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Coactions of $E(n)$ on Clifford Algebras

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Contents

Introduction

One of the interesting features that motivates the study of Hopf algebras is the fact that their representation theory is particularly nice: given an Hopf algebra H, the category ${}_H\mathcal{M}$ of (left) H-modules is rigid monoidal. This means that the dual M^* of an H-module M is again an H-module and so is the tensor product $M \otimes N$ of two H-modules. The study of mathematical objects through their representations has proved to be an effective strategy bearing significant results in many fields: Group theory [**FH, Se**], Associative algebra theory [**CR, Pi**], Lie algebra theory [**H, Ki**], etc. Hopf algebra theory makes no exception. A fundamental result in this sense is the so-called *Tannaka-Krein duality*, which describes the interaction between an object and its category of representations, and that was first discovered in the area of Algebraic Topology and has been lately generalized in different branches of algebra.

With this motivation in mind, one could immediately decide to focus their efforts on the investigation of the category ${}_H\mathcal{M}$, in order to better understand the properties of H, but it turns out that this is not always the best option. There are contexts in which *dualizing* the problem makes it much easier to tackle it. Sometimes one should consider the dual space H^* (which is still an Hopf algebra when H is finite-dimensional) and consider the category $\mathcal{M}^{H^*}$ of (right) H^*-comodules (or *corepresentations* of H^*). The notion of comodule arises naturally by "reversing the arrows" in the diagrams that define a module, but, despite the naivety of the procedure, the new definition we obtain is of great importance. Indeed, when H is a finite-dimensional Hopf algebra we have that the categories of (left) H-modules and (right) H^*-comodules are isomorphic ${}_H\mathcal{M} \cong \mathcal{M}^{H^*}$ ([**CMZ**, p.10], [**Ma**, p.23]). On the other hand, comodules are not just a mere dualization of the pre-existing notion of module and some concrete examples can be found quite easily. For instance, let V be a graded vector space with index set I. Then we can consider the free vector space U with basis $\{u_i\}_{i\in I}$. U becomes a coalgebra with comultiplication $\Delta(u_i) = u_i \otimes u_i$ and $\varepsilon(u_i) = 1$ for every $i \in I$ and V becomes a U-comodule via $\rho(v) = \sum_{i\in I} v_i \otimes u_i$, where $v = \sum_{i\in I} v_i$ is the decomposition of v into homogeneous components.

Depending on the presented problem, the properties of H and the particular structure of the involved objects, one could decide to investigate one (or more than one) of the categories${}_H\mathcal{M}$, ${}^*_H\mathcal{M}$, $\mathcal{M}^H$, $\mathcal{M}^{H^*}$ and possibly the intervening relations among them. Since these are all monoidal categories, it is also possibile to search for (co)monoids (or (co)algebra objects) in such categories and determine their properties as well. These objects are what is usually called an *H-(co)module algebra* (or a (co)action of H). Two kinds of comodule algebras with additional properties are particularly important: cleft extensions and Hopf-Galois extensions (see e.g. [**Mo**, Ch.8]). We will not really take into account these notions in our thesis, but it is worth mentioning that the study on $E(n)$-comodule algebras herein contained also stemmed from [**PVO2**], where it is proved that all $E(n)$-cleft (resp. Hopf-Galois) extension of the ground field k are isomorphic to a Clifford algebra (see Corollary 2 and Remark 5 ibid.). The study of actions and coactions of Hopf algebras on rings and associative algebras has been a matter of interest for mathematician all around the world for at least four decades [**BeCo, CY, DT, Mas, MS**] and in our particular case, has been motivated by two main reasons.

The first was to answer to a question that was brought to my attention by C. Menini and B. Torrecillas and that originated from an example contained in their article [**MT1**]. In that paper Menini and Torrecillas were concerned with the study of (h-)separable coalgebras in monoidal categories and gave non-trivial examples of (h-)separable coalgebras in the monoidal category $\mathcal{T}^{\#}_{A\otimes H^{op}}$ (see Def. 3.2) for $H = H_4$ the four-dimensional Sweedler's Hopf algebra and $A = Cl(\alpha, \beta, \gamma)$ a Clifford algebra. These particular coalgebras are called *cowreath* and can be constructed in a natural way when we have a Hopf algebra H and an H-comodule algebra A. The Clifford algebra $A = Cl(\alpha, \beta, \gamma)$ has a canonical H_4-comodule algebra structure that was described in [**PVO2**]. The cowreath induced by this coaction is (h-)separable provided α, β and γ satisfy the conditions described in [**MT1**, Thm. 6.1]. Nevertheless, such conditions were forced by the particular form of the Casimir element used to realize (h-)separability and that was chosen in order to simplify calculations. At the time we did not know whether such a cowreath could be proved (h-)separable with no restriction whatsoever on the scalars α, β and γ by just changing the H-comodule algebra structure of A or the form of the Casimir element, so we decided to explore both possibilities. Now we know that even if we change the comodule algebra structure of $A = Cl(\alpha, \beta, \gamma)$, the induced cowreath is still (h-)separable via a Casimir element of the form fixed in [**MT1**, Thm. 6.1] only if $\gamma^2 - 4\alpha\beta = 0$, i.e. only if A is non-semisimple (see Theorem 6.2). On the other hand, Menini and Torrecillas were able to prove (with remarkable effort) that if we keep on A the canonical H_4-comodule algebra structure the induced cowreath is always (h-)separable, though via a Casimir element of a much more general form [**MT2**, Thms. 1,2]. In order to prove Theorem 6.2 we needed to obtain a complete classification of H_4-coactions on $A = Cl(\alpha, \beta, \gamma)$. A first step in this direction had already been taken in [**FR**], where a result similar to [**MT1**, Thm. 6.1] was obtained using a different coaction (see Thm. 6.2, Ch.1), but we were clearly far from proving a general result. Thus, inspired by [**CY, MS**] we began to understand that each H_4-coaction on a finite-dimensional algebra is completely determined by the choice of an involution φ and a φ-derivation d satisfying appropriate conditions, and that to obtain a full classification of H_4-coactions on a finite-dimensional algebra was equivalent to have a full perspective of its involutions and skew-derivations (Theorem 1.8). This gave us a second reason to pursue our goal: to gain further knowledge of Clifford algebras.

Outline of the Thesis. Chapter 1 is mostly introductory and contains all the preliminaries needed to fully understand the sequel. In the first section we briefly recall the definitions of (co)algebras and (co)modules in a monoidal category. Section 2 is devoted to the concept of separability and its categorical interpretation. We recall that the separability of the forgetful functor $F : \mathcal{M}^C \to \mathcal{M}$ is equivalent to the existence of a normalized Casimir morphism for the coalgebra C (Theorem 2.9), i.e. to the coseparability of the coalgebra C. Similarly we point out that h-separability of the forgetful functor F is equivalent to h-coseparability of C (Theorem 2.11). Section 3 starts with the definition of cowreath and proceeds with a list of analogous results for the case where $F : \mathcal{M}(\psi)_A^X \to \mathcal{M}_A$ is the forgetful functor from the category of entwined modules over a cowreath (A, X, ψ) to the category of A-modules. In Section 4 we focus on a cowreath $(A \otimes H^{op}, H, \psi)$ obtained considering a Hopf algebra H and a H-comodule algebra (A, ρ). As proved in Proposition 4.11, H can be endowed with the structure of $H \otimes H^{op}$-module coalgebra and $A \otimes H^{op}$ with the structure of $H \otimes H^{op}$-comodule algebra. By defining $\psi : H \otimes A \otimes H^{op} \to A \otimes H^{op} \otimes H$ as in (66), this gives rise to a cowreath $(A \otimes H^{op}, H, \psi)$ (see Proposition 4.12). We recall that the separability (resp. h-separability) of such a cowreath via a Casimir morphism $B : H \otimes H \to A \otimes H^{op}$ of the form $h \otimes h' \mapsto B^A(h \otimes h') \otimes 1_H$ is equivalent to the set of conditions (74), (75), (76) (resp. to separability and the further condition (77)). All these results were proved in [**MT1**] and are essential to understand how we obtained our own results on separable cowreaths in Chapter 3. Section 5 is about Clifford algebras. After a brief introduction on the subject we show that the family of algebras defined in [**PVO2**] called "of Clifford-type" are actually classical Clifford algebras and we discuss about orthogonal generators, semisimplicity and the Jacobson radical of these algebras. We conclude this section by proving that the only Clifford algebras admitting a Hopf algebra structure are the Hopf algebras $E(n)$ (Theorem 5.23). The last section of this chapter contains the main original result of my Master's thesis (Theorem 6.2), [**MT1**, Thm. 6.1] and a couple of corollaries that further corroborate our motivation to investigate $E(n)$-coactions on Clifford algebras.

Chapter 2 contains all the steps we followed to obtain a complete classification of H_4-coactions on a four-dimensional Clifford algebra $A = Cl(\alpha, \beta, \gamma)$. In Section 1 we show that there is a bijective correspondence between H_4-coactions and pairs (φ, d) where φ is an involution of A an d is a φ-derivation of A such that $d^2 \equiv 0$ and $d\varphi = -\varphi d$. Sections 2 and 3 are devoted to the individuation of such maps. Our approach was that of regarding involutions and skew-derivations as k-linear maps verifying some further algebraic relations. For instance, an involution φ on a four-dimensional Clifford algebra $A = Cl(\alpha, \beta, \gamma)$ can be interpreted as a k-linear map satisfying $\varphi^2 = \mathrm{Id}$ and further conditions (100)-(104). In this sense, involutions can be regarded as matrices whose eigenvalues are ± 1 and whose entries also satisfy a family of quadratic relations. The calculations carried out in this section were simplified with the tools developed in Appendix A and verified with the help of a computer software. In the last part of Section 2 we also obtain a refined classification of involutions of A, that make a distinction between inner and non-inner automorphisms. In Section 4 we determine the space of coinvariants for every H_4-coactions as a straightforward application and we conclude the chapter with Section 5, where a summarizing table is displayed together with some examples of H_4-coactions.

Chapter 3 presents some applications of the classification found in Chapter 2. In Section 1 we study the separability of the cowreath $(A \otimes H_4^{op}, H_4, \psi)$ induced by any H_4-coaction on $A = Cl(\alpha, \beta, \gamma)$, by means of Proposition 4.14. It is fundamental to recall that in this proposition the Casimir element that realizes separability is of a specific form. The fact that a cowreath is not separable via a Casimir element of this peculiar form does not imply that it is not separable in general (cf. [**MT2**]) and that is why we introduced the notion of rt-separability (see Def. 4.16) to distinguish between the cases. Section 2 is the natural continuation of the former and is concerned with the study of rth-separability. In both sections examples of coactions inducing rt(h-)separable cowreaths are presented. Section 3 contains summarizing tables with every coaction that makes $(A \otimes H_4^{op}, H_4, \psi)$ rt(h-)separable. As an additional application we also derive a complete characterization of H_4-coactions that allow us to construct a Frobenius cowreath (see Sections 4 and 5). In Section 6 we give a negative answer to Question 6.6 and we use it to prove Theorem 6.2 which was our first goal. We close Chapter 3 with a section where we address the case of a cowreath that is both Frobenius and rt-separable. We prove that there is no H_4-coaction on $A = Cl(\alpha, \beta, \gamma)$ such that the induced cowreath is both Frobenius and rt-separable (Theorem 7.2).

In Chapter 4 we extend some of the results obtained in Chapter 2 to algebras in higher dimension. In Section 1 we show that $E(n)$-coactions over a finite-dimensional algebra A are classified by tuples $(\varphi, d_1, \ldots, d_n)$ consisting of an involution φ and a family $(d_i)_{i=1,\ldots,n}$ of φ-derivations satisfying appropriate conditions (see Thm. 1.6). Since a complete classification of involutions and skew-derivations seems currently out of reach for a general Clifford algebra $A = Cl(\alpha, \beta_i, \gamma_i, \lambda_{ij})$, in Section 2 we specialize our main result for the case when A is a simple algebra (Thm. 2.2). In this instance a full classification of $E(n)$-coactions becomes roughly equivalent to the understanding of the structure of a particular subset of A that contains every element whose square is contained in $\mathcal{Z}(A)$.

In Section 1 of Chapter 5 we introduce the notion of *equivalent tuples*, which is the appropriate counterpart of isomorphic comodule algebras, and that will be used in the following sections to determine a much more refined classification of H_4-coactions in the four-dimensional case. In Section 2 we prove that for every Clifford algebra $A = Cl(\alpha, \beta, \gamma)$ there are only at most two coactions (up to isomorphism) such that the associated involution is of type $\mathfrak{F}_1$ (which means that the associated involution has determinant -1). In Section 3 we deal with coactions of type 2 (i.e. with associated involution $\varphi \neq \mathrm{Id}$ and such that $\det \varphi = 1$). We show that when $A = Cl(0, 0, 0)$ is the Exterior algebra the number of (non-isomorphic) H-coactions one can define is *finite* (see Table 2). By investigating the rest of the non-semisimple cases we prove that the list of (non-isomorphic) coactions on A is still infinite, but of a much more compact form (see Tables 4, 6 and 8). Unfortunately we do not have a complete answer for the semisimple case. Nonetheless it should be noted that when A is semisimple it is either a central simple algebra or the product of two isomorphic central simple algebras (see Thm. 5.16). By the Skolem-Noether Theorem, every involution and every skew-derivation of a simple algebra is inner and therefore of a more manageable form. To define a coaction, by means of Theorem 1.6, using only inner involutions and skew-derivations is indeed the easiest part of the problem. What is really challenging is to determine those that do not correspond to tuples with inner entries. With this motivation in mind we focused our last efforts on determining all the equivalent pairs in the non-semisimple case.

CHAPTER 1

Structures and properties involved: a categorical approach

Most of the contents of this introductory chapter are taken from [**BT, BCT1, MT1, FR**]. Proofs of results are usually not given, the reader interested in details is referred to these papers.

1. The setting of monoidal categories

In order to fully understand the motivation that led us in this research work we need to approach things from the perspective of Category Theory. For the basic definitions, as those of category, covariant and contravariant functor, natural transformation, adjunctions, etc. we refer the reader to [**Bo**], while we explicitly recall that of a monoidal category.

Definition 1.1. A *monoidal category* is a category $\mathcal{M}$ endowed with an object $\underline{1} \in \mathcal{M}$ (called the *unit object* of $\mathcal{M}$), a bifunctor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ (called *tensor product*) and natural equivalences

$$\begin{aligned} a_{-,-,-} : (- \otimes -) \otimes - &\xrightarrow{\cong} - \otimes (- \otimes -), \\ l_- : \underline{1} \otimes - &\xrightarrow{\cong} - \\ r_- : - \otimes \underline{1} &\xrightarrow{\cong} - \end{aligned}$$

such that

$$\begin{aligned} a_{X,Y,Z} : (X \otimes Y) \otimes Z &\xrightarrow{\cong} X \otimes (Y \otimes Z), \\ l_X : \underline{1} \otimes X &\xrightarrow{\cong} X \\ r_X : X \otimes \underline{1} &\xrightarrow{\cong} X \end{aligned}$$

for every X, Y, $Z \in \mathcal{M}$. The natural transformation $a_{-,-,-}$ is called the *associativity constraint* and satisfies the so-called

Pentagon axiom. The diagram

$$\begin{array}{ccccc} & & (W \otimes (X \otimes Y)) \otimes Z & & \\ & \swarrow^{a_{W,X,Y} \otimes \mathrm{Id}_Z} & & \searrow^{a_{W \otimes X,Y,Z}} & \\ ((W \otimes X) \otimes Y) \otimes Z & & & & (W \otimes X) \otimes (Y \otimes Z) \\ \downarrow{\scriptstyle a_{W,X \otimes Y,Z}} & & & & \downarrow{\scriptstyle a_{W,X,Y \otimes Z}} \\ W \otimes ((X \otimes Y) \otimes Z) & & \xrightarrow{\mathrm{Id}_W \otimes a_{X,Y,Z}} & & W \otimes (X \otimes (Y \otimes Z)) \end{array}$$

commutes for every X, Y, Z, $W \in \mathcal{M}$. The natural transformations l_- and r_- are called *left* and *right unit constraint* and satisfy the so-called

Triangle axiom. The diagram

$$\begin{array}{ccc} (X \otimes \underline{1}) \otimes Y & \xrightarrow{a_{X,\underline{1},Y}} & X \otimes (\underline{1} \otimes Y) \\ & \searrow_{r_X \otimes Y} & \downarrow{\scriptstyle X \otimes l_Y} \\ & & X \otimes Y \end{array}$$

commutes for every X, $Y \in \mathcal{M}$.

A monoidal category $\mathcal{M}$ is called *strict* when the natural equivalences $a_{-,-,-}$, l_- and r_- are the identity natural transformation.

One of the reasons why monoidal categories are of interest is that the notions of algebra, module over an algebra, coalgebra and comodule over a coalgebra (as known in the context of vector spaces) can be introduced in the general setting of monoidal categories. As it is noticed in [**Ma**, p. 420], the Pentagon Axiom solves the consistency problem that appears because there are two ways to go from $((U \otimes V) \otimes W) \otimes X$ to $U \otimes (V \otimes (W \otimes X))$. The *Coherence Theorem*, due to S. Mac Lane [**Mac**, Chapter VII, Section 2], solves the similar problem for the tensor product of an arbitrary number of objects in $\mathcal{M}$. Accordingly with this theorem, we can always omit all brackets and simply write $X_1 \otimes \ldots \otimes X_n$ for any object obtained from $X_1, \ldots, X_n$ by using $\otimes$ and brackets. Also as a consequence of the coherence theorem, the morphisms $a_{-,-,-}$, l_-, r_- take care of themselves, so they can be omitted in any computation involving morphisms in $\mathcal{M}$. Thus, for sake of simplicity, from now on we will omit the associativity and unit constraints unless needed to clarify the context.

1.1. Algebras and coalgebras in monoidal categories.

Definition 1.2. An *algebra* in a monoidal category $\mathcal{M}$ is a triple (A, m, u), where A is an object in $\mathcal{M}$ and $m : A \otimes A \to A$ (the *multiplication*) and $u : \underline{1} \to A$ (the *unit*) are morphisms in $\mathcal{M}$ such that the following diagrams commute

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes A} & A \otimes A \\ {\scriptstyle A \otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow{m} & A \end{array} \qquad \begin{array}{ccccc} \underline{1} \otimes A & \xrightarrow{u \otimes A} & A \otimes A & \xleftarrow{A \otimes u} & A \otimes \underline{1} \\ & {\scriptstyle l_A}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle r_A} & \\ & & A & & \end{array}$$

i.e. such that m and u satisfy the *associativity* and *unit conditions*

$$m \circ (m \otimes A) = m \circ (A \otimes m) \tag{1}$$

and

$$m \circ (u \otimes A) = l_A, \quad m \circ (A \otimes u) = r_A. \tag{2}$$

Definition 1.3. Given two algebras (A, m_A, u_A) and $(A', m_{A'}, u_{A'})$, a morphism of algebras $f : A \to A'$ is a morphism in $\mathcal{M}$ such that the following diagrams commute

$$\begin{array}{ccc} A \otimes A & \xrightarrow{f \otimes f} & A' \otimes A' \\ {\scriptstyle m_A}\downarrow & & \downarrow{\scriptstyle m_{A'}} \\ A & \xrightarrow{f} & A' \end{array} \qquad \begin{array}{ccc} \underline{1} & \xrightarrow{u_A} & A \\ & {\scriptstyle u_{A'}}\searrow & \downarrow{\scriptstyle f} \\ & & A' \end{array}$$

i.e. such that the following conditions hold

$$f \circ m_A = m_{A'} \circ (f \otimes f) \tag{3}$$

and

$$f \circ u_A = u_{A'}. \tag{4}$$

Definition 1.4. Given an algebra (A, m, u) in a monoidal category $\mathcal{M}$, a *right A-module* is an object M in $\mathcal{M}$, equipped with a morphism

$$\mu_M : M \otimes A \to M$$

such that the following diagrams commute

$$\begin{array}{ccc} M \otimes A \otimes A & \xrightarrow{\mu_M \otimes A} & M \otimes A \\ {\scriptstyle M \otimes m}\downarrow & & \downarrow{\scriptstyle \mu_M} \\ M \otimes A & \xrightarrow{\mu_M} & M \end{array} \qquad \begin{array}{ccc} M \otimes \underline{1} & \xrightarrow{M \otimes u} & M \otimes A \\ {\scriptstyle r_M}\searrow & & \swarrow{\scriptstyle \mu_M} \\ & M & \end{array}$$

i.e.

$$\mu_M \circ (\mu_M \otimes A) = \mu_M \circ (M \otimes m) \tag{5}$$

and

$$\mu_M \circ (M \otimes u) = r_M. \tag{6}$$

Definition 1.5. Given a morphism $f : M \to N$ between two right A-modules M and N, f is called *A-linear* if the following diagram commutes

$$\begin{array}{ccc} M \otimes A & \xrightarrow{\mu_M} & M \\ {\scriptstyle f\otimes A}\downarrow & & \downarrow{\scriptstyle f} \\ N \otimes A & \xrightarrow{\mu_N} & N \end{array}$$

i.e. if

$$f \circ \mu_M = \mu_N \circ (f \otimes A). \tag{7}$$

Given an algebra A, the category of all right A-modules in $\mathcal{M}$ together with A-linear morphisms will be denoted $\mathcal{M}_A$.

Remark 1.6. One can define in an similar way the category ${}_A\mathcal{M}$ of left A-modules with (left) A-linear morphisms.

Remark 1.7. For every object $M \in \mathcal{M}$, and for every algebra $A \in \mathcal{M}$ we have $(M \otimes A, M \otimes m) \in \mathcal{M}_A$.

Definition 1.8. Let A, A', be algebras in a monoidal category $\mathcal{M}$. A left A-module M which is also a right A'-module and such that $(am) \cdot_{A'} a' = a \cdot_A (ma')$ for every $a \in A$, $m \in M$ and $a' \in A'$, will be called a A-A'-bimodule. The category of all A-A'-bimodules in $\mathcal{M}$ together with left A-linear and right A'-linear morphisms will be denoted ${}_A\mathcal{M}_{A'}$.

Definition 1.9. A *coalgebra* in a monoidal category $\mathcal{M}$ is a triple (C, Δ, ε), where C is an object in $\mathcal{M}$ and $\Delta : C \to C \otimes C$ (the *comultiplication*) and $\varepsilon : C \to \underline{1}$ (the *counit*) are morphisms in $\mathcal{M}$ such that the following diagrams commute

$$\begin{array}{ccc} C \otimes C \otimes C & \xleftarrow{\Delta\otimes C} & C \otimes C \\ {\scriptstyle C\otimes\Delta}\uparrow & & \uparrow{\scriptstyle \Delta} \\ C \otimes C & \xleftarrow{\Delta} & C \end{array} \qquad \begin{array}{ccccc} \underline{1} \otimes C & \xleftarrow{\varepsilon\otimes C} & C \otimes C & \xrightarrow{C\otimes\varepsilon} & C \otimes \underline{1} \\ & \searrow{\scriptstyle l_C} & \uparrow{\scriptstyle \Delta} & \swarrow{\scriptstyle r_C} & \\ & & C & & \end{array}$$

i.e. such that they satisfy the *coassociativity* and *counit conditions*

$$(\Delta \otimes C) \circ \Delta = \Delta \circ (C \otimes \Delta) \tag{8}$$

and

$$l_C \circ (\varepsilon \otimes C) \circ \Delta = \mathrm{Id}_C = r_C \circ (C \otimes \varepsilon) \circ \Delta. \tag{9}$$

Definition 1.10. Given two coalgebras $(C, \Delta_C, \varepsilon_C)$ and $(C', \Delta_{C'}, \varepsilon_{C'})$, a morphism of coalgebras $f : C \to C'$ is a morphism in $\mathcal{M}$ such that the following diagrams commute

$$\begin{array}{ccc} C' \otimes C' & \xleftarrow{f\otimes f} & C \otimes C \\ {\scriptstyle \Delta_{C'}}\uparrow & & \uparrow{\scriptstyle \Delta_C} \\ C' & \xleftarrow{f} & C \end{array} \qquad \begin{array}{ccc} \underline{1} & \xleftarrow{\varepsilon_{C'}} & C' \\ & {\scriptstyle \varepsilon_C}\nwarrow & \uparrow{\scriptstyle f} \\ & & C \end{array}$$

i.e. such that the following conditions hold

$$(f \otimes f) \circ \Delta_C = \Delta_{C'} \circ f \tag{10}$$

and

$$\varepsilon_{C'} \circ f = \varepsilon_C. \tag{11}$$

Definition 1.11. Given a coalgebra (C, Δ, ε) in a monoidal category $\mathcal{M}$, a *right C-comodule* is an object M in $\mathcal{M}$, equipped with a morphism

$$\rho_M : M \to M \otimes C$$

such that the following diagrams commute

$$\begin{array}{ccc} M & \xrightarrow{\rho_M} & M \otimes C \\ {\scriptstyle \rho_M}\downarrow & & \downarrow{\scriptstyle \rho_M \otimes C} \\ M \otimes C & \xrightarrow{M \otimes \Delta} & M \otimes C \otimes C \end{array} \qquad \begin{array}{ccc} M \otimes \underline{1} & \xleftarrow{M \otimes \varepsilon} & M \otimes C \\ & {\scriptstyle r_M}\searrow \quad \nearrow{\scriptstyle \rho_M} & \\ & M & \end{array}$$

i.e.

$$(\rho_M \otimes C) \circ \rho_M = (M \otimes \Delta) \circ \rho_M \tag{12}$$

and

$$r_M \circ (M \otimes \varepsilon) \circ \rho_M = \mathrm{Id}_M. \tag{13}$$

Definition 1.12. Given a morphism $f : M \to N$ between two right C-comodules M and N, f is called *C-colinear* if the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{\rho_M} & M \otimes C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \otimes C} \\ N & \xrightarrow{\rho_N} & N \otimes C \end{array}$$

i.e. if

$$\rho_N \circ f = (f \otimes C) \circ \rho_M. \tag{14}$$

Given a coalgebra C, the category of all right C-comodules in $\mathcal{M}$ together with C-colinear morphisms will be denoted $\mathcal{M}^C$.

Remark 1.13. One can define in an similar way the category ${}^C\mathcal{M}$ of left C-comodules with C-colinear morphisms.

Definition 1.14. Let C, C', be coalgebras in a monoidal category $\mathcal{M}$. A left C-comodule (M, ρ) which is also a right C'-comodule - with structure map ρ' - and such that

$$(\mathrm{Id}_C \otimes \rho') \circ \rho = (\rho \otimes \mathrm{Id}_{C'}) \circ \rho'$$

will be called a C-C'-bicomodule. The category of all C-C'-bicomodules in $\mathcal{M}$ together with left C-linear and right C'-linear morphisms will be denoted ${}^C\mathcal{M}^{C'}$.

2. Separability and Coseparability

The concept of *separable* algebras is a generalization of that of separable field extensions and as such was first defined in the context of vector spaces (cf. [**AG**]). In fact it is usually said that a k-algebra A is separable if the multiplication map $m_A : A \otimes A \to A$ admits a section $\gamma : A \to A \otimes A$ which is also a morphism of A-A-bimodules. This is easily generalized to the case of algebras in a monoidal category.

2.1. Separability in monoidal categories. As introduced in [**NBO**] and reported in [**BT**]:

Definition 2.1. [**BT**, Def. 4.1] Let $\mathcal{M}$ be a monoidal category and A an algebra in $\mathcal{M}$. A is called *separable* if there exists a morphism $\gamma : A \to A \otimes A$ of A-A-bimodules such that $m_A \circ \gamma = \mathrm{Id}_A$, where both A and $A \otimes A$ are viewed as A-bimodules via the multiplication m_A of A.

Moreover, when adopting the categorical approach, separability of an object can be interpreted also as a property of a certain forgetful functor. For this we need to recall the notion of separable functor, also introduced in [**NBO**].

Definition 2.2. [**NBO**, p. 398] For every functor $F : \mathcal{M} \to \mathcal{D}$ we set, for every X, $Y \in \mathcal{M}$,

$$\begin{array}{rccc} F_{X,Y}: & \mathrm{Hom}_{\mathcal{M}}(X, Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(FX, FY) \\ & f & \longmapsto & Ff \end{array}$$

The functor F is called *separable* if the canonical map $F_{X,Y}$ *cosplits naturally* i.e. there is a natural transformation

$$P_{-,-} := P^F_{-,-} : \mathrm{Hom}_{\mathcal{D}}(F,F) \to \mathrm{Hom}_{\mathcal{M}}(\mathrm{Id}_{\mathcal{M}}, \mathrm{Id}_{\mathcal{M}})$$

such that for every $X, Y \in \mathcal{M}$ the map

$$P_{X,Y} := P^F_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(FX,FY) \to \mathrm{Hom}_{\mathcal{M}}(X,Y)$$

satisfies $P_{X,Y} \circ F_{X,Y} = \mathrm{Id}_{\mathrm{Hom}_{\mathcal{M}}}$.

Now, given some additional property on the unit object $\underline{1}$ of the category, it is possible to derive a series of equivalent condition to separability of an algebra A.

DEFINITION 2.3. [**BT**, Def. 3.1] An object P of a monoidal category $\mathcal{M}$ is a *left $\otimes$-generator* of $\mathcal{M}$ if, given morphisms $f, g : Y \otimes Z \to W$ in $\mathcal{M}$ such that $f \circ (\lambda \otimes Z) = g \circ (\lambda \otimes Z)$ for all $\lambda : P \to Y$ in $\mathcal{M}$, we then have $f = g$.

PROPOSITION 2.4. [**BT**, Prop. 4.3] *Let $\mathcal{M}$ be a monoidal category such that $\underline{1}$ is a left $\otimes$-generator for $\mathcal{M}$. Then the following are equivalent*

(1) A is separable.
(2) The forgetful functor $F : \mathcal{M}_A \to \mathcal{M}$ is separable.
(3) There exists a morphism $e : \underline{1} \to A \otimes A$ in $\mathcal{M}$, such that

$$(m_A \otimes Id_A) \circ (Id_A \otimes e) = (Id_A \otimes m_A) \circ (e \otimes Id_A) \quad and \quad m_A \circ e = u_A.$$

Such morphism is called the separability morphism *of A.*

Similarly, if one introduces the definition of projective object in a category, separability of an algebra A can be characterized in yet another way.

DEFINITION 2.5. An object P in a category $\mathcal{M}$ is projective if for any morphism $f : P \to B$ and any epimorphism $q : A \to B$, there exists a morphism $p : P \to A$ such that $f = q \circ p$.

PROPOSITION 2.6. [**BT**, Prop. 4.5] *Let $\mathcal{M}$ be a monoidal category such that $\underline{1}$ is a projective object in $\mathcal{M}$. Then an algebra A in $\mathcal{M}$ is separable if, and only if, A is projective as an A-A-bimodule.*

If one wants to dualize Definition 2.3 and Proposition 2.4 to the case of coalgebras, the definition of coseparable coalgebra needs to be presented. We give the one introduced by Larson in [**Lar**] and also reported in [**BCT1**].

DEFINITION 2.7. [**Lar**, p.262] A coalgebra C is coseparable if it is a relative injective C-bicomodule in C, which comes down to the following property. If $i : M \to N$ in ${}^C\mathcal{M}^C$ has a left inverse $p : M \to N$ in $\mathcal{M}$, then every $f : M \to C$ in ${}^C\mathcal{M}^C$ factors through i in ${}^C\mathcal{M}^C$: there exists a C-bicolinear morphism $g : N \to C$ such that $g \circ i = f$.

Bulacu, Caenepeel and Torrecillas outline the fact that a dual result to Proposition 2.4 can be proved.

PROPOSITION 2.8. [**BCT1**, Prop. 7.3] *Let $\mathcal{M}$ be a monoidal category. Then the following are equivalent*

(1) C is coseparable.
(2) There exists a morphism $\gamma : C \otimes C \to C$ of bicomodules such that $\gamma \circ \Delta = Id_C$.
(3) There exists a morphism $B : C \otimes C \to \underline{1}$ in $\mathcal{M}$, such that

$$(Id_C \otimes B) \circ (\Delta \otimes Id_C) = (B \otimes Id_C) \circ (Id_C \otimes \Delta) \quad and \quad B \circ \Delta = \varepsilon. \tag{15}$$

A morphism $B : C \otimes C \to \underline{1}$ satisfying the first of the two conditions (15) *is called a* Casimir morphism *for C. A Casimir morphism satisfying also $B \circ \Delta = \varepsilon$ is called a* normalized *Casimir morphism.*

Notice that Proposition 2.8 is not completely dual to Proposition 2.4, in that it lacks an interpretation of coseparability of C in terms of a functorial property and the fact that $\underline{1}$ be a left $\otimes$-generator is not required. The picture becomes complete once we take into account a fundamental result contained in [**MT1**].

THEOREM 2.9. [**MT1**, Thm. 2.5] *Assume that $\underline{1}$ is a left $\otimes$-generator for a monoidal category $\mathcal{M}$, and let (C, Δ, ε) be a coalgebra in $\mathcal{M}$. The forgetful functor $F : \mathcal{M}^C \to \mathcal{M}$ is separable if, and only if, there exists a morphism $B : C \otimes C \to \underline{1}$ in $\mathcal{M}$ which fulfills* (15) *i.e. if, and only if, the coalgebra C is coseparable.*

2.2. h-separability. Starting from an example related to the tensor algebra, Ardizzoni and Menini introduced in [**AM**] a stronger version of the notion of separability for functors, called *heavy* separability (or, for short, h-separability). This also led them to obtain a Rafael-type theorem for h-separable functors [**AM**, Theorem 2.1].

DEFINITION 2.10. [**MT1**, Def. 2.7] A separable functor $F : \mathcal{M} \to \mathcal{D}$ is called *heavily separable* (*h-separable* for short) if the natural transformation

$$P_{-,-} = P^F_{-,-} : \mathrm{Hom}_{\mathcal{D}}(F, F) \to \mathrm{Hom}_{\mathcal{M}}(\mathrm{Id}_{\mathcal{M}}, \mathrm{Id}_{\mathcal{M}})$$

of Definition 2.2 makes the following diagram commutative for every X, Y, $Z \in \mathcal{M}$.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(FX, FY) \times \mathrm{Hom}_{\mathcal{D}}(FY, FZ) & \xrightarrow{P^F_{X,Y} \times P^F_{Y,Z}} & \mathrm{Hom}_{\mathcal{M}}(X, Y) \times \mathrm{Hom}_{\mathcal{M}}(Y, Z) \\ \circ\downarrow & & \downarrow\circ \\ \mathrm{Hom}_{\mathcal{D}}(FX, FZ) & \xrightarrow{\quad P^F_{X,Z} \quad} & \mathrm{Hom}_{\mathcal{M}}(X, Z) \end{array}$$

where the vertical arrows are the obvious compositions and

$$P_{X,Y} := P^F_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(FX, FY) \to \mathrm{Hom}_{\mathcal{M}}(X, Y)$$

as in Definition 2.2.

The fact that the functor $F : \mathcal{M}^C \to \mathcal{M}$ of Theorem 2.9 is h-separable now becomes equivalent to (15) plus an additional condition.

THEOREM 2.11. [**MT1**, Thm. 2.8] *Assume that $\underline{1}$ is a left $\otimes$-generator for a monoidal category $\mathcal{M}$, and let (C, Δ, ε) be a coalgebra in $\mathcal{M}$. The forgetful functor $F : \mathcal{M}^C \to \mathcal{M}$ is heavily separable if, and only if, there exists a morphism $B : C \otimes C \to \underline{1}$ which fulfills* (15) *and*

$$(B \otimes B) \circ (C \otimes \Delta \otimes C) = B \circ (C \otimes \varepsilon \otimes C).$$

This result justify the next definition.

DEFINITION 2.12. [**MT1**, Def. 2.2] Let (C, Δ, ε) be a coalgebra in a monoidal category $\mathcal{M}$. We will say C is *heavily coseparable* (*h-coseparable* for short) if C is coseparable, endowed with a normalized Casimir morphism B that also satisfies

$$(B \otimes B) \circ (C \otimes \Delta \otimes C) = B \circ (C \otimes \varepsilon \otimes C) \quad \text{(h-coseparability condition).} \tag{16}$$

3. Entwined structures: cowreaths

Wreaths and cowreaths are generalized entwined structures that can be introduced in the context of 2-categories. These are categories where one can define 2-morphisms, i.e. "morphisms between 1-morphisms". The typical example is $\mathcal{CAT}$ the category of all categories with functors as 1-morphisms and natural transformations as 2-morphisms. The definition of wreath was introduced in [**LS**, p.256], while the dual notion of cowreath in [**BC**, p. 1056].

DEFINITION 3.1. [**LS, BC**] Let $\mathcal{M}$ be a 2-category. Then a (co)wreath in $\mathcal{M}$ is a (co)monad in the Eilenberg-Moore category $\mathrm{EM}(\mathcal{M})$ of $\mathcal{M}$.

In [**BC**] the authors show that when $\mathcal{M}$ is a monoidal category a (co)wreath in $\mathcal{M}$ is equivalently defined by the choice of an algebra A in $\mathcal{M}$, an object X in $\mathcal{M}$ and an entwining map $\psi : X \otimes A \to A \otimes X$ satisfying appropriate relations (see Definition 3.3). Among these relations is the fact that (X, ψ) is a coalgebra in a suitable monoidal category associated to the algebra A. This ultimately leads to the notion of corepresentation of a cowreath, i.e. of entwined module over a cowreath (see Definition 3.4). Further details about (co)wreaths in a 2-category, their (co)representations and motivations for their investigation can be found in [**BC, BCT1, BCT2**]. Given that cowreaths can be regarded as (co)algebras in a suitable monoidal category, the notion of separable and Frobenius cowreaths can also be introduced and examples of such structures can be presented (see [**BCT2, MT1**]).

3.1. Cowreaths in a monoidal category.

DEFINITION 3.2. [**BCT1**, Subsec. 3.1] Let $\mathcal{M}$ be a (strict) monoidal category and let (A, m, u) be an algebra in $\mathcal{M}$. A (*right*) *transfer morphism* through A is a pair (X, ψ) with $X \in \mathcal{M}$ and $\psi : X \otimes A \to A \otimes X$ in $\mathcal{M}$ such that

$$\psi \circ (X \otimes m) = (m \otimes X) \circ (A \otimes \psi) \circ (\psi \otimes A) \tag{17}$$

$$\psi \circ (X \otimes u) = u \otimes X. \tag{18}$$

The category of all right transfer morphism through A will be denoted by $\mathcal{T}_A^\#$ and a morphism $f : X \to Y$ for this category is a morphism $f : X \to A \otimes Y$ in $\mathcal{M}$ such that

$$(m \otimes Y) \circ (A \otimes f) \circ \psi_X = (m \otimes Y) \circ (A \otimes \psi_Y) \circ (f \otimes A) \tag{19}$$

The composition of two morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{T}_A^\#$ is

$$g \odot f = (m \otimes Z) \circ (A \otimes g) \circ f \tag{20}$$

and we have

$$\mathrm{Id}_{(X,\psi)} = u \otimes X.$$

The tensor product of (X, ψ_X) and (Y, ψ_Y) is

$$X \circledast Y = (X \otimes Y, \psi_X \odot \psi_Y = (\psi_X \otimes Y) \circ (X \otimes \psi_Y)). \tag{21}$$

The tensor product of morphisms $f : X \to X'$ and $g : Y \to Y'$ is

$$f \circledast g = (m \otimes X' \otimes Y') \circ (A \otimes \psi_X \otimes Y') \circ (f \otimes g). \tag{22}$$

The unit object of $\mathcal{T}_A^\#$ is

$$(\underline{1}, r_A^{-1} \circ l_A : \underline{1} \otimes A \to A \otimes \underline{1}).$$

DEFINITION 3.3. [**BCT1**, Subsec. 3.2] A *cowreath* in $\mathcal{M}$ is a triple (A, X, ψ) where A is an algebra in $\mathcal{M}$ and (X, ψ) is a coalgebra in $\mathcal{T}_A^\#$. This means that $(X, \psi) \in \mathcal{T}_A^\#$ and there are morphisms $\delta : X \to A \otimes X \otimes X$ and $\varepsilon : X \to A$ such that

$$(m \otimes X^2) \circ (A \otimes \psi \otimes X) \circ (A \otimes X \otimes \psi) \circ (\delta \otimes A) = (m \otimes X^2) \circ (A \otimes \delta) \circ \psi \tag{23}$$

(δ is a morphism in $\mathcal{T}_A^\#$),

$$(m \otimes X^3) \circ (A \otimes \delta \otimes X) \circ \delta = (m \otimes X^3) \circ (A \otimes \psi \otimes X^2) \circ (A \otimes X \otimes \delta) \circ \delta \tag{24}$$

(coassociativity),

$$m \circ (A \otimes \varepsilon) \circ \psi = m \circ (\varepsilon \otimes A) \tag{25}$$

(ε is a morphism in $\mathcal{T}_A^\#$),

$$(m \otimes X) \circ (A \otimes \varepsilon \otimes X) \circ \delta = u \otimes X \tag{26}$$

(left counit property),

$$(m \otimes X) \circ (A \otimes \psi) \circ (A \otimes X \otimes \varepsilon) \circ \delta = u \otimes X \tag{27}$$

(right counit property).

DEFINITION 3.4. [**BCT1**, Def. 3.2] An *entwined module* over a cowreath (A, X, ψ) is a triple (M, μ_M, ρ_M), where $(M, \mu_M) \in \mathcal{M}_A$ and $\rho_M : M \to M \otimes X$, satisfying

$$(\rho_M \otimes X) \circ \rho_M = (\mu_M \otimes X^2) \circ (M \otimes \delta) \circ \rho_M \quad \text{(coassociativity)} \tag{28}$$

$$\mu_M \circ (M \otimes \varepsilon) \circ \rho_M = \mathrm{Id}_M \quad \text{(counitality)} \tag{29}$$

$$\rho_M \circ \mu_M = (\mu_M \otimes X) \circ (M \otimes \psi) \circ (\rho_M \otimes A) \quad \text{(A-linearity)} \tag{30}$$

A morphism between entwined modules is a A-linear morphism $f : M \to N$ such that $(f \otimes X) \circ \rho_M = \rho_M \circ f$. The category of entwined modules will be denoted by $\mathcal{M}(\psi)_A^X$.

We recall the following result, concerning the conditions for which $B : X \otimes X \to A \otimes \underline{1}$ is a Casimir morphism for a coalgebra (X, ψ) in $\mathcal{T}_A^\#$. A detailed proof can be found in [**FR**].

PROPOSITION 3.5. [**MT1**, Sec. 3] *Let (A, X, ψ) be a cowreath in a (strict) monoidal category $\mathcal{M}$. A Casimir morphism for the coalgebra (X, ψ) is a morphism $B : X \otimes X \to A \otimes \underline{1}$ in $\mathcal{M}$ such that*

$$m \circ (A \otimes B) \circ (\psi \otimes X) \circ (X \otimes \psi) = m \circ (B \otimes A) \tag{31}$$

and

$$(m \otimes X) \circ (A \otimes \psi) \circ (A \otimes X \otimes B) \circ (\delta \otimes X) = (m \otimes X) \circ (A \otimes B \otimes X) \circ (\psi \otimes X^2) \circ (X \otimes \delta). \tag{32}$$

hold.

REMARK 3.6. Let (A, X, ψ) be a cowreath in $\mathcal{M}$. For every object $(M, \mu_M) \in \mathcal{M}_A$ we can define a functor $G : \mathcal{M}_A \to \mathcal{M}(\psi)_A^X$ such that

$$\begin{aligned} GM &= M \otimes X \\ \mu_{GM} &= (\mu_M \otimes X) \circ (M \otimes \psi) \\ \rho_{GM} &= (\mu_M \otimes X^2) \circ (M \otimes \delta) \end{aligned}$$

for every $(M, \mu_M) \in \mathcal{M}_A$ and $G(f) = f \otimes X$ for every morphism $f : M \to M'$. This functor is a right adjoint for the forgetful functor $F : \mathcal{M}(\psi)_A^X \to \mathcal{M}_A$. The unit η of this adjunction is defined, for every $(M, \mu_M, \rho_M) \in \mathcal{M}(\psi)_A^X$, by $\eta(M, \mu_M, \rho_M) = \rho_M$. The counit is defined, for any $M \in \mathcal{M}_A$, by $\epsilon M = \mu_M \circ (M \otimes \varepsilon)$.

3.2. Separable Cowreaths. This subsection contains most of the steps needed to obtain similar results to Proposition 2.8 and Theorem 2.9 in the context of cowreaths. Complete proofs can be found in [**MT1, FR**]. Recall the functors $F : \mathcal{M}(\psi)_A^X \to \mathcal{M}_A$ and $G : \mathcal{M}_A \to \mathcal{M}(\psi)_A^X$, defined in Remark 3.6.

LEMMA 3.7. [**MT1**, Lemma 3.2, case n=2] *Let (A, X, ψ) be a cowreath in $\mathcal{M}$ and assume that $\underline{1}$ is a left $\otimes$-generator for $\mathcal{M}$. Let $n \in \mathbb{N}$, $\Omega = (GF)^n : \mathcal{M}(\psi)_X^A \to \mathcal{M}(\psi)_X^A$ and $\Theta : \Omega \to Id_{\mathcal{M}(\psi)_X^A}$ be a natural transformation. Then, for every $(M, \mu_M, \rho_M) \in \mathcal{M}(\psi)_X^A$, we have*

$$\Theta(M, \mu_M, \rho_M) = \mu_M \circ (M \otimes \mathbb{T}) \circ (\rho_M \otimes X^n) \tag{33}$$

where

$$\mathbb{T} = m \circ (A \otimes \varepsilon) \circ \Theta(A \otimes X) \circ (u \otimes X^{n+1}) : X^{n+1} \to A. \tag{34}$$

Conversely, given a morphism $\mathbb{T} : X^{n+1} \to A$ in $\mathcal{M}$, set, for any $(M, \mu_M, \rho_M) \in \mathcal{M}(\psi)_X^A$,

$$\Theta(M, \mu_M, \rho_M) = \mu_M \circ (M \otimes \mathbb{T}) \circ (\rho_M \otimes X^n).$$

Then the collection $(\Theta(M, \rho_M))_{(M, \mu_M, \rho_M) \in \mathcal{M}(\psi)_X^A}$, defines a natural transformation $\Theta : \Omega \to Id_{\mathcal{M}(\psi)_X^A}$ if and only if the $\Theta(M, \mu_M, \rho_M)$ are morphisms in $\mathcal{M}(\psi)_X^A$ if and only if the following conditions are satisfied:

$$m \circ (\mathbb{T} \otimes A) = m \circ (A \otimes \mathbb{T}) \circ \psi^{\# n+1} \tag{35}$$

and

$$(m \otimes X) \circ (A \otimes \psi) \circ (A \otimes X \otimes \mathbb{T}) \circ (\delta \otimes X^n) = (m \otimes X) \circ (A \otimes \mathbb{T} \otimes X) \circ (\psi^{\# n} \otimes X^2) \circ (X^n \otimes \delta). \tag{36}$$

THEOREM 3.8. [**MT1**, Thm. 3.3] *Assume that $\underline{1}$ is a left $\otimes$-generator of the (strict) monoidal category $\mathcal{M}$, and let (A, X, ψ) be a cowreath in $\mathcal{M}$. The forgetful functor $F : \mathcal{M}(\psi)_A^X \to \mathcal{M}_A$ is separable if and only if there exists a Casimir morphism $B : X \otimes X \to A$ for the coalgebra (X, ψ) in $\mathcal{T}_A^{\#}$ such that*

$$m \circ (A \otimes B) \circ \delta = \varepsilon. \tag{37}$$

Namely if $\theta : GF \to \mathcal{M}(\psi)_A^X$ is a natural transformation such that

$$\theta \circ \eta = \mathrm{Id}_{\mathcal{M}(\psi)_A^X}, \tag{38}$$

then

$$B = m \circ (A \otimes \varepsilon) \circ \theta_{A \otimes X} \circ (u \otimes X^2) \tag{39}$$

and

$$\theta(M) = \mu_M \circ (M \otimes B) \circ (\rho_M \otimes X), \text{ for every } (M, \mu_M, \rho_M) \in \mathcal{M}(\psi)_A^X. \tag{40}$$

REMARK 3.9. We will use the assumptions and notations of Theorem 3.8. As stated in Definition 3.2, we will write $\circledast$ for the tensor product in $\mathcal{T}_A^{\#}$ and $\odot$ for the composition of two morphisms in $\mathcal{T}_A^{\#}$. A straightforward computation shows that equality (37) can be written in $\mathcal{T}_A^{\#}$ as

$$B \odot \delta = \varepsilon.$$

Thus from Theorem 3.8 follows

COROLLARY 3.10. [**BCT1**, Thm. 7.5] *Assume that $\underline{1}$ is a left $\otimes$-generator of the (strict) monoidal category $\mathcal{M}$, and let (A, X, ψ) be a cowreath in $\mathcal{M}$. The forgetful functor $F : \mathcal{M}(\psi)_A^X \to \mathcal{M}_A$ is separable if and only if the coalgebra (X, ψ) is coseparable in $\mathcal{T}_A^{\#}$.*

THEOREM 3.11. [**MT1**, Thm. 3.1] *Assume that $\underline{1}$ is a left $\otimes$-generator of the (strict) monoidal category $\mathcal{M}$ and let (A, X, ψ) be a cowreath in $\mathcal{M}$. Then the following are equivalent.*

(a) The forgetful functor $F : \mathcal{M}(\psi)_A^X \to \mathcal{M}_A$ is separable.
(b) (X, ψ) is a coseparable coalgebra in $\mathcal{T}_A^{\#}$.

Moreover if $(\underline{1}, A)$ is a left $\otimes$-generator of the category $\mathcal{T}_A^{\#}$, they are also equivalent to

(c) The forgetful functor $F^{\#} : \left(\mathcal{T}_A^{\#}\right)^{(X,\psi)} \to \mathcal{M}$ is separable.

THEOREM 3.12. [**MT1**, Thm. 3.4] *Assume that $\underline{1}$ is a left $\otimes$-generator of the (strict) monoidal category $\mathcal{M}$, and let (A, X, ψ) be a cowreath in $\mathcal{M}$. The forgetful functor $F : \mathcal{M}(\psi)_A^X \to \mathcal{M}_A$ is h-separable if, and only if, there exists a Casimir morphism $B : X \otimes X \to A$ for the coalgebra (X, ψ) in $\mathcal{T}_A^{\#}$ such that*

$$m \circ (A \otimes B) \circ \delta = \varepsilon \tag{41}$$

and

$$m \circ (m \otimes A) \circ (A \otimes B \otimes B) \circ \left(\psi \otimes X^3\right) \circ (X \otimes \delta \otimes X) = m \circ (A \otimes B) \circ (\psi \otimes X) \circ (X \otimes \varepsilon \otimes X). \tag{42}$$

REMARK 3.13. We will use the assumptions and notations of Theorem 3.12. Recalling Definition 3.2, a straightforward computation shows that our equality (42) can be written in $\mathcal{T}_A^{\#}$ as

$$(B \circledast B) \odot \left(\mathrm{Id}_{(X,\psi)} \circledast \delta \circledast \mathrm{Id}_{(X,\psi)}\right) = B \odot \left(\mathrm{Id}_{(X,\psi)} \circledast \varepsilon \circledast \mathrm{Id}_{(X,\psi)}\right),$$

while the equality (41) rewrites as

$$B \odot \delta = \varepsilon.$$

Now we introduce the following definition, from [**BCT2**].

DEFINITION 3.14. [**BCT2**, p. 240] A cowreath (A, X, ψ) is called *separable* if (X, ψ) is a coseparable coalgebra in the monoidal category $\mathcal{T}_A^{\#}$. Similarly we will say that a cowreath (A, X, ψ) is *heavily separable* (*h-separable* for short) if (X, ψ) is a heavily coseparable coalgebra in the monoidal category $\mathcal{T}_A^{\#}$.

In view of the foregoing, Theorem 3.12 can be rewritten in the following form.

THEOREM 3.15. [**MT1**, Thm. 3.6] *Assume that $\underline{1}$ is a left $\otimes$-generator of the (strict) monoidal category $\mathcal{M}$, and let (A, X, ψ) be a cowreath in $\mathcal{M}$. Then the following are equivalent.*

(1) The forgetful functor $F : \mathcal{M}(\psi)_A^X \to \mathcal{M}_A$ is h-separable.
(2) (X, ψ) is a heavily coseparable coalgebra in the monoidal category $\mathcal{T}_A^{\#}$ i.e. the cowreath (A, X, ψ) is h-separable.

In the next section we will specialize to the case of two-sided Hopf modules who was first studied in [**MT1**].

4. The case of two-sided Hopf modules

From now on we will consider $\mathcal{M} = \mathrm{Vec}_k$ the monoidal category of vector spaces over a fixed field k. We recall some useful definitions.

DEFINITION 4.1. A *bialgebra* in Vec_k is a 5-tuple $(B, m, u, \Delta, \varepsilon)$, where (B, m, u) is an algebra in Vec_k, (B, Δ, ε) is a coalgebra in Vec_k and such that the following conditions hold

$$(m \otimes m) \circ (B \otimes \tau_{B,B} \otimes B) \circ (\Delta \otimes \Delta) = \Delta \circ m \tag{43}$$

$$\Delta \circ u \circ l_k = u \otimes u \tag{44}$$

$$\varepsilon \circ m = l_k \circ (\varepsilon \otimes \varepsilon) \tag{45}$$

$$\varepsilon \circ u = \mathrm{Id}_k, \tag{46}$$

where $\tau_{B,B} : B \otimes B \to B \otimes B$ denotes the usual flip.

REMARK 4.2. Observe that the conditions (43)-(46) are equivalent to the fact that Δ and ε are algebra morphisms and also to the fact that m and u are coalgebra morphisms.

DEFINITION 4.3. Let (C, Δ, ε) be a coalgebra in Vec_k and let (A, m, u) be an algebra in Vec_k. Then

$$\mathrm{Hom}_k(C, A)$$

is always an algebra, called *convolution algebra*. The multiplication $*$ (*convolution product*) of this algebra is defined by setting, for every $f, g \in \mathrm{Hom}_k(C, A)$ and $c \in C$

$$(f * g)(c) = f(c_1) \cdot g(c_2).$$

DEFINITION 4.4. A *Hopf algebra* in Vec_k is a 6-tuple $(H, m, u, \Delta, \varepsilon, S)$, where $(H, m, u, \Delta, \varepsilon)$ is a bialgebra in Vec_k and $S : H \to H$ is an inverse for Id_H in the *convolution algebra* $\mathrm{Hom}(H, H)$.

REMARK 4.5. The map S is unique for every Hopf algebra and it is called *the antipode* of H.

DEFINITION 4.6. Let (A, m_A, u_A) be an algebra in Vec_k and let $(H, m_H, u_H, \Delta_H, \varepsilon_H, S_H)$ be a *Hopf algebra* in Vec_k. Given $\mu_A : H \otimes A \to A$, the pair (A, μ_A) will be called a (left) *H-module algebra* if the following conditions are satisfied.

$$(A, \mu_A) \quad \text{is a (left) } H\text{-module.} \tag{47}$$

$$\mu(h \otimes ab) = \mu(h_1 \otimes a)\mu(h_2 \otimes b) \tag{48}$$

$$\mu(h \otimes 1_A) = \varepsilon(h)1_A \tag{49}$$

for all $h \in H$ and $a, b \in A$. The map μ_A will also be called a (right) H-action on A.

DEFINITION 4.7. Let (A, m_A, u_A) be an algebra in Vec_k and let $(H, m_H, u_H, \Delta_H, \varepsilon_H, S_H)$ be a *Hopf algebra* in Vec_k. Given $\rho_A : A \to A \otimes H$, the pair (A, ρ_A) will be called a (right) *H-comodule algebra* if the following conditions are satisfied.

$$(A, \rho_A) \quad \text{is a (right) } H\text{-comodule.} \tag{50}$$

$$\rho_A(ab) = a_0 b_0 \otimes a_1 b_1. \tag{51}$$

$$\rho_A(1_A) = 1_A \otimes 1_H. \tag{52}$$

for all $a, b \in A$. The map ρ_A will also be called a (right) H-coaction on A.

REMARK 4.8. Notice that a left H-module algebra is an algebra in the category ${}_H\mathrm{Vec}_k$ of (left) H-modules, while a (right) H-comodule algebra A is an algebra in the category Vec_k^H of (right) H-comodules.

DEFINITION 4.9. Let $(C, \Delta_C, \varepsilon_C)$ be a coalgebra in Vec_k and let $(H, m_H, u_H, \Delta_H, \varepsilon_H, S_H)$ be a *Hopf algebra* in Vec_k. Given $\mu_C : C \otimes H \to C$, the pair (C, μ_C) will be called a (right) *H-module coalgebra* if the following conditions are satisfied.

$$(C, \mu_C) \quad \text{is a (right) } H\text{-module.} \tag{53}$$

$$(\Delta_C \circ \mu_C)(c \otimes h) = \mu_C(c_1 \otimes h_1) \otimes \mu_C(c_2 \otimes h_2). \tag{54}$$

$$(\varepsilon_C \circ \mu_C)(c \otimes h) = \varepsilon_C(c)\varepsilon_H(h). \tag{55}$$

for all $c \in C$ and $h \in H$.

REMARK 4.10. Notice that a (right) H-module coalgebra C is just a coalgebra in the category $(\mathrm{Vec}_k)_H$ of (right) H-modules.

PROPOSITION 4.11. *Let $(H, m_H, u_H, \Delta_H, \varepsilon_H, S_H)$ be a Hopf algebra and let (A, ρ_A) be a right H-comodule algebra. Then*

- *$H \otimes H^{op}$ is a Hopf algebra via*

$$m_{H\otimes H^{op}} = (m_H \otimes m_{H^{op}}) \circ (H \otimes \tau_{H,H^{op}} H^{op}) \tag{56}$$
$$u_{H\otimes H^{op}} = (u_H \otimes u_H) \circ l_k^{-1} \tag{57}$$
$$\Delta_{H\otimes H^{op}} = (H \otimes \tau_{H,H^{op}} \otimes H^{op}) \circ (\Delta_H \otimes \Delta_H) \tag{58}$$
$$\varepsilon_{H\otimes H^{op}} = l_k \otimes (\varepsilon_H \otimes \varepsilon_H) \tag{59}$$
$$S_{H\otimes H^{op}} = S_H \otimes S_H. \tag{60}$$

- *H is a right $H \otimes H^{op}$-module coalgebra via*

$$\mu_H(g \otimes (h \otimes h')) = h'gh \quad \text{for every } g, h, h' \in H. \tag{61}$$

- *$A \otimes H^{op}$ is a right $H \otimes H^{op}$-comodule algebra with*

$$\begin{array}{rccc} \rho_{A\otimes H^{op}}: & A \otimes H^{op} & \to & A \otimes H^{op} \otimes H \otimes H^{op} \\ & a \otimes h & \mapsto & (a_0 \otimes h_1) \otimes (a_1 \otimes h_2) \end{array} \tag{62}$$

where

$$\rho_A(a) = a_0 \otimes a_1 \quad \text{for all } a \in A \text{ and}$$
$$\Delta_H(h) = h_1 \otimes h_2 \quad \text{for all } h \in H.$$

PROPOSITION 4.12. *Let $(H, m_H, u_H, \Delta_H, \varepsilon_H, S_H)$ be a Hopf algebra, (A, ρ_A) be a right H-comodule algebra and (X, μ_X) a right H-module coalgebra. Let $\psi : X \otimes A \to A \otimes X$ defined by*

$$\psi(x \otimes a) = a_0 \otimes xa_1 \tag{63}$$

for every $x \in X$ and every $a \in A$. Then $(X, \psi) \in \mathcal{T}_A^{\#}$ and (X, ψ) is a coalgebra in $\mathcal{T}_A^{\#}$, with comultiplication $\delta_X : X \to A \otimes X^2$ and counit $\epsilon_X : X \to A$ given by

$$\delta_X(x) = 1_A \otimes x_1 \otimes x_2 \tag{64}$$
$$\epsilon_X(x) = \varepsilon_X(x)1_A. \tag{65}$$

In view of Proposition 4.11, we can apply Proposition 4.12 by choosing "H"$= H \otimes H^{op}$, "A"$= A \otimes H^{op}$ and "X"$= H$ and by defining

$$\begin{array}{rccc} \psi: & H \otimes A \otimes H^{op} & \to & A \otimes H^{op} \otimes H \\ & h \otimes a \otimes l & \mapsto & a_0 \otimes l_1 \otimes l_2 h a_1 \end{array} \tag{66}$$

Then $(H, \psi) \in \mathcal{T}_{A\otimes H^{op}}^{\#}$ and (H, ψ) is a coalgebra in $\mathcal{T}_{A\otimes H^{op}}^{\#}$ via

$$\begin{array}{rcl} \delta_H & : & H \to A \otimes H^{op} \otimes H \otimes H \\ \delta_H(h) & = & 1_A \otimes 1_H \otimes h_1 \otimes h_2 \end{array} \tag{67}$$

and

$$\begin{array}{rcl} \epsilon_H & : & H \to A \otimes H^{op} \\ \epsilon_H(h) & = & \varepsilon_H(h) 1_A \otimes 1_H. \end{array} \tag{68}$$

We can state the following Theorem. For a proof see [**MT1, FR**].

THEOREM 4.13. [**MT1**, Theorem 5.1] *A cowreath $(A \otimes H^{op}, H, \psi)$ is separable via a Casimir morphism*

$$B : H \otimes H \to A \otimes H^{op}$$

given by

$$h \otimes h' \to B^A(h \otimes h') \otimes B^H(h \otimes h')$$

if and only if B satisfies the following conditions

$$a_0 B^A(g_2 h a_1 \otimes g_3 h' a_2) \otimes B^H(g_2 h a_1 \otimes g_3 h' a_2) g_1 = B^A(h \otimes h') a \otimes g B^H(h \otimes h') \tag{69}$$

$$B^A(h_2 \otimes h')_0 \otimes B^H(h_2 \otimes h')_1 \otimes B^H(h_2 \otimes h')_2 h_1 B^A(h_2 \otimes h')_1 = B^A(h \otimes h'_1) \otimes B^H(h \otimes h'_1) \otimes h'_2 \tag{70}$$

$$B^A(h_1 \otimes h_2) \otimes B^H(h_1 \otimes h_2) = \varepsilon(h) 1_A \otimes 1_H \tag{71}$$

Moreover it is h-separable if and only if B satisfies the further condition

$$B^A(h \otimes h'_1) B^A(h'_2 \otimes h'') \otimes B^H(h'_2 \otimes h'') B^H(h \otimes h'_1) = \varepsilon_H(h') B^A(h \otimes h'') \otimes B^H(h \otimes h'') \tag{72}$$

Asking that the Casimir morphism B of the last proposition be of a particular form results in easier conditions to check.

PROPOSITION 4.14. [**MT1**, Prop. 5.2] *A cowreath $(A \otimes H^{op}, H, \psi)$ is separable via a Casimir element*

$$B : H \otimes H \to A \otimes H^{op}$$

of the form

$$h \otimes h' \to B^A(h \otimes h') \otimes 1_H$$

if and only if B^A satisfies the following conditions

$$a_0 B^A(h a_1 \otimes h' a_2) = B^A(h \otimes h') a \tag{73}$$

$$B^A(1 \otimes h)_0 \otimes B^A(1 \otimes h)_1 = B^A(1 \otimes h_1) \otimes h_2 \tag{74}$$

$$B^A(h_1 h' \otimes h_2 h'') = \varepsilon_H(h) B^A(h' \otimes h'') \text{ and } B^A(1_H \otimes 1_H) = 1_A \tag{75}$$

Moreover, whenever S is invertible, it is h-separable if and only if B satisfies the further condition

$$B^A(h \otimes 1_H) \cdot B^A(1_H \otimes h') = B^A(h \otimes h') \tag{76}$$

PROOF. The equality (69) rewrites as

$$a_0 B^A(g_2 h a_1 \otimes g_3 h' a_2) \otimes g_1 = B^A(h \otimes h') a \otimes g. \tag{77}$$

which for $g = 1_H$ gives us (73). Now (70) rewrites as

$$B^A(h_2 \otimes h')_0 \otimes h_1 B^A(h_2 \otimes h')_1 = B^A(h \otimes h'_1) \otimes h'_2, \tag{78}$$

which for $h = 1_H$, gives (74).

Now (71) rewrites as

$$B^A(h_1 \otimes h_2) = \varepsilon_H(h) 1_A. \tag{79}$$

For $h = 1_H$ we get $B^A(1_H \otimes 1_H) = 1_A$, while by applying $r_A \circ (\mathrm{Id}_A \otimes \varepsilon)$ to (77) we get

$$a_0 B^A(g_1 h a_1 \otimes g_2 h' a_2) = \varepsilon(g) B^A(h \otimes h') a$$

which for $a = 1_A$ gives the first equality of (75).

Conversely assume that equalities (73), (74) and (75) hold. We have

$$\begin{aligned} B^A(h \otimes h') &= B^A(h_1 \otimes \varepsilon_H(h_2) h') \\ &= B^A(h_1 \otimes h_2 S_H(h_3) h') \\ &\overset{(75)}{=} \varepsilon_H(h) B^A(1_H \otimes S_H(h_3) h') \\ &= B^A(1_H \otimes S_H(\varepsilon_H(h) h_3) h') \\ &= B^A(1_H \otimes S_H(h_1 S_H(h_2) h_3) h') \\ &= B^A(1_H \otimes S_H(h_1 \varepsilon_H(h_2)) h') \\ &= B^A(1_H \otimes S_H(h) h'), \end{aligned}$$

i.e.

$$B^A(h\otimes h') = B^A\left(1_H\otimes S_H\left(h\right)h'\right). \tag{80}$$

Therefore we obtain

$$\begin{aligned}B^A(h_2\otimes h')_0\otimes h_1B^A(h_2\otimes h')_1 &\overset{(80)}{=} B^A\left(1_H\otimes S_H\left(h_2\right)h'\right)_0\otimes h_1B^A\left(1_H\otimes S_H\left(h_2\right)h'\right)_1\\ &\overset{(74)}{=} B^A\left(1_H\otimes\left(S_H\left(h_3\right)h'_1\right)\right)\otimes h_1\left(S_H\left(h_2\right)h'_2\right) = B^A\left(1_H\otimes\left(S_H\left(h_3\varepsilon_H(h)\right)h'_1\right)\right)\otimes h'_2\\ &= B^A\left(1_H\otimes\left(S_H\left(h\right)h'_1\right)\right)\otimes h'_2 \overset{(80)}{=} B^A(h\otimes h'_1)\otimes h'_2\end{aligned}$$

which is (78). Now (75) implies (79). Finally we get

$$\begin{aligned}&a_0B^A(g_2ha_1\otimes g_3h'a_2)\otimes g_1 \overset{(75)}{=} a_0\varepsilon\left(g_2\right)B^A(ha_1\otimes h'a_2)\otimes g_1\\ =\ & a_0B^A(ha_1\otimes h'a_2)\otimes g \overset{(73)}{=} B^A(h\otimes h')a\otimes g\end{aligned}$$

which gives us (77).

Let us consider the h-separablity in the case S is invertible. We have

$$B^A(h\otimes h') = B^A(h'_2S_H^{-1}\left(h'_1\right)h\otimes h'_3) \overset{(75)}{=} B^A(S_H^{-1}\left(h'\right)h\otimes 1_H). \tag{81}$$

From (72) we have to show that

$$B^A\left(h\otimes h'_1\right)\cdot B^A\left(h'_2\otimes h''\right) = \varepsilon_H\left(h'\right)B^A\left(h\otimes h''\right) \tag{82}$$

is equivalent to (76). Clearly (82) implies (76). Conversely, using (80) and (81)

$$\begin{aligned}B^A\left(h\otimes h'_1\right)\cdot B^A\left(h'_2\otimes h''\right) &\overset{(81)+(80)}{=} B^A\left(S_H^{-1}(h'_1)h\otimes 1_H\right)\cdot B^A\left(1_H\otimes S_H(h'_2)h''\right)\\ &\overset{(76)}{=} B^A\left(S_H^{-1}(h'_1)h\otimes S_H(h'_2)h''\right)\\ &\overset{(80)}{=} B^A\left(1_H\otimes S_H\left(S_H^{-1}(h'_1)h\right)S_H(h'_2)h''\right)\\ &= B^A\left(1_H\otimes S_H(h)h'_1S_H(h'_2)h''\right)\\ &= \varepsilon_H\left(h'\right)B^A\left(1_H\otimes S(h)h''\right)\\ &\overset{(80)}{=} \varepsilon_H\left(h'\right)B^A\left(h\otimes h''\right).\end{aligned}$$

□

Remark 4.15. Proposition 4.14 was proved in [**MT1**] in order to construct examples of separable cowreaths which are not h-separable or not Frobenius (see Remarks 6.2 in [**MT1**]). Notice that a cowreath $(A\otimes H^{op},H,\psi)$ for which there is no map $B:H\otimes H\to A\otimes H^{op}$ of the form $h\otimes h'\mapsto B^A(h\otimes h')\otimes 1_H$ satisfying (73)-(75), is not necessarily non-separable. Separability for such a cowreath could be attained via a morphism of a much more complicated form. The conditions under which a cowreath $(A\otimes H^{op},H,\psi)$ is separable via a Casimir morphism of a general form are described in [**MT2**].

In view of the previous remark, to make a clear distinction we will say that

Definition 4.16. A cowreath $(A\otimes H^{op},H,\psi)$ for which there is a map $B:H\otimes H\to A\otimes H^{op}$ of the form $h\otimes h'\mapsto B^A(h\otimes h')\otimes 1_H$ satisfying (73)-(75), will be called *right-trivially separable* (rt-separable, for short). Similarly, a cowreath $(A\otimes H^{op},H,\psi)$ for which there is a map $B:H\otimes H\to A\otimes H^{op}$ of the form $h\otimes h'\mapsto B^A(h\otimes h')\otimes 1_H$ satisfying (73)-(76) will be called *right-trivially h-separable* (rth-separable, for short).

Remark 4.17. As already observed in Remark 4.15, a cowreath can be separable even if it is not rt-separable. Clearly, a rt-separable cowreath is separable.

With this remark we conclude the preliminaries on cowreaths that are needed to fully understand the machinery that we will use in Chapter 3 to give equivalent condition in order for a cowreath $(A\otimes H^{op},H,\psi)$ to be rt(h)-separable and/or Frobenius. Such cowreath is built using two main ingredients: Sweedler's Hopf algebra $H=E(1)$ and a four-dimensional Clifford algebra $A = Cl(\alpha, \beta, \gamma)$. For sake of completeness, the next section will be devoted to the description of Clifford algebras and some of their fundamental properties.

5. Clifford algebras

We start by first recalling the definition of a Clifford algebra, as given by C. Chevalley in [**C**]. We consider a quadratic form $q : V \to k$ on a vector space V over a field k. This means that q satisfies

$$q(mv) = m^2 q(v) \quad \text{for any } m \in k \text{ and any } v \in V$$
$$\text{The mapping } \beta_q(u, v) := q(u + v) - q(u) - q(v) \text{ is bilinear.}$$

We can now build the tensor algebra $T(V)$ over V and consider the ideal I_q generated by elements

$$v \otimes v - q(v)1$$

for $v \in V$. The Clifford algebra $Cl(V, q)$ associated to V and q is then the quotient algebra $\frac{T(V)}{I_q}$. Let $\iota : V \to Cl(V, q)$ be the map defined by the composition of the inclusion of V into $T(V)$ with the canonical projection $\pi : T(V) \to Cl(V, q)$. Clifford algebras satisfy the following universal property.

Theorem 5.1. [**C**, Theorem 3.1] *Let A be a k-algebra and $f : V \to A$ be a k-linear map such that $(f(v))^2 = q(v)1_A$ for all $v \in V$. Then there exists an algebra homomorphism $\varphi : Cl(V, q) \to A$ such that*

$$\varphi(\iota(v)) = f(v) \qquad \text{for all } v \in V.$$

If V has finite dimension n and k has characteristic char $k \neq 2$, it is known that V admits an orthogonal basis with respect to $\beta_q(\cdot, \cdot)$, i.e. a n-uple of vectors $(e_1, e_2, \dots, e_n)$ such that

$$\beta_q(e_i, e_j) = 0, \; j \neq i, \qquad \beta_q(e_i, e_i) = q(e_i), \qquad \text{for every } i = 1, \dots, n.$$

It is also known that in this case a basis for $Cl(V, q)$ is given by the linearly independent elements $e_{i_1} \cdots e_{i_h}$ with $i_1 < \ldots < i_h$. It is also immediate to see that $\dim_k Cl(V, q) = 2^n$. For a further insight into Clifford algebras we refer to [**C, Lo, Lam**].

Example 5.2. Some well-known examples of Clifford algebras (over the field $\mathbb{R}$) are $\mathbb{C}$, $\mathbb{H}$, $\mathbb{R} \oplus \mathbb{R}$, $M_2(\mathbb{R})$.

In [**PVO2**] a family of algebras called “of Clifford-type” were introduced in order to describe H-cleft extensions of the ground field k, where H denotes Sweedler’s Hopf algebra. These Clifford-type algebras are actually usual Clifford algebras whose presentation is given by generators and relations, as we will see in the next subsection.

Definition 5.3. [**PVO2**, Def. 1] Let $\alpha, \beta_i, \gamma_i \in k$ for $i = 1, \dots, n$ and $\lambda_{ij} \in k$ for $i, j \in \{1, \dots, n\}$, $i < j$. The Clifford-type algebra $Cl(\alpha, \beta_i, \gamma_i, \lambda_{ij})$ is the unital associative algebra generated by elements $G, X_1, \dots, X_n$ such that $G^2 = \alpha$, $X_i^2 = \beta_i$, $GX_i + X_iG = \gamma_i$ for all $i = 1, \dots, n$ and $X_iX_j + X_jX_i = \lambda_{ij}$ for all $i, j \in \{1, \dots, n\}$ with $i < j$. A k-basis for this algebra is given by $\{G^j X_P\}$, where $j = 0, 1$ and $X_P = X_{i_1} \cdots X_{i_s}$ with $P = \{i_1 < i_2 < \ldots < i_s\} \subseteq \{1, \dots, n\}$.

In order to see that the two given definition are equivalent let us start by showing how each four-dimensional algebra given by Definition 5.3 is isomorphic to a suitable Clifford algebra.

5.1. An orthogonal basis for Clifford-type algebras.

Let $\alpha, \beta, \gamma \in k$ and define $Cl(\alpha, \beta, \gamma)$ according to Definition 5.3. Consider the two dimensional vector space V with basis (e_1, e_2) and take the matrix

$$Q = \begin{pmatrix} \alpha & \frac{\gamma}{2} \\ \frac{\gamma}{2} & \beta \end{pmatrix}$$

which is then associated to the quadratic form $q : V \to k$ defined by $q(v) = v^t Q v$ for every $v \in V$. It is clear that whenever $\gamma = 0$, then (e_1, e_2) is an orthogonal basis for V and that

$$Cl(V, q) \cong Cl(\alpha, \beta, 0).$$

In fact let $\Phi : V \to Cl(\alpha, \beta, 0)$ be the k-linear map defined by $\Phi(e_1) = G$, $\Phi(e_2) = X$. This map can be extended in a unique way to an algebra map whose domain is $T(V)$, by the fundamental property of the tensor algebra. Furthermore

$$\Phi(v \otimes v - q(v)1) = \Phi(v \otimes v) - q(v) = (\Phi(v))^2 - q(v) = (\lambda_1 \Phi(e_1) + \lambda_2 \Phi(e_2))^2 - \lambda_1^2 \alpha - \lambda_2^2 \beta = \lambda_1 \lambda_2 \gamma = 0,$$

for every $v = \lambda_1 e_1 + \lambda_2 e_2 \in V$, which means $\Phi(I_q) = 0$, i.e. that Φ induces an algebra map $\overline{\Phi}$ between $Cl(V, q)$ and $Cl(\alpha, \beta, 0)$. Since $\overline{\Phi}(1) = 1$ and $\overline{\Phi}(e_1 e_2) = GX$ we see that $\overline{\Phi}$ sends a basis into a basis and thus that it is invertible.

Now, what happens when $\gamma \neq 0$? When $\gamma \neq 0$, then (e_1, e_2) is not an orthogonal basis and therefore we need to build one to give a standard presentation of $Cl(V, q)$.

5.1.1. *Assume* $\alpha \neq 0$. Then we can set $v_1 := e_1$ and $v_2 := -\frac{\gamma}{2\alpha} e_1 + e_2$. Observe that the two vectors are linearly independent and thus form a basis of V. Moreover $q(v_1) = q(e_1) = \alpha \neq 0$, $q(v_2) = v_2^t Q v = \frac{4\alpha\beta - \gamma^2}{4\alpha}$ and

$$\beta_q(v_1, v_2) = \begin{pmatrix} 1, 0 \end{pmatrix} \begin{pmatrix} \alpha & \frac{\gamma}{2} \\ \frac{\gamma}{2} & \beta \end{pmatrix} \begin{pmatrix} -\frac{\gamma}{2\alpha} \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha, \frac{\gamma}{2} \end{pmatrix} \begin{pmatrix} -\frac{\gamma}{2\alpha} \\ 1 \end{pmatrix} = 0,$$

so that if again we define $\Phi(v_1) = \Phi(e_1) = G$ and $\Phi\left(\frac{2\alpha v_2 + \gamma v_1}{2\alpha}\right) = \Phi(e_2) = X$ we see that $Cl(\alpha, \beta, \gamma)$ is isomorphic to $Cl(V, q)$ where (v_1, v_2) is an orthogonal basis of V such that $q(v_1) = \alpha$ and $q(v_2) = \frac{4\alpha\beta - \gamma^2}{4\alpha}$ or in other words

$$Cl(\alpha, \beta, \gamma) \cong Cl\left(\alpha, \frac{4\alpha\beta - \gamma^2}{4\alpha}, 0\right) \text{ if } \alpha \neq 0.$$

Example 5.4. Take $\alpha = -1$, $\beta = -2$ and $\gamma = -2$. Then we find that the free algebra generated by e_1 and e_2 subjected to relations

$$e_1^2 = -1, \quad e_2^2 = -2, \quad e_1 e_2 + e_2 e_1 = -2$$

is isomorphic to the free algebra generated by v_1 and v_2 subjected to relations

$$v_1^2 = -1, \quad v_2^2 = -1, \quad v_1 v_2 + v_2 v_1 = 0,$$

and so is in fact a generalized quaternion algebra. In the special case when $k = \mathbb{R}$, these are just two equivalent presentations of Hamilton's quaternion, namely v_1 and v_2 correspond with classical generators i and j, while e_1 and e_2 identify with generators i and $i + j$.

5.1.2. *Assume* $\beta \neq 0$. Then we can set $v_1 := e_1 - \frac{\gamma}{2\beta} e_2$ and $v_2 := e_2$. Again these vectors are linearly independent and they form an orthogonal basis of V:

$$\beta_q(v_1, v_2) = \begin{pmatrix} 1, -\frac{\gamma}{2\beta} \end{pmatrix} \begin{pmatrix} \alpha & \frac{\gamma}{2} \\ \frac{\gamma}{2} & \beta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha - \frac{\gamma^2}{4\beta}, 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0.$$

Since $q(v_1) = v_1^t Q v_1 = \frac{4\alpha\beta - \gamma^2}{4\beta}$ and $q(v_2) = \beta \neq 0$, by using the same isomorphism Φ as before, we see that $Cl(\alpha, \beta, \gamma)$ is isomorphic to $Cl(V, q)$ where (v_1, v_2) is an orthogonal basis of V such that $q(v_1) = \frac{4\alpha\beta - \gamma^2}{4\beta}$ and $q(v_2) = \beta$ or in other words

$$Cl(\alpha, \beta, \gamma) \cong Cl\left(\frac{4\alpha\beta - \gamma^2}{4\beta}, \beta, 0\right) \text{ if } \beta \neq 0.$$

Example 5.5. One can refer to Example 5.4 and swap the roles of α and β, i.e. of e_1 and e_2 (and consequently of v_1 and v_2).

5.1.3. *Assume* $\alpha = \beta = 0$. In this case one can take $v_1 = e_1 + e_2$ and $v_2 = e_1 - e_2$. (v_1, v_2) is an orthogonal basis of V, because

$$\beta_q(v_1, v_2) = \begin{pmatrix} 1, 1 \end{pmatrix} \begin{pmatrix} \alpha & \frac{\gamma}{2} \\ \frac{\gamma}{2} & \beta \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \alpha + \frac{\gamma}{2}, \frac{\gamma}{2} + \beta \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \alpha - \beta = 0.$$

Since $q(v_1) = v_1^t Q v_1 = \gamma$ and $q(v_2) = v_2^t Q v_2 = -\gamma$, Φ gives an isomorphism between $Cl(0, 0, \gamma)$ and $Cl(V, q)$, where (v_1, v_2) is an orthogonal basis of V such that $q(v_1) = \gamma$ and $q(v_2) = -\gamma$, i.e.

$$Cl(0, 0, \gamma) \cong Cl(\gamma, -\gamma, 0).$$

Example 5.6. We can consider the algebra $M_2(\mathbb{R})$ of 2 by 2 matrices with coefficients in $\mathbb{R}$. This is generated by elements

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

whose product is $e_1e_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Since $e_1^2 = e_2^2$ both coincide with the trivial matrix and $e_1e_2 + e_2e_1 = \mathrm{Id}_2$ we can see that $M_2(\mathbb{R}) \cong Cl(0,0,1)$. Now take $v_1 = e_1 - e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $v_2 = e_1 + e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, whose squares are respectively $-\mathrm{Id}_2$ and Id_2. We see that

$$v_1v_2 + v_2v_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

so that the algebra generated by v_1 and v_2 is of type $Cl(1,-1,0)$. It is clear that v_1 and v_2 generate once again $M_2(\mathbb{R})$, since Id_2, v_1, v_2 and v_1v_2 are a basis of $M_2(\mathbb{R})$.

We have proved the following proposition.

Proposition 5.7. *Let $A = Cl(\alpha,\beta,\gamma)$ be a four-dimensional Clifford-type algebra on a field k with $\operatorname{char} k \neq 2$. Then there exists a classical Clifford algebra isomorphic to A. Namely*

- *If $\alpha \neq 0$, then $A \cong Cl\left(\alpha, \frac{4\alpha\beta-\gamma^2}{4\alpha}, 0\right)$*
- *If $\beta \neq 0$, then $A \cong Cl\left(\frac{4\alpha\beta-\gamma^2}{4\beta}, \beta, 0\right)$*
- *If $\alpha = \beta = 0$, then $A \cong Cl(\gamma,-\gamma,0)$.*

After having fully examined the case when A is a four-dimensional algebra, we try to understand how to proceed in the general case. We set $i := n \geq 1$ and we consider the Clifford-type algebra $Cl(\alpha,\beta_i,\gamma_i,\lambda_{ij})$. We take the $n+1$-dimensional vector space V with basis $(e_0, e_1, \dots, e_n)$ and denote

$$Q = \begin{pmatrix} \alpha & \frac{\gamma_1}{2} & \cdots & \cdots & \frac{\gamma_n}{2} \\ \frac{\gamma_1}{2} & \beta_1 & \cdots & \cdots & \frac{\lambda_{2n}}{2} \\ \vdots & \vdots & \ddots & & \\ & & & & \vdots \\ \vdots & \vdots & & \beta_{n-1} & \frac{\lambda_{n-1n}}{2} \\ \frac{\gamma_n}{2} & \frac{\lambda_{2n}}{2} & \cdots & \frac{\lambda_{n-1n}}{2} & \beta_n \end{pmatrix} \tag{83}$$

the symmetric matrix associated to the quadratic form $q : V \to k$ such that $q(v) = v^tQv$ for every $v \in V$. We have $q(e_0) = \alpha$ and $q(e_i) = \beta_i$ for all $i = 1 \dots n$, while $2\beta_q(e_0, e_i) = \gamma_i$ for all $i = 1, \dots n$ and $2\beta_q(e_i, e_j) = \lambda_{ij}$ for all $i, j \in \{1, \dots, n\}$. Whenever $\gamma_i = 0 = \lambda_{ij}$ for all $i, j \in \{1, \dots, n\}$ we find the classical presentation of a Clifford algebra with orthogonal generators:

$$Cl(V,q) \cong Cl(\alpha, \beta_i, 0, 0).$$

If at least one of the γ_i's or λ_{ij}'s is not zero, then we need the matrix Q to be diagonalizable. It is actually a known fact that a symmeytric matrix with entries in a field with characteristic different from 2 can be diagonalized or equivalently, that we can always find a basis of V which is orthogonal with respect to the quadratic form q. Proofs of this result are usually carried out by induction and are not constructive (see e.g. [**St**, pp. 114-115]). As a consequence, we can state the following result.

Theorem 5.8. *Let $A = Cl(\alpha,\beta_i,\gamma_i,\lambda_{ij})$ be a Clifford-type algebra on a field k with $\operatorname{char} k \neq 2$. Then there exists a Clifford algebra $Cl(V,q) \cong Cl(\alpha', \beta_i', 0, 0)$ which is isomorphic to A. To find a presentation that makes this clear, one must find an orthogonal basis for the quadratic form $q : V \to k$ defined via the matrix Q in* (83).

It seems that there is no hope of finding an expression for an orthogonal basis (that is, for α', β_i' in terms of α, β_i) for the general case. Obviously one can proceed by repeatedly applying suitable modification of the steps shown for the four-dimensional case (we just need to focus on the 2-by-2 leading principal submatrix of Q and inspect its diagonal entries), but this method does not yield an explicit description of the orthogonal basis we are looking for, nor of the diagonal form of Q.

Nonetheless there are peculiar cases in which we are able to determine a classical presentation even when Q has unspecified entries. Suppose for example we have a matrix Q with $Q_{11} \neq 0$. Then we can start our process of orthogonal diagonalization by means of Gaussian elimination (both on the first row and column) to find a matrix Q' where $Q_{1i} = 0$ and $Q_{i1} = 0$ for all $i = 2, \dots n+1$. Now if $Q'_{22} \neq 0$ we can repeat this process again and reduce also the second row and the second column of Q'. It is clear that we can fully diagonalize Q, provided at the i-th step the (i,i) entry of our matrix is different from 0. This procedure actually corresponds to the description of a well-known algorithm used in linear algebra to find the LDL^T decomposition of a symmetric matrix Q.

Theorem 5.9 (Adapted from [**HoJo**, Cor. 3.5.6]). *Let Q be a square symmetric matrix of order n. Suppose that $Q[\{1,\dots,i\}]$ is nonsingular*[1] *for all $i = 1,\dots,n$. Then $Q = LDL^T$, in which L, D are of order n, L is unit lower triangular, $D = diag(D_1,\dots,D_n)$ is diagonal, $D_1 = Q_{11}$, and*

$$D_i = \frac{\det Q[\{1,\dots,i\}]}{\det Q[\{1,\dots,i-1\}]}, \quad i = 2,\dots,n$$

The factors L, D are uniquely determined.

This theorem admits a generalization for singular square matrices. Let $\det Q[\{1,\dots,j-1,i\},\{1,\dots,j\}]$ denote the j-th leading principal submatrix of Q where the j-th row has been changed with the i-th row of Q.

Theorem 5.10 (Adapted from [**Ga**, Thm. 2, Ch. II]). *Let Q be a square symmetric matrix of order n and rank $r \leq n$, where $\det Q[\{1,\dots,i\}] \neq 0$ for all $i = 1,\dots,r$. Then there exist a lower unit triangular matrix L and a diagonal matrix D such that $Q = LDL^T$, where*

$$L_{ij} = \frac{\det Q[\{1,\dots,j-1,i\},\{1,\dots,j\}]}{\det Q[\{1,\dots,j\}]}, \text{ for all } i = j+1,\dots,n,\ j = 1,\dots,r$$

$$D_i = \begin{cases} Q_{11} & \text{if } i = 1 \\ \dfrac{\det Q[\{1,\dots,i\}]}{\det Q[\{1,\dots,i-1\}]} & \text{if } i = 2,\dots,r \\ 0 & \text{if } i = r+1,\dots n \end{cases}.$$

L and D are uniquely determined when $r = n$, otherwise the value of L_{ij} can be arbitrarily chosen for $i = j+1,\dots,n$ and $j = r+1,\dots,n$.

If we denote $\Delta_i := \det Q[\{1,\dots,i\}]$, we can state the following theorem.

Theorem 5.11. *Let $A = Cl(\alpha,\beta_i,\gamma_i,\lambda_{ij})$ be a Clifford algebra on a field k with $\operatorname{char} k \neq 2$. Suppose there exists a relabeling of the generators $G, X_1,\dots X_n$, such that when one defines the matrix Q introduced in (83), $\Delta_i \neq 0$ for all $i = 1,\dots,r$, where $r = \operatorname{rank} Q$. Then there exists a Clifford algebra*

$$Cl(V,q) \cong Cl\left(\underbrace{\Delta_1}_{\alpha'}, \underbrace{\frac{\Delta_2}{\Delta_1}, \frac{\Delta_3}{\Delta_2}, \dots, \frac{\Delta_r}{\Delta_{r-1}}, 0, \dots, 0}_{\beta'_i}, \underbrace{0}_{\gamma'_i}, \underbrace{0}_{\lambda'_{ij}}\right)$$

which is isomorphic to A.

Remark 5.12. Unfortunately the existence of a relabeling (which amounts to swap simultaneously the same rows and columns) such that $\Delta_i \neq 0$ for i up to $\operatorname{rank} Q$ is not always guaranteed – not even when Q is non-singular.

Take for example

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Q has full rank, but any relabeling of the generators does not change the form of Q. In fact there are no unit lower triangular matrix L and no diagonal matrix D such that $Q = LDL^T$. If we suppose there are, then it is not hard to check that D must be the zero matrix and so LDL^T has rank 0, contradiction.

[1] $Q[\{1,\dots,i\}]$ indicates the i-th leading principal submatrix of Q.

Nonetheless, Theorem 5.8 guarantees that any Clifford-type algebra can be regarded as a *classical* Clifford algebra where the generators have not been chosen to be (necessarily) orthogonal. This enable us to use both the presentation of A by generators and relations (i.e. Definition 5.3) and the vast literature on Clifford algebras without making a distinction depending on the values of the γ_i's and the λ_{ij}'s.

The next subsection helps us to better understand how the semisimplicity of a Clifford algebra $A = Cl(\alpha, \beta_i, \gamma_i, \lambda_{ij})$ is related to its associated quadratic form q, or equivalently to the values of its defining scalars.

5.2. Semisimplicity. The following theorem gives us a complete classification of semisimple Clifford algebras and also gives us information on the structure of the Jacobson radical in the non-semisimple case.

THEOREM 5.13. [**Sh**] *Consider a Clifford algebra $A = Cl(\alpha, \beta_i, 0, 0)$ and let $J = Jac(A)$ be its Jacobson radical.*

- *If $\alpha = 0$, then $G \in J$ and similarly if $\beta_i = 0$, then $X_i \in J$.*
- *If $\alpha \neq 0$ and $\beta_i \neq 0$ for all $i = 1, \dots n$, then $J = 0$.*
- *A is semisimple if and only if $\alpha \neq 0$ and $\beta_i \neq 0$ for all $i = 1, \dots, n$.*

PROOF. First of all observe that $\alpha = 0$ means that $G^2 = 0$ and this is equivalent to the fact that for any $a \in A$ the element $1 - aG$ is invertible and its inverse is $1 + aG$. This is equivalent to $G \in J$. The proof for the statement with X_i is identical. Now let $\mathcal{B} = \{G^j X_P\}$ be the k-basis of A fixed at the beginning. Note that, since $\alpha \neq 0$ and $\beta_i \neq 0$ for all i, G, each X_i, and hence each element of $\mathcal{B}$, is an invertible element of A. Now, consider the k-algebra homomorphism $\rho : A \to \mathrm{End}_k(A)$ defined by $\rho(a)(z) = az$ for all $a, z \in A$. Define the map $\Phi : A \to k$ by $\Phi(a) = \mathrm{tr}(\rho(a)),\ a \in A$, where $\mathrm{tr}(\rho(a))$ is the trace of the matrix corresponding to the linear map $\rho(a)$ with respect to the ordered basis $\mathcal{B}$. We now make three simple observations.

1. $\Phi(1) = |\mathcal{B}| = 2^{n+1}$ because $\rho(1)$ is the identity map of A.
2. If $1 \neq a \in \mathcal{B}$, then $\Phi(a) = 0$. That's because $\rho(a)(z) = az \neq z$ for all $z \in \mathcal{B} \setminus \{a\}$ and so the diagonal entries of the matrix of $\rho(a)$ are all zero hence $\Phi(a) = \mathrm{tr}(\rho(a)) = 0$.
3. If $a \in A$ is nilpotent, then $\Phi(a) = 0$. That's because $a^m = 0$ for some m and so $(\rho(a))^m = \rho(a^m) = 0$. Thus $\rho(a)$ is nilpotent and we know that the trace of a nilpotent matrix is zero.

Now let $a \in J$. Then $a \neq 1$ and since A is Artinian, a is nilpotent hence $\Phi(a) = 0$, by 3). Let $a = \sum_{i=1}^{2^{n+1}} c_i z_i$, where $c_i \in k$, $z_i \in \mathcal{B}$, $z_1 = 1$. So, by 1), 2),

$$0 = \Phi(a) = \sum_{i=1}^{2^{n+1}} c_i \Phi(z_i) = c_1 \Phi(z_1) = c_1 \Phi(1) = 2^{n+1} c_1$$

and hence $c_1 = 0$ because char $k \neq 2$. So the coefficient of z_1 of every element in J is zero. But for every i, the coefficient of $z_1 = 1$ of the element $z_i^{-1} a \in J$ is c_i and so $c_i = 0$ for all i (recall that the z_i's are invertibles). Hence $r = 0$ and so $J = 0$. Finally recall that a ring is semisimple if and only if it is Artinian and its Jacobson radical is zero. □

Thanks to Theorem 5.8 we know that, given a Clifford algebra $A = Cl(\alpha, \beta_i, \gamma_i, \lambda_{ij})$ whose associated quadratic form q has matrix

$$Q = \begin{pmatrix} \alpha & \frac{\gamma_1}{2} & \cdots & & \cdots & \frac{\gamma_n}{2} \\ \frac{\gamma_1}{2} & \beta_1 & \cdots & & \cdots & \frac{\lambda_{2n}}{2} \\ \vdots & \vdots & \ddots & & & \\ & & & & & \vdots \\ \vdots & \vdots & & & \beta_{n-1} & \frac{\lambda_{n-1n}}{2} \\ \frac{\gamma_n}{2} & \frac{\lambda_{2n}}{2} & & \cdots & \frac{\lambda_{n-1n}}{2} & \beta_n \end{pmatrix}$$

with respect to the standard basis, we can always find a basis so that q can be represented by a diagonal matrix Q'. It is also well known that $\det Q$ and $\det Q'$ differ by a non-zero square scalar and therefore that $\det Q \neq 0$ if, and only if, $\det Q' \neq 0$. Therefore we find an immediate corollary to Theorem 5.13.

COROLLARY 5.14. *Consider a Clifford algebra $A = Cl(\alpha, \beta_i, \gamma_i, \lambda_{ij})$ and let Q be the matrix defined in* (83). *Then A is semisimple if and only if* $\det Q \neq 0$.

Example 5.15. If we take a four-dimensional Clifford algebra $A = Cl(\alpha, \beta, \gamma)$, then A is semisimple if, and only if, $\det Q = \alpha\beta - \frac{\gamma^2}{4} \neq 0$.

Corollary 5.14 agrees with the more detailed classification of semisimple central Clifford algebras found in [**Lam**]. In fact we have

THEOREM 5.16. [**Lam**, Thms. V.2.4 - V.2.5] *Let A be a 2^{n+1}-dimensional Clifford algebra with associated quadratic form Q. Assume $d := \det Q \neq 0$. Then*

- *If n is odd, then $\mathcal{Z}(A) = k$ and A is a central simple algebra over k.*
- *If n is even and $d \notin k^2$, then $\mathcal{Z}(A) = k(\sqrt{d})$ and A is a central simple algebra over $k(\sqrt{d})$.*
- *If n is even and $d \in k^2$, then $\mathcal{Z}(A) = k \times k$ and $A \cong A_0 \times A_0$, where A_0 is the even part of A. Notice that, since $\dim_k A_0 = 2^n$, A_0 is central simple over k and thus A is semisimple over k.*

Theorem 5.13 can be generalized to the case of a Clifford algebra $A = Cl(\alpha, \beta_i, \gamma_i, \lambda_{ij})$, where the generators of A are not necessarily chosen to be orthogonal. Let us fix the vector space $V = \{G, X_1, \dots, X_n\}$ and the usual matrix

$$Q = \begin{pmatrix} \alpha & \frac{\gamma_1}{2} & \cdots & & \cdots & \frac{\gamma_n}{2} \\ \frac{\gamma_1}{2} & \beta_1 & \cdots & & \cdots & \frac{\lambda_{2n}}{2} \\ \vdots & \vdots & \ddots & & & \\ & & & & & \vdots \\ \vdots & \vdots & & & \beta_{n-1} & \frac{\lambda_{n-1n}}{2} \\ \frac{\gamma_n}{2} & \frac{\lambda_{2n}}{2} & & \cdots & \frac{\lambda_{n-1n}}{2} & \beta_n \end{pmatrix},$$

associated to the quadratic form $q : V \to k$, $q(x) = x^t Q x$ for all $x \in V$, so that $A = Cl(\alpha, \beta_i, \gamma_i, \lambda_{ij}) \cong Cl(V, q)$. Remember that there is a symmetric bilinear form associated to q defined as $\beta_q(v, w) = q(v + w) - q(v) - q(w)$ for all $v, w \in V$. The kernel of β_q (also called the radical of β_q) is just the kernel of the matrix Q. The bilinear form β_q induces a non-degenerate bilinear form $\overline{\beta_q} : \overline{V} \times \overline{V} \to k$ on $\overline{V} := \frac{V}{\ker Q}$, defined by

$$\overline{\beta_q}(\overline{v}, \overline{w}) := \beta(v, w), \qquad \text{for every } \overline{v} = v + \ker Q,\ \overline{w} = w + \ker Q \in \overline{V}.$$

Then clearly we can consider the Clifford algebra $Cl(\overline{V}, \overline{q})$, where $\overline{q} : \overline{V} \to k$ is simply the quadratic form associated to $\overline{\beta_q}$, defined by $\overline{q}(\overline{v}) = \frac{1}{2}\overline{\beta_q}(\overline{v}, \overline{v})$ for every $\overline{v} \in \overline{V}$.

Now consider the diagram

$$\begin{array}{ccc} V & \xrightarrow{\ \iota\ } & Cl(V, q) \\ {\scriptstyle p_{\ker Q}}\downarrow & & \downarrow {\scriptstyle \exists!\ \overline{p}} \\ \frac{V}{\ker Q} & \xrightarrow{\ \overline{\iota}\ } & Cl(\overline{V}, \overline{q}) \end{array}.$$

where ι and $\overline{\iota}$ are the canonical inclusions of each generating vector space into the respective Clifford algebra. Since we have

$$(\overline{\iota} \circ p_{\ker Q}(v))^2 = (\overline{v})^2 = \overline{q}(\overline{v}) = \frac{1}{2}\overline{\beta_q}(\overline{v}, \overline{v}) = \frac{1}{2}\beta_q(v, v) = q(v),$$

by the universal property of Clifford algebras (Theorem 5.1), there exists a unique algebra morphism $\overline{p} : Cl(V, q) \to Cl(\overline{V}, \overline{q})$, such that $\overline{\iota} \circ p_{\ker Q} = \overline{p} \circ \iota$. From this, with abuse of notation, we deduce that $\overline{p}(v) = \overline{v}$ for every $v \in V$. Now we can consider an element of the basis of $Cl(\overline{V}, \overline{q})$, say $\overline{z} = \overline{v_{i_1}}\,\overline{v_{i_2}} \cdots \overline{v_{i_r}}$, for some generators $\overline{v_{i_1}}, \overline{v_{i_2}}, \dots, \overline{v_{i_r}} \in \overline{V}$, and show that

$$z = \overline{v_{i_1}}\,\overline{v_{i_2}} \cdots \overline{v_{i_r}} = \overline{p}(v_{i_1})\overline{p}(v_{i_2}) \cdots \overline{p}(v_{i_r}) = \overline{p}(v_{i_1} v_{i_2} \cdots v_{i_r}).$$

Since this holds for any element z of the basis of $Cl(\overline{V}, \overline{q})$, this means that $\overline{p}$ is a surjective morphism of algebras. Now we consider the ideal $(\ker Q)$ generated in $Cl(V, q)$. If $a \in (\ker Q)$, then $a = \sum_i s_i q_i$ with $s_i \in Cl(V, q)$ and $q_i \in \ker Q$. Then we have

$$\overline{p}(a) = \sum\nolimits_i \overline{p}(s_i)\overline{p}(q_i) = \sum\nolimits_i \overline{p}(s_i)(\overline{\iota} p_{\ker Q})(q_i) = \sum\nolimits_i \overline{p}(s_i) \cdot 0 = 0,$$

which means that $(\ker Q) \subseteq \ker \overline{p}$. Let us indicate once again with $\mathcal{B} = \{G^j X_P\}$ the usual basis of $Cl(V,q)$. Then, if we pick an element a in $\ker \overline{p}$, we can write it $a = \sum_{i=1}^{2^{n+1}} c_i z_i$, with $c_i \in k$, $z_i \in \mathcal{B}$, $z_1 = 1$. Clearly we have

$$0 = \overline{p}(a) = \sum_{i=1}^{2^{n+1}} c_i \overline{p}(z_i) = \sum_{i=1}^{2^{n+1}} c_i \overline{p}(v_{i_1}) \overline{p}(v_{i_2}) \cdots \overline{p}(v_{i_r}),$$

where we have just split every element z_i of the basis into a product of generators contained in V. Now we can use the fact that $\overline{\iota} \circ p_{\ker Q} = \overline{p} \circ \iota$ to write

$$0 = \sum_{i=1}^{2^{n+1}} c_i \overline{p}(v_{i_1}) \overline{p}(v_{i_2}) \cdots \overline{p}(v_{i_r}) = \sum_{i=1}^{2^{n+1}} c_i \overline{v_{i_1}} \, \overline{v_{i_2}} \cdots \overline{v_{i_r}}.$$

Each product $\overline{v_{i_1}} \, \overline{v_{i_2}} \cdots \overline{v_{i_r}}$ is either 0 (when at least one of the v_{i_j}'s is in $\ker Q$) or an element of the basis of $Cl(\overline{V}, \overline{q})$, therefore we find that $c_i = 0$ if $z_i \notin (\ker Q)$, so that $a \in (\ker Q)$. From this we can conclude that

$$\frac{Cl(V,q)}{(\ker Q)} \cong Cl(\overline{V}, \overline{q}).$$

Moreover, if we denote $A := Cl(V,q)$ and $\overline{A} := Cl(\overline{V}, \overline{q})$, then, thanks to the surjectivity of $\overline{p} : A \to \overline{A}$, we have

$$\overline{p}(J(A)) \subseteq J(\overline{A}).$$

But $\overline{A}$ is a semisimple Clifford algebra, because $\overline{q}$ is non-degenerate (see Corollary 5.14), thus $J(\overline{A}) = 0$ and this immediately implies $J(A) \subseteq \ker \overline{p} = (\ker Q)$. Finally consider $a \in \ker Q$. For any $v \in V$ we have

$$av + va = a^t Q v + v^t Q a = 0 \text{ and } a^2 = 0,$$

where we have identified a and v with their coordinate vectors on the usual basis in A. If we consider $z \in \mathcal{B} = \{G^j X_P\}$, such that $z = v_{i_1} v_{i_2} \cdot v_{i_r}$, then it is clear that

$$az = a v_{i_1} v_{i_2} \cdot v_{i_r} = -v_{i_1} a v_{i_2} \cdot v_{i_r} = v_{i_1} v_{i_2} a v_{i_3} \cdots v_{i_r} = (-1)^r z a,$$

i.e. $az + (-1)^{r+1} za = 0$ for any $z \in \mathcal{B}$. Then, for every element $A \ni b = \sum_{i=1}^{2^{n+1}} \alpha_{j,P} G^j X_P$ it is clear that

$$(1 - ba)(1 - a \sum_{i=1}^{2^{n+1}} (-1)^{j+|P|+1} \alpha_{j,P} G^j X_P) = 1 - ba + ba - ba^2 \sum_{i=1}^{2^{n+1}} (-1)^{j+|P|+1} \alpha_{j,P} G^j X_P = 1,$$

i.e. $1 - ba$ is invertible for any $b \in A$. This is equivalent to $a \in J(A)$ and therefore we can conclude that

$$J(A) = (\ker Q).$$

Thus we have proved the following

THEOREM 5.17. *Let $A = Cl(\alpha, \beta_i, \gamma_i, \lambda_{ij})$ be a Clifford algebra. Let Q be the matrix defined in (83) and q the associated quadratic form on the generating space V, so that $A = Cl(V,q)$. Then there is an algebra isomorphism*

$$\frac{Cl(V,q)}{(\ker Q)} \cong Cl(\overline{V}, \overline{q}),$$

where $\overline{V} = \frac{V}{\ker Q}$ and $\overline{q}$ is the (non-degenerate) quadratic form induced by q on $\overline{V}$. Moreover

$$J(A) = (\ker Q)$$

and the Clifford algebra $Cl(\overline{V}, \overline{q})$ is semisimple.

Since $\dim_k Cl(V,q) = 2^{\dim_k V}$ for any Clifford algebra (see, e.g., [**Lam**, Thm. V.1.8]), we get an immediate corollary.

COROLLARY 5.18. *Let $n+1 = \dim_k V$ and $h = \dim_k \ker Q$. Then*

$$\dim_k J(A) = 2^{n+1} - 2^{n+1-h} = 2^{n+1-h}(2^h - 1).$$

Example 5.19. If we take $A = Cl(0,0,0,0)$, the Grassmann algebra (also called the Exterior algebra $\Lambda(V)$), then $h = n+1$ and we find

$$\dim_k J(\Lambda(V)) = 2^{n+1} - 1.$$

In this case $Cl(\overline{V}, \overline{q}) \cong k$.

REMARK 5.20. We can also calculate the cardinality of a basis for $J(A) = (\ker Q)$ making use of combinatorics. Clearly, a basis for $J(A)$ is given by all elements

$$q_1^{k_1} q_2^{k_2} \cdots q_h^{k_h} z_{h+1}^{k_{h+1}} \cdots z_{n+1}^{k_{n+1}},$$

where $\{z_i\}_{i=1,\dots n+1}$ is the usual basis of V, q_i is just used to indicate which of the z_i's are in $\ker Q$, and $k_i = 0, 1$ for all $i = 1, \dots, n+1$. There are clearly 2^{n+1} of these elements, namely the ones giving the basis $\mathcal{B} = \{G^j X_P\}$ of A. If we want to consider only those generating $J(A)$, then we must focus on those for which $k_i = 1$ for at least one $i \in \{1, \dots, h\}$. If we suppose $k_1 = 1$, then any choice for the other k_i's gives an element in $(\ker Q) = J(A)$ and we find 2^n elements therein contained. If $k_1 = 0$, but $k_2 = 1$, by a similar argument we identify 2^{n-1} additional elements belonging to $(\ker Q)$. Generalizing, if $k_1 = k_2 = \dots = k_{i-1} = 0$ and $k_i = 1$, then there are 2^{n+1-i} elements of $\mathcal{B}$ that are contained in $(\ker Q)$ and this is true for every $i = 1, \dots, h$. Therefore

$$\dim_k J(A) = \dim_k(\ker Q) = \sum\nolimits_{i=1}^{h} 2^{n+1-i} = 2^{n+1}\left(\frac{\frac{1}{2} - \left(\frac{1}{2}\right)^{h+1}}{1 - \frac{1}{2}}\right) = 2^{n+1}(1 - 2^{-h}) = 2^{n+1-h}(2^h - 1).$$

REMARK 5.21. We can use the same strategy to calculate the dimension of the space $J^2(A)$. An element of the basis of $J^2(A)$ must have $k_i \neq 0 \neq k_j$ for at least two different i and j in $\{1, \dots, h\}$. One can easily see that, for fixed $i, j \in \{1, \dots, h\}$ such that $i < j$, $k_i = k_j = 1$ and $k_r = 0$ for all $r < j$, $r \neq i$, the number of elements in $\mathcal{B}$ of this form is 2^{n+1-j}. For any $i = 1, \dots, h-1$, then j can be chosen in $\{i+1, \dots, h\}$, so that the above requirements are fulfilled. Hence

$$\dim_k J^2(A) = \sum\nolimits_{i=1}^{h-1} \sum\nolimits_{j=i+1}^{h} 2^{n+1-j} = 2^{n+1} \sum\nolimits_{i=1}^{h-1} \sum\nolimits_{j=i+1}^{h} 2^{-j} = 2^{n+1} \sum\nolimits_{i=1}^{h-1} (2^{-i} - 2^{-h}) = 2^{n+1-h}(2^h - h - 1).$$

5.3. Hopf algebra structures on Clifford algebras. In general, Clifford algebras are not Hopf algebras. The only members of this family who admit an Hopf algebra structure are those isomorphic (as associative algebras) to $E(n) := Cl(1, 0, 0, 0)$. This family of Hopf algebras generalizes Sweedler's Hopf algebra $E(1)$ and were introduced in [**BDG**, p. 755] and studied in [**CD, PVO, CC**].

DEFINITION 5.22. [**CD**, p.18] We denote by $E(n)$ the 2^{n+1}-dimensional Hopf algebra over a field k of characteristic $\mathrm{char}(k) \neq 2$ generated by elements g and x_i, for $i = 1, \dots, n$, such that $g^2 = 1$, $x_i^2 = 0$ and $gx_i = -x_i g$ for any $i = 1, \dots, n$ and $x_i x_j = -x_j x_i$ for $i, j = 1, \dots, n$, $i < j$.

The canonical Hopf algebra structure of $E(n)$ is given by

$$\Delta(g) = g \otimes g, \quad \Delta(x_i) = x_i \otimes g + 1 \otimes x_i, \; i = 1, \dots, n$$
$$\varepsilon(g) = 1, \quad \varepsilon(x_i) = 0, \; i = 1, \dots, n$$
$$S(g) = g^{-1} = g, \quad S(x_i) = -gx_i, \; i = 1, \dots, n.$$

Now let us prove that these are essentially the only Clifford algebras with a Hopf structure.

Consider $A = Cl(\alpha, \beta_i, \gamma_i, \lambda_{ij})$ a Clifford algebra on a field k with char $k \neq 2$. As stated in Theorem 5.8, we can always assume that $\gamma_i = 0$ and $\lambda_{ij} = 0$ for all $i, j \in \{1, \dots, n\}$, $i < j$. Suppose we define a coalgebra structure (A, Δ, ε) on A, such that A becomes a bialgebra. In this case ε is an algebra map and we have

$$0 = \gamma_i = \varepsilon(\gamma_i) = \varepsilon(GX_i + X_i G) = \varepsilon(GX_i) + \varepsilon(X_i G) = 2\varepsilon(G)\varepsilon(X_i)$$

and

$$0 = \lambda_{ij} = \varepsilon(\lambda_{ij}) = \varepsilon(X_i X_j + X_j X_i) = \varepsilon(X_i X_j) + \varepsilon(X_j X_i) = 2\varepsilon(X_i)\varepsilon(X_j)$$

for every $i, j \in \{1, \dots, n\}$, $i < j$.

Now suppose there exists an $r \in \{1, \dots, n\}$ such that $\varepsilon(X_r) \neq 0$. Then $2\varepsilon(G)\varepsilon(X_r) = 0$ and $2\varepsilon(X_r)\varepsilon(X_j) = 0$ for every $j = 1, \dots, n$, $j \neq r$ imply that $\varepsilon(G) = 0$ and $\varepsilon(X_j) = 0$ for every $j = 1, \dots, n$, $j \neq r$. By swapping the roles of G and X_r we can suppose that $\varepsilon(G) \neq 0$ and $\varepsilon(X_i) = 0$ for every $i = 1, \dots, n$. Hence we obtain that $\alpha = \varepsilon(\alpha) = \varepsilon(G^2) = \varepsilon(G)^2 \neq 0$ and $\beta_i = \varepsilon(\beta_i) = \varepsilon(X_i^2) = \varepsilon(X_i)^2 = 0$ for every $i = 1, \dots, n$. In this case we see that A can be identified with $Cl(\alpha, 0, 0, 0)$ with $\alpha \neq 0$.

On the other hand, if $\varepsilon(G) = \varepsilon(X_i) = 0$ for all $i = 1, \dots, n$, then we can call $X_{n+1} := G$ and we get that

$$\alpha = \varepsilon(\alpha) = \varepsilon(X_{n+1}^2) = \varepsilon(X_{n+1})^2 = 0 \quad \text{and} \quad \beta_i = \varepsilon(\beta_i) = \varepsilon(X_i^2) = \varepsilon(X_i)^2 = 0 \quad \text{for every } i = 1, \dots, n,$$

i.e. $A = Cl(0, 0, 0, 0)$ is the 2^{n+1}-dimensional Exterior algebra.

We are going to prove that the latter case does not admit a bialgebra structure. Let us write $\Delta(X_1) = \sum_{P,Q\subseteq\{1,\dots n+1\}} c_{P,Q}X_P \otimes X_Q$, where as usual $X_R = X_{i_1}\cdots X_{i_{|R|}}$ with $R = \{i_1 < i_2 < \ldots < i_{|R|}\} \subseteq \{1,\dots,n+1\}$. Then the counit axioms give

$$X_1 = (A\otimes\varepsilon)\Delta(X_1) = \sum\nolimits_{P,Q\subseteq\{1,\dots n+1\}} c_{P,Q}X_P\varepsilon(X_Q) = \sum\nolimits_{P\subseteq\{1,\dots n+1\}} c_{P,\varnothing}X_P$$

and

$$X_1 = (\varepsilon\otimes A)\Delta(X_1) = \sum\nolimits_{P,Q\subseteq\{1,\dots n+1\}} c_{P,Q}\varepsilon(X_P)X_Q = \sum\nolimits_{Q\subseteq\{1,\dots n+1\}} c_{\varnothing,Q}X_Q.$$

These force $c_{\varnothing,\varnothing} = 0$, $c_{\{1\},\varnothing} = 1$, $c_{\varnothing,\{1\}} = 1$ and $c_{P,\varnothing} = 0$, $c_{\varnothing,Q} = 0$ for every $\varnothing \neq P, Q \neq \{1\}$, i.e.

$$\Delta(X_1) = 1\otimes X_1 + X_1\otimes 1 + \sum\nolimits_{\varnothing\neq P,Q\subseteq\{1,\dots n+1\}} c_{P,Q}X_P\otimes X_Q.$$

If we square this equality we find

$$\begin{aligned}\Delta(X_1)^2 &= 2X_1\otimes X_1 + \sum\nolimits_{\varnothing\neq P,Q\subseteq\{1,\dots n+1\}} c_{P,Q}[X_P\otimes(X_1X_Q + X_QX_1) + (X_1X_P + X_PX_1)\otimes X_Q] + \\ &+ \sum\nolimits_{\varnothing\neq P,P',Q,Q'\subseteq\{1,\dots n+1\}} c_{P,Q}c_{P',Q'}X_PX_{P'}\otimes X_QX_{Q'}.\end{aligned}$$

Notice that $2X_1 \otimes X_1$ and any other term appearing in the RHS are linearly independent, since the former has no tensorand that is contained in $(\ker\varepsilon)^2$. On the other hand $\Delta(X_1)^2 = \Delta(X_1^2) = \Delta(\beta_1) = \Delta(0) = 0$, which is a contradiction. This proves that the exterior algebra $A = Cl(0,0,0,0)$ does not admit any bialgebra structure.

Therefore any Clifford algebra A admitting a bialgebra structure must be isomorphic to a Clifford algebra of the form $Cl(\alpha,0,0,0)$ with $\alpha \neq 0$. Furthermore, if k does not contain a square root of α we cannot define a bialgebra structure on A, because $\varepsilon(G)$ cannot be defined (remember that $\varepsilon(G)^2 = \alpha$). Conversely, if $\sqrt{\alpha} \in k$, then we can substitute the generator G with $\frac{G}{\sqrt{\alpha}}$ and we get that A is isomorphic to the Hopf algebra $Cl(1,0,0,0) = E(n)$.

PROPOSITION 5.23. *Let $A = Cl(\alpha,\beta_i,\gamma_i,\lambda_{ij})$ be a Clifford algebra on a field k with char $k \neq 2$. Then A is a Hopf algebra if, and only if, it is isomorphic to the algebra $E(n) = Cl(1,0,0,0)$ (not necessarily as Hopf algebras).*

6. Examples of separable cowreaths using Clifford algebras

In this section we collect a couple of examples where separability of the cowreath $(A\otimes H^{op}, H, \psi)$ is studied, when $A = Cl(\alpha,\beta,\gamma)$ is a four-dimensional Clifford algebra and H is Sweedler's four-dimensional Hopf algebra. These examples provided some motivation for a complete study of the coactions of H on A, in order to further understand how the choice of the comodule structure of A determined the properties of the induced cowreath. We outline that the main difference between these examples lies indeed in the choice of the H-comodule algebra structure of A. The coaction chosen in the first is the canonical one, while in the second, we decided to let H coact in the same way on both generators G and X of A.

6.1. Using the canonical H-comodule structure. We consider Sweedler's four-dimensional Hopf algebra H over a field k of characteristic $char(k) \neq 2$. As a k-algebra it is generated by elements g and x such that $g^2 = 1$, $x^2 = 0$ and $gx = -xg$ and has basis $\{1_H, g, x, gx\}$ over k. The coalgebra structure is given by $\Delta(g) = g\otimes g$, $\Delta(x) = x\otimes g + 1_H\otimes x$, $\varepsilon(g) = 1$, and $\varepsilon(x) = 0$. The antipode is such that $S(g) = g$ and $S(x) = gx$. We also consider the Clifford algebra $A = Cl(\alpha,\beta,\gamma)$, generated by elements G, X such that

$$\begin{aligned}G^2 &= \alpha\in k,\\ X^2 &= \beta\in k,\\ XG + GX &= \gamma\in k.\end{aligned}$$

This algebra admits a canonical H-comodule algebra structure. In fact A can be regarded as a H-cleft extension of k, as it has been proved in [**PVO2**], when endowed with the k-linear map $\rho : A \to A\otimes H$ such that

$$\rho(1_A) = 1_A\otimes 1_H, \tag{84}$$
$$\rho(G) = G\otimes g, \tag{85}$$
$$\rho(X) = X\otimes g + 1_A\otimes x, \tag{86}$$
$$\rho(GX) = GX\otimes 1_H + G\otimes gx. \tag{87}$$

As already seen in a previous section, by making use of Proposition 4.11 and Proposition 4.12 we can build a cowreath $(A\otimes H^{op},H,\psi)$ whose rt-separability is completely described by conditions (73)-(76). Exploiting these equations, in [**MT1**], Menini and Torrecillas proved

THEOREM 6.1. [**MT1**, Thm. 6.1] *Assume that* $\mathrm{char}\,(k)\neq 2$. *The cowreath* $(A\otimes H^{op},H,\psi)$ *is separable with Casimir element*

$$B_1 : H\otimes H\to A\otimes H^{op}$$

given by

$$h\otimes h'\to B(h\otimes h')\otimes 1_H$$

if and only if either

i) $\alpha\neq 0$ *and then* $\gamma^2=4\alpha\beta$. *We get* A *of the type* $Cl(\alpha,\frac{\gamma^2}{4\alpha},\gamma)$ *with* $\alpha\in k^\times$ *and* $\gamma\in k$,

or

ii) $\alpha=0$, *then* $\beta=\gamma=0$ *and* $b,b'\in k$, *we get* A *of the type* $Cl(0,0,0)$.

Moreover, in this case $(A\otimes H^{op},H)$ *is h-separable if and only if* $\alpha\in(k^\times)^2$ *and* $\gamma^2=4\alpha\beta$ *so that* A *is of type* $Cl(\alpha^2,\left(\frac{\gamma}{4\alpha}\right)^2,\gamma)$ *where* $\alpha\in k^\times$ *and* $\gamma\in k$.

Therefore in cases *i)* and *ii)* the considered cowreath is rt-separable (hence separable). In every other case, the cowreath $(A\otimes H^{op},H,\psi)$ is not rt-separable. Nonetheless it is still separable and also h-separable, as proved in [**MT2**], though via a Casimir morphism of a different kind (see Thms. 1 and 2 ibid.).

One could still ask when $(A\otimes H^{op},H,\psi)$ is rt-separable, once the canonical H-comodule algebra structure of A is replaced with a different one. A first step in this direction was taken in [**FR**] and is the content of the following subsection.

6.2. Using a different H-comodule structure. We want to endow $A=Cl(\alpha,\beta,\gamma)$ with a different structure of right H-comodule algebra. Thus we define a k-linear map $\rho:A\to A\otimes H$ by setting

$$\rho(1_A) = 1_A\otimes 1_H, \tag{88}$$
$$\rho(G) = G\otimes g+1_A\otimes x, \tag{89}$$
$$\rho(X) = X\otimes g+1_A\otimes x, \tag{90}$$
$$\rho(GX) = GX\otimes 1_H+G\otimes gx+X\otimes xg. \tag{91}$$

It is easily verified that (88)-(91) define an H-comodule algebra structure on A. By considering the induced cowreath $(A\otimes H^{op},H,\psi)$ (see Proposition 4.12) and using Proposition 4.14, a set of conditions under which this cowreath is rt-separable and rth-separable were found in [**FR**].

THEOREM 6.2. [**FR**, Thm. 5.0.2] *The cowreath* $(A\otimes H^{op},H,\psi)$ *is separable via a Casimir element*

$$B^1 : H\otimes H\to A\otimes H^{op}$$

of the form

$$h\otimes h'\mapsto B(h\otimes h')\otimes 1_H$$

if, and only if, the following conditions hold:

$$B(1_H\otimes 1_H)=B(g\otimes g)=1_A$$
$$B(1_H\otimes x)=B(g\otimes gx)=-B(x\otimes g)=-B(gx\otimes 1_H)=s'G+(1-s')X$$
$$B(1_H\otimes g)=B(g\otimes 1_H)=s''G-s''X$$
$$B(1_H\otimes gx)=B(g\otimes x)=B(x\otimes 1_H)=B(gx\otimes g)=-\frac{s''}{2}\gamma+s''GX$$
$$B(x\otimes x)=B(gx\otimes gx)=B(gx\otimes x)=B(x\otimes gx)=0,$$

where $\alpha,\beta,\gamma,s',s''\in k$ *satisfy one of the following conditions:*

i) $\alpha=\beta=\gamma=0$.

ii) $\alpha\neq 0$, $\beta=\gamma=0$ *and* $s'=0$.

iii) $\beta\neq 0$, $\alpha=\frac{\gamma^2}{4\beta}$ *and* $s'=\frac{2\beta}{2\beta-\gamma}$.

Moreover it is h-separable if, and only if,

$$(s'')^2(\alpha+\beta-\gamma)=1.$$

REMARK 6.3. Notice that both in Theorem 6.1 and Theorem 6.2, a necessary condition for separability of the cowreath $(A\otimes H^{op}, H, \psi)$ is that $\gamma^2-4\alpha\beta=0$. This condition is equivalent to the fact that the Clifford algebra $A=Cl(\alpha,\beta,\gamma)$ is not semisimple, i.e. it is associated to a degenerate quadratic form (see Example 5.15).

6.3. The quaternionic case. Let us consider A the *generalized quaternion algebra* over the field k. This is the four-dimensional k-vector space with basis $\{1,i,j,ij\}$ endowed with the algebra structure given by

$$i^2=a,\quad j^2=b,\quad ij=-ji,$$

where a and b are invertible elements of the field k. Hamilton's quaternion are obtained considering $k=\mathbb{R}$ and $a=b=-1$. Thus, in our previous notations, A is exactly the Clifford algebra $Cl(a,b,0)$. We can see that the cowreath $(\mathbb{H}\otimes H^{op}, H, \psi)$, obtained by using $Cl(a,b,0)=\mathbb{H}$, is not separable, since $\gamma=0$ would imply either $a=0$ or $b=0$, by Remark 6.3. This is not possible since both a and b must be invertible.

COROLLARY 6.4. [**FR**, Cor. 5.0.5] *Let (A,ρ) be a generalized quaternion algebra, endowed with the H-comodule algebra structure defined by* (84)-(87). *The cowreath $(A\otimes H^{op}, H, \psi)$ cannot be rt-separable.*

COROLLARY 6.5. [**FR**, Cor. 5.0.3] *Let (A,ρ) be a generalized quaternion algebra, endowed with the H-comodule algebra structure defined by* (88)-(91). *The cowreath $(A\otimes H^{op}, H, \psi)$ cannot be rt-separable.*

The previous corollaries led us to the following questions.

QUESTION 6.6. Is it possible to define an H-comodule algebra structure on the generalized quaternion algebra $A=Cl(a,b,0)$ such that the induced cowreath $(A\otimes H^{op}, H, \psi)$ is rt-separable?

QUESTION 6.7. Can we obtain a complete classification of the H-comodule algebra structures on a four-dimensional Clifford algebra $A=Cl(\alpha,\beta,\gamma)$ and determine which of these yield an induced cowreath $(A\otimes H^{op}, H, \psi)$ that is rt-separable?

The answer to the former is given by Theorem 6.1 in Chapter 3, where it is proved that there is no H-coaction that makes the cowreath $(A\otimes H^{op}, H, \psi)$ rt-separable, when $A=Cl(a,b,0)$ is a generalized quaternion algebra. This is ultimately due to the fact that A is a semisimple algebra (see Example 5.15). Notice that this theorem is a generalization of Corollaries 6.4 and 6.5. A complete answer to Question 6.7 is obtained in Chapter 3: the table displayed in Section 3 characterizes every possible H-coaction that yields an rt(h)-separable cowreath and the corresponding Casimir element.

In order to obtain the aforementioned results we started by determining a complete classification of the H-comodule algebra structures on a four-dimensional Clifford algebra $A=Cl(\alpha,\beta,\gamma)$. This is the content of the next chapter.

CHAPTER 2

A study of H-coactions on Clifford algebras

In this chapter we explain in detail the steps followed to classify all H-coactions on a four-dimensional Clifford algebra $A = Cl(\alpha, \beta, \gamma)$ when H is Sweedler's Hopf algebra. In the first section we show how any such coaction is completely determined by the choice of a couple (φ, d), where φ is an algebra involution of A and d is a φ-derivation of A such that $d^2 \equiv 0$ and φ and d anticommute (this is actually true for any finite-dimensional algebra A). In the next sections we display all the calculations needed to determine all the algebra involutions of A and the corresponding derivations, thus achieving in the end the desired classification through the explicit correspondence between H-coactions and couples (φ, d) (see Theorem 1.8). This classification is contained in a table in Section 5.

1. An equivalent description of coactions

Let once again H be Sweedler's Hopf algebra over a field k of characteristic $\mathrm{char}(k) \neq 2$ and let $A = Cl(\alpha, \beta, \gamma)$ be the four-dimensional Clifford algebra generated by elements G, X such that

$$G^2 = \alpha \in k, \tag{92}$$

$$X^2 = \beta \in k, \tag{93}$$

$$XG + GX = \gamma \in k. \tag{94}$$

We already know that A admits a canonical H-comodule algebra structure, which makes it a cleft extension of k over H (see [**PVO2**]). The aim of this section is to determine a complete characterization of all the possible H-comodule algebra structures on A. These coactions can be used to build different cowreaths $(A \otimes H^{op}, H, \psi)$ whose properties extend the results contained in Theorems 6.1 and 6.2 in the previous chapter.

1.1. A useful correspondence. First of all we want to show that there is an isomorphism of categories

$$\mathrm{Vec}_k^H \cong {}_{H^{cop}}\mathrm{Vec}_k$$

that preserves algebras, so to make clear that each right H-coaction corresponds to a unique left H^{cop}-action. This will allow us to determine a full characterization of H-comodule structures of A by determining the corresponding H^{cop}-actions.

Let us write $\rho(m) = m_0 \otimes m_1$ for every $m \in M$. It is known (cf. [**CMZ**], p. 10) that when H is finite-dimensional there is an isomorphism of categories

$$F' : \mathrm{Vec}_k^H \to {}_{H^*}\mathrm{Vec}_k$$

given by

$$F'(M, \rho) = (M, \mu_\rho),$$

where

$$\mu_\rho(h^* \otimes m) = m_0 h^*(m_1)$$

for every $h \in H$ and every $m \in M$. The inverse of this functor is given by $G : {}_{H^*}\mathrm{Vec}_k \to \mathrm{Vec}_k^H$, $G(M, \mu) = (M, \rho_\mu)$, where

$$\rho_\mu(m) = \mu(h_i^* \otimes m) \otimes h_i$$

for every $m \in M$. Here h_i denotes the elements of a basis of H and h_i^* their duals. Both the functor F' and its inverse G send a map to itself, i.e. a map is H-colinear if, and only if, is H^*-linear.

In this particular case we also have that H is self dual, i.e. there exists an Hopf algebra map $\psi : H \to H^*$ that gives an isomorphism $H \cong H^*$. It is defined by (cf. [**PVO**], Prop. 1)

$$\psi(1) = 1^* + g^* = \varepsilon_H, \quad \psi(g) = 1^* - g^*, \quad \psi(x) = x^* + (gx)^*, \quad \psi(gx) = x^* - (gx)^*.$$

This could immediately be used to determine an isomorphism of categories

$$\mathrm{Vec}_k^H \cong {}_H\mathrm{Vec}_k,$$

but since in the sequel we work best with a flipped comultiplication, we are going to modify ψ in order to define a new Hopf algebra isomorphism $\varphi : H^{cop} \to H^*$.

LEMMA 1.1. *The k-linear map $\varphi : H^{cop} \to H^*$ defined by*

$$\varphi(1) = 1^* + g^* = \varepsilon_H, \quad \varphi(g) = 1^* - g^*, \quad \varphi(x) = -x^* + (gx)^*, \quad \varphi(gx) = -x^* - (gx)^*$$

is an Hopf algebra isomorphism.

PROOF. By using the following multiplication table one can easily prove that φ is an algebra morphism.

$\star$	1^*	g^*	x^*	$(gx)^*$
1^*	1^*	0	x^*	0
g^*	0	g^*	0	$(gx)^*$
x^*	0	x^*	0	0
$(gx)^*$	$(gx)^*$	0	0	0

TABLE 1. Multiplication table in H^*

Similarly one can use

$$\Delta(1^*) = 1^* \otimes 1^* + g^* \otimes g^* \tag{95}$$

$$\Delta(g^*) = 1^* \otimes g^* + g^* \otimes 1^*, \tag{96}$$

$$\Delta(x^*) = x^* \otimes 1^* - (gx)^* \otimes g^* + 1^* \otimes x^* + g^* \otimes (gx)^*, \tag{97}$$

$$\Delta((gx)^*) = (gx)^* \otimes 1^* - x^* \otimes g^* + g^* \otimes x^* + 1^* \otimes (gx)^*, \tag{98}$$

to show that φ is a coalgebra map. Finally, to check that φ is invertible, one can write its matrix with respect to the basis $(h_i)_{i=1,\dots,4}$ and $(h_i^*)_{i=1,\dots,4}$ and calculate its determinant:

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{vmatrix} = -4.$$

□

Hence we can use the isomorphism φ to define a mapping

$$U : {}_{H^{cop}}\mathrm{Vec}_k \to \mathrm{Vec}_k^H, \quad U(M,\mu) = (M, \rho_\mu),$$

where

$$\rho_\mu(m) = \mu(\varphi^{-1}(h_i^*) \otimes m) \otimes h_i$$

and $U(f) = f$ for every H^{cop}-linear f. In order to show that U is a functor, we will make use of Table 1 again. We have

$$\begin{aligned} (\rho_\mu \otimes H)\rho_\mu(m) &= \mu(\varphi^{-1}(h_j'^*) \otimes \mu(\varphi^{-1}(h_i^*) \otimes m)) \otimes h_j' \otimes h_i \\ &= \mu(\varphi^{-1}(h_j'^*) \cdot \varphi^{-1}(h_i^*) \otimes m) \otimes h_j' \otimes h_i \\ &= \mu(\varphi^{-1}(h_j'^* \star h_i^*) \otimes m) \otimes h_j' \otimes h_i \end{aligned}$$

which, by using Table 1, we can explicitly write as

$$\mu(\varphi^{-1}(1^*) \otimes m) \otimes 1 \otimes 1 + \mu(\varphi^{-1}(x^*) \otimes m) \otimes 1 \otimes x + \mu(\varphi^{-1}(g^*) \otimes m) \otimes g \otimes g +$$
$$+\mu(\varphi^{-1}((gx)^*) \otimes m) \otimes g \otimes gx + \mu(\varphi^{-1}(x^*) \otimes m) \otimes x \otimes g + \mu(\varphi^{-1}((gx)*) \otimes m) \otimes gx \otimes 1,$$

which is equal to

$$\mu(\varphi^{-1}(h_i^*)\otimes m)\otimes\Delta(h_i)=(M\otimes\Delta)(\mu(\varphi^{-1}(h_i^*)\otimes m)\otimes h_i)=(M\otimes\Delta)\rho_\mu(m)$$

for every $m\in M$. Furthermore

$$\begin{aligned}(M\otimes\varepsilon_H)\rho_\mu(m) &= \mu(\varphi^{-1}(h_i^*)\otimes m)\otimes\varepsilon_H(h_i)\\ &= \mu(\varphi^{-1}(\varepsilon_H(h_i)h_i^*)\otimes m)\\ &= \mu(\varphi^{-1}(\varepsilon_H)\otimes m)\\ &= \mu(1_H\otimes m)\\ &= m\end{aligned}$$

for every $m\in M$ and thus we can conclude that (M,ρ_μ) is an object in Vec_k^H. Now let us consider two H^{cop}-modules M,N and an H^{cop}-linear map $f:M\to N$. We prove that f is automatically H-colinear. We have

$$\begin{aligned}(f\otimes H)\rho_M(m) &= (f\otimes H)(\mu_M(\varphi^{-1}(h_i^*)\otimes m)\otimes h_i)\\ &= (f\circ\mu_M)(\varphi^{-1}(h_i^*)\otimes m)\otimes h_i\\ &= \mu_N(\varphi^{-1}(h_i^*)\otimes f(m))\otimes h_i\\ &= (\rho_N\circ f)(m)\end{aligned}$$

which is what we wanted. Finally, since U is the identity on morphisms, we can conclude that U is a functor.

U has an inverse $V:\mathrm{Vec}_k^H\to{}_{H^{cop}}\mathrm{Vec}_k$ defined by

$$V:\mathrm{Vec}_k^H\to{}_{H^{cop}}\mathrm{Vec}_k,\quad V(M,\rho)=(M,\mu_\rho),$$

where

$$\mu_\rho(h\otimes m)=(\varphi(h)(m_1))m_0$$

and $V(f)=f$ for every H-colinear f.

We have

$$\begin{aligned}\mu_\rho(H\otimes\mu_\rho)(h'\otimes h\otimes m) &= \mu_\rho(h'\otimes(\varphi(h)(m_1))m_0)\\ &= (\varphi(h)(m_1))(\varphi(h')(m_{0_1}))m_{0_0}\\ &= (\varphi(h)(m_{1_2}))(\varphi(h')(m_{1_1}))m_0\\ &= (\varphi(h')\star\varphi(h))(m_1)m_0\\ &= \varphi(h'h)(m_1)m_0\\ &= \mu_\rho(m\otimes M)(h'\otimes h\otimes m)\end{aligned}$$

for every $h,h'\in H$ and every $m\in M$. Moreover

$$\begin{aligned}\mu_\rho(1_H\otimes m) &= (\varphi(1_H)(m_1))(m_0)\\ &= \varepsilon(m_1)m_0\\ &= m\end{aligned}$$

for every $m\in M$ and so we can conclude that (M,μ_ρ) is an object in ${}_{H^{cop}}\mathrm{Vec}_k$. Now we consider two H-comodules M,N and an H-colinear map $f:M\to N$. We prove that f is automatically H^{cop}-linear. We have

$$\begin{aligned}(f\circ\mu_M)(h\otimes m) &= (\varphi(h)(m_1))f(m_0)\\ &= (\varphi(h)(f(m)_1)f(m)_0\\ &= \mu_N(h\otimes f(m))\\ &= \mu_N(H\otimes f)(h\otimes m)\end{aligned}$$

for every $h\in H$ and every $m\in M$. Since V is the identity on morphisms, we conclude that V is a functor.

Finally we want to prove that $UV=\mathrm{Id}_{\mathrm{Vec}_k^H}$ and $VU=\mathrm{Id}_{{}_{H^{cop}}\mathrm{Vec}_k}$. We have $UV(M,\rho)=(M,\rho_{\mu_\rho})$

$$\begin{aligned}\rho_{\mu_\rho}(m) &= \mu_\rho(\varphi^{-1}(h_i^*)\otimes m)\otimes h_i\\ &= (\varphi\varphi^{-1}(h_i^*))(m_1)m_0\otimes h_i\end{aligned}$$

$$
\begin{aligned}
&= h_i^*(m_1)m_0 \otimes h_i \\
&= m_0 \otimes h_i^*(m_1)h_i \\
&= m_0 \otimes m_1 \\
&= \rho(m)
\end{aligned}
$$

for every $m \in M$. Moreover $VU(M,\mu) = (M, \mu_{\rho_\mu})$ and

$$
\begin{aligned}
\mu_{\rho_\mu}(h \otimes m) &= (\varphi(h)(h_i))\mu(\varphi^{-1}(h_i^*) \otimes m) \\
&= \mu([\varphi(h)(h_i)]\varphi^{-1}(h_i^*) \otimes m) \\
&= \mu(\varphi^{-1}([\varphi(h)(h_i)]h_i^*) \otimes m).
\end{aligned}
$$

Observe that the $\varphi(h)(h_i)$'s are the coordinates of the vector $\varphi(h) \in H^*$ on the dual basis h_i^*, which means that $[\varphi(h)(h_i)]h_i^* = \varphi(h)$. This allows to conclude that

$$\mu_{\rho_\mu}(h \otimes m) = \mu(\varphi^{-1}(\varphi(h)) \otimes m) = \mu(h \otimes m)$$

for any $h \in H$ and any $m \in M$. Therefore we have proved that the functors U and V previously defined give an isomorphism of categories.

Another crucial observation is that both U and V send algebras to algebras, i.e. the image (A, ρ_μ) of an H^{cop}-module algebra (A,μ) is an H-comodule algebra and, conversely, the image $(V(A), \mu_\rho)$ of an H-comodule algebra (A,ρ) is an H^{cop}-module algebra.

Suppose (A,μ) is an H^{cop}-module algebra. This means that μ satisfies (48)-(49). To prove that A is also an H-comodule algebra we need to show (51) and (52) hold.

If we write the LHS of (51) we get

$$\rho_\mu(a)\rho_\mu(b) \quad = \quad [\mu(\varphi^{-1}(h_i^*) \otimes a) \otimes h_i][\mu(\varphi^{-1}(h_j'^*) \otimes b) \otimes h_j']$$

that explicitly reads

$$
\begin{aligned}
&[\mu(\varphi^{-1}(1_H^*) \otimes a)\mu(\varphi^{-1}(1_H^*) \otimes b) + \mu(\varphi^{-1}(g^*) \otimes a)\mu(\varphi^{-1}(g^*) \otimes b)] \otimes 1_H + \\
&+[\mu(\varphi^{-1}(1_H^*) \otimes a)\mu(\varphi^{-1}(g^*) \otimes b) + \mu(\varphi^{-1}(g^*) \otimes a)\mu(\varphi^{-1}(1_H^*) \otimes b)] \otimes g + \\
&+[\mu(\varphi^{-1}(1^*) \otimes a)\mu(\varphi^{-1}(x^*) \otimes b) + \mu(\varphi^{-1}(g^*) \otimes a)\mu(\varphi^{-1}((gx)^*) \otimes b) + \\
&-\mu(\varphi^{-1}((gx)^*) \otimes a)\mu(\varphi^{-1}(g^*) \otimes b) + \mu(\varphi^{-1}(x^*) \otimes a)\mu(\varphi^{-1}(1^*) \otimes b)] \otimes x + \\
&+[\mu(\varphi^{-1}(1^*) \otimes a)\mu(\varphi^{-1}((gx)^*) \otimes b) + \mu(\varphi^{-1}(g^*) \otimes a)\mu(\varphi^{-1}(x^*) \otimes b) + \\
&-\mu(\varphi^{-1}(x^*) \otimes a)\mu(\varphi^{-1}(g^*) \otimes b) + \mu(\varphi^{-1}((gx)^*) \otimes a)\mu(\varphi^{-1}(1^*) \otimes b)] \otimes gx.
\end{aligned}
$$

By using equalities (95)-(98) we can check that this amounts to $[\mu(\varphi^{-1}((h_i^*)_1) \otimes a)\mu(\varphi^{-1}((h_i^*)_2) \otimes b)] \otimes h_i$. Therefore we can conclude that

$$
\begin{aligned}
\rho_\mu(a)\rho_\mu(b) &= [\mu(\varphi^{-1}(h_i^*) \otimes a) \otimes h_i][\mu(\varphi^{-1}(h_j'^*) \otimes b) \otimes h_j'] \\
&= [\mu(\varphi^{-1}((h_i^*)_1) \otimes a)\mu(\varphi^{-1}((h_i^*)_2) \otimes b)] \otimes h_i \\
&=^{1} [\mu((\varphi^{-1}(h_i^*))_2 \otimes a)\mu((\varphi^{-1}(h_i^*))_1 \otimes b)] \otimes h_i \\
&\overset{(48)}{=} \mu(\varphi^{-1}(h_i^*) \otimes ab) \otimes h_i \\
&= \rho_\mu(ab),
\end{aligned}
$$

for every $a, b \in A$, i.e. that (51) holds. Furthermore

$$
\begin{aligned}
\rho_\mu(1_A) &= \mu(\varphi^{-1}(h_i^*) \otimes 1_A) \otimes h_i \\
&\overset{(49)}{=} \varepsilon_H(\varphi^{-1}(h_i^*))1_A \otimes h_i \\
&= \varepsilon_{H^*}(h_i^*)1_A \otimes h_i
\end{aligned}
$$

[1] Remember that $\varphi : H^{cop} \to H^*$, i.e. comultiplication of the domain is flipped.

$$\begin{aligned} &= h_i^*(1_H)1_A \otimes h_i \\ &= 1_A \otimes 1_H \end{aligned}$$

and thus (A, ρ_μ) is an H-comodule algebra.

Now suppose (A, ρ) is an H-comodule algebra. This means that (51) and (52) hold. To prove that (A, μ_ρ) is an H^{cop}-module algebra we need to show that (48) and (49) hold.

We have

$$\begin{aligned} \mu_\rho(h \otimes ab) &= (\varphi(h)((ab)_1))(ab)_0 \\ &\overset{(51)}{=} (\varphi(h)(a_1 b_1))a_0 b_0 \\ &= [(\varphi(h))_1(a_1)(\varphi(h))_2(b_1)]a_0 b_0 \\ &= [\varphi(h_2)(a_1)\varphi(h_1)(b_1)]a_0 b_0 \\ &= (\varphi(h_2)(a_1)a_0)(\varphi(h_1)(b_1)b_0) \\ &= \mu_\rho(h_2 \otimes a)\mu_\rho(h_1 \otimes b) \end{aligned}$$

for every $h \in H$ and every $a, b \in A$. Moreover

$$\begin{aligned} \mu_\rho(h \otimes 1_A) &\overset{(52)}{=} (\varphi(h)(1_H))1_A \\ &= \varepsilon_{H^*}(\varphi(h))1_A \\ &= \varepsilon(h)1_A, \end{aligned}$$

for every $h \in H$.

We gather all these results in one proposition.

PROPOSITION 1.2. *Let $\varphi : H^{cop} \to H^*$ be the Hopf algebra isomorphism defined in Lemma 1.1. The assignment*

$$U : {}_{H^{cop}}\mathrm{Vec}_k \to \mathrm{Vec}_k^H, \quad U(M, \mu) = (M, \rho_\mu),$$

where

$$\rho_\mu(m) = \mu(\varphi^{-1}(h_i^*) \otimes m) \otimes h_i$$

for every $m \in M$, and $U(f) = f$ for every H^{cop}-linear f defines an invertible functor. Its inverse is given by the assignment

$$V : \mathrm{Vec}_k^H \to {}_{H^{cop}}\mathrm{Vec}_k, \quad V(M, \rho) = (M, \mu_\rho),$$

where

$$\mu_\rho(h \otimes m) = (\varphi(h)(m_1))m_0$$

for every $h \in H$ and every $m \in M$, and $V(f) = f$ for every H-colinear f. Moreover both U and V preserve algebras.

This means that each H-coaction ρ can be expressed in terms of a unique H^{cop}-action:

$$\rho(a) = \mu(\varphi^{-1}(h_i^*) \otimes a) \otimes h_i$$

and explicitly

$$\begin{aligned} \rho(a) &= \mu(\varphi^{-1}(1^*) \otimes a) \otimes 1 + \mu(\varphi^{-1}(g^*) \otimes a) \otimes g + \mu(\varphi^{-1}(x^*) \otimes a) \otimes x + \mu(\varphi^{-1}((gx)^*) \otimes a) \otimes gx \\ &= \frac{1}{2}\left[\mu((1+g) \otimes a) \otimes 1 + \mu((1-g) \otimes a) \otimes g + \mu((-x-gx) \otimes a) \otimes x + \mu((x-gx) \otimes a) \otimes gx\right] \\ &= a \otimes \frac{1+g}{2} + \mu(g \otimes a) \otimes \frac{1-g}{2} - \mu(x \otimes a) \otimes \frac{x-gx}{2} - \mu(gx \otimes a) \otimes \frac{x+gx}{2}. \end{aligned}$$

REMARK 1.3. It is clear that, since $\mu(gx \otimes a) = \mu(g \otimes \mu(x \otimes a))$, ρ is completely determined once we know $\mu(g \otimes a)$ and $\mu(x \otimes a)$ for every $a \in A$, i.e. how g and x act on each element of A.

Let us show that actually H^{cop}-actions on a finite-dimensional algebra A are in bijective correspondence with pairs (φ, d), of suitable maps.

1.2. **H^{cop}-actions, involutions and derivations.** We have seen that each H^{cop}-action is completely determined by the action of elements g and x on the elements of A. We will show that these correspond to the choice of an involution φ and of a φ-derivation d on the algebra A, using a well-established approach (see, e.g., [**CY, MS**] and Examples 4.1.6 and 4.1.8 in [**Mo**]). We start by giving the following definition.

DEFINITION 1.4. Let V be a k-vector space. We call a map $\varphi : V \to V$ a *k-linear involution* if $\varphi^2 = \mathrm{Id}_V$. If A is an algebra and $\varphi : A \to A$ is a k-linear involution which is also an algebra endomorphism, we will call it an *A-linear involution* (or an *algebra involution* or simply an *involution*).

Fix an algebra morphism φ. A k-linear map $d : A \to A$ such that

$$d(ab) = d(a)b + \varphi(a)d(b)$$

for every $a, b \in A$ is called a *φ-derivation* (or a *skew-derivation*).

Let us consider an H^{cop}-action $\mu : H \otimes A \to A$ and let us set $\varphi(a) := \mu(g \otimes a)$. Clearly $\varphi : A \to A$ is a k-linear map and furthermore, for every $a, b \in A$,

$$\varphi(ab) = \mu(g \otimes ab) \overset{(48)}{=} \mu(g_2 \otimes a)\mu(g_1 \otimes b) = \mu(g \otimes a)\mu(g \otimes b) = \varphi(a)\varphi(b),$$
$$\varphi(1_A) \overset{(49)}{=} \mu(g \otimes 1_A) = \varepsilon(g)1_A = 1_A,$$

which means that φ is an algebra endomorphism. Finally

$$\varphi(\varphi(a)) = \mu(g \otimes \mu(g \otimes a)) = \mu(g^2 \otimes a) = \mu(1_H \otimes a) = a,$$

for every $a \in A$, so $o(\varphi) \leq 2$, that is, φ is an involution on A. Next, if we set $d(a) := \mu(x \otimes a)$, we have a k-linear map $d : A \to A$ such that

$$d(ab) = \mu(x \otimes ab) \overset{(48)}{=} \mu(x \otimes a)\mu(1_H \otimes b) + \mu(g \otimes a)\mu(x \otimes b) = d(a)b + \varphi(a)d(b),$$

for every $a, b \in A$, that is, $d : A \to A$ is a φ-derivation. Furthermore, for any $a \in A$,

$$d(d(a)) = \mu(x \otimes \mu(x \otimes a)) = \mu(x^2 \otimes a) = \mu(0 \otimes a) = 0,$$

which means that d^2 is the zero map. Finally

$$d(\varphi(a)) = \mu(x \otimes \mu(g \otimes a)) = \mu(xg \otimes a) = -\mu(gx \otimes a) = -\mu(g \otimes \mu(x \otimes a)) = -\varphi(d(a)),$$

for every $a \in A$, which means that φ and d must anticommute.

In this way we have established an assignment

$$(\mu : H^{cop} \otimes A \to A) \overset{\Phi}{\longmapsto} (\varphi := \mu(g \otimes -), d := \mu(x \otimes -)),$$

so that to each H^{cop}-action corresponds a couple of maps (φ, d), where φ is an involution, d is a φ-derivation such that $d^2 \equiv 0$ and $\varphi d = -d\varphi$.

Now let us fix a couple of k-linear maps $(\varphi : A \to A, d : A \to A)$ satisfying said properties. We are going to show that we can define an H^{cop}-action on A, say $\mu : H^{cop} \otimes A \to A$, by setting $\mu(1 \otimes a) = a$, $\mu(g \otimes a) := \varphi(a)$, $\mu(x \otimes a) := d(a)$ and $\mu(gx \otimes a) = \varphi(d(a))$ for every $a \in A$. We need to show that (A, μ) is an H^{cop}-module and that (48)-(49) hold.

By definition $\mu(1_H \otimes a) = a$, therefore we only need to prove that

$$\mu(h' \otimes \mu(h \otimes a)) = \mu(h'h \otimes a)$$

for every $h, h' \in H^{cop}$ and every $a \in A$, to show that (A, μ) is an H^{cop}-module. It is sufficient to pick h and h' among elements of the basis of H.

Set $h' = 1_H$. Then, by definition, $\mu(1_H \otimes \mu(h \otimes a)) = \mu(h \otimes a) = \mu(1_H \cdot h \otimes a)$ for every $a \in A$.

Set $h' = g$. If $h = 1_H$ we get $\mu(g \otimes \mu(1_H \otimes a)) = \mu(g \otimes a)$. If $h = x$ we get $\mu(g \otimes \mu(x \otimes a)) = \varphi(d(a)) = \mu(gx \otimes a)$, which is true by definition. If $h = gx$, then $\mu(g \otimes \mu(gx \otimes a)) = \varphi(\varphi(d(a)) = d(a) = \mu(x \otimes a) = \mu(ggx \otimes a)$.

Now set $h' = x$. If $h = 1_H$ we get $\mu(x \otimes \mu(1_H \otimes a)) = \mu(x \otimes a)$. If $h = g$ we get $\mu(x \otimes \mu(g \otimes a)) = d(\varphi(a)) = -\varphi(d(a)) = -\mu(gx \otimes a) = \mu(xg \otimes a)$. If $h = gx$ we get $\mu(x \otimes \mu(gx \otimes a)) = d(\varphi(d(a))) = -d^2(\varphi(a)) = 0 = -\mu(xxg \otimes a) = \mu(xgx \otimes a)$.

Finally set $h' = gx$. If $h = 1_H$ we get $\mu(gx \otimes \mu(1_H \otimes a)) = \mu(gx \otimes a)$. If $h = g$, then $\mu(gx \otimes \mu(g \otimes a)) = \varphi(d(\varphi(a))) = -d(\varphi(\varphi(a))) = -d(a) = -\mu(x \otimes a) = -\mu(xgg \otimes a) = \mu(gxg \otimes a)$. If $h = x$, then $\mu(gx \otimes \mu(x \otimes a)) = \varphi(d(d(a))) = -\varphi(d^2(a)) = 0 = \mu(gxx \otimes a)$. If $h = gx$, then $\mu(gx \otimes \mu(gx \otimes a)) = \varphi(d(\varphi(d(a)))) = -d^2(a) = 0 = -\mu(gxxg \otimes a) = \mu(gxgx \otimes a)$.

Therefore (A, μ) is an H^{cop}-module. Now we prove that it is an H^{cop}-module algebra. We have

$$\begin{aligned}
\mu(1_H \otimes 1_A) &= 1_A = \varepsilon(1_H)1_A, \\
\mu(g \otimes 1_A) &= \varphi(1_A) = 1_A = \varepsilon(g)1_A, \\
\mu(x \otimes 1_A) &= d(1_A) = 0 = \varepsilon(x)1_A, \\
\mu(gx \otimes 1_A) &= \varphi(d(1_A)) = \varphi(0) = 0 = \varepsilon(gx)1_A,
\end{aligned}$$

therefore (49) holds.

To prove (48), it is sufficient to prove that $\mu(h \otimes ab) = \mu(h_2 \otimes a)\mu(h_1 \otimes b)$ for every choice of h among the elements of the basis of H and every $a, b \in A$. We proceed by cases.

Set $h = 1_H$. We get $\mu(1_H \otimes ab) = ab = \mu(1_H \otimes a)\mu(1_H \otimes b)$ for every $a, b \in A$.

Now set $h = g$. We get $\mu(g \otimes ab) = \varphi(ab) = \varphi(a)\varphi(b) = \mu(g \otimes a)\mu(g \otimes b)$ for every $a, b \in A$, since φ is an algebra morphism.

Set $h = x$. We get $\mu(x \otimes ab) = d(ab) = d(a)b + \varphi(a)d(b) = \mu(x \otimes a)\mu(1_A \otimes b) + \mu(g \otimes a)\mu(x \otimes b)$ for every $a, b \in A$.

Finally set $h = gx$. We get $\mu(gx \otimes ab) = \varphi(d(ab)) = \varphi(d(a)b + \varphi(a)d(b)) = \varphi(d(a))\varphi(b) + a\varphi(d(b)) = \mu(gx \otimes a)\mu(g \otimes b) + \mu(1_H \otimes a)\mu(gx \otimes b)$ for every $a, b \in A$, so that (48) is proved.

In this way we have established an assignment

$$\begin{aligned}
(\varphi, d) \quad \overset{\Psi}{\longmapsto} \quad \mu : H^{cop} \otimes A &\longrightarrow A \\
1 \otimes a &\longmapsto a \\
g \otimes a &\longmapsto \varphi(a) \\
x \otimes a &\longmapsto d(a) \\
gx \otimes a &\longmapsto \varphi(d(a))
\end{aligned}$$

so that to each couple of maps (φ, d), where φ is an involution, d is a φ-derivation such that $d^2 \equiv 0$ and $\varphi d = -d\varphi$, corresponds an H^{cop}-action. It is straightforward that the assignments Φ and Ψ are inverse to each other, and therefore the correspondence between H^{cop}-actions and pairs (φ, d) such that $\varphi^2 = \mathrm{Id}_A$, $d^2 \equiv 0$ and $\varphi d = -d\varphi$ is bijective.

PROPOSITION 1.5. *Let A be a finite-dimensional algebra over a field k of characteristic $char(k) \neq 2$, then an H^{cop}-action on A is completely determined by a choice of:*

(1) an automorphism φ of A of order $o(\varphi) \leq 2$ (i.e. an involution or the identity),
(2) a φ-derivation such that $d^2 \equiv 0$ and $\varphi d = -d\varphi$.

REMARK 1.6. In [**CY**] a more general result (see Prop. 4 ibid.) was proved, that characterizes the action of a Taft's algebra H_m on a finite-dimensional algebra A in terms of automorphisms and skew-derivations on A. Sweedler's algebra H is one of Taft's algebras, therefore our proposition can be derived directly as a corollary of Centrone and Yasumura's one. We included all details for sake of clarity and for further generalizations, not included in [**CY**], that will be shown in the sequel.

1.3. $\mathbb{Z}_2$-gradings. Since φ is an automorphism of order $o(\varphi) \leq 2$ this means that its eigenvalues are 1 and -1 and A admits a decomposition $A = A_+ \oplus A_-$, where

$$A_\pm = \{a \in A \mid \varphi(a) = \pm a\}.$$

The ground field k is always contained in A_+, while A_- can reduce to $\{0\}$ (clearly if, and only if, $\varphi = \mathrm{Id}$). Notice that $A_iA_j \subseteq A_{ij}$, i.e. φ induces a $\mathbb{Z}_2$-grading. In this case A is usually called a *superalgebra*.

Conversely if we suppose that A admits a $\mathbb{Z}_2$-grading $A = A_+ \oplus A_-$, then we can define a k-algebra map $\varphi : A \to A$ so that $\varphi(a) = a$ for every $a \in A_+$ and $\varphi(b) = -b$ for every $b \in A_-$, which is easily seen to be an involution. This correspondence is one to one.

Next we observe that if $\varphi d = -d\varphi$, then, for every $a \in A_\pm$ we have

$$\varphi(d(a)) = (\varphi d)(a) = -(d\varphi)(a) = -d(\varphi(a)) = \mp d(a),$$

which means that $d(A_\pm) \subseteq A_\mp$. Conversely, if the above equality holds then we have that $\varphi d = -d\varphi$, therefore we can restate Proposition 1.5 in the following way.

Proposition 1.7. *Let A be a finite-dimensional algebra over a field k of characteristic $char(k) \neq 2$, then an action of H on A is completely determined by a choice of:*

(1) a $\mathbb{Z}_2$ grading $A = A_+ \oplus A_-$,
(2) a φ-derivation d (where φ defines the above grading) such that $d^2 \equiv 0$ and $d(A_\pm) \subseteq A_\mp$.

1.4. The explicit correspondence. To conclude this section we are going to write down the explicit form of a coaction ρ in term of the corresponding pair (φ, d) given by Proposition 1.5. We have seen that every H-coaction ρ on A is defined by

$$\rho(a) \quad = \quad a \otimes \frac{1+g}{2} + \mu(g \otimes a) \otimes \frac{1-g}{2} - \mu(x \otimes a) \otimes \frac{x-gx}{2} - \mu(gx \otimes a) \otimes \frac{x+gx}{2} \tag{99}$$

for every $a \in A$, where μ is an H^{cop}-action. Since each H^{cop}-action is in bijective correspondence with a pair (φ, d) where $\varphi : A \to A$ is an involution and $d : A \to A$ is a φ-derivation such that $d^2 \equiv 0$ and $\varphi d = -d\varphi$, (99) rewrites as

$$\rho(a) = a \otimes \frac{1+g}{2} + \varphi(a) \otimes \frac{1-g}{2} - d(a) \otimes \frac{x-gx}{2} - \varphi(d(a)) \otimes \frac{x+gx}{2}$$

The main result can be stated in the following way.

Theorem 1.8. *Let A be a finite-dimensional algebra over a field k of characteristic $char(k) \neq 2$, then a right H-comodule algebra structure on A is given by:*

$$\rho(a) = a \otimes \frac{1+g}{2} + \varphi(a) \otimes \frac{1-g}{2} - d(a) \otimes \frac{(1-g)x}{2} - \varphi(d(a)) \otimes \frac{(1+g)x}{2},$$

where

(1) φ is an automorphism of A of order $o(\varphi) \leq 2$ (i.e. an involution),
(2) d is a φ-derivation such that $d^2 \equiv 0$ and $\varphi(d(a)) = -d(\varphi(a))$

or equivalently by:

(1) a $\mathbb{Z}_2$-grading $A = A_+ \oplus A_-$,
(2) a φ-derivation d (where φ defines the above grading) such that $d^2 \equiv 0$ and $d(A_\pm) \subseteq A_\mp$.

Remark 1.9. φ and d are completely determined once we have fixed $\varphi(G)$, $\varphi(X)$, $d(G)$ and $d(X)$.

It is now clear that to have a classification of all H-coactions on a finite-dimensional algebra A is equivalent to have a classification of all the involutions of A and the corresponding skew-derivations satisfying the hypothesis of Theorem 1.8. We start by classifying involutions.

2. Involutions: an approach with eigenvalues

From now on we will consider $A = Cl(\alpha, \beta, \gamma)$ a four-dimensional Clifford algebra on a fixed field k with $\text{char}(k) \neq 2$. Since we want to determine all the involutions of A, it is convenient to proceed by considering them as k-linear involutions that satisfy some further conditions. What these conditions are is the content of the next proposition.

Proposition 2.1. *Let $A = Cl(\alpha, \beta, \gamma)$ be a Clifford algebra and let $\varphi : A \to A$ be a k-linear map. Then φ is an algebra map if, and only if,*

$$\varphi(1_A) = 1_A \tag{100}$$

$$\varphi(G)\varphi(G) = \alpha, \tag{101}$$

$$\varphi(X)\varphi(X) = \beta, \tag{102}$$

$$\varphi(G)\varphi(X) = \varphi(GX), \tag{103}$$

$$\varphi(G)\varphi(X) + \varphi(X)\varphi(G) = \gamma. \tag{104}$$

PROOF. ($\Longleftarrow$) If φ is an algebra map, clearly (100)-(104) are satisfied.

($\Longrightarrow$) Since φ is k-linear, we need to prove that $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in \{1_A, G, X, GX\}$. When $a = 1_A$ this is trivially true, given (100). Now set $a = G$. Then for $b = 1_A$ we get $\varphi(G) = \varphi(G)\varphi(1_A)$ which is again satisfied, in view of (100). For $b = G$ we get

$$\varphi(GG) = \alpha\varphi(1_A) \overset{(100)}{=} \alpha \overset{(101)}{=} \varphi(G)\varphi(G).$$

For $b = X$ we have

$$\varphi(GX) \overset{(103)}{=} \varphi(G)\varphi(X).$$

Finally for $b = GX$ we have

$$\begin{aligned} \varphi(G \cdot GX) &= \varphi(\alpha X) \\ &= \alpha\varphi(X) \\ &\overset{(101)}{=} \varphi(G)\varphi(G)\varphi(X) \\ &\overset{(103)}{=} \varphi(G)\varphi(GX). \end{aligned}$$

Now set $a = X$. Then for $b = 1_A$ we get $\varphi(X) = \varphi(X)\varphi(1_A)$ which is satisfied, in view of (100). For $b = G$ we have

$$\begin{aligned} \varphi(XG) &= \varphi(\gamma - GX) \\ &\overset{(100)}{=} \gamma - \varphi(GX) \\ &\overset{(103)}{=} \gamma - \varphi(G)\varphi(X) \\ &\overset{(104)}{=} \varphi(X)\varphi(G). \end{aligned}$$

For $b = X$ we have

$$\varphi(XX) = \beta\varphi(1_A) \overset{(100)}{=} \beta \overset{(102)}{=} \varphi(X)\varphi(X).$$

Finally, for $b = GX$ we see that

$$\begin{aligned} \varphi(X \cdot GX) &= \varphi(\gamma X - \beta G) \\ &= \gamma\varphi(X) - \beta\varphi(G) \\ &\overset{(102)}{=} \gamma\varphi(X) - \varphi(X)\varphi(X)\varphi(G) \\ &= \varphi(X)(\gamma - \varphi(X)\varphi(G)) \\ &\overset{(104)}{=} \varphi(X)\varphi(G)\varphi(X) \\ &\overset{(103)}{=} \varphi(X)\varphi(GX). \end{aligned}$$

Set $a = GX$. Then for $b = 1_A$ we get $\varphi(GX) = \varphi(GX)\varphi(1_A)$ which is satisfied, in view of (100). For $b = G$ we have

$$\begin{aligned} \varphi(GX \cdot G) &= \varphi(\gamma G - \alpha X) \\ &= \gamma\varphi(G) - \alpha\varphi(X) \\ &\overset{(101)}{=} \gamma\varphi(G) - \varphi(X)\varphi(G)\varphi(G) \\ &= (\gamma - \varphi(X)\varphi(G))\varphi(G) \\ &\overset{(104)}{=} \varphi(G)\varphi(X)\varphi(G) \\ &\overset{(103)}{=} \varphi(GX)\varphi(G). \end{aligned}$$

For $b = X$ we get

$$\begin{aligned}\varphi(GX \cdot X) &= \beta\varphi(G) \\ &\overset{(102)}{=} \varphi(G)\varphi(X)\varphi(X) \\ &\overset{(103)}{=} \varphi(GX)\varphi(X)\end{aligned}$$

and finally for $b = GX$ we get

$$\begin{aligned}\varphi(GX \cdot GX) &= \varphi(\gamma GX - \alpha\beta) \\ &= \gamma\varphi(GX) - \alpha\beta \\ &\overset{(101)+(102)}{=} \gamma\varphi(GX) - \varphi(G)\varphi(G)\varphi(X)\varphi(X) \\ &\overset{(104)}{=} \gamma\varphi(GX) - \varphi(G)(\gamma - \varphi(X)\varphi(G))\varphi(X) \\ &= \varphi(G)\varphi(X)\varphi(G)\varphi(X) \\ &\overset{(103)}{=} \varphi(GX)\varphi(GX).\end{aligned}$$

□

Thanks to Proposition 2.1 we can determine all k-linear involutions of A and then further impose conditions (100)-(104).

Let $\mathrm{Inv}_k(A) = \{\varphi \in \mathrm{End}_k(A) \mid \varphi^2 = \mathrm{Id}\}$ be the set of all k-linear involutions of A. We want to determine the structure of $\mathrm{Inv}_k(A)$ in order to be able to study in detail its subset $\mathrm{Inv}_{\mathrm{Alg}}(A) = \{\varphi \in \mathrm{End}_{\mathrm{Alg}}(A) \mid \varphi^2 = \mathrm{Id}\}$ which contains algebra involutions. Notice that in both cases the considered morphisms are invertible, i.e. they are automorphisms. For sake of completeness we prove the following easy result.

Proposition 2.2. *For a k-vector space A, a k-linear map φ is contained in $Inv_k(A)$ if, and only if, it is diagonalizable with eigenvalues -1 and 1.*

Proof. ($\Longrightarrow$) Suppose $\varphi \in \mathrm{Inv}_k(A)$ and let $M_\varphi \in GL_4(k)$ be its associated matrix. Since $\varphi^2 = \mathrm{Id}$, also $M_\varphi^2 - I = 0$ and we have that the minimal polynomial of M_φ is either $X^2 - 1 = (X-1)(X+1)$, $X - 1$ or $X + 1$. These are all products of distinct linear factors over k, thus M_φ is diagonalizable and so is φ. Now it is clear that if λ is an eigenvalue for M_φ and $v \neq 0$ is an associated eigenvector, we have $\lambda^2 v = M_\varphi^2 v = Iv = v$, so that $\lambda^2 = 1$.

($\Longleftarrow$) Suppose that M_φ is diagonalizable with eigenvalues -1 and 1. Then there exists an invertible matrix $P \in GL_4(k)$ such that $PM_\varphi P^{-1}$ is diagonal with only -1 and 1 on the diagonal. Then it is clear that

$$I = (PM_\varphi P^{-1})(PM_\varphi P^{-1}) = PM_\varphi^2 P^{-1},$$

which is equivalent to $M_\varphi^2 = I$, that is $\varphi^2 = \mathrm{Id}$. □

We can thus identify $\mathrm{Inv}_k(A) = \{PDP^{-1} \mid P \in GL_4(k),\ D \text{ is diagonal},\ \sigma(D) \subseteq \{-1, 1\}\}$, where $\sigma(D)$ indicates the spectrum of D. The diagonal matrices in $GL_4(k)$ with $\sigma(D) \subseteq \{-1, 1\}$ are the following (up to rearrangement of columns and rows):

$$D_0 := \mathrm{Id} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, D_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, D_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$D_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, D_4 := -\mathrm{Id} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

so we can further write $\mathrm{Inv}_k(A) = \{\mathrm{Id}, -\mathrm{Id}, PD_iP^{-1} \mid P \in GL_4(k), i = 1, 2, 3\}$. Now we move our attention to $\mathrm{Inv}_{\mathrm{Alg}}(A)$, i.e. we search for the elements φ in $\mathrm{Inv}_k(A)$ that respect conditions (100)-(104). Since we want (100) to hold, we see that the eigenspace of φ relative to the eigenvalue 1 must have dimension at least 1. This immediately implies that $-\mathrm{Id} \notin \mathrm{Inv}_{\mathrm{Alg}}(A)$. Furthermore suppose that this eigenspace have dimension exactly 1. This means that the 3-dimensional space $\langle G, X, GX\rangle$ coincides with A_-, the eigenspace of φ relative to the eigenvalue -1, and so $-GX = \varphi(GX) = \varphi(G)\varphi(X) = (-G)(-X) = GX$, contradiction. Therefore

$$\mathrm{Inv}_{\mathrm{Alg}}(A) \subseteq \{\mathrm{Id}, PD_iP^{-1} \mid P \in GL_4(k), i = 1, 2\}.$$

We deal with each case separately.

2.1. The case D_1. Suppose $M_\varphi = PD_1P^{-1}$ for some $P = (p_{ij}) \in GL_4(k)$. Clearly P is the matrix whose column are the eigenvectors of A relative to the eigenvalues of M_φ. Let us denote $(q_{ij}) = Q := P^{-1}$ and E_{ij} the matrix with $e_{ij} = 1$ and 0 elsewhere. Then $M_\varphi = P(I - 2E_{44})Q = I - 2PE_{44}Q$. Let ψ be the k-linear map whose matrix is P. We notice that

$$PE_{44}Q = \begin{pmatrix} p_{14}q_{41} & p_{14}q_{42} & p_{14}q_{43} & p_{14}q_{44} \\ p_{24}q_{41} & p_{24}q_{42} & p_{24}q_{43} & p_{24}q_{44} \\ p_{34}q_{41} & p_{34}q_{42} & p_{34}q_{43} & p_{34}q_{44} \\ p_{44}q_{41} & p_{44}q_{42} & p_{44}q_{43} & p_{44}q_{44} \end{pmatrix}$$

and that $(p_{14}, p_{24}, p_{34}, p_{44})^t = \psi(GX)$. Therefore we can rewrite $PE_{44}Q$ as

$$PE_{44}Q = (q_{41}\psi(GX), q_{42}\psi(GX), q_{43}\psi(GX), q_{44}\psi(GX)),$$

provided we interpret a row vector whose coordinates are column vectors as a matrix. Then, if we want (100) to hold we must have $q_{41}\psi(GX) = 0$, i.e. $q_{41} = 0$, since P is invertible and so is ψ.

Now, since $M_\varphi = I - 2PE_{44}Q$, we find $\varphi(G) = G - 2q_{42}\psi(GX)$ and $\varphi(G)^2 = \alpha - 2q_{42}(G\psi(GX) + \psi(GX)G) + 4q_{42}^2\psi(GX)^2$, so that (101) is equivalent to

$$2q_{42}^2\psi(GX)^2 - q_{42}(G\psi(GX) + \psi(GX)G) = 0.$$

Similarly one sees that $\varphi(X) = X - 2q_{43}\psi(GX)$ and that (102) is equivalent to

$$2q_{43}^2\psi(GX)^2 - q_{43}(X\psi(GX) + \psi(GX)X) = 0.$$

Next

$$\varphi(G)\varphi(X) = (G - 2q_{42}\psi(GX))(X - 2q_{43}\psi(GX)) = GX - 2(q_{42}\psi(GX)X + q_{43}G\psi(GX)) + 4q_{42}q_{43}\psi(GX)^2$$

and

$$\varphi(X)\varphi(G) = (X - 2q_{43}\psi(GX))(G - 2q_{42}\psi(GX)) = XG - 2(q_{42}X\psi(GX) + q_{43}\psi(GX)G) + 4q_{42}q_{43}\psi(GX)^2,$$

while $\varphi(GX) = GX - 2q_{44}\psi(GX)$, so (103) reads

$$q_{42}\psi(GX)X + q_{43}G\psi(GX) - 2q_{42}q_{43}\psi(GX)^2 - q_{44}\psi(GX) = 0$$

and (104) becomes

$$4q_{42}q_{43}\psi(GX)^2 - q_{42}(\psi(GX)X + X\psi(GX)) - q_{43}(G\psi(GX) + \psi(GX)G) = 0.$$

In the end we see that morphism conditions (100)-(104) are equivalent to

$$q_{41} = 0 \tag{105}$$

$$2q_{42}^2\psi(GX)^2 - q_{42}(G\psi(GX) + \psi(GX)G) = 0 \tag{106}$$

$$2q_{43}^2\psi(GX)^2 - q_{43}(X\psi(GX) + \psi(GX)X) = 0 \tag{107}$$

$$2q_{42}q_{43}\psi(GX)^2 - q_{42}\psi(GX)X - q_{43}G\psi(GX) + q_{44}\psi(GX) = 0 \tag{108}$$

$$4q_{42}q_{43}\psi(GX)^2 - q_{42}(\psi(GX)X + X\psi(GX)) - q_{43}(G\psi(GX) + \psi(GX)G) = 0 \tag{109}$$

We now use our results on multiplications as linear maps contained in Appendix A to rewrite this set of equations. In particular we employ the formula for anticommutators contained in Example 0.1 to simplify $(G\psi(GX) + \psi(GX)G)$ and $(X\psi(GX) + \psi(GX)X)$, and the one for squares denoted by (262) to calculate $\psi(GX)^2$. We set $\Lambda := p_{14}^2 + \alpha p_{24}^2 + \beta p_{34}^2 + \gamma p_{24}p_{34} - \alpha\beta p_{44}^2$. Then (106) becomes

$$2q_{42}^2 \begin{pmatrix} \Lambda \\ p_{24}(2p_{14} + \gamma p_{44}) \\ p_{34}(2p_{14} + \gamma p_{44}) \\ p_{44}(2p_{14} + \gamma p_{44}) \end{pmatrix} - q_{42} \begin{pmatrix} 2\alpha p_{24} + \gamma p_{34} \\ 2p_{14} + \gamma p_{44} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{110}$$

and similarly (107) yields

$$2q_{43}^2\begin{pmatrix}\Lambda\\ p_{24}(2p_{14}+\gamma p_{44})\\ p_{34}(2p_{14}+\gamma p_{44})\\ p_{44}(2p_{14}+\gamma p_{44})\end{pmatrix} - q_{43}\begin{pmatrix}\gamma p_{24}+2\beta p_{34}\\ 0\\ 2p_{14}+\gamma p_{44}\\ 0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}. \tag{111}$$

If we suppose $(2p_{14}+\gamma p_{44})q_{42}\neq 0$, then the last three equations of (110) give $p_{34}=0=p_{44}$ and $2p_{24}q_{24}=1$. This is equivalent to $\psi(GX)=-2p_{14}q_{42}$ which is a contradiction, since $\psi(GX)$ and $1_A=\psi(1_A)$ are linearly independent (ψ is invertible). This shows that $(2p_{14}+\gamma p_{44})q_{42}=0$. We can prove that $(2p_{14}+\gamma p_{44})q_{43}=0$ by performing analogous steps on (111). Now assume $2p_{14}+\gamma p_{44}\neq 0$, so that $q_{42}=0=q_{43}$. Then (108) yields $q_{44}=0$, since $\psi(GX)\neq 0$. This leads to a contradiction, because Q is invertible and it cannot have a zero column. Therefore we deduce that $2p_{14}+\gamma p_{44}=0$.

Now let us consider (108):

$$2q_{42}q_{43}\begin{pmatrix}\Lambda\\ p_{24}(2p_{14}+\gamma p_{44})\\ p_{34}(2p_{14}+\gamma p_{44})\\ p_{44}(2p_{14}+\gamma p_{44})\end{pmatrix} - q_{42}\begin{pmatrix}\beta p_{34}\\ \beta p_{44}\\ p_{14}\\ p_{24}\end{pmatrix} - q_{43}\begin{pmatrix}\alpha p_{24}\\ p_{14}\\ \alpha p_{44}\\ p_{34}\end{pmatrix} + q_{44}\begin{pmatrix}p_{14}\\ p_{24}\\ p_{34}\\ p_{44}\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix},$$

that is

$$2q_{42}q_{43}\begin{pmatrix}\Lambda\\0\\0\\0\end{pmatrix} - q_{42}\begin{pmatrix}\beta p_{34}\\ \beta p_{44}\\ -\frac{\gamma}{2}p_{44}\\ p_{24}\end{pmatrix} - q_{43}\begin{pmatrix}\alpha p_{24}\\ -\frac{\gamma}{2}p_{44}\\ \alpha p_{44}\\ p_{34}\end{pmatrix} + q_{44}\begin{pmatrix}-\frac{\gamma}{2}p_{44}\\ p_{24}\\ p_{34}\\ p_{44}\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}. \tag{112}$$

Since $QP=I$ and $q_{41}=0$ we know that $p_{24}q_{42}+p_{34}q_{43}+p_{44}q_{44}=1$, so the last row of (112) gives $p_{24}q_{42}+p_{34}q_{43}=\frac{1}{2}=p_{44}q_{44}$, which also implies $p_{44}\neq 0\neq q_{44}$.

Finally (109) rewrites as

$$4q_{42}q_{43}\begin{pmatrix}\Lambda\\0\\0\\0\end{pmatrix} - q_{42}\begin{pmatrix}\gamma p_{24}+2\beta p_{34}\\0\\0\\0\end{pmatrix} - q_{43}\begin{pmatrix}2\alpha p_{24}+\gamma p_{34}\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}.$$

Altogether we recover the following set of scalar equations (where we also used the fact that $2p_{44}q_{44}=1$).

$$q_{41} = 0 \tag{113}$$
$$2q_{42}^2\Lambda - q_{42}(\gamma p_{34}+2\alpha p_{24}) = 0 \tag{114}$$
$$2q_{43}^2\Lambda - q_{43}(\gamma p_{24}+2\beta p_{34}) = 0 \tag{115}$$
$$8q_{42}q_{43}\Lambda - 4\alpha p_{24}q_{43} - 4\beta p_{34}q_{42} - \gamma = 0 \tag{116}$$
$$4p_{24}q_{44}^2 + \gamma q_{43} - 2\beta q_{42} = 0 \tag{117}$$
$$4p_{34}q_{44}^2 + \gamma q_{42} - 2\alpha q_{43} = 0 \tag{118}$$
$$2p_{24}q_{42} + 2p_{34}q_{43} = 1 \tag{119}$$
$$4q_{42}q_{43}\Lambda - \gamma(p_{24}q_{42}+p_{34}q_{43}) - 2\alpha p_{24}q_{43} - 2\beta p_{34}q_{42} = 0 \tag{120}$$

First we notice that (120) is equivalent to (116) thanks to (119), then we divide (117) and (118) by $4q_{44}^2\neq 0$ so to isolate p_{24} and p_{34}:

$$4p_{24}q_{44}^2+\gamma q_{43}-2\beta q_{42}=0 \iff p_{24}=-\frac{\gamma q_{43}-2\beta q_{42}}{4q_{44}^2},$$
$$4p_{34}q_{44}^2+\gamma q_{42}-2\alpha q_{43}=0 \iff p_{34}=-\frac{\gamma q_{42}-2\alpha q_{43}}{4q_{44}^2}.$$

Substituting into (114) and (115) gives

$$q_{42}^2[8q_{44}^2\Lambda+(\gamma^2-4\alpha\beta)]=0,$$

$$q_{43}^2[8q_{44}^2\Lambda + (\gamma^2 - 4\alpha\beta)] = 0,$$

so $8q_{44}^2\Lambda + (\gamma^2 - 4\alpha\beta) = 0$, because q_{42} and q_{43} cannot be zero at the same time (see (119)). Replacing p_{24} and p_{34} with their expression into 119 and (116) yields

$$\alpha q_{43}^2 + \beta q_{42}^2 - \gamma q_{42}q_{43} = q_{44}^2$$

and

$$[8q_{44}^2\Lambda + (\gamma^2 - 4\alpha\beta)]q_{42}q_{43} = 0.$$

We are thus left with

$$\begin{cases} 8q_{44}^2\Lambda + (\gamma^2 - 4\alpha\beta) = 0 \\ p_{24} = -\dfrac{\gamma q_{43} - 2\beta q_{42}}{4q_{44}^2} \\ p_{34} = -\dfrac{\gamma q_{42} - 2\alpha q_{43}}{4q_{44}^2} \\ \alpha q_{43}^2 + \beta q_{42}^2 - \gamma q_{42}q_{43} = q_{44}^2. \end{cases}$$

Now we can make use of the explicit expression of Λ. The first equation is

$$2[(\gamma^2 - 4\alpha\beta)p_{44}^2 + 4\alpha p_{24}^2 + 4\beta p_{34}^2 + 4\gamma p_{24}p_{34}]q_{44}^2 + (\gamma^2 - 4\alpha\beta) = 0,$$

that rewrites as

$$\alpha(\gamma q_{43} - 2\beta q_{42})^2 + \beta(\gamma q_{42} - 2\alpha q_{43})^2 + \gamma(\gamma q_{43} - 2\beta q_{42})(\gamma q_{42} - 2\alpha q_{43}) + 3(\gamma^2 - 4\alpha\beta)q_{44}^2 = 0,$$

and, after a few calculations,

$$(\gamma^2 - 4\alpha\beta)q_{44}^2 = 0 \iff (\gamma^2 - 4\alpha\beta) = 0.$$

We conclude that the only automorphisms in $\mathrm{Inv}_{\mathrm{Alg}}(A)$ that are similar to D_1 are given by

$$\begin{aligned} \varphi(G) &= \frac{\gamma}{2}\cdot\frac{q_{42}}{q_{44}} - \frac{\gamma q_{42}q_{43} - 2\alpha q_{43}^2}{2q_{44}^2}G + \frac{\gamma q_{42}^2 - 2\alpha q_{42}q_{43}}{2q_{44}^2}X - \frac{q_{42}}{q_{44}}GX \\ \varphi(X) &= \frac{\gamma}{2}\cdot\frac{q_{43}}{q_{44}} + \frac{\gamma q_{43}^2 - 2\beta q_{42}q_{43}}{2q_{44}^2}G - \frac{\gamma q_{43}q_{42} - 2\beta q_{42}^2}{2q_{44}^2}X - \frac{q_{43}}{q_{44}}GX \\ \varphi(GX) &= \frac{\gamma}{2} + \frac{\gamma q_{43} - 2\beta q_{42}}{2q_{44}}G + \frac{\gamma q_{42} - 2\alpha q_{43}}{2q_{44}}X \end{aligned}$$

with

$$\begin{cases} \gamma^2 - 4\alpha\beta = 0 \\ \alpha q_{43}^2 + \beta q_{42}^2 - \gamma q_{42}q_{43} = q_{44}^2 \\ q_{44} \neq 0. \end{cases}$$

If we define $\mathbf{A} := \frac{q_{42}}{q_{44}}$ and $\mathbf{B} := \frac{q_{43}}{q_{44}}$, then the involutions similar to D_1 have matrix

$$M_\varphi = \begin{pmatrix} 1 & \frac{\gamma}{2}\mathbf{A} & \frac{\gamma}{2}\mathbf{B} & \frac{\gamma}{2} \\ 0 & \frac{2\alpha\mathbf{B}-\gamma\mathbf{A}}{2}\mathbf{B} & \frac{\gamma\mathbf{B}-2\beta\mathbf{A}}{2}\mathbf{B} & \frac{\gamma\mathbf{B}-2\beta\mathbf{A}}{2} \\ 0 & \frac{\gamma\mathbf{A}-2\alpha\mathbf{B}}{2}\mathbf{A} & \frac{2\beta\mathbf{A}-\gamma\mathbf{B}}{2}\mathbf{A} & \frac{\gamma\mathbf{A}-2\alpha\mathbf{B}}{2} \\ 0 & -\mathbf{A} & -\mathbf{B} & 0 \end{pmatrix},$$

with

$$\begin{cases} \gamma^2 - 4\alpha\beta = 0 \\ \alpha\mathbf{B}^2 + \beta\mathbf{A}^2 - \gamma\mathbf{A}\mathbf{B} = 1. \end{cases}$$

It can be verified by computation (with the help of a software) that any such map is contained in $\mathrm{Inv}_{\mathrm{Alg}}(A)$ and that the associated decomposition of the space A is explicitly given by

$$A_+ = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\mathbf{B} \\ \mathbf{A} \\ 0 \end{pmatrix}, \begin{pmatrix} -\gamma \\ \gamma\mathbf{B} - 2\beta\mathbf{A} \\ \gamma\mathbf{A} - 2\alpha\mathbf{B} \\ 2 \end{pmatrix} \rangle, \quad A_- = \langle \begin{pmatrix} \gamma \\ \gamma\mathbf{B} - 2\beta\mathbf{A} \\ \gamma\mathbf{A} - 2\alpha\mathbf{B} \\ -2 \end{pmatrix} \rangle.$$

2.2. The case D_2. To study the automorphisms with associated matrix similar to D_2 we mimic the steps of the former case. Suppose $M_\varphi = PD_2P^{-1}$ for some $P = (p_{ij}) \in GL_4(k)$. Let us denote $(q_{ij}) = Q := P^{-1}$ and E_{ij} the matrix with $e_{ij} = 1$ and 0 elsewhere. Then $M_\varphi = P(I - 2E_{33} - 2E_{44})Q = I - 2PE_{33}Q - 2PE_{44}Q$. Let ψ be the k-linear map whose matrix is P. We notice that this time

$$PEQ = \begin{pmatrix} p_{13}q_{31} + p_{14}q_{41} & p_{13}q_{32} + p_{14}q_{42} & p_{13}q_{33} + p_{14}q_{43} & p_{13}q_{34} + p_{14}q_{44} \\ p_{23}q_{31} + p_{24}q_{41} & p_{23}q_{32} + p_{24}q_{42} & p_{23}q_{33} + p_{24}q_{43} & p_{23}q_{34} + p_{24}q_{44} \\ p_{33}q_{31} + p_{34}q_{41} & p_{33}q_{32} + p_{34}q_{42} & p_{33}q_{33} + p_{34}q_{43} & p_{33}q_{34} + p_{34}q_{44} \\ p_{43}q_{31} + p_{44}q_{41} & p_{43}q_{32} + p_{44}q_{42} & p_{43}q_{33} + p_{44}q_{43} & p_{43}q_{34} + p_{44}q_{44} \end{pmatrix}.$$

and that $(p_{13}, p_{23}, p_{33}, p_{43})^t = \psi(X)$, $(p_{14}, p_{24}, p_{34}, p_{44})^t = \psi(GX)$. Therefore we can rewrite PEQ as

$$PEQ = (q_{31}\psi(X) + q_{41}\psi(GX), q_{32}\psi(X) + q_{42}\psi(GX), q_{33}\psi(X) + q_{43}\psi(GX), q_{34}\psi(X) + q_{44}\psi(GX)),$$

provided we interpret a row vector whose coordinates are column vectors as a matrix. If we want (100) to hold we must have $q_{31}\psi(X) + q_{41}\psi(GX) = 0$, i.e. $q_{31} = q_{41} = 0$, since P is invertible and so $\psi(X)$ and $\psi(GX)$ are linearly independent.

Since (101) and (102) imply $\varphi(G)^2 \in k \ni \varphi(X)^2$, we deduce that either $\varphi(G)$ and $\varphi(X)$ are elements of the ground field or their coordinates satisfy $2\lambda_1 + \gamma\lambda_4 = 0$ (see Remark 0.2). Remember that φ is invertible (since it is an involution) and thus sends linearly independent vectors to linearly independent vectors. This means that $\varphi(G)$ and $\varphi(X)$ cannot be contained in k (which already contains $1_A = \varphi(1_A)$). We conclude that

$$2(p_{13}q_{32} + p_{14}q_{42}) + \gamma(p_{43}q_{32} + p_{44}q_{42}) = 0, \tag{121}$$

$$2(p_{13}q_{33} + p_{14}q_{43}) + \gamma(p_{43}q_{33} + p_{44}q_{43}) = 0. \tag{122}$$

This observation is of fundamental importance for the sequel. In fact let

$$\begin{aligned} \mathbf{A} &:= q_{11}q_{23} - q_{13}q_{21} \\ \mathbf{B} &:= q_{32}q_{44} - q_{34}q_{42} \\ \mathbf{C} &:= q_{11}q_{24} - q_{14}q_{21} \\ \mathbf{D} &:= q_{32}q_{43} - q_{33}q_{42} \\ \mathbf{E} &:= q_{11}q_{22} - q_{12}q_{21} \\ \mathbf{F} &:= q_{33}q_{44} - q_{34}q_{43} \\ \mathbf{G} &:= q_{12}q_{23} - q_{13}q_{22} \\ \mathbf{H} &:= q_{12}q_{24} - q_{14}q_{22} \\ \mathbf{J} &:= q_{13}q_{24} - q_{14}q_{23} \end{aligned}$$

Then using the well-known expressions for the inverse of a matrix in terms of cofactors, (121) and (122) read[2]

$$2[\mathbf{BG} - \mathbf{DH}] - \gamma\mathbf{DE} = 0, \tag{123}$$

$$2[\mathbf{FG} - \mathbf{DJ}] - \gamma\mathbf{AD} = 0, \tag{124}$$

while $P(E_{33} + E_{44})Q$ can be written in the following form

$$P(E_{33} + E_{44})Q = \frac{1}{\det Q} \begin{pmatrix} 0 & \frac{\gamma}{2}\mathbf{DE} & \frac{\gamma}{2}\mathbf{AD} & \mathbf{FH} - \mathbf{BJ} \\ 0 & \mathbf{CD} - \mathbf{AB} & -\mathbf{AF} & -\mathbf{CF} \\ 0 & \mathbf{BE} & \mathbf{CD} + \mathbf{EF} & \mathbf{BC} \\ 0 & -\mathbf{DE} & -\mathbf{AD} & \mathbf{EF} - \mathbf{AB} \end{pmatrix},$$

[2] Again, a lot of the following rearrangements are best verified with the help of a software.

so that

$$M_\varphi = \frac{1}{\det Q}\begin{pmatrix} \det Q & -\gamma\mathbf{DE} & -\gamma\mathbf{AD} & 2(\mathbf{BJ}-\mathbf{FH}) \\ 0 & \det Q + 2(\mathbf{AB}-\mathbf{CD}) & 2\mathbf{AF} & 2\mathbf{CF} \\ 0 & -2\mathbf{BE} & \det Q - 2(\mathbf{CD}+\mathbf{EF}) & -2\mathbf{BC} \\ 0 & 2\mathbf{DE} & 2\mathbf{AD} & \det Q + 2(\mathbf{AB}-\mathbf{EF}) \end{pmatrix}.$$

Finally, a computation shows that $\det Q = -\mathbf{AB}+\mathbf{CD}+\mathbf{EF}$ and thus

$$M_\varphi = \frac{1}{\det Q}\begin{pmatrix} -\mathbf{AB}+\mathbf{CD}+\mathbf{EF} & -\gamma\mathbf{DE} & -\gamma\mathbf{AD} & 2(\mathbf{BJ}-\mathbf{FH}) \\ 0 & \mathbf{AB}-\mathbf{CD}+\mathbf{EF} & 2\mathbf{AF} & 2\mathbf{CF} \\ 0 & -2\mathbf{BE} & -\mathbf{AB}-\mathbf{CD}-\mathbf{EF} & -2\mathbf{BC} \\ 0 & 2\mathbf{DE} & 2\mathbf{AD} & \mathbf{AB}+\mathbf{CD}-\mathbf{EF} \end{pmatrix}.$$

Now we use again the tools developed in Appendix A to calculate

$$\varphi(G)\varphi(X) = \frac{1}{\det Q}[-(\gamma\mathbf{DE})I + (\mathbf{AB}-\mathbf{CD}+\mathbf{EF})L_G - (2\mathbf{BE})L_X + (2\mathbf{DE})L_G L_X]\varphi(X) =$$

$$= \frac{1}{(\det Q)^2}\begin{pmatrix} -\gamma\mathbf{DE} & \alpha(\mathbf{AB}-\mathbf{CD}+\mathbf{EF})-2\gamma\mathbf{BE} & -2\beta\mathbf{BE} & -2\alpha\beta\mathbf{DE} \\ \mathbf{AB}-\mathbf{CD}+\mathbf{EF} & \gamma\mathbf{DE} & 2\beta\mathbf{DE} & 2\beta\mathbf{BE} \\ -2\mathbf{BE} & -2\alpha\mathbf{DE} & -\gamma\mathbf{DE} & \alpha(\mathbf{AB}-\mathbf{CD}+\mathbf{EF})-2\gamma\mathbf{BE} \\ 2\mathbf{DE} & 2\mathbf{BE} & \mathbf{AB}-\mathbf{CD}+\mathbf{EF} & \gamma\mathbf{DE} \end{pmatrix}\begin{pmatrix} -\gamma\mathbf{AD} \\ 2\mathbf{AF} \\ -\mathbf{AB}-\mathbf{CD}-\mathbf{EF} \\ 2\mathbf{AD} \end{pmatrix} =$$

$$= \frac{1}{(\det Q)^2}\begin{pmatrix} (\gamma^2-4\alpha\beta)\mathbf{AD}^2\mathbf{E} + 2\alpha\mathbf{AF}(\mathbf{AB}-\mathbf{CD}+\mathbf{EF}) - 4\gamma\mathbf{ABEF} + 2\beta\mathbf{BE}(\mathbf{AB}+\mathbf{CD}+\mathbf{EF}) \\ (\gamma\mathbf{AD}-2\beta\mathbf{DE})(-\mathbf{AB}+\mathbf{CD}+\mathbf{EF}) \\ (\gamma\mathbf{DE}-2\alpha\mathbf{AD})(-\mathbf{AB}+\mathbf{CD}+\mathbf{EF}) \\ 4\mathbf{ABEF} - (\mathbf{AB}-\mathbf{CD}+\mathbf{EF})(\mathbf{AB}+\mathbf{CD}+\mathbf{EF}) \end{pmatrix}.$$

It is then clear that (103) is equivalent to

$$(\gamma^2-4\alpha\beta)\mathbf{AD}^2\mathbf{E} + 2\alpha\mathbf{AF}(\mathbf{AB}-\mathbf{CD}+\mathbf{EF}) - 4\gamma\mathbf{ABEF} + 2\beta\mathbf{BE}(\mathbf{AB}+\mathbf{CD}+\mathbf{EF}) = 2(\mathbf{BJ}-\mathbf{FH})(-\mathbf{AB}+\mathbf{CD}+\mathbf{EF}) \tag{125}$$

$$(\gamma\mathbf{AD}-2\beta\mathbf{DE})(-\mathbf{AB}+\mathbf{CD}+\mathbf{EF}) = 2\mathbf{CF}(-\mathbf{AB}+\mathbf{CD}+\mathbf{EF}) \tag{126}$$

$$(\gamma\mathbf{DE}-2\alpha\mathbf{AD})(-\mathbf{AB}+\mathbf{CD}+\mathbf{EF}) = -2\mathbf{BC}(-\mathbf{AB}+\mathbf{CD}+\mathbf{EF}) \tag{127}$$

$$4\mathbf{ABEF} - (\mathbf{AB}-\mathbf{CD}+\mathbf{EF})(\mathbf{AB}+\mathbf{CD}+\mathbf{EF}) = (\mathbf{AB}+\mathbf{CD}-\mathbf{EF})(-\mathbf{AB}+\mathbf{CD}+\mathbf{EF}). \tag{128}$$

It is easy to see that (128) holds trivially. Indeed

$$4\mathbf{ABEF} - (\mathbf{AB}+\mathbf{EF})^2 + \mathbf{C}^2\mathbf{D}^2 = \mathbf{C}^2\mathbf{D}^2 - (\mathbf{AB}-\mathbf{EF})^2 \iff 4\mathbf{ABEF} - (\mathbf{AB}+\mathbf{EF})^2 = -(\mathbf{AB}-\mathbf{EF})^2,$$

which is an identity. Then, since Q is invertible $\det Q \neq 0$ and (126) and (127) become

$$\gamma\mathbf{AD} = 2\mathbf{CF} + 2\beta\mathbf{DE} \tag{129}$$

$$\gamma\mathbf{DE} = -2\mathbf{BC} + 2\alpha\mathbf{AD}. \tag{130}$$

Now let us focus on (101). Thanks to (123) we already know that $\varphi(G)^2 \in k$, thus we only need to recall what is the explicit formulation of its first component (see (262)) and equate it to α:

$$\frac{1}{(\det Q)^2}[(\gamma^2-4\alpha\beta)\mathbf{D}^2\mathbf{E}^2 + \alpha(\mathbf{AB}-\mathbf{CD}+\mathbf{EF})^2 + 4\beta\mathbf{B}^2\mathbf{E}^2 - 2\gamma\mathbf{BE}(\mathbf{AB}-\mathbf{CD}+\mathbf{EF})] = \alpha.$$

Now we multiply both sides by $(\det Q)^2 = (-\mathbf{AB}+\mathbf{CD}+\mathbf{EF})^2$ and we move everything to the LHS, so to get

$$\begin{aligned} 0 &= (\gamma^2-4\alpha\beta)\mathbf{D}^2\mathbf{E}^2 + \alpha[(\mathbf{AB}-\mathbf{CD}+\mathbf{EF})^2 - (-\mathbf{AB}+\mathbf{CD}+\mathbf{EF})^2] + 4\beta\mathbf{B}^2\mathbf{E}^2 + \\ &- 2\gamma\mathbf{BE}(\mathbf{AB}-\mathbf{CD}+\mathbf{EF}) \\ &= (\gamma^2-4\alpha\beta)\mathbf{D}^2\mathbf{E}^2 + 4\alpha\mathbf{EF}(\mathbf{AB}-\mathbf{CD}) + 4\beta\mathbf{B}^2\mathbf{E}^2 - 2\gamma\mathbf{BE}(\mathbf{AB}-\mathbf{CD}+\mathbf{EF}) \\ &\overset{(129)+(130)}{=} (\gamma^2-4\alpha\beta)\mathbf{D}^2\mathbf{E}^2 + 4\alpha\mathbf{ABEF} - 2\alpha\mathbf{DE}(\gamma\mathbf{AD}-2\beta\mathbf{DE}) + 4\beta\mathbf{B}^2\mathbf{E}^2 + \\ &- 2\gamma\mathbf{AB}^2\mathbf{E} + \gamma\mathbf{DE}(-\gamma\mathbf{DE}+2\alpha\mathbf{AD}) - 2\gamma\mathbf{BE}^2\mathbf{F} \\ &= 2\mathbf{BE}(2\alpha\mathbf{AF} + 2\beta\mathbf{BE} - \gamma\mathbf{AB} - \gamma\mathbf{EF}) \end{aligned}$$

Now we perform the same steps for (102). We need to impose that the first coordinate of $\varphi(X)^2$ be β :

$$\frac{1}{(\det Q)^2}[(\gamma^2-4\alpha\beta)\mathbf{A}^2\mathbf{D}^2 + 4\alpha\mathbf{A}^2\mathbf{F}^2 + \beta(\mathbf{AB}+\mathbf{CD}+\mathbf{EF})^2 + 2\gamma\mathbf{AF}(\mathbf{AB}+\mathbf{CD}+\mathbf{EF})] = \beta.$$

Again we multiply both sides by $(\det Q)^2$ and we move everything to the LHS, so to get

$$\begin{aligned}
0 &= (\gamma^2-4\alpha\beta)\mathbf{A}^2\mathbf{D}^2+4\alpha\mathbf{A}^2\mathbf{F}^2+\beta[(\mathbf{AB}+\mathbf{CD}+\mathbf{EF})^2-(-\mathbf{AB}+\mathbf{CD}+\mathbf{EF})^2]+\\
&- 2\gamma\mathbf{AF}(\mathbf{AB}+\mathbf{CD}+\mathbf{EF})\\
&= (\gamma^2-4\alpha\beta)\mathbf{A}^2\mathbf{D}^2+4\alpha\mathbf{A}^2\mathbf{F}^2+4\beta\mathbf{AB}(\mathbf{CD}+\mathbf{EF})-2\gamma\mathbf{AF}(\mathbf{AB}+\mathbf{CD}+\mathbf{EF})\\
&\overset{(129)+(130)}{=} (\gamma^2-4\alpha\beta)\mathbf{A}^2\mathbf{D}^2+4\alpha\mathbf{A}^2\mathbf{F}^2+2\beta\mathbf{AD}(-\gamma\mathbf{DE}+2\alpha\mathbf{AD})+4\beta\mathbf{ABEF}+\\
&- 2\gamma\mathbf{A}^2\mathbf{BF}-\gamma\mathbf{AD}(\mathbf{AD}-2\beta\mathbf{DE})-2\gamma\mathbf{AEF}^2\\
&= 2\mathbf{AF}(2\alpha\mathbf{AF}+2\beta\mathbf{BE}-\gamma\mathbf{AB}-\gamma\mathbf{EF})
\end{aligned}$$

Finally we make use once again of the machinery on multiplications contained in Appendix A to observe that

$$\varphi(G)\varphi(X)+\varphi(X)\varphi(G)=(L_{\varphi(G)}+R_{\varphi(G)})\varphi(X)=$$

$$=\frac{1}{(\det Q)^2}[-2\gamma(\mathbf{DE})I+(\mathbf{AB}-\mathbf{CD}+\mathbf{EF})(L_G+R_G)-(2\mathbf{BE})(L_X+R_X)+(2\mathbf{DE})(L_GL_X+R_XR_G)]\varphi(X)=$$

$$=\frac{2}{(\det Q)^2}\begin{pmatrix} -\gamma\mathbf{DE} & \alpha(\mathbf{AB}-\mathbf{CD}+\mathbf{EF})-\gamma\mathbf{BE} & \frac{\gamma}{2}(\mathbf{AB}-\mathbf{CD}+\mathbf{EF})-2\beta\mathbf{BE} & -2\alpha\beta\mathbf{DE}\\ \mathbf{AB}-\mathbf{CD}+\mathbf{EF} & 0 & 0 & \frac{\gamma}{2}(\mathbf{AB}-\mathbf{CD}+\mathbf{EF})\\ -2\mathbf{BE} & 0 & 0 & -\gamma\mathbf{BE}\\ 2\mathbf{DE} & 0 & 0 & \gamma\mathbf{DE}\end{pmatrix}\begin{pmatrix}-\gamma\mathbf{AD}\\ 2\mathbf{AF}\\ -\mathbf{AB}-\mathbf{CD}-\mathbf{EF}\\ 2\mathbf{AD}\end{pmatrix}=$$

$$=\frac{2}{(\det Q)^2}\begin{pmatrix}(\gamma^2-4\alpha\beta)\mathbf{AD}^2\mathbf{E}+2\alpha\mathbf{AF}(\mathbf{AB}-\mathbf{CD}+\mathbf{EF})-2\gamma\mathbf{ABEF}+2\beta\mathbf{BE}(\mathbf{AB}+\mathbf{CD}+\mathbf{EF})-\frac{\gamma}{2}[(\mathbf{AB}+\mathbf{EF})^2-\mathbf{C}^2\mathbf{D}^2]\\ 0\\ 0\\ 0\end{pmatrix},$$

so that (104) is equivalent to the first coordinate of this vector being equal to γ. From this we can carry out the following calculations:

$$\begin{aligned}
0 &= 2(\gamma^2-4\alpha\beta)\mathbf{AD}^2\mathbf{E}+4\alpha\mathbf{AF}(\mathbf{AB}-\mathbf{CD}+\mathbf{EF})-4\gamma\mathbf{ABEF}+4\beta\mathbf{BE}(\mathbf{AB}+\mathbf{CD}+\mathbf{EF})+\\
&- \gamma[(\mathbf{AB}+\mathbf{EF})^2-\mathbf{C}^2\mathbf{D}^2]-\gamma(-\mathbf{AB}+\mathbf{CD}+\mathbf{EF})^2\\
&= 2(\gamma^2-4\alpha\beta)\mathbf{AD}^2\mathbf{E}+4\alpha\mathbf{AF}(\mathbf{AB}-\mathbf{CD}+\mathbf{EF})-4\gamma\mathbf{ABEF}+4\beta\mathbf{BE}(\mathbf{AB}+\mathbf{CD}+\mathbf{EF})+\\
&- \gamma[2\mathbf{A}^2\mathbf{B}^2+2\mathbf{E}^2\mathbf{F}^2-2\mathbf{ABCD}+2\mathbf{CDEF}]\\
&\overset{(129)+(130)}{=} 4\alpha\mathbf{AF}(\mathbf{AB}+\mathbf{EF})-4\gamma\mathbf{ABEF}+4\beta\mathbf{BE}(\mathbf{AB}+\mathbf{EF})-2\gamma[\mathbf{A}^2\mathbf{B}^2+\mathbf{E}^2\mathbf{F}^2]\\
&= 2(\mathbf{AB}+\mathbf{EF})(2\alpha\mathbf{AF}+2\beta\mathbf{BE}-\gamma\mathbf{AB}-\gamma\mathbf{EF}).
\end{aligned}$$

We obtain a result which will be useful in the sequel:

$$(\mathbf{AB}+\mathbf{EF})(2\alpha\mathbf{AF}+2\beta\mathbf{BE})=\gamma(\mathbf{AB}+\mathbf{EF})^2. \tag{131}$$

We also simplify (125)

$$\begin{aligned}
0 &= (\gamma^2-4\alpha\beta)\mathbf{AD}^2\mathbf{E}+2\alpha\mathbf{AF}(\mathbf{AB}-\mathbf{CD}+\mathbf{EF})-4\gamma\mathbf{ABEF}+2\beta\mathbf{BE}(\mathbf{AB}+\mathbf{CD}+\mathbf{EF})+\\
&- 2(\mathbf{BJ}-\mathbf{FH})(-\mathbf{AB}+\mathbf{CD}+\mathbf{EF})\\
&\overset{(123)+(124)}{=} (\gamma^2-4\alpha\beta)\mathbf{AD}^2\mathbf{E}+2\alpha\mathbf{AF}(\mathbf{AB}-\mathbf{CD}+\mathbf{EF})-4\gamma\mathbf{ABEF}+2\beta\mathbf{BE}(\mathbf{AB}+\mathbf{CD}+\mathbf{EF})+\\
&- 2(\mathbf{BJ}-\mathbf{FH})(-\mathbf{AB}+\mathbf{EF})-\mathbf{BC}(2\mathbf{FG}-\gamma\mathbf{AD})+\mathbf{CF}(2\mathbf{BG}-\gamma\mathbf{DE})\\
&\overset{(129)+(130)}{=} 2\alpha\mathbf{AF}(\mathbf{AB}+\mathbf{EF})-4\gamma\mathbf{ABEF}+2\beta\mathbf{BE}(\mathbf{AB}+\mathbf{EF})-2(\mathbf{BJ}-\mathbf{FH})(-\mathbf{AB}+\mathbf{EF})\\
&= (2\alpha\mathbf{AF}+2\beta\mathbf{BE})(\mathbf{AB}+\mathbf{EF})-4\gamma\mathbf{ABEF}-2(\mathbf{BJ}-\mathbf{FH})(-\mathbf{AB}+\mathbf{EF}).
\end{aligned}$$

and we use (131) to conclude that

$$\gamma(\mathbf{AB}+\mathbf{EF})^2-4\gamma\mathbf{ABEF}-2(\mathbf{BJ}-\mathbf{FH})(-\mathbf{AB}+\mathbf{EF})=0,$$

that is

$$[\gamma(\mathbf{AB}-\mathbf{EF})+2(\mathbf{BJ}-\mathbf{FH})](\mathbf{AB}-\mathbf{EF})=0.$$

In addition

$$[\gamma(\mathbf{AB}-\mathbf{EF})+2(\mathbf{BJ}-\mathbf{FH})]\mathbf{CD} \quad = \quad \gamma\mathbf{ABCD}-\gamma\mathbf{CDEF}+2\mathbf{BCDJ}-2\mathbf{FCDH}$$

$$
\begin{aligned}
&= \quad \mathbf{BC}(2\mathbf{DJ} + \gamma\mathbf{AD}) - \mathbf{CF}(\gamma\mathbf{DE} + 2\mathbf{DH}) \\
&\overset{(123)+(124)}{=} \quad 2\mathbf{BCFG} - 2\mathbf{BCFG} \\
&= \quad 0,
\end{aligned}
$$

so that we have both

$$
\begin{aligned}
[\gamma(\mathbf{AB} - \mathbf{EF}) + 2(\mathbf{BJ} - \mathbf{FH})](\mathbf{AB} - \mathbf{EF}) &= 0 \\
[\gamma(\mathbf{AB} - \mathbf{EF}) + 2(\mathbf{BJ} - \mathbf{FH})]\mathbf{CD} &= 0
\end{aligned}
$$

If $\gamma(\mathbf{AB} - \mathbf{EF}) + 2(\mathbf{BJ} - \mathbf{FH}) \neq 0$, we get $\mathbf{AB} - \mathbf{EF} = 0 = \mathbf{CD}$, which in turn implies $\det Q = 0$, contradiction. Hence we are able to rewrite conditions (100)-(104) in the following equivalent way:

$$
\begin{aligned}
q_{41} = 0 &= q_{31} \\
2[\mathbf{BG} - \mathbf{DH}] - \gamma\mathbf{DE} &= 0, \\
2[\mathbf{FG} - \mathbf{DJ}] - \gamma\mathbf{AD} &= 0, \\
\gamma(\mathbf{AB} - \mathbf{EF}) + 2(\mathbf{BJ} - \mathbf{FH}) &= 0, \\
\gamma\mathbf{AD} - 2\mathbf{CF} - 2\beta\mathbf{DE} &= 0, \\
\gamma\mathbf{DE} + 2\mathbf{BC} - 2\alpha\mathbf{AD} &= 0, \\
\mathbf{BE}(2\alpha\mathbf{AF} + 2\beta\mathbf{BE} - \gamma\mathbf{AB} - \gamma\mathbf{EF}) &= 0 \\
\mathbf{AF}(2\alpha\mathbf{AF} + 2\beta\mathbf{BE} - \gamma\mathbf{AB} - \gamma\mathbf{EF}) &= 0 \\
(\mathbf{AB} + \mathbf{EF})(2\alpha\mathbf{AF} + 2\beta\mathbf{BE} - \gamma\mathbf{AB} - \gamma\mathbf{EF}) &= 0
\end{aligned}
$$

Notice that if we assume $2\alpha\mathbf{AF} + 2\beta\mathbf{BE} - \gamma\mathbf{AB} - \gamma\mathbf{EF} \neq 0$, then $\mathbf{BE} = \mathbf{AF} = 0 = \mathbf{AB} + \mathbf{EF}$ which leads to immediate contradiction, so in conclusion we are left with

$$
\begin{aligned}
q_{31} = q_{41} &= 0 \\
2[\mathbf{BG} - \mathbf{DH}] - \gamma\mathbf{DE} &= 0, \\
2[\mathbf{FG} - \mathbf{DJ}] - \gamma\mathbf{AD} &= 0, \\
\gamma(\mathbf{AB} - \mathbf{EF}) + 2(\mathbf{BJ} - \mathbf{FH}) &= 0, \\
\gamma\mathbf{AD} - 2\mathbf{CF} - 2\beta\mathbf{DE} &= 0, \\
\gamma\mathbf{DE} + 2\mathbf{BC} - 2\alpha\mathbf{AD} &= 0, \\
2\alpha\mathbf{AF} + 2\beta\mathbf{BE} - \gamma\mathbf{AB} - \gamma\mathbf{EF} &= 0.
\end{aligned}
$$

In particular the map φ results associated to a matrix

$$
M_\varphi = \begin{pmatrix}
1 & \frac{-\gamma\mathbf{DE}}{-\mathbf{AB}+\mathbf{CD}+\mathbf{EF}} & \frac{-\gamma\mathbf{AD}}{-\mathbf{AB}+\mathbf{CD}+\mathbf{EF}} & \frac{-\gamma(\mathbf{AB}-\mathbf{EF})}{-\mathbf{AB}+\mathbf{CD}+\mathbf{EF}} \\
0 & \frac{\mathbf{AB}-\mathbf{CD}+\mathbf{EF}}{-\mathbf{AB}+\mathbf{CD}+\mathbf{EF}} & \frac{2\mathbf{AF}}{-\mathbf{AB}+\mathbf{CD}+\mathbf{EF}} & \frac{2\mathbf{CF}}{-\mathbf{AB}+\mathbf{CD}+\mathbf{EF}} \\
0 & \frac{-2\mathbf{BE}}{-\mathbf{AB}+\mathbf{CD}+\mathbf{EF}} & \frac{-\mathbf{AB}-\mathbf{CD}-\mathbf{EF}}{-\mathbf{AB}+\mathbf{CD}+\mathbf{EF}} & \frac{-2\mathbf{BC}}{-\mathbf{AB}+\mathbf{CD}+\mathbf{EF}} \\
0 & \frac{2\mathbf{DE}}{-\mathbf{AB}+\mathbf{CD}+\mathbf{EF}} & \frac{2\mathbf{AD}}{-\mathbf{AB}+\mathbf{CD}+\mathbf{EF}} & \frac{\mathbf{AB}+\mathbf{CD}-\mathbf{EF}}{-\mathbf{AB}+\mathbf{CD}+\mathbf{EF}}
\end{pmatrix}, \tag{132}
$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F} \in k$ satisfy

$$
\begin{cases}
\gamma\mathbf{AD} - 2\mathbf{CF} - 2\beta\mathbf{DE} = 0, \\
\gamma\mathbf{DE} + 2\mathbf{BC} - 2\alpha\mathbf{AD} = 0, \\
2\alpha\mathbf{AF} + 2\beta\mathbf{BE} - \gamma\mathbf{AB} - \gamma\mathbf{EF} = 0, \\
-\mathbf{AB} + \mathbf{CD} + \mathbf{EF} \neq 0.
\end{cases} \tag{133}
$$

Moreover it can be easily checked that a linear map $\varphi : A \to A$ whose matrix in the fixed basis $(1, G, X, GX)$ is (132) and where $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F} \in k$ satisfy (133) is contained in $\mathrm{Inv}_{\mathrm{Alg}}(A)$.

We conclude that each and every automorphism in $\mathrm{Inv}_{\mathrm{Alg}}(A)$ whose matrix is similar to D_2 is given by

$$\varphi(G) = \frac{1}{-\mathbf{AB} + \mathbf{CD} + \mathbf{EF}}[-\gamma\mathbf{DE} + (\mathbf{AB} - \mathbf{CD} + \mathbf{EF})G - (2\mathbf{BE})X + (2\mathbf{DE})GX]$$

$$\varphi(X) = \frac{1}{-\mathbf{AB} + \mathbf{CD} + \mathbf{EF}}[-\gamma\mathbf{AD} + (2\mathbf{AF})G + (-\mathbf{AB} - \mathbf{CD} - \mathbf{EF})X + (2\mathbf{AD})GX]$$

$$\varphi(GX) = \frac{1}{-\mathbf{AB} + \mathbf{CD} + \mathbf{EF}}[-\gamma(\mathbf{AB} - \mathbf{EF}) + (2\mathbf{CF})G - (2\mathbf{BC})X + (\mathbf{AB} + \mathbf{CD} - \mathbf{EF})GX]$$

with

$$\begin{cases} \gamma\mathbf{AD} - 2\mathbf{CF} - 2\beta\mathbf{DE} = 0 \\ \gamma\mathbf{DE} + 2\mathbf{BC} - 2\alpha\mathbf{AD} = 0 \\ 2\alpha\mathbf{AF} + 2\beta\mathbf{BE} - \gamma\mathbf{AB} - \gamma\mathbf{EF} = 0 \\ -\mathbf{AB} + \mathbf{CD} + \mathbf{EF} \neq 0. \end{cases}$$

The associated decomposition of the space A is explicitly given by

$$A_+ = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mathbf{F} \\ -\mathbf{B} \\ \mathbf{D} \end{pmatrix} \rangle, \quad A_- = \langle \begin{pmatrix} 0 \\ \mathbf{A} \\ -\mathbf{E} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\gamma}{2}\mathbf{E} \\ \mathbf{C} \\ 0 \\ -\mathbf{E} \end{pmatrix} \rangle, \quad \text{if } \mathbf{E} \neq 0$$

and

$$A_+ = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mathbf{F} \\ -\mathbf{B} \\ \mathbf{D} \end{pmatrix} \rangle, \quad A_- = \langle \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\gamma}{2}\mathbf{A} \\ 0 \\ \mathbf{C} \\ -\mathbf{A} \end{pmatrix} \rangle, \quad \text{if } \mathbf{E} = 0.$$

As of now, we have a complete classification of the algebra involutions φ on a four dimensional Clifford algebra $Cl(\alpha, \beta, \gamma)$, gathered with respect to their spectrum. In the next subsection we are going to refine this first classification, distinguishing between inner and non-inner involutions. This refinement will later help us determine the class of isomorphic H-coactions of $A = Cl(\alpha, \beta, \gamma)$, in the non-semisimple case, i.e. when $\gamma^2 - 4\alpha\beta = 0$ (see Sec. 3, Ch. 5).

2.3. A refined classification for the case $\mathfrak{F}_2$. The family of matrices of the form (132), where coefficients satisfy (133), can be split into subfamilies that are easier to deal with. In particular this further subdivision helps to determine which of the involutions of A are inner, as we will see in the sequel. We proceed as follows.

Notice that (133) can be rewritten as

$$\begin{pmatrix} \gamma\mathbf{D} & -2\mathbf{F} & -2\beta\mathbf{D} \\ -2\alpha\mathbf{D} & 2\mathbf{B} & \gamma\mathbf{D} \\ 2\alpha\mathbf{F} - \gamma\mathbf{B} & 0 & 2\beta\mathbf{B} - \gamma\mathbf{F} \end{pmatrix} \begin{pmatrix} \mathbf{A} \\ \mathbf{C} \\ \mathbf{E} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with the additional condition

$$-\mathbf{AB} + \mathbf{CD} + \mathbf{EF} \neq 0. \tag{134}$$

Let us call $\mathcal{T} := \begin{pmatrix} \gamma\mathbf{D} & -2\mathbf{F} & -2\beta\mathbf{D} \\ -2\alpha\mathbf{D} & 2\mathbf{B} & \gamma\mathbf{D} \\ 2\alpha\mathbf{F} - \gamma\mathbf{B} & 0 & 2\beta\mathbf{B} - \gamma\mathbf{F} \end{pmatrix}$. It is immediate to check that $\det\mathcal{T} = 0$ and therefore $\dim_k \ker\mathcal{T} > 1$.

2.3.1. *Assume* $\dim_k \ker \mathcal{T} = 3$. This means that $\mathcal{T}$ is the zero matrix, so $\mathbf{B} = \mathbf{F} = 0 = \gamma\mathbf{D} = \alpha\mathbf{D} = \beta\mathbf{D}$. Since $\mathbf{D} = 0$ contradicts (134), we must have $\alpha = \beta = \gamma = 0$ and $\mathbf{D} \neq 0$. One sees that in this case

$$M_\varphi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{2\mathbf{E}}{\mathbf{C}} & \frac{2\mathbf{A}}{\mathbf{C}} & 1 \end{pmatrix},$$

with $\mathbf{A}, \mathbf{C}, \mathbf{E} \in k$, $\mathbf{C} \neq 0$.

2.3.2. *Assume* $\dim_k \ker \mathcal{T} = 2$. Then all 2-by-2 minors of $\mathcal{T}$ must be 0. This is equivalent to

$$\begin{cases} 2\beta\mathbf{B} - \gamma\mathbf{F} = 0 \\ 2\alpha\mathbf{F} - \gamma\mathbf{B} = 0 \\ (\gamma^2 - 4\alpha\beta)\mathbf{D} = 0. \end{cases}$$

If we suppose $\gamma^2 - 4\alpha\beta \neq 0$, then we immediately find $\mathbf{B} = \mathbf{D} = \mathbf{F} = 0$, which contradicts (134), thus we deduce $\gamma^2 - 4\alpha\beta = 0$.

If $\gamma = \alpha = \beta = 0$, since $\operatorname{rank} \mathcal{T} = 1$, either $\mathbf{B} \neq 0$ or $\mathbf{F} \neq 0$ and $\ker \mathcal{T} = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle$. We find that $(\mathbf{A}, \mathbf{C}, \mathbf{E})^t = (\mathbf{A}, 0, \mathbf{E})^t$, with $\mathbf{AB} - \mathbf{EF} \neq 0$ by (134). Therefore when $\gamma = \alpha = \beta = 0$ we find that M_φ must be of the form:

$$M_\varphi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{\mathbf{AB}+\mathbf{EF}}{\mathbf{AB}-\mathbf{EF}} & -\frac{2\mathbf{AF}}{\mathbf{AB}-\mathbf{EF}} & 0 \\ 0 & \frac{2\mathbf{BE}}{\mathbf{AB}-\mathbf{EF}} & \frac{\mathbf{AB}+\mathbf{EF}}{\mathbf{AB}-\mathbf{EF}} & 0 \\ 0 & -\frac{2\mathbf{DE}}{\mathbf{AB}-\mathbf{EF}} & -\frac{2\mathbf{AD}}{\mathbf{AB}-\mathbf{EF}} & -1 \end{pmatrix},$$

with $\mathbf{A}, \mathbf{B}, \mathbf{D}, \mathbf{E}, \mathbf{F} \in k$ and $\mathbf{AB} - \mathbf{EF} \neq 0$.

If we assume $\gamma = \alpha = 0$ and $\beta \neq 0$, then $\mathbf{B} = 0$ and again, since $\operatorname{rank} \mathcal{T} = 1$, either $\mathbf{F} \neq 0$ or $\mathbf{D} \neq 0$. Then $\ker \mathcal{T} = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta\mathbf{D} \\ -\mathbf{F} \end{pmatrix} \rangle$. In this case $(\mathbf{A}, \mathbf{C}, \mathbf{E})^t = (\mathbf{A}, \lambda\beta\mathbf{D}, -\lambda\mathbf{F})$ with $\lambda(\beta\mathbf{D}^2 - \mathbf{F}^2) \neq 0$ and M_φ can only be of the form

$$M_\varphi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{\beta\mathbf{D}^2+\mathbf{F}^2}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{2\mathbf{AF}}{\lambda(\beta\mathbf{D}^2-\mathbf{F}^2)} & \frac{2\beta\mathbf{DF}}{\beta\mathbf{D}^2-\mathbf{F}^2} \\ 0 & 0 & -1 & 0 \\ 0 & -\frac{2\mathbf{DF}}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{2\mathbf{AD}}{\lambda(\beta\mathbf{D}^2-\mathbf{F}^2)} & \frac{\beta\mathbf{D}^2+\mathbf{F}^2}{\beta\mathbf{D}^2-\mathbf{F}^2} \end{pmatrix},$$

with $\mathbf{A}, \mathbf{D}, \mathbf{F}, \lambda \in k$ and $\lambda(\beta\mathbf{D}^2 - \mathbf{F}^2) \neq 0$.

Similarly, if we suppose $\gamma = 0$ but $\alpha \neq 0$, then $\beta = 0$, since $\gamma^2 - 4\alpha\beta = 0$, and also $\mathbf{F} = 0$. $\operatorname{rank} \mathcal{T} = 1$ implies either $\mathbf{D} \neq 0$ or $\mathbf{B} \neq 0$. In this case $\ker \mathcal{T} = \langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \mathbf{B} \\ \alpha\mathbf{D} \\ 0 \end{pmatrix} \rangle$, $(\mathbf{A}, \mathbf{C}, \mathbf{E})^t = (\lambda\mathbf{B}, \lambda\alpha\mathbf{D}, \mathbf{E})^t$ with $\lambda(\mathbf{B}^2 - \alpha\mathbf{D}^2) \neq 0$ and M_φ is of the form

$$M_\varphi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -\frac{2\mathbf{BE}}{\lambda(-\mathbf{B}^2+\alpha\mathbf{D}^2)} & \frac{-\mathbf{B}^2-\alpha\mathbf{D}^2}{-\mathbf{B}^2+\alpha\mathbf{D}^2} & -\frac{2\alpha\mathbf{BD}}{-\mathbf{B}^2+\alpha\mathbf{D}^2} \\ 0 & \frac{2\mathbf{DE}}{\lambda(-\mathbf{B}^2+\alpha\mathbf{D}^2)} & \frac{2\mathbf{BD}}{-\mathbf{B}^2+\alpha\mathbf{D}^2} & \frac{\mathbf{B}^2+\alpha\mathbf{D}^2}{-\mathbf{B}^2+\alpha\mathbf{D}^2} \end{pmatrix},$$

with $\mathbf{B}, \mathbf{D}, \mathbf{E}, \lambda \in k$ and $\lambda(\mathbf{B}^2 - \alpha\mathbf{D}^2) \neq 0$.

Finally, if $\gamma \neq 0$, then both $\alpha \neq 0$ and $\beta \neq 0$. If $\mathbf{B} = 0$, then $\mathbf{F} = 0$ and $\mathbf{D}$ must be non-zero, since $\dim_k \ker \mathcal{T} = 2$. In this case $\ker \mathcal{T} = \langle \begin{pmatrix} \gamma \\ 0 \\ 2\alpha \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rangle$. We have $(\mathbf{A}, \mathbf{C}, \mathbf{E})^t = (\lambda\gamma, \mathbf{C}, 2\lambda\alpha)^t$ with $\mathbf{CD} \neq 0$, by (134). M_φ is of the form

$$M_\varphi = \begin{pmatrix} 1 & -\frac{2\lambda\alpha\gamma}{\mathbf{C}} & -\frac{\lambda\gamma^2}{\mathbf{C}} & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{4\lambda\alpha}{\mathbf{C}} & \frac{2\lambda\gamma}{\mathbf{C}} & 1 \end{pmatrix}$$

with $\gamma^2 - 4\alpha\beta = 0$, $\lambda, \mathbf{C} \in k$ and $\mathbf{C} \neq 0$.

If $\mathbf{B} \neq 0$, also $\mathbf{F} = 2\beta\frac{\mathbf{B}}{\gamma} \neq 0$ and then $\ker \mathcal{T} = \langle \begin{pmatrix} \mathbf{B} \\ \alpha\mathbf{D} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\gamma\mathbf{D} \\ 2\mathbf{B} \end{pmatrix} \rangle$. Thus $(\mathbf{A}, \mathbf{C}, \mathbf{E})^t = (\lambda\mathbf{B}, (\lambda\alpha - \mu\gamma)\mathbf{D}, 2\mu\mathbf{B})$ with $(\lambda\gamma - 4\mu\beta)(\mathbf{B}^2 - \alpha\mathbf{D}^2) \neq 0$. We conclude that M_φ is of the form

$$M_\varphi = \begin{pmatrix} 1 & \frac{2\mu\gamma^2\mathbf{BD}}{(4\mu\beta-\lambda\gamma)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{\lambda\gamma^2\mathbf{BD}}{(4\mu\beta-\lambda\gamma)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{-\gamma\mathbf{B}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & -1-\frac{8\mu\beta\mathbf{B}^2}{(4\mu\beta-\lambda\gamma)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{4\lambda\beta\mathbf{B}^2}{(4\mu\beta-\lambda\gamma)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{\gamma\mathbf{BD}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & \frac{4\mu\gamma\mathbf{B}^2}{(4\mu\beta-\lambda\gamma)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -1+\frac{2\lambda\gamma\mathbf{B}^2}{(4\mu\beta-\lambda\gamma)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{2\alpha\mathbf{BD}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & -\frac{4\mu\gamma\mathbf{BD}}{(4\mu\beta-\lambda\gamma)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{2\lambda\gamma\mathbf{BD}}{(4\mu\beta-\lambda\gamma)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{\alpha\mathbf{D}^2+\mathbf{B}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} \end{pmatrix}$$

when $\gamma^2 - 4\alpha\beta = 0$, $\mathbf{D}, \lambda, \mu \in k$ and $(\lambda\gamma - 4\mu\beta)(\alpha\mathbf{D}^2 - \mathbf{B}^2) \neq 0$.

2.3.3. *Assume* $\dim_k \ker \mathcal{T} = 1$. It is not hard to check that

$$v = \begin{pmatrix} 2(\gamma\mathbf{F} - 2\beta\mathbf{B}) \\ (\gamma^2 - 4\alpha\beta)\mathbf{D} \\ 2(2\alpha\mathbf{F} - \gamma\mathbf{B}) \end{pmatrix} \in \ker_k \mathcal{T}$$

and that $v = 0$ is equivalent to $\operatorname{rank} \mathcal{T} \leq 1$ (by looking at its 2-by-2 minors). Therefore in our case v is a generator of $\ker \mathcal{T}$ and we find that M_φ is of the form

$$M_\varphi = \begin{pmatrix} 1 & \frac{2\gamma\mathbf{D}(\gamma\mathbf{B}-2\alpha\mathbf{F})}{4\alpha\mathbf{F}^2-4\gamma\mathbf{FB}+4\beta\mathbf{B}^2+(\gamma^2-4\alpha\beta)\mathbf{D}^2} & -\frac{2\gamma\mathbf{D}(\gamma\mathbf{F}-2\beta\mathbf{B})}{4\alpha\mathbf{F}^2-4\gamma\mathbf{FB}+4\beta\mathbf{B}^2+(\gamma^2-4\alpha\beta)\mathbf{D}^2} & \gamma-\frac{\gamma(\gamma^2-4\alpha\beta)\mathbf{D}^2}{4\alpha\mathbf{F}^2-4\gamma\mathbf{FB}+4\beta\mathbf{B}^2+(\gamma^2-4\alpha\beta)\mathbf{D}^2} \\ 0 & -1-\frac{4\mathbf{F}(\gamma\mathbf{B}-2\alpha\mathbf{F})}{4\alpha\mathbf{F}^2-4\gamma\mathbf{FB}+4\beta\mathbf{B}^2+(\gamma^2-4\alpha\beta)\mathbf{D}^2} & \frac{4\mathbf{F}(\gamma\mathbf{F}-2\beta\mathbf{B})}{4\alpha\mathbf{F}^2-4\gamma\mathbf{FB}+4\beta\mathbf{B}^2+(\gamma^2-4\alpha\beta)\mathbf{D}^2} & \frac{2(\gamma^2-4\alpha\beta)\mathbf{DF}}{4\alpha\mathbf{F}^2-4\gamma\mathbf{FB}+4\beta\mathbf{B}^2+(\gamma^2-4\alpha\beta)\mathbf{D}^2} \\ 0 & \frac{4\mathbf{B}(\gamma\mathbf{B}-2\alpha\mathbf{F})}{4\alpha\mathbf{F}^2-4\gamma\mathbf{FB}+4\beta\mathbf{B}^2+(\gamma^2-4\alpha\beta)\mathbf{D}^2} & -1-\frac{4\mathbf{B}(\gamma\mathbf{F}-2\beta\mathbf{B})}{4\alpha\mathbf{F}^2-4\gamma\mathbf{FB}+4\beta\mathbf{B}^2+(\gamma^2-4\alpha\beta)\mathbf{D}^2} & \frac{2(4\alpha\beta-\gamma^2)\mathbf{BD}}{4\alpha\mathbf{F}^2-4\gamma\mathbf{FB}+4\beta\mathbf{B}^2+(\gamma^2-4\alpha\beta)\mathbf{D}^2} \\ 0 & -\frac{4\mathbf{D}(\gamma\mathbf{B}-2\alpha\mathbf{F})}{4\alpha\mathbf{F}^2-4\gamma\mathbf{FB}+4\beta\mathbf{B}^2+(\gamma^2-4\alpha\beta)\mathbf{D}^2} & \frac{4\mathbf{D}(\gamma\mathbf{F}-2\beta\mathbf{B})}{4\alpha\mathbf{F}^2-4\gamma\mathbf{FB}+4\beta\mathbf{B}^2+(\gamma^2-4\alpha\beta)\mathbf{D}^2} & -1+\frac{2(\gamma^2-4\alpha\beta)\mathbf{D}^2}{4\alpha\mathbf{F}^2-4\gamma\mathbf{FB}+4\beta\mathbf{B}^2+(\gamma^2-4\alpha\beta)\mathbf{D}^2} \end{pmatrix},$$

where

$$4\alpha\mathbf{F}^2 - 4\gamma\mathbf{BF} + 4\beta\mathbf{B}^2 + (\gamma^2 - 4\alpha\beta)\mathbf{D}^2 \neq 0. \tag{135}$$

Remark 2.3. In this case M_φ can be rewritten as $L_{a^{-1}}R_a = (L_a)^{-1}R_a$ where $a = (\gamma\mathbf{D}, -2\mathbf{F}, 2\mathbf{B}, -2\mathbf{D})^t$, and (135) is equivalent to ask that a be invertible (see Prop 0.5). In particular every involution of this form is inner.

The refined classification of involutions we have obtained is contained in Table 2.

2.4. Inner and non-inner involutions. Remark 2.3 raises the question of whether some of the subfamilies we have obtained contain inner involutions. To answer to this question we first write the matrix M_{φ_a} of an inner automorphism in $A = Cl(\alpha, \beta, \gamma)$. Let $a = (a_1, a_2, a_3, a_4)^t$ be an invertible element of A. Then

$$M_{\varphi_a} = L_{a^{-1}}R_a = \begin{pmatrix} 1 & -\frac{\gamma(a_1a_3-\alpha a_2a_4)}{|a|} & \frac{\gamma(a_1a_2+\beta a_3a_4+\gamma a_2a_4)}{|a|} & -\frac{\gamma(\alpha a_2^2+\beta a_3^2+\gamma a_2a_3)}{|a|} \\ 0 & \frac{a_1^2-\alpha a_2^2+\beta a_3^2-\alpha\beta a_4^2}{|a|} & -\frac{2\beta(a_1a_4+a_2a_3)+\gamma(a_2^2+\beta a_4^2)}{|a|} & \frac{2\beta(a_1a_3+\alpha a_2a_4)+\gamma(a_1a_2+\beta a_3a_4)}{|a|} \\ 0 & \frac{2\alpha(a_1a_4-a_2a_3)+\gamma(\alpha a_4^2-a_3^2)}{|a|} & \frac{a_1^2+2\gamma a_1a_4+\alpha a_2^2-\beta a_3^2+(\gamma^2-\alpha\beta)a_4^2}{|a|} & \frac{-2\alpha a_1a_2-\gamma a_1a_3-\alpha\gamma a_2a_4+(2\alpha\beta-\gamma^2)a_3a_4}{|a|} \\ 0 & \frac{2(a_1a_3-\alpha a_2a_4)}{|a|} & -\frac{2(a_1a_2+\beta a_3a_4+\gamma a_2a_4)}{|a|} & \frac{a_1^2+\gamma a_1a_4+\alpha a_2^2+\gamma a_2a_3+\beta a_3^2+\alpha\beta a_4^2}{|a|} \end{pmatrix}.$$

Here $|a| = a_1^2 + \gamma a_1 a_4 - \alpha a_2^2 - \gamma a_2 a_3 - \beta a_3^2 + \alpha\beta a_4^2 \neq 0$ as in Proposition 0.5. If we suppose that φ_a is an involution, we also have that $a^{-2}ba^2 = \varphi_a^2(b) = b$, that is $ba^2 = a^2 b$ for all $b \in A$. This means that $a^2 \in \mathcal{Z}(A)$. The Clifford algebras $Cl(\alpha, \beta, \gamma)$ are central (see Thm. 5.16), therefore we must have $a^2 \in k$. By Remark 0.2 this forces either $a \in k$ or $2a_1 + \gamma a_4 = 0$. Clearly $a \in k$ means $\varphi_a \equiv \mathrm{Id}$, while $2a_1 + \gamma a_4 = 0$ yields

$$M_{\varphi_a} = \begin{pmatrix} 1 & \frac{\gamma a_4\left(\frac{\gamma}{2}a_3+\alpha a_2\right)}{|a|} & \frac{\gamma a_4(\frac{\gamma}{2}a_2+\beta a_3)}{|a|} & \gamma + \frac{\gamma(\frac{\gamma^2}{4}-\alpha\beta)a_4^2}{|a|} \\ 0 & -1-\frac{a_2(\gamma a_3+2\alpha a_2)}{|a|} & -\frac{a_2(\gamma a_2+2\beta a_3)}{|a|} & \frac{(2\alpha\beta-\frac{\gamma^2}{2})a_2a_4}{|a|} \\ 0 & -\frac{a_3(\gamma a_3+2\alpha a_2)}{|a|} & -1-\frac{a_3(\gamma a_2+2\beta a_3)}{|a|} & \frac{(2\alpha\beta-\frac{\gamma^2}{2})a_3a_4}{|a|} \\ 0 & -\frac{a_4(\gamma a_3+2\alpha a_2)}{|a|} & -\frac{a_4(\gamma a_2+2\beta a_3)}{|a|} & -1+\frac{(2\alpha\beta-\frac{\gamma^2}{2})a_4^2}{|a|} \end{pmatrix}, \tag{136}$$

provided $|a| = -\alpha a_2^2 - \gamma a_2 a_3 - \beta a_3^2 + \left(\alpha\beta - \frac{\gamma^2}{4}\right) a_4^2 \neq 0$.

The invertibility of a immediately implies that when $\alpha = \beta = \gamma = 0$, there are no involution of this form (i.e. there are no inner involutions other than the identity). Next, if $\alpha = \gamma = 0$ and $\beta \neq 0$, an inner involution must have matrix

$$M_{\varphi_a} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2\frac{a_2}{a_3} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2\frac{a_4}{a_3} & -1 \end{pmatrix}.$$

If we suppose that M_{φ_a} is also in the subfamily

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{\beta\mathbf{D}^2+\mathbf{F}^2}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{2\mathbf{AF}}{\lambda(\beta\mathbf{D}^2-\mathbf{F}^2)} & \frac{2\beta\mathbf{DF}}{\beta\mathbf{D}^2-\mathbf{F}^2} \\ 0 & 0 & -1 & 0 \\ 0 & -\frac{2\mathbf{DF}}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{2\mathbf{AD}}{\lambda(\beta\mathbf{D}^2-\mathbf{F}^2)} & \frac{\beta\mathbf{D}^2+\mathbf{F}^2}{\beta\mathbf{D}^2-\mathbf{F}^2} \end{pmatrix},$$

by equating the entries in the third row and third column, we get $-1 = 1$, contradiction. Therefore all the involutions in this subfamily are non-inner. Similarly when $\beta = \gamma = 0$ and $\alpha \neq 0$ the entry in the second row and second column of (136) becomes 1 and therefore no member of the subfamily

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -\frac{2\mathbf{BE}}{\lambda(-\mathbf{B}^2+\alpha\mathbf{D}^2)} & \frac{-\mathbf{B}^2-\alpha\mathbf{D}^2}{-\mathbf{B}^2+\alpha\mathbf{D}^2} & -\frac{2\alpha\mathbf{BD}}{-\mathbf{B}^2+\alpha\mathbf{D}^2} \\ 0 & \frac{2\mathbf{DE}}{\lambda(-\mathbf{B}^2+\alpha\mathbf{D}^2)} & \frac{2\mathbf{BD}}{-\mathbf{B}^2+\alpha\mathbf{D}^2} & \frac{\mathbf{B}^2+\alpha\mathbf{D}^2}{-\mathbf{B}^2+\alpha\mathbf{D}^2} \end{pmatrix}$$

can be inner. If $\gamma \neq 0$ and $\gamma^2 - 4\alpha\beta = 0$, then (136) becomes

$$M_{\varphi_a} = \begin{pmatrix} 1 & -\frac{\gamma^2 a_4}{\gamma a_2+2\beta a_3} & -\frac{2\beta\gamma a_4}{\gamma a_2+2\beta a_3} & \gamma \\ 0 & -1+\frac{2\gamma a_2}{\gamma a_2+2\beta a_3} & \frac{4\beta a_2}{\gamma a_2+2\beta a_3} & 0 \\ 0 & \frac{2\gamma a_3}{\gamma a_2+2\beta a_3} & -1+\frac{4\beta a_3}{\gamma a_2+2\beta a_3} & 0 \\ 0 & \frac{2\gamma a_4}{\gamma a_2+2\beta a_2} & \frac{4\beta a_4}{\gamma a_2+2\beta a_3} & -1 \end{pmatrix},$$

with $0 \neq |a| = -\frac{(\gamma a_3+2\alpha a_2)^2}{4\alpha} = -\frac{(\gamma a_2+2\beta a_3)^2}{4\beta}$. We immediately deduce that all involutions with matrix

$$\begin{pmatrix} 1 & -\frac{2\lambda\alpha\gamma}{\mathbf{C}} & -\frac{\lambda\gamma^2}{\mathbf{C}} & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{4\lambda\alpha}{\mathbf{C}} & \frac{2\lambda\gamma}{\mathbf{C}} & 1 \end{pmatrix}$$

are non-inner. Finally, looking at the last column, we see that an element of the subfamily

$$\begin{pmatrix} 1 & \frac{2\mu\gamma^2\mathbf{BD}}{(4\mu\beta-\lambda\gamma)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{\lambda\gamma^2\mathbf{BD}}{(4\mu\beta-\lambda\gamma)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{-\gamma\mathbf{B}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & -1-\frac{8\mu\beta\mathbf{B}^2}{(4\mu\beta-\lambda\gamma)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{4\lambda\beta\mathbf{B}^2}{(4\mu\beta-\lambda\gamma)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{\gamma\mathbf{BD}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & \frac{4\mu\gamma\mathbf{B}^2}{(4\mu\beta-\lambda\gamma)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -1+\frac{2\lambda\gamma\mathbf{B}^2}{(4\mu\beta-\lambda\gamma)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{2\alpha\mathbf{BD}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & -\frac{4\mu\gamma\mathbf{BD}}{(4\mu\beta-\lambda\gamma)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{2\lambda\gamma\mathbf{BD}}{(4\mu\beta-\lambda\gamma)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{\alpha\mathbf{D}^2+\mathbf{B}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} \end{pmatrix}$$

is an inner morphism only if $\mathbf{D}=0$ (remember that for an element in this family $\alpha,\beta,\gamma,\mathbf{B}$ are all non-zero), i.e. only if it is of the form

$$\begin{pmatrix} 1 & 0 & 0 & \gamma \\ 0 & -1+\frac{8\mu\beta}{4\mu\beta-\lambda\gamma} & \frac{4\lambda\beta}{4\mu\beta-\lambda\gamma} & 0 \\ 0 & -\frac{4\mu\gamma}{4\mu\beta-\lambda\gamma} & -1-\frac{2\lambda\gamma}{4\mu\beta-\lambda\gamma} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

with $\lambda\gamma-4\mu\beta\neq 0$. Then equality

$$\begin{pmatrix} 1 & 0 & 0 & \gamma \\ 0 & -1+\frac{8\mu\beta}{4\mu\beta-\lambda\gamma} & \frac{4\lambda\beta}{4\mu\beta-\lambda\gamma} & 0 \\ 0 & -\frac{4\mu\gamma}{4\mu\beta-\lambda\gamma} & -1-\frac{2\lambda\gamma}{4\mu\beta-\lambda\gamma} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{\gamma^2 a_4}{\gamma a_2+2\beta a_3} & -\frac{2\beta\gamma a_4}{\gamma a_2+2\beta a_3} & \gamma \\ 0 & -1+\frac{2\gamma a_2}{\gamma a_2+2\beta a_3} & \frac{4\beta a_2}{\gamma a_2+2\beta a_3} & 0 \\ 0 & \frac{2\gamma a_3}{\gamma a_2+2\beta a_3} & -1+\frac{4\beta a_3}{\gamma a_2+2\beta a_3} & 0 \\ 0 & \frac{2\gamma a_4}{\gamma a_2+2\beta a_2} & \frac{4\beta a_4}{\gamma a_2+2\beta a_3} & -1 \end{pmatrix} \tag{137}$$

forces $a_4=0$. If $\mu=0$, (137) gives

$$\begin{pmatrix} 1 & 0 & 0 & \gamma \\ 0 & -1 & -\frac{4\beta}{\gamma} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \gamma \\ 0 & -1+\frac{2\gamma a_2}{\gamma a_2+2\beta a_3} & \frac{4\beta a_2}{\gamma a_2+2\beta a_3} & 0 \\ 0 & \frac{2\gamma a_3}{\gamma a_2+2\beta a_3} & -1+\frac{4\beta a_3}{\gamma a_2+2\beta a_3} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

which immediately leads to contradiction. Hence $\mu\neq 0$ and from (137) we obtain

$$-1+\frac{8\mu\beta}{4\mu\beta-\lambda\gamma} = -1+\frac{2\gamma a_2}{\gamma a_2+2\beta a_3} \iff a_3 = -\frac{\lambda\gamma^2}{8\mu\beta^2}a_2.$$

Now (137) becomes

$$\begin{pmatrix} 1 & 0 & 0 & \gamma \\ 0 & -1+\frac{8\mu\beta}{4\mu\beta-\lambda\gamma} & \frac{4\lambda\beta}{4\mu\beta-\lambda\gamma} & 0 \\ 0 & -\frac{4\mu\gamma}{4\mu\beta-\lambda\gamma} & -1-\frac{2\lambda\gamma}{4\mu\beta-\lambda\gamma} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \gamma \\ 0 & -1+\frac{8\mu\beta}{4\mu\beta-\lambda\gamma} & \frac{16\mu\beta^2}{\gamma(4\mu\beta-\lambda\gamma)} & 0 \\ 0 & -\frac{\lambda\gamma^2}{\beta(4\mu\beta-\lambda\gamma)} & -1-\frac{2\lambda\gamma}{4\mu\beta-\lambda\gamma} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

that is

$$\begin{cases} 4\lambda\beta = 16\mu\dfrac{\beta^2}{\gamma} \\ -4\mu\gamma = -\dfrac{\lambda\gamma^2}{\beta} \end{cases} \iff \lambda\gamma = 4\mu\beta,$$

and this is again a contradiction. We conclude that also the last subfamily of involutions contains no inner morphisms. For sake of completeness we point out that $\mathfrak{F}_0=\mathrm{Id}_A$ is clearly an inner involution, while any involution with matrix of type $\mathfrak{F}_1$ is non-inner, since $\det\mathfrak{F}_1=(-1)^3=-1$ and inner morphisms have determinant 1, by Proposition 0.6.

The following table contains the refined classification of involutions on $A=Cl(\alpha,\beta,\gamma)$.

Family	Matrix M_φ of φ	Conditions	Inner
$\mathfrak{F}_0$	$M_\varphi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathrm{Id}_A$		Yes
$\mathfrak{F}_1$	$M_\varphi = \begin{pmatrix} 1 & \frac{\gamma}{2}\mathbf{A} & \frac{\gamma}{2}\mathbf{B} & \frac{\gamma}{2} \\ 0 & \frac{2\alpha\mathbf{B}-\gamma\mathbf{A}}{2}\mathbf{B} & \frac{\gamma\mathbf{B}-2\beta\mathbf{A}}{2}\mathbf{B} & \frac{\gamma\mathbf{B}-2\beta\mathbf{A}}{2} \\ 0 & \frac{\gamma\mathbf{A}-2\alpha\mathbf{B}}{2}\mathbf{A} & \frac{2\beta\mathbf{A}-\gamma\mathbf{B}}{2}\mathbf{A} & \frac{\gamma\mathbf{A}-2\alpha\mathbf{B}}{2} \\ 0 & -\mathbf{A} & -\mathbf{B} & 0 \end{pmatrix}$	$\gamma^2-4\alpha\beta=0$ $\alpha\mathbf{B}^2+\beta\mathbf{A}^2-\gamma\mathbf{AB}=1$	Never
$\mathfrak{F}_2$	$M_\varphi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{2\mathbf{E}}{\mathbf{C}} & \frac{2\mathbf{A}}{\mathbf{C}} & 1 \end{pmatrix}$	$\alpha=\beta=\gamma=0$ $\mathbf{C}\neq 0$	Never
	$M_\varphi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{\mathbf{AB}+\mathbf{EF}}{\mathbf{AB}-\mathbf{EF}} & -\frac{2\mathbf{AF}}{\mathbf{AB}-\mathbf{EF}} & 0 \\ 0 & \frac{2\mathbf{BE}}{\mathbf{AB}-\mathbf{EF}} & \frac{\mathbf{AB}+\mathbf{EF}}{\mathbf{AB}-\mathbf{EF}} & 0 \\ 0 & -\frac{2\mathbf{DE}}{\mathbf{AB}-\mathbf{EF}} & -\frac{2\mathbf{AD}}{\mathbf{AB}-\mathbf{EF}} & -1 \end{pmatrix}$	$\alpha=\beta=\gamma=0$ $\mathbf{AB}-\mathbf{EF}\neq 0$	Never
	$M_\varphi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{\beta\mathbf{D}^2+\mathbf{F}^2}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{2\mathbf{AF}}{\lambda(\beta\mathbf{D}^2-\mathbf{F}^2)} & \frac{2\beta\mathbf{DF}}{\beta\mathbf{D}^2-\mathbf{F}^2} \\ 0 & 0 & -1 & 0 \\ 0 & -\frac{2\mathbf{DF}}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{2\mathbf{AD}}{\lambda(\beta\mathbf{D}^2-\mathbf{F}^2)} & \frac{\beta\mathbf{D}^2+\mathbf{F}^2}{\beta\mathbf{D}^2-\mathbf{F}^2} \end{pmatrix}$	$\gamma=\alpha=0,\ \beta\neq 0$ $\lambda(\beta\mathbf{D}^2-\mathbf{F}^2)\neq 0$	Never
	$M_\varphi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -\frac{2\mathbf{BE}}{\lambda(-\mathbf{B}^2+\alpha\mathbf{D}^2)} & \frac{-\mathbf{B}^2-\alpha\mathbf{D}^2}{-\mathbf{B}^2+\alpha\mathbf{D}^2} & -\frac{2\alpha\mathbf{BD}}{-\mathbf{B}^2+\alpha\mathbf{D}^2} \\ 0 & \frac{2\mathbf{DE}}{\lambda(-\mathbf{B}^2+\alpha\mathbf{D}^2)} & \frac{2\mathbf{BD}}{-\mathbf{B}^2+\alpha\mathbf{D}^2} & \frac{\mathbf{B}^2+\alpha\mathbf{D}^2}{-\mathbf{B}^2+\alpha\mathbf{D}^2} \end{pmatrix}$	$\gamma=\beta=0,\ \alpha\neq 0$ $\lambda(\alpha\mathbf{D}^2-\mathbf{B}^2)\neq 0$	Never
	$M_\varphi = \begin{pmatrix} 1 & -\frac{2\lambda\alpha\gamma}{\mathbf{C}} & -\frac{\lambda\gamma^2}{\mathbf{C}} & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{4\lambda\alpha}{\mathbf{C}} & \frac{2\lambda\gamma}{\mathbf{C}} & 1 \end{pmatrix}$	$\gamma\neq 0,\ \gamma^2-4\alpha\beta=0$ $\mathbf{C}\neq 0$	Never
	$M_\varphi = \begin{pmatrix} 1 & \frac{2\mu\gamma^2\mathbf{BD}}{(4\mu\beta-\lambda\gamma)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{\lambda\gamma^2\mathbf{BD}}{(4\mu\beta-\lambda\gamma)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{-\gamma\mathbf{B}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & -1-\frac{8\mu\beta\mathbf{B}^2}{(4\mu\beta-\lambda\gamma)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{4\lambda\beta\mathbf{B}^2}{(4\mu\beta-\lambda\gamma)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{\gamma\mathbf{BD}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & \frac{4\mu\gamma\mathbf{B}^2}{(4\mu\beta-\lambda\gamma)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -1+\frac{2\lambda\gamma\mathbf{B}^2}{(4\mu\beta-\lambda\gamma)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{2\alpha\mathbf{BD}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & -\frac{4\mu\gamma\mathbf{BD}}{(4\mu\beta-\lambda\gamma)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{2\lambda\gamma\mathbf{BD}}{(4\mu\beta-\lambda\gamma)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{\alpha\mathbf{D}^2+\mathbf{B}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} \end{pmatrix}$	$\gamma\neq 0,\ \gamma^2-4\alpha\beta=0$ $(\lambda\gamma-4\mu\beta)(\alpha\mathbf{D}^2-\mathbf{B}^2)\neq 0$	Never
	$M_{\varphi_a} = L_{a^{-1}}R_a$	$a=(\gamma\mathbf{D},-2\mathbf{F},2\mathbf{B},-2\mathbf{D})^t$ $\lvert a\rvert\neq 0$	Always

TABLE 2. Involutions in $A=Cl(\alpha,\beta,\gamma)$.

3. Skew-derivations anticommuting with involutions

The aim of this section is to find all possible φ-derivations d that anticommute with a fixed involution φ and such that $d^2 \equiv 0$. d is a φ-derivation if and only if $d(ab) = d(a)b + \varphi(a)d(b)$ for every $a, b \in A$, but we can dramatically reduce the number of instances to verify.

Proposition 3.1. *Let $A = Cl(\alpha, \beta, \gamma)$ be a Clifford algebra, let $\varphi : A \to A$ be a fixed algebra morphism and let $d : A \to A$ be a k-linear map. Then d is a φ-derivation if, and only if,*

$$d(1_A) = 0, \tag{138}$$
$$d(G)G + \varphi(G)d(G) = 0, \tag{139}$$
$$d(X)X + \varphi(X)d(X) = 0, \tag{140}$$
$$d(GX) - d(G)X - \varphi(G)d(X) = 0, \tag{141}$$
$$d(G)X + \varphi(G)d(X) + d(X)G + \varphi(X)d(G) = 0. \tag{142}$$

Proof. ($\Longleftarrow$) Assume d is a φ-derivation. Then (138)-(142) are easily verified.

($\Longrightarrow$) Assume (138)-(142) hold. Since d is k-linear, we need to prove that $d(ab) = d(a)b + \varphi(a)d(b)$ for all $a, b \in \{1_A, G, X, GX\}$. When $a = 1_A$ this is trivially true, given (138) and (100). Assume $a = G$, then (138), (139) and (141) guarantee that the formula holds respectively for $b = 1, G, X$. We only need to show that $d(G \cdot GX) = d(G)GX + \varphi(G)d(GX)$. We have

$$\begin{aligned} d(G)GX + \varphi(G)d(GX) &\overset{(141)}{=} d(G)GX + \varphi(G)d(G)X + \varphi(G)\varphi(G)d(X) \\ &= (d(G)G + \varphi(G)d(G))X + \varphi(G)^2 d(X) \\ &\overset{(139)}{=} \alpha d(X) \\ &= d(\alpha X) \\ &= d(G \cdot GX). \end{aligned}$$

Let $a = X$. For $b = 1$ we get $d(X) = d(X)1_A + \varphi(X)d(1) \overset{(138)}{=} d(X)$. For $b = G$ we need to prove that $d(XG) = d(X)G + \varphi(X)d(G)$:

$$\begin{aligned} d(XG) &= d(\gamma - GX) \\ &= d(\gamma) - d(GX) \\ &\overset{(138)}{=} -d(GX) \\ &\overset{(141))}{=} -d(G)X - \varphi(G)d(X) \\ &\overset{(142)}{=} d(X)G + \varphi(X)d(G). \end{aligned}$$

For $b = GX$, we find

$$\begin{aligned} d(X)GX + \varphi(X)d(GX) &\overset{(141)}{=} d(X)GX + \varphi(X)(d(G)X + \varphi(G)d(X)) \\ &\overset{(142)}{=} d(X)GX + \varphi(X)(-d(X)G - \varphi(X)d(G)) \\ &= d(X)GX - \varphi(X)d(X)G - \beta d(G) \\ &\overset{(140)}{=} d(X)GX + d(X)XG - \beta d(G) \\ &= \gamma d(X) - \beta d(G) \\ &= d(\gamma X - \beta G) \\ &= d(X(\gamma - XG)) \\ &= d(X \cdot GX). \end{aligned}$$

Finally let $a = GX$. For $b = 1$ we have $d(GX) = d(GX)1_A + \varphi(GX)d(1_A) \overset{(138)}{=} d(GX)$.

For $b = G$ we have

$$d(GX)G + \varphi(GX)d(G) \overset{(141)}{=} d(G)XG + \varphi(G)d(X)G + \varphi(G)\varphi(X)d(G)$$

$$\begin{aligned}
&\overset{(142)}{=} d(G)XG+\varphi(G)(-d(G)X-\varphi(G)d(X))\\
&= d(G)XG-\varphi(G)d(G)X-\alpha d(X)\\
&\overset{(139)}{=} d(G)XG+d(G)GX-\alpha d(X)\\
&= \gamma d(G)-\alpha d(X)\\
&= d(\gamma G-\alpha X)\\
&= d(G(\gamma-GX))\\
&= d(GX\cdot G).
\end{aligned}$$

For $b=X$,

$$\begin{aligned}
d(GX)X+\varphi(GX)d(X) &\overset{(141)}{=} d(G)X^2+\varphi(G)d(X)X+\varphi(G)\varphi(X)d(X)\\
&= \beta d(G)+\varphi(G)(d(X)X+\varphi(X)d(X))\\
&\overset{(140)}{=} \beta d(G)\\
&= d(\beta G)\\
&= d(GX\cdot X)
\end{aligned}$$

and finally, for $b=GX$,

$$\begin{aligned}
d(GX)GX+\varphi(GX)d(GX) &\overset{(141)}{=} d(GX)GX+\varphi(GX)(d(G)X+\varphi(G)d(X))\\
&\overset{(142)}{=} d(GX)GX+\varphi(G)\varphi(X)(-d(X)G-\varphi(X)d(G))\\
&\overset{(140)}{=} d(GX)GX+\varphi(G)d(X)XG-\beta\varphi(G)d(G)\\
&\overset{(139)}{=} d(GX)GX+\varphi(G)d(X)XG+\beta d(G)G\\
&= d(GX)GX+\varphi(G)d(X)XG+d(G)XXG\\
&\overset{(141)}{=} d(GX)GX+d(GX)XG\\
&= \gamma d(GX)\\
&\overset{(138)}{=} d(\gamma GX-\alpha\beta)\\
&= d(G(\gamma-GX)X)\\
&= d(GX\cdot GX)
\end{aligned}$$

□

Since we have split all elements of $\mathrm{Inv}_{\mathrm{Alg}}(A)$ into three families according to their spectrum, we will treat each case picking a representative for each family and then discuss their form depending on the involved coefficients.

3.1. The case Id. Clearly if we want $d\circ\mathrm{Id}+\mathrm{Id}\circ d=0$, then the only possible choice for d is the trivial one $d\equiv 0$.

3.2. The case D_1. Consider all involutions φ whose matrix is similar to D_1. This means that there is an invertible matrix P such that $M_\varphi = PD_1P^{-1}$ (where M_φ is the matrix of φ with respect to the canonical basis $(1,G,X,GX)$). We have seen in the previous section that P has to be of the form

$$P=\begin{pmatrix}1 & 0 & -\gamma & \gamma\\ 0 & -\mathbf{B} & \gamma\mathbf{B}-2\beta\mathbf{A} & \gamma\mathbf{B}-2\beta\mathbf{A}\\ 0 & \mathbf{A} & \gamma\mathbf{A}-2\alpha\mathbf{B} & \gamma\mathbf{A}-2\alpha\mathbf{B}\\ 0 & 0 & 2 & -2\end{pmatrix}$$

with $\mathbf{A},\mathbf{B}\in k$, satisfying the following conditions

$$\begin{cases}\gamma^2-4\alpha\beta=0\\ \alpha\mathbf{B}^2+\beta\mathbf{A}^2-\gamma\mathbf{AB}=1.\end{cases}\tag{143}$$

Suppose to know all possible matrices Δ such that $\Delta D_1 + D_1\Delta = 0$. Then it is clear that the anticommuting φ-derivation we are looking for will have matrix Δ with respect to the eigenbasis for M_φ and therefore their matrix with respect to the canonical basis will be $P\Delta P^{-1}$. It is convenient to look at D_1 in block form: $D_1 = \begin{pmatrix} I_3 & \\ & -1 \end{pmatrix}$, where I_n denotes the identity matrix of order n and blank entries are suitable sets of zeros. Then, if we partion Δ accordingly $\Delta = \begin{pmatrix} \Delta^1 & \Delta^2 \\ \Delta^3 & \Delta_{44} \end{pmatrix}$, we can easily calculate

$$\Delta D_1 + D_1\Delta = \begin{pmatrix} 2\Delta^1 & \\ & -2\Delta_{44} \end{pmatrix},$$

so that we immediately see that Δ must be of the form $\Delta = \begin{pmatrix} & \Delta^2 \\ \Delta^3 & \end{pmatrix}$, where again blanks denote zeros. Done with anticommutativity we can proceed with conditions of Proposition 3.1. (138) yields

$$d(1) = 0 \iff \Delta P^{-1}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \iff \Delta_{41} = 0.$$

By employing the tools of Appendix A,(139) can be rewritten as $(R_G + L_{\varphi(G)})P\Delta P^{-1}(0,1,0,0)^t = (0,0,0,0)^t$. We rearrange things using Proposition 0.3, since $\varphi \in \mathrm{Inv}_{\mathrm{Alg}}(A)$:

$$(R_G + L_{\varphi(G)})P\Delta P^{-1}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \overset{\text{Remark 0.3}}{\iff} (R_G + M_\varphi L_G M_\varphi)P\Delta P^{-1}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \overset{\text{Apply } D_1P^{-1}}{\iff}$$

$$(D_1P^{-1}R_G + P^{-1}L_GM_\varphi)P\Delta P^{-1}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \iff (D_1P^{-1}R_GP + P^{-1}L_GPD_1)\Delta P^{-1}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}.$$

The term $(D_1P^{-1}R_GP + P^{-1}L_GPD_1)$ can be easily calculated to be

$$D_1P^{-1}R_GP + P^{-1}L_GPD_1 = \begin{pmatrix} 0 & \gamma\mathbf{A} - 2\alpha\mathbf{B} & 0 & 0 \\ \gamma\mathbf{A} - 2\alpha\mathbf{B} & 0 & 0 & 0 \\ -\frac{\mathbf{A}}{2} & 0 & 0 & 0 \\ 0 & -\frac{\mathbf{A}}{2} & 0 & 0 \end{pmatrix} =: T.$$

Then

$$T\Delta P^{-1}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix} 0 & \frac{\Delta_{24}\mathbf{A}(2\mathbf{B}-\gamma\mathbf{A})}{4} & \frac{\Delta_{24}\mathbf{B}(2\mathbf{B}-\gamma\mathbf{A})}{4} & \frac{\Delta_{24}\mathbf{A}(2\mathbf{B}-\gamma\mathbf{A})}{4} \\ 0 & \frac{\Delta_{14}\mathbf{A}(2\mathbf{B}-\gamma\mathbf{A})}{4} & \frac{\Delta_{14}\mathbf{B}(2\mathbf{B}-\gamma\mathbf{A})}{4} & \frac{\Delta_{14}\mathbf{A}(2\mathbf{B}-\gamma\mathbf{A})}{4} \\ 0 & \frac{\Delta_{14}\mathbf{A}^2}{8} & \frac{\Delta_{14}\mathbf{AB}}{8} & \frac{\Delta_{14}\mathbf{A}}{8} \\ 0 & \frac{\Delta_{24}\mathbf{A}^2}{8} & \frac{\Delta_{24}\mathbf{AB}}{8} & \frac{\Delta_{24}\mathbf{A}}{8} \end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix} \frac{\Delta_{24}\mathbf{A}(2\mathbf{B}-\gamma\mathbf{A})}{4} \\ \frac{\Delta_{14}\mathbf{A}(2\mathbf{B}-\gamma\mathbf{A})}{4} \\ \frac{\Delta_{14}\mathbf{A}^2}{8} \\ \frac{\Delta_{24}\mathbf{A}^2}{8} \end{pmatrix}.$$

In particular we must have $\Delta_{14}\mathbf{A}^2 = 0 = \Delta_{24}\mathbf{A}^2$. If one performs similar calculations starting from (140), it can be shown that also $\Delta_{14}\mathbf{B}^2 = 0 = \Delta_{24}\mathbf{B}^2$. If we suppose $\Delta_{14} \neq 0 \neq \Delta_{24}$, then $\mathbf{A} = \mathbf{B} = 0$, but this is clearly a contradiction (see (143)), thus $\Delta_{14} = \Delta_{24} = 0$. Observe that now both (139) and (140) result verified. Now let us rewrite (142):

$$d(G)X + \varphi(G)d(X) + d(X)G + \varphi(X)d(G) = 0 \iff$$

$$\iff (R_X + L_{\varphi(X)})P\Delta P^{-1}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + (R_G + L_{\varphi(G)})P\Delta P^{-1}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix},$$

or equivalently, by applying D_1P^{-1} on the left,

$$(D_1P^{-1}R_XP + P^{-1}L_XP)\Delta P^{-1}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + (D_1P^{-1}R_GP + P^{-1}L_GPD_1)\Delta P^{-1}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}. \tag{144}$$

Once checked that $(D_1P^{-1}R_XP + P^{-1}L_XPD_1)\Delta$ and $(D_1P^{-1}R_GP + P^{-1}L_GPD_1)\Delta$ are both the zero matrix, it follows that (142) is trivially satisfied. Finally we consider (141), which can be rewritten as

$$D_1\Delta P^{-1}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = D_1P^{-1}R_XP\Delta P^{-1}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + P^{-1}L_GPD_1\Delta P^{-1}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}.$$

Explicitly we find

$$\begin{pmatrix}0\\0\\-\frac{\Delta_{34}}{4}\\-\frac{\Delta_{43}}{4}\end{pmatrix} = \begin{pmatrix}0\\0\\\frac{\Delta_{34}}{4}\\\frac{\Delta_{43}}{4}\end{pmatrix},$$

which implies $\Delta_{34} = \Delta_{43} = 0$. It follows that $\Delta^2 = 0$. With the help of a computer software it is not difficult to check that these necessary conditions are also sufficient for d to be a φ-derivation such that $d^2 \equiv 0$ and $d\varphi = -\varphi d$. We can conclude that all the φ-derivations we are looking for, are given by matrices

$$P\Delta P^{-1} = \begin{pmatrix}0 & \gamma\Delta_{42}\frac{\gamma\mathbf{A}-2\alpha\mathbf{B}}{2} & \gamma\Delta_{42}\frac{2\beta\mathbf{A}-\gamma\mathbf{B}}{2} & 0\\ 0 & -\gamma\Delta_{42} & -2\beta\Delta_{42} & 0\\ 0 & 2\alpha\Delta_{42} & \gamma\Delta_{42} & 0\\ 0 & \Delta_{42}(2\alpha\mathbf{B}-\gamma\mathbf{A}) & \Delta_{42}(\gamma\mathbf{B}-2\beta\mathbf{A}) & 0\end{pmatrix},$$

where $\Delta_{42} \in k$ and

$$\begin{cases}\gamma^2 - 4\alpha\beta = 0\\ \alpha\mathbf{B}^2 + \beta\mathbf{A}^2 - \gamma\mathbf{A}\mathbf{B} = 1.\end{cases}$$

We have

$$d(G) = \gamma\Delta_{42}\frac{\gamma\mathbf{A}-2\alpha\mathbf{B}}{2} - \gamma\Delta_{42}G + 2\alpha\Delta_{42}X + \Delta_{42}(2\alpha\mathbf{B}-\gamma\mathbf{A})GX$$
$$d(X) = \gamma\Delta_{42}\frac{2\beta\mathbf{A}-\gamma\mathbf{B}}{2} - 2\beta\Delta_{42}G + \gamma\Delta_{42}X + \Delta_{42}(\gamma\mathbf{B}-2\beta\mathbf{A})GX$$
$$d(GX) = d(1) = 0,$$

while the image of the eigenbasis is given by

$$P\Delta P^{-1}P = P\Delta = \begin{pmatrix}0 & \gamma\Delta_{42} & 0 & 0\\ 0 & \Delta_{42}(\gamma\mathbf{B}-2\beta\mathbf{A}) & 0 & 0\\ 0 & \Delta_{42}(\gamma\mathbf{A}-2\alpha\mathbf{B}) & 0 & 0\\ 0 & -2\Delta_{42} & 0 & 0\end{pmatrix}.$$

REMARK 3.2. Since A_- is the space generated by the last column of P, we immediately see that $d(A_-) = \{0\}$.

3.3. The case D_2. In this case, unfortunately, the form of the change-of-basis matrix P depends on the invertibility of $\mathbf{E}$. Recall that

$$P = \begin{pmatrix}1 & 0 & 0 & \frac{\gamma}{2}\mathbf{E}\\ 0 & \mathbf{F} & \mathbf{A} & \mathbf{C}\\ 0 & -\mathbf{B} & -\mathbf{E} & 0\\ 0 & \mathbf{D} & 0 & -\mathbf{E}\end{pmatrix} \text{ if } \mathbf{E} \neq 0 \quad\text{and}\quad P = \begin{pmatrix}1 & 0 & 0 & \frac{\gamma}{2}\mathbf{A}\\ 0 & \mathbf{F} & 1 & 0\\ 0 & -\mathbf{B} & 0 & \mathbf{C}\\ 0 & \mathbf{D} & 0 & -\mathbf{A}\end{pmatrix} \text{ if } \mathbf{E} = 0,$$

where involved coefficients need to satisfy

$$\begin{cases} \gamma\mathbf{AD} - 2\mathbf{CF} - 2\beta\mathbf{DE} = 0 \\ \gamma\mathbf{DE} + 2\mathbf{BC} - 2\alpha\mathbf{AD} = 0 \\ 2\alpha\mathbf{AF} + 2\beta\mathbf{BE} - \gamma\mathbf{AB} - \gamma\mathbf{EF} = 0 \\ -\mathbf{AB} + \mathbf{CD} + \mathbf{EF} \neq 0. \end{cases} \tag{145}$$

If again we denote by Δ the matrix of skew-derivation d with respect to the eigenbasis of M_φ there is a first simplification that can be carried out both when $\mathbf{E} \neq 0$ and $\mathbf{E} = 0$. Since we want to find all possible matrices Δ such that $\Delta D_2 + D_2\Delta = 0$, we write $D_2 = \begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix}$ and partition Δ accordingly $\Delta = \begin{pmatrix} \Delta^1 & \Delta^2 \\ \Delta^3 & \Delta^4 \end{pmatrix}$. Notice that in this case each block Δ^i is of order 2. We find

$$\Delta D_2 + D_2\Delta = \begin{pmatrix} 2\Delta^1 & \\ & -2\Delta^4 \end{pmatrix},$$

so that $\Delta = \begin{pmatrix} & \Delta^2 \\ \Delta^3 & \end{pmatrix}$. If we impose (138):

$$d(1) = 0 \iff \Delta P^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} 0 \\ 0 \\ \Delta_{31} \\ \Delta_{41} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

we can conclude that also $\Delta_{31} = \Delta_{41} = 0$. As already pointed out, this is true both when $\mathbf{E} \neq 0$ and $\mathbf{E} = 0$, but to further proceed we need to discuss the two cases separately. For sake of clarity we display the present form of Δ and Δ^2 which will be needed in the sequel

$$\Delta = \begin{pmatrix} 0 & 0 & \Delta_{13} & \Delta_{14} \\ 0 & 0 & \Delta_{23} & \Delta_{24} \\ 0 & \Delta_{32} & 0 & 0 \\ 0 & \Delta_{42} & 0 & 0 \end{pmatrix}, \tag{146}$$

$$\Delta^2 = \begin{pmatrix} 0 & \Delta_{13}\Delta_{32} + \Delta_{14}\Delta_{42} & 0 & 0 \\ 0 & \Delta_{23}\Delta_{32} + \Delta_{24}\Delta_{42} & 0 & 0 \\ 0 & 0 & \Delta_{23}\Delta_{32} & \Delta_{24}\Delta_{32} \\ 0 & 0 & \Delta_{23}\Delta_{42} & \Delta_{24}\Delta_{42} \end{pmatrix}. \tag{147}$$

3.3.1. *Assume* $\boldsymbol{E} \neq 0$. We take a look at (147) and see that $d^2 \equiv 0$ forces $\Delta_{23}\Delta_{32} = 0 = \Delta_{23}\Delta_{42}$. Let us suppose $\Delta_{23} \neq 0$. Then $\Delta_{32} = 0 = \Delta_{42}$ and $d^2 \equiv 0$ is guaranteed. Since (139) explicitly becomes

$$PD_2(D_2P^{-1}R_GP + P^{-1}L_GPD_2)\Delta P^{-1} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

then $d(G)G + \varphi(G)d(G) = 0$ if, and only if, the second column of $(D_2P^{-1}R_GP + P^{-1}L_GPD_2)\Delta P^{-1}$ has every entry equal to 0. Focusing our attention on the third and fourth entries of this column we see that we need

$$\frac{2\alpha\mathbf{D}(\Delta_{23}\mathbf{B} - \Delta_{24}\mathbf{D})}{\mathbf{E}(-\mathbf{AB} + \mathbf{CD} + \mathbf{EF})} = 0, \tag{148}$$

$$\frac{2\mathbf{B}(\Delta_{23}\mathbf{B} - \Delta_{24}\mathbf{D})}{\mathbf{E}(-\mathbf{AB} + \mathbf{CD} + \mathbf{EF})} = 0. \tag{149}$$

Let us suppose $\mathbf{B} \neq 0$. Then (149) implies $\Delta_{23}\mathbf{B} - \Delta_{24}\mathbf{D} = 0$. By applying D_2P^{-1} on the left, (142) can be rewritten as

$$(D_2P^{-1}R_XP + P^{-1}L_XP)\Delta P^{-1}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + (D_2P^{-1}R_GP + P^{-1}L_GPD_2)\Delta P^{-1}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}. \tag{150}$$

The equation involving the last entries is

$$\Delta_{24}(\mathbf{ABD} + \mathbf{DEF}) - \Delta_{23}(\mathbf{BCD} + 2\mathbf{BEF}) = 0,$$

i.e.[3]

$$\Delta_{23}\mathbf{B}(\mathbf{AB} - \mathbf{EF} - \mathbf{CD}) = 0.$$

This is clearly a contradiction since none of the factors involved can be zero (cf. (145)). Therefore we deduce that our assumption is wrong and we necessarily have $\mathbf{B} = 0$. Consequently the last entry of (150) becomes $\frac{2\Delta_{24}\mathbf{DF}}{\mathbf{E}(\mathbf{CD}+\mathbf{EF})}$ and we get the necessary condition $\Delta_{24}\mathbf{DF} = 0$. If we suppose $\mathbf{F} \neq 0$, then $\Delta_{24}\mathbf{D} = 0$. By applying D_2P^{-1} on the left, (140) can be rewritten as

$$(D_2P^{-1}R_XP + P^{-1}L_XP)\Delta P^{-1}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}.$$

The last of this four equations is

$$\frac{\Delta_{23}\mathbf{F}}{\mathbf{E}^2} = 0,$$

which gives a contradiction. As a consequence $\mathbf{F} = 0$. The last equation in (145) gives $\mathbf{C} \neq 0 \neq \mathbf{D}$, therefore (148) becomes $\alpha\Delta_{24} = 0$. If we suppose $\alpha \neq 0$, then $\Delta_{24} = 0$ and the third entry in (150) reads

$$2\frac{\alpha\Delta_{23}\mathbf{D}}{\mathbf{E}^2} = 0,$$

which again gives a contradiction. Therefore we must have $\alpha = 0$ and the second equation of (145) gives $\gamma = 0$, while the first reduces to $\beta = 0$. (139) becomes equivalent to $\Delta_{14} = 0$ and (142) to $\Delta_{13} = 0$.

So, whenever $\mathbf{E} \neq 0$ and

$$\begin{cases} \mathbf{B} = \mathbf{F} = 0 \\ \mathbf{C} \neq 0 \neq \mathbf{D} \\ \alpha = \beta = \gamma = 0 \end{cases}$$

there is a family of φ-derivation with matrix of the form

$$P\Delta P^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\Delta_{24}\mathbf{D}}{\mathbf{C}} & \frac{\mathbf{D}(\Delta_{24}\mathbf{A} - \Delta_{23}\mathbf{C})}{\mathbf{CE}} & 0 \end{pmatrix},$$

with $\Delta_{24}, \Delta_{23} \in k$

Now assume $\Delta_{24} \neq 0$. Then again $d^2 \equiv 0$ forces $\Delta_{32} = 0 = \Delta_{42}$ (see (147)). As in the previous case, we find that (139) is satisfied only if (148) and (149) are. Again, let us suppose $\mathbf{B} \neq 0$. Then (149) implies $\Delta_{23}\mathbf{B} - \Delta_{24}\mathbf{D} = 0$ and (142) rewrites as (150). The equation involving the last entries is

$$\Delta_{24}(\mathbf{ABD} + \mathbf{DEF}) - \Delta_{23}(\mathbf{BCD} + 2\mathbf{BEF}) = 0,$$

i.e.

$$\Delta_{24}\mathbf{D}(\mathbf{AB} - \mathbf{CD} - \mathbf{EF}) = 0,$$

[3] by using $\Delta_{23}\mathbf{B} - \Delta_{24}\mathbf{D} = 0$

which forces $\mathbf{D} = 0$. $\Delta_{23}\mathbf{B} - \Delta_{24}\mathbf{D} = 0$ yields $\Delta_{23} = 0$ and (145) implies $\mathbf{C} = 0$. (139) becomes equivalent to

$$2\frac{\Delta_{13}\mathbf{B}^2\mathbf{E}}{(1\mathbf{AB} - \mathbf{EF})^2} = 0,$$

i.e. $\Delta_{13} = 0$, while (141) rewrites

$$\begin{pmatrix} -\frac{\Delta_{14}}{\mathbf{E}} \\ -\frac{\Delta_{24}\mathbf{F}}{\mathbf{E}} \\ \frac{\Delta_{24}\mathbf{B}}{\mathbf{E}} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

that gives a contradiction since $\Delta_{24} \neq 0 \neq \mathbf{B}$. This shows that the assumption $\mathbf{B} \neq 0$ is wrong, hence $\mathbf{B} = 0$. Again the last entry of (150) becomes $\frac{2\Delta_{24}\mathbf{DF}}{\mathbf{E}(\mathbf{CD}+\mathbf{EF})}$ and we get the necessary condition $\mathbf{DF} = 0$. If we suppose $\mathbf{F} \neq 0$, then $\mathbf{D} = 0$ and (145) yields also $\mathbf{C} = 0$. (140) becomes

$$\begin{pmatrix} -\frac{\gamma\Delta_{23}\mathbf{F}}{\mathbf{E}} \\ -2\frac{\Delta_{13}\mathbf{A}}{\mathbf{E}^2\mathbf{F}} \\ 0 \\ -2\frac{\Delta_{23}\mathbf{F}}{\mathbf{E}^2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which forces $\Delta_{23} = 0$. Consequently (150) becomes equivalent to $\Delta_{13} = 0$ and finally (141) boils down to

$$\begin{pmatrix} -\frac{\Delta_{14}}{\mathbf{E}} \\ -\frac{\Delta_{24}}{\mathbf{E}} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which gives a contradiction, since $\Delta_{24} \neq 0 \neq \mathbf{F}$. This shows that the assumption $\mathbf{F} \neq 0$ is wrong and therefore $\mathbf{F} = 0$. The last equation in (145) gives $\mathbf{C} \neq 0 \neq \mathbf{D}$, thus (148) becomes $\alpha\Delta_{24} = 0$, i.e. $\alpha = 0$. The second equation of (145) gives $\gamma = 0$, while the first reduces to $\beta = 0$. (139) becomes equivalent to $\Delta_{14} = 0$ and (142) to $\Delta_{13} = 0$. Notice that this family of φ-derivations coincides with the one we have already found.

Finally if $\Delta_{23} = 0 = \Delta_{24}$, (139) is equivalent to

$$\begin{cases} 2\Delta_{13}\mathbf{B} - 2\Delta_{14}\mathbf{D} - 2\Delta_{32}\mathbf{CE} - \Delta_{42}(2\alpha\mathbf{A} - \gamma\mathbf{E})\mathbf{E} = 0 \\ \gamma(\Delta_{13}\mathbf{BD} - \Delta_{14}\mathbf{D}^2 - \Delta_{32}\mathbf{CDE} - \Delta_{42}\mathbf{BCE}) = 0. \end{cases}$$

If we substitute $2\Delta_{13}\mathbf{B} = 2\Delta_{14}\mathbf{D} + 2\Delta_{32}\mathbf{CE} + \Delta_{42}(2\alpha\mathbf{A} - \gamma\mathbf{E})\mathbf{E}$ in the second equation, we find

$$\gamma\Delta_{42}[(2\alpha\mathbf{A} - \gamma\mathbf{E})\mathbf{D} - 2\mathbf{BC}] = 0,$$

which is true, in view of (145). Therefore

$$(139) \iff 2\Delta_{13}\mathbf{B} = 2\Delta_{14}\mathbf{D} + 2\Delta_{32}\mathbf{CE} + \Delta_{42}(2\alpha\mathbf{A} - \gamma\mathbf{E})\mathbf{E}.$$

The matrix of d is of the form

$$P\Delta P^{-1} = \begin{pmatrix} 0 & \frac{\gamma\Delta_{42}\mathbf{E}^2 - 2\Delta_{13}\mathbf{B} + 2\Delta_{14}\mathbf{D}}{2(-\mathbf{AB}+\mathbf{CD}+\mathbf{EF})} & \frac{\gamma\Delta_{42}\mathbf{AE}^2 - 2\Delta_{13}\mathbf{EF} + 2\Delta_{14}\mathbf{AD} - 2\Delta_{13}\mathbf{CD}}{2\mathbf{E}(-\mathbf{AB}+\mathbf{CD}+\mathbf{EF})} & \frac{\gamma\Delta_{42}\mathbf{CE}^2 - 2\Delta_{14}\mathbf{EF} + 2\Delta_{14}\mathbf{AB} - 2\Delta_{13}\mathbf{BC}}{2\mathbf{E}(-\mathbf{AB}+\mathbf{CD}+\mathbf{EF})} \\ 0 & \frac{-\mathbf{E}(\Delta_{42}\mathbf{C}+\Delta_{32}\mathbf{A})}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\mathbf{A}(\Delta_{42}\mathbf{C}+\Delta_{32}\mathbf{A})}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\mathbf{C}(\Delta_{42}\mathbf{C}+\Delta_{32}\mathbf{A})}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} \\ 0 & \frac{\Delta_{32}\mathbf{E}^2}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\Delta_{32}\mathbf{AE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\Delta_{32}\mathbf{CE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} \\ 0 & \frac{\Delta_{42}\mathbf{E}^2}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\Delta_{42}\mathbf{AE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\Delta_{42}\mathbf{CE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} \end{pmatrix},$$

thus, if we substitute $2\Delta_{13}\mathbf{B} = 2\Delta_{14}\mathbf{D} + 2\Delta_{32}\mathbf{CE} + \Delta_{42}(2\alpha\mathbf{A} - \gamma\mathbf{E})\mathbf{E}$ we find

$$P\Delta P^{-1} = \begin{pmatrix} 0 & \frac{\mathbf{E}[\Delta_{42}(\alpha\mathbf{A}-\gamma\mathbf{E})+\Delta_{32}\mathbf{C}]}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\gamma\Delta_{42}\mathbf{AE}^2-2\Delta_{13}\mathbf{EF}+2\Delta_{14}\mathbf{AD}-2\Delta_{13}\mathbf{CD}}{2\mathbf{E}(-\mathbf{AB}+\mathbf{CD}+\mathbf{EF})} & \frac{\gamma\Delta_{42}\mathbf{CE}-\Delta_{32}\mathbf{C}^2-\alpha\Delta_{42}\mathbf{AC}}{-\mathbf{AB}+\mathbf{CD}+\mathbf{EF}} - \frac{\Delta_{14}}{\mathbf{E}} \\ 0 & \frac{-\mathbf{E}(\Delta_{42}\mathbf{C}+\Delta_{32}\mathbf{A})}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\mathbf{A}(\Delta_{42}\mathbf{C}+\Delta_{32}\mathbf{A})}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\mathbf{C}(\Delta_{42}\mathbf{C}+\Delta_{32}\mathbf{A})}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} \\ 0 & \frac{\Delta_{32}\mathbf{E}^2}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\Delta_{32}\mathbf{AE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\Delta_{32}\mathbf{CE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} \\ 0 & \frac{\Delta_{42}\mathbf{E}^2}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\Delta_{42}\mathbf{AE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\Delta_{42}\mathbf{CE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} \end{pmatrix}.$$

Now (140) explicitly reads

$$\begin{pmatrix} -\frac{\gamma\mathbf{D}\Pi}{2(\mathbf{AB}-\mathbf{CD}-\mathbf{EF})^2} \\ \frac{\mathbf{F}\Pi}{(\mathbf{AB}-\mathbf{CD}-\mathbf{EF})^2} \\ -\frac{\mathbf{B}\Pi}{(\mathbf{AB}-\mathbf{CD}-\mathbf{EF})^2} \\ \frac{\mathbf{D}\Pi}{(\mathbf{AB}-\mathbf{CD}-\mathbf{EF})^2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where $\Pi = -2\Delta_{13}(\mathbf{DC}+\mathbf{EF}) + 2\Delta_{14}\mathbf{AD} + 2\Delta_{32}\mathbf{ACE} + \Delta_{42}(2\mathbf{C}^2 - 2\beta\mathbf{E}^2 + \gamma\mathbf{AE})\mathbf{E}$. Since $\Pi \neq 0$ forces $\mathbf{B} = \mathbf{D} = \mathbf{F} = 0$ (that is a contradiction in view of (145)) then $\Pi = 0$. If we use $2\Delta_{13}(\mathbf{DC}+\mathbf{EF}) = 2\Delta_{14}\mathbf{AD} + 2\Delta_{32}\mathbf{ACE} + \Delta_{42}(2\mathbf{C}^2 - 2\beta\mathbf{E}^2 + \gamma\mathbf{AE})\mathbf{E}$ for the entries of the matrix of d we get

$$P\Delta P^{-1} = \begin{pmatrix} 0 & \frac{\mathbf{E}[\Delta_{42}(\alpha\mathbf{A}-\gamma\mathbf{E})+\Delta_{32}\mathbf{C}]}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\Delta_{42}(\mathbf{C}^2-\beta\mathbf{E}^2)+\Delta_{32}\mathbf{AC}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\gamma\Delta_{42}\mathbf{CE}-\Delta_{32}\mathbf{C}^2-\alpha\Delta_{42}\mathbf{AC}}{-\mathbf{AB}+\mathbf{CD}+\mathbf{EF}} - \frac{\Delta_{14}}{\mathbf{E}} \\ 0 & \frac{-\mathbf{E}(\Delta_{42}\mathbf{C}+\Delta_{32}\mathbf{A})}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{-\mathbf{A}(\Delta_{42}\mathbf{C}+\Delta_{32}\mathbf{A})}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\mathbf{C}(\Delta_{42}\mathbf{C}+\Delta_{32}\mathbf{A})}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} \\ 0 & \frac{\Delta_{32}\mathbf{E}^2}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\Delta_{32}\mathbf{AE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\Delta_{32}\mathbf{CE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} \\ 0 & \frac{\Delta_{42}\mathbf{E}^2}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\Delta_{42}\mathbf{AE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\Delta_{42}\mathbf{CE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} \end{pmatrix}.$$

Now we calculate $d(G)X + \varphi(G)d(X)$ and we get

$$R_X P\Delta P^{-1}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + PD_2P^{-1}\Delta PD_2P^{-1}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{(\Delta_{42}\mathbf{C}+\Delta_{32}\mathbf{A})(\alpha\mathbf{A}-\gamma\mathbf{E})+\beta\Delta_{32}\mathbf{E}^2}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} \\ \frac{\mathbf{C}(\Delta_{42}\mathbf{C}+\Delta_{32}\mathbf{A})}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} \\ \frac{\Delta_{32}\mathbf{CE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} \\ \frac{\Delta_{42}\mathbf{CE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} \end{pmatrix}$$

and therefore we conclude that d must have matrix of the form

$$P\Delta P^{-1} = \begin{pmatrix} 0 & \frac{\mathbf{E}[\Delta_{42}(\alpha\mathbf{A}-\gamma\mathbf{E})+\Delta_{32}\mathbf{C}]}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\Delta_{42}(\mathbf{C}^2-\beta\mathbf{E}^2)+\Delta_{32}\mathbf{AC}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{(\Delta_{42}\mathbf{C}+\Delta_{32}\mathbf{A})(\alpha\mathbf{A}-\gamma\mathbf{E})+\beta\Delta_{32}\mathbf{E}^2}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} \\ 0 & \frac{-\mathbf{E}(\Delta_{42}\mathbf{C}+\Delta_{32}\mathbf{A})}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{-\mathbf{A}(\Delta_{42}\mathbf{C}+\Delta_{32}\mathbf{A})}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\mathbf{C}(\Delta_{42}\mathbf{C}+\Delta_{32}\mathbf{A})}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} \\ 0 & \frac{\Delta_{32}\mathbf{E}^2}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\Delta_{32}\mathbf{AE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\Delta_{32}\mathbf{CE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} \\ 0 & \frac{\Delta_{42}\mathbf{E}^2}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\Delta_{42}\mathbf{AE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\Delta_{42}\mathbf{CE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} \end{pmatrix}.$$

with $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F}, \alpha, \beta, \gamma, \Delta_{32}, \Delta_{42} \in k$ satisfying (145). One checks by computation that condition $d^2 \equiv 0$ is automatically satisfied.

3.3.2. *Assume* $\boldsymbol{E} = 0$. In this case (145) becomes

$$\begin{cases} \gamma\mathbf{AD} - 2\mathbf{CF} = 0 \\ \mathbf{BC} - \alpha\mathbf{AD} = 0 \\ \gamma\mathbf{B} - 2\alpha\mathbf{F} = 0 \\ \mathbf{AB} - \mathbf{CD} \neq 0. \end{cases} \tag{151}$$

Once again, if we write (139) explicitly we have

$$PD_2(D_2P^{-1}R_GP + P^{-1}L_GPD_2)\Delta P^{-1}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix},$$

which is equivalent to

$$(D_2P^{-1}R_GP + P^{-1}L_GPD_2)\Delta P^{-1}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}.$$

The LHS is easily calculated:

$$(D_2P^{-1}R_GP + P^{-1}L_GPD_2)\Delta P^{-1}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\-\frac{\Delta_{23}(2\mathbf{BCF}-\gamma\mathbf{CD}^2+\gamma\mathbf{ABD}-2\alpha\mathbf{ADF})}{\mathbf{AB}-\mathbf{CD}}\\-\frac{2\Delta_{23}(\alpha\mathbf{D}^2-\mathbf{B}^2)}{\mathbf{AB}-\mathbf{CD}}\end{pmatrix}.$$

(139) is thus equivalent to

$$\begin{cases}\gamma\Delta_{23}\mathbf{D}(\mathbf{CD}-\mathbf{AB}) = 0\\ \Delta_{23}(\alpha\mathbf{D}^2-\mathbf{B}^2) = 0.\end{cases} \tag{152}$$

Suppose $\Delta_{23} \neq 0$. Again $d^2 \equiv 0$ implies $\Delta_{32} = 0 = \Delta_{42}$ (see (147)). Moreover (152) gives $\alpha\mathbf{D}^2 - \mathbf{B}^2 = 0$. If $\mathbf{B} \neq 0$, then $0 \overset{(151)}{\neq} \mathbf{B}(\mathbf{AB}-\mathbf{CD}) = \mathbf{AB}^2 - \mathbf{BCD} \overset{(151)}{=} \mathbf{AB}^2 - \alpha\mathbf{AD}^2 = \mathbf{A}(\mathbf{B}^2 - \alpha\mathbf{D}^2) = 0$, contradiction. This means that $\mathbf{B} = 0$. The last condition in (151) implies $\mathbf{C} \neq 0 \neq \mathbf{D}$ and consequently (152) forces $\alpha = 0 = \gamma$. Now the first equation in (151) gives $\mathbf{F} = 0$. (142) is again equivalent to (150), which explicitly reads

$$\begin{pmatrix}0\\ \frac{2\Delta_{13}\mathbf{A}}{\mathbf{CD}}\\ -2\beta\Delta_{23}\mathbf{D}\\ 0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}.$$

This immediately implies $\beta = 0$. Finally (141) reads

$$P\Delta P^{-1}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} - R_XP\Delta P^{-1}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} - PD_2P^{-1}\Delta PD_2P^{-1}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix},$$

that is

$$\begin{pmatrix}0\\ \Delta_{14}\\ -\frac{\Delta_{13}}{\mathbf{C}}\\ 0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}.$$

Therefore $\Delta_{13} = 0 = \Delta_{14}$.

So, whenever

$$\begin{cases}\mathbf{B} = \mathbf{F} = 0\\ \mathbf{C} \neq 0 \neq \mathbf{D}\\ \alpha = \beta = \gamma = 0\end{cases}$$

there is a family of φ-derivation with matrix of the form

$$P\Delta P^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \Delta_{23}\mathbf{D} & \frac{\Delta_{24}\mathbf{D}}{\mathbf{C}} & 0 \end{pmatrix},$$

with $\Delta_{23}, \Delta_{24} \in k$.

Now suppose $\Delta_{24} \neq 0$. Again $d^2 \equiv 0$ forces $\Delta_{32} = 0 = \Delta_{42}$. (139) is still equivalent to (152). If $\mathbf{B} \neq 0$, then $0 \overset{(151)}{\neq} \mathbf{B}(\mathbf{AB} - \mathbf{CD}) = \mathbf{AB}^2 - \mathbf{BCD} \overset{(151)}{=} \mathbf{AB}^2 - \alpha\mathbf{AD}^2 = \mathbf{A}(\mathbf{B}^2 - \alpha\mathbf{D}^2)$, and therefore $\mathbf{A} \neq 0 \neq \mathbf{B}^2 - \alpha\mathbf{D}^2$. (151) forces $\Delta_{23} = 0$, while (142), being equivalent to (150), boils down to

$$\begin{pmatrix} \frac{2\Delta_{13}\mathbf{CF}}{\mathbf{AB}-\mathbf{CD}} \\ -\frac{2\Delta_{13}\mathbf{A}}{\mathbf{AB}-\mathbf{CD}} \\ -\frac{\Delta_{24}\mathbf{CD}(\gamma\mathbf{D}^2-2\mathbf{BF})}{(\mathbf{AB}-\mathbf{CD})^2} \\ \frac{2\Delta_{24}\mathbf{D}(\alpha\mathbf{D}^2-\mathbf{B}^2)}{(\mathbf{AB}-\mathbf{CD})^2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

From this we get $\Delta_{13} = 0 = \mathbf{D}$. From (151) we get also $\mathbf{C} = 0$. Then (141) becomes

$$\begin{pmatrix} -\frac{\Delta_{14}}{\mathbf{A}} \\ -\frac{\Delta_{24}\mathbf{F}}{\mathbf{A}} \\ \frac{\Delta_{24}\mathbf{B}}{\mathbf{A}} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which gives a contradiction, since $\Delta_{24} \neq 0 \neq \mathbf{B}$. This shows that the inital assumption $\mathbf{B} \neq 0$ is wrong and therefore that $\mathbf{B} = 0$. The last condition in (151) gives $\mathbf{C} \neq 0 \neq \mathbf{D}$. Now suppose $\mathbf{F} \neq 0$. Then (151) forces $\alpha = 0$. By applying D_2P^{-1} on the left, (140) can be rewritten as

$$(D_2P^{-1}R_XP + P^{-1}L_XP)\Delta P^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The last of this four equations is

$$\frac{\gamma\Delta_{24}\mathbf{D}}{\mathbf{C}^2} = 0$$

and so we must have $\gamma = 0$. The first equation in (151) becomes $\mathbf{CF} = 0$, which is a contradiction. This shows that the assumption $\mathbf{F} \neq 0$ was wrong and therefore that $\mathbf{F} = 0$. (140) yields

$$\begin{pmatrix} 0 \\ 2\frac{\Delta_{14}\mathbf{A}}{\mathbf{C}^2\mathbf{D}} \\ -2\frac{\beta\Delta_{24}\mathbf{D}}{\mathbf{C}} \\ \frac{\gamma\Delta_{24}\mathbf{D}}{\mathbf{C}^2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

so that $\beta = 0 = \gamma$. (142) reads

$$\begin{pmatrix} 0 \\ \frac{2\Delta_{13}\mathbf{A}}{\mathbf{CD}} \\ 0 \\ \frac{2\alpha\Delta_{24}\mathbf{D}}{\mathbf{C}^2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

and we conclude that $\alpha = 0$. Finally (141) yields $\Delta_{13} = 0 = \Delta_{14}$. Notice that this family of φ-derivations coincides with the one we have already found.

If $\Delta_{23} = 0 = \Delta_{24}$, (139) is trivially satisfied. (142), being equivalent to (150), reads

$$\begin{pmatrix} \frac{\gamma\mathbf{AD}(-\alpha\Delta_{42}\mathbf{A}^2+\Delta_{13}(\mathbf{AB}-\mathbf{CD})+\Delta_{42}\mathbf{C}^2)}{(\mathbf{AB}-\mathbf{CD})^2} \\ -2\frac{\mathbf{A}(-\alpha\Delta_{42}\mathbf{A}^2+\Delta_{13}(\mathbf{AB}-\mathbf{CD})+\Delta_{42}\mathbf{C}^2}{(\mathbf{AB}-\mathbf{CD})^2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

If we suppose

$$-\alpha\Delta_{42}\mathbf{A}^2 + \Delta_{13}(\mathbf{AB} - \mathbf{CD}) + \Delta_{42}\mathbf{C}^2 \neq 0, \tag{153}$$

then $\mathbf{A} = 0$. (151) yields $\mathbf{B} = 0 = \mathbf{F}$ and $\mathbf{C} \neq 0 \neq \mathbf{D}$ and consequently (141) becomes

$$\begin{pmatrix} 0 \\ \frac{\Delta_{14}}{\mathbf{C}} + \frac{\Delta_{32}}{\mathbf{D}} \\ \frac{\Delta_{42}\mathbf{C}}{\mathbf{D}} - \Delta_{13} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

that forces $\Delta_{13} = \frac{\Delta_{42}\mathbf{C}}{\mathbf{D}}$. Observe that (153) changes into

$$-\Delta_{13}\mathbf{CD} + \Delta_{42}\mathbf{C}^2 \neq 0,$$

that is $\Delta_{13} \neq \frac{\Delta_{42}\mathbf{C}}{\mathbf{D}}$. This means that (153) gives rise to a contradiction. Hence we must have

$$\Delta_{13} = \frac{\Delta_{42}(\alpha\mathbf{A}^2 - \mathbf{C}^2)}{\mathbf{AB} - \mathbf{CD}}.$$

The matrix of d is of the form

$$P\Delta P^{-1} = \begin{pmatrix} 0 & \Delta_{13} & -\frac{\gamma\Delta_{42}\mathbf{A}^2-2\Delta_{13}\mathbf{AF}+2\Delta_{14}\mathbf{D}}{2(\mathbf{AB}-\mathbf{CD})} & -\frac{2\Delta_{14}\mathbf{B}-2\Delta_{13}\mathbf{CF}+\gamma\Delta_{42}\mathbf{AC}}{2(\mathbf{AB}-\mathbf{CD})} \\ 0 & 0 & \frac{-\Delta_{32}\mathbf{A}}{\mathbf{AB}-\mathbf{CD}} & \frac{-\Delta_{32}\mathbf{C}}{\mathbf{AB}-\mathbf{CD}} \\ 0 & 0 & \frac{-\Delta_{42}\mathbf{AC}}{\mathbf{AB}-\mathbf{CD}} & \frac{-\Delta_{42}\mathbf{C}^2}{\mathbf{AB}-\mathbf{CD}} \\ 0 & 0 & \frac{\Delta_{42}\mathbf{A}^2}{\mathbf{AB}-\mathbf{CD}} & \frac{\Delta_{42}\mathbf{AC}}{\mathbf{AB}-\mathbf{CD}} \end{pmatrix},$$

therefore, if we substitute the value of Δ_{13} and we use (151), we get

$$P\Delta P^{-1} = \begin{pmatrix} 0 & \frac{\Delta_{42}(\alpha\mathbf{A}^2-\mathbf{C}^2)}{\mathbf{AB}-\mathbf{CD}} & -\frac{\Delta_{14}\mathbf{D}}{\mathbf{AB}-\mathbf{CD}} & -\frac{\Delta_{14}\mathbf{B}}{\mathbf{AB}-\mathbf{CD}} \\ 0 & 0 & \frac{-\Delta_{32}\mathbf{A}}{\mathbf{AB}-\mathbf{CD}} & \frac{-\Delta_{32}\mathbf{C}}{\mathbf{AB}-\mathbf{CD}} \\ 0 & 0 & \frac{-\Delta_{42}\mathbf{AC}}{\mathbf{AB}-\mathbf{CD}} & \frac{-\Delta_{42}\mathbf{C}^2}{\mathbf{AB}-\mathbf{CD}} \\ 0 & 0 & \frac{\Delta_{42}\mathbf{A}^2}{\mathbf{AB}-\mathbf{CD}} & \frac{\Delta_{42}\mathbf{AC}}{\mathbf{AB}-\mathbf{CD}} \end{pmatrix},$$

Now (140) becomes

$$\begin{pmatrix} -\frac{\gamma\mathbf{AD}(\Delta_{32}\mathbf{C}+\Delta_{14}\mathbf{D})}{(\mathbf{AB}-\mathbf{CD})^2} \\ 2\frac{\mathbf{AF}(\Delta_{32}\mathbf{C}+\Delta_{14}\mathbf{D})}{(\mathbf{AB}-\mathbf{CD})^2} \\ -2\frac{\mathbf{AB}(\Delta_{32}\mathbf{C}+\Delta_{14}\mathbf{D})}{(\mathbf{AB}-\mathbf{CD})^2} \\ 2\frac{\mathbf{AD}(\Delta_{32}\mathbf{C}+\Delta_{14}\mathbf{D})}{(\mathbf{AB}-\mathbf{CD})^2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Suppose $\Delta_{32}\mathbf{C} + \Delta_{14}\mathbf{D} \neq 0$. Then $\mathbf{AB} = \mathbf{AD} = \mathbf{AF} = 0$ which, in view of (151), yields $\mathbf{A} = 0$. Again (151) forces $\mathbf{F} = 0 = \mathbf{B}$ and $\mathbf{C} \neq 0 \neq \mathbf{D}$.

(141) becomes

$$\begin{pmatrix} 0 \\ \frac{\Delta_{14}}{\mathbf{C}} + \frac{\Delta_{32}}{\mathbf{D}} \\ \frac{\Delta_{42}\mathbf{C}}{\mathbf{D}} - \Delta_{13} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which gives rise to a contradiction. This means that $\Delta_{32}\mathbf{C} + \Delta_{14}\mathbf{D} = 0$. By usign $\Delta_{14}\mathbf{D} = -\Delta_{32}\mathbf{C}$ into the explicit expression of $P\Delta P^{-1}$ we obtain

$$P\Delta P^{-1} = \begin{pmatrix} 0 & \frac{\Delta_{42}(\alpha\mathbf{A}^2 - \mathbf{C}^2)}{\mathbf{AB}-\mathbf{CD}} & \frac{\Delta_{32}\mathbf{C}}{\mathbf{AB}-\mathbf{CD}} & -\frac{\Delta_{14}\mathbf{B}}{\mathbf{AB}-\mathbf{CD}} \\ 0 & 0 & \frac{-\Delta_{32}\mathbf{A}}{\mathbf{AB}-\mathbf{CD}} & \frac{-\Delta_{32}\mathbf{C}}{\mathbf{AB}-\mathbf{CD}} \\ 0 & 0 & \frac{-\Delta_{42}\mathbf{AC}}{\mathbf{AB}-\mathbf{CD}} & \frac{-\Delta_{42}\mathbf{C}^2}{\mathbf{AB}-\mathbf{CD}} \\ 0 & 0 & \frac{\Delta_{42}\mathbf{A}^2}{\mathbf{AB}-\mathbf{CD}} & \frac{\Delta_{42}\mathbf{AC}}{\mathbf{AB}-\mathbf{CD}} \end{pmatrix}.$$

Finally, we calculate $d(G)X + \varphi(G)d(X)$:

$$P\Delta P^{-1} = \begin{pmatrix} \frac{\alpha\Delta_{32}\mathbf{A}}{\mathbf{AB}-\mathbf{CD}} \\ \frac{-\Delta_{32}\mathbf{C}}{\mathbf{AB}-\mathbf{CD}} \\ \frac{-\Delta_{42}\mathbf{C}^2}{\mathbf{AB}-\mathbf{CD}} \\ \frac{\Delta_{42}\mathbf{AC}}{\mathbf{AB}-\mathbf{CD}} \end{pmatrix}.$$

We conclude that d has matrix of the form

$$P\Delta P^{-1} = \begin{pmatrix} 0 & \frac{\Delta_{42}(\alpha\mathbf{A}^2 - \mathbf{C}^2)}{\mathbf{AB}-\mathbf{CD}} & \frac{\Delta_{32}\mathbf{C}}{\mathbf{AB}-\mathbf{CD}} & \frac{\alpha\Delta_{32}\mathbf{A}}{\mathbf{AB}-\mathbf{CD}} \\ 0 & 0 & \frac{-\Delta_{32}\mathbf{A}}{\mathbf{AB}-\mathbf{CD}} & \frac{-\Delta_{32}\mathbf{C}}{\mathbf{AB}-\mathbf{CD}} \\ 0 & 0 & \frac{-\Delta_{42}\mathbf{AC}}{\mathbf{AB}-\mathbf{CD}} & \frac{-\Delta_{42}\mathbf{C}^2}{\mathbf{AB}-\mathbf{CD}} \\ 0 & 0 & \frac{\Delta_{42}\mathbf{A}^2}{\mathbf{AB}-\mathbf{CD}} & \frac{\Delta_{42}\mathbf{AC}}{\mathbf{AB}-\mathbf{CD}} \end{pmatrix}$$

with $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{F}, \alpha, \beta, \gamma, \Delta_{32}, \Delta_{42}$ satisfying (151).

3.4. Recapitulation. We have seen that if $\Delta_{23} \neq 0$ or $\Delta_{24} \neq 0$, then this forces d to be of a very specific form. Namely if

$$\begin{cases} \mathbf{B} = \mathbf{F} = 0 \\ \mathbf{C} \neq 0 \neq \mathbf{D} \\ \alpha = \beta = \gamma = 0 \end{cases}$$

then there are admissible skew-derivations with matrix of the form

$$P\Delta P^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\Delta_{24}\mathbf{D}}{\mathbf{C}} & \frac{\mathbf{D}(\Delta_{24}\mathbf{A} - \Delta_{23}\mathbf{C})}{\mathbf{CE}} & 0 \end{pmatrix}, \text{ if } \mathbf{E} \neq 0 \quad \text{and} \quad P\Delta P^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \Delta_{24}\mathbf{D} & \frac{\Delta_{23}\mathbf{D}}{\mathbf{C}} & 0 \end{pmatrix}, \text{ if } \mathbf{E} = 0.$$

If both Δ_{23} and Δ_{24} are zero, then the coefficients defining d need to obey (145) and we see that admissible skew-derivations have matrix of the form

$$P\Delta P^{-1} = \begin{pmatrix} 0 & \frac{\mathbf{E}[\Delta_{42}(\alpha\mathbf{A}-\gamma\mathbf{E})+\Delta_{32}\mathbf{C}]}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\Delta_{42}(\mathbf{C}^2-\beta\mathbf{E}^2)+\Delta_{32}\mathbf{AC}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{(\Delta_{42}\mathbf{C}+\Delta_{32}\mathbf{A})(\alpha\mathbf{A}-\gamma\mathbf{E})+\beta\Delta_{32}\mathbf{E}^2}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} \\ 0 & \frac{-\mathbf{E}(\Delta_{42}\mathbf{C}+\Delta_{32}\mathbf{A})}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{-\mathbf{A}(\Delta_{42}\mathbf{C}+\Delta_{32}\mathbf{A})}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\mathbf{C}(\Delta_{42}\mathbf{C}+\Delta_{32}\mathbf{A})}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} \\ 0 & \frac{\Delta_{32}\mathbf{E}^2}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\Delta_{32}\mathbf{AE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\Delta_{32}\mathbf{CE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} \\ 0 & \frac{\Delta_{42}\mathbf{E}^2}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\Delta_{42}\mathbf{AE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\Delta_{42}\mathbf{CE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} \end{pmatrix}, \text{ if } \mathbf{E} \neq 0$$

and

$$P\Delta P^{-1} = \begin{pmatrix} 0 & \frac{\Delta_{42}(\alpha\mathbf{A}^2-\mathbf{C}^2)}{\mathbf{AB}-\mathbf{CD}} & \frac{\Delta_{32}\mathbf{C}}{\mathbf{AB}-\mathbf{CD}} & \frac{\alpha\Delta_{32}\mathbf{A}}{\mathbf{AB}-\mathbf{CD}} \\ 0 & 0 & \frac{-\Delta_{32}\mathbf{A}}{\mathbf{AB}-\mathbf{CD}} & \frac{-\Delta_{32}\mathbf{C}}{\mathbf{AB}-\mathbf{CD}} \\ 0 & 0 & \frac{-\Delta_{42}\mathbf{AC}}{\mathbf{AB}-\mathbf{CD}} & \frac{-\Delta_{42}\mathbf{C}^2}{\mathbf{AB}-\mathbf{CD}} \\ 0 & 0 & \frac{\Delta_{42}\mathbf{A}^2}{\mathbf{AB}-\mathbf{CD}} & \frac{\Delta_{42}\mathbf{AC}}{\mathbf{AB}-\mathbf{CD}} \end{pmatrix}, \text{ if } \mathbf{E} = 0.$$

Now that we have a full classification of the pairs (φ, d) of involutions and skew-derivations such that $d^2 \equiv 0$ and $d\varphi = -\varphi d$, by means of Theorem 1.8, we also have a comprehensive list of H-coactions on $A = Cl(\alpha, \beta, \gamma)$. In the next chapter we will describe some applications, among which are a couple of theorems that answer to Questions 6.6 and 6.7. Before we display a summarising table and we turn our attention to much harder calculation let us show a straightforward result: we can easily determine the space of coinvariants of each coaction, by simply looking at the eigenspace A_+ of φ and at the kernel of d.

4. The space of coinvariants

We recall that given an H-comodule algebra A with structure $\rho : A \to A \otimes H$, then the space of coinvariants A^{coH} is given by

$$A^{coH} = \{a \in A \mid \rho(a) = a \otimes 1_H\}.$$

Theorem 1.8 tells us that each comodule algebra structure for $A = Cl(\alpha, \beta, \gamma)$ is given by

$$\rho(a) = \left(\frac{a + \varphi(a)}{2}\right) \otimes 1_H + \left(\frac{a - \varphi(a)}{2}\right) \otimes g + \left(\frac{d(\varphi(a)) - d(a)}{2}\right) \otimes x + \left(\frac{d(\varphi(a)) + d(a)}{2}\right) \otimes gx,$$

where φ is an involution and d is a φ-derivation such that $d^2 \equiv 0$ and $d\varphi = -\varphi d$.

Remark 4.1. It is very easy to prove that $A^{coH} = A_+ \cap \ker d$. Notice that each coinvariant element must be such that $\varphi(a) = a$ and $d(a) = 0$.

If we consider the case when $\varphi = \mathrm{Id}$, then we already know that $d \equiv 0$ and so $A^{coH} = A_+ = A$. Next, if φ has matrix similar to D_1 we know that

$$A_+ = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\mathbf{B} \\ \mathbf{A} \\ 0 \end{pmatrix}, \begin{pmatrix} -\gamma \\ \gamma\mathbf{B} - 2\beta\mathbf{A} \\ \gamma\mathbf{A} - 2\alpha\mathbf{B} \\ 2 \end{pmatrix} \rangle, \text{ with } \begin{cases} \gamma^2 - 4\alpha\beta = 0 \\ \alpha\mathbf{B}^2 + \beta\mathbf{A}^2 - \gamma\mathbf{AB} = 1 \end{cases}$$

and since

$$P\Delta P^{-1}P = P\Delta = \begin{pmatrix} 0 & \Delta_{42}\gamma & 0 & 0 \\ 0 & \Delta_{42}(\gamma\mathbf{B} - 2\beta\mathbf{A}) & 0 & 0 \\ 0 & \Delta_{42}(\gamma\mathbf{A} - 2\alpha\mathbf{B}) & 0 & 0 \\ 0 & -2\Delta_{42} & 0 & 0 \end{pmatrix},$$

then

$$A^{coH} = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\gamma \\ \gamma\mathbf{B} - 2\beta\mathbf{A} \\ \gamma\mathbf{A} - 2\alpha\mathbf{B} \\ 2 \end{pmatrix} \rangle, \text{ if } \Delta_{42} \neq 0 \ (\iff d \not\equiv 0)$$

and $A^{coH} = A_+$ if $\Delta_{42} = 0 \iff d \equiv 0$. Finally if we consider the case when φ has matrix similar to D_2 we know that

$$A_+ = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mathbf{F} \\ -\mathbf{B} \\ \mathbf{D} \end{pmatrix} \rangle,$$

therefore we can determine the space of coinvariants by looking at the value of $d(\mathbf{F}G - \mathbf{B}X + \mathbf{D}GX)$. This element corresponds to

$$\begin{pmatrix} \frac{\gamma}{2}\Delta_{42}\mathbf{E} \\ \Delta_{32}\mathbf{A} + \Delta_{42}\mathbf{C} \\ -\Delta_{32}\mathbf{E} \\ -\Delta_{42}\mathbf{E} \end{pmatrix}, \text{ if } \mathbf{E} \neq 0 \qquad \text{and to} \qquad \begin{pmatrix} \frac{\gamma}{2}\Delta_{42}\mathbf{A} \\ \Delta_{32} \\ \Delta_{42}\mathbf{C} \\ -\Delta_{42}\mathbf{A} \end{pmatrix}, \text{ if } \mathbf{E} = 0.$$

Remember that when $\mathbf{E} = 0$, $\mathbf{A}$ and $\mathbf{C}$ cannot be both 0 in view of (151). It is now clear that, in both cases, $A^{coH} = A_+$ if $\Delta_{32} = \Delta_{42} = 0$ and $A^{coH} = k$ otherwise.

5. Summary and examples

We gather here all the results of the last three sections.

Elements in $\mathbf{Inv}_{\mathbf{Alg}}(A)$

Eigenvalues	Matrix $\mathfrak{F}_i$ of φ w.r.t. to the standard basis	$\mathbb{Z}_2$-grading	Conditions
$1,1,1,1$	$\mathfrak{F}_0 = I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$A_+ = A$ $A_- = \{0\}$	
$1,1,1,-1$	$\mathfrak{F}_1 = \begin{pmatrix} 1 & \frac{\gamma}{2}\mathbf{A} & \frac{\gamma}{2}\mathbf{B} & \frac{\gamma}{2} \\ 0 & \frac{2\alpha\mathbf{B}-\gamma\mathbf{A}}{2}\mathbf{B} & \frac{\gamma\mathbf{B}-2\beta\mathbf{A}}{2}\mathbf{B} & \frac{\gamma\mathbf{B}-2\beta\mathbf{A}}{2} \\ 0 & \frac{\gamma\mathbf{A}-2\alpha\mathbf{B}}{2}\mathbf{A} & \frac{2\beta\mathbf{A}-\gamma\mathbf{B}}{2}\mathbf{A} & \frac{\gamma\mathbf{A}-2\alpha\mathbf{B}}{2} \\ 0 & -\mathbf{A} & -\mathbf{B} & 0 \end{pmatrix}$	$A_+ = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\mathbf{B} \\ \mathbf{A} \\ 0 \end{pmatrix}, \begin{pmatrix} -\gamma \\ \gamma\mathbf{B}-2\beta\mathbf{A} \\ \gamma\mathbf{A}-2\alpha\mathbf{B} \\ 2 \end{pmatrix} \rangle$ $A_- = \langle \begin{pmatrix} \gamma \\ \gamma\mathbf{B}-2\beta\mathbf{A} \\ \gamma\mathbf{A}-2\alpha\mathbf{B} \\ -2 \end{pmatrix} \rangle$	$\begin{cases} \gamma^2 - 4\alpha\beta = 0 \\ \alpha\mathbf{B}^2 + \beta\mathbf{A}^2 - \gamma\mathbf{AB} = 1 \end{cases}$
$1,1,-1,-1$	$\mathfrak{F}_2 = \begin{pmatrix} 1 & \frac{-\gamma\mathbf{DE}}{\mathfrak{D}} & \frac{-\gamma\mathbf{AD}}{\mathfrak{D}} & \frac{-\gamma(\mathbf{AB}-\mathbf{EF})}{\mathfrak{D}} \\ 0 & \frac{-\mathfrak{D}+2\mathbf{EF}}{\mathfrak{D}} & \frac{2\mathbf{AF}}{\mathfrak{D}} & \frac{2\mathbf{CF}}{\mathfrak{D}} \\ 0 & \frac{-2\mathbf{BE}}{\mathfrak{D}} & \frac{-\mathfrak{D}-2\mathbf{AB}}{\mathfrak{D}} & \frac{-2\mathbf{BC}}{\mathfrak{D}} \\ 0 & \frac{2\mathbf{DE}}{\mathfrak{D}} & \frac{2\mathbf{AD}}{\mathfrak{D}} & \frac{-\mathfrak{D}+2\mathbf{CD}}{\mathfrak{D}} \end{pmatrix}$	$A_+ = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mathbf{F} \\ -\mathbf{B} \\ \mathbf{D} \end{pmatrix} \rangle$ $A_- = \langle \begin{pmatrix} 0 \\ \mathbf{A} \\ -\mathbf{E} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\gamma}{2}\mathbf{E} \\ \mathbf{C} \\ 0 \\ -\mathbf{E} \end{pmatrix} \rangle$	$\begin{cases} \gamma\mathbf{AD} - 2\mathbf{CF} - 2\beta\mathbf{DE} = 0 \\ \gamma\mathbf{DE} + 2\mathbf{BC} - 2\alpha\mathbf{AD} = 0 \\ 2(\alpha\mathbf{AF} + \beta\mathbf{BE}) - \gamma(\mathbf{AB} + \mathbf{EF}) = 0 \\ \mathfrak{D} = -\mathbf{AB} + \mathbf{CD} + \mathbf{EF} \neq 0 \\ \mathbf{E} \neq 0 \end{cases}$
		$A_+ = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mathbf{F} \\ -\mathbf{B} \\ \mathbf{D} \end{pmatrix} \rangle$ $A_- = \langle \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\gamma}{2}\mathbf{A} \\ 0 \\ \mathbf{C} \\ -\mathbf{A} \end{pmatrix} \rangle$	$\begin{cases} \gamma\mathbf{AD} - 2\mathbf{CF} = 0 \\ \mathbf{BC} - \alpha\mathbf{AD} = 0 \\ \gamma\mathbf{B} - 2\alpha\mathbf{F} = 0 \\ \mathfrak{D} = -\mathbf{AB} + \mathbf{CD} \neq 0 \\ \mathbf{E} = 0 \end{cases}$

Admissible φ-derivations associated to each $\mathfrak{F}_i$

$\mathfrak{F}_i$	Matrix $\mathfrak{D}_i$ of admissible φ-derivation d w.r.t. to the canonical basis	Conditions	Space of coinvariants
$\mathfrak{F}_0$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$		$A = A_+$
$\mathfrak{F}_1$	$\mathfrak{a} \cdot \begin{pmatrix} 0 & \gamma\frac{\gamma\mathbf{A}-2\alpha\mathbf{B}}{2} & \gamma\frac{2\beta\mathbf{A}-\gamma\mathbf{B}}{2} & 0 \\ 0 & -\gamma & -2\beta & 0 \\ 0 & 2\alpha & \gamma & 0 \\ 0 & 2\alpha\mathbf{B}-\gamma\mathbf{A} & \gamma\mathbf{B}-2\beta\mathbf{A} & 0 \end{pmatrix}$	$\begin{cases} \gamma^2 - 4\alpha\beta = 0 \\ \alpha\mathbf{B}^2 + \beta\mathbf{A}^2 - \gamma\mathbf{AB} = 1 \end{cases}$	A_+ if $\mathfrak{a} = 0$, $\langle 1, \begin{pmatrix} -\gamma \\ \gamma\mathbf{B}-2\beta\mathbf{A} \\ \gamma\mathbf{A}-2\alpha\mathbf{B} \\ 2 \end{pmatrix} \rangle$ if $\mathfrak{a} \neq 0$
$\mathfrak{F}_2$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\mathfrak{a}\mathbf{D}}{\mathbf{C}} & \frac{\mathbf{D}(\mathfrak{a}\mathbf{A}-\mathfrak{b}\mathbf{C})}{\mathbf{CE}} & 0 \end{pmatrix}$	$\begin{cases} \mathbf{C}, \mathbf{D}, \mathbf{E} \neq 0 \\ \mathbf{B} = \mathbf{F} = 0 \\ \alpha = \beta = \gamma = 0 \end{cases}$	A_+

$\mathfrak{F}_i$	Matrix $\mathfrak{D}_i$ of admissible φ-derivation d w.r.t. to the canonical basis	Conditions	Space of coinvariants
$\mathfrak{F}_2$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mathfrak{a}\mathbf{D} & \frac{\mathfrak{b}\mathbf{D}}{\mathbf{C}} & 0 \end{pmatrix}$	$\begin{cases} \mathbf{C}, \mathbf{D} \neq 0 \\ \mathbf{B} = \mathbf{E} = \mathbf{F} = 0 \\ \alpha = \beta = \gamma = 0 \end{cases}$	A_+
$\mathfrak{F}_2$	$\begin{pmatrix} 0 & \frac{\mathbf{E}[\mathfrak{a}(\alpha\mathbf{A}-\gamma\mathbf{E})+\mathfrak{b}\mathbf{C}]}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\mathfrak{a}(\mathbf{C}^2-\beta\mathbf{E}^2)+\mathfrak{b}\mathbf{AC}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{(\mathfrak{a}\mathbf{C}+\mathfrak{b}\mathbf{A})(\alpha\mathbf{A}-\gamma\mathbf{E})+\beta\mathfrak{b}\mathbf{E}^2}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} \\ 0 & \frac{-\mathbf{E}(\mathfrak{a}\mathbf{C}+\mathfrak{b}\mathbf{A})}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{-\mathbf{A}(\mathfrak{a}\mathbf{C}+\mathfrak{b}\mathbf{A})}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{-\mathbf{C}(\mathfrak{a}\mathbf{C}+\mathfrak{b}\mathbf{A})}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} \\ 0 & \frac{\mathfrak{b}\mathbf{E}^2}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\mathfrak{b}\mathbf{AE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\mathfrak{b}\mathbf{CE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} \\ 0 & \frac{\mathfrak{a}\mathbf{E}^2}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\mathfrak{a}\mathbf{AE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} & \frac{\mathfrak{a}\mathbf{CE}}{\mathbf{AB}-\mathbf{CD}-\mathbf{EF}} \end{pmatrix}$	$\begin{cases} \gamma\mathbf{AD} - 2\mathbf{CF} - 2\beta\mathbf{DE} = 0 \\ \gamma\mathbf{DE} + 2\mathbf{BC} - 2\alpha\mathbf{AD} = 0 \\ 2(\alpha\mathbf{AF} + \beta\mathbf{BE}) - \gamma(\mathbf{AB} + \mathbf{EF}) = 0 \\ \mathbf{AB} - \mathbf{CD} - \mathbf{EF} \neq 0 \\ \mathbf{E} \neq 0 \\ \mathfrak{a}, \mathfrak{b} \in k \end{cases}$	A_+ if $\mathfrak{a} = \mathfrak{b} = 0$, k otherwise
$\mathfrak{F}_2$	$\begin{pmatrix} 0 & \frac{\mathfrak{a}(\alpha\mathbf{A}^2-\mathbf{C}^2)}{\mathbf{AB}-\mathbf{CD}} & \frac{\mathfrak{b}\mathbf{C}}{\mathbf{AB}-\mathbf{CD}} & \frac{\alpha\mathfrak{b}\mathbf{A}}{\mathbf{AB}-\mathbf{CD}} \\ 0 & 0 & \frac{-\mathfrak{b}\mathbf{A}}{\mathbf{AB}-\mathbf{CD}} & \frac{-\mathfrak{b}\mathbf{C}}{\mathbf{AB}-\mathbf{CD}} \\ 0 & 0 & \frac{-\mathfrak{a}\mathbf{AC}}{\mathbf{AB}-\mathbf{CD}} & \frac{-\mathfrak{a}\mathbf{C}^2}{\mathbf{AB}-\mathbf{CD}} \\ 0 & 0 & \frac{\mathfrak{a}\mathbf{A}^2}{\mathbf{AB}-\mathbf{CD}} & \frac{\mathfrak{a}\mathbf{AC}}{\mathbf{AB}-\mathbf{CD}} \end{pmatrix}$	$\begin{cases} \gamma\mathbf{AD} - 2\mathbf{CF} = 0 \\ \mathbf{BC} - \alpha\mathbf{AD} = 0 \\ \gamma\mathbf{B} - 2\alpha\mathbf{F} = 0 \\ \mathbf{AB} - \mathbf{CD} \neq 0 \\ \mathbf{E} = 0 \\ \mathfrak{a}, \mathfrak{b} \in k \end{cases}$	A_+ if $\mathfrak{a} = \mathfrak{b} = 0$, k otherwise

For sake of completeness we include a couple of examples. In every example $A = Cl(\alpha, \beta, \gamma)$ is a generic four-dimensional Clifford algebra if not specified otherwise.

Example 5.1. If we choose $\varphi = \mathrm{Id}$ and $d \equiv 0$ we get the trivial coaction $\rho(a) = a \otimes 1$ for every $a \in A$.

Example 5.2. Let $A = Cl(0, 0, 0)$ be the four-dimensional exterior algebra. Then we can choose an involution of the form $\mathfrak{F}_2$ and a skew-derivation from the fourth row of our table, with $\mathbf{A} = \mathbf{B} = \mathbf{E} = \mathbf{F} = 0$ and $\mathbf{C} = \mathbf{D} = \mathfrak{a} = \mathfrak{b} = 1$. We find

$$\mathfrak{F}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathfrak{D}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

and, by means of Theorem 1.8, the corresponding coaction

$$\rho(G) = G \otimes g - GX \otimes x, \quad \rho(X) = X \otimes g - GX \otimes x.$$

Example 5.3. Again, let $A = Cl(0, 0, 0)$ be the four-dimensional exterior algebra. Then we can choose an involution of the form $\mathfrak{F}_2$ and a skew-derivation from the fifth row of our table, with $\mathbf{A} = \mathbf{B} = \mathbf{F} = \mathfrak{a} = 0$ and $\mathbf{E} = \mathbf{C} = \mathbf{D} = \mathfrak{b} = 1$. We find

$$\mathfrak{F}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathfrak{D}_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and the corresponding coaction

$$\rho(G) = G \otimes g + (1 + X) \otimes x, \quad \rho(X) = X \otimes g.$$

REMARK 5.4. The canonical H-comodule algebra structure of $A = Cl(\alpha, \beta, \gamma)$ (see (84)-(87)) can be obtained from the last row of our table by choosing $\mathbf{A} = \mathbf{B} = \mathbf{E} = \mathbf{F} = \mathfrak{a} = 0$, $\mathbf{C} = \mathbf{D} = \mathfrak{b} = 1$. The associated matrices are

$$\mathfrak{F}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathfrak{D}_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Similarly, the H-comodule algebra structure defined by (88)-(94) is found once we choose $\mathbf{A} = \mathbf{B} = \mathbf{E} = \mathbf{F} = \mathfrak{a} = 0$, $\mathbf{C} = \mathbf{D} = \mathfrak{b} = 1 = -\mathfrak{a}$. Its associated matrices are

$$\mathfrak{F}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathfrak{D}_2 = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Example 5.5. Let $A = Cl(\alpha, \beta, \gamma)$ with $\beta, \gamma \neq 0$. We can choose an involution of the form $\mathfrak{F}_2$ and a skew-derivation from the fifth row of our table and set $\mathbf{A} = \mathbf{C} = \mathbf{D} = \mathfrak{b} = 0$, $\mathbf{B} = \mathbf{E} = \mathfrak{a} = 1$ and $\mathbf{F} = \frac{2\beta}{\gamma}$. We obtain the matrices

$$\mathfrak{F}_2 = \begin{pmatrix} 1 & 0 & 0 & \gamma \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{\gamma}{\beta} & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \mathfrak{D}_2 = \begin{pmatrix} 0 & 0 & 0 & -\frac{\gamma}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\gamma}{2\beta} & 0 & 0 \end{pmatrix},$$

and the corresponding coaction

$$\rho(G) = \left(G - \frac{\gamma}{2\beta}\right) \otimes 1 + \frac{\gamma}{2\beta} X \otimes g - \frac{\gamma}{2\beta} X \otimes gx, \quad \rho(X) = X \otimes g.$$

Example 5.6. Assume that $\alpha \neq 0$ and that k contains a square root of α. Let $A = Cl\left(\alpha, \frac{\gamma^2}{4\alpha}, \gamma\right)$. Then we can consider an involution of the form $\mathfrak{F}_1$ and an associated skew-derivation with $\mathbf{A} = 0$, $\mathbf{B} = \frac{1}{\sqrt{\alpha}}$ and $\mathfrak{a}$. Their matrices are

$$\mathfrak{F}_1 = \begin{pmatrix} 1 & 0 & \frac{\gamma}{2\sqrt{\alpha}} & \frac{\gamma}{2} \\ 0 & 1 & \frac{\gamma}{2\alpha} & \frac{\gamma}{2\sqrt{\alpha}} \\ 0 & 0 & 0 & -\sqrt{\alpha} \\ 0 & 0 & -\frac{1}{\sqrt{\alpha}} & 0 \end{pmatrix}, \quad \mathfrak{D}_1 = \begin{pmatrix} 0 & -\sqrt{\alpha}\gamma & -\frac{\gamma^2}{2\sqrt{\alpha}} & 0 \\ 0 & -\gamma & -\frac{\gamma^2}{2\alpha} & 0 \\ 0 & 2\alpha & \gamma & 0 \\ 0 & 2\sqrt{\alpha} & \frac{\gamma}{\sqrt{\alpha}} & 0 \end{pmatrix}$$

and the corresponding coaction results

$$\rho(G) = G \otimes 1 - \left(\sqrt{\alpha}\gamma + \gamma G - 2\alpha X - 2\sqrt{\alpha} G X\right) \otimes gx,$$

$$\rho(X) = X \otimes \frac{1+g}{2} + \left(\frac{\gamma}{2\sqrt{\alpha}} + \frac{\gamma}{2\alpha} G - \frac{1}{\sqrt{\alpha}} G X\right) \otimes \frac{1-g}{2} +$$

$$+\left(\frac{\gamma^2}{2\sqrt{\alpha}} + \frac{\gamma^2}{2\alpha} G - \gamma X - \frac{\gamma}{\sqrt{\alpha}} G X\right) \otimes \frac{(1-g)x}{2} + \frac{\gamma}{2\alpha}\left(\frac{\gamma^2}{2\sqrt{\alpha}} + \frac{\gamma^2}{2\alpha} G - \gamma X - \frac{\gamma}{\sqrt{\alpha}} G X\right) \otimes \frac{(1+g)x}{2}.$$

REMARK 5.7. In contrast to previous examples, the form of the coaction in this last one looks rather complicated. As a matter of fact it is not clear at all when coactions of this type exist – hence how to provide easier examples – without making further assumption on the ground field k. It is also worth mentioning that these coactions only exist when $\gamma^2 - 4\alpha\beta = 0$, i.e. when the Clifford algebra A is not semisimple. To construct coactions associated to involutions with matrix of the form $\mathfrak{F}_1$, one needs to solve the defining condition

$$\alpha\mathbf{B}^2 + \beta\mathbf{A}^2 - \gamma\mathbf{A}\mathbf{B} = 1$$

for the coefficients $\mathbf{A}, \mathbf{B}$ of the coaction. Notice that in some cases this can be regarded as a diophantine equation (take e.g. $k = \mathbb{Q}$).

Remark 5.8. It will be shown in the sequel (see Subsec. 2.2, Ch. 5) that coactions of this type can be gathered into two distinct classes of isomorphism. Namely for every Clifford algebra $A = Cl(\alpha, \beta, \gamma)$ such that $\gamma^2 - 4\alpha\beta = 0$ there are at most two coactions (up to isomorphism) such that the associated involution is of type $\mathfrak{F}_1$. They are the one described in Example 5.6 and

$$\rho(G) = G \otimes 1 - \left(\sqrt{\alpha}\gamma + \gamma G - 2\alpha X - 2\sqrt{\alpha} GX\right) \otimes gx,$$
$$\rho(X) = X \otimes \frac{1+g}{2} + \left(\frac{\gamma}{2\sqrt{\alpha}} + \frac{\gamma}{2\alpha} G - \frac{1}{\sqrt{\alpha}} GX\right) \otimes \frac{1-g}{2}.$$

CHAPTER 3

Applications

In this chapter we will make use of the classification of H-coactions of $A = Cl(\alpha, \beta, \gamma)$ we have obtained in the previous one, in order to see which ones ensure the induced cowreath $(A \otimes H^{op}, H, \psi)$ a set of nice properties. We recall how to build the cowreath $(A \otimes H^{op}, H, \psi)$ starting from a Hopf algebra H and an H-comodule algebra A.

PROPOSITION 4.12. *Let $(H, m_H, u_H, \Delta_H, \varepsilon_H, S_H)$ be a Hopf algebra, (A, ρ_A) be a right H-comodule algebra and (X, μ_X) a right H-module coalgebra. Let $\psi : X \otimes A \to A \otimes X$ defined by*

$$\psi(x \otimes a) = a_0 \otimes xa_1 \tag{154}$$

for every $x \in X$ and every $a \in A$. Then $(X, \psi) \in \mathcal{T}_A^{\#}$ and (X, ψ) is a coalgebra in $\mathcal{T}_A^{\#}$, with comultiplication $\delta_X : X \to A \otimes X^2$ and counit $\epsilon_X : X \to A$ given by

$$\delta_X(x) = 1_A \otimes x_1 \otimes x_2 \tag{155}$$

$$\epsilon_X(x) = \varepsilon_X(x)1_A. \tag{156}$$

In view of Proposition 4.11, we can apply Proposition 4.12 by choosing "H"$= H \otimes H^{op}$, "A"$= A \otimes H^{op}$ and "X"$= H$ and by defining

$$\begin{array}{rccc} \psi : & H \otimes A \otimes H^{op} & \to & A \otimes H^{op} \otimes H \\ & h \otimes a \otimes l & \mapsto & a_0 \otimes l_1 \otimes l_2 h a_1 \end{array} \tag{157}$$

Then $(H, \psi) \in \mathcal{T}_{A \otimes H^{op}}^{\#}$ and (H, ψ) is a coalgebra in $\mathcal{T}_{A \otimes H^{op}}^{\#}$ via

$$\begin{array}{rcl} \delta_H & : & H \to A \otimes H^{op} \otimes H \otimes H \\ \delta_H(h) & = & 1_A \otimes 1_H \otimes h_1 \otimes h_2 \end{array} \tag{158}$$

and

$$\begin{array}{rcl} \epsilon_H & : & H \to A \otimes H^{op} \\ \epsilon_H(h) & = & \varepsilon_H(h) 1_A \otimes 1_H. \end{array} \tag{159}$$

We start by studying rt-separability of this cowreath.

1. Rt-separability

We recall a fundamental result contained in [**MT1**] that gives equivalent conditions for the cowreath $(A \otimes H^{op}, H, \psi)$ to be rt-separable.

PROPOSITION 4.14. [**MT1**, Prop. 5.2] *A cowreath $(A \otimes H^{op}, H, \psi)$ is separable via a Casimir element*

$$B : H \otimes H \to A \otimes H^{op}$$

of the form

$$h \otimes h' \to B^A(h \otimes h') \otimes 1_H$$

if and only if B^A satisfies (73), (74), (75). *Moreover, whenever S is invertible, it is h-separable if and only if B satisfies the further condition* (76).

REMARK 1.2. From now on we will write B in place of B^A for sake of brevity.

Following the same steps performed in the proof of [**MT1**, Theorem 6.1] we can see that (75) is equivalent to the set of conditions

(160) $$B(1_H \otimes 1_H) = B(g \otimes g) = 1_A$$

(161) $$B(1_H \otimes x) = B(g \otimes gx) = -B(x \otimes g) = -B(gx \otimes 1_H) =: \mathbf{Y}$$

(162) $$B(1_H \otimes g) = B(g \otimes 1_H) =: \mathbf{Z}$$

(163) $$B(1_H \otimes gx) = B(g \otimes x) = B(x \otimes 1_H) = B(gx \otimes g) =: \mathbf{X}$$

(164) $$B(x \otimes x) = B(gx \otimes gx) = B(gx \otimes x) = B(x \otimes gx) = 0.$$

We recall that a generic H-comodule algebra structure is given by

$$\rho(a) = \left(\frac{a + \varphi(a)}{2}\right) \otimes 1_H + \left(\frac{a - \varphi(a)}{2}\right) \otimes g - d\left(\frac{a - \varphi(a)}{2}\right) \otimes x + d\left(\frac{a + \varphi(a)}{2}\right) \otimes gx,$$

for an admissible pair (φ, d). Let us introduce notation a_+ and a_- for the element decomposition $a = a_+ + a_-$ corresponding to the eigendecomposition $A = A_+ \oplus A_-$. We see that the comodule algebra structure can be rewritten as

$$\rho(a) = a_+ \otimes 1_H + a_- \otimes g - d(a_-) \otimes x + d(a_+) \otimes gx.$$

Since ρ is a comodule structure we get

$$a_0 \otimes a_1 \otimes a_2 = a_+ \otimes 1_H \otimes 1_H + a_- \otimes g \otimes g - d(a_-) \otimes (x \otimes g + 1_H \otimes x) + d(a_+) \otimes (gx \otimes 1_H + g \otimes gx).$$

Now we consider (73) and we will verify it for the generators G, X of A and the basis $(1_H, g, x, gx)$ of H. Let $a = G$. Then we have

$$B(h \otimes h')G = G_+B(h \otimes h') + G_-B(hg \otimes h'g) - d(G_-)B(hx \otimes h'g + h \otimes h'x) + d(G_+)B(hgx \otimes h' + hg \otimes h'gx)$$

for every $h, h' \in \{1_H, g, x, gx\}$.

By means of (160)-(164) we get

h	h'	$B(h \otimes h')G = G_+B(h \otimes h') + G_-B(hg \otimes h'g) - d(G_-)B(hx \otimes h'g + h \otimes h'x) + d(G_+)B(hgx \otimes h' + hg \otimes h'gx)$	
1_H	1_H	$B(1_H \otimes 1_H)G = G_+B(1_H \otimes 1_H) + G_-B(g \otimes g) - d(G_-)B(x \otimes g + 1_H \otimes x) + d(G_+)B(gx \otimes 1_H + g \otimes gx)$	$G = G$
1_H	g	$B(1_H \otimes g)G = G_+B(1_H \otimes g) + G_-B(g \otimes 1_H) - d(G_-)B(x \otimes 1_H + 1_H \otimes gx) + d(G_+)B(gx \otimes g + g \otimes x)$	$\mathbf{Z}G = G\mathbf{Z} + 2d(\varphi(G))\mathbf{X}$
1_H	x	$B(1_H \otimes x)G = G_+B(1_H \otimes x) + G_-B(g \otimes xg) - d(G_-)B(x \otimes xg + 1_H \otimes x^2) + d(G_+)B(gx \otimes x + g \otimes xgx)$	$\mathbf{Y}G = \varphi(G)\mathbf{Y}$
1_H	gx	$B(1_H \otimes gx)G = G_+B(1_H \otimes gx) - G_-B(g \otimes x) - d(G_-)B(-x \otimes x + 1_H \otimes gx^2) + d(G_+)B(gx \otimes gx - g \otimes x^2)$	$\mathbf{X}G = \varphi(G)\mathbf{X}$
g	1_H	$B(g \otimes 1_H)G = G_+B(g \otimes 1_H) + G_-B(1_H \otimes g) - d(G_-)B(gx \otimes g + g \otimes x) + d(G_+)B(x \otimes 1_H + 1_H \otimes gx)$	$\mathbf{Z}G = G\mathbf{Z} + 2d(\varphi(G))\mathbf{X}$
g	g	$B(g \otimes g)G = G_+B(g \otimes g) + G_-B(1_H \otimes 1_H) - d(G_-)B(gx \otimes 1_H + g \otimes gx) + d(G_+)B(x \otimes g + 1_H \otimes x)$	$G = G$
g	x	$B(g \otimes x)G = G_+B(g \otimes x) + G_-B(1_H \otimes xg) - d(G_-)B(gx \otimes xg + g \otimes x^2) + d(G_+)B(x \otimes x + 1_H \otimes xgx)$	$\mathbf{X}G = \varphi(G)\mathbf{X}$
g	gx	$B(g \otimes gx)G = G_+B(g \otimes gx) - G_-B(1_H \otimes x) - d(G_-)B(-gx \otimes x + g \otimes gx^2) + d(G_+)B(x \otimes gx - 1_H \otimes x^2)$	$\mathbf{Y}G = \varphi(G)\mathbf{Y}$
x	1_H	$B(x \otimes 1_H)G = G_+B(x \otimes 1_H) + G_-B(xg \otimes g) - d(G_-)B(x^2 \otimes g + x \otimes x) - d(G_+)B(x^2 \otimes 1_H + gx \otimes gx)$	$\mathbf{X}G = \varphi(G)\mathbf{X}$
x	g	$B(x \otimes g)G = G_+B(x \otimes g) + G_-B(xg \otimes 1_H) - d(G_-)B(x^2 \otimes 1_H + x \otimes gx) + d(G_+)B(xgx \otimes g + xg \otimes x)$	$-\mathbf{Y}G = -\varphi(G)\mathbf{Y}$
x	x	$B(x \otimes x)G = G_+B(x \otimes x) + G_-B(xg \otimes xg) - d(G_-)B(x^2 \otimes xg + x \otimes x^2) + d(G_+)B(xgx \otimes x + xg \otimes xgx)$	$0 = 0$
x	gx	$B(x \otimes gx)G = G_+B(x \otimes gx) + G_-B(xg \otimes gxg) - d(G_-)B(-x^2 \otimes x + x \otimes gx^2) - d(G_+)B(x^2 \otimes gx - gx \otimes x^2)$	$0 = 0$
gx	1_H	$B(gx \otimes 1_H)G = G_+B(gx \otimes 1_H) - G_-B(x \otimes g) - d(G_-)B(gx^2 \otimes g + gx \otimes x) - d(G_+)B(x^2 \otimes 1_H + x \otimes gx)$	$-\mathbf{Y}G = -\varphi(G)\mathbf{Y}$
gx	g	$B(gx \otimes g)G = G_+B(gx \otimes g) - G_-B(x \otimes 1_H) - d(G_-)B(gx^2 \otimes 1_H + gx \otimes gx) - d(G_+)B(x^2 \otimes g + x \otimes x)$	$\mathbf{X}G = \varphi(G)\mathbf{X}$
gx	x	$B(gx \otimes x)G = G_+B(gx \otimes x) + G_-B(gxg \otimes xg) - d(G_-)B(gx^2 \otimes xg + gx \otimes x^2) + d(G_+)B(gxgx \otimes x + x \otimes x^2)$	$0 = 0$
gx	gx	$B(gx \otimes gx)G = G_+B(gx \otimes gx) + G_-B(x \otimes x) - d(G_-)B(-gx^2 \otimes x + gx \otimes gx^2) + d(G_+)B(-x^2 \otimes gx + x \otimes x^2)$	$0 = 0$

i.e.

(165) $$\mathbf{Z}G = G\mathbf{Z} + 2d(\varphi(G))\mathbf{X}$$

(166) $$\mathbf{Y}G = \varphi(G)\mathbf{Y}$$

(167) $$\mathbf{X}G = \varphi(G)\mathbf{X}.$$

By performing the same calculations with X in place of G we obtain that

(168) $$\mathbf{Z}X = X\mathbf{Z} + 2d(\varphi(X))\mathbf{X}$$

$$\mathbf{Y}X = \varphi(X)\mathbf{Y} \tag{169}$$

$$\mathbf{X}X = \varphi(X)\mathbf{X} \tag{170}$$

also hold true. So (73) is equivalent to equalities from (165) to (170).

Now we focus on (74). It is enough to check that it holds for every $h \in \{1_H, g, x, gx\}$. For $h = 1_H$ the equality

$$\rho\left(B(1_H \otimes 1_H)\right) = B(1_H \otimes 1_H) \otimes 1_H \iff \rho\left(1_A\right) = 1_A \otimes 1_H$$

holds thanks to (160). For $h = g$ we get

$$\rho\left(B(1_H \otimes g)\right) = B(1_H \otimes g) \otimes g \iff \rho(\mathbf{Z}) = \mathbf{Z} \otimes g.$$

Since $\rho(\mathbf{Z}) = \mathbf{Z}_+ \otimes 1_H + \mathbf{Z}_- \otimes g - d(\mathbf{Z}_-) \otimes x + d(\mathbf{Z}_+) \otimes gx$ we get

$$\begin{cases} \mathbf{Z}_+ = 0 \\ \mathbf{Z}_- = \mathbf{Z} \\ d(\mathbf{Z}_-) = 0 \end{cases}$$

or equivalently $\mathbf{Z} \in A_- \cap \ker d$. For $h = x$ we get

$$\rho\left(B(1_H \otimes x)\right) = B(1_H \otimes x) \otimes g + B(1_H \otimes 1_H) \otimes x \iff \rho(\mathbf{Y}) = \mathbf{Y} \otimes g + 1_A \otimes x.$$

Since $\rho(\mathbf{Y}) = \mathbf{Y}_+ \otimes 1_H + \mathbf{Y}_- \otimes g - d(\mathbf{Y}_-) \otimes x + d(\mathbf{Y}_+) \otimes gx$ we get

$$\begin{cases} \mathbf{Y}_+ = 0 \\ \mathbf{Y}_- = \mathbf{Y} \\ d(\mathbf{Y}_-) = -1_A. \end{cases}$$

Finally for $h = gx$ we obtain

$$\rho\left(B(1_H \otimes gx)\right) = B(1_H \otimes gx) \otimes 1_H + B(1_H \otimes g) \otimes gx \iff \rho(\mathbf{X}) = \mathbf{X} \otimes 1_H + \mathbf{Z} \otimes gx.$$

Since $\rho(\mathbf{X}) = \mathbf{X}_+ \otimes 1_H + \mathbf{X}_- \otimes g - d(\mathbf{X}_-) \otimes x + d(\mathbf{X}_+) \otimes gx$ we find

$$\begin{cases} \mathbf{X}_+ = \mathbf{X} \\ \mathbf{X}_- = 0 \\ d(\mathbf{X}_+) = \mathbf{Z}. \end{cases}$$

We find that (74) is equivalent to

$$\begin{cases} \mathbf{X} \in A_+, \ d(\mathbf{X}) = \mathbf{Z} \\ \mathbf{Y} \in A_-, \ d(\mathbf{Y}) = -1_A \\ \mathbf{Z} \in A_-, \ d(\mathbf{Z}) = 0. \end{cases} \tag{171}$$

Since $d(\mathbf{Y}) = -1_A$ with $\mathbf{Y} \in A_-$ implies that $d(A_-) \neq \{0\}$, by looking at the tables of Section 7, we can deduce that rt-separability can be attained only when $\dim_k A_+ = \dim_k A_- = 2$ (i.e. only when φ has matrix similar to D_2). If we further inspect the four cases of admissible φ-derivations when $M_\varphi \sim D_2$, we see that the first two must be rejected, because in those cases we have $d(A_-) \subseteq \langle GX \rangle$ In the remaining we have

$$d(A_-) = \langle \begin{pmatrix} -\mathfrak{a}(\alpha\mathbf{A}^2 + \beta\mathbf{E}^2 - \gamma\mathbf{A}\mathbf{E} - \mathbf{C}^2) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \mathfrak{b}(\alpha\mathbf{A}^2 + \beta\mathbf{E}^2 - \gamma\mathbf{A}\mathbf{E} - \mathbf{C}^2) \\ 0 \\ 0 \\ 0 \end{pmatrix} \rangle \subseteq k.$$

In particular we see that $d(A_-)$ is either k or the trivial subspace $\{0\}$. Given that rt-separability implies $d(A_-) \neq \{0\}$ we see that

$$\alpha\mathbf{A}^2 + \beta\mathbf{E}^2 - \gamma\mathbf{A}\mathbf{E} - \mathbf{C}^2 \neq 0. \tag{172}$$

For the same reason $\mathfrak{a}$ and $\mathfrak{b}$ cannot be both zero and this ultimately leads to $d(A_-) = k$.

1.0.1. *Assume* $\boldsymbol{E} \neq 0$. Since $\mathbf{Y} \in A_-$, then $\mathbf{Y} = r' \begin{pmatrix} 0 \\ \mathbf{A} \\ -\mathbf{E} \\ 0 \end{pmatrix} + s' \begin{pmatrix} \frac{\gamma}{2}\mathbf{E} \\ \mathbf{C} \\ 0 \\ -\mathbf{E} \end{pmatrix}$. $d(\mathbf{Y}) = -1_A$ becomes

$$-1_A = d(\mathbf{Y}) = r'd(\mathbf{A}G - \mathbf{E}X) + s'd\left(\frac{\gamma}{2}\mathbf{E} + \mathbf{C}G - \mathbf{E}GX\right) = \mathbf{E}\frac{\alpha\mathbf{A}^2 + \beta\mathbf{E}^2 - \gamma\mathbf{AE} - \mathbf{C}^2}{\mathbf{AB} - \mathbf{CD} - \mathbf{EF}}(r'\mathfrak{a} - s'\mathfrak{b}),$$

thus

$$\mathbf{E}(\alpha\mathbf{A}^2 + \beta\mathbf{E}^2 - \gamma\mathbf{AE} - \mathbf{C}^2)(r'\mathfrak{a} - s'\mathfrak{b}) = -\mathbf{AB} + \mathbf{CD} + \mathbf{EF}. \tag{173}$$

Next, since $\mathbf{X} \in A_+$, there exist $r, s \in k$ such that $\mathbf{X} = \begin{pmatrix} r \\ s\mathbf{F} \\ -s\mathbf{B} \\ s\mathbf{D} \end{pmatrix}$. By using the explicit form of d we see that $d(\mathbf{X}) = s \begin{pmatrix} \frac{\gamma}{2}\mathfrak{a}\mathbf{E} \\ \mathfrak{b}\mathbf{A} + \mathfrak{a}\mathbf{C} \\ -\mathfrak{b}\mathbf{E} \\ -\mathfrak{a}\mathbf{E} \end{pmatrix}$ which, in view of (171), must be also equal to $\mathbf{Z}$. Notice that $\mathbf{Z} \in A_-$ and $d(\mathbf{Z}) = 0$ trivially hold (since $\mathbf{X} \in A_+$ and $d^2 \equiv 0$).

Now we go back to (167), which becomes

$$\mathbf{X}G - \varphi(G)\mathbf{X} = 0 \iff (R_G - L_{\varphi(G)}) \begin{pmatrix} r \\ s\mathbf{F} \\ -s\mathbf{B} \\ s\mathbf{D} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \iff \frac{2r + \gamma s\mathbf{D}}{-\mathbf{AB} + \mathbf{CD} + \mathbf{EF}} \begin{pmatrix} \frac{\gamma}{2}\mathbf{DE} \\ \mathbf{CD} - \mathbf{AB} \\ \mathbf{BE} \\ -\mathbf{DE} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

while (170) changes into

$$\mathbf{X}X = \varphi(X)\mathbf{X} \iff (R_X - L_{\varphi(X)}) \begin{pmatrix} r \\ s\mathbf{F} \\ -s\mathbf{B} \\ s\mathbf{D} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \iff \frac{2r + \gamma s\mathbf{D}}{-\mathbf{AB} + \mathbf{CD} + \mathbf{EF}} \begin{pmatrix} \frac{\gamma}{2}\mathbf{AD} \\ -\mathbf{AF} \\ \mathbf{CD} + \mathbf{EF} \\ -\mathbf{AD} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

If we suppose $2r + \gamma s\mathbf{D} \neq 0$, then we must have

$$\begin{cases} \mathbf{CD} - \mathbf{AB} = 0 \\ \mathbf{BE} = 0 \\ \mathbf{DE} = 0 \\ \mathbf{AF} = 0 \\ \mathbf{CD} + \mathbf{EF} = 0 \\ \mathbf{AD} = 0. \end{cases}$$

In this case if $\mathbf{D} \neq 0$, then $\mathbf{A} = \mathbf{E} = \mathbf{C} = 0$, which contradicts (172). Therefore $\mathbf{D} = 0$. This in turn gives $\mathbf{AB} = 0 = \mathbf{EF}$ which is again a contradiction (see "Conditions" on the last row of the first table in Sec. 5, Ch. 2). We can conclude that $2r + \gamma s\mathbf{D} = 0$. If we want (166) and (169) to be verified we need

$$r'(\mathbf{AB} - \mathbf{EF}) + s'\mathbf{BC} = 0. \tag{174}$$

In fact the last entry in $\mathbf{Y}G - \varphi(G)\mathbf{Y}$ is $\frac{-2\mathbf{E}[r'(\mathbf{AB}-\mathbf{EF})+s'\mathbf{BC}]}{-\mathbf{AB}+\mathbf{CD}+\mathbf{EF}}$. Given (174), second and third entries of $\mathbf{Y}G - \varphi(G)\mathbf{Y}$ change into $\frac{-\mathbf{E}[2r'\mathbf{CF}+s'(-2\beta\mathbf{BE}+\gamma\mathbf{CD}+\gamma\mathbf{EF})]}{-\mathbf{AB}+\mathbf{CD}+\mathbf{EF}}$ and $\frac{\mathbf{E}[2r'\mathbf{BC}+s'(2\alpha\mathbf{CD}-\gamma\mathbf{BE}+2\alpha\mathbf{EF})]}{-\mathbf{AB}+\mathbf{CD}+\mathbf{EF}}$, so that (166) and (169) also imply

$$2r'\mathbf{CF} + s'[\gamma(\mathbf{CD} + \mathbf{EF}) - 2\beta\mathbf{BE}] = 0 \tag{175}$$

$$2r'\mathbf{BC} + s'[2\alpha(\mathbf{CD} + \mathbf{EF}) - \gamma\mathbf{BE}] = 0. \tag{176}$$

Finally consider (167) and apply d to both sides of the equality:

$$d(\mathbf{X}G) = d(\varphi(G)\mathbf{X}) \iff d(\mathbf{X})G + \varphi(\mathbf{X})d(G) = d(\varphi(G))\mathbf{X} + Gd(\mathbf{X}) \iff d(\mathbf{X})G = Gd(\mathbf{X}) + d(\varphi(G))\mathbf{X} - \varphi(\mathbf{X})d(G).$$

Since $d(\mathbf{X}) = \mathbf{Z}$ it is sufficent to prove that $-\varphi(\mathbf{X})d(G) = d(\varphi(G))\mathbf{X}$ to show that (165) holds. We will make use of the following lemma.

LEMMA 1.3. *Suppose $v \in A$ is such that*

$$vG - \varphi(G)v = 0 \tag{177}$$
$$vX - \varphi(X)v = 0. \tag{178}$$

Then $va - \varphi(a)v = 0$ for any $a \in A$.

PROOF. Let $a = \lambda_1 + \lambda_2 G + \lambda_3 X + \lambda_4 GX$. We have

$$\begin{aligned}
va - \varphi(a)v &= (R_a - L_{\varphi(a)})v \\
&\overset{0.3}{=} (R_a - M_\varphi L_a M_\varphi)v \\
&= [\lambda_2(R_G - M_\varphi L_G M_\varphi) + \lambda_3(R_X - M_\varphi L_X M_\varphi) + \lambda_4(R_X R_G - M_\varphi L_G L_X M_\varphi)]v \\
&\overset{(177)+(178)}{=} \lambda_4(R_X R_G - M_\varphi L_G L_X M_\varphi)v \\
&\overset{(178)}{=} \lambda_4(R_X R_G - M_\varphi L_G M_\varphi R_X)v \\
&\overset{(L_a \text{ and } R_b \text{ commute})}{=} \lambda_4(R_X R_G - R_X M_\varphi L_G M_\varphi)v \\
&\overset{(177)}{=} \lambda_4(R_X R_G - R_X R_G)v \\
&= 0.
\end{aligned}$$

□

We can apply Lemma(1.3) with $v = \mathbf{X}$ and $a = d(G)$, thanks to (167) and (170) and get that $\mathbf{X}d(G) - \varphi(d(G))\mathbf{X} = 0$. Since $\varphi(\mathbf{X}) = \mathbf{X}$ and $\varphi d = -d\varphi$ this equality rewrites as $\varphi(\mathbf{X})d(G) + d(\varphi(G))\mathbf{X} = 0$, which is what we needed. Starting from (170), applying d to both sides, and performing the same calculations replacing each G with an X shows that also (168) holds unconditionally. We can conclude that, when $\mathbf{E} \neq 0$, the cowreath $(A \otimes H^{op}, H, \psi)$ is rt-separable if, and only if, the following equalities hold:

$$\begin{aligned}
B(1_H \otimes 1_H) = B(g \otimes g) &= 1_A \\
B(1_H \otimes x) = B(g \otimes gx) = -B(x \otimes g) = -B(gx \otimes 1_H) &=: \mathbf{Y} \\
B(1_H \otimes g) = B(g \otimes 1_H) &=: \mathbf{Z} \\
B(1_H \otimes gx) = B(g \otimes x) = B(x \otimes 1_H) = B(gx \otimes g) &=: \mathbf{X} \\
B(x \otimes x) = B(gx \otimes gx) = B(gx \otimes x) = B(x \otimes gx) &= 0,
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{X} &= s\left(-\frac{\gamma}{2}\mathbf{D} + \mathbf{F}G - \mathbf{B}X + \mathbf{D}GX\right) \\
\mathbf{Y} &= \frac{\gamma}{2}s'\mathbf{E} + (r'\mathbf{A} + s'\mathbf{C})G - r'\mathbf{E}X - s'\mathbf{E}GX \\
\mathbf{Z} &= s\left(\frac{\gamma}{2}\mathfrak{a}\mathbf{E} + (\mathfrak{a}\mathbf{C} + \mathfrak{b}\mathbf{A})G - \mathfrak{b}\mathbf{E}X - \mathfrak{a}\mathbf{E}GX\right)
\end{aligned}$$

with

$$(179)\quad \begin{cases} \mathbf{E}(\alpha\mathbf{A}^2+\beta\mathbf{E}^2-\gamma\mathbf{AE}-\mathbf{C}^2)(r'\mathfrak{a}-s'\mathfrak{b}) = -\mathbf{AB}+\mathbf{CD}+\mathbf{EF} \\ r'(\mathbf{AB}-\mathbf{EF})+s'\mathbf{BC}=0 \\ 2r'\mathbf{CF}+s'[\gamma(\mathbf{CD}+\mathbf{EF})-2\beta\mathbf{BE}]=0 \\ 2r'\mathbf{BC}+s'[2\alpha(\mathbf{CD}+\mathbf{EF})-\gamma\mathbf{BE}]=0 \\ \gamma\mathbf{AD}-2\mathbf{CF}-2\beta\mathbf{DE}=0 \\ \gamma\mathbf{DE}+2\mathbf{BC}-2\alpha\mathbf{AD}=0 \\ 2\alpha\mathbf{AF}+2\beta\mathbf{BE}-\gamma(\mathbf{AB}+\mathbf{EF})=0 \\ \mathbf{AB}-\mathbf{CD}-\mathbf{EF}\neq 0 \\ \mathbf{E}\neq 0. \end{cases}$$

Example 1.4. A family of solutions of this system is given by $\mathbf{A}=\mathbf{B}=\mathbf{C}=\mathbf{D}=0$, $\mathbf{E}=\mathbf{F}=1$, $\alpha=\gamma=0$, $\beta=1$, $\mathfrak{b}=-1$, $r'=0$, $s'=1$, $\mathfrak{a},s\in k$.

If $\mathfrak{a}=0$, this H-coaction is given explicitly by

$$\begin{aligned} \rho(G) &= G\otimes 1_H + X\otimes gx, \\ \rho(X) &= X\otimes g. \end{aligned}$$

and the Casimir element is defined by

$$\begin{aligned} B(1_H\otimes 1_H)=B(g\otimes g)=1_A \\ B(1_H\otimes x)=B(g\otimes gx)=-B(x\otimes g)=-B(gx\otimes 1_H)=-GX \\ B(1_H\otimes g)=B(g\otimes 1_H)=sX \\ B(1_H\otimes gx)=B(g\otimes x)=B(x\otimes 1_H)=B(gx\otimes g)=sG \\ B(x\otimes x)=B(gx\otimes gx)=B(gx\otimes x)=B(x\otimes gx)=0 \end{aligned}$$

with $s\in k$.

Example 1.5. The coaction described in Example 5.5, Ch.2, does not yield a solution of the previous system, since the first two equations become $r'=\frac{2}{\gamma}$ and $r'\left(-\frac{2\beta}{\gamma}\right)=0$, which contradicts $\beta\neq 0$.

1.0.2. *Assume* $\boldsymbol{E}=0$. Since $\mathbf{Y}\in A_-$, then $\mathbf{Y}=r'\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+s'\begin{pmatrix}\frac{\gamma}{2}\mathbf{A}\\0\\\mathbf{C}\\-\mathbf{A}\end{pmatrix}$. $d(\mathbf{Y})=-1_A$ becomes

$$-1_A=d(\mathbf{Y})=r'd(G)+s'd\left(\frac{\gamma}{2}\mathbf{A}+\mathbf{C}X-\mathbf{A}GX\right)=\frac{\alpha\mathbf{A}^2-\mathbf{C}^2}{\mathbf{AB}-\mathbf{CD}}(r'\mathfrak{a}-s'\mathfrak{b}),$$

thus

$$(180)\quad (\alpha\mathbf{A}^2-\mathbf{C}^2)(r'\mathfrak{a}-s'\mathfrak{b})=-\mathbf{AB}+\mathbf{CD}.$$

Next, since $\mathbf{X}\in A_+$, there exist $r,s\in k$ such that $\mathbf{X}=\begin{pmatrix}r\\s\mathbf{F}\\-s\mathbf{B}\\s\mathbf{D}\end{pmatrix}$. Then we have that $d(\mathbf{X})=s\begin{pmatrix}\frac{\gamma}{2}\mathfrak{a}\mathbf{A}\\\mathfrak{b}\\\mathfrak{a}\mathbf{C}\\-\mathfrak{a}\mathbf{A}\end{pmatrix}$ which, in view of (171), must be also equal to $\mathbf{Z}$. Notice that $\mathbf{Z}\in A_-$ and $d(\mathbf{Z})=0$ again trivially hold.

Now we go back to (167), which becomes

$$\mathbf{X}G-\varphi(G)\mathbf{X}=0 \iff (R_G-L_{\varphi(G)})\begin{pmatrix}r\\s\mathbf{F}\\-s\mathbf{B}\\s\mathbf{D}\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \iff \begin{pmatrix}0\\2r+\gamma s\mathbf{D}\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},$$

while (170) changes into

$$\mathbf{X}X = \varphi(X)\mathbf{X} \iff (R_X - L_{\varphi(X)}) \begin{pmatrix} r \\ s\mathbf{F} \\ -s\mathbf{B} \\ s\mathbf{D} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \iff \frac{1}{\mathbf{AB}-\mathbf{CD}} \begin{pmatrix} -\mathbf{CF}(2r+\gamma s\mathbf{D}) \\ 2\mathbf{F}(r\mathbf{A}+s\mathbf{CF}) \\ -\mathbf{CD}(2r+\gamma s\mathbf{D}) \\ 2\mathbf{D}(r\mathbf{A}+s\mathbf{CF}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It is straightforward that $2r+\gamma s\mathbf{D} = 0$. This in turn gives $2r\mathbf{A}+2s\mathbf{CF} = s(-\gamma\mathbf{AD}+2\mathbf{CF}) = 0$ (see "Conditions" on the last row of the first table in Sec. 5, Ch. 2), so that (170) is satisfied. (166) is equivalent to $2\alpha r' + \gamma s'\mathbf{C} = 0$ and the first entry in $\mathbf{Y}G - \varphi(G)\mathbf{Y}$ is $\gamma r' + 2\beta s'\mathbf{C}$, so that (169) implies

$$\begin{cases} \gamma r' + 2\beta s'\mathbf{C} = 0 \\ \mathbf{A}(r'\mathbf{B} + s'\mathbf{CF}) = 0 \\ s'\mathbf{A}(\gamma\mathbf{F} - 2\beta\mathbf{B}) = 0. \end{cases}$$

Since $\mathbf{A} = 0$ forces $\mathbf{B} = 0$ and $\mathbf{F} = 0$[1], then the system actually reduces to

$$\begin{cases} \gamma r' + 2\beta s'\mathbf{C} = 0 \\ r'\mathbf{B} + s'\mathbf{CF} = 0 \\ s'(\gamma\mathbf{F} - 2\beta\mathbf{B}) = 0. \end{cases}$$

Moreover, if $\gamma\mathbf{F} - 2\beta\mathbf{B} \neq 0$, then $s' = 0$, so that either $r' = 0$ or $\gamma = \mathbf{B} = 0$. The second case goes against our assumption, while the first changes (180) into $\mathbf{AB} - \mathbf{CD} = 0$, contradiction. Thus the last equation of the system is equivalent to $\gamma\mathbf{F} - 2\beta\mathbf{B} = 0$. To conclude we can prove that (165) and (168) are satisfied using Lemma 1.3.

We can deduce that, if $\mathbf{E} = 0$, the cowreath $(A \otimes H^{op}, H, \psi)$ is rt-separable if, and only if, the following equalities hold:

$$\begin{aligned} B(1_H \otimes 1_H) = B(g \otimes g) &= 1_A \\ B(1_H \otimes x) = B(g \otimes gx) = -B(x \otimes g) = -B(gx \otimes 1_H) &=: \mathbf{Y} \\ B(1_H \otimes g) = B(g \otimes 1_H) &=: \mathbf{Z} \\ B(1_H \otimes gx) = B(g \otimes x) = B(x \otimes 1_H) = B(gx \otimes g) &=: \mathbf{X} \\ B(x \otimes x) = B(gx \otimes gx) = B(gx \otimes x) = B(x \otimes gx) &= 0, \end{aligned}$$

where

$$\begin{aligned} \mathbf{X} &= s\left(-\frac{\gamma}{2}\mathbf{D} + \mathbf{F}G - \mathbf{B}X + \mathbf{D}GX\right) \\ \mathbf{Y} &= s'\frac{\gamma}{2}\mathbf{A} + r'G + s'\mathbf{C}X - s'\mathbf{A}GX \\ \mathbf{Z} &= s\left(\frac{\gamma}{2}\mathfrak{a}\mathbf{A} + \mathfrak{b}G + \mathfrak{a}\mathbf{C}X - \mathfrak{a}\mathbf{A}GX\right) \end{aligned}$$

with

$$(181)\qquad \begin{cases} (\alpha\mathbf{A}^2 - \mathbf{C}^2)(r'\mathfrak{a} - s'\mathfrak{b}) = -\mathbf{AB} + \mathbf{CD} \\ 2\alpha r' + \gamma s'\mathbf{C} = 0 \\ \gamma r' + 2\beta s'\mathbf{C} = 0 \\ r'\mathbf{B} + s'\mathbf{CF} = 0 \\ \gamma\mathbf{F} - 2\beta\mathbf{B} = 0 \\ \gamma\mathbf{AD} - 2\mathbf{CF} = 0 \\ \mathbf{BC} - \alpha\mathbf{AD} = 0 \\ \gamma\mathbf{B} - 2\alpha\mathbf{F} = 0 \\ \mathbf{AB} - \mathbf{CD} \neq 0. \end{cases}$$

[1]See again the table in Sec. 5, Ch. 2.

Example 1.6. Two solutions of this system are given by $\mathbf{A} = \mathbf{B} = \mathbf{F} = 0$, $\mathbf{C} = \mathbf{D} = 1$, $\mathfrak{a} = 0$, $\mathfrak{b} = 1$, $s' = 1$ and either

- $\alpha \in k^{\times}$, $\beta, \gamma \in k$ with $\gamma^2 - 4\alpha\beta = 0$, $r' = -\frac{\gamma}{2\alpha}$, $s, \in k$.
- $\alpha = \beta = \gamma = 0$, $r', s \in k$.

with $s \in k$. The Casimir element is defined by

$$\begin{aligned} B(1_H \otimes 1_H) &= B(g \otimes g) = 1_A \\ B(1_H \otimes x) = B(g \otimes gx) = -B(x \otimes g) &= -B(gx \otimes 1_H) = sG \\ B(1_H \otimes g) &= B(g \otimes 1_H) = r'G + X \\ B(1_H \otimes gx) = B(g \otimes x) = B(x \otimes 1_H) &= B(gx \otimes g) = s\left(-\frac{\gamma}{2} + GX\right) \\ B(x \otimes x) = B(gx \otimes gx) &= B(gx \otimes x) = B(x \otimes gx) = 0. \end{aligned}$$

These are exactly the solutions found by Menini and Torrecillas in [**MT1**, Thm. 6.1], when ρ is the canonical H-coaction on A.

Example 1.7. Another set of solutions is reported in [**FR**, Thm. 5.0.2]. For these, take $\mathbf{A} = \mathbf{B} = \mathbf{F} = 0$, $\mathbf{C} = \mathbf{D} = 1$, $\mathfrak{a} = -1$, $\mathfrak{b} = 1$, and either

- $\alpha = \beta = \gamma = 0$, $r', s, s' \in k$ with $s' = 1 - r'$.
- $\alpha \in k^{\times}$, $\beta = \gamma = 0$, $r' = 0$, $s' = 1$ and $s \in k$.
- $\beta \in k^{\times}$, $\alpha, \gamma \in k$ with $\gamma^2 - 4\alpha\beta = 0$, $s' = \frac{2\beta}{2\beta - \gamma}$, $r' = 1 - s'$, $s \in k$.

The Casimir element is defined by

$$\begin{aligned} B(1_H \otimes 1_H) &= B(g \otimes g) = 1_A \\ B(1_H \otimes x) = B(g \otimes gx) = -B(x \otimes g) &= -B(gx \otimes 1_H) = s\,(G - X) \\ B(1_H \otimes g) &= B(g \otimes 1_H) = r'G + s'X \\ B(1_H \otimes gx) = B(g \otimes x) = B(x \otimes 1_H) &= B(gx \otimes g) = s\left(-\frac{\gamma}{2} + GX\right) \\ B(x \otimes x) = B(gx \otimes gx) &= B(gx \otimes x) = B(x \otimes gx) = 0. \end{aligned}$$

2. Rth-separability

If $(A \otimes H^{op}, H, \psi)$ is rt-separable, rth-separability of $(A \otimes H^{op}, H, \psi)$ means that further condition (76) must hold for all h, $h' \in \{1, g, x, gx\}$. These equalities can be summarized as follows

h	h'	$B(h \otimes 1_H) \cdot B(1_H \otimes h') = B(h \otimes h')$	
1_H	1_H	$B(1_H \otimes 1_H) \cdot B(1_H \otimes 1_H) = B(1_H \otimes 1_H)$	$1 = 1$
1_H	g	$B(1_H \otimes 1_H) \cdot B(1_H \otimes g) = B(1_H \otimes g)$	$1 \cdot \mathbf{Z} = \mathbf{Z}$
1_H	x	$B(1_H \otimes 1_H) \cdot B(1_H \otimes x) = B(1_H \otimes x)$	$1 \cdot \mathbf{Y} = \mathbf{Y}$
1_H	gx	$B(1_H \otimes 1_H) \cdot B(1_H \otimes gx) = B(1_H \otimes gx)$	$1 \cdot \mathbf{X} = \mathbf{X}$
g	1_H	$B(g \otimes 1_H) \cdot B(1_H \otimes 1_H) = B(g \otimes 1_H)$	$\mathbf{Z} \cdot 1 = \mathbf{Z}$
g	g	$B(g \otimes 1_H) \cdot B(1_H \otimes g) = B(g \otimes g)$	$\mathbf{Z}^2 = 1$
g	x	$B(g \otimes 1_H) \cdot B(1_H \otimes x) = B(g \otimes x)$	$\mathbf{ZY} = \mathbf{X}$
g	gx	$B(g \otimes 1_H) \cdot B(1_H \otimes gx) = B(g \otimes gx)$	$\mathbf{ZX} = \mathbf{Y}$
x	1_H	$B(x \otimes 1_H) \cdot B(1_H \otimes 1_H) = B(x \otimes 1_H)$	$\mathbf{X} \cdot 1 = \mathbf{X}$
x	g	$B(x \otimes 1_H) \cdot B(1_H \otimes g) = B(x \otimes g)$	$\mathbf{XZ} = -\mathbf{Y}$

x	x	$B(x \otimes 1_H) \cdot B(1_H \otimes x) = B(x \otimes x)$	$\mathbf{XY} = 0$
x	gx	$B(x \otimes 1_H) \cdot B(1_H \otimes gx) = B(x \otimes gx)$	$\mathbf{X}^2 = 0$
gx	1_H	$B(gx \otimes 1_H) \cdot B(1_H \otimes 1_H) = B(gx \otimes 1_H)$	$-\mathbf{Y} \cdot 1 = -\mathbf{Y}$
gx	g	$B(gx \otimes 1_H) \cdot B(1_H \otimes g) = B(gx \otimes g)$	$-\mathbf{YZ} = \mathbf{X}$
gx	x	$B(gx \otimes 1_H) \cdot B(1_H \otimes x) = B(gx \otimes x)$	$-\mathbf{Y}^2 = 0$
gx	gx	$B(gx \otimes 1_H) \cdot B(1_H \otimes gx) = -B(gx \otimes gx)$	$\mathbf{YX} = 0$

Non-trivial equalities are

$$\mathbf{X}^2 = 0,\ \mathbf{Y}^2 = 0,\ \mathbf{Z}^2 = 1, \mathbf{XY} = \mathbf{YX} = 0,\ -\mathbf{XZ} = \mathbf{ZX} = \mathbf{Y},\ \mathbf{ZY} = -\mathbf{YZ} = \mathbf{X}.$$

Now suppose $\mathbf{X}^2 = 0$, $\mathbf{Z}^2 = 1$ and $\mathbf{ZY} = -\mathbf{YZ} = \mathbf{X}$. Then

$$\mathbf{Y} = \mathbf{Z}(\mathbf{ZY}) = \mathbf{ZX}$$

and

$$\mathbf{Y} = (\mathbf{YZ})\mathbf{Z} = -\mathbf{XZ}.$$

Moreover

$$\mathbf{YX} = \mathbf{Z}^2\mathbf{YX} = \mathbf{Z}(\mathbf{ZY})\mathbf{X} = \mathbf{ZXX} = 0$$

and

$$\mathbf{XY} = \mathbf{XYZ}^2 = \mathbf{X}(\mathbf{YZ})\mathbf{Z} = -\mathbf{XXZ} = 0.$$

Finally

$$\mathbf{Y}^2 = \mathbf{Y}^2\mathbf{Z}^2 = \mathbf{Y}(\mathbf{YZ})\mathbf{Z} = \mathbf{Y}(-\mathbf{X})\mathbf{Z} = -\mathbf{YXZ} = 0.$$

Therefore a rt-separable cowreath $(A \otimes H^{op}, H, \psi)$ is rth-separable if and only if $\mathbf{X}^2 = 0$, $\mathbf{Z}^2 = 1$ and $\mathbf{ZY} = -\mathbf{YZ} = \mathbf{X}$. Now

$$\mathbf{X}^2 = 0 \overset{(262)}{\iff} s^2 \left(\frac{\gamma^2}{4}\mathbf{D}^2 + \alpha\mathbf{F}^2 + \beta\mathbf{B}^2 - \gamma\mathbf{FB} - \alpha\beta\mathbf{D}^2 \right) = 0.$$

We can further simplify this expression by making use of a claim. The fact that the cowreath $(A \otimes H^{op}, H, \psi)$ is rt-separable actually implies that $\gamma^2 - 4\alpha\beta = 0$. This claim will be proved later in Sec. (see Prop. 6.2). Therefore

$$\mathbf{X}^2 = 0 \iff s^2(\alpha\mathbf{F}^2 + \beta\mathbf{B}^2 - \gamma\mathbf{FB}) = 0.$$

Since $\mathbf{Y}$ and $\mathbf{Z}$ differ according to whether $\mathbf{E} \neq 0$ or $\mathbf{E} = 0$ we continue distinguishing between the two cases.

2.0.1. *Assume* $\boldsymbol{E} \neq 0$. Then

$$\mathbf{Z}^2 = 1 \overset{(262)}{\iff} s^2(\alpha(\mathfrak{a}\mathbf{C} + \mathfrak{b}\mathbf{A})^2 + \beta\mathfrak{b}^2\mathbf{E}^2 - \gamma\mathfrak{b}(\mathfrak{a}\mathbf{C} + \mathfrak{b}\mathbf{A})\mathbf{E}) = 1.$$

Observe that this implies $s \neq 0$, so that

$$\mathbf{X}^2 = 0 \iff \alpha\mathbf{F}^2 + \beta\mathbf{B}^2 - \gamma\mathbf{FB} = 0. \tag{182}$$

Moreover

$$\mathbf{YZ} = \frac{s}{2}\begin{pmatrix} 2(\alpha(r'\mathbf{A} + s'\mathbf{C}) - \gamma r'\mathbf{E})(\mathfrak{a}\mathbf{C} + \mathfrak{b}\mathbf{A}) + 2\beta r'\mathfrak{b}\mathbf{E}^2 \\ \mathbf{E}(\gamma\mathbf{A} - 2\beta\mathbf{E})(r'\mathfrak{a} - s'\mathfrak{b}) \\ \mathbf{E}(\gamma\mathbf{E} - 2\alpha\mathbf{A})(r'\mathfrak{a} - s'\mathfrak{b}) \\ 2\mathbf{CE}(r'\mathfrak{a} - s'\mathfrak{b}) \end{pmatrix}$$

and

$$\mathbf{ZY} = \frac{s}{2}\begin{pmatrix} 2(r'\mathbf{A} + s'\mathbf{C})(\alpha(\mathfrak{a}\mathbf{C} + \mathfrak{b}\mathbf{A}) - \gamma\mathfrak{b}\mathbf{E}) + 2\beta r'\mathfrak{b}\mathbf{E}^2 \\ \mathbf{E}(2\beta\mathbf{E} - \gamma\mathbf{A})(r'\mathfrak{a} - s'\mathfrak{b}) \\ \mathbf{E}(2\alpha\mathbf{A} - \gamma\mathbf{E})(r'\mathfrak{a} - s'\mathfrak{b}) \\ -2\mathbf{CE}(r'\mathfrak{a} - s'\mathfrak{b}) \end{pmatrix},$$

therefore $\mathbf{YZ} = -\mathbf{ZY}$ is equivalent to

$$2\alpha(r'\mathbf{A} + s'\mathbf{C})(\mathfrak{a}\mathbf{C} + \mathfrak{b}\mathbf{A}) - \gamma r'\mathbf{E}(\mathfrak{a}\mathbf{C} + \mathfrak{b}\mathbf{A}) - \gamma\mathfrak{b}\mathbf{E}(r'\mathbf{A} + s'\mathbf{C}) + 2\beta r'\mathfrak{b}\mathbf{E}^2 = 0$$

while $\mathbf{ZY} = \mathbf{X}$ is equivalent to

$$2\alpha(r'\mathbf{A} + s'\mathbf{C})(\mathfrak{a}\mathbf{C} + \mathfrak{b}\mathbf{A}) - 2\gamma\mathfrak{b}\mathbf{E}(r'\mathbf{A} + s'\mathbf{C}) + 2\beta r'\mathfrak{b}\mathbf{E}^2 = -\gamma\mathbf{D} \tag{183}$$

$$\mathbf{E}(2\beta\mathbf{E} - \gamma\mathbf{A})(r'\mathfrak{a} - s'\mathfrak{b}) = 2\mathbf{F} \tag{184}$$

$$\mathbf{E}(2\alpha\mathbf{A} - \gamma\mathbf{E})(r'\mathfrak{a} - s'\mathfrak{b}) = -2\mathbf{B} \tag{185}$$

$$-\mathbf{CE}(r'\mathfrak{a} - s'\mathfrak{b}) = \mathbf{D}. \tag{186}$$

We can conclude that, when $\mathbf{E} \neq 0$, the cowreath $(A \otimes H^{op}, H, \psi)$ is rth-separable if, and only if,

$$\begin{cases} \mathbf{E}(\alpha\mathbf{A}^2 + \beta\mathbf{E}^2 - \gamma\mathbf{AE} - \mathbf{C}^2)(r'\mathfrak{a} - s'\mathfrak{b}) = -\mathbf{AB} + \mathbf{CD} + \mathbf{EF} \\ r'(\mathbf{AB} - \mathbf{EF}) + s'\mathbf{BC} = 0 \\ 2r'\mathbf{CF} + s'[\gamma(\mathbf{CD} + \mathbf{EF}) - 2\beta\mathbf{BE}] = 0 \\ 2r'\mathbf{BC} + s'[2\alpha(\mathbf{CD} + \mathbf{EF}) - \gamma\mathbf{BE}] = 0 \\ \gamma\mathbf{AD} - 2\mathbf{CF} - 2\beta\mathbf{DE} = 0 \\ \gamma\mathbf{DE} + 2\mathbf{BC} - 2\alpha\mathbf{AD} = 0 \\ 2\alpha\mathbf{AF} + 2\beta\mathbf{BE} - \gamma(\mathbf{AB} + \mathbf{EF}) = 0 \\ \mathbf{AB} - \mathbf{CD} - \mathbf{EF} \neq 0 \\ \mathbf{E} \neq 0 \\ \alpha\mathbf{F}^2 + \beta\mathbf{B}^2 - \gamma\mathbf{FB} = 0 \\ s^2(\alpha(\mathfrak{a}\mathbf{C} + \mathfrak{b}\mathbf{A})^2 + \beta\mathfrak{b}^2\mathbf{E}^2 - \gamma\mathfrak{b}(\mathfrak{a}\mathbf{C} + \mathfrak{b}\mathbf{A})\mathbf{E}) = 1 \\ 2\alpha(r'\mathbf{A} + s'\mathbf{C})(\mathfrak{a}\mathbf{C} + \mathfrak{b}\mathbf{A}) - \gamma r'\mathbf{E}(\mathfrak{a}\mathbf{C} + \mathfrak{b}\mathbf{A}) - \gamma\mathfrak{b}\mathbf{E}(r'\mathbf{A} + s'\mathbf{C}) + 2\beta r'\mathfrak{b}\mathbf{E}^2 = 0 \\ 2\alpha(r'\mathbf{A} + s'\mathbf{C})(\mathfrak{a}\mathbf{C} + \mathfrak{b}\mathbf{A}) - 2\gamma\mathfrak{b}\mathbf{E}(r'\mathbf{A} + s'\mathbf{C}) + 2\beta r'\mathfrak{b}\mathbf{E}^2 = -\gamma\mathbf{D} \\ \mathbf{F} = \frac{\mathbf{E}}{2}(2\beta\mathbf{E} - \gamma\mathbf{A})(r'\mathfrak{a} - s'\mathfrak{b}) \\ \mathbf{B} = -\frac{\mathbf{E}}{2}(2\alpha\mathbf{A} - \gamma\mathbf{E})(r'\mathfrak{a} - s'\mathfrak{b}) \\ \mathbf{D} = -\mathbf{CE}(r'\mathfrak{a} - s'\mathfrak{b}). \end{cases}$$

which boils down to

$$\begin{cases} r'(2\beta\mathbf{E} - \gamma\mathbf{A}) - s'\gamma\mathbf{C} = 0 \\ r'(2\alpha\mathbf{A} - \gamma\mathbf{E}) + 2s'\alpha\mathbf{C} = 0 \\ \alpha\mathbf{A}^2 + \beta\mathbf{E}^2 - \gamma\mathbf{AE} - \mathbf{C}^2 \neq 0 \\ \mathbf{E} \neq 0 \\ r'\mathfrak{a} - s'\mathfrak{b} \neq 0 \\ s^2(\alpha(\mathfrak{a}\mathbf{C} + \mathfrak{b}\mathbf{A})^2 + \beta\mathfrak{b}^2\mathbf{E}^2 - \gamma\mathfrak{b}(\mathfrak{a}\mathbf{C} + \mathfrak{b}\mathbf{A})\mathbf{E}) = 1 \\ \mathbf{F} = \frac{\mathbf{E}}{2}(2\beta\mathbf{E} - \gamma\mathbf{A})(r'\mathfrak{a} - s'\mathfrak{b}) \\ \mathbf{B} = -\frac{\mathbf{E}}{2}(2\alpha\mathbf{A} - \gamma\mathbf{E})(r'\mathfrak{a} - s'\mathfrak{b}) \\ \mathbf{D} = -\mathbf{CE}(r'\mathfrak{a} - s'\mathfrak{b}). \end{cases}$$

Example 2.1. A family of solutions is given by $\mathbf{A} = \mathbf{B} = \mathbf{E} = 1$, $\mathbf{C} = \mathbf{D} = \mathbf{F} = 0$, $\alpha = 1$, $\beta = \gamma = 0$, $\mathfrak{b} = 1$, $r' = 0$, $s' = 1$, $s = 1$ and $\mathfrak{a} \in k$. If $\mathfrak{a} = 0$, this H-coaction is given explicitly by

$$\begin{aligned} \rho(G) &= G \otimes g - G \otimes gx + X \otimes 1_H - X \otimes g + X \otimes gx, \\ \rho(X) &= X \otimes 1_H - X \otimes g + X \otimes gx. \end{aligned}$$

and the Casimir element is defined by

$$\begin{aligned} B(1_H\otimes 1_H) &= B(g\otimes g)=1_A\\ B(1_H\otimes x) &= B(g\otimes gx)=-B(x\otimes g)=-B(gx\otimes 1_H)=-GX\\ B(1_H\otimes g) &= B(g\otimes 1_H)=sG\\ B(1_H\otimes gx) &= B(g\otimes x)=B(x\otimes 1_H)=B(gx\otimes g)=-sX\\ B(x\otimes x) &= B(gx\otimes gx)=B(gx\otimes x)=B(x\otimes gx)=0 \end{aligned}$$

with $s\in k$.

2.0.2. *Assume* $\boldsymbol{E}=0$. Then

$$\mathbf{Z}^2 = \overset{(262)}{\Longleftrightarrow} s^2\left(\alpha\mathfrak{b}^2+\beta\mathfrak{a}^2\mathbf{C}^2+\gamma\mathfrak{a}\mathfrak{b}\mathbf{C}\right)=1.$$

This implies that $s\neq 0$, hence

$$\mathbf{X}^2=0 \iff \alpha\mathbf{F}^2+\beta\mathbf{B}^2-\gamma\mathbf{F}\mathbf{B}=0. \tag{187}$$

Moreover

$$\mathbf{YZ}=\frac{s}{2}\begin{pmatrix}-\gamma\mathbf{C}(r'\mathfrak{a}-s'\mathfrak{b})\\ \gamma\mathbf{A}(r'\mathfrak{a}-s'\mathfrak{b})\\ -2\alpha\mathbf{A}(r'\mathfrak{a}-s'\mathfrak{b})\\ 2\mathbf{C}(r'\mathfrak{a}-s'\mathfrak{b})\end{pmatrix}$$

and

$$\mathbf{ZY}=\frac{s}{2}\begin{pmatrix}\gamma\mathbf{C}(r'\mathfrak{a}-s'\mathfrak{b})\\ -\gamma\mathbf{A}(r'\mathfrak{a}-s'\mathfrak{b})\\ 2\alpha\mathbf{A}(r'\mathfrak{a}-s'\mathfrak{b})\\ -2\mathbf{C}(r'\mathfrak{a}-s'\mathfrak{b})\end{pmatrix},$$

so that $\mathbf{YZ}=-\mathbf{ZY}$ holds trivially, while $\mathbf{ZY}=\mathbf{X}$ is equivalent to

$$-\gamma\mathbf{A}(r'\mathfrak{a}-s'\mathfrak{b})=2\mathbf{F} \tag{188}$$

$$\alpha\mathbf{A}(r'\mathfrak{a}-s'\mathfrak{b})=-\mathbf{B} \tag{189}$$

$$-\mathbf{C}(r'\mathfrak{a}-s'\mathfrak{b})=\mathbf{D}. \tag{190}$$

We can conclude that, when $\mathbf{E}=0$, the cowreath $(A\otimes H^{op},H,\psi)$ is rth-separable if, and only if,

$$\begin{cases} (\alpha\mathbf{A}^2-\mathbf{C}^2)(r'\mathfrak{a}-s'\mathfrak{b})=-\mathbf{AB}+\mathbf{CD}\\ 2\alpha r'+\gamma s'\mathbf{C}=0\\ \gamma r'+2\beta s'\mathbf{C}=0\\ r'\mathbf{B}+s'\mathbf{CF}=0\\ \gamma\mathbf{F}-2\beta\mathbf{B}=0\\ \gamma\mathbf{AD}-2\mathbf{CF}=0\\ \mathbf{BC}-\alpha\mathbf{AD}=0\\ \gamma\mathbf{B}-2\alpha\mathbf{F}=0\\ \mathbf{AB}-\mathbf{CD}\neq 0\\ \alpha\mathbf{F}^2+\beta\mathbf{B}^2-\gamma\mathbf{BF}=0\\ s^2\left(\alpha\mathfrak{b}^2+\beta\mathfrak{a}^2\mathbf{C}^2+\gamma\mathfrak{a}\mathfrak{b}\mathbf{C}\right)=1\\ \mathbf{F}=-\frac{\gamma}{2}\mathbf{A}(r'\mathfrak{a}-s'\mathfrak{b})\\ \mathbf{B}=-\alpha\mathbf{A}(r'\mathfrak{a}-s'\mathfrak{b})\\ \mathbf{D}=-\mathbf{C}(r'\mathfrak{a}-s'\mathfrak{b}), \end{cases}$$

or equivalently if, and only if,

$$\begin{cases} 2\alpha r' + \gamma s'\mathbf{C} = 0 \\ \gamma r' + 2\beta s'\mathbf{C} = 0 \\ \alpha\mathbf{A}^2 - \mathbf{C}^2 \neq 0 \\ r'\mathfrak{a} - s'\mathfrak{b} \neq 0 \\ s^2\left(\alpha\mathfrak{b}^2 + \beta\mathfrak{a}^2\mathbf{C}^2 + \gamma\mathfrak{a}\mathfrak{b}\mathbf{C}\right) = 1 \\ \mathbf{F} = -\frac{\gamma}{2}\mathbf{A}(r'\mathfrak{a} - s'\mathfrak{b}) \\ \mathbf{B} = -\alpha\mathbf{A}(r'\mathfrak{a} - s'\mathfrak{b}) \\ \mathbf{D} = -\mathbf{C}(r'\mathfrak{a} - s'\mathfrak{b}). \end{cases}$$

Example 2.2. A solution of this system is given by $\mathbf{A} = \mathbf{B} = \mathbf{F} = 0$, $\mathbf{C} = \mathbf{D} = 1$, $\alpha = 1$, $\beta = \gamma = 0$, $\mathfrak{a} = 0$, $\mathfrak{b} = 1$, $r' = 0$, $s = s' = 1$. Notice that this is a particular case of the values recovered by Menini and Torrecillas in [**MT1**, Thm. 6.1], when ρ is the canonical H-coaction on A.

Example 2.3. Another solution is listed in [**FR**, Thm. 5.0.2]. For this, take $\mathbf{A} = \mathbf{B} = \mathbf{F} = 0$, $\mathbf{C} = \mathbf{D} = 1$, $\alpha = 1$, $\beta = \gamma = 0$, $\mathfrak{a} = -1$, $\mathfrak{b} = 1$, $r' = 0$, $s' = 1$ and $s = -1$.

3. Summary on rt-separability

We recapitulate the results on rt-separability in the following tables

Rt-separability conditions

$\mathfrak{F}_i$	Casimir element	Explicit definition	rt-separability onditions
$\mathfrak{F}_2$	$\begin{cases} B(1_H \otimes 1_H) = 1_A \\ B(g \otimes g) = 1_A \\ B(1_H \otimes x) = \mathbf{Y} \\ B(g \otimes gx) = \mathbf{Y} \\ B(x \otimes g) = -\mathbf{Y} \\ B(gx \otimes 1_H) = -\mathbf{Y} \\ B(1_H \otimes g) = \mathbf{Z} \\ B(g \otimes 1_H) = \mathbf{Z} \\ B(1_H \otimes gx) = \mathbf{X} \\ B(g \otimes x) = \mathbf{X} \\ B(x \otimes 1_H) = \mathbf{X} \\ B(gx \otimes g) = \mathbf{X} \\ B(x \otimes x) = 0 \\ B(gx \otimes gx) = 0 \\ B(gx \otimes x) = 0 \\ B(x \otimes gx) = 0 \end{cases}$	$\begin{cases} \mathbf{X} = s\left(-\frac{\gamma}{2}\mathbf{D} + \mathbf{F}G - \mathbf{B}X + \mathbf{D}GX\right) \\ \mathbf{Y} = \frac{\gamma}{2}s'\mathbf{E} + (r'\mathbf{A} + s'\mathbf{C})G - r'\mathbf{E}X - s'\mathbf{E}GX \\ \mathbf{Z} = s\left(\frac{\gamma}{2}\mathfrak{a}\mathbf{E} + (\mathfrak{a}\mathbf{C} + \mathfrak{b}\mathbf{A})G - \mathfrak{b}\mathbf{E}X - \mathfrak{a}\mathbf{E}GX\right) \end{cases}$	$\begin{cases} \mathbf{E}(\alpha\mathbf{A}^2 + \beta\mathbf{E}^2 - \gamma\mathbf{AE} - \mathbf{C}^2)(r'\mathfrak{a} - s'\mathfrak{b}) = -\mathbf{AB} + \mathbf{CD} + \mathbf{EF} \\ r'(\mathbf{AB} - \mathbf{EF}) + s'\mathbf{BC} = 0 \\ 2r'\mathbf{CF} + s'[\gamma(\mathbf{CD} + \mathbf{EF}) - 2\beta\mathbf{BE}] = 0 \\ 2r'\mathbf{BC} + s'[2\alpha(\mathbf{CD} + \mathbf{EF}) - \gamma\mathbf{BE}] = 0 \\ \gamma\mathbf{AD} - 2\mathbf{CF} - 2\beta\mathbf{DE} = 0 \\ \gamma\mathbf{DE} + 2\mathbf{BC} - 2\alpha\mathbf{AD} = 0 \\ 2\alpha\mathbf{AF} + 2\beta\mathbf{BE} - \gamma(\mathbf{AB} + \mathbf{EF}) = 0 \\ \mathbf{AB} - \mathbf{CD} - \mathbf{EF} \neq 0 \\ \mathfrak{a}, \mathfrak{b} \in k, \mathbf{E} \neq 0 \end{cases}$
		$\begin{cases} \mathbf{X} = s\left(-\frac{\gamma}{2}\mathbf{D} + \mathbf{F}G - \mathbf{B}X + \mathbf{D}GX\right) \\ \mathbf{Y} = s'\frac{\gamma}{2}\mathbf{A} + r'G + s'\mathbf{C}X - s'\mathbf{A}GX \\ \mathbf{Z} = s\left(\frac{\gamma}{2}\mathfrak{a}\mathbf{A} + \mathfrak{b}G + \mathfrak{a}\mathbf{C}X - \mathfrak{a}\mathbf{A}GX\right) \end{cases}$	$\begin{cases} (\alpha\mathbf{A}^2 - \mathbf{C}^2)(r'\mathfrak{a} - s'\mathfrak{b}) = -\mathbf{AB} + \mathbf{CD} \\ 2\alpha r' + \gamma s'\mathbf{C} = 0 \\ \gamma r' + 2\beta s'\mathbf{C} = 0 \\ r'\mathbf{B} + s'\mathbf{CF} = 0 \\ \gamma\mathbf{F} - 2\beta\mathbf{B} = 0 \\ \gamma\mathbf{AD} - 2\mathbf{CF} = 0 \\ \mathbf{BC} - \alpha\mathbf{AD} = 0 \\ \gamma\mathbf{B} - 2\alpha\mathbf{F} = 0 \\ \mathbf{AB} - \mathbf{CD} \neq 0 \\ \mathfrak{a}, \mathfrak{b} \in k, \mathbf{E} = 0 \end{cases}$

Rth-separability conditions

$\mathfrak{F}_i$	Casimir element	Explicit definition	rth-separability conditions
$\mathfrak{F}_2$	$\begin{cases} B(1_H \otimes 1_H) = 1_A \\ B(g \otimes g) = 1_A \\ B(1_H \otimes x) = \mathbf{Y} \\ B(g \otimes gx) = \mathbf{Y} \\ B(x \otimes g) = -\mathbf{Y} \\ B(gx \otimes 1_H) = -\mathbf{Y} \\ B(1_H \otimes g) = \mathbf{Z} \\ B(g \otimes 1_H) = \mathbf{Z} \\ B(1_H \otimes gx) = \mathbf{X} \\ B(g \otimes x) = \mathbf{X} \\ B(x \otimes 1_H) = \mathbf{X} \\ B(gx \otimes g) = \mathbf{X} \\ B(x \otimes x) = 0 \\ B(gx \otimes gx) = 0 \\ B(gx \otimes x) = 0 \\ B(x \otimes gx) = 0 \end{cases}$	$\begin{cases} \mathbf{X} = s\left(-\frac{\gamma}{2}\mathbf{D} + \mathbf{F}G - \mathbf{B}X + \mathbf{D}GX\right) \\ \mathbf{Y} = \frac{\gamma}{2}s'\mathbf{E} + (r'\mathbf{A} + s'\mathbf{C})G - r'\mathbf{E}X - s'\mathbf{E}GX \\ \mathbf{Z} = s\left(\frac{\gamma}{2}\mathfrak{a}\mathbf{E} + (\mathfrak{a}\mathbf{C} + \mathfrak{b}\mathbf{A})G - \mathfrak{b}\mathbf{E}X - \mathfrak{a}\mathbf{E}GX\right) \end{cases}$	$\begin{cases} r'(2\beta\mathbf{E} - \gamma\mathbf{A}) - s'\gamma\mathbf{C} = 0 \\ r'(2\alpha\mathbf{A} - \gamma\mathbf{E}) + 2s'\alpha\mathbf{C} = 0 \\ \alpha\mathbf{A}^2 + \beta\mathbf{E}^2 - \gamma\mathbf{A}\mathbf{E} - \mathbf{C}^2 \neq 0 \\ \mathbf{E} \neq 0 \\ r'\mathfrak{a} - s'\mathfrak{b} \neq 0 \\ s^2(\alpha(\mathfrak{a}\mathbf{C} + \mathfrak{b}\mathbf{A})^2 + \beta\mathfrak{b}^2\mathbf{E}^2 - \gamma\mathfrak{b}(\mathfrak{a}\mathbf{C} + \mathfrak{b}\mathbf{A})\mathbf{E}) = 1 \\ \mathbf{F} = \frac{\mathbf{E}}{2}(2\beta\mathbf{E} - \gamma\mathbf{A})(r'\mathfrak{a} - s'\mathfrak{b}) \\ \mathbf{B} = -\frac{\mathbf{E}}{2}(2\alpha\mathbf{A} - \gamma\mathbf{E})(r'\mathfrak{a} - s'\mathfrak{b}) \\ \mathbf{D} = -\mathbf{C}\mathbf{E}(r'\mathfrak{a} - s'\mathfrak{b}). \end{cases}$
		$\begin{cases} \mathbf{X} = s\left(-\frac{\gamma}{2}\mathbf{D} + \mathbf{F}G - \mathbf{B}X + \mathbf{D}GX\right) \\ \mathbf{Y} = s'\frac{\gamma}{2}\mathbf{A} + r'G + s'\mathbf{C}X - s'\mathbf{A}GX \\ \mathbf{Z} = s\left(\frac{\gamma}{2}\mathfrak{a}\mathbf{A} + \mathfrak{b}G + \mathfrak{a}\mathbf{C}X - \mathfrak{a}\mathbf{A}GX\right) \end{cases}$	$\begin{cases} 2\alpha r' + \gamma s'\mathbf{C} = 0 \\ \gamma r' + 2\beta s'\mathbf{C} = 0 \\ \alpha\mathbf{A}^2 - \mathbf{C}^2 \neq 0 \\ r'\mathfrak{a} - s'\mathfrak{b} \neq 0 \\ s^2\left(\alpha\mathfrak{b}^2 + \beta\mathfrak{a}^2\mathbf{C}^2 + \gamma\mathfrak{a}\mathfrak{b}\mathbf{C}\right) = 1 \\ \mathbf{F} = -\frac{\gamma}{2}\mathbf{A}(r'\mathfrak{a} - s'\mathfrak{b}) \\ \mathbf{B} = -\alpha\mathbf{A}(r'\mathfrak{a} - s'\mathfrak{b}) \\ \mathbf{D} = -\mathbf{C}(r'\mathfrak{a} - s'\mathfrak{b}). \end{cases}$

4. Frobenius property

As a coalgebra in the monoidal category $\mathcal{T}_A^{\#}$, a cowreath can respect a kind of Frobenius property that is a natural generalization of that for classical (co)algebras. The definition of Frobenius coalgebra was introduced by dualization in [**BCT1**] and then extended to cowreaths in [**BCT2**, p.240]. In [**BT2**, Example 6.8] a classification of Frobenius cowreaths $(A \otimes H^{op}, H, \psi)$ was given for $A = Cl(\alpha, \beta, \gamma)$ and H Sweedler's Hopf algebra, when A is endowed with the canonical H-comodule algebra structure we can find in literature. In this section we extend the result of Bulacu and Torrecillas by determining all the H-coactions that ensure that the cowreath $(A \otimes H^{op}, H, \psi)$ is Frobenius.

To check for which cases the previous cowreath is Frobenius we are going to use Proposition 6.7 contained in [**BT2**]. It involves the linear map $\mu \in H^*$ (called the modular element) defined by $(x + gx)h = \mu(h)(x + gx)$ for all $h \in H$. It is not hard to see that, in our case, $\mu(1) = 1$, $\mu(g) = -1$ and $\mu(x) = \mu(gx) = 0$, i.e. $\mu = 1^* - g^*$. The statement of the Proposition is the following.

PROPOSITION 4.1. [**BT2**, Prop. 6.7] *If A is a right H-comodule algebra then the cowreath $(A \otimes H^{op}, H, \psi)$ is Frobenius if, and only if, there exists an invertible element $\mathcal{A} \in A$ such that*

$$a\mathcal{A} = \mu(a_1)\mathcal{A}a_0 \tag{191}$$

for all $a \in A$.

First of all we notice that, in view of Theorem 1.8, (191) is equivalent to

$$a\mathcal{A} = \mu\left(\frac{1+g}{2}\right)\mathcal{A}a + \mu\left(\frac{1-g}{2}\right)\mathcal{A}\varphi(a) - \mu\left(\frac{(1-g)x}{2}\right)\mathcal{A}d(a) + \mu\left(\frac{(1+g)x}{2}\right)\mathcal{A}d(\varphi(a)),$$

that is

$$a\mathcal{A} = \mu\left(\frac{1+g}{2}\right)\mathcal{A}a + \mu\left(\frac{1-g}{2}\right)\mathcal{A}\varphi(a) = \mathcal{A}\varphi(a). \tag{192}$$

It is enough to verify (192) on generators G, X. Therefore we need to check for which φ there exists an $\mathcal{A}$ such that

$$G\mathcal{A} = \mathcal{A}\varphi(G) \tag{193}$$

$$X\mathcal{A} = \mathcal{A}\varphi(X). \tag{194}$$

As usal we proceed by cases.

4.1. $\varphi = \mathbf{Id}$**.** When we consider $\varphi = \text{Id}$, we can take $\mathcal{A} = 1_A$ and deduce immediately that $(A \otimes H^{op}, H, \psi)$ is a Frobenius cowreath.

4.2. $\varphi \sim D_1$**.** Next we take φ with matrix of type $\mathfrak{F}_1$. Recall that in this case

$$\begin{cases} \gamma^2 - 4\alpha\beta = 0 \\ \alpha\mathbf{B}^2 + \beta\mathbf{A}^2 - \gamma\mathbf{A}\mathbf{B} = 1 \end{cases}$$

hold. Equalities (193) and (194) are equivalent to

$$(L_G - R_{\varphi(G)})\mathcal{A} = 0 \qquad \text{and} \qquad (L_X - R_{\varphi(X)})\mathcal{A} = 0,$$

and, by applying $D_1 P^{-1}$ on the left, to

$$(D_1 P^{-1} L_G P - P^{-1} R_G P D_1) P^{-1}\mathcal{A} = 0 \tag{195}$$

$$(D_1 P^{-1} L_X P - P^{-1} R_X P D_1) P^{-1}\mathcal{A} = 0. \tag{196}$$

Since

$$D_1 P^{-1} L_G P - P^{-1} R_G P D_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\mathbf{A}}{2} & \gamma\mathbf{A} - 2\alpha\mathbf{B} & 0 \\ \frac{\mathbf{A}}{2} & 0 & 0 & \gamma\mathbf{A} - 2\alpha\mathbf{B} \end{pmatrix}$$

and

$$D_1 P^{-1} L_X P - P^{-1} R_X P D_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\mathbf{B}}{2} & 2\beta\mathbf{A} - \gamma\mathbf{B} & 0 \\ \frac{\mathbf{B}}{2} & 0 & 0 & 2\beta\mathbf{A} - \gamma\mathbf{B} \end{pmatrix}$$

we finally see that (195) and (196) yield

$$\mu_2 \frac{\mathbf{A}}{2} + \mu_3(\gamma\mathbf{A} - 2\alpha\mathbf{B}) = 0, \tag{197}$$

$$\mu_1 \frac{\mathbf{A}}{2} + \mu_4(\gamma\mathbf{A} - 2\alpha\mathbf{B}) = 0, \tag{198}$$

$$\mu_2 \frac{\mathbf{B}}{2} + \mu_3(2\beta\mathbf{A} - \gamma\mathbf{B}) = 0, \tag{199}$$

$$\mu_1 \frac{\mathbf{B}}{2} + \mu_4(2\beta\mathbf{A} - \gamma\mathbf{B}) = 0, \tag{200}$$

where $P^{-1}\mathcal{A} = \mu_1 + \mu_2 G + \mu_3 X + \mu_4 GX$. Observe that (197) and (199) can be rewritten as

$$\begin{pmatrix} \frac{\mathbf{A}}{2} & \gamma\mathbf{A} - 2\alpha\mathbf{B} \\ \frac{\mathbf{B}}{2} & 2\beta\mathbf{A} - \gamma\mathbf{B} \end{pmatrix} \begin{pmatrix} \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We have $\frac{\mathbf{A}}{2}(2\beta\mathbf{A} - \gamma\mathbf{B}) - \frac{\mathbf{B}}{2}(\gamma\mathbf{A} - 2\alpha\mathbf{B}) = \beta\mathbf{A}^2 - \gamma\mathbf{A}\mathbf{B} + \alpha\mathbf{B}^2 = 1 \neq 0$, therefore (197) and (199) force $\mu_2 = \mu_3 = 0$. Likewise it can be shown that (198) and (200) imply $\mu_1 = \mu_4 = 0$. This means that $\mathcal{A}$ must be zero and cannot be invertible: the cowreath $(A \otimes H^{op}, H, \psi)$ is never Frobenius when ρ is defined via a couple (φ, d) with M_φ similar to D_1.

4.3. $\varphi \sim D_2$**.** To conclude, let us consider the case when φ has matrix of type $\mathfrak{F}_2$. In this case M_φ must be such that

$$\begin{cases} \gamma\mathbf{AD} - 2\mathbf{CF} - 2\beta\mathbf{DE} = 0 \\ \gamma\mathbf{DE} + 2\mathbf{BC} - 2\alpha\mathbf{AD} = 0 \\ 2\alpha\mathbf{AF} + 2\beta\mathbf{BE} - \gamma\mathbf{AB} - \gamma\mathbf{EF} = 0 \\ -\mathbf{AB} + \mathbf{CD} + \mathbf{EF} \neq 0. \end{cases}$$

We proceed by steps.

1. Assume $\gamma\mathbf{B} - 2\alpha\mathbf{F} \neq 0$. Then from the first table in Sec. 5, Ch. 2 we see that necessarily $\mathbf{E} \neq 0$. If we take $\mathcal{A} = -\frac{\gamma}{2}\mathbf{D} + \mathbf{F}G - \mathbf{B}X + \mathbf{D}GX$ we get that (195) and (196) give

$$\begin{aligned} 2\mathbf{BC} - 2\alpha\mathbf{AD} + \gamma\mathbf{DE} &= 0 \\ 2\mathbf{CF} - \gamma\mathbf{AD} + 2\beta\mathbf{DE} &= 0, \end{aligned}$$

that are clearly satisfied. Notice that in this case

$$|\mathcal{A}| = -\frac{\gamma^2}{4}\mathbf{D}^2 - \alpha\mathbf{F}^2 + \gamma\mathbf{BF} - \beta\mathbf{B}^2 + \alpha\beta\mathbf{D}^2 = \frac{1}{2\mathbf{E}}(\gamma\mathbf{B} - 2\alpha\mathbf{F})(-\mathbf{AB} + \mathbf{CD} + \mathbf{EF}) \neq 0,$$

thus $\mathcal{A}$ is invertible[2]. Therefore when $\gamma\mathbf{B} - 2\alpha\mathbf{F} \neq 0$ the cowreath $(A \otimes H^{op}, h, \psi)$ is always Frobenius.

Example 4.2. An example is given by $\mathbf{A} = \mathbf{B} = \mathbf{C} = \mathbf{D} = 0$, $\mathbf{E} = \mathbf{F} = 1$, $\alpha = 1$, $\beta = \gamma = 0$. Notice that there is no restriction on the choice of $\mathfrak{a}, \mathfrak{b}$, therefore we choose them to be both zero. The H-coaction is given explicitly by

$$\begin{aligned} \rho(G) &= G \otimes 1_H, \\ \rho(X) &= X \otimes g. \end{aligned}$$

The distinguished element is $\mathcal{A} = G$.

2. Suppose $\gamma\mathbf{B} - 2\alpha\mathbf{F} = 0$ and $\mathbf{E} \neq 0$. Then

$$\begin{cases} \gamma\mathbf{AD} - 2\mathbf{CF} - 2\beta\mathbf{DE} = 0 \\ \gamma\mathbf{DE} + 2\mathbf{BC} - 2\alpha\mathbf{AD} = 0 \\ 2\beta\mathbf{B} - \gamma\mathbf{F} = 0 \\ -\mathbf{AB} + \mathbf{CD} + \mathbf{EF} \neq 0. \end{cases} \tag{201}$$

By setting $P^{-1}\mathcal{A} = \mu_1 + \mu_2 G + \mu_3 X + \mu_4 GX$, (195) and (196) become equivalent to

$$2\mu_3\alpha\mathbf{A} - \mu_3\gamma\mathbf{E} + 2\mu_4\alpha\mathbf{C} = 0 \tag{202}$$
$$2\mu_1\mathbf{B} + \mu_2\gamma\mathbf{BD} - 2\mu_3\mathbf{BC} - 2\mu_4\alpha\mathbf{CD} = 0 \tag{203}$$
$$2\mu_1\mathbf{D} + \mu_2\gamma\mathbf{D}^2 - 2\mu_3(\mathbf{AB} - \mathbf{EF}) - 2\mu_4\mathbf{BC} = 0 \tag{204}$$
$$\mu_3\gamma\mathbf{A} - 2\mu_3\beta\mathbf{E} + \mu_4\gamma\mathbf{C} = 0 \tag{205}$$
$$2\mu_1(\mathbf{CD} + \mathbf{EF}) + \mu_2\gamma\mathbf{D}(\mathbf{CD} + \mathbf{EF}) - 2\mu_3\mathbf{ABC} - \mu_4\mathbf{C}(\gamma\mathbf{DE} + 2\mathbf{BC}) = 0. \tag{206}$$

Notice that we can use (202) to rewrite (203) and (204), and (205) to rewrite (206). We find

$$\mathbf{B}(2\mu_1 + \mu_2\gamma\mathbf{D}) = 0, \quad \text{and} \quad \mathbf{F}(2\mu_1 + \mu_2\gamma\mathbf{D}) = 0.$$

It is straightforward that $2\mu_1 + \mu_2\gamma\mathbf{D} = 0$, otherwise $\mathbf{B} = \mathbf{F} = 0$ and (204) forces $\mathbf{D} = 0$, contradiction. Condition (202)-(206) change into

$$2\mu_3\alpha\mathbf{A} - \mu_3\gamma\mathbf{E} + 2\mu_4\alpha\mathbf{C} = 0 \tag{207}$$
$$2\mu_1 + \mu_2\gamma\mathbf{D} = 0 \tag{208}$$
$$-2\mu_3(\mathbf{AB} - \mathbf{EF}) - 2\mu_4\mathbf{BC} = 0 \tag{209}$$
$$\mu_3\gamma\mathbf{A} - 2\mu_3\beta\mathbf{E} + \mu_4\gamma\mathbf{C} = 0. \tag{210}$$

In this case $\mathcal{A} = \frac{1}{2}\gamma(-\mu_2\mathbf{D} + \mu_4\mathbf{E}) + (\mu_2\mathbf{F} + \mu_3\mathbf{A} + \mu_4\mathbf{C})G - (\mu_2\mathbf{B} + \mu_3\mathbf{E})X + (\mu_2\mathbf{D} - \mu_4\mathbf{E})GX$ and

[2] We are using notation of Proposition 0.5.

$$\begin{aligned}
|\mathcal{A}| &= \left(\alpha\beta - \frac{1}{4}\gamma^2\right)(-\mu_2\mathbf{D}+\mu_4\mathbf{E})^2 - \alpha(\mu_2\mathbf{F}+\mu_3\mathbf{A}+\mu_4\mathbf{C})^2 + \gamma(\mu_2\mathbf{F}+\mu_3\mathbf{A}+\mu_4\mathbf{C})(\mu_2\mathbf{B}+\mu_3\mathbf{E}) + \\
&- \beta(\mu_2\mathbf{B}+\mu_3\mathbf{E})^2 \\
&\overset{(210)}{=} \left(\alpha\beta - \frac{1}{4}\gamma^2\right)(-\mu_2\mathbf{D}+\mu_4\mathbf{E})^2 - \alpha(\mu_2\mathbf{F}+\mu_3\mathbf{A}+\mu_4\mathbf{C})^2 + (\mu_2\gamma\mathbf{F}+2\mu_3\beta\mathbf{E})(\mu_2\mathbf{B}+\mu_3\mathbf{E}) + \\
&- \beta(\mu_2\mathbf{B}+\mu_3\mathbf{E})^2 \\
&= \left(\alpha\beta - \frac{1}{4}\gamma^2\right)(-\mu_2\mathbf{D}+\mu_4\mathbf{E})^2 - \alpha(\mu_2\mathbf{F}+\mu_3\mathbf{A}+\mu_4\mathbf{C})^2 + \beta(\mu_2\mathbf{B}+\mu_3\mathbf{E})^2 \\
&\overset{(207)}{=} \left(\alpha\beta - \frac{1}{4}\gamma^2\right)(-\mu_2\mathbf{D}+\mu_4\mathbf{E})^2 - \frac{1}{2}(2\mu_2\alpha\mathbf{F}+\mu_3\gamma\mathbf{E})(\mu_2\mathbf{F}+\mu_3\mathbf{A}+\mu_4\mathbf{C}) + \beta(\mu_2\mathbf{B}+\mu_3\mathbf{E})^2 \\
&\overset{2\alpha\mathbf{F}=\gamma\mathbf{B}}{=} \left(\alpha\beta - \frac{1}{4}\gamma^2\right)(-\mu_2\mathbf{D}+\mu_4\mathbf{E})^2 - \frac{1}{2}(\mu_2\mathbf{B}+\mu_3\mathbf{E})(\mu_2\gamma\mathbf{F}+\mu_3\gamma\mathbf{A}+\mu_4\gamma\mathbf{C}) + \beta(\mu_2\mathbf{B}+\mu_3\mathbf{E})^2 \\
&\overset{(210)+(201)}{=} \left(\alpha\beta - \frac{1}{4}\gamma^2\right)(-\mu_2\mathbf{D}+\mu_4\mathbf{E})^2 - \frac{1}{2}(\mu_2\mathbf{B}+\mu_3\mathbf{E})(2\mu_2\beta\mathbf{B}+2\mu_3\beta\mathbf{E}) + \beta(\mu_2\mathbf{B}+\mu_3\mathbf{E})^2 \\
&= \left(\alpha\beta - \frac{1}{4}\gamma^2\right)(-\mu_2\mathbf{D}+\mu_4\mathbf{E})^2.
\end{aligned}$$

We see that $\mathcal{A}$ is invertible if, and only if, $\left(\alpha\beta - \frac{1}{4}\gamma^2\right)(-\mu_2\mathbf{D}+\mu_4\mathbf{E})^2 \neq 0$. Since this forces $\gamma^2 - 4\alpha\beta \neq 0$, it is clear that conditions

$$\begin{cases} \gamma\mathbf{B} - 2\alpha\mathbf{F} = 0 \\ 2\beta\mathbf{B} - \gamma\mathbf{F} = 0 \end{cases}$$

imply $\mathbf{B} = \mathbf{F} = 0$. One look at (201) makes it clear that $\mathbf{C} \neq 0 \neq \mathbf{D}$, $\gamma\mathbf{A} - 2\beta\mathbf{E} = 0$ and $\gamma\mathbf{E} - 2\alpha\mathbf{A} = 0$. Then, given that $\gamma^2 - 4\alpha\beta \neq 0$, we must have $\mathbf{A} = \mathbf{E} = 0$, but this is a contradiction ($\mathbf{E} \neq 0$). Therefore when $\gamma\mathbf{B} - 2\alpha\mathbf{F} = 0$ and $\mathbf{E} \neq 0$, the cowreath $(A \otimes H^{op}, h, \psi)$ is never Frobenius.

3. Finally assume $\gamma\mathbf{B} - 2\alpha\mathbf{F} = 0$ and $\mathbf{E} = 0$. Then

$$\begin{cases} \gamma\mathbf{AD} - 2\mathbf{CF} = 0 \\ \mathbf{BC} - \alpha\mathbf{AD} = 0 \\ \gamma\mathbf{B} - 2\alpha\mathbf{F} = 0 \\ \mathbf{AB} - \mathbf{CD} \neq 0. \end{cases} \tag{211}$$

By setting $P^{-1}\mathcal{A} = \mu_1 + \mu_2 G + \mu_3 X + \mu_4 GX$, (195) and (196) become equivalent to

$$2\mu_3\alpha + \mu_4\gamma\mathbf{C} = 0 \tag{212}$$
$$2\mu_1(\mathbf{AB} - \mathbf{CD}) - \mu_2\mathbf{C}(\gamma\mathbf{D}^2 - 2\mathbf{BF}) = 0 \tag{213}$$
$$\mu_3\gamma + 2\mu_4\beta\mathbf{C} = 0 \tag{214}$$
$$2\mu_1\mathbf{D} + \mu_2\gamma\mathbf{D}^2 - 2\mu_3\mathbf{B} - 2\mu_4\mathbf{CF} = 0 \tag{215}$$
$$2\mu_1\mathbf{AF} + 2\mu_2\mathbf{CF}^2 - \mu_4\mathbf{A}^2(\gamma\mathbf{F} - 2\beta\mathbf{B}) = 0. \tag{216}$$

Since $\gamma\mathbf{AD} = 2\mathbf{CF}$ we can rewrite (213) as

$$2\mu_1(\mathbf{AB} - \mathbf{CD}) - \mu_2\gamma(\mathbf{CD} - \mathbf{AB})\mathbf{D} = 0,$$

which is equivalent to $2\mu_1 + \mu_2\gamma\mathbf{D} = 0$. Condition (212)-(216) change into

$$2\mu_3\alpha + \mu_4\gamma\mathbf{C} = 0 \tag{217}$$
$$2\mu_1 + \mu_2\gamma\mathbf{D} = 0 \tag{218}$$
$$\mu_3\gamma + 2\mu_4\beta\mathbf{C} = 0 \tag{219}$$
$$2\mu_3\mathbf{B} + \mu_4\gamma\mathbf{AD} = 0 \tag{220}$$
$$-\mu_4\mathbf{A}(\gamma\mathbf{F} - 2\beta\mathbf{B}) = 0. \tag{221}$$

In this case $\mathcal{A} = \frac{1}{2}\gamma(-\mu_2\mathbf{D} + \mu_4\mathbf{A}) + (\mu_2\mathbf{F} + \mu_3)G - (\mu_2\mathbf{B} - \mu_4\mathbf{C})X + (\mu_2\mathbf{D} - \mu_4\mathbf{A})GX$

$$\begin{aligned}
|\mathcal{A}| &= \left(\alpha\beta - \frac{1}{4}\gamma^2\right)(-\mu_2\mathbf{D} + \mu_4\mathbf{A})^2 - \alpha(\mu_2\mathbf{F} + \mu_3)^2 + \gamma(\mu_2\mathbf{F} + \mu_3)(\mu_2\mathbf{B} - \mu_4\mathbf{C}) - \beta(\mu_2\mathbf{B} - \mu_4\mathbf{C})^2 \\
&= \left(\alpha\beta - \frac{1}{4}\gamma^2\right)(-\mu_2\mathbf{D} + \mu_4\mathbf{A})^2 - (\mu_2\mathbf{F} + \mu_3)(\mu_2\alpha\mathbf{F} + \mu_3\alpha - \mu_2\gamma\mathbf{B} + \mu_4\gamma\mathbf{C}) - \beta(\mu_2\mathbf{B} - \mu_4\mathbf{C})^2 \\
&\overset{(217)+(2\alpha\mathbf{F}=\gamma\mathbf{B})}{=} \left(\alpha\beta - \frac{1}{4}\gamma^2\right)(-\mu_2\mathbf{D} + \mu_4\mathbf{A})^2 - \frac{1}{2}(\mu_2\mathbf{F} + \mu_3)(-\mu_2\gamma\mathbf{B} + \mu_4\gamma\mathbf{C}) - \beta(\mu_2\mathbf{B} - \mu_4\mathbf{C})^2 \\
&\overset{(219)}{=} \left(\alpha\beta - \frac{1}{4}\gamma^2\right)(-\mu_2\mathbf{D} + \mu_4\mathbf{A})^2 + \frac{1}{2}(\mu_2\mathbf{B} - \mu_4\mathbf{C})(\mu_2\gamma\mathbf{F} - 2\beta\mu_2\mathbf{B}) \\
&\overset{(211)}{=} \left(\alpha\beta - \frac{1}{4}\gamma^2\right)(-\mu_2\mathbf{D} + \mu_4\mathbf{A})^2 + \frac{1}{2}\mu_2^2(\gamma\mathbf{B}\mathbf{F} - 2\beta\mathbf{B}^2) - \frac{1}{4}\mu_4\mu_2(\gamma^2 - 4\alpha\beta)\mathbf{A}\mathbf{D} \\
&= \left(\alpha\beta - \frac{1}{4}\gamma^2\right)(\mu_2\mathbf{D} + \mu_4\mathbf{A})^2 + \frac{1}{2}\mu_2^2\mathbf{B}(\gamma\mathbf{F} - 2\beta\mathbf{B}).
\end{aligned}$$

i) If we suppose $\mathbf{A} = 0$, then (211) gives $\mathbf{C} \neq 0 \neq \mathbf{D}$ and $\mathbf{B} = \mathbf{F} = 0$. Then $\mathcal{A}$ is invertible if, and only if, $\mu_2(\gamma^2 - 4\alpha\beta) \neq 0$. One can take $\mu_2 = 1$, $\mu_3 = \mu_4 = 0$, i.e. $\mathcal{A} = -\frac{\gamma}{2}\mathbf{D} + \mathbf{D}GX$ to show that the cowreath $(A \otimes H^{op}, h, \psi)$ is Frobenius, provided $\gamma^2 - 4\alpha\beta \neq 0$. On the other hand, if $\gamma^2 - 4\alpha\beta = 0$ any $\mathcal{A}$ satisfying (217)-(221) is not invertible.
ii) Now let $\mathbf{A} \neq 0$ and suppose $\gamma\mathbf{F} - 2\beta\mathbf{B} = 0$. Then $|\mathcal{A}| = \left(\alpha\beta - \frac{1}{4}\gamma^2\right)(\mu_2\mathbf{D} + \mu_4\mathbf{A})^2$, therefore we must have $\gamma^2 - 4\alpha\beta \neq 0$ in order for $\mathcal{A}$ to be invertible. Since $\gamma\mathbf{F} - 2\beta\mathbf{B} = 0$ and $\gamma\mathbf{B} - 2\alpha\mathbf{F} = 0$, we must have $\mathbf{B} = \mathbf{F} = 0$. Hence (211) forces $\mathbf{C} \neq 0 \neq \mathbf{D}$ and $\alpha = \gamma = 0$, which is a contradiction.
iii) Finally let $\mathbf{A} \neq 0 \neq \gamma\mathbf{F} - 2\beta\mathbf{B}$. Conditions (217)-(221) give $\mu_4 = \mu_3 = 0$ (otherwise $\gamma = \mathbf{B} = 0$, contradiction). Then one can verify that $|\mathcal{A}| = \mu_2^2\frac{(\mathbf{AB}-\mathbf{CD})(\gamma\mathbf{F}-2\beta\mathbf{B})}{2\mathbf{A}} \neq 0$ and therefore, by choosing $\mu_2 = 1$ one concludes that the cowreath $(A \otimes H^{op}, h, \psi)$ is Frobenius under the following conditions:

$$\begin{cases} \gamma\mathbf{A}\mathbf{D} - 2\mathbf{C}\mathbf{F} = 0 \\ \mathbf{B}\mathbf{C} - \alpha\mathbf{A}\mathbf{D} = 0 \\ \gamma\mathbf{B} - 2\alpha\mathbf{F} = 0 \\ \mathbf{A}\mathbf{B} - \mathbf{C}\mathbf{D} \neq 0 \\ \mathbf{A} \neq 0 \\ \gamma\mathbf{F} - 2\beta\mathbf{B} \neq 0. \end{cases}$$

Remark 4.3. Notice that case i) is exactly the one described in [**BT2**, Example 6.8].

Example 4.4. An example in case i) is given by $\mathbf{A} = \mathbf{B} = \mathbf{E} = \mathbf{F} = 0$, $\mathbf{C} = \mathbf{D} = 1$, $\alpha = \beta = 0$, $\gamma = 1$. Once again $\mathfrak{a}$, $\mathfrak{b}$ can be chosen freely, so we choose them both to be zero. The H-coaction is given explicitly by

$$\begin{aligned} \rho(G) &= G \otimes g, \\ \rho(X) &= X \otimes g. \end{aligned}$$

The distinguished element is $\mathcal{A} = -\frac{1}{2} + GX$.

Example 4.5. An example in case iii) is given by $\mathbf{A} = \mathbf{B} = 1$, $\mathbf{C} = \mathbf{D} = \mathbf{F} = 0$, $\alpha = \gamma = 0$, $\beta = 1$. Again $\mathfrak{a}$, $\mathfrak{b}$ can be chosen freely, so we choose them both to be zero. The H-coaction is given explicitly by

$$\begin{aligned} \rho(G) &= G \otimes g, \\ \rho(X) &= X \otimes 1_H. \end{aligned}$$

The distinguished element is $\mathcal{A} = -X$.

Remark 4.6. Notice that whether the cowreath $(A \otimes H^{op}, H, \psi)$ is Frobenius or not, does not depend on the choice of d in the definition of the H-coaction. This is made clear by the form in which (193) and (194) appear.

5. Summary on Frobenius property

We recapitulate the results on Frobenius property in the following table

Frobenius cowreaths

$\mathfrak{F}_i$	Distinguished element $\mathcal{A}$	Conditions
Id	$\mathcal{A} = 1_A$	
$\mathfrak{F}_2$	$\mathcal{A} = -\frac{\gamma}{2}\mathbf{D} + \mathbf{F}G - \mathbf{B}X + \mathbf{D}GX$	$\begin{cases} \gamma\mathbf{AD} - 2\mathbf{CF} - 2\beta\mathbf{DE} = 0 \\ \gamma\mathbf{DE} + 2\mathbf{BC} - 2\alpha\mathbf{AD} = 0 \\ 2\alpha\mathbf{AF} + 2\beta\mathbf{BE} - \gamma\mathbf{AB} - \gamma\mathbf{EF} = 0 \\ -\mathbf{AB} + \mathbf{CD} + \mathbf{EF} \neq 0 \\ \gamma\mathbf{B} - 2\alpha\mathbf{F} \neq 0 \\ \mathbf{E} \neq 0. \end{cases}$
		$\begin{cases} \mathbf{C}, \mathbf{D} \neq 0 \\ \mathbf{A} = \mathbf{B} = \mathbf{E} = \mathbf{F} = 0 \\ \gamma^2 - 4\alpha\beta \neq 0. \end{cases}$
		$\begin{cases} \gamma\mathbf{AD} - 2\mathbf{CF} = 0 \\ \mathbf{BC} - \alpha\mathbf{AD} = 0 \\ \gamma\mathbf{B} - 2\alpha\mathbf{F} = 0 \\ \mathbf{AB} - \mathbf{CD} \neq 0 \\ \mathbf{A} \neq 0 \\ \gamma\mathbf{F} - 2\beta\mathbf{B} \neq 0 \end{cases}$

REMARK 5.1. We see that any Clifford algebra can be used to construct a Frobenius cowreath $(A \otimes H^{op}, H, \psi)$, using the trivial comodule algebra structure $\rho(a) = a \otimes 1_H$, for every $a \in A$. On the other hand this cowreath is not rt-separable. There is in fact a discrepancy between Frobenius cowreaths and separable cowreaths since in order for the cowreath $(A \otimes H^{op}, H, \psi)$ to be Frobenius, $\gamma^2 - 4\alpha\beta \neq 0$ is a necessary condition, with the exception of a few cases. This is shown in Theorem 7.1.

6. Semisimplicity and rt-separability

In this section we give a negative answer to Question 6.6. This result is contained in Theorem 6.1 and it is used to prove the much more general statement of Theorem 6.2 which describes the interaction between semisimplicity of the algebra $A = Cl(\alpha, \beta, \gamma)$ and the rt(h)-separability of the cowreath $(A \otimes H^{op}, H, \psi)$.

Let us consider A the generalized quaternion algebra over the field k. This is the four-dimensional k-vector space with basis $1, i, j, ij$ endowed with the algebra structure given by

$$i^2 = a, \quad j^2 = b, \quad ij = -ji,$$

where a and b are invertible elements of the field k. Thus A is exactly the Clifford algebra $Cl(a, b, 0)$. Suppose A is given an H-comodule algebra structure by (φ, d) with $\mathbf{E} \neq 0$. Then (φ, d) is such that

$$\begin{cases} \mathbf{CF} + b\mathbf{DE} = 0 \\ \mathbf{BC} - a\mathbf{AD} = 0 \\ a\mathbf{AF} + b\mathbf{BE} = 0 \\ \mathbf{AB} - \mathbf{CD} - \mathbf{EF} \neq 0 \\ \mathbf{E}, a, b \neq 0 \\ \mathfrak{a}, \mathfrak{b} \in k. \end{cases}$$

Since $b, \mathbf{E} \neq 0$, from the first equality we get $\mathbf{D} = -\frac{\mathbf{CF}}{b\mathbf{E}}$ and from the third $\mathbf{B} = -\frac{a\mathbf{AF}}{b\mathbf{E}}$. $\mathbf{BC} - a\mathbf{AD} = 0$ is trivially satisfied. We see that $\mathbf{F} \neq 0$, otherwise $\mathbf{B} = \mathbf{D} = \mathbf{F} = 0$, contradiction. Thus the system reduces to

$$\begin{cases} \mathbf{D} = -\dfrac{\mathbf{CF}}{b\mathbf{E}} \\ \mathbf{B} = -\dfrac{a\mathbf{AF}}{b\mathbf{E}} \\ a\mathbf{A}^2 + b\mathbf{E}^2 - \mathbf{C}^2 \neq 0 \\ \mathbf{E}, \mathbf{F}, a, b \neq 0 \\ \mathfrak{a}, \mathfrak{b} \in k. \end{cases}$$

If we also impose rt-separability we need (φ, d) to satisfy

$$\begin{cases} b\mathbf{E}^2(r'\mathfrak{a} - s'\mathfrak{b}) = \mathbf{F} \\ r'(-a\mathbf{A}^2 - b\mathbf{E}^2) - s'a\mathbf{AC} = 0 \\ r'\mathbf{C} + s'a\mathbf{A} = 0 \\ -r'a\mathbf{AC} + s'a(-\mathbf{C}^2 + b\mathbf{E}^2) = 0 \\ \mathbf{D} = -\dfrac{\mathbf{CF}}{b\mathbf{E}} \\ \mathbf{B} = -\dfrac{a\mathbf{AF}}{b\mathbf{E}} \\ a\mathbf{A}^2 + b\mathbf{E}^2 - \mathbf{C}^2 \neq 0 \\ \mathbf{E}, \mathbf{F}, a, b \neq 0 \\ \mathfrak{a}, \mathfrak{b} \in k \end{cases}$$

If we use $r'\mathbf{C} + s'a\mathbf{A} = 0$ in the second and fourth equalities we get

$$r'(a\mathbf{A}^2 + b\mathbf{E}^2 - \mathbf{C}^2) = 0, \quad s'a(a\mathbf{A}^2 - \mathbf{C}^2 + b\mathbf{E}^2) = 0$$

which imply $r' = s' = 0$. This result clearly contradicts the first equation, therefore, when $A = Cl(a, b, 0)$, $a, b \in k^\times$ and $\mathbf{E} \neq 0$, the cowreath $(A \otimes H^{op}, H, \psi)$ is never rt-separable.

Now suppose A is given an H-comodule algebra structure (φ, d) with $\mathbf{E} = 0$. Then rt-separability is attained if, and only if,

$$\begin{cases} (a\mathbf{A}^2 - \mathbf{C}^2)(r'\mathfrak{a} - s'\mathfrak{b}) = -\mathbf{AB} + \mathbf{CD} \\ ar' = 0 \\ bs'\mathbf{C} = 0 \\ r'\mathbf{B} + s'\mathbf{CF} = 0 \\ b\mathbf{B} = 0 \\ \mathbf{CF} = 0 \\ \mathbf{BC} - a\mathbf{AD} = 0 \\ a\mathbf{F} = 0 \\ \mathbf{AB} - \mathbf{CD} \neq 0. \end{cases}$$

Since a, b are invertible we immediately get $\mathbf{F} = \mathbf{B} = r' = 0$. Then $\mathbf{C} \neq 0 \neq \mathbf{D}$ and so the third equality gives $s' = 0$, contradiction (see the first equation).

We have proved the following

THEOREM 6.1. *Let (A, ρ) be a generalized quaternion algebra, endowed with any H-comodule algebra structure. The cowreath $(A \otimes H^{op}, H, \psi)$ cannot be rt-separable.*

Actually this theorem can be regarded as a consequence of a much more general result – though here we use the former to prove the latter.

THEOREM 6.2. *Let $A = Cl(\alpha,\beta,\gamma)$ be a Clifford algebra endowed with an H-comodule algebra structure $\rho : A \to A\otimes H$. If the cowreath $(A\otimes H^{op}, H, \psi)$ is rt-separable, then $\gamma^2 - 4\alpha\beta = 0$.*

PROOF. Let us start by supposing $\gamma = 0$. In this case either α or β must be 0, because if they are both invertible then A is a generalized quaternion algebra and the cowreath $(A\otimes H^{op}, H, \psi)$ is not rt-separable, by Thm. 6.1. Thus if A is rt-separable and $\gamma = 0$ we must have $\gamma^2 - 4\alpha\beta = 0$.

Now assume $\gamma \neq 0$ and $\mathbf{E} = 0$. Let us show that $\gamma^2 - 4\alpha\beta \neq 0$ leads to a contradiction. In this case rt-separability implies that

$$\begin{cases} (\alpha\mathbf{A}^2 - \mathbf{C}^2)(r'\mathfrak{a} - s'\mathfrak{b}) = -\mathbf{AB} + \mathbf{CD} \\ 2\alpha r' + \gamma s'\mathbf{C} = 0 \\ \gamma r' + 2\beta s'\mathbf{C} = 0 \\ r'\mathbf{B} + s'\mathbf{CF} = 0 \\ \gamma\mathbf{F} - 2\beta\mathbf{B} = 0 \\ \gamma\mathbf{AD} - 2\mathbf{CF} = 0 \\ \mathbf{BC} - \alpha\mathbf{AD} = 0 \\ \gamma\mathbf{B} - 2\alpha\mathbf{F} = 0 \\ \mathbf{AB} - \mathbf{CD} \neq 0 \\ \mathfrak{a}, \mathfrak{b} \in k, \mathbf{E} = 0 \end{cases}$$

In particular $\gamma\mathbf{F} - 2\beta\mathbf{B} = 0$ and $\gamma\mathbf{B} - 2\alpha\mathbf{F} = 0$ give $\mathbf{B} = \mathbf{F} = 0$, because $\gamma^2 - 4\alpha\beta \neq 0$. $\mathbf{B} = 0$ forces $\mathbf{C} \neq 0 \neq \mathbf{D}$. Moreover $2\alpha r' + \gamma s'\mathbf{C} = 0$, $\gamma r' + 2\beta s'\mathbf{C} = 0$ yield $r' = s' = 0$, since $(\gamma^2 - 4\alpha\beta)\mathbf{C} \neq 0$, but this clearly contradicts

$$(\alpha\mathbf{A}^2 - \mathbf{C}^2)(r'\mathfrak{a} - s'\mathfrak{b}) = -\mathbf{AB} + \mathbf{CD} \neq 0,$$

thus we must have $\gamma^2 - 4\alpha\beta = 0$.

Finally suppose $\gamma \neq 0 \neq \mathbf{E}$ and let us show that $\gamma^2 - 4\alpha\beta \neq 0$ leads again to a contradiction. This time, rt-separability implies that

$$\begin{cases} \mathbf{E}(\alpha\mathbf{A}^2 + \beta\mathbf{E}^2 - \gamma\mathbf{AE} - \mathbf{C}^2)(r'\mathfrak{a} - s'\mathfrak{b}) = -\mathbf{AB} + \mathbf{CD} + \mathbf{EF} \\ r'(\mathbf{AB} - \mathbf{EF}) + s'\mathbf{BC} = 0 \\ 2r'\mathbf{CF} + s'[\gamma(\mathbf{CD} + \mathbf{EF}) - 2\beta\mathbf{BE}] = 0 \\ 2r'\mathbf{BC} + s'[2\alpha(\mathbf{CD} + \mathbf{EF}) - \gamma\mathbf{BE}] = 0 \\ \gamma\mathbf{AD} - 2\mathbf{CF} - 2\beta\mathbf{DE} = 0 \\ \gamma\mathbf{DE} + 2\mathbf{BC} - 2\alpha\mathbf{AD} = 0 \\ 2\alpha\mathbf{AF} + 2\beta\mathbf{BE} - \gamma(\mathbf{AB} + \mathbf{EF}) = 0 \\ \mathbf{AB} - \mathbf{CD} - \mathbf{EF} \neq 0 \\ \mathfrak{a}, \mathfrak{b} \in k, \mathbf{E} \neq 0. \end{cases} \tag{222}$$

Notice that, in view of the first equation, r' and s' cannot be both zero. The two equations

$$\begin{aligned} 2r'\mathbf{CF} + s'[\gamma(\mathbf{CD} + \mathbf{EF}) - 2\beta\mathbf{BE}] &= 0 \\ 2r'\mathbf{BC} + s'[2\alpha(\mathbf{CD} + \mathbf{EF}) - \gamma\mathbf{BE}] &= 0, \end{aligned}$$

admit a non-trivial solution $(r', s') \neq (0,0)$ if, and only if,

$$\mathbf{CF}[2\alpha(\mathbf{CD} + \mathbf{EF}) - \gamma\mathbf{BE}] - \mathbf{BC}[\gamma(\mathbf{CD} + \mathbf{EF}) - 2\beta\mathbf{BE}] = 0,$$

i.e.

$$\mathbf{C}(-\mathbf{AB} + \mathbf{CD} + \mathbf{EF})(2\alpha\mathbf{F} - \gamma\mathbf{B}) = 0,$$

therefore the cowreath $(A\otimes H^{op}, H, \psi)$ is rt-separable if, and only if, either $\mathbf{C} = 0$ or $2\alpha\mathbf{F} - \gamma\mathbf{B} = 0$.

If we suppose $2\alpha\mathbf{F}-\gamma\mathbf{B}\neq 0$, then $\mathbf{C}$ must be zero and thus (222) becomes

$$\begin{cases} \mathbf{E}(\alpha\mathbf{A}^2+\beta\mathbf{E}^2-\gamma\mathbf{AE})(r'\mathfrak{a}-s'\mathfrak{b})=-\mathbf{AB}+\mathbf{EF} \\ r'(\mathbf{AB}-\mathbf{EF})=0 \\ s'(\gamma\mathbf{F}-2\beta\mathbf{B})=0 \\ s'(2\alpha\mathbf{F}-\gamma\mathbf{B})=0 \\ \gamma\mathbf{AD}-2\beta\mathbf{DE}=0 \\ \gamma\mathbf{DE}-2\alpha\mathbf{AD}=0 \\ 2\alpha\mathbf{AF}+2\beta\mathbf{BE}-\gamma(\mathbf{AB}+\mathbf{EF})=0 \\ \mathbf{AB}-\mathbf{EF}\neq 0 \\ \mathfrak{a},\mathfrak{b}\in k, \mathbf{E}\neq 0. \end{cases}$$

From the second an fourth equations we get that $r'=0$ and $s'=0$, contradiction. This forces $\gamma\mathbf{B}-2\alpha\mathbf{F}=0$. Then the seventh equality in (222) becomes $2\beta\mathbf{B}-\gamma\mathbf{F}=0$ and so, given that $\gamma\mathbf{B}-2\alpha\mathbf{F}=0$ and $\gamma^2-4\alpha\beta\neq 0$, we get $\mathbf{B}=\mathbf{F}=0$. This leads to a contradiction, because $\mathbf{AB}-\mathbf{EF}\neq 0$. We can conclude that if the cowreath $(A\otimes H^{op},H,\psi)$ is separable, then $\gamma^2-4\alpha\beta=0$. ☐

7. Rt-separability vs Frobenius property

The aim of this section is to the determine for which H-comodule structure, the induced cowreath is both rt-separable and Frobenius. Since Theorem 6.2 shows that rt-separability implies $\gamma^2-4\alpha\beta=0$, we start by determining for which Frobenius cowreath this condition is satisfied.

THEOREM 7.1. *Let (A,ρ) be a Clifford algebra $Cl(\alpha,\beta,\gamma)$ endowed with any non-trivial H-comodule algebra structure. The cowreath $(A\otimes H^{op},H,\psi)$ is Frobenius and $\gamma^2-4\alpha\beta=0$ only if $\boldsymbol{C}=0$ and either*

- *$\boldsymbol{E},\alpha\neq 0$. Then $\gamma=2\alpha\frac{\boldsymbol{A}}{\boldsymbol{E}}$, $\beta=\alpha\frac{\boldsymbol{A}^2}{\boldsymbol{E}^2}$ and $\boldsymbol{A},\boldsymbol{B},\boldsymbol{D},\boldsymbol{F}\in k$ with $\boldsymbol{AB}-\boldsymbol{EF}\neq 0$.*
- *$\boldsymbol{E}=\alpha=\gamma=0$. Then $\beta,\boldsymbol{A},\boldsymbol{B}\neq 0$ and $\boldsymbol{D},\boldsymbol{F}\in k$.*

PROOF. **1.** Let $\mathbf{E}\neq 0$ and consider the second row of the table for Frobenius cowreaths. In this case, if we further assume $\gamma^2-4\alpha\beta=0$, we must have

$$\begin{cases} \gamma\mathbf{AD}-2\mathbf{CF}-2\beta\mathbf{DE}=0 \\ \gamma\mathbf{DE}+2\mathbf{BC}-2\alpha\mathbf{AD}=0 \\ 2\alpha\mathbf{AF}+2\beta\mathbf{BE}-\gamma\mathbf{AB}-\gamma\mathbf{EF}=0 \\ -\mathbf{AB}+\mathbf{CD}+\mathbf{EF}\neq 0 \\ \gamma\mathbf{B}-2\alpha\mathbf{F}\neq 0 \\ \mathbf{E}\neq 0 \\ \gamma^2-4\alpha\beta=0. \end{cases}$$

If $\alpha=0$, then $\gamma\mathbf{B}\neq 0$, but also $\gamma=0$, which is a contradiction. Thus $\alpha\neq 0$ and $\beta=\frac{\gamma^2}{4\alpha}$. The first equation of the system becomes $2\alpha\gamma\mathbf{AD}-4\alpha\mathbf{CF}-\gamma^2\mathbf{DE}=0$, that can be further simplified using the second and the fifth ones:

$$2\alpha\gamma\mathbf{AD}-4\alpha\mathbf{CF}-\gamma(-2\mathbf{BC}+2\alpha\mathbf{AD})=0 \iff \mathbf{C}(\gamma\mathbf{B}-2\alpha\mathbf{F})=0 \iff \mathbf{C}=0.$$

Now the third equation becomes $(\gamma\mathbf{E}-2\alpha\mathbf{A})(\gamma\mathbf{B}-2\alpha\mathbf{F})=0$, that is $\gamma\mathbf{E}-2\alpha\mathbf{A}=0$, and the initial system reduces to

$$\begin{cases} \mathbf{C}=0 \\ \gamma=2\alpha\dfrac{\mathbf{A}}{\mathbf{E}} \\ \beta=\alpha\dfrac{\mathbf{A}^2}{\mathbf{E}^2} \\ \mathbf{AB}-\mathbf{EF}\neq 0 \\ \mathbf{E},\alpha\neq 0. \end{cases}$$

2. Now consider the last row of the table for Frobenius cowreaths. By supposing $\gamma^2 - 4\alpha\beta = 0$, we must have

$$\begin{cases} \gamma\mathbf{AD} - 2\mathbf{CF} = 0 \\ \mathbf{BC} - \alpha\mathbf{AD} = 0 \\ \gamma\mathbf{B} - 2\alpha\mathbf{F} = 0 \\ \mathbf{AB} - \mathbf{CD} \neq 0 \\ \mathbf{A} \neq 0 \\ \gamma\mathbf{F} - 2\beta\mathbf{B} \neq 0 \\ \gamma^2 - 4\alpha\beta = 0. \end{cases}$$

If $\beta = 0$, then $\gamma = 0$, but also $\gamma\mathbf{F} \neq 0$, which is a contradiction. Thus $\beta \neq 0$ and $\alpha = \frac{\gamma^2}{4\beta}$. As a consequence $\gamma\mathbf{B} - 2\alpha\mathbf{F} = 0$ becomes $2\gamma\beta\mathbf{B} - \gamma^2\mathbf{F} = 0$, that is $\gamma = 0$. It is straightforward that $\alpha = \mathbf{C} = 0$ and $\mathbf{B} \neq 0$. □

As a consequence we see that it is not possible to build a cowreath $(A \otimes H^{op}, H, \psi)$ that is both Frobenius and rt-separable.

Corollary 7.2. *Let (A, ρ) be a Clifford algebra $Cl(\alpha, \beta, \gamma)$ endowed with any H-comodule algebra structure. The cowreath $(A \otimes H^{op}, H, \psi)$ cannot be both Frobenius and rt-separable.*

Proof. If the comodule algebra structure is trivial, then we know from previous results that the cowreath $(A \otimes H^{op}, H, \psi)$ is Frobenius but not rt-separable (see Remark 5.1). Now let us suppose that ρ is non-trivial and that $(A \otimes H^{op}, H, \psi)$ is Frobenius. If we want $(A \otimes H^{op}, H, \psi)$ also to be rt-separable, then, by Theorem 6.2, $\gamma^2 - 4\alpha\beta = 0$, so that we must focus our attention on the two cases of Theorem 7.1.

In the first case, since $\mathbf{C} = 0$, $\mathbf{E} \neq 0$ and $\gamma = 2\alpha\frac{\mathbf{A}}{\mathbf{E}}$, rt-separability requires that

$$\begin{cases} \mathbf{E}(\beta\mathbf{E}^2 - \alpha\mathbf{A}^2)(r'\mathfrak{a} - s'\mathfrak{b}) = -\mathbf{AB} + \mathbf{EF} \\ r'(\mathbf{AB} - \mathbf{EF}) = 0 \\ s'(2\alpha\mathbf{AF} - 2\beta\mathbf{BE}) = 0 \\ s'\alpha(\mathbf{EF} - \mathbf{AB}) = 0 \\ \gamma\mathbf{AD} - 2\beta\mathbf{DE} = 0 \\ 2\alpha\mathbf{AF} + 2\beta\mathbf{BE} - \gamma(\mathbf{AB} + \mathbf{EF}) = 0 \\ \mathbf{AB} - \mathbf{EF} \neq 0 \\ \mathfrak{a}, \mathfrak{b} \in k, \mathbf{E} \neq 0. \end{cases}$$

Given that furthermore $\alpha \neq 0$ and $\mathbf{AB} - \mathbf{EF} \neq 0$ (by Frobenius property) we see that $r' = s' = 0$ is forced, but this is against the first equality.

If we consider the second case of Theorem 7.1, we find that, since $\mathbf{C} = \mathbf{E} = \alpha = \gamma = 0$, rt-separability requires

$$\begin{cases} 0 = -\mathbf{AB} \\ r'\mathbf{B} = 0 \\ -2\beta\mathbf{B} = 0 \\ \mathbf{AB} \neq 0 \\ \mathfrak{a}, \mathfrak{b} \in k, \mathbf{E} = 0 \end{cases},$$

which is clearly inconsistent. □

CHAPTER 4

Partial results in higher dimension

In this chapter we will extend some of the results obtained in Chapter 2 to algebras in higher dimension. Namely we will show that $E(n)$-coactions over a finite-dimensional algebra A are classified by tuples $(\varphi, d_1, \ldots, d_n)$ consisting of an involution φ and a family $(d_i)_{i=1,\ldots,n}$ of φ-derivations satisfying appropriate conditions (see Thm. 1.6). Since a complete classification of involutions and skew-derivations seems currently out of reach for a Clifford algebra $A = Cl(\alpha, \beta_i, \gamma_i, \lambda_{ij})$ we specialized our main result for the case when A is a simple algebra (Thm. 2.2). In this instance a full classification of $E(n)$-coactions becomes roughly equivalent to the understanding of the structure of a particular subset of A that contains every element whose square is contained in $\mathcal{Z}(A)$.

In order to reach our goal we first need to recall the definition of the Hopf algebras $E(n)$ given in Section 5 of Chapter 1.

Definition 5.22. We denote by $E(n)$ the 2^{n+1}-dimensional Hopf algebra over a field k of characteristic $\mathrm{char}(k) \neq 2$ generated by elements g and x_i, for $i = 1, \ldots, n$, such that $g^2 = 1$, $x_i^2 = 0$ and $gx_i = -x_i g$ for any $i = 1, \ldots, n$ and $x_i x_j = -x_j x_i$ for $i, j = 1, \ldots, n$, $i < j$.

Remark 0.2. Remember that as an associative algebra $E(n)$ can be regarded also as $Cl(1, 0, 0, 0)$.

The Hopf algebra structure is given by

$$\Delta(g) = g \otimes g, \quad \Delta(x_i) = x_i \otimes g + 1 \otimes x_i, \; i = 1, \ldots, n$$

$$\varepsilon(g) = 1, \quad \varepsilon(x_i) = 0, \; i = 1, \ldots, n$$

$$S(g) = g^{-1} = g, \quad S(x_i) = -gx_i, \; i = 1, \ldots, n.$$

Notice that $E(1)$ is Sweedler's Hopf algebra. For $P = \{i_1, i_2, \ldots, i_s\} \subseteq \{1, 2, \ldots, n\}$ such that $i_1 < i_2 < \cdots < i_s$, we denote $x_P = x_{i_1} x_{i_2} \cdots x_{i_s}$. If $P = \varnothing$ then $x_\varnothing = 1$. The set $\{g^j x_P \mid P \subseteq \{1, \ldots, n\}, j \in \{0, 1\}\}$ is a basis of $E(n)$. Let $F = \{i_{j_1}, i_{j_2}, \ldots, i_{j_r}\}$ be a subset of P and define

$$S(F, P) = (j_1 + \cdots + j_r) - \frac{r(r+1)}{2} \text{ and } S(\varnothing, P) = 0.$$

Then computations show that

$$\Delta(g^j x_P) = \sum\nolimits_{F \subseteq P} (-1)^{S(F,P)} g^j x_F \otimes g^{|F|+j} x_{P \setminus F}, \tag{223}$$

$$S(g^j x_P) = (-1)^{|P|(j+1)} g^{|P|+j} x_P. \tag{224}$$

To gain further insight into these family of algebras we refer to [**BDG, CD, PVO, PVO2, CC**]. In [**PVO2**] it is shown that a 2^{n+1}-dimensional Clifford algebra $A = Cl(\alpha, \beta_i, \gamma_i, \lambda_{ij})$ admits a canonical $E(n)$-comodule algebra structure $\rho : A \to A \otimes E(n)$ given by

$$\begin{aligned}
\rho(1_A) &= 1_A \otimes 1_{E(n)} \\
\rho(G) &= G \otimes g \\
\rho(X_i) &= X_i \otimes g + 1_A \otimes x_i, \quad i = 1, \ldots, n \\
\rho(GX_i) &= GX_i \otimes 1_{E(n)} + G \otimes gx_i, \quad i = 1, \ldots, n.
\end{aligned}$$

Our first goal is to understand how to characterize all the possible $E(n)$-comodule algebra structures that the Clifford algebra A admits. It turns out that, as it happens in the four-dimensional case, $E(n)$-coactions are in bijective correspondence with tuples $(\varphi, d_1, \ldots, d_n)$.

1. Coactions, involutions and skew-derivations

In this section we are going to prove that each $E(n)$-coaction of $A = Cl(\alpha, \beta_i, \gamma_i, \lambda_{ij})$ is again completely determined by the choice of an involution $\varphi : A \to A$ and of a family of φ-derivations $\{d_1, \dots d_n\}$. Most of the steps performed comes as a natural generalization of what we did for the four-dimensional case.

1.1. The category isomorphism in higher dimension. In the first place, we want to show that there is an isomorphism of categories

$$\mathrm{Vec}_k^{E(n)} \cong {}_{E(n)^{cop}}\mathrm{Vec}_k$$

that preserves algebras, so to make clear that each right $E(n)$-coaction corresponds to a unique left $E(n)^{cop}$-action.

Again we recall that when H is finite-dimensional there is an isomorphism of categories

$$F' : \mathrm{Vec}_k^H \to {}_{H^*}\mathrm{Vec}_k$$

given by

$$F'(M, \rho) = (M, \mu_\rho),$$

where

$$\mu_\rho(h^* \otimes m) = m_0 h^*(m_1)$$

for every $h \in H$ and every $m \in M$ (cf. [**CMZ**], p. 10). The inverse of this functor is given by $G : {}_{H^*}\mathrm{Vec}_k \to \mathrm{Vec}_k^H$, $G(M, \mu) = (M, \rho_\mu)$, where

$$\rho_\mu(m) = \mu(h_i^* \otimes m) \otimes h_i$$

for every $m \in M$. Both the functor F' and its inverse G send a map to itself, i.e. a map is H-colinear if, and only if, is H^*-linear. Furthermore, in our case we also have that $H = E(n)$ is self dual, i.e. there exists an Hopf algebra map $\psi : E(n) \to E(n)^*$ that gives an isomorphism $E(n) \cong E(n)^*$. It is defined by (cf. [**PVO**], Prop. 1)

$$\psi(1) = 1^* + g^* = \varepsilon_H, \quad \psi(g) = 1^* - g^*, \quad \psi(x_i) = x_i^* + (gx_i)^*, \quad i = 1, \dots, n.$$

We are going to modify ψ in order to define a new Hopf algebra isomorphism $\varphi : E(n)^{cop} \to E(n)^*$, because we work best with a flipped comultiplication.

LEMMA 1.1. *The algebra map $\varphi : E(n)^{cop} \to E(n)^*$ defined by*

$$\varphi(1) = 1^* + g^* = \varepsilon_{E(n)}, \quad \varphi(g) = 1^* - g^*, \quad \varphi(x_i) = -x_i^* + (gx_i)^*, \quad i = 1, \dots, n$$

is an Hopf algebra isomorphism.

PROOF. One shows by induction that

$$\varphi(g^j x_P) = (-1)^{\lfloor \frac{|P|+1}{2} \rfloor}(x_P)^* + (-1)^{\lfloor \frac{|P|}{2} \rfloor + j}(gx_P)^* \tag{225}$$

for $j \in \{0, 1\}$ and $P \subseteq \{1, \dots n\}$. Hence we have that

$$(x_P)^* = (-1)^{\lfloor \frac{|P|+1}{2} \rfloor}\varphi\left(\frac{x_P + gx_P}{2}\right), \quad (gx_P)^* = (-1)^{\lfloor \frac{|P|}{2} \rfloor}\varphi\left(\frac{x_P - gx_P}{2}\right),$$

for every $P \subseteq \{1, \dots n\}$. This proves that φ is a surjective and therefore bijective. The comultiplication on $E(n)^*$ is given by

$$\Delta((g^j x_P)^*) = (-1)^{j|P|}\sum\nolimits_{F \subseteq P}(-1)^{S(F,P)+(j+|P|)|F|}\left[(-1)^{|F|}(x_{P\setminus F})^* \otimes (g^j x_F)^* + (-1)^{|P|}(gx_{P\setminus F})^* \otimes (g^{j+1}x_F)^*\right].$$

To show that φ is a coalgebra map, one calculates

$$\varepsilon_{E(n)^*}(\varphi(g^j x_P)) = \varepsilon_{E(n)^*}\left((-1)^{\lfloor \frac{|P|+1}{2} \rfloor}(x_P)^* + (-1)^{\lfloor \frac{|P|}{2} \rfloor + j}(gx_P)^*\right) = \delta_{P,\varnothing} = \varepsilon_{E(n)^{cop}}(g^j x_P)$$

and

$$\begin{aligned}(\varphi \otimes \varphi)\Delta^{cop}(g^j x_P) &= (\varphi \otimes \varphi)\left(\sum\nolimits_{F \subseteq P}(-1)^{S(F,P)}g^{|F|+j}x_{P\setminus F} \otimes g^j x_F\right) \\ &= \sum\nolimits_{F \subseteq P}(-1)^{S(F,P)}\left[(-1)^{\lfloor \frac{|P|-|F|+1}{2} \rfloor}(x_{P\setminus F})^* + (-1)^{\lfloor \frac{|P|-|F|}{2} \rfloor + |F| + j}(gx_{P\setminus F})^*\right] \otimes \\ &\otimes \quad \left[(-1)^{\lfloor \frac{|F|+1}{2} \rfloor}(x_F)^* + (-1)^{\lfloor \frac{|F|}{2} \rfloor + j}(gx_F)^*\right]\end{aligned}$$

$$\begin{aligned}
&= \textstyle\sum_{F\subseteq P}(-1)^{S(F,P)}\Big[(-1)^{\lfloor\frac{|P|-|F|+1}{2}\rfloor+\lfloor\frac{|F|+1}{2}\rfloor}(x_{P\setminus F})^*\otimes(x_F)^*+\\
&+ (-1)^{\lfloor\frac{|P|-|F|}{2}\rfloor+|F|+\lfloor\frac{|F|}{2}\rfloor}(gx_{P\setminus F})^*\otimes(gx_F)^*+\\
&+ (-1)^{\lfloor\frac{|P|-|F|+1}{2}\rfloor+\lfloor\frac{|F|}{2}\rfloor+j}(x_{P\setminus F})^*\otimes(gx_F)^*+\\
&+ (-1)^{\lfloor\frac{|P|-|F|}{2}\rfloor+|F|+j+\lfloor\frac{|F|+1}{2}\rfloor}(gx_{P\setminus F})^*\otimes(x_F)^*\Big]\\
&\overset{(\star)}{=} \textstyle\sum_{F\subseteq P}(-1)^{S(F,P)}\Big[(-1)^{\lfloor\frac{|P|+1}{2}\rfloor+(1+|P|)|F|}(x_{P\setminus F})^*\otimes(x_F)^*+\\
&+ (-1)^{\lfloor\frac{|P|+1}{2}\rfloor+(1+|F|)|P|}(gx_{P\setminus F})^*\otimes(gx_F)^*+\\
&+ (-1)^{\lfloor\frac{|P|}{2}\rfloor+(1+|F|)|P|+j}(x_{P\setminus F})^*\otimes(gx_F)^*+\\
&+ (-1)^{\lfloor\frac{|P|}{2}\rfloor+(1+|P|)|F|+j}(gx_{P\setminus F})^*\otimes(x_F)^*\Big]\\
&= (-1)^{\lfloor\frac{|P|+1}{2}\rfloor}\Delta((x_P)^*)+(-1)^{\lfloor\frac{|P|}{2}\rfloor+j}\Delta((gx_P)^*)\\
&= \Delta(\varphi(g^jx_P)).
\end{aligned}$$

Equality $(\star)$ can be checked case by case, fixing the parity of $|P|$ and $|F|$. For example when both $|P|$ and $|F|$ are even, one has

$$\begin{aligned}
&\textstyle\sum_{F\subseteq P}(-1)^{S(F,P)}\Big[(-1)^{\lfloor\frac{|P|-|F|+1}{2}\rfloor+\lfloor\frac{|F|+1}{2}\rfloor}(x_{P\setminus F})^*\otimes(x_F)^*+\\
+\ &(-1)^{\lfloor\frac{|P|-|F|}{2}\rfloor+|F|+\lfloor\frac{|F|}{2}\rfloor}(gx_{P\setminus F})^*\otimes(gx_F)^*+\\
+\ &(-1)^{\lfloor\frac{|P|-|F|+1}{2}\rfloor+\lfloor\frac{|F|}{2}\rfloor+j}(x_{P\setminus F})^*\otimes(gx_F)^*+\\
+\ &(-1)^{\lfloor\frac{|P|-|F|}{2}\rfloor+|F|+j+\lfloor\frac{|F|+1}{2}\rfloor}(gx_{P\setminus F})^*\otimes(x_F)^*\Big]\\
=\ &\textstyle\sum_{F\subseteq P}(-1)^{S(F,P)+\frac{|P|}{2}}\big[(x_{P\setminus F})^*\otimes(x_F)^*+(gx_{P\setminus F})^*\otimes(gx_F)^*+\\
+\ &(-1)^j(x_{P\setminus F})^*\otimes(gx_F)^*+(-1)^j(gx_{P\setminus F})^*\otimes(x_F)^*\big]
\end{aligned}$$

and

$$\begin{aligned}
&\textstyle\sum_{F\subseteq P}(-1)^{S(F,P)}\Big[(-1)^{\lfloor\frac{|P|+1}{2}\rfloor+(1+|P|)|F|}(x_{P\setminus F})^*\otimes(x_F)^*+\\
+\ &(-1)^{\lfloor\frac{|P|+1}{2}\rfloor+(1+|F|)|P|}(gx_{P\setminus F})^*\otimes(gx_F)^*+\\
+\ &(-1)^{\lfloor\frac{|P|}{2}\rfloor+(1+|F|)|P|+j}(x_{P\setminus F})^*\otimes(gx_F)^*+\\
+\ &(-1)^{\lfloor\frac{|P|}{2}\rfloor+(1+|P|)|F|+j}(gx_{P\setminus F})^*\otimes(x_F)^*\Big]\\
=\ &\textstyle\sum_{F\subseteq P}(-1)^{S(F,P)+\frac{|P|}{2}}\big[(x_{P\setminus F})^*\otimes(x_F)^*+(gx_{P\setminus F})^*\otimes(gx_F)^*+\\
+\ &(-1)^j(x_{P\setminus F})^*\otimes(gx_F)^*+(-1)^j(gx_{P\setminus F})^*\otimes(x_F)^*\big]
\end{aligned}$$

□

Let h_i denote the i-th element of the basis of $E(n)$ and h_i^* its dual element in $E(n)^*$. We can use the isomorphism φ to define a mapping

$$U : {}_{E(n)^{cop}}\mathrm{Vec}_k \to \mathrm{Vec}_k^E(n), \quad U(M,\mu) = (M,\rho_\mu),$$

where

$$\rho_\mu(m) = \mu(\varphi^{-1}(h_i^*)\otimes m)\otimes h_i$$

and $U(f) = f$ for every $E(n)^{cop}$-linear f.

To show that U is a functor, we will make use of Lemma 1 contained in [**CMZ**]. We have, for every $f, g \in E(n)^*$ and $m \in M$,

$$\begin{aligned}
(\mathrm{Id}_M \otimes f \otimes g)(\rho_\mu \otimes \mathrm{Id}_{E(n)})\rho_\mu(m) &= (\mathrm{Id}_M \otimes f \otimes g)(\mu(\varphi^{-1}(h_j'^*) \otimes \mu(\varphi^{-1}(h_i^*) \otimes m)) \otimes h_j' \otimes h_i) \\
&= \mu(\varphi^{-1}(h_j'^*) \cdot \varphi^{-1}(h_i^*) \otimes m) f(h_j') g(h_i) \\
&= \mu(\varphi^{-1}(h_j'^* \star h_i^*) \otimes m) f(h_j') g(h_i) \\
&= \mu(\varphi^{-1}(f(h_j') h_j'^* \star g(h_i) h_i^*) \otimes m) \\
&= \mu(\varphi^{-1}(f \star g) \otimes m) \\
&= \mu(\varphi^{-1}((f \star g)(h_i) h_i^*) \otimes m) \\
&= \mu(\varphi^{-1}(h_i^*) \otimes m)(f \star g)(h_i) \\
&= \mu(\varphi^{-1}(h_i^*) \otimes m) f(h_{i_1}) g(h_{i_2}) \\
&= (\mathrm{Id}_M \otimes f \otimes g)\mu(\varphi^{-1}(h_i^*) \otimes m) \otimes h_{i_1} \otimes h_{i_2} \\
&= (\mathrm{Id}_M \otimes f \otimes g)(\mathrm{Id}_M \otimes \Delta)\mu(\varphi^{-1}(h_i^*) \otimes m) \otimes h_i \\
&= (\mathrm{Id}_M \otimes f \otimes g)(\mathrm{Id}_M \otimes \Delta)\rho_\mu(m).
\end{aligned}$$

By [**CMZ**, Lemma 1], it follows that $(\rho_\mu \otimes \mathrm{Id}_{E(n)})\rho_\mu = (\mathrm{Id}_M \otimes \Delta)\rho_\mu$ (cf. proof of Prop. 3 ibid.). Furthermore

$$\begin{aligned}
(M \otimes \varepsilon_{E(n)})\rho_\mu(m) &= \mu(\varphi^{-1}(h_i^*) \otimes m) \otimes \varepsilon_{E(n)}(h_i) \\
&= \mu(\varphi^{-1}(\varepsilon_{E(n)}(h_i) h_i^*) \otimes m) \\
&= \mu(\varphi^{-1}(\varepsilon_{E(n)}) \otimes m) \\
&= \mu(1_{E(n)} \otimes m) \\
&= m
\end{aligned}$$

for every $m \in M$ and thus we can conclude that (M, ρ_μ) is an object in $\mathrm{Vec}_k^{E(n)}$. Now let us consider two $E(n)^{cop}$-modules M, N and an $E(n)^{cop}$-linear map $f : M \to N$. We prove that f is automatically $E(n)$-colinear. We have

$$\begin{aligned}
(f \otimes \mathrm{Id}_{E(n)})\rho_M(m) &= (f \otimes \mathrm{Id}_{E(n)})(\mu_M(\varphi^{-1}(h_i^*) \otimes m) \otimes h_i) \\
&= (f \circ \mu_M)(\varphi^{-1}(h_i^*) \otimes m) \otimes h_i \\
&= \mu_N(\varphi^{-1}(h_i^*) \otimes f(m)) \otimes h_i \\
&= (\rho_N \circ f)(m).
\end{aligned}$$

Finally, since U is the identity on morphisms, we can conclude that U is a functor.

U has an inverse $V : \mathrm{Vec}_k^{E(n)} \to {}_{E(n)^{cop}}\mathrm{Vec}_k$ defined by

$$V : \mathrm{Vec}_k^{E(n)} \to {}_{E(n)^{cop}}\mathrm{Vec}_k, \quad V(M, \rho) = (M, \mu_\rho),$$

where

$$\mu_\rho(h \otimes m) = (\varphi(h)(m_1))m_0$$

and $V(f) = f$ for every $E(n)$-colinear f. We have

$$\begin{aligned}
\mu_\rho(\mathrm{Id}_{E(n)} \otimes \mu_\rho)(h' \otimes h \otimes m) &= \mu_\rho(h' \otimes (\varphi(h)(m_1))m_0) \\
&= (\varphi(h)(m_1))(\varphi(h')(m_{0_1}))m_{0_0} \\
&= (\varphi(h)(m_{1_2}))(\varphi(h')(m_{1_1}))m_0 \\
&= (\varphi(h') \star \varphi(h))(m_1)m_0 \\
&= \varphi(h'h)(m_1)m_0 \\
&= \mu_\rho(m \otimes M)(h' \otimes h \otimes m)
\end{aligned}$$

for every $h, h' \in E(n)$ and every $m \in M$. Moreover

$$\mu_\rho(1_{E(n)} \otimes m) \quad = \quad (\varphi(1_{E(n)})(m_1))(m_0)$$

$$\begin{aligned} &= \varepsilon(m_1)m_0 \\ &= m \end{aligned}$$

for every $m \in M$ and so we can conclude that (M, μ_ρ) is an object in ${}_{E(n)^{cop}}\mathrm{Vec}_k$.

Then we consider two $E(n)$-comodules M, N and an $E(n)$-colinear map $f : M \to N$. We prove that f is automatically $E(n)^{cop}$-linear. We have

$$\begin{aligned} (f \circ \mu_M)(h \otimes m) &= (\varphi(h)(m_1))f(m_0) \\ &= (\varphi(h)(f(m)_1)f(m)_0 \\ &= \mu_N(h \otimes f(m)) \\ &= \mu_N(\mathrm{Id}_{E(n)} \otimes f)(h \otimes m) \end{aligned}$$

for every $h \in E(n)$ and every $m \in M$. Since V is the identity on morphisms, we conclude that V is a functor.

Finally we want to prove that $UV = \mathrm{Id}_{\mathrm{Vec}_k^{E(n)}}$ nd $VU = \mathrm{Id}_{{}_{E(n)^{cop}}\mathrm{Vec}_k}$. We have $UV(M, \rho) = (M, \rho_{\mu_\rho})$ and

$$\begin{aligned} \rho_{\mu_\rho}(m) &= \mu_\rho(\varphi^{-1}(h_i^*) \otimes m) \otimes h_i \\ &= (\varphi\varphi^{-1}(h_i^*))(m_1)m_0 \otimes h_i \\ &= h_i^*(m_1)m_0 \otimes h_i \\ &= m_0 \otimes h_i^*(m_1)h_i \\ &= m_0 \otimes m_1 \\ &= \rho(m) \end{aligned}$$

for every $m \in M$. Moreover $VU(M, \mu) = (M, \mu_{\rho_\mu})$ and

$$\begin{aligned} \mu_{\rho_\mu}(h \otimes m) &= (\varphi(h)(h_i))\mu(\varphi^{-1}(h_i^*) \otimes m) \\ &= \mu([\varphi(h)(h_i)]\varphi^{-1}(h_i^*) \otimes m) \\ &= \mu(\varphi^{-1}([\varphi(h)(h_i)]h_i^*) \otimes m). \end{aligned}$$

Remember that the $\varphi(h)(h_i)$'s are the coordinates of the vector $\varphi(h) \in E(n)^*$ on the dual basis h_i^*, which means that $[\varphi(h)(h_i)]h_i^* = \varphi(h)$. Hence

$$\mu_{\rho_\mu}(h \otimes m) = \mu(\varphi^{-1}(\varphi(h)) \otimes m) = \mu(h \otimes m)$$

for any $h \in E(n)$ and any $m \in M$. Therefore we have proved that the functors U and V previously defined give an isomorphism of categories

$${}_{E(n)^{cop}}\mathrm{Vec}_k \cong \mathrm{Vec}_k^{E(n)}.$$

Again, the functors U and V send algebras to algebras, i.e. the image (A, ρ_μ) of an $E(n)^{cop}$-module algebra (A, μ) is an $E(n)$-comodule algebra and, conversely, the image $(V(A), \mu_\rho)$ of an $E(n)$-comodule algebra (A, ρ) is an $E(n)^{cop}$-module algebra.

Suppose (A, μ) is an $E(n)^{cop}$-module algebra. This means that (48) and (49) hold. To prove that A is also an $E(n)$-comodule algebra we need to show (51) and (52) are satisfied. For any $f \in E(n)^*$ we have

$$\begin{aligned} (\mathrm{Id}_A \otimes f)(\rho_\mu(a)\rho_\mu(b)) &= (\mathrm{Id}_A \otimes f)([\mu(\varphi^{-1}(h_i^*) \otimes a) \otimes h_i][\mu(\varphi^{-1}(h_j'^*) \otimes b) \otimes h_j']) \\ &= \mu(\varphi^{-1}(h_i^*) \otimes a)\mu(\varphi^{-1}(h_j'^*) \otimes b)f(h_i h_j') \\ &= \mu(\varphi^{-1}(h_i^*) \otimes a)\mu(\varphi^{-1}(h_j'^*) \otimes b)f_1(h_i)f_2(h_j') \\ &= \mu(\varphi^{-1}(f_1(h_i)h_i^*) \otimes a)\mu(\varphi^{-1}(f_2(h_j')h_j'^*) \otimes b) \\ &= \mu(\varphi^{-1}(f_1) \otimes a)\mu(\varphi^{-1}(f_2) \otimes b) \\ &= \mu(\varphi^{-1}(f)_2 \otimes a)\mu(\varphi^{-1}(f)_1 \otimes b) \\ &\overset{(48)}{=} \mu(\varphi^{-1}(f) \otimes ab) \\ &= \mu(\varphi^{-1}(f(h_i)h_i^*) \otimes ab) \\ &= \mu(\varphi^{-1}(h_i^*) \otimes ab)f(h_i) \end{aligned}$$

$$\begin{aligned} &= (\mathrm{Id}_A \otimes f)(\mu(\varphi^{-1}(h_i^*) \otimes ab) \otimes h_i) \\ &= (\mathrm{Id}_A \otimes f)\rho_\mu(ab). \end{aligned}$$

By [**CMZ**, Lemma 1], it follows that $\rho_\mu(a)\rho_\mu(b) = \rho_\mu(ab)$ for any $a, b \in A$.

Furthermore

$$\begin{aligned} \rho_\mu(1_A) &= \mu(\varphi^{-1}(h_i^*) \otimes 1_A) \otimes h_i \\ &\overset{(49)}{=} \varepsilon_{E(n)}(\varphi^{-1}(h_i^*))1_A \otimes h_i \\ &= \varepsilon_{E(n)^*}(h_i^*)1_A \otimes h_i \\ &= h_i^*(1_H)1_A \otimes h_i \\ &= 1_A \otimes 1_{E(n)} \end{aligned}$$

and thus (A, ρ_μ) is an $E(n)$-comodule algebra.

Now suppose (A, ρ) is an $E(n)$-comodule algebra. This means that (51) and (52) hold. To prove that (A, μ_ρ) is an $E(n)^{cop}$-module algebra we need to show that (48) and (49) hold. We have

$$\begin{aligned} \mu_\rho(h \otimes ab) &= (\varphi(h)((ab)_1))(ab)_0 \\ &\overset{(51)}{=} (\varphi(h)(a_1 b_1))a_0 b_0 \\ &= [(\varphi(h))_1(a_1)(\varphi(h))_2(b_1)]a_0 b_0 \\ &= [\varphi(h_2)(a_1)\varphi(h_1)(b_1)]a_0 b_0 \\ &= (\varphi(h_2)(a_1)a_0)(\varphi(h_1)(b_1)b_0) \\ &= \mu_\rho(h_2 \otimes a)\mu_\rho(h_1 \otimes b) \end{aligned}$$

for every $h \in E(n)$ and every $a, b \in A$. Then

$$\begin{aligned} \mu_\rho(h \otimes 1_A) &\overset{(52)}{=} (\varphi(h)(1_{E(n)}))1_A \\ &= \varepsilon_{E(n)^*}(\varphi(h))1_A \\ &= \varepsilon_{E(n)}(h)1_A, \end{aligned}$$

for every $h \in E(n)$. We obtain a generalization of Proposition 1.2 of Chapter 2.

PROPOSITION 1.2. *Let $\varphi : E(n)^{cop} \to E(n)^*$ be the Hopf algebra isomorphism defined by (225). The assignment*

$$U : {}_{E(n)^{cop}}\mathrm{Vec}_k \to \mathrm{Vec}_k^{E(n)}, \quad U(M, \mu) = (M, \rho_\mu),$$

where

$$\rho_\mu(m) = \mu(\varphi^{-1}(h_i^*) \otimes m) \otimes h_i$$

for every $m \in M$, and $U(f) = f$ for every $E(n)^{cop}$-linear f defines an invertible functor. Its inverse is given by the assignment

$$V : \mathrm{Vec}_k^{E(n)} \to {}_{E(n)^{cop}}\mathrm{Vec}_k, \quad V(M, \rho) = (M, \mu_\rho),$$

where

$$\mu_\rho(h \otimes m) = (\varphi(h)(m_1))m_0$$

for every $h \in E(n)$ and every $m \in M$, and $V(f) = f$ for every $E(n)$-colinear f. Moreover both U and V preserve algebras.

This means that each $E(n)$-coaction ρ on a finite dimensional algebra A can be expressed in terms of a unique $E(n)^{cop}$-action:

$$\rho(a) = \mu(\varphi^{-1}(h_i^*) \otimes a) \otimes h_i$$

and explicitly

$$\begin{aligned} \rho(a) &= \sum_{\substack{j=0,1 \\ P}} \mu(\varphi^{-1}((g^j x_P)^*) \otimes a) \otimes g^j x_P \\ &= \sum_{\substack{j=0,1 \\ P}} \mu\left(\varphi^{-1}\left((-1)^{\lfloor \frac{|P|+1-j}{2} \rfloor}\varphi\left(\frac{x_P + (-1)^j g x_P}{2}\right)\right) \otimes a\right) \otimes g^j x_P \end{aligned}$$

$$= \sum_{\substack{j=0,1\\P}} (-1)^{\lfloor \frac{|P|+1-j}{2} \rfloor} \mu\left(\frac{x_P + (-1)^j g x_P}{2} \otimes a\right) \otimes g^j x_P$$

$$= \sum_P (-1)^{\lfloor \frac{|P|+1}{2} \rfloor} \mu\left(\frac{x_P + g x_P}{2} \otimes a\right) \otimes x_P + (-1)^{\lfloor \frac{|P|}{2} \rfloor} \mu\left(\frac{x_P - g x_P}{2} \otimes a\right) \otimes g x_P$$

$$= \sum_P (-1)^{\lfloor \frac{|P|+1}{2} \rfloor} \left[\mu(x_P \otimes a) \otimes \frac{x_P + (-1)^{|P|} g x_P}{2} + \mu(g x_P \otimes a) \otimes \frac{x_P + (-1)^{|P|+1} g x_P}{2}\right],$$

where P is taken to run over all subset of $\{1, \ldots, n\}$. It is clear that, since $\mu(g x_P \otimes a) = \mu(g \otimes \mu(x_P \otimes a))$, and $\mu(x_P \otimes a) = \mu(x_{i_1} \otimes \mu(x_{i_2} \otimes \ldots \mu(x_{i_s} \otimes a)))$, when $P = \{i_1 < i_2 < \ldots < i_s\}$, then ρ is completely determined once we know how g and each x_i act on the elements of A. As a matter of fact $E(n)^{cop}$-actions on a finite-dimensional algebra A are in bijective correspondence with $n+1$-uples $(\varphi, d_1, \ldots, d_n)$, of suitable maps.

1.2. $E(n)^{cop}$-actions, involutions and derivations. We have seen that each $E(n)^{cop}$-action on a finite dimensional algebra A is completely determined by the choice of an action of $g, x_1, \ldots, x_n$ on the elements of A. This, in turn, is equivalent to the choice of an $n+1$-uple $(\varphi, d_1, \ldots, d_n)$, where φ is an involution of A, the d_i's are φ-derivations on A and $d_i^2 \equiv 0$, $\varphi d_i = -d_i \varphi$, $d_i d_j = -d_j d_i$ for every $i, j = 1, \ldots, n$. The steps needed to prove this result are exactly those followed in Subsection 1.2, Chapter 2.

In fact, given an $E(n)^{cop}$-action $\mu : E(n) \otimes A \to A$, set $\varphi(a) := \mu(g \otimes a)$. As previously shown, $\varphi : A \to A$ is easily proved to be an algebra map and an involution on A. Next, if we set $d_i(a) := \mu(x_i \otimes a)$ for every $i = 1, \ldots, n$, we obtain a family of n φ-derivations $d_i : A \to A$ that satisfy $d_i^2 \equiv 0$ and $d_i \varphi = -\varphi d_i$. Furthermore, it is not hard to see that $d_i d_j = d_j d_i$ for every $i, j = 1, \ldots, n$. In this way we have established an assignment

$$(\mu : E(n)^{cop} \otimes A \to A) \overset{\Phi}{\longmapsto} (\varphi := \mu(g \otimes -), d_i := \mu(x_i \otimes -)),$$

so that to each $E(n)^{cop}$-action corresponds a $(n+1)$-uple of maps (φ, d_i), where φ is an involution and the d_i's are φ-derivations such that $d_i^2 \equiv 0$, $\varphi d_i = -d_i \varphi$ and $d_i d_j = d_j d_i$ for every $i, j = 1, \ldots, n$.

Conversely let us fix an $n+1$-uple of k-linear maps $(\varphi : A \to A, d_i : A \to A)$, $i = 1, \ldots, n$, satisfying the previous properties. We are going to show that we can define an $E(n)^{cop}$-action on A. We define a k-linear map $\mu : E(n)^{cop} \otimes A \to A$, by setting

$$\mu(g^j x_P \otimes a) := \varphi^j(d_{i_1} d_{i_2} \cdots d_{i_s}(a)), \quad \mu(g^j x_\varnothing \otimes a) = \mu(g^j \otimes a) := \varphi^j(a) \tag{226}$$

for every $j \in \{0,1\}$, $P \subseteq \{1, \ldots, n\}$ and $a \in A$. Here $x_P = x_{i_1} x_{i_2} \cdots x_{i_s}$, $x_\varnothing = 1_{E(n)}$ as usual. We need to show that (A, μ) is an $E(n)^{cop}$-module and that (48)-(49) hold.

By definition $\mu(1_{E(n)} \otimes a) = a$, therefore we only need to prove that

$$\mu(h' \otimes \mu(h \otimes a)) = \mu(h'h \otimes a) \tag{227}$$

for every $h, h' \in E(n)^{cop}$ and every $a \in A$, to show that (A, μ) is an $E(n)^{cop}$-module. It is sufficient to pick h and h' among elements of the basis of $E(n)$. Let $x_P = x_{i_1} x_{i_2} \cdots x_{i_s}$ and $x_Q = x_{j_1} x_{j_2} \cdots x_{j_t}$. We have

$$\begin{aligned} \mu(g^k x_Q \otimes \mu(g^j x_P \otimes a)) &\overset{(226)}{=} \mu(g^k x_Q \otimes \varphi^j(d_{i_1} d_{i_2} \cdots d_{i_s}(a))) \\ &\overset{(226)}{=} \varphi^k(d_{j_1} d_{j_2} \cdots d_{j_t}(\varphi^j(d_{i_1} d_{i_2} \cdots d_{i_s}(a))). \end{aligned}$$

Since $\varphi d_i = -d_i \varphi$ for every $i = 1, \ldots, n$, it follows that

$$\varphi^k(d_{j_1} d_{j_2} \cdots d_{j_t}(\varphi^j(d_{i_1} d_{i_2} \cdots d_{i_s}(a))) = (-1)^{js} \varphi^{j+k}(d_{j_1} d_{j_2} \cdots d_{j_t}(d_{i_1} d_{i_2} \cdots d_{i_s}(a))).$$

Suppose there are $l, m \in \mathbb{N}$ such that $j_l = i_m$. Since the d_i's anticommute and $d_i^2 \equiv 0$, by changing sign accordingly, one can bring d_{j_l} and d_{i_m} together and show that the whole term must vanish. In this case, it is clear that $P \cap Q \neq \varnothing$, and $g^j x_Q g^j x_P = 0$, since $x_{j_l} x_{i_m} = x_{i_m}^2 = 0$. Therefore

$$\mu(g^k x_Q \otimes \mu(g^j x_P \otimes a)) = 0 = \mu(0 \otimes a) = \mu(g^j x_Q g^j x_P \otimes a),$$

i.e. (227) holds.

Now suppose that $P \cap Q = \varnothing$. Then we have

$$\mu(g^k x_Q \otimes \mu(g^j x_P \otimes a)) = (-1)^{js} \varphi^{j+k}(d_{j_1} d_{j_2} \cdots d_{j_t} d_{i_1} d_{i_2} \cdots d_{i_s}(a)),$$

where all the involved derivations are different. Let $\alpha_{Q,P} \in \{-1,1\}$ be defined by $x_Q x_P = \alpha_{Q,P} x_{Q\cup P}$. If we denote by d_R the ordered composition of derivations indexed by the ordered set R (e.g. $d_P = d_{i_1} d_{i_2} \cdots d_{i_s}$, where $P = \{i_1 < i_2 < \ldots < i_s\}$), it is clear that also $d_Q d_P = \alpha_{Q,P} d_{Q\cup P}$. We can conclude that

$$\begin{aligned}
\mu(g^k x_Q \otimes \mu(g^j x_P \otimes a)) &= (-1)^{js}\varphi^{j+k}(d_{j_1} d_{j_2} \cdots d_{j_t} d_{i_1} d_{i_2} \cdots d_{i_s}(a)) \\
&= (-1)^{js}\varphi^{j+k}(d_Q d_P(a)) \\
&= (-1)^{js}\alpha_{Q,P}\varphi^{j+k}(d_{Q\cup P}(a)) \\
&\overset{(226)}{=} (-1)^{js}\alpha_{Q,P}\mu(g^{j+k} x_{Q\cup P} \otimes a) \\
&= (-1)^{js}\mu(g^{j+k} x_Q x_P \otimes a) \\
&= \mu(g^k x_Q g^j x_P \otimes a)
\end{aligned}$$

Therefore (A,μ) is an $E(n)^{cop}$-module. Now we prove that it is an $E(n)^{cop}$-module algebra, i.e. that (48) and (49) hold.

Set $d_\varnothing := \mathrm{Id}_A$. We have

$$\mu(g^j x_P \otimes 1_A) = \varphi^j(d_P(1_A)) = \delta_{P,\varnothing} = \varepsilon(g^j x_P)1_A$$

therefore (49) holds.

Next

$$\begin{aligned}
\mu(x_i \otimes ab) &= d_i(ab) \\
&= d_i(a)b + \varphi(a)d_i(b) \\
&= \mu(x_i \otimes a)\mu(1_{E(n)} \otimes b) + \mu(g \otimes a)\mu(x_i \otimes b) \\
&= \mu((x_i)_2 \otimes a)\mu((x_i)_1 \otimes b)
\end{aligned}$$

for every $a,b \in A$ and every $i = 1,\ldots,n$. Now suppose that $\mu(x_P \otimes ab) = \mu((x_P)_2 \otimes a)\mu((x_P)_1 \otimes b)$ holds true for every $a,b \in A$ and every P with cardinality $m \geq 1$. Take $Q = \{i_1, i_2, \ldots i_{m+1}\}$ such that $|Q| = m+1$ and let $Q' = Q \setminus \{i_1\} = \{i_2, \ldots i_{m+1}\}$. Then

$$\begin{aligned}
\mu(x_Q \otimes ab) &= \mu(x_{i_1} \otimes \mu(x_{Q'} \otimes ab)) \\
&\overset{ind.hyp.}{=} \mu(x_{i_1} \otimes \mu((x_{Q'})_2 \otimes a)\mu((x_{Q'})_1 \otimes b)) \\
&= \mu((x_{i_1})_2 \otimes \mu((x_{Q'})_2 \otimes a))\mu((x_{i_1})_1 \otimes \mu((x_{Q'})_1 \otimes b)) \\
&= \mu((x_{i_1})_2(x_{Q'})_2 \otimes a))\mu((x_{i_1})_1(x_{Q'})_1 \otimes b)) \\
&= \mu((x_Q)_2 \otimes a))\mu((x_Q)_1 \otimes b)).
\end{aligned}$$

Thus we have proved $\mu(x_P \otimes ab) = \mu((x_P)_2 \otimes a))\mu((x_P)_1 \otimes b))$ by induction on $m = |P|$. Clearly $\mu(1_{E(n)} \otimes ab) = ab = \mu(1_{E(n)} \otimes a)\mu(1_{E(n)} \otimes b)$ and $\mu(g \otimes ab) = \varphi(ab) = \varphi(a)\varphi(b) = \mu(g \otimes a)\mu(g \otimes b)$ for every $a,b \in A$. Finally

$$\begin{aligned}
\mu(g x_P \otimes ab) &= \mu(g \otimes \mu(x_P \otimes ab)) \\
&= \mu(g \otimes \mu((x_P)_2 \otimes a)\mu((x_P)_1 \otimes b)) \\
&= \varphi(\mu((x_P)_2 \otimes a)\mu((x_P)_1 \otimes b)) \\
&= \varphi(\mu((x_P)_2 \otimes a))\varphi(\mu((x_P)_1 \otimes b)) \\
&= \mu(g \otimes \mu((x_P)_2 \otimes a))\mu(g \otimes \mu((x_P)_1 \otimes b)) \\
&= \mu(g(x_P)_2 \otimes a))\mu(g(x_P)_1 \otimes b)) \\
&= \mu((g x_P)_2 \otimes a))\mu((g x_P)_1 \otimes b))
\end{aligned}$$

for every $a,b \in A$. We have proved that (48) holds for every $a,b \in A$ and every h of the canonical basis of $E(n)$. Since involved maps are k-linear, this implies that (48) holds for every $h \in E(n)$ and every $a,b \in A$.

In this way we have established an assignment

$$(\varphi, d_1, \ldots, d_n) \quad \overset{\Psi}{\longmapsto} \quad \mu : E(n)^{cop} \otimes A \longrightarrow A$$
$$g^j x_P \otimes a \longmapsto \varphi^j(d_P(a))$$

so that an $E(n)^{cop}$-action corresponds to each $n+1$-uple of maps $(\varphi, d_1, \ldots, d_n)$, where φ is an involution, the d_i's are φ-derivations such that $d_i^2 \equiv 0$, $\varphi d_i = -d_i\varphi$ and $d_i d_j = -d_j d_i$ for every $i, j = 1, \ldots, n$. It is straightforward to check that the assignments Φ and Ψ are inverse to each other, and therefore the correspondence between $E(n)^{cop}$-actions and such $n+1$-uples (φ, d_i) is bijective.

Proposition 1.3. *Let A be a finite-dimensional algebra over a field k of characteristic $char(k) \neq 2$, then an $E(n)^{cop}$-action on A is completely determined by a choice of:*

(1) an automorphism φ of A of order $o(\varphi) \leq 2$ (i.e. an involution or the identity),
(2) a family $\{d_i\}$ of φ-derivations such that $d_i^2 \equiv 0$, $\varphi d_i = -d_i\varphi$ and $d_i d_j = -d_j d_i$ for $i, j = 1, \ldots, n$, $i < j$.

Remark 1.4. This result is a generalization of Proposition 1.5 of Chapter 2. Unlike that statement, this cannot be deduced directly by any result contained in [**CY**], where the authors studied the actions of Taft's algebras on finite dimensional algebras. Notice that these algebras are mostly unrelated to the $E(n)$'s: the intersection between the two families consists only of Sweedler's algebra.

As in the four-dimensional case, this statement can be expressed in terms of $\mathbb{Z}_2$-gradings.

Proposition 1.5. *Let A be a finite-dimensional algebra over a field k of characteristic $char(k) \neq 2$, then an action of $E(n)^{cop}$ on A is completely determined by a choice of:*

(1) a $\mathbb{Z}_2$ grading $A = A_+ \oplus A_-$,
(2) a family $\{d_i\}$ of φ-derivations (where φ defines the above grading) such that $d_i^2 \equiv 0$, $d_i d_j = -d_j d_i$ and $d_i(A_\pm) \subseteq A_\mp$ for every $i, j = 1, \ldots, n$.

1.3. The explicit correspondence. Finally we can employ the established correspondence to write down the explcit expression of a coaction ρ in terms of the associated involution φ and derivations $d_1, \ldots, d_n$.

We have seen that every $E(n)$-coaction ρ on A is defined by

$$\rho(a) = \sum_{P \subseteq \{1,\ldots,n\}} (-1)^{\lfloor \frac{|P|+1}{2} \rfloor} \left[\mu(x_P \otimes a) \otimes \frac{x_P + (-1)^{|P|} g x_P}{2} + \mu(g x_P \otimes a) \otimes \frac{x_P + (-1)^{|P|+1} g x_P}{2} \right]. \tag{228}$$

for every $a \in A$, where μ is an $E(n)^{cop}$-action. Since each $E(n)^{cop}$-action is in bijective correspondence with an $n+1$-uple (φ, d_i) where $\varphi : A \to A$ is an involution and each $d_i : A \to A$ is a φ-derivation such that $d_i^2 \equiv 0$, $\varphi d_i = -d_i \varphi$ and $d_i d_j = -d_j d_i$ for every $i, j = 1, \ldots, n$, (228) rewrites as

$$\rho(a) = \sum_{P \subseteq \{1,\ldots,n\}} (-1)^{\lfloor \frac{|P|+1}{2} \rfloor} \left[d_P(a) \otimes \frac{x_P + (-1)^{|P|} g x_P}{2} + \varphi(d_P(a)) \otimes \frac{x_P + (-1)^{|P|+1} g x_P}{2} \right],$$

where again $d_P = d_{i_1} d_{i_2} \cdots d_{i_s}$ for $P = \{i_1 < i_2 < \ldots < i_s\}$. The main result can be restated in the following way.

Theorem 1.6. *Let A be a finite dimensional algebra over a field k of characteristic $char(k) \neq 2$, then an $E(n)$-comodule algebra structure on A is given by:*

$$\rho(a) = \sum_{P \subseteq \{1,\ldots,n\}} (-1)^{\lfloor \frac{|P|+1}{2} \rfloor} \left[d_P(a) \otimes \frac{x_P + (-1)^{|P|} g x_P}{2} + \varphi(d_P(a)) \otimes \frac{x_P + (-1)^{|P|+1} g x_P}{2} \right]$$

where

(1) φ is an automorphism of A of order $o(\varphi) \leq 2$ (i.e. an involution),
(2) $d_\varnothing = Id_A$ and $d_P = d_{i_1} d_{i_2} \cdots d_{i_{|P|}}$ is a composition of φ-derivations such that $d_i^2 \equiv 0$, $\varphi(d_i(a)) = -d_i(\varphi(a))$ and $d_i(d_j(a)) = -d_j(d_i(a))$ for any $a \in A$.

or equivalently by:

(1) a $\mathbb{Z}_2$ grading $A = A_+ \oplus A_-$,

(2) a family $\{d_i\}$ of φ-derivations (where φ defines the above grading) such that $d_i^2 \equiv 0$, $d_i d_j = -d_j d_i$ and $d_i(A_\pm) \subseteq A_\mp$ for every $i, j = 1, \dots, n$.

REMARK 1.7. φ and each d_i are completely determined once we have fixed $\varphi(G)$, $\varphi(X)$, $d_i(G)$ and $d_i(X)$.

It is now clear that to have a classification of all $E(n)$-coactions on a finite-dimensional algebra A is equivalent to have a classification of all the involutions of A and the corresponding skew-derivations satisfying the hypothesis of Theorem 1.6.

2. A first insight into the case A simple

To reach complete knowledge of involutions and skew-derivations for a Clifford algebra $A = Cl(\alpha, \beta_i, \gamma_i, \lambda_{ij})$ of dimension greater than four seems not an easy task at all. The machinery developed in Appendix A is of no use when dealing with the general case and, in fact, performs quite poorly even in the simplest of the remaining cases – when $\dim_k A = 8$ matrices have 64 entries and quadratic equations are not easy to solve even with the help of a software. In order to simplify our task we can add the further assumption that A be simple. In this case the Skolem-Noether Theorem [**Lam**, Thm. IV.1.8] ensures that every automorphism of A, and thus every involution of A, is inner. It can be proved that the simplicity of A also forces every φ-derivation d to be inner, but in our case, since we are interested in skew-derivation anticommuting with involutions, we will see that the fact that d is inner comes as an immediate consequence of this request. We will prove that in the case A is simple, Theorem 1.6 can be restated and tuples containing an involution φ and n skew-derivations d_i can be substituted by tuples $(c, u_1, u_2, \dots, u_n)$ of suitable elements of A. Notice that once again we can work with a finite-dimensional algebra A which does not have to be necessarily a Clifford one.

Let us start by giving the definition of inner automorphism and inner skew-derivation.

DEFINITION 2.1. Let A be an algebra and $\varphi : A \to A$ an automorphism of A. φ is said to be an *inner* automorphism if there exists an invertible element $c \in A$ such that

$$\varphi(b) = c^{-1}bc \qquad \text{for all } b \in A.$$

A φ-derivation $d : A \to A$ is said to be *inner* if there exists an element $u \in A$ such that

$$d(b) = ub - \varphi(b)u \qquad \text{for all } b \in A.$$

Theorem 1.6 takes into account an involution φ of order $o(\varphi) \leq 2$ and φ-derivations d_i such that $d_i^2 \equiv 0$, $d_i\varphi = -\varphi d_i$ and $d_i d_j = -d_j d_i$ for every $i, j = 1, \dots, n$. Let us show what these conditions translate into when φ is an inner involution. We will write φ_c to indicate the automorphism such that $\varphi_c(b) = c^{-1}bc$ for every $b \in A$ and d_i to indicate the skew-derivation such that $d_i(b) = u_i b - \varphi(b)u_i$ for every $b \in A$.

$$\varphi_c^2 \equiv \mathrm{Id} \iff c^{-2}bc^2 = b \quad \text{for all } b \in A \iff bc^2 = c^2 b \quad \text{for all } b \in A \iff c^2 \in \mathcal{Z}(A).$$

Next we want to prove that a φ_c-derivations d anticommute with φ_c if and only if d is of a very specific form.

$$d\varphi_c = -\varphi_c d \iff d(\varphi_c(b)) = -\varphi_c(d(b)) \quad \text{for all } b \in A$$

The LHS is

$$\begin{aligned} d(\varphi_c(b)) &= d(c^{-1}bc) \\ &= d(c^{-1})bc + \varphi_c(c^{-1})d(bc) \\ &= d(c^{-1})bc + c^{-1}c^{-1}c[d(b)c + \varphi_c(b)d(c)] \\ &= d(c^{-1})bc + c^{-1}d(b)c + c^{-1}\varphi_c(b)d(c). \end{aligned}$$

The RHS gives $-\varphi_c(d(b)) = -c^{-1}d(b)c$. Thus we can easily conclude that d anticommutes with φ_c if, and only if,

$$-c^{-1}d(b)c = d(c^{-1})bc + c^{-1}d(b)c + c^{-1}\varphi_c(b)d(c) \iff d(b) = \frac{-cd(c^{-1})b - \varphi_c(b)d(c)c^{-1}}{2}.$$

It is immediate to see that $d(c)c^{-1} + cd(c^{-1}) = d(c)c^{-1} + \varphi_c(c)d(c^{-1}) = d(cc^{-1}) = d(1) = 0$ and therefore that $cd(c^{-1}) = -d(c)c^{-1}$, hence

$$d(b) = \frac{d(c)c^{-1}}{2}b - \varphi_c(b)\frac{d(c)c^{-1}}{2}. \tag{229}$$

We have proved that d anticommutes with φ_c if, and only if, $d(b) = ub - \varphi_c(b)u$ with $u = \frac{d(c)c^{-1}}{2}$. In particular d must be an inner φ_c-derivation. Notice that (229) for $b = c$ gives

$$d(c) = uc - \varphi_c(c)u = uc - cu \iff 2uc = uc - cu \iff uc + cu = 0$$

We also point out that

$$\varphi_c(u) = \varphi_c\left(\frac{d(c)c^{-1}}{2}\right) = \frac{1}{2}\varphi_c(d(c))\varphi_c(c^{-1}) = -\frac{1}{2}d(\varphi_c(c))c^{-1} = -\frac{1}{2}d(c)c^{-1} = -u \tag{230}$$

$$d(u) = u^2 - \varphi_c(u)u = u^2 + u^2 = 2u^2. \tag{231}$$

Since we are mostly interested in skew-derivations satisfying all of the requests of Theorem 1.6, from now on we will assume that $\varphi_c^2 \equiv \mathrm{Id}$ and that $\varphi_c d_i = -d_i\varphi_c$ for every $i = 1, \ldots, n$. We have that $d_i^2 \equiv 0$ if, and only if, for every $b \in A$

$$d_i(u_ib - \varphi_c(b)u_i) = 0 \iff u_i^2 b - \varphi_c(u_ib)u_i - u_i\varphi_c(b)u_i + \varphi_c(\varphi_c(b)u_i)u_i = 0$$
$$\iff u_i^2 b - \varphi_c(u_i)\varphi_c(b)u_i - u_i\varphi_c(b)u_i + \varphi_c^2(b)\varphi_c(u_i)u_i = 0 \overset{(230)}{\iff} u_i^2 b + u_i\varphi_c(b)u_i - u_i\varphi_c(b)u_i - bu_i^2 = 0$$
$$\iff u_i^2 b - bu_i^2 = 0.$$

This is equivalent to $u_i^2 \in \mathcal{Z}(A)$.

Finally let $d_i \neq d_j$ be φ_c-derivations. Then $d_id_j = -d_jd_i$ if, and only if, for every $b \in A$

$$d_i(d_j(b)) = -d_j(d_i(b)) \iff u_id_j(b) - \varphi_c(d_j(b))u_i = -u_jd_i(b) + \varphi_c(d_i(b))u_j$$
$$\iff u_iu_jb - u_i\varphi_c(b)u_j - \varphi_c(u_jb - \varphi_c(b)u_j)u_i = -u_ju_ib + u_j\varphi_c(b)u_i + \varphi_c(u_ib - \varphi_c(b)u_i)u_j$$
$$\overset{(230)}{\iff} u_iu_jb - u_i\varphi_c(b)u_j + u_j\varphi_c(b)u_i - bu_ju_i = -u_ju_ib + u_j\varphi_c(b)u_i - u_i\varphi_c(b)u_j + bu_iu_j$$
$$\iff u_iu_jb - bu_ju_i = -u_ju_ib + bu_iu_j \iff (u_iu_j + u_ju_i)b = b(u_iu_j + u_ju_i).$$

This is equivalent to $u_iu_j + u_ju_i \in \mathcal{Z}(A)$.

In this way we have shown that if the involution φ that determines the coaction in the statement of Theorem 1.6 is inner, say φ_c, then also the φ_c-derivations d_i need to be inner with $u_i = \frac{d(c)c^{-1}}{2}$ and the elements c and u_i defining each map must satisfy

$$c^2 \in \mathcal{Z}(A) \tag{232}$$

$$u_ic + cu_i = 0 \text{ for every } i = 1, \tag{233}$$

$$u_i^2 \in \mathcal{Z}(A) \text{ for every } i = 1, \ldots, n \tag{234}$$

$$u_iu_j + u_ju_i \in \mathcal{Z}(A) \text{ for every } i, j = 1, \ldots, n. \tag{235}$$

Conversely, if $\varphi_c : A \to A$ is defined by $\varphi_c(b) = c^{-1}bc$ with c satisfying (232), then it is immediate to check that φ_c is an automorphism such that $\varphi_c^2 \equiv \mathrm{Id}$. Moreover, given the map $d_i : A \to A$ defined by $d_i(b) = u_ib - \varphi_c(b)u_i$ with $u_ic = -cu_i$, we can easily show that it satisfies $\varphi_cd_i = -d_i\varphi_c$. In fact we have

$$\begin{aligned}
\varphi_c(d_i(b)) &= \varphi_c(u_ib - \varphi_c(b)u_i) \\
&\overset{(232)}{=} \varphi_c(u_i)\varphi_c(b) - b\varphi_c(u_i) \\
&= c^{-1}u_ibc - bc^{-1}u_ic \\
&\overset{(233)}{=} -u_ic^{-1}bc + bu_i \\
&= -d_i(c^{-1}bc) \\
&= -d_i\varphi_c(b)
\end{aligned}$$

for every $b \in B$. If (235) is also satisfied for d_i and d_j defined as before, then

$$d_i(d_j(b)) = u_id_j(b) - \varphi_c(d_j(b))u_i$$

$$\begin{aligned}
&= u_i(u_jb-\varphi_c(b)u_j)-c^{-1}(u_jb-\varphi_c(b)u_j)cu_i\\
&= u_iu_jb-u_i\varphi_c(b)u_j-c^{-1}u_jbcu_i-c^{-1}\varphi_c(b)u_jcu_i\\
&= u_iu_jb-u_ic^{-1}bcu_j-c^{-1}u_jbcu_i-c^{-2}bcu_jcu_i\\
&\overset{(232)+(233)}{=} u_iu_jb+c^{-1}u_ibcu_j+u_jc^{-1}bcu_i+bu_ju_i\\
&\overset{(235)}{=} -u_ju_ib+c^{-1}u_ibcu_j+u_jc^{-1}bcu_i+bu_iu_j\\
&\overset{(232)+(233)}{=} -u_ju_ib+u_jc^{-1}bcu_i+c^{-1}u_ibcu_j-c^{-2}bcu_icu_j\\
&= -u_ju_ib+u_j\varphi_c(b)u_i+c^{-1}u_ibcu_j-c^{-1}\varphi_c(b)u_icu_j\\
&= -u_j(u_ib-\varphi_c(b)u_i)+c^{-1}(u_ib-\varphi_c(b)u_i)cu_j\\
&= -d_j(d_i(b))
\end{aligned}$$

for every $b \in B$. Finally, if we assume (234) to hold too,

$$\begin{aligned}
d_i^2(b) &= d_i(u_ib-\varphi_c(b)u_i)\\
&= u_i^2b-u_i\varphi_c(b)u_i-\varphi_c(u_ib)u_i+\varphi_c^2(b)\varphi_c(u_i)u_i\\
&\overset{(232)}{=} u_i^2b-u_ic^{-1}bcu_i-c^{-1}u_ibcu_i+bc^{-1}u_icu_i\\
&\overset{(233)}{=} u_i^2b-bu_i^2\\
&\overset{(234)}{=} 0.
\end{aligned}$$

Therefore any map φ_c and any tuple of maps d_i, $i=1,\dots,n$ with c and u_i satisfying (232)-(235) are maps that respect the required properties of Theorem 1.6. If we assume that the algebra A is simple, then by the Skolem-Noether Theorem, every involution of A is inner and we can state the following theorem.

THEOREM 2.2. *Let A be a simple finite dimensional algebra over a field k of characteristic $char(k) \neq 2$, then an $E(n)$-comodule algebra structure on A is given by:*

$$\rho(a)=\sum_{P\subseteq\{1,\dots,n\}}(-1)^{\lfloor\frac{|P|+1}{2}\rfloor}\left[d_P(a)\otimes\frac{x_P+(-1)^{|P|}gx_P}{2}+c^{-1}d_P(a)c\otimes\frac{x_P+(-1)^{|P|+1}gx_P}{2}\right]$$

where

(1) c is an invertible element of A such that $c^2 \in \mathcal{Z}(A)$,
(2) $d_\varnothing = Id_A$ and $d_P = d_{i_1}d_{i_2}\cdots d_{i_{|P|}}$ is a composition of maps $d_i : A \to A$ defined by $d_i(A) = u_ia - c^{-1}acu_i$ such that $u_ic + cu_i = 0$, $u_i^2 \in \mathcal{Z}(A)$ and $u_iu_j + u_ju_i \in \mathcal{Z}(A)$.

This theorem tells us that to have a classification of all $E(n)$-coactions on a finite-dimensional *simple* algebra A one needs to understand the structure of the set

$$\mathcal{Z}^2(A)=\{a\in A \mid a^2\in\mathcal{Z}(A)\}.$$

REMARK 2.3. Obviously $\mathcal{Z}(A) \subseteq \mathcal{Z}^2(A)$, but in most cases equality does not hold. For example, when $A = Cl(\alpha,\beta,\gamma)$ is a four-dimensional Clifford algebra, the elements $a = \lambda_1+\lambda_2G+\lambda_3X+\lambda_4GX$ contained in $\mathcal{Z}^2(A)\setminus\mathcal{Z}(A)$ are exactly those for which $2\lambda_1 + \gamma\lambda_4 = 0$ (see Remark 0.2).

When $A = Cl(\alpha,\beta_i,\gamma_i,\lambda_{ij})$ is a simple algebra, it is central simple over k or over $k(\sqrt{\det Q})$ (see Theorem 5.16). When $k=\mathbb{R}$, $\mathcal{Z}(A)$ can either be $\mathbb{R}$ or $\mathbb{C}$, while when $k=\mathbb{C}$, $\mathcal{Z}(A)$ is clearly $\mathbb{C}$. Altough the center of A is trivial in these cases, it is still not completely obvious how to determine the nature of the set $\mathcal{Z}^2(A)$.

CHAPTER 5

Isomorphic coactions

In this final chapter we will introduce the notion of *equivalent tuples*, which is the appropriate counterpart of isomorphic comodule algebras, and use it to determine a much more refined classification of $E(1)$-coactions in the four-dimensional case. Since the list of distinct pairs one can define using the table in Section 5 of Chapter 2 is clearly infinite, we started wondering whether some of this coactions were actually isomorphic and if it was possible to perfect our classification by identifying equivalent pairs. We were able to show that when $A = Cl(0,0,0)$ is the Exterior algebra the number of (non-isomorphic) H-coactions one can define is *finite*. By investigating the rest of the non-semisimple cases (i.e. when $\gamma^2 - 4\alpha\beta = 0$) we proved that the list of (non-isomorphic) coactions on A is still infinite, but of a much more compact form. Unfortunately we do not have a complete answer for the semisimple case, as it seems that the existence of isomorphisms between different coactions strongly relies on the structure of the ground field k. Nonetheless it should be noted that when A is semisimple it is either a central simple algebra or the product of two isomorphic central simple algebras (see Thm. 5.16). By the Skolem-Noether Theorem, every involution and every skew-derivation of a simple algebra is inner and therefore of a much simpler form. To define a coaction, by means of Theorem 1.6, using only inner involutions and skew-derivations is indeed the easiest part of the problem when one wants to understand all the $E(n)$-coactions on a finite-dimensional algebra. What is really challenging is to determine those that do not correspond to tuples with inner entries. With this motivation in mind we focused our last efforts on determining all the equivalent pairs in the non-semisimple case in hope of possibly recognising a pattern shared also by Clifford algebra with higher dimension.

1. Equivalent tuples

We have already established that $E(n)$-coactions on a finite-dimensional algebra A are in bijective correspondence with $n+1$-tuples $(\varphi, d_1, d_2, \dots, d_n)$ described in Theorem 1.6. If (A, ρ_A) and (B, ρ_B) are two isomorphic $E(n)$-comodule algebras, then there is an invertible algebra map $f : A \to B$ that also makes the following diagram commutative

$$\begin{array}{ccc} A & \xrightarrow{\rho_A} & A \otimes E(n) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f\otimes \mathrm{Id}_{E(n)}} \\ B & \xrightarrow{\rho_B} & B \otimes E(n) \end{array}.$$

In other words f is such that

$$\rho_B = (f \otimes \mathrm{Id}_{E(n)}) \circ \rho_A \circ f^{-1}.$$

If we explicitly write ρ_A and ρ_B in terms of their corresponding tuples we get that (A, ρ_A) and (B, ρ_B) are isomorphic as $E(n)$-comodule algebras if, and only if,

$$\sum_{P\subseteq\{1,\dots,n\}} (-1)^{\lfloor \frac{|P|+1}{2} \rfloor} \left[d_P^B(b) \otimes \frac{x_P + (-1)^{|P|} g x_P}{2} + \varphi^B(d_P^B(b)) \otimes \frac{x_P + (-1)^{|P|+1} g x_P}{2} \right] =$$

$$\sum_{P\subseteq\{1,\dots,n\}} (-1)^{\lfloor \frac{|P|+1}{2} \rfloor} \left[f d_P^A f^{-1}(b) \otimes \frac{x_P + (-1)^{|P|} g x_P}{2} + f\varphi^A d_P^A f^{-1}(b) \otimes \frac{x_P + (-1)^{|P|+1} g x_P}{2} \right]$$

for every $b \in B$. By linear independence of the terms appearing on the second leg of the tensor products, this is equivalent to

$$d_P^B = f d_P^A f^{-1} \tag{236}$$

$$\varphi^B d_P^B = f \varphi^A d_P^A f^{-1} \tag{237}$$

for all $P \subseteq \{1,2,\dots,n\}$. Choosing $P=\{i\}$ for each $i \in \{1,2,\dots,n\}$ gives $d_i^B = f d_i^A f^{-1}$ for every $i=1,\dots,n$, while setting $P=\varnothing$ in (237) yields $\varphi^B = f\varphi^A f^{-1}$. Conversely, if $d_i^B = f d_i^A f^{-1}$ for every $i=1,\dots,n$, by composition, we immediately find that (236) holds for every $P \subseteq \{1,2,\dots,n\}$. If also $\varphi^B = f\varphi^A f^{-1}$, then (237) follows.

We have proved the following result

PROPOSITION 1.1. *Let (A,ρ_A) and (B,ρ_B) be $E(n)$-comodule algebras and let $(\varphi^A, d_1^A, d_2^A, \dots, d_n^A)$ and $(\varphi^B, d_1^B, d_2^B, \dots, d_n^B)$ be the tuples defining their structures as in Theorem 1.6. Then A and B are isomorphic as $E(n)$-comodule algebras if, and only if, there exists an algebra isomorphism $f: A \to B$ such that*

$$(\varphi^B, d_1^B, d_2^B, \dots, d_n^B) = (f\varphi^A f^{-1}, f d_1^A f^{-1}, f d_2^A f^{-1}, \dots, f d_n^A f^{-1}).$$

This motivates the following definition.

DEFINITION 1.2. Two tuples $(\varphi^A, d_1^A, d_2^A, \dots, d_n^A)$ and $(\varphi^B, d_1^B, d_2^B, \dots, d_n^B)$ will be called *equivalent* if there exists an algebra isomorphism $f: A \to B$ such that $(\varphi^B, d_1^B, d_2^B, \dots, d_n^B) = (f\varphi^A f^{-1}, f d_1^A f^{-1}, f d_2^A f^{-1}, \dots, f d_n^A f^{-1})$.

REMARK 1.3. An interesting special case occurs when we choose $B = A$. In this case we see that all isomorphic $E(n)$-coactions on a fixed algebra A are obtained by conjugating a fixed tuple $(\varphi^A, d_1^A, d_2^A, \dots, d_n^A)$ by elements $f \in \mathrm{Aut}(A)$.

2. Isomorphic coactions of type 0 and 1

The last remark suggests that we can further inspect the classification of H-comodule algebra structures of $A = Cl(\alpha,\beta,\gamma)$ contained in Section 5, Chapter 2, underlining the conjugacy classes that are contained in each family of involutions and skew-derivations. A coaction defined by a pair (φ, d), where φ is in the family $\mathfrak{F}_i$ will be called a *coaction of type i* $(i = 0,1,2)$.

2.1. The case $\mathfrak{F}_0$. The family $\mathfrak{F}_0$ consists only of the identity map Id_H and also coincides with the conjugacy class of this element. The only skew-derivation corresponding to Id_H is the zero map and again it is the unique element belonging to its conjugacy class. Clearly there is only one H-coaction of type 0, namely the trivial one

$$\rho(a) = a \otimes 1_H \quad \text{for every } a \in A.$$

2.2. The case $\mathfrak{F}_1$. Let us define

$$P_{A,B} := \begin{pmatrix} 1 & 0 & -\gamma & \gamma \\ 0 & -\mathbf{B} & \gamma\mathbf{B} - 2\beta\mathbf{A} & \gamma\mathbf{B} - 2\beta\mathbf{A} \\ 0 & \mathbf{A} & \gamma\mathbf{A} - 2\alpha\mathbf{B} & \gamma\mathbf{A} - 2\alpha\mathbf{B} \\ 0 & 0 & 2 & -2 \end{pmatrix}$$

for every $\mathbf{A}, \mathbf{B} \in k$. Remember that we have

$$P_{A,B} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} (P_{A,B})^{-1} = \begin{pmatrix} 1 & \frac{\gamma}{2}\mathbf{A} & \frac{\gamma}{2}\mathbf{B} & \frac{\gamma}{2} \\ 0 & \frac{2\alpha\mathbf{B}-\gamma\mathbf{A}}{2}\mathbf{B} & \frac{\gamma\mathbf{B}-2\beta\mathbf{A}}{2}\mathbf{B} & \frac{\gamma\mathbf{B}-2\beta\mathbf{A}}{2} \\ 0 & \frac{\gamma\mathbf{A}-2\alpha\mathbf{B}}{2}\mathbf{A} & \frac{2\beta\mathbf{A}-\gamma\mathbf{B}}{2}\mathbf{A} & \frac{\gamma\mathbf{A}-2\alpha\mathbf{B}}{2} \\ 0 & -\mathbf{A} & -\mathbf{B} & 0 \end{pmatrix} = \mathfrak{F}_1^{A,B},$$

provided $\gamma^2 - 4\alpha\beta = 0$ and $\alpha\mathbf{B}^2 + \beta\mathbf{A}^2 - \gamma\mathbf{A}\mathbf{B} - 1 = 0$. If we set $\mathcal{P}_{A',B'}^{A,B} := P_{A',B'}(P_{A,B}^{-1})$, then it is clear that

$$\mathcal{P}_{A',B'}^{A,B}\mathfrak{F}_1^{A,B}(\mathcal{P}_{A',B'}^{A,B})^{-1} = \mathfrak{F}_1^{A',B'}, \tag{238}$$

when $\mathbf{A}, \mathbf{A}', \mathbf{B}, \mathbf{B}' \in k$ and

$$\begin{cases} \gamma^2 - 4\alpha\beta = 0 \\ \alpha\mathbf{B}^2 + \beta\mathbf{A}^2 - \gamma\mathbf{A}\mathbf{B} = 1 \\ \alpha\mathbf{B}'^2 + \beta\mathbf{A}'^2 - \gamma\mathbf{A}'\mathbf{B}' = 1. \end{cases} \tag{239}$$

The matrix

$$\mathcal{P}^{A,B}_{A',B'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha\mathbf{B}\mathbf{B}' + \beta\mathbf{A}\mathbf{A}' - \gamma\mathbf{A}\mathbf{B}' & -\beta(\mathbf{A}\mathbf{B}' - \mathbf{A}'\mathbf{B}) & 0 \\ 0 & \alpha(\mathbf{A}\mathbf{B}' - \mathbf{A}'\mathbf{B}) & \alpha\mathbf{B}\mathbf{B}' + \beta\mathbf{A}\mathbf{A}' - \gamma\mathbf{A}'\mathbf{B} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

has determinant $\det \mathcal{P}^{A,B}_{A',B'} = (\alpha\mathbf{B}^2 + \beta\mathbf{A}^2 - \gamma\mathbf{A}\mathbf{B})(\alpha\mathbf{B}'^2 + \beta\mathbf{A}'^2 - \gamma\mathbf{A}'\mathbf{B}') \overset{(239)}{=} 1$ and therefore corresponds to a k-linear map $f : A \to A$ which is invertible. By direct computation one checks that (100)-(104) are satisfied and concludes that f is an automorphism of A. This shows that any two matrices in the family $\mathfrak{F}_1$ are conjugate.

Now define

$$D_{A,B} := \begin{pmatrix} 0 & \gamma\frac{\gamma\mathbf{A}-2\alpha\mathbf{B}}{2} & \gamma\frac{2\beta\mathbf{A}-\gamma\mathbf{B}}{2} & 0 \\ 0 & -\gamma & -2\beta & 0 \\ 0 & 2\alpha & \gamma & 0 \\ 0 & 2\alpha\mathbf{B} - \gamma\mathbf{A} & \gamma\mathbf{B} - 2\beta\mathbf{A} & 0 \end{pmatrix}$$

for every $\mathbf{A}, \mathbf{B}$ satisfying $\gamma^2 - 4\alpha\beta = 0$ and $\alpha\mathbf{B}^2 + \beta\mathbf{A}^2 - \gamma\mathbf{A}\mathbf{B} - 1 = 0$. We know that

$$D_{A',B'} = \mathcal{P}^{A,B}_{A',B'} D_{A,B} (\mathcal{P}^{A,B}_{A',B'})^{-1}, \tag{240}$$

when $\mathbf{A}, \mathbf{A}', \mathbf{B}, \mathbf{B}' \in k$ and (239) are satisfied.

Now recall that any $\mathfrak{F}_1^{A,B}$-derivation $d_{A,B,\mathfrak{a}} : A \to A$ has matrix given by

$$\mathfrak{a} \cdot D_{A,B} = \mathfrak{a} \cdot \begin{pmatrix} 0 & \gamma\frac{\gamma\mathbf{A}-2\alpha\mathbf{B}}{2} & \gamma\frac{2\beta\mathbf{A}-\gamma\mathbf{B}}{2} & 0 \\ 0 & -\gamma & -2\beta & 0 \\ 0 & 2\alpha & \gamma & 0 \\ 0 & 2\alpha\mathbf{B} - \gamma\mathbf{A} & \gamma\mathbf{B} - 2\beta\mathbf{A} & 0 \end{pmatrix}.$$

If $\mathfrak{a} = 0$, then $d_{A,B,0}$ is the zero map and the pair $(\mathfrak{F}_1^{A,B}, 0)$ is equivalent to any other pair $(\mathfrak{F}_1^{A',B'}, 0)$, where $\mathbf{A}, \mathbf{A}', \mathbf{B}, \mathbf{B}' \in k$ satisfy (239). Now let $\mathfrak{a}, \mathfrak{a}' \neq 0$ and consider the invertible matrix

$$W_{\mathfrak{a}',\mathfrak{a}} = P \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{\mathfrak{a}'}{\mathfrak{a}} \end{pmatrix} P^{-1} = \begin{pmatrix} 1 & \gamma\mathbf{A}\frac{\mathfrak{a}-\mathfrak{a}'}{4\mathfrak{a}} & \gamma\mathbf{B}\frac{\mathfrak{a}-\mathfrak{a}'}{4\mathfrak{a}} & \gamma\frac{\mathfrak{a}-\mathfrak{a}'}{4\mathfrak{a}} \\ 0 & 1 + \mathbf{A}\frac{(\gamma\mathbf{B}-2\beta\mathbf{A})(\mathfrak{a}-\mathfrak{a}')}{4\mathfrak{a}} & \mathbf{B}\frac{(\gamma\mathbf{B}-2\beta\mathbf{A})(\mathfrak{a}-\mathfrak{a}')}{4\mathfrak{a}} & \frac{(\gamma\mathbf{B}-2\beta\mathbf{A})(\mathfrak{a}-\mathfrak{a}')}{4\mathfrak{a}} \\ 0 & \mathbf{A}\frac{(\gamma\mathbf{A}-2\alpha\mathbf{B})(\mathfrak{a}-\mathfrak{a}')}{4\mathfrak{a}} & 1 + \mathbf{B}\frac{(\gamma\mathbf{A}-2\alpha\mathbf{B})(\mathfrak{a}-\mathfrak{a}')}{4\mathfrak{a}} & \frac{(\gamma\mathbf{A}-2\alpha\mathbf{B})(\mathfrak{a}-\mathfrak{a}')}{4\mathfrak{a}} \\ 0 & -\mathbf{A}\frac{\mathfrak{a}-\mathfrak{a}'}{2\mathfrak{a}} & -\mathbf{B}\frac{\mathfrak{a}-\mathfrak{a}'}{2\mathfrak{a}} & 1 - \frac{\mathfrak{a}-\mathfrak{a}'}{2\mathfrak{a}} \end{pmatrix}.$$

We have

$$W_{\mathfrak{a}',\mathfrak{a}}(\mathfrak{a} \cdot D_{A,B})W^{-1}_{\mathfrak{a}',\mathfrak{a}} = P \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{\mathfrak{a}'}{\mathfrak{a}} \end{pmatrix} P^{-1} P \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mathfrak{a} & 0 & 0 \end{pmatrix} P^{-1} P \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{\mathfrak{a}}{\mathfrak{a}'} \end{pmatrix} P^{-1} =$$

$$= P \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mathfrak{a}' & 0 & 0 \end{pmatrix} P^{-1} = \mathfrak{a}' \cdot D_{A,B},$$

therefore matrices $\mathfrak{a} \cdot D_{A,B}$ and $\mathfrak{a}' \cdot D_{A,B}$ are always conjugate, provided both $\mathfrak{a}$ and $\mathfrak{a}'$ are non-zero. One checks again by direct calculation that (100)-(104) are satisfied, i.e. that $W_{\mathfrak{a}',\mathfrak{a}}$ is the matrix of an algebra automorphism. It follows from (240) that

$$\mathfrak{a}' \cdot D_{A',B'} = (\mathcal{P}^{A,B}_{A',B'} W_{\mathfrak{a}',\mathfrak{a}})(\mathfrak{a} \cdot D_{A,B})(\mathcal{P}^{A,B}_{A',B'} W_{\mathfrak{a}',\mathfrak{a}})^{-1}$$

when $\mathbf{A}, \mathbf{A}', \mathbf{B}, \mathbf{B}' \in k$ satisfy (239) and $\mathfrak{a}, \mathfrak{a}' \in k^\times$. Moreover

$$W_{\mathfrak{a}',\mathfrak{a}} \mathfrak{F}_1^{A,B} W_{\mathfrak{a}',\mathfrak{a}}^{-1} = P \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{\mathfrak{a}'}{\mathfrak{a}} \end{pmatrix} P^{-1} P \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} P^{-1} P \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{\mathfrak{a}}{\mathfrak{a}'} \end{pmatrix} P^{-1} =$$

$$= P \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} P^{-1} = \mathfrak{F}_1^{A,B},$$

thus

$$\mathfrak{F}_1^{A',B'} = \mathcal{P}_{A',B'}^{A,B} \mathfrak{F}_1^{A,B} (\mathcal{P}_{A',B'}^{A,B})^{-1} = \mathcal{P}_{A',B'}^{A,B} W_{\mathfrak{a}',\mathfrak{a}} \mathfrak{F}_1^{A,B} W_{\mathfrak{a}',\mathfrak{a}}^{-1} (\mathcal{P}_{A',B'}^{A,B})^{-1} = (\mathcal{P}_{A',B'}^{A,B} W_{\mathfrak{a}',\mathfrak{a}}) \mathfrak{F}_1^{A,B} (\mathcal{P}_{A',B'}^{A,B} W_{\mathfrak{a}',\mathfrak{a}})^{-1}.$$

Since the k-linear map $f' : A \to A$ whose matrix is $\mathcal{P}_{A',B'}^{A,B} W_{\mathfrak{a}',\mathfrak{a}}$ is an algebra automorphism (it is the composition of two algebra automorphisms), we get that the pair $(\mathfrak{F}_1^{A',B'}, d_{A',B',\mathfrak{a}'})$ is equivalent to any other pair $(\mathfrak{F}_1^{A,B}, d_{A,B,\mathfrak{a}})$ where $\mathbf{A}, \mathbf{A}', \mathbf{B}, \mathbf{B}' \in k$ satisfy (239) and $\mathfrak{a}, \mathfrak{a}' \in k^\times$. This shows that there are at most 2 non-isomorphic H-coactions of type 1 on $A = Cl(\alpha, \beta, \gamma)$.

The study of coactions of type 2 will require much more work and we will deal with four cases separately, depending on the values of α, β and γ. In each of this cases A is supposed to be non-semisimple, i.e. $\gamma^2 - 4\alpha\beta = 0$.

3. Isomorphic coactions of type 2: the non-semisimple case

In this section we are going to determine a minimal list of non-isomorphic H-coactions for every four-dimensional Clifford algebra $A = Cl(\alpha, \beta, \gamma)$ such that $\gamma^2 - 4\alpha\beta = 0$. In each case we proceed more or less following the same strategy: to show that two pairs (φ_1, d_1) and (φ_2, d_2) with associated matrices M_1, $\mathfrak{D}_1$ and M_2, $\mathfrak{D}_2$ are equivalent we exhibit an invertible matrix Z such that $M_2 = Z^{-1} M_1 Z$ and $\mathfrak{D}_2 = Z^{-1} \mathfrak{D}_1 Z$. Z needs to be the matrix of an algebra automorphism, that is must satisfy (100)-(104), and therefore will always be of the form described in Lemma 3.2. When not clearly stated, every one of these matrices will be considered well-defined according to appropriate conditions contained in a table of coactions at the beginning of each subsection. For instance: let us consider a pair (φ_1, d_1) with matrices

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{2\mathbf{E}}{\mathbf{C}} & \frac{2\mathbf{A}}{\mathbf{C}} & 1 \end{pmatrix}, \quad \mathfrak{D}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\mathfrak{a}\mathbf{D}}{\mathbf{C}} & \frac{\mathbf{D}(\mathfrak{a}\mathbf{A} - \mathfrak{b}\mathbf{C})}{\mathbf{CE}} & 0 \end{pmatrix}$$

belonging to the first row of Table 1. By definition we have $\mathbf{C}, \mathbf{D}, \mathbf{E} \neq 0$. Then, if we want to show that such a pair is equivalent to a pair from the second row of Table 1, we need to define an invertible matrix Z that satisfies (100)-(104) and such that

$$Z^{-1} M_1 Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{2\mathbf{E}'}{\mathbf{C}'} & \frac{2\mathbf{A}'}{\mathbf{C}'} & 1 \end{pmatrix} =: M_2, \quad Z\mathfrak{D}_1 Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mathfrak{a}'\mathbf{D}' & \frac{\mathfrak{b}'\mathbf{D}'}{\mathbf{C}'} & 0 \end{pmatrix} =: \mathfrak{D}_2.$$

Clearly the entries of Z are going to depend from $\mathbf{C}', \mathbf{D}', \mathfrak{a}', \mathfrak{b}'$ and these scalars are forced to respect the conditions reported in Table 1, otherwise M_2 and $\mathfrak{D}_2$ will not define an appropriate pair. In this particular case $\mathbf{C}', \mathbf{D}' \neq 0$ and such conditions allow us, for example, to define Z using fractions where $\mathbf{C}'$ and $\mathbf{D}'$ are denominators.

In case we want to show that two pairs (φ_1, d_1) and (φ_2, d_2) cannot be equivalent we assume the existence of an algebra automorphism with associated matrix Z such that $M_2 = Z^{-1} M_1 Z$ and $\mathfrak{D}_2 = Z^{-1} \mathfrak{D}_1 Z$, and show that this leads to a contradiction. Since equivalent pairs define a conjugacy class, when suitable for our purposes, we will make use of representatives, by specifying the values of $\mathbf{A}, \mathbf{B}, \mathbf{C}$, etc.

3.1. The case $\mathfrak{F}_2$ when $\alpha = \beta = \gamma = 0$. When $\alpha = \beta = \gamma = 0$ every coaction of type $\mathfrak{F}_2$ corresponds to a pair (φ, d) where φ and d are of the following form[1].

Matrix M_φ of φ	Matrix of d	Conditions
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{2\mathbf{E}}{\mathbf{C}} & \frac{2\mathbf{A}}{\mathbf{C}} & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\mathfrak{a}\mathbf{D}}{\mathbf{C}} & \frac{\mathbf{D}(\mathfrak{a}\mathbf{A}-\mathfrak{b}\mathbf{C})}{\mathbf{CE}} & 0 \end{pmatrix}$	$\alpha = \beta = \gamma = 0$ $\mathbf{C}, \mathbf{D}, \mathbf{E} \neq 0$
	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mathfrak{a}\mathbf{D} & \frac{\mathfrak{b}\mathbf{D}}{\mathbf{C}} & 0 \end{pmatrix}$	$\alpha = \beta = \gamma = \mathbf{E} = 0$ $\mathbf{C}, \mathbf{D} \neq 0$
	$\begin{pmatrix} 0 & -\frac{\mathfrak{b}\mathbf{E}}{\mathbf{D}} & -\frac{\mathfrak{a}\mathbf{C}+\mathfrak{b}\mathbf{A}}{\mathbf{D}} & 0 \\ 0 & \frac{\mathbf{E}(\mathfrak{a}\mathbf{C}+\mathfrak{b}\mathbf{A})}{\mathbf{CD}} & \frac{\mathbf{A}(\mathfrak{a}\mathbf{C}+\mathfrak{b}\mathbf{A})}{\mathbf{CD}} & \frac{\mathfrak{a}\mathbf{C}+\mathfrak{b}\mathbf{A}}{\mathbf{D}} \\ 0 & -\frac{\mathfrak{b}\mathbf{E}^2}{\mathbf{CD}} & -\frac{\mathfrak{b}\mathbf{AE}}{\mathbf{CD}} & -\frac{\mathfrak{b}\mathbf{E}}{\mathbf{D}} \\ 0 & -\frac{\mathfrak{a}\mathbf{E}^2}{\mathbf{CD}} & -\frac{\mathfrak{a}\mathbf{AE}}{\mathbf{CD}} & -\frac{\mathfrak{a}\mathbf{E}}{\mathbf{D}} \end{pmatrix}$	$\alpha = \beta = \gamma = 0$ $\mathbf{C}, \mathbf{D}, \mathbf{E} \neq 0$
	$\begin{pmatrix} 0 & \frac{\mathfrak{a}\mathbf{C}}{\mathbf{D}} & -\frac{\mathfrak{b}}{\mathbf{D}} & 0 \\ 0 & 0 & \frac{\mathfrak{b}\mathbf{A}}{\mathbf{CD}} & \frac{\mathfrak{b}}{\mathbf{D}} \\ 0 & 0 & \frac{\mathfrak{a}\mathbf{A}}{\mathbf{D}} & \frac{\mathfrak{a}\mathbf{C}}{\mathbf{D}} \\ 0 & 0 & -\frac{\mathfrak{a}\mathbf{A}^2}{\mathbf{CD}} & -\frac{\mathfrak{a}\mathbf{A}}{\mathbf{D}} \end{pmatrix}$	$\alpha = \beta = \gamma = \mathbf{E} = 0$ $\mathbf{C}, \mathbf{D} \neq 0$
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{\mathbf{AB}+\mathbf{EF}}{\mathbf{AB}-\mathbf{EF}} & -\frac{2\mathbf{AF}}{\mathbf{AB}-\mathbf{EF}} & 0 \\ 0 & \frac{2\mathbf{BE}}{\mathbf{AB}-\mathbf{EF}} & \frac{\mathbf{AB}+\mathbf{EF}}{\mathbf{AB}-\mathbf{EF}} & 0 \\ 0 & -\frac{2\mathbf{DE}}{\mathbf{AB}-\mathbf{EF}} & -\frac{2\mathbf{AD}}{\mathbf{AB}-\mathbf{EF}} & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{\mathfrak{b}\mathbf{AE}}{\mathbf{AB}-\mathbf{EF}} & -\frac{\mathfrak{b}\mathbf{A}^2}{\mathbf{AB}-\mathbf{EF}} & 0 \\ 0 & \frac{\mathfrak{b}\mathbf{E}^2}{\mathbf{AB}-\mathbf{EF}} & \frac{\mathfrak{b}\mathbf{AE}}{\mathbf{AB}-\mathbf{EF}} & 0 \\ 0 & \frac{\mathfrak{a}\mathbf{E}^2}{\mathbf{AB}-\mathbf{EF}} & \frac{\mathfrak{a}\mathbf{AE}}{\mathbf{AB}-\mathbf{EF}} & 0 \end{pmatrix}$	$\alpha = \beta = \gamma = 0$ $\mathbf{AB} - \mathbf{EF}, \mathbf{E} \neq 0$
	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\mathfrak{b}}{\mathbf{B}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mathfrak{a}\mathbf{A}}{\mathbf{B}} & 0 \end{pmatrix}$	$\alpha = \beta = \gamma = \mathbf{E} = 0$ $\mathbf{A}, \mathbf{B} \neq 0$

TABLE 1. Pairs defining coactions on $A = Cl(0, 0, 0)$.

REMARK 3.1. It is not hard to prove that $d \equiv 0$ is equivalent to $\mathfrak{a} = \mathfrak{b} = 0$ in each of the displayed instances.

Our aim is to understand what pairs contained in this table are equivalent and what are not. In order to reach this goal we will make use of the following lemma.

[1]This table and the following are obtained by looking at Table 2 and Table 5 of Chapter 2.

LEMMA 3.2. *Let $A = Cl(\alpha,\beta,\gamma)$. Then $f \in Aut(A)$ only if it has matrix of the form*

$$Z = \begin{pmatrix} 1 & -\frac{\gamma}{2}Z_{42} & -\frac{\gamma}{2}Z_{43} & \alpha Z_{22}Z_{23} + \beta Z_{32}Z_{33} + \gamma Z_{23}Z_{32} + \left(\frac{\gamma^2}{4} - \alpha\beta\right) Z_{42}Z_{43} \\ 0 & Z_{22} & Z_{23} & \beta(Z_{33}Z_{42} - Z_{32}Z_{43}) - \frac{\gamma}{2}(Z_{22}Z_{43} - Z_{23}Z_{42}) \\ 0 & Z_{32} & Z_{33} & \alpha(Z_{22}Z_{43} - Z_{23}Z_{42}) - \frac{\gamma}{2}(Z_{32}Z_{43} - Z_{33}Z_{42}) \\ 0 & Z_{42} & Z_{43} & Z_{22}Z_{33} - Z_{23}Z_{32} \end{pmatrix}. \tag{241}$$

PROOF. Since f is a k-linear map defined on a four-dimensional algebra it can be represented with a matrix $Z = (Z_{ij})_{i,j=1,\dots,4}$. Given that $f \in \mathrm{Aut}(A)$, we have that it is in particular an algebra map and therefore satisfies (100)-(104). (100) is equivalent to the fact that the first column of Z must be $(1,0,0,0)^t$. (101) can be rewritten as

$$\begin{pmatrix} Z_{12}^2 + \alpha Z_{22}^2 + \beta Z_{32}^2 + \gamma Z_{22}Z_{32} - \alpha\beta Z_{42}^2 \\ Z_{22}(2Z_{12} + \gamma Z_{42}) \\ Z_{32}(2Z_{12} + \gamma Z_{42}) \\ Z_{42}(2Z_{12} + \gamma Z_{42}) \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which forces $Z_{12} = -\frac{\gamma}{2}Z_{42}$, since Z is invertible and cannot have linearly dependent columns. Similarly one proves that (102) $\Longrightarrow$ $Z_{12} = -\frac{\gamma}{2}Z_{42}$. (103) can be rewritten as

$$\begin{pmatrix} Z_{14} \\ Z_{24} \\ Z_{34} \\ Z_{44} \end{pmatrix} = \begin{pmatrix} \alpha Z_{22}Z_{23} + \beta Z_{32}Z_{33} + \gamma Z_{23}Z_{32} + \left(\frac{\gamma^2}{4} - \alpha\beta\right) Z_{42}Z_{43} \\ \beta(Z_{33}Z_{42} - Z_{32}Z_{43}) - \frac{\gamma}{2}(Z_{22}Z_{43} - Z_{23}Z_{42}) \\ \alpha(Z_{22}Z_{43} - Z_{23}Z_{42}) - \frac{\gamma}{2}(Z_{32}Z_{43} - Z_{33}Z_{42}) \\ Z_{22}Z_{33} - Z_{23}Z_{32} \end{pmatrix}.$$

□

3.1.1. *Coactions with trivial skew-derivation.* Let us start our investigation by considering a coaction $\rho_1 : A \to A \otimes H$ with associated pair $(\varphi_1, 0)$, where φ_1 has matrix

$$M_{\mathbf{A},\mathbf{C},\mathbf{E}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{2\mathbf{E}}{\mathbf{C}} & \frac{2\mathbf{A}}{\mathbf{C}} & 1 \end{pmatrix}, \tag{242}$$

with $\mathbf{C} \neq 0$. It is straightforward to check that $M_{\mathbf{A},\mathbf{C},\mathbf{E}} = Z M_{\mathbf{A}',\mathbf{C}',\mathbf{E}'} Z^{-1}$ for

$$Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{\mathbf{E}'}{\mathbf{C}'} - \frac{\mathbf{E}}{\mathbf{C}} & \frac{\mathbf{A}'}{\mathbf{C}'} - \frac{\mathbf{A}}{\mathbf{C}} & 1 \end{pmatrix}$$

and that Z defines a k-linear map $f : A \to A$ satisfying (100)-(104). Moreover $\det Z = 1$, therefore two involutions with matrices $M_{\mathbf{A},\mathbf{C},\mathbf{E}}$ and $M_{\mathbf{A}',\mathbf{C}',\mathbf{E}'}$ are always conjugate via an $f \in \mathrm{Aut}(A)$. Hence we conclude that all pairs $(\varphi, 0)$ with M_φ of the form (242) are equivalent.

Similarly, we can prove that all coactions with associated pair $(\varphi_1, 0)$, where φ_1 has matrix

$$M_{\mathbf{A},\mathbf{B},\mathbf{D},\mathbf{E}\,\mathbf{F}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{\mathbf{AB}+\mathbf{EF}}{\mathbf{AB}-\mathbf{EF}} & -\frac{2\mathbf{AF}}{\mathbf{AB}-\mathbf{EF}} & 0 \\ 0 & \frac{2\mathbf{BE}}{\mathbf{AB}-\mathbf{EF}} & \frac{\mathbf{AB}+\mathbf{EF}}{\mathbf{AB}-\mathbf{EF}} & 0 \\ 0 & -\frac{2\mathbf{DE}}{\mathbf{AB}-\mathbf{EF}} & -\frac{2\mathbf{AD}}{\mathbf{AB}-\mathbf{EF}} & -1 \end{pmatrix} \tag{243}$$

with $\mathbf{AB}-\mathbf{EF}\neq 0$, are equivalent. Let Z be the matrix

$$Z := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\mathbf{AB'}-\mathbf{E'F}}{\mathbf{AB}-\mathbf{EF}} & \frac{\mathbf{AF'}-\mathbf{A'F}}{\mathbf{AB}-\mathbf{EF}} & 0 \\ 0 & \frac{\mathbf{BE'}-\mathbf{B'E}}{\mathbf{AB}-\mathbf{EF}} & \frac{\mathbf{A'B}-\mathbf{EF'}}{\mathbf{AB}-\mathbf{EF}} & 0 \\ 0 & \frac{\mathbf{E'(D'-D)}}{\mathbf{AB}-\mathbf{EF}} & \frac{\mathbf{A'(D'-D)}}{\mathbf{AB}-\mathbf{EF}} & \frac{\mathbf{A'B'}-\mathbf{E'F'}}{\mathbf{AB}-\mathbf{EF}} \end{pmatrix}.$$

Then Z defines a k-linear map satisfying (100)-(104) and we have that $\det Z = \left(\frac{\mathbf{A'B'}-\mathbf{E'F'}}{\mathbf{AB}-\mathbf{EF}}\right)^2 \neq 0$ if, and only if, $\mathbf{A'B'}-\mathbf{E'F'}\neq 0$. It is easy to check that $M_{\mathbf{A},\mathbf{B},\mathbf{D},\mathbf{E}\,\mathbf{F}} = ZM_{\mathbf{A'},\mathbf{B'},\mathbf{D'},\mathbf{E'}\,\mathbf{F'}}Z^{-1}$. Therefore $M_{\mathbf{A},\mathbf{B},\mathbf{D},\mathbf{E}\,\mathbf{F}}$ and $M_{\mathbf{A'},\mathbf{B'},\mathbf{D'},\mathbf{E'}\,\mathbf{F'}}$ are conjugate via an $f\in \mathrm{Aut}(A)$. From this we can conclude that all the coactions with associated pair $(\varphi,0)$, where M_φ is of form (243), are equivalent.

Finally we show that two pairs $(\varphi_1,0)$ and $(\varphi_2,0)$ with

$$M_{\varphi_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{2\mathbf{E}}{\mathbf{C}} & \frac{2\mathbf{A}}{\mathbf{C}} & 1 \end{pmatrix}, \qquad M_{\varphi_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{\mathbf{A'B'}+\mathbf{E'F'}}{\mathbf{A'B'}-\mathbf{E'F'}} & -\frac{2\mathbf{A'F'}}{\mathbf{A'B'}-\mathbf{E'F'}} & 0 \\ 0 & \frac{2\mathbf{B'E'}}{\mathbf{A'B'}-\mathbf{E'F'}} & \frac{\mathbf{A'B'}+\mathbf{E'F'}}{\mathbf{A'B'}-\mathbf{E'F'}} & 0 \\ 0 & -\frac{2\mathbf{D'E'}}{\mathbf{A'B'}-\mathbf{E'F'}} & -\frac{2\mathbf{A'D'}}{\mathbf{A'B'}-\mathbf{E'F'}} & -1 \end{pmatrix}$$

are never equivalent. We take the representatives

$$M_{\varphi_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad M_{\varphi_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{244}$$

i.e. we choose $\mathbf{A}=\mathbf{E}=\mathbf{A'}=\mathbf{B'}=\mathbf{D'}=0$ and $\mathbf{E'}=\mathbf{F'}=1$. Suppose there is an algebra automorphism $f: A\to A$ whose matrix Z is such that $M_{\varphi_1}Z - ZM_{\varphi_2}=0$. By Lemma 3.2 this equality reads

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2Z_{22} & 0 & 0 \\ 0 & -2Z_{32} & 0 & 0 \\ 0 & 0 & 2Z_{43} & 2(Z_{22}Z_{33}-Z_{23}Z_{32}) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which gives a contradiction, since Z is invertible.

In conclusion we have found that, up to isomorphism, there are only two coactions of type 2 on $A = Cl(0,0,0)$ corresponding to pairs with trivial skew-derivation. These are given by $\rho(G)=G\otimes g$, $\rho(X)=X\otimes g$ and $\rho(G)=G\otimes 1$, $\rho(X)=X\otimes g$ (to obtain them one simply substitute the representatives in (244) in the explicit correspondence described in Theorem 1.8).

3.1.2. *Coactions with non-trivial skew-derivation.* Let us consider a coaction with associated pair (φ_1,d_1) and corresponding matrices

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{2\mathbf{E}}{\mathbf{C}} & \frac{2\mathbf{A}}{\mathbf{C}} & 1 \end{pmatrix}, \qquad \mathfrak{D}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\mathfrak{a}\mathbf{D}}{\mathbf{C}} & \frac{\mathbf{D}(\mathfrak{a}\mathbf{A}-\mathfrak{b}\mathbf{C})}{\mathbf{CE}} & 0 \end{pmatrix}. \tag{245}$$

The skew-derivation d_1 is taken to be non-trivial, which means that either $\mathfrak{a}\neq 0$ or $\mathfrak{b}\neq 0$ by Remark 3.1. Let (φ_2,d_2) be a pair equivalent to (φ_1,d_1) whose corresponding matrices are denoted by M_2 and $\mathfrak{D}_2$. This means that there is an invertible matrix of the form (241) such that $M_1Z-ZM_2=0$ and $\mathfrak{D}_1Z-Z\mathfrak{D}_2=0$. We have already proved that if such a Z exists, then M_2 must be of the same form of M_1 and therefore we will write

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{2\mathbf{E}'}{\mathbf{C}'} & \frac{2\mathbf{A}'}{\mathbf{C}'} & 1 \end{pmatrix}.$$

Hence we obtain that

$$M_1 Z - Z M_2 = 0 \iff \begin{cases} Z_{42} = -\dfrac{\mathbf{A}Z_{32}}{\mathbf{C}} - \dfrac{\mathbf{E}Z_{22}}{\mathbf{C}} + \dfrac{\mathbf{E}'(Z_{22}Z_{33} - Z_{23}Z_{32})}{\mathbf{C}'} \\ Z_{43} = -\dfrac{\mathbf{A}Z_{33}}{\mathbf{C}} - \dfrac{\mathbf{E}Z_{23}}{\mathbf{C}} + \dfrac{\mathbf{A}'(Z_{22}Z_{33} - Z_{23}Z_{32})}{\mathbf{C}'}. \end{cases}$$

We update the entries of Z and we calculate

$$\mathfrak{D}_2 = Z^{-1}\mathfrak{D}_1 Z = \frac{1}{Z_{22}Z_{33} - Z_{23}Z_{32}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\mathfrak{a}\mathbf{D}Z_{22}}{\mathbf{C}} + \frac{\mathbf{D}(\mathfrak{a}\mathbf{A} - \mathfrak{b}\mathbf{C})Z_{32}}{\mathbf{C}\mathbf{E}} & \frac{\mathfrak{a}\mathbf{D}Z_{23}}{\mathbf{C}} + \frac{\mathbf{D}(\mathfrak{a}\mathbf{A} - \mathfrak{b}\mathbf{C})Z_{33}}{\mathbf{C}\mathbf{E}} & 0 \end{pmatrix}. \tag{246}$$

We want to show that $\mathfrak{D}_2$ can only be of the first two forms contained in Table 1. We proceed by cases.

Assume $\mathfrak{a} = 0$. Then $\mathfrak{b} \neq 0$ (or $\mathfrak{D}_1$ would be trivial) and we can choose

$$\begin{cases} Z_{32} = -\dfrac{\mathfrak{a}'\mathbf{D}'\mathbf{E}}{\mathfrak{b}\mathbf{C}'\mathbf{D}} \\ Z_{33} = -\dfrac{\mathbf{D}'(\mathfrak{a}'\mathbf{A}' - \mathfrak{b}'\mathbf{C}')\mathbf{E}}{\mathfrak{b}\mathbf{C}'\mathbf{D}\mathbf{E}'} \end{cases}$$

so that

$$\mathfrak{D}_2 = Z^{-1}\mathfrak{D}_1 Z = \frac{\mathfrak{b}\mathbf{C}'\mathbf{D}\mathbf{E}'}{-\mathfrak{a}'\mathbf{A}'\mathbf{D}'\mathbf{E}Z_{22} + \mathfrak{b}'\mathbf{C}'\mathbf{D}'\mathbf{E}Z_{22} + \mathfrak{a}'\mathbf{D}'\mathbf{E}\mathbf{E}'Z_{23}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\mathfrak{a}'\mathbf{D}'}{\mathbf{C}'} & \frac{\mathbf{D}'(\mathfrak{a}'\mathbf{A}' - \mathfrak{b}'\mathbf{C}')}{\mathbf{C}'\mathbf{E}'} & 0 \end{pmatrix}.$$

If $\mathfrak{a}' = 0$, then $\mathfrak{b}' \neq 0$ (or else $\mathfrak{D}_2$ would be trivial) and we can choose $Z_{22} = \frac{\mathfrak{b}\mathbf{D}\mathbf{E}'}{\mathfrak{b}'\mathbf{D}'\mathbf{E}}$ and $Z_{23} = 0$. If $\mathfrak{a}' \neq 0$ then we can choose $Z_{22} = 0$ and $Z_{23} = \frac{\mathfrak{b}\mathbf{C}'\mathbf{D}}{\mathfrak{a}'\mathbf{D}'\mathbf{E}}$. In each of the presented cases we find that

$$\mathfrak{D}_2 = Z^{-1}\mathfrak{D}_1 Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\mathfrak{a}'\mathbf{D}'}{\mathbf{C}'} & \frac{\mathbf{D}'(\mathfrak{a}'\mathbf{A}' - \mathfrak{b}'\mathbf{C}')}{\mathbf{C}'\mathbf{E}'} & 0 \end{pmatrix}, \tag{247}$$

that Z satisfies (100)-(104), and $\det Z = 1$.

Suppose now $\mathfrak{a} \neq 0$. If we fix a $\mathfrak{b}' \neq 0$ and choose $Z_{22} = -\frac{(\mathfrak{a}\mathbf{A} - \mathfrak{b}\mathbf{C})\mathbf{D}\mathbf{E}'}{\mathfrak{b}'\mathbf{C}\mathbf{D}'\mathbf{E}}$, $Z_{23} = 1$, $Z_{32} = \frac{\mathfrak{a}\mathbf{D}\mathbf{E}'}{\mathfrak{b}'\mathbf{C}\mathbf{D}'}$, $Z_{33} = 0$, we get

$$\mathfrak{D}_2 = Z^{-1}\mathfrak{D}_1 Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\mathfrak{b}'\mathbf{D}'}{\mathbf{E}'} & 0 \end{pmatrix},$$

for an algebra morphism Z whose determinant is $\det Z = \left(\frac{\mathfrak{a}\mathbf{D}\mathbf{E}'}{\mathfrak{b}'\mathbf{C}\mathbf{D}'}\right)^2 \neq 0$. Notice that this is exactly (247) when $\mathfrak{a}' = 0$. Finally, if we fix a $\mathfrak{a}' \neq 0$, then we can choose $Z_{22} = 1$, $Z_{23} = \frac{\mathfrak{a}'\mathbf{A}' - \mathfrak{b}'\mathbf{C}'}{\mathfrak{a}'\mathbf{E}'} - \frac{\mathbf{C}'\mathbf{D}(\mathfrak{a}\mathbf{A} - \mathfrak{b}\mathbf{C})}{\mathfrak{a}'\mathbf{C}\mathbf{D}'\mathbf{E}}$, $Z_{32} = 0$, $Z_{33} = \frac{\mathfrak{a}\mathbf{C}'\mathbf{D}}{\mathfrak{a}'\mathbf{C}\mathbf{D}'}$ and we get again

$$\mathfrak{D}_2 = Z^{-1}\mathfrak{D}_1 Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\mathfrak{a}'\mathbf{D}'}{\mathbf{C}'} & \frac{\mathbf{D}'(\mathfrak{a}'\mathbf{A}' - \mathfrak{b}'\mathbf{C}')}{\mathbf{C}'\mathbf{E}'} & 0 \end{pmatrix},$$

for an algebra automorphism Z whose determinant is $\det Z = \left(\frac{\mathfrak{a}\mathbf{C}'\mathbf{D}}{\mathfrak{a}'\mathbf{C}\mathbf{D}'}\right)^2 \neq 0$.

We can conclude that all pairs (φ, d) with associated matrices of the form

$$M_\varphi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{2\mathbf{E}}{\mathbf{C}} & \frac{2\mathbf{A}}{\mathbf{C}} & 1 \end{pmatrix}, \qquad \mathfrak{D} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\mathfrak{a}\mathbf{D}}{\mathbf{C}} & \frac{\mathbf{D}(\mathfrak{a}\mathbf{A}-\mathfrak{b}\mathbf{C})}{\mathbf{C}\mathbf{E}} & 0 \end{pmatrix}$$

with $\mathbf{C}, \mathbf{D}, \mathbf{E} \neq 0$ are equivalent.

Now let us consider a pair (φ_2, d_2) whose associated matrices are

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{2\mathbf{E}'}{\mathbf{C}'} & \frac{2\mathbf{A}'}{\mathbf{C}'} & 1 \end{pmatrix}, \qquad \mathfrak{D}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mathfrak{a}'\mathbf{D}' & \frac{\mathfrak{b}'\mathbf{D}'}{\mathbf{C}'} & 0 \end{pmatrix},$$

i.e. a pair belonging to the second row of Table 1. Then we can show that this is equivalent to the pair (φ_1, d_1) with associated matrices

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \qquad \mathfrak{D}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

This is a representative of the family of equivalent pairs of the form (245) (take $\mathbf{A} = \mathfrak{a} = 0$, $\mathbf{C} = 2$, $\mathbf{D} = \mathbf{E} = 1$, $\mathfrak{b} = -2$). By transitivity we can conclude that all pairs belonging to the first and the second row of Table 1 are equivalent. To prove our claim we proceed by cases.

Assume $\mathfrak{a}' = 0$. If we fix a $\mathfrak{b}' \neq 0$ and consider

$$Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{\mathbf{C}'}{\mathfrak{b}'\mathbf{D}'} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{\mathbf{C}'-2\mathbf{E}'}{2\mathfrak{b}'\mathbf{D}'} & -\frac{\mathbf{A}'}{\mathfrak{b}'\mathbf{D}'} & -\frac{\mathbf{C}'}{\mathfrak{b}'\mathbf{D}'} \end{pmatrix},$$

then we can easily check that $ZM_2Z^{-1} = M_1$, $Z\mathfrak{D}_2Z^{-1} = \mathfrak{D}_1$, and that Z is an invertible algebra map.

Now suppose $\mathfrak{a}' \neq 0$. Then, for

$$Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\mathfrak{a}'\mathbf{D}'} & 0 \\ 0 & 1 & \frac{\mathfrak{b}'}{\mathfrak{a}'\mathbf{C}'} & 0 \\ 0 & -\frac{\mathbf{E}'}{\mathfrak{a}'\mathbf{C}'\mathbf{D}'} & -\frac{2\mathbf{A}'+\mathbf{C}'}{2\mathbf{C}'\mathbf{D}'\mathfrak{a}'} & -\frac{1}{\mathfrak{a}'\mathbf{D}'} \end{pmatrix},$$

we have that $ZM_2Z^{-1} = M_1$, $Z\mathfrak{D}_2Z^{-1} = \mathfrak{D}_1$ and $\det Z = \left(\frac{1}{\mathfrak{a}'\mathbf{D}'}\right)^2 \neq 0$.

We have proved that all pairs contained in the first and the second row of Table 1 are equivalent, but what about the other rows? Remember that if we conjugate the matrix $\mathfrak{D}_1$ defined in (245) via an algebra automorphism Z such that $M_1Z - ZM_2 = 0$, then we get a matrix of the form (246), where the first row consists of zeros. If this were one of the matrices of the families of the third or fourth row of Table 1, then this would imply that $\mathfrak{a} = \mathfrak{b} = 0$, i.e. that $\mathfrak{D}_1$ would be trivial, which is a contradiction. As a matter of fact, pairs whose skew-derivation belongs to the third and fourth rows of Table 1 form another equivalence class as we will show in the sequel. The coaction defined by an element of the class we have just obtained is (isomorphic to) $\rho(G) = G \otimes g - GX \otimes x$, $\rho(X) = X \otimes g$ (choose $\mathbf{A} = \mathbf{E} = \mathfrak{b} = 0$, $\mathbf{C} = \mathbf{D} = \mathfrak{a} = 1$).

Now we will prove that every pair contained in the third and fourth row of Table 1 is equivalent to the one whose associated matrices are

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}, \qquad \mathfrak{D}_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}.$$

It can be proved that all the matrices Z defined hereafter satisfy (100)-(104), i.e. are algebra maps.

We fix $\mathfrak{b} \neq 0$ and define

$$Z := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{\mathfrak{b}\mathbf{E}}{\mathbf{D}} & \frac{\mathfrak{b}\mathbf{A}+\mathfrak{a}\mathbf{C}}{\mathbf{D}} & 0 \\ 0 & -\frac{\mathfrak{b}\mathbf{E}^2}{\mathbf{CD}} & -\frac{\mathfrak{b}\mathbf{AE}}{\mathbf{CD}} - 1 & -\frac{\mathfrak{b}\mathbf{E}}{\mathbf{D}} \end{pmatrix}.$$

We have $\det Z = \left(\frac{\mathfrak{b}\mathbf{E}}{\mathbf{D}}\right)^2 \neq 0$ and it is easy to check that

$$Z^{-1}M_1Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{2\mathbf{E}}{\mathbf{C}} & \frac{2\mathbf{A}}{\mathbf{C}} & 1 \end{pmatrix}, \qquad Z^{-1}\mathfrak{D}_1Z = \begin{pmatrix} 0 & -\frac{\mathfrak{b}\mathbf{E}}{\mathbf{D}} & -\frac{\mathfrak{a}\mathbf{C}+\mathfrak{b}\mathbf{A}}{\mathbf{D}} & 0 \\ 0 & \frac{\mathbf{E}(\mathfrak{a}\mathbf{C}+\mathfrak{b}\mathbf{A})}{\mathbf{CD}} & \frac{\mathbf{A}(\mathfrak{a}\mathbf{C}+\mathfrak{b}\mathbf{A})}{\mathbf{CD}} & \frac{\mathfrak{a}\mathbf{C}+\mathfrak{b}\mathbf{A}}{\mathbf{D}} \\ 0 & -\frac{\mathfrak{b}\mathbf{E}^2}{\mathbf{CD}} & -\frac{\mathfrak{b}\mathbf{AE}}{\mathbf{CD}} & -\frac{\mathfrak{b}\mathbf{E}}{\mathbf{D}} \\ 0 & -\frac{\mathfrak{a}\mathbf{E}^2}{\mathbf{CD}} & -\frac{\mathfrak{a}\mathbf{AE}}{\mathbf{CD}} & -\frac{\mathfrak{a}\mathbf{E}}{\mathbf{D}} \end{pmatrix}.$$

Similarly, if we fix $\mathfrak{a} \neq 0$ and define

$$Z := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\mathfrak{a}\mathbf{C}}{\mathbf{D}} & 0 \\ 0 & \frac{\mathfrak{a}\mathbf{E}}{\mathbf{D}} - 1 & \frac{\mathfrak{a}\mathbf{A}}{\mathbf{D}} & \frac{\mathfrak{a}\mathbf{C}}{\mathbf{D}} \end{pmatrix}$$

we have that Z is invertible and furthermore

$$Z^{-1}M_1Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{2\mathbf{E}}{\mathbf{C}} & \frac{2\mathbf{A}}{\mathbf{C}} & 1 \end{pmatrix}, \qquad Z^{-1}\mathfrak{D}_1Z = \begin{pmatrix} 0 & 0 & -\frac{\mathfrak{a}\mathbf{C}}{\mathbf{D}} & 0 \\ 0 & \frac{\mathfrak{a}\mathbf{E}}{\mathbf{D}} & \frac{\mathfrak{a}\mathbf{A}}{\mathbf{D}} & \frac{\mathfrak{a}\mathbf{C}}{\mathbf{D}} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\mathfrak{a}\mathbf{E}^2}{\mathbf{CD}} & -\frac{\mathfrak{a}\mathbf{AE}}{\mathbf{CD}} & -\frac{\mathfrak{a}\mathbf{E}}{\mathbf{D}} \end{pmatrix}$$

which are exactly the matrices defining a pair in the third row of Table 1 when $\mathfrak{b} = 0$.

Next, if we fix $\mathfrak{b} \neq 0$ we can define

$$Z := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{\mathfrak{a}\mathbf{C}}{\mathbf{D}} & \frac{\mathfrak{b}}{\mathbf{D}} & 0 \\ 0 & -1 & \frac{\mathfrak{b}\mathbf{A}}{\mathbf{CD}} & \frac{\mathfrak{b}}{\mathbf{D}} \end{pmatrix}$$

and check that $\det Z = \left(\frac{\mathfrak{b}}{\mathbf{D}}\right)^2 \neq 0$ and

$$Z^{-1}M_1Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & \frac{2\mathbf{A}}{\mathbf{C}} & 1 \end{pmatrix}, \qquad Z^{-1}\mathfrak{D}_1Z = \begin{pmatrix} 0 & \frac{\mathfrak{a}\mathbf{C}}{\mathbf{D}} & -\frac{\mathfrak{b}}{\mathbf{D}} & 0 \\ 0 & 0 & \frac{\mathfrak{b}\mathbf{A}}{\mathbf{CD}} & \frac{\mathfrak{b}}{\mathbf{D}} \\ 0 & 0 & \frac{\mathfrak{a}\mathbf{A}}{\mathbf{D}} & \frac{\mathfrak{a}\mathbf{C}}{\mathbf{D}} \\ 0 & 0 & -\frac{\mathfrak{a}\mathbf{A}^2}{\mathbf{CD}} & -\frac{\mathfrak{a}\mathbf{A}}{\mathbf{D}} \end{pmatrix}.$$

Finally, given $\mathfrak{a} \neq 0$, we set

$$Z := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{\mathfrak{a}\mathbf{C}}{\mathbf{D}} & \frac{\mathfrak{b}}{\mathbf{D}} & 0 \\ 0 & 0 & \frac{\mathfrak{a}\mathbf{A}}{\mathbf{D}} - 1 & \frac{\mathfrak{a}\mathbf{C}}{\mathbf{D}} \end{pmatrix}$$

and see that $\det Z = \left(\frac{\mathfrak{a}\mathbf{C}}{\mathbf{D}}\right)^2 \neq 0$

$$Z^{-1}M_1Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & \frac{2\mathbf{A}}{\mathbf{C}} & 1 \end{pmatrix}, \qquad Z^{-1}\mathfrak{D}_1Z = \begin{pmatrix} 0 & \frac{\mathfrak{a}\mathbf{C}}{\mathbf{D}} & -\frac{\mathfrak{b}}{\mathbf{D}} & 0 \\ 0 & 0 & \frac{\mathfrak{b}\mathbf{A}}{\mathbf{C}\mathbf{D}} & \frac{\mathfrak{b}}{\mathbf{D}} \\ 0 & 0 & \frac{\mathfrak{a}\mathbf{A}}{\mathbf{D}} & \frac{\mathfrak{a}\mathbf{C}}{\mathbf{D}} \\ 0 & 0 & -\frac{\mathfrak{a}\mathbf{A}^2}{\mathbf{C}\mathbf{D}} & -\frac{\mathfrak{a}\mathbf{A}}{\mathbf{D}} \end{pmatrix}.$$

By transitivity we can conclude that all pairs belonging to the third and the fourth row of Table 1 are equivalent. The coaction defined by an element of this class is (isomorphic to) $\rho(G) = G \otimes g$, $\rho(X) = X \otimes g + 1 \otimes x$ (choose $\mathbf{A} = \mathbf{E} = \mathfrak{a} = 0$, $\mathbf{C} = \mathbf{D} = \mathfrak{b} = 1$).

Now we consider a pair (φ_1, d_1) belonging to the fifth row of Table 1. Its associated matrices are

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{\mathbf{AB}+\mathbf{EF}}{\mathbf{AB}-\mathbf{EF}} & -\frac{2\mathbf{AF}}{\mathbf{AB}-\mathbf{EF}} & 0 \\ 0 & \frac{2\mathbf{BE}}{\mathbf{AB}-\mathbf{EF}} & \frac{\mathbf{AB}+\mathbf{EF}}{\mathbf{AB}-\mathbf{EF}} & 0 \\ 0 & -\frac{2\mathbf{DE}}{\mathbf{AB}-\mathbf{EF}} & -\frac{2\mathbf{AD}}{\mathbf{AB}-\mathbf{EF}} & -1 \end{pmatrix}, \qquad \mathfrak{D}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{\mathfrak{b}\mathbf{AE}}{\mathbf{AB}-\mathbf{EF}} & -\frac{\mathfrak{b}\mathbf{A}^2}{\mathbf{AB}-\mathbf{EF}} & 0 \\ 0 & \frac{\mathfrak{b}\mathbf{E}^2}{\mathbf{AB}-\mathbf{EF}} & \frac{\mathfrak{b}\mathbf{AE}}{\mathbf{AB}-\mathbf{EF}} & 0 \\ 0 & \frac{\mathfrak{a}\mathbf{E}^2}{\mathbf{AB}-\mathbf{EF}} & \frac{\mathfrak{a}\mathbf{AE}}{\mathbf{AB}-\mathbf{EF}} & 0 \end{pmatrix}, \tag{248}$$

with $\mathbf{AB} - \mathbf{EF} \neq 0 \neq \mathbf{E}$. We take a Z of the form (241) and we calculate

$$Z^{-1}\mathfrak{D}_1Z = \frac{1}{(\mathbf{AB}-\mathbf{EF})Z_{44}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\mathfrak{b}(\mathbf{A}Z_{32}+\mathbf{E}Z_{22})(\mathbf{A}Z_{33}+\mathbf{E}Z_{23}) & -\mathfrak{b}(\mathbf{A}Z_{33}+\mathbf{E}Z_{23})^2 & 0 \\ 0 & \mathfrak{b}(\mathbf{A}Z_{32}+\mathbf{E}Z_{22})^2 & \mathfrak{b}(\mathbf{A}Z_{32}+\mathbf{E}Z_{22})(\mathbf{A}Z_{33}+\mathbf{E}Z_{23}) & 0 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

This immediately shows that if $\mathfrak{b} = 0$, any $\mathfrak{D}_2$ conjugate to $\mathfrak{D}_1$ must satisfy $\mathfrak{b}' = 0$ and therefore that pairs belonging to the fifth row of Table 1 are not all equivalent.

Assume $\mathfrak{b} = 0$. It follows that $\mathfrak{a} \neq 0$. Then we fix an $\mathfrak{a}' \neq 0$ and we define

$$Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\mathfrak{a}\mathbf{AB}'\mathbf{E}-\mathfrak{a}'\mathbf{E}'^2\mathbf{F}}{\mathfrak{a}'\mathbf{E}'(\mathbf{AB}-\mathbf{EF})} & \frac{\mathfrak{a}\mathbf{AEF}'-\mathfrak{a}'\mathbf{A}'\mathbf{E}'\mathbf{F}}{\mathfrak{a}'\mathbf{E}'(\mathbf{AB}-\mathbf{EF})} & 0 \\ 0 & \frac{\mathfrak{a}'\mathbf{BE}'^2-\mathfrak{a}\mathbf{B}'\mathbf{E}^2}{\mathfrak{a}'\mathbf{E}'(\mathbf{AB}-\mathbf{EF})} & \frac{\mathfrak{a}'\mathbf{A}'\mathbf{BE}'-\mathfrak{a}\mathbf{E}^2\mathbf{F}'}{\mathfrak{a}'\mathbf{E}'(\mathbf{AB}-\mathbf{EF})} & 0 \\ 0 & \frac{\mathfrak{a}\mathbf{D}'\mathbf{E}-\mathfrak{a}'\mathbf{DE}'}{\mathfrak{a}'(\mathbf{AB}-\mathbf{EF})} & \frac{\mathbf{A}'(\mathfrak{a}\mathbf{D}'\mathbf{E}-\mathfrak{a}'\mathbf{DE}')}{\mathfrak{a}'\mathbf{E}'(\mathbf{AB}-\mathbf{EF})} & \frac{\mathfrak{a}\mathbf{E}(\mathbf{A}'\mathbf{B}'-\mathbf{E}'\mathbf{F}')}{\mathfrak{a}'\mathbf{E}'(\mathbf{AB}-\mathbf{EF})} \end{pmatrix}.$$

It is straightforward to check that $\det Z = \left(\frac{\mathfrak{a}\mathbf{E}(\mathbf{A}'\mathbf{B}'-\mathbf{E}'\mathbf{F}')}{\mathfrak{a}'\mathbf{E}'(\mathbf{AB}-\mathbf{EF})}\right)^2 \neq 0$ and that

$$Z^{-1}M_1Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{\mathbf{A}'\mathbf{B}'+\mathbf{E}'\mathbf{F}'}{\mathbf{A}'\mathbf{B}'-\mathbf{E}'\mathbf{F}'} & -\frac{2\mathbf{A}'\mathbf{F}'}{\mathbf{A}'\mathbf{B}'-\mathbf{E}'\mathbf{F}'} & 0 \\ 0 & \frac{2\mathbf{B}'\mathbf{E}'}{\mathbf{A}'\mathbf{B}'-\mathbf{E}'\mathbf{F}'} & \frac{\mathbf{A}'\mathbf{B}'+\mathbf{E}'\mathbf{F}'}{\mathbf{A}'\mathbf{B}'-\mathbf{E}'\mathbf{F}'} & 0 \\ 0 & -\frac{2\mathbf{D}'\mathbf{E}'}{\mathbf{A}'\mathbf{B}'-\mathbf{E}'\mathbf{F}'} & -\frac{2\mathbf{A}'\mathbf{D}'}{\mathbf{A}'\mathbf{B}'-\mathbf{E}'\mathbf{F}'} & -1 \end{pmatrix}, \qquad Z^{-1}\mathfrak{D}_1Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\mathfrak{a}'\mathbf{E}'^2}{\mathbf{A}'\mathbf{B}'-\mathbf{E}'\mathbf{F}'} & \frac{\mathfrak{a}'\mathbf{A}'\mathbf{E}'}{\mathbf{A}'\mathbf{B}'-\mathbf{E}'\mathbf{F}'} & 0 \end{pmatrix}.$$

We can conclude that all pairs (φ, d) in the fifth row of Table 1 and with $\mathfrak{b} = 0$ are equivalent. Next, we can show that also each pair in the last row and with $\mathfrak{b} = 0$ is in the same equivalence class. If we take the representative

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \mathfrak{D}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

we fix $\mathfrak{a}, \mathbf{A}, \mathbf{B} \neq 0$ and we define the invertible matrix

$$Z := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{\mathbf{B}}{\mathfrak{a}\mathbf{A}} & \frac{\mathbf{F}}{\mathfrak{a}\mathbf{A}} & 0 \\ 0 & 0 & -\frac{\mathbf{D}}{\mathfrak{a}\mathbf{A}} & -\frac{\mathbf{B}}{\mathfrak{a}\mathbf{A}} \end{pmatrix},$$

we can check that

$$Z^{-1}M_1Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -\frac{2\mathbf{F}}{\mathbf{B}} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{2\mathbf{D}}{\mathbf{B}} & -1 \end{pmatrix}, \qquad Z^{-1}\mathfrak{D}_1Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mathfrak{a}\mathbf{A}}{\mathbf{B}} & 0 \end{pmatrix}.$$

The coaction defined by an element of this class is (isomorphic to) $\rho(G) = G \otimes 1 - GX \otimes gx$, $\rho(X) = X \otimes g$ (choose $\mathbf{A} = \mathbf{B} = \mathbf{D} = \mathfrak{b} = 0$, $\mathbf{E} = \mathbf{F} = \mathfrak{a} = 1$).

Finally let us fix the pair (φ_1, d_1)

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \mathfrak{D}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{249}$$

which is a pair contained in the fifth row of Table 1 and is such that $\mathfrak{b} = 1 \neq 0$. We are going to prove that every pair of the fifth and sixth row with $\mathfrak{b} \neq 0$ is equivalent to (249). We start by defining the matrix

$$Z := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{\mathbf{A}}{\mathbf{E}} & 0 \\ 0 & \frac{\mathbf{B}}{\mathfrak{b}\mathbf{E}} & \frac{\mathbf{F}}{\mathfrak{b}\mathbf{E}} & 0 \\ 0 & -\frac{\mathfrak{a}\mathbf{B}+\mathfrak{b}\mathbf{D}}{\mathfrak{b}^2\mathbf{E}} & -\frac{\mathfrak{b}\mathbf{AD}+\mathfrak{a}\mathbf{EF}}{\mathfrak{b}^2\mathbf{E}^2} & -\frac{\mathbf{AB}-\mathbf{EF}}{\mathfrak{b}\mathbf{E}^2} \end{pmatrix},$$

that has determinant $\det Z = \left(\frac{\mathbf{AB}-\mathbf{EF}}{\mathfrak{b}\mathbf{E}^2}\right)^2 \neq 0$. In this case we have

$$Z^{-1}M_1Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{\mathbf{AB}+\mathbf{EF}}{\mathbf{AB}-\mathbf{EF}} & -\frac{2\mathbf{AF}}{\mathbf{AB}-\mathbf{EF}} & 0 \\ 0 & \frac{2\mathbf{BE}}{\mathbf{AB}-\mathbf{EF}} & \frac{\mathbf{AB}+\mathbf{EF}}{\mathbf{AB}-\mathbf{EF}} & 0 \\ 0 & -\frac{2\mathbf{DE}}{\mathbf{AB}-\mathbf{EF}} & -\frac{2\mathbf{AD}}{\mathbf{AB}-\mathbf{EF}} & -1 \end{pmatrix}, \qquad Z^{-1}\mathfrak{D}_1Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{\mathfrak{b}\mathbf{AE}}{\mathbf{AB}-\mathbf{EF}} & -\frac{\mathfrak{b}\mathbf{A}^2}{\mathbf{AB}-\mathbf{EF}} & 0 \\ 0 & \frac{\mathfrak{b}\mathbf{E}^2}{\mathbf{AB}-\mathbf{EF}} & \frac{\mathfrak{b}\mathbf{AE}}{\mathbf{AB}-\mathbf{EF}} & 0 \\ 0 & \frac{\mathfrak{a}\mathbf{E}^2}{\mathbf{AB}-\mathbf{EF}} & \frac{\mathfrak{a}\mathbf{AE}}{\mathbf{AB}-\mathbf{EF}} & 0 \end{pmatrix},$$

therefore the first part of our claim is proved. Lastly we define

$$Z := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mathfrak{b}}{\mathbf{B}} & 0 \\ 0 & 1 & \frac{\mathbf{F}}{\mathbf{B}} & 0 \\ 0 & -\frac{\mathfrak{a}\mathbf{A}}{\mathbf{B}} & -\frac{\mathfrak{a}\mathbf{AF}+\mathfrak{b}\mathbf{D}}{\mathbf{B}^2} & -\frac{\mathfrak{b}}{\mathbf{B}} \end{pmatrix}$$

and we see that $\det Z = \frac{\mathfrak{b}^2}{\mathbf{B}^2} \neq 0$ and

$$Z^{-1}M_1Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -\frac{2\mathbf{F}}{\mathbf{B}} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{2\mathbf{D}}{\mathbf{B}} & -1 \end{pmatrix}, \qquad Z^{-1}\mathfrak{D}_1Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\mathfrak{b}}{\mathbf{B}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mathfrak{a}\mathbf{A}}{\mathbf{B}} & 0 \end{pmatrix}.$$

The coaction defined by an element of this class is (isomorphic to) $\rho(G) = G \otimes 1 - X \otimes gx$, $\rho(X) = X \otimes g$ (choose $\mathbf{A} = \mathbf{B} = \mathbf{D} = \mathfrak{a} = 0$, $\mathbf{E} = \mathbf{F} = \mathfrak{b} = 1$).

We gather all these results in the following table.

Type of Coaction	Definition	Skew-derivation
0	$\rho(G) = G \otimes 1,\ \rho(X) = X \otimes 1$	Trivial
1	None	
2	$\rho(G) = G \otimes g,\ \rho(X) = X \otimes g$	Trivial
	$\rho(G) = G \otimes 1,\ \rho(X) = X \otimes g$	Trivial
	$\rho(G) = G \otimes g - GX \otimes x,\ \rho(X) = X \otimes g$	Non-trivial
	$\rho(G) = G \otimes g,\ \rho(X) = X \otimes g + 1 \otimes x$	Non-trivial
	$\rho(G) = G \otimes 1 - GX \otimes gx,\ \rho(X) = X \otimes g$	Non-trivial
	$\rho(G) = G \otimes 1 - X \otimes gx,\ \rho(X) = X \otimes g$	Non-trivial

TABLE 2. Non-isomorphic coactions on $A = Cl(0,0,0)$.

REMARK 3.3. Notice that the list in Table 2 is finite.

3.2. The case $\mathfrak{F}_2$ when $\alpha = \gamma = 0$, $\beta \neq 0$. When $\alpha = \gamma = 0$, $\beta \neq 0$ every coaction of type $\mathfrak{F}_2$ corresponds to a pair (φ, d) where φ and d are of the following form.

Matrix M_φ of φ	Matrix of d	Conditions
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{\beta\mathbf{D}^2+\mathbf{F}^2}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{2\mathbf{AF}}{\lambda(\beta\mathbf{D}^2-\mathbf{F}^2)} & \frac{2\beta\mathbf{DF}}{\beta\mathbf{D}^2-\mathbf{F}^2} \\ 0 & 0 & -1 & 0 \\ 0 & -\frac{2\mathbf{DF}}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{2\mathbf{AD}}{\lambda(\beta\mathbf{D}^2-\mathbf{F}^2)} & \frac{\beta\mathbf{D}^2+\mathbf{F}^2}{\beta\mathbf{D}^2-\mathbf{F}^2} \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{\mathfrak{b}\lambda\beta\mathbf{DF}}{\beta\mathbf{D}^2-\mathbf{F}^2} & -\mathfrak{a}\lambda\beta - \frac{\mathfrak{b}\beta\mathbf{AD}}{\beta\mathbf{D}^2-\mathbf{F}^2} & -\frac{\mathfrak{b}\lambda\beta\mathbf{F}^2}{\beta\mathbf{D}^2-\mathbf{F}^2} \\ 0 & -\frac{\mathbf{F}(\mathfrak{a}\lambda\beta\mathbf{D}+\mathfrak{b}\mathbf{A})}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{\mathbf{A}(\mathfrak{a}\lambda\beta\mathbf{D}+\mathfrak{b}\mathbf{A})}{\lambda(\beta\mathbf{D}^2-\mathbf{F}^2)} & \frac{\beta\mathbf{D}(\mathfrak{a}\lambda\beta\mathbf{D}+\mathfrak{b}\mathbf{A})}{\beta\mathbf{D}^2-\mathbf{F}^2} \\ 0 & -\frac{\mathfrak{b}\lambda\mathbf{F}^2}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{\mathfrak{b}\mathbf{AF}}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{\mathfrak{b}\lambda\beta\mathbf{DF}}{\beta\mathbf{D}^2-\mathbf{F}^2} \\ 0 & -\frac{\mathfrak{a}\lambda\mathbf{F}^2}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{\mathfrak{a}\mathbf{AF}}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{\mathfrak{a}\lambda\beta\mathbf{DF}}{\beta\mathbf{D}^2-\mathbf{F}^2} \end{pmatrix}$	$\alpha = \gamma = 0,\ \beta \neq 0$ $\lambda, \mathbf{F}, (\beta\mathbf{D}^2 - \mathbf{F}^2) \neq 0$
	$\begin{pmatrix} 0 & \mathfrak{a}\lambda\beta & -\frac{\mathfrak{b}}{\mathbf{D}} & 0 \\ 0 & 0 & \frac{\mathfrak{b}\mathbf{A}}{\lambda\beta\mathbf{D}^2} & \frac{\mathfrak{b}}{\mathbf{D}} \\ 0 & 0 & \frac{\mathfrak{a}\mathbf{A}}{\mathbf{D}} & \mathfrak{a}\lambda\beta \\ 0 & 0 & -\frac{\mathfrak{a}\mathbf{A}^2}{\lambda\beta\mathbf{D}^2} & -\frac{\mathfrak{a}\mathbf{A}}{\mathbf{D}} \end{pmatrix}$	$\alpha = \gamma = \mathbf{F} = 0,\ \beta \neq 0$ $\lambda, \mathbf{D} \neq 0$
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -\frac{2\mathbf{F}}{\mathbf{B}} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{2\mathbf{D}}{\mathbf{B}} & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\mathfrak{b}}{\mathbf{B}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4\mathfrak{a}\beta & 0 \end{pmatrix}$	$\alpha = \gamma = 0,\ \beta \neq 0$ $\mathbf{B} \neq 0$

TABLE 3. Pairs defining coactions on $A = Cl(0,\beta,0)$.

Remark 3.4. Again we have that $d \equiv 0$ is equivalent to $\mathfrak{a} = \mathfrak{b} = 0$ in each of the displayed instances.

As in the previous subsection we start by checking what pairs $(\varphi, 0)$ with trivial skew-derivation are equivalent.
3.2.1. *Coactions with trivial skew-derivation.* We can fix the pair $(\varphi_1, 0)$ with associate matrix

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and define

$$Z := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{\mathbf{A}}{\lambda \mathbf{F}} & \frac{\beta \mathbf{D}}{\mathbf{F}} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{\mathbf{D}}{\mathbf{F}} & 0 & 1 \end{pmatrix},$$

with $\lambda, \mathbf{F}, \beta\mathbf{D}^2 - \mathbf{F}^2 \neq 0$. One checks that Z defines an algebra map, that $\det Z = -\left(\frac{\beta\mathbf{D}^2 - \mathbf{F}^2}{\mathbf{F}}\right)^2 \neq 0$ and

$$Z^{-1} M_1 Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{\beta\mathbf{D}^2+\mathbf{F}^2}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{2\mathbf{AF}}{\lambda(\beta\mathbf{D}^2-\mathbf{F}^2)} & \frac{2\beta\mathbf{DF}}{\beta\mathbf{D}^2-\mathbf{F}^2} \\ 0 & 0 & -1 & 0 \\ 0 & -\frac{2\mathbf{DF}}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{2\mathbf{AD}}{\lambda(\beta\mathbf{D}^2-\mathbf{F}^2)} & \frac{\beta\mathbf{D}^2+\mathbf{F}^2}{\beta\mathbf{D}^2-\mathbf{F}^2} \end{pmatrix}.$$

Similarly we can fix the pair $(\varphi_2, 0)$ with associate matrix

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and define the invertible algebra map

$$Z := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{\mathbf{F}}{\mathbf{B}} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{\mathbf{D}}{\mathbf{B}} & 0 & 1 \end{pmatrix},$$

with $\mathbf{B} \neq 0$. Then

$$Z M_2 Z^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -\frac{2\mathbf{F}}{\mathbf{B}} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{2\mathbf{D}}{\mathbf{B}} & -1 \end{pmatrix}.$$

Finally we show that $(\varphi_1, 0)$ and $(\varphi_2, 0)$ cannot be equivalent. Suppose there is an invertible algebra map with matrix Z such that $M_1 Z - Z M_2 = 0$. Then, by Lemma 3.2, we have

$$M_1 Z - Z M_2 = \begin{pmatrix} 0 & 0 & 0 & 2\beta Z_{32} Z_{33} \\ 0 & 2Z_{22} & 0 & -2\beta(Z_{32}Z_{43} - Z_{33}Z_{42}) \\ 0 & 0 & -2Z_{33} & 0 \\ 0 & 0 & -2Z_{43} & 0 \end{pmatrix},$$

which forces $Z_{22} = Z_{33} = Z_{43} = 0$. As a consequence (101) becomes equivalent to $Z_{32} = 0$ and it follows that $\det Z = 0$, contradiction.

Remark 3.5. This last result can also be obtained just by observing that involutions in the first and second row of Table 3 are not inner, while those in the third are (cf. Table 2 in Chapter 2).

In conclusion we have found that, up to isomorphism, there are only two coactions of type 2 on $A = Cl(0, \beta, 0)$ corresponding to pairs with trivial skew-derivation. These are given by $\rho(G) = G \otimes 1$, $\rho(X) = X \otimes g$ and $\rho(G) = G \otimes g$, $\rho(X) = X \otimes 1$.

3.2.2. *Coactions with non-trivial skew-derivation.* Let us fix a pair (φ_1, d_1) with associated matrices

(250)
$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{\beta\mathbf{D}^2+\mathbf{F}^2}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{2\mathbf{A}\mathbf{F}}{\lambda(\beta\mathbf{D}^2-\mathbf{F}^2)} & \frac{2\beta\mathbf{D}\mathbf{F}}{\beta\mathbf{D}^2-\mathbf{F}^2} \\ 0 & 0 & -1 & 0 \\ 0 & -\frac{2\mathbf{D}\mathbf{F}}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{2\mathbf{A}\mathbf{D}}{\lambda(\beta\mathbf{D}^2-\mathbf{F}^2)} & \frac{\beta\mathbf{D}^2+\mathbf{F}^2}{\beta\mathbf{D}^2-\mathbf{F}^2} \end{pmatrix}, \quad \mathfrak{D}_1 = \begin{pmatrix} 0 & \frac{\mathfrak{b}\lambda\beta\mathbf{D}\mathbf{F}}{\beta\mathbf{D}^2-\mathbf{F}^2} & -\mathfrak{a}\lambda\beta - \frac{\mathfrak{b}\beta\mathbf{A}\mathbf{D}}{\beta\mathbf{D}^2-\mathbf{F}^2} & -\frac{\mathfrak{b}\lambda\beta\mathbf{F}^2}{\beta\mathbf{D}^2-\mathbf{F}^2} \\ 0 & -\frac{\mathbf{F}(\mathfrak{a}\lambda\beta\mathbf{D}+\mathfrak{b}\mathbf{A})}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{\mathbf{A}(\mathfrak{a}\lambda\beta\mathbf{D}+\mathfrak{b}\mathbf{A})}{\lambda(\beta\mathbf{D}^2-\mathbf{F}^2)} & \frac{\beta\mathbf{D}(\mathfrak{a}\lambda\beta\mathbf{D}+\mathfrak{b}\mathbf{A})}{\beta\mathbf{D}^2-\mathbf{F}^2} \\ 0 & -\frac{\mathfrak{b}\lambda\mathbf{F}^2}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{\mathfrak{b}\mathbf{A}\mathbf{F}}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{\mathfrak{b}\lambda\beta\mathbf{D}\mathbf{F}}{\beta\mathbf{D}^2-\mathbf{F}^2} \\ 0 & -\frac{\mathfrak{a}\lambda\mathbf{F}^2}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{\mathfrak{a}\mathbf{A}\mathbf{F}}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{\mathfrak{a}\lambda\beta\mathbf{D}\mathbf{F}}{\beta\mathbf{D}^2-\mathbf{F}^2} \end{pmatrix}$$

and $\lambda, \mathbf{F}, \beta\mathbf{D}^2 - \mathbf{F}^2 \neq 0$.

Suppose $\mathfrak{b} \neq 0$ and define the matrix

$$Z := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\mathfrak{b}'\lambda'\mathbf{F}'(\beta\mathbf{D}\mathbf{D}'+\mathbf{F}\mathbf{F}')}{\mathfrak{b}\lambda\mathbf{F}(\beta\mathbf{D}'^2-\mathbf{F}'^2)} & -\frac{\mathbf{A}}{\lambda\mathbf{F}} - \frac{\mathfrak{a}\beta\mathbf{D}}{\mathfrak{b}\mathbf{F}} - \frac{\mathfrak{a}'\lambda'\beta\mathbf{D}}{\mathfrak{b}\lambda\mathbf{F}} - \frac{\mathfrak{b}'\mathbf{A}'(\beta\mathbf{D}\mathbf{D}'+\mathbf{F}\mathbf{F}')}{\mathfrak{b}\lambda\mathbf{F}(\beta\mathbf{D}'^2-\mathbf{F}'^2)} & -\frac{\mathfrak{b}'\lambda'\beta\mathbf{F}'(\mathbf{D}'\mathbf{F}+\mathbf{D}\mathbf{F}')}{\mathfrak{b}\lambda\mathbf{F}(\beta\mathbf{D}'^2-\mathbf{F}'^2)} \\ 0 & 0 & -1 & 0 \\ 0 & \frac{\mathfrak{b}'\lambda'\mathbf{F}'(\mathbf{D}'\mathbf{F}+\mathbf{D}\mathbf{F}')}{\mathfrak{b}\lambda\mathbf{F}(\beta\mathbf{D}'^2-\mathbf{F}'^2)} & -\frac{\mathfrak{a}'\lambda'}{\mathfrak{b}\lambda} - \frac{\mathfrak{a}}{\mathfrak{b}} - \frac{\mathfrak{b}'\mathbf{A}'(\mathbf{D}'\mathbf{F}+\mathbf{D}\mathbf{F}')}{\mathfrak{b}\lambda\mathbf{F}(\beta\mathbf{D}'^2-\mathbf{F}'^2)} & -\frac{\mathfrak{b}'\lambda'\mathbf{F}'(\beta\mathbf{D}\mathbf{D}'+\mathbf{F}\mathbf{F}')}{\mathfrak{b}\lambda\mathbf{F}(\beta\mathbf{D}'^2-\mathbf{F}'^2)} \end{pmatrix},$$

with $\mathfrak{b}', \lambda, \mathbf{F}', \beta\mathbf{D}'^2 - \mathbf{F}^2 \neq 0$. Then we have

$$Z^{-1}M_1Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{\beta\mathbf{D}'^2+\mathbf{F}'^2}{\beta\mathbf{D}'^2-\mathbf{F}'^2} & \frac{2\mathbf{A}'\mathbf{F}'}{\lambda(\beta\mathbf{D}'^2-\mathbf{F}'^2)} & \frac{2\beta\mathbf{D}'\mathbf{F}'}{\beta\mathbf{D}'^2-\mathbf{F}'^2} \\ 0 & 0 & -1 & 0 \\ 0 & -\frac{2\mathbf{D}'\mathbf{F}'}{\beta\mathbf{D}'^2-\mathbf{F}'^2} & \frac{2\mathbf{A}'\mathbf{D}'}{\lambda(\beta\mathbf{D}'^2-\mathbf{F}'^2)} & \frac{\beta\mathbf{D}'^2+\mathbf{F}'^2}{\beta\mathbf{D}'^2-\mathbf{F}'^2} \end{pmatrix}, Z^{-1}\mathfrak{D}_1Z = \begin{pmatrix} 0 & \frac{\mathfrak{b}'\lambda'\beta\mathbf{D}'\mathbf{F}'}{\beta\mathbf{D}'^2-\mathbf{F}'^2} & -\mathfrak{a}'\lambda'\beta - \frac{\mathfrak{b}'\beta\mathbf{A}'\mathbf{D}'}{\beta\mathbf{D}'^2-\mathbf{F}'^2} & -\frac{\mathfrak{b}'\lambda'\beta\mathbf{F}'^2}{\beta\mathbf{D}'^2-\mathbf{F}'^2} \\ 0 & -\frac{\mathbf{F}'(\mathfrak{a}'\lambda'\beta\mathbf{D}'+\mathfrak{b}'\mathbf{A}')}{\beta\mathbf{D}'^2-\mathbf{F}'^2} & \frac{\mathbf{A}'(\mathfrak{a}'\lambda'\beta\mathbf{D}'+\mathfrak{b}'\mathbf{A}')}{\lambda'(\beta\mathbf{D}'^2-\mathbf{F}'^2)} & \frac{\beta\mathbf{D}'(\mathfrak{a}'\lambda'\beta\mathbf{D}'+\mathfrak{b}'\mathbf{A}')}{\beta\mathbf{D}'^2-\mathbf{F}'^2} \\ 0 & -\frac{\mathfrak{b}'\lambda'\mathbf{F}'^2}{\beta\mathbf{D}'^2-\mathbf{F}'^2} & \frac{\mathfrak{b}'\mathbf{A}'\mathbf{F}'}{\beta\mathbf{D}'^2-\mathbf{F}'^2} & \frac{\mathfrak{b}'\lambda'\beta\mathbf{D}'\mathbf{F}'}{\beta\mathbf{D}'^2-\mathbf{F}'^2} \\ 0 & -\frac{\mathfrak{a}'\lambda'\mathbf{F}'^2}{\beta\mathbf{D}'^2-\mathbf{F}'^2} & \frac{\mathfrak{a}'\mathbf{A}'\mathbf{F}'}{\beta\mathbf{D}'^2-\mathbf{F}'^2} & \frac{\mathfrak{a}'\lambda'\beta\mathbf{D}'\mathbf{F}'}{\beta\mathbf{D}'^2-\mathbf{F}'^2} \end{pmatrix}.$$

Furthermore Z satisfies (100)-(104) and $\det Z = \frac{\mathfrak{b}'^2\lambda'^2\mathbf{F}'^2(\beta\mathbf{D}^2-\mathbf{F}^2)}{\mathfrak{b}^2\lambda^2\mathbf{F}^2(\beta\mathbf{D}'^2-\mathbf{F}'^2)} \neq 0$. This shows that all pairs with matrices (250) and $\mathfrak{b} \neq 0$ are equivalent.

By setting $\mathbf{A} = \mathbf{D} = \mathfrak{a} = 0$, $\mathbf{F} = \lambda = \mathfrak{b} = 1$, i.e. by considering the representative

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \mathfrak{D}_1 = \begin{pmatrix} 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

we can show that pairs from the first row of Table 3 with $\mathfrak{b} = 0$ cannot be in the same class. We take

$$\mathfrak{D}_2 = \begin{pmatrix} 0 & 0 & -\mathfrak{a}\lambda\beta & 0 \\ 0 & -\frac{\mathfrak{a}\lambda\beta\mathbf{D}\mathbf{F}}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{\mathfrak{a}\beta\mathbf{A}\mathbf{D}}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{\mathfrak{a}\lambda\beta^2\mathbf{D}^2}{\beta\mathbf{D}^2-\mathbf{F}^2} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\mathfrak{a}\lambda\mathbf{F}^2}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{\mathfrak{a}\mathbf{A}\mathbf{F}}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{\mathfrak{a}\lambda\beta\mathbf{D}\mathbf{F}}{\beta\mathbf{D}^2-\mathbf{F}^2} \end{pmatrix}$$

and an invertible Z of the form (241). Since (101) forces $Z_{32} = 0$, we get

$$0 = \mathfrak{D}_1Z - Z\mathfrak{D}_2 = \begin{pmatrix} 0 & \beta Z_{42} & \dots \\ 0 & 0 & \dots \\ 0 & Z_{22} & \dots \\ 0 & \dots & \end{pmatrix},$$

i.e. $Z_{22} = Z_{42} = 0$, contradiction.

We can show that also pairs from the second row of Table 3 with $\mathfrak{a} \neq 0$ are in the class formerly obtained. In fact, we define the invertible matrix

$$Z := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mathfrak{a}\mathbf{A}}{\mathbf{D}} & \mathfrak{a}\lambda\beta \\ 0 & 0 & 1 & 0 \\ 0 & \mathfrak{a}\lambda & \frac{-\mathfrak{b}}{\beta\mathbf{D}} & 0 \end{pmatrix}$$

and check that

$$Z^{-1}M_1Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & \frac{2\mathbf{A}}{\lambda\beta\mathbf{D}} & 1 \end{pmatrix}, \qquad Z^{-1}\mathfrak{D}_1 Z = \begin{pmatrix} 0 & \mathfrak{a}\lambda\beta & -\frac{\mathfrak{b}}{\mathbf{D}} & 0 \\ 0 & 0 & \frac{\mathfrak{b}\mathbf{A}}{\lambda\beta\mathbf{D}^2} & \frac{\mathfrak{b}}{\mathbf{D}} \\ 0 & 0 & \frac{\mathfrak{a}\mathbf{A}}{\mathbf{D}} & \mathfrak{a}\lambda\beta \\ 0 & 0 & -\frac{\mathfrak{a}\mathbf{A}^2}{\lambda\beta\mathbf{D}^2} & -\frac{\mathfrak{a}\mathbf{A}}{\mathbf{D}} \end{pmatrix}$$

and that Z satisfies (100)-(104). Pairs from the second row of Table 3 with $\mathfrak{a} = 0$ cannot be in the same class. In fact if we consider

$$\mathfrak{D}_2 = \begin{pmatrix} 0 & 0 & -\frac{\mathfrak{b}}{\mathbf{D}} & 0 \\ 0 & 0 & \frac{\mathfrak{b}\mathbf{A}}{\lambda\beta\mathbf{D}^2} & \frac{\mathfrak{b}}{\mathbf{D}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and we take an invertible Z of the form (241) we can see that

$$0 = \mathfrak{D}_1 Z - Z\mathfrak{D}_2 = \begin{pmatrix} 0 & \beta Z_{42} & \dots \\ 0 & 0 & \dots \\ 0 & Z_{22} & \dots \\ 0 & 0 & \dots \end{pmatrix},$$

which forces $Z_{22} = Z_{42} = 0$. Then (101) yields $Z_{32} = 0$, contradiction. The coaction defined by an element of this class is (isomorphic to) $\rho(G) = G \otimes 1 + X \otimes gx$, $\rho(X) = X \otimes g$ (choose $\mathbf{A} = \mathbf{D} = \mathfrak{a} = 0$, $\mathbf{F} = \lambda = \mathfrak{b} = 1$).

Now we fix a pair (φ_1, d_1) with associated matrices

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{\beta\mathbf{D}^2+\mathbf{F}^2}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{2\mathbf{AF}}{\lambda(\beta\mathbf{D}^2-\mathbf{F}^2)} & \frac{2\beta\mathbf{DF}}{\beta\mathbf{D}^2-\mathbf{F}^2} \\ 0 & 0 & -1 & 0 \\ 0 & -\frac{2\mathbf{DF}}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{2\mathbf{AD}}{\lambda(\beta\mathbf{D}^2-\mathbf{F}^2)} & \frac{\beta\mathbf{D}^2+\mathbf{F}^2}{\beta\mathbf{D}^2-\mathbf{F}^2} \end{pmatrix}, \quad \mathfrak{D}_1 = \begin{pmatrix} 0 & 0 & -\mathfrak{a}\lambda\beta & 0 \\ 0 & -\frac{\mathfrak{a}\lambda\beta\mathbf{DF}}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{\mathfrak{a}\beta\mathbf{AD}}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{\mathfrak{a}\lambda\beta^2\mathbf{D}^2}{\beta\mathbf{D}^2-\mathbf{F}^2} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\mathfrak{a}\lambda\mathbf{F}^2}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{\mathfrak{a}\mathbf{AF}}{\beta\mathbf{D}^2-\mathbf{F}^2} & \frac{\mathfrak{a}\lambda\beta\mathbf{DF}}{\beta\mathbf{D}^2-\mathbf{F}^2} \end{pmatrix} \tag{251}$$

and $\lambda, \mathbf{F}, \mathfrak{a}, \beta\mathbf{D}^2 - \mathbf{F}^2 \neq 0$. This is a pair belonging to the first row of Table 3 with $\mathfrak{b} = 0$. If we consider a second pair (φ_2, d_2) of the same kind and suppose that there is an invertible algebra map with matrix Z such that $\mathfrak{D}_1 Z - Z\mathfrak{D}_2 = 0$, by using Lemma 3.2 and (101), we find

$$0 = \mathfrak{D}_1 Z - Z\mathfrak{D}_2 = \begin{pmatrix} 0 & 0 & \beta(\mathfrak{a}'\lambda' - Z_{33}\mathfrak{a}\lambda) & 0 \\ \dots & & & \end{pmatrix},$$

i.e. $\mathfrak{a}'\lambda' - Z_{33}\mathfrak{a}\lambda = 0$. We know that $\lambda, \lambda' \neq 0$ and since $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are non-trivial skew-derivation, also $\mathfrak{a}, \mathfrak{a}' \neq 0$. This means that if (φ_1, d_1) and (φ_2, d_2) are equivalent, then $Z_{33} = \frac{\mathfrak{a}'\lambda'}{\mathfrak{a}\lambda}$. As a consequence, (102) becomes equivalent to $\left(\frac{\mathfrak{a}'\lambda'}{\mathfrak{a}\lambda}\right)^2 = 1$ and we find that (φ_1, d_1) and (φ_2, d_2) are equivalent and only if $\mathfrak{a}' = \pm\frac{\mathfrak{a}\lambda}{\lambda'}$. By defining a suitable invertible algebra map Z we can prove that this condition is also sufficient. Let

$$Z_{\pm} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathbf{FF}' \mp \beta\mathbf{DD}' & -\frac{\lambda\mathbf{A}'\mathbf{F}^2 \mp \lambda'\mathbf{A}}{\lambda\lambda'\mathbf{F}} & -\beta(\mathbf{D}'\mathbf{F} \mp \mathbf{DF}') \\ 0 & 0 & \pm 1 & 0 \\ 0 & \mathbf{DF}' \mp \mathbf{D}'\mathbf{F} & -\frac{\mathbf{A}'\mathbf{D}}{\lambda'} & -(\beta\mathbf{DD}' \mp \mathbf{FF}') \end{pmatrix}.$$

Then we have $M_1 Z_{\pm} - Z_{\pm}M_2 = 0 = \mathfrak{D}_1 Z_{\pm} - Z_{\pm}\mathfrak{D}_2$, provided $\mathfrak{a}' = \pm\frac{\mathfrak{a}\lambda}{\lambda'}$. $Z_{\pm}$ satisfies (100)-(104) and $\det Z_{\pm} = (\beta\mathbf{D}^2 - \mathbf{F}^2)(\beta\mathbf{D}'^2 - \mathbf{F}'^2) \neq 0$. It follows that there is an infinite family of non-isomorphic coactions on $A = Cl(0, \beta, 0)$ given by

$$\rho(G) = G \otimes 1 + \omega(GX \otimes gx), \quad \rho(X) = X \otimes g + \omega\beta(1 \otimes x) \tag{252}$$

with $\omega \in k^{>0}$. Here $k^{>0}$ is used to denote the largest subset of k with the property that if an element ω is contained in k, then its opposite $-\omega$ is not. To get (252) one chooses $\mathbf{A} = \mathbf{D} = 0$, $\mathbf{F} = 1$ and set $\omega := \mathfrak{a}\lambda$.

Finally let us consider a pair (φ_2, d_2) from the second row of Table 3 with $\mathfrak{a}' = 0$. Its associated matrices are

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & \frac{2\mathbf{A}'}{\lambda'\beta\mathbf{D}'} & 1 \end{pmatrix}, \qquad \mathfrak{D}_2 = \begin{pmatrix} 0 & 0 & -\frac{\mathfrak{b}'}{\mathbf{D}'} & 0 \\ 0 & 0 & \frac{\mathfrak{b}'\mathbf{A}'}{\lambda'\beta\mathbf{D}'^2} & \frac{\mathfrak{b}'}{\mathbf{D}'} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with $\mathfrak{b}' \neq 0$ since d_2 is non-trivial. We can define the invertible algebra map

$$Z := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mathbf{A}'}{\lambda'\mathbf{D}'} & \beta \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and check that

$$ZM_2Z^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad Z\mathfrak{D}_2Z^{-1} = \begin{pmatrix} 0 & 0 & -\frac{\mathfrak{b}'}{\mathbf{D}'} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\mathfrak{b}'}{\beta\mathbf{D}'} & 0 & 0 \end{pmatrix}.$$

The coaction defined by these matrices is the one described in (252) when $\omega = \frac{\mathfrak{b}'}{\beta\mathbf{D}'}$.

To conclude this subsection let us fix the pair

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \mathfrak{D}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4\beta & 0 \end{pmatrix},$$

obtained from the third row of Table 3 by choosing $\mathbf{D} = \mathbf{F} = \mathfrak{b} = 0$, $\mathbf{B} = \mathfrak{a} = 1$. For every $\mathfrak{a}', \mathfrak{b}', \mathbf{B}', \mathbf{D}', \mathbf{F}' \in k$ such that $\mathbf{B}' \neq 0$ and $\mathfrak{b}'^2 - 16\mathfrak{a}'^2\beta^3\mathbf{B}'^2 \neq 0$ we can define

$$Z := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{16\mathfrak{a}'\beta^3\mathbf{B}'^2}{\mathfrak{b}'^2-16\mathfrak{a}'^2\beta^3\mathbf{B}'^2} & -\frac{4\beta^2(\mathfrak{b}'\mathbf{D}'-4\mathfrak{a}'\beta\mathbf{B}'\mathbf{F}')}{\mathfrak{b}'^2-16\mathfrak{a}'^2\beta^3\mathbf{B}'^2} & \frac{4\mathfrak{b}'\beta^2\mathbf{B}'}{\mathfrak{b}'^2-16\mathfrak{a}'^2\beta^3\mathbf{B}'^2} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{4\mathfrak{b}'\beta\mathbf{B}'}{\mathfrak{b}'^2-16\mathfrak{a}'^2\beta^3\mathbf{B}'^2} & \frac{4\beta(-4\mathfrak{a}'\beta^2\mathbf{B}'\mathbf{D}'+\mathfrak{b}'\mathbf{F}')}{\mathfrak{b}'^2-16\mathfrak{a}'^2\beta^3\mathbf{B}'^2} & -\frac{16\mathfrak{a}'\beta^3\mathbf{B}'^2}{\mathfrak{b}'^2-16\mathfrak{a}'^2\beta^3\mathbf{B}'^2} \end{pmatrix},$$

check that (100)-(104) are satisfied, $\det Z = -\frac{16\beta^3\mathbf{B}'^2}{\mathfrak{b}'^2-16\mathfrak{a}'^2\beta^3\mathbf{B}'^2} \neq 0$, and

$$Z^{-1}M_1Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -\frac{2\mathbf{F}'}{\mathbf{B}'} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{2\mathbf{D}'}{\mathbf{B}'} & -1 \end{pmatrix}, \qquad Z^{-1}\mathfrak{D}_1Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\mathfrak{b}'}{\mathbf{B}'} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4\mathfrak{a}'\beta & 0 \end{pmatrix}.$$

Notice that if the field k does not contain a square root of β then the condition $\mathfrak{b}'^2 - 16\mathfrak{a}'^2\beta^3\mathbf{B}'^2 \neq 0$ is always satisfied and there are no other coactions to consider. The coaction defined by an element of the class we have just found is (isomorphic to) $\rho(G) = G \otimes g$, $\rho(X) = X \otimes 1 - 4\beta(GX \otimes gx)$ (choose $\mathbf{D} = \mathbf{F} = \mathfrak{b} = 0$, $\mathbf{B} = \mathfrak{a} = 1$).

On the other hand, if $\sqrt{\beta} \in k$, then there are pairs in the third row of Table 3, for which such condition is true and it turns out that these pairs are all equivalent. Let us fix one:

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \mathfrak{D}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -4\beta^{\frac{3}{2}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4\beta & 0 \end{pmatrix}.$$

Then, given $\mathbf{D}, \mathbf{F} \in k$ and $\mathbf{B}, \mathfrak{a} \neq 0$, we define the invertible algebra map

$$Z_{\pm} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\mathfrak{a}} & \pm\frac{\mathbf{F}}{\mathfrak{a}\mathbf{B}} & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & \pm\frac{\mathbf{D}}{\mathfrak{a}\mathbf{B}} & \pm\frac{1}{\mathfrak{a}} \end{pmatrix}$$

and check that

$$Z_{\pm}^{-1} M_1 Z_{\pm} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -\frac{2\mathbf{F}}{\mathbf{B}} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{2\mathbf{D}}{\mathbf{B}} & -1 \end{pmatrix}, \qquad Z_{\pm}^{-1} \mathfrak{D}_1 Z_{\pm} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \mp 4\mathfrak{a}\beta^{\frac{3}{2}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4\mathfrak{a}\beta & 0 \end{pmatrix}.$$

Therefore, all pairs for which $\mathfrak{b}^2 = 16\mathfrak{a}^2\beta^3\mathbf{B}^2$ are equivalent to the pair defined by M_1 and $\mathfrak{D}_1$. The coaction defined by an element of this class is (isomorphic to) $\rho(G) = G \otimes g$, $\rho(X) = X \otimes 1 - 4\beta G(\sqrt{\beta} + X) \otimes gx$ (choose $\mathbf{D} = \mathbf{F} = 0$, $\mathbf{B} = \mathfrak{a} = 1$, $\mathfrak{b} = 4\beta^{\frac{3}{2}}$).

We gather all these results in the following table.

Type of Coaction	Definition	Skew-derivation
0	$\rho(G) = G \otimes 1$, $\rho(X) = X \otimes 1$	Trivial
1	$\rho(G) = \frac{G(\sqrt{\beta}-X)}{2\sqrt{\beta}} \otimes 1 + \frac{G(\sqrt{\beta}+X)}{2\sqrt{\beta}} \otimes g$, $\rho(X) = X \otimes 1$	Trivial
	$\rho(G) = \frac{G(\sqrt{\beta}-X)}{2\sqrt{\beta}} \otimes 1 + \frac{G(\sqrt{\beta}+X)}{2\sqrt{\beta}} \otimes g$, $\rho(X) = X \otimes 1 - 2\sqrt{\beta}(\sqrt{\beta} + G)X \otimes gx$	Non-trivial
2	$\rho(G) = G \otimes 1$, $\rho(X) = X \otimes g$	Trivial
	$\rho(G) = G \otimes g$, $\rho(X) = X \otimes 1$	Trivial
	$\rho(G) = G \otimes 1 + X \otimes gx$, $\rho(X) = X \otimes g$	Non-trivial
	$\rho(G) = G \otimes 1 + \omega(GX \otimes gx)$, $\rho(X) = X \otimes g + \omega\beta(1 \otimes x)$ $(\omega \in k \setminus \{0\})$ (The coactions in this family are all non-isomorphic, unless $\omega = \pm\omega'$)	Non-trivial
	$\rho(G) = G \otimes g$, $\rho(X) = X \otimes 1 - 4\beta(GX \otimes gx)$	Non-trivial
	$\rho(G) = G \otimes g$, $\rho(X) = X \otimes 1 - 4\beta G(\sqrt{\beta} + X) \otimes gx$	Non-trivial

TABLE 4. Non-isomorphic coactions on $A = Cl(0, \beta, 0)$.

REMARK 3.6. Coactions of type 1 and the last class of coactions of type 2 on $Cl(0, \beta, 0)$ actually exist only if k contains a square root of β.

3.3. The case $\mathfrak{F}_2$ when $\alpha \neq 0$, $\beta = \gamma = 0$. When $\alpha \neq 0$, $\beta = \gamma = 0$ every coaction of type $\mathfrak{F}_2$ corresponds to a pair (φ, d) where φ and d are of the following form.

Matrix M_φ of φ	Matrix of d	Conditions
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -\frac{2\mathbf{B}\mathbf{E}}{\lambda(-\mathbf{B}^2+\alpha\mathbf{D}^2)} & \frac{-\mathbf{B}^2-\alpha\mathbf{D}^2}{-\mathbf{B}^2+\alpha\mathbf{D}^2} & -\frac{2\alpha\mathbf{B}\mathbf{D}}{-\mathbf{B}^2+\alpha\mathbf{D}^2} \\ 0 & \frac{2\mathbf{D}\mathbf{E}}{\lambda(-\mathbf{B}^2+\alpha\mathbf{D}^2)} & \frac{2\mathbf{B}\mathbf{D}}{-\mathbf{B}^2+\alpha\mathbf{D}^2} & \frac{\mathbf{B}^2+\alpha\mathbf{D}^2}{-\mathbf{B}^2+\alpha\mathbf{D}^2} \end{pmatrix}$	$\begin{pmatrix} 0 & -\frac{\alpha\mathbf{E}(\mathfrak{a}\mathbf{B}+\mathfrak{b}\mathbf{D})}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{\lambda\alpha\mathbf{D}(\mathfrak{a}\alpha\mathbf{D}+\mathfrak{b}\mathbf{B})}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{\lambda\alpha\mathbf{B}(\mathfrak{a}\alpha\mathbf{D}+\mathfrak{b}\mathbf{B})}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & \frac{\mathbf{E}(\mathfrak{a}\alpha\mathbf{D}+\mathfrak{b}\mathbf{B})}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{\lambda\mathbf{B}(\mathfrak{a}\alpha\mathbf{D}+\mathfrak{b}\mathbf{B})}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{\lambda\alpha\mathbf{D}(\mathfrak{a}\alpha\mathbf{D}+\mathfrak{b}\mathbf{B})}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & -\frac{\mathfrak{b}\mathbf{E}^2}{\lambda(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{\mathfrak{b}\mathbf{B}\mathbf{E}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{\mathfrak{b}\alpha\mathbf{D}\mathbf{E}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & -\frac{\mathfrak{a}\mathbf{E}^2}{\lambda(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{\mathfrak{a}\mathbf{B}\mathbf{E}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{\mathfrak{a}\alpha\mathbf{D}\mathbf{E}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \end{pmatrix}$	$\alpha \neq 0$, $\beta = \gamma = 0$ $\lambda, (\alpha\mathbf{D}^2 - \mathbf{B}^2), \mathbf{E} \neq 0$
	$\begin{pmatrix} 0 & \mathfrak{a}\lambda\alpha & -\frac{\mathfrak{b}\alpha\mathbf{D}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{\mathfrak{b}\alpha\mathbf{B}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & 0 & \frac{\mathfrak{b}\mathbf{B}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{\mathfrak{b}\alpha\mathbf{D}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & 0 & \frac{\mathfrak{a}\lambda\alpha\mathbf{B}\mathbf{D}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{\mathfrak{a}\lambda\alpha^2\mathbf{D}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & 0 & -\frac{\mathfrak{a}\lambda\mathbf{B}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{\mathfrak{a}\lambda\alpha\mathbf{B}\mathbf{D}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \end{pmatrix}$	$\alpha \neq 0$, $\beta = \gamma = \mathbf{E} = 0$ $\lambda, (\alpha\mathbf{D}^2 - \mathbf{B}^2) \neq 0$
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{2\mathbf{B}}{\mathbf{F}} & -1 & 0 \\ 0 & \frac{2\mathbf{D}}{\mathbf{F}} & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -4\mathfrak{b}\alpha & 0 & 0 \\ 0 & -4\mathfrak{a}\alpha & 0 & 0 \end{pmatrix}$	$\alpha \neq 0$, $\beta = \gamma = 0$ $\mathbf{F} \neq 0$

TABLE 5. Pairs defining coactions on $A = Cl(\alpha, 0, 0)$.

REMARK 3.7. Again we have that $d \equiv 0$ is equivalent to $\mathfrak{a} = \mathfrak{b} = 0$ in each of the displayed instances.

3.3.1. *Coactions with trivial skew-derivation.* We can fix the pair $(\varphi_1, 0)$ from the first row with associate matrix

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and define

$$Z := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\alpha\mathbf{D} & -\alpha\mathbf{B} \\ 0 & -\frac{\mathbf{E}}{\lambda} & -\mathbf{B} & -\alpha\mathbf{D} \end{pmatrix},$$

with $\lambda, \alpha\mathbf{D}^2 - \mathbf{B}^2 \neq 0$. One checks that Z defines an algebra map, that $\det Z = \alpha(\alpha\mathbf{D}^2 - \mathbf{B}^2) \neq 0$ and

$$Z^{-1}M_1Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -\frac{2\mathbf{B}\mathbf{E}}{\lambda(-\mathbf{B}^2+\alpha\mathbf{D}^2)} & \frac{-\mathbf{B}^2-\alpha\mathbf{D}^2}{-\mathbf{B}^2+\alpha\mathbf{D}^2} & -\frac{2\alpha\mathbf{B}\mathbf{D}}{-\mathbf{B}^2+\alpha\mathbf{D}^2} \\ 0 & \frac{2\mathbf{D}\mathbf{E}}{\lambda(-\mathbf{B}^2+\alpha\mathbf{D}^2)} & \frac{2\mathbf{B}\mathbf{D}}{-\mathbf{B}^2+\alpha\mathbf{D}^2} & \frac{\mathbf{B}^2+\alpha\mathbf{D}^2}{-\mathbf{B}^2+\alpha\mathbf{D}^2} \end{pmatrix}.$$

Similarly we can fix the pair $(\varphi_2, 0)$ from the last row with associate matrix

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and define the invertible algebra map

$$Z := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{\alpha\mathbf{D}}{\mathbf{F}} & 0 & \alpha \\ 0 & \frac{\mathbf{B}}{\mathbf{F}} & 1 & 0 \end{pmatrix},$$

with $\mathbf{F} \neq 0$. Then

$$ZM_2Z^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{2\mathbf{B}}{\mathbf{F}} & -1 & 0 \\ 0 & \frac{2\mathbf{D}}{\mathbf{F}} & 0 & -1 \end{pmatrix}.$$

Finally we show that $(\varphi_1, 0)$ and $(\varphi_2, 0)$ cannot be equivalent. Suppose there is an invertible algebra map with matrix Z such that $M_1Z - ZM_2 = 0$. Then, by Lemma 3.2, we have

$$M_1Z - ZM_2 = \begin{pmatrix} 0 & 0 & 0 & 2\alpha Z_{22}Z_{23} \\ 0 & -2Z_{22} & 0 & 0 \\ 0 & -2Z_{32} & 0 & 0 \\ 0 & 0 & 2Z_{43} & 2(Z_{22}Z_{33} - Z_{23}Z_{32}) \end{pmatrix},$$

which forces $Z_{22} = Z_{32} = Z_{43} = 0$. As a consequence (101) becomes equivalent to $\alpha = 0$, contradiction.

REMARK 3.8. This can also be obtained as a consequence of the fact that involutions in the first and second row of Table 3 are not inner, while those in the third are (cf. Table 2 in Chapter 2).

In conclusion we have found that, up to isomorphism, there are only two coactions of type 2 on $A = Cl(\alpha, 0, 0)$ corresponding to pairs with trivial skew-derivation. These are given by $\rho(G) = G \otimes g$, $\rho(X) = X \otimes g$ and $\rho(G) = G \otimes 1$, $\rho(X) = X \otimes g$.

3.3.2. *Coactions with non-trivial skew-derivation.* Let us fix the pair (φ_1, d_1) with associated matrices

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}, \quad \mathfrak{D}_1 = \begin{pmatrix} 0 & 0 & -\alpha & 0 \\ 0 & \alpha & 0 & \alpha \\ 0 & 0 & 0 & 0 \\ 0 & -\alpha & 0 & -\alpha \end{pmatrix} \tag{253}$$

and define the matrix

$$Z := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{\mathbf{E}(\mathfrak{a}\mathbf{B}+\mathfrak{b}\mathbf{D})}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{\lambda\mathbf{D}(\mathfrak{b}\mathbf{B}+\mathfrak{a}\alpha\mathbf{D})}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{\lambda\mathbf{B}(\mathfrak{b}\mathbf{B}+\mathfrak{a}\alpha\mathbf{D})}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & \frac{\mathbf{E}(\mathfrak{b}\mathbf{B}+\mathfrak{a}\alpha\mathbf{D})}{\alpha(\alpha\mathbf{D}^2-\mathbf{B}^2)} - 1 & \frac{\lambda\mathbf{B}(\mathfrak{b}\mathbf{B}+\mathfrak{a}\alpha\mathbf{D})}{\alpha(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{\lambda\mathbf{D}(\mathfrak{b}\mathbf{B}+\mathfrak{a}\alpha\mathbf{D})}{\alpha\mathbf{D}^2-\mathbf{B}^2} \end{pmatrix},$$

under the condition that $\mathbf{E}, \mathfrak{b}\mathbf{B}+\mathfrak{a}\alpha\mathbf{D} \neq 0$. It's easy to check that Z satisfies (100)-(104), that $\det Z = \frac{\lambda^2(\mathfrak{b}\mathbf{B}+\mathfrak{a}\alpha\mathbf{D})^2}{\alpha(\alpha\mathbf{D}^2-\mathbf{B}^2)} \neq 0$ and that

$$Z^{-1}M_1Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -\frac{2\mathbf{B}\mathbf{E}}{\lambda(-\mathbf{B}^2+\alpha\mathbf{D}^2)} & \frac{-\mathbf{B}^2-\alpha\mathbf{D}^2}{-\mathbf{B}^2+\alpha\mathbf{D}^2} & -\frac{2\alpha\mathbf{B}\mathbf{D}}{-\mathbf{B}^2+\alpha\mathbf{D}^2} \\ 0 & \frac{2\mathbf{D}\mathbf{E}}{\lambda(-\mathbf{B}^2+\alpha\mathbf{D}^2)} & \frac{2\mathbf{B}\mathbf{D}}{-\mathbf{B}^2+\alpha\mathbf{D}^2} & \frac{\mathbf{B}^2+\alpha\mathbf{D}^2}{-\mathbf{B}^2+\alpha\mathbf{D}^2} \end{pmatrix}, \quad Z^{-1}\mathfrak{D}_1Z = \begin{pmatrix} 0 & -\frac{\alpha\mathbf{E}(\mathfrak{a}\mathbf{B}+\mathfrak{b}\mathbf{D})}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{\lambda\alpha\mathbf{D}(\mathfrak{a}\alpha\mathbf{D}+\mathfrak{b}\mathbf{B})}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{\lambda\alpha\mathbf{B}(\mathfrak{a}\alpha\mathbf{D}+\mathfrak{b}\mathbf{B})}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & \frac{\mathbf{E}(\mathfrak{a}\alpha\mathbf{D}+\mathfrak{b}\mathbf{B})}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{\lambda\mathbf{B}(\mathfrak{a}\alpha\mathbf{D}+\mathfrak{b}\mathbf{B})}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{\lambda\alpha\mathbf{D}(\mathfrak{a}\alpha\mathbf{D}+\mathfrak{b}\mathbf{B})}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & -\frac{\mathfrak{b}\mathbf{E}^2}{\lambda(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{\mathfrak{b}\mathbf{B}\mathbf{E}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{\mathfrak{b}\alpha\mathbf{D}\mathbf{E}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & -\frac{\mathfrak{a}\mathbf{E}^2}{\lambda(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{\mathfrak{a}\mathbf{B}\mathbf{E}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{\mathfrak{a}\alpha\mathbf{D}\mathbf{E}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \end{pmatrix}.$$

This means that all pairs from the first row of Table 5 with $\mathfrak{b}\mathbf{B} + \mathfrak{a}\alpha\mathbf{D} \neq 0$ are equivalent. Pairs not satisfying this conditions are not in the same class. To prove this, we consider an invertible Z of the form (241). It is easy to check that (102) is equivalent to $Z_{23} = 0$. Thus a $\mathfrak{D}_2$ associated to an equivalent pair is of the form

$$Z^{-1}\mathfrak{D}_1 Z = \begin{pmatrix} 0 & -\alpha Z_{32} & -\alpha Z_{33} & -\alpha^2 Z_{22} Z_{43} \\ 0 & \dots & \frac{\alpha Z_{43}}{\dots} & \alpha Z_{33} \\ 0 & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots \end{pmatrix}. \tag{254}$$

A pair in the first row of Table 5 with $\mathfrak{b}\mathbf{B} + \mathfrak{a}\alpha\mathbf{D} = 0$ has a skew-derivation associated to a matrix of the form

$$\mathfrak{D}_2 = \begin{pmatrix} 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots \end{pmatrix},$$

therefore, by equating the two matrices, we find $Z_{33} = Z_{43} = 0$. This is a contradiction, since Z is invertible and cannot have a zero column.

We can show that also pairs from the second row of Table 3 with $\mathfrak{b} \neq 0$ are in the class formerly obtained. In fact when $\mathfrak{b} \neq 0$, we can define the invertible matrix

$$Z := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\mathfrak{a}\lambda & \frac{\mathfrak{b}\mathbf{D}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{\mathfrak{b}\mathbf{B}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & -1 & \frac{\mathfrak{b}\mathbf{B}}{\alpha(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{\mathfrak{b}\mathbf{D}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \end{pmatrix}$$

and check that

$$Z^{-1}M_1 Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{-\mathbf{B}^2-\alpha\mathbf{D}^2}{-\mathbf{B}^2+\alpha\mathbf{D}^2} & -\frac{2\alpha\mathbf{B}\mathbf{D}}{-\mathbf{B}^2+\alpha\mathbf{D}^2} \\ 0 & 0 & \frac{2\mathbf{B}\mathbf{D}}{-\mathbf{B}^2+\alpha\mathbf{D}^2} & \frac{\mathbf{B}^2+\alpha\mathbf{D}^2}{-\mathbf{B}^2+\alpha\mathbf{D}^2} \end{pmatrix}, \qquad Z^{-1}\mathfrak{D}_1 Z = \begin{pmatrix} 0 & \mathfrak{a}\lambda\alpha & -\frac{\mathfrak{b}\alpha\mathbf{D}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{\mathfrak{b}\alpha\mathbf{B}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & 0 & \frac{\mathfrak{b}\mathbf{B}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{\mathfrak{b}\alpha\mathbf{D}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & 0 & \frac{\mathfrak{a}\lambda\alpha\mathbf{B}\mathbf{D}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{\mathfrak{a}\lambda\alpha^2\mathbf{D}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & 0 & -\frac{\mathfrak{a}\lambda\mathbf{B}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{\mathfrak{a}\lambda\alpha\mathbf{B}\mathbf{D}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \end{pmatrix}$$

Z satisfies (100)-(104). A pair from the second row of Table 3 with $\mathfrak{b} = 0$ is defined by a skew-derivation of the form

$$\mathfrak{D}_2 = \begin{pmatrix} 0 & \mathfrak{a}\lambda\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mathfrak{a}\lambda\alpha\mathbf{B}\mathbf{D}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{\mathfrak{a}\lambda\alpha^2\mathbf{D}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & 0 & -\frac{\mathfrak{a}\lambda\mathbf{B}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{\mathfrak{a}\lambda\alpha\mathbf{B}\mathbf{D}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \end{pmatrix}.$$

Suppose $\mathfrak{D}_2 = Z^{-1}\mathfrak{D}_1 Z$ for some invertible algebra map Z. Then $\mathfrak{D}_2$ is equal to (254) and we find $Z_{33} = Z_{43} = 0$. Since once again (102) yields $Z_{32} = 0$, we encounter a contradiction and we conclude that pairs from the second row of Table 3 with $\mathfrak{b} = 0$ are not equivalent to the pair defined by (253). The coaction defined by an element of this class is (isomorphic to) $\rho(G) = GX \otimes 1 + G(1-X) \otimes g - \alpha G(1-X) \otimes x$, $\rho(X) = X \otimes g + \alpha(1 \otimes x)$.

Now, for any $\mathfrak{a}, \lambda \neq 0$, we can fix the pair (φ_1, d_1) associated to the matrices

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathfrak{D}_1 = \begin{pmatrix} 0 & \mathfrak{a}\lambda\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathfrak{a}\lambda\alpha \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{255}$$

By using Lemma 3.2 and (102) we can see that any pair of the second row of Table 5 which is equivalent to (φ_1, d_1) must have skew-derivation of the form

$$Z^{-1}\mathfrak{D}_1 Z = \begin{pmatrix} 0 & Z_{22}\mathfrak{a}\lambda\alpha & 0 & 0 \\ \dots & & & \\ \dots & & & \\ \dots & & & \end{pmatrix}.$$

Since (101) reduces to $Z_{22}^2 = 1$, it becomes clear that any pair equivalent to (φ_1, d_1) must have $\mathfrak{a}'\lambda' = \pm\mathfrak{a}\lambda$. This condition is also sufficient for the equivalence to hold, because we can define the invertible matrix

$$Z_\pm := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm\alpha\mathbf{D} & \pm\alpha\mathbf{B} \\ 0 & 0 & \mathbf{B} & \alpha\mathbf{D} \end{pmatrix}$$

and check that

$$Z_\pm^{-1}\mathfrak{D}_1 Z_\pm = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{-\mathbf{B}^2-\alpha\mathbf{D}^2}{-\mathbf{B}^2+\alpha\mathbf{D}^2} & -\frac{2\alpha\mathbf{B}\mathbf{D}}{-\mathbf{B}^2+\alpha\mathbf{D}^2} \\ 0 & 0 & \frac{2\mathbf{B}\mathbf{D}}{-\mathbf{B}^2+\alpha\mathbf{D}^2} & \frac{\mathbf{B}^2+\alpha\mathbf{D}^2}{-\mathbf{B}^2+\alpha\mathbf{D}^2} \end{pmatrix}, \quad Z_\pm^{-1}\mathfrak{D}_2 Z_\pm = \begin{pmatrix} 0 & \mathfrak{a}\lambda\alpha & -\frac{\mathfrak{b}\alpha\mathbf{D}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{\mathfrak{b}\alpha\mathbf{B}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & 0 & \frac{\mathfrak{b}\mathbf{B}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{\mathfrak{b}\alpha\mathbf{D}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & 0 & \frac{\mathfrak{a}\lambda\alpha\mathbf{B}\mathbf{D}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{\mathfrak{a}\lambda\alpha^2\mathbf{D}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & 0 & -\frac{\mathfrak{a}\lambda\mathbf{B}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{\mathfrak{a}\lambda\alpha\mathbf{B}\mathbf{D}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \end{pmatrix}$$

and $Z_\pm$ verify (100)-(104).

It follows that there is an infinite family of non-isomorphic coactions on $A = Cl(\alpha, 0, 0)$ given by

$$\rho(G) = G \otimes g - \omega\alpha(1 \otimes x), \quad \rho(X) = X \otimes g \tag{256}$$

with $\omega \in k^{>0}$. Here $k^{>0}$ is again used to denote the largest subset of k with the property that if an element ω is contained in k, then its opposite $-\omega$ is not. To get (256) one uses (255) with $\omega := \mathfrak{a}\lambda$.

To complete this first part we need to understand in which class are contained the pairs of the first row of Table 5 with $\mathfrak{b}\mathbf{B} + \mathfrak{a}\alpha\mathbf{D} = 0$. Notice that, in this case, when $\mathfrak{b} = 0$, then $\mathbf{D} = 0$ (since we are dealing with non-trivial skew-derivations) and $\mathbf{B}, \mathfrak{a} \neq 0$.

Let us fix $\mathbf{B}, \mathbf{E}, \lambda \neq 0$ and define the invertible matrix

$$Z := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha\mathbf{B}}{\mathbf{E}} \\ 0 & \frac{1}{\lambda} & \frac{\mathbf{B}}{\mathbf{E}} & 0 \end{pmatrix}.$$

Then

$$Z^{-1}M_1 Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & \frac{2\mathbf{E}}{\lambda\mathbf{B}} & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad Z^{-1}\mathfrak{D}_1 Z = \begin{pmatrix} 0 & \omega\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\omega\mathbf{E}}{\lambda\mathbf{B}} & \omega & 0 \end{pmatrix}$$

and Z satisfies (100)-(104). If we take $\omega = \frac{\mathfrak{a}\mathbf{E}}{\mathbf{B}}$ we get exactly a pair of the first row of Table 5 with $\mathfrak{b}\mathbf{B} + \mathfrak{a}\alpha\mathbf{D} = 0$ and $\mathfrak{b} = 0$ (i.e. $\mathfrak{b} = \mathbf{D} = 0$). Next, if we fix $\mathbf{D}, \mathbf{E}, \lambda, \mathfrak{b} \neq 0$ and $\mathfrak{a} \in k$ such that $\alpha\mathfrak{a}^2 - \mathfrak{b}^2 \neq 0$ and define the invertible matrix

$$Z := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\frac{\lambda}{\mathbf{E}} & \frac{\mathfrak{a}\alpha\lambda}{\mathfrak{b}\mathbf{E}} \\ 0 & \frac{1}{\alpha\mathbf{D}} & -\frac{\mathfrak{a}\lambda}{\mathfrak{b}\mathbf{E}} & \frac{\lambda}{\mathbf{E}} \end{pmatrix}$$

we see that

$$Z^{-1}M_1 Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -\frac{2\mathfrak{a}\mathfrak{b}\mathbf{E}}{\lambda\mathbf{D}(\alpha\mathfrak{a}^2-\mathfrak{b}^2)} & \frac{\alpha\mathfrak{a}^2+\mathfrak{b}^2}{\alpha\mathfrak{a}^2-\mathfrak{b}^2} & -\frac{2\mathfrak{a}\mathfrak{b}\alpha}{\alpha\mathfrak{a}^2-\mathfrak{b}^2} \\ 0 & -\frac{2\mathfrak{b}^2\mathbf{E}}{\lambda\alpha\mathbf{D}(\alpha\mathfrak{a}^2-\mathfrak{b}^2)} & \frac{2\mathfrak{a}\mathfrak{b}}{\alpha\mathfrak{a}^2-\mathfrak{b}^2} & -\frac{\alpha\mathfrak{a}^2+\mathfrak{b}^2}{\alpha\mathfrak{a}^2-\mathfrak{b}^2} \end{pmatrix}, \qquad Z^{-1}\mathfrak{D}_1 Z = \begin{pmatrix} 0 & -\omega\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\omega\mathfrak{b}^2\mathbf{E}}{\lambda\mathbf{D}(\alpha\mathfrak{a}^2-\mathfrak{b}^2)} & -\frac{\omega\mathfrak{a}\mathfrak{b}\alpha}{\alpha\mathfrak{a}^2-\mathfrak{b}^2} & \frac{\omega\alpha\mathfrak{b}^2}{\alpha\mathfrak{a}^2-\mathfrak{b}^2} \\ 0 & \frac{\omega\mathfrak{a}\mathfrak{b}\mathbf{E}}{\lambda\mathbf{D}(\alpha\mathfrak{a}^2-\mathfrak{b}^2)} & -\frac{\omega\mathfrak{a}^2\alpha}{\alpha\mathfrak{a}^2-\mathfrak{b}^2} & \frac{\omega\mathfrak{a}\mathfrak{b}\alpha}{\alpha\mathfrak{a}^2-\mathfrak{b}^2} \end{pmatrix}$$

and that Z satisfies (100)-(104). If we take $\omega = \frac{\mathfrak{b}\mathbf{E}}{\alpha\mathbf{D}}$ we find a pair of the first row of Table 5 with $\mathfrak{b} \neq 0$ and $\mathbf{B} = -\frac{\mathfrak{a}\alpha\mathbf{D}}{\mathfrak{b}}$.

To conclude this subsection let us consider the pair

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \mathfrak{D}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -4\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{257}$$

For any $\mathbf{B}, \mathbf{D}, \mathbf{F}, \mathfrak{a}, \mathfrak{b} \in k$ such that $\mathbf{F}, \alpha\mathfrak{a}^2 - \mathfrak{b}^2 \neq 0$ we can define the invertible matrix

$$Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{\mathfrak{b}\mathbf{B}+\mathfrak{a}\alpha\mathbf{D}}{\mathbf{F}(\alpha\mathfrak{a}^2-\mathfrak{b}^2)} & -\frac{\mathfrak{b}}{\alpha\mathfrak{a}^2-\mathfrak{b}^2} & \frac{\mathfrak{a}\alpha}{\alpha\mathfrak{a}^2-\mathfrak{b}^2} \\ 0 & \frac{\mathfrak{a}\mathbf{B}+\mathfrak{b}\mathbf{D}}{\mathbf{F}(\alpha\mathfrak{a}^2-\mathfrak{b}^2)} & \frac{\mathfrak{a}}{\alpha\mathfrak{a}^2-\mathfrak{b}^2} & -\frac{\mathfrak{b}}{\alpha\mathfrak{a}^2-\mathfrak{b}^2} \end{pmatrix}$$

We can easily check that Z verify (100)-(104) and that

$$Z^{-1}M_1Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{2\mathbf{B}}{\mathbf{F}} & -1 & 0 \\ 0 & \frac{2\mathbf{D}}{\mathbf{F}} & 0 & -1 \end{pmatrix}, \qquad Z^{-1}\mathfrak{D}_1Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -4\mathfrak{b}\alpha & 0 & 0 \\ 0 & -4\mathfrak{a}\alpha & 0 & 0 \end{pmatrix}.$$

If the field k does not contain any square root of α, then $\alpha\mathfrak{a}^2 - \mathfrak{b}^2 \neq 0$ always holds true and there are no more coactions to consider. The coaction defined by an element of the class we have just considered is (isomorphic to) $\rho(G) = G \otimes 1 - 4\alpha(X \otimes gx)$, $\rho(X) = X \otimes g$.

Finally, any pair belonging to the last row of Table 5 with $\alpha\mathfrak{a}^2 - \mathfrak{b}^2 = 0$ is equivalent to the one defined by

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \mathfrak{D}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -4\alpha^{\frac{3}{2}} & 0 & 0 \\ 0 & -4\alpha & 0 & 0 \end{pmatrix}.$$

To prove this, fix a $\mathfrak{a} \neq 0$ and define the invertible matrix

$$Z_\pm := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mp 1 & 0 & 0 \\ 0 & \mp\frac{\sqrt{\alpha}\mathbf{D}}{\mathfrak{a}\mathbf{F}} & 0 & \mp\frac{\sqrt{\alpha}}{\mathfrak{a}} \\ 0 & \frac{\mathbf{B}}{\mathfrak{a}\sqrt{\alpha}\mathbf{F}} & \frac{1}{\mathfrak{a}\sqrt{\alpha}} & 0 \end{pmatrix}.$$

Then we have that Z satisfies (100)-(104) and

$$Z_\pm^{-1}M_1Z_\pm = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{2\mathbf{B}}{\mathbf{F}} & -1 & 0 \\ 0 & \frac{2\mathbf{D}}{\mathbf{F}} & 0 & -1 \end{pmatrix}, \qquad Z_\pm^{-1}\mathfrak{D}_1Z_\pm = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mp 4\mathfrak{a}\alpha^{\frac{3}{2}} & 0 & 0 \\ 0 & -4\mathfrak{a}\alpha & 0 & 0 \end{pmatrix}.$$

Notice that these are the matrices that define a pair from the last row of Table 5 when $\mathfrak{b} = \pm\sqrt{\alpha}\mathfrak{a}$. We don't need to check the case $\mathfrak{a} = 0$, since $\alpha\mathfrak{a}^2 - \mathfrak{b}^2 = 0$ would imply that the skew-derivation we are considering is trivial. The coaction defined by an element of this class is (isomorphic to) $\rho(G) = G \otimes 1 - 4\alpha(\sqrt{\alpha} + G)X \otimes gx$, $\rho(X) = X \otimes g$.

We gather all these results in the following table.

Type of Coaction	Definition	Skew-derivation
0	$\rho(G) = G \otimes 1$, $\rho(X) = X \otimes 1$	Trivial

1	$\rho(G) = G \otimes g,\ \rho(X) = \frac{(\sqrt{\alpha}-G)X}{2\sqrt{\alpha}} \otimes 1 + \frac{(\sqrt{\alpha}+G)X}{2\sqrt{\alpha}} \otimes g$	Trivial
	$\rho(G) = G \otimes g - 2\sqrt{\alpha}(\sqrt{\alpha}+G)X \otimes x,\ \rho(X) = \frac{(\sqrt{\alpha}-G)X}{2\sqrt{\alpha}} \otimes 1 + \frac{(\sqrt{\alpha}+G)X}{2\sqrt{\alpha}} \otimes g$	Non-trivial
2	$\rho(G) = G \otimes g,\ \rho(X) = X \otimes g$	Trivial
	$\rho(G) = G \otimes 1,\ \rho(X) = X \otimes g$	Trivial
	$\rho(G) = GX \otimes 1 + G(1-X) \otimes g - \alpha G(1-X) \otimes x,\ \rho(X) = X \otimes g + \alpha(1 \otimes x)$	Non-trivial
	$\rho(G) = G \otimes g + \omega\alpha(1 \otimes x),\ \rho(X) = X \otimes g \quad (\omega \in k \setminus \{0\})$ (The coactions in this family are all non-isomorphic, unless $\omega = \pm\omega'$)	Non-trivial
	$\rho(G) = G \otimes 1 - 4\alpha(X \otimes gx),\ \rho(X) = X \otimes g$	Non-trivial
	$\rho(G) = G \otimes 1 - 4\alpha(\sqrt{\alpha}+G)X \otimes gx,\ \rho(X) = X \otimes g$	Non-trivial

TABLE 6. Non-isomorphic coactions on $A = Cl(\alpha, 0, 0)$.

REMARK 3.9. Coactions of type 1 and the last class of coactions of type 2 on $Cl(\alpha, 0, 0)$ actually exist only if k contains a square root of α.

3.4. The case $\mathfrak{F}_2$ when $\gamma \neq 0$, $\gamma^2 - 4\alpha\beta = 0$. When $\gamma \neq 0$ and $\gamma^2 - 4\alpha\beta = 0$ every coaction of type $\mathfrak{F}_2$ corresponds to a pair (φ, d) where φ and d are of the following form.

Matrix M_φ of φ	Matrix of d	Conditions
$\begin{pmatrix} 1 & -\frac{2\lambda\alpha\gamma}{\mathbf{C}} & -\frac{\lambda\gamma^2}{\mathbf{C}} & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{4\lambda\alpha}{\mathbf{C}} & \frac{2\lambda\gamma}{\mathbf{C}} & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{2\lambda\alpha(\mathfrak{a}\lambda\alpha\gamma-\mathfrak{b}\mathbf{C})}{\mathbf{CD}} & -\frac{\mathfrak{a}\mathbf{C}+\mathfrak{b}\lambda\gamma}{\mathbf{D}} + \frac{\mathfrak{a}\lambda^2\alpha\gamma^2}{\mathbf{CD}} & \frac{\mathfrak{a}\lambda\alpha\gamma}{\mathbf{D}} \\ 0 & \frac{2\lambda\alpha(\mathfrak{a}\mathbf{C}+\mathfrak{b}\lambda\gamma)}{\mathbf{CD}} & \frac{\lambda\gamma(\mathfrak{a}\mathbf{C}+\mathfrak{b}\lambda\gamma)}{\mathbf{CD}} & \frac{\mathfrak{a}\mathbf{C}+\mathfrak{b}\lambda\gamma}{\mathbf{D}} \\ 0 & -\frac{4\mathfrak{b}\lambda^2\alpha^2}{\mathbf{CD}} & -\frac{2\mathfrak{b}\lambda^2\alpha\gamma}{\mathbf{CD}} & -\frac{2\mathfrak{b}\lambda\alpha}{\mathbf{D}} \\ 0 & -\frac{4\mathfrak{a}\lambda^2\alpha^2}{\mathbf{CD}} & -\frac{2\mathfrak{a}\lambda^2\alpha\gamma}{\mathbf{CD}} & -\frac{2\mathfrak{a}\lambda\alpha}{\mathbf{D}} \end{pmatrix}$	$\gamma \neq 0,\ \beta = \frac{\gamma^2}{4\alpha}$ $\mathbf{C}, \mathbf{D}, \lambda \neq 0$
	$\begin{pmatrix} 0 & \frac{\mathfrak{a}\mathbf{C}}{\mathbf{D}} & -\frac{\mathfrak{b}}{\mathbf{D}} & 0 \\ 0 & 0 & 0 & \frac{\mathfrak{b}}{\mathbf{D}} \\ 0 & 0 & 0 & \frac{\mathfrak{a}\mathbf{C}}{\mathbf{D}} \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\gamma \neq 0,\ \beta = \frac{\gamma^2}{4\alpha}$ $\lambda = 0,\ \mathbf{C}, \mathbf{D}, \neq 0$
$\begin{pmatrix} 1 & \frac{2\mu\alpha\gamma\mathbf{BD}}{\mathbf{RQ}} & \frac{\lambda\alpha\gamma\mathbf{BD}}{\mathbf{RQ}} & \frac{-\gamma\mathbf{B}^2}{\mathbf{Q}} \\ 0 & -\frac{\mathbf{RQ}+2\mu\gamma\mathbf{B}^2}{\mathbf{RQ}} & -\frac{\lambda\gamma\mathbf{B}^2}{\mathbf{RQ}} & \frac{\gamma\mathbf{BD}}{\mathbf{Q}} \\ 0 & \frac{4\mu\alpha\mathbf{B}^2}{\mathbf{RQ}} & \frac{2\lambda\alpha\mathbf{B}^2-\mathbf{RQ}}{\mathbf{RQ}} & -\frac{2\alpha\mathbf{BD}}{\mathbf{Q}} \\ 0 & -\frac{4\mu\alpha\mathbf{BD}}{\mathbf{RQ}} & -\frac{2\lambda\alpha\mathbf{BD}}{\mathbf{RQ}} & \frac{\alpha\mathbf{D}^2+\mathbf{B}^2}{\mathbf{Q}} \end{pmatrix}$	$\begin{pmatrix} 0 & \star & \star\star & \star\star\star \\ 0 & \frac{2\mu\alpha\mathbf{B}(\mathfrak{a}\mathbf{DR}-\mathfrak{b}\lambda\mathbf{B})}{\mathbf{RQ}} & \frac{\lambda\alpha\mathbf{B}(\mathfrak{a}\mathbf{DR}-\mathfrak{b}\lambda\mathbf{B})}{\mathbf{RQ}} & \frac{-\alpha\mathbf{D}(\mathfrak{a}\mathbf{DR}-\mathfrak{b}\lambda\mathbf{B})}{\mathbf{Q}} \\ 0 & \frac{4\mathfrak{b}\mu^2\alpha\mathbf{B}^2}{\mathbf{RQ}} & \frac{2\mathfrak{b}\lambda\mu\alpha\mathbf{B}^2}{\mathbf{RQ}} & \frac{-2\mathfrak{b}\mu\alpha\mathbf{BD}}{\mathbf{Q}} \\ 0 & \frac{4\mathfrak{a}\mu^2\alpha\mathbf{B}^2}{\mathbf{RQ}} & \frac{2\mathfrak{a}\lambda\mu\alpha\mathbf{B}^2}{\mathbf{RQ}} & \frac{-2\mathfrak{a}\mu\alpha\mathbf{BD}}{\mathbf{Q}} \end{pmatrix}$ $\star = -\frac{2\mu\alpha\mathbf{B}[\mathfrak{a}\mu\gamma\mathbf{B}+(\mathfrak{a}\mathbf{B}+\mathfrak{b}\mathbf{D})\mathbf{R}]}{\mathbf{RQ}}$ $\star\star = \frac{\mathfrak{a}\mu^2\gamma^2}{\mathbf{R}} - \frac{\lambda\alpha\mathbf{D}[(\mathfrak{a}\alpha\mathbf{D}+\mathfrak{b}\mathbf{B})\mathbf{R}+\mathfrak{a}\mu\alpha\gamma\mathbf{D}]}{\mathbf{RQ}}$ $\star\star\star = \frac{\mathbf{B}[(\mathfrak{a}\alpha\mathbf{D}+\mathfrak{b}\mathbf{B})\mathbf{R}+\mathfrak{a}\mu\alpha\gamma\mathbf{D}]}{\mathbf{Q}}$	$\gamma \neq 0,\ \beta = \frac{\gamma^2}{4\alpha}$ $\mathbf{B}, \mu \neq 0$ $\mathbf{Q} = \alpha\mathbf{D}^2 - \mathbf{B}^2 \neq 0$ $\mathbf{R} = \mu\gamma - \lambda\alpha \neq 0$
	$\begin{pmatrix} 0 & \mathfrak{a}\lambda\alpha & -\frac{\mathfrak{b}\alpha\mathbf{D}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{\mathfrak{b}\alpha\mathbf{B}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & 0 & \frac{\mathfrak{b}\mathbf{B}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{\mathfrak{b}\alpha\mathbf{D}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & 0 & \frac{\mathfrak{a}\lambda\alpha\mathbf{BD}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{\mathfrak{a}\lambda\alpha^2\mathbf{D}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & 0 & -\frac{\mathfrak{a}\lambda\mathbf{B}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{\mathfrak{a}\lambda\alpha\mathbf{BD}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \end{pmatrix}$	$\gamma \neq 0,\ \beta = \frac{\gamma^2}{4\alpha}$ $\mathbf{B} \neq 0,\ \mu = 0$ $\mathbf{Q} = \alpha\mathbf{D}^2 - \mathbf{B}^2 \neq 0$ $\mathbf{R} = -\lambda\alpha \neq 0$

$\begin{pmatrix} 1 & \frac{2\alpha\gamma\mathbf{D}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & \frac{\gamma^2\mathbf{D}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & \gamma \\ 0 & -\frac{\gamma\mathbf{B}+2\alpha\mathbf{F}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & -\frac{2\gamma\mathbf{F}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & 0 \\ 0 & \frac{4\alpha\mathbf{B}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & \frac{\gamma\mathbf{B}+2\alpha\mathbf{F}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & 0 \\ 0 & -\frac{4\alpha\mathbf{D}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & -\frac{2\gamma\mathbf{D}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 2\mathfrak{a}\alpha\gamma & \mathfrak{a}\gamma^2 & 0 \\ 0 & 2\mathfrak{b}\gamma & \frac{\mathfrak{b}\gamma^2}{\alpha} & 0 \\ 0 & -4\mathfrak{b}\alpha & -2\mathfrak{b}\gamma & 0 \\ 0 & -4\mathfrak{a}\alpha & -2\mathfrak{a}\gamma & 0 \end{pmatrix}$	$\gamma \neq 0,\ \beta = \frac{\gamma^2}{4\alpha}$ $\gamma\mathbf{B} - 2\alpha\mathbf{F} \neq 0$

TABLE 7. Pairs defining coactions on $A = Cl\left(\alpha, \frac{\gamma^2}{4\alpha}, \gamma\right)$.

REMARK 3.10. Again we have that $d \equiv 0$ is equivalent to $\mathfrak{a} = \mathfrak{b} = 0$ in each of the displayed instances.

3.4.1. *Coactions with trivial skew-derivation.* Let us fix a pair $(\varphi_1, 0)$ from the first row of Table 7 with associated matrix

$$M_1 = \begin{pmatrix} 1 & -\frac{2\lambda\alpha\gamma}{\mathbf{C}} & -\frac{\lambda\gamma^2}{\mathbf{C}} & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{4\lambda\alpha}{\mathbf{C}} & \frac{2\lambda\gamma}{\mathbf{C}} & 1 \end{pmatrix}$$

The matrix

$$Z = \begin{pmatrix} 1 & -\frac{\alpha\gamma(\lambda'\mathbf{C}-\lambda\mathbf{C}')}{\mathbf{C}\mathbf{C}'} & -\frac{\gamma^2(\lambda'\mathbf{C}-\lambda\mathbf{C}')}{2\mathbf{C}\mathbf{C}'} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{2\alpha(\lambda'\mathbf{C}-\lambda\mathbf{C}')}{\mathbf{C}\mathbf{C}'} & \frac{\gamma(\lambda'\mathbf{C}-\lambda\mathbf{C}')}{\mathbf{C}\mathbf{C}'} & 1 \end{pmatrix}$$

with $\mathbf{C}, \mathbf{C}' \neq 0$ has determinant $\det Z = 1$, defines an algebra map and is such that

$$Z^{-1}M_1Z = \begin{pmatrix} 1 & -\frac{2\lambda'\alpha\gamma}{\mathbf{C}'} & -\frac{\lambda'\gamma^2}{\mathbf{C}'} & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{4\lambda'\alpha}{\mathbf{C}'} & \frac{2\lambda'\gamma}{\mathbf{C}'} & 1 \end{pmatrix}.$$

Similarly we can fix a pair $(\varphi_2, 0)$ with associated matrix

$$M_2 = \begin{pmatrix} 1 & \frac{2\mu\alpha\gamma\mathbf{BD}}{(\mu\gamma-\lambda\alpha)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{\lambda\alpha\gamma\mathbf{BD}}{(\mu\gamma-\lambda\alpha)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{\gamma\mathbf{B}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & -1-\frac{2\mu\gamma\mathbf{B}^2}{(\mu\gamma-\lambda\alpha)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{\lambda\gamma\mathbf{B}^2}{(\mu\gamma-\lambda\alpha)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{\gamma\mathbf{BD}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & \frac{4\mu\alpha\mathbf{B}^2}{(\mu\gamma-\lambda\alpha)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -1+\frac{2\lambda\alpha\mathbf{B}^2}{(\mu\gamma-\lambda\alpha)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{2\alpha\mathbf{BD}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & -\frac{4\mu\alpha\mathbf{BD}}{(\mu\gamma-\lambda\alpha)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{2\lambda\alpha\mathbf{BD}}{(\mu\gamma-\lambda\alpha)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{\alpha\mathbf{D}^2+\mathbf{B}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} \end{pmatrix},$$

with $\gamma, \mathbf{B}, (\mu\gamma - \lambda\alpha), (\alpha\mathbf{D}^2 - \mathbf{B}^2) \neq 0$. Then we can define

$$Z := \begin{pmatrix} 1 & \frac{\gamma(\mu'\alpha(\lambda\alpha-\mu\gamma)\mathbf{B}'\mathbf{D}-\mathbf{B})}{(\lambda\alpha-\mu\gamma)(\lambda'\alpha-\mu'\gamma)} & -\frac{\gamma(\gamma\mathbf{B}+\lambda'\alpha^2(\lambda\alpha-\mu\gamma)(\mathbf{BD}'-\mathbf{B}'\mathbf{D})-\mu'\alpha\gamma(\lambda\alpha-\mu\gamma)\mathbf{BD}')}{2\alpha(\lambda\alpha-\mu\gamma)(\lambda'\alpha-\mu'\gamma)} & \frac{\gamma}{2}(\alpha\mathbf{DD}'-\mathbf{BB}'+1) \\ 0 & \frac{\gamma\mathbf{D}+\lambda\alpha(\lambda'\alpha-\mu'\gamma)-\mu'\gamma(\lambda\alpha-\mu\gamma)\mathbf{BB}'}{(\lambda\alpha-\mu\gamma)(\lambda'\alpha-\mu'\gamma)} & \frac{\gamma(\alpha\mathbf{DD}'-\mathbf{BB}')}{2\alpha}+\frac{\gamma(\gamma\mathbf{D}+\lambda\alpha(\lambda'\alpha-\mu'\gamma)+\mu'\gamma(\mu\gamma-\lambda\alpha)\mathbf{BB}')}{2\alpha(\lambda\alpha-\mu\gamma)(\lambda'\alpha-\mu'\gamma)} & -\frac{\gamma}{2}(\mathbf{BD}'-\mathbf{B}'\mathbf{D}) \\ 0 & -\frac{2\alpha(\mathbf{D}+\mu(\lambda'\alpha-\mu'\gamma)-\mu'(\lambda\alpha-\mu\gamma)\mathbf{BB}')}{(\lambda\alpha-\mu\gamma)(\lambda'\alpha-\mu'\gamma)} & -\alpha\mathbf{DD}'+\mathbf{BB}'-\frac{\gamma\mathbf{D}+\mu\gamma(\lambda'\alpha-\mu'\gamma)-\mu'\gamma(\lambda\alpha-\mu\gamma)\mathbf{BB}'}{(\lambda\alpha-\mu\gamma)(\lambda'\alpha-\mu'\gamma)} & \alpha(\mathbf{BD}'-\mathbf{B}'\mathbf{D}) \\ 0 & \frac{-2(\mu'\alpha(\lambda\alpha-\mu\gamma)\mathbf{B}'\mathbf{D}-\mathbf{B})}{(\lambda\alpha-\mu\gamma)(\lambda'\alpha-\mu'\gamma)} & \frac{2(\gamma\mathbf{B}+\lambda'\alpha^2(\lambda\alpha-\mu\gamma)(\mathbf{BD}'-\mathbf{B}'\mathbf{D})-\mu'\alpha\gamma(\lambda\alpha-\mu\gamma)\mathbf{BD}')}{2\alpha(\lambda\alpha-\mu\gamma)(\lambda'\alpha-\mu'\gamma)} & \mathbf{BB}'-\alpha\mathbf{DD}' \end{pmatrix},$$

with the further condition $(\lambda'\gamma - 4\mu'\beta)(\alpha\mathbf{D}'^2 - \mathbf{B}'^2) \neq 0$. Z defines a k-linear map satisfying (100)-(104) and is invertible, since $\det Z = (\alpha\mathbf{D}^2 - \mathbf{B}^2)(\alpha\mathbf{D}'^2 - \mathbf{B}'^2) \neq 0$. Furthermore

$$Z^{-1}M_2Z = \begin{pmatrix} 1 & \frac{2\mu'\alpha\gamma\mathbf{B}'\mathbf{D}'}{(\mu'\gamma-\lambda'\alpha)(\alpha\mathbf{D}'^2-\mathbf{B}'^2)} & \frac{\lambda'\alpha\gamma\mathbf{B}'\mathbf{D}'}{(\mu'\gamma-\lambda'\alpha)(\alpha\mathbf{D}'^2-\mathbf{B}'^2)} & -\frac{\gamma\mathbf{B}'^2}{\alpha\mathbf{D}'^2-\mathbf{B}'^2} \\ 0 & -1-\frac{2\mu'\gamma\mathbf{B}'^2}{(\mu'\gamma-\lambda'\alpha)(\alpha\mathbf{D}'^2-\mathbf{B}'^2)} & -\frac{\lambda'\gamma\mathbf{B}'^2}{(\mu'\gamma-\lambda'\alpha)(\alpha\mathbf{D}'^2-\mathbf{B}'^2)} & \frac{\gamma\mathbf{B}'\mathbf{D}'}{\alpha\mathbf{D}'^2-\mathbf{B}'^2} \\ 0 & \frac{4\mu'\alpha\mathbf{B}'^2}{(\mu'\gamma-\lambda'\alpha)(\alpha\mathbf{D}'^2-\mathbf{B}'^2)} & -1+\frac{2\lambda'\alpha\mathbf{B}'^2}{(\mu'\gamma-\lambda'\alpha)(\alpha\mathbf{D}'^2-\mathbf{B}'^2)} & -\frac{2\alpha\mathbf{B}'\mathbf{D}'}{\alpha\mathbf{D}'^2-\mathbf{B}'^2} \\ 0 & -\frac{4\mu'\alpha\mathbf{B}'\mathbf{D}'}{(\mu'\gamma-\lambda'\alpha)(\alpha\mathbf{D}'^2-\mathbf{B}'^2)} & -\frac{2\lambda'\alpha\mathbf{B}'\mathbf{D}'}{(\mu'\gamma-\lambda'\alpha)(\alpha\mathbf{D}'^2-\mathbf{B}'^2)} & \frac{\alpha\mathbf{D}'^2+\mathbf{B}'^2}{\alpha\mathbf{D}'^2-\mathbf{B}'^2} \end{pmatrix}.$$

These two equivalence classes are actually the same, since the representative

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

from the first, and the representative

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & \gamma \\ 0 & -1 & -\frac{\gamma}{\alpha} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

from the second are conjugate via an the invertible algebra map

$$Z := \begin{pmatrix} 1 & 0 & -\frac{\gamma}{2} & \frac{\gamma}{2} \\ 0 & 1 & \frac{\gamma}{2\alpha} & -\frac{\gamma}{2} \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

It is easy to check that Z satisfies (100)-(104), $\det Z = -\alpha \neq 0$ and $M_1Z - ZM_2 = 0$.

Finally let us fix the pair $(\varphi_1, 0)$ from the last row of Table 7 with associated matrix

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & \gamma \\ 0 & -1 & 0 & 0 \\ 0 & \frac{4\alpha}{\gamma} & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We set

$$Z := \begin{pmatrix} 1 & -\frac{\alpha\gamma\mathbf{D}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & -\frac{\gamma^2\mathbf{D}}{2(\gamma\mathbf{B}-2\alpha\mathbf{F})} & 0 \\ 0 & \frac{\gamma\mathbf{B}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & \frac{\gamma\mathbf{F}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & 0 \\ 0 & -\frac{4\alpha^2\mathbf{F}}{\gamma(\gamma\mathbf{B}-2\alpha\mathbf{F})} & 1-\frac{2\alpha\mathbf{F}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & 0 \\ 0 & \frac{2\alpha\mathbf{D}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & \frac{\gamma\mathbf{D}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & 1 \end{pmatrix}.$$

The matrix Z defines an invertible algebra map such that

$$Z^{-1}M_1Z = \begin{pmatrix} 1 & \frac{2\alpha\gamma\mathbf{D}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & \frac{\gamma^2\mathbf{D}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & \gamma \\ 0 & -\frac{\gamma\mathbf{B}+2\alpha\mathbf{F}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & -\frac{2\gamma\mathbf{F}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & 0 \\ 0 & \frac{4\alpha\mathbf{B}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & \frac{\gamma\mathbf{B}+2\alpha\mathbf{F}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & 0 \\ 0 & -\frac{4\alpha\mathbf{D}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & -\frac{2\gamma\mathbf{D}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & -1 \end{pmatrix},$$

therefore all pairs $(\varphi, 0)$ from the last row of Table 7 are equivalent.

REMARK 3.11. The equivalence class of pairs belonging to the first and second row of Table 7 is different from the class of pairs contained in the third, since the involutions considered in the latter case are inner, while those in the former are not.

In conclusion we have found that, up to isomorphism, there are only two coactions of type 2 on $A = Cl(\alpha, 0, 0)$ corresponding to pairs with trivial skew-derivation. These are given by $\rho(G) = G \otimes g$, $\rho(X) = X \otimes g$ and $\rho(G) = G \otimes g + \frac{2\alpha}{\gamma}(X \otimes 1) - \frac{2\alpha}{\gamma}(X \otimes g)$, $\rho(X) = X \otimes 1$.

3.4.2. *Coactions with non-trivial skew-derivation.* Let us fix the pair (φ_1, d_1) from the first row of Table 7 with associated matrices

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathfrak{D}_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{258}$$

For every $\mathfrak{a} \neq 0$ we can define the matrix

$$Z := \begin{pmatrix} 1 & -\frac{\mathfrak{a}\lambda\alpha\gamma}{\mathbf{D}} & -\frac{\mathfrak{a}\lambda\gamma^2}{2\mathbf{D}} & \frac{\gamma(\mathbf{D}-\mathfrak{a}\mathbf{C})}{2\mathbf{D}} \\ 0 & \frac{\mathbf{D}-\mathfrak{b}\lambda\gamma}{\mathbf{D}} & -\frac{\gamma(\mathfrak{a}\mathbf{C}+\mathfrak{b}\lambda\gamma-\mathbf{D})}{2\alpha\mathbf{D}} & 0 \\ 0 & \frac{2\mathfrak{b}\lambda\alpha}{\mathbf{D}} & \frac{\mathfrak{a}\mathbf{C}+\mathfrak{b}\lambda\gamma}{\mathbf{D}} & 0 \\ 0 & \frac{2\mathfrak{a}\lambda\alpha}{\mathbf{D}} & \frac{\mathfrak{a}\lambda\gamma}{\mathbf{D}} & \frac{\mathfrak{a}\mathbf{C}}{\mathbf{D}} \end{pmatrix},$$

whose determinant is $\det Z = \left(\frac{\mathfrak{a}\mathbf{C}}{\mathbf{D}}\right)^2 \neq 0$. Z satisfies (100)-(104) and

$$Z^{-1}M_1Z = \begin{pmatrix} 1 & -\frac{2\lambda\alpha\gamma}{\mathbf{C}} & -\frac{\lambda\gamma^2}{\mathbf{C}} & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{4\lambda\alpha}{\mathbf{C}} & \frac{2\lambda\gamma}{\mathbf{C}} & 1 \end{pmatrix}, \qquad Z^{-1}\mathfrak{D}_1Z = \begin{pmatrix} 0 & \frac{2\lambda\alpha(\mathfrak{a}\lambda\alpha\gamma-\mathfrak{b}\mathbf{C})}{\mathbf{CD}} & -\frac{\mathfrak{a}\mathbf{C}+\mathfrak{b}\lambda\gamma}{\mathbf{D}} + \frac{\mathfrak{a}\lambda^2\alpha\gamma^2}{\mathbf{CD}} & \frac{\mathfrak{a}\lambda\alpha\gamma}{\mathbf{D}} \\ 0 & \frac{2\lambda\alpha(\mathfrak{a}\mathbf{C}+\mathfrak{b}\lambda\gamma)}{\mathbf{CD}} & \frac{\lambda\gamma(\mathfrak{a}\mathbf{C}+\mathfrak{b}\lambda\gamma)}{\mathbf{CD}} & \frac{\mathfrak{a}\mathbf{C}+\mathfrak{b}\lambda\gamma}{\mathbf{D}} \\ 0 & -\frac{4\mathfrak{b}\lambda^2\alpha^2}{\mathbf{CD}} & -\frac{2\mathfrak{b}\lambda^2\alpha\gamma}{\mathbf{CD}} & -\frac{2\mathfrak{b}\lambda\alpha}{\mathbf{D}} \\ 0 & -\frac{4\mathfrak{a}\lambda^2\alpha^2}{\mathbf{CD}} & -\frac{2\mathfrak{a}\lambda^2\alpha\gamma}{\mathbf{CD}} & -\frac{2\mathfrak{a}\lambda\alpha}{\mathbf{D}} \end{pmatrix}.$$

Therefore all pairs (φ, d) with $\mathfrak{a} \neq 0$ are in the same equivalence class.

Pairs from the second row of Table 7 with $\mathfrak{b} \neq -\frac{\mathfrak{a}\gamma\mathbf{C}}{2\alpha}$ are contained in the same class of (258), since

$$Z := \begin{pmatrix} 1 & 0 & 0 & -\frac{\gamma(2\mathfrak{b}\alpha+\mathfrak{a}\gamma\mathbf{C}-2\alpha\mathbf{D})}{4\alpha\mathbf{D}} \\ 0 & \frac{\mathfrak{a}\gamma\mathbf{C}+2\alpha\mathbf{D}}{2\alpha\mathbf{D}} & \frac{\gamma(\mathbf{D}-\mathfrak{b})}{2\alpha\mathbf{D}} & 0 \\ 0 & -\frac{\mathfrak{a}\mathbf{C}}{\mathbf{D}} & \frac{\mathfrak{b}}{\mathbf{D}} & 0 \\ 0 & 0 & 0 & \frac{2\mathfrak{b}\alpha+\mathfrak{a}\gamma\mathbf{C}}{2\alpha\mathbf{D}} \end{pmatrix}$$

defines an invertible algebra map such that

$$ZM_1Z^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad Z^{-1}\mathfrak{D}_1Z = \begin{pmatrix} 0 & \frac{\mathfrak{a}\mathbf{C}}{\mathbf{D}} & -\frac{\mathfrak{b}}{\mathbf{D}} & 0 \\ 0 & 0 & 0 & \frac{\mathfrak{b}}{\mathbf{D}} \\ 0 & 0 & 0 & \frac{\mathfrak{a}\mathbf{C}}{\mathbf{D}} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The coaction defined by an element of this class is (isomorphic to) $\rho(G) = G \otimes g$, $\rho(X) = X \otimes g + 1 \otimes x$.

Pairs from the first row of Table 7 with $\mathfrak{a} = 0$ do not belong in this class. They determine a family of classes, each distinguished by the scalar $\frac{\mathfrak{b}\lambda}{\mathbf{D}}$: two pairs (φ_1, d_1), (φ_2, d_2) with $\mathfrak{a} = 0$ are equivalent if, and only if, $\frac{\mathfrak{b}\lambda}{\mathbf{D}} = \pm\frac{\mathfrak{b}'\lambda'}{\mathbf{D}'}$.

To prove it one checks that, in this case, the condition $\mathfrak{D}_1 Z - Z\mathfrak{D}_2 = 0$ forces Z to be of the form

$$Z := \begin{pmatrix} 1 & \frac{\mathfrak{b}'\lambda'\alpha\mathbf{D}[\lambda\gamma\mathbf{C}'+\lambda'\mathbf{C}(2Z_{23}\alpha-Z_{22}\gamma)]}{\mathfrak{b}\lambda\mathbf{C}\mathbf{C}'\mathbf{D}'} & \frac{\mathfrak{b}'\lambda'\gamma\mathbf{D}[\lambda\gamma\mathbf{C}'+\lambda'\mathbf{C}(2Z_{23}\alpha-Z_{22}\gamma)]}{2\mathfrak{b}\lambda\mathbf{C}\mathbf{C}'\mathbf{D}'} & \frac{\mathfrak{b}'\lambda'\mathbf{D}[\mathfrak{b}'\lambda'\gamma\mathbf{D}+\mathfrak{b}\lambda\mathbf{D}'(2Z_{23}\alpha-Z_{22}\gamma)}{2\mathfrak{b}^2\lambda^2\mathbf{D}'^2} \\ 0 & Z_{22} & Z_{23} & 0 \\ 0 & \frac{2\alpha(\mathfrak{b}'\lambda'\mathbf{D}-Z_{22}\mathfrak{b}\lambda\mathbf{D}')}{\mathfrak{b}\lambda\gamma\mathbf{D}'} & \frac{\mathfrak{b}'\lambda'\gamma\mathbf{D}-2Z_{23}\mathfrak{b}\lambda\alpha\mathbf{D}'}{\mathfrak{b}\lambda\gamma\mathbf{D}'} & 0 \\ 0 & -\frac{2\mathfrak{b}'\lambda'\alpha\mathbf{D}[\lambda\gamma\mathbf{C}'+\lambda'\mathbf{C}(2Z_{23}\alpha-Z_{22}\gamma)]}{\mathfrak{b}\lambda\gamma\mathbf{C}\mathbf{C}'\mathbf{D}'} & -\frac{\mathfrak{b}'\lambda'\mathbf{D}[\lambda\gamma\mathbf{C}'+\lambda'\mathbf{C}(2Z_{23}\alpha-Z_{22}\gamma)]}{\mathfrak{b}\lambda\mathbf{C}\mathbf{C}'\mathbf{D}'} & -\frac{\mathfrak{b}'\lambda'\mathbf{D}(2Z_{23}\alpha-Z_{22}\gamma)}{\mathfrak{b}\lambda\gamma\mathbf{D}'} \end{pmatrix}$$

so that (101) yields $\left(\frac{\mathfrak{b}'\lambda'\mathbf{D}}{\mathfrak{b}\lambda\mathbf{D}'}\right)^2 = 1$. Next, we write down the matrices associated to (φ_1, d_1)

$$M_1 = \begin{pmatrix} 1 & -\frac{2\lambda\alpha\gamma}{\mathbf{C}} & -\frac{\lambda\gamma^2}{\mathbf{C}} & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{4\lambda\alpha}{\mathbf{C}} & \frac{2\lambda\gamma}{\mathbf{C}} & 1 \end{pmatrix}, \qquad \mathfrak{D}_1 = \begin{pmatrix} 0 & \frac{-2\mathfrak{b}\lambda\alpha}{\mathbf{D}} & -\frac{\mathfrak{b}\lambda\gamma}{\mathbf{D}} & 0 \\ 0 & \frac{2\mathfrak{b}\lambda^2\alpha\gamma}{\mathbf{C}\mathbf{D}} & \frac{\mathfrak{b}\lambda^2\gamma^2}{\mathbf{C}\mathbf{D}} & \frac{\mathfrak{b}\lambda\gamma}{\mathbf{D}} \\ 0 & -\frac{4\mathfrak{b}\lambda^2\alpha^2}{\mathbf{C}\mathbf{D}} & -\frac{2\mathfrak{b}\lambda^2\alpha\gamma}{\mathbf{C}\mathbf{D}} & -\frac{2\mathfrak{b}\lambda\alpha}{\mathbf{D}} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

and, for any $\mathbf{C}', \lambda' \neq 0$, we define

$$Z_{\pm} = \begin{pmatrix} 1 & \mp\frac{\alpha\gamma(\lambda'\mathbf{C}-\lambda\mathbf{C}')}{\mathbf{C}\mathbf{C}'} & \mp\frac{\gamma^2(\lambda'\mathbf{C}-\lambda\mathbf{C}')}{2\mathbf{C}\mathbf{C}'} & \gamma(1\mp 1) \\ 0 & 1 & 0 & 0 \\ 0 & -(1\mp 1)\frac{2\alpha}{\gamma} & \mp 1 & 0 \\ 0 & \mp\frac{2\alpha(\lambda'\mathbf{C}-\lambda\mathbf{C}')}{\mathbf{C}\mathbf{C}'} & \mp\frac{\gamma(\mathbf{C}\lambda'-\lambda\mathbf{C}')}{\mathbf{C}\mathbf{C}'} & \mp 1 \end{pmatrix}.$$

$Z_{\pm}$ satisfies (100)-(104), $\det Z = 1 \neq 0$ and

$$Z_{\pm}^{-1} M_1 Z_{\pm} = \begin{pmatrix} 1 & -\frac{2\lambda'\alpha\gamma}{\mathbf{C}'} & -\frac{\lambda'\gamma^2}{\mathbf{C}'} & 0 \\ 0 & -1 & -0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{4\lambda'\alpha}{\mathbf{C}'} & \frac{2\lambda'\gamma}{\mathbf{C}'} & 1 \end{pmatrix}, \qquad Z_{\pm}^{-1}\mathfrak{D}_1 Z_{\pm} = \begin{pmatrix} 0 & \mp\frac{2\mathfrak{b}\lambda\alpha}{\mathbf{D}} & \mp\frac{\mathfrak{b}\lambda\gamma}{\mathbf{D}} & 0 \\ 0 & \pm\frac{2\mathfrak{b}\lambda\lambda'\alpha\gamma}{\mathbf{C}'\mathbf{D}} & \pm\frac{\mathfrak{b}\lambda\lambda'\gamma^2}{\mathbf{C}'\mathbf{D}} & \pm\frac{\mathfrak{b}\lambda\gamma}{\mathbf{D}} \\ 0 & \mp\frac{4\mathfrak{b}\lambda\lambda'\alpha^2}{\mathbf{C}'\mathbf{D}} & \mp\frac{2\mathfrak{b}\lambda\lambda'\alpha\gamma}{\mathbf{C}'\mathbf{D}} & \mp\frac{2\mathfrak{b}\lambda\alpha}{\mathbf{D}} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Notice that these are the matrices of a pair (φ_2, d_2) when $\mathfrak{b}' = \pm\frac{\mathfrak{b}\lambda\mathbf{D}'}{\lambda'\mathbf{D}}$. It follows that if $\frac{\mathfrak{b}\lambda}{\mathbf{D}} = \pm\frac{\mathfrak{b}'\lambda'}{\mathbf{D}'}$ the two pairs (φ_1, d_1), (φ_2, d_2) are equivalent. By setting $\omega := \frac{\mathfrak{b}\lambda}{\mathbf{D}}$ and by choosing $\mathbf{C} = \mathbf{D} = \lambda = 1$ we can write a representative for each equivalence class:

(259) $$M_\omega = \begin{pmatrix} 1 & -2\alpha\gamma & -\gamma^2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 4\alpha & 2\gamma & 1 \end{pmatrix}, \qquad \mathfrak{D}_\omega = \begin{pmatrix} 0 & -2\omega\alpha & -\omega\gamma & 0 \\ 0 & 2\omega\alpha\gamma & \omega\gamma^2 & \omega\gamma \\ 0 & -4\omega\alpha^2 & -2\omega\alpha\gamma & -2\omega\alpha \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Pairs from the second row of Table 7 with $\mathfrak{b} = -\frac{\mathfrak{a}\gamma\mathbf{C}}{2\alpha}$ are a part of this family of classes. Namely, a pair of this kind is equivalent to (259) if $\omega = -\frac{\mathfrak{a}\mathbf{C}}{2\alpha\mathbf{D}}$. In fact let

$$Z := \begin{pmatrix} 1 & \alpha\gamma & \frac{\gamma^2}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2\alpha & -\gamma & 1 \end{pmatrix}.$$

Then $\det Z = 1$, Z satisfies (100)-(104) and

$$Z^{-1}M_\omega Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Z^{-1}M_\omega Z = \begin{pmatrix} 0 & -2\omega\alpha & -\omega\gamma & 0 \\ 0 & 0 & 0 & \omega\gamma \\ 0 & 0 & 0 & -2\omega\alpha \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{\omega=-\frac{\mathfrak{a}\mathbf{C}}{2\alpha\mathbf{D}}}{=} \begin{pmatrix} 0 & \frac{\mathfrak{a}\mathbf{C}}{\mathbf{D}} & \frac{\mathfrak{a}\gamma\mathbf{C}}{2\alpha\mathbf{D}} & 0 \\ 0 & 0 & 0 & -\frac{\mathfrak{a}\gamma\mathbf{C}}{2\alpha\mathbf{D}} \\ 0 & 0 & 0 & \frac{\mathfrak{a}\mathbf{C}}{\mathbf{D}} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Notice that this defines a pair of the second row of Table 7 when $\mathfrak{b} = -\frac{\mathfrak{a}\gamma\mathbf{C}}{2\alpha}$.

It follows that there is an infinite family of non-isomorphic coactions on $A = Cl\left(\alpha, \frac{\gamma^2}{4\alpha}, \gamma\right)$ given by

$$\rho(G) = G \otimes g + 2\omega\alpha(1 \otimes x), \quad \rho(X) = X \otimes g + \omega\gamma(1 \otimes x).$$

with $\omega \in k^{>0}$.

Next we define the matrix

$$Z := \begin{pmatrix} 1 & -\frac{\mu\gamma\mathbf{B}(\mathfrak{b}\mathbf{B}+\mathfrak{a}\alpha\mathbf{D})}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{\lambda\gamma\mathbf{B}(\mathfrak{b}\mathbf{B}+\mathfrak{a}\alpha\mathbf{D})}{2(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{\gamma}{2} + \frac{\gamma\mathbf{D}(\mu\gamma-\lambda\alpha)(\mathfrak{b}\mathbf{B}+\mathfrak{a}\alpha\mathbf{D})}{2(\alpha\mathbf{D}^2-\mathbf{B}^2)} \\ 0 & 1 - \frac{\mu\gamma\mathbf{B}(\mathfrak{a}\mathbf{B}+\mathfrak{b}\mathbf{D})}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{\gamma(1+\mathfrak{a}\mu\gamma)}{2\alpha} - \frac{\lambda\gamma\mathbf{D}(\mathfrak{b}\mathbf{B}+\mathfrak{a}\alpha\mathbf{D})}{2(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{\gamma\mathbf{B}(\mu\gamma-\lambda\alpha)(\mathfrak{b}\mathbf{B}+\mathfrak{a}\alpha\mathbf{D})}{2\alpha(\alpha\mathbf{D}^2-\mathbf{B}^2)} \\ 0 & \frac{2\mu\alpha\mathbf{B}(\mathfrak{a}\mathbf{B}+\mathfrak{b}\mathbf{D})}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\mathfrak{a}\mu\gamma + \frac{\lambda\alpha\mathbf{D}(\mathfrak{b}\mathbf{B}+\mathfrak{a}\alpha\mathbf{D})}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{\mathbf{B}(\mu\gamma-\lambda\alpha)(\mathfrak{b}\mathbf{B}+\mathfrak{a}\alpha\mathbf{D})}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & \frac{2\mu\mathbf{B}(\mathfrak{b}\mathbf{B}+\mathfrak{a}\alpha\mathbf{D})}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{\lambda\mathbf{B}(\mathfrak{b}\mathbf{B}+\mathfrak{a}\alpha\mathbf{D})}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{\mathbf{D}(\mu\gamma-\lambda\alpha)(\mathfrak{b}\mathbf{B}+\mathfrak{a}\alpha\mathbf{D})}{\alpha\mathbf{D}^2-\mathbf{B}^2} \end{pmatrix}$$

which defines an invertible algebra map under the condition that

$$\begin{cases} \mathfrak{b}\mathbf{B} + \mathfrak{a}\alpha\mathbf{D} \neq 0 \\ \alpha\mathbf{D}^2 - \mathbf{B}^2 \neq 0 \\ \mu\gamma - \lambda\alpha \neq 0. \end{cases}$$

One can check that for M_1 and $\mathfrak{D}_1$ defined in (258) we have that $Z^{-1}M_1Z = M_2$, $Z^{-1}\mathfrak{D}_1 Z = \mathfrak{D}_2$ with M_2, $\mathfrak{D}_2$ associated to a pair of the third row of Table 7 with the further restriction $\mathfrak{b}\mathbf{B} + \mathfrak{a}\alpha\mathbf{D} \neq 0$.

On the other hand, if $\mathfrak{b}\mathbf{B} + \mathfrak{a}\alpha\mathbf{D} = 0$, a pair from this row is equivalent to (259) for $\omega = -\mathfrak{a}\mu$. To prove it, we define the matrix

$$Z = \begin{pmatrix} 1 & -\frac{\mu\gamma}{2} & -\frac{\lambda\gamma}{4} & \frac{\gamma[2\mathbf{B}+\mathbf{D}(\mu\gamma-\lambda\alpha)]}{4\mathbf{B}} \\ 0 & -\frac{\mathbf{D}(\mu\gamma-\lambda\alpha)}{2\mathbf{B}} & 0 & \frac{\gamma(\mu\gamma-\lambda\alpha)}{4\alpha} \\ 0 & \frac{\alpha[2\mathbf{B}+\mathbf{D}(\mu\gamma-\lambda\alpha)]}{\gamma\mathbf{B}} & 1 & -\frac{\mu\gamma-\lambda\alpha}{2} \\ 0 & \mu & \frac{\lambda}{2} & -\frac{\mathbf{D}(\mu\gamma-\lambda\alpha)}{2\mathbf{B}} \end{pmatrix}$$

for $\mathbf{B} \neq 0$. It is an invertible algebra map if $\alpha\mathbf{D}^2 - \mathbf{B}^2 \neq 0 \neq \mu\gamma - \lambda\alpha$. Moreover

$$Z^{-1}M_\omega Z = \begin{pmatrix} 1 & \frac{2\mu\alpha\gamma\mathbf{B}\mathbf{D}}{(\mu\gamma-\lambda\alpha)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{\lambda\alpha\gamma\mathbf{B}\mathbf{D}}{(\mu\gamma-\lambda\alpha)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{-\gamma\mathbf{B}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & 1 - \frac{2\mu\gamma\mathbf{B}^2}{(\mu\gamma-\lambda\alpha)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{\lambda\gamma\mathbf{B}^2}{(\mu\gamma-\lambda\alpha)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{\gamma\mathbf{B}\mathbf{D}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & \frac{4\mu\alpha\mathbf{B}^2}{(\mu\gamma-\lambda\alpha)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -1 + \frac{2\lambda\alpha\mathbf{B}^2}{(\mu\gamma-\lambda\alpha)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{2\alpha\mathbf{B}\mathbf{D}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & -\frac{4\mu\alpha\mathbf{B}\mathbf{D}}{(\mu\gamma-\lambda\alpha)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{2\lambda\alpha\mathbf{B}\mathbf{D}}{(\mu\gamma-\lambda\alpha)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{\alpha\mathbf{D}^2+\mathbf{B}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} \end{pmatrix}$$

and

$$Z^{-1}\mathfrak{D}_\omega Z = \begin{pmatrix} 0 & \frac{-2\omega\alpha(\mu\gamma-\lambda\alpha)(\alpha\mathbf{D}^2-\mathbf{B}^2)+2\omega\mu\alpha\gamma\mathbf{B}^2}{(\mu\gamma-\lambda\alpha)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{\omega\gamma[\mu\gamma\mathbf{B}^2-\alpha\mathbf{D}^2(\mu\gamma-\lambda\alpha)]}{(\mu\gamma-\lambda\alpha)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{\omega\alpha\gamma\mathbf{B}\mathbf{D}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & -\frac{2\omega\mu\alpha\gamma\mathbf{B}\mathbf{D}}{(\mu\gamma-\lambda\alpha)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{\omega\lambda\alpha\gamma\mathbf{B}\mathbf{D}}{(\mu\gamma-\lambda\alpha)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{\omega\alpha\gamma\mathbf{D}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & \frac{4\omega\mu\alpha^2\mathbf{B}\mathbf{D}}{(\mu\gamma-\lambda\alpha)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{2\omega\lambda\alpha^2\mathbf{B}\mathbf{D}}{(\mu\gamma-\lambda\alpha)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{2\omega\alpha^2\mathbf{D}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & -\frac{4\omega\mu\alpha\mathbf{B}^2}{(\mu\gamma-\lambda\alpha)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{2\omega\lambda\alpha\mathbf{B}^2}{(\mu\gamma-\lambda\alpha)(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{2\omega\alpha\mathbf{B}\mathbf{D}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \end{pmatrix}.$$

If we choose $\omega = -\mathfrak{a}\mu$ we get exactly a pair belonging the third row of Table 7 with $\mathfrak{b}\mathbf{B} + \mathfrak{a}\alpha\mathbf{D} = 0$.

Also pairs from the fourth row do not determine new equivalence classes. In fact when $\mathfrak{b} \neq -\frac{\mathfrak{a}\lambda\gamma\mathbf{D}}{2}$ the matrix

$$Z = \begin{pmatrix} 1 & 0 & \frac{\gamma\mathbf{B}(2\mathfrak{b}+\mathfrak{a}\lambda\gamma\mathbf{D})}{4(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{\gamma}{2} + \frac{\alpha\gamma\mathbf{D}(2\mathfrak{b}+\mathfrak{a}\lambda\gamma\mathbf{D})}{4(\alpha\mathbf{D}^2-\mathbf{B}^2)} \\ 0 & \frac{\mathfrak{a}\lambda\gamma}{2}-1 & \frac{\gamma(\mathfrak{a}\lambda\gamma-2)}{4\alpha} - \frac{\gamma\mathbf{D}(2\mathfrak{b}+\mathfrak{a}\lambda\gamma\mathbf{D})}{4(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{\gamma\mathbf{B}(2\mathfrak{b}+\mathfrak{a}\lambda\gamma\mathbf{D})}{4(\alpha\mathbf{D}^2-\mathbf{B}^2)} \\ 0 & -\mathfrak{a}\lambda\alpha & \frac{\mathfrak{a}\lambda\gamma\mathbf{B}^2+2\mathfrak{b}\alpha\mathbf{D}}{2(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{\alpha\mathbf{B}(2\mathfrak{b}+\mathfrak{a}\lambda\gamma\mathbf{D})}{4(\alpha\mathbf{D}^2-\mathbf{B}^2)} \\ 0 & 0 & -\frac{\mathbf{B}(2\mathfrak{b}+\mathfrak{a}\lambda\gamma\mathbf{D})}{2(\alpha\mathbf{D}^2-\mathbf{B}^2)} & -\frac{\alpha\mathbf{D}(2\mathfrak{b}+\mathfrak{a}\lambda\gamma\mathbf{D})}{4(\alpha\mathbf{D}^2-\mathbf{B}^2)} \end{pmatrix}$$

defines an invertible algebra map such that

$$Z^{-1}M_1Z = \begin{pmatrix} 1 & 0 & -\frac{\gamma\mathbf{BD}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{-\gamma\mathbf{B}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & -1 & \frac{\gamma\mathbf{B}^2}{\alpha(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{\gamma\mathbf{BD}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & 0 & -\frac{\alpha\mathbf{D}^2+\mathbf{B}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{2\alpha\mathbf{BD}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & 0 & \frac{2\mathbf{BD}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{\alpha\mathbf{D}^2+\mathbf{B}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} \end{pmatrix}, \quad Z^{-1}\mathfrak{D}_1Z = \begin{pmatrix} 0 & \mathfrak{a}\lambda\alpha & -\frac{\mathfrak{b}\alpha\mathbf{D}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{\mathfrak{b}\alpha\mathbf{B}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & 0 & \frac{\mathfrak{b}\mathbf{B}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{\mathfrak{b}\alpha\mathbf{D}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & 0 & \frac{\mathfrak{a}\lambda\alpha\mathbf{BD}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{\mathfrak{a}\lambda\alpha^2\mathbf{D}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & 0 & -\frac{\mathfrak{a}\lambda\mathbf{B}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{\mathfrak{a}\lambda\alpha\mathbf{BD}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \end{pmatrix},$$

where $M_1, \mathfrak{D}_1$ are those defined in (258).

On the other hand, when $\mathbf{B} \neq 0 \neq \alpha\mathbf{D}^2 - \mathbf{B}^2$ we have that

$$Z = \begin{pmatrix} 1 & \alpha\gamma & \frac{\gamma(\gamma-1)}{2} & \frac{\gamma(\mathbf{B}-\alpha\mathbf{D})}{2\mathbf{B}} \\ 0 & \frac{\alpha\mathbf{D}}{\mathbf{B}} & 0 & -\frac{\gamma}{2} \\ 0 & \frac{2\alpha(\mathbf{B}-\alpha\mathbf{D})}{\gamma\mathbf{B}} & 1 & \alpha \\ 0 & -2\alpha & 1-\gamma & \frac{\alpha\mathbf{D}}{\mathbf{B}} \end{pmatrix}$$

defines an invertible algebra map such that

$$Z^{-1}M_\omega Z = \begin{pmatrix} 1 & 0 & -\frac{\gamma\mathbf{BD}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{-\gamma\mathbf{B}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & -1 & \frac{\gamma\mathbf{B}^2}{\alpha(\alpha\mathbf{D}^2-\mathbf{B}^2)} & \frac{\gamma\mathbf{BD}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & 0 & -\frac{\alpha\mathbf{D}^2+\mathbf{B}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{2\alpha\mathbf{BD}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & 0 & \frac{2\mathbf{BD}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{\alpha\mathbf{D}^2+\mathbf{B}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} \end{pmatrix}, \qquad Z^{-1}M_\omega Z = \begin{pmatrix} 0 & -2\omega\alpha & -\frac{\omega\alpha\gamma\mathbf{D}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{\omega\alpha\gamma\mathbf{BD}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & 0 & \frac{\omega\gamma\mathbf{BD}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{\omega\alpha\gamma\mathbf{D}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & 0 & -\frac{2\omega\alpha\mathbf{BD}}{\alpha\mathbf{D}^2-\mathbf{B}^2} & -\frac{2\omega\alpha^2\mathbf{D}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} \\ 0 & 0 & \frac{2\omega\mathbf{B}^2}{\alpha\mathbf{D}^2-\mathbf{B}^2} & \frac{2\omega\alpha\mathbf{BD}}{\alpha\mathbf{D}^2-\mathbf{B}^2} \end{pmatrix},$$

which is a pair from the fourth row of Table 7 when $\mathfrak{b} = -\frac{\mathfrak{a}\lambda\gamma\mathbf{D}}{2}$ and $\mathfrak{a}\lambda = -2\omega$.

To conclude this subsection we study the equivalence classes determined by the last row of the table. Notice that these pairs cannot be equivalent to the previous ones, as the involutions we are considering now are inner, while those in the first four rows are not.

We fix the pair

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & \gamma \\ 0 & -1 & 0 & 0 \\ 0 & \frac{4\alpha}{\gamma} & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \mathfrak{D}_1 = \begin{pmatrix} 0 & 2\alpha\gamma & \gamma^2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -4\alpha & -2\gamma & 0 \end{pmatrix}. \tag{260}$$

Provided $\mathfrak{b}^2 - \alpha\mathfrak{a}^2$, $\gamma\mathbf{B} - 2\alpha\mathbf{F} \neq 0$, we can define the matrix

$$Z := \begin{pmatrix} 1 & -\frac{\alpha\gamma(\mathfrak{b}\mathbf{B}+\mathfrak{a}\alpha\mathbf{D})}{(\alpha\mathfrak{a}^2-\mathfrak{b}^2)(\gamma\mathbf{B}-2\alpha\mathbf{F})} & -\frac{\alpha\gamma(2\mathfrak{b}\mathbf{F}+\mathfrak{a}\gamma\mathbf{D})}{2(\alpha\mathfrak{a}^2-\mathfrak{b}^2)(\gamma\mathbf{B}-2\alpha\mathbf{F})} & \frac{\gamma}{2} - \frac{\mathfrak{a}\alpha\gamma}{2(\alpha\mathfrak{a}^2-\mathfrak{b}^2)} \\ 0 & \frac{\alpha\gamma(\mathfrak{a}\mathbf{B}+\mathfrak{b}\mathbf{D})}{(\alpha\mathfrak{a}^2-\mathfrak{b}^2)(\gamma\mathbf{B}-2\alpha\mathbf{F})} & \frac{\gamma(2\mathfrak{a}\alpha\mathbf{F}+\mathfrak{b}\gamma\mathbf{D})}{2(\alpha\mathfrak{a}^2-\mathfrak{b}^2)(\gamma\mathbf{B}-2\alpha\mathbf{F})} & \frac{\mathfrak{b}\gamma}{2(\alpha\mathfrak{a}^2-\mathfrak{b}^2)} \\ 0 & \frac{2\alpha}{\gamma} - \frac{2\alpha^2(\mathfrak{a}\mathbf{B}+\mathfrak{b}\mathbf{D})}{(\alpha\mathfrak{a}^2-\mathfrak{b}^2)(\gamma\mathbf{B}-2\alpha\mathbf{F})} & 1 - \frac{\alpha(2\mathfrak{a}\alpha\mathbf{F}+\mathfrak{b}\gamma\mathbf{D})}{(\alpha\mathfrak{a}^2-\mathfrak{b}^2)(\gamma\mathbf{B}-2\alpha\mathbf{F})} & -\frac{\mathfrak{b}\alpha}{\alpha\mathfrak{a}^2-\mathfrak{b}^2} \\ 0 & \frac{2\alpha(\mathfrak{b}\mathbf{B}+\mathfrak{a}\alpha\mathbf{D})}{(\alpha\mathfrak{a}^2-\mathfrak{b}^2)(\gamma\mathbf{B}-2\alpha\mathbf{F})} & \frac{\alpha(2\mathfrak{b}\mathbf{F}+\mathfrak{a}\gamma\mathbf{D})}{(\alpha\mathfrak{a}^2-\mathfrak{b}^2)(\gamma\mathbf{B}-2\alpha\mathbf{F})} & \frac{\mathfrak{a}\alpha}{\alpha\mathfrak{a}^2-\mathfrak{b}^2} \end{pmatrix}.$$

One can easily check that $\det Z = \frac{\alpha}{\alpha\mathfrak{a}^2-\mathfrak{b}^2} \neq 0$, that Z satisfies (100)-(104) and that

$$Z^{-1}M_1Z = \begin{pmatrix} 1 & \frac{2\alpha\gamma\mathbf{D}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & \frac{\gamma^2\mathbf{D}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & \gamma \\ 0 & -\frac{\gamma\mathbf{B}+2\alpha\mathbf{F}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & -\frac{2\gamma\mathbf{F}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & 0 \\ 0 & \frac{4\alpha\mathbf{B}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & \frac{\gamma\mathbf{B}+2\alpha\mathbf{F}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & 0 \\ 0 & -\frac{4\alpha\mathbf{D}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & -\frac{2\gamma\mathbf{D}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & -1 \end{pmatrix}, \qquad Z^{-1}\mathfrak{D}_1Z = \begin{pmatrix} 0 & 2\mathfrak{a}\alpha\gamma & \mathfrak{a}\gamma^2 & 0 \\ 0 & 2\mathfrak{b}\gamma & \frac{\mathfrak{b}\gamma^2}{\alpha} & 0 \\ 0 & -4\mathfrak{b}\alpha & -2\mathfrak{b}\gamma & 0 \\ 0 & -4\mathfrak{a}\alpha & -2\mathfrak{a}\gamma & 0 \end{pmatrix}.$$

Notice that if the field k does not contain a square root of α then the condition $\mathfrak{b}^2 - \alpha\mathfrak{a}^2 \neq 0$ is always satisfied and there are no other coactions to consider. The coaction defined by an element of the class we have just found is (isomorphic to) $\rho(G) = \frac{2\alpha}{\gamma}X\otimes 1+\left(G - \frac{2\alpha}{\gamma}X\right)\otimes g+2\alpha\gamma(1\otimes gx)-4\alpha GX\otimes gx$, $\rho(X) = X\otimes 1+\gamma^2(1\otimes gx)-2\gamma GX\otimes gx$ (see (260)).

On the other hand, if $\sqrt{\alpha} \in k$, then there are pairs in the last row of Table 7, for which $\alpha\mathfrak{a}^2 - \mathfrak{b}^2 = 0$. We are going to show that any such pair is equivalent to the one defined by

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & \gamma \\ 0 & -1 & 0 & 0 \\ 0 & \frac{4\alpha}{\gamma} & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \mathfrak{D}_1 = \begin{pmatrix} 0 & 2\alpha\gamma & \gamma^2 & 0 \\ 0 & 2\sqrt{\alpha}\gamma & \frac{\gamma^2}{\sqrt{\alpha}} & 0 \\ 0 & -4\alpha^{\frac{3}{2}} & -2\sqrt{\alpha}\gamma & 0 \\ 0 & -4\alpha & -2\gamma & 0 \end{pmatrix}.$$

Since we want $d_2 \not\equiv 0$, we can assume $\mathfrak{a} \neq 0$ (or else $\mathfrak{b}^2 - \alpha\mathfrak{a}^2 = 0$ forces also $\mathfrak{b} = 0$) and thus we can define

$$Z_\pm := \begin{pmatrix} 1 & \frac{\sqrt{\alpha}\gamma\mathbf{B}}{\mathfrak{a}(\gamma\mathbf{B}-2\alpha\mathbf{F})} & \frac{\sqrt{\alpha}\gamma\mathbf{F}}{\mathfrak{a}(\gamma\mathbf{B}-2\alpha\mathbf{F})} & \frac{\gamma}{2} \\ 0 & \mp\frac{\sqrt{\alpha}\gamma\mathbf{D}}{\mathfrak{a}(\gamma\mathbf{B}-2\alpha\mathbf{F})} & \mp\frac{\gamma^2\mathbf{D}^2}{2\mathfrak{a}\sqrt{\alpha}(\gamma\mathbf{B}-2\alpha\mathbf{F})} & \mp\frac{\gamma}{2\mathfrak{a}\sqrt{\alpha}} \\ 0 & \pm\frac{2\alpha}{\gamma} \pm \frac{2\alpha^{\frac{3}{2}}\mathbf{D}}{\mathfrak{a}(\gamma\mathbf{B}-2\alpha\mathbf{F})} & \pm 1 \pm \frac{\sqrt{\alpha}\gamma\mathbf{D}}{\mathfrak{a}(\gamma\mathbf{B}-2\alpha\mathbf{F})} & \pm\mathfrak{a}\sqrt{\alpha} \\ 0 & -\frac{2\sqrt{\alpha}\mathbf{B}}{\mathfrak{a}(\gamma\mathbf{B}-2\alpha\mathbf{F})} & -\frac{2\sqrt{\alpha}\mathbf{F}}{\mathfrak{a}(\gamma\mathbf{B}-2\alpha\mathbf{F})} & 0 \end{pmatrix}.$$

One checks that $\det Z_\pm = -\frac{1}{\mathfrak{a}^2} \neq 0$ and that it satisfies (100)-(104). Furthermore

$$Z_\pm^{-1}M_1Z_\pm = \begin{pmatrix} 1 & \frac{2\alpha\gamma\mathbf{D}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & \frac{\gamma^2\mathbf{D}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & \gamma \\ 0 & -\frac{\gamma\mathbf{B}+2\alpha\mathbf{F}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & -\frac{2\gamma\mathbf{F}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & 0 \\ 0 & \frac{4\alpha\mathbf{B}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & \frac{\gamma\mathbf{B}+2\alpha\mathbf{F}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & 0 \\ 0 & -\frac{4\alpha\mathbf{D}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & -\frac{2\gamma\mathbf{D}}{\gamma\mathbf{B}-2\alpha\mathbf{F}} & -1 \end{pmatrix}, \qquad Z_\pm^{-1}\mathfrak{D}_1Z_\pm = \begin{pmatrix} 0 & 2\mathfrak{a}\alpha\gamma & \mathfrak{a}\gamma^2 & 0 \\ 0 & \pm 2\mathfrak{a}\sqrt{\alpha}\gamma & \pm\frac{\mathfrak{a}\gamma^2}{\sqrt{\alpha}} & 0 \\ 0 & \mp 4\mathfrak{a}\alpha^{\frac{3}{2}} & \mp 2\mathfrak{a}\sqrt{\alpha}\gamma & 0 \\ 0 & -4\mathfrak{a}\alpha & -2\mathfrak{a}\gamma & 0 \end{pmatrix},$$

which are the matrices associated to a pair from the last row of Table 7 when $\mathfrak{b} = \pm\sqrt{\alpha}\mathfrak{a}$. The coaction defined by an element of this class is (isomorphic to) $\rho(G) = \frac{2\alpha}{\gamma}(X\otimes 1) + \left(G - \frac{2\alpha}{\gamma}\right)\otimes g + 2\sqrt{\alpha}(\sqrt{\alpha}\gamma + \gamma G - 2\alpha X - 4\sqrt{\alpha}GX)\otimes gx$, $\rho(X) = X\otimes 1 + \frac{\gamma}{\sqrt{\alpha}}(\gamma\sqrt{\alpha} + \gamma G - 2\alpha X - 2\sqrt{\alpha}GX)\otimes gx$.

We gather all these results in the following table.

Type of Coaction	Definition	Skew-derivation
0	$\rho(G) = G\otimes 1$, $\rho(X) = X\otimes 1$	Trivial
1	$\rho(G) = G\otimes 1$ $\rho(X) = X\otimes\frac{1+g}{2} + \left(\frac{\gamma}{2\sqrt{\alpha}} + \frac{\gamma}{2\alpha}G - \frac{1}{\sqrt{\alpha}}GX\right)\otimes\frac{1-g}{2}$	Trivial

	$\rho(G) = G\otimes 1 - (\sqrt{\alpha}\gamma + \gamma G - 2\alpha X - 2\sqrt{\alpha}GX)\otimes gx$ $\rho(X) = X\otimes \frac{1+g}{2} + \left(\frac{\gamma}{2\sqrt{\alpha}} + \frac{\gamma}{2\alpha}G - \frac{1}{\sqrt{\alpha}}GX\right)\otimes \frac{1-g}{2} + \left(\frac{\gamma^2}{2\sqrt{\alpha}} + \frac{\gamma^2}{2\alpha}G + \right.$ $\left. -\gamma X - \frac{\gamma}{\sqrt{\alpha}}GX\right)\otimes \frac{(1-g)x}{2} + \frac{\gamma}{2\alpha}\left(\frac{\gamma^2}{2\sqrt{\alpha}} + \frac{\gamma^2}{2\alpha}G - \gamma X - \frac{\gamma}{\sqrt{\alpha}}GX\right)\otimes \frac{(1+g)x}{2}$	Non-trivial
2	$\rho(G) = G\otimes g,\ \rho(X) = X\otimes g$	Trivial
	$\rho(G) = G\otimes g + \frac{2\alpha}{\gamma}(X\otimes 1) - \frac{2\alpha}{\gamma}(X\otimes g),\ \rho(X) = X\otimes 1$	Trivial
	$\rho(G) = G\otimes g,\ \rho(X) = X\otimes g + 1\otimes x.$	Non-trivial
	$\rho(G) = G\otimes g + 2\omega\alpha(1\otimes x), \quad \rho(X) = X\otimes g + \omega\gamma(1\otimes x) \qquad (\omega\in k\setminus\{0\})$ (The coactions in this family are all non-isomorphic, unless $\omega = \pm\omega'$)	Non-trivial
	$\rho(G) = \frac{2\alpha}{\gamma}X\otimes 1 + \left(G - \frac{2\alpha}{\gamma}X\right)\otimes g + 2\alpha\gamma(1\otimes gx) - 4\alpha GX\otimes gx$ $\rho(X) = X\otimes 1 + \gamma^2(1\otimes gx) - 2\gamma GX\otimes gx$	Non-trivial
	$\rho(G) = \frac{2\alpha}{\gamma}(X\otimes 1) + \left(G - \frac{2\alpha}{\gamma}\right)\otimes g + 2\sqrt{\alpha}(\sqrt{\alpha}\gamma + \gamma G - 2\alpha X - 4\sqrt{\alpha}GX)\otimes gx$ $\rho(X) = X\otimes 1 + \frac{\gamma}{\sqrt{\alpha}}(\gamma\sqrt{\alpha} + \gamma G - 2\alpha X - 2\sqrt{\alpha}GX)\otimes gx$	Non-trivial

TABLE 8. Non-isomorphic coactions on $A = Cl\left(\alpha, \frac{\gamma^2}{4\alpha}, \gamma\right)$.

REMARK 3.12. Coactions of type 1 and the last class of coactions of type 2 on $Cl\left(\alpha, \frac{\gamma^2}{4\alpha}, \gamma\right)$ actually exist only if k contains a square root of α (or equivalently of β).

APPENDIX A

Multiplications as linear maps

In this appendix we wish to spend a few words in order to explain how we developed some tools that helped us in identifying the involutions and skew-derivations of Section 5, Chapter 2. One of the greatest obstacles when dealing with algebraic conditions such as (101)-(104) is that they give rise to quadratic equations in many variables, which are notoriously hard to solve. The main idea behind our strategy was to "try to keep things as linear as possible" by treating multiplications by fixed elements as linear maps.

Let us consider the four-dimensional Clifford algebra $A = Cl(\alpha, \beta, \gamma)$ generated by elements G, X such that $G^2 = \alpha \in k$, $X^2 = \beta \in k$ and $XG + GX = \gamma \in k$. We can fix the canonical basis $(1_A, G, X, GX) = (a_i)_{i=1,\dots 4}$. Given an element a_i in the fixed basis of A we can see left and right multiplication by a_i as k-linear maps l_{a_i} and r_{a_i} with corresponding matrices L_{a_i} and R_{a_i}. This can help in dealing with "algebra conditions for maps and elements" such as $\varphi(ab) = \varphi(a)\varphi(b)$ and allow us to use linear algebra softwares, such as MATLAB, to speed up or to verify our computations.

Since A is non-commutative, left and right multiplication (and thus their associated matrices) will differ. An easy computation shows that

$$L_G = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; R_G = \begin{pmatrix} 0 & \alpha & \gamma & 0 \\ 1 & 0 & 0 & \gamma \\ 0 & 0 & 0 & -\alpha \\ 0 & 0 & -1 & 0 \end{pmatrix}, \; L_X = \begin{pmatrix} 0 & \gamma & \beta & 0 \\ 0 & 0 & 0 & -\beta \\ 1 & 0 & 0 & \gamma \\ 0 & -1 & 0 & 0 \end{pmatrix}, \; R_X = \begin{pmatrix} 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and clearly $L_{1_A} = R_{1_A} = I$, $L_{GX} = L_G L_X$, $R_{GX} = R_X R_G$ (pay attention to the order of the product!).

Example 0.1. As a first and rather simple application we can determine the image of anticommutators of G and X with any element $v = (v_1, v_2, v_3, v_4)^t \in A$.

$$Gv + vG = (L_G + R_G)v = \left(\begin{pmatrix} 0 & \alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha & \gamma & 0 \\ 1 & 0 & 0 & \gamma \\ 0 & 0 & 0 & -\alpha \\ 0 & 0 & -1 & 0 \end{pmatrix} \right) v = \begin{pmatrix} 0 & 2\alpha & \gamma & 0 \\ 2 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} v = \begin{pmatrix} 2\alpha v_2 + \gamma v_3 \\ 2v_1 + \gamma v_4 \\ 0 \\ 0 \end{pmatrix}$$

and similarly

$$Xv + vX = (L_X + R_X)v = \left(\begin{pmatrix} 0 & \gamma & \beta & 0 \\ 0 & 0 & 0 & -\beta \\ 1 & 0 & 0 & \gamma \\ 0 & -1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right) v = \begin{pmatrix} 0 & \gamma & 2\beta & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \end{pmatrix} v = \begin{pmatrix} \gamma v_2 + 2\beta v_3 \\ 0 \\ 2v_1 + \gamma v_4 \\ 0 \end{pmatrix}.$$

Once L_G, R_G, L_X, and R_X are defined it is possible to describe left or right multiplication by any fixed element $a \in A$ as a linear map. In fact let $a = \lambda_1 + \lambda_2 G + \lambda_3 X + \lambda_4 GX$, $\lambda_i \in k$. Then it is not hard to see that

$$L_a = \lambda_1 I + \lambda_2 L_G + \lambda_3 L_X + \lambda_4 L_G L_X = \begin{pmatrix} \lambda_1 & \alpha\lambda_2 + \gamma\lambda_3 & \beta\lambda_3 & -\alpha\beta\lambda_4 \\ \lambda_2 & \lambda_1 + \gamma\lambda_4 & \beta\lambda_4 & -\beta\lambda_3 \\ \lambda_3 & -\alpha\lambda_4 & \lambda_1 & \alpha\lambda_2 + \gamma\lambda_3 \\ \lambda_4 & -\lambda_3 & \lambda_2 & \lambda_1 + \gamma\lambda_4 \end{pmatrix}. \tag{261}$$

This tool can also be used to calculate squares in A. For example $a^2 = a^2 \cdot 1_A = L_a^2(1,0,0,0)^t$ and

$$L_a^2 = \begin{pmatrix} \Lambda & (\alpha\lambda_2 + \gamma\lambda_3)(2\lambda_1 + \gamma\lambda_4) & \beta\lambda_3(2\lambda_1 + \gamma\lambda_4) & -\alpha\beta\lambda_4(2\lambda_1 + \gamma\lambda_4) \\ \lambda_2(2\lambda_1 + \gamma\lambda_4) & \Lambda + 2\gamma\lambda_1\lambda_4 + \gamma\lambda_4^2 & \beta\lambda_4(2\lambda_1 + \gamma\lambda_4) & -\beta\lambda_3(2\lambda_1 + \gamma\lambda_4) \\ \lambda_3(2\lambda_1 + \gamma\lambda_4) & -\alpha\lambda_4(2\lambda_1 + \gamma\lambda_4) & \Lambda & (\alpha\lambda_2 + \gamma\lambda_3)(2\lambda_1 + \gamma\lambda_4) \\ \lambda_4(2\lambda_1 + \gamma\lambda_4) & -\lambda_3(2\lambda_1 + \gamma\lambda_4) & \lambda_2(2\lambda_1 + \gamma\lambda_4) & \Lambda + 2\gamma\lambda_1\lambda_4 + \gamma\lambda_4^2 \end{pmatrix},$$

where $\Lambda = \lambda_1^2 + \alpha\lambda_2^2 + \beta\lambda_3^2 + \gamma\lambda_2\lambda_3 - \alpha\beta\lambda_4^2$. We can conclude that

$$a^2 = \begin{pmatrix} \lambda_1^2 + \alpha\lambda_2^2 + \beta\lambda_3^2 + \gamma\lambda_2\lambda_3 - \alpha\beta\lambda_4^2 \\ \lambda_2(2\lambda_1 + \gamma\lambda_4) \\ \lambda_3(2\lambda_1 + \gamma\lambda_4) \\ \lambda_4(2\lambda_1 + \gamma\lambda_4) \end{pmatrix}. \tag{262}$$

REMARK 0.2. From (262) it is clear that if $a^2 \in k$, then a is either in the ground field k or is of the form

$$a = -\frac{\gamma}{2}\lambda_4 + \lambda_2 G + \lambda_3 X + \lambda_4 GX.$$

In fact, if $a \notin k$, at least one among $\lambda_2, \lambda_3, \lambda_4$ is non-zero and thus $a^2 \in k$ forces $2\lambda_1 + \gamma\lambda_4 = 0$.

Another interesting result we can prove is the following. If the element we want to multiply by is the image of an invertible morphism of algebras, then the associated matrix $L_{\varphi(a)}$ is a suitable conjugate of L_a.

PROPOSITION 0.3. *Let $\varphi : A \to A$ be an invertible k-linear map and denote by M_φ its matrix. Then φ is an algebra map if, and only if,*

$$L_{\varphi(a)} = M_\varphi L_a M_\varphi^{-1}$$

for every $a \in A$.

PROOF. ($\implies$) Suppose φ is an algebra map. Then, for every $a, b \in A$

$$L_{\varphi(a)} b = \varphi(a) \cdot b = \varphi(a \cdot \varphi^{-1}(b)) = M_\varphi L_a M_\varphi^{-1} b$$

which means $L_{\varphi(a)} = M_\varphi L_a M_\varphi^{-1}$ for every $a \in A$.

($\impliedby$) Now suppose that $L_{\varphi(a)} = M_\varphi L_a M_\varphi^{-1}$ for every $a \in A$. We have

$$\varphi(ab) = M_\varphi L_a b = L_{\varphi(a)} M_\varphi b = \varphi(a)\varphi(b)$$

for every $b \in A$. □

REMARK 0.4. The same result can be proved with right multiplications in place of left ones.

Finally we can easily prove the following useful results on Clifford algebras, using our new toolbox.

PROPOSITION 0.5. *The invertible elements in $Cl(\alpha, \beta, \gamma)$ are exactly those of the form $a = \lambda_1 + \lambda_2 G + \lambda_3 X + \lambda_4 GX$, $\lambda_i \in k$, where*

$$|a| := \lambda_1^2 + \gamma\lambda_1\lambda_4 - \alpha\lambda_2^2 - \gamma\lambda_2\lambda_3 - \beta\lambda_3^2 + \alpha\beta\lambda_4^2 \neq 0. \tag{263}$$

In this case we have

$$a^{-1} := \frac{1}{|a|}[\lambda_1 + \gamma\lambda_4 - \lambda_2 G - \lambda_3 X - \lambda_4 GX].$$

The set

$$\Xi = \{a \in A \mid |a| = 0\}$$

is the set of all the zero divisors of A.

PROOF. Assume $a \neq 0$ and $|a| = 0$. Then, in view of (262), we have

$$\begin{aligned} a^2 &= \lambda_1(2\lambda_1 + \gamma\lambda_4) + \lambda_2(2\lambda_1 + \gamma\lambda_4)G + \lambda_3(2\lambda_1 + \gamma\lambda_4)X + \lambda_4(2\lambda_1 + \gamma\lambda_4)GX \\ &= (2\lambda_1 + \gamma\lambda_4)a, \end{aligned}$$

i.e. $a(a - 2\lambda_1 - \gamma\lambda_4) = 0$. If $a = 2\lambda_1 + \gamma\lambda_4 \in k$, then $\lambda_2 = \lambda_3 = \lambda_4 = 0$ and this forces also $\lambda_1 = 0$, contradiction. Then $a - 2\lambda_1 - \gamma\lambda_4 \neq 0$ and a must be a zero divisor, hence not invertible. If $|a| \neq 0$, then one can use (261) and calculate

$$L_a a^{-1} = \frac{1}{|a|}\begin{pmatrix} \lambda_1 & \alpha\lambda_2 + \gamma\lambda_3 & \beta\lambda_3 & -\alpha\beta\lambda_4 \\ \lambda_2 & \lambda_1 + \gamma\lambda_4 & \beta\lambda_4 & -\beta\lambda_3 \\ \lambda_3 & -\alpha\lambda_4 & \lambda_1 & \alpha\lambda_2 + \gamma\lambda_3 \\ \lambda_4 & -\lambda_3 & \lambda_2 & \lambda_1 + \gamma\lambda_4 \end{pmatrix}\begin{pmatrix} \lambda_1 + \gamma\lambda_4 \\ -\lambda_2 \\ -\lambda_3 \\ -\lambda_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and similarly

$$R_a a^{-1} = \frac{1}{|a|} \begin{pmatrix} \lambda_1 & \alpha\lambda_2 & \beta\lambda_3 + \gamma\lambda_2 & -\alpha\beta\lambda_4 \\ \lambda_2 & \lambda_1 & -\beta\lambda_4 & \beta\lambda_3 + \gamma\lambda_2 \\ \lambda_3 & \alpha\lambda_4 & \lambda_1 + \gamma\lambda_4 & -\alpha\lambda_2 \\ \lambda_4 & \lambda_3 & -\lambda_2 & \lambda_1 + \gamma\lambda_4 \end{pmatrix} \begin{pmatrix} \lambda_1 + \gamma\lambda_4 \\ -\lambda_2 \\ -\lambda_3 \\ -\lambda_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Finally, since every finite dimensional algebra is Artinian, then each non-invertible element in A must be a zero divisor. □

PROPOSITION 0.6. *Let $A = Cl(\alpha, \beta, \gamma)$ be a Clifford algebra. For every $a \in A$ we have* $\det L_a = \det R_a$.

PROOF. Let $a = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^t$. Remember that

$$L_a = \lambda_1 I + \lambda_2 L_G + \lambda_3 L_X + \lambda_4 L_G L_X = \begin{pmatrix} \lambda_1 & \alpha\lambda_2 + \gamma\lambda_3 & \beta\lambda_3 & -\alpha\beta\lambda_4 \\ \lambda_2 & \lambda_1 + \gamma\lambda_4 & \beta\lambda_4 & -\beta\lambda_3 \\ \lambda_3 & -\alpha\lambda_4 & \lambda_1 & \alpha\lambda_2 + \gamma\lambda_3 \\ \lambda_4 & -\lambda_3 & \lambda_2 & \lambda_1 + \gamma\lambda_4 \end{pmatrix}$$

and similarly

$$R_a = \lambda_1 I + \lambda_2 R_G + \lambda_3 R_X + \lambda_4 R_X R_G = \begin{pmatrix} \lambda_1 & \alpha\lambda_2 & \gamma\lambda_2 + \beta\lambda_3 & -\alpha\beta\lambda_4 \\ \lambda_2 & \lambda_1 & -\beta\lambda_4 & \gamma\lambda_2 + \beta\lambda_3 \\ a_3 & \alpha\lambda_4 & \lambda_1 + \gamma\lambda_4 & -\alpha\lambda_2 \\ \lambda_4 & \lambda_3 & -\lambda_2 & \lambda_1 + \gamma\lambda_4 \end{pmatrix}.$$

Then the truth of the statement follows from computation. □

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