form of adaptive time meshing [65], where the time-step size is adjusted according to the behavior of the solution, thereby maintaining numerical accuracy while avoiding excessive computational cost.

As expected, the discrepancy between the two methods depends on the value of  $\alpha$ . Figure 2.9 reports the maximum percentage difference between the GL and L1 approximations at the end of the observation interval of 100s. The maximum deviation is observed around the two limits  $\alpha = 0$  and  $\alpha = 1$ , whereas the MAPE is minimum when  $\alpha = 0.6$ . Recall that the limit case  $\alpha = 0$  corresponds to a purely elastic interlayer, while  $\alpha = 1$  represents a standard viscoelastic interlayer, since the fractional derivative reduces to the ordinary first-order derivative. In both of these limit cases, the hereditary (memory) effects characteristic of viscoelasticity disappear.

If, instead of considering  $f(t) = \arctan(t)$ , one were to adopt functions without a horizontal asymptote, for instance,  $f(t) = t^2$ , the error would be larger. Nevertheless, when the number of time steps is kept fixed for both methods, the qualitative trend remains consistent with that shown in Figure 2.9, namely a minimum discrepancy at  $\alpha \simeq 0.6$  and a maximum for  $\alpha = 0$  or  $\alpha = 1$ .

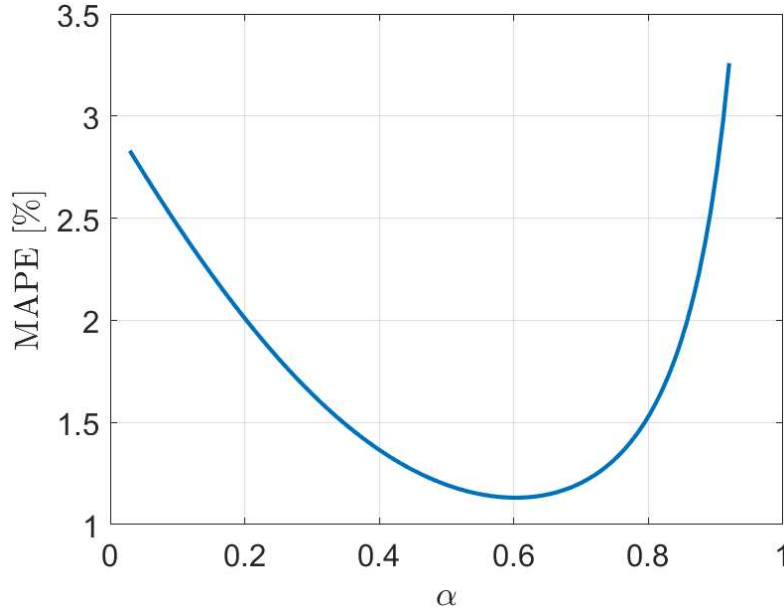


Figure 2.9: Mean Absolute Percentage Error between the two fractional derivatives numerical approximation methods as a function of the different derivative orders  $\alpha$ .

In general, the number of time steps required to achieve convergence—defined as the point at which the solution no longer changes significantly upon further refinement of the time discretization—may differ between the two methods. However, a meaningful comparison should be based on computational time rather than on the number of time steps alone, as will be discussed in the next Section.

### Final Remarks

In the first part of the chapter, two ways to fit the experimental relaxation curve of polymeric materials are presented. The classical Prony series model is compared with the innovative Power Law's interpolation, which bring several advantages over the classical method, like an easier and faster parameters tuning and a more precise fitting with experimental evidences as underlined by Nutting. From the analysis, it comes out that the Power law fitting it is not just more precise, but it allows to interpret very complex relaxation curves with very few parameters, the relaxation function is a piecewise function composed by a certain number of power laws, depending to the curve's shape. In the bi-log representation, the power law is a line identified by two constant parameters: the slope and the height in the graph. Those two parameters can be obtained with a very simple geometrical interpolation, it does not require complex ad-hoc algorithm like with the Prony Series. Using the Power law fitting, the constitutive equation of the polymeric material becomes a relationship that links the stress to the strain with a derivative of fractional order between 0 and 1 (the slope), multiplied by a constant. This is true within

a specific branch of the relaxation curve described by one power law, the relationship is also extended to relaxation function composed by a generic number of power laws. In the second part of the chapter the two main numerical approximation technique for fractional derivatives are presented and compared, underlining the advantages of the L1 formula over the classical Grünwald-Letnikov method. In particular, the L1 formula allows a non-uniform time mesh and this significantly reduce computational time. This advantage will be quantified with a numerical example in chapter 3. The Medium Absolute Percentage Error (MAPE) between the two numerical method is also calculated by varying the fractional derivative order  $\alpha$ . It comes out that the error vary between 3 % and 1 %, showing the maximum error for  $\alpha = 1$  and  $\alpha = 0$ , and the minimum error for approximately  $\alpha = 0.6$ . This allows to employ the L1 formula for the numerical approximation of the fractional derivative in the structural models later proposed. Further comparisons with the Grünwald-Letnikov method will be showed.

## Chapter 3

# Inflexed Laminated Beam

Laminated glass is a composite structure consisting of two or more glass plies bonded together by one or more thin thermoplastic polymeric interlayers. These interlayers are generally considered *soft*, as they do not provide axial or flexural stiffness, but rather contribute to the overall behavior of the system by enabling shear coupling between the glass plies. The most common commercial interlayers are polyvinyl butyral (PVB), ionoplasts such as SentryGlas, and ethylene-vinyl acetate (EVA). Numerous variations of these products exist, depending on the amount of plasticizers or metal salts incorporated, as well as the processing methods employed.

Despite their inherently viscoelastic nature, engineering design is often based on the quasi-elastic approximation [3], where the polymeric film is modeled as a linear elastic material characterized by an *effective shear modulus*. This modulus is typically calibrated as a secant value in relaxation tests, depending on the duration of loading and the operating temperature. Such an approach is well established in design practice, codified in international Standards, and serves as the basis for analytical formulations and simplified engineering methods [67].

However, the quasi-elastic approximation neglects the hereditary effects of viscoelasticity, which can become significant under certain conditions. This is particularly relevant in the presence of impulsive loading, such as impact or blast waves [68], in processes that induce permanent deformation of the glass, such as cold bending [59] or cold-lamination bending [69], and more generally whenever the applied load is non-monotonic. In these cases, a complete viscoelastic analysis is necessary to accurately describe the structural response.

The effectiveness of the interlayer in coupling the glass plies can vary between two idealized limits: the *monolithic limit*, corresponding to fully bonded plies, and the *layered limit*, corresponding to completely uncoupled plies that slide freely relative to one another. Real viscoelastic interlayers provide a time-dependent condition that continuously evolves between these two extreme cases, reflecting the intrinsic rheological properties of the material.

### 3.1 Simply-Supported Beam

Consider the model problem illustrated in Figure 3.1. The system consists of a simply supported three-layer beam composed of two glass plies of thickness  $h$ , separated by a thin polymeric interlayer of thickness  $s = rh$ , with  $r \ll 1$ . The beam has length  $L$  and width  $b$ . The cross-sectional area of each glass ply, treated as an Euler–Bernoulli beam, is  $A = bh$ , while the sum of the second moments of area of the two plies is  $I = 2 \cdot \frac{bh^3}{12}$ , which corresponds to the total beam inertia in the layered limit.

The total beam mass per unit length is  $\mu$  [Kg/m], obtained by multiplying the glass density  $\rho = 2500$  Kg/m<sup>3</sup> by the cross-sectional area  $2A$  [m<sup>2</sup>] of the two glass plies (the mass of the interlayer is neglected). A reference frame  $(y, z)$  is introduced as shown in Figure 3.1. The beam is subjected to a distributed load per unit length  $q(z, t)$ , varying with time.

The fractional viscoelastic modeling of Euler–Bernoulli and Timoshenko beams has been studied in [70–72]. However, in the present case a laminated composite configuration must be considered. A similar model was proposed in [62, 59], where the relaxation function was described by a single power law. In the following, the derivation of the model is recalled and extended to the case of piecewise power-law relaxation functions.

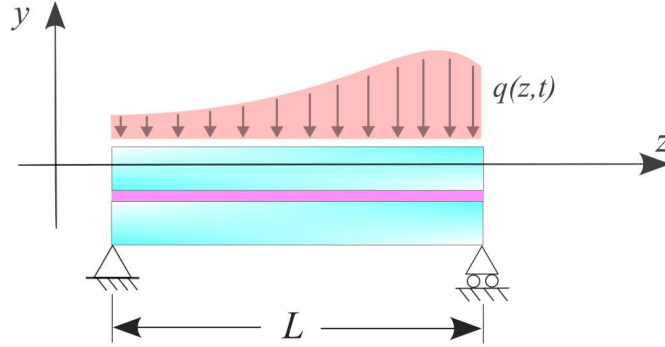


Figure 3.1: Structural model problem of a simply supported laminated beam composed of two linear elastic glass plies bonded by a viscoelastic interlayer.

The polymeric interlayer is considered “soft”: its axial and bending stiffness are neglected, while its role is to provide the shear coupling between the two glass plies. Furthermore, the interlayer is assumed to be “thin”, so that the shear stress is constant across its thickness, although it may vary along the beam axis.

Due to symmetry, it is sufficient to analyze the equilibrium of the upper half of the beam (i.e., the extrados ply plus half of the interlayer thickness), as shown in Figure 3.2. In this representation,  $V(z, t)$  denotes the shear force,  $N(z, t)$  the axial force, and  $M(z, t)$  the bending moment, while  $\tau_{yz}(z, t)$  represents the shear stress transmitted by the interlayer, considered positive according to the sign convention in the figure.

The contribution of the interlayer is equivalent to an action characterized by:

$$n(z, t) = b \tau_{yz}(z, t), \quad m(z, t) = \frac{1}{2}(1 + r)h \tau_{yz}(z, t),$$

where  $n(z, t)$  is the shear-induced force per unit length and  $m(z, t)$  the corresponding moment per unit length.

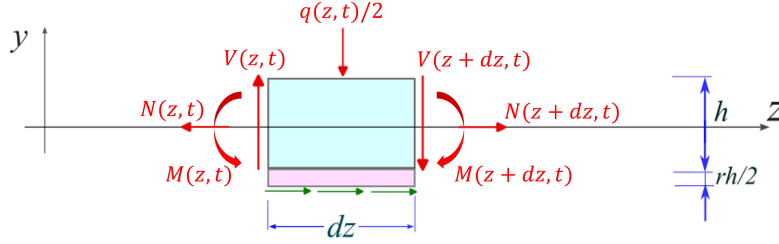


Figure 3.2: Free body diagram of a segment of the upper half of the beam, comprising the extrados ply and half of the thickness of the interlayer.

Let  $v(z, t)$  denote the transverse displacement of the beam at position  $z$  and time  $t$ , considered positive in the  $y$ -direction. The equations of dynamic equilibrium for the half-beam element introduced above can be written as

$$\frac{\partial N(z, t)}{\partial z} = -n(z, t) = -b \tau_{yz}(z, t), \quad (3.1a)$$

$$\frac{\partial V(z, t)}{\partial z} = -\frac{1}{2}q(z, t) + \frac{1}{2}\mu \frac{\partial^2 v(z, t)}{\partial t^2}, \quad (3.1b)$$

$$\frac{\partial M(z, t)}{\partial z} = V(z, t) - \frac{(1 + r)hb}{2} \tau_{yz}(z, t). \quad (3.1c)$$

According to Figure 3.3,  $\varphi(z, t)$  is the rotation of the cross-section, while  $u_A$ ,  $u_B$  and  $u_C$  are the axial displacement of the points  $A$ ,  $B$  and  $C$ , respectively. The constitutive equations of the upper glass ply take the form

$$\frac{\partial u_C(z, t)}{\partial z} = \frac{N(z, t)}{EA}, \quad (3.2a)$$

$$\frac{\partial^2 v(z, t)}{\partial z^2} = -\frac{M(z, t)}{EI}. \quad (3.2b)$$

On the other hand, in the viscoelastic interlayer, from Boltzmann superposition from Equation (2.4) principle holds.

With reference to the scheme of Figure 3.3, the shear strain of the interlayer can be written as

$$\gamma_{yz}(z, t) = -\frac{2}{rh}u_B(z, t) - \varphi(z, t) = -\left[\frac{2}{rh}u_C(z, t) + \frac{1+r}{r}\varphi(z, t)\right]. \quad (3.3)$$

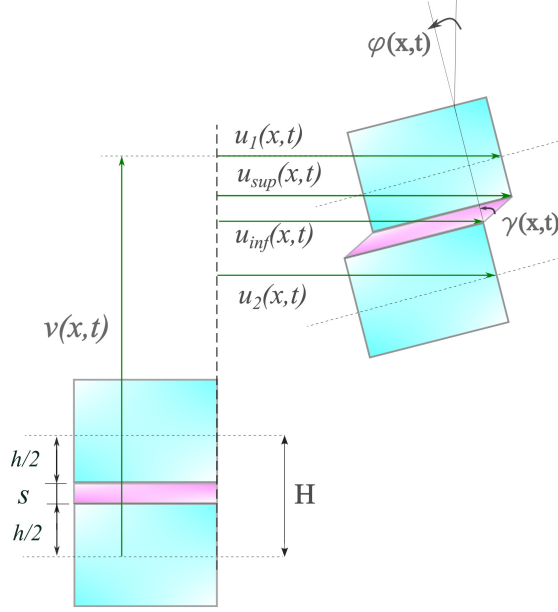


Figure 3.3: Rotation and axial displacements of representative points of the cross-section of the laminated glass beam.

Therefore, substituting (3.3) in (2.4) and recalling (3.1a), it is possible to write

$$\frac{\partial N(z, t)}{\partial z} = -b\tau_{yz}(z, t) = b \int_0^t \frac{\partial}{\partial \bar{t}} \left[ \frac{2}{rh} u_C(z, \bar{t}) + \frac{1+r}{r} \varphi(z, \bar{t}) \right] R(t - \bar{t}) d\bar{t}. \quad (3.4)$$

Using (3.2a), posing  $\frac{\partial v(z, t)}{\partial z} = -\varphi(z, t)$  (assumption of small rotations), it is possible to obtain, after differentiation, the first governing equation

$$\frac{\partial^2 N(z, t)}{\partial z^2} = b \int_0^t \frac{\partial}{\partial \bar{t}} \left[ \frac{2}{rh} \frac{N(z, \bar{t})}{EA} - \frac{1+r}{r} \frac{\partial^2 v(z, \bar{t})}{\partial z^2} \right] R(t - \bar{t}) d\bar{t}. \quad (3.5)$$

The second governing equation is obtained considering (3.2b) combined with equation (3.1c). After double differentiation, one finds

$$\frac{\partial^4 v(z, t)}{\partial z^4} = -\frac{1}{EI} \left( -\frac{1}{2} q(z, t) + \frac{1}{2} \mu \frac{\partial^2 v(z, t)}{\partial t^2} \right) + \frac{1}{EI} \left( \frac{(1+r)hb}{2} \frac{\partial \tau_{yz}(z, t)}{\partial z} \right). \quad (3.6)$$

Then, observing from (3.1a) that  $\frac{\partial \tau_{yz}(z, t)}{\partial z} = -b \frac{\partial^2 N(z, t)}{\partial z^2}$ , one finally obtains

$$\frac{\partial^4 v(z, t)}{\partial z^4} = -\frac{1}{EI} \left( -\frac{1}{2} q(z, t) + \frac{1}{2} \mu \frac{\partial^2 v(z, t)}{\partial t^2} \right) - \frac{1}{EI} \left( \frac{(1+r)h}{2} \frac{\partial^2 N(z, t)}{\partial z^2} \right). \quad (3.7)$$

In a more general setting, consider a relaxation function piecewise defined by power laws and of the type (2.3). Now the shear stress  $\tau_{y,z}(z, t)$  is related to the strain  $\gamma_{y,z}(z, t)$  as explicitly stated in 2.6 and 2.7.

In the domain  $z \in (0, L)$  and for  $t \geq 0$ , the governing equations (3.5) and (3.7) become

$$\frac{\partial^2 N(z, t)}{\partial z^2} = b C_{\tilde{\alpha}} {}^C D_t^{\tilde{\alpha}} \left[ \frac{2}{r h E A} N(z, \cdot) - \frac{1+r}{r} \frac{\partial^2 v(z, \cdot)}{\partial z^2} \right] (t), \quad (3.8a)$$

$$\frac{\partial^4 v(z, t)}{\partial z^4} = -\frac{1}{EI} \left( -\frac{1}{2} q(z, t) + \frac{1}{2} \mu \frac{\partial^2 v(z, t)}{\partial t^2} \right) - \frac{1}{EI} \left( \frac{(1+r) h}{2} \frac{\partial^2 N(z, t)}{\partial z^2} \right). \quad (3.8b)$$

The boundary conditions read

$$N(0, t) = 0, \quad N(L, t) = 0, \quad (3.9a)$$

$$\begin{aligned} v(0, t) &= 0, \quad v(L, t) = 0, \\ \frac{\partial^2 v}{\partial z^2}(0, t) &= 0, \quad \frac{\partial^2 v}{\partial z^2}(L, t) = 0. \end{aligned} \quad (3.9b)$$

with initial conditions

$$N(z, 0) = 0, \quad v(z, 0) = 0, \quad \frac{\partial v}{\partial t}(z, 0) = 0. \quad (3.10)$$

Following an approach à la Galerkin, the vertical displacement, the axial force and the load can be expressed through Fourier Series satisfying the boundary conditions (3.9), in the form

$$v(z, t) = \sum_{n=1}^N v_n(t) \sin\left(\frac{n\pi z}{L}\right), \quad (3.11a)$$

$$N(z, t) = \sum_{n=1}^N N_n(t) \sin\left(\frac{n\pi z}{L}\right), \quad (3.11b)$$

$$q(z, t) = \sum_{n=1}^N q_n(t) \sin\left(\frac{n\pi z}{L}\right). \quad (3.11c)$$

Substituting into (3.8b), it is possible to obtain

$$N_n(t) = \frac{L^2}{n^2 \pi^2} \frac{\mu}{(1+r) h} \frac{d^2 v_n(t)}{dt^2} + \frac{n^2 \pi^2}{L^2} \frac{2EI}{h(1+r)} v_n(t) - \frac{L^2}{n^2 \pi^2} \frac{1}{(1+r) h} q_n(t). \quad (3.12)$$

Further substitution in (3.8a), provides one fractional differential equation in the only unknown  $v_n(t)$ , i.e.,

$$\begin{aligned}
& \frac{L^2}{n^2\pi^2} \frac{C_{\tilde{\alpha}}}{r} \frac{2\mu}{h^2(1+r)} \frac{1}{EA} {}^C D_t^{2+\tilde{\alpha}} [v_n(\cdot)](t) + \frac{\mu}{h(1+r)} \frac{d^2 v_n(t)}{dt^2} \\
& + \frac{L^2}{n^2\pi^2} \frac{C_{\tilde{\alpha}} b}{r} \left[ \frac{4I}{h^2(1+r)A} + (1+r) \right] {}^C D_t^{\tilde{\alpha}} [v_n(\cdot)](t) + \frac{n^4\pi^4}{L^4} \frac{2EI}{h(1+r)} v_n(t) \quad (3.13) \\
& = \frac{L^2}{n^2\pi^2} \frac{C_{\tilde{\alpha}} b}{r} \frac{2}{h^2(1+r)} \frac{1}{EA} {}^C D_t^{\tilde{\alpha}} [q_n(\cdot)](t) + \frac{1}{h(1+r)} q_n(t) .
\end{aligned}$$

The coefficients  $C_{\tilde{\alpha}}$  and  $\tilde{\alpha}$  vary over time according to the piecewise definition of the relaxation function. The problem can be numerically solved via a step-by-step integration.

### 3.1.1 L1 approximation with variable time step

The fractional derivatives in Eq. (3.13) can be reformulated using the finite difference scheme presented in the previous chapter (Eq. (2.19)). This formulation is then directly applied to the governing equation of the present problem, as discussed in the following section of this chapter.

The second derivative of  $v_n(t)$ , for a non-constant time discretization, can obviously be written as

$$\begin{aligned}
\frac{d^2 v_n}{dt^2}(t_j) &= \frac{\frac{v_n(t_j) - v_n(t_{j-1})}{t_j - t_{j-1}} - \frac{v_n(t_{j-1}) - v_n(t_{j-2})}{t_{j-1} - t_{j-2}}}{(t_j - t_{j-2})/2} \\
&= v_n(t_j) \frac{2}{(t_j - t_{j-1})(t_j - t_{j-2})} \\
&\quad - v_n(t_{j-1}) \frac{2}{(t_j - t_{j-1})(t_{j-1} - t_{j-2})} \\
&\quad + v_n(t_{j-2}) \frac{2}{(t_{j-1} - t_{j-2})(t_j - t_{j-2})} . \quad (3.14)
\end{aligned}$$

Setting for brevity  $v_n(t_j) = v_{n,j}$ , considering that the vertical displacement at the midspan and the load are schematized through the vectors  $\mathbf{v}_n = [v_{n,1}, v_{n,2}, \dots, v_{n,s^*}]$ ,  $\mathbf{q}_n = [q_{n,1}, q_{n,2}, \dots, q_{n,s^*}]$ , the governing fractional equation (3.13) can be approximated as

$$\begin{aligned}
& \frac{K_1}{n^2} \sum_{k=1}^j \frac{C_{\alpha_i}}{\Gamma(\alpha_i)} T_{j+1,k+1}^{\alpha_i+2} (v_{n,k+1} - v_{n,k}) \\
& + K_2 \left[ v_{n,j+1} \frac{2}{(t_{j+1} - t_j)(t_{j+1} - t_{j-1})} \right. \\
& \left. - v_{n,j} \frac{2}{(t_{j+1} - t_j)(t_j - t_{j-1})} + v_{n,j-1} \frac{2}{(t_j - t_{j-1})(t_{j+1} - t_{j-1})} \right] \\
& + n^2 K_3 \sum_{k=1}^j \frac{C_{\alpha_i}}{\Gamma(2 - \alpha_i)} T_{j+1,k+1}^{\alpha_i} (v_{n,k+1} - v_{n,k}) + n^4 K_4 v_{n,j+1} \\
& = \frac{K_5}{n^2} \sum_{k=1}^j \frac{C_{\alpha_i}}{\Gamma(2 - \alpha_i)} T_{j+1,k+1}^{\alpha_i} (q_{n,k+1} - q_{n,k}) + K_6 q_{n,j+1},
\end{aligned} \tag{3.15}$$

where  $i$  is a function of  $j$  through (2.18),  $T_{j+1,k+1}^{\alpha_i}$  is related to the time step (analogous relations hold when  $\alpha_i + 2$  is in place of  $\alpha_i$ ) as

$$T_{j+1,k+1}^{\alpha_i} = \frac{(t_{j+1} - t_k)^{1-\alpha_i} - (t_{j+1} - t_{k+1})^{1-\alpha_i}}{t_{k+1} - t_k}, \tag{3.16}$$

whereas the other constants depend on geometry and mechanical properties via

$$\begin{aligned}
K_1 &= \frac{L^2 b}{\pi^2 r} \frac{2\mu}{h^2 (1+r) EA}, & K_2 &= \frac{\mu}{h (1+r)}, \\
K_3 &= \frac{\pi^2 b}{L^2 r} \left[ \frac{4I}{h^2 (1+r) A} + (1+r) \right], & K_4 &= \frac{\pi^4}{L^4} \frac{2EI}{h(1+r)}, \\
K_5 &= \frac{L^2 b}{\pi^2 r} \frac{2}{h^2 (1+r) EA}, & K_6 &= \frac{1}{h (1+r)}.
\end{aligned}$$

The initial conditions (3.10) imply that  $v_{n,0} = v_{n,1} = 0, \forall n$ .

To solve the governing equation using a finite difference approach, the equation (3.15) shall be transformed in implicit form. Observe first that, for any temporal mesh on  $[0, T]$  and for any  $j = 1, \dots, s^*$ , from the definition (3.16), since  $0 < \alpha_i < 1$ , one has that (analogous conditions hold when  $\alpha_i + 2$  is in place of  $\alpha_i$ )

$$T_{j,k}^{\alpha_i} > 0, \quad T_{j,k}^{\alpha_i} > T_{j,k-1}^{\alpha_i}, \quad \text{with } 1 \leq k \leq j. \tag{3.17}$$

It is then possible to obtain the first term of the summation through the identity

$$\sum_{k=1}^j T_{j+1,k+1}^{\alpha_i} (v_{n,k+1} - v_{n,k}) = T_{j+1,j+1}^{\alpha_i} v_{n,j+1} - \sum_{k=1}^j v_{n,k} (T_{j+1,k+1}^{\alpha_i} - T_{j+1,k}^{\alpha_i}). \tag{3.18}$$

The expression is analogous when  $\alpha_i + 2$  is in place of  $\alpha_i$ . Hence, one obtains the implicit finite difference scheme

$$\begin{aligned}
v_{n,j+1} & \left( \frac{K_1}{n^2} \frac{C_{\alpha_i}}{\Gamma(\alpha_i)} T_{j+1,j+1}^{\alpha_i+2} - K_2 \frac{2}{(t_{j+1}-t_j)(t_{j+1}-t_{j-1})} - n^2 K_3 \frac{C_{\alpha_i}}{\Gamma(2-\alpha_i)} T_{j+1,j+1}^{\alpha_i} - n^4 K_4 \right) \\
& = v_{n,j} K_2 \frac{2}{(t_{j+1}-t_j)(t_j-t_{j-1})} - v_{n,j-1} K_2 \frac{2}{(t_j-t_{j-1})(t_{j+1}-t_{j-1})} \\
& + \frac{K_1}{n^2} \sum_{k=1}^j \frac{C_{\alpha_i}}{\Gamma(\alpha_i)} v_{n,k} (T_{j+1,j}^{\alpha_i+2} - T_{j+1,k-1}^{\alpha_i+2}) - n^2 K_3 \sum_{k=1}^j \frac{C_{\alpha_i}}{\Gamma(2-\alpha_i)} v_{n,k} (T_{j+1,k}^{\alpha_i} - T_{j+1,k-1}^{\alpha_i}) \\
& - \frac{K_5}{n^2} \sum_{k=1}^j \frac{C_{\alpha_i}}{\Gamma(2-\alpha_i)} T_{j+1,k+1}^{\alpha_i} (q_{n,k+1} - q_{n,k}) - K_6 q_{n,j+1}.
\end{aligned} \tag{3.19}$$

This form allows to find  $v_{n,j+1}$  at each step. The complete solution is obtained by summing all the term of the Fourier expansion (3.11a). A good approximation is obtained by setting  $n = 5$ , i.e, by considering  $n = 1, 3, 5$  (recall that the even terms are null), but in general the term  $n = 1$  is dominant.

### 3.1.2 Solution with the Grunwald-Letnikov (GL) method

Using the GL approach illustrated in 2.4, one finds the algebraic system of equations 3.13 as follow

$$\begin{aligned}
& \frac{L^2}{n^2 \pi^2} \frac{b}{r} \frac{2\mu}{h^2 (1+r)} \frac{1}{EA} \mathbf{A}_s^{(2+\tilde{\alpha})} \mathbf{v}_n + \frac{\mu}{h (1+r)} \mathbf{A}_s^{(2)} \mathbf{v}_n \\
& + \frac{L^2}{n^2 \pi^2} \frac{b}{r} \left[ \frac{4I}{h^2 (1+r) A} + (1+r) \right] \mathbf{A}_s^{\tilde{\alpha}} \mathbf{v}_n + \frac{k^4 \pi^4}{L^4} \frac{2EI}{h (1+r)} \mathbf{I}_s \mathbf{v}_n \\
& = \frac{L^2}{n^2 \pi^2} \frac{b}{r} \frac{2}{h^2 (1+r)} \frac{1}{EA} \mathbf{A}_s^{\tilde{\alpha}} \mathbf{q}_n + \frac{1}{h (1+r)} \mathbf{I}_s \mathbf{q}_n,
\end{aligned} \tag{3.20}$$

where  $\mathbf{I}_s$  is the  $s \times s$  identity matrix.

## 3.2 Variational principle of Laminated beams with viscoelastic interlayer

The kinematics is thus fully described by the vertical displacement  $v(x, t)$ , common to both glass plies, and by the horizontal displacements  $u_1(x, t)$  and  $u_2(x, t)$  of their centroidal lines, as shown in Figure 3.4.

Let  $A_i = bh_i$  and  $I_i = bh_i^3/12$  denote, respectively, the cross-sectional area and the second moment of inertia of the  $i$ -th glass ply ( $i = 1, 2$ ). The mass per unit length  $\mu$  [kg/m] is computed assuming a glass density of  $2500 \text{ kg/m}^3$ , while the mass contribution of the interlayer is neglected.

### 3.2. Variational principle of Laminated beams with viscoelastic interlayer 63

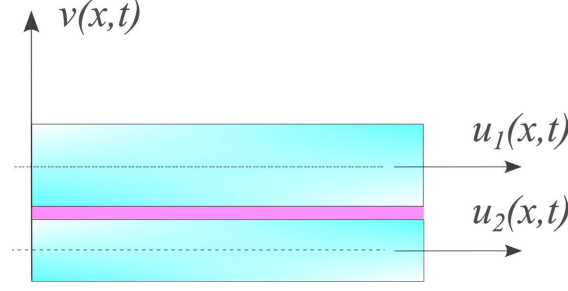


Figure 3.4: Vertical and horizontal displacement fields describing the kinematics of a laminated beam composed of two linear elastic Euler–Bernoulli beams bonded by a thin polymeric interlayer.

The weak form of the governing equations is derived via Hamilton’s principle, namely

$$\delta\Pi = \int_{t_1}^{t_2} [\delta\Theta + \delta\Gamma + \delta\Delta + \delta\Upsilon - \delta W_{\text{ext}}] dt = 0, \quad (3.21)$$

where all the components of Equation 3.21 are:

$$\delta\Theta = \mu \int_0^L \dot{v}(x, t) \delta\dot{v}(x, t) dx, \quad (3.22)$$

which is the variation of the kinetic energy;

$$\delta\Gamma = \int_0^L E(I_1 + I_2) v''(x, t) \delta v''(x, t) dx, \quad (3.23)$$

corresponds to the variation of the bending strain energy in the glass plies;

$$\delta\Delta = \int_0^L [EA_1 u_1'(x, t) \delta u_1'(x, t) + EA_2 u_2'(x, t) \delta u_2'(x, t)] dx, \quad (3.24)$$

represents the variation of the axial strain energy in the glass plies.

is associated with the extensional strain energy of the glass plies;

$$\delta\Upsilon = \int_0^L \tau_{yz}(x, t) \delta\gamma_{yz}(x, t) dx, \quad (3.25)$$

represents the non-conservative contribution of the viscoelastic interlayer, expressed in terms of its shear stress  $\tau_{yz}(x, t)$  and the work-conjugate virtual shear strain  $\delta\gamma_{yz}(x, t)$ ;

Figure 3.3 illustrates the reference and deformed configurations of an infinitesimal portion of the laminated structure. Let  $\varphi(x, t)$  denote the cross-sectional rotation, while  $u_{\text{sup}}(x, t)$  and  $u_{\text{inf}}(x, t)$  represent the horizontal displacements at the intrados of the upper ply and the extrados of the lower ply, respectively, at the interfaces with the interlayer. The shear strain in the interlayer is then expressed as

$$\gamma_{yz}(x, t) = \frac{1}{s} [u_{\text{sup}}(x, t) - u_{\text{inf}}(x, t) + \varphi(x, t) s]. \quad (3.26)$$

Under the assumption of small strains and moderate rotations, such that  $\varphi(x, t) \simeq v'(x, t)$ , the horizontal displacements at the interlayer interfaces can be approximated as

$$u_{\text{sup}}(x, t) \simeq v'(x, t) \frac{h_1}{2} + u_1(x, t), \quad u_{\text{inf}}(x, t) \simeq v'(x, t) \frac{h_2}{2} + u_2(x, t).$$

Substituting into (3.26), the shear strain in the interlayer becomes

$$\gamma_{yz}(x, t) = \frac{1}{s} [u_1(x, t) - u_2(x, t) + v'(x, t)H], \quad (3.27)$$

where  $H = s + \frac{1}{2}(h_1 + h_2)$  denotes the distance between the centroids of the two glass plies.

Accordingly, the virtual work contribution of the interlayer in (3.25) can be rewritten as

$$\delta \Upsilon = \int_0^L \frac{C_{\tilde{\alpha}} b}{s} {}_0^C D_t^\alpha [u_1(x, t) - u_2(x, t) + v'(x, t)H] \delta [u_1(x, t) - u_2(x, t) + v'(x, t)H] dx. \quad (3.28)$$

Equation (3.28) thus represents the viscoelastic constitutive relation of the interlayer, formulated in the framework of fractional calculus and consistently embedded into the weak form.

A key limitation of the proposed model lies in the assumption of geometric linearity. This simplification is, however, justified by the fact that it provides the most straightforward framework for simulating a four-point bending configuration with cylindrical deformation, where membrane stresses induced by second-order effects are inherently less significant than in plate or shell structures. Although edge effects near specimen boundaries may introduce stress concentrations that influence the measured strength, the present analysis focuses on the pre-breakage stiffness regime, where such localized phenomena have negligible impact. Recent developments, such as the solid-shell formulation proposed in [73], allow for the modeling of large deformations in initially curved laminated glass, but they do not account for viscoelasticity. Extending the model to incorporate both nonlinear geometry and time-dependent interlayer behavior would represent a natural direction for future research. Nevertheless, the present formulation provides an adequate and efficient description for the objectives of this study.

### Final Remarks

The analytical model for the inflexed laminated beam, when the interlayer's relaxation curve is fitted with power laws, is presented in this chapter. The simply supported laminated beam under distributed vertical load is developed using the two numerical approximation for the fractional derivatives, included in the constitutive equation of the polymeric interlayer, already introduced in the previous chapter. The two semi analytical solutions are presented in a finite difference scheme that is solved in the next chapters. In the second part of the chapter the variational principle for the same problem is also

### **3.2. Variational principle of Laminated beams with viscoelastic interlayer****65**

presented, that allows to solve finite element problem of the laminated beam, that will be applied to the four point bending test in chapter 5, obtaining the numerical solution and comparing it with experimental data.



## Chapter 4

# Numerical Solution for the semi-analytical problem

### 4.1 The simply supported beam problem

In this section, the problem of the simply supported laminated beam with a power law fitting of the interlayer's relaxation curve is numerically solved using the two approximation methods for the fractional derivative before introduced: the Grünwald-Letnikov (GL) method and with the L1 formula. The two methods are compared to determine the advantages, on a computational side, of the L1 formula over the GL method with an easy and practical example.

#### 4.1.1 Numerical Solution and validation

As a benchmark, we consider the simply supported laminated glass beam described in Section 3.1 with the following numerical parameters:

$$\begin{aligned} L &= 1.5 \text{ m}, & b &= 1 \text{ m}, & h &= 8 \text{ mm}, & s &= rh = 1.52 \text{ mm} \ (r = 0.19), \\ E &= 72 \text{ GPa}, & \mu &= 40 \text{ kg/m}. \end{aligned}$$

The deformation of interest is the mid-span transverse displacement  $v(L/2, t)$ . The structure is loaded by a uniformly distributed pressure  $p(t)$  which increases linearly from zero and reaches a constant plateau of 3 kPa in approximately one hour (see Figure 4.1). The corresponding load per unit length is therefore

$$q(z, t) = b p(t).$$

A loading history of this type was employed in the experimental campaign of Centelles et al. [56].

The governing equations (3.20) and (3.19) are solved using the first five terms of the Fourier sine series (3.11). Owing to geometric and load symmetry, only the odd modes

$n = 1, 3, 5$  are nonzero. The observation interval is chosen very long (of the order of  $10^8$  s, i.e. roughly three years) to highlight potential differences in the long-term creep response.

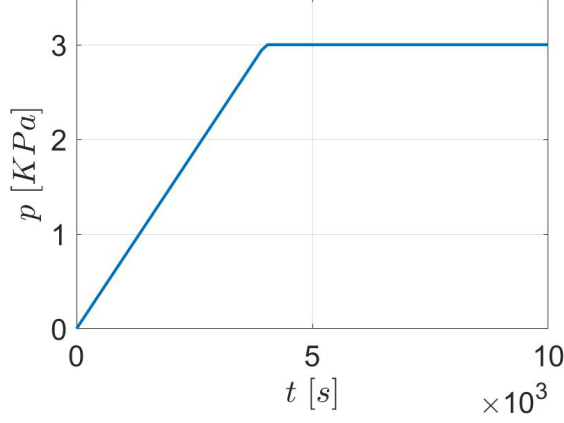


Figure 4.1: Pressure history used in the numerical experiments.

The mid-span deflection at the layered and monolithic limits can be expressed by the classical formula for a simply supported beam under a uniformly distributed load:

$$v_{\max}(t) = \frac{5}{384} \frac{qL^4}{EI},$$

with  $q = bp(t)$ . For the two limiting inertias we set

$$I_L = 2 \cdot \frac{bh^3}{12}, \quad I_M = 2 \cdot \frac{bh^3}{12} + \frac{(1+r)^2 bh^3}{2},$$

corresponding respectively to the *layered* (free sliding plies) and *monolithic* (fully coupled plies) limits. Using the benchmark parameters and  $p_{\max} = 3$  kPa one obtains

$$v_{\max,L} \approx 32 \text{ mm}, \quad v_{\max,M} \approx 6.1 \text{ mm}.$$

The layered limit is approached as  $C_\alpha \rightarrow 0$ , while the monolithic limit corresponds to  $C_\alpha \rightarrow \infty$ . Hence, the ability of the polymeric interlayer to shear-couple the glass plies can be assessed qualitatively by comparing the computed mid-span displacement  $v(L/2, t)$  with the two bounds above: values close to  $v_{\max,L}$  indicate weak coupling, whereas values approaching  $v_{\max,M}$  indicate strong coupling. In general, a relaxation curve that remains higher (i.e., larger instantaneous shear stiffness or slower decay) yields a smaller mid-span displacement; accordingly, from Figure 4.3 one expects Conventional PVB to produce larger displacements than Acoustic Monolayer PVB under the same loading history.

Our goal is to compare the numerical solutions obtained using either the GL or L1 approximations, when the observation interval is divided into the same number of time steps. In the GL scheme, the time step is constant; consequently, the evaluation of

fractional derivatives may become inaccurate if the total observation interval is large. In contrast, the L1 approach employs a geometrically progressing time mesh, which provides smaller time steps at the beginning of the analysis. This feature allows for a more accurate representation of the initial response of the beam to the applied load. Furthermore, when the relaxation function is simple and can be described by a single power law, the discrepancy between the two methods is expected to be small. However, if the relaxation function consists of multiple branches, the differences between the GL and L1 solutions tend to increase, reflecting the greater complexity in capturing the viscoelastic response accurately.

As an illustrative example, Figure 4.2 shows the solutions of the governing equations (3.20) and (3.19) obtained using the GL (red curve) and L1 (blue curve) schemes. In this case, the interlayer is modeled by a fictitious relaxation function consisting of a single power law, with parameters  $C_\alpha = 54.9561 \text{ MPa s}^\alpha$  and two values of  $\alpha$ , namely  $\alpha = 0.0957$  and  $\alpha = 0.4886$ .

All numerical experiments were performed by subdividing the observation interval into the same number of time steps,  $s = 1000$ . The solutions obtained with the two methods coincide almost perfectly, even for  $\alpha$  close to 0.5, which was identified as a critical value in the example of Section 2.6; the maximum difference is less than 0.02%. The mid-span deflection increases with  $\alpha$ , as discussed in [59], because larger  $\alpha$  corresponds to a more pronounced viscous contribution, leading to greater time-dependent deformation.

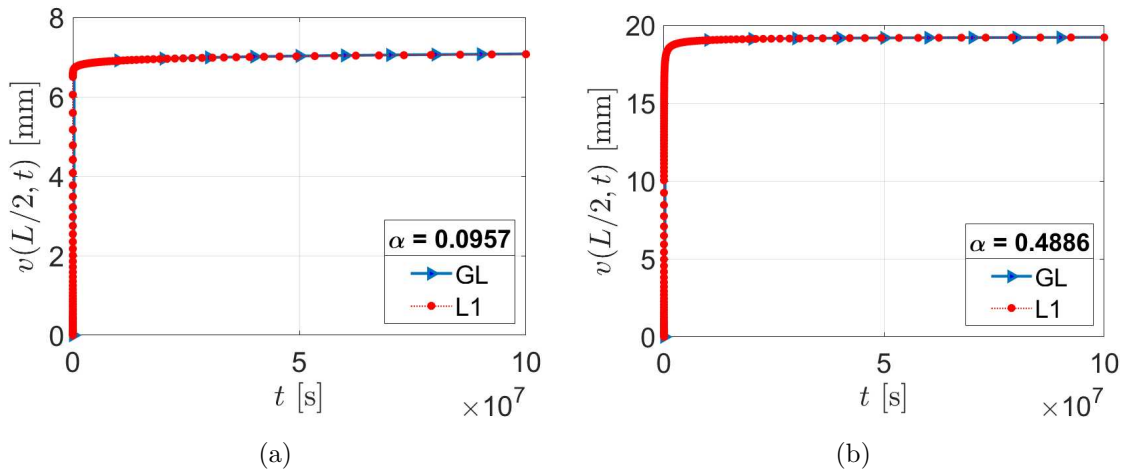


Figure 4.2: Vertical displacement at the beam mid span obtained with the GL method and constant time step (red curve) and with the L1 approach and variable time step (blue curve), for a fictitious relaxation function of only one power law with  $C_\alpha = 54.9561 \text{ MPa}$  and (a)  $\alpha = 0.0957$  and (b)  $\alpha = 0.4886$ .

The situation changes when the relaxation function is composed of multiple power-law branches. Figure 4.4 shows the numerical solutions obtained with the GL (red curve)

and L1 (blue curve) methods for three different interlayers—Stiff PVB, Conventional PVB, and Acoustic Monolayer PVB at 20° C—whose relaxation curves are reported in Figure 4.3. In all cases, the observation interval was divided into the same number of time steps,  $s = 1000$ .

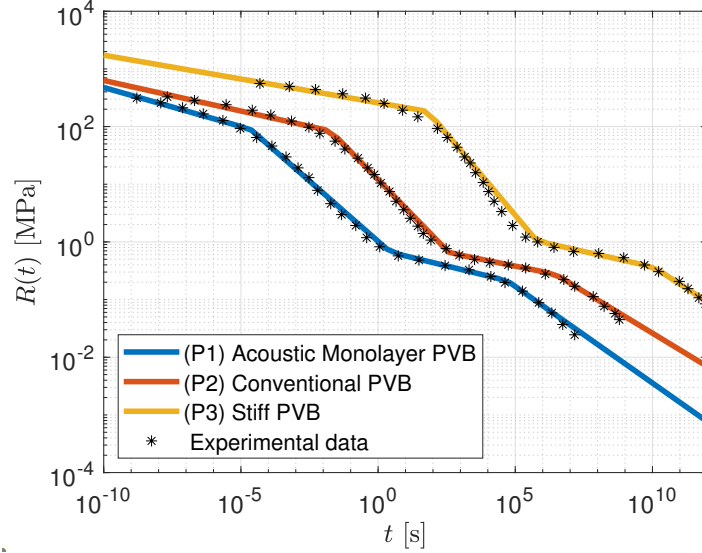


Figure 4.3: Bi-logarithmic plot of the relaxation curves in terms of secant shear modulus  $R(t)$  [MPa] as a function of time  $t$  [s] for the three polymeric materials (acoustic monolayer PVB, conventional PVB and stiff PVB tested at 20° C). Approximation with a polyline in the bilogarithmic plane.

Figure 4.3 presents the bi-logarithmic representation of the relaxation functions together with their corresponding interpolations using polylines for three polymeric materials, all provided by Kuraray Europe GmbH: acoustic monolayer PVB, conventional PVB, and stiff PVB. A polyline with four branches of the form given in Eq. (2.3) provides an excellent fit to the experimental data points. The corresponding fitting parameters are summarized in Table 4.1.

The maximum difference between the solutions reaches approximately 3% for the Acoustic Monolayer and Stiff PVB interlayers, and about 7% for the Conventional PVB. This indicates that the GL scheme is not sufficiently accurate for the present cases. Indeed, as seen in Figure 4.3, the mid-span displacement of the beam with Conventional PVB as interlayer is predicted to be higher than with Acoustic Monolayer PVB, which contradicts physical intuition: the Conventional PVB is “stiffer,” as reflected by its higher relaxation curve. This inconsistency does not appear in the solution based on the L1 formula.

The reason lies in the time-stepping strategy. In the GL scheme, the constant time step is too large at the beginning of the load history, preventing an accurate evaluation of

|                                       | Acoustic monolayer PVB | Conventional PVB | Stiff PVB |
|---------------------------------------|------------------------|------------------|-----------|
| $C_{\alpha_1}$ [MPa s $^{\alpha_1}$ ] | 21.1367                | 54.9561          | 258.1     |
| $\alpha_1$                            | 0.1352                 | 0.1060           | 0.0826    |
| $C_{\alpha_2}$ [MPa s $^{\alpha_2}$ ] | 1.0245                 | 12.0010          | 1946.2    |
| $\alpha_2$                            | 0.4180                 | 0.4886           | 0.5652    |
| $C_{\alpha_3}$ [MPa s $^{\alpha_3}$ ] | 0.7468                 | 1.1452           | 4.3       |
| $\alpha_3$                            | 0.1159                 | 0.0957           | 0.1063    |
| $C_{\alpha_4}$ [MPa s $^{\alpha_4}$ ] | 8.0046                 | 20.1653          | 453.6     |
| $\alpha_4$                            | 0.3350                 | 0.2887           | 0.3081    |

Table 4.1: Coefficients defining the power-law approximation given in Eq. (2.3) for the materials shown in Fig. 4.3.

the fractional derivative and failing to capture the initial steep increase in displacement, particularly when the relaxation function exhibits abrupt changes in slope. In contrast, the L1 scheme uses very short time steps at the start of the loading, enlarging them only as the displacement approaches its asymptotic limit, thereby providing a more accurate representation of the viscoelastic response.

#### 4.1.2 Convergence analysis of the numerical solution

Further advantages of the L1 method based on variable time step over the GL approximation with constant time step are evidenced by the convergence analysis. Here, the number of time steps is varied and the difference in the solutions evaluated: convergence is achieved when such difference becomes less than a target value.

For the same materials considered in Figure 4.4 (Stiff PVB, Conventional PVB, Acoustic Monolayer PVB at 20° C), Figure 4.5 evidences how the solution changes when the GL method with constant time step is used. A number of time steps of the order of 20000 is needed to achieve a difference less than 0.2 % in the interval of observation. This aspect regards all the considered materials.

Figure 4.6 is the counterpart of the previous Figure corresponding to the case in which the L1 formula, with time mesh in geometric progression, is used. Remarkably, only 200 time steps are needed to reach the target value 0.2 % of accuracy. This great advantage with respect to the GL method is due to the adaptive time mesh, which provides for shorter time steps where needed.

The fact that the approximation based on the L1 formula requires less time step for convergence, implies a lower calculation time. However, the L1 algorithm is based on

| L1 formula |                    | Grünwald-Letnikov |                    |
|------------|--------------------|-------------------|--------------------|
| Time steps | Computational Time | Time steps        | Computational Time |
| 1000       | 1.6986 s           | 20000             | 122.3978 s         |
| 500        | 0.7613 s           | 15000             | 37.4455 s          |
| 200        | 0.3675 s           | 8000              | 7.6806 s           |
| 50         | 0.2796 s           | 1000              | 1.0090 s           |

Table 4.2: Computational time as a function of the number of time steps for the L1 and the Grünwald-Letnikov methods of approximation.

finite difference and the solution is calculated step by step; on the other hand, the GL approach requires the calculation of a triangular strip matrix, as per (2.9) and (2.11), which can be readily inverted. Therefore, kept fixed the number of time steps, the GL approach is in general faster than the L1 method, although this advantage is annihilated by the fact that a higher number of steps (two orders of magnitude) is required. To illustrate, Table 4.2 shows the computational time required by the two methods. For 1000 times steps, the GL method requires 1.0090 s but, to achieve convergence, 20000 time steps require more than 122 s. On the other hand, for 1000 times steps the computational time for the L1 method is higher (1.6986 s), but convergence is reached for 200 time steps, requiring 0.3675 s.

## 4.2 Cold-Bending

The same model of the simply supported laminated beam subjected to a vertical distributed load is here adapted to an industrial problem; the cold bending lamination process. The work is developed to show the potentiality of the new approach on an industrial problem and it is compared with an Abaqus simulation to assess its reliability in the numerical solution.

### 4.2.1 Mechanical Model

To make the calculations explicit for the case of cold-bending of laminated glass in a single direction, we still consider the model problem of a simply supported three-layer beam composed of two glass plies of thickness  $h$ , sandwiching a thin polymeric interlayer of thickness  $s = rh$ , with  $r \ll 1$ . The beam has length  $L$  and width  $b$ . The cross-sectional area of each ply, modeled as an Euler–Bernoulli beam, is  $A = bh$ , while the corresponding second moment of inertia of the two plies is  $I = 2bh^3/12$ , which represents the effective inertia of the layered configuration.

The total beam mass per unit length is denoted by  $\mu$  [kg/m] and is obtained by multiplying the glass density  $2500 \text{ kg/m}^3$  by the total glass cross-sectional area  $2A$  [m<sup>2</sup>].

The contribution of the interlayer mass is neglected. This simplified model has already been discussed in [62, 59], but is here specialized to the case of cold bending.

The external action applied to the beam is schematically represented in Figure 4.7. With respect to the reference frame  $(y, z)$  defined therein, let  $v(z, t)$  denote the transverse displacement, in the  $y$ -direction, of the point with coordinate  $z$  at time  $t$ . Under the assumption of linearized infinitesimal deformations (small strains and moderate rotations), we prescribe  $v(z, t)$  for  $z \in [0, L]$  and  $t \geq 0$ , with the condition  $v(z, t) = 0$  for  $t < 0$ . In other words, the bending deformation is prescribed as a time-dependent input, while axial deformations remain unconstrained.

To visualize this scenario, one may imagine that the beam rests on a frictionless mould that gradually changes its curvature with time. The contact pressure exerted by the mould onto the beam can then be modeled as a distributed transverse load per unit length,  $q(z, t)$ , taken as positive in the  $y$ -direction.

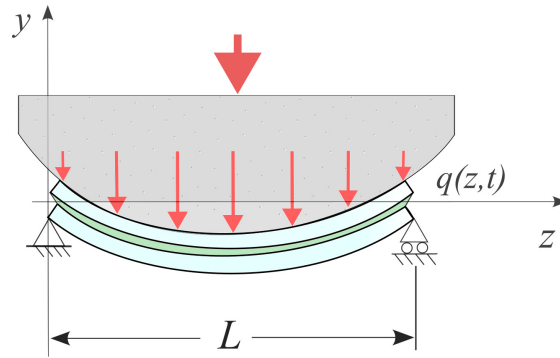


Figure 4.7: The model problem of a simply-supported three-layered laminated glass beam, constrained to deform on a slippery mould, transmitting the contact trasversal force per unit length  $q(z, t)$ .

By symmetry, it suffices to analyze the equilibrium of the upper half of the beam (i.e., the extrados glass ply together with one half of the interlayer), as illustrated in Figure 3.2. Let  $N(z, t)$ ,  $V(z, t)$ , and  $M(z, t)$  denote, respectively, the axial force, shear force, and bending moment acting in the upper glass ply, while  $\tau_{yz}(z, t)$  represents the shear stress transmitted by the interlayer, taken as positive according to the convention in Figure 3.2.

The equations of the dynamic equilibrium are recalled from Eq. (3.8).

According to Figure 3.3, let  $\varphi(z, t)$  denote the rotation of the cross-section, while  $u_A$ ,  $u_B$  and  $u_C$  are the axial displacement of the points  $A$ ,  $B$  and  $C$ , respectively.

The constitutive equations for the upper glass ply take are the one before mentioned (Eq. (3.2)).

After further manipulations and still considering the constitutive equation for the polymeric interlayer (Equation (2.8)) illustrated in the previous sections, for  $z \in (0, L)$  and  $t \geq 0$ , the expression (3.6) and (3.1a) become

$$\frac{\partial^2 N(z, t)}{\partial z^2} = b C_{\tilde{\alpha}} {}^C D_t^{\tilde{\alpha}} \left[ \frac{2}{rh EA} N(z, \cdot) - \frac{1+r}{r} \frac{\partial^2 v(z, \cdot)}{\partial z^2} \right] (t), \quad (4.1a)$$

$$\frac{\partial^4 v(z, t)}{\partial z^4} = -\frac{1}{EI} \left( -\frac{1}{2} q(z, t) + \frac{1}{2} \mu \frac{\partial^2 v(z, t)}{\partial t^2} + \frac{(1+r)h}{2} \frac{\partial^2 N(z, t)}{\partial z^2} \right). \quad (4.1b)$$

In these equations the unknowns functions are the axial force  $N(z, t)$  and the mold constraint reaction  $q(z, t)$ , since the displacement  $v(z, t)$  is assigned. Clearly, we shall assume that  $v(z, t)$  fulfills the boundary conditions (3.9), and initial conditions (3.10)

The axial force  $N(z, t)$  can be obtained by integrating (3.8a) with the prescribed boundary conditions. This solution can be further used in (3.8b) to obtain the contact forces  $q(z, t)$ .

### The quasi-elastic approximation

A widely used practical approach to describe the constitutive behavior of the polymeric interlayer  $\tau_{yz}(z, t)$  relies on the *quasi-elastic* approximation. In this framework, the polymer is treated as a linear elastic material whose equivalent (secant) elastic shear modulus is assumed to coincide with the value of the relaxation function at the considered time and at the given operating temperature. Accordingly, the shear stress  $\tau_{yz}(z, t)$  is expressed as

$$\tau_{yz}(z, t) = R(t) \gamma_{yz}(z, t). \quad (4.2)$$

This approximation neglects the memory effects intrinsic to viscoelastic materials. In practice, this means that the shear stress transmitted by the interlayer to the glass plies at each instant  $t$  is the same as if the strain  $\gamma_{yz}(z, t)$  had remained constant from the initial time up to  $t$ . Consequently, the effective shear modulus depends only on the elapsed time and the operating temperature.

This approach is well consolidated in engineering practice and is reported in Standards [74], being generally applicable under a wide range of conditions. However, it has been shown [75] that, when the applied loading is not monotonic, the hereditary effects of viscoelasticity—which are neglected in the quasi-elastic approximation—may become significant. Since in cold-bent glass the stress state is not a monotonic function of time, it is of particular interest to assess the validity of this approximation.

Recalling Eq. (3.6) and making use of Eq. (3.2a), together with the kinematic relation  $\frac{\partial v(z, t)}{\partial z} = -\varphi(z, t)$ , one obtains the quasi-elastic counterpart of Eq. (4.1), namely

$$\frac{\partial^2 N(z, t)}{\partial z^2} = b R(t) \left[ \frac{2}{rh EA} N(z, t) - \frac{1+r}{r} \frac{\partial^2 v(z, t)}{\partial z^2} \right], \quad (4.3a)$$

$$EI \frac{\partial^4 v(z, t)}{\partial z^4} = \frac{1}{2} q(z, t) - \frac{1}{2} \mu \frac{\partial^2 v(z, t)}{\partial t^2} - \frac{(1+r)hb}{2} R(t) \left[ \frac{2}{rh EA} N(z, t) - \frac{1+r}{r} \frac{\partial^2 v(z, t)}{\partial z^2} \right]. \quad (4.3b)$$

Since  $v(z, t)$  is prescribed, Eq. (4.3a) allows one to determine  $N(z, t)$  from the boundary conditions (3.9). The resulting axial force is then used in Eq. (4.3b) to compute the constraint reaction  $q(z, t)$ .

### The model problem of a sinusoidally cold-bent beam

It has been demonstrated in [76, 77] that the optimal deformation shape for single-curvature cold-bending of laminated glass is sinusoidal. By contrast, the circular shape (constant curvature), which is often assumed in design practice, can only be realized at the cost of constraint reactions in the form of concentrated moments at the beam ends. These reactions lead to stress concentrations in the interlayer, potentially promoting delamination. The absence of reported failures in circular cold-bent panes installed in practice may be conjecturally attributed to the actual boundary conditions, which enforce a deformation that is only approximately circular. From a theoretical perspective, however, a perfectly circular deformation always induces stress concentrations at the extremities, which can be only partially mitigated by the viscoelasticity of the polymer.

If instead the imposed deformation is sinusoidal, no stress intensification occurs, even when the interlayer shear modulus is large. For this reason, and because it allows a significant simplification of the governing equations, the following discussion focuses on the sinusoidal case.

The imposed transverse displacement is assumed in the form

$$v(z, t) = f(t) v_0 \sin\left(\frac{\pi z}{L}\right), \quad (4.4)$$

where  $v_0$  denotes the maximum deflection at midspan, and  $f(t) : [0, \infty) \rightarrow [0, 1]$  prescribes the deformation history.

Figure 4.8 shows three different histories  $f(t)$  that will be compared in the worked examples. The first deformation history, denoted D-A, follows a parabolic law and reaches the maximum unitary value after 120 s. The second history, D-B, reaches the target value in one hour. The third history, D-C, consists of three loading steps, each one corresponding to one third of the maximum value, which is fully attained after three days.

The unknown axial force  $N(z, t)$  and the constraint reactions  $q(z, t)$  are then obtained by solving the governing equations (4.1) for the full viscoelastic model, or their quasi-elastic counterpart (4.3).

### Fractional differential problem with variable time-step

Adopting a Galerkin-type approach, the axial force  $N(z, t)$  and the constraint reaction  $q(z, t)$  can be expressed as Fourier series, satisfying the boundary conditions, in the form

$$N(z, t) = \sum_{n=1}^N N_n(t) \sin\left(\frac{n\pi z}{L}\right), \quad q(z, t) = \sum_{n=1}^N q_n(t) \sin\left(\frac{n\pi z}{L}\right).$$

Thanks to the sinusoidal shape of the imposed deformation (4.4), only the first mode ( $n = 1$ ) contributes. For brevity, we omit the mode index and, with a slight abuse of notation, write

$$N(z, t) = N(t) \sin\left(\frac{\pi z}{L}\right), \quad q(z, t) = q(t) \sin\left(\frac{\pi z}{L}\right). \quad (4.5)$$

Substituting (4.5) into the governing equilibrium equation (4.1b), one obtains the reduced fractional differential problem for the axial force  $N(t)$ :

$$\frac{\pi^2}{L^2} N(t) + \frac{2b}{rhEA} C_{\tilde{\alpha}} {}^C D_t^{\tilde{\alpha}} [N(\cdot)](t) = -v_0 \frac{(1+r)b}{r} \frac{\pi^2}{L^2} C_{\tilde{\alpha}} {}^C D_t^{\tilde{\alpha}} [f(\cdot)](t). \quad (4.6)$$

To integrate the fractional differential equation (4.6), it is convenient to employ a variable time step. The rationale is that, at early times, when the derivative of  $f(t)$  is large, the discretization must be fine, whereas at later times, when the relaxation response tends to zero, a coarser discretization is sufficient.

Since each branch of the relaxation curve follows a power-law behavior, i.e. a straight line in bi-logarithmic coordinates, it is natural to construct the discretized time vector so that consecutive points are uniformly spaced on a logarithmic scale:

$$\text{Log}(t_{i+1}) - \text{Log}(t_i) = c = \text{const.}, \quad \text{equivalently,} \quad t_{i+1} = 10^c t_i.$$

This corresponds to a geometric progression of time instants, which yields a uniform partition of the relaxation curve in the bi-logarithmic plane.

As an illustrative case, Figure 4.9 shows the time mesh obtained in this way for the relaxation function of stiff PVB at 20°C (material P3 in Figure 4.3), over an observation horizon of  $10^7$  s. Each power-law branch is homogeneously sampled, in the sense that the discretization captures with equal fidelity the variations of the relaxation curve in the bi-logarithmic representation.

A variable time step can then be embedded in the L1 discretization scheme for the Caputo fractional derivative of a generic function is defined in Equation (2.21). In particular, one has [64–66]:

The L1 formula can be extended to the case of a piecewise-defined relaxation function, composed of four temporal branches represented by power-laws, as shown in Figure 4.9. In this case, both the fractional order  $\alpha$  and the scaling coefficient  $C_\alpha$  vary in time.

The observation horizon is divided into  $s$  intervals, distributed as follows:  $p$  intervals in the range  $[0, t_A]$ ,  $u$  intervals in  $[t_A, t_B]$ ,  $d$  intervals in  $[t_B, t_C]$ , and  $g = s - p - u - d$  intervals in  $[t_C, t]$ . Within each temporal branch, the parameters  $C_\alpha$  and  $\alpha$  are constant but assume different values from one branch to another. To this end, we introduce the arrays  $\tilde{\alpha}$  and  $C_{\tilde{\alpha}}$ , whose elements  $\alpha_i$  and  $C_{\alpha_i}$  depend on the interval index  $j$  according to Equation (2.18).

The fractional operator  ${}_0^C D_t^{\tilde{\alpha}} [y(\cdot)](t)$ , defined in (2.8), can then be approximated with Equation (2.21).

Denoting  $y(t_j) = y_j$  and introducing the vectors of discrete values for midspan displacement, constraint reaction, and axial force,

$$\mathbf{v} = [v_1, v_2, \dots, v_s], \quad \mathbf{q} = [q_1, q_2, \dots, q_s], \quad \mathbf{N} = [N_1, N_2, \dots, N_s],$$

the governing fractional equation (4.6) becomes

$$\begin{aligned} \frac{\pi^2}{L^2} N_{j+1} + \frac{2b}{rhEA} \sum_{k=1}^j \frac{C_{\alpha_i}}{\Gamma(2-\alpha_i)} T_{j+1,k+1}^{\alpha_i} (N_{k+1} - N_k) \\ = -v_0 \frac{(1+r)b}{r} \frac{\pi^2}{L^2} \sum_{k=1}^j \frac{C_{\alpha_i}}{\Gamma(2-\alpha_i)} T_{j+1,k+1}^{\alpha_i} (f_{k+1} - f_k), \end{aligned} \quad (4.7)$$

with  $i = i(j)$  as given in (2.18), and with the weighted temporal step defined in equation (3.16). The initial conditions (3.10) impose  $N_0 = N_1 = 0$ .

To proceed with the finite difference solution, it is convenient to reformulate (4.7) in implicit form. First, note that for any temporal mesh on  $[0, T]$  and for any  $j = 1, \dots, s$ , from the definition (3.16), since  $0 < \alpha_i < 1$ , one has

$$T_{j,k}^{\alpha_i} > 0, \quad T_{j,k}^{\alpha_i} > T_{j,k-1}^{\alpha_i}, \quad 1 \leq k \leq j. \quad (4.8)$$

This property allows us to rewrite the summation as

$$\sum_{k=1}^j T_{j+1,k+1}^{\alpha_i} (y_{k+1} - y_k) = T_{j+1,j+1}^{\alpha_i} y_{j+1} - \sum_{k=1}^j y_k (T_{j+1,k+1}^{\alpha_i} - T_{j+1,k}^{\alpha_i}). \quad (4.9)$$

The implicit finite difference scheme for the governing equation then reads

$$\begin{aligned} \frac{\pi^2}{L^2} N_{j+1} + \frac{2b}{rhEA} \frac{C_{\alpha_i}}{\Gamma(2-\alpha_i)} T_{j+1,j+1}^{\alpha_i} N_{j+1} \\ = \frac{2b}{rhEA} \sum_{k=1}^j \frac{C_{\alpha_i}}{\Gamma(2-\alpha_i)} N_k (T_{j+1,k+1}^{\alpha_i} - T_{j+1,k}^{\alpha_i}) \\ - v_0 \frac{(1+r)b}{r} \frac{\pi^2}{L^2} \sum_{k=1}^j \frac{C_{\alpha_i}}{\Gamma(2-\alpha_i)} T_{j+1,k+1}^{\alpha_i} (f_{k+1} - f_k). \end{aligned} \quad (4.10)$$

This implicit formulation allows the recursive computation of the axial force  $N_{j+1}$  at each time step, which is then used as the input for the following step  $t_{j+2}$ .

### Solution under the quasi-elastic approach

Within the *quasi-elastic* approximation, the governing equation can be simplified by substituting the assumed forms of the deflection (4.4) and the Fourier expansion (3.11) into the constitutive relation (4.3a). This leads to the following expression:

$$\frac{\pi^2}{L^2} N(t) = -bR(t) \left[ \frac{2}{rhEA} N(t) + \frac{(1+r)}{r} \frac{\pi^2}{L^2} v_0 f(t) \right]. \quad (4.11)$$

Equation (4.11) is a purely algebraic relation, which can be directly solved for the unknown axial force  $N(t)$ :

$$N(t) = -\frac{\frac{(1+r)\pi^2}{rL^2}v_0 bR(t)f(t)}{\frac{\pi^2}{L^2} + \frac{2b}{rhEA}R(t)}. \quad (4.12)$$

Unlike the full viscoelastic formulation, no recursive or integral computations are required, and the solution at each instant of time is obtained in closed form. This highlights the computational efficiency of the quasi-elastic approximation, albeit at the expense of neglecting the full memory effects intrinsic to fractional viscoelastic models.

#### 4.2.2 Numerical Solution and validation

Consider, as a benchmark problem, the simply supported laminated glass beam described in Section 3.1, subjected to the imposed deformation (4.4). The following parameter values are adopted:

$$L = 3 \text{ m}, \quad b = 1 \text{ m}, \quad h = 10 \text{ mm}, \quad E = 72 \text{ GPa}, \quad \mu = 50 \text{ kg/m}, \quad \frac{v_0}{L} = \frac{1}{30} \text{ (} v_0 = 0.1 \text{ m)}.$$

The layered and monolithic limits are obtained by respectively setting  $C_\alpha = 0$  (no shear coupling) and  $C_\alpha \rightarrow \infty$  (perfect shear coupling). In practice the monolithic limit is approximated by a very large value, e.g.  $C_\alpha = 10^4 \text{ MPa s}^\alpha$ .

The maximum tensile stress occurs at midspan  $z = L/2$  and is given by

$$\sigma_{\max} = \sigma\left(\frac{L}{2}, t\right) = \frac{M\left(\frac{L}{2}, t\right)}{W_M} + \frac{N\left(\frac{L}{2}, t\right)}{A}, \quad (4.13)$$

where the bending moment reads

$$M(z, t) = -EI \frac{\partial^2 v}{\partial z^2} = EI \frac{\pi^2}{L^2} v_0 f(t) \sin\left(\frac{\pi z}{L}\right), \quad (4.14)$$

the section modulus is  $W_M = bh^2/6$ , the cross-sectional area of each ply is  $A = bh$ , and for a single glass ply  $I = 1/12 bh^3$ .

Hence, the midspan stress and its layered and monolithic bounds are expressed as

$$\sigma\left(\frac{L}{2}, t\right) = \frac{6EI v_0 \pi^2 f(t)}{h^2 b L^2} + \frac{N\left(\frac{L}{2}, t\right)}{bh}, \quad I = \frac{1}{12} bh^3, \quad (4.15a)$$

$$\sigma_L\left(\frac{L}{2}, t\right) = \frac{6EI_L v_0 \pi^2 f(t)}{2h^2 b L^2}, \quad I_L = 2\frac{1}{12} bh^3, \quad (4.15b)$$

$$\sigma_M\left(\frac{L}{2}, t\right) = \frac{6EI_M v_0 \pi^2 f(t)}{(2h + rh)^2 b L^2}, \quad I_M = 2\left[\frac{1}{12} bh^3 + \left(\frac{h}{2} + \frac{rh}{2}\right)^2 bh\right]. \quad (4.15c)$$

The mechanical response of cold bent monolithic glass plates during the bending process is also deeply analysed in [78]. The two expressions  $\sigma_L$  and  $\sigma_M$  provide useful

comparison limits:  $\sigma_L$  corresponds to the fully layered (no shear transfer) lower bound, while  $\sigma_M$  corresponds to the monolithic (full shear transfer) upper bound. The deviation of the actual midspan stress  $\sigma(L/2, t)$  from these bounds quantifies the degree of shear coupling provided by the polymeric interlayer.

The fractional viscoelastic model is validated through comparison with the results obtained using the commercial finite element software ABAQUS 2021. This software implements a full viscoelastic analysis based on the numerical solution of the integral–differential equations derived from Boltzmann’s superposition principle, but it does not rely on the fractional calculus formulation [79]. The relaxation function can only be defined in terms of a Prony series of exponential terms, expressed as

$$R(t) = R_0 + \sum_i R_i e^{-\frac{t}{\theta_i}}, \quad (4.16)$$

where  $R_0$  denotes the residual stiffness for  $t \rightarrow \infty$ . Each exponential contribution is characterized by two positive parameters:  $R_i$ , the relaxation shear modulus, and  $\theta_i$ , the corresponding characteristic relaxation time. The numerical solver adopts an adaptive time-stepping scheme, automatically adjusting the time increment in order to minimize the discretization error while ensuring convergence. This feature is essential because, as discussed in [80], numerical errors may arise at specific times if the time increment is larger than the smallest  $\theta_i$ . On the other hand, the use of a very small constant time step is computationally unfeasible for long observation windows.

It is not straightforward to approximate a relaxation function governed by power-law behaviour with a finite series of exponential terms. A good approximation can only be achieved over a finite time interval; the longer the time window of interest, the larger the required number of Prony terms. Moreover, the calibration of the parameters  $R_i$  and  $\theta_i$  does not benefit from the geometric interpretation available in the fractional model, and therefore requires specific numerical identification techniques.

For the three types of interlayer, P1, P2, and P3, illustrated in Figure 4.3, Table 4.3 reports the values of the material parameters  $R_0$ ,  $R_i$ , and  $\theta_i$ , with  $i = 1, \dots, 12$ , obtained through the Domain of Influence method [60] implemented in MATLAB. In this case, 23 terms of the Prony series were considered sufficient to guarantee accurate fitting over the time interval  $10^{-3} \text{ s} < t < 10^9 \text{ s}$ .

The beam was modelled in ABAQUS using 8-node solid elements available in the standard element library. The discretization of the finite element mesh is shown in Figure 4.10a. Glass was assumed linear elastic, while the polymeric interlayer was modelled as an incompressible linear viscoelastic material, with its shear relaxation function given by the Prony series parameters in Table 4.3. The interlayer thickness was set to  $s = 1.52 \text{ mm}$ , and the prescribed deformation history corresponded to case D–A in Figure 4.8. The deformed configuration, showing the distribution of the normal bending stress component (“S11”), at time  $t = 10^2 \text{ s}$ , is represented in Figure 4.10b for the case of interlayer type P2.

| P1: Acoustic Monolayer PVB |                | P2: Conventional PVB   |                | P3: Stiff PVB          |                |
|----------------------------|----------------|------------------------|----------------|------------------------|----------------|
| $R_i$ [MPa]                | $\theta_i$ [s] | $R_i$ [MPa]            | $\theta_i$ [s] | $R_i$ [MPa]            | $\theta_i$ [s] |
| 16.0642                    | 0.0034         | 33.7229                | 0.0044         | 107.4573               | 0.0044         |
| 4.2838                     | 0.0417         | 58.7581                | 0.0442         | 70.8787                | 0.0543         |
| 1.4665                     | 0.5138         | 21.3411                | 0.4682         | 56.6252                | 0.6698         |
| 0.2691                     | 4.9786         | 6.2431                 | 5.7722         | 45.9465                | 8.2574         |
| 0.1311                     | 97.9191        | 1.8297                 | 71.1626        | 119.3430               | 115.9269       |
| 0.0980                     | 1.2072e+03     | 0.2345                 | 668.8633       | 60.4867                | 779.7002       |
| 0.0732                     | 1.4883e+04     | 0.1120                 | 1.5243e+04     | 14.6241                | 9.6125e+03     |
| 0.1202                     | 1.7279e+05     | 0.0881                 | 1.8792e+05     | 3.5357                 | 1.1851e+05     |
| 0.0549                     | 1.9833e+06     | 0.1073                 | 3.6693e+06     | 0.3214                 | 1.7177e+06     |
| 0.0237                     | 2.4450e+07     | 0.1118                 | 2.5638e+07     | 0.1888                 | 2.8212e+07     |
| 0.0102                     | 3.0144e+08     | 0.0541                 | 3.1607e+08     | 0.1446                 | 3.4781e+08     |
| $R_0 = 22.6033$ [MPa]      |                | $R_0 = 122.6579$ [MPa] |                | $R_0 = 480.0389$ [MPa] |                |

Table 4.3: Parameters  $R_0$ ,  $R_i$  and  $\theta_i$  defining the Prony series representation of the relaxation function for polymeric interlayers P1, P2, and P3 (cf. Figure 4.3).

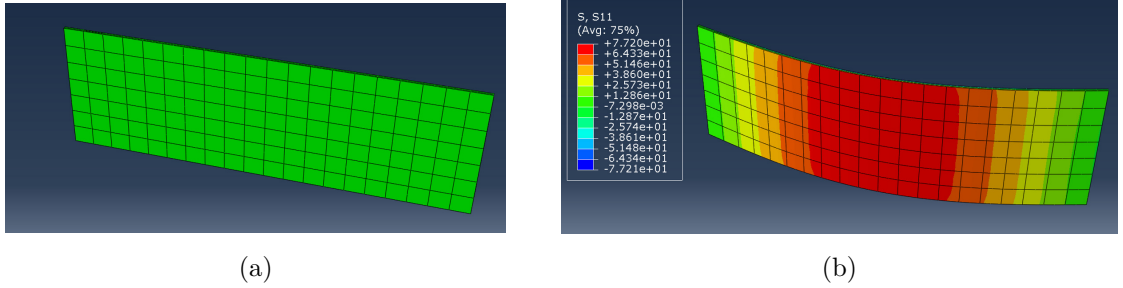


Figure 4.10: Laminated beam model implemented in ABAQUS. (a) mesh used in the numerical simulations; (b) deformed configuration with indication of the bending stress “S11” at time  $t = 10^2$  s, for interlayer of type P2.

Figure 4.11 represents a comparison, in terms of the time-history of the maximum bending stress at the midspan  $\sigma(L/2, t)$ , between the viscoelastic solutions obtained with the fractional approach (F-V) and ABAQUS. The response at the layered (L-L) and monolithic (M-L) limits is also indicated: since these cases are not affected by the interlayer viscosity, their trends exactly follow the deformation history. For the sake of completeness, the graphs corresponding to the quasi-elastic approximation are juxtaposed to the others. The three pictures respectively refer to the three interlayers P1, P2 and P3. It is evident that, for all these materials, the proposed fractional viscoelas-

tic model presents results in excellent agreement with ABAQUS. The memory effect of viscoelasticity is evidenced by the difference with the quasi-elastic solution.

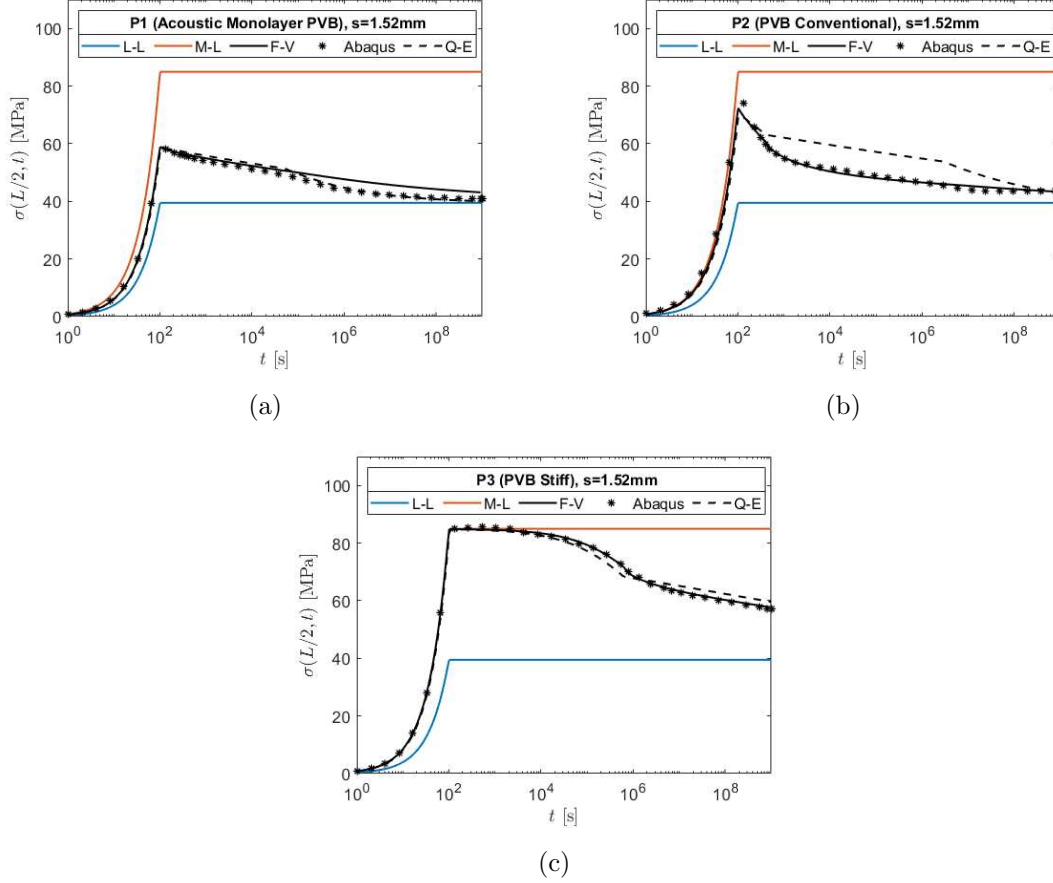


Figure 4.11: Maximum bending stress  $\sigma(L/2, t)$  at mid-span, as a function of time  $t$ . Comparison of the responses obtained with the full viscoelastic analysis with the fractional approach (F-V) and the ABAQUS simulations (Abaqus), juxtaposed to the layered limit (L-L), the monolithic limit (M-L) and the quasi-elastic approximation (Q-E), for polymeric interlayers of the type: (a) P1; (b) P2; (c) P3.

This comparison confirms the reliability of the proposed approach. It should be mentioned that, whereas the proposed method requires about 0.2s for a full viscoelastic analysis, the time required by ABAQUS for the same type of simulation is of the order of a few minutes. This comparison in terms of computational times can only be considered at a qualitative level, because the fractional viscoelastic approach is based on a 1-D beam model, whereas ABAQUS uses a complex 3D mesh. In any case, the characterization via Prony series obviously requires the calculation of a greater number of terms and, in general, calls for shorter time steps of integration [80] with respect to the

fractional viscoelastic approach.

### 4.2.3 Parametric analysis

The parametric investigation addresses three different polymer interlayers, denoted as P1, P2 and P3 (see Figure 4.3), combined with three thickness values  $s = r h$  ( $S1 = 2.28$  mm,  $S2 = 1.52$  mm,  $S3 = 0.76$  mm). In addition, three time-dependent loading histories (D-A, D-B and D-C), reported in Figure 4.8, are considered. Possible variations due to changes in the service temperature are also accounted for. In all cases, the computed stress  $\sigma(L/2, t)$  is compared with the reference monolithic (M-L) and layered (L-L) limits.

Figure 4.12 illustrates the evolution of  $\sigma(L/2, t)$  for beams laminated with the three interlayers P1, P2 and P3, all with thickness  $S2 = 1.52$  mm, under loading history D-A at  $20^\circ$  C. The stiffest interlayer P3 initially follows the monolithic limit, but the subsequent relaxation produces a decay with an inflection point at about  $t = 10^6$  s (1–2 days). On the other hand, the softest polymer P1 provides a significantly lower stress level, with a fast decay towards the L-L condition. The intermediate material P2 yields a response that falls between the two extremes.

The effect of varying the interlayer thickness is examined in Figure 4.13, where the three values  $S1 = 2.28$  mm,  $S2 = 1.52$  mm and  $S3 = 0.76$  mm are considered for polymer P2 at  $20^\circ$  C, still under loading history D-A. While the layered limit remains unaffected, the monolithic limit depends on the thickness, as predicted by (4.15c). With all other parameters fixed, thinner interlayers provide stronger shear coupling and thus higher stresses for a given deformation. As a consequence, the peak load decreases as the interlayer becomes thicker, and this hierarchy is preserved throughout the relaxation process.

The influence of the loading history is depicted in Figure 4.14, where the three laws D-A, D-B and D-C of Figure 4.8 are applied to a laminated beam with interlayer P2, thickness  $S2 = 1.52$  mm, at  $20^\circ$  C. Although the long-term response in all cases tends to the same asymptote, the loading rate significantly affects the peak stress: the slower the deformation is imposed, the smaller the stress attained at the instant when the target deformation is reached.

These examples highlight the capability of the proposed model to predict the time-dependent stress evolution in laminated glass. The results can be effectively exploited to optimize both the design of the laminate and the imposed deformation history, in accordance with service requirements.

### 4.2.4 Effect of temperature

Temperature influences laminated glass mainly through two mechanisms: (i) the mismatch in thermal expansion between glass and polymer, and (ii) the strong temperature dependence of the viscoelastic properties of the interlayer.

The first mechanism is generally negligible in practice. Although the thermal expansion coefficient of glass is  $\alpha_g = 9.1 \cdot 10^{-6} \text{ K}^{-1}$  and that of polymers is of the order of  $\alpha_p = 10^{-4} \text{ K}^{-1}$ , the polymer interlayer is thin and relatively soft. Its thickness is

one order of magnitude smaller than that of the glass plies, and its elastic modulus, which decreases with increasing temperature, is typically of the order  $E_p = 1\text{--}10$  MPa. Assuming that glass and polymer undergo the same strain, a temperature variation of, for example,  $30^\circ\text{C}$  would induce stresses of the order of  $10^{-3}$  MPa, which are negligible compared to mechanical stresses due to external loads.

By contrast, the second mechanism — the dependence of the polymer relaxation function on temperature — plays a key role. The relaxation function is generally defined at a reference temperature  $\tau_0$ , often chosen to coincide with the glass transition temperature  $T_g$  of the polymer. The response at a different operating temperature  $\tau$  can be obtained through the Time–Temperature Superposition (TTS) principle, which states that changing the temperature effectively shifts the time scale of the viscoelastic response. In this framework, the relaxation function  $R_{\tau_0}(t)$  at  $\tau_0$  is related to that at temperature  $\tau$  by

$$R_\tau(t) = R_{\tau_0}(t/a_\tau),$$

where  $a_\tau$  is the temperature-dependent shift factor.

A widely used expression for  $a_\tau$  was proposed by Williams, Landel and Ferry (WLF) [49], which reads

$$\log(a_\tau) = \frac{-C_1(\tau - \tau_0)}{C_2 - (\tau - \tau_0)}, \quad (4.17)$$

where  $C_1$  and  $C_2$  are material constants determined experimentally.

The experiments carried out by Schuster et al. [81] using Dynamical Mechanical Thermal Analysis (DMTA) on different polymers have shown that the glass transition temperature  $T_g$  may vary significantly within the range  $294 \div 320$  K, depending primarily on the plasticizer content. The situation is further complicated by the frequent use of multilayer polymer interlayers, especially in applications where a single-layer interlayer cannot provide the required balance of properties, for example in enhancing acoustic insulation. Reported values of  $C_1$  lie in the range  $7.5 \div 23.9$ , while  $C_2$  values (expressed in  $\text{K}^{-1}$ ) span from 61 to 142.1, depending on both polymer type and, importantly, on the considered temperature range. This evidence suggests that the WLF equation should be regarded as an approximate representation of the temperature dependence, and that accurate quantification of temperature effects requires specific experimental campaigns.

In the present work, the aim is limited to providing a qualitative illustration of how temperature variations affect the response of cold-bent laminated glass beams, rather than pursuing precise quantitative predictions, which would need experimental validation. Therefore, for all polymers considered, we assume the empirical values  $C_1 = 9$  and  $C_2 = 100$  K, as originally proposed in [49]. The reference temperature is taken as  $\tau_0 = 293$  K, consistent with the relaxation curves plotted in Figure 4.3. In the bi-logarithmic plane, the relaxation curves preserve their shape, being shifted along the time axis: the fractional exponents  $\alpha$  remain unchanged, while the coefficients  $C_\alpha$  are modified accordingly.

Figure 4.15 illustrates this effect by comparing the relaxation curve of conventional PVB (polymer P2) at the reference temperature  $\tau_0 = 20^\circ\text{C}$  with those at  $10^\circ\text{C}$  and

40° C, obtained via the Williams–Landel–Ferry (WLF) equation.

The relaxation curves discussed above have been employed to compute the maximum stress in the beam, as reported in Figure 4.16. The analysis refers to conventional PVB (polymer P2) with thickness  $S2 = 1.52$  mm, subjected to the deformation history D-A. As expected, increasing the temperature reduces the stiffness of the polymer, and consequently the maximum stress decreases with rising environmental temperature. At 10° C the response is close to the monolithic limit, while at 40° C the layered limit is rapidly approached.

Since the peak stress occurs as soon as the target deformation is reached, this observation suggests a possible practical strategy to mitigate the effects of cold bending with respect to the strength limit of glass. Specifically, raising the temperature of the laminated panes (for example by using heating blankets) during the deformation process can lower the induced stresses and reduce the risk of premature cracking.

#### 4.2.5 Comparison with the quasi-elastic approximation

It is instructive to compare the results obtained using the full viscoelastic characterization of the polymer with those from the *quasi-elastic* approximation. As outlined in Section 4.2.1, this approach accounts for the time dependence of the secant shear modulus of the polymer but neglects the viscoelastic memory effects.

In cold bending, the deformation is imposed, while the reaction forces evolve over time according to the rheological behavior of the interlayer. In the schematic model problem of Figure 4.7, the laminated beam is gradually pressed against a sinusoidal mould, which generates contact forces that drive the beam deformation. Figure 4.17 shows the time evolution of the maximum contact force per unit length,  $q(L/2, t)$ , at mid-span. The response is non-monotonic: the force initially increases until reaching a peak—corresponding to the instant of maximum deformation—and then decreases due to interlayer relaxation.

Although the effective applied load is non-monotonic, the shear strain in the interlayer remains essentially constant once the target deformation is reached. Consequently, the memory effect of viscoelasticity primarily influences the initial stages of the loading history, when the strain is still evolving in time. Since this phase is of limited duration, the differences between the full viscoelastic and quasi-elastic analyses are expected to be small.

This is confirmed by the graphs in Figure 4.18, which show the maximum stress  $\sigma(L/2, t)$  as a function of time for a beam with an interlayer thickness of  $S2 = 1.52$  mm at 20° C, under the deformation history D-A. The three subfigures correspond to the three polymer types P1, P2, and P3, and compare the results obtained using the *quasi-elastic* approximation (Q-E) with those from the full viscoelastic characterization (F-V).

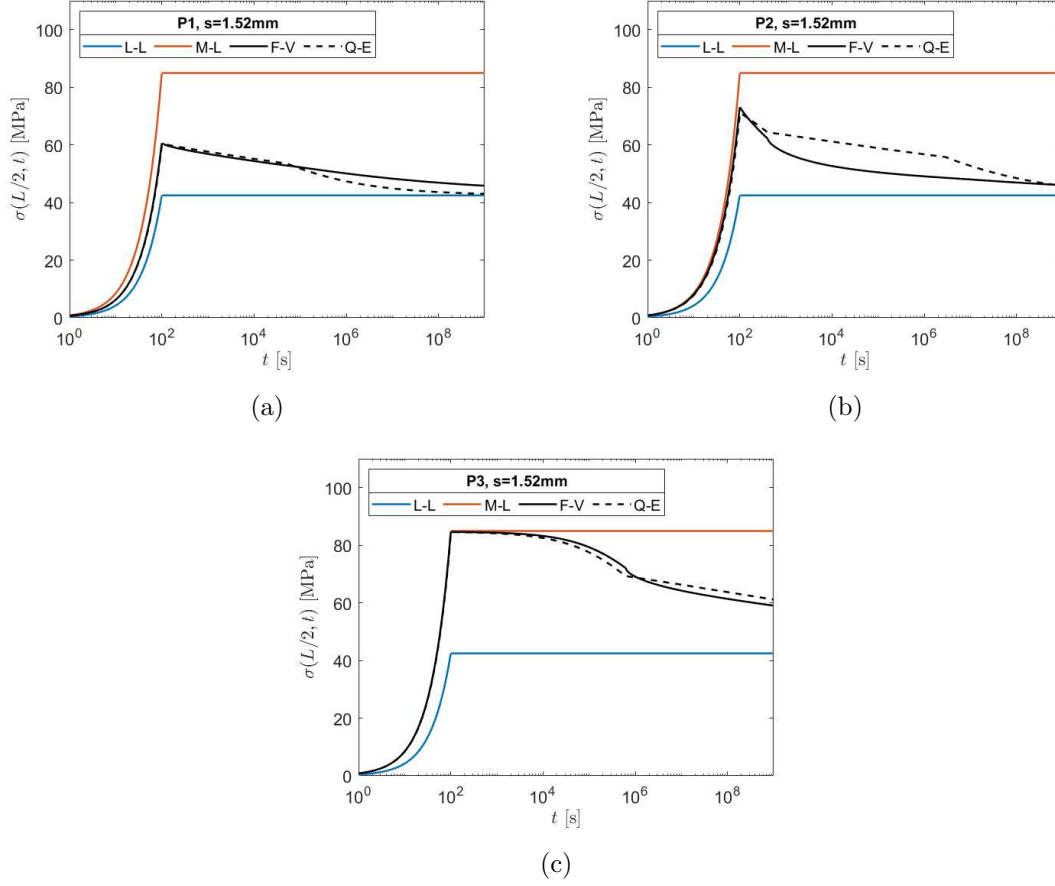


Figure 4.18: Comparison of the stress  $\sigma(L/2, t)$  obtained with the full viscoelastic analysis (F-V) and the quasi-elastic approximation (Q-E) for the three polymeric interlayers P1 (a), P2 (b), and P3 (c), all of thickness  $S2 = 1.52$  mm, at an operating temperature of  $20^\circ\text{C}$ , under the deformation history D-A.

The two solutions yield, in practice, the same peak load and the same asymptotic behavior. Differences are, however, observable: they are relatively small for the softest interlayer P1, approximately 7% as shown in Figure 4.18a, and for the stiffest interlayer P3, about 4% according to Figure 4.18c, corresponding respectively to the layered (L-L) and monolithic (M-L) limits. The largest discrepancies in stress occur during the relaxation phase for the intermediate polymer P2, reaching around 15–20%, as indicated in Figure 4.18b.

Overall, this analysis confirms that the quasi-elastic approximation provides a sufficiently accurate estimate of both the peak stress and its asymptotic value at the end of the relaxation process. The approximation is particularly reliable when the beam response is close to either the layered or monolithic limit.

### Final Remarks

In summary, a fractional calculus-based model has been developed to describe the viscoelastic response of laminated glass beams in cold bending. This approach allows for an accurate prediction of the time-dependent stress evolution under prescribed deformation histories, efficiently capturing both the short-term transient behavior and the long-term relaxation of the interlayer. The problem is solved using the Galerkin method combined with the L1 approximation for the fractional derivative, enabling variable time-stepping to reduce computational effort while maintaining high accuracy.

Parametric studies considering different polymer types, interlayer thicknesses, loading histories, and environmental temperatures indicate that, because the interlayer strain remains nearly constant after the target deformation is reached, the quasi-elastic approximation reproduces the peak stress and asymptotic behavior with good accuracy. The largest discrepancies occur only during the early transient stages, particularly for intermediate-stiffness interlayers. Although the current analysis focuses on beams, the findings are expected to qualitatively extend to laminated glass plates, providing guidance for optimizing cold-bending processes and minimizing stress during transient periods.

Moreover, the methodology shows significant advantages over classical Prony-series-based approaches and could be generalized to other laminated or composite structures, including glulam, sandwich panels, and biological materials. The recovery behavior of laminated glass after load release can also be straightforwardly analyzed using the proposed model. Future work will include experimental validation and a more detailed study of temperature-dependent effects on interlayer stiffness to further refine the predictive capability of the model.

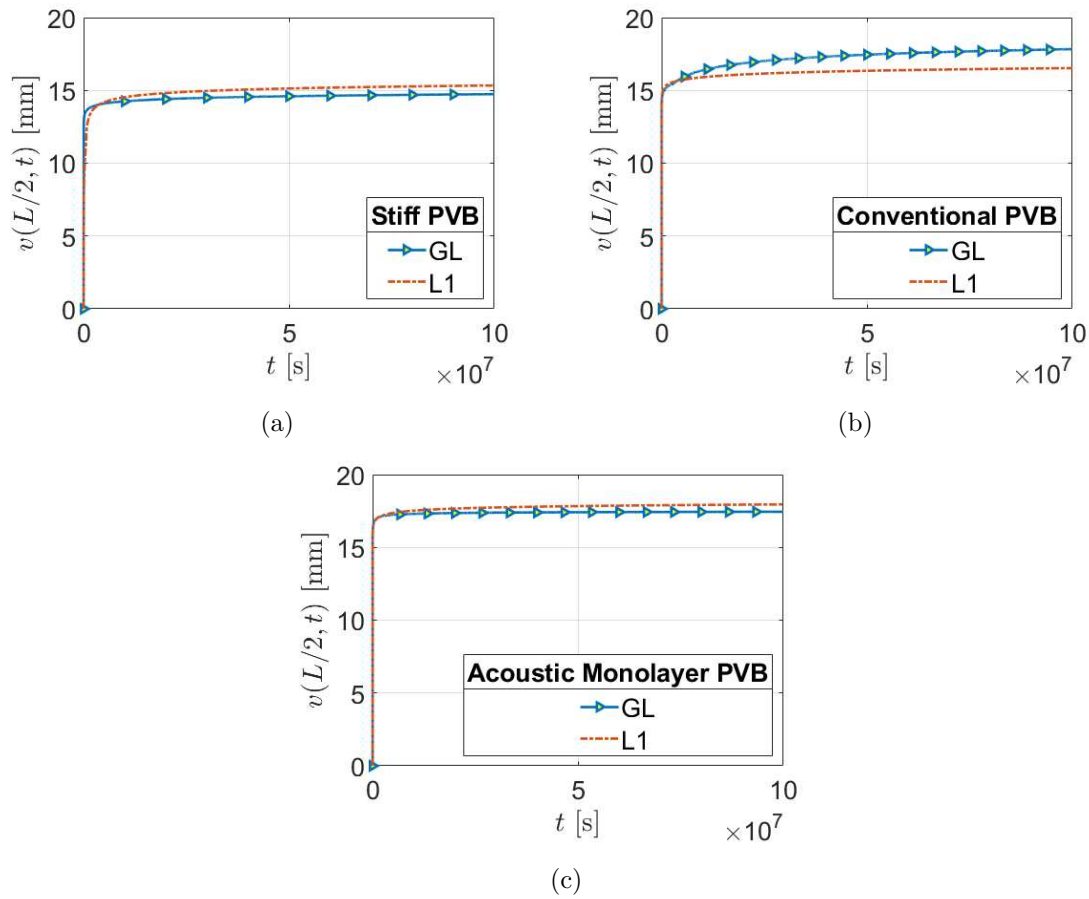


Figure 4.4: Vertical displacement at the beam mid span obtained with the GL method and constant time step (red curve) and with the L1 approach and variable time step (blue curve), for the three polymers whose relaxation curves are as in Figure 4.3: (a) Stiff PVB, (b) Conventional PVB, (c) Acoustic Monolayer PVB, all tested at 20° C.

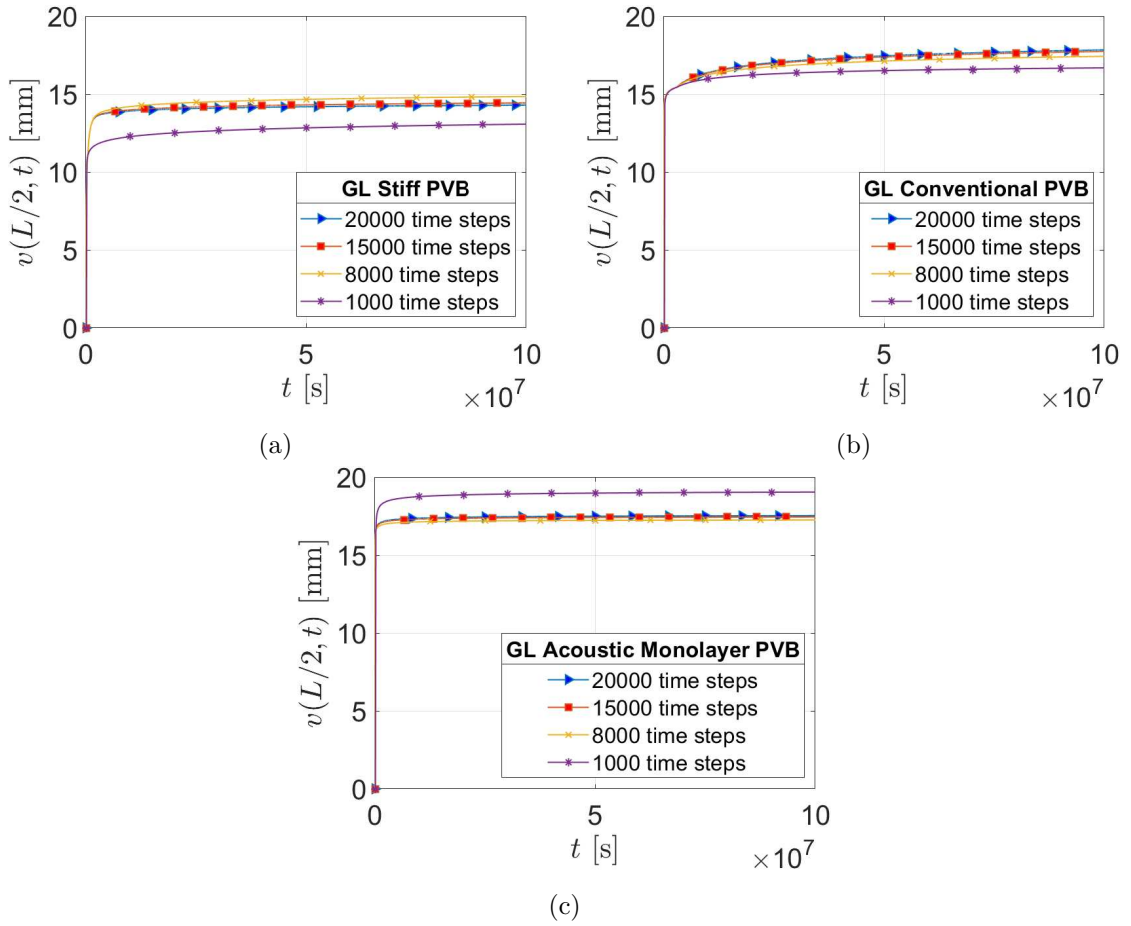


Figure 4.5: Convergence analysis of the long creep solution obtained through the GL approximation with constant time step, the for three different interlayers of Figure 4.3: (a) Stiff PVB, (b) Conventional PVB, (c) Acoustic Monolayer PVB, at 20° C.

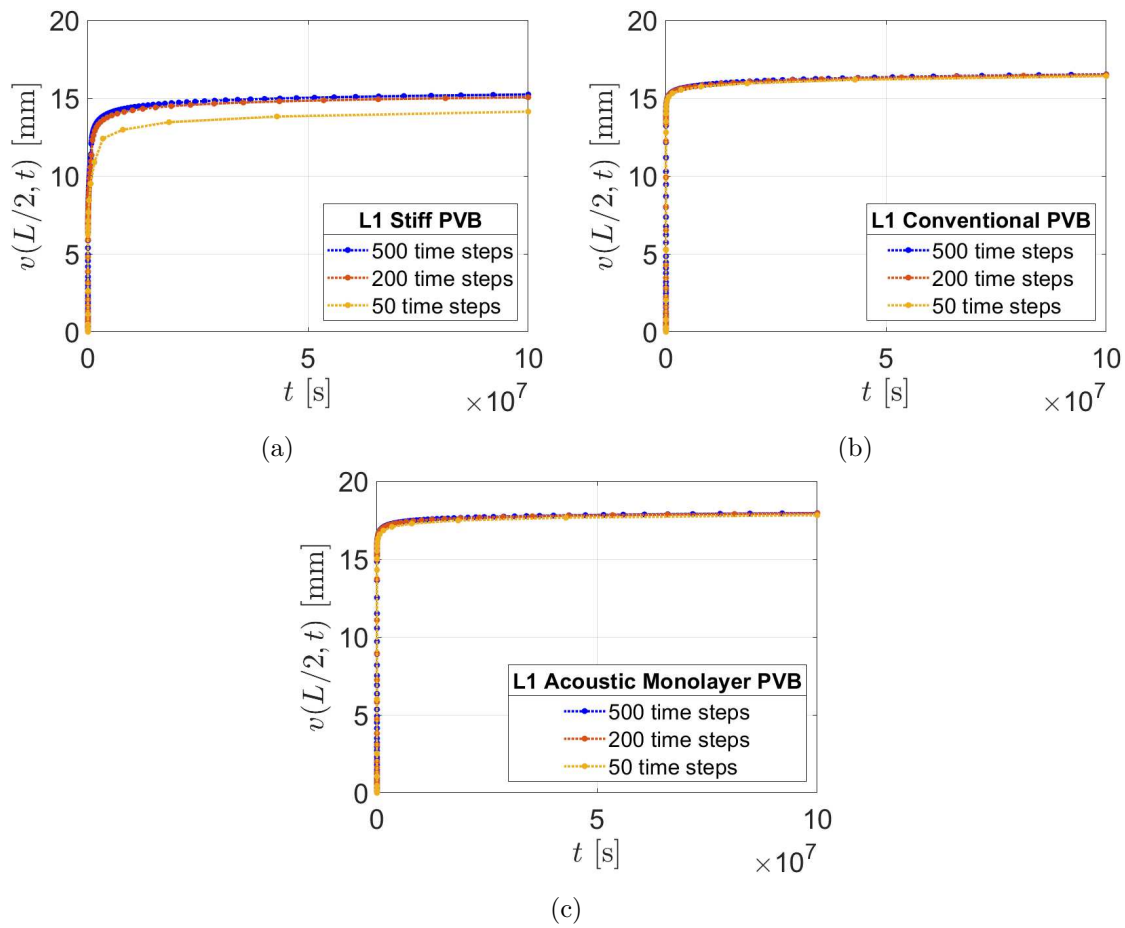


Figure 4.6: Convergence analysis of the long creep solution obtained through the L1 formula and time mesh in geometric progression, for the three different interlayers of Figure 4.3: (a) Stiff PVB, (b) Conventional PVB, (c) Acoustic Monolayer PVB, all tested at 20° C.

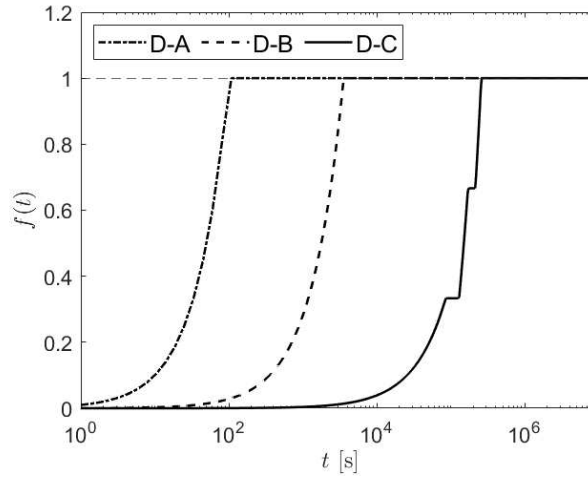


Figure 4.8: Different types of deformation history. The target deflection is reached at different times: 120 s (D-A), 1 hour (D-B), and 3 days in three equal steps (D-C).

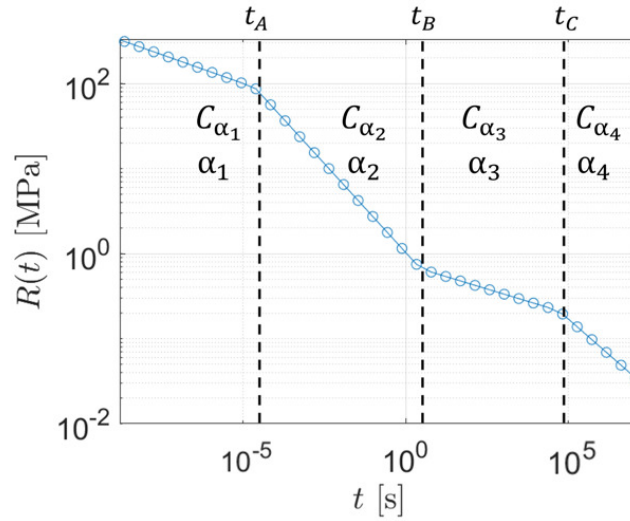


Figure 4.9: Bi-logarithmic representation of the stiff PVB relaxation curve at 20°C, approximated by four power-law branches. The dots indicate the geometric-progression time mesh, which ensures a homogeneous representation of the relaxation behavior across the entire time domain.

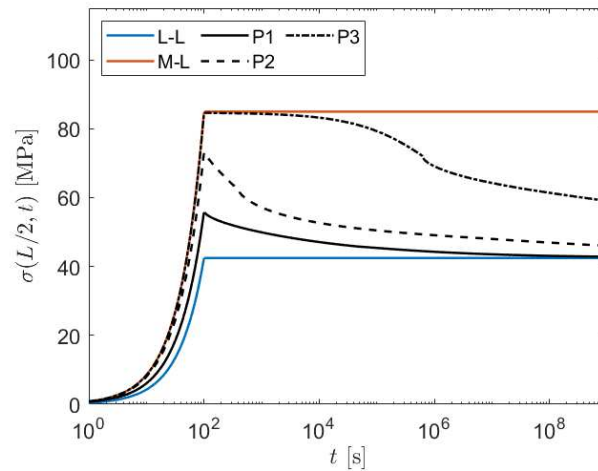


Figure 4.12: Maximum stress at mid-span  $\sigma(L/2, t)$  as a function of time  $t$ , for beams laminated with interlayers P1, P2 and P3 at 20° C, all of thickness  $S2 = 1.52$  mm, under loading history D-A; comparison with the monolithic (M-L, red line) and layered (L-L, blue line) limits.

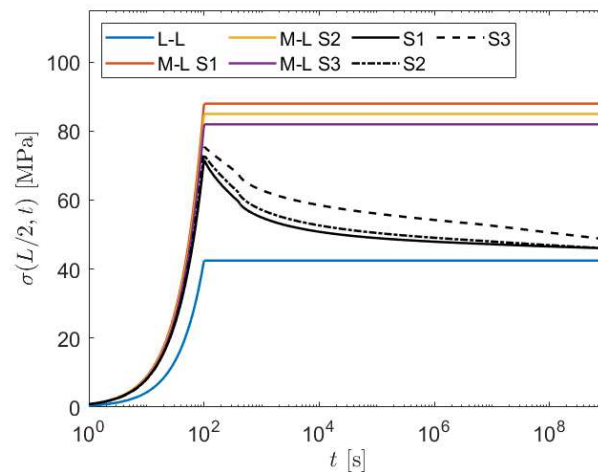


Figure 4.13: Maximum stress at mid-span  $\sigma(L/2, t)$  as a function of time  $t$  under the deformation history D-A, for interlayer P2 at 20° C and thicknesses  $S1 = 2.28$  mm,  $S2 = 1.52$  mm,  $S3 = 0.76$  mm. Comparison with layered (L-L) and monolithic (M-L) limits.

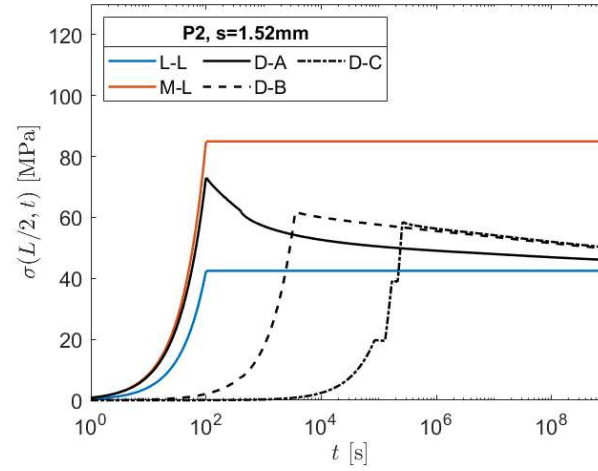


Figure 4.14: Maximum stress at mid-span  $\sigma(L/2, t)$  as a function of time  $t$  for interlayer P2 at 20° C of thickness  $S2 = 1.52$  mm, under the three loading histories D-A, D-B and D-C of Figure 4.8. Comparison with layered (L-L) and monolithic (M-L) limits for deformation history D-A.

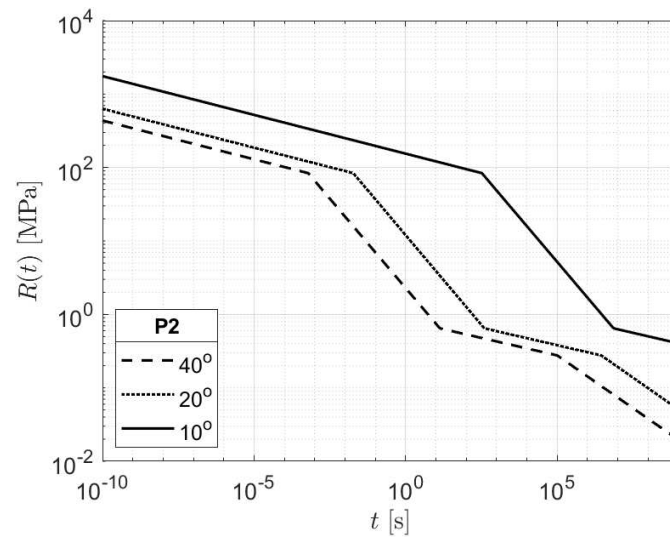


Figure 4.15: Bi-logarithmic representation of the relaxation curve for conventional PVB (polymer P2) at different temperature (10° C, 20° C and 40° C) obtained via the Williams-Landel-Ferry equation.

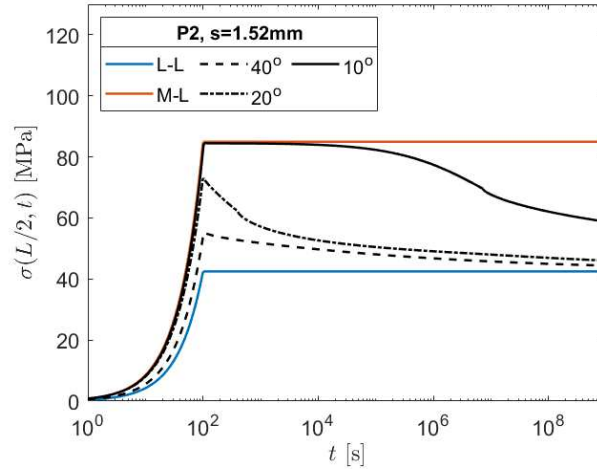


Figure 4.16: Maximum stress at mid-span  $\sigma(L/2, t)$  as a function of time  $t$  for interlayer P2 of thickness  $S2 = 1.52$  mm, under deformation history D-A, at the three operating temperatures:  $10^\circ\text{C}$ ,  $20^\circ\text{C}$  and  $40^\circ\text{C}$ . Comparison with layered (L-L) and monolithic (M-L) limits.

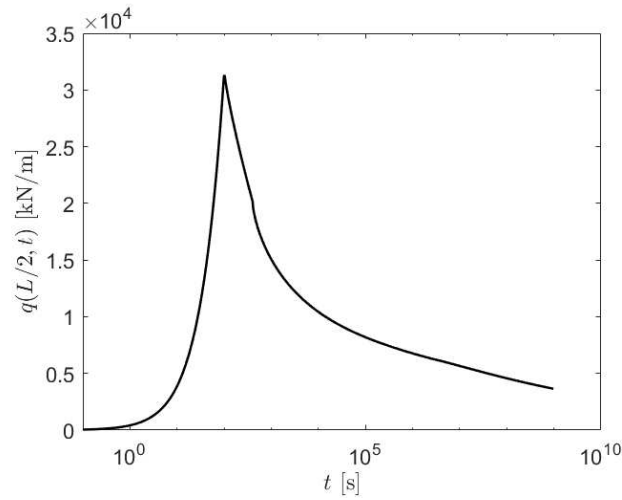


Figure 4.17: Maximum contact forces per unit length at mid-span,  $q(L/2, t)$ , for interlayer P2 at  $20^\circ\text{C}$  with thickness  $S2 = 1.52$  mm, under the deformation history D-A.



## Chapter 5

# Fractional-Viscoelastic-Finite-Element Model

In this chapter, a one-dimensional finite element implementation of the model proposed in [63, 82] for simulating laminated glass beams under various boundary and loading conditions is presented. The approach employs linear and Hermite shape functions for spatial discretization, integrated into the weak formulation of the dynamic viscoelastic problem derived from Hamilton's principle. Time integration is performed using a finite difference scheme based on the L1 formula with a variable time step progressing in logarithmic scale.

Although the L1 approximation is only first-order accurate, it is well-suited for analyzing laminated glass in architectural contexts. Its application to other sectors, such as automotive or aerospace, where faster response times and higher accuracy are critical, would require alternative numerical approaches. The method is fully compatible with standard finite element formulations and has been developed for relaxation functions composed of four power-law branches, which is sufficient to characterize most commercial polymers. Including additional or fewer branches poses no conceptual challenge.

Code validation is carried out through comparisons with experimental data from four-point bending tests on laminated glass beams under different loading rates and temperatures. The relaxation curve for a stiff PVB interlayer, taken as a paradigmatic example, was obtained via Dynamic Mechanical Thermal Analysis (DMTA) [50] and used as the constitutive input in the FEM code. The numerical simulations demonstrated excellent agreement with displacement-driven experiments spanning four orders of magnitude in strain rate, and a very good fit was also observed with creep tests conducted at various temperatures.

Given its focus on architectural applications, this study does not address post-fracture behavior, fracture mechanics, interlayer bridging, or delamination effects. While these mechanisms are critical in safety-sensitive applications such as automotive or aerospace glazing, they are generally not considered in architectural design, where structural assessment rarely extends beyond initial glass fracture. Consequently, the present

work is confined to the pre-fracture mechanical response of laminated glass, aiming to characterize its viscoelastic behavior under varying loading rates and thermal conditions. Investigations into fracture initiation, crack propagation, and post-fracture load redistribution via interlayer bridging, as reviewed in [83], remain topics for future research.

The main contribution of this study lies in demonstrating, through experimental validation, that approximating the relaxation curve with no more than four power-law segments sufficiently captures the mechanical behavior of standard polymeric interlayers used in glass lamination. This representation naturally facilitates the use of fractional calculus to describe viscoelastic behavior. The numerical FE implementation of the fractional viscoelastic model follows the same procedures as traditional structural models, without introducing additional complexity. Comparisons with bending experiments confirm the model's accuracy in predicting the flexural response of laminated glass, using the relaxation function of the polymeric interlayer, independently determined through DMTA, as constitutive input. These tools provide essential capabilities for engineers and researchers designing laminated glass structures across a wide range of architectural applications.

## 5.1 The four point bending problem

The four-point bending (FPB) test is a widely used experimental method to investigate the flexural behavior of beams and laminated structures, including glass laminates. In this configuration, the specimen is supported at two points near its ends, while two loading points are symmetrically applied within the span, creating a region of constant bending moment between the load application points. This arrangement allows for a uniform stress distribution in the central portion of the beam, minimizing the influence of shear stresses and providing a clear characterization of flexural properties.

For laminated glass beams, the FPB test is particularly suitable because it highlights the contribution of the polymeric interlayer to the overall structural response. The test captures both the short-term elastic behavior and the time-dependent viscoelastic effects, as the interlayer governs the shear coupling between glass plies. By controlling the displacement or load rate, the test can probe the material response across multiple time scales, making it ideal for validating numerical models based on fractional viscoelasticity.

In addition to its ability to provide accurate bending data, the FPB setup reduces the complexity of boundary conditions compared to three-point bending, where shear and bending effects are more closely coupled. This simplification is advantageous for both experimental repeatability and numerical modeling, as it allows the central region of the beam to experience nearly pure bending. Consequently, the FPB test represents a robust and practical approach for investigating the viscoelastic behavior of laminated glass under controlled laboratory conditions.

The following sections describe the experimental apparatus, specimen geometry, and loading protocols used in this study, which provide the basis for comparison with the Finite Element Fractional Viscoelastic (FEFV) model presented in this chapter.

The viscoelastic structural model for laminated glass beams is presented in this work.

In the four-point bending configuration adopted here, the deformation of the beam is predominantly cylindrical, and the resulting displacements remain moderate, making geometric nonlinearities negligible. While membrane effects can become significant in plate or shell structures, their influence is minimal in the current setup. Edge effects near the specimen boundaries may affect the ultimate bending strength; however, these localized phenomena are not the focus of this study. Instead, our attention is directed toward the viscoelastic response of the interlayer, which governs the laminate's pre-fracture stiffness. In this context, the effects of boundaries and localized geometric nonlinearities are insignificant, allowing the model to accurately capture the time-dependent behavior of the laminated glass under typical bending conditions. Moreover, this simplified representation provides a robust framework for incorporating fractional viscoelasticity without introducing unnecessary computational complexity.

We consider a laminated glass beam of length  $L$  and width  $b$ , composed of two glass plies of thickness  $h_1$  and  $h_2$ , bonded by a thin polymeric interlayer of total thickness  $s$ , which may consist of a single sheet or a stack of multiple layers. A general mechanical model is developed, with particular focus here on reproducing the four-point bending test configuration illustrated in Figure 5.1.

In this setup, the beam is supported at two points spaced by a distance  $d_{BC}$ , while the load  $F$  is applied through two rollers separated by a distance  $d_R$ . A reference frame  $(x, z)$  is introduced as shown in the same figure, where  $x$  denotes the longitudinal axis and  $z$  the transverse coordinate.

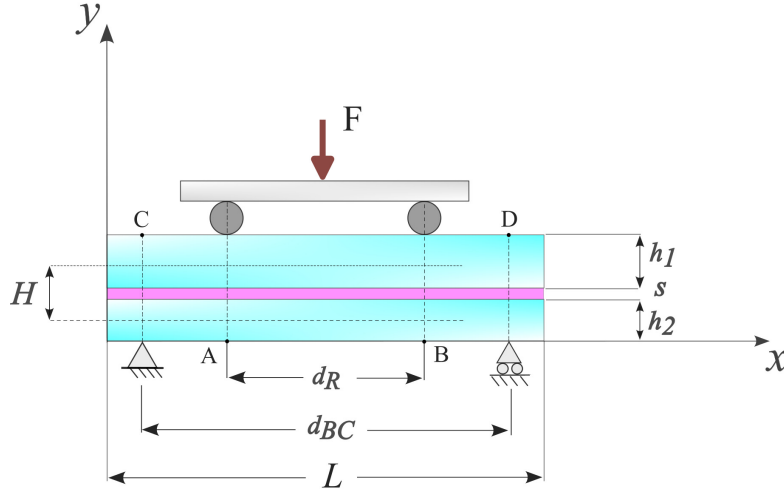


Figure 5.1: Four-point bending test configuration. Laminated glass plate simply supported at a distance  $d_{BC}$  and loaded by a vertical force  $F$  applied through two rollers positioned  $d_R$  apart.

The glass plies are modeled as Euler–Bernoulli beams, while the polymeric interlayer provides their shear coupling. This modeling choice is justified by two main considerations:

- **Stiffness contrast:** There is a large difference between the shear modulus of glass ( $\sim 70$  GPa) and the secant shear modulus of the polymeric interlayer (1–100 MPa, depending on load duration and temperature).
- **Structural proportions:** Laminated glass elements are typically slender (lengths of meters, thicknesses of centimeters), which results in relatively low shear stresses, necessarily below the polymer's shear strength and adhesion strength ( $\sim$  few MPa).

This assumption is consistent with established analytical models and design methodologies for laminated glass elements with conventional aspect ratios. Because the interlayer is much softer, shear strain is concentrated almost entirely within it, producing a distinctive zig-zag distortion of the cross-section—a key feature explicitly captured by the present model. In contrast, the shear strain in the much stiffer glass plies is smaller by orders of magnitude and can be neglected. While higher-order shear theories could in principle be applied [84], they generally fail to reproduce the zig-zag sectional deformation accurately. Within this framework, the proposed formulation achieves the minimum level of complexity required to correctly describe the structural behavior.

In this framework, it is possible to add to the variational principles illustrated in 3.2 the virtual work  $\delta W_{\text{ext}}$  of the external load  $F(t)$ , applied through the rollers at positions  $x_A$  and  $x_B$  as shown in Figure 5.1

$$\delta W_{\text{ext}} = \frac{1}{2}F(t) \delta v(x_A, t) + \frac{1}{2}F(t) \delta v(x_B, t). \quad (5.1)$$

### 5.1.1 Finite element discretization

The weak form of the governing equations naturally lends itself to a finite element discretization. The resulting formulation is here referred to as the *Finite Element Fractional Viscoelastic method* (FEFV). The unknown fields are collected in the vector

$$\mathbf{u}(x, t) = \begin{bmatrix} v(x, t) & \theta(x, t) & u_1(x, t) & u_2(x, t) \end{bmatrix}^T, \quad (5.2)$$

where  $v(x, t)$  is the transverse displacement,  $\theta(x, t)$  denotes the cross-section rotation, and  $u_1(x, t)$ ,  $u_2(x, t)$  represent the axial displacements of the upper and lower glass plies, respectively. Although  $\theta(x, t)$  is kinematically related to the derivative of the transverse displacement ( $\theta(x, t) \simeq v'(x, t)$ ), it is here introduced as an independent variable, following the standard finite element procedure.

A one-dimensional two-node beam element is adopted, with four degrees of freedom per node: the transverse displacement  $v_i(t)$ , the infinitesimal rotation  $\theta_i(t)$ , and the axial displacements  $u_{1,i}(t)$  and  $u_{2,i}(t)$  of the glass plies. The corresponding nodal displacement vector for an element of length  $l_e$  reads

$$\mathbf{d}(t) = \begin{bmatrix} \mathbf{d}_1(t) \\ \mathbf{d}_2(t) \end{bmatrix}, \quad (5.3)$$

with

$$\mathbf{d}_j(t) = \begin{bmatrix} v_j(t) & \theta_j(t) & u_{1,j}(t) & u_{2,j}(t) \end{bmatrix}^T, \quad j = 1, 2. \quad (5.4)$$

It is then possible to separate the spatial and temporal dependence of the fields by interpolating the nodal degrees of freedom with suitable shape functions, i.e.,

$$\mathbf{u}(x, t) = \mathbf{N}(x) \mathbf{d}(t). \quad (5.5)$$

The interpolation matrix is partitioned as

$$\mathbf{N}(x) = \begin{bmatrix} \mathbf{N}_1(x) & \mathbf{N}_2(x) \end{bmatrix}, \quad (5.6)$$

where the sub-matrices  $\mathbf{N}_i(x)$  are defined as

$$\mathbf{N}_i(x) = \begin{bmatrix} N_{1,i}(x) & N_{2,i}(x) & 0 & 0 \\ 0 & 0 & N_{3,i}(x) & 0 \\ 0 & 0 & 0 & N_{4,i}(x) \end{bmatrix}, \quad i = 1, 2. \quad (5.7)$$

In the element domain  $0 \leq x \leq l_e$ , the adopted shape functions read

$$\begin{aligned} N_{3,1}(x) &= N_{4,1}(x) = 1 - \frac{x}{l_e}, & N_{3,2}(x) &= N_{4,2}(x) = \frac{x}{l_e}, \\ N_{1,1}(x) &= 1 - 3\frac{x^2}{l_e^2} + 2\frac{x^3}{l_e^3}, & N_{2,1}(x) &= x - 2\frac{x^2}{l_e} + \frac{x^3}{l_e^2}, \\ N_{1,2}(x) &= 3\frac{x^2}{l_e^2} - 2\frac{x^3}{l_e^3}, & N_{2,2}(x) &= -\frac{x^2}{l_e} + \frac{x^3}{l_e^2}. \end{aligned} \quad (5.8)$$

After assembly one obtains the discretized formulation of the problem at the element level, in the form

$$\mathbf{M}\ddot{\mathbf{d}}(t) + \mathbf{C} {}^C_0D_t^\alpha[\mathbf{d}(\cdot)](t) + \mathbf{K}\mathbf{d}(t) = \mathbf{f}(t), \quad (5.9)$$

where  $\mathbf{M}$  is the mass matrix,  $\mathbf{C}$  is the damping matrix, and  $\mathbf{K}$  is the stiffness matrix. Observe that the damping matrix multiplies the fractional derivative of order  $\alpha$  of the displacement vector, denoted by  ${}^C_0D_t^\alpha[\mathbf{d}(\cdot)](t)$ .

The element matrices are obtained with standard arguments. With the notation

$$\mathbf{M}_e = \int_0^{l_e} \mathbf{N}^T(x) \mathbf{m}(x) \mathbf{N}(x) dx, \quad \mathbf{K}_e = \int_0^{l_e} \mathbf{B}^T(x) \mathbf{k}(x) \mathbf{B}(x) dx, \quad (5.10)$$

$$\mathbf{C}_e = \int_0^{l_e} \mathbf{B}^T(x) \mathbf{c}(x) \mathbf{B}(x) dx, \quad (5.11)$$

where  $\mathbf{N}(x)$  is the interpolation matrix,  $\mathbf{B}(x)$  is the strain-displacement matrix, and  $\mathbf{m}(x)$ ,  $\mathbf{k}(x)$ ,  $\mathbf{c}(x)$  are the local material constitutive operators for inertia, elasticity, and fractional viscoelasticity, respectively.

$$\mathbf{N}_{v,i}(x) = [N_{1,i}(x) \quad N_{2,i}(x) \quad 0 \quad 0], \quad (5.12a)$$

$$\mathbf{N}_{u_1,i}(x) = [0 \quad 0 \quad N_{3,i}(x) \quad 0], \quad (5.12b)$$

$$\mathbf{N}_{u_2,i}(x) = [0 \quad 0 \quad 0 \quad N_{4,i}(x)], \quad (5.12c)$$

one obtains from (3.22), (3.23) (3.24) and (3.25) that

$$\mathbf{M} = \mu \int_0^{l_e} \mathbf{N}_v(x) \mathbf{N}_v^T(x) dx, \quad (5.13)$$

$$\begin{aligned} \mathbf{C} = & \int_0^{l_e} \frac{C\bar{\alpha}b}{s} \left[ \mathbf{N}_{u_1}(x) \mathbf{N}_{u_1}^T(x) - \mathbf{N}_{u_1}(x) \mathbf{N}_{u_2}^T(x) + H \mathbf{N}_{u_1}(x) \mathbf{N}_v'^T(x) \right] dx \\ & + \int_0^{l_e} \frac{C\bar{\alpha}b}{s} \left[ -\mathbf{N}_{u_2}(x) \mathbf{N}_{u_1}^T(x) + \mathbf{N}_{u_2}(x) \mathbf{N}_{u_2}^T(x) - H \mathbf{N}_{u_2}(x) \mathbf{N}_v'^T(x) \right] dx \\ & + \int_0^{l_e} H \frac{C\bar{\alpha}b}{s} \left[ \mathbf{N}_v'(x) \mathbf{N}_{u_1}^T(x) - \mathbf{N}_v'(x) \mathbf{N}_{u_2}^T(x) + H \mathbf{N}_v'(x) \mathbf{N}_v'^T(x) \right] dx, \end{aligned} \quad (5.14)$$

$$\begin{aligned} K = & \int_0^{l_e} \left[ E(I_1 + I_2) \mathbf{N}_v''(x) \mathbf{N}_v''^T(x) - p(t) \mathbf{N}_v'(x) \mathbf{N}_v'^T(x) \right] dx \\ & + \int_0^{l_e} \left[ EA_1 \mathbf{N}_{u_1}'(x) \mathbf{N}_{u_1}'^T(x) + EA_2 \mathbf{N}_{u_2}'(x) \mathbf{N}_{u_2}'^T(x) \right] dx. \end{aligned} \quad (5.15)$$

The work done by the vertical external forces, corresponding to equation (5.1), can be readily calculated if a node is placed where the forces are applied.

### 5.1.2 Time discretization with variable time steps

To solve the system (5.9), it is convenient to employ a finite difference approximation. As demonstrated in previous work [63], using a variable time step is particularly effective for capturing the long-term response of the polymeric inner layers. Specifically, a logarithmic geometric progression of the time steps distributes the observation points evenly on a logarithmic scale, giving comparable importance to the different power-law branches of the relaxation function. The observation interval extends from  $t_0 = 0$  to  $t = T$  and is subdivided into  $s$  time steps. The piecewise linear approximation of the bilogarithmic plot of the relaxation function, illustrated in Figure 2.6, motivates the use of a progressively increasing time step: on a logarithmic scale, the time steps align with equally spaced points, enabling a balanced representation of the various branches of the relaxation function.

For  $1 \leq j \leq s$ , the second time derivative  $\ddot{\mathbf{d}}(t_j)$  can be approximated as described following the finite difference scheme stated in Equation (3.14).

The Grünwald-Letnikov method [85] is highly efficient for approximating fractional derivatives; however, it requires constant time steps. For variable time steps, it is more convenient [63] to use the “L1 formula” from Equation (2.16).

The L1 formula can be extended to a piecewise relaxation function composed of four temporal branches, each represented by a power-law and defined by the coefficients  $\alpha_i$  and  $C_{\alpha_i}$ . The observation time is divided into  $s$  intervals, which are evenly spaced on the logarithmic time scale of the relaxation function, as illustrated in Figure 2.6. The number of time steps is  $p$  in the first interval  $[t_0, t_A]$ ;  $u$  in  $[t_A, t_B]$ ;  $d$  in  $[t_B, t_C]$ ; and  $g = s - p - u - d$  in the final interval  $[t_C, T]$ . This partition ensures that within each subinterval, the relaxation function is characterized by the same coefficients  $C_\alpha$  and  $\alpha$ .

The fractional operator  ${}_0^C D_t^\alpha[\mathbf{d}(\cdot)](t)$  can then be schematically approximated with Equation (2.21).

The system (5.9) can then be solved using a finite difference approach, by approximating the second derivative with (3.14) and the fractional derivative with (2.21). A more detailed description of the procedure is provided in [63].

The initial conditions for this problem are

$$\mathbf{d}(t_0) = \mathbf{0}, \quad \dot{\mathbf{d}}(t_0) = \mathbf{0}. \quad (5.16)$$

The geometric boundary conditions for the configuration of Figure 5.1 are

$$v(x_C, t) = v(x_D, t) = 0, \quad (5.17a)$$

$$u_1(L/2, t) = u_2(L/2, t) = 0. \quad (5.17b)$$

The vertical displacement  $x_C$  and  $x_D$  is zero because here the beam is supported by the rollers. Condition (5.17b) results from the symmetry of the problem and is introduced to rule out rigid body displacements.

It should be noted that the L1 approximation for fractional derivatives, although computationally efficient, is only first-order accurate. Despite these inherent limitations, this approach is particularly suitable for simulating loading scenarios on building envelopes, which typically involve moderate-duration wind gusts (3–5 s) superimposed on decades-long dead load effects. In such cases, a variable time step is necessary to reduce computational effort: finer time resolution can be assigned to accurately capture the response to the wind, corresponding to the first branch of the master curve, while coarser time steps can be used for long-term behavior, where stiffness decays rapidly.

The L1-based integration scheme may produce significant errors in scenarios such as long-term sinusoidal loading, due to error accumulation from progressively coarser time discretization; however, such cases are uncommon in building envelope applications. Inaccuracies can also arise during rapid transients in short-term responses, when steep changes in the relaxation behavior occur. Such situations, typical of impact or blast loading, are more efficiently handled by the Grünwald-Letnikov method [59, 86].

We emphasize that the present work specifically addresses four-point bending tests under moderate stress-rate variations and environmental-temperature fluctuations.

## 5.2 Experimental campaign and comparisons with the FEM analysis

An experimental campaign was conducted to assess the response of layered glass beams laminated with PVB Extra Stiff PRO (B231) interlayer, produced by Kuraray Europe GmbH, at various operating temperatures: 20°C, 30°C, 40°C, and 50°C. Figure 2.3 presents the relaxation curves<sup>1</sup> for this material at the different temperatures over a time range from one second to fifty years, obtained via Dynamic Mechanical Thermal Analysis (DMTA) following the procedure described in [50].

Table 2.1 reports the parameters describing the power-law branches approximating the relaxation curves at the various temperatures. These parameters are subsequently employed in the numerical simulations.

Referring to the scheme in Figure 5.1, the geometric parameters are  $d_R = 150$  mm,  $d_{BC} = 250$  mm, and  $L = 300$  mm, while the width is  $b = 150$  mm. The laminated glass specimens consist of two plates of thickness  $h_1 = h_2 = 3$  mm, bonded together by a polymeric interlayer of thickness  $s = 0.76$  mm.

### 5.2.1 Validation of the code

A first series of tests was conducted at a temperature of 40°C under a load history consisting of a displacement-driven ramp, linear in time, reaching a maximum load of 1 kN at displacement rates ranging from 0.01 to 10 mm/s. These tests therefore examine behavior across multiple time scales, from 1.2 seconds to 5.5 hours. The finite element simulations were carried out using the proposed methodology, with all analyses employing 200 time steps to ensure computational accuracy. For any given relaxation curve, temporal branches corresponding to times  $\tau_i$  (as listed in Table 2.1) exceeding the observation period can be omitted.

Figure 5.2 compares the load  $F$  [kN] versus midspan displacement  $v(L/2)$  [mm] obtained from experiments (coloured solid lines, FPBT: Four-Point Bending Test) and from numerical simulations (black dashed lines, FEFV: Finite Element Fractional Viscoelastic). The agreement is excellent, with discrepancies consistently below 2%.

The second series of tests consists of creep experiments. The load history includes an initial linearly increasing ramp reaching a maximum load of 1 kN over 10 s, followed by a constant load maintained for an observation period of 18 h. Figure 5.3 presents semilogarithmic plots of the vertical displacement  $v(L/2)$  [mm] as a function of time  $t$  [s] for tests conducted at temperatures of (a) 20°C, (b) 30°C, (c) 40°C, and (d) 50°C. Experimental curves are shown as continuous red lines (FPBT), while numerical results are indicated by black dashed lines (FEFV).

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<sup>1</sup>Kuraray's technical portal (<https://www.trosifol.com>) provides complete product specifications across their range, including elastic property datasheets and standardized classifications. For this study, we examined the commercially available product PVB Extra Stiff PRO (Kuraray internal reference B231). While general material data are available on the website, the complete relaxation curves required for our analysis were not publicly provided and were specifically characterized for this research.

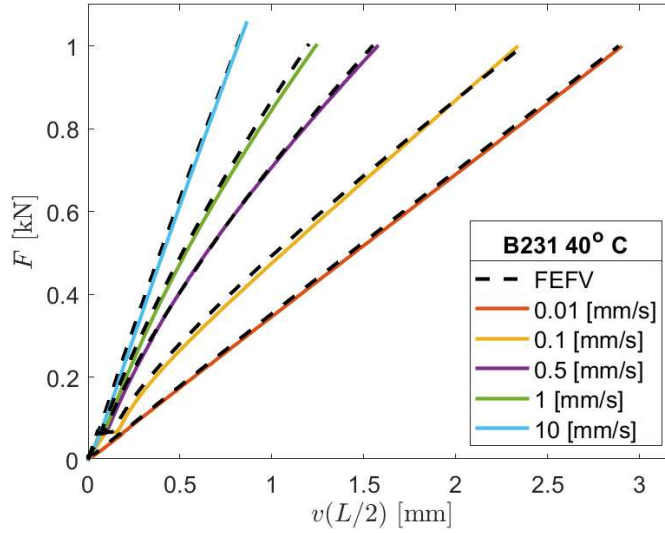


Figure 5.2: Plots of the load  $F$  [kN] against midspan displacement  $v(L/2)$  [mm], in displacement driven four-point bending test at different rates. Comparison between the results obtained experimentally (continues lines) and with the FEFV numerical model (dashed lines).

In these simulations, a total of 200 time steps is employed. Consistent with the relaxation curves in Figure 2.3, the shear coupling of the glass plies through the interlayer weakens as temperature increases, resulting in larger vertical displacements.

The creep tests also demonstrate very strong agreement between experiments and the model, particularly at temperatures of 20°C and 50°C, where the error remains below 1%. At 30°C and 40°C, viscoelastic effects of the polymeric interlayer become more pronounced, as the glass transition region is encountered within the 18 h observation period, as indicated by the relaxation curves in Figure 2.3. Even in these more critical cases, the maximum deviation between experimental and theoretical curves does not exceed 5%. This slightly larger discrepancy may be attributed to thickness tolerances in the interlayers and minor changes in the material's viscoelastic properties following the lamination process. Nonetheless, the results are considered highly satisfactory.

### 5.2.2 Computational time and convergence analysis

The FEFV model for the four-point bending test was solved using 10 elements of equal length along the beam and 200 time steps over the observation period, for both validation tests. These parameters ensure full numerical convergence, meaning that further increases in spatial or temporal resolution result in negligible changes to the solution. The computation time for this configuration is approximately 2.5 seconds.

A convergence analysis is summarized in Figure 5.4. This corresponds to the case of a simply supported laminated glass beam under a uniformly distributed load, with the

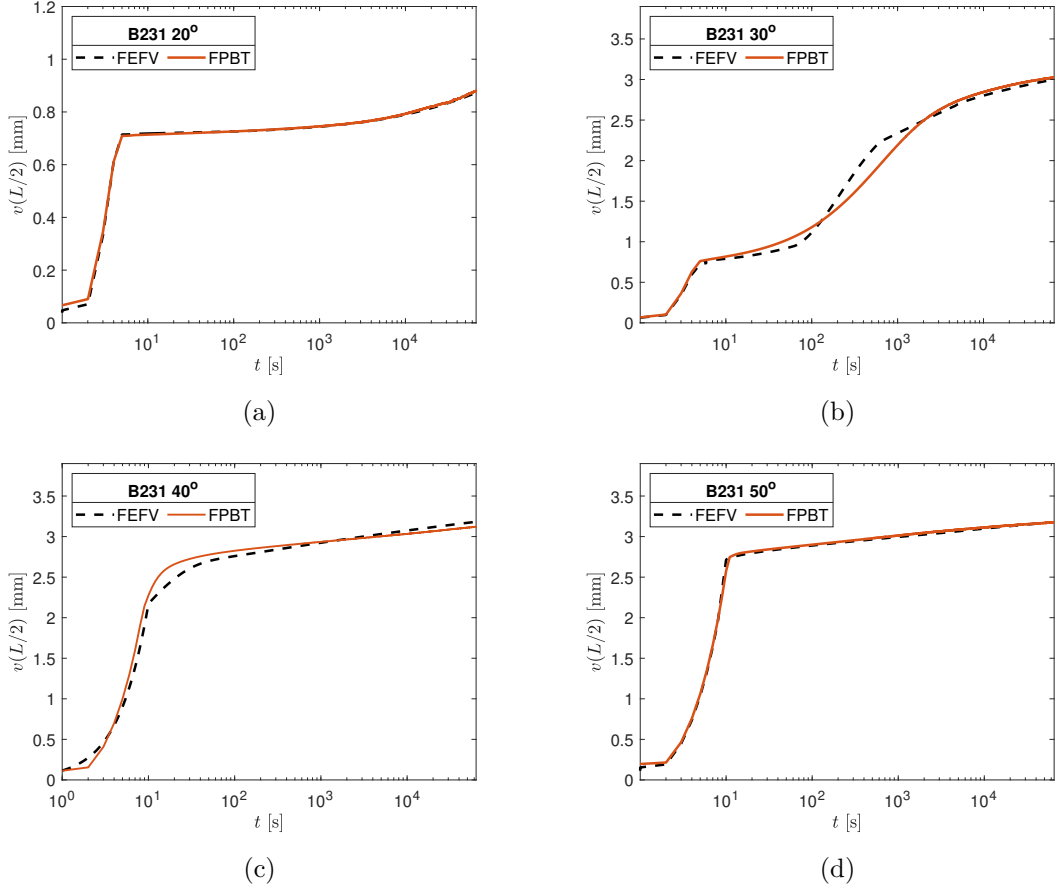


Figure 5.3: Four-point bending creep test on specimens laminated with PVB Extra Stiff PRO (B231) interlayer at temperatures (a) 20°C, (b) 30°C, (c) 40°C and (d) 50°C, for an observation time of 18 h.

same geometry and layer composition as the specimens considered in Section 5.2. The load follows the same time history as the four-point bending creep test shown in Figure 5.3: after the initial loading ramp, it is held constant at a resultant of 1 kN for  $10^9$  s (approximately 30 h). The polymer is B231 at 40°C.

In Figure 5.4, the left-hand-side y-axis shows the maximum deflection  $v(L/2)$  [mm] of the beam as a function of the number of time steps used in the numerical solution, which are equally spaced in logarithmic scale to capture the various branches of the relaxation curve. The different curves correspond to different values of the number  $N$  of elements used in the spatial discretization, uniformly distributed along the beam length. The right-hand-side y-axis reports the corresponding computational time ( $et$ ) required to obtain the solution. Only a single curve for computational time is plotted because, for the considered case, it is practically independent of  $N$ .

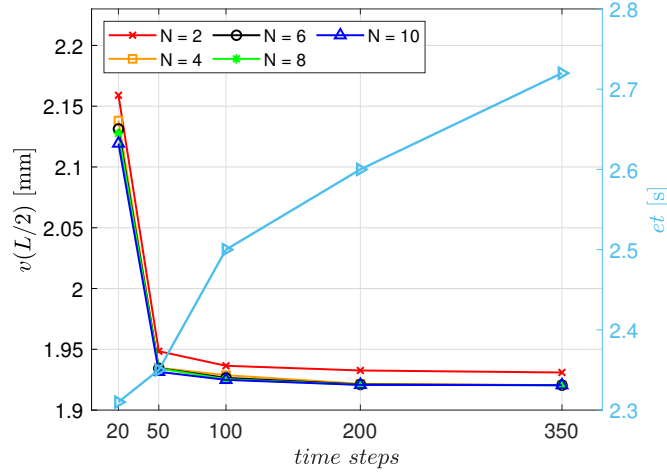


Figure 5.4: Convergence analysis for a simply supported glass beam laminated with B231, under uniformly distributed load at 40° C. Computed maximum displacement  $v(L/2)$  for different values  $N$  of the number of used elements, plotted as a function of the number of time steps in the observation time of  $\sim 30$  h. Corresponding computational time  $et$ .

This analysis shows that the maximum deflection converges to the expected value of 1.9205 mm, even with a coarse mesh of only four elements and 100 time steps. Furthermore, it highlights that the primary computational effort lies in the time integration, as the spatial mesh has negligible impact on the computational time.

Although the computational time is acceptable for the presented scenarios, it should be noted that the linear scaling of cost with the number of time steps may become a bottleneck for long-duration simulations, such as extended creep analyses or high-resolution dynamic assessments. This limitation can be further amplified by the L1 time integration scheme, which may become computationally demanding under fine temporal discretization, potentially affecting the model's scalability. It should be emphasized, however, that the proposed model is not intended as a universal tool, but rather as a minimal approach specifically designed to replicate the experimental test conditions.

### Final Remarks

We have presented a numerical approach for modeling the viscoelastic behavior of polymeric materials in laminated glass, based on a power-law approximation of the relaxation function rather than the traditional Prony series. The power-law approximation naturally incorporates fractional derivatives into the constitutive equations of the interlayer material and offers several advantages, including simpler parameter calibration via geometrical interpolation of experimental data and reduced parameterization requirements, without compromising accuracy in fitting the relaxation curves.

The spatial finite element discretization is incorporated into the weak formulation of

the dynamic viscoelastic problem, derived from Hamilton's principle. Fractional derivatives in time are numerically approximated using the L1 formula, which allows for non-uniform time meshes. This approach provides several key benefits. First, a logarithmically spaced time mesh accurately captures each branch of the relaxation function over a wide range of time scales; in contrast, constant time steps fail to interpolate all branches, leading to reduced accuracy. Second, the variable time-step approach significantly reduces computational effort by requiring fewer steps to represent the relaxation function, enabling efficient computation over extensive observation periods. For instance, solutions can be obtained in seconds for observation times ranging from  $10^{-10}$  s to  $10^{15}$  s, capturing all transitions in the relaxation function—a task infeasible with constant time steps.

The relaxation curves at various temperatures for an unprocessed, stiff PVB polymer were preliminarily determined experimentally via Dynamic Mechanical Thermal Analysis (DMTA) and used as constitutive input for the model.

The model was validated against results from four-point bending tests, conducted under variable strain rates as well as creep tests at different environmental temperatures. In the former case, deviations between experimental data and numerical simulations were below 1%, while in the latter, differences did not exceed 5%. These results are considered highly satisfactory, given the model approximations, geometric tolerances, and the slight modification of the polymer's rheological properties during autoclave lamination.

The one-dimensional model presented here is straightforward, yet effectively demonstrates that incorporating the fractional description of viscoelasticity introduces no additional complexity. While tailored for the four-point bending test, it is also applicable to other typical laminated glass beam configurations, such as balustrades and floors/roofs supported on two sides. One limitation is its linear-theory foundation. Although geometric nonlinearities have limited impact when deflections are small (a few centimeters), they may become significant for plates. Extending the model to Kirchhoff-Love plates is straightforward; however, more advanced structural theories are required to account for membrane effects, zig-zag warping due to interlayer compliance, and initially curved shell configurations.

Moreover, while the present study employs the L1 approximation for time discretization, future developments could explore advanced numerical strategies—including adaptive time-stepping, predictor-corrector algorithms, and higher-order schemes—to enhance computational accuracy within the fractional viscoelastic framework.

## Chapter 6

# Multi-Material Polymeric Interlayers

Multi-Material Polymeric Interlayers (MMPI) for laminated glass are often employed when a single-foil interlayer fails to deliver the desired combination of properties. The primary application of MMPI is the improvement of acoustic insulation. In such configurations, a soft polyvinyl butyral (PVB) core layer, specifically optimized for acoustic damping, is sandwiched between standard PVB layers designed for strong adhesion to glass, toughness, and ease of lamination processing. Naturally, the mechanical behaviour of MMPI differs from that of single-layer products. Estimating the mechanical performance of such interlayers is challenging because each polymer exhibits its own long-term behaviour, and when combined, the overall relaxation curve assumes an intermediate form. Furthermore, the thickness of each layer in the composite significantly influences the global mechanical response.

The relaxation curve of MMPIs can in principle be obtained through Dynamic Mechanical Thermal Analysis (DMTA). However, this method is typically expensive and time-consuming, and errors may occur in the experimental data due to the complexity of the testing machine and the need for precise calibration. A numerical simulation capable of predicting the relaxation curve with good accuracy, starting from the relaxation functions of the single materials, would therefore be highly beneficial for the design and optimization of MMPIs.

In this work, we present a numerical tool based on the power-law interpolation of the relaxation curve of polymeric materials. The method predicts the relaxation response of MMPI by using as input data the relaxation curves of the individual interlayers, which are usually available from manufacturers. Specific reference is made to three main products by Kuraray GmbH: Ionoplast *SentryGlas*<sup>®</sup>, Stiff PVB, and Acoustic PVB, at various operating temperatures (20°C, 30°C, and 40°C).

## 6.1 Shear Response of Multi-Material Interlayers

To predict the mechanical behavior of a multi-material polymer interlayer (MMPI) in a laminated glass structure, it is necessary to investigate the coupling effect of the interlayer on the two glass plies it connects. The relaxation functions of the individual materials that compose the MMPI are well known, and they represent the starting point of the analysis.

The objective is to predict a single relaxation function that characterizes the coupling effect of the entire package, i.e., the evolution of the effective stiffness over time. Such a relaxation function can be extrapolated through Dynamic Mechanical Thermal Analysis (DMTA) and subsequently employed for model validation by comparing numerical predictions with the experimental results.

The proposed Fractional Viscoelastic (FV) model consists of a laminated glass specimen with an MMPI composed of two outer layers of acoustic PVB and one inner sheet of stiff PVB used as the core. In Figure 6.1, the relaxation functions  $R_1(t)$  for acoustic PVB (yellow) and  $R_2(t)$  for stiff PVB (violet) are represented according to a power-law interpolation. The black curve corresponds to the combination of the two materials, while the experimental points are reported as red triangles.

The curves are shown for three different temperatures, 20°C (a), 30°C (b), and 40°C (c), which were obtained by controlling the temperature of the testing machine platen.

The laminated glass package is subjected to a shear stress on the top surface, while the bottom surface is fixed by preventing displacement in the  $x$ -direction, as shown in Figure 6.2.

The proposed system can be simplified by considering a single layer of material “2” with a total thickness equal to the sum of the original layers’ thicknesses, and with the corresponding relaxation function  $R_1$ . The total displacements in upper glass ply of the two materials  $\varepsilon$  in the  $x$ -direction in a two spring pot in series scheme is given by

$$\varepsilon = \varepsilon_1 + \varepsilon_2$$

where  $\varepsilon_1$  and  $\varepsilon_2$  are the two material’s displacement. This system is then compared with an equivalent system where, instead of considering two distinct materials, the interlayer is replaced by a single material with thickness  $h_m$  equal to the sum of the thicknesses of layers “1” and “2” in the previous scheme, i.e.,

$$h_m = h_1 + h_2,$$

and with an equivalent relaxation function  $R_m$ , obtained from DMTA by considering the contribution of both materials.

The total displacement of the upper glass ply should therefore be nearly the same in both systems:

$$\varepsilon \cong \varepsilon_m.$$

It should be noted that uncertainties in the results are inevitable, as the relaxation functions employed as input data are derived from experiments and are therefore subject to measurement errors.

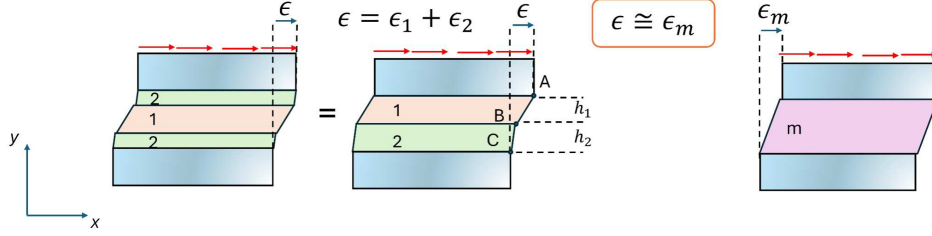


Figure 6.2: Laminated glass sandwich package composed of MMPI. The green material “1” used in the core is Trosifol Extra Stiff PVB, while the orange material “1” is Acoustic PVB. The structure is subjected to shear stress on the top surface.

Considering the power-law approximation of the experimental relaxation function, the constitutive law of the polymeric interlayer can be described by Equation (1.53), where the shear stress  $\tau(z, t)$  in the material is equal to the  $\alpha$ -order time derivative of the shear strain multiplied by the stiffness constant  $C_\alpha$ . The shear deformation is therefore:

$$\varepsilon_1(z, t) = \frac{1}{h_1} [u_A(z, t) - u_B(z, t)], \quad (6.1)$$

with reference to interlayer “1”; the relationship for material “2” is analogous.

By considering the illustrated system, we can express the condition  $\varepsilon = \varepsilon_1 + \varepsilon_2$  as follows:

$$\varepsilon(z, t) = \frac{u_A(z, t)}{h_1} + u_B(z, t) \left( \frac{1}{h_2} - \frac{1}{h_1} \right) - \frac{u_C(z, t)}{h_2}, \quad (6.2)$$

with the limit case  $h_1 = h_2$ :

$$\varepsilon(z, t) = \frac{u_A(z, t) - u_C(z, t)}{h}. \quad (6.3)$$

This assumption leads to the conceptualization of the *Spring-Pot* model, schematically represented in Figure 6.3.

The constitutive equation of the Spring-Pot can be expressed through Caputo’s fractional derivative in the generalized form from Equation (2.8).

Although the Spring-Pot serves as an analytical model for simulating viscoelastic behavior, it lacks a straightforward physical interpretation. In classical models, one can distinguish the contribution of the solid phase from that of the fluid phase, but in the Spring-Pot this distinction is lost. Indeed, the parameter  $C_\alpha$ , which represents the proportionality between stress and the fractional derivative of strain, cannot be directly interpreted as either an elastic modulus or a viscosity.

The laminated glass package with MMPI can therefore be schematized as two Spring–Pots in series, where the total displacement of the system is equal to the sum of their displacements and the stress is the same in each element, as depicted in Figure 6.3. In this representation, the entire laminated glass interlayer is modeled as a single equivalent Spring–Pot. The constants required for the constitutive equation can be determined from the experimental data of the relaxation curve, by interpolating the experimental points with two linear branches in the log–log representation. The slope of the lines provides the values of  $\alpha$ , while their vertical position at  $t = 1$  s defines the mechanical branch constant.

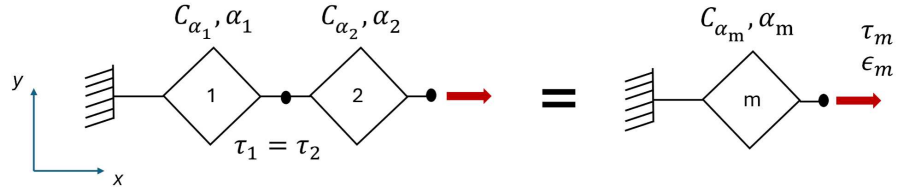


Figure 6.3: Scheme of the laminated glass package modeled as multiple Spring–Pots in series. Each material of the interlayer is represented as a fractional viscoelastic element with constitutive law governed by fractional derivatives.

According to Figure 6.3, it is possible to write the total displacement of the two spring pots in series using Equation 1.58:

$$\varepsilon(t) = \frac{1}{C_{\alpha_1}} {}^C_0 I_t^{\alpha_1} \tau(t) + \frac{1}{C_{\alpha_2}} {}^C_0 I_t^{\alpha_2} \tau(t). \quad (6.4)$$

The notation  $\varepsilon(z, t)$  is substituted for brevity with  $\varepsilon(t)$ , and it is considered  $\sigma_1(t) = \sigma_2(t) = \sigma(t)$ , because the two Spring pots are in series. Equation 6.4 can be rewritten using the formulation 1.58 proposed by G. L. SLONIMSKY and introduced in the subparagraph 1.2.5

$$\varepsilon(t) = \frac{1}{C_{\alpha_1}} {}^C_0 D_t^{-\alpha_1} \tau(t) + \frac{1}{C_{\alpha_2}} {}^C_0 D_t^{-\alpha_2} \tau(t). \quad (6.5)$$

An analogous expression for the only spring-pot representing the response of the material if used the relaxation function, from DTMA, using the two materials combined together. In this case we do not have the two relaxation functions of the two materials that represent the response of the composite, but just one equivalent curve that is supposed to represent the full package mechanical properties.

$$\varepsilon_m(t) = \frac{1}{C_{\alpha_m}} {}^C_0 D_t^{-\alpha_m} \tau_m(t). \quad (6.6)$$

The fractional derivatives in time of the constitutive equation (Equation (1.53)) are numerically approximated using the L1 formula, which allows the use of nonuniform time meshes. This approach provides several advantages.

It is also possible to write the stress, which is the same in the two spring-pots element by definition, as:

$$\tau(t) = C_{\alpha_1} {}^C D_t^{\alpha_1} \varepsilon_1(t), \quad (6.7a)$$

$$\tau(t) = C_{\alpha_2} {}^C D_t^{\alpha_2} [\varepsilon(t) - \varepsilon_1(t)], \quad (6.7b)$$

with the deformation in the second spring-pot  $\varepsilon_2(t) = \varepsilon(t) - \varepsilon_1(t)$ .

The numerical FV model has been implemented in MATLAB and PYTHON. The formulation incorporates the constitutive equation of the polymeric interlayer expressed in terms of fractional derivatives, approximated via the L1 scheme, and based on the experimental relaxation functions represented with power laws.

This mathematical framework allows the calculation of the deformation of the two different systems represented in Figure 6.3 and enables their comparison. Once the deformation of the system with two Spring-Pots in series is known, it is possible to compute the stress trend over the observation time and, by reversing Equation (2.4), the effective relaxation function  $R(t)$  of the laminated glass structure with MMPI can be obtained.

For validation purposes, the effective relaxation function  $R(t)$  derived from the FV model is compared against the experimental relaxation curve previously obtained through DMTA and presented in Figure 6.5.

It should be emphasized that this mathematical framework provides a full viscoelastic characterization of the stress in the polymeric interlayer. This implies that the strain at a given observation time step does not depend solely on the current stress state, but also on the history of all previous stress applications since the beginning of loading. This property is mathematically described by the convolution integral of the Boltzmann superposition principle (Equation (1.9)).

## 6.2 Numerical results and model validation

The time-dependent deformations obtained by solving Equation 6.5 are shown in Figure 6.4, where three cases at different environmental temperatures (20°C, 30°C, 40°C) are reported.

The results are compared using:

- the relaxation functions of the two interlayer materials ( $R_1(t)$  for Stiff PVB and  $R_2(t)$  for Acoustic PVB), modeled in series with Spring-Pots (black dashed line),
- and the relaxation function of the equivalent single material, obtained by combining the two materials, as represented in Figure 6.3.

The two deformation trends are expected to be nearly identical ( $\varepsilon \cong \varepsilon_m$ ). From Figure 6.4, it can be observed that this agreement is verified across the three considered cases, although with some unavoidable uncertainty. These discrepancies originate from inevitable measurement errors in the DTMA analysis of the relaxation function, which is used as the input data for the numerical model.

The obtained results confirm that the FV model yields reliable predictions. The second step of the analysis consists of calculating the relaxation curve from the previously determined strain trends. This is possible by inverting Equation (2.4), using as input data the deformation and the stress, which describes the mechanical response of the laminated glass structure with MMPI.

The corresponding results are reported in Figure 6.5, showing the relaxation function at three different environmental temperatures (20°C, 30°C, 40°C). These curves are directly compared with the experimental relaxation results obtained for the same MMPI through DTMA measurements.

It can therefore be concluded that the FV model is capable of accurately predicting the long-term viscoelastic response of the composite interlayer system. In particular, given the known relaxation functions of the two constitutive materials and their thickness ratio, the simulated relaxation function demonstrates excellent agreement with the experimental points represented by the red triangles.

### 6.2.1 Comparison with the elastic model

The proposed FV model is based on a full viscoelastic analysis of the polymeric interlayer. In order to emphasize the importance of adopting such a material model in this type of analysis, a comparison is now made with the simplified elastic model previously proposed in [81]. In the simplified approach, the relaxation function is computed as the inverse of the multilayer stiffness at time  $t$ , which is approximated by the inverse sum of the individual stiffness contributions at the same time, namely:

$$\frac{h_1 + h_2}{R_m(t)} = \frac{h_1}{R_1(t)} + \frac{h_2}{R_2(t)}, \quad (6.8)$$

where  $h_1$  and  $h_2$  are the thicknesses of the two polymeric layers, and  $R_1(t)$ ,  $R_2(t)$  their respective relaxation functions.

The results of this comparison are presented in Figure 6.6. It is observed that in the early stages of the observation time, both approaches yield nearly identical results, as the memory effect of the polymer has not yet become significant. At longer times, however, the difference between the two approaches becomes increasingly evident. The elastic model provides reliable predictions only up to approximately  $10^2$  seconds, beyond which it fails to capture the long-term response of the MMPI. In contrast, the FV model remains in good agreement with the experimental results throughout the entire observation time.

### 6.2.2 Influence of different interlayers

Since the model has been validated, a final analysis is now presented in order to demonstrate the potential of this approach in predicting the mechanical behaviour of MMPIs. A comparison is made between two different interlayer configurations:

- the first, already analysed, composed of Stiff PVB and Acoustic PVB,

- the second, where the Stiff PVB is replaced by a Sentry Glass sheet of the same thickness.

It is important to point out that the example with Sentry Glass is purely theoretical, in practical it is not possible to laminate together Sentry Glass and Acoustic PVB because they have different thermal properties in autoclave and due to their different chemical composition it is not possible to reach the standard adhesion characteristics. The objective is to evaluate the influence of different interlayers on the global mechanical response of the laminated package at three environmental temperatures (20°C, 30°C, 40°C), to show the potentiality of the proposed FV model.

The results of this comparison are illustrated in Figure 6.7. For completeness, the elastic model described in the previous section is also included in the analysis.

In the first column of Figure 6.7 (panels a, c, e), the two approaches are compared for the interlayer composed of Stiff PVB and Acoustic PVB, juxtaposed with experimental data. It is observed that Sentry Glass does not significantly affect the global mechanical behaviour of the MMPI at 20°C, as the two solutions nearly overlap. At higher temperatures, however, the influence becomes more evident due to the drop in stiffness of Stiff PVB (see Figure 6.7).

Conversely, Sentry Glass tends to maintain higher stability at elevated temperatures, as seen in the second column of Figure 6.7 (panels b, d, f), where the interlayer is composed of Sentry Glass and Acoustic PVB. Here, the comparison between FV and the elastic model clearly shows that the elastic model fails to reproduce the relaxation function for long observation times, whereas the FV model continues to provide accurate agreement with the experimental data.

The results confirm that the proposed FV model can successfully capture the viscoelastic behaviour of the MMPI for both material configurations. When Stiff PVB is used in combination with Acoustic PVB (left column, Figure 6.7a,c,e), the model predictions agree very well with the experimental data, especially at short and intermediate time scales. At higher temperatures, however, the relaxation function of Stiff PVB decreases rapidly, leading to a notable deviation from the elastic model predictions.

In contrast, when Sentry Glass is employed as the stiffer component (Figure 6.7b,d,f), the interlayer maintains greater stability, particularly at elevated temperatures. The FV model is again able to accurately describe the relaxation behaviour over the entire observation window, whereas the elastic model fails to reproduce the long-term viscoelastic response.

This demonstrates that the replacement of Stiff PVB with Sentry Glass in the MMPI improves the long-term stability of the laminated glass package, and that the FV model provides a consistent and reliable predictive framework for different interlayer material combinations.

## Final Remarks

A novel approach has been proposed to characterize the mechanical performance of a MMPI and to accurately predict its corresponding relaxation function. The methodol-

ogy relies on an innovative interpolation of the experimental relaxation curves obtained from DTMA analysis, represented through piecewise power-law functions. This allows the constitutive equation of the polymeric interlayer to be expressed in terms of fractional derivatives, which naturally capture the viscoelastic response. In particular, the stress is expressed as a real-order derivative of strain, with order ranging between 0 and 1, thereby describing a behaviour intermediate between perfectly elastic and perfectly viscous materials.

The proposed approach incorporates the full viscoelastic characterization of the polymeric material, which is crucial in this type of analysis. The relaxation functions predicted with this model show very good agreement with the experimental data across all tested cases, thus validating its accuracy.

Conversely, when an elastic model is adopted, the predicted relaxation functions are not reliable, particularly in the long-term regime. The strength of the proposed method is its general applicability: it can be used to predict the relaxation function of any MMPI composed of two or more polymers with different thicknesses, and enables a rapid evaluation of the mechanical performance without the need for costly and time-consuming experimental campaigns.

This modelling framework is therefore a powerful design tool, particularly in the development of innovative interlayers for laminated glass systems. For instance, it can support the design of MMPIs with tailored mechanical and acoustic properties. Ultimately, the proposed method furnishes a reliable and interpretable solution, helping designers and manufacturers to identify the most suitable MMPI configuration for specific applications.

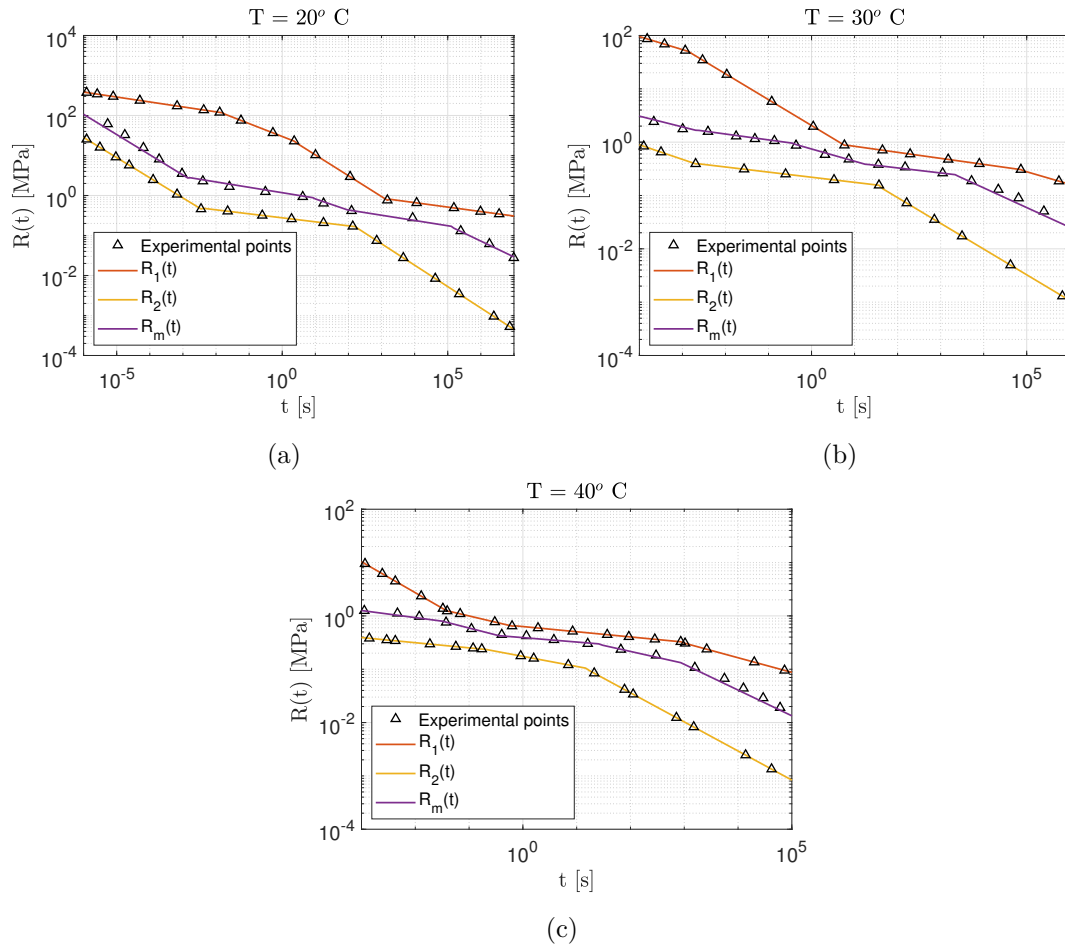


Figure 6.1: Power-law approximation of the relaxation function for two materials:  $R_1(t)$  for acoustic PVB (yellow) and  $R_2(t)$  for stiff PVB (violet), together with the curve obtained from their combination (black) and the experimental data (red triangles). The three curves correspond to different environmental temperatures:  $20^\circ \text{C}$  (a),  $30^\circ \text{C}$  (b), and  $40^\circ \text{C}$  (c).

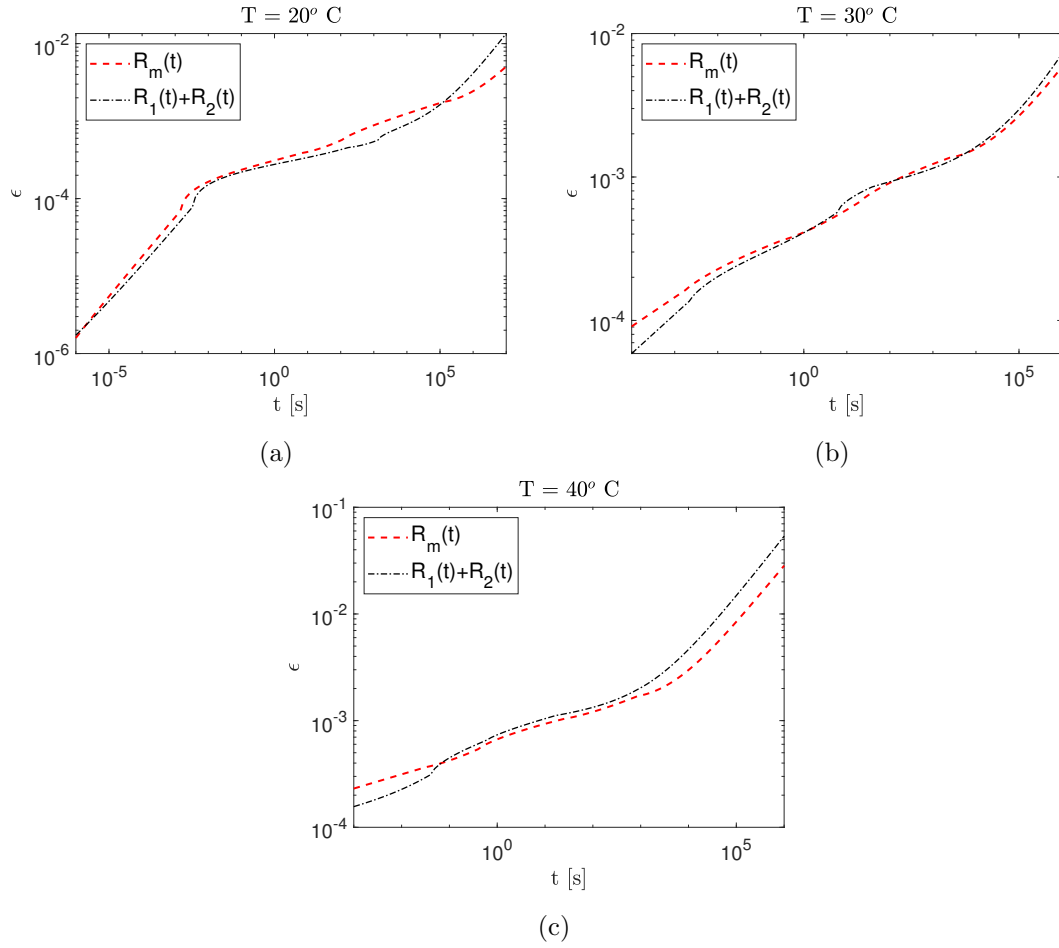


Figure 6.4: Deformation  $\epsilon(t)$  for the two different cases described in Figure 6.3, for the three environmental temperatures:  $20^\circ\text{C}$  (a),  $30^\circ\text{C}$  (b),  $40^\circ\text{C}$  (c).

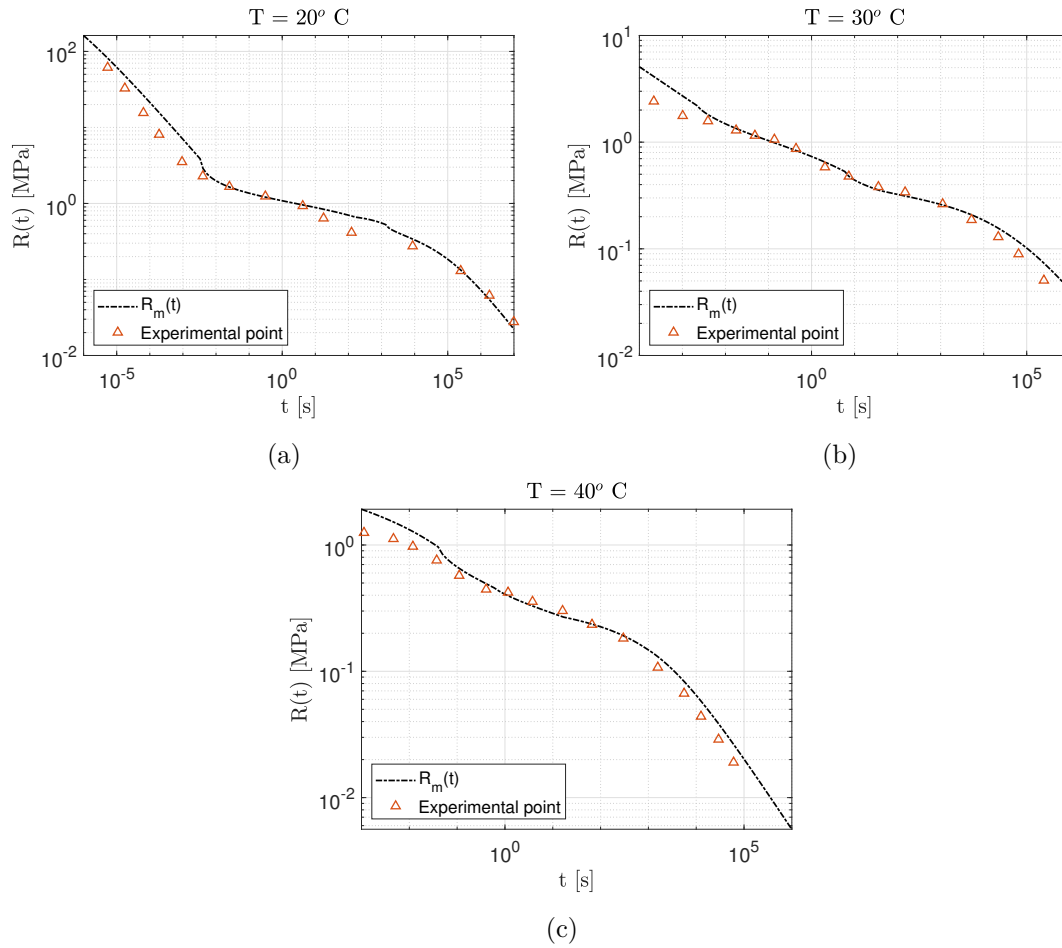


Figure 6.5: Simulated relaxation function of the MMPI obtained through the FV model (black dashed line), compared with the experimental points (red triangles) of the relaxation function of the same interlayer obtained through DTMA, at three different temperatures:  $20^\circ\text{C}$  (a),  $30^\circ\text{C}$  (b),  $40^\circ\text{C}$  (c). The polymeric materials used are Stiff PVB and Acoustic PVB.

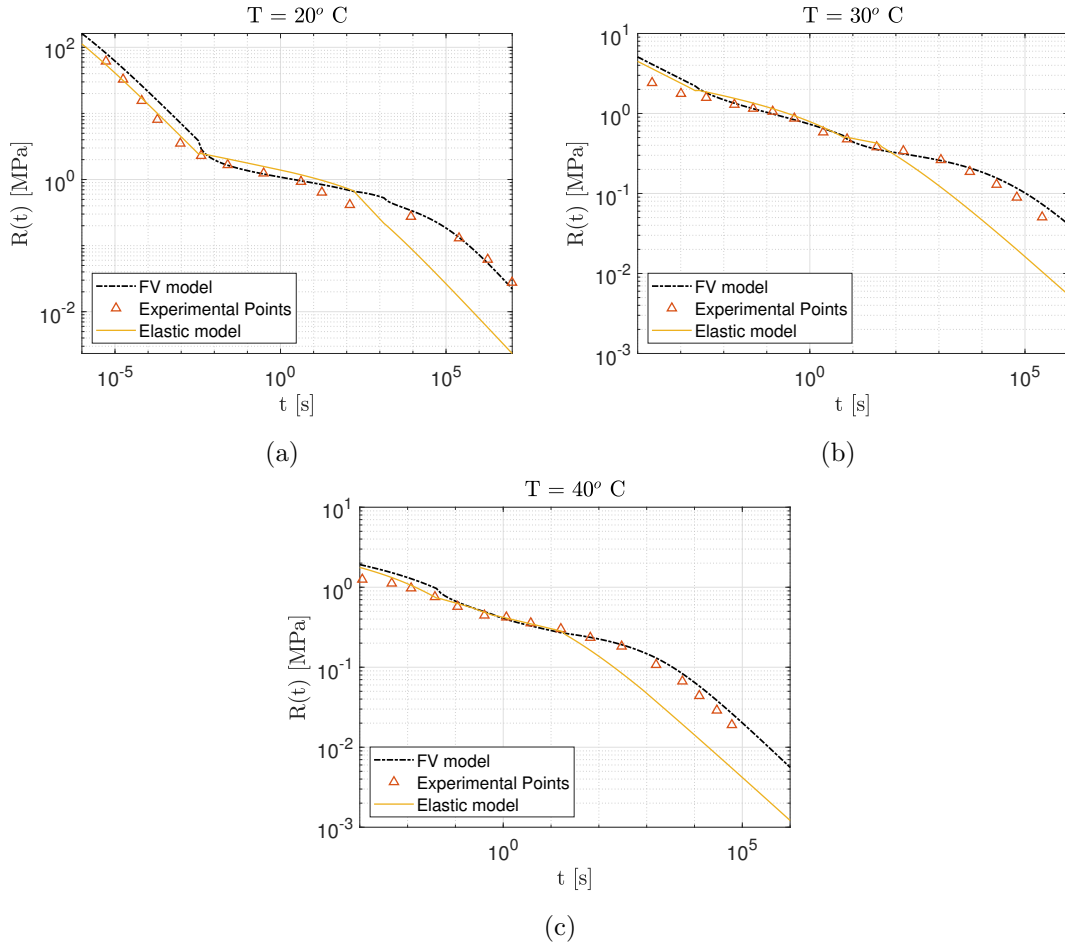


Figure 6.6: Simulated relaxation function of the MMPI obtained through the FV model (black dashed line), compared with the elastic model prediction (yellow line), and with the experimental points (red triangles) of the relaxation function of the same interlayer obtained via DTMA, at three different temperatures:  $20^\circ\text{C}$  (a),  $30^\circ\text{C}$  (b),  $40^\circ\text{C}$  (c). The polymeric materials used are Stiff PVB and Acoustic PVB.

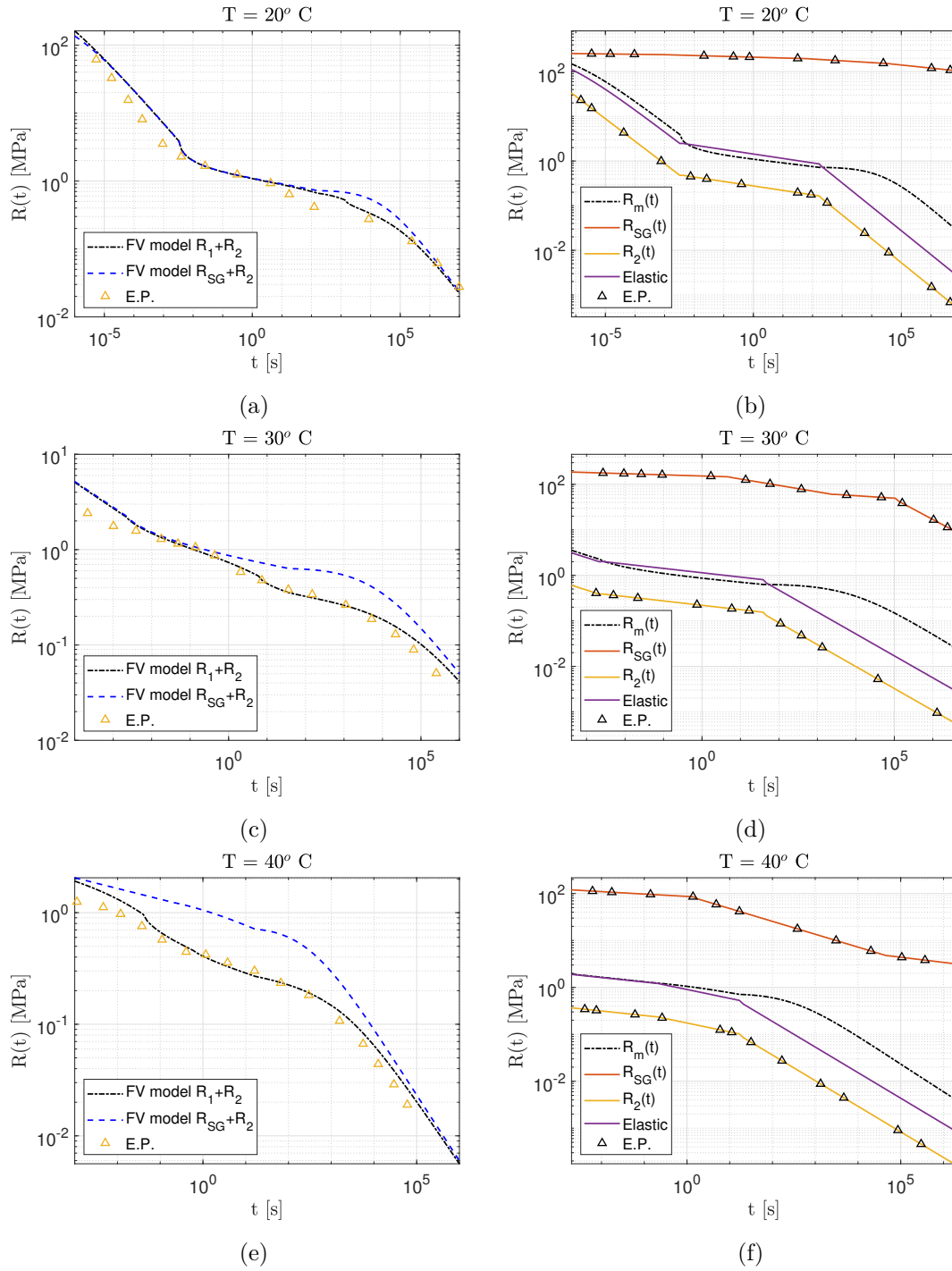


Figure 6.7: Simulated relaxation function of the MMPI obtained through the FV model, represented with black and blue dashed lines, and the simulated relaxation function obtained with the elastic model, represented with purple line. Comparison is made with the experimental points of the relaxation function of the same interlayer obtained through DTMA (yellow triangles), at different temperatures: 20°C (a, b), 30°C (c, d), 40°C (e, f).



## Chapter 7

# Conclusions

This research has addressed the complex mechanical behaviour of laminated glass structures by developing, validating, and implementing a fractional viscoelastic framework for the characterization and numerical modelling of polymeric interlayers. The work originated from the practical need to improve predictive capabilities in the design of glass components where the time-, temperature-, and rate-dependent response of interlayers plays a crucial role. Conventional approaches based on discrete Prony series or exponential kernels, although widely used, often require an excessive number of parameters and exhibit poor extrapolation properties outside the calibration range. In contrast, the present work has used a fractional constitutive formulation that captures the essential viscoelastic features of interlayers through compact, physically meaningful parameters.

The thesis has provided a comprehensive investigation that bridges theoretical formulation, numerical implementation, and experimental validation. Two different approaches for fitting the experimental relaxation curves of polymeric materials are presented and compared. The first is the classical *Prony series* model, while the second is an interpolation based on *power laws*. The latter offers several advantages over the conventional method, such as simpler and faster parameter calibration, as well as improved agreement with experimental observations, as already highlighted by Nutting. The analysis shows that the power-law fitting is not only more accurate but also remarkably efficient, as it can represent complex relaxation behaviors using only a few parameters. In this formulation, the relaxation function is expressed as a piecewise function composed of several power-law branches, whose number depends on the shape of the experimental curve. In a bi-logarithmic representation, each power-law branch corresponds to a straight line characterized by two parameters: its slope and its intercept. These two parameters can be determined through a straightforward geometric interpolation, eliminating the need for complex optimization algorithms typically required by the Prony series approach. By adopting the power-law fitting, the constitutive equation of the polymeric material can be expressed as a relationship between stress and strain involving a fractional derivative of order between 0 and 1 (corresponding to the slope of the power law), multiplied by a constant factor. This relation holds within each branch of the relaxation function,

but it can also be extended to a general relaxation law composed of multiple power-law segments. In the second part of this chapter, two main numerical techniques for approximating fractional derivatives are presented and compared, highlighting the advantages of the L1 formula over the classical Grünwald–Letnikov (GL) method. In particular, the L1 scheme enables the use of a non-uniform time discretization, which significantly reduces computational effort while maintaining high accuracy. This computational benefit will be quantified through a numerical example discussed in Chapter 3. The Mean Absolute Percentage Error (MAPE) between the two numerical methods is evaluated as a function of the fractional derivative order  $\alpha$ . The results show that the error ranges from 3% to 1%, reaching its maximum for  $\alpha = 0$  and  $\alpha = 1$ , and its minimum around  $\alpha \approx 0.6$ . These findings confirm the suitability of the L1 scheme for the numerical approximation of fractional derivatives in the structural models developed in subsequent chapters, where additional comparisons with the Grünwald–Letnikov method will also be discussed.

A numerical model for the simply supported inflexed beam problem, based on fractional calculus to describe the long-term viscoelastic response, is here proposed with particular emphasis on cold-bending applications. The approach exploits the fact that the relaxation functions of commercial interlayers are accurately represented by power laws, or by continuously connected branches of power laws. By employing the L1 formula for the approximation of fractional derivatives, the method allows for the adoption of variable time steps, finely resolving the early stages of transient deformations while efficiently capturing the long-term asymptotic response. This yields accurate predictions with a reduced computational effort compared to classical approaches, such as those relying on Prony series or the Grünwald–Letnikov scheme, which are penalized by the need for constant time discretization. The efficiency and robustness of the proposed method have been demonstrated in benchmark problems of laminated glass beams, where direct comparisons with ABAQUS simulations and quasi-elastic approximations confirmed the accuracy of the fractional model in predicting both short-term and long-term responses. Parametric studies, varying polymer type, interlayer thickness, load history and temperature, highlight the potential of this approach to guide optimization of cold-bending processes by minimizing stresses during transient stages. While the analyses presented were limited to beams, the methodology can be extended to laminated glass plates and, more generally, to other laminated and sandwich structures. Overall, this study confirms that fractional viscoelastic models combined with variable time-step integration represent a powerful and versatile framework, providing substantial advantages over classical formulations. They not only improve the predictive capability for laminated glass but also offer a broader potential for application to complex laminated systems in engineering and material science.

The proposed model is subsequently implemented within a finite element framework (called Finite Element Fractional Viscoelastic FEFV model, comparing its numerical solution with results from a creep four-point-bending test. In this case, the spatial discretization is treated within a finite element weak formulation derived from Hamilton's principle, while time integration of fractional derivatives is performed using the L1 for-

mula on non-uniform meshes, with all the advantages previously demonstrated. The relaxation curves for polymeric interlayers were experimentally determined with DTMA analysis. The results demonstrated the model's capability to reproduce both short-term elastic responses and long-term viscoelastic behaviour with high accuracy. Deviations between experiments and simulations were consistently small—below 1% in bending tests and within 5% in creep tests—despite geometric tolerances, model simplifications, and the modifications in polymer properties induced by autoclave lamination. These results highlight the robustness of the proposed fractional approach. The study demonstrates that the fractional formulation offers a natural and efficient means of describing the rheological behaviour of laminated glass, preserving both accuracy and computational efficiency across short- and long-term responses. These findings confirm the potential of fractional viscoelastic models to replace traditional Prony-based formulations and pave the way for their integration into next-generation design and analysis standards for laminated glass structures.

Beyond the basic mechanical characterization, the thesis extended the fractional framework to account for more advanced configurations relevant to industrial and architectural applications. These included the modelling of multi-material polymeric interlayers MMPI with temperature-dependent relaxation functions. The methodology is developed to evaluate the mechanical response of interlayers composed by different materials and to predict their effective relaxation functions with high accuracy. The strategy builds on the interpolation of experimental relaxation curves obtained from DTMA, which are represented by piecewise power-law segments. This formulation makes it possible to express the constitutive behaviour of the interlayer directly in terms of fractional derivatives, thereby offering a natural description of viscoelastic phenomena. Within this framework, stress is related to strain through a derivative of fractional order, lying between zero and one, which reflects the intermediate character of polymeric interlayers between purely elastic solids and purely viscous fluids. Beyond validation, the model demonstrates its potential as a practical design tool. It provides engineers and manufacturers with a flexible instrument for developing innovative laminated glass systems, including interlayers optimized for specific mechanical or acoustic requirements. Ultimately, the methodology offers a robust and interpretable framework for selecting and tailoring MMPI configurations, supporting the advancement of next-generation laminated glass technologies.

The methodological advances presented in this work have several noteworthy implications. From a scientific perspective, the adoption of fractional calculus provides a rigorous yet efficient framework for describing viscoelasticity across multiple time scales. From an engineering standpoint, the resulting model provides a tool capable of improving the accuracy of design simulations for laminated glass components, reducing the uncertainty associated with material characterization, and enabling more reliable safety and serviceability assessments. The formulation is fully compatible with standard finite element procedures, making its integration into industrial workflows straightforward. Overall, the research outcomes have met the objectives established at the outset of the project. The developed fractional viscoelastic model has proven capable of accurately describing

the time-dependent response of polymeric interlayers with minimal parameterization. Its implementation into the finite element framework enables predictive simulations of laminated glass structures under realistic conditions of load duration, temperature variation, and geometric complexity. The thesis has also demonstrated that fractional modelling is not only a theoretical alternative to classical approaches but a practical and scalable solution for engineering analysis.

## Future developments

While the work has established a solid foundation for the application of fractional viscoelasticity to laminated glass, several directions remain open for future exploration. One promising avenue concerns the full thermo-viscoelastic coupling of the model, incorporating temperature-dependent fractional operators to account for the simultaneous influence of thermal gradients and mechanical loads. This would extend the applicability of the framework to scenarios such as fire resistance analysis, automotive glazing, and photovoltaic glass systems. Another important development involves the extension of the formulation to nonlinear geometrical effects and large deformations, which are relevant in cold-bent or prestressed glass panels where strain levels exceed the linear range.

Further research could also focus on the stochastic characterization of material variability. Since the fractional parameters encapsulate complex molecular phenomena, integrating probabilistic descriptions could improve the reliability assessment of laminated systems under long-term environmental exposure. In addition, the model could be embedded into multiphysics simulations, coupling viscoelastic behaviour with fracture mechanics or moisture diffusion to predict durability and failure.

On the numerical side, there is room to optimize the adaptive L1 scheme for parallel computing architectures and to explore reduced-order formulations that preserve accuracy while minimizing computational cost. Such developments would support large-scale industrial simulations and enable real-time predictive tools for design optimization. Finally, the conceptual framework developed here — combining fractional operators with experimental calibration and finite element implementation — could be transferred to other material systems exhibiting hereditary behaviour, including polymer composites, adhesives, and biological tissues.

In summary, this research has demonstrated that fractional viscoelastic modelling provides a powerful, unified, and efficient tool for describing the time-dependent behaviour of laminated glass. The outcomes of this thesis contribute both to the advancement of the scientific understanding of polymeric interlayers and to the practical enhancement of design methodologies in modern glass engineering. The developed framework establishes a solid basis for future investigations and industrial applications aimed at improving the performance, safety, and sustainability of transparent structural systems.

## Chapter 8

# Appendix

### 8.1 Laplace Transform of Fractional Operators

The Laplace transform is a powerful mathematical tool widely employed in the analysis of linear systems and differential equations. By transforming a time-domain function  $f(t)$ , defined for  $t \geq 0$ , into a function  $F_{\mathcal{L}}(s)$  in the complex frequency domain, the Laplace operator simplifies the treatment of differential and integro-differential equations into algebraic relations. The Laplace transform of a function  $f(t)$  is defined as:

$$F_{\mathcal{L}}(s) = \mathcal{L}\{f(t); s\} = \int_0^{\infty} e^{-st} f(t) dt, \quad s \in \mathbb{C}, \Re(s) > 0. \quad (8.1)$$

This representation provides several advantages, such as the ability to handle initial conditions in a straightforward manner and to characterize system behavior in terms of poles and singularities of  $F_{\mathcal{L}}(s)$ .

In the framework of fractional calculus, the Laplace transform plays a fundamental role, as it allows the fractional derivatives and integrals of a function to be expressed in closed form. This makes it possible to extend classical methods of linear system analysis to models governed by fractional-order operators, which are frequently encountered in viscoelasticity, control theory, and anomalous diffusion phenomena.

#### 8.1.1 Laplace Transform of the Riemann–Liouville Fractional Derivative

By exploiting the properties of the Laplace transform  $\mathcal{L}$ , and assuming the lower bound of the fractional derivative to be  $a = 0$  with order  $\Re(\alpha) > 0$ , one can demonstrate that the Laplace transform of the Riemann–Liouville derivative is given by:

$$\mathcal{L}\{ {}_0D_t^{\alpha} f(t); s\} = s^{\alpha} F_{\mathcal{L}}(s) - \sum_{r=0}^{n-1} s^r \left[ {}_0D_t^{\alpha-r-1} f(t) \right]_{t=0}, \quad (n-1 \leq \alpha < n), \quad (8.2)$$

where  $F_{\mathcal{L}}(s)$  denotes the Laplace transform of  $f(t)$ .

This formulation has only limited applicability in engineering problems, mainly due to the lack of a clear physical interpretation of the fractional derivative at  $t = 0$ . This difficulty is particularly relevant when dealing with initial conditions, as the Riemann–Liouville derivative requires them to be expressed in terms of fractional integrals, which do not always have an immediate physical meaning.

### 8.1.2 Laplace Transform of the Caputo Fractional Derivative

In contrast to the Riemann–Liouville definition, the Caputo fractional derivative has the important advantage of retaining a physically interpretable meaning also at  $t = 0$ . For instance,  $f(0)$  represents the initial displacement,  $f'(0)$  the initial velocity, and  $f''(0)$  the initial acceleration. This feature makes the Caputo formulation particularly well suited for the solution of practical engineering problems involving fractional differential equations with constant coefficients, where initial conditions are naturally expressed in the classical form.

The Laplace transform of the Caputo fractional derivative is given by:

$$\mathcal{L}\left\{{}_0^C D_t^\alpha f(t); s\right\} = s^\alpha F_{\mathcal{L}}(s) - \sum_{r=0}^{n-1} s^{\alpha-r-1} f^{(r)}(0), \quad (n-1 < \alpha \leq n), \quad (8.3)$$

where  $F_{\mathcal{L}}(s)$  denotes the Laplace transform of  $f(t)$ , and  $f^{(r)}(0)$  are the integer-order derivatives of the function evaluated at  $t = 0$ .

It can be observed that, unlike the Riemann–Liouville case (Equation 8.2), here the initial conditions appear in the form of classical derivatives of integer order. This makes the Caputo formulation much more practical and widely used in applied problems, since it avoids the need for fractional initial conditions, which are often not physically measurable or meaningful.

### 8.1.3 Laplace Transform of the Grünwald–Letnikov Fractional Derivative

By applying the Laplace transform to the Grünwald–Letnikov fractional derivative, the following expression is obtained:

$$\mathcal{L}\left\{{}_0^{GL} D_t^\alpha f(t); s\right\} = \frac{f(0)}{s^{1-\alpha}} + \frac{1}{s^{1-\alpha}} [sF_{\mathcal{L}}(s) - f(0)] = s^\alpha F_{\mathcal{L}}(s), \quad (0 \leq \alpha < 1), \quad (8.4)$$

where  $F_{\mathcal{L}}(s)$  is the Laplace transform of  $f(t)$ .

In this formulation, the lower limit of integration has been assumed equal to zero ( $a = 0$ ), and the derivative order  $\alpha$  is considered within the interval  $0 \leq \alpha < 1$ , since no classical Laplace transform exists for values of  $\alpha > 1$ .

The Grünwald–Letnikov definition is often regarded as the most intuitive extension of the finite-difference scheme to fractional order, and its Laplace transform highlights the formal consistency of this operator with those of Riemann–Liouville and Caputo for the case  $0 < \alpha < 1$ .

## 8.2 Euler's Gamma Function

One of the fundamental functions of fractional calculus is Euler's Gamma Function, denoted by  $\Gamma(z)$ . This function generalizes the concept of factorials  $n!$ , extending their definition to non-integer and even complex values of  $n$ .

The introduction of the Gamma function can be traced back to a problem of interpolation posed in a letter from *Christian Goldbach* (1690–1764) to *Leonhard Euler* (1707–1783). The challenge was to “find a simple formula for computing factorials that could be extended to non-integer values”.

The Gamma function is defined in the right half-plane of positive  $z$  by the following integral:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \Re(z) > 0. \quad (8.5)$$

If  $z = x + iy$  is complex, then the function can be expressed as:

$$\Gamma(x + iy) = \int_0^\infty e^{-t} t^{x-1} [\cos(y \log t) + i \sin(y \log t)] dt. \quad (8.6)$$

This representation highlights that the Gamma function is well-defined in the right half-plane ( $\Re(z) > 0$ ). The integral converges for all real  $t > 0$  since the exponential term  $e^{-t}$  ensures decay at infinity, while near  $t = 0$  convergence is guaranteed when  $\Re(z) > 0$ .

The Gamma function is a *meromorphic* function, with simple poles at the non-positive integers. It plays a central role in fractional calculus because it naturally appears in the generalization of factorials and binomial coefficients, and in the definition of fractional integrals and derivatives.

In addition to the integral representation given in Equation 8.5, Euler's Gamma function can also be expressed in an alternative form provided by Gauss:

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{z(z+1) \dots (z+n)}. \quad (8.7)$$

From the integral definition, several remarkable identities follow directly, which are widely used in the evaluation of integrals. One of the most important is:

$$\sqrt{\pi} = \Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt. \quad (8.8)$$

Table 8.1 summarizes some notable values of the Gamma function for arguments between  $-\frac{3}{2}$  and 2.

|                                                           |                                                          |
|-----------------------------------------------------------|----------------------------------------------------------|
| $\Gamma\left(-\frac{3}{2}\right) = \frac{4}{3}\sqrt{\pi}$ | $\Gamma(1) = 1$                                          |
| $\Gamma(-1) = \pm\infty$                                  | $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$ |
| $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$          | $\Gamma(2) = 1$                                          |
| $\Gamma(0) = \pm\infty$                                   | $\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$ |
| $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$             | $\Gamma(3) = 2$                                          |

Table 8.1: Notable values of the Gamma function  $\Gamma(x)$  for  $-\frac{3}{2} \leq x \leq 2$ .

### Recursive Property

A fundamental property of the Gamma function is its recursive relation:

$$\Gamma(z+1) = z\Gamma(z). \quad (8.9)$$

This identity can be proved by integration by parts:

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^z dt = \left[ -e^{-t} t^z \right]_{t=0}^{t=\infty} + z \int_0^\infty e^{-t} t^{z-1} dt = z\Gamma(z). \quad (8.10)$$

From this property, and recalling that  $\Gamma(1) = 1$ , it follows that:

$$\Gamma(2) = 1 \cdot \Gamma(1) = 1!, \quad (8.11)$$

$$\Gamma(3) = 2 \cdot \Gamma(2) = 2!, \quad (8.12)$$

$$\Gamma(4) = 3 \cdot \Gamma(3) = 3!, \quad (8.13)$$

$$\vdots$$

$$\Gamma(n+1) = n\Gamma(n) = n!. \quad (8.14)$$

Thus, the Gamma function provides a natural extension of the factorial function to non-integer and even complex values.

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## ITALIAN VERSION

- **Ente finanziatore:** progetto finanziato dall'Unione Europea - NextGenerationEU – Piano Nazionale Ripresa e Resilienza (PNRR) - Missione 4 Componente 2 Investimento 3.3 “Introduzione di dottorati innovativi che rispondono ai fabbisogni di innovazione delle imprese e promuovono l’assunzione dei ricercatori dalle imprese” – Avviso N. 352 del 09/04/2022 del Ministero dell’Università e della Ricerca;
- **Numero Borsa:** 38-033-21-DOT13SJY60-2678, Decreto N. 352 in DATA adottata dal Ministero Italiano 09/04/2022, D92B22000670005.

