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Holomorphic Dynamics in Fatou Components in Two Complex Variables

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Contents

Abstract

Complex dynamics, also called holomorphic dynamics, is a branch of dynamical systems that intersects deeply with complex geometry and complex analysis. It investigates the behavior and evolution of a complex manifold M under the repeated application of a holomorphic function $F : M \longrightarrow M$.

This thesis explores holomorphic dynamics from the prospective of the stable set, known as the Fatou set, and its connected components, called Fatou components. The work begins with an overview of key findings in one-dimensional holomorphic dynamics, then extends to higher-dimensional settings, with a particular focus on two-dimensional holomorphic dynamics.

The novel contribution of this research lies in the construction of transcendental Hénon maps that exhibit cycles of escaping Fatou components with rank-1 limit functions. The most notable result is the fact that these Fatou components have disjoint and hyperbolic limit sets, offering a deeper understanding of the intricate and rich dynamics in higher dimensions.

Introduction

Complex dynamics, also called holomorphic dynamics, is a branch of dynamical systems deeply related to complex geometry and complex analysis. It focuses on the study of dynamical systems generated by iterating holomorphic functions on complex manifolds.

Please refer to the Appendix 5, Section 5.1 for the definition of holomorphic maps from $\mathbb{C}^k$ to $\mathbb{C}^k$ as well as for the definition of complex manifolds.

The field of holomorphic dynamics combines aspects of geometry, topology, and complex analysis to explore how points in the complex plane or in higher-dimensional complex manifold evolve under repeated application of a given holomorphic map. Thus, at the core of holomorphic dynamics lies the study of iterated maps.

The process of iterating a function F involves taking a point z_0 in the complex plane or more generally, in a complex manifold, applying the function F to it to get a new point $z_1 = F(z_0)$, and repeating this process to form a sequence of points $z_n = F^n(z_0)$, where F^n denotes the n-th iterate of the function F.

The central question of holomorphic dynamics is to understand how this sequence behaves as n increases. Does it converge to a fixed point? Does it wander to "infinity"? Does it exhibit more complicated, possibly chaotic behavior?

This recent line of research emerged in the early 20th century thanks to the work of two French mathematicians, Pierre Fatou (see [Fat20b] and [Fat26]) and Gaston Julia (see [Jul18]). Their research on the complex plane $\mathbb{C}$ laid the foundation for the entire discipline of holomorphic dynamics.

In recognition of their foundational contributions to the study of complex dynamical systems, the region of the complex plane (or, more generally, of a manifold) where the iterates of a holomorphic function exhibit stable and predictable behavior is now known as *the Fatou set.* On the other hand, the complementary of the Fatou set, that is generally characterized by chaotic and unpredictable behavior, and often has fractal properties is now known as *the Julia set.*

Fatou also devoted significant attention to the classification of periodic points and introduced the concept of what are now called Fatou components, which are open subsets of the Fatou set. Within these components, the iterates of a holomorphic function exhibit stable and predictable behavior. Fatou's work emphasized the importance of understanding the structure of these components in $\mathbb{C}$, which can be classified into several types, depending on their behavior under iteration (see [Fat26]).

With regard to Gaston Julia, at the age of 25, in 1918, he rose to prominence for his mathematical contributions when his work, *Mémoire sur l'itération des fonctions rationnelles* [Jul18], was published in the *Journal de Mathématiques Pures et Appliquées.* This paper garnered significant recognition within the mathematical community and earned him the *Grand Prix des Sciences Mathématiques* from the French Academy of Sciences that same year.

However, after this initial fame, his work was partly forgotten until 1975 when Benoit Mandelbrot revived interest in it through his research on fractals, particularly in his French book *Les Objets Fractals: Forme, Hasard et Dimension* [Man75].

One of the most well-known and extensively studied examples in the literature of holomorphic dynamics pertains to the iteration of complex quadratic polynomials, typically written in the form $f_c(z) = z^2 + c$, where c is a complex parameter.

The dynamical behavior of these polynomials can vary dramatically depending on the value of c. For some values of c, the iterates of most points tend to infinity, while for other values of c, a significant portion of the complex plane $\mathbb{C}$ exhibits fractal structures.

The study of these polynomials leads to the famous Mandelbrot set, which is the set of all complex numbers c for which the orbit of 0 under iteration of f_c remains bounded. The Mandelbrot set is a rich object of study and its boundary has a fractal shape.

Beyond quadratic polynomials, holomorphic dynamics in $\mathbb{C}$ extends to more general classes of maps, such as higher-degree polynomials, rational maps, and transcendental entire functions.

Furthermore, holomorphic dynamics also extends to higher-dimensional manifold, and this generalization brings with it a host of new challenges and phenomena, as the geometry and topology of higher-dimensional spaces introduce additional layers of complexity to the dynamics.

In general, each class of functions associated with a complex manifold brings with it a new set of dynamical phenomena and challenges.

One-dimensional holomorphic dynamics also has applications of practical importance. Following [Aba08], an example of a problem with a centuries-long history recently solved using holomorphic dynamics techniques concerns Newton's method for determining roots of polynomials.

A classic problem in mathematics is, given a polynomial $p(x)$, to find the roots of $p(x)$, that is values of x for which $p(x) = 0$. Formulas to find the roots of first and second degree polynomials were known since antiquity; and in the 16th century, Del Ferro, Tartaglia, Cardano and Ferrari extended these results by deriving analogous formulas for the roots of third and fourth degree polynomials. In the 19th century Abel and Galois proved that for polynomials of the fifth degree and higher, similar formulas do not exist, indeed it is not possible to obtain the roots of a polynomial of the fifth degree, or higher, by simply doing algebraic operations on the coefficients of the polynomial.

Since in most applications an approximate value of the root is just as useful as the exact value, an alternative classical procedure, known as Newton's (or Newton-Raphson) method consists in considering an auxiliary rational function

$$f(x) = x - \frac{p(x)}{p'(x)}$$

which provides approximations of the roots of p. More precisely, if x_1 is a number sufficiently close to the unknown root x_0 of p, the orbit of x_1 converges to x_0: $f^n(x_1)$ tends to x_0. In other words the roots of p are attracting fixed points of f and applying the function f a finite number of times to the initial value x_1, called the *seed*, we can approximate the root x_0 with the desired precision.

All this was already known in the 17th century, and can be demonstrated by elementary techniques of mathematical analysis, without disturbing dynamic systems. However Newton showed that, starting from a value sufficiently close to a root, this method allows us to approximate the root with the precision we want. But if we do not know where the roots of the polynomial are, how can we be sure to start close enough? And if we start far from a root, what happens then? Do we still approximate a root, or does the procedure, when iterated, lead to a different outcome? In other words, what is the behavior of the orbits of this dynamic system?

To answer these questions, it is convenient to consider the action of the dynamical system on the complex plane, even if the polynomial has real coefficients. In fact, any complex roots of the polynomial significantly influence the system's behavior. In the 19th century Cayley succeeded in solving the problem for polynomials of degree two, showing that in this case Newton's method applied to a seed x_1 converges to the nearest root. The only exception is when x_1 is equidistant from the two roots; in that case Newton's method does not work, the orbit of x_1 does not converge to a root. But the set of points equidistant from the two roots is a straight line, which has zero area; therefore taking a seed at random, Newton's method converges to one of the two roots, and the set of points where Newton's method does not work is negligible. Cayley was unable to say anything about polynomials of degree three or higher.

Julia and Fatou, in the 1920s, translated the problem in terms of holomorphic dynamical systems, and noted that the set of seeds for which Newton's method works coincides with the components of the Fatou set that are attracted to the roots of the polynomial. Consequently, the set of seeds where Newton's method fails consists of the Julia set and potentially other components of the Fatou set. If this latter have zero area, Newton's method will almost certainly work when starting from a randomly chosen seed. However, if they have positive area, there is a significant risk that a randomly chosen seed may fail to converge to a root.

In the 1980s, numerous examples were found of polynomials (even simply of third degree) in which the Fatou set has components of positive area that are not attracted by the roots of the polynomial; for example, they are attracted by a periodic orbit that has nothing to do with the roots. Consequently, Newton's method on a random seed may not work in general. To overcome this problem, Hubbard, Schleicher and Sutherland in 2001 proposed not to choose a seed at random. Taking full advantage of recent results in holomorphic dynamics, for each natural number m Hubbard, Schleicher and Sutherland were able to construct a finite set of seeds from which all the roots of any polynomial of degree m can be found using Newton's method. Thus, holomorphic dynamics has provided a safe method for approximating the roots of any polynomial.

One can see also [DZ20], which further discusses the Newton method. Moreover this article is a survey of some of the connections linking complex dynamics to other fields of mathematics and science.

The principles of complex dynamics extend beyond mathematics into various areas of research such as fluid dynamics, biology and physics, ans these concepts are useful in modeling natural phenomena.

For instance in fluid dynamics, holomorphic functions can be used to model potential flows and analyze the behavior of fluid systems. More precisely, potential flows are idealized flows that are irrotational and incompressible and holomorphic maps are particularly useful for analyzing the behavior of fluid systems in two dimensions.

Moreover holomorphic dynamics is applied in biology to model population dynamics and the spread of biological species. The iterative processes in holomorphic dynamics can represent the growth and interaction of populations over time. A simple model of population dynamics is given by the well known logistic map:

$$x_{n+1} = r x_n (1 - x_n)\,,$$

where r is the growth rate, and x_n is the population size at generation n. This map can exhibit behaviors such as fixed points, cycles, and chaos depending on the value of r.

Additionally in physics, holomorphic functions are utilized to solve problems in quantum mechanics and statistical mechanics, where complex functions often describe physical phenomena and their evolution over time.

Returning to the area of pure mathematics, let M be a complex manifold and F a holomorphic function from M into itself, the goal is studying the evolution of the points of M under the iteration of F. Therefore, for a given holomorphic function F, we denote by F^n the function F composed with itself n times, that is $F^n = F \circ ... \circ F$.

As mentioned before, holomorphic dynamics can be naturally divided into two main areas of study. One focuses on the Julia set, that is the region of M where most of the unstable and chaotic dynamics take place. The other area of research examines the dynamically stable region of M, commonly referred to as the Fatou set. The Julia and Fatou sets are complementary to each other, and, in this dissertation, we will focus on the stable set and on its connected components.

When M is the complex plane $\mathbb{C}$ we talk about 1-dimensional complex dynamics, on the other hand when M has complex dimension grater then 1 we talk about complex dynamics in several complex variables.

While significant progress has been made in understanding the theory of one-dimensional complex dynamics, the transition to higher dimensions still presents difficult challenges since the situation is vastly different from the one-dimensional case and the construction of significant examples is an active area of research.

Even only the study of the dynamics of automorphisms, that is holomorphc maps injective and surjective, already poses deep difficulties, also when M is $\mathbb{C}^2$.

Therefore, in this thesis the main focus will be studying the dynamics of automorphisms of $\mathbb{C}^2$.

Structure of the Thesis

In Chapter 1, we introduce the theoretical background. We provide an introduction to the fundamental definitions and concepts that underpin holomorphic dynamics.

This part serves to establish a solid foundation upon which subsequent discussions will be built.

Specifically, we begin by explaining the concept of a normal family, which is crucial for understanding the behavior of sequences of holomorphic functions and their convergence properties.

Next, we delve into the definition of the Fatou set, as well as the concept of invariant Fatou component, periodic Fatou component, escaping Fatou component and wandering domain.

We expand our discussion by incorporating other essential definitions. To be specific, we define key concepts such as limit functions, which describe the behavior of sequences of iterates in the limit, limit sets, and rank of a limit function.

Than we proceed with an overview of the current state of knowledge in one-dimensional dynamics. This overview is based on an examination of the significant results and theorems that have been documented in the existing literature. More in detail, we focus on classification theorems concerning Fatou components in both the polynomial and transcendental cases.

In Chapter 2, our discussion shifts to the dynamics of several complex variables, specifically exploring the nature and behavior of the dynamics of automorphisms that map from $\mathbb{C}^2$ to $\mathbb{C}^2$.

Additionally, we will delineate the distinction between recurrent and non-recurrent Fatou components.

In this chapter, we provide an overview of the existing knowledge regarding the dynamics of automorphisms of $\mathbb{C}^k$, with $k \geq 2$ distinguishing between the polynomial and non-polynomial cases. In particular, we examine what is currently known about Fatou components in both the recurrent and non-recurrent cases, and finally on wandering domains. We introduce the concepts of polynomial and transcendental Hénon maps with related theorems and we briefly discuss skew products. Additionally, we will highlight some open problems that remain unresolved to date.

The following two chapters (Chapter 3 and Chapter 4) are dedicated to the research I conducted during my PhD, which culminated in two publications. The first one, see [BBS24], is a joint work with Anna Miriam Benini and Alberto Saracco and it is published on *Mathematische Zeitschrift*. The second one, see [Bel24], is published on the *International Journal of Mathematics*.

In particular, the results of [BBS24] are presented in Chapter 3, while those of [Bel24] can be found in Chapter 4 .

Delving deeper into the specifics, there are, in general, very few known examples of non-polynomial automorphisms that display non-recurrent Fatou components with limit functions of rank 1. Hence, one of the goals of this two chapters is to introduce new examples of transcendental Hénon maps that have escaping Fatou components characterized by limit functions of rank 1.

Additionally, this work aims to address two specific open problems within this area. The first objective is to investigate whether the limit sets corresponding to escaping Fatou components can exhibit hyperbolicity. The second is to examine the possibility of having distinct and disjoint limit sets. In this context, when we refer to a hyperbolic set, we mean a set that is biholomorphic to the unit disk.

The main results of Chapter 3 are summarized in the following theorem. Let

$$F(z,w) := (e^{-z^2} - \delta w, z) \tag{0.0.1}$$

with $\delta \in \mathbb{R}$, $\delta > 2$.

Let $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$ denotes the right half plane and $-\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Re}(z) < 0\}$ denotes the left half plane.

0.0.1 Theorem (Main Theorem of [BBS24])
Let F as above. Then F has a cycle of four Fatou components Ω^{ab} with $a, b \in \{+,-\}$, each of which is biholomorphic to $\mathbb{H} \times \mathbb{H}$. There are exactly two limit functions h_1, h_2, both of rank 1, such that

$$h_1(\Omega^{aa}) = h_2(\Omega^{a(-a)}) = \mathbb{H} \text{ and } h_1(\Omega^{a(-a)}) = h_2(\Omega^{aa}) = -\mathbb{H} \text{ for all } a.$$

Moreover, F is conjugate to its linear part on every Ω^{ab}.

On the other hand, the findings presented in Chapter 4 are collected in the following theorem. Let

$$F(z,w) = (e^{-z^m} + \delta e^{\frac{2\pi}{m}i}\, w\, ,\, z) \tag{0.0.2}$$

with $m \geq 2$ a natural number and $\mathbb{R} \ni \delta > 2$.

0.0.2 Theorem (Main Theorem of [Bel24])
Let F as above, then

- *There are m^2 distinct Fatou components that exhibit cyclic behavior, more precisely*
    - *If m is even there are $\frac{m}{2}$ cycles of escaping Fatou components of period $2m$.*
    - *If m is odd there are $\frac{m-1}{2}$ cycles of escaping Fatou components of period $2m$ and one cycle of escaping Fatou components of period m.*
- *Each cycle has exactly two distinct limit functions h_1, h_2, both of which have generic rank 1.*
- *Each Fatou component in each cycle has two disjoint and hyperbolic limit sets, with the exception of the Fatou components belonging to the cycle of period m (the one occurring when m is odd), which have the same hyperbolic limit set.*
- *Denote the union of the m^2 components with Ω, then F is conjugate to the linear map $L(z,w) = (\delta e^{\frac{2\pi}{m}i} w, z)$ on Ω.*
- *Each Fatou component in each cycle is biholomorphic to $\mathbb{H} \times \mathbb{H}$.*

Observe that the function provided in 0.0.1 falls within the scope of function 0.0.2 when setting $m = 2$.

Theorem 0.0.2 can be seen as a generalization of Theorem 0.0.1, where the introduction of the parameter m allows for the examination of a broader and more complex scenario. The dynamics differs depending on whether m is even or odd, leading to a combinatorial dynamical behavior. Notably, in

Theorem 0.0.2, new dynamical features arise with respect to Theorem 0.0.1, particularly when m is odd.

I would like to highlight the fact that the content of Chapter 3 represents a specific case of the broader framework developed in Chapter 4, that is, the material in Chapter 4 encompasses and extends that of Chapter 3. However, these two chapters have been intentionally kept separate to reflect the chronological progression of my doctoral research. Initially, the simpler case of $m = 2$ was analyzed, allowing for a deeper understanding of the dynamics of this specific scenario before approaching the problem from a more general perspective. This structure was also adopted to facilitate the reader's comprehension, following the common approach of starting with a simpler and more intuitive case before delving into its generalization.

Finally, we include Appendix 5, divided into five sections.

Chapter 1

General Set Up and Dynamics in One Complex Variable

This chapter offers an overview of the fundamental principles of holomorphic dynamics and it introduces the key definitions.

Moreover it examines the categorization of Fatou components in the case where the manifold is the complex plane. The primary focus of the chapter is on entire maps (holomorphic maps over the entire complex plane $\mathbb{C}$), dividing the discussion into polynomial and transcendental cases, then it offers some insights into meromorphic maps (quotients of entire functions). By presenting examples and theorems, the chapter sets a solid foundation for more advanced topics in later chapters.

1.1 Foundational Concepts

In this section, we will explore the base concepts of holomorphic dynamics, introducing key definitions that form the foundation of the field. These concepts will establish a framework that will be referenced throughout the thesis, preparing the reader for the advanced topics in the following chapters.

Let M be a complex manifold and F a holomophic function from M into itself. The goal of holomorphic dynamics is studying the evolution of M under the iteration of F. The idea is to analyze the iterative behavior of F, exploring how complex structures and patterns emerge from repeated application of F on M. This thesis aims to understand the stability properties of points of M and the nature of their orbits.

Holomorphic dynamics seeks to classify and describe the behavior of these iterative processes, providing insights into the stability, geometric and topological structure, and long-term dynamics of these functions.

With this in mind we denote the n-th iterate of F, i.e. F composed with itself n times, by F^n and we look at what happens to the orbits of points of M after many iterates of F.

We will now proceed to introduce the fundamental definitions of holomorphic dynamics, which are crucial for developing a thorough understanding of the subject.

The first concept to introduce is that of invariance. In this sense, a set $A \subset M$ is *completely invariant* under F if $F(A) \subseteq A$ and $F^{-1}(A) \subseteq A$.

When you study the iteration of a holomorphic map F on a manifold M, there is a natural division of M into two completely invariant subsets that are complementary sets, called the *Fatou set* and the *Julia set*.

Informally the Fatou set consists of points of M with the property that all nearby points behave similarly under repeated iteration of F. The Julia set consists of points of M such that, chosen a point in it, an arbitrary small perturbation of that point can cause drastic changes in the sequence of iterated function values.

Thus the Fatou set is also called the "stable set", while the Julia is the "chaotic set". In this thesis we will focus on the stable set.

Before giving the formal definition of the Fatou set, we need to introduce the definition of *normal families*. Let $\overline{M}$ be a compactification of M and we require that $\overline{M}$ is equipped with a complex structure itself compatible with that of M.

1.1.1 Definition. A family of holomorphic functions on a domain $\Omega \subset M$ is *normal* if every sequence has a subsequence which converges uniformly on compact subsets of Ω to a holomorphic function $h : \Omega \longrightarrow \overline{M}$.

Now we can give the definitions of the Fatou set and Fatou components.

1.1.2 Definition. The *Fatou set* of F is the set of points of M that have a neighborhood U such that $\{F^n_{|_U}\}_{n\in\mathbb{N}}$ forms a normal family.

1.1.3 Definition. A *Fatou component* is a connected component of the Fatou set of F.

Moreover, we give the following two definitions.

1.1.4 Definition. A Fatou component Ω is *invariant* if $F(\Omega) \subseteq \Omega$.

1.1.5 Definition. A Fatou component Ω is *periodic of period p* if p is the smallest positive integer such that $F^p(\Omega) \subseteq \Omega$.

Notice that an invariant Fatou component is nothing other than a periodic component with period $p = 1$.

For simplicity, we primarily focus on the invariant case, that is on the invariant Fatou components. Actually, this choice does not affect generality up to taking iterates, as periodic components merely cycle through different regions before eventually returning to the original set after p iterations.

At this point, we move to a particular class of Fatou components that will play a central role and be a recurring focus throughout the remainder of this thesis.

1.1.6 Definition. An *invariant escaping Fatou component* is an invariant Fatou component such that all its points have orbits that converges to the boundary of M.

To gain a clearer understanding of the previous definitions, let us examine the following easy example. We consider the complex plane, denoted by $\mathbb{C}$, as the manifold M. And let $\hat{\mathbb{C}}$ be the compactification of $\mathbb{C}$ by including the point at infinity, which is known as the Riemann sphere: $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$.

As a holomorphic function, we consider the basic squaring function:

$$\begin{aligned} f : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto z^2 \end{aligned}$$

It is easy to compute the direct form of f^n, that is $f^n(z) = z^{2^n}$.

Let $\mathbb{S}^1$ denote the unit circle, $\mathbb{D}$ the open unit disk and $\overline{\mathbb{D}}$ the closure of the open unit disk.

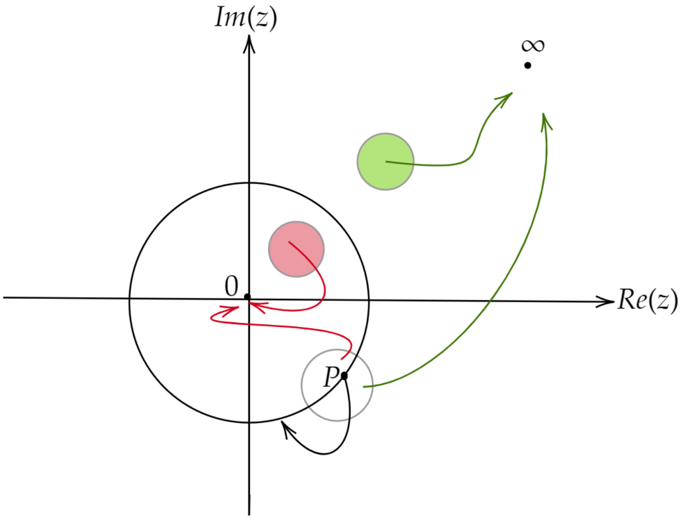

Figure 1.1: How f acts on $\mathbb{C}$.

Look at Figure 1.1 and consider a generic open green small ball inside the complement of the closed unit disk. All its points converge, under repeated iterations of f, to the point at infinity ∞, hence $\mathbb{C} \setminus \overline{\mathbb{D}}$ is in the stable set.

Similarly, if we take a generic open red small ball of points inside the unit disk, we have that all points in it converge uniformly to the origin 0, hence $\mathbb{D}$ is also in the Fatou set.

If, on the other hand, we consider a point P that lies on $\mathbb{S}^1$, the dynamics changes. Specifically, that point P always remain on $\mathbb{S}^1$ when repeatedly applying f. However, examining all small open neighborhoods U around P, we find that points of U in the complement of the closed unit disk diverge to infinity; points of U in the open unit disk converge to zero, and points on $\mathbb{S}^1$ stay on $\mathbb{S}^1$, showing no hope of convergence. Therefore, $\mathbb{S}^1$ must be part of the Julia set.

Notably, the Fatou set is composed of two distinct Fatou components, $\mathbb{C} \setminus \overline{\mathbb{D}}$ and $\mathbb{D}$, while the entire unit circle $\mathbb{S}^1$ represents the Julia set.

It is worth noting that the Fatou component $\mathbb{C} \setminus \overline{\mathbb{D}}$ is, in particular, an escaping component. This is due to the fact that every point in $\mathbb{C} \setminus \overline{\mathbb{D}}$ tends towards the point at infinity under repeated iterations of f.

1.2 One-Dimensional Case

Having established a general framework for holomorphic dynamics in the preliminary section, this section will shift our attention to a more specific scenario: the one-dimensional case. Here, we will delve into the dynamics when the manifold is the complex plane $\mathbb{C}$, exploring the behaviors and properties that arise in this context.

To distinguish this one-dimensional setting from the multidimensional case, we will denote the maps using lowercase italic font rather than uppercase. Specifically, if the map is a polynomial function, we will denote it by $p(z)$, whereas if it is a transcendental function, we will denote it by $f(z)$.

When the manifold is $\mathbb{C}$, there exists a complete classification of Fatou components due to Fatou and Julia developed in the 1920s, see [Fat20a], [Fat26] and [Jul18].

Initially, our focus will be on the entire functions, starting from the well-documented results and classifications of Fatou and Julia in the case of polynomial maps. In this regard, let us consider the following general polynomial map $p(z)$ of degree $d \geq 2$.

$$\begin{aligned} p : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto a_d z^d + a_{d-1} z^{d-1} + ... + a_0 \end{aligned}$$

We consider $d \geq 2$ because for constant maps and for polynomial maps of degree 1, the dynamics is straightforwardfor and intuitive, lacking the complexity and richness that arise in higher-degree cases, which often have complex and interesting dynamics.

From now on let $p(z)$ be a polynomial of degree $d \geq 2$, we first focus on the specific case of escaping Fatou components.

When we are close to infinity, meaning when the modulus of z is very large, the behavior of the map $p(z)$ can be approximated by $a_d z^d$. This causes the modulus of a point to grow rapidly.

Specifically, if the modulus of z is R, then the modulus of its image is approximately $|a_d| R^d$, provided that R is sufficiently large.

As a result, all points in a neighborhood of infinity are pushed closer to infinity with each iteration of $p(z)$. Consequently, every neighborhood of infinity is contained within a Fatou component that maps onto itself, continually contracting toward the point at infinity.

This phenomenon defines what is known as the attracting basin of infinity, which is the only escaping Fatou component that exists in the polynomial case and it is present for every polynomial map.

Geometrically, this basin of attraction might also not be simply connected (see for example [Ben21] at minute 12:34). The observations and results discussed above can be formally summarized in the following theorem:

1.2.1 Theorem (Fatou and Julia)
Let $p(z)$ be a polynomial map in $\mathbb{C}$. Then there is exactly one escaping Fatou component Ω, which is a basin of attraction for the point at infinity.

We now proceed to explore the classification of all non-escaping Fatou components in the polynomial scenario. We denote with p' the derivative of the polynomial map p.

1.2.2 Theorem (Fatou and Julia)
Let $p(z)$ be a polynomial map in $\mathbb{C}$. Let Ω be an invariant Fatou component, but non-escaping. Then Ω is bounded, biholomorphic to $\mathbb{D}$ and falls in one of the following types:

- *Attracting basin: $p^n(z) \longrightarrow z_0$ for some point $z_0 \in \Omega$. Moreover p is conjugate to $z \longmapsto \lambda z$ with $|\lambda| < 1$, where $\lambda = p'(z_0)$ (attractive case); or to $z \longmapsto z^d$ ($p'(z_0) = 0$, superattractive case).*

- *Parabolic basin: $p^n(z) \longrightarrow z_0$ for some point $z_0 \in \partial\Omega$. Moreover p is conjugate to $z \longmapsto z + 1$. In this case $p'(z_0) = e^{2\pi i\theta}$, with $\theta \in \mathbb{Q}$.*

- *Siegel disk: $p : \Omega \longrightarrow \Omega$ is conformal and p is conjugate to an irrational rotation of the disk. Moreover $p'(z_0) = e^{2\pi i\theta}$, with $\theta \in \mathbb{R} \setminus \mathbb{Q}$.*

In the superattractive case, the Fatou component contains a very “strong” fixed point, where the speed of attraction is more intense than in the case of a simple attractive fixed point.

To appreciate the classification of the non-escaping Fatou components in the polynomial case more thoroughly, the Figure 1.2 has been provided.

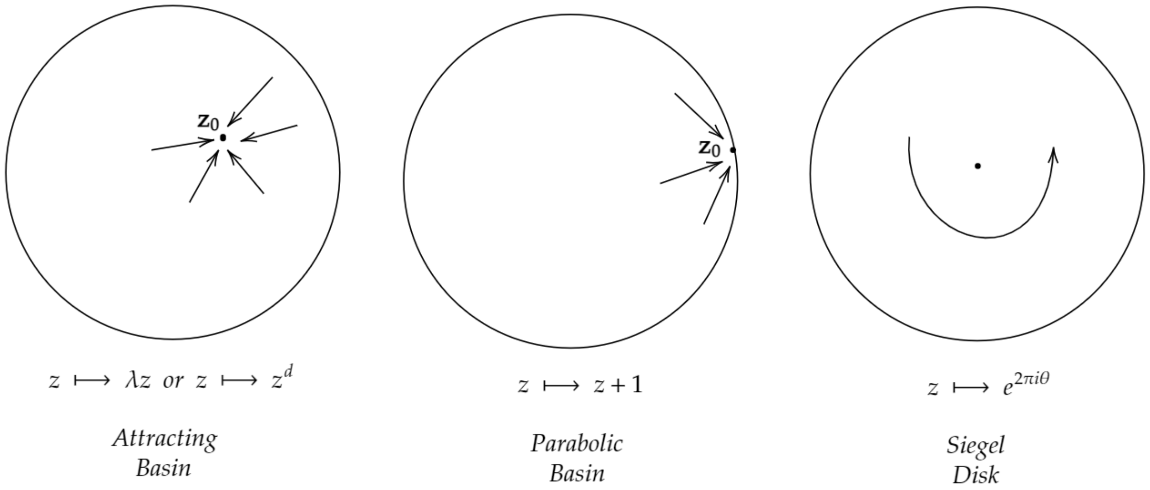

Figure 1.2: Types of non-escaping Fatou components in the polynomial case.

Note that, from a geometric perspective, Fatou components in the polynomial case are all biholomorphic to the disk, except for the attracting basin at infinity in the escaping case, which might also not be simply connected.

In contrast, from a dynamical viewpoint, there is significant diversity among these components.

We will now proceed to analyze Fatou components in the context of transcendental maps $f(z)$ from $\mathbb{C}$ to $\mathbb{C}$. It is important to note that transcendental maps are entire functions with an essential singularity at infinity.

Depending on the direction in which one approaches infinity, the points under the iteration of f will be mapped back into the complex plane, which precludes the possibility of having an attracting basin at infinity.

Consequently, in the transcendental case, we encounter new types of escaping Fatou components known as Baker domains.

1.2.3 Definition. For a transcendental function in $\mathbb{C}$, a Baker domain is a periodic Fatou component on which the orbits converge to the point at infinity.

In essence, Considering the dynamics on $\mathbb{C}$, a Baker domain is an escaping Fatou component which is not an attracting basin.

There is a well known classification of all the types of Baker domains developed by Cowen [Cow81].

Geometrically, all of these domains are biholomorphic to the disk $\mathbb{D}$. However, from a dynamical perspective, there are several possibilities.

In the classification outlined in the theorem below, we consider the right half-plane $\mathbb{H}$ instead of the disk $\mathbb{D}$ to better observe the dynamics of these components. It is a well known fact that $\mathbb{H}$ is biholomorphic to $\mathbb{D}$ under the map $w \longrightarrow \frac{i-w}{i+w}$ with inverse $z \longrightarrow \frac{i(1-z)}{1+z}$, where $z \in \mathbb{D}$ and $w \in \mathbb{H}$.

1.2.4 Theorem (Cowen)
Let $f(z)$ be a transcendental map and let Ω be Baker domain. Then Ω is biholomorphic to $\mathbb{H}$ and falls in one of the following types:

- *Hyperbolic: $f^n(z) \longrightarrow \infty$ and f is conjugate to $z \longmapsto az$ with $a > 1$;*

- *Simply parabolic:* $f^n(z) \longrightarrow \infty$ *and* f *is conjugate to* $z \longmapsto z \pm i$*;*

- *Doubly parabolic:* $f^n(z) \longrightarrow \infty$ *and* f *is conjugate to* $z \longmapsto z + 1$*.*

To gain a clearer understanding of this classification, the Figure 1.3 has been included both in the half-plane $\mathbb{H}$ and in the disk $\mathbb{D}$.

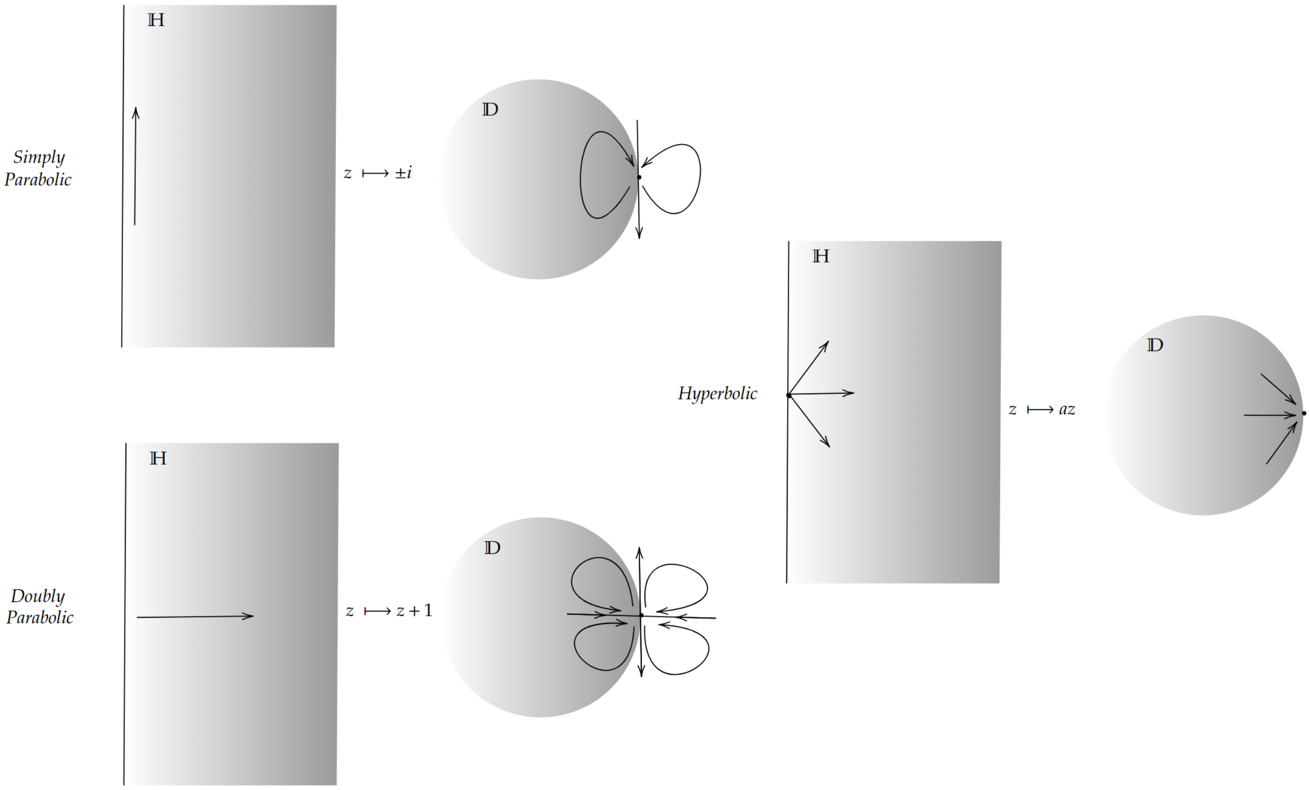

Figure 1.3: Types of Baker domains.

We now turn our attention to the case of non-escaping Fatou components for transcendental maps. In this scenario, the classification of these components is analogous to that in the polynomial case, that is Theorem 1.2.2 remains valid if we replace p with a transcendental map f.

A key difference is that, unlike in the polynomial case, the non-escaping components in the transcendental setting can be unbounded.

To gain a visual understanding of non-escaping Fatou components in the case of transcendental maps, refer to Figure 1.2. The only difference, as mentioned earlier, is that such components may not necessarily be bounded.

Thus far, we have presented the complete classification of invariant Fatou components for entire functions, which are holomorphic over the entire complex plane $\mathbb{C}$. This classification encompasses both polynomial maps and transcendental entire maps.

For the sake of completeness, we now present Theorem 1.2.5 from article [Ber93] (which references the classifications of Cremer [Cre32] and Fatou [Fat20b]) that outlines all possible periodic Fatou components in the case of meromorphic maps. In this theorem Bergweiler assumes that f is a meromorphic function from $\mathbb{C}$ to $\hat{\mathbb{C}}$, neither constant nor a linear. Recall that a meromorphic function is a holomorphic function defined on the entire complex plane $\mathbb{C}$, except for a discrete set of isolated points, which are singularities of pole type. These functions do not exhibit any singularities other than poles. Meromorphic functions from $\mathbb{C}$ to $\hat{\mathbb{C}}$ can be expressed as the quotient of two entire functions and so entire functions can be regarded as a subset of meromorphic functions.

A rational map is a meromorphic map of the form

$$r(z) = \frac{p(z)}{q(z)},$$

where $p(z)$ and $q(z)$ are polynomials in one variable z with complex coefficients, and $q(z)$ is not the zero polynomial. The degree of a rational map is determined by the highest degree of the polynomials $p(z)$ and $q(z)$. Obviously a polynomial map can be expressed as a rational map by considering it as a ratio where the denominator is equal to 1.

Therefore, what holds for rational maps also applies to polynomial maps.

If f is a rational function, then it possesses a meromorphic extension to $\hat{\mathbb{C}}$; thus, by denoting this extension again by f, we observe that f^n is defined and meromorphic in $\hat{\mathbb{C}}$. However, if f is transcendental, there is, naturally, no (reasonable) method to define $f(\infty)$.

1.2.5 Theorem ([Ber93])
Let Ω be a periodic Fatou component of period p. Then we have one of the following possibilities:

- *Ω contains an attracting periodic point z_0 of period p. Then $f^{np}(z) \to z_0$ for $z \in \Omega$ as $n \to \infty$, and Ω is called the immediate attractive basin of z_0.*
- *$\partial\Omega$ contains a periodic point z_0 of period p and $f^{np}(z) \to z_0$ for $z \in \Omega$ as $n \to \infty$. Then $(f^p)'(z_0) = 1$ if $z_0 \in \mathbb{C}$. In this case, Ω is called a Leau domain.*
- *There exists an analytic homeomorphism $\varphi : \Omega \to \mathbb{D}$ where $\mathbb{D}$ is the unit disk such that $\varphi(f^p(\varphi^{-1}(z))) = e^{2\pi i\alpha} z$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. In this case, Ω is called a Siegel disk.*
- *There exists an analytic homeomorphism $\varphi : \Omega \to \mathbb{A}$ where $\mathbb{A}$ is an annulus defined by $\mathbb{A} = \{z : 1 < |z| < r\}, r > 1$, such that $\varphi(f^p(\varphi^{-1}(z))) = e^{2\pi i\alpha} z$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. In this case, Ω is called a Herman ring.*
- *There exists $z_0 \in \partial\Omega$ such that $f^{np}(z) \to z_0$ for $z \in \Omega$ as $n \to \infty$, but $f^p(z_0)$ is not defined. In this case, Ω is called a Baker domain.*

If f is rational, then Baker domains do not exist. If f is transcendental, then Baker domains are possible only for $z_0 = \infty$.

We now introduce an important definition. In the context of holomorphic dynamics on $\mathbb{C}$, a *limit function* is a function obtained as the limit of a subsequence of iterations of a holomorphic map on a Fatou component Ω. Specifically,

1.2.6 Definition. Given a Fatou component Ω for f, we define a *limit function* for Ω as a holomorphic function $h : \Omega \longrightarrow \hat{\mathbb{C}}$ such that there exists a subsequence n_j such that $f^{n_j} \to h$ uniformly on compact subsets of Ω.

Fatou in [Fat20b] on page 249 established that if the sequence $\{f^n(\Omega)\}$ possesses only constant limit functions, then Ω must be classified as either an immediate attractive basin or a Leau domain, provided f is a rational function. His proof indicates that the only additional possibility for transcendental functions is the existence of a Baker domain. Furthermore, Cremer, in [Cre32] on page 317, demonstrated that if the sequence $\{f^n(\Omega)\}$ exhibits non-constant limit functions, then Ω is either a Siegel disk or a Herman ring.

We note that the classification of an immediate attractive basin can be further specified based on whether the attracting periodic point it contains is superattracting.

The term superattracting refers to an attracting periodic point where the derivative of the map at the point in absolute value is not just less than 1 (as for a regular attracting point), but is actually zero. This means the map's behavior near the point exhibits stronger attraction compared to a typical attracting point.

If the attracting point is superattracting, then Ω is designated as a *Böttcher domain*; if it is not, Ω is referred to as a *Schröder domain*.

Additionally, the terminology used in the literature is often inconsistent: *Leau domains* are also known as *parabolic domains*; *Herman rings* are frequently associated with *Arnold*; and *Baker domains* may be referred to as *infinite Fatou components* [Hay85], *essentially parabolic domains* [BKL92], or *domains at infinity* [DK89]. The term *Baker domain* seems to have originated in references [EL84] and [EL92].

If a Fatou component Ω is neither periodic nor preperiodic, meaning that it does not become periodic after a finite number of iterations, then $f^i(\Omega) \cap f^j(\Omega) = \varnothing$ for all $i, j \geq 0$ with $i \neq j$. In this case, Ω is called a *wandering domain*.

Wandering domains for which the only limit function is the point at infinity are called *escaping wandering domains*. On the other hand, if infinity is a limit function along with at least one other finite value, the domain is called *oscillating wandering domain*. Finally, when all limit functions are finite points in the complex plane, the domain is referred to as *dynamically bounded wandering domain*.

For example in [Ber93], Bergweiler shows that the function $f(z) = z + \lambda sin(2\pi z) + 1$ for suitable λ has escaping wandering domains. In [EL87], Eremenko and Lyubich provide an example of oscillating wandering domains.

Nevertheless, a significant open question in transcendental dynamics, is whether dynamically bounded wandering domains actually exist.

It is a well-known result, dating back over a century to Pierre Fatou, that in a wandering domain, all limit functions are necessarily constant [Fat20b]. Moreover a notable result by Sullivan in this area is the absence of wandering domains for rational maps [Sul85], see Theorem 1.2.7.

Notice that, in the polynomial (and rational) case, Fatou's classification of invariant Fatou components also provides a classification of periodic Fatou components, since they are invariant for an iterate of the rational map, and even preperiodic Fatou components, that is, components Ω such that there exist $m \geq 0$ and $p \geq 1$ with $f^{m+p}(\Omega) = f^m(\Omega)$. The following fundamental result, due to Sullivan [Sul85], completes the description for rational maps of degree $d \geq 2$.

1.2.7 Theorem

Let $f : \mathbb{C} \to \mathbb{C}$ be a rational map of degree $d \geq 2$. Then every Fatou component of f is preperiodic.

Since the dynamical behavior in periodic components is well understood as showed before (see also [Bea91], [CG93], [Mil06]) this leads to a complete description of the dynamics of rational functions on their Fatou sets.

Of course, wandering domains do not exist for polynomial maps as well.

Baker was the first who constructed in [Bak76], a transcendental entire function exhibiting a nested sequence of multiply connected Fatou components, each mapped onto the next, whose orbits escaped to infinity, thus proving that wandering domains can occur.

Baker demonstrated in [Bak75] that if Ω is any multiply connected Fatou component of a transcendental entire function f, then Ω is a wandering domain and possesses the following characteristics:

(a) each Ω_n is bounded and multiply connected,

(b) there exists $N \in \mathbb{N}$ such that Ω_n and 0 reside in a bounded complementary component of Ω_{n+1} for $n \geq N$,

(c) $\operatorname{dist}(\Omega_n, 0) \to \infty$ as $n \to \infty$.

For a significant class of functions known as finite-type maps (those with a finite number of singular values), every Fatou component is either periodic or preperiodic. As a matter of fact, a notable result in this area is the absence of wandering domains for transcendental entire functions of finite-type [EL92] and [GH62].

In summary, the discussion above establishes that the dynamical behavior of polynomial maps, rational maps and of transcendental entire functions of finite-type on the Fatou set was completely classified since wandering domains do not exist in these cases.

The discovery that wandering domains do not exist for transcendental finite-type maps was particularly remarkable, given that, as said before, Baker [Bak76] had constructed a transcendental entire function with wandering domains, proving that they can occur. While Baker's example involved multiply connected wandering domains, subsequent research has provided numerous examples of simply connected wandering domains.

A comprehensive analysis of the dynamics of entire functions within multiply connected wandering domains is provided in [BRS13]. The main result in [BRS13] is that it has been shown that within these wandering domains, all orbits exhibit similar behavior, eventually settling into and remaining within a sequence of progressively larger, nested circular annuli.

With regard to the possible behaviors of orbits within simply connected wandering domains, a thorough study has been conducted in [BEF$^+$21]. The idea followed in [BEF$^+$21] is the following.

Consider any holomorphic self-map of $\mathbb{C} \setminus \{0\}$, or an entire map $g : \mathbb{C} \to \mathbb{C}$ for which $z = 0$ is either an omitted value or has itself as its only preimage. For example, consider $g_\lambda(z) = \lambda z^d \exp(z)$ with $d \in \mathbb{N}$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Such a map g can be lifted via the exponential map to a transcendental entire function $f : \mathbb{C} \to \mathbb{C}$ that satisfies $\exp(f(z)) = g(\exp(z))$. Note that f is not uniquely defined, as any map of the form $f_k(z) = f(z) + 2k\pi i$ for $k \in \mathbb{Z}$ will also satisfy this property.

Now, observe that if g has an attracting component Ω (which does not contain $z = 0$), then any logarithmic of Ω, denoted Ω' ($p' := \log p \in \Omega'$ for $p \in \Omega$), would be a wandering domain for f_k (for an appropriate choice of k). Nevertheless, the orbits of points in Ω' would still "remember" their origin from the attracting component, meaning that the iterates of any given point would move progressively closer to the orbit of $p' := \log p \in \Omega'$, where p is the fixed point of g in Ω. Similarly, if Ω were a Siegel disk, the iterates of points in successive images of Ω' would "rotate" around a center (which is actually an orbit), with the iterates of p' following the same behavior.

Utilizing this lifting procedure, in [BEF$^+$21] it is underlined that it is possible to create examples of simply connected escaping wandering domains that display three distinct types of internal dynamics, corresponding to the dynamics observed within a periodic component: attracting, parabolic, or rotation-like. Consequently, this presents a clear contrast with multiply connected wandering domains, where only a single type of dynamical behavior is permitted, as previously mentioned.

In [BEF$^+$21], the behavior of points within wandering domains is analyzed from two distinct perspectives. As they emphasized, while points must navigate with the wandering domain that contains them (analogous to how passengers on a cruise ship must follow the ship's course), on the one hand they may or may not cluster together as they move along (as happens when lifting an attracting component but not when lifting a Siegel disk), and on the other hand orbits may stay away from the boundaries of their domains (as happens when lifting an attracting basin but not when lifting a parabolic basin). The results in [BEF$^+$21] explore both of these perspectives. We now report their main theorems.

They consider hyperbolic distances between pairs of corresponding points from two orbits and they focus on evaluating how these hyperbolic distances evolve under iteration.

Recall that a domain $\Omega \subset \mathbb{C}$ is hyperbolic if its boundary in $\mathbb{C}$ contains at least two points. For a hyperbolic domain Ω, let $\rho_\Omega(z)$ denote the hyperbolic density at $z \in \Omega$, and for $z, z' \in \Omega$, let $\operatorname{dist}_\Omega(z, z')$ denote the hyperbolic distance in Ω between z and z'.

Notice that for both hyperbolic density and distance, one can directly refer to those on the unit disk. Indeed, since we are working in $\mathbb{C}$, if Ω is simply-connected, then by the Uniformization Theorem (see Theorem 2.3.11), it is biholomorphically equivalent to the unit disk. If Ω is not simply-connected, its universal covering is the unit disk, and thus its hyperbolic density and distance are precisely those inherited from the disk.

Also recall that if Ω and V are hyperbolic domains, and $f : \Omega \to V$ is a holomorphic map, the Schwarz–Pick Lemma ensures that f is a contraction with respect to the hyperbolic distance. Therefore, if $\Omega \subset \mathbb{C}$ is a wandering domain of a transcendental entire function f, and we define Ω_n as the Fatou component containing $f^n(\Omega)$ for $n \in \mathbb{N}$, then for any two points $z, z' \in \Omega$, the sequence of hyperbolic distances

$$\operatorname{dist}_{\Omega_n}(f^n(z), f^n(z'))$$

is decreasing, and thus converges to a value that we denote as

$$c(z, z') = c_\Omega(z, z') := \lim_{n\to\infty} \operatorname{dist}_{\Omega_n}(f^n(z), f^n(z')) \geq 0.$$

The first classification result of [BEF$^+$21] shows that whether or not $c(z, z')$ is zero does not actually depend on the chosen pair (z, z'), provided that the two points have distinct orbits. They also provide a criterion to distinguish between these cases, based on the concept of hyperbolic distortion.

1.2.8 Definition (Hyperbolic Distortion)**.** If $f : \Omega \to V$ is a holomorphic map between two hyperbolic domains Ω and V, then the hyperbolic distortion of f at z is defined as:

$$||Df(z)||_V^\Omega := \lim_{z'\to z} \frac{\operatorname{dist}_V(f(z'), f(z))}{\operatorname{dist}_\Omega(z', z)},$$

and this equals the modulus of the hyperbolic derivative of f at z, given by:

$$\frac{\rho_V(f(z))f'(z)}{\rho_\Omega(z)}.$$

1.2.9 Theorem (Theorem A, First Classification Theorem)
Let Ω be a simply connected wandering domain associated with a transcendental entire function f, and let Ω_n denote the Fatou component that contains $f^n(\Omega)$ for $n \geq 0$. Consider the countable set of pairs

$$E = \{(z, z') \in \Omega \times \Omega : f^k(z) = f^k(z') \text{ for some } k \in \mathbb{N}\}.$$

Then, precisely one of the following conditions holds:

- $\operatorname{dist}_{\Omega_n}(f^n(z), f^n(z')) \xrightarrow{n\to\infty} c(z, z') = 0$ *for all* $z, z' \in \Omega$*, in which case we say that* Ω *is (hyperbolically) contracting;*
- $\operatorname{dist}_{\Omega_n}(f^n(z), f^n(z')) \xrightarrow{n\to\infty} c(z, z') > 0$ *and* $\operatorname{dist}_{\Omega_n}(f^n(z), f^n(z')) \neq c(z, z')$ *for all* $(z, z') \in (\Omega \times \Omega) \setminus E$*,* $n \in \mathbb{N}$*, and we say that* Ω *is (hyperbolically) semi-contracting; or*

- *There exists $N > 0$ such that for all $n \geq N$, $\operatorname{dist}_{\Omega_n}(f^n(z), f^n(z')) = c(z, z') > 0$ for all $(z, z') \in (\Omega \times \Omega) \setminus E$, and we say that Ω is (hyperbolically) eventually isometric.*

Additionally, for $z \in \Omega$ and $n \in \mathbb{N}$, let $\lambda_n(z)$ be the hyperbolic distortion $||Df(f^{n-1}(z))||_{\Omega_{n-1}}^{\Omega_n}$. Then, the following statements hold:

1. *Ω is contracting if and only if $\sum_{n=1}^{\infty}(1 - \lambda_n(z)) = \infty$;*
2. *Ω is eventually isometric if and only if $\lambda_n(z) = 1$ for sufficiently large n.*

Next, in [BEF+21], they provide sufficient criteria for a wandering domain to be strongly contracting or super-contracting, expressed in terms of the long-term average values of the hyperbolic distortion along the orbit of a point $z_0 \in \Omega$. They also show that this quantity is independent of the choice of z_0.

1.2.10 Definition (Rate of contraction)**.** Let Ω be a simply connected wandering domain of a transcendental entire function f and Ω_n be the Fatou component containing $f^n(\Omega)$, for $n \geq 0$. We say that Ω is strongly contracting if there exists $c \in (0, 1)$ such that

$$\operatorname{dist}_{\Omega_n}(f^n(z), f^n(z')) = O(c^n), \quad \text{for } z, z' \in \Omega.$$

We say that Ω is super-contracting if it satisfies the stronger condition that

$$\lim_{n\to\infty} \left(\operatorname{dist}_{\Omega_n}(f^n(z), f^n(z'))\right)^{1/n} = 0, \quad \text{for } z, z' \in \Omega.$$

1.2.11 Theorem (Theorem B)
Let Ω be a simply connected wandering domain of a transcendental entire function f and let Ω_n be the Fatou component containing $f^n(\Omega)$, for $n \geq 0$. Fix a point $z_0 \in \Omega$ and, for $z \in \Omega$ and $n \in \mathbb{N}$, let $\lambda_n(z) = ||Df(f^{n-1}(z))||_{\Omega_{n-1}}^{\Omega_n}$. Then the following hold:

(a) If $\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \lambda_k(z_0) < 1$, then Ω is strongly contracting.

(b) If $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \lambda_k(z_0) = 0$, then Ω is super-contracting.

(c) If $z \in \Omega$, then $\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \lambda_k(z) = \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \lambda_k(z_0)$.

1.2.12 Theorem (Theorem C, Second classification theorem)
Let Ω be a simply connected wandering domain of a transcendental entire function f and let Ω_n be the Fatou component containing $f^n(\Omega)$, for $n \geq 0$. Then exactly one of the following holds:

(a) $\liminf_{n\to\infty} dist(f^n(z), \partial\Omega_n) > 0$ for all $z \in \Omega$, that is, all orbits stay away from the boundary;

(b) there exists a subsequence $n_k \to \infty$ for which $dist(f^{n_k}(z), \partial\Omega_{n_k}) \to 0$ for all $z \in \Omega$, while for a different subsequence $m_k \to \infty$ we have that

$$\liminf_{k\to\infty} dist(f^{m_k}(z), \partial\Omega_{m_k}) > 0, \quad \text{for } z \in \Omega;$$

(c) $dist(f^n(z), \partial\Omega_n) \to 0$ for all $z \in \Omega$, that is, all orbits converge to the boundary.

These theorems provide a wandering version of the classification theorem for periodic components. Theorems A and C give rise to nine distinct dynamical types of simply connected wandering domains. In [BEF+21] it is proven that all of these cases exist due a criteria which establishes the existence of wandering components for a given map.

1.2.13 Definition (Surrounding a Set)**.** A curve σ *surrounds* a set B if and only if B is contained in a bounded complementary component of σ. Also, for a Jordan curve η, we denote by $int\eta$ the bounded component of $\mathbb{C} \setminus \eta$ and by $ext\eta$ the unbounded component of $\mathbb{C} \setminus \eta$.

1.2.14 Theorem (Existence criteria for wandering domains)
Let f be a transcendental entire function and suppose that there exist Jordan curves γ_n and Γ_n, $n \geq 0$, a bounded domain D, a subsequence $n_k \to \infty$, and compact sets L_k (associated with Γ_n) such that

1. *Γ_n surrounds γ_n, for $n \geq 0$;*
2. *for every k, n, $m \geq 0$, $m \neq n$, the sets L_k, D, Γ_m are in ext(Γ_n);*
3. *γ_{n+1} surrounds $f(\gamma_n)$, for $n \geq 0$;*
4. *$f(\Gamma_n)$ surrounds Γ_{n+1}, for $n \geq 0$;*
5. *$f(D \cup \bigcup_{k \geq 0} L_k) \subset D$;*
6. *$\max\{\mathit{dist}(z, L_k) : z \in \Gamma_{n_k}\} = o(\mathit{dist}(\gamma_{n_k}, \Gamma_n))$ as $k \to \infty$.*

Then there exists an orbit of simply connected wandering domains Ω_n such that int(γ_n) $\subset \Omega_n \subset$ int(Γ_n), for $n \geq 0$. Moreover, if there exists $z_n \in$ int(γ_n) such that both $f(\gamma_n)$ and $f(\Gamma_n)$ wind d_n times around $f(z_n)$, then $f : \Omega_n \to \Omega_{n+1}$ has degree d_n, for $n \geq 0$.

Chapter 2

Dynamics in Two Complex Variables and Beyond

This chapter explores the dynamics of holomorphic maps on the complex manifold $\mathbb{C}^k$, with $k \geq 2$, focusing mainly on what is known on the classification of their Fatou components. In particular we will consider automorphisms of $\mathbb{C}^2$ and polynomial and transcendent Hénon maps.

This chapter highlights the differences in dynamical behavior between one-dimensional and higher-dimensional systems, highlighting the complexities that arise in the transition to higher dimensions.

We introduce new fundamental definitions, such as the concept of limit function in $\mathbb{C}^2$ and the distinction between recurrent and non-recurrent Fatou components.

Additionally, the chapter raises open questions regarding this area of research.

2.1 Background and Motivation

In Chapter 1 we focused on one-dimensional complex dynamics, while in this chapter we consider the case when M has a complex dimension greater than one, that is the area of complex dynamics in several complex variables.

The understanding of one-dimensional complex dynamics is significantly more developed compared to what is currently known about complex dynamics in several complex variables. The transition to studying dynamics in higher dimensions still presents numerous difficult challenges, as there are new phenomena arising in the iteration of functions of several complex variables.

Even the study of the dynamics of automorphisms (that is, holomorphic maps that are both injective and surjective) in the setting of two complex variables already introduces profound difficulties. Constructing new examples of such automorphisms in the context of several complex variables remains a highly active area of research in this field.

For this reason, in this chapter, we focus mainly on the case where the manifold M is $\mathbb{C}^2$, and where F is an automorphism of $\mathbb{C}^2$.

It is important to notice that the definitions given in the previous chapter of normal families and, consequently, of Fatou sets depend on the chosen compactification of the manifold M.

Depending on which compactification of M we choose, we may have different Fatou sets, and in general we do have different sets of stability. For instance, see Remark 2.1.2 in which we report Example 2.6 of [ABFP19] where we have different Fatou sets for the same function.

It is important to highlight that for an automorphism of $\mathbb{C}^2$, $F : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$, there are at least two natural definitions of the stable set, which correspond to compactifying $\mathbb{C}^2$ with $\hat{\mathbb{C}}^2 := \mathbb{C}^2 \cup \{\infty\}$ or with $\mathbb{P}^2 := \mathbb{C}^2 \cup \ell_\infty$, that are respectively the compactification of $\mathbb{C}^2$ with the point at infinity or with the line at infinity. The main difference is that $\hat{\mathbb{C}}^2$ is not a complex manifold, whereas $\mathbb{P}^2$ is.

More generally, following [ABFP19], let $n \in \mathbb{N}$, $n \geq 1$ and let M be a complex manifold. As said before, there are at least two natural definitions of what it means for a family $\mathcal{F}$ of holomorpfic functions from M into $\mathbb{C}^n$ to be normal. For simplicity we write $\mathcal{F} \subset \mathrm{Hol}(M, \mathbb{C}^n)$. We denote by $\hat{\mathbb{C}}^n$ the one-point compactification of $\mathbb{C}^n$.

2.1.1 Definition. A family $\mathcal{F} \subset \mathrm{Hol}(M, \mathbb{C}^n)$ is $\mathbb{P}^n$*-normal* if for every sequence $(F_n) \in \mathcal{F}$ there exists a subsequence (F_{n_k}) converging uniformly on compact subsets to $h \in \mathrm{Hol}(X, \mathbb{P}^n)$.

A family $\mathcal{F} \subset \mathrm{Hol}(X, \mathbb{C}^n)$ is $\hat{\mathbb{C}}^n$*-normal* if for every sequence $(F_n) \in F$ which is not divergent on compact subsets, there exists a subsequence (F_{n_k}) converging uniformly on compact subsets to $h \in \mathrm{Hol}(X, \mathbb{C}^n)$.

Notice that, in contrast to Definition 1.1.1, where we require the compactification to possess a complex structure because the limit functions are holomorphic and map into the compactification, this condition is not imposed in [ABFP19]. The key difference lies in the fact that, in the case of compactification with $\hat{\mathbb{C}}^n$, which does not have a complex structure if $n \geq 2$, the holomorphic limit functions h map indeed into $\mathbb{C}^n$, and not into the compactification.

As observed in [ABFP19], if we compactify $\mathbb{C}^2$ with $\hat{\mathbb{C}}^2$, any open subset of $\mathbb{C}^2$ on which the sequence of iterates F^n diverges uniformly on compact subsets would be in the Fatou set regardless of "how" the orbits go to infinity. This definition appears to be too weak for two complex variables; therefore, they also define the Fatou set by compactifying $\mathbb{C}^2$ with $\mathbb{P}^2$, which also offers the benefit of being a complex manifold.

Observe that in one dimension the two compactification coincide and so the Fatou sets are the same.

2.1.2 Remark. Given an increasing sequence $N_j \in \mathbb{N}$, consider the sequence of polynomials

$$f_j(z) = (z - 5(j-1))^{N_j},$$

defined respectively on the disks $\mathbb{D}_j = \mathbb{D}(5(j-1), 2)$, where $j \geq 1$. Given a sequence $\varepsilon_j \searrow 0$, the Runge approximation guarantees that we can find an entire function f that is ε_j-close to f_j on $\overline{\mathbb{D}}_j$ for all j.

It is defined the automorphism of $\mathbb{C}^2$, F, by

$$F(z, w) = (z + 5, w + f(z)).$$

It follows immediately from the first coordinate that the forward orbit of any point (z_0, w_0) converges to infinity, i.e., $\|F^n(z_0, w_0)\| \to \infty$, hence the $\hat{\mathbb{C}}^2$-Fatou set equals all of $\mathbb{C}^2$. Moreover, if $|z_0| < 1$, then $F^n(z_0, w_0) \to [1 : 0 : 0]$, uniformly on compact subsets. Thus, the domain $\mathbb{D} \times \mathbb{C}$ is contained in a $\mathbb{P}^2$-Fatou component.

On the other hand, if the sequence N_j increases sufficiently fast, then for $1 < |z_0| \leq 2$, we have $F^n(z_0, w_0) \longrightarrow [0 : 1 : 0] \in \ell_\infty$, again uniformly on compact subsets. It follows that $\mathbb{D} \times \mathbb{C}$ is a $\mathbb{P}^2$-Fatou component. Therefore in this example the single $\mathbb{C}^2$-Fatou component contains infinitely many distinct $\mathbb{P}^2$-Fatou components.

Based on the aforementioned definitions, in what follows, we will only consider $\mathbb{P}^2$-normality. We will call the $\mathbb{P}^2$-Fatou set simply the Fatou set.

We will now present a crucial definition. Drawing on the concept of normal families introduced in Chapter 1, the set of all holomorphic functions h that arise as limits of subsequences of the iterates of the given function F are called *limit functions*. More precisely, we extend the definition 1.2.6 as follows.

2.1.3 Definition. Given a Fatou component Ω for F, we define a *limit function* for Ω as a holomorphic function $h : \Omega \longrightarrow \mathbb{P}^2$ such that there exists a subsequence n_j such that $F^{n_j} \to h$ uniformly on compact subsets of Ω.

Observe that by definition, the limit maps h are initially considered to map from the Fatou component to $\mathbb{P}^2$. However, in invariant Fatou components, since all orbits of the Fatou component are contained within the component itself, when taking the limit of subsequences, continuity ensures that the limit maps h actually map from the Fatou component Ω into the closure of the Fatou component $\overline{\Omega}$.

The image of Ω under h, which, as previously observed, lies in the closure of Ω if Ω is invariant, is referred to as the *limit set* of Ω and we denote it by $h(\Omega)$. Furthermore we define the *rank* of a limit function h as the maximal rank of its differential.

2.1.4 Remark. From a practical standpoint, thanks to the Sard's theorem, the rank of a limit function for Ω effectively coincides with the complex dimension of the limit set of Ω, that is

$$\operatorname{rank}(h) = \dim_{\mathbb{C}} h(\Omega)$$

.

A result about limit sets, proved in Lemma 2.4 of [ABFP19], asserts the following.

2.1.5 Proposition
Let Ω be a Fatou component for F. If $h : \Omega \longrightarrow \mathbb{P}^2$ is a limit function for Ω and $h(\Omega) \cap \ell_\infty \neq \varnothing$ then $h(\Omega) \subset \ell_\infty$.

Thanks to this proposition, it becomes almost natural to define the *escaping Fatou components* in $\mathbb{C}^2$, which are precisely those on which we will focus our attention in the last two chapters of this dissertation. We recall the definition of an escaping Fatou component given in the previous chapter (see Definition 1.1.6), now adapted to the case of $M = \mathbb{C}^2$.

2.1.6 Definition. An invariant Fatou component Ω is called *escaping* if for any of its limit functions h, $h(\Omega) \subset \ell_\infty$, that is Ω is a Fatou components for which all limit sets are in the line at infinity.

In a certain sense the escaping Fatou components in $\mathbb{C}^2$ can be seen as the analogous of the Baker domains in one-dimensional transcendental dynamics. Indeed they both have points whose orbits converge to the boundary of the compactification of the manifold. Recall that for a transcendental function from $\mathbb{C}$ to $\mathbb{C}$, a Baker domain is a periodic Fatou component on which the orbits converge locally uniformly to the point at infinity.

As seen in the previous chapter, in one complex variables there is a complete classification for invariant and periodic Fatou components both in the case the map is a polynomial and a transcendental function. However, as you can imagine, in more than one variable there are many more possibilities. We will now delve into understanding the current knowledge about complex dynamics in the case where F is an automorphism from $\mathbb{C}^2$ to $\mathbb{C}^2$, and in some instances, an automorphism from $\mathbb{C}^k$ to $\mathbb{C}^k$ with $k \geq 2$.

To accomplish this, we first consider the polynomial case and then the non-polynomial case.

2.2 Polynomial Scenario

We begin this section by defining what is meant by a *polynomial Hénon map*.

2.2.1 Definition. A *polynomial Hénon map* P is a polynomial automorphism of $\mathbb{C}^2$ of the form

$$P(z,w) := (p(z) + \alpha w\, ,\, z)$$

where $p : \mathbb{C} \longrightarrow \mathbb{C}$ is a polynomial of degree greater than 1 and $\alpha \in \mathbb{C} \setminus \{0\}$ is a complex constant.

Notice that these maps have the Jacobian determinant equals to $-\alpha$, so constant. In the polynomial case, there is a well-known theorem by Friedland and Milnor from 1989 [FM89] that proved that every polynomial automorphisms of $\mathbb{C}^2$ is conjugate to either an affine map, an elementary map, or a finite compositions of polynomial Hénon maps.

We recall that a map P is *elementary* if and only if it can be written as

$$P(z,w) = (az + p(w), bw + c)$$

for some constants a, b and c with $ab \neq 0$, and for some polynomial function $p(w)$.

To be precise, in the context of this thesis, two holomorphic maps f and g are said to be conjugate if there exists a third holomorphic map φ such that

$$\varphi \circ f \circ \varphi^{-1} = g.$$

The dynamical behavior of affine and elementary maps is easy to describe (see [FM89]). Therefore we will only look at finite compositions of polynomial Hénon maps. So we state the theorem of [FM89] as it is relevant to our discussion.

2.2.2 Theorem (Friedland and Milnor '89)
Any polynomial automorphisms of $\mathbb{C}^2$ with non-trivial dynamical behavior is conjugate to a finite composition of polynomial Hénon maps.

This theorem tells us that studying the class of polynomial Hénon maps (and their finite compositions) runs out all possible interesting dynamical behaviors in the case of polynomial automorphisms of $\mathbb{C}^2$.

For polynomial Hénon maps the matter of existence and properties of escaping Fatou components is essentially settled thanks to the work of Bedforf and Smillie [BS91b] from 1991: there is always only one escaping Fatou component, which is an attracting basin of $[1:0:0]$. More precisely,

2.2.3 Theorem (Bedford and Smillie '91 escaping case)
Let P be a polynomial Hénon map, the domain $U^+ := \big\{(z,w) \in \mathbb{C}^2 \,:\, |z| > |w|,$ with $|z|$ large enough$\big\}$ is invariant under P and $P^n_{|_{U^+}}$ converges to the point $[1:0:0] \in \ell_\infty$. Hence U^+ is part of an escaping Fatou component Ω, which we can think of as an attracting basin of $[1:0:0]$. Every escaping orbit is eventually contained in U^+ and no other escaping Fatou component is possible. This escaping Fatou component Ω is biholomorphic to the solenoid.

Note that, by this theorem and Remark 2.1.4, escaping Fatou components with rank 1 limit functions cannot occur for polynomial Hénon maps; only rank 0 is possible.

Another significant element of research in the field of complex dynamics in two complex variables concerns the duality between recurrent Fatou components and non-recurrent Fatou components. This duality is based on the distinction between Fatou components where the orbits accumulate within the interior of the Fatou component (like attracting basin in $\mathbb{C}$), versus those where the orbits accumulate on the boundary of the Fatou component (like parabolic basin in $\mathbb{C}$).

To further explore this aspect, we need to introduce the following formal definitions.

2.2.4 Definition. An invariant Fatou component Ω is *recurrent* if there exists a point in Ω whose orbit accumulates in Ω.

2.2.5 Definition. An invariant Fatou component Ω is *non-recurrent* if all orbits converge to the boundary of Ω.

Notice that the escaping Fatou components are a subset of the non-recurrent Fatou components, in which all orbits converge to the line at infinity, that is to the boundary of the manifold.

Due to Bedford and Smillie [BS91b], we are provided with a comprehensive classification of recurrent Fatou components that can arise in the context of polynomial Hénon maps.

2.2.6 Theorem (Bedford and Smillie '91 recurrent case)
Let Ω be a recurrent Fatou component for a polynomial Hénon map P. The following three cases are possible:

- *Ω is an attracting basin of an attracting fixed point in Ω, and it is biholomorphic to $\mathbb{C}^2$ (rank 0 case);*
- *All orbits converge to a Riemann surface $\Sigma \subset \Omega$ which is biholomorphic to either a disk $\mathbb{D}$ or an annulus $\mathbb{A}$. Either $P^n_{|\Sigma}$ is the identity, or $P^n_{|\Sigma}$ is conjugate to an irrational rotation. Ω is biholomorphic to $\mathbb{C} \times \Sigma$ (rank 1 case);*
- *There exists a subsequence P^{n_k} converging to the identity on Ω. Ω is a rotation domain, also called Siegel disk (rank 2 case).*

To better visualize the case of recurrent and escaping Fatou components in the polynomial setting, the Figure 2.1 has been provided.

Nevertheless, regarding Theorem 2.2.6, to date, there are no known examples where the Riemann surface Σ is biholomorphic to an annulus.

After Theorem 2.2.6 and Theorem 2.2.3, the problem in the polynomial scenario is reduced to the classification of non-recurrent (not-escaping) Fatou components.

In [LP14] M. Lyubich and H. Peters solve the problem under an additional assumption on the Jacobian determinant:

2.2.7 Theorem
Let $P : \mathbb{C}^2 \to \mathbb{C}^2$ be a non-elementary polynomial automorphism of degree $d \geq 2$, and let $\alpha = \det DP$ be its Jacobian determinant. Assume that P is moderately dissipative, i.e.,

$$|\alpha| < \frac{1}{d^2}.$$

Let Ω be an invariant non-recurrent Fatou component of P with bounded forward orbits. Then all the orbits in Ω converge to a parabolic point $\alpha \in \partial\Omega$ with multiplier 1.

In this case all orbits converge to a parabolic attracting fixed point, the non-recurrent component Ω is biholomorphic to $\mathbb{C}^2$, and P is conjugate on Ω to a map $(z, w) \longmapsto (z + 1, w)$. Thus, for moderately dissipative automorphisms, the answer turns out to be the same as in the one-dimensional case.

As said in [LP14], if Ω is a non-recurrent Fatou component then all orbits converge to the boundary $\partial\Omega$. By normality there exists a sequence P^{n_j} that converges, uniformly on compact subsets of Ω, to a limit map $h : \Omega \longrightarrow \partial\Omega$. In general the map h is not unique and depends on the sequence (n_j). The main difficulty lies in the fact that a priori it is not even clear whether the limit set $h(\Omega)$ is always unique. In [LP14] there is a precise classification of non-recurrent Fatou components under the assumption that $h(\Omega)$ is unique. We now report that classification in the following theorem.

2.2.8 Theorem
Let P be a polynomial Hénon map and Ω be a non-recurrent invariant Fatou component. Suppose that

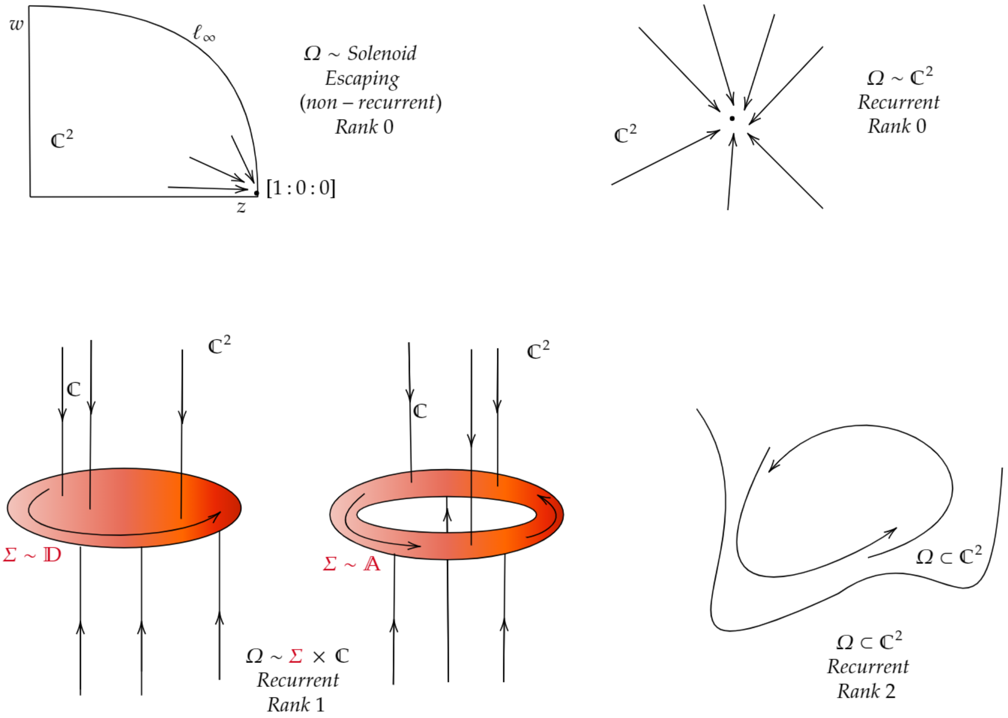

Figure 2.1: In the polynomial scenario, the first image illustrates the escaping case, while the other three images represent the types of recurrent Fatou components.

the limit set $h(\Omega)$ is unique. Then there exists a sequence $\{P^{n_j}\}$ that converges uniformly on compact subsets of Ω to a fixed point $p \in \partial\Omega$. If the entire sequence $\{P^n\}$ converges to p, then the eigenvalues λ_1 and λ_2 of DP_p satisfy $|\lambda_1| < 1$ and $\lambda_2 = 1$, and Ω is biholomorphically equivalent to $\mathbb{C}^2$.

They also obtain a result that lie outside the class of Hénon maps:

2.2.9 Theorem

Let F be a holomorphic endomorphism of $\mathbb{P}^2$ and let Ω be a non-recurrent, invariant Fatou component. Suppose that the limit set $h(\Omega)$ is unique. Then $h(\Omega)$ either consists of one point, or $h(\Omega)$ is an injectively immersed Riemann surface, conformally equivalent to either the unit disk, the punctured unit disk, or an annulus, and F acts on $h(\Omega)$ as an irrational rotation.

Regarding the rank of limits functions for non-recurrent Fatou components for polynomial Hénon maps, there are no examples of rank 1 limit functions. Observe that rank 1 limit functions cannot exist for escaping Fatou components as mentioned before as a consequence of Theorem 2.2.3.

More precisely in 2.2.7 the existence of rank 1 limit functions for non-recurrent Fatou components is excluded if $|\alpha|$ is small enough. Indeed, under this assumption, if P is a polynomial Hénon map and Ω a non-recurrent Fatou component for P, then all orbits in Ω converge to a point in $\partial\Omega$. So by Remark 2.1.4, non-recurrent Fatou components of rank 1 cannot exist for moderately dissipative polynomials.

When $|\alpha|$ is large, the problem remains unresolved and continues to be an open question in the field.

In the following section, we will shift our focus to the non-polynomial case.

2.3 Non-Polynomial Scenario

We now turn our attention to the dynamics of non-polynomial automorphisms of $\mathbb{C}^k$ with $k \geq 2$, considering mostly the case of $k = 2$. To date there is no classification for recurrent and non-recurrent Fatou components for general non-polynomial automorphisms of $\mathbb{C}^k$, so the construction of significant new examples is interesting in view of a future classification.

The most noteworthy results, which are of particular interest in the context of general non-polynomial automorphisms of $\mathbb{C}^k$ with $k \geq 2$, are presented below.

Following [BRS21], we call an invariant Fatou component Ω for a map F *attracting* if there exists a point $p \in \overline{\Omega}$ with $\lim_{n\to\infty} F^n(z) = p$ for all $z \in \Omega$. Note that, in particular, p is a fixed point for F. Observe that if $p \in \Omega$, then Ω is recurrent, and it is non-recurrent if $p \in \partial\Omega$.

For simplicity, we may assume that $p = 0$, that is the origin is a fixed point for F, and so $\Omega = \{z \in \mathbb{C}^k \mid F^n(z) \longrightarrow 0 \text{ as } n \longrightarrow \infty\}$.

In the case where $k = 1$, it is a well-known fact that if F has a basin of attraction Ω, then Ω is a disjoint union of simply connected regions in $\mathbb{C}$ (see [Mil06]). This is no longer true when $k \geq 2$.

Every attracting recurrent Fatou component of a holomorphic automorphism F of $\mathbb{C}^k$ is biholomorphic to $\mathbb{C}^k$. In fact it is the basin of attraction of F at 0, which is an attracting fixed point, that is all eigenvalues of DF_0 have modulus strictly less than 1 (see [RR88] and [PVW08]). In particular in [PVW08] there is the following theorem.

2.3.1 Theorem
Let $F : \mathbb{C}^k \to \mathbb{C}^k$ be a holomorphic map such that $F(0) = 0$, and let Ω be the attracting set. If Ω contains a neighborhood of the origin, then 0 is an attracting fixed point.

This is the simplest case and, in other words, it means that if 0 is an interior point of Ω, then Ω must consist of a single simply connected region. Therefore, it is necessary to seek examples among the maps where 0 lies on the boundary of Ω, the non-recurrent case.

For a more comprehensive understanding of attracting sets based on the eigenvalues of DF_0 obtained in [PVW08], please refer to Appendix 5, Section 5.2.

Shifting to the non-recurrent case, B. Stensønes and L. Vivas, in [SV14], show that there are automorphisms of $\mathbb{C}^3$, tangent to the identity, having attracting non-recurrent Fatou components biholomorphic to $\mathbb{C}^2 \times \mathbb{C}^*$. We will now delve into the details. To be clear, F is tangent to the identity if $DF_0 = Id$ and $\mathbb{C}^*$ denote the punctured plane.

The main theorem of [SV14], that is a more general version of what was stated above, is as follows.

2.3.2 Theorem
For any $k \geq 3$, there exists an automorphism F of $\mathbb{C}^k$ tangent to the identity at 0 whose basin Ω is biholomorphic to $(\mathbb{C}^*)^{k-2} \times \mathbb{C}^2$.

To prove this theorem, Stensønes and Vivas draw inspiration from a well-known example in complex dynamics in one variable. They consider the function $f(z) = z + az^2 = z(1 + az)$, where a is a non-zero real number. Observe that $f(0) = 0$ and $f'(0) = 1$.

The basins of attraction $\mathcal{C}_a$ for these maps are simply connected, bounded sets, and we have $0 \in \partial\mathcal{C}_a$.

The map $z \mapsto z(1 + \frac{1}{2}z)^2$ exhibits similar behavior. In $\mathbb{C}^2$, they define the map

$$F(z, w) = \left(z + \frac{1}{2}z^2w,\, w + \frac{1}{2}zw^2\right) = (z, w)\left(1 + \frac{1}{2}zw\right).$$

Examining how this map acts on the product zw, they see that $zw \mapsto zw\left(1 + \frac{1}{2}zw\right)^2$. From this, it follows that if $(z_n, w_n) = F^n(z, w)$, then

$$\prod_{j=0}^{n}\left(1+\frac{1}{2}z_jw_j\right)^2 \to 0 \quad \text{as } n\to\infty,$$

provided that $zw \in \tilde{C}$, where $\tilde{C}$ is the same region as the domains $\mathcal{C}_a$ for $a = 1$. In particular, we can observe that $\tilde{C}$ includes the region

$$U = \left\{0 < |zw| < \varepsilon \,|\, |\mathrm{Arg}(zw) - \pi| < \frac{\pi}{4}\right\}$$

(and more precisely, $\tilde{C} = \bigcup_k F^{-k}(U)$). If the product $zw \notin \tilde{C}$, then z_nw_n does not converge to 0. Thus, the basin of attraction for F is the region

$$\{(z, w) \in \mathbb{C}^2 \,|\, zw \in \tilde{C}\}.$$

Since the map acts as the identity on the axes, it follows that the map $(z, w) \mapsto (z, zw)$ is one-to-one on Ω, which implies that Ω is biholomorphic to $\mathbb{C}^* \times \tilde{C}$.

However, the map F is not an automorphism, and they do not identify an automorphism of $\mathbb{C}^2$ with similar behavior.

The situation changes when we allow k to be greater than or equal to 3. In [SV14] it is presented the construction for the case $k = 3$ that, however, can be extended to larger values of k.

It is shown that for any given $a, b, c \neq 0$, there exists an automorphism $F = (F_1, F_2, F_3) : \mathbb{C}^3 \to \mathbb{C}^3$ tangent to the identity of the form:

$$F_1(z,t,w) = z\left[1 - a\zeta + O(\zeta^2, \zeta w)\right],$$

$$F_2(z,t,w) = t\left[1 - b\zeta + O(\zeta^2, \zeta w)\right],$$

$$F_3(z,t,w) = w\left[1 - c\zeta + O(\zeta^2, \zeta w)\right] + O(\zeta^3),$$

where $\zeta = zt$.

Stensønes and Vivas choose and fix a, b, c to be positive real numbers with $a = b$ and $c > 2a$. Note that, with this choice, we have $tF_1(z, t, w) = zF_2(z, t, w)$. Furthermore, it is important to observe that $F(z, 0, w) = (z, 0, w)$ and $F(0, t, w) = (0, t, w)$. They demonstrate that F has a basin of attraction that is biholomorphic to $\mathbb{C}^* \times \mathbb{C}^2$.

A more recent article of Bracci, Raissy and Stensønes from 2021 (see [BRS21]) shows examples of non-recurrent Fatou components which are biholomorhic to $\mathbb{C} \times (\mathbb{C}^*)^{k-1}$.

The main result of this paper is the following theorem.

2.3.3 Theorem
Let $k \geq 2$. There exist holomorphic automorphisms of $\mathbb{C}^k$ having an invariant, non-recurrent, attracting Fatou component biholomorphic to $\mathbb{C} \times (\mathbb{C}^*)^{k-1}$.

In particular, this shows that there exist (non-polynomial) automorphisms of $\mathbb{C}^2$ having a non-simply connected attracting non-recurrent Fatou component.

Moreover there are examples of automorphisms with non-constant Jacobian and with a non-recurrent Fatou component Ω (non-escaping), whose limit functions have rank 1. In these cases, Ω may be biholomorphic to $\mathbb{C}^2$ [JL03] and [BTBP21] or to $\mathbb{C} \times \mathbb{C}^*$ [Rep19]. We will now delve into the substance of these articles.

The first example of an automorphism of $\mathbb{C}^2$ with a non-recurrent Fatou component on which all limit maps have rank 1 was given in [JL03]. There, an explicit map of the form

$$G(z,w) = \left(z + z^2 + O(z^3, z^4w, z^6w^2),\, w - \frac{z^2w}{2} + O(z^3, z^3w, z^3w^2)\right),$$

is constructed, and it is shown that G exhibits a Fatou component on which the orbits converge to the fixed plane $\{0\} \times \mathbb{C}$. By composing G with a rotation $(z,w) \mapsto (z, e^{2\pi i\theta}w)$, one obtains a non-recurrent Fatou component where the orbits converge to the rotating w-axis. One can also obtain the normal form $(z,w) \mapsto (z+1,w)$ as shown in [BTBP21].

Indeed in 2021, L. Boc-Thaler, F. Bracci and H. Peters [BTBP21], studied a specific type of non-recurrent Fatou component, which they refer to as a *parabolic cylinder*.

2.3.4 Definition. Let F be an automorphism of $\mathbb{C}^2$. The ω-limit set $\omega_F(p)$ of a point $p \in \mathbb{C}^2$ or $\omega_F(\Omega)$ of an open set $\Omega \subseteq \mathbb{C}^2$ under F is the set of all accumulation points of orbits under F starting in p or Ω, respectively.

Observe that a Fatou component Ω of F is non-recurrent if and only if $\omega_F(\Omega) \cap \Omega = \varnothing$.

2.3.5 Definition. Let F be an automorphism of $\mathbb{C}^2$. An invariant non-recurrent Fatou component Ω is called a *parabolic cylinder* if:

1. the closure of $\omega_F(\Omega)$ contains an isolated fixed point,
2. there exists an injective holomorphic map $\varphi : \Omega \to \mathbb{C}^2$, conjugating F to the translation $(z,w) \mapsto (z+1,w)$,
3. all limit maps of $F_{n\in\mathbb{N}}$ on Ω have dimension one.

The main result of their work is summarized in the following theorem.

2.3.6 Theorem
Let F be an automorphism of $\mathbb{C}^2$ of the form

$$F(z,w) = \left(z + f(w)z^2 + O(z^3, z^3w),\, e^{2\pi i\theta}w + g(w)z + O(z^2, z^2w)\right),$$

where $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is Diophantine, $f(0) \neq 0$, and $g(w) = O(w^2)$. Then, there exists a parabolic cylinder Ω for F that is biholomorphically equivalent to $\mathbb{C}^2$. Moreover, every limit map of F on Ω has image $\{0\} \times \mathbb{C}$.

They consider $\lambda = e^{2\pi i\theta}$ where $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is Diophantine, that is, there exist $c, r > 0$ such that $|\lambda^n - 1| \geq cn^{-r}$ for every $n \geq 1$. Such numbers form a dense subset of the unit circle with full measure. Note that if λ is Diophantine, then λ^{-1} is also Diophantine and satisfies the same estimates.

Reppekus, in [Rep19], extend the example by Bracci, Raissy, and Stensønes [BRS21] mentiod before. Based on [BRS21], it is not difficult to construct automorphisms of $\mathbb{C}^d$ with non-recurrent attracting invariant Fatou components biholomorphic to $\mathbb{C}^{d-m} \times (\mathbb{C}^*)^m$ for $m < d$. In [Rep19] there is a construction of automorphisms of $\mathbb{C}^d$ with an arbitrary finite number of non-recurrent Fatou components. Each of these components is biholomorphic to $\mathbb{C} \times (\mathbb{C}^*)^{d-1}$ and all attracted to a common boundary fixed point. Furthermore, these automorphisms can be constructed such that each Fatou component is invariant, or the components can be organized into periodic cycles of any common period. For more details about [Rep19] see 5, Section 5.4.

After this digression on what is currently known for general automorphisms of $\mathbb{C}^k$ with $k \geq 2$, we now focus on a special class of non-polynomial automorphisms of $\mathbb{C}^2$, called *transcendental Hénon maps*.

Indeed, in this work, our focus is on these particular maps, that represent essentially a natural generalization of the polynomial Hénon maps. They are characterized by the following definition.

2.3.7 Definition. A *transcendental Hénon map* is a non-polynomial automorphism of $\mathbb{C}^2$ of the form

$$F(z,w) := (f(z) + \alpha w\,,\, z)$$

where $f : \mathbb{C} \longrightarrow \mathbb{C}$ is a transcendental map in $\mathbb{C}$ and $\alpha \in \mathbb{C} \setminus \{0\}$ is a complex constant.

Again notice that these maps have constant Jacobian determinant equals to $-\alpha$.

It is a natural question to consider whether a theorem similar to that of Friedland and Milnor, Theorem 2.2.2, could be established in the context of non-polynomial automorphisms and transcendental Hénon maps. However, the possibility of proving such a theorem appears unlikely, particularly because the class of non-polynomial automorphisms of $\mathbb{C}^2$ is exceedingly broad and diverse.

Nevertheless, the class of transcendental Hénon maps is particularly interesting because, on the one hand, it is sufficiently small to offer hope for obtaining general classification results, yet on the other hand, it is large enough to yield new insights into dynamic behaviors that differ from those observed in the polynomial case.

The investigation of these maps from a dynamical viewpoint commenced in 2019, with an article authored by Arosio, Benini, Fornaess and Peters (referred to as [ABFP19]).

Regarding the case of recurrent Fatou components for transcendental Hénon maps, an analogous theorem to that of Bedford and Smillie, Theorem 2.2.6, holds. Specifically, in [ABFP19] there is the classification of the invariant recurrent components of the Fatou set of a transcendental Hénon map. More precisely, this classification holds not only for transcendental Hénon maps, but also for the larger class of holomorphic automorphisms with constant Jacobian determinant.

2.3.8 Theorem

Let F be a holomorphic automorphism of $\mathbb{C}^2$ with constant Jacobian determinant α, and let Ω be an invariant recurrent Fatou component for F. Then there exists a holomorphic retraction ρ from Ω to a closed complex submanifold $\Sigma \subset \Omega$, called the limit manifold, such that for all limit functions h there exists an automorphism η of Σ such that $h = \eta \circ \rho$. Every orbit converges to Σ, and $F|_\Sigma : \Sigma \to \Sigma$ is an automorphism. Moreover:

- *If $\dim \Sigma = 0$, then Ω is the basin of an attracting fixed point, and is biholomorphically equivalent to $\mathbb{C}^2$.*
- *If $\dim \Sigma = 1$, either Σ is biholomorphic to a circular domain A, and there exists a biholomorphism from Ω to $A \times \mathbb{C}$ which conjugates the map F to*

  $$(z,w) \mapsto \left(e^{i\theta} z, \frac{\alpha}{e^{i\theta}} w\right),$$

  where θ is irrational, or there exists $j \in \mathbb{N}$ such that $F^j|_\Sigma = \mathrm{id}_\Sigma$, and there exists a biholomorphism from Ω to $\Sigma \times \mathbb{C}$ which conjugates the map F^j to

  $$(z,w) \mapsto (z, \alpha^j w).$$

- *$\dim \Sigma = 2$ if and only if $|\alpha| = 1$. In this case, there exists a sequence of iterates converging to the identity on Ω.*

By a circular domain we mean either the disk, the punctured disk, an annulus, the complex plane, or the punctured plane.

To date, no classification has been established for non-recurrent Fatou components in the context of transcendental Hénon maps, including the specific case of escaping Fatou components.

Below, we will outline the most significant results obtained so far regarding transcendental Hénon maps.

The 2019 article by Arosio, Benini, Fornaess, and Peters (for references see [ABFP19]) includes the following example.

2.3.9 Theorem
Let F be a transcendental Hénon map defined as $F(z,w) := (e^{-z} + 2z - w, z)$. It has an escaping Fatou component Ω on which the orbits converge to the point $[1:0:0] \in \ell_\infty$, moreover F is conjugate to the linear map $(z,w) \longrightarrow (2z-w,z)$ on Ω and Ω is biholomoprphic to $\mathbb{H} \times \mathbb{C}$.

Observe that, given the fact that the limit set of Ω is a single point on the line at infinity, it follows that this is a case of rank 0 limit functions. This conclusion is a direct consequence of Remark 2.1.4, as has been repeatedly emphasized.

An additional relevant example is discussed in the 2023 article authored by Benini, Saracco, and Zedda (see [BSZ23]). This example provides further insight to the context of the current discussion.

2.3.10 Theorem
Let F be a transcendental Hénon map defined as $F(z,w) := (e^{-z} + 2w, z)$. It has an escaping Fatou component Ω with two distinct limit functions, h_1 and h_2, from Ω into ℓ_∞, both of which have rank 1. Moreover $h_1(\Omega), h_2(\Omega) = \ell_\infty \setminus \{[1:0:0],[0:1:0]\}$. F is conjugate to the linear map $(z,w) \longrightarrow (2w,z)$ on Ω, and Ω is biholomorphic to $\mathbb{H} \times \mathbb{H}$.

It should be noted that, in this example, the rank of the limit functions is 1.

In general, there are only a few known examples of non-polynomial automorphisms that exhibit non-recurrent Fatou components with limit functions of rank 1. Therefore, in this thesis, one of the objectives is to provide new examples of transcendental Hénon maps that exhibit escaping Fatou components with limit functions of rank 1.

Moreover in the following chapters we aim to investigate and provide answers to two specific open questions in this context. There are no examples of hyperbolic limit sets, thus, more precisely, we first seek to determine whether the limit sets associated with escaping Fatou components can be hyperbolic.

It should be noted that the term "hyperbolic" is associated with the limit sets of escaping Fatou components, which therefore lie on the line at infinity. Hence, it is sufficient to define hyperbolicity in one complex variable and, in this sense, we recall the Uniformization Theorem.

2.3.11 Theorem (Uniformization Theorem)
Every simply-connected Riemann surface is biholomorphic to

- $\mathbb{C}$ *(Euclidean case), or*
- $\mathbb{D} \subset \mathbb{C}$ *(hyperbolic case), or*
- $\hat{\mathbb{C}} := \mathbb{C} \cup \infty$ *the Riemann sphere (spherical case).*

Remember that $\mathbb{H}$ is biholomorphic to $\mathbb{D}$ as already observed in Chapter 1.

Regarding the second open question, this work aims to explore the possibility of having disjoint and distinct limit sets.

In [LP14], this issue is highlighted, as stated: "...The main difficulty lies in the fact that a priori it is not even clear whether the limit set $h(\Omega)$ is always unique...".

It is also stated in [BTBP21] that "...Another natural open question concerns the uniqueness of limit sets, that is, whether all limit maps must have the same image...". We answer to these open questions in Chapter 3 and in Chapter 4.

2.4 Wandering Domains

We now provide an overview of what is known in the case of wandering domains.

Remember that a Fatou component Ω is a *wandering domain* if it is neither periodic nor preperiodic, meaning that it does not become periodic after a finite number of iterations. This means that $F^i(\Omega) \cap F^j(\Omega) = \varnothing$ for all $i, j \geq 0$ with $i \neq j$.

As said in the previous chapter, that wandering domains do not exist in one-dimensional polynomials and rational maps, although they do appear in transcendental maps.

In higher dimensions, wandering domains are known to exist for holomorphic automorphisms of $\mathbb{C}^2$. Ideed in [FS98] Fornæss and Sibony construct a biholomorphism F of $\mathbb{C}^2$ with a wandering Fatou component Ω, such that a subsequence (F^{n_i}) converges to ∞ on Ω and another subsequence (F^{m_i}) has a finite limit on Ω. They call such a component an oscillating Fatou component. To find that F they use the concept of an r-shift that consists of a sequence of regions converging to infinity and a shift between them (see section 6 of [FS98] for more details).

Wandering domains also exist for polynomial maps of $\mathbb{C}^2$. It is an interesting results since, as mentioned in Chapter 1, this dynamical phenomenon cannot occur for polynomial maps in one complex dimension. Indeed, in [ABD$^+$16], it is proven the following theorem. See the Appendix 5, Section 5.3, for the definition of Lavaurs maps.

2.4.1 Theorem
Let $f : \mathbb{C} \to \mathbb{C}$ and $g : \mathbb{C} \to \mathbb{C}$ be polynomials of the form

$$f(z) = z + z^2 + O(z^3) \quad \text{and} \quad g(w) = w - w^2 + O(w^3).$$

If the Lavaurs map $L_f : B_f \to \mathbb{C}$ has an attracting fixed point, then the map $P : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$P(z, w) := \left(f(z) + \frac{\pi^2}{4} w, g(w) \right)$$

admits a wandering Fatou component.

B_f denotes the parabolic basin of 0, that is

$$B_f := \{ z \in \mathbb{C} \mid f^n(z) \to 0 \text{ as } n \to +\infty, \text{ with } f^n(z) \neq 0 \}.$$

Notice that since f and g have the same degree, F extends to a holomorphic endomorphism of $\mathbb{P}^2$.

A more general statement of Theorem 2.4.1 is the following.

2.4.2 Theorem
There exists an endomorphism $P : \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$, induced by a polynomial skew-product mapping $P : \mathbb{C}^2 \to \mathbb{C}^2$, possessing a wandering Fatou component.

We will delve further into skew-products at the end of this section; for the definition, please refer to Definition 2.4.5. Let us point out that the orbits in these wandering domains are bounded.

Moreover, in [HP21], Hahn and Peters constructed polynomial automorphisms of $\mathbb{C}^4$ with wandering Fatou components.

In the case of polynomial and transcendental Hénon maps, we provide the following definition as stated in [ABFP19].

2.4.3 Definition. Let F be a transcendental or polynomial Hénon map. A wandering domain Ω

1. is *escaping* if all orbits converge to the line at infinity,

2. is *oscillating* if there exists an unbounded orbit and an orbit with a bounded subsequence,

3. is *orbitally bounded* if every orbit is bounded.

As stated in [ABFP19] it is known that polynomial Hénon maps cannot have escaping and oscillating wandering domain, but the existence of orbitally bounded wandering domains is an open question for polynomial Hénon maps. However, more recently, Pierre Berger and Sébastien Biebler constructed an example of a polynomial Hénon map with a wandering Fatou component (see [BB22]).

In [ABFP19] there are examples of both escaping and oscillating wandering domains for transcendental Hénon maps. However, also in this case, the possibility of having transcendental Hénon maps with orbitally bounded wandering domains is an open problem in the field.

In [ABFP19] the construction of the Hénon maps with escaping wandering takes inspiration from the function in example [Ber93] in one complex variables.

Indeed in [ABFP19] they prove that the transcendental Hénon map

$$F(z,w) = (f(z) + \delta(z-1) - \delta w, z)$$

with $f(z) = z + sin(2\pi z) + \lambda$ and δ a suitable constant has an escaping wandering domain. Moreover that wandering domain is biholomorphic to $\mathbb{C}^2$, and all orbits converge to the point $[1:1:0]$ at the line at infinity (escaping).

Again in [ABFP19] there is an example of a transcendental Hénon map F that has an oscillating wandering domain Ω, biholomophic to $\mathbb{C}^2$. They use the fact that the map F, up to a linear change of variables, can be seen as the limit, as $k \longrightarrow \infty$, of automorphisms of $\mathbb{C}^2$ of the form

$$F_k(z,w) = \Big(f_k(z) + \frac{1}{2}w\,,\,\frac{1}{2}z\Big)$$

all with a hyperbolic fixed point at the origin. The following theorem implies the existence of a wandering Fatou component.

2.4.4 Theorem
There exists a sequence of entire maps

$$F_k(z,w) = \Big(f_k(z) + \frac{1}{2}w, \frac{1}{2}z\Big), \quad f_k(z) = bz + O(|z|^2), \quad k = 0,1,2,\ldots$$

a sequence of points $P_n = (z_n, w_n)$*, where* $n = 0,1,2,\ldots,$ *sequences* $R_k \to \infty$*,* $0 < \varepsilon_k \leq \frac{1}{2^k}$ *and* $\beta_n \to 0$*, a decreasing sequence* $\theta_k \to 0$*, and strictly increasing sequences of integers* $\{n_k\}, \{n'_k\}$ *with* $n_k < n'_k < n_{k+1}$*, such that the following five properties are satisfied:*

(i) $\|F_k - F_{k-1}\|_{D(0,R_{k-1})\times\mathbb{C}} \leq \varepsilon_k$ *for all* $k \geq 1$*;*

(ii) $F_k(P_n) = P_{n+1}$ *for all* $0 \leq n < n_k$*;*

(iii) $F_k(B(P_n,\beta_n)) \subset\subset B(P_{n+1},\beta_{n+1})$ *for all* $0 \leq n < n_k$*, where* $\beta_n \leq \frac{\theta_\rho}{2}$ *for all* $0 \leq n \leq n_k$*;*

(iv) $|z_n| < R_k - \theta_\rho$ *for all* $0 \leq n < n_k$*, and* $|z_{n_k}| > R_k + 5\theta_k$*;*

(v) $P_j \in B(0,\frac{1}{k})$ *for some* j *with* $n_{k-1} \leq j \leq n_k$*, for all* $k \geq 1$*.*

Here $\rho := \rho(n)$ *denotes the unique integer for which* $n'_{\rho-1} \leq n < n'_\rho$*, using* $n'_{-1} = 0$*.*

The family (F_k) is built inductively through the use of Runge approximation in one variable, allowing to define an entire function f_{k+1} which remains sufficiently close to f_k on increasingly larger disks. This construction ensures that the orbit of an open set $U_0 \subset \mathbb{C}^2$ repeatedly approaches the origin along the stable manifold of F_k, and then moves outward along the unstable manifold of F_k, for every $k \in \mathbb{N}$.

Notice that since $\beta_n \longrightarrow 0$ as $n \longrightarrow \infty$, by *(iii)* of Theorem 2.4.4 and by identity principle, it follows that all limit functions are constant, so of rank 0. It remains an open question whether oscillating wandering domains can have rank 1 limit functions.

As previously mentioned, we conclude this chapter with a discussion of skew products. To introduce this topic, we will begin with a brief digression.

In general, holomorphic maps from $\mathbb{C}^2$ to $\mathbb{C}^2$ have the form

$$F(z,w) = (q(z,w), g(z,w))$$

where q and g are holomorphic maps from $\mathbb{C}^2$ to $\mathbb{C}$.

Hénon maps, as said in Definitions 2.2.1 and 2.3.7, represent a special class of these maps, as their structure can be seen as

$$F(z,w) = (q(z,w), g(z))$$

where

$$q(z,w) = f(z) + \alpha w$$

and

$$g(z) = z \ .$$

Specifically, the first component is a function of a single variable, $f(z)$, with an added linear term of w, while the second component depends only on z.

Because of this specific form, Hénon maps can almost be seen as maps in "one and a half variables". In a certain sense, their form allows us to apply techniques from one-dimensional complex dynamics and adapt them to gain insights into the dynamics of Hénon maps in two complex variables.

In the field of holomorphic dynamics on $\mathbb{C}^2$, there is an approach similar to that of Hénon maps, which also aims to exploit the idea of using a special subclass of holomorphic maps of $\mathbb{C}^2$ that can be regarded as maps in "one and a half variables".

Specifically, this approach refers to what is known in the literature as *skew products*. We now provide the definition of skew product in complex dimension two.

2.4.5 Definition. A skew product is a holomorphic map of the form:

$$F(z,w) = (q(z,w), g(w))$$

where $q : \mathbb{C}^2 \longrightarrow \mathbb{C}$ is a holomorphic map that depends on z and w, acting on the first coordinate and $g : \mathbb{C} \longrightarrow \mathbb{C}$ is a holomorphic map acting on the second coordinate.

One could also define a skew product as $F(z,w) = (q(z), g(z,w))$.

Because of their special form, also the dynamics of skew products is simplified in the sense that it can be viewed as an intermediate case between one-dimensional and two-dimensional dynamics.

Moreover notice that Hénon maps are not skew products, since both variables mix in both coordinates. In addition, in general, skew products on $\mathbb{C}^2$ are not automorphisms; whereas, as already noted, Hénon maps are indeed automorphisms.

To my knowledge, the existing literature on the study of the dynamics of skew products focuses exclusively on the polynomial case.

This means that in Definition 2.4.5, q is a complex polynomial in two variables and g is a complex polynomial in one variable. The key characteristics of the dynamics of polynomial skew products have been researched in articles [Hei98], [Jon99], [Roe11] and [FG01].

We now provide an explanation of what is meant by deriving results from one-dimensional dynamics and extending them to two-dimensional dynamics in the case of polynomial skew-products. To accomplish this, we take into account the recent article [BR24], in which there is a survey of the recent results on Fatou components for polynomial skew-products in complex dimension two. Until the end of the chapter, when referring to skew-products F, we will specifically mean the polynomial case.

Note that F leaves invariant the fiber $\{w = w_0\}$, moreover the n-th iteration of a skew product has the form $F^n(z, w) = (\dots, g^n(w))$, this means that if Ω is a Fatou component for F, its projection on the second coordinate $\pi_w(\Omega)$ is a Fatou component for g. Thanks to Sullivan's non-wandering Theorem 1.2.7, up to considering an iterate of F, $\pi_w(\Omega)$ has to fall into one of the three cases given by Theorem 1.2.2: attracting basin, parabolic basin, or Siegel disk.

In addition, since (pre)periodic points for g correspond to (pre)periodic fibers for F, up to considering an iterate of F, we can just consider what happens in the neighborhoods of invariant fibers of the form $\{w = w_0\}$. We can assume, without loss of generality, that $g(0) = 0$ and consider $\{w = 0\}$ the invariant fiber.

The one-dimensional theory is also applied to the dynamics on the invariant fiber, since it is represented by the action of the one-dimensional polynomial $F(z, 0) = (q(z, 0), 0) =: q_0(z)$. Consequently, the Fatou components of q_0 are, once again, all (pre)periodic and, up to considering an iterate of F, we can assume that they fall into one of three categories of Theorem 1.2.2.

At this point, the natural question that arises is whether all one-dimensional Fatou components Ω_{q_0} of q_0 "extend" to two-dimensional Fatou components Ω of F. We will refer to this question with the symbol $\mathcal{Q}_1$.

Another natural question is whether it is possible to have wandering Fatou components for F in a neighborhood of an invariant fiber. We will refer to this question with the symbol $\mathcal{Q}_2$.

Following [BR24], we call the fiber $\{w = 0\}$ as attracting, parabolic or elliptic respectively, depending on whether 0 is an attracting, parabolic or elliptic fixed point for g, that is respectively if $|g'(0)| < 1$, $g'(0) = 0$ or $g'(0) = e^{2\pi i\theta}$, with $\theta \in \mathbb{R} \setminus \mathbb{Q}$.

Moreover an "extending" Fatou component is a Fatou component Ω of F such that $\Omega \cap \{w = 0\}$ is a one-dimensional Fatou component of q_0 on the invariant fiber $\{w = 0\}$. Finally, we say that a Fatou component Ω_{q_0} of q_0 on the invariant fiber $\{w = 0\}$ "is extending" if there exists an extended Fatou component Ω of F so that $\Omega_{q_0} = \Omega \cap \{w = 0\}$.

Lilov in [Lil04] answers question $\mathcal{Q}_1$ in the superattracting case, that is when $g'(0) = 0$:

2.4.6 Theorem
Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial skew-product of degree $d \geq 2$. Let $\{w = 0\}$ be a superattracting invariant fiber for F. Then all one-dimensional Fatou components of q_0 extend to Fatou components of F.

Moreover, again in [Lil04], Lilov answers also question $\mathcal{Q}_2$; he proves the non-existence of wandering Fatou components in a neighborhood of a superattracting invariant fiber.

Theorem 2.4.6 can be adjusted and generalized to the attracting case, obtaining the following statement answering question $\mathcal{Q}_1$.

2.4.7 Theorem
Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial skew-product of degree $d \geq 2$. Let $\{w = 0\}$ be an attracting invariant fiber for F. Then all one-dimensional Fatou components of q_0 extend to Fatou components of F.

The attracting case and question $\mathcal{Q}_2$ has been further investigated by Peters and Smit in [PS18].

They focused their investigation on polynomial skew-products such that the action on the invariant attracting fiber is subhyperbolic, that is the polynomial does not have parabolic periodic points and all critical points lying on the Julia set are preperiodic. They proved the following result.

2.4.8 Theorem

Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial skew-product. Assume that the origin is an attracting fixed point for g with corresponding basin B_g, and the polynomial $q_0(z)$ is subhyperbolic. Then F has no wandering Fatou component over B_g.

Moreover, Ji in [Ji20] extends the previous theorem to polynomial skew-products, again with an invariant attracting fiber, just under the hypothesis that the multiplier of the invariant fiber is sufficiently small.

In [ABD$^+$16], as said in Theorem 2.4.1, they consider a polynomial skew-product having a parabolic invariant fiber. They prove the existence of polynomial skew-products of $\mathbb{C}^2$, extending to holomorphic endomorphisms of $\mathbb{P}^2$, having a wandering Fatou component. It is an important example since it is the first construction of non-invertible polynomial maps with wandering domains.

Moreover, referring to Theorem 2.4.1 of [ABD$^+$16], Astorg, Boc-Thaler and Peters in [ABTP23] provide an example of a wandering domain arising from a Lavaurs map with a fixed point of Siegel type instead of an attracting fixed point as in [ABD$^+$16]. See Appendix 5, Section 5.3 for the definition of Lavaurs maps.

Again in the parabolic case, in [ABT22] they study the dynamics of polynomial skew products tangent to the identity, that, as said before, are maps with Jacobian matrix, in 0, equals to the identity matrix. They prove the existence of countably many wandering domains with rank 1 limit functions.

In [PR19] they study the case of invariant fibers at the center of a Siegel disk. More precisely, they consider a polynomial skew-product having an elliptic invariant fiber $\{w = 0\}$.

In this case there is only a partial answer to question $\mathcal{Q}_1$. In fact, while we already know that the attracting Fatou components of q_0 always extend, the general situation appears to be more complicated. They prove that all parabolic Fatou components of a polynomial skew-product with an elliptic invariant fiber extend if the rotation number satisfies the Brjuno condition (for the definition of the Brjuno condition see Appendix 5, Section 5.4).

They also have an answer to question $\mathcal{Q}_2$, under the assumption that the multiplier at the elliptic invariant fiber is Brjuno and all critical points of the polynomial acting on the invariant fiber lie in basins of attracting or parabolic cycles.

2.4.9 Theorem

Let F be a polynomial skew-product and let $\{w = 0\}$ be an elliptic invariant fiber with multiplier λ. If λ is Brjuno and all critical points of the polynomial q_0 lie in basins of attracting or parabolic cycles, then all Fatou components of q_0 extend, and there is a neighborhood of the invariant fiber $\{w = 0\}$ in which the only Fatou components of F are the extending Fatou components of q_0. In particular, there are no wandering Fatou components in this neighborhood.

We recall that a complex number λ is called Brjuno if

$$\sum_{k=0}^{\infty} 2^{-k} \log\left(\frac{1}{\omega(2k+1)}\right) < \infty, \tag{2.4.1}$$

where $\omega(m) = \min_{2 \le k \le m} |\lambda^k - \lambda|$ for any $m \ge 2$.

To conclude, one can also see articles [PV16], [Viv18], [Uen19] and [Uen24] for further insights related to the study of polynomial skew products.

Chapter 3

Rank-1 Limit Functions, Hyperbolicity and Disjointness of Limit Sets

This chapter presents the outcomes of my research, carried out in collaboration with Anna Miriam Benini and Alberto Saracco. These findings are also accessible in article [BBS24].

3.1 Framework and Goals

In this chapter, we will focus on constructing automorphisms of $\mathbb{C}^2$ of constant Jacobian determinant, that will be transcendental Hénon maps, with a cycle of escaping Fatou components, on which there are exactly two limit functions, both of rank 1. For each such Fatou component, the limit sets are two disjoint hyperbolic subsets of the line at infinity.

In the literature there are currently very few examples of general automorphisms of $\mathbb{C}^2$ with rank one limit functions, as said in the previous chapter. In addition, to my knowledge, this is the first example in which the limit sets are hyperbolic, and moreover different limit sets, coming from rank 1 limit functions, coexist.

As said before, transcendental Hénon maps are automorphisms of $\mathbb{C}^2$ with constant Jacobian determinant of the form

$$F(z,w) := (f(z) + \alpha w, z) \text{ with } f : \mathbb{C} \to \mathbb{C} \text{ entire transcendental and } \mathbb{C} \ni \alpha \neq 0.$$

In analogy with classical complex polynomial Hénon maps, for which f is assumed to be a polynomial (see for example [BS91a], [BS91b], [BLS93], [HOV94], [FS95] and [FS98]), the dynamical investigation of transcendental Hénon maps can rely on tools and knowledge from one dimensional complex dynamics, which is better understood than its higher dimensional counterpart.

As said in the previous chapter, general properties of transcendental Hénon maps were established in [ABFP19],[ABFP21] and [ABFP23].

In this chapter we investigate escaping Fatou components. Recall that escaping Fatou components are Fatou components for which all limit sets lie in the line at infinity (see Definition 2.1.6).

More precisely we construct a transcendental Hénon map with a cycle of escaping Fatou components satisfying the following properties. Let $\mathbb{H}$ denote the right half plane, $-\mathbb{H}$ denote the left half plane.

3.1.1 Theorem (Main Theorem of [BBS24])
Let

$$F(z,w) := (e^{-z^2} - \delta w, z), \ \mathbb{R} \ni \delta > 2.$$

Then F has a cycle of four Fatou components Ω^{ab} with $a, b \in \{+,-\}$, each of which is biholomorphic to $\mathbb{H}\times\mathbb{H}$. There are exactly two limit functions h_1, h_2, both of rank 1, such that

$$h_1(\Omega^{aa}) = h_2(\Omega^{a(-a)}) = \mathbb{H} \text{ and } h_1(\Omega^{a(-a)}) = h_2(\Omega^{aa}) = -\mathbb{H} \text{ for all } a.$$

Moreover, F is conjugate to its linear part on every Ω^{ab}.

Notice that δ is the modulus of the Jacobian determinant of F, hence the latter is expansive since $\delta > 1$. For convenience we take $\delta > 2$.

The main points of interest of this result are that the limit functions have rank one, that each Fatou component has two disjoint limit sets, and that the limit sets are hyperbolic.

3.2 Computing Limit Functions

From now on let F be as in Theorem 3.1.1, that is

$$F(z,w) = (e^{-z^2} - \delta w, z) \text{ with } \delta > 2. \tag{3.2.1}$$

Notice that, following Definition 2.3.7, $\alpha = -\delta \in \mathbb{R}$.

Recall that we denote the n-th iterate of F, that is F composed with itself n times, by F^n and given a point $P = (z_0, w_0) \in \mathbb{C}^2$ and $n \in \mathbb{N}$ we denote its iterates by $F^n(P) =: (z_n, w_n)$.

In this section we give an explicit expression for the iterates of F and their formal limit. A direct computation (compare with [BSZ23]) shows that

$$F^{2n}(z_0,w_0) = (-\delta)^n\Big(z_0 + \sum_{j=1}^{n}(-\delta)^{-j} f(z_{2j-1}), w_0 + \sum_{j=1}^{n}(-\delta)^{-j} f(z_{2j-2})\Big)$$

$$F^{2n+1}(z_0,w_0) = (-\delta)^n\Big(-\delta\big(w_0 + \sum_{j=1}^{n+1}(-\delta)^{-j} f(z_{2j-2})\big), z_0 + \sum_{j=1}^{n}(-\delta)^{-j} f(z_{2j-1})\Big).$$

For $n \in \mathbb{N}$, define the following holomorphic functions from $\mathbb{C}^2$ to $\hat{\mathbb{C}}$

$$\Delta_1^n(z_0,w_0) := \sum_{j=1}^{n}(-\delta)^{-j} f(z_{2j-1})$$

$$\Delta_2^n(z_0,w_0) := \sum_{j=1}^{n}(-\delta)^{-j} f(z_{2j-2})$$

With this notation the iterates of F take the form

$$F^{2n}(z_0,w_0) = (-\delta)^n\Big((z_0 + \Delta_1^n(z_0,w_0), w_0 + \Delta_2^n(z_0,w_0)\Big) \tag{3.2.2}$$

$$F^{2n+1}(z_0,w_0) = (-\delta)^n\Big(-\delta w_0 - \delta\Delta_2^{n+1}(z_0,w_0), z_0 + \Delta_1^n(z_0,w_0)\Big). \tag{3.2.3}$$

Let

$$\Delta_1(z,w) = \Delta_1^\infty(z,w) := \lim_{n\to\infty} \Delta_1^n(z,w)$$

$$\Delta_2(z,w) = \Delta_2^\infty(z,w) := \lim_{n\to\infty} \Delta_2^n(z,w)$$

$$\Delta(z,w) = \max\left(|\Delta_1(z,w)|, |\Delta_2(z,w)|\right).$$

Notice that Δ_1, Δ_2 are holomorphic functions to $\hat{\mathbb{C}}$ on open sets on which they are well defined.

We can deduce the following formal limits.

$$h_1(z,w) := \lim_{n\to\infty} \frac{z_{2n}}{w_{2n}} = \frac{z+\Delta_1(z,w)}{w+\Delta_2(z,w)} \tag{3.2.4}$$

$$h_2(z,w) := \lim_{n\to\infty} \frac{z_{2n+1}}{w_{2n+1}} = \frac{-\delta(w+\Delta_2(z,w))}{z+\Delta_1(z,w)} = -\frac{\delta}{h_1(z,w)}. \tag{3.2.5}$$

We have that h_1, h_2 are holomorphic functions to $\hat{\mathbb{C}}$ on open sets on which Δ_1 and Δ_2 are holomorphic functions to $\hat{\mathbb{C}}$. We will show in Proposition 3.3.7 that $h_1 \neq h_2$.

3.3 Existence of Fatou Components and Rank of the Limit Functions

In this section we construct a forward invariant open set W on which the even and the odd iterates converge, from which we deduce the existence of Fatou components. We then show that the limit functions have rank 1 on such Fatou components.

For $A \subseteq \mathbb{C}^2$ and $a, b \in \{+,-\}$ define

$$A^{ab} := A \cap \{(z,w) \in \mathbb{C}^2 : a\operatorname{Re}(z) > 0,\ b\operatorname{Re}(w) > 0\}. \tag{3.3.1}$$

If $A \cap (\{\operatorname{Re} z = 0\} \cup \{\operatorname{Re} w = 0\}) = \varnothing$ then $A = \bigcup_{a,b\in\{+,-\}} A^{ab}$.

We start by defining a set on which we have control on the dynamics. Let

$$\mathcal{S} := \{z \in \mathbb{C} : |\operatorname{Im}(z)| < |\operatorname{Re}(z)|\} \subset \mathbb{C}$$
$$S := \mathcal{S} \times \mathcal{S} \subset \mathbb{C}^2$$

A representation of $\mathcal{S}$, highlighted in red, is provided in Figure 3.1 .

3.3.1 Lemma
Let $z \in \mathcal{S}$, then $|f(z)| = |e^{-z^2}| < 1$.

Proof. If $z \in \mathcal{S}$, then $z^2 \in \mathbb{H}$ and hence $\operatorname{Re}(z^2) > 0$ from which we have $|e^{-z^2}| = e^{-\operatorname{Re} z^2} < 1$. □

3.3.2 Lemma (Orbits contained in S)
For any $P = (z_0, w_0) \in S^{ab}$ such that $F(P) \in S$ and $|\operatorname{Re} w_0| > \frac{1}{\delta}$ we have that $F(P) \in S^{(-b)a}$.

From now on assume that $F^n(P) \in S$ for all $n \in \mathbb{N}$. Then

$$F^{2n}(z_0, w_0) \to h_1(z_0, w_0)$$
$$F^{2n+1}(z_0, w_0) \to h_2(z_0, w_0).$$

Fix $\lambda > 0$ and assume also that $|\operatorname{Re} z_0|, |\operatorname{Re} w_0| > \frac{1+\lambda}{\delta-1}$. Then

$$|\operatorname{Re} z_{2n-1}| = |\operatorname{Re} w_{2n}| > |\operatorname{Re} w_0| + n\lambda \tag{3.3.2}$$
$$|\operatorname{Re} z_{2n}| = |\operatorname{Re} w_{2n+1}| > |\operatorname{Re} z_0| + n\lambda. \tag{3.3.3}$$

Proof. By hypothesis, $F(P) \in S$ hence $F(P) \in S^{\tilde{a}\tilde{b}}$ for some $\tilde{a}, \tilde{b} \in \{+,-\}$. Since $\operatorname{Re} w_1 = \operatorname{Re} z_0$ we have that $\tilde{b} = a$. Moreover $\operatorname{Re} z_1 = -\delta \operatorname{Re} w_0 + \operatorname{Re}(e^{-z_0^2})$ and since $P \in S$, $|\operatorname{Re}(e^{-z_0^2})| < 1$ by

Lemma 3.3.1. Hence the sign of $\operatorname{Re} z_1$ is opposite to the sign of $\operatorname{Re} w_0$ provided $|\operatorname{Re} w_0| > \frac{1}{\delta}$, and $\tilde{a} = -b$ as required.

Assume from now on that $F^n(P) \in S$ for all $n \in \mathbb{N}$. It follows that $z_n \in \mathcal{S}$ for all $n \in \mathbb{N}$ and hence by Lemma 3.3.1 $|f(z_n)| < 1$ for all $n \in \mathbb{N}$. Since $\delta > 2$ this implies

$$\Delta(z_0, w_0) < \sum_{j=1}^{\infty} \delta^{-j} < 1 \text{ whenever } F^n(z_0, w_0) \in S \text{ for all } n \in \mathbb{N}, \tag{3.3.4}$$

which implies convergence of the even and odd iterates of F according to the expression in (3.2.2), (3.2.3).

We now prove (3.3.2), (3.3.3). Using the expression of F and since $P \in S$, by Lemma 3.3.1 we have $|\operatorname{Re} z_1| \geq \delta|\operatorname{Re} w_0| - |e^{-z_0^2}| \geq \delta|\operatorname{Re} w_0| - 1$ which is larger than $|\operatorname{Re} w_0| + \lambda$ if $|\operatorname{Re} w_0| > \frac{1+\lambda}{\delta-1}$. It follows that

$$|\operatorname{Re} z_1| > |\operatorname{Re} w_0| + \lambda \tag{3.3.5}$$

$$|\operatorname{Re} z_2| > |\operatorname{Re} w_1| + \lambda = |\operatorname{Re} z_0| + \lambda, \tag{3.3.6}$$

where the claim for z_2 follows because $w_1 = z_0$. The more general formula follows by induction, using that $F^n(P) \in S$ for all $n \in \mathbb{N}$. □

As a corollary, the following conclusion is derived.

3.3.3 Corollary

Let $A \subset S$ be forward invariant. If $P = (z_0, w_0) \in A$ such that $|\operatorname{Re} w_0| > \frac{1}{\delta}$ then Lemma 3.3.2 holds for $\lambda = 1$, in particular, if $P \in A^{ab}$ then $F(P) \in A^{(-b)a}$.

For $R > 0$ and $0 < k < 1$ define the sets (See Figure 3.1.)

$$\mathcal{W}_{k,R} := \{z \in \mathbb{C} : |\operatorname{Im} z| < k|\operatorname{Re} z|,\ |\operatorname{Re} z| > R\} \subset \mathbb{C}$$
$$W_{k,R_1,R_2} := \mathcal{W}_{k,R_1} \times \mathcal{W}_{k,R_2} \subset \mathbb{C}^2.$$

Observe that $\mathcal{W}_{k,R} \subset \mathcal{S}$ and that $\mathcal{W}_{1,0} = \mathcal{S}$.

3.3.4 Lemma

Let $n \in \mathbb{N}$, and let $(z_0, w_0) \in W_{k,R_1,R_2}$. Let $0 < k < \tilde{k} < 1$. If $R_2 > \frac{2}{\delta(\tilde{k}-k)}$ then

$$\left|\frac{\operatorname{Im} z_1}{\operatorname{Re} z_1}\right| < \tilde{k} \text{ and } \left|\frac{\operatorname{Im} w_1}{\operatorname{Re} w_1}\right| < k.$$

Proof. Let $(z_0, w_0) \in W_{k,R_1,R_2}$. The claim for w_1 is immediate because $w_1 = z_0$. Using the expression of F, the triangular inequality, the estimate in Lemma 3.3.1 and the fact that $|\operatorname{Im} w_0| < k|\operatorname{Re} w_0|$ we have

$$\left|\frac{\operatorname{Im} z_1}{\operatorname{Re} z_1}\right| < \frac{\delta|\operatorname{Im} w_0| + 1}{\delta|\operatorname{Re} w_0| - 1} < \frac{k\delta|\operatorname{Re} w_0| + 1}{\delta|\operatorname{Re} w_0| - 1}.$$

Setting the resulting expression to be less than $\tilde{k}$ we get $|\operatorname{Re} w_0| > \frac{1+\tilde{k}}{\delta(\tilde{k}-k)}$. Since $\tilde{k} < 1$, it is enough to take $|\operatorname{Re} w_0| > \frac{2}{\delta(\tilde{k}-k)}$ as required. □

Let $k_n := 1 - \frac{1}{n+2}$ and $R_n := (\frac{\delta}{2})^{\frac{n}{2}} R_0$ for $R_0 > 2$ sufficiently large depending only on δ (see (3.3.7)). Let $R_{-1} := R_0$ and set

$$W_n := \begin{cases} W^{++}_{k_n, R_n, R_{n-1}} & \text{if } n = 0 \mod 4 \\ W^{-+}_{k_n, R_n, R_{n-1}} & \text{if } n = 1 \mod 4 \\ W^{--}_{k_n, R_n, R_{n-1}} & \text{if } n = 2 \mod 4 \\ W^{+-}_{k_n, R_n, R_{n-1}} & \text{if } n = 3 \mod 4 \end{cases}.$$

and define

$$W := \bigcup_{n \in \mathbb{N}} W_n.$$

Refer to Figure 3.1 for a visual representation.

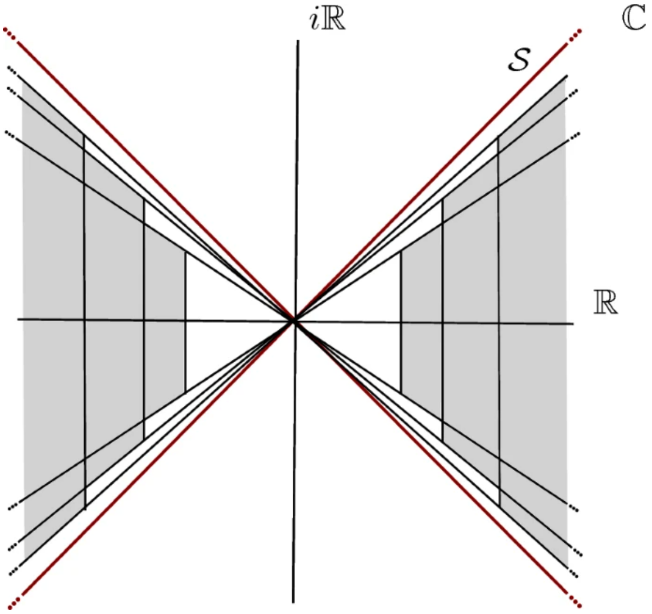

Figure 3.1: A sketch in $\mathbb{C}$ of the set $\mathcal{S}$ and three of the sets $\mathcal{W}_{k_n, R_n}$. The set W is obtained by taking appropriate products of (parts of) the sets $\mathcal{W}_{k_n, R_n}$.

3.3.5 Proposition (Invariance of W)
The set W is open and $W \subset S$. For any $n \in \mathbb{N}$ we have that $F(W_n) \subset W_{n+1}$, hence W is forward invariant. The set W consists of four connected components W^{ab} with $a, b \in \{+, -\}$ and $F(W^{ab}) \subset W^{(-b)a}$.

Proof. The fact that W is open and $W \subset S$ follows from the definition. Fix $n \in \mathbb{N}$. Let $(z_0, w_0) \in W_n$ and let (z_1, w_1) be its image. Since $w_1 = z_0$, the signs of $\operatorname{Re} w_1, \operatorname{Re} z_0$ are the same, and we have that $|\operatorname{Re} w_1| = |\operatorname{Re} z_0| > R_n$ and that

$$\left| \frac{\operatorname{Im} w_1}{\operatorname{Re} w_1} \right| = \left| \frac{\operatorname{Im} z_0}{\operatorname{Re} z_0} \right| < k_n < k_{n+1}.$$

Hence to show that $F(W_n) \subset W_{n+1}$ it is enough to see that $|\operatorname{Re} z_1| > R_{n+1}$ and that

$$\left| \frac{\operatorname{Im} z_1}{\operatorname{Re} z_1} \right| < k_{n+1}.$$

Let $\lambda_n := R_{n+1} - R_{n-1}$. Since $P \in S$, by (3.3.5) we have that

$$|\operatorname{Re} z_1| > |\operatorname{Re} w_0| + \lambda_n > R_{n-1} + \lambda_n = R_{n+1}$$

provided $R_{n-1} > \frac{1+\lambda_n}{\delta-1}$. Substituting the expression for λ_n we get $R_{n+1} < \delta R_{n-1} - 1$. Substituting the expression for R_{n+1} and R_{n-1} we get

$$\delta^{\frac{n+1}{2}} R_0 > 2^{\frac{n+1}{2}}$$

which is satisfied because $\delta > 2$, provided $R_0 \geq 1$. This gives $|\operatorname{Re} z_1| > R_{n+1}$.

We now prove $\left|\frac{\operatorname{Im} z_1}{\operatorname{Re} z_1}\right| < k_{n+1}$. By Lemma 3.3.4, it is enough to check that $R_{n-1} > \frac{2}{\delta(k_{n+1}-k_n)} = \frac{2(n+2)(n+3)}{\delta}$, that is

$$R_0 > 2^{\frac{n+1}{2}} \delta^{-\frac{n+1}{2}} (n+2)(n+3) \text{ for all } n \in \mathbb{N}. \tag{3.3.7}$$

Since the function on the right hand side is bounded in n for any $\delta > 2$ (in fact, it tends to 0 as $n \to \infty$), such R_0 exists and depends only on δ.

Finally, for any $(z, w) \in W$ we have $\operatorname{Re} z, \operatorname{Re} w \neq 0$, so the sets W^{ab} are well defined. By construction, $W_n \cap W_{n+4} \neq \varnothing$ so each W^{ab} is connected. It follows that W consists of 4 connected components W^{ab}. Since W is forward invariant and contained in S, the orbits of points in W are contained in S hence Corollary 3.3.3 applies. □

Before the following proposition, we refer the reader to the equations defining h_1 and h_2, given in 3.2.4 and 3.2.5 respectively.

3.3.6 Proposition (Existence of Fatou components)
On each W^{ab} we have that

$$F^{2n} \to h_1, \ F^{2n+1} \to h_2 \text{ uniformly on compact subsets of } W^{ab}.$$

It follows that each W^{ab} is contained in a Fatou component that we denote by Ω^{ab}.

Proof. Since $W \subset S$ and since W is forward invariant by Proposition 3.3.5, (3.3.4) holds. Hence F^{2n} and F^{2n+1} converge uniformly on W to h_1, h_2 respectively, hence W is contained in the Fatou set. Since each W^{ab} is open and connected it is contained in a unique Fatou component that we denote by Ω^{ab}. □

We will see in Proposition 3.5.2 that in fact the components Ω^{ab} are all distinct and that the notation Ω^{ab} matches the definition of A^{ab} given in 3.3.1 for a general set A.

3.3.7 Proposition
Both h_1 and h_2 have rank 1 on W, and $h_1 \neq h_2$.

Proof. Recall that $\Delta(z, w) < 1$ on W by (3.3.4). It follows that by the explicit expression of F^{2n}, F^{2n+1}, the iterates of any point in W converge to the line at infinity. So $h_i(W) \subset \ell_\infty$, and h_1, h_2 either have generic rank 1 or are constants. Suppose by contradiction that $h_1 = c$ is constant. If $|c| \neq \infty$, then one has:

$$|z_0| - \Delta(z_0, w_0) \leq |z_0 + \Delta_1(z_0, w_0)| = |c|\,|w_0 + \Delta_2(z_0, w_0))| \leq |c||w_0| + |c|\Delta(z_0, w_0),$$

hence

$$|z_0| \leq |c||w_0| + (|c| + 1),$$

contradicting the fact that (z_0, w_0) could be any point in W, which is unbounded in the z direction for any choice of w. If $c = \infty$, we have $|w_0| \leq 1$, again a contradiction. It follows that $h_1 \neq h_2$. Indeed, $h_1 \cdot h_2 = -\delta$ is constant, if we had $h_1 = h_2$ it would follow that h_1^2 (and hence h_1) would be constant as well, contradicting the argument above. □

3.4 Construction of an Absorbing Set

Let Ω^{ab} with $a, b \in \{+, -\}$ be the Fatou components defined in Proposition 3.3.6 and let

$$\Omega := \bigcup_{ab} \Omega^{ab}.$$

Since each Ω^{ab} is connected, Ω consists of at most 4 Fatou components. This section is devoted to find an absorbing set W_I for Ω under F.

An absorbing set intuitively serves as a "buffer zone" that captures and confines the behavior of points of Ω under the iteration of F.

An absorbing set ensures that as points are repeatedly transformed under F, they eventually enter this set and remain within it.

This means that any point in the larger domain Ω will, after sufficient iterations, be drawn into the absorbing set. Consequently, the absorbing set provides a controlled environment where we can study the dynamics, stability, and structure of the system.

The existence of the absorbing set W_I will be used in Section 3.6 to show that the Fatou components Ω^{ab} are all distinct and to describe both their limit sets and their geometric structure. We use an argument based on harmonic functions used also in [For04], [ABFP19], [BSZ23].

We will now provide the formal definition of an absorbing set.

3.4.1 Definition (Absorbing sets)**.** A set A is *absorbing* for an open set $\Omega \supset A$ under a map F if for any compact $K \subset \Omega$ there exists $N > 0$ such that

$$F^n(K) \subset A \text{ for all } n \geq N.$$

If A is absorbing for Ω, then $\Omega = \bigcup_n F^{-n}(A)$.

Fix $C \geq 1$ and let

$$I = I(C) := \{z \in \mathbb{C} \mid |\operatorname{Im} z|^2 < |\operatorname{Re} z|^2 - C^2\} = \{z \in \mathbb{C} \,:\, \operatorname{Re}(z^2) > C^2\} \subset \mathcal{S}.$$

Notice that if $z \in I$, then $|\operatorname{Re} z| > C$.

Define

$$W_I = W_I(C) := \Omega \cap \{(z, w) \in \mathbb{C}^2 : F^n(z, w) \in I \times I \text{ for all } n \geq 0\}.$$

3.4.2 Proposition
We have that $W_I^{ab} \neq \varnothing$ for all $a, b \in \{+, -\}$. For every $a, b \in \{+, -\}$,

$$F(W_I^{ab}) \subset W_I^{(-b)a}.$$

The sets $W_I^{++} \cup W_I^{--}$, $W_I^{-+} \cup W_I^{+-}$ are both forward invariant under F^2. Moreover F^{2n} and F^{2n+1} are convergent on W_I.

Proof. Each W_I^{ab} contains the set $\{(z, w) \in \mathbb{C}^2 : a \operatorname{Re} z > M,\ b \operatorname{Re} w > M,\ \operatorname{Im} z = \operatorname{Im} w = 0\}$ for M sufficiently large. The set $W_I \subset S$ is forward invariant hence Corollary 3.3.3 applies. Convergence of even and odd iterates follows by (3.3.4). ☐

It will turn out that W_I is open as well (Proposition 3.4.9).

The rest of this section is devoted to proving the following proposition.

3.4.3 Proposition
The set W_I is absorbing for Ω under F, that is,

$$\Omega = \bigcup_n F^{-n}(W_I) =: \mathcal{A}_I.$$

Let

$$\mathcal{X} := \{(z,w) \in \Omega \,:\, h_1(z,w) = 0, \infty\}.$$

Since $\mathcal{X}$ is an analytic set, being the union of the 0-set and the ∞-set of a meromorphic function, it is locally a finite union of 1-complex-dimensional varieties (see [Chi89]).

Let K be a compact subset of $\Omega \setminus \mathcal{X}$, hence $h_i(P) \neq 0, \infty$ for all $P \in K$, and $i = 1, 2$. Define

$$M := \max_{K,i} |h_i| < \infty. \tag{3.4.1}$$

Note that $M > 1$ because $h_2 = -\frac{\delta}{h_1}$ and $\delta > 1$. By Corollary 2.3 in [BSZ23] if $\varepsilon > 0$ is sufficiently small there exists a constant c such that for every $(z_0, w_0) \in K$

$$|z_n| \leq c(M+\varepsilon)^n. \tag{3.4.2}$$

$$|w_n| = |z_{n-1}| \leq c(M+\varepsilon)^{n-1}. \tag{3.4.3}$$

The proof of Proposition 3.4.3 relies on the following technical lemma. Recall that for $P = (z_0, w_0)$, we write $F^n(P) = (z_n, w_n)$.

3.4.4 Lemma
Define the sequence of harmonic functions u_n from Ω to $\mathbb{R}$ as

$$u_n(z_0, w_0) := \frac{-\operatorname{Re}(z_n^2)}{n}. \tag{3.4.4}$$

Then

1. *Let $K \subset \Omega$ compact. Then there exists $M = M(K)$ and $N \in \mathbb{N}$ such that $u_n \leq \log M$ on K for $n > N$;*
2. *$u_n \to -\infty$ uniformly on compact subsets of W;*
3. *If $P \in \Omega \setminus \mathcal{A}_I$, for every $\varepsilon > 0$ there is a subsequence $n_k \to \infty$ such that $u_{n_k}(P) \geq -\varepsilon$.*

3.4.5 Lemma
Let $z \in \mathbb{C}$, $k < 1$. If

$$\left|\frac{\operatorname{Im} z}{\operatorname{Re} z}\right| \leq k < 1 \textit{ then} \tag{3.4.5}$$

$$\left|\frac{\operatorname{Im} z^2}{\operatorname{Re} z^2}\right| \leq \frac{2k}{1-k^2}. \tag{3.4.6}$$

Proof. Let $z = re^{i\theta}$ satisfying (3.4.5); then $|\tan\theta| \leq k < 1$. Hence since $z^2 = r^2 e^{2i\theta}$,

$$\left|\frac{\operatorname{Im} z^2}{\operatorname{Re} z^2}\right| = |\tan(2\theta)| = \left|\frac{2\tan\theta}{1-\tan^2\theta}\right| \leq \frac{2k}{1-k^2}.$$

□

The following fact is certainly known, however we give a proof in the Appendix 5, Section 5.5. Given a set A, let $\mathring{A}$ denote its interior.

3.4.6 Proposition

Let L be a compact set and H be an analytic subset of dimension one of $\mathbb{C}^2$. For any compact K s.t. $K \subset \mathring{L}$ there exists $\eta = \eta(K, L, H)$ such that for any u harmonic defined in a neighborhood of L and such that

$$u \leq \alpha < \infty \text{ on } L \setminus (\eta\text{−neighborhood of } H)$$

we have

$$u \leq \alpha \text{ on } K$$

Proof of Lemma 3.4.4.

1. Let K be a compact subset of Ω. Let η as obtained by applying Proposition 3.4.6 to a slightly larger compact set $L \subset \Omega$ and to the analytic set $\mathcal{X}$. Let $U_\eta(\mathcal{X})$ be an η-neighborhood of $\mathcal{X}$. In view of Proposition 3.4.6 it is enough to prove that there exists $N \in \mathbb{N}$ such that $u_n \leq \log M$ for $n > N$ and for some M on the set
$$\tilde{K} := K \setminus U_\eta(\mathcal{X})$$
which is a compact subset of $\Omega \setminus \mathcal{X}$. Hence it is enough to prove the claim for any K compact subset of $\Omega \setminus \mathcal{X}$.

   Fix $\varepsilon > 0$ sufficiently small and let M, c be as in (3.4.2) and (3.4.3) for K. Suppose that there exists a subsequence (n_j) and points $(z, w) = (z(j), w(j)) \in K$ such that
$$-\frac{\operatorname{Re}(z_{n_j}^2)}{n_j} > \beta$$
for some β. We will show that $\beta \leq M$.

   Using (3.4.2) and (3.4.3) we have that
$$c(M+\varepsilon)^{n_j+1} \geq |z_{n_j+1}| = |e^{-z_{n_j}^2} - \delta w_{n_j}| \geq |e^{-z_{n_j}^2}| - \delta |w_{n_j}| \geq$$
$$\geq e^{-\operatorname{Re}(z_{n_j}^2)} - \delta c(M+\varepsilon)^{n_j-1} \geq e^{\beta n_j} - \delta c(M+\varepsilon)^{n_j-1}.$$
Hence, using $M > 1$ and $\varepsilon > 0$ sufficiently small,
$$e^{\beta n_j} \leq \delta c(M+\varepsilon)^{n_j-1} + c(M+\varepsilon)^{n_j+1} \leq c(\delta+1)(M+\varepsilon)^{n_j+1}.$$
Then
$$\beta \leq \frac{\log\big(c(\delta+1)\big)}{n_j} + \frac{n_j+1}{n_j}\log(M+\varepsilon) \to \log M$$
as $n_j \longrightarrow \infty$ and $\varepsilon \longrightarrow 0$.

2. It is enough to show that $u_n(z_0, w_0) \to -\infty$ for any point $(z_0, w_0) \in W$ and it will follow for any compact subset of W. Since W is forward invariant, $F^n(z_0, w_0) \subset W \subset S$ for all $n \in \mathbb{N}$ and $\Delta(z_0, w_0) < 1$ by (3.3.4). Using the explicit expression for iterates of F given by (3.2.2), (3.2.3) we have
$$|z_n^2| = |z_n|^2 = \begin{cases} \delta^n |z_0 + \Delta_1^{n/2}(z_0, w_0)|^2 \geq \delta^n |z_0 - 1|^2 & \text{if n even;} \\ \delta^{(n+1)} |w_0 + \Delta_2^{(n+1)/2}(z_0, w_0)|^2 \geq \delta^{n+1} |w_0 - 1|^2 & \text{if n odd.} \end{cases}$$

In both cases, since $|z_0|, |w_0| > R_0 > 2$ we obtain $|z_n^2| \geq \delta^n$. Since $W = \bigcup_j W_j$ as defined in Section 3.3, $(z_0, w_0) \in W_j$ for some j, hence by Proposition 3.3.5,

$$F^n(z_0, w_0) \in W_{j+n} \text{ for all } n \in \mathbb{N},$$

hence $\left|\frac{\operatorname{Im} z_n}{\operatorname{Re} z_n}\right| \leq k_{j+n} < 1$ and by Lemma 3.4.5 we obtain

$$\left|\frac{\operatorname{Im} z_n^2}{\operatorname{Re} z_n^2}\right| \leq \frac{2k_{j+n}}{1-k_{j+n}^2} =: \alpha_n \sim n \text{ as } n \to \infty,$$

where the estimate $\alpha_n \sim n$ as $n \to \infty$ is computed using the explicit expression for k_{j+n}. It follows that

$$\delta^n \leq |z_n^2| = \sqrt{(\operatorname{Re}(z_n^2))^2 + (\operatorname{Im}(z_n^2))^2} \leq \operatorname{Re}(z_n^2)\sqrt{1+\alpha_n^2}$$

hence $\operatorname{Re}(z_n^2) \geq \frac{\delta^n}{\sqrt{1+\alpha_n^2}} \sim \frac{\delta^n}{n} \geq \delta^{n/2}$ for n large. Finally

$$u_n(z_0, w_0) = -\frac{\operatorname{Re}(z_n^2)}{n} \leq -\frac{\delta^{n/2}}{n} \to -\infty \text{ as } n \to \infty$$

3. Suppose by contradiction that there exists $P = (z_0, w_0) \in \Omega \setminus \mathcal{A}_I$, $\varepsilon > 0$ and $N \in \mathbb{N}$ such that

$$u_n(z_0, w_0) = \frac{-\operatorname{Re} z_n^2}{n} < -\varepsilon \text{ for all } n \geq N.$$

Hence there exists $N' > N$ depending on ε, C (where C is the constant used to define W_I) such that

$$\operatorname{Re}(z_n^2) > \varepsilon n > C^2 \text{ for all } n \geq N'.$$

Since $w_n = z_{n-1}$ and since $P \in \Omega$ for hypothesis, we have that $F^n(P) \in I \times I$ for all $n \geq N'$ hence $P \in F^{-n}(W_I) \subset \mathcal{A}_I$, a contradiction.

This concludes the proof. □

3.4.7 Lemma (Good holomorphic disks)
Let $P \in \Omega$, W as before. Then there exists $\varphi : \overline{\mathbb{D}} \to \Omega$ holomorphic in a neighborhood of $\overline{\mathbb{D}}$ such that

- $\varphi(0) = P$
- $\varphi(\mathbb{D}) \Subset \Omega$ *and* $\partial\varphi(\mathbb{D})$ *is analytic*
- *The one-dimensional Lebesgue measure of* $\partial\varphi(\mathbb{D}) \cap W$ *is greater than* 0.

Proof. Since W is open it is enough to have $\varphi(\mathbb{D}) \cap W \neq \varnothing$ to ensure that the one-dimensional Lebesgue measure of $\partial\varphi(\mathbb{D}) \cap W$ is greater than 0. Let $a, b \in \{+, -\}$ such that $P \in \Omega^{ab}$. Since $W^{ab} \neq \varnothing$ for all $a, b \in \{+, -\}$ there exists $Q \in W^{ab}$. Since Ω^{ab} is connected and open there exists a simple real analytic curve passing through P and Q in Ω^{ab}. Complexifying this curve we obtain a holomorphic disc passing through P that we can write as $\varphi(\mathbb{D})$ for some φ holomorphic defined in a neighborhood of $\overline{\mathbb{D}}$. Up to precomposing φ with a Moebius transformation we can assume that $P = \varphi(0)$. □

In our proof, we are going to use the mean value property for the harmonic functions u_N.

3.4.8 Lemma (Mean value property for holomorphic disks)
Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk and $\varphi : \overline{\mathbb{D}} \to \Omega$ be a holomorphic map. Let u be harmonic on the holomorphic open disk $D = \varphi(\mathbb{D})$ and continuous up to the boundary of $\mathbb{D}$. Let $P_0 := \varphi(0)$. Then

$$u(P_0) = \frac{1}{2\pi}\int_{\partial\mathbb{D}} u(\zeta)|\varphi'(\zeta)|^{-1} d\zeta$$

Proof. Consider the function $u \circ \varphi : \overline{\mathbb{D}} \to \mathbb{R}$. First, note that it is harmonic on $\mathbb{D}$ and continuous up to the boundary. Indeed if $u : D \to \mathbb{R}$ is $\mathcal{C}^2$-smooth, then we can explicitly compute its Laplacian

$$\nabla^2(u \circ \varphi) = \nabla^2(u)|\varphi'|^2 = 0$$

while if u is not $\mathcal{C}^2$-smooth, the result follows by approximating u with harmonic smooth functions.

Hence for $u \circ \varphi$ the classical Mean Value Property holds. By computing $u(P_0)$ we get

$$u(P_0) \,=\, u(\varphi(0)) \,=\, \frac{1}{2\pi}\int_{\partial\mathbb{D}} u(\varphi(\eta))\,d\eta \,=\, \frac{1}{2\pi}\int_{\partial\mathbb{D}} u(\zeta)\,|\varphi'(\zeta)|^{-1}\,d\zeta. \tag{3.4.7}$$

□

Proof of Proposition 3.4.3. Let $P \in \Omega \setminus \mathcal{A}_I$ and $D := \varphi(\mathbb{D})$ where φ is given by Lemma 3.4.7. Let μ be the pushforward under φ of the one-dimensional Lebesgue measure on $\partial\mathbb{D}$. Let K be a compact subset of W such that $\mu(K \cap \partial D) > 0$.

Let $\mu_{\text{good}} = \mu(\partial D \cap K) > 0$ and $\mu_{\text{bad}} = \mu(\partial D \cap (\Omega \setminus K))$. Since Ω contains D, $\partial D = (\partial D \cap K) \cup (\partial D \cap (\Omega \setminus K))$, and since K is compact and Ω is open, the sets in question are measurable.

By Lemma 3.4.4 for any given $\mathcal{M} > 0$ there exists N such that $u_N \le -\mathcal{M}$ on K, $u_N(P) \ge -\varepsilon$ for some $\varepsilon > 0$ since $P \in \Omega \setminus \mathcal{A}_I$, and $u_N \le \log M$ on $\overline{D}$ (with $M = M(\overline{D})$). By the Mean value property (3.4.7) for u_N we have

$$\begin{aligned}-\varepsilon \le u_N(P) = \frac{1}{2\pi}\int_{\partial D} u_N(\zeta)|\varphi'(\zeta)|d\zeta = \frac{1}{2\pi}\int_{\partial D\cap K} u_N(\zeta)|\varphi'(\zeta)|d\zeta + \frac{1}{2\pi}\int_{\partial D\cap(\Omega\setminus K)} u_N(\zeta)|\varphi'(\zeta)|d\zeta \le \\ \le \frac{1}{2\pi}\left(-\mathcal{M}\mu_{\text{good}} + \log M \mu_{\text{bad}}\right)\cdot \sup_{\partial\mathbb{D}} |\varphi'|^{-1}.\end{aligned}$$

Since $\mathcal{M}$ is arbitrarily large, this gives a contradiction. □

3.4.9 Proposition

The set W_I is open.

Proof. Let $P \in W_I$. We want to find $V \subset W_I$ neighborhood of P. Since $W_I \subset \Omega \cap (I \times I)$ which is open there is a neighborhood U of P which is compactly contained in $\Omega \cap (I \times I)$. Since W_I is absorbing for Ω under F there exists $N > 0$ such that

$$F^n(\overline{U}) \subset W_I \text{ for all } n \ge N. \tag{3.4.8}$$

As usual let us define $P_j := F^j(P)$; by definition of W_I, $P_j \subset I \times I$ for all $j \ge 0$, which is an open set. Hence for each $j \ge 0$ there is a neighborhood $U_j \subset I \times I$ of P_j. So up to making the U_j smaller, we can assume that $U_j \subset F^j(U)$.

Let

$$V := \bigcap_{j=0}^{N} F^{-j}(U_j) \subset U.$$

The set V is open since it is a finite intersection of open sets. We only need to check that $V \subset W_I$, or equivalently, that $F^j(V) \subset I \times I$ for all $j \ge 0$. For $j \le N-1$, this is true by definition, since $F^j(V) \subset U_j \subset I \times I$. For $j \ge N$, this is true by (3.4.8). Since $P \in V$ by construction, V is a neighborhood of P in W_I as required. □

In the following two sections, we explore the geometric structure of the domain Ω, showing that it consists of four distinct Fatou components, each biholomorphic to $\mathbb{H} \times \mathbb{H}$. Additionally, we examine the limit sets of Ω, analyzing their behavior with respect to the limit functions h_1 and h_2. This study offers insights into the topological and geometric properties of the space Ω.

3.5 Geometric Structure of Ω

In this section we show that Ω is the union of four disjoint Fatou components Ω^{ab}, $a, b \in \{+,-\}$, each of which is biholomorphic to $\mathbb{H} \times \mathbb{H}$.

We first show conjugacy of F to its linear part on Ω, and estimate the distance between the conjugacy and the identity map.

3.5.1 Proposition (Conjugacy)
F is conjugate to the linear map $L(z,w) = (-\delta w, z)$ on Ω via an injective holomorphic map φ. If P is such that $F^n(P) \in S$ for all $n \in \mathbb{N}$, then $\|(\varphi - Id)(P)\| < \sqrt{2}$. Finally, $\varphi(\Omega) \subset S$.

Proof. We first show that F is conjugate to L on W_I.

For $n \in \mathbb{N}$ let $\varphi_n : \mathbb{C}^2 \to \mathbb{C}^2$ be the automorphisms defined as

$$\varphi_n := L^{-n} \circ F^n.$$

If we show that the φ_n converge to a map φ uniformly on W_I we obtain that φ satisfies the functional equation $\varphi = L^{-1} \circ \varphi \circ F$ and hence is a conjugacy between F and L.

Computing L^{-n} and using the explicit expressions for the iterates of F we obtain

$$\varphi_{2n}(z,w) = \Big(z + \Delta_1^n(z,w), w + \Delta_2^n(z,w)\Big), \tag{3.5.1}$$

$$\varphi_{2n+1}(z,w) = \Big(z + \Delta_1^n(z,w), w + \Delta_2^{n+1}(z,w)\Big). \tag{3.5.2}$$

Both have the same formal limit

$$\varphi(z,w) = \Big(z + \Delta_1(z,w), w + \Delta_2(z,w)\Big).$$

If $P = (z,w) \in W_I$, then $F^n(P) = (z_n, w_n) \subset I \times I \subset S$ for all j, hence, by (3.3.4), we have that $\Delta(z,w) < 1$; in particular, $\Delta_1(z,w)$ and $\Delta_2(z,w)$ are convergent. Hence φ is a holomorphic map from W_I to $\varphi(W_I)$ (W_I is open by Proposition 3.4.9). Moreover, for any point (z,w) whose orbit is contained in S,

$$\Big\|(\varphi - \mathrm{Id})(z,w)\Big\| = \Big\|\Big(\Delta_1(z,w), \Delta_2(z,w)\Big)\Big\| < \sqrt{2}\Delta(z,w) < \sqrt{2}. \tag{3.5.3}$$

It follows that φ is open because W_I is an unbounded set, hence if φ had rank 0 or 1, $\|(\varphi - \mathrm{Id})\|$ could not be bounded on W_I. Hence the map φ is injective by Hurwitz Theorem (see [Kra01], Exercise 3 on page 310) because the maps φ_n are injective and their limit has rank 2. It follows that φ is a biholomorphism between W_I and $\varphi(W_I)$.

To extend φ to all of Ω recall that W_I is absorbing for Ω. So if $P \in \Omega$, we have that $F^k(P) \in W_I$ for some $k \in \mathbb{N}$, hence we can define $\varphi(P) = L^{-k} \circ \varphi \circ F^k(P)$. Since F is an automorphism, φ extends as a biholomorphism to Ω.

It remains to show that $\varphi(\Omega) \subseteq S$. By (3.5.3) we have that $\varphi(W_I)$ is contained in a $\sqrt{2}$ neighborhood U of W_I. Suppose by contradiction that there exists $Q = (z,w) \in \varphi(W_I) \setminus S$. Since W_I is forward invariant under F and φ is a conjugacy we have that $\varphi(W_I)$ is forward invariant under L. Up to considering $L(Q)$ if necessary, and since θ is such that $re^{i\theta} \notin S$, we can assume that $z = re^{i\theta} \notin \mathcal{S}$. By forward invariance $L^{2n}(Q) = ((-\delta^n)re^{i\theta}, (-\delta)^n w) \in \varphi(W_I)$.

Since $(-\delta)^n r$ tends to infinity, the distance of $L^{2n}(Q)$ from the boundary of S tends to infinity, hence so does the distance of $L^{2n}(Q)$ from $W_I \subset S$, contradicting $\varphi(W_I) \subset U$. Hence $\varphi(W_I) \subset S$. Since W_I is an absorbing set for Ω under F, $\varphi \circ F = L \circ \varphi$, and $\varphi(W_I)$ is completely invariant under L, we have that

$$\varphi(\Omega) = \varphi(\bigcup_{n \geq 0} F^{-n}(W_I)) = \bigcup_{n \geq 0} L^{-n}(\varphi(W_I)) \subset \varphi(W_I) \subset S. \tag{3.5.4}$$

□

We now present the geometric structure of Ω in Proposition 3.5.2, the proof of which will be provided later on.

3.5.2 Proposition (Geometry of Ω)
Ω consists of four distinct connected components, each of which is biholomorphic to $\mathbb{H} \times \mathbb{H}$, and which form a cycle of period 4.

We recall the following simple topological lemma. Here ∂ denotes the topological boundary.

3.5.3 Lemma
Let $A, B \subset \mathbb{C}^n$ be open, A connected. If $A \cap B \neq \varnothing$ and $\partial B \cap A = \varnothing$ then $A \subseteq B$.

Proof. Since $A \cap \partial B = \varnothing$ we can write

$$A = (A \cap B) \cup (A \setminus \overline{B}).$$

Both $A \cap B$ and $A \setminus \overline{B}$ are open and $A \cap B \neq \varnothing$ by assumption, so since A is connected, $A \setminus \overline{B} = \varnothing$. □

Recall also that if a set A is invariant under a map F, by continuity of the latter we have $F(\overline{A}) \subset \overline{A}$. The following lemma is also known.

3.5.4 Lemma
Let Ω_1, Ω_2 be two Fatou components for an automorphism F of $\mathbb{C}^2$. Then if $F(\Omega_1) \cap \Omega_2 \neq \varnothing$, $F(\Omega_1) = \Omega_2$.

Proof. We have that $F(\Omega_1) \subset \Omega_2$, indeed otherwise, $F(\Omega_1)$ would intersect the boundary of Ω_2 which is contained in the forward Julia set, and this is impossible because the Fatou set is completely invariant. On the other hand suppose for a contradiction that there is $P \in \Omega_2 \setminus F(\Omega_1)$. Then since $F(\Omega_1) \cap \Omega_2 \neq \varnothing$ and both $\Omega_1, F(\Omega_1)$ are connected there exists $Q \in \Omega_2 \cap F(\partial \Omega_1)$, which is impossible because $\partial \Omega_1$ is contained in the forward Julia set which is forward invariant. □

Observe that we could not simply use the same argument applied to F^{-1}, since the Fatou components for F and F^{-1} are, in general, different sets.

Proof of Proposition 3.5.2. We prove the claim by showing that Ω is biholomorphic to S. Since S has four connected components S^{ab} each of which is biholomorphic to $\mathbb{H} \times \mathbb{H}$, the same holds for Ω. Since by definition $\Omega = \bigcup_{a,b \in \{+,-\}} \Omega^{ab}$ and each Ω^{ab} is connected, these are exactly the connected components of Ω.

Recall the definition of the set $W \subset S$ from Section 3.3 and recall that it is forward invariant and contained in S. Hence (3.5.3) holds. Also recall that by (3.5.4) $\varphi(\Omega) \subset S$.

Let U be a $2\sqrt{2}$-neighborhood of ∂W. Fix $a, b \in \{+,-\}$. We want to apply Lemma 3.5.3 to the sets $A = W^{ab} \setminus \overline{U}$ and $B = \varphi(W^{ab})$. So we need to show that

- $A \cap B = (W^{ab} \setminus \overline{U}) \cap \varphi(W^{ab}) \neq \varnothing$
- $\partial B \cap A = \partial \varphi(W^{ab}) \cap (W^{ab} \setminus \overline{U}) = \varnothing$.

The second item is true because $\partial(\varphi(W^{ab})) \subset \varphi(\partial W^{ab}) \Subset U$ by (3.5.3). So we now show that $(W^{ab} \setminus \overline{U}) \cap \varphi(W^{ab}) \neq \varnothing$. Let $P \in W^{ab}$ such that the ball of radius $\sqrt{2}$ centered at P is contained in $W^{ab} \setminus \overline{U}$. This is possible because this set contains arbitrarily large balls. By (3.5.3), $\|P - \varphi(P)\| < \sqrt{2}$ hence $\varphi(P) \in W^{ab} \setminus \overline{U}$.

Hence, applying Lemma 3.5.3 we obtain that for each $a, b \in \{+,-\}$ we have $\varphi(W^{ab}) \supset (W^{ab} \setminus U)$ hence

$$\varphi(\Omega) \supset \varphi(W) \supset (W \setminus U).$$

We now show that this implies that $\varphi(\Omega) \supset S$. Notice that S can be written as

$$S = \{(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \in \mathbb{C}^2 : r_1, r_2 > 0, \text{ and for each } i = 1,2 \text{ either } |\theta_i| < \frac{\pi}{4} \text{ or } |\theta_i - \pi| < \frac{\pi}{4}\}.$$

Fix $\alpha < \frac{\pi}{4}$. By definition of W there exists $R = R(\alpha)$ such that $W \setminus U$ contains the set

$$W \setminus U \supset X_{\alpha,R} := \{(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) : r_1, r_2 > R \text{ and for each } i = 1,2 \text{ either } |\theta_i| < \alpha \text{ or } |\theta_i - \pi| < \alpha\}.$$

Hence $\varphi(\Omega) \supset \varphi(W) \supset (W \setminus U) \supset X_{\alpha,R}$.

By the explicit form of L, $\bigcup_{j \geq 0} L^{-j} X_{\alpha,R} = X_{\alpha,0}$. Hence by backward invariance of $\varphi(\Omega)$ under L we have that

$$\varphi(\Omega) \supset \bigcup_{j \geq 0} L^{-j} X_{\alpha,R} = X_{\alpha,0} \text{ for every } \alpha < \frac{\pi}{4}.$$

It follows that

$$\varphi(\Omega) \supset \bigcup_{\alpha < \frac{\pi}{4}} X_{\alpha,0} = S.$$

Hence $\varphi(\Omega) = S$.

It remains to show that the Fatou components Ω^{ab} with $a, b \in \{+,-\}$ form a cycle of period four, more precisely, that

$$F(\Omega^{ab}) = \Omega^{(-b)a} \text{ for all } a, b \in \{+,-\}. \tag{3.5.5}$$

By definition $\Omega^{ab} \supset W^{ab}$ and by Lemma 3.3.2, $F(W^{ab}) \cap W^{(-b)a} \neq \varnothing$. Hence $F(\Omega^{ab}) \cap \Omega^{(-b)a} \neq \varnothing$. By Lemma 3.5.4, $F(\Omega^{ab}) = \Omega^{(-b)a}$.

□

3.6 Limit Sets

Let $\mathbb{H}$ and $-\mathbb{H}$ denote the right and left half plane respectively. In this section we show the following.

3.6.1 Proposition (Limit set for Ω)
We have that

$$\begin{array}{llll} h_1(\Omega^{ab}) = \mathbb{H} & \textit{and} & h_2(\Omega^{ab}) = -\mathbb{H} & \textit{if } a = b \\ h_1(\Omega^{ab}) = -\mathbb{H} & \textit{and} & h_2(\Omega^{ab}) = \mathbb{H} & \textit{if } a \neq b. \end{array}$$

Let W be as defined in Section 3.3 and W_I as defined in Section 3.4. Since both are forward invariant and contained in S we have that, for any $a, b \in \{+,-\}$, $F(W^{ab}) \subset W^{(-b)a}$ and $F(W_I^{ab}) \subset W_I^{(-b)a}$. Compare with Lemma 3.3.2 and Corollary 3.3.3.

We first study the image of W_I^{ab} under h_1, h_2.

3.6.2 Lemma

$$\begin{array}{llll} h_1(W_I^{ab}) \subset \mathbb{H} & \textit{and} & h_2(W_I^{ab}) \subset -\mathbb{H} & \textit{if } a = b \\ h_1(W_I^{ab}) \subset -\mathbb{H} & \textit{and} & h_2(W_I^{ab}) \subset \mathbb{H} & \textit{if } a \neq b \end{array}$$

Proof. Recall that $h_1(z_0, w_0) = \lim_{n\to\infty} \frac{z_{2n}}{w_{2n}}$. Let $I_+ := \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ and $I_- := \left(\frac{3}{4}\pi, \frac{5}{4}\pi\right)$.

For $a, b \in \{+, -\}$ and $(z, w) \in W_I^{ab}$, then $\arg(z) \in I_a$ and $\arg(w) \in I_b$. Hence $\arg\left(\frac{z}{w}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ if $a = b$, and $\arg\left(\frac{z}{w}\right) \in \left(\frac{\pi}{2}, \frac{3}{2}\pi\right)$ if $a \neq b$. Since $F^2(W_I^{++} \cup W_I^{--}) \subset W_I^{++} \cup W_I^{--}$, If $(z, w) \in W_I^{++} \cup W_I^{--}$ then all of its even iterates $(z_{2n}, w_{2n}) \in W_I^{++} \cup W_I^{--}$, hence by taking the limit $h_1(W_I^{++}), h_1(W_I^{--}) \subset \overline{\mathbb{H}}$. Similarly if $(z, w) \in W_I^{+-} \cup W_I^{-+}$ then all of its even iterates $(z_{2n}, w_{2n}) \in W_I^{+-} \cup W_I^{-+}$, hence $h_1(W_I^{+-}), h_1(W_I^{-+}) \subset \overline{-\mathbb{H}}$. The analogous results for h_2 hold by observing that $h_2 = \frac{-\delta}{h_1}$. Since W_I is open by Proposition 3.4.9, its image under a holomorphic map of maximal rank is open, hence we can replace $\overline{\mathbb{H}}, \overline{-\mathbb{H}}$ by $\mathbb{H}, -\mathbb{H}$. ☐

3.6.3 Lemma

$$\begin{array}{cccc} \mathbb{H} \subset h_1(W^{ab}) & \textit{and} & \mathbb{H} \subset h_2(W^{ab}) & \textit{if } a = b \\ -\mathbb{H} \subset h_1(W^{ab}) & \textit{and} & \mathbb{H} \subset h_2(W^{ab}) & \textit{if } a \neq b. \end{array}$$

Before proving Lemma 3.6.3 let us see how Lemmas 3.6.2 and 3.6.3 imply Proposition 3.6.1.

Proof of Proposition 3.6.1. We prove the claims for h_1; for $h_2 = \frac{-\delta}{h_1}$, it follows by symmetry.

Clearly $h_1(\Omega^{ab}) \supset h_1(W^{ab})$ for any $a, b \in \{+, -\}$ since $\Omega^{ab} \supset W^{ab}$. So in view of Lemma 3.6.3, $h_1(\Omega^{ab}) \supset -\mathbb{H}$ or $h_1(\Omega^{ab}) \supset \mathbb{H}$ depending on whether $a = b$.

We now consider limit sets for Ω^{++} and Ω^{--}; the other cases are analogous. By (3.5.5),

$$F^2(\Omega^{++}) \subset \Omega^{--} \text{ and } F^2(\Omega^{--}) \subset \Omega^{++}.$$

It follows that for any $n > 0$,

$$F^{2n}(\Omega^{++} \cup \Omega^{--}) \subset \Omega^{++} \cup \Omega^{--}.$$

In view of this, and since W_I is absorbing for Ω, we have that for any $P \in (\Omega^{++} \cup \Omega^{--})$

$$F^{2n}(P) \subset W_I \cap (\Omega^{++} \cup \Omega^{--}) = W_I^{++} \cup W_I^{--} \text{ for any } n \text{ large enough.}$$

Hence $h_1(P) \in h_1(W_I^{++} \cup W_I^{--}) \subset \mathbb{H}$ for every $P \in (\Omega^{++} \cup \Omega^{--})$, hence $h_1(\Omega^{++} \cup \Omega^{--}) \subset \mathbb{H}$. It follows that $h_1(\Omega^{++}) = h_1(\Omega^{--}) = \mathbb{H}$.

☐

We devote the rest of this section to proving Lemma 3.6.3. We first give a version of Rouché's Theorem which relies on one of the many versions of Rouché's Theorem existing in one variable (compare with Theorem 3.4 in [BSZ23]; we will use it with the spherical instead of the Euclidean metric). This is certainly known to experts in the field but we are not aware of a reference. In this section ∂ denotes the topological boundary, and $\mathrm{dist}_{\mathrm{spher}}$ denotes the spherical distance.

3.6.4 Theorem (Rouché's Theorem in $\mathbb{C}^2$)
Let $B \subset \mathbb{C}^2$ be a polydisk, F, G be holomorphic maps defined in a neighborhood of $\overline{B}$ which take values in $\hat{\mathbb{C}}$. Let $c \in G(B)$, let $\varepsilon = \mathrm{dist}_{\mathrm{spher}}(c, G(\partial B)) > 0$ and assume

$$\mathrm{dist}_{\mathrm{spher}}(F, G) < \varepsilon \textit{ on } \partial B.$$

Then $c \in F(B)$.

Notice that the assumptions imply that F, G have generic rank 1: They cannot have rank 2 because the target is $\hat{\mathbb{C}}$, and G cannot be constant otherwise there could not be $c \in G(B)$ with positive distance from $G(\partial B)$. One can check that F cannot be constant either.

Proof. Let D be a horizontal disk passing through a point $P_c \in G^{-1}(c) \cap B$, such that $\partial D \subset \partial B$. Let $g := G|_D$, $f := F_D$. They are holomorphic in a slightly larger horizontal disk. Notice that $\mathrm{dist}_{\mathrm{spher}}(g, f) < \varepsilon$ on ∂D, and that $\mathrm{dist}_{\mathrm{spher}}(G(\partial D), c) \geq \varepsilon$ because $\partial D \subset \partial B$. By Rouché's Theorem in one variable, $c \in f(D) \subset F(B)$ as required. ☐

3.6.5 Remark. Unless P_c is an isolated point in $G^{-1}(c)$ we obtain a curve of points in $F^{-1}(c)$. Indeed, the proof gives a point in $F^{-1}(c)$ for any Euclidean disk passing through points in $G^{-1}(c)$, for example, a family of disks passing through P_c along different complex directions. The points obtained for $F^{-1}(c)$ are distinct unless they always coincide with P_c. On the other hand, if P_c is an isolated point in $G^{-1}(c)$ then $P_c \in F^{-1}(c)$ is also isolated in $F^{-1}(c)$. Indeed otherwise we could reverse the role of F and G and obtain one point in $G^{-1}(c)$ for any Euclidean disk passing through any point in $F^{-1}(c)$ and obtain a curve of points in $G^{-1}(c)$. The proof as it is works when B is any $\mathbb{C}$-convex set instead of a polydisk, and can certainly be generalized further.

Proof of Lemma 3.6.3. We show $\mathbb{H} \subset h_1(W_I^{++})$. The other cases are analogous. Recall that orbits of points in W are contained in S, hence (3.3.4) holds. Since

$$\frac{z_{2n}}{w_{2n}} = \frac{z_0 + \Delta_1^n(z_0, w_0)}{w_0 + \Delta_2^n(z_0, w_0)},$$

dividing the numerator and the denominator by w_0 and using $\frac{1}{1+x} = 1 + \sum_{j=1}^{\infty}(-x)^j$ for $|x| < 1$ we have that

$$\frac{z_{2n}}{w_{2n}} - \frac{z_0}{w_0} = \left(\frac{z_0}{w_0} + \frac{\Delta_1^n(z_0, w_0)}{w_0}\right)\left(1 + \sum_{j=1}^{\infty}\left(\frac{-\Delta_2^n(z_0, w_0)}{w_0}\right)^j\right) - \frac{z_0}{w_0} \tag{3.6.1}$$

$$= \frac{\Delta_1^n(z_0, w_0)}{w_0} + \left(\frac{z_0}{w_0} + \frac{\Delta_1^n(z_0, w_0)}{w_0}\right)\sum_{j=1}^{\infty}\left(\frac{-\Delta_2^n(z_0, w_0)}{w_0}\right)^j \quad \forall n \geq 0. \tag{3.6.2}$$

This expression makes sense for $|x| = \left|\frac{-\Delta_2^n(z_0, w_0)}{w_0}\right| < 1$, hence, in view of (3.3.4), for $|w_0| > 1$. Recall also that $|\sum_{j=1}^{\infty} x^j| = \frac{|x|}{1-x} \leq 2|x|$ if $|x| < \frac{1}{2}$. Let $K \subset \hat{\mathbb{C}}$ be a compact set and suppose that $\frac{z_0}{w_0}$ takes values in K. By (3.6.1) and using (3.3.4), for any $\varepsilon > 0$ there exists $M = M(K, \varepsilon)$ such that

$$\left|\frac{z_{2n}}{w_{2n}} - \frac{z_0}{w_0}\right| < \varepsilon \text{ for } |w_0| > M \text{ and } \frac{z_0}{w_0} \in K. \tag{3.6.3}$$

Consider the function $G(z, w) := \frac{z}{w}$ Observe that

$$G^{-1}(re^{i\theta}) = \left\{(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \in \mathbb{C}^2 : \frac{r_1}{r_2} = r, \theta = \theta_1 - \theta_2\right\}.$$

Let $c \in \mathbb{H}$. By the shape of W we have that $G(W^{++}) = \mathbb{H}$, that $\varepsilon := \frac{1}{2}\mathrm{dist}_{\mathrm{spher}}(c, G(\partial W)) > 0$, and that we can choose $Q = (z_0, w_0) \in W^{++} \in G^{-1}(c)$ such that $|w_0|$ is arbitrarily large. By taking a limit in n in equation (3.6.1) and on a sufficiently small polydisk centered at Q we can ensure that $\mathrm{dist}_{\mathrm{spher}}(h_1, G) < \varepsilon$, hence the claim follows by Rouché's Theorem. ☐

The Main Theorem 3.1.1 is a direct consequence of Propositions 3.3.6, 3.5.1, 3.5.2, 3.6.1.

Chapter 4

Generalized Approach with cycles of Escaping Fatou Components

In this chapter, the results obtained from my latest research will be presented, which are also available in article [Bel24].

4.1 Extension and Outcome

This chapter starts by extending the map F defined in 3.2.1, trough a map G, to uncover its broader implications, followed by an analysis of its cyclic behavior.

More precisely, the map F defined in 3.2.1 can be extended through the map G defined in the following way:

$$G(z,w) = (e^{-z^m} + \delta e^{\frac{2\pi}{m}i}\, w\,,\, z) \tag{4.1.1}$$

with $m \geq 2$ a natural number and $\mathbb{R} \ni \delta > 2$. Specifically, F is equivalent to G in the case where $m = 2$.

This perspective allows us to explore the behaviors of F in relation to G and vice versa and we will conduct a thorough analysis of this extension, examining its properties.

From this point forward, we will denote G by F.

The central theorem, which will be thoroughly explored and proven in this chapter, is structured as follows. Again $\mathbb{H}$ denotes the right half plane.

4.1.1 Theorem (Main Theorem of [Bel24])
Let F be defined as in (4.1.1), then

- *There are m^2 distinct Fatou components that exhibit cyclic behavior, more precisely*
  - *If m is even there are $\frac{m}{2}$ cycles of escaping Fatou components of period $2m$.*
  - *If m is odd there are $\frac{m-1}{2}$ cycles of escaping Fatou components of period $2m$ and one cycle of escaping Fatou components of period m.*
- *Each cycle has exactly two distinct limit functions h_1, h_2, both of which have generic rank 1.*

- *Each Fatou component in each cycle has two disjoint and hyperbolic limit sets, with the exception of the Fatou components belonging to the cycle of period m (the one occurring when m is odd), which have the same hyperbolic limit set.*

- *Denote the union of the m^2 components with Ω, then F is conjugate to the linear map $L(z,w) = (\delta e^{\frac{2\pi}{m}i} w, z)$ on Ω.*

- *Each Fatou component in each cycle is biholomorphic to $\mathbb{H} \times \mathbb{H}$.*

One of the points of greatest interest is that we get a combinatorial dynamic behavior, indeed there are cycles of escaping Fatou components where the dynamics vary depending on whether m is even or odd.

Moreover we have two distinct limit functions on each cycle, both of which have generic rank 1.

Additionally, each Fatou component in each cycle has two disjoint and hyperbolic limit sets on the line at infinity, with the exception of the Fatou components belonging to the short cycle of period m (that occurs if m is odd) which have the same hyperbolic limit set.

New dynamic behaviors emerge that were not observed in the study presented in Chapter 3, particularly in the case where m is odd.

4.2 Cyclic Behavior

From now on throughout the article, F denotes that defined in (4.1.1), that is

$$F(z,w) := \left(e^{-z^m} + \delta e^{\frac{2\pi}{m}i} w\,,\, z\right) \text{ with } m \in \mathbb{N},\ m \geq 2 \text{ and } \delta \in \mathbb{R},\ \delta > 2.$$

Notice that, following Definition 2.3.7, $\alpha = \delta e^{\frac{2\pi}{m}i}$. Moreover the modulus of the Jacobian determinant is equals to δ, hence F is an expansive map since $\delta > 2$.

In this section we give an explicit expression for the iterates of F and their formal limit, we demonstrate that there is a cyclic behavior and in Proposition 4.2.5 we show how the real part grows under the iteration of F.

Again, for $P = (z_0, w_0) \in \mathbb{C}^2$ and $n \in \mathbb{N}$, we define $(z_n, w_n) := F^n(P)$ the n-th iterate of the point P under the action of F. Using the expression of F, we can compute explicitly the iterates F^{2n} and F^{2n+1}:

$$F^{2n}(z_0, w_0) = \left(\alpha^n z_0 + \alpha^n \sum_{j=1}^{n} \alpha^{-j} f(z_{2j-1})\,,\, \alpha^n w_0 + \alpha^n \sum_{j=1}^{n} \alpha^{-j} f(z_{2j-2})\right) \tag{4.2.1}$$

$$F^{2n+1}(z_0, w_0) = \left(\alpha^{n+1} w_0 + \alpha^{n+1} \sum_{j=1}^{n+1} \alpha^{-j} f(z_{2j-2})\,,\, \alpha^n z_0 + \alpha^n \sum_{j=1}^{n} \alpha^{-j} f(z_{2j-1})\right). \tag{4.2.2}$$

Define the following m open subsets of $\mathbb{C}$:

$$\mathcal{S}_a := \left\{ z \in \mathbb{C} \,:\, \left| \operatorname{Im}\left(z\, e^{\frac{2(m-a)}{m}\pi i}\right) \right| < \tan\left(\frac{\pi}{2m}\right) \operatorname{Re}\left(z\, e^{\frac{2(m-a)}{m}\pi i}\right) \right\},$$

with $a \in \mathbb{Z}_m$. And let

$$\mathcal{S} = \bigcup_{a \in \mathbb{Z}_m} \mathcal{S}_a\,.$$

Observe the following simple lemma asserting that f is bounded on $\mathcal{S}$.

4.2.1 Lemma
Let $z \in \mathcal{S}$, then

$$|f(z)| = |e^{-z^m}| < 1. \tag{4.2.3}$$

Proof. If $z \in \mathcal{S}_a$, then $z^m \in \mathbb{H}$ and hence $\operatorname{Re} z^m > 0$, from which we have $|e^{-z^m}| = e^{-\operatorname{Re} z^m} < 1$. $\square$

Consider also the following m^2 open subsets of $\mathbb{C}^2$

$$S_{ab} := \mathcal{S}_a \times \mathcal{S}_b$$

with $a, b \in \mathbb{Z}_m$, and let

$$S := \bigcup_{a,b} S_{ab}, \quad \text{with } a, b \in \mathbb{Z}_m.$$

Now define

$$\Delta_1(z_0, w_0) := \sum_{j=1}^{\infty} \alpha^{-j} f(z_{2j-1}) \tag{4.2.4}$$

$$\Delta_2(z_0, w_0) := \sum_{j=1}^{\infty} \alpha^{-j} f(z_{2j-2}) \tag{4.2.5}$$

$$\Delta := \sum_{j=1}^{\infty} |\alpha|^{-j} = \sum_{j=1}^{\infty} \delta^{-j}. \tag{4.2.6}$$

Notice that $\Delta = \frac{\delta}{\delta-1} - 1$, and since $\delta > 2$, $\Delta < 1$. Using Lemma 4.2.1, we have the following.

4.2.2 Remark. Let $P = (z_0, w_0) \in S$ such that $F^n(z_0, w_0) \in S$ for all $n \in \mathbb{N}$, then

$$|\Delta_1(z_0, w_0)|, \, |\Delta_2(z_0, w_0)| < \Delta < 1.$$

For the rest of this section, we shall consider P to be a point in S such that $F^n(P) \in S$ for all $n \in \mathbb{N}$.

For such $P = (z_0, w_0)$ we can also deduce the following formal limits:

$$h_1(z_0, w_0) := \lim_{n\to\infty} \frac{z_{2n}}{w_{2n}} = \frac{z_0 + \Delta_1(z_0, w_0)}{w_0 + \Delta_2(z_0, w_0)} \tag{4.2.7}$$

$$h_2(z_0, w_0) := \lim_{n\to\infty} \frac{z_{2n+1}}{w_{2n+1}} = \frac{\alpha w_0 + \alpha\Delta_2(z_0, w_0)}{z_0 + \Delta_1(z_0, w_0)} = \frac{\alpha}{h_1} \tag{4.2.8}$$

and we will later show in Proposition 4.4.3 that $h_1 \neq h_2$.

Notice that such a P exists, indeed each S_{ab} contains the set

$$A_{ab} := \left\{ \operatorname{Re}\left(z e^{\frac{2(m-a)}{m}\pi i}\right), \operatorname{Re}\left(w e^{\frac{2(m-b)}{m}\pi i}\right) > M \,, \operatorname{Im}\left(z e^{\frac{2(m-a)}{m}\pi i}\right), \operatorname{Im}\left(w e^{\frac{2(m-b)}{m}\pi i}\right) = 0 \right\} \tag{4.2.9}$$

for M sufficiently large, and

$$A := \bigcup_{a,b \in \mathbb{Z}_m} A_{ab} \subset S$$

is forward invariant under F, so for example each $P \in A$ satisfies the requirement.

4.2.3 Lemma (Forever in S implies convergence)
Let $P = (z_0, w_0) \in S$ such that $F^n(P) \in S$ for all $n \in \mathbb{N}$, then

$$\begin{aligned} F^{2n}(z_0, w_0) &\to h_1(z_0, w_0) \\ F^{2n+1}(z_0, w_0) &\to h_2(z_0, w_0) \end{aligned}$$

Proof. Since $F^n(P) \in S$ for all $n \in \mathbb{N}$ we have that $z_n \in \mathcal{S}$ for all $n \in \mathbb{N}$ and hence by Lemma 4.2.1 $|f(z_n)| < 1$ for all $n \in \mathbb{N}$. This implies that $|\Delta_1(z_0, w_0)|$, $|\Delta_2(z_0, w_0)| < \Delta < 1$, which implies convergence of the even and odd iterates of F.

□

With the following proposition we show that there is a cyclic behavior.

4.2.4 Proposition
Let $P = (z_0, w_0) \in S_{ab}$, with $a, b \in \mathbb{Z}_m$, such that $F^n(P) \in S$ for all $n \in \mathbb{N}$. If $\operatorname{Re} z_0, \operatorname{Re} w_0$ are sufficiently large, then $F(P) \in S_{(b+1)a}$.

Proof. We prove the claim for $P = (z_0, w_0) \in S_{00}$, the other cases are analogous.

By hypothesis we know that $F(P) = (z_1, w_1) \in S_{ab}$ for some $a, b \in \mathbb{Z}_m$. Since $P \in S_{00}$, we have

$$|\operatorname{Im} z_0| < \tan\left(\frac{\pi}{2m}\right) \operatorname{Re} z_0$$

and

$$|\operatorname{Im} w_0| < \tan\left(\frac{\pi}{2m}\right) \operatorname{Re} w_0,$$

moreover $w_1 = z_0$, so $|\operatorname{Im} w_1| < \tan\left(\frac{\pi}{2m}\right) \operatorname{Re} w_1$, and so $b = 0$.

Recall that $z_1 = e^{-z_0^m} + \delta e^{\frac{2\pi}{m} i} w_0$, and notice that $\delta e^{\frac{2\pi}{m} i} w_0 \in \mathcal{S}_1$. Furthermore $|e^{-z_0^m}| < 1$, then $z_1\, e^{\frac{2(m-1)}{m}\pi i}$ belongs to the 1-neighborhood of $\mathcal{S}_0$. Choosing $\operatorname{Re} w_0$ sufficiently large, this 1-neighborhood of $\mathcal{S}_0$ intersects $\mathcal{S}$ only in $\mathcal{S}_0$ and so $a = 1$. □

To better understand the cycling behavior inside the sectors S_{ab}, with $a, b \in \mathbb{Z}_m$, let us consider the following application:

$$\gamma : \mathbb{Z}_m \times \mathbb{Z}_m \longrightarrow \mathbb{Z}_m \times \mathbb{Z}_m \quad \text{defined as} \quad \gamma(a, b) := (b + 1, a)\,.$$

It is easy to check that

$$\gamma^{2n}(a, b) = (a + n, b + n)$$

and

$$\gamma^{2n+1}(a, b) = (b + n + 1, a + n)\,.$$

In order to have cycles we need to set the following equations:

$$\gamma^{2n}(a, b) = (a, b)$$

$$\gamma^{2n+1}(a, b) = (a, b)$$

and we obtain respectively $2n = 2m$ and $2n + 1 = m$. So we can have cycles of period $2m$ or of period m. Since we have m^2 sectors, to understand how many and which cycles have a period $2m$ or m, we first need to solve the equation

$$m^2 = A2m + Bm$$

that is $m = 2A + B$, with $A, B \in \mathbb{N}$.

Additionally, notice that for each cycle, we can take $(0, b)$ as a representative. Therefore, let us see after how many iterations, in each cycle, we obtain $(0, \tilde{b})$. We again consider

$$\gamma^{2n}(0, b) = (0, \tilde{b})$$

from which we obtain, after $2m$ iterations, $\tilde{b} = b$, and

$$\gamma^{2n+1}(0, b) = (0, \tilde{b})$$

from which we have, after $2m - (2b + 1)$ iterations, $\tilde{b} = m - (b + 1)$ or, after m iterations, $\tilde{b} = b = \frac{m-1}{2}$. This means that we can have at most one cycle of period m, the one represented by $(0, \frac{m-1}{2})$.

If m is even we can not have the cycle of period m, since $\frac{m-1}{2} \notin \mathbb{N}$, so in this case we get $A = \frac{m}{2}$ and $B = 0$, that is we have $\frac{m}{2}$ cycles of period $2m$ and zero of period m. Moreover in each cycle there is $(0, b)$ and after $2m - (2b + 1)$ iterations also $(0, m - (b + 1))$.

If m is odd, we have $B = 1$ and $A = \frac{m-1}{2}$, that is there are $\frac{m-1}{2}$ cycles of period $2m$ and one cycle of period m, which we refer to as the *short cycle*, and it is the one represented by $(0, \frac{m-1}{2})$. In each cycle of period $2m$, as in the even case, there is $(0, b)$ and after $2m - (2b + 1))$ iterations also $(0, m - (b + 1))$.

To better understand these cycles, let us consider two examples: $m = 5$ and $m = 6$.

$$m = 5 \qquad \left\{\begin{array}{ll}
\mathbf{00} & \textbf{iteration 0} \\
10 & \text{iteration 1} \\
11 & \text{iteration 2} \\
21 & \text{iteration 3} \\
22 & \text{iteration 4} \\
32 & \text{iteration 5} \\
33 & \text{iteration 6} \\
43 & \text{iteration 7} \\
44 & \text{iteration 8} \\
\mathbf{04} & \textbf{iteration 9} \\
\mathbf{00} & \textbf{iteration 10}
\end{array}\right.
\qquad
\left\{\begin{array}{ll}
\mathbf{01} & \textbf{iteration 0} \\
20 & \text{iteration 1} \\
12 & \text{iteration 2} \\
31 & \text{iteration 3} \\
23 & \text{iteration 4} \\
42 & \text{iteration 5} \\
34 & \text{iteration 6} \\
\mathbf{03} & \textbf{iteration 7} \\
40 & \text{iteration 8} \\
14 & \text{iteration 9} \\
\mathbf{01} & \textbf{iteration 10}
\end{array}\right.
\qquad
\left\{\begin{array}{ll}
\mathbf{02} & \textbf{iteration 0} \\
30 & \text{iteration 1} \\
13 & \text{iteration 2} \\
41 & \text{iteration 3} \\
24 & \text{iteration 4} \\
\mathbf{02} & \textbf{iteration 5}
\end{array}\right.$$

$$m = 6 \qquad \left\{\begin{array}{ll}
\mathbf{00} & \textbf{iteration 0} \\
10 & \text{iteration 1} \\
11 & \text{iteration 2} \\
21 & \text{iteration 3} \\
22 & \text{iteration 4} \\
32 & \text{iteration 5} \\
33 & \text{iteration 6} \\
43 & \text{iteration 7} \\
44 & \text{iteration 8} \\
54 & \text{iteration 9} \\
55 & \text{iteration 10} \\
\mathbf{05} & \textbf{iteration 11} \\
\mathbf{00} & \textbf{iteration 12}
\end{array}\right.
\qquad
\left\{\begin{array}{ll}
\mathbf{01} & \textbf{iteration 0} \\
20 & \text{iteration 1} \\
12 & \text{iteration 2} \\
31 & \text{iteration 3} \\
23 & \text{iteration 4} \\
42 & \text{iteration 5} \\
34 & \text{iteration 6} \\
53 & \text{iteration 7} \\
45 & \text{iteration 8} \\
\mathbf{04} & \textbf{iteration 9} \\
50 & \text{iteration 10} \\
15 & \text{iteration 11} \\
\mathbf{01} & \textbf{iteration 12}
\end{array}\right.
\qquad
\left\{\begin{array}{ll}
\mathbf{02} & \textbf{iteration 0} \\
30 & \text{iteration 1} \\
13 & \text{iteration 2} \\
41 & \text{iteration 3} \\
24 & \text{iteration 4} \\
52 & \text{iteration 5} \\
35 & \text{iteration 6} \\
\mathbf{03} & \textbf{iteration 7} \\
40 & \text{iteration 8} \\
14 & \text{iteration 9} \\
51 & \text{iteration 10} \\
25 & \text{iteration 11} \\
\mathbf{02} & \textbf{iteration 12}
\end{array}\right.$$

In the following proposition we analyze how the real part of P increases.

4.2.5 Proposition (Growth of the real part)
Let $P = (z_0, w_0) \in S_{00}$, such that $F^n(P) \in S$ for all $n \in \mathbb{N}$. Then for all $\lambda > 0$, if $\operatorname{Re} w_0, \operatorname{Re} z_0 > \frac{1+\lambda}{\delta-1}$

$$\operatorname{Re}\left(z_{2n-1}\, e^{\frac{2(m-(2n-1))}{m}\pi i}\right) = \operatorname{Re}\left(w_{2n}\, e^{\frac{2(m-(2n-1))}{m}\pi i}\right) > \operatorname{Re} w_0 + n\lambda\,,$$

$$\operatorname{Re}\left(z_{2n}\, e^{\frac{2(m-(2n-1))}{m}\pi i}\right) = \operatorname{Re}\left(w_{2n+1}\, e^{\frac{2(m-(2n-1))}{m}\pi i}\right) > \operatorname{Re} z_0 + n\lambda\,.$$

Proof. Let $P = (z_0, w_0)$ as in the hypothesis. Since $P \in S$, then by Lemma 4.2.1, we have

$$\operatorname{Re}\left(z_1\, e^{\frac{2(m-1))}{m}\pi i}\right) = \operatorname{Re}\left(e^{-z_0^m}\, e^{\frac{2(m-1))}{m}\pi i}\right) + \delta \operatorname{Re} w_0 >$$

$$> \delta \operatorname{Re} w_0 - \left|\operatorname{Re}\left(e^{-z_0^m}\, e^{\frac{2(m-1))}{m}\pi i}\right)\right| > \delta \operatorname{Re} w_0 - 1$$

so

$$\operatorname{Re}\left(z_1\, e^{\frac{2(m-1))}{m}\pi i}\right) > \delta \operatorname{Re} w_0 - 1 \tag{4.2.10}$$

which is larger than $\operatorname{Re} w_0 + \lambda$ if $\operatorname{Re} w_0 > \frac{1+\lambda}{\delta-1}$ as required. The claim for z_2 follows because $w_1 = z_0$ and the more general formula follows by induction. ☐

If we substitute S_{00} with a generic S_{ab}, with $a, b \in \mathbb{Z}_m$, in the Proposition 4.2.5, we obtain the following.

$$\operatorname{Re}\left(z_{2n-1}\, e^{\frac{2(m-b-(2n-1))}{m}\pi i}\right) = \operatorname{Re}\left(w_{2n}\, e^{\frac{2(m-b-(2n-1))}{m}\pi i}\right) > \operatorname{Re} w_0 + n\lambda\,,$$

$$\operatorname{Re}\left(z_{2n}\, e^{\frac{2(m-a-(2n-1))}{m}\pi i}\right) = \operatorname{Re}\left(w_{2n+1}\, e^{\frac{2(m-a-(2n-1))}{m}\pi i}\right) > \operatorname{Re} z_0 + n\lambda\,.$$

4.3 Construction of a Forward Invariant Open Set W

The purpose of this section is to construct a forward invariant open set $W \subset S$, that is $F(W) \subset W$, and so Lemma 4.2.1, Lemma 4.2.3, Proposition 4.2.4 and Proposition 4.2.5 hold on W. With this in mind let us introduce the following m subsets of $\mathbb{C}$:

$$\left(\mathcal{W}_{\sigma,R}\right)_a := \left\{ z \in \mathbb{C} \,:\, \left|\operatorname{Im}\left(z\, e^{\frac{2(m-a)}{m}\pi i}\right)\right| < \sigma \operatorname{Re}\left(z\, e^{\frac{2(m-a)}{m}\pi i}\right)\right\} \subset \mathcal{S}_a,$$

with $0 < \sigma < \tan(\pi/2m)$, $\operatorname{Re}\left(z\, e^{\frac{2(m-a)}{m}\pi i}\right) > R$ and $a \in \mathbb{Z}_m$.

Define also the following m^2 subsets of $\mathbb{C}^2$:

$$\left(W_{\sigma,R_1,R_2}\right)_{ab} := \left(\mathcal{W}_{\sigma,R}\right)_a \times \left(\mathcal{W}_{\sigma,R}\right)_b \subset S_{ab}$$

with $a, b \in Z_m$ and let

$$W_{\sigma,R_1,R_2} := \bigcup_{a,b} \left(W_{\sigma,R_1,R_2}\right)_{ab}\,.$$

4.3.1 Proposition (Amplitude)
Let $(z_0, w_0) \in \left(W_{\sigma,R_1,R_2}\right)_{00}$ and let $0 < \sigma < \tilde{\sigma} < \tan(\pi/2m) < 1$. if $R_2 > \frac{2}{\delta(\tilde{\sigma}-\sigma)}$, then

$$\frac{\left|\operatorname{Im}\left(z_1 e^{\frac{2(m-1)}{m}\pi i}\right)\right|}{\operatorname{Re}\left(z_1 e^{\frac{2(m-1)}{m}\pi i}\right)} < \tilde{\sigma}\,.$$

Proof. Let $(z_0, w_0) \in (W_{\sigma,R_1,R_2})_{00}$, using Lemma 4.2.1 and the fact that $|\operatorname{Im} w_0| < \sigma \operatorname{Re} w_0$ we have

$$\frac{\left|\operatorname{Im}\left(z_1 e^{\frac{2(m-1)}{m}\pi i}\right)\right|}{\operatorname{Re}\left(z_1 e^{\frac{2(m-1)}{m}\pi i}\right)} < \frac{\delta|\operatorname{Im} w_0| + 1}{\delta \operatorname{Re} w_0 - 1} < \frac{\delta\sigma \operatorname{Re} w_0 + 1}{\delta \operatorname{Re} w_0 - 1}$$

which is less than $\tilde{\sigma}$ if $\operatorname{Re} w_0 > \frac{1+\tilde{\sigma}}{\delta(\tilde{\sigma}-\sigma)}$. Since $\tilde{\sigma} < 1$, it is enough to take $\operatorname{Re} w_0 > \frac{2}{\delta(\tilde{\sigma}-\sigma)}$ as required. □

If we substitute S_{00} with a generic S_{ab} with $a, b \in \mathbb{Z}_m$, in the Proposition 4.3.1, we obtain the following:

$$\frac{\left|\operatorname{Im}\left(z_1 e^{\frac{2(m-1-b)}{m}\pi i}\right)\right|}{\operatorname{Re}\left(z_1 e^{\frac{2(m-1-b)}{m}\pi i}\right)} < \tilde{\sigma}\,.$$

Now observe that, fixing $b = 0, 1, ..., \frac{m}{2}$ if m is even or $b = 0, 1, ..., \frac{m+1}{2}$ if m is odd, the pair $(0, b)$ identifies one of the cycles. Thus, to simplify notation, fixed b as above, we define $\mu_b : \mathbb{N} \to \mathbb{Z}_m \times \mathbb{Z}_m$, which denotes the cyclic behavior:

$$\mu_b(n) := \begin{cases} 0\,b & \text{if } n = 0 \mod 2m \\ (b+1)\,0 & \text{if } n = 1 \mod 2m \\ 1\,(b+1) & \text{if } n = 2 \mod 2m \\ (b+2)\,1 & \text{if } n = 3 \mod 2m \\ 2\,(b+2) & \text{if } n = 4 \mod 2m \\ . \\ . \\ . \\ 0\,(m-1) & \text{if } n = 2m-1 \mod 2m \\ 0\,b & \text{if } n = 2m \mod 2m \end{cases}$$

Notice that for the short cycle, that is when $b = \frac{m-1}{2}$, we can consider the following

$$\mu_b(n) = \begin{cases} 0\,b & \text{if } n = 0 \mod m \\ (b+1)\,0 & \text{if } n = 1 \mod m \\ 1\,(b+1) & \text{if } n = 2 \mod m \\ (b+2)\,1 & \text{if } n = 3 \mod m \\ 2\,(b+2) & \text{if } n = 4 \mod m \\ . \\ . \\ . \\ 0\,(m-1) & \text{if } n = m-1 \mod m \\ 0\,b & \text{if } n = m \mod m \end{cases}$$

Let $\sigma_n := \left(\frac{n+1}{n+2}\right)\tan(\frac{\pi}{2m})$ and $R_n := (\frac{\delta}{2})^{\frac{n}{2}} R_0$ for R_0 sufficiently large depending only on δ. Notice that $\sigma_n \in \left(0, \tan(\frac{\pi}{2m})\right)$ for all $n \in \mathbb{N}$. Fixed b as above, set

$$W_{b,n} := \left(W_{\sigma_n,R_n,R_{n-1}}\right)_{\mu_b(n)}\,, \tag{4.3.1}$$

and define

$$W := \bigcup_{\substack{b \\ n\in\mathbb{N}}} W_{b,n}\,, \tag{4.3.2}$$

where b varies in $\{0, 1, ..., \frac{m}{2}\}$ if m is even or in $\{0, 1, ..., \frac{m+1}{2}\}$ if m is odd.

Observe that $W \subset S$, it is open and consists of m^2 connected components. We define W_{ab} with $a, b \in \mathbb{Z}_m$, the component of W contained in S_{ab}.

4.3.2 Proposition (Invariance of W)
Fixed b as above, we have that $F(W_{b,n}) \subset W_{b,n+1}$. In particular, W is forward invariant.

This also implies that if W_{ab} does not belong to the short cycle it is forward invariant under F^{2m}, otherwise it is forward invariant under F^m.

Proof. Let $(z_0, w_0) \in W_{b,n}$, with $b = 0$ and $n = 0 \mod 2m$, the other cases are analogous; and let (z_1, w_1) be its image. Notice that $w_1 = z_0$, so $\operatorname{Re} w_1 = \operatorname{Re} z_0 > R_n$, and

$$\frac{|\operatorname{Im} w_1|}{\operatorname{Re} w_1} = \frac{|\operatorname{Im} z_0|}{\operatorname{Re} z_0} < \sigma_n < \sigma_{n+1}\ ,$$

hence to show that $F(W_{b,n}) \subset W_{b,n+1}$ it is enough to prove that

1. $\operatorname{Re}\left(z_1 e^{\frac{2(m-1)}{m}\pi i}\right) > R_{n+1}$
2. $\dfrac{\left|\operatorname{Im}\left(z_1 e^{\frac{2(m-1)}{m}\pi i}\right)\right|}{\operatorname{Re}\left(z_1 e^{\frac{2(m-1)}{m}\pi i}\right)} < \sigma_{n+1}$.

Let $\lambda_n := R_{n+1} - R_{n-1}$. Since $(z_0, w_0) \in S$, from (4.2.10) we have $\operatorname{Re}\left(z_1 e^{\frac{2(m-1)}{m}\pi i}\right) > \operatorname{Re} w_0 + \lambda_n > R_{n-1} + \lambda_n = R_{n+1}$ provided $R_{n-1} > \frac{1+\lambda_n}{\delta-1}$. Substituting the expression for λ_n we get $R_{n+1} < \delta R_{n-1} - 1$. Substituting the expression for R_{n+1} and R_{n-1} we get

$$\delta^{\frac{n+1}{2}} R_0 > 2^{\frac{n+1}{2}}$$

which is satisfied because $\delta > 2$, provided $R_0 \geq 1$. This gives $\operatorname{Re}\left(z_1 e^{\frac{2(m-1)}{m}\pi i}\right) > R_{n+1}$.

We now show $\dfrac{\left|\operatorname{Im}\left(z_1 e^{\frac{2(m-1)}{m}\pi i}\right)\right|}{\operatorname{Re}\left(z_1 e^{\frac{2(m-1)}{m}\pi i}\right)} < \sigma_{n+1}$. In view of Proposition 4.3.1, it is enough to check that $R_{n-1} > \frac{2}{\delta(\sigma_{n+1}-\sigma_n)} = \frac{2(n+2)(n+3)}{\delta}\tan(\frac{\pi}{2m})$, that is

$$R_0 > \frac{2^{\frac{n+1}{2}}}{\delta^{\frac{n+1}{2}}}(n+2)(n+3)\tan\left(\frac{\pi}{2m}\right)\ .$$

Since the function on the right hand side is bounded in n for any $\delta > 2$ (moreover, it tends to 0 as $n \to \infty$), such R_0 exists and depends only on δ. □

4.4 Fatou Components and Rank 1 Limit Functions

In this section, we establish that W is contained in the Fatou set. This is achieved by leveraging Lemma 4.2.3 in conjunction with the observation that W possesses the characteristics of being non-empty, open, forward invariant, and contained in S. Additionally, we demonstrate that the functions h_1 and h_2, as defined in (4.2.7) and (4.2.8) respectively, have generic rank 1, and further, that $h_1 \neq h_2$.

4.4.1 Proposition (Existence of Fatou components)
On each W_{ab} we have that

$$F^{2n} \to h_1, F^{2n+1} \to h_2 \text{ uniformly on compact subsets of } W_{ab}.$$

It follows that each W_{ab} is contained in a Fatou component Ω_{ab}.

Proof. Points in each W_{ab} never leave S by Proposition 4.3.2. Hence the even and odd iterates of F converge according to Lemma 4.2.3 on compact subsets of each W_{ab}. Since each W_{ab} is open and connected it is contained in a Fatou component Ω_{ab}. □

Notice that in Proposition 4.4.1 we define Ω_{ab} to be the Fatou component containing W_{ab} with $a, b \in \mathbb{Z}_m$. Let

$$\Omega := \bigcup_{ab} \Omega_{ab}, .$$

The following corollary is a direct consequence of Proposition 4.4.1.

4.4.2 Corollary
The set Ω consists of at most m^2 connected components.

We will see in Proposition 4.6.5 that the components Ω_{ab} are in fact all distinct, so Ω consists of exactly m^2 connected components.

We now show that h_1, h_2 are distinct and have generic rank 1.

4.4.3 Proposition
Both h_1 and h_2 have (generic) rank 1, and $h_1 \neq h_2$.

Proof. Notice that Proposition 4.2.5 implies that $h_i(W)$ is contained in the line at infinity and so, by Sard's Theorem, h_1 and h_2 have generic rank at most 1. We now show that h_1 and h_2 are non-constant, so we can conclude that they have rank 1. Suppose by contradiction that $|h_1| = c$ is constant.

If $c \neq 0, \infty$, then one has:

$$|z_0| - \Delta \leq \left| z_0 + \sum_{j=1}^{\infty} \alpha^{-j} f(z_{2j-1}) \right| = c \left| w_0 + \sum_{j=1}^{\infty} \alpha^{-j} f(z_{2j-2}) \right| \leq c|w_0| + c\Delta,$$

hence

$$|z_0| \leq c|w_0| + (c+1)\Delta,$$

contradicting the fact that (z_0, w_0) could be any point in W, which is unbounded in the z direction for any choice of w.
If $c = 0$, we have $|z_0| \leq \Delta$, while if $c = \infty$, we have $|w_0| \leq \Delta$; in either case we have a contradiction.

This also implies that $h_1 \neq h_2$. Indeed, $h_1 \cdot h_2$ is constant, so if we had $h_1 = h_2$ we would have that h_1^2 is constant and hence so is h_1. □

4.5 Construction of an Absorbing Set W_I for Ω

This section is dedicated to the construction of the absorbing set W_I for Ω under F, as stated in Proposition 4.5.1.

The formal definition of an absorbing set has been provided in Definition 3.4.1.

Remember that in Proposition 4.4.1 we define Ω_{ab} to be the Fatou component containing W_{ab} with $a, b \in \mathbb{Z}_m$, and

$$\Omega := \bigcup_{ab} \Omega_{ab}.$$

Fix $C \geq 1$ and let

$$I = I(C) := \{z \in \mathbb{C} \,:\, \mathrm{Re}(z^m) > C^m\} \subset \mathcal{S}.$$

Notice that I consists of m connected component, each of which is contained in one of the $\mathcal{S}_a$, so we define I_a the component of I contained in $\mathcal{S}_a$, with $a \in \mathbb{Z}_m$.

Notice that if $z \in I_a$, then $\mathrm{Re}\left(ze^{\frac{2(m-a)}{m}\pi i}\right) > C$.

Now define the following subset of S

$$W_I = W_I(C) := \{(z, w) \in \mathbb{C}^2 : F^n(z, w) \in I \times I \text{ for all } n \in \mathbb{N}\} \cap \Omega \;, \tag{4.5.1}$$

and let

$$\mathcal{A}_I = \mathcal{A}_I(C) := \bigcup_n F^{-n}(W_I) \;.$$

Our next goal is to show that W_I is an absorbing set for Ω under F, that is $\mathcal{A}_I = \Omega$.

Since $W_I \subset S$, we define $(W_I)_{ab}$ the subset of W_I contained in S_{ab}. Notice that W_I is forward invariant by construction and that each $(W_I)_{ab}$ contains the set A_{ab} already defined in (4.2.9), for M sufficiently large; hence that they are all not empty. It will turn out in Corollary 4.5.6 that W_I is also an open set.

Since $W_I \subset S$ and forward invariant, Proposition 4.2.4 holds and hence the sets $(W_I)_{ab}$ are mapping to each other: $F\left((W_I)_{ab}\right) \subset (W_I)_{(b+1)a}$.

Moreover $(W_I)_{ab}$ is forward invariant under F^{2m}, in particular if $(W_I)_{ab}$ belongs to the short cycle, it is forward invariant under F^m. By Lemma 4.2.3 we have convergence of even and odd iterates of F on W_I.

From now on the entire section is devoted to prove the following proposition:

4.5.1 Proposition (W_I is absorbing for Ω)
The set W_I is absorbing for Ω under F, that is, $\mathcal{A}_I = \Omega$.

Define

$$\mathcal{X} := \{(z, w) \in \Omega \,:\, h_1(z, w) = 0 \text{ or } \infty\} \;,$$

and observe that since $\mathcal{X}$ is an analytic set, being the union of the 0-set and the ∞-set of a meromorphic function, it is locally a finite union of 1-complex-dimensional varieties.

Let K be a compact subset of $\Omega \setminus \mathcal{X}$, that is $h_1(P) \neq 0, \infty$ for all $P \in K$. We can define the quantities

$$M := \max_K(\max(|h_1|, |h_2|)) < \infty$$
$$m := \min_K(\min(|h_1|, |h_2|)) > 0.$$

Note that $M > 1$ because $|h_2| = \frac{\delta}{|h_1|}$ and $\delta > 2$. By Corollary 2.3 in [BSZ23], if $0 < \varepsilon < m$ there exists a constant c such that for every $(z_0, w_0) \in K$,

$$|z_n| \leq c(M + \varepsilon)^n. \tag{4.5.2}$$

Recall that $w_n = z_{n-1}$, hence

$$|w_n| \leq c(M + \varepsilon)^{n-1}. \tag{4.5.3}$$

Now consider the following remarks.

4.5.2 Remark. Recall that $\cos(m\theta) = T_m(\cos(\theta))$, with $m \in \mathbb{N}$, where T_m are the Chebyshev polynomials of the first kind defined as

$$T_m(x) = \sum_{h=0}^{\left[\frac{m}{2}\right]} (-1)^h \binom{m}{2h} x^{m-2h}(1-x^2)^h,$$

where $\left[\frac{m}{2}\right]$ is the integer part of $\frac{m}{2}$.

The proof of Proposition 4.5.1 relies on the following technical lemma. Recall that for a point $P = (z_0, w_0)$, we define $(z_n, w_n) := F^n(P)$.

4.5.3 Lemma

Define the sequence of harmonic functions u_n from Ω to $\mathbb{R}$ as $u_n(z_0, w_0) := \frac{-\operatorname{Re}(z_n^m)}{n}$. Then

1. *Let $K \subset \Omega$ be a compact set, then there exists $M = M(K)$ and $N \in \mathbb{N}$ such that $u_n \leq \log M$ on K for $n > N$.;*
2. *$u_n \to -\infty$ uniformly on compact subsets of W;*
3. *If $P \in \Omega \setminus \mathcal{A}_I$, then for all $\varepsilon > 0$ there exists a subsequence $n_k \longrightarrow \infty$ such that $u_{n_k}(P) \geq -\varepsilon$.*

We will later show that such a $P \in \Omega \setminus \mathcal{A}_I$ as in point *3* leads to a contradiction.

Proof. 1. Let $K \subset \Omega$ compact. Let η as in Proposition 3.4.6 applied to a slightly larger compact set $L \subset \Omega$ and to the analytic set $\mathcal{X}$. Let $U_\eta(\mathcal{X})$ be an η-neighborhood of $\mathcal{X}$. Because of Proposition 3.4.6 it is enough to show that there exists M and $N \in \mathbb{N}$ such that $u_n \leq \log M$ for $n > N$ on the set

$$K \setminus U_\eta(\mathcal{X})$$

which is a compact subset of $\Omega \setminus \mathcal{X}$. Hence it is enough to prove the claim for any compact subset K of $\Omega \setminus \mathcal{X}$.

Fix $\varepsilon \in (0, m)$ and let c as in (4.5.2) and (4.5.3). Suppose that there exists a subsequence (n_j) and points $(z, w) = (z(j), w(j)) \in K$ such that

$$-\frac{\operatorname{Re}(z_{n_j}^m)}{n_j} > \beta$$

for some β. We will show that $\beta \leq M$.

We have that

$$\begin{aligned} |z_{n_j+1}| = |e^{-z_{n_j}^m} + \delta e^{\frac{2\pi}{m} i} w_{n_j}| &\geq |e^{-z_{n_j}^m}| - \delta |w_{n_j}| \geq \\ &\geq e^{-\operatorname{Re}(z_{n_j}^m)} - \delta c (M+\varepsilon)^{n_j - 1} \geq \\ &\geq e^{\beta n_j} - \delta c (M+\varepsilon)^{n_j-1} \ . \end{aligned}$$

Furthermore

$$|z_{n_{j+1}}| \leq c(M+\varepsilon)^{n_j+1} \ .$$

Then we have

$$e^{\beta n_j} - \delta c (M+\varepsilon)^{n_j-1} \leq c(M+\varepsilon)^{n_j+1} \ ,$$

that is

$$e^{\beta n_j} \leq \delta c(M+\varepsilon)^{n_j-1} + c(M+\varepsilon)^{n_j+1} \ .$$

Since $M > 1$ and $\varepsilon > 0$, we have that $(M+\varepsilon) > 1$ and hence

$$e^{\beta n_j} < \delta c(M+\varepsilon)^{n_j+1} + c(M+\varepsilon)^{n_j+1} = c(\delta+1)(M+\varepsilon)^{n_j+1} \ .$$

Then

$$\beta < \frac{\log\big(c(\delta+1)\big)}{n_j} + \frac{n_j+1}{n_j} \log(M+\varepsilon)$$

which implies, using $n_j \longrightarrow \infty$ and $\varepsilon \longrightarrow 0$, that $\beta \leq \log M$.

2. Let K be a compact subset of W, since W is forward invariant, $F^n(K) \subset W$ for all $n \in \mathbb{N}$. Moreover there exist b and j such that $K \subset W_{b,j}$ defined in (4.3.1) and, by Proposition 4.3.2, we have that $F^n(K) \subset W_{b,n+j}$. Let $P = (z_0, w_0) \in K \subset W_{b,j}$ and observe that $z_n^m \in \mathbb{H}^+$, so $\mathrm{Re}(z_n^m) > 0$, hence our goal is to prove that

$$\frac{\mathrm{Re}(z_n^m)}{n} = \frac{|\mathrm{Re}(z_n^m)|}{n} \longrightarrow \infty \qquad \text{for } n \to \infty.$$

Since we are interested in $|\,\mathrm{Re}(z_n^m)|$, we can consider $\tilde{z}_n = z_n\, e^{\frac{2(m-j-n)}{m}\pi i} = |z_n| e^{i\tilde{\theta}_n}$ (with $-\frac{\pi}{2m} < \tilde{\theta}_n < \frac{\pi}{2m}$) instead of z_n, as

$$|\mathrm{Re}(z_n^m)| = |\mathrm{Re}(\tilde{z}_n^m)| = |z_n|^m \left|\cos(m\tilde{\theta}_n)\right| .$$

Denote by α_n the angle such that $|\tan(\alpha_n)| = C\ \frac{j+n+1}{j+n+2}$ with $C = \tan(\frac{\pi}{2m})$, and using (4.3.1), we have that $\cos(\tilde{\theta}_n) > \cos(\alpha_n)$ and

$$\cos(m\tilde{\theta}_n) > \cos(m\alpha_n). \tag{4.5.4}$$

It is easy to check that

$$|\cos(\alpha_n)| = \frac{j+n+1}{\sqrt{C^2(n+j+1)^2 + (n+j+2)^2}} =: U(n)$$

and by Remark 4.5.2,

$$|\cos(m\alpha_n)| = T_m(U(n)). \tag{4.5.5}$$

Using the explicit expressions of the iterates of F, that is equations (4.2.1) and (4.2.2), and the fact that $\delta > 2$, we have that

$$|z_n|^m \geq \begin{cases} \delta^{\frac{mn}{2}}\ |z_0 - 1|^m & \text{if } n \text{ is even} \\ \delta^{\frac{mn}{2}}\ |w_0 - 1|^m & \text{if } n \text{ is odd.} \end{cases}$$

Since $|z_0| > R_0 > 2$, we have that $|z_0 - 1| \geq |z_0| - 1 \geq 1$ and the same holds true for w_0, hence

$$|z_n|^m \geq \delta^{\frac{mn}{2}}. \tag{4.5.6}$$

By (4.5.4), (4.5.5) and (4.5.6), we conclude

$$\frac{|\mathrm{Re}(z_n)^m|}{n} = \frac{|z_n|^m |\cos(m\tilde{\theta}_n)|}{n} \geq \delta^{\frac{mn}{2}} \frac{T_m(U(n))}{n} \longrightarrow \infty$$

since the dominating term is $\delta^{\frac{mn}{2}}$ and $\delta > 2$.

3. Let $P = (z_0, w_0) \in \Omega \setminus \mathcal{A}_I$, and suppose by contradiction that there is $\varepsilon > 0$ and $N \in \mathbb{N}$ s.t.

$$u_n(P) < -\varepsilon \qquad \forall\, n \geq N\ .$$

   So we have that

$$-\frac{\mathrm{Re}(z_n^m)}{n} < -\varepsilon \qquad \forall\, n \geq N\ ,$$

   that is

$$\mathrm{Re}(z_n^m) > \varepsilon\, n \qquad \forall\, n \geq N\ .$$

   Since $\varepsilon > 0$ we have that there exists $N' > N$ such that

$$\mathrm{Re}(z_n^m) > C^m \qquad \forall\, n \geq N'\ ,$$

   where C is the constant fixed in (4.5.1).

   Since $w_n = z_{n-1}$ and since $P \in \Omega$ for hypothesis, we have that $F^n(P) = (z_n, w_n) \in W_I\ \forall\, n \geq N'$, so $P = (z_0, w_0) \in F^{-n}(W_I) \subset \mathcal{A}_I$, hence the contradiction.

□

Now consider the following lemma, compare with Lemma 3.4.7.

4.5.4 Lemma (Good holomorphic disks)
Let $P \in \Omega$, then there exists $\varphi : \overline{\mathbb{D}} \to \Omega$ holomorphic such that

- $\varphi(0) = P$.
- $D := \varphi(\mathbb{D}) \Subset \Omega$ *and* ∂D *is analytic.*
- *The one-dimensional Lebesgue measure of* $\partial\varphi(\mathbb{D})$ *intersected with* W *is bigger than* 0.

Proof. Since W is open it is enough to take $\varphi(\mathbb{D}) \cap W \neq \varnothing$ to get positive one- dimensional Lebesgue measure of $\partial\varphi(\mathbb{D}) \cap W$.

Let $P \in \Omega_{ab}$, with $a, b \in \mathbb{Z}_m$. Since W_{ab} is not empty for all $a, b \in \mathbb{Z}_m$ there exists $Q \in W_{ab}$. Moreover Ω_{ab} is open and connected, so there exists a simple real analytic curve in Ω_{ab} passing through P and Q. Complexifying this curve we get a holomorphic disk passing through P that we can write as $\varphi(\mathbb{D})$ for some φ holomorphic defined in a neighborhood of $\mathbb{D}$. Up to precomposing φ with a Moebius transformation we can assume that $P = \varphi(0)$. □

We recall the mean value property, for a proof see Lemma 3.4.8 for harmonic functions.

4.5.5 Remark (Mean value property)**.** Let $\mathbb{D} \subset \mathbb{C}$ be an open unit disk and $\varphi : \overline{\mathbb{D}} \to \Omega$ a holomorphic map. Let u be harmonic on $D = \varphi(\mathbb{D})$ and continuous up to the boundary of D. Let $P_0 := \varphi(0)$, then

$$u(P_0) = \frac{1}{2\pi}\int_{\partial\mathbb{D}} u(\zeta)|\varphi'(\zeta)|^{-1} d\zeta.$$

Proof of Proposition 4.5.1. Let $P \in \Omega \backslash \mathcal{A}_I$ and $D := \varphi(\mathbb{D})$ as in Lemma 4.5.4. Let μ be the pushforward under φ of the one-dimensional Lebesgue measure on $\partial\mathbb{D}$. Let $K \subset W$ compact such that $\mu(\partial D \cap K)$ is strictly positive.

Let $\mu_{\mathrm{good}} = \mu(\partial D \cap K) > 0$ and $\mu_{\mathrm{bad}} = \mu(\partial D \cap (\Omega \setminus K))$. Since $D \subset \Omega$, then $\partial D = (\partial D \cap K) \cup (\partial D \cap (\Omega \setminus K))$, moreover K is compact and Ω is open, then all these sets are measurable.

By Lemma 4.5.3 for any $\mathcal{M} > 0$ there exists N such that $u_N \leq -\mathcal{M}$ on K, $u_N(P) \geq -\varepsilon$ for some $\varepsilon > 0$ since $P \in \Omega \setminus \mathcal{A}_I$, and $u_N \leq \log M$ on $\overline{D}$ (with $M = M(\overline{D})$). Using the Mean value property we have

$$-\varepsilon \leq u_N(P) = \frac{1}{2\pi}\int_{\partial D} u_N(\zeta)|\varphi'(\zeta)|d\zeta = \frac{1}{2\pi}\int_{\partial D\cap K} u_N(\zeta)|\varphi'(\zeta)|d\zeta + \frac{1}{2\pi}\int_{\partial D\cap(\Omega\setminus K)} u_N(\zeta)|\varphi'(\zeta)|d\zeta \leq$$
$$\leq \frac{1}{2\pi}\left(-\mathcal{M}\mu_{\text{good}} + \log M\mu_{\text{bad}}\right)\cdot \sup_{\partial\mathbb{D}}|\varphi'|^{-1}.$$

Since $\mathcal{M}$ is arbitrarily large, this gives a contradiction. □

As a corollary of Proposition 4.5.1 we obtain what follows.

4.5.6 Corollary
W_I is an open set.

Proof. Let $P \in W_I$, our goal is to find an open neighborhood U of P such that $U \subset W_I$. Since $W_I \subset \Omega \cap (I \times I)$ which is open, there exist an open neighborhood U_P of P compactly contained in $\Omega \cap (I \times I)$. Recall that W_I is absorbing for Ω, then

$$\exists\ N > 0 \quad \text{such that} \quad F^n(\overline{U_P}) \subset W_I \quad \forall\, n \geq N\ . \tag{4.5.7}$$

As usual let $P_j := F^j(P)$ and notice that by definition of W_I, we have that $P_j \in W_I \subset I \times I$, which is an open set. Hence there is an open neighborhood U_j of P_j such that $U_j \subset I \times I$.

Define

$$U := \bigcap_{j=1}^{N} F^{-j}(U_j) \cap U_P\ ,$$

it is clear that $P \in U$ and U is an open set since it is a finite intersection of open sets. We only need to prove that $U \subset W_I$.

Notice that $U \subset U_P$, hence U is in the Fatou set and moreover $U \subset I \times I$. So we only need to check that $F^j(U) \subset I \times I$ for all $j \geq 0$. If $j \geq N$, this is true by (4.5.7); while if $j < N$, this is true by definition since $F^j(U) \subset U_j \subset I \times I$. □

4.6 Limit Sets and Geometric Structure of Ω

We first study the image of $(W_I)_{ab}$ under h_1, h_2 and then use the fact that W_I is absorbing for Ω to understand $h_1(\Omega_{ab})$ and $h_1(\Omega_{ab})$. Moreover we show that Ω consists of m^2 connected components Ω_{ab}, each of which is biholomorphic to $\mathbb{H} \times \mathbb{H}$.

Define the following m open slices of $\mathbb{C}$ defined in terms of angles, all of amplitude $\frac{2\pi}{m}$:

$$
\begin{aligned}
U_0 &:= \left(-\frac{\pi}{m}, \frac{\pi}{m}\right)\\
U_1 &:= \left(\frac{\pi}{m}, \frac{\pi}{m} + \frac{2\pi}{m}\right)\\
U_2 &:= \left(\frac{\pi}{m} + \frac{2\pi}{m}, \frac{\pi}{m} + 2\frac{2\pi}{m}\right)\\
&.\\
&.\\
U_j &:= \left(\frac{\pi}{m} + (j-1)\frac{2\pi}{m}, \frac{\pi}{m} + j\frac{2\pi}{m}\right)\\
&.\\
&.\\
U_{m-1} &:= \left(\frac{\pi}{m} + (m-2)\frac{2\pi}{m}, 2\pi - \frac{\pi}{m}\right)
\end{aligned}
\tag{4.6.1}
$$

Observe that

$$\mathbb{C} = \bigcup_{j \in \mathbb{Z}_m} \overline{U}_j \,.$$

Notice that if $(z, w) \in (W_I)_{ab}$, the ratio $\frac{z}{w} \in U_{a-b}$ with $a-b \mod m$. Remember that we can take $(W_I)_{0b}$ as the representative of the cycle. With this in mind we consider the following lemma.

4.6.1 Lemma (Limit set for W_I)
Let U_J defined as in (4.6.1), with $j \in \mathbb{Z}_m$. Then

$$h_1\left((W_I)_{0b}\right) \subseteq U_{m-b} \quad \textit{and} \quad h_2\left((W_I)_{0b}\right) \subseteq U_{b+1}\,, \qquad \textit{if } b \neq \frac{m-1}{2}$$

and

$$h_1\left((W_I)_{0b}\right), h_2\left((W_I)_{0b}\right) \subseteq U_{\frac{m+1}{2}}\,, \qquad \textit{if } b = \frac{m-1}{2}.$$

Proof. Remember that $h_1(z_0, w_0) = \lim_{n\to\infty} \frac{z_{2n}}{w_{2n}}$ and $h_2(z_0, w_0) = \lim_{n\to\infty} \frac{z_{2n+1}}{w_{2n+1}}$. Hence if $(z_0, w_0) \in (W_I)_{0b}$, with $b \neq \frac{m-1}{2}$, then $\frac{z_{2n}}{w_{2n}} \in U_{m-b}$ and $\frac{z_{2n+1}}{w_{2n+1}} \in U_{b+1}$. Taking the limit we get $h_1(z_0, w_0) \in \overline{U}_{m-b}$ and $h_2(z_0, w_0) \in \overline{U}_{b+1}$.

If $m - b = b + 1$, that is if $b = \frac{m-1}{2}$, we have that $\frac{z_k}{w_k} \in U_{\frac{m+1}{2}}$ and taking the limit we get $h_1(z_0, w_0), h_2(z_0, w_0) \in \overline{U}_{\frac{m+1}{2}}$.
Since W_I is open by Corollary 4.5.6 its image under a holomorphic map of maximal rank is open, hence we can replace each $\overline{U_j}$ by U_j. □

To better understand Lemma 4.6.1, let us consider two examples: $m = 5$ and $m = 6$. In the following examples, to simplify notation, instead of writing $(W_I)_{ab}$ in the first column, we simply write ab.

$m = 5$

(z,w)	$\frac{z}{w}$	*iteration*
00	U_0	0
10	U_1	1
11	U_0	2
21	U_1	3
22	U_0	4
32	U_1	5
33	U_0	6
43	U_1	7
44	U_0	8
04	U_1	9
00	U_0	10

(z,w)	$\frac{z}{w}$	*iteration*
01	U_4	0
20	U_2	1
12	U_4	2
31	U_2	3
23	U_4	4
42	U_2	5
34	U_4	6
03	U_2	7
40	U_4	8
14	U_2	9
01	U_4	10

(z,w)	$\frac{z}{w}$	*iteration*
02	U_3	0
30	U_3	1
13	U_3	2
41	U_3	3
24	U_3	4
02	U_3	5

$m = 6$

(z,w)	$\frac{z}{w}$	*iteration*
00	U_0	0
10	U_1	1
11	U_0	2
21	U_1	3
22	U_0	4
32	U_1	5
33	U_0	6
43	U_1	7
44	U_0	8
54	U_1	9
55	U_0	10
05	U_1	11
00	U_0	12

(z,w)	$\frac{z}{w}$	*iteration*
01	U_5	0
20	U_2	1
12	U_5	2
31	U_2	3
23	U_5	4
42	U_2	5
34	U_5	6
53	U_2	7
45	U_5	8
04	U_2	9
50	U_5	10
15	U_2	11
01	U_5	12

(z,w)	$\frac{z}{w}$	*iteration*
02	U_4	0
30	U_3	1
13	U_4	2
41	U_3	3
24	U_4	4
52	U_3	5
35	U_4	6
03	U_3	7
40	U_4	8
14	U_3	9
51	U_4	10
25	U_3	11
02	U_4	12

To obtain a visual representation, see Figure 4.1 and observe that the components of W_I belonging to the same cycle are mapped, under h_1 and h_2, into two distinct sectors U_j and U_k, such that $j+k = 1 \mod m$; with the exception of the components of W_I that belong to the short cycle (that occurs if m is odd), in which case they are mapped, under h_1 and h_2, into the same sector $U_{\frac{m+1}{2}}$.

Consider the following proposition in which we show the conjugacy φ between F and its linear part L on Ω, and then in the remark we estimate the distance between the conjugacy and the identity map.

4.6.2 Proposition (Conjugacy)
F is conjugate to the linear map $L(z,w) = (\delta e^{\frac{2\pi}{m}i}w, z)$ on the set Ω through a biholomorphism φ.

Proof. Let $\alpha = \delta e^{\frac{2\pi}{m}i}$, it is easy to show that $L^{-n}(z,w) = (\frac{z}{\alpha^{n/2}}, \frac{w}{\alpha^{n/2}})$ if n is even and $L^{-n}(z,w) = (\frac{w}{\alpha^{(n-1)/2}}, \frac{z}{\alpha^{(n+1)/2}})$ if n is odd.

Let $\varphi_n : \mathbb{C}^2 \to \mathbb{C}^2$ be the automorphisms defined as

$$\varphi_n := L^{-n} \circ F^n.$$

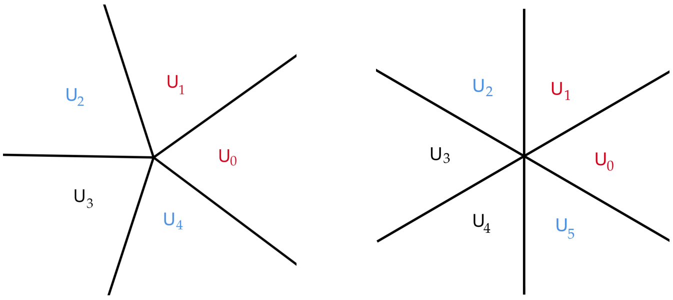

Figure 4.1: Case $m = 5$ and $m = 6$ on the line at infinity without the point at infinity (on $\mathbb{P}^1 \setminus \{\infty\}$).

We first show that F is conjugate to L on W_I.
Our goal is to prove that φ_n converge to a map $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$ uniformly on W_I so we obtain that φ_n satisfy the functional equation $\varphi_{n+1} = L^{-1} \circ \varphi_n \circ F$, and so the map φ is a conjugacy between F and L.

Using the explicit expressions for the iterates of F, we compute

$$\varphi_{2k}(z, w) = \left(z + \sum_{j=1}^{k} \alpha^{-j} f(z_{2j-1}), w + \sum_{j=1}^{k} \alpha^{-j} f(z_{2j-2}) \right), \tag{4.6.2}$$

$$\varphi_{2k+1}(z, w) = \left(z + \sum_{j=1}^{k} \alpha^{-j} f(z_{2j-1}), w + \sum_{j=1}^{k+1} \alpha^{-j} f(z_{2j-2}) \right), \tag{4.6.3}$$

and taking the limit we obtain, using the definitions (4.2.4) and (4.2.5),

$$\varphi(z, w) = (z + \Delta_1(z, w), w + \Delta_2(z, w)),$$

If $P = (z, w) \in W_I$, then $F^n(P) = (z_n, w_n) \subset I \times I \subset S$ for all j, so $\Delta_1(z, w)$ and $\Delta_2(z, w)$ are convergent. Hence φ is a holomorphic map from W_I to $\varphi(W_I)$. Recall that W_I is open by Corollary 4.5.6. Moreover on W_I, using (4.2.4) and (4.2.5), we get

$$\Big\|(\varphi - Id)(z, w)\Big\| = \Big\|\Big(\Delta_1(z, w), \Delta_2(z, w)\Big)\Big\| < \sqrt{2}\Delta(z, w) < \sqrt{2}. \tag{4.6.4}$$

It follows that φ is open because W_I is an unbounded set, hence if φ had rank 0 or 1, $\|(\varphi - Id)\|$ could not be bounded on W_I. Hence the map φ is injective by Hurwitz Theorem (see [Kra01], Exercise 3 on page 310) because the maps φ_n are injective and their limit has rank 2. It follows that φ is a biholomorphism between W_I and $\varphi(W_I)$.

To extend φ to all of Ω recall that W_I is absorbing for Ω. So if $P \in \Omega$, we have that $F^k(P) \in W_I$ for some $k \in \mathbb{N}$, hence we can define $\varphi(P) = L^{-k} \circ \varphi \circ F^k(P)$. Since F is an automorphism, φ extends as a biholomorphism from Ω to $\varphi(\Omega)$.

□

4.6.3 Remark. More generally, from (4.6.4) it follows that if $P = (z, w) \in S$ is such that $F^n(P) \in S$ for all $n \in \mathbb{N}$, then

$$\Big\|(\varphi - Id)(z, w)\Big\| = \Big\|\Big(\Delta_1(z, w), \Delta_2(z, w)\Big)\Big\| < \sqrt{2}\Delta(z, w) < \sqrt{2}.$$

4.6.4 Lemma
$\varphi(\Omega) \subset S$.

Proof. We first prove that $\varphi(W_I) \subset S$, then, using Proposition 4.5.1, we extend this result to Ω.

Since W_I is forward invariant and contained in S, using Remark 4.6.3, it follows that $\varphi(W_I)$ is contained in a $\sqrt{2}$-neighborhood V of W_I.

Assume by contradiction that there exists $Q \in \varphi(W_I) \setminus S$. We can assume, without loss of generality, that $Q = (z_0, w_0) \in \varphi((W_I)_{00})$, with $z_0 = re^{i\theta} \notin \mathcal{S}_0$. Notice that since W_I is forward invariant under F and φ is a conjugacy, also $\varphi(W_I)$ is forward invariant under L, so $L^{2mn}(Q) = (\alpha^{mn} z, \alpha^{mn} w) = (\delta^{mn} re^{i\theta}, \delta^{mn} w) \in \varphi((W_I)_{00})$.

Since $\delta^{nm} r$ tends to infinity and since θ is such that $re^{i\theta} \notin \mathcal{S}_0$, the distance of $L^{2mn}(Q)$ from the boundary of S_{00} tends to infinity, hence so does the distance of $L^{2mn}(Q)$ from $(W_I)_{00} \subset S_{00}$, contradicting $\varphi(W_I) \subset V$. Hence $\varphi(W_I) \subset S$.

Since W_I is absorbing for Ω under F, $\varphi \circ F = L \circ \varphi$, and $\varphi(W_I)$ is completely invariant under L, we have that

$$\varphi(\Omega) = \varphi(\bigcup_{n \geq 0} F^{-n}(W_I)) = \bigcup_{n \geq 0} L^{-n}(\varphi(W_I)) \subset \varphi(W_I) \subset S. \tag{4.6.5}$$

□

4.6.5 Proposition
Ω consists of m^2 distinct connected components.

Proof. We will prove that $\varphi(\Omega)$ consists of exactly m^2 connected components, so, using the fact that φ is a biholomorphism, the same is true for Ω.

By Corollary 4.4.2, we have that Ω consists of at most m^2 connected components and again since φ is a biholomorphism, the same is true for $\varphi(\Omega)$. Using Lemma 4.6.4, we have that $\varphi(\Omega) \subset S$ and as usual let us define $\varphi(\Omega)_{ab}$ the component of $\varphi(\Omega)$ contained in S_{ab}, with $a, b \in \mathbb{Z}_m$. We conclude if we prove that $\varphi(\Omega)_{ab} \neq \varnothing$ for all $a, b \in \mathbb{Z}_m$.

Since the sets A_{ab} defined in (4.2.9) are contained in Ω for M sufficiently large, and since (using Remark 4.6.3) a $\sqrt{2}$-neighborhood of A_{ab} is contained in S_{ab} for M sufficiently large, we have that $\varphi(A_{ab}) \subset \varphi(\Omega)_{ab}$ for M sufficiently large, hence $\varphi(\Omega)_{ab} \neq \varnothing$. □

We now recall a simple topological fact, for a proof see Lemma 3.5.3 of Chapter 3 that we will use in Proposition 4.6.7.

4.6.6 Remark. Let $A, B \subset \mathbb{C}^n$ open and A is connected. If $A \cap B \neq \varnothing$ and $\partial B \cap A = \varnothing$, then $A \subseteq B$.

4.6.7 Proposition (Geometric structure of Ω)
Ω is biholomorphic to S.

Proof. Let W defined in (4.3.2). Since $W \subset \Omega$ is invariant, we have that

$$\bigcup_{n \in \mathbb{N}} F^{-n}(W) \subset \Omega\,,$$

moreover, since φ is defined on Ω, it is also defined on W, so

$$\bigcup_{n \in \mathbb{N}} L^{-n}(\varphi(W)) \subset \varphi(\Omega)\ . \tag{4.6.6}$$

Let $U \subset S$ such that S is a $\sqrt{2}$-neighborhood of U.
Let $Q = (z, w) \in U$ and notice that $\lambda Q = (\lambda z, \lambda w) \in U$ for all $\lambda > 1$; furthermore

$$\bigcup_{\substack{n \in \mathbb{N} \\ \lambda > 1}} L^{-n}(\lambda Q) = \mu Q = (\mu z, \mu w) \quad \text{with} \quad \mu > 0\,.$$

By varying $Q \in U$, we can cover the entire set S:

$$S = \bigcup_{\substack{n\in\mathbb{N}\\ \lambda>1\\ Q\in U}} L^{-n}(\lambda Q),$$

that is

$$S = \bigcup_{n\in\mathbb{N}} L^{-n}(U).$$

In view of Remark 4.6.6, let $B = \varphi(W)$ and $A = U^*$, where U^* is U with $\operatorname{Re} z, \operatorname{Re} w$ sufficiently large such that $\partial(\varphi(W)) \cap U^* = \varnothing$. Notice that $U^* \cap \varphi(W) \neq \varnothing$, so by Remark 4.6.6, we have

$$\varphi(W) \subseteq U^*. \tag{4.6.7}$$

Moreover

$$S = \bigcup_{n\in\mathbb{N}} L^{-n}(U^*). \tag{4.6.8}$$

Using Remark 4.6.3, equation (4.6.8), equation (4.6.7) and Lemma 4.6.4, we obtain

$$S = \bigcup_{n\in\mathbb{N}} L^{-n}(U^*) \subseteq \bigcup_{n} L^{-n}(\varphi(W)) \subset \varphi(\Omega) \subset S\ ,$$

so $\varphi(\Omega) = S$. Again since φ is a biholomorphism, the claim follows. ☐

As a corollary we have what follows.

4.6.8 Corollary
Each Fatou component of Ω is biholomorphic to $\mathbb{H} \times \mathbb{H}$.

Proof. By Proposition 4.6.7, Ω is biholomorphic to S and since S has m^2 connected components S_{ab}, each of which is biholomorphic to $\mathbb{H} \times \mathbb{H}$, the same is true for Ω. ☐

We now study the limit set of Ω.

4.6.9 Proposition (Hyperbolic limit sets)
$h_1(\Omega_{ab})$ and $h_2(\Omega_{ab})$ are hyperbolic.

Proof. By Proposition 4.5.1, W_I is absorbing for Ω under F, hence by Proposition 4.6.5 and because of the sets $(W_I)_{ab}$ are mapping to each other, each $(W_I)_{ab}$ is absorbing for Ω_{ab} (Fatou components of F) under F^{2m}, in particular if $(W_I)_{ab}$ belongs to the short cycle, it is absorbing for Ω_{ab} under F^m. Consequently, $\bigcup_{k\in\mathbb{Z}_m}(W_I)_{(a+k)\,(b+k)}$ is absorbing for $\bigcup_{k\in\mathbb{Z}_m}\Omega_{(a+k)\,(b+k)}$ under F^2.

Using Lemma 4.6.1, the fact that W_I is open, and considering that for each cycle we can take Ω_{0b} as the representative of the cycle, we have

$$h_1(\Omega_{0b}) \subset h_1(\bigcup_{k\in\mathbb{Z}_m} \Omega_{k\,(b+k)}) = h_1(\bigcup_{k\in\mathbb{Z}_m} (W_I)_{k\,(b+k)}) \subseteq \begin{cases} U_{m-b} & \text{if } b \neq \frac{m-1}{2} \\ U_{\frac{m+1}{2}} & \text{if } b = \frac{m-1}{2} \end{cases}$$

and

$$h_2(\Omega_{0b}) \subset h_2(\bigcup_{k\in\mathbb{Z}_m} \Omega_{k\,(b+k)}) = h_2(\bigcup_{k\in\mathbb{Z}_m} (W_I)_{k\,(b+k)}) \subseteq \begin{cases} U_{b+1} & \text{if } b \neq \frac{m-1}{2} \\ U_{\frac{m+1}{2}} & \text{if } b = \frac{m-1}{2} \end{cases}$$

where U_j are defined in (4.6.1).

So $h_1(\Omega_{ab})$ with $a, b \in \mathbb{Z}_m$ are hyperbolic sets. ☐

We devote the rest of this section to proving the following proposition. Again we only consider $h_i(\Omega_{0b})$ to simplify notation.

4.6.10 Proposition (Limit set for Ω)
Let U_j defined in (4.6.1), then

$$h_1(\Omega_{0b}) = \begin{cases} U_{m-b} & \text{if } b \neq \frac{m-1}{2} \\ U_{\frac{m+1}{2}} & \text{if } b = \frac{m-1}{2} \end{cases}$$

and

$$h_2(\Omega_{0b}) = \begin{cases} U_{b+1} & \text{if } b \neq \frac{m-1}{2} \\ U_{\frac{m+1}{2}} & \text{if } b = \frac{m-1}{2} \end{cases}$$

To prove Proposition 4.6.10 we shall use the following lemma.

4.6.11 Lemma

$$h_1\left((W)_{0b}\right) \supseteq U_{m-b} \;\; \text{and} \;\; h_2\left((W)_{0b}\right) \supseteq U_{b+1}\,, \quad \text{if } b \neq \frac{m-1}{2}$$

and

$$h_1\left((W)_{0b}\right), h_2\left((W)_{0b}\right) \supseteq U_{\frac{m+1}{2}}\,, \quad \text{if } b = \frac{m-1}{2}.$$

Before proving Lemma 4.6.11 let us see how Lemma 4.6.11 and Proposition 4.6.9 imply Proposition 4.6.10.

Proof of Proposition 4.6.10. We prove the claim for h_1; for $h_2 = \frac{a}{h_1}$, it follows by symmetry. Since $\Omega_{ab} \supset W_{ab}$ for any $a, b \in \mathbb{Z}_m$, it follows that $h_1(\Omega_{ab}) \supset h_1(W_{ab})$. So in view of Lemma 4.6.11, $h_1(\Omega_{ab}) \supseteq U_j$ for some $j \in \mathbb{Z}_m$. By Proposition 4.6.9, we have that $h_1(\Omega_{ab}) \subseteq U_j$, and so $h_1(\Omega_{ab}) = U_j$. ☐

For convenience, we will now restate the extended version of Rouché's theorem, which was previously demonstrated in Chapter 3 (see Theorem 3.6.4). Here ∂ denotes the topological boundary, and $\text{dist}_{\text{spher}}$ denotes the spherical distance.

4.6.12 Theorem (Rouché's Theorem in $\mathbb{C}^2$)
Let $B \subset \mathbb{C}^2$ be a polydisk, F, G be holomorphic maps defined in a neighborhood of $\overline{B}$ which take values in $\hat{\mathbb{C}}$. Let $c \in G(B)$, let $\varepsilon = \text{dist}_{\text{spher}}(c, G(\partial B)) > 0$ and assume

$$\text{dist}_{\text{spher}}(F, G) < \varepsilon \;\; \text{on } \partial B.$$

Then $c \in F(B)$.

Proof of Lemma 4.6.11. We show that $U_0 \subset h_1(W_{00})$, the other cases are analogous. Recall that orbits of points in W are contained in S, hence Remark 4.2.2 holds. Since

$$\frac{z_{2n}}{w_{2n}} = \frac{z_0 + \Delta_1^n(z_0, w_0)}{w_0 + \Delta_2^n(z_0, w_0)},$$

dividing the numerator and the denominator by w_0 and using the fact that $\frac{1}{1+x} = \sum_{j=0}^{\infty}(-x)^j$ for

$|x| < 1$, considering $x = \frac{\Delta_2^n(z_0,w_0)}{w_0}$, we obtain

$$\begin{aligned}\frac{z_{2n}}{w_{2n}} &= \left(\frac{z_0}{w_0} + \frac{\Delta_1^n(z_0,w_0)}{w_0}\right)\frac{1}{1+\frac{\Delta_2^n(z_0,w_0)}{w_0}} = \left(\frac{z_0}{w_0} + \frac{\Delta_1^n(z_0,w_0)}{w_0}\right)\sum_{j=0}^{\infty}\left(-\frac{\Delta_2^n(z_0,w_0)}{w_0}\right)^j = \\ &= \left(\frac{z_0}{w_0} + \frac{\Delta_1^n(z_0,w_0)}{w_0}\right)\left(1+\sum_{j=1}^{\infty}\left(-\frac{\Delta_2^n(z_0,w_0)}{w_0}\right)^j\right) = \\ &= \frac{z_0}{w_0} + \frac{\Delta_1^n(z_0,w_0)}{w_0} + \left(\frac{z_0}{w_0} + \frac{\Delta_1^n(z_0,w_0)}{w_0}\right)\sum_{j=1}^{\infty}\left(-\frac{\Delta_2^n(z_0,w_0)}{w_0}\right)^j.\end{aligned}$$

That is

$$\frac{z_{2n}}{w_{2n}} - \frac{z_0}{w_0} = \frac{\Delta_1^n(z_0,w_0)}{w_0} + \left(\frac{z_0}{w_0} + \frac{\Delta_1^n(z_0,w_0)}{w_0}\right)\sum_{j=1}^{\infty}\left(\frac{-\Delta_2^n(z_0,w_0)}{w_0}\right)^j \quad \forall n \geq 0. \tag{4.6.9}$$

This expression makes sense for $|x| = \left|\frac{-\Delta_2^n(z_0,w_0)}{w_0}\right| < 1$, hence, in view of Remark 4.2.2, for $|w_0| > 1$. Recall also that $|\sum_{j=1}^{\infty} x^j| = \frac{|x|}{1-x} \leq 2|x|$ if $|x| < \frac{1}{2}$. Let $K \subset \hat{\mathbb{C}}$ be a compact set and suppose that $\frac{z_0}{w_0}$ takes values in K. By (4.6.9) and using Remark 4.2.2, for any $\varepsilon > 0$ there exists $M = M(K,\varepsilon)$ such that

$$\left|\frac{z_{2n}}{w_{2n}} - \frac{z_0}{w_0}\right| < \varepsilon \text{ for } |w_0| > M \text{ and } \frac{z_0}{w_0} \in K. \tag{4.6.10}$$

Consider the function $G(z,w) := \frac{z}{w}$ Observe that

$$G^{-1}(re^{i\theta}) = \{(r_1e^{i\theta_1}, r_2e^{i\theta_2}) \in \mathbb{C}^2 : \frac{r_1}{r_2} = r, \theta = \theta_1 - \theta_2\}.$$

Let $c \in U_0$. By the shape of W we have that $G(W_{00}) = U_0$, that $\varepsilon := \frac{1}{2}\operatorname{dist}_{\text{spher}}(c, G(\partial W)) > 0$, and that we can choose $Q = (z_0, w_0) \in W_{00} \in G^{-1}(c)$ such that $|w_0|$ is arbitrarily large. By taking a limit in n in equation (4.6.9) and on a sufficiently small polydisk centered at Q we can ensure that $\operatorname{dist}_{\text{spher}}(h_1, G) < \varepsilon$, hence the claim follows by Rouché's Theorem. ☐

The Main Theorem 4.1.1 stated in the introduction is a direct consequence of Propositions 4.4.1, Proposition 4.4.3, Proposition 4.6.2, Proposition 4.6.5, Proposition 4.6.7 and Proposition 4.6.10.

Chapter 5

Appendix

5.1 Holomorphic Functions and Complex Manifolds

Now, we recall the definition of a holomorphic function and of a complex manifold following [DSSST06].

Let $D \subset \mathbb{C}^n$ be a domain (that is D is open and connected). A function $f : D \to \mathbb{C}$ is called $\mathbb{C}$-*differentiable* at $z^0 \in D$ if there exist $l_1, \ldots, l_n \in \mathbb{C}$ such that

$$\left| f(z) - f(z^0) - \sum_{\alpha=1}^{n} l_\alpha (z_\alpha - z_\alpha^0) \right| = O_2(|z - z^0|).$$

If a function f is $\mathbb{C}$-differentiable in z^0, then, for every $\alpha = 1, \ldots, n$, we have

$$l_\alpha = \frac{\partial f}{\partial z_\alpha}(z^0),$$

where

$$\frac{\partial f}{\partial z_\alpha}(z^0) = \frac{1}{2}\left(\frac{\partial f}{\partial x_\alpha} - i \frac{\partial f}{\partial y_\alpha} \right)(z^0) = \lim_{h \to 0} \frac{f(z_1^0, \ldots, z_\alpha^0 + h, \ldots, z_n^0) - f(z^0)}{h}.$$

Moreover,

$$l_\alpha = \frac{\partial f}{\partial z_\alpha}(z^0) = \frac{\partial f}{\partial x_\alpha}(z^0) = -i \frac{\partial f}{\partial y_\alpha}(z^0).$$

The expression $\frac{\partial f}{\partial z_\alpha}(z^0)$ is called the partial derivative of f at z^0.

5.1.1 Remark. A function f is $\mathbb{C}$-differentiable at z^0 if and only if it is $\mathbb{R}$-differentiable at z^0 and the following Cauchy-Riemann conditions are fulfilled:

$$\frac{\partial f}{\partial \overline{z}_\alpha}(z^0) = 0, \tag{5.1.1}$$

where

$$\frac{\partial f}{\partial \overline{z}_\alpha}(z^0) = \frac{1}{2}\left(\frac{\partial f}{\partial x_\alpha} + i \frac{\partial f}{\partial y_\alpha} \right)(z^0). \tag{5.1.2}$$

The differential operators $\frac{\partial}{\partial z_\alpha}$, $\frac{\partial}{\partial \overline{z}_\alpha}$ are defined for all differentiable functions f, and the following notations are universally adopted:

$$\partial f = \sum_{\alpha=1}^{n} \frac{\partial f}{\partial z_\alpha} dz_\alpha, \tag{5.1.3}$$

$$\overline{\partial} f = \sum_{\alpha=1}^{n} \frac{\partial f}{\partial \overline{z}_\alpha} d\overline{z}_\alpha, \tag{5.1.4}$$

where $dz_\alpha = dx_\alpha + idy_\alpha$ and $d\overline{z}_\alpha = dx_\alpha - idy_\alpha$. It follows that:

$$df = \partial f + \overline{\partial} f. \tag{5.1.5}$$

A function f, $\mathbb{C}$-differentiable at every point of a domain D, is called *holomorphic* in D.

Now let D and D' be open sets in $\mathbb{C}^n$. A map $F(z) = (f_1(z), ..., f_n(z)) : D \longrightarrow D'$ is said to be holomorphic if $f_1, ..., f_n$ are holomorphic functions.

Before defining what a complex manifold is, we must first introduce the definition of a topological manifold.

5.1.2 Definition. A *topological manifold* M of real dimension n is a Hausdorf space and 2-numerable in which every point has an open neighborhood that is homeomorphic to $\mathbb{R}^n$.

If $\varphi : U \to V$ is a homeomorphism between an open set of X and an open set of $\mathbb{R}^n$, then the pair (U, φ) is called a chart. Thus, if X is a topological manifold, there exists a family of charts

$$\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$$

that cover X, meaning that

$$X = \bigcup_{\alpha \in A} U_\alpha.$$

Such a family of charts is called an atlas.

If $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are two charts such that $U_\alpha \cap U_\beta \neq \varnothing$, then the map

$$\varphi_\alpha^\beta : \varphi_\beta(U_\alpha \cap U_\beta) \longrightarrow \varphi_\alpha(U_\alpha \cap U_\beta),$$
$$x \longmapsto \varphi_\alpha \circ \varphi_\beta^{-1}(x)$$

is called the transition function. Transition functions are homeomorphisms.

In the definition, one can equivalently require that X is locally homeomorphic to an open set in $\mathbb{R}^n$.

A *complex manifold* M of complex dimension n is a topological manifold of real dimension $2n$ whose transition functions, seen as maps from open subsets of $\mathbb{C}^n$ through the natural identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, are holomorphic (not only homeomorphic as in the definition of topological manifold).

5.2 Attracting Set Based on the Eigenvalues of DF_0

Let F be an automorphism of $\mathbb{C}^k$ with a fixed point at the origin. Even if the origin is not an attracting fixed point, there can still be points whose orbits converge to the origin. In [PVW08] they study how large such an attracting set can be and they give an overview of what is known on attracting sets based on the eigenvalues of DF_0. We now report what is stated in [PVW08].

They study the attracting set

$$\Omega = \{z \in \mathbb{C}^k \mid F^n(z) \to p, \text{ as } n \to \infty\}.$$

The nature of an attracting set depends significantly on the eigenvalues of DF_0. If all eigenvalues have a modulus strictly less than 1, we say that 0 is an attracting fixed point and we generally called

Ω the *attracting basin* . This is the simplest case, and it was studied by J.P. Rosay and W. Rudin in [RR88]. What occurs is that the attracting set must include a neighborhood of the origin and is biholomorphic to $\mathbb{C}^k$.

5.2.1 Theorem
Let $F : \mathbb{C}^k \to \mathbb{C}^k$ be a holomorphic map such that $F(0) = 0$, and let Ω be the attracting set. If Ω contains a neighborhood of the origin, then 0 is an attracting fixed point.

It is the recurrent case as seen in Chapter 2, at the beginning of Section 2.3.

When all eigenvalues of DF_0 have modulus strictly greater than 1, one can consider the inverse mapping, resulting in a situation analogous to that where all eigenvalues of DF_0 have modulus strictly less than 1.

If none of the eigenvalues of DF_0 have modulus 1, and if there are eigenvalues with modulus both greater than 1 and less than 1, the fixed point 0 is referred to as *hyperbolic*. It follows again from [RR88] that, in this case, the attracting set is biholomorphic to $\mathbb{C}^m$, where m denotes the number of eigenvalues with modulus less than 1.

The complex structure of attracting sets has also been investigated in the semi-attracting case (where some eigenvalues of DF_0 have modulus smaller than or equal to 1), as well as for automorphisms tangent to the identity (where $DF_0 = \mathrm{Id}$). In both instances, the attracting set can also be biholomorphic to (potentially lower-dimensional) complex Euclidean space. See, for example, [Ued86] for the semiattracting case, and [Wei97],[Hak98] and [Hak97] for automorphisms tangent to the identity. The automorphisms of $\mathbb{C}^k$ constructed by Ueda, Hakim and Weickert that have a neutral or semi-attractive fixed point with an attracting set biholomorphic to $\mathbb{C}^k$ all have the fixed point lying in the boundary of the basin (non-recurrent case).

In [PVW08], H. Peters, L. R. Vivas and E. F. Wold explore how large such an attracting set can be. More precisely, suppose an automorphism has a fixed point that is not attracting but does have a non-trivial attracting set Ω. They do not study the complex structure of attracting sets but instead they look at topological properties. They are interested in exploring three related questions.

(a) Can Ω be dense?

(b) Can Ω have interior points?

(c) Can Ω contain a neighborhood of the origin?

The main result of [PVW08] provides an affirmative answer to question (a).

The answer to question (b) is straightforward, as it is possible to have an attracting set that is biholomorphic to $\mathbb{C}^k$, where k is the dimension of the ambient space [Ued86], [Wei97], [Hak97]. However, the attracting set of a volume-preserving automorphism cannot have interior points. They also demonstrate that the answer to question (c) is negative. If the attracting set contains a neighborhood of the origin, then the fixed point must be attracting.

5.3 Lavaurs Map

As in [ABD$^+$16], to explain what a Lavaurs map is, we need to recall some facts on parabolic dynamics. Let f be a polynomial of the form

$$f(z) = z + z^2 + az^3 + O(z^4)$$

for some $a \in \mathbb{C}$.

We denote by

$$B_f := \{z \in \mathbb{C} \mid f^n(z) \to 0 \text{ as } n \to +\infty \text{ , with } f^n(z) \neq 0\}$$

the parabolic basin of 0. It is known that there is an attracting Fatou coordinate $\varphi_f : B_f \to \mathbb{C}$ that conjugates f to the translation T_1:

$$\varphi_f \circ f = T_1 \circ \varphi_f.$$

This Fatou coordinate may be normalized by

$$\varphi_f(z) = -\frac{1}{z} - (1-a)\log\left(-\frac{1}{z}\right) + o(1) \quad \text{as } \mathrm{Re}\left(-\frac{1}{z}\right) \to +\infty.$$

Likewise, there is a repelling Fatou parametrization $\psi_f : \mathbb{C} \to \mathbb{C}$ satisfying

$$\psi_f \circ T_1 = f \circ \psi_f,$$

which may be normalized by

$$-\frac{1}{\psi_f(Z)} = Z + (1-a)\log(-Z) + o(1) \quad \text{as } \mathrm{Re}(Z) \to -\infty.$$

The (phase 0) Lavaurs map L_f is defined by

$$L_f := \psi_f \circ \varphi_f : B_f \to \mathbb{C}.$$

Mappings of this kind appear when considering high iterates of small perturbations of f. This phenomenon is known as parabolic implosion.

5.4 The Brjuno Condition

In [Rep19], Reppekus studies germs F of automorphisms of $\mathbb{C}^d$ at the origin of the form:

$$F(z^1, \dots, z^d) = \left(\lambda_1 z^1, \dots, \lambda_d z^d\right)\left(1 - \frac{(z^1 \cdots z^d)^k}{kd}\right) + O(||z||^l), \tag{*}$$

where $\lambda_1, \dots, \lambda_d$ are of unit modulus, not roots of unity, such that F is one-resonant via $\lambda_1 \cdots \lambda_d = 1$, i.e., $\lambda_1^{m_1} \cdots \lambda_d^{m_d} = \lambda_j$ for $m_1, \dots, m_d \in \mathbb{N}$ and $j \in \{1, \dots, d\}$ if and only if $(m_1, \dots, m_d) = (q, \dots, q) + e_j$ for some $q \in \mathbb{N}$ (see Definition 5.4.1), and $l > 2kd + 1$. In some parts, we will additionally assume all subsets $\{\lambda_1, \dots, \lambda_d\} \setminus \{\lambda_j\}$, $j = 1, \dots, d$, to satisfy the Brjuno condition (Definition 5.4.3). For $k = 1$, this is precisely the set-up of [BRS21].

5.4.1 Definition. A germ F of endomorphisms of $\mathbb{C}^d$ at the origin such that $F(0) = 0$ and $DF_0 = \mathrm{diag}(\lambda_1, \dots, \lambda_d)$ is called *one-resonant* of index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, if $\lambda_j = \lambda_1^{m_1} \cdots \lambda_d^{m_d}$ for some $j \leq d$ and $m = (m_1, \dots, m_d) \in \mathbb{N}^d$ if and only if $m = k\alpha + e_j$ for some $k \in \mathbb{N}$ (where e_j denotes the j-th unit vector).

For $1 \leq j \leq d$, one-resonance of index $\alpha \neq n \cdot e_j$ for every $n \in \mathbb{N}$ implies, in particular, that λ_j is not a root of unity.

5.4.2 Definition. Let F be a germ of endomorphisms of $\mathbb{C}^d$ with $dF_0 = \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_d)$. A set $A \subset \mathbb{N}^d$ is a Brjuno set (of exponents) for $(\lambda_1, \dots, \lambda_d)$ (or for F), if

$$\sum_{k \geq 1} 2^{-k} \log\left(\frac{1}{\omega_A(2k)}\right) < \infty, \tag{5.4.1}$$

where

$$\omega_A(k) := \min\left\{|\lambda^\alpha - \lambda_i| \,:\, \alpha \in A, 2 \leq |\alpha| \leq k,\, 1 \leq i \leq d\right\} \cup \{1\}, \tag{5.4.2}$$

for $k \geq 2$.

Subsets and finite unions of Brjuno sets are Brjuno sets.

5.4.3 Definition. Let $\lambda_1, \ldots, \lambda_d \in \mathbb{C}$.

1. $\{\lambda_1, \ldots, \lambda_d\}$ satisfies the Brjuno condition if $A = \mathbb{N}^d$ is a Brjuno set for $(\lambda_1, \ldots, \lambda_d)$.
2. $L \subseteq \{\lambda_1, \ldots, \lambda_d\}$ satisfies the partial Brjuno condition (with respect to $(\lambda_1, \ldots, \lambda_d)$) if $A = \{\alpha \in \mathbb{N}^d \mid \alpha_j = 0 \text{ for } \lambda_j \notin L\}$ is a Brjuno set for $(\lambda_1, \ldots, \lambda_d)$.

The main results in [Rep19] are the following.

5.4.4 Theorem
Let F be a germ of automorphisms of $\mathbb{C}^d$ at the origin as in (*). Then, F admits k disjoint, completely invariant, attracting basins $\Omega_0, \ldots, \Omega_{k-1}$ such that

(1) If each subset $\{\lambda_1, \ldots, \lambda_d\} \setminus \{\lambda_j\}$, $j = 1, \ldots, d$, satisfies the Brjuno condition, then:

- *(a) Ω_h is a union of Fatou components for each $h = 0, \ldots, k-1$,*
- *(b) F admits Siegel hypersurfaces (i.e., invariant hypersurfaces on which F acts as a rotation) tangent to each coordinate hyperplane,*
- *(c) All stable orbits of F near the origin are contained in one of the above.*

(2) If F is a global automorphism of $\mathbb{C}^d$, then for each $h = 0, \ldots, k-1$, there exists a biholomorphic map $\varphi_h : \Omega_h \to \mathbb{C} \times (\mathbb{C}^*)^{d-1}$ conjugating F to

$$(\zeta^1, \ldots, \zeta^d) \mapsto (\zeta^1 + 1, \zeta^2, \ldots, \zeta^d).$$

Moreover, there exist automorphisms of the form (*) for each admissible choice of $\lambda_1, \ldots, \lambda_d$ and $l > 2kd + 1$.

5.4.5 Theorem
Let $p \in \mathbb{N}^*$ divide k. Then, there exist automorphisms G of $\mathbb{C}^d$ such that G^p has the form (*) and $G(\Omega_h) = \Omega_{h+p \mod k}$ for $h = 0, \ldots, k-1$. In particular, G admits $\frac{k}{p}$ disjoint p-cycles of non-recurrent, attracting Fatou components biholomorphic to $\mathbb{C} \times (\mathbb{C}^*)^{d-1}$, that are all attracted to the origin.

As a direct consequence, we get automorphisms exhibiting cycles of non-recurrent attracting Fatou components that are biholomorphic to any product of copies of $\mathbb{C}$ and $\mathbb{C}^*$ (possessing admissible cohomology):

5.4.6 Theorem
Let $d, k \in \mathbb{N}^*$, $p \in \mathbb{N}^*$ divide k, and $0 \leq m < d$. Then, there exist holomorphic automorphisms of $\mathbb{C}^d$ possessing $\frac{k}{p}$ disjoint p-cycles of non-recurrent, attracting, invariant Fatou components biholomorphic to $\mathbb{C}^{d-m} \times (\mathbb{C}^*)^m$ and attracted to the origin.

5.5 Proof of Proposition 3.4.6

We split the proof of Proposition 3.4.6 over several lemmas.

5.5.1 Definition. Let $E \subset \mathbb{C}^n$. A vector $v \in \mathbb{C}^n$ is called tangent to E at a point $P \in \overline{E}$ if there exist a sequence of points $P_J \in E$ and real numbers $t_j > 0$ such that $P_i \to P$ and $t_j(P_j - P) \to v$ as $j \to \infty$. The set of all such tangent vectors is the *tangent cone* to E at P.

The tangent cone is indeed a cone in $\mathbb{C}^n = T_P\mathbb{C}^n$. If the set E is a $\mathcal{C}^1$-smooth manifold, the tangent cone coincides with the tangent space.

For complex analytic sets of dimension one, the following is a well known fact. For a proof, see [Chi89], Corollary on page 80.

5.5.2 Lemma
Let $H \subset \mathbb{C}^n$ be an analytic set of dimension one. For all $x \in H$ the tangent cone of H at x consists of a finite union of complex lines (whose number is not greater than the number of irreducible components of H at x).

5.5.3 Definition. Let $B \subset \mathbb{C}^n$ be a polidisk. The torus $\mathbb{T}$ with same center and same poliradius as B is called its *Šilov boundary.* We will denote it by $\partial_S B$.

The Šilov boundary is a very general notion, for Banach algebras, but we will not need it here in all generality. For details, we refer to [DSSST06], from page 325.

5.5.4 Lemma
Let B be a polydisk, $\partial_S B$ be its Šilov boundary, and $u : U \to \mathbb{R}$ be a harmonic function defined on a neighborhood U of $\overline{B}$. Then

$$\max_{\overline{B}} u = \max_{\partial_S B} u.$$

Recall that ∂ denotes the topological boundary.

Proof. For every $P \in \partial B \setminus \partial_S B$ there is a horizontal Euclidean disc D through P which is contained in ∂B whose boundary is in $\partial_S B$. Being u harmonic in a neighborhood of $\overline{B}$, u is harmonic on such a closed disc, hence its value at P is less or equal to the maximum of u at its boundary $\partial D \subset \partial_S B$. Hence

$$\max_{\partial B} u = \max_{\partial_S B} u\,.$$

If $P \in B$, we can find a disc through P with boundary in ∂B and repeat the argument, getting the conclusion

$$\max_{\overline{B}} u = \max_{\partial B} u = \max_{\partial_S B} u\,.$$

□

5.5.5 Lemma
Let $H \subset \mathbb{C}^2$ be an analytic set of dimension one. Then for every $P \in H$ there exists an arbitrarily small torus $\mathbb{T}_P$ centered in P such that $\mathbb{T}_P \cap H = \varnothing$.

Proof. Let $P \in H$. Consider the tangent cone C_P of H at P. By Lemma 5.5.2, C_P is a finite set of directions $\alpha_1, \ldots, \alpha_k \in \hat{\mathbb{C}}$. Up to a rotation, we can suppose all directions to be in $\mathbb{C}$. Up to choosing $\eta > 0$ small enough, we can ensure that the polidisk B_η of poliradius η centered in P intersects only one connected component of H. Moreover, by the definition of tangent cone, we can choose a small neighborhood $K \subset \mathbb{C}$ of all α_j such that

$$H \cap U_\eta \ \subset\ \cup_{\alpha \in K}(P + (z, \alpha z))\,.$$

We can suppose K to be small enough that there is $0 < b < 1$ such that $K \cap \{\beta \in \mathbb{C} \mid |\beta| = b\} = \varnothing$.

For any $0 < \delta < \eta$, defining

$$\mathbb{T}_P = \{|z - z_P| = \delta\} \times \{|w - w_P| = b\delta\}$$

we have that $\mathbb{T}_P \subset U_\delta$ and if $(z, w) \in \mathbb{T}_P$, $\frac{w - w_P}{z - z_P} = \beta$ with $|\beta| = b$. So

$$\mathbb{T}_P \cap H \ = \ T_P \cap U_\eta \cap H \ = \ \varnothing.$$

□

Proof of Proposition 3.4.6. Let K and L be compact sets as in the statement. For each $P \in H \cap K$, by Lemma 5.5.5 there exists a torus $\mathbb{T}_P \subset L$ centered in P such that $\mathbb{T}_P \cap H = \varnothing$. Each torus $\mathbb{T}_P$ is the Šilov boundary of a polidisk B_P centered in P. Since $\{B_P\}_{P \in H \cap K}$ is a covering of $H \cap K$, by compactness we can extract a finite covering $\{B_1, \ldots, B_k\}$.

There is a η-neighborhood U_η of H such that $U_\eta \cap K \subset \cup B_j$. If the harmonic function u satisfies $u \leq \alpha$ on $L \setminus U_\eta$ then it satisfies the same estimate on all tori $\mathbb{T}_j$, and by Lemma 5.5.4 the same estimate holds on all B_j. Hence $u \leq \alpha$ on K. □

Bibliography

[Aba08] M. Abate, *Sistemi dinamici e sistemi caotici*, Dipartimento di Matematica, Università di Pisa (2008).

[ABD$^+$16] M. Astorg, X. Buff, R. Dujardin, H. Peters, and J. Raissy, *A two-dimensional polynomial mapping with a wandering Fatou component*, Ann. of Math. (2) **184** (2016), no. 1, 263–313.

[ABFP19] L. Arosio, A. M. Benini, J. E. Fornæss, and H. Peters, *Dynamics of transcendental Hénon maps*, Math. Ann. **373** (2019), no. 1-2, 853–894.

[ABFP21] ———, *Dynamics of transcendental Hénon maps III: Infinite entropy*, J. Mod. Dyn. **17** (2021), 465–479.

[ABFP23] ———, *Dynamics of transcendental Hénon maps-II*, Math. Ann. **385** (2023), no. 3-4, 975–999.

[ABT22] M. Astorg and L. Boc Thaler, *Dynamics of skew-products tangent to the identity*, Preprint ArXiv: 2204.02644 (2022).

[ABTP23] M. Astorg, L. Boc Thaler, and H. Peters, *Wandering domains arising from Lavaurs maps with siegel disks*, Analysis & PDE **16** (2023), no. 1, 35–88.

[Bak75] I. N. Baker, *The domains of normality of an entire function*, Ann. Acad. Sci. Fenn. Ser. AI Math. **1** (1975), 277–283.

[Bak76] ———, *An entire function which has wandering domains*, J. Austral. Math. Soc. Ser. A **22** (1976), no. 2, 173–176.

[BB22] P. Berger and S. Biebler, *Emergence of wandering stable components*, Journal of the American Mathematical Society **36** (2022).

[BBS24] V. Beltrami, A. M. Benini, and A. Saracco, *Escaping fatou components with disjoint hyperbolic limit sets*, Mathematische Zeitschrift **307** (2024), no. 37.

[Bea91] A. F. Beardon, *Iteration of rational functions*, Graduate Texts in Mathematics, vol. 132, Springer-Verlag, New York, 1991, Complex analytic dynamical systems.

[BEF$^+$21] A. M. Benini, V. Evdoridou, N. Fagella, G. Stallard, and P. Rippon, *Classifying simply connected wandering domains*, Mathematische Annalen **383** (2021), 1127–1178.

[Bel24] V. Beltrami, *Automorphisms of $\mathbb{C}^2$ with cycles of escaping fatou components with hyperbolic limit sets*, International Journal of Mathematics (2024).

[Ben21] A. M. Benini, *Polynomial versus transcendental dynamics*, ABCD Conference in Luminy 20-26 Sept. 2021, `https://www.youtube.com/watch?v=eqbhHAI1b6w`.

[Ber93] W. Bergweiler, *Iteration of meromorphic functions*, Bull. Amer. Math. Soc. **29** (1993), no. 2, 151–188.

[BKL92] I. N. Baker, J. Kotus, and I. Lü, *Iterates of meromorphic functions IV: Critically finite functions*, Results Math. **22** (1992), 651–656.

[BLS93] E. Bedford, M. J. Lyubich, and J. Smillie, *Polynomial diffeomorphisms of* $\mathbb{C}^2$*. IV: The measure of maximal entropy and laminar currents.*, Inventiones mathematicae **112** (1993), no. 1, 77–126.

[BR24] X. Buff and J. Raissy, *Introduction to fatou components in holomorphic dynamics*, Modern Aspects of Dynamical Systems. Lecture Notes in Mathematics **2347** (2024), 59–103.

[BRS13] W. Bergweiler, P. J. Rippon, and G. M. Stallard, *Multiply connected wandering domains of entire functions*, Proc. Lond. Math. Soc. (3) **107** (2013), no. 6, 1261–1301. MR 3149847

[BRS21] F. Bracci, J. Raissy, and B. Stensønes, *Automorphisms of* $\mathbb{C}^k$ *with an invariant non-recurrent attracting Fatou component biholomorphic to* $\mathbb{C} \times (\mathbb{C}^*)^{k-1}$, J. Eur. Math. Soc. (JEMS) **23** (2021), no. 2, 639–666.

[BS91a] E. Bedford and J. Smillie, *Polynomial diffeomorphisms of* $\mathbb{C}^2$*: currents, equilibrium measure and hyperbolicity*, Invent. Math. **103** (1991), no. 1, 69–99.

[BS91b] ______, *Polynomial diffeomorphisms of* $\mathbb{C}^2$*. II. Stable manifolds and recurrence*, J. Amer. Math. Soc. **4** (1991), no. 4, 657–679.

[BSZ23] A. M. Benini, A. Saracco, and M. Zedda, *Invariant escaping fatou components with two rank-one limit functions for automorphisms of* $\mathbb{C}^2$, Ergodic Theory and Dynamical Systems **43** (2023), no. 2, 401–416.

[BTBP21] L. Boc Thaler, F. Bracci, and H. Peters, *Automorphisms of* $\mathbb{C}^2$ *with parabolic cylinders*, J. Geom. Anal. **31** (2021), no. 4, 3498–3522. MR 4236533

[CG93] L. Carleson and T. W. Gamelin, *Complex dynamics*, Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993. MR 1230383

[Chi89] E. M. Chirka, *Tangent cones and intersection theory*, pp. 79–153, Springer Netherlands, Dordrecht, 1989.

[Cow81] C. C. Cowen, *Iteration and the solution of functional equations for functions analytic in the unit disk*, Trans. Amer. Math. Soc. **265** (1981), no. 1, 69–95.

[Cre32] H. Cremer, *Über die schrödersche funktionalgleichung und das schwarzsche eckenabbildungsproblem*, Ber. Verh. Sachs. Akad. Wiss. Leipzig, Math.-Phys. Kl **84** (1932), 291–324.

[DK89] R. L. Devaney and L. Keen, *Dynamics of meromorphic maps with polynomial schwarzian derivative*, Ann. Sei. École Norm. Sup. **22** (1989), no. 4, 55–81.

[DSSST06] G. Della Sala, A. Saracco, A. Simioniuc, and G. Tomassini, *Lectures on complex analysis and analytic geometry*, Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)], vol. 3, Edizioni della Normale, Pisa, 2006. MR 2260365

[DZ20] A. De Zotti, *Some connections of complex dynamics*, Preprint ArXiv:2006.16386v1 (2020).

[EL84] A. E. Eremenko and M. J. Lyubich, *Iterates of entire functions*, Soviet Math. Dokl. **30** (1984), 592–594.

[EL87] ______, *Examples of entire function with pathological dynamics*, J. Lond. Math. Soc. **36** (1987), 458–468.

[EL92] ______, *Dynamical properties of some classes of entire functions*, Ann. Inst. Fourier (Grenoble) **42** (1992), no. 4, 989–1020.

[Fat20a] P. Fatou, *Sur les équations fonctionnelles*, Bull. Soc. Math. France **48** (1920), 208–314. MR 1504797

[Fat20b] ______, *Sur les équations fonctionnelles*, Bulletin de la Société Mathématique de France **48** (1920), 208–314.

[Fat26] ______, *Sur l'itération des fonctions transcendantes entières*, Acta Math. **47** (1926), 337–370.

[FG01] C. Favre and V. Guedj, *Dynamique des applications rationnelles des espaces multiprojectifs*, Indiana Univ. Math. J. **50** (2001), 881–934.

[FM89] S. Friedland and J. Milnor, *Dynamical properties of plane polynomial automorphisms*, Ergodic Theory Dynam. Systems **9** (1989), no. 1, 67–99.

[For04] J. E. Fornæss, *Short* $\mathbb{C}^k$, Complex analysis in several variables—Memorial Conference of Kiyoshi Oka's Centennial Birthday, Adv. Stud. Pure Math. Math. Soc. Japan, Tokyo **42** (2004), no. 4, 95–108.

[FS95] J. E. Fornæss and N. Sibony, *Classification of recurrent domains for some holomorphic maps*, Math. Ann. **301** (1995), no. 4, 813–820.

[FS98] ______, *Fatou and Julia sets for entire mappings in* $\mathbb{C}^k$, Math. Ann. **311** (1998), no. 1, 27–40.

[GH62] F. W. Gehring and W. K. Hayman, *An inequality in the theory of conformal mapping*, J. Math. Pures Appl. (9) **41** (1962), 353–361.

[Hak97] M. Hakim, *Transformations tangent to the identity.*, Stable pieces of manifolds (1997).

[Hak98] ______, *Analytic transformations of* $(\mathbb{C}^p, 0)$ *tangent to the identity*, Duke Math. J. **92** (1998), no. 2, 403–428.

[Hay85] W. K. Hayman, *Are there critical points on the boundary of singular domains?*, Comm. Math. Phys **99** (1985), 593–612.

[Hei98] S. M. Heinemann, *Julia sets of skew-products in* $\mathbb{C}^2$, Kyushu J. Math. **52** (1998), no. 2, 299–329.

[HOV94] J. H. Hubbard and R. W. Oberste-Vorth, *Hénon mappings in the complex domain. I. The global topology of dynamical space*, Inst. Hautes Études Sci. Publ. Math. (1994), no. 79, 5–46.

[HP21] D. Hahn and H. Peters, *A polynomial automorphism with a wandering fatou component*, Adv. Math. **382** (2021).

[Ji20] Z. Ji, *Non-wandering fatou components for strongly attracting polynomial skew products*, J. Geom. Anal. **30** (2020), no. 1, 124–152.

[JL03] D. Jupiter and K. Lilov, *Invariant nonrecurrent Fatou components of automorphisms of* $\mathbb{C}^2$, Preprint ArXiv: 0309253 (2003).

[Jon99] M. Jonsson, *Dynamics of polynomial skew products on* $\mathbb{C}^2$, Indiana Univ. Math. J. **314** (1999), 403–447.

[Jul18] G. Julia, *Mémoire sur l'itération des fonctions rationnelles*, Mathématiques Pures et Appliquées **1** (1918), 47–245.

[Kra01] S. G. Krantz, *Function theory of several complex variables*, AMS Chelsea Publishing, Providence, RI, 2001, Reprint of the 1992 edition.

[Lil04] K. Lilov, *Fatou theory in two dimensions*, University of Michigan ProQuest Dissertations & PhD Theses (2004).

[LP14] M. J. Lyubich and H. Peters, *Classification of invariant Fatou components for dissipative Hénon maps*, Geom. Funct. Anal. **24** (2014), no. 3, 887–915.

[Man75] B. Mandelbrot, *Les objets fractals : forme, hasard et dimension*, Paris : Flammarion, 1975.

[Mil06] J.W. Milnor, *Dynamics in one complex variable*, third ed., Annals of Mathematics Studies, vol. 160, Princeton University Press, Princeton, 2006.

[PR19] H. Peters and J. Raissy, *Fatou components of elliptic polynomial skew products*, Ergodic Theory and Dynamical Systems **39** (2019), 2235–2247.

[PS18] H. Peters and I. M. Smit, *Fatou components of attracting skew products*, The Journal of Geometric Analysis **28** (2018), 84–110.

[PV16] H. Peters and L. Vivas, *Polynomial skew-products with wandering fatou-disks*, Math. Z. **283** (2016), no. 1-2, 349–366.

[PVW08] H. Peters, . R. Vivas, and E. F. Wold, *Attracting basins of volume preserving automorphisms of* $\mathbb{C}^k$, Internat. J. Math. **19** (2008), no. 7, 801–810.

[Rep19] J. Reppekus, *Periodic cycles of attracting fatou components of type* $\mathbb{C} \times (\mathbb{C}^*)^{d-1}$ *in automorphisms of* $\mathbb{C}^d$, 2019.

[Roe11] R. K. W. Roeder, *A dichotomy for fatou components of polynomial skew products*, Conformal Geometry and Dynamics **15** (2011), 7–19.

[RR88] J.P. Rosay and W. Rudin, *Holomorphic maps from* $\mathbb{C}^n$ *to* $\mathbb{C}^n$, Trans. Amer. Math. Soc. **310** (1988), no. 1, 47–86.

[Sul85] D. Sullivan, *Quasiconformal homeomorphisms and dynamics. I. Solution of the Fatou-Julia problem on wandering domains*, Ann. of Math. (2) **122** (1985), no. 3, 401–418.

[SV14] B. Stensønes and L. Vivas, *Basins of attraction of automorphisms in* $\mathbb{C}^3$, Ergodic Theory Dynam. Systems **34** (2014), 689–692.

[Ued86] T. Ueda, *Local structure of analytic transformations of two complex variables. I*, J. Math. Kyoto Univ. **26** (1986), no. 2, 233–261.

[Uen19] K. Ueno, *A construction of böttcher coordinates for holomorphic skew products*, Nonlinearity **32** (2019), 2694–2720.

[Uen24] ———, *Attraction rates for iterates of a superattracting skew product*, Journal of Mathematical Analysis and Applications **539** (2024), 2694–2720.

[Viv18] L. Vivas, *Parametrization of unstable manifolds and fatou disks for parabolic skew products*, Complex Anal. Synerg. **4** (2018), no. 1.

[Wei97] B. Weickert, *Automorphisms of* $\mathbb{C}^n$, PhD thesis, University of Michigan (1997).

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