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On algorithmic applications of sim-width and mim-width of (H_1, H_2)-free graphs ☆

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A B S T R A C T

Mim-width and sim-width are among the most powerful graph width parameters, with sim-width more powerful than mim-width, which is in turn more powerful than clique-width. While several NP-hard graph problems become tractable for graph classes whose mim-width is bounded and quickly computable, no algorithmic applications of boundedness of sim-width are known. In Kang et al. (2017) [32], it is asked whether Independent Set and 3-Colouring are NP-complete on graphs of sim-width at most 1. We observe that, for each $k \in \mathbb{N}$, List k-Colouring is polynomial-time solvable for graph classes whose sim-width is bounded and quickly computable. Moreover, we show that if the same holds for Independent Set, then Independent $\mathcal{H}$-Packing is polynomial-time solvable for graph classes whose sim-width is bounded and quickly computable. This problem is a common generalisation of Independent Set, Induced Matching, Dissociation Set and k-Separator.

We also make progress toward classifying the mim-width of (H_1, H_2)-free graphs in the case H_1 is complete or edgeless. Our results solve some open problems in Brettell et al. (2022) [6].

1. Introduction

Over the last decades, graph width parameters have proven to be an extremely successful tool in algorithmic graph theory. Arguably the most important reason explaining the jump from computational hardness of a graph problem to tractability, after restricting the input to some graph class $\mathcal{G}$, is that $\mathcal{G}$ has bounded "width", for some width parameter p. That is, there exists a constant c such that, for each graph $G \in \mathcal{G}$, $p(G) \leq c$. A large number of width parameters have been introduced, and these parameters typically differ in strength. We say that a width parameter p *dominates* a width parameter q if there is a function f such that $p(G) \leq f(q(G))$ for all graphs G. If p dominates q but q does not dominate p, then p is said to be *more powerful* than q. If both p and q dominate each other, then p and q are *equivalent*. For instance, the equivalent parameters boolean-width, clique-width, module-width, NLC-width and rank-width [8,37,26,38] are more powerful than the equivalent parameters branch-width, treewidth and mm-width [16,31,39,41] but less powerful than mim-width [41], which is less powerful than sim-width [32]. We also mention that the recently introduced tree-independence number [19] is more powerful than treewidth, less powerful than sim-width and incomparable with both clique-width and

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mim-width (see Section 1.1). The *tree-independence number of a graph* G, denoted tree-$\alpha(G)$, is defined as the minimum independence number over all tree decompositions of G, where the independence number of a tree decomposition of G is the maximum independence number over all subgraphs of G induced by some bag of the tree decomposition.

In this paper, we focus on mim-width and sim-width, both defined using the framework of branch decompositions. A *branch decomposition* of a graph G is a pair (T, δ), where T is a subcubic tree and δ is a bijection from $V(G)$ to the leaves of T. Every edge $e \in E(T)$ partitions the leaves of T into two classes, L_e and $\overline{L_e}$, depending on which component of $T - e$ they belong to. Hence, e induces a partition $(A_e, \overline{A_e})$ of $V(G)$, where $\delta(A_e) = L_e$ and $\delta(\overline{A_e}) = \overline{L_e}$. We let $G[A_e, \overline{A_e}]$ denote the bipartite subgraph of G induced by the edges with one endpoint in A_e and the other in $\overline{A_e}$. A matching $F \subseteq E(G)$ of G is *induced* if there is no edge in G between vertices of different edges of F. We let $\text{cutmim}_G(A_e, \overline{A_e})$ denote the maximum size of an induced matching in $G[A_e, \overline{A_e}]$ and $\text{cutsim}_G(A_e, \overline{A_e})$ denote the maximum size of an induced matching between A_e and $\overline{A_e}$ in G (equivalently, $\text{cutsim}_G(A_e, \overline{A_e})$ is the maximum size of an induced matching in $G[A_e, \overline{A_e}]$ such that in addition there are no edges in G between any two endpoints of matching edges that both belong to either A_e or $\overline{A_e}$). The *mim-width* of (T, δ), denoted $\text{mimw}_G(T, \delta)$, is the maximum value of $\text{cutmim}_G(A_e, \overline{A_e})$ over all edges $e \in E(T)$ and the *mim-width* of G, denoted $\text{mimw}(G)$, is the minimum value of $\text{mimw}_G(T, \delta)$ over all branch decompositions (T, δ) of G. Similarly, the *sim-width* of (T, δ), denoted $\text{simw}_G(T, \delta)$, is the maximum value of $\text{cutsim}_G(A_e, \overline{A_e})$ over all edges $e \in E(T)$ and the *sim-width* of G, denoted $\text{simw}(G)$, is the minimum value of $\text{simw}_G(T, \delta)$ over all branch decompositions (T, δ) of G. Clearly, $\text{simw}(G) \leq \text{mimw}(G)$, for any graph G.

We now briefly review the algorithmic implications of boundedness of mim-width, sim-width and tree-independence number. We begin with a recent and remarkable meta-theorem provided by Bergougnoux et al. [3]. They showed that all problems expressible in A&C DN logic, an extension of existential MSO_1 logic, can be solved in XP time parameterized by the mim-width of a given branch decomposition of the input graph. This result, which can be viewed as the mim-width analogue of the famous meta-theorems for treewidth [15] and clique-width [14], generalises essentially all the previously known XP algorithms parameterized by mim-width, as A&C DN logic captures both local and non-local problems. Just to name few problems falling into this framework, we have all Locally Checkable Vertex Subset and Vertex Partitioning problems [1,9], their distance versions [28] and their connectivity and acyclicity versions [2], Longest Induced Path and Induced Disjoint Paths [29], Feedback Vertex Set [30], Semitotal Dominating Set [20]. Boundedness of tree-independence number has interesting algorithmic implications as well. Dallard et al. [19] showed that, for any fixed finite set $\mathcal{H}$ of connected graphs, Maximum Weight Independent $\mathcal{H}$-Packing, a common generalisation of Maximum Weight Independent Set and Maximum Weight Induced Matching first defined in [10], can be solved in XP time parameterized by the independence number of a given tree decomposition of the input graph. They also showed that k-Clique and List k-Colouring admit linear-time algorithms for every graph class with bounded tree-independence number. This result holds more generally for every (tw, ω)-bounded graph class admitting a computable binding function, as shown by Chaplick and Zeman [11], where a graph class $\mathcal{G}$ is (tw, ω)-*bounded* if there exists a function f (called a binding function) such that the treewidth of any graph $G \in \mathcal{G}$ is at most $f(\omega(G))$ and the same holds for all induced subgraphs of G. In [19], it was observed that in every graph class with bounded tree-independence number, the treewidth is bounded by an explicit polynomial function of the clique number, and hence bounded tree-independence number implies (tw, ω)-boundedness.

The trade-off of working with a more powerful width parameter is that, typically, fewer problems admit a polynomial-time algorithm when the parameter is bounded. Consider, for example, mim-width and the more powerful sim-width. Dominating Set is in XP parameterized by mim-width [9]. However, Dominating Set is NP-complete on chordal graphs, a class of graphs of sim-width at most 1 [32]. On the other hand, it is known that one can solve Independent Set and 3-Colouring in polynomial time on both chordal graphs and co-comparability graphs, two classes of sim-width at most 1, as shown by Kang et al. [32]. This led them to ask whether any of Independent Set and 3-Colouring is NP-complete on graphs of sim-width at most 1 [32, Question 2]. For convenience, we reformulate this question as follows:

Open Problem 1. Is any of Independent Set and 3-Colouring in XP parameterized by the sim-width of a given branch decomposition of the input graph?

To the best of our knowledge, no problem NP-complete on general graphs is known to be in XP parameterized by the sim-width of a given branch decomposition of the input graph.

In view of the discussion above, if we are interested in the computational complexity of a certain graph problem restricted to a special graph class, it is useful to know whether the mim-width of the class is bounded or not and, in the case of a positive answer to Open Problem 1, the same is true for sim-width. A systematic study on the boundedness of mim-width for hereditary graph classes, comparable to similar studies on the boundedness of clique-width (see, e.g., [18]) and treewidth [35], was recently initiated in [6] (see also [5]). Recall that a graph class is *hereditary* if it is closed under vertex deletion. It is well known that hereditary graph classes are exactly those classes characterised by a (unique) set $\mathcal{F}$ of minimal forbidden induced subgraphs. If $|\mathcal{F}| = 1$ or $|\mathcal{F}| = 2$, we say that the hereditary graph class is *monogenic* or *bigenic*, respectively. In [6], boundedness or unboundedness of mim-width has been determined for all monogenic classes and a large number of bigenic classes.

In general, computing the mim-width is NP-hard, deciding if the mim-width is at most k is W[1]-hard when parameterized by k, and there is no polynomial-time algorithm for approximating the mim-width of a graph to within a constant factor of the optimal unless NP = ZPP [40]. Moreover, it remains a challenging open problem to obtain, for fixed k, a polynomial-time algorithm for computing a branch decomposition with mim-width $f(k)$ of a graph with mim-width k; a similar problem for sim-width is open as well (see, e.g., [27]). Therefore, in contrast to algorithms for graph classes of bounded treewidth or rank-width [4,26], algorithms for classes of bounded mim-width require a branch decomposition of constant mim-width as part of the input. Obtaining such branch decompositions in polynomial time has been shown possible for several special graph classes $\mathcal{G}$ (see, e.g., [1,6]). In this case, we say that the mim-width of $\mathcal{G}$ is *quickly computable*.

Mim-width has proven to be particularly effective in tackling colouring problems. For instance, Kwon [34] showed the following (see also [7]):

Theorem 1 *(Kwon [34]). For every $k \geq 1$,* List k-Colouring *is polynomial-time solvable for every graph class whose mim-width is bounded and quickly computable.*

Notice however that Colouring (and hence List Colouring) is NP-complete for circular-arc graphs [21], a class of graphs of mim-width at most 2 and for which mim-width is quickly computable [1]. The complexity of k-Colouring restricted to H-free graphs has not yet been settled and there are infinitely many open cases when H is a *linear forest*, that is, a disjoint union of paths. An extensive body of work has been devoted to studying whether forbidding certain linear forests makes k-Colouring and its generalisation List k-Colouring easy. We refer to [23] for a survey and to [12,24,33] for updated summaries and briefly highlight below the connections with mim-width.

For $r \geq 1$ and $s \geq 1$, let $K_{r,s}$ denote the complete bipartite graph with partition classes of size r and s. The 1-*subdivision* of a graph G is the graph obtained from G by subdividing each edge exactly once. The 1-subdivision of $K_{1,s}$ is denoted by $K^1_{1,s}$; in particular $K^1_{1,2} = P_5$. Brettell et al. [7] showed that a number of known polynomial-time results for k-Colouring and List k-Colouring on hereditary classes [12,17,22,25] can be obtained, and strengthened, by combining Theorem 1 with the following:

Theorem 2 *(Brettell et al. [7]). For every $r \geq 1$, $s \geq 1$ and $t \geq 1$, the mim-width of the class of $(K_r, K^1_{1,s}, P_t)$-free graphs is bounded and quickly computable.*

The trivial but useful observation is that each yes-instance of List k-Colouring is K_{k+1}-free, and so we obtain that, for every $k \geq 1$, $s \geq 1$ and $t \geq 1$, List k-Colouring is polynomial-time solvable for $(K^1_{1,s}, P_t)$-free graphs [7]. Hence, in the context of colouring problems on hereditary classes, it makes sense to investigate the mim-width of subclasses of K_r-free graphs. A first step is to consider the mim-width of (K_r, H)-free graphs, for some graph H. For any H such that the mim-width of (K_r, H)-free graphs is bounded and quickly computable, List k-Colouring is polynomial-time solvable for all $k < r$. More generally, for problems admitting polynomial-time algorithms when mim-width is bounded and quickly computable, we obtain XP algorithms parameterized by $\omega(G)$ when restricted to H-free graphs. For example, Chudnovsky et al. [13] showed that for P_5-free graphs, there exists an $n^{O(\omega(G))}$-time algorithm for Max Partial H-Colouring (a common generalisation of Maximum Independent Set and Odd Cycle Transversal which is polynomial-time solvable when mim-width is bounded and quickly computable). Theorem 2 allows to generalise this, although with a worse running time (see [7,13]).

From a merely structural point of view, the study of the mim-width of (K_r, H)-free graphs falls into the systematic study of the mim-width of bigenic classes mentioned above. For each $r \geq 4$, Brettell et al. [6] completely classified the mim-width of the class of (K_r, H)-free graphs, except for one infinite family, and asked the following:

Open Problem 2 *(Brettell et al. [6]).* For each $r \geq 4$, and for each $t \geq 0$ and $u \geq 1$ such that $t + u \geq 2$, determine the (un)boundedness of mim-width of $(K_r, tP_2 + uP_3)$-free graphs.

Consider now the class of (rP_1, H)-free graphs. If the mim-width of such a class is bounded and quickly computable, we obtain, for many problems, XP algorithms parameterized by $\alpha(G)$ for the class of H-free graphs. For $r \geq 5$, Brettell et al. [6] completely classified the mim-width of the class of (rP_1, H)-free graphs, except for one infinite family, and asked the following:

Open Problem 3 *(Brettell et al. [6]).* For each $r \geq 4$, and for each $s, t \geq 2$, determine the (un)boundedness of mim-width of $(rP_1, \overline{K_{s,t} + P_1})$-free graphs.

1.1. Our results

In this paper we observe that List k-Colouring is polynomial-time solvable for every graph class whose sim-width is bounded and quickly computable, thus answering in the positive one half of Open Problem 1. We also show that if Independent Set is polynomial-time solvable for a given graph class whose sim-width is bounded and quickly computable, then the same is true for its generalisation Independent $\mathcal{H}$-packing. Finally, we completely resolve Open Problem 3 and make considerable progress toward solving Open Problem 2.

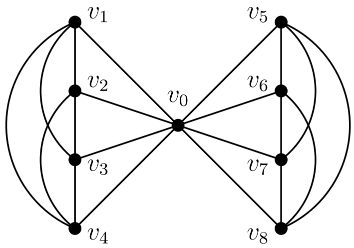


Fig. 1. The graph $\overline{K_{4,4} + P_1}$.

1.1.1. Algorithmic implications of boundedness of sim-width

Let us begin by discussing our results related to Open Problem 1. Let $K_t \boxminus K_t$ be the graph obtained from $2K_t$ by adding a perfect matching and let $K_t \boxminus S_t$ be the graph obtained from $K_t \boxminus K_t$ by removing all the edges in one of the complete graphs. Combining Theorem 1 with [32, Proposition 4.2] stated below, we observe that List k-Colouring is in XP when parameterized by the sim-width of a given branch decomposition of the input graph.

Proposition 3 *(see Proof of Proposition 4.2 in [32]). Let G be a graph with no induced subgraph isomorphic to $K_t \boxminus K_t$ and $K_t \boxminus S_t$ and let (T, δ) be a branch decomposition of G with $\mathrm{simw}_G(T, \delta) = w$. Then $\mathrm{mimw}_G(T, \delta) \leq R(R(w+1, t), R(t, t))$.*

Theorem 4. *For every $k \geq 1$, List k-Colouring is polynomial-time solvable for every graph class whose sim-width is bounded and quickly computable.*

Proof. Given an instance consisting of a graph G and a k-list assignment L, together with a branch decomposition (T, δ) of G with $\mathrm{simw}_G(T, \delta) = w$, we proceed as follows. We check in polynomial time whether G contains a copy of K_{k+1}. If it does, then we have a no-instance. Otherwise, G is K_{k+1}-free. Then, by Proposition 3, (T, δ) has mim-width at most $R(R(w+1, k+1), R(k+1, k+1))$, and we simply apply Theorem 1. This concludes the proof. □

It is worth noticing that Theorem 4 does not really give wider applicability when compared to Theorem 1. Indeed, input graphs of List k-Colouring can always be assumed to be K_{k+1}-free and every subclass of K_{k+1}-free graphs has bounded sim-width if and only if it has bounded mim-width: This follows from Proposition 3 and the fact that $\mathrm{simw}(G) \leq \mathrm{mimw}(G)$ for any graph G. Nevertheless, Theorem 4 has interesting consequences. Besides answering in the positive one half of Open Problem 1, it extends the result in [19] that List k-Colouring is polynomial-time solvable for every graph class whose tree-independence number is bounded and quickly computable. This is because of the following unpublished observation of Dallard, Krnc, Kwon, Milanič, Munaro and Štorgel, which is part of a work in progress and whose proof we sketch for convenience.

Lemma 5. *Let G be a graph. Then $\mathrm{simw}(G) \leq \text{tree-}\alpha(G)$.*

Proof sketch. Given a tree decomposition $(F, \{B_t\}_{t \in V(F)})$ of G, the proof of Proposition 3.1 in [32] shows how to construct a branch decomposition (T, δ) of G such that, for each $e \in E(T)$, either $N_G(A_e) \cap \overline{A_e}$ or $N_G(\overline{A_e}) \cap A_e$ is contained in a bag in $\{B_t\}_{t \in V(F)}$. Consider then a tree decomposition $(F, \{B_t\}_{t \in V(F)})$ of G with tree-independence number tree-$\alpha(G)$ and the corresponding branch decomposition (T, δ) of G satisfying the property above. Fix $e \in E(T)$ and suppose without loss of generality that $N_G(A_e) \cap \overline{A_e} \subseteq B_t$, for some $t \in V(F)$. This implies that the independence number of $G[N_G(A_e) \cap \overline{A_e}]$ is at most tree-$\alpha(G)$ and so $\mathrm{cutsim}_G(A_e, \overline{A_e}) \leq \text{tree-}\alpha(G)$. Since this holds for every $e \in E(T)$, we have that $\mathrm{simw}_G(T, \delta) \leq \text{tree-}\alpha(G)$ and so $\mathrm{simw}(G) \leq \text{tree-}\alpha(G)$. □

Together with the fact that complete bipartite graphs have bounded sim-width (in fact, bounded clique-width) but unbounded tree-independence number [19], Lemma 5 implies that sim-width is more powerful than tree-independence number. Note also that graph classes of bounded sim-width are not necessarily (tw, ω)-bounded and so Theorem 4 cannot be deduced from the results in [11]. Indeed, it is easy to see that complete bipartite graphs, which have bounded sim-width, are not (tw, ω)-bounded. However, we do not know whether a (tw, ω)-bounded graph class has necessarily bounded sim-width.

In Section 3, we show that a positive answer to the other half of Open Problem 1 would have important algorithmic implications for Maximum Weight Independent $\mathcal{H}$-Packing, a problem studied for example in [10,19]. Before formulating it, we state some definitions and results. Let $\mathcal{H}$ be a set of connected graphs. Given a graph G, let $\mathcal{H}_G$ be the set of all subgraphs of G isomorphic to a member of $\mathcal{H}$. The *$\mathcal{H}$-graph of G*, denoted $\mathcal{H}(G)$, is defined in [10] as follows: the vertex set is $\mathcal{H}_G$ and two distinct subgraphs of G isomorphic to a member of $\mathcal{H}$ are adjacent if and only if they either have a vertex in common or there is an edge in G connecting them. Cameron and Hell [10] showed that, for any set $\mathcal{H}$ of connected graphs, the $\mathcal{H}$-graph of any chordal graph is chordal. Dallard et al. [19] generalised this by showing that mapping any graph G to its $\mathcal{H}$-graph does not increase the tree-independence number. We show that this operation does not increase the sim-width either.

Theorem 6. *Let $\mathcal{H}$ be a non-empty finite set of connected non-null graphs and let r be the maximum number of vertices of a graph in $\mathcal{H}$. Let G be a graph and let (T, δ) be a branch decomposition of G. If $|V(\mathcal{H}(G))| > 1$, then we can obtain in $O(|V(G)|^{r+1})$ time a branch decomposition (T', δ') of $\mathcal{H}(G)$ such that $\mathrm{simw}_{\mathcal{H}(G)}(T', \delta') \leq \mathrm{simw}_G(T, \delta)$.*

Two subgraphs H_1 and H_2 of a graph G are *independent* if they are vertex-disjoint and no edge of G joins a vertex of H_1 with a vertex of H_2. An *independent $\mathcal{H}$-packing* in G is a set of pairwise independent subgraphs from $\mathcal{H}_G$. Given a graph G, a weight function $w\colon \mathcal{H}_G \to \mathbb{Q}_+$ on the subgraphs in $\mathcal{H}_G$, and an independent $\mathcal{H}$-packing P in G, the weight of P is defined as $\sum_{H \in P} w(H)$. Given a graph G and a weight function $w\colon \mathcal{H}_G \to \mathbb{Q}_+$, the Maximum Weight Independent $\mathcal{H}$-Packing problem asks to find an independent $\mathcal{H}$-packing in G of maximum weight. If all subgraphs in $\mathcal{H}_G$ have weight 1, we obtain the special case Independent $\mathcal{H}$-Packing. Maximum Weight Independent $\mathcal{H}$-Packing is a common generalisation of several problems studied in the literature, including Maximum Weight Independent Set, Maximum Weight Induced Matching, Dissociation Set and k-Separator (we refer to [19] for a comprehensive literature review).

Cameron and Hell [10] showed that Independent $\mathcal{H}$-packing is polynomial-time solvable, among others, for the following graph classes: weakly chordal graphs and hence chordal graphs, AT-free graphs and hence co-comparability graphs, circular-arc graphs, circle graphs. Dallard et al. [19] showed that Maximum Weight Independent $\mathcal{H}$-Packing is polynomial-time solvable for every graph class whose tree-independence number is bounded and quickly computable. With the aid of Theorem 6, we show the following.

Corollary 7. *Let $\mathcal{H}$ be a non-empty finite set of connected non-null graphs such that each graph in $\mathcal{H}$ has at most r vertices. Let $\mathcal{G}$ be a graph class whose sim-width is bounded and quickly computable. If* Maximum Weight Independent Set *is polynomial-time solvable for $\mathcal{G}$, then* Maximum Weight Independent $\mathcal{H}$-Packing *is polynomial-time solvable for $\mathcal{G}$. Similarly, if* Independent Set *is polynomial-time solvable for $\mathcal{G}$, then* Independent $\mathcal{H}$-Packing *is polynomial-time solvable for $\mathcal{G}$.*

1.1.2. Mim-width of (H_1, H_2)-free graphs

We now address the classification of (un)boundedness of mim-width of (H_1, H_2)-free graphs, where H_1 is either rP_1 or K_r.

In Section 4, we completely resolve Open Problem 3 by showing the following.

Theorem 8. *Let $r \geq 3$ and $s, t \geq 2$ be integers. Then the mim-width of the class of $(rP_1, \overline{K_{s,t} + P_1})$-free graphs is bounded if and only if:*

- *$r = 3$ and one of s and t is at most 3;*
- *$r = 4$ and one of s and t is at most 2.*

In all these cases, the mim-width is also quickly computable.

In Section 5, we finally address the case $H_1 = K_r$, related to Open Problem 2, by showing the following two results.

Theorem 9. *Let $r \geq 5$ be an integer and let $H = sP_1 + tP_2 + uP_3$, for $s, t, u \geq 0$. Then exactly one of the following holds:*

- *$H \subseteq_i sP_1 + tP_2$, or $H \subseteq_i sP_1 + P_3$, and the mim-width of the class of (K_r, H)-free graphs is bounded and quickly computable;*
- *$H \supseteq_i P_3 + P_2 + P_1$, and the mim-width of the class of (K_r, H)-free graphs is unbounded;*
- *$H = 2P_3$, or $H = P_3 + P_2$.*

Theorem 10. *Let $r = 4$ and let $H = sP_1 + tP_2 + uP_3$, for $s, t, u \geq 0$. Then exactly one of the following holds:*

- *$H \subseteq_i sP_1 + tP_2$, or $H \subseteq_i sP_1 + P_3$, and the mim-width of the class of (K_r, H)-free graphs is bounded and quickly computable;*
- *$H \supseteq_i P_3 + 2P_2 + P_1$, or $2P_3 + P_2$, and the mim-width of the class of (K_r, H)-free graphs is unbounded;*
- *$H = P_3 + 2P_2$, or $H = P_3 + P_2 + sP_1$, or $H = 2P_3 + sP_1$.*

Our results are related to the class of uP_3-free graphs. Recently, Hajebi et al. [24] showed that, for every $u \geq 1$, List 5-Colouring is polynomial-time solvable for uP_3-free graphs. Since an instance of List 5-Colouring can always be assumed to be K_6-free, in view of Theorem 4 an alternative approach to obtaining the aforementioned result might pass through studying the sim-width of (K_6, uP_3)-free graphs. Unfortunately, Theorem 9 readily shows that, with the possible exception of the case $u = 2$, this is not possible: For each $u \geq 3$, the mim-width of (K_6, uP_3)-free graphs is unbounded and, by [32, Proposition 4.2], the same must be true for sim-width.

2. Preliminaries

We consider only finite graphs $G = (V, E)$ with no loops and no multiple edges. A graph is *null* if it has no vertices. For a vertex $v \in V$, the *neighbourhood* $N(v)$ is the set of vertices adjacent to v in G. The *degree* $d(v)$ of a vertex $v \in V$ is the size $|N(v)|$ of its neighbourhood. A graph is *subcubic* if every vertex has degree at most 3. For disjoint $S, T \subseteq V$, we say that S is *complete to* T if every vertex of S is adjacent to every vertex of T, and S is *anticomplete to* T if there are no edges between S and T. The *distance* from a vertex u to a vertex v in G is the length of a shortest path between u and v. A set $S \subseteq V$ *induces* the subgraph $G[S] = (S, \{uv : u, v \in S, uv \in E\})$. If G' is an induced subgraph of G, we write $G' \subseteq_i G$. The *complement* of G is the graph $\overline{G}$ with vertex set $V(G)$, such that $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

The *k-subdivision* of an edge uv in a graph replaces uv by k new vertices $w_1, \dots, w_k$ with edges uw_1, w_kv and w_iw_{i+1} for each $i \in \{1, \dots, k-1\}$, i.e. the edge is replaced by a path of length $k+1$. The *disjoint union* $G + H$ of graphs G and H has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. We denote the disjoint union of k copies of G by kG. For a graph H, a graph G is *H-free* if G has no induced subgraph isomorphic to H. For a set of graphs $\{H_1, \dots, H_k\}$, a graph G is $(H_1, \dots, H_k)$*-free* if G is H_i-free for every $i \in \{1, \dots, k\}$.

Let T be a tree and let v be a leaf of T. Let u be a vertex of degree at least 3 having shortest distance in T from v and let P be the v, u-path in T. The operation of *trimming* the leaf v consists in deleting from T the vertex set $V(P) \setminus \{u\}$.

An *independent set* of a graph G is a set of pairwise non-adjacent vertices and the maximum size of an independent set of G is denoted by $\alpha(G)$. A *clique* of a graph G is a set of pairwise adjacent vertices and the maximum size of a clique of G is denoted by $\omega(G)$. A *matching* of a graph is a set of edges with no shared endpoints.

The path and the complete graph on n vertices are denoted by P_n and K_n, respectively. A graph is *r-partite*, for $r \geq 2$, if its vertex set admits a partition into r classes such that every edge has its endpoints in different classes. An r-partite graph in which every two vertices from different partition classes are adjacent is a *complete r-partite graph* and a 2-partite graph is also called *bipartite*. The complete bipartite graph with partition classes of size t and s is denoted by $K_{t,s}$. A graph is *co-bipartite* if it is the complement of a bipartite graph.

For $\ell \geq 1$, an *ℓ-caterpillar* is a subcubic tree T on 2ℓ vertices with $V(T) = \{s_1, \dots, s_\ell, t_1, \dots, t_\ell\}$, such that $E(T) = \{s_it_i : 1 \leq i \leq \ell\} \cup \{s_is_{i+1} : 1 \leq i \leq \ell - 1\}$. The vertices $t_1, t_2, \dots, t_\ell$ are the leaves and the path $s_1s_2 \cdots s_\ell$ is the *backbone* of the caterpillar.

A *colouring* of a graph $G = (V, E)$ is a mapping $c\colon V \to \{1, 2, \dots\}$ that gives each vertex $u \in V$ a *colour* $c(u)$ in such a way that, for every two adjacent vertices u and v, we have that $c(u) \neq c(v)$. If for every $u \in V$ we have $c(u) \in \{1, \dots, k\}$, then we say that c is a *k-colouring* of G. The Colouring problem is to decide whether a given graph G has a k-colouring for some given integer $k \geq 1$. If k is *fixed*, that is, not part of the input, we call this the k-Colouring problem. It is well known that k-Colouring is NP-complete for each $k \geq 3$. A generalisation of k-Colouring is the following. For an integer $k \geq 1$, a *k-list assignment* of a graph $G = (V, E)$ is a function L that assigns each vertex $u \in V$ a *list* $L(u) \subseteq \{1, 2, \dots, k\}$ of *admissible* colours for u. A colouring c of G *respects* L if $c(u) \in L(u)$ for every $u \in V$. For a fixed integer $k \geq 1$, the List k-Colouring problem is to decide whether a given graph G with a k-list assignment L admits a colouring that respects L. By setting $L(u) = \{1, \dots, k\}$ for every $u \in V$, we obtain the k-Colouring problem.

3. Sim-width and independent packings

In this section we show Theorem 6 and Corollary 7. Let $\mathcal{H}$ be a finite set of connected non-null graphs. Given a graph G, let $\mathcal{H}_G$ be the set of all subgraphs of G isomorphic to a member of $\mathcal{H}$. Recall that the $\mathcal{H}$-graph of G, denoted $\mathcal{H}(G)$, is defined as follows: the vertex set is $\mathcal{H}_G$ and two distinct subgraphs of G isomorphic to a member of $\mathcal{H}$ are adjacent if and only if they either have a vertex in common or there is an edge in G connecting them. We begin by showing Theorem 6: mapping a graph G to its $\mathcal{H}$-graph does not increase the sim-width.

Theorem 6. *Let $\mathcal{H}$ be a non-empty finite set of connected non-null graphs and let r be the maximum number of vertices of a graph in $\mathcal{H}$. Let G be a graph and let (T, δ) be a branch decomposition of G. If $|V(\mathcal{H}(G))| > 1$, then we can obtain in $O(|V(G)|^{r+1})$ time a branch decomposition (T', δ') of $\mathcal{H}(G)$ such that $\mathrm{simw}_{\mathcal{H}(G)}(T', \delta') \leq \mathrm{simw}_G(T, \delta)$.*

Proof. Observe that if G is edgeless, then $\mathcal{H}(G)$ is edgeless as well and the statement trivially holds. Therefore, we assume that G is not edgeless, and hence $\mathrm{simw}_G(T, \delta) \geq 1$.

Let $\mathcal{H} = \{H_1, \dots, H_n\}$. Let h be an arbitrary vertex of $\mathcal{H}(G)$. Hence, h corresponds to a subgraph of G isomorphic to H_i, for some $i \in \{1, \dots, n\}$. This means there exists a unique vertex set $S(h) \subseteq V(G)$ such that $|S(h)| = |V(H_i)|$ and $G[S(h)]$ contains a copy of H_i as a subgraph ($S(h)$ is just the vertex set of the subgraph of G corresponding to h). We compute all $S(h)$, for $h \in \mathcal{H}(G)$, in $O(|V(G)|^r)$ time as follows. We enumerate all $O(|V(G)|^r)$ subsets of vertices of G of size at most r. For each such set S and each $H \in \mathcal{H}$ with $|S|$ vertices, we iterate over all $|S|! \leq r! = O(1)$ possible bijections $g\colon V(H) \to S$. We then keep the subsets S for which one such bijection maps every pair of adjacent vertices in H to a pair of adjacent vertices in $G[S]$. We now arbitrarily order $V(G)$ and let $f(h)$ be the smallest vertex in $S(h)$ with respect to this ordering. For $v \in V(G)$, let $F(v) = \{h \in V(\mathcal{H}(G)) : f(h) = v\}$. Note that $F(v)$ is a clique in $\mathcal{H}(G)$. We can compute all sets $F(v)$, for $v \in V(G)$, in $O(|V(G)| \cdot |V(\mathcal{H}(G))|) = O(|V(G)|^{r+1})$ time.

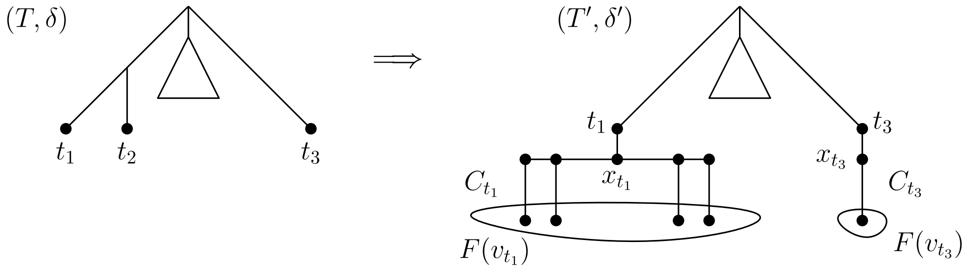


Fig. 2. How to construct a branch decomposition (T', δ') of $\mathcal{H}(G)$ from a branch decomposition (T, δ) of G. We distinguish vertices t_i such that $|F(v_{t_i})| = 0$ $(i = 2)$, $|F(v_{t_i})| = 1$ $(i = 3)$ and $|F(v_{t_i})| \geq 2$ $(i = 1)$.

We are now ready to construct (T', δ') from (T, δ) as follows (see Fig. 2). For each leaf $t \in V(T)$, we let $v_t = \delta^{-1}(t)$, and do the following. If $F(v_t) \neq \varnothing$, we distinguish two cases. Suppose first that $|F(v_t)| = 1$. In this case, build a $|F(v_t)|$-caterpillar C_t and add the edge connecting the single vertex x_t in the backbone of C_t with the node t. Suppose now that $|F(v_t)| \geq 2$. In this case, build a $|F(v_t)|$-caterpillar C_t, subdivide an arbitrary edge of the backbone of C_t by adding a new vertex x_t and add the edge $x_t t$. Finally, if $F(v_t) = \varnothing$, trim the leaf t of T, as defined in Section 2. Observe that, since $|V(\mathcal{H}(G))| > 1$, either there exists a leaf $t \in V(T)$ such that $|F(v_t)| \geq 2$ or there exist at least two leaves $t_1, t_2 \in V(T)$ such that $|F(v_{t_1})| \geq 1$ and $|F(v_{t_2})| \geq 1$. This implies that each leaf t of T such that $F(v_t) = \varnothing$ can be trimmed. Moreover, by definition, no new leaf is created after an application of trimming. Let T' be the tree obtained by the procedure above. Let δ' be the map from $V(\mathcal{H}(G))$ to the leaves of T' which restricted to $F(v_t)$ is an arbitrary bijection from $F(v_t)$ to the leaves of C_t. It is easy to see that (T', δ') is a branch decomposition of $\mathcal{H}(G)$ and that it can be computed in $O(|V(G)|^2)$ time.

We now show that $\text{simw}_{\mathcal{H}(G)}(T', \delta') \leq \text{simw}_G(T, \delta)$. Suppose that $\text{simw}_{\mathcal{H}(G)}(T', \delta') = k$. Since the statement is trivially true if $k \leq 1$, we may assume $k \geq 2$. Each $e' \in E(T')$ naturally induces a partition $(A_{e'}, \overline{A_{e'}})$ of $V(\mathcal{H}(G))$. Consider $e \in E(T')$ such that $\text{cutsim}_{\mathcal{H}(G)}(A_e, \overline{A_e}) = \text{simw}_{\mathcal{H}(G)}(T', \delta') = k$. Then, there is a matching $\{x'_1 y'_1, \ldots, x'_k y'_k\}$ of size k such that $\{x'_1, \ldots, x'_k\} \subseteq A_e$ and $\{y'_1, \ldots, y'_k\} \subseteq \overline{A_e}$ are independent sets of $\mathcal{H}(G)$. Suppose first that e is an edge of C_t or the edge $x_t t$, for some leaf $t \in V(T)$. Then, one of A_e and $\overline{A_e}$ is a subset of $F(v_t)$, where $v_t = \delta^{-1}(t)$. Since each $F(v_t)$ is a clique in $\mathcal{H}(G)$, we have that $k \leq 1$. Hence, we may assume that $e \in E(T') \cap E(T)$. Then, for any $h \in V(\mathcal{H}(G))$, $\delta'(h)$ and $\delta(f(h))$ belong to the same component of $T' - e$, and so e naturally induces a partition $(A_e, \overline{A_e})$ of $V(\mathcal{H}(G))$ and a partition $(B_e, \overline{B_e})$ of $V(G)$ satisfying the following property: For any $h \in V(\mathcal{H}(G))$, $h \in A_e$ if and only if $f(h) \in B_e$.

We claim that, for $i \neq j$, $S(x'_i) \cup S(y'_i)$ and $S(x'_j) \cup S(y'_j)$ are disjoint and anticomplete in G. Indeed, suppose without loss of generality that $S(x'_i)$ shares a vertex with either $S(x'_j)$ or $S(y'_j)$. Then, x'_i is adjacent to either x'_j or y'_j in $\mathcal{H}(G)$, a contradiction. Similarly, if there is an edge between $S(x'_i)$ and either $S(x'_j)$ or $S(y'_j)$ in G, then x'_i is adjacent to either x'_j or y'_j in $\mathcal{H}(G)$, a contradiction again.

We now claim that $G[S(x'_i) \cup S(y'_i)]$ is connected. Since $G[S(x'_i)]$ contains a copy of a connected graph $H_s \in \mathcal{H}$, with $|S(x'_i)| = |V(H_s)|$, as a subgraph, we have that $G[S(x'_i)]$ is connected. Similarly, $G[S(y'_i)]$ is connected. Moreover, since x'_i is adjacent to y'_i, either $S(x'_i)$ shares a vertex with $S(y'_i)$ or there is an edge in G between $S(x'_i)$ and $S(y'_i)$. In either case we obtain that $G[S(x'_i) \cup S(y'_i)]$ is connected.

Therefore, for each $i \in \{1, \ldots, k\}$, there is a path P_i in $G[S(x'_i) \cup S(y'_i)]$ from $f(x'_i)$ to $f(y'_i)$ in G, say $P_i = v_0 v_1 \cdots v_\ell$ where $v_0 = f(x'_i)$ and $v_\ell = f(y'_i)$. Since $x'_i \in A_e$ and $y'_i \in \overline{A_e}$, it follows that $f(x'_i) \in B_e$ and $f(y'_i) \in \overline{B_e}$. Since the path P_i must cross the cut $(B_e, \overline{B_e})$ of G, there exists $q \in \{0, \ldots, \ell - 1\}$ such that $v_q \in B_e$ and $v_{q+1} \in \overline{B_e}$. We let $x_i = v_q$ and $y_i = v_{q+1}$. Clearly, $x_i y_i \in E(G)$. We now claim that, for each $i \neq j$, $\{x_i, y_i\}$ and $\{x_j, y_j\}$ are disjoint and anticomplete in G. This simply follows from the fact that, for $p \in \{i, j\}$, $\{x_p, y_p\} \subseteq G[S(x'_p) \cup S(y'_p)]$ and $G[S(x'_i) \cup S(y'_i)]$ and $G[S(x'_j) \cup S(y'_j)]$ are disjoint and anticomplete in G.

Let now $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_k\}$. By the previous paragraph, $X \subseteq B_e$ and $Y \subseteq \overline{B_e}$, X and Y are independent sets and $G[X, Y] \cong kP_2$. Therefore, $\text{simw}_G(T, \delta) \geq \text{cutsim}_G(B_e, \overline{B_e}) \geq k = \text{simw}_{\mathcal{H}(G)}(T', \delta')$. □

Recall that two subgraphs H_1 and H_2 of a graph G are independent if they are vertex-disjoint and no edge of G joins a vertex of H_1 with a vertex of H_2. An independent $\mathcal{H}$-packing in G is a set of pairwise independent subgraphs from $\mathcal{H}_G$. Given a graph G, a weight function $w\colon \mathcal{H}_G \to \mathbb{Q}_+$ on the subgraphs in $\mathcal{H}_G$, and an independent $\mathcal{H}$-packing P in G, the weight of P is defined as the sum $\sum_{H \in P} w(H)$. Given a graph G and a weight function $w\colon \mathcal{H}_G \to \mathbb{Q}_+$, Maximum Weight Independent $\mathcal{H}$-Packing is the problem of finding an independent $\mathcal{H}$-packing in G of maximum weight. Besides Theorem 6, in order to show Corollary 7, we need the following two results.

Theorem 11 *([19]). Let $\mathcal{H}$ be a non-empty finite set of connected non-null graphs and let r be the maximum number of vertices of a graph in $\mathcal{H}$. Then there exists an algorithm that takes as input a graph G and computes the graph $\mathcal{H}(G)$ in $O(|V(G)|^{2r})$ time.*

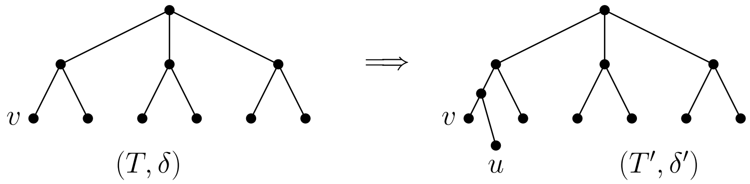


Fig. 3. How to construct a branch decomposition (T', δ') of G' from a branch decomposition (T, δ) of G, where G' is obtained from G by adding a leaf vertex u adjacent to v.

Observation 12 *([19]). Let $\mathcal{H}$ be a finite set of connected non-null graphs. Let G be a graph and let $w\colon \mathcal{H}_G \to \mathbb{Q}_+$. Let I be an independent set in $\mathcal{H}(G)$ of maximum weight with respect to the weight function w. Then I is an independent $\mathcal{H}$-packing in G of maximum weight.*

Corollary 7. *Let $\mathcal{H}$ be a non-empty finite set of connected non-null graphs such that each graph in $\mathcal{H}$ has at most r vertices. Let $\mathcal{G}$ be a graph class whose sim-width is bounded and quickly computable. If* Maximum Weight Independent Set *is polynomial-time solvable for $\mathcal{G}$, then* Maximum Weight Independent $\mathcal{H}$-Packing *is polynomial-time solvable for $\mathcal{G}$. Similarly, if* Independent Set *is polynomial-time solvable for $\mathcal{G}$, then* Independent $\mathcal{H}$-Packing *is polynomial-time solvable for $\mathcal{G}$.*

Proof. Given the input graph $G \in \mathcal{G}$, we compute in polynomial time a branch decomposition of G of sim-width at most k, for some integer k. We then compute $\mathcal{H}(G)$ in polynomial time using Theorem 11. If $|V(\mathcal{H}(G))| \leq 1$, we immediately conclude thanks to Observation 12. Otherwise, by Theorem 6, we compute in polynomial time a branch decomposition of $\mathcal{H}(G)$ of sim-width at most k. Finally, using the assumed algorithm, we compute in polynomial time a maximum-weight independent set in $\mathcal{H}(G)$ which, by Observation 12, is an independent $\mathcal{H}$-packing in G of maximum weight. □

4. Mim-width of $(rP_1, \overline{K_{t,s} + P_1})$-free graphs

In this section we show the mim-width dichotomy for the class of $(rP_1, \overline{K_{t,s} + P_1})$-free graphs stated in Theorem 8. We begin by identifying the cases of bounded mim-width (Section 4.1) and then pass to the cases of unbounded mim-width (Section 4.2). These results are then combined to prove Theorem 8 (Section 4.3).

4.1. Boundedness results

In this section we show that, for each $t \geq 4$, the mim-width of $(3P_1, \overline{K_{3,t} + P_1})$-free graphs and the mim-width of $(4P_1, \overline{K_{2,t} + P_1})$-free graphs are bounded and quickly computable (Theorems 16 and 19, respectively). The proofs are based on the following common strategy. We find t pairwise non-adjacent vertices $v_1, \dots v_t$ in the input graph G ($t = 2$ in Theorem 16 and $t = 3$ in Theorem 19). We then obtain a partition of $V(G)$ where one partition class is $\{v_1, \dots, v_t\}$ and the remaining ones are the sets of private neighbours of subsets of $\{v_1, \dots, v_t\}$ with respect to $\{v_1, \dots, v_t\}$. We finally construct an appropriate branch decomposition of G and use the following simple observation.

Observation 13. *Let $V_1, \dots, V_m$ be a partition of $V(G)$ and let (T, δ) be a branch decomposition of G. Then,*

$$\mathrm{mimw}_G(T,\delta) = \max_{e \in E(T)} \mathrm{cutmim}_G(A_e, \overline{A_e}) \leq \max_{e \in E(T)} \sum_{1 \leq i,j \leq m} \mathrm{cutmim}_G(A_e \cap V_i, \overline{A_e} \cap V_j).$$

We will need two auxiliary results. The first one below is left as an easy exercise (see Fig. 3).

Lemma 14. *Let G be a graph and let (T, δ) be a branch decomposition of G with $\mathrm{mimw}_G(T, \delta) \leq k$, with $k \geq 1$. Let G' be the graph obtained from G by adding a vertex of degree at most 1. Then we can construct in $O(1)$ time a branch decomposition (T', δ') of G' with $\mathrm{mimw}_{G'}(T', \delta') \leq k$.*

The second one is essentially stated in the proof of [41, Corollary 3.7.4]. We provide its short proof for completeness.

Lemma 15 *(Vatshelle [41]). Let G be a graph with $|V(G)| > 1$ and maximum degree at most 2. Then $\mathrm{mimw}(G) \leq 2$ and a branch decomposition (T, δ) of G with $\mathrm{mimw}_G(T, \delta) \leq 2$ can be constructed in $O(n)$ time.*

Proof. Suppose that G has k components, $C_1, \dots, C_k$, where each C_i is a path or a cycle with vertex set $\{v_{i,1}, \dots, v_{i,|C_i|}\}$. For $1 < j < |C_i|$, each $v_{i,j}$ is adjacent to $v_{i,j-1}$ and $v_{i,j+1}$ and, if C_i is a cycle, $v_{i,1}$ is adjacent to $v_{1,|C_i|}$. For each component C_i, we construct a $|C_i|$-caterpillar T_i with leaves $\ell_{i,1}, \dots, \ell_{i,|C_i|}$ and subdivide an arbitrary edge of the backbone of T_i with a new vertex t_i, unless the backbone of T_i has size 1, in which case we let t_i be the unique vertex of the backbone. We then construct a k-caterpillar T_0 with leaves $\ell_{0,1}, \dots, \ell_{0,k}$. Let T be the subcubic tree obtained from the disjoint union of $T_0, T_1, \dots, T_k$ by adding the edges $\ell_{0,1}t_1, \dots, \ell_{0,k}t_k$ and, if $k = 1$, by additionally deleting $V(T_0)$. Let δ be the bijection from the vertices of G to the leaves of T given by $\delta(v_{i,j}) = \ell_{i,j}$. Clearly, (T, δ) is a branch decomposition of G and it can be constructed in $O(n)$ time.

We now show that $\text{mimw}_G(T, \delta) \leq 2$. Let $e \in E(T)$ and consider the partition $(A_e, \overline{A_e})$ of $V(G)$ induced by e. Suppose first that e belongs to $E(T_0)$ or $e = \ell_{0,j}t_j$ for some j. Then, for each component C_i of G, $V(C_i)$ is fully contained in either A_e or $\overline{A_e}$ and so $\text{cutmim}_G(A_e, \overline{A_e}) = 0$. Suppose now that e belongs to the backbone of T_i, for some $i > 0$. Then, it is easy to see that there are at most two edges across the cut $(A_e, \overline{A_e})$, from which $\text{cutmim}_G(A_e, \overline{A_e}) \leq 2$. Suppose finally that e is incident to a leaf of T. Then $\text{cutmim}_G(A_e, \overline{A_e}) = 1$. These observations imply that $\text{mimw}_G(T, \delta) \leq 2$. $\square$

We can finally provide our two boundedness results. In both proofs, we make repeated implicit use of Ramsey's theorem: there exists a least positive integer $R(r, s)$ for which every graph with at least $R(r, s)$ vertices either contains an independent set of size r or a clique of size s. Observe that, for $r, s > 1$, $R(r, s) \geq s$.

Theorem 16. *Let $t \geq 4$ and let G be a $(3P_1, \overline{K_{3,t} + P_1})$-free graph. Then $\text{mimw}(G) < 5R(3, t) + 8t + 46$ and a branch decomposition (T, δ) of G with $\text{mimw}_G(T, \delta) < 5R(3, t) + 8t + 46$ can be constructed in $O(n^2)$ time.*

Proof. We assume that G contains two non-adjacent vertices v_a and v_b, or else G is a complete graph and the statement is trivially true. Let $S_z = \{v_a, v_b\}$. Since G is $3P_1$-free, all remaining vertices are adjacent to at least one of v_a and v_b and we partition them into three classes S_a, S_b and S_{ab} as follows: S_a is the set of vertices that are adjacent to v_a but not v_b, S_b is the set of vertices that are adjacent to v_b but not v_a and S_{ab} is the set of vertices that are adjacent to both v_a and v_b. Note that S_a is a clique, or else two non-adjacent vertices in S_a together with v_b would induce a copy of $3P_1$. Similarly, S_b is a clique.

We now proceed to the construction of a branch decomposition of G by distinguishing two cases. We say that G is *good* (w.r.t. $\{v_a, v_b\}$) if every vertex in S_a has at most two neighbours in S_b and every vertex in S_b has at most two neighbours in S_a. Otherwise, we say that G is *bad* (w.r.t. $\{v_a, v_b\}$).

Suppose first that G is good. Then, $G[S_a, S_b]$ has maximum degree at most 2 and, if $G[S_a, S_b]$ contains at least two vertices, Lemma 15 allows us to construct a branch decomposition (T_1, δ_1) of $G[S_a, S_b]$ with mim-width at most 2. Let u be a leaf of T_1 and let e be the edge of T_1 incident to u. We subdivide e by introducing a new vertex x and obtain a new tree T_1'. If however $G[S_a, S_b]$ contains exactly one vertex, let x be this vertex. We now let $\ell = |V(G) \setminus (S_a \cup S_b)|$ and consider an ℓ-caterpillar T_2 (notice that $\ell \geq 2$). We subdivide one of the edges of the backbone of T_2 by introducing a new vertex y and obtain a new tree T_2'. Let δ_2 be any bijection from $V(G) \setminus (S_a \cup S_b)$ to the set of leaves of T_2'. We finally add the edge xy in order to obtain a subcubic tree T, unless $G[S_a, S_b]$ is the null graph, in which case we let $T = T_2'$. Clearly, the set of leaves L of T is the disjoint union of the set of leaves of T_1 and the set of leaves of T_2. Considering the map $\delta \colon V(G) \to L$ which coincides with δ_1 when restricted to $S_a \cup S_b$ and with δ_2 when restricted to $V(G) \setminus (S_a \cup S_b)$, we obtain a branch decomposition (T, δ) of G. If G is bad, we simply let (T, δ) be any branch decomposition of G.

The branch decomposition (T, δ) of G defined above can be constructed in $O(n^2)$ time. Indeed, we first find two non-adjacent vertices v_a and v_b in $O(n^2)$ time and check whether $G[S_a, S_b]$ has maximum degree at most 2 in linear time. If so, G is good and we then construct (T, δ) in $O(n)$ time thanks to Lemma 15. Otherwise, G is bad, and we trivially construct (T, δ) in linear time.

Claim 17. *Let S_P and S_Q be subsets of vertices of G, not necessarily disjoint. If there exists a vertex that is complete to both S_P and S_Q, then $\text{cutmim}_G(A_e \cap S_P, \overline{A_e} \cap S_Q) < R(3, t) + 6$, for any $e \in E(T)$.*

Proof of Claim 17. Let $v \in V(G)$ be complete to S_P and S_Q. Suppose, to the contrary, that $\text{cutmim}_G(A_e \cap S_P, \overline{A_e} \cap S_Q) \geq R(3, t) + 6$ for some $e \in E(T)$ and let $\{p_1q_1, \dots, p_{R(3,t)+6}q_{R(3,t)+6}\}$ be an induced matching witnessing this, where $\{p_1, \dots, p_{R(3,t)+6}\} \subseteq A_e \cap S_P$ and $\{q_1, \dots, q_{R(3,t)+6}\} \subseteq \overline{A_e} \cap S_Q$. Since G is $3P_1$-free, $\{q_1, \dots, q_{R(3,t)}\}$ contains a clique of size at least t. Without loss of generality, $\{q_1, \dots, q_t\}$ induces such a clique. Observe now that $\{p_{R(3,t)+1}, \dots, p_{R(3,t)+6}\}$ contains a clique of size 3, as $R(3, 3) = 6$. Without loss of generality, $\{p_{R(3,t)+1}, p_{R(3,t)+2}, p_{R(3,t)+3}\}$ induces such a clique. But then we have that $G[p_{R(3,t)+1}, p_{R(3,t)+2}, p_{R(3,t)+3}, q_1, q_2, \dots, q_t, v] \cong \overline{K_{3,t} + P_1}$, a contradiction. $\diamond$

Claim 18. *Suppose that G is bad. Then $\text{cutmim}_G(A_e \cap S_a, \overline{A_e} \cap S_b) < 4t$ and $\text{cutmim}_G(A_e \cap S_b, \overline{A_e} \cap S_a) < 4t$, for any $e \in E(T)$.*

Proof of Claim 18. By symmetry, it is enough to show the first statement. Since G is bad, $G[S_a, S_b]$ contains a vertex u of degree at least 3. Without loss of generality, $u \in S_a$. Suppose, to the contrary, that $\text{cutmim}_G(A_e \cap S_a, \overline{A_e} \cap S_b) \geq 4t$ for some $e \in E(T)$ and let $\{a_1b_1, \dots, a_{4t}b_{4t}\}$ be an induced matching witnessing this, where $\{a_1, \dots, a_{4t}\} \subseteq A_e \cap S_a$ and $\{b_1, \dots, b_{4t}\} \subseteq \overline{A_e} \cap S_b$. Let $v_1, v_2, v_3 \in S_b$ be distinct neighbours of $u \in S_a$. Observe now that all except possibly $t - 1$ vertices in $\{a_1, \dots, a_{4t}\}$ are adjacent to at least one of v_1, v_2, v_3, or else there are t vertices in $\{a_1, \dots, a_{4t}\}$, say without loss of generality $a_1, \dots, a_t$, non-adjacent to any of v_1, v_2, v_3 and so, since S_a and S_b are cliques, $G[v_1, v_2, v_3, a_1, \dots, a_t, u] \cong \overline{K_{3,t} + P_1}$, a contradiction. Hence, there is a vertex in $\{v_1, v_2, v_3\}$ with at least t neighbours in $\{a_1, \dots, a_{4t}\}$, say without loss of generality v_1 is adjacent to $a_1, \dots, a_t$, and so $G[b_{t+1}, b_{t+2}, b_{t+3}, a_1, \dots, a_t, v_1] \cong \overline{K_{3,t} + P_1}$, a contradiction. $\diamond$

We can finally show that $\text{mimw}_G(T, \delta) < 5R(3,t) + 8t + 46$. Let $D = \{a, b, ab, z\}$. Since S_a, S_b, S_{ab}, S_z is a partition of $V(G)$, Observation 13 implies that

$$\text{mimw}_G(T, \delta) \le \max_{e \in E(T)} \sum_{i,j \in D} \text{cutmim}_G(A_e \cap S_i, \overline{A_e} \cap S_j).$$

It is then enough to estimate the terms in the sum. Since S_a and S_b are cliques, $\text{cutmim}_G(A_e \cap S_a, \overline{A_e} \cap S_a) \le 1$ and $\text{cutmim}_G(A_e \cap S_b, \overline{A_e} \cap S_b) \le 1$. Moreover, since v_a is complete to S_a and S_{ab}, and v_b is complete to S_b and S_{ab}, Claim 17 implies that $\text{cutmim}_G(A_e \cap S_a, \overline{A_e} \cap S_{ab})$, $\text{cutmim}_G(A_e \cap S_b, \overline{A_e} \cap S_{ab})$, $\text{cutmim}_G(A_e \cap S_{ab}, \overline{A_e} \cap S_a)$, $\text{cutmim}_G(A_e \cap S_{ab}, \overline{A_e} \cap S_b)$, $\text{cutmim}_G(A_e \cap S_{ab}, \overline{A_e} \cap S_{ab}) < R(3,t) + 6$. Observe now that, for any $i \in D$, $\text{cutmim}_G(A_e \cap S_z, \overline{A_e} \cap S_i) \le 2$, $\text{cutmim}_G(A_e \cap S_i, \overline{A_e} \cap S_z) \le 2$.

It remains to bound $\text{cutmim}_G(A_e \cap S_a, \overline{A_e} \cap S_b)$ and $\text{cutmim}_G(A_e \cap S_b, \overline{A_e} \cap S_a)$. If G is bad then, by Claim 18, $\text{cutmim}_G(A_e \cap S_a, \overline{A_e} \cap S_b) < 4t$ and $\text{cutmim}_G(A_e \cap S_b, \overline{A_e} \cap S_a) < 4t$. If G is good, we proceed as follows. Suppose first that either $e = xy$ or $e \in E(T_2')$. Then all vertices of S_a and S_b belong to the same partition class of $V(G)$ induced by e and so $\text{cutmim}_G(A_e \cap S_a, \overline{A_e} \cap S_b) = \text{cutmim}_G(A_e \cap S_b, \overline{A_e} \cap S_a) = 0$. Suppose finally that $e \in E(T_1')$. Then e induces a partition $(A_e', \overline{A_e'})$ of $S_a \cup S_b$ with respect to (T_1, δ_1), and $(A_e', \overline{A_e'})$ coincides with $(A_e, \overline{A_e})$ restricted to $S_a \cup S_b$. Consequently, $\text{cutmim}_G(A_e \cap S_a, \overline{A_e} \cap S_b) = \text{cutmim}_G(A_e' \cap S_a, \overline{A_e'} \cap S_b) \le 2$ as $\text{cutmim}_G(T_1, \delta_1) \le 2$. The same holds for $\text{cutmim}_G(A_e \cap S_b, \overline{A_e} \cap S_a)$.

By the previous paragraphs, $\text{mimw}_G(T, \delta) < 2 \cdot 1 + 5 \cdot (R(3,t) + 6) + 7 \cdot 2 + 2 \cdot 4t = 5R(3,t) + 8t + 46$. $\square$

Theorem 19. *Let $t \ge 4$ and let G be a $(4P_1, \overline{K_{2,t} + P_1})$-free graph. Then $\text{mimw}(G) < 43R(4,t) + 24t + 208$ and a branch decomposition (T, δ) of G with $\text{mimw}_G(T, \delta) < 43R(4,t) + 24t + 214$ can be computed in $O(n^3)$ time.*

Proof. We assume that G contains three pairwise non-adjacent vertices v_a, v_b and v_c, or else G is $3P_1$-free and the statement follows from Theorem 16. Since G is $4P_1$-free, all remaining vertices are adjacent to at least one of v_a, v_b and v_c. For a subset $\alpha \subseteq \{a, b, c\}$, let $S_\alpha = \cap_{i \in \alpha} N(v_i) \setminus \cup_{j \in \{a,b,c\} \setminus \alpha} N(v_j)$. In words, S_α is the set of private neighbours of $\{v_i : i \in \alpha\}$ with respect to $\{v_a, v_b, v_c\}$. Note that S_a, S_b and S_c are cliques, or else, for distinct $i, j, k \in \{a, b, c\}$, two non-adjacent vertices in S_i together with v_j and v_k would induce a copy of $4P_1$. This fact will be repeatedly used in the claims below. For $\alpha, \beta \subseteq \{a, b, c\}$ and an integer $s \ge 1$, we say that the vertex set S_α is $3s$*-almost-complete* to the vertex set S_β if there are at most two vertices in S_α non-adjacent to at least $3s$ vertices in S_β.

Claim 20. *Let $p, q \in \{a, b, c\}$ with $p \ne q$. If a vertex in S_p is adjacent to at least two vertices in S_q, then S_q is $3t$-almost-complete to S_p.*

Proof of Claim 20. Note that v_p is complete to S_p but anticomplete to S_q and v_q is complete to S_q but anticomplete to S_p. Suppose that $x \in S_p$ is adjacent to two distinct vertices y_1 and y_2 of S_q. Then $\{y_1, y_2\} \cap \{v_q\} = \varnothing$.

We claim that there are at most $t-1$ vertices in S_p anticomplete to $\{y_1, y_2\}$. Indeed, if there are t vertices in S_p anticomplete to $\{y_1, y_2\}$, then these t vertices together with $\{x, y_1, y_2\}$ induce a copy of $\overline{K_{2,t} + P_1}$, as S_p and S_q are cliques, a contradiction.

Let now $y \in S_q$ be a vertex distinct from y_1 and y_2. We claim that y is anticomplete to at most $t-1$ vertices in $S_p \cap N(y_i)$, for each $i \in \{1, 2\}$. Indeed, if there are t vertices in $S_p \cap N(y_i)$ anticomplete to y, then these t vertices together with $\{y_i, v_q, y\}$ induce a copy of $\overline{K_{2,t} + P_1}$, a contradiction.

Let $A_1 = S_p \cap N(y_1)$, $A_2 = S_p \cap N(y_2)$ and let $y \in S_q$ be a vertex distinct from y_1 and y_2. Clearly, $S_p = A_1 \cup A_2 \cup (S_p \setminus (A_1 \cup A_2))$. By the second paragraph, $|S_p \setminus (A_1 \cup A_2)| \le t-1$ and so y is anticomplete to at most $t-1$ vertices in $S_p \setminus (A_1 \cup A_2)$. By the third paragraph, y is anticomplete to at most $t-1$ vertices in A_1 and at most $t-1$ vertices in A_2. Therefore, y is anticomplete to at most $3(t-1) < 3t$ vertices in S_p and so S_q is $3t$-almost-complete to S_p. $\diamond$

We now proceed to the construction of a branch decomposition of G. Consider first the graph G_1 with vertex set $V(G_1) = S_a \cup S_b \cup S_c$ and edge set $E(G_1) = \{uv : uv \in E(G), u \in S_\alpha, v \in S_\beta, \alpha, \beta \in \{a, b, c\}, \alpha \ne \beta, S_\alpha$ is not $3t$-almost-complete to S_β, S_β is not $3t$-almost complete to $S_\alpha\}$. We claim that each vertex v of G_1 has degree at most 2. By symmetry, suppose that $v \in S_a$. By definition of G_1, v has no neighbours in S_a. If S_b is $3t$-almost-complete to S_a, then v has no neighbours in S_b. Otherwise, S_b is not $3t$-almost-complete to S_a and, by Claim 20, v has at most one neighbour in S_b. Similarly, v has at most one neighbour in S_c. Therefore, G_1 has maximum degree at most 2 and so, by Lemma 15, if G_1 contains at least two vertices, then we can construct in $O(n)$ time a branch decomposition (T_1, δ_1) of G_1 with $\text{mimw}_{G_1}(T_1, \delta_1) \le 2$.

For $x \in \{a, b, c\}$ and $Y = \{a, b, c\} \setminus \{x\}$, a vertex $v \in S_Y$ is S_x*-good* if it has at most one neighbour in S_x, and S_x*-bad* otherwise. Let $S_Y^\star$ be the set of vertices in S_Y that are S_x-bad. We now build a graph G_2 as follows. Start with $G_2 = G_1$.

For each $x \in \{a,b,c\}$, let $Y = \{a,b,c\} \setminus \{x\}$. For each vertex $v \in S_Y$, if v is S_x-good, then add v to $V(G_2)$ and, if v has a neighbour u in S_x, add uv to $E(G_2)$. In other words, we grow G_1 by adding leaf vertices or isolated vertices.

Now, if G_2 is the null graph, let T_2' be the null tree, and if G_2 consists of one vertex, let T_2' be the tree with a single vertex r. Otherwise, G_2 contains at least two vertices and, given (T_1, δ_1), we can construct a branch decomposition (T_2, δ_2) of G_2 with $\text{mimw}_{G_2}(T_2, \delta_2) \le 2$ in $O(n)$ time thanks to Lemma 14, unless G_1 contains at most one vertex, in which case G_2 has maximum degree at most 1 and we let (T_2, δ_2) be any branch decomposition of G_2. We then subdivide one of the edges of T_2 by introducing a new vertex r to obtain a new tree T_2'. Clearly, $\text{mimw}_{G_2}(T_2', \delta_2) = \text{mimw}_{G_2}(T_2, \delta_2) \le 2$. Let now $\ell = |V(G) \setminus V(G_2)|$ and consider an ℓ-caterpillar T_3 (notice that $\ell \ge 3$). Let δ_3 be any bijection from $V(G) \setminus V(G_2)$ to the set of leaves of T_3. We subdivide one of the edges of the backbone of T_3 by introducing a new vertex s and obtain a new tree T_3'. We finally add the edge rs in order to obtain a tree T. Observe that the set of leaves L of T is the disjoint union of the set of leaves L_2 of T_2' and the set of leaves L_3 of T_3'. Considering the map δ which coincides with δ_i when restricted to L_i (for $i = 2, 3$), we obtain a branch decomposition (T, δ) of G.

We now analyse the running time to construct (T, δ). Finding three pairwise non-adjacent vertices v_a, v_b and v_c and computing S_α for each $\alpha \subseteq \{a,b,c\}$ can be done in $O(n^3)$ time. Checking for $3t$-almost-completeness and constructing G_1 can be done in $O(n)$ time. Finding the S_x-good vertices and constructing G_2 can be done in $O(n)$ time. Therefore, constructing (T, δ) can be done in $O(n^3)$ time.

Claim 21. *Let $\alpha, \beta \subseteq \{a,b,c\}$. If S_α is $3t$-almost-complete to S_β, then* $\text{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S_\beta) < 3t + 1$ *and* $\text{cutmim}_G(A_e \cap S_\beta, \overline{A_e} \cap S_\alpha) < 3t + 1$*, for any $e \in E(T)$.*

Proof of Claim 21. Suppose that there exist $V_\alpha \subseteq A_e \cap S_\alpha$ and $V_\beta \subseteq \overline{A_e} \cap S_\beta$ such that $G[V_\alpha, V_\beta] \cong (3t+1)P_2$. Then, each of the $3t + 1$ vertices in V_α is non-adjacent to at least $3t$ vertices in V_β, contradicting the fact that S_α is $3t$-almost-complete to S_β. The proof of the other inequality is similar. ◇

Claim 22. *Let $x \in \{a,b,c\}$ and $Y = \{a,b,c\} \setminus \{x\}$. Then* $\text{cutmim}_G(A_e \cap S_x, \overline{A_e} \cap S_Y^\star) < R(4,t) + t + 1$ *and* $\text{cutmim}_G(A_e \cap S_Y^\star, \overline{A_e} \cap S_x) < R(4,t) + t + 1$*, for any $e \in E(T)$.*

Proof of Claim 22. We show the first inequality, the proof of the other being similar. Suppose, to the contrary, that there exists $e \in E(T)$ such that $\text{cutmim}_G(A_e \cap S_x, \overline{A_e} \cap S_Y^\star) \ge R(4,t) + t + 1$. Let $\{p_1q_1, \dots, p_{R(4,t)+t+1}q_{R(4,t)+t+1}\}$ be an induced matching witnessing this, where $P = \{p_1, \dots, p_{R(4,t)+t+1}\} \subseteq S_x$ and $Q = \{q_1, \dots, q_{R(4,t)+t+1}\} \subseteq S_Y^\star$. Since q_1 is S_x-bad, let $u_1 \in S_x$ be one of its neighbours distinct from p_1. Suppose that q_1 has at least $R(4,t)$ neighbours in Q. Then, at least t of these neighbours induce a clique. Without loss of generality, suppose that $\{q_2, \dots, q_{t+1}\}$ are neighbours of q_1 inducing a clique. If $\{q_2, \dots, q_{t+1}\}$ is anticomplete to u_1, then these t vertices together with $\{q_1, p_1, u_1\}$ induce a copy of $\overline{K_{2,t} + P_1}$, a contradiction. Hence, u_1 has at least one neighbour in $\{q_2, \dots, q_{t+1}\}$, say without loss of generality q_2. But then, $G[q_1, q_2, p_3, \dots, p_{t+2}, u_1] \cong \overline{K_{2,t} + P_1}$, a contradiction.

Hence, q_1 has less than $R(4,t)$ neighbours in Q. Without loss of generality, suppose that $q_{R(4,t)+1}, \dots, q_{R(4,t)+t+1}$ are non-neighbours of q_1. Then, these $t + 1$ vertices form a clique, or else two non-adjacent vertices v and v' among them would give $G[v, v', q_1, v_x] \cong 4P_1$, a contradiction. Next, since $q_{R(4,t)+1}$ is S_x-bad, it has another neighbour $u_2 \in S_x$ distinct from $p_{R(4,t)+1}$. Suppose that $\{q_{R(4,t)+2}, \dots, q_{R(4,t)+t+1}\}$ is anticomplete to u_2. Then, we have that $G[p_{R(4,t)+1}, u_2, q_{R(4,t)+2}, \dots, q_{R(4,t)+t+1}, q_{R(4,t)+1}] \cong \overline{K_{2,t} + P_1}$, a contradiction. Therefore, u_2 has at least one neighbour in $\{q_{R(4,t)+2}, \dots, q_{R(4,t)+t+1}\}$, say without loss of generality $q_{R(4,t)+2}$. Then, $G[q_{R(4,t)+1}, q_{R(4,t)+2}, p_1, \dots, p_t, u_2] \cong \overline{K_{2,t} + P_1}$, a contradiction. ◇

Claim 23. *Let $\alpha, \beta \subseteq \{a,b,c\}$ with $\alpha \cap \beta \neq \varnothing$. Then* $\text{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S_\beta) < R(4,t) + 4$*, for any $e \in E(T)$.*

Proof of Claim 23. Let $i \in \alpha \cap \beta$. Then v_i is complete to S_α and S_β. Suppose, to the contrary, that there exists $e \in E(T)$ such that $\text{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S_\beta) \ge R(4,t) + 4$. Let $\{p_1q_1, \dots, p_{R(4,t)+4}q_{R(4,t)+4}\}$ be an induced matching witnessing this, where $P = \{p_1, \dots, p_{R(4,t)+4}\} \subseteq S_\alpha$ and $Q = \{q_1, \dots, q_{R(4,t)+4}\} \subseteq S_\beta$. Since G is $4P_1$-free, Q contains a clique of size at least t. Without loss of generality, suppose that $\{q_1, \dots, q_t\}$ induces a clique. Observe now that $\{p_{R(4,t)+1}, \dots, p_{R(4,t)+4}\}$ contains a pair of adjacent vertices, as G is $4P_1$-free. Without loss of generality, suppose that $p_{R(4,t)+1}$ is adjacent to $p_{R(4,t)+2}$. But then, $G[p_{R(4,t)+1}, p_{R(4,t)+2}, q_1, q_2, \dots, q_t, v_i] \cong \overline{K_{2,t} + P_1}$, a contradiction. ◇

We can finally show that $\text{mimw}_G(T, \delta) < 43R(4,t) + 24t + 214$. Let $S_z = \{v_a, v_b, v_c\}$ and let $D = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}, \{z\}\}$. Since $\{S_\alpha : \alpha \in D\}$ is a partition of $V(G)$, Observation 13 implies that

$$\text{mimw}_G(T, \delta) \le \max_{e \in E(T)} \sum_{\alpha, \beta \in D} \text{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S_\beta). \tag{1}$$

For any $\alpha \in D$, we have that $\text{cutmim}_G(A_e \cap S_z, \overline{A_e} \cap S_\alpha) \le 3$ and $\text{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S_z) \le 3$, for any $e \in E(T)$. For $\alpha, \beta \neq \{z\}$, there are 49 distinct pairs (α, β). 12 of such pairs are such that $\alpha \cap \beta = \varnothing$: $(\{a\}.\{b\})$, $(\{b\}, \{c\})$, $(\{a\}, \{c\})$,

$(\{a\},\{b,c\}), (\{b\},\{a,c\}), (\{c\},\{a,b\})$ and those obtained by swapping α and β. The remaining 37 pairs are such that $\alpha \cap \beta \neq \varnothing$. In this case, by Claim 23, $\operatorname{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S_\beta) \leq R(4,t)+4$, for any $e \in E(T)$.

We now estimate the terms in the sum above corresponding to pairs (α,β) such that $\alpha \cap \beta = \varnothing$. Suppose first that (α,β) is one of $(\{a\},\{b\}), (\{b\},\{c\}), (\{a\},\{c\}), (\{b\},\{a\}), (\{c\},\{b\}), (\{c\},\{a\})$. If S_α is $3t$-almost-complete to S_β or S_β is $3t$-almost-complete to S_α then, by Claim 21, $\operatorname{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S_\beta) < 3t+1$ and $\operatorname{cutmim}_G(A_e \cap S_\beta, \overline{A_e} \cap S_\alpha) < 3t+1$. Otherwise, S_α is not $3t$-almost-complete to S_β and S_β is not $3t$-almost-complete to S_α. By definition of G_1 and G_2, this implies that $G[S_\alpha, S_\beta] = G_1[S_\alpha, S_\beta] = G_2[S_\alpha, S_\beta]$. If either $e = rs$ or e belongs to T'_3, then all vertices of S_α and S_β belong to the same partition class of $V(G)$ induced by e and so $\operatorname{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S_\beta) = \operatorname{cutmim}_G(A_e \cap S_\beta, \overline{A_e} \cap S_\alpha) = 0$. Otherwise, e must belong to T'_2. The edge e then induces a partition $(A'_e, \overline{A'_e})$ of the vertices of G_2 with respect to (T'_2, δ_2), and $(A'_e, \overline{A'_e})$ coincides with $(A_e, \overline{A_e})$ restricted to $S_\alpha \cup S_\beta$. Hence, $\operatorname{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S_\beta) = \operatorname{cutmim}_{G_2}(A'_e \cap S_\alpha, \overline{A'_e} \cap S_\beta) \leq 2$.

Suppose finally that (α,β) is one of $(\{a\},\{b,c\}), (\{b\},\{a,c\}), (\{c\},\{a,b\}), (\{b,c\},\{a\}), (\{a,c\},\{b\}), (\{a,b\},\{c\})$. Clearly, $\operatorname{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S_\beta) \leq \operatorname{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S^\star_\beta) + \operatorname{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap (S_\beta \setminus S^\star_\beta))$. Note that $G[S_\alpha, S_\beta \setminus S^\star_\beta] = G_2[S_\alpha, S_\beta \setminus S^\star_\beta]$. Thus, by the same reasoning as in the previous paragraph, $\operatorname{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap (S_\beta \setminus S^\star_\beta)) \leq 2$. On the other hand, by Claim 22, $\operatorname{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S^\star_\beta) \leq R(4,t)+t+1$. Therefore, $\operatorname{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S_\beta) \leq R(4,t)+t+3$.

Combining these bounds with (1), we obtain $\operatorname{mimw}_G(T,\delta) < 14 \cdot 3 + 37 \cdot (R(4,t)+4) + 6 \cdot (3t+1) + 6 \cdot (R(4,t)+t+3) = 43R(4,t) + 24t + 214$. □

4.2. Unboundedness results

All the unboundedness results of this section are obtained by applying the same strategy. The class of walls plays a crucial role. A *wall of height h and width r* (an *$(h \times r)$-wall* for short) is the graph obtained from the grid of height h and width $2r$ as follows. Let $C_1, \ldots, C_{2r}$ be the set of vertices in each of the $2r$ columns of the grid, in their natural left-to-right order. For each column C_j, let $e^j_1, e^j_2, \ldots, e^j_{h-1}$ be the edges between two vertices of C_j, in their natural top-to-bottom order. If j is odd, we delete all edges e^j_i with i even. If j is even, we delete all edges e^j_i with i odd. We then remove all vertices of the resulting graph whose degree is 1. This final graph is an *elementary $(h \times r)$-wall* (see Fig. 4). We denote by $\mathcal{W}$ the class of all elementary $2n \times 2n$ walls, for $n \geq 1$.

Theorem 24 *(Brettell et al. [6]). Let W be an elementary $n \times n$ wall with $n \geq 7$. Then $\operatorname{mimw}(W) \geq \frac{\sqrt{n}}{50}$. Hence, $\mathcal{W}$ has unbounded mim-width.*

The idea is to start from an elementary wall, find an appropriate vertex colouring, and repeatedly apply the following result (the case $k=2$ was first proved in [36]).

Lemma 25 *(Brettell et al. [6]). Let G be a k-partite graph with partition classes $V_1, \ldots, V_k$ and let G' be a graph obtained from G by adding edges where, for each added edge, there exists some i such that both endpoints are in V_i. Then $\operatorname{mimw}(G') \geq \frac{1}{k} \cdot \operatorname{mimw}(G)$.*

Theorem 26. *The class of $(3P_1, \overline{K_{4,4}+P_1})$-free graphs has unbounded mim-width.*

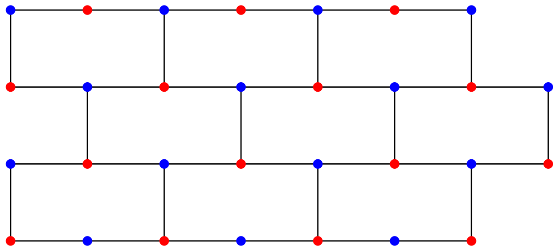
Proof. Let W be an elementary $2n \times 2n$ wall and consider its proper 2-colouring depicted in Fig. 4(a). We add edges within each colour class to make them cliques. Let $f(W)$ be the graph obtained and let $\mathcal{W}_1 = \{f(W) : W \in \mathcal{W}\}$. By Theorem 24 and Lemma 25, $\mathcal{W}_1$ has unbounded mim-width.

Note that, for the graph $f(W)$, every two vertices of the same colour are adjacent, and every two vertices of different colours are adjacent if and only if they are adjacent in W. Clearly, $f(W)$ is $3P_1$-free. It remains to show that $f(W)$ is $\overline{K_{4,4}+P_1}$-free.

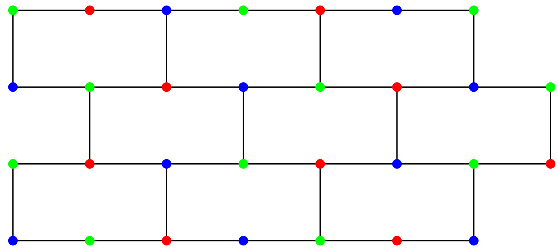
Claim 27. *Any copy of K_5 in $f(W)$ is monochromatic.*

Proof of Claim 27. Let $u_1, \ldots, u_5$ be the vertices of a copy of K_5. Since $f(W)$ is obtained from W by adding edges within each colour class, if an edge $uv \in E(f(W))$ is not monochromatic, then uv belongs to $E(W)$ as well. Hence, there cannot be one blue vertex and four red vertices in $\{u_1, \ldots, u_5\}$, since this would imply that in W there is a vertex with four neighbours. Also, there cannot be exactly two blue vertices in $\{u_1, \ldots, u_5\}$, for otherwise these two blue vertices share three common red neighbours in W, contradicting the fact that in W any two vertices have at most one common neighbour. By symmetry, there cannot be exactly one or two red vertices, and so $\{u_1, \ldots, u_5\}$ is monochromatic. ◇

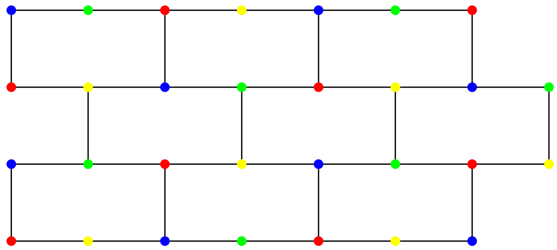
Suppose, to the contrary, that $f(W)$ contains an induced copy of $\overline{K_{4,4}+P_1}$ with vertex set $\{v_0, \ldots, v_8\}$ as depicted in Fig. 1. By Claim 27, the two copies of K_5 induced by $\{v_0, v_1, v_2, v_3, v_4\}$ and $\{v_0, v_5, v_6, v_7, v_8\}$ must both be monochromatic. Hence, $v_1, \ldots, v_8$ must be of the same colour. This implies that $v_1, \ldots, v_8$ must form a clique in $f(W)$, a contradiction. □



(a) A 2-colouring of the elementary 4×4 wall.



(b) A 3-colouring of the elementary 4×4 wall.



(c) A 4-colouring of the elementary 4×4 wall.

Fig. 4. The different colourings of elementary walls used in the proofs of Theorems 26, 28 and 30. (For interpretation of the colours in the figures, the reader is referred to the web version of this article.)

Theorem 28. *The class of* $(4P_1, \overline{K_{3,3} + P_1})$*-free graphs has unbounded mim-width.*

Proof. Let W be an elementary $2n \times 2n$ wall and consider its proper 3-colouring depicted in Fig. 4(b). We add edges within each colour class to make them cliques. Let $g(W)$ be the graph obtained and let $\mathcal{W}_2 = \{g(W) : W \in \mathcal{W}\}$. By Theorem 24 and Lemma 25, $\mathcal{W}_2$ has unbounded mim-width. Clearly, $g(W)$ is $4P_1$-free. It remains to show that $g(W)$ is $\overline{K_{3,3} + P_1}$-free.

Claim 29. *Any copy of* K_4 *in* $g(W)$ *is monochromatic.*

Proof of Claim 29. Let $u_1, \dots, u_4$ be the vertices of a copy of K_4. At least two such vertices have the same colour, say colour c. Since $g(W)$ is obtained from W by adding edges within each colour class, if an edge $uv \in E(g(W))$ is not monochromatic, then uv belongs to $E(W)$ as well. Observe first that there cannot be exactly two vertices with colour c in $\{u_1, \dots, u_4\}$, for otherwise these two vertices coloured c have two common neighbours coloured different from c, contradicting the fact that in W any two vertices have at most one common neighbour. Moreover, there cannot be exactly three vertices coloured c in $\{u_1, \dots, u_4\}$, since this would imply that in W there is a vertex not coloured c adjacent to three vertices coloured c. However, in the 3-colouring of W depicted in Fig. 4(b), no vertex has three monochromatic neighbours. $\diamond$

Suppose, to the contrary, that $g(W)$ contains an induced copy of $\overline{K_{3,3} + P_1}$ with vertex set $\{v_0, \dots, v_6\}$, where v_0 is the universal vertex and $\{v_1, v_2, v_3\}$ and $\{v_4, v_5, v_6\}$ induce disjoint cliques. By Claim 29, the two copies of K_4 induced by $\{v_0, v_1, v_2, v_3\}$ and $\{v_0, v_4, v_5, v_6\}$ must both be monochromatic. Hence, $v_1, \dots, v_6$ must be of the same colour. This implies that $v_1, \dots, v_6$ must form a clique in $g(W)$, a contradiction. $\square$

Theorem 30. *The class of* $(5P_1, \overline{K_{2,2} + P_1})$*-free graphs has unbounded mim-width.*

Proof. Let W be an elementary $2n \times 2n$ wall and consider its proper 4-colouring depicted in Fig. 4(c). We add edges within each colour class to make them cliques. Let $h(W)$ be the graph obtained and let $\mathcal{W}_3 = \{h(W) : W \in \mathcal{W}\}$. By Theorem 24 and Lemma 25, $\mathcal{W}_3$ has unbounded mim-width. Clearly, $h(W)$ is $5P_1$-free. It remains to show that $h(W)$ is $\overline{K_{2,2} + P_1}$-free.

Claim 31. *Any copy of* K_3 *in* $h(W)$ *is monochromatic.*

Proof of Claim 31. Let u_1, u_2, u_3 be the vertices of a copy of K_3. Firstly, there cannot be exactly two vertices in $\{u_1, u_2, u_3\}$ of the same colour, say colour c, since this would imply that in W there is a vertex coloured different from c which is adjacent to two vertices coloured c, contradicting the 4-colouring of W depicted in Fig. 4(c). Moreover, the vertices in $\{u_1, u_2, u_3\}$ cannot be coloured with distinct colours, for otherwise these three vertices would induce a K_3 in W. $\diamond$

Similarly to Theorems 26 and 28, it is now easy to see that $h(W)$ is $\overline{K_{2,2} + P_1}$-free. □

4.3. Dichotomy

Combining the results of Sections 4.1 and 4.2, we can finally show Theorem 8, which we restate for convenience.

Theorem 8. *Let $r \geq 3$ and $s, t \geq 2$ be integers. Then the mim-width of the class of $(rP_1, \overline{K_{s,t} + P_1})$-free graphs is bounded if and only if:*

- *$r = 3$ and one of s and t is at most 3;*
- *$r = 4$ and one of s and t is at most 2.*

In all these cases, the mim-width is also quickly computable.

Proof. If $r \geq 5$, the mim-width is unbounded by Theorem 30. Suppose now that $r = 4$. If both s and t are at least 3, the mim-width is unbounded by Theorem 28, whereas if one of s and t is at most 2, the mim-width is bounded and quickly computable by Theorem 19. Finally, suppose that $r = 3$. If both s and t are at least 4, the mim-width is unbounded by Theorem 26, whereas if one of s and t is at most 3, the mim-width is bounded and quickly computable by Theorem 16. □

5. Mim-width of $(K_r, sP_1 + tP_2 + uP_3)$-free graphs

In this section we address Open Problem 2 and show Theorems 9 and 10. Both results are obtained by identifying new $(K_r, sP_1 + tP_2 + uP_3)$-free classes of unbounded mim-width.

Open Problem 2 was formulated in [6] starting from [6, Theorem 35]. We remark that there is a typo in the formulation of this statement. For completeness we provide the correct formulation, whose proof is essentially identical to that of [6, Theorem 35].

Theorem 32 *(Brettell et al. [6]). Let H be a graph and let $r \geq 4$ be an integer. Let $\mathcal{S}$ be the class of graphs every component of which is either a subdivided claw or a path. Then exactly one of the following holds:*

- *$H \subseteq_i sP_1 + P_5$ or tP_2, and the mim-width of the class of (K_r, H)-free graphs is bounded and quickly computable;*
- *$H \notin \mathcal{S}$, or $H \supseteq_i K_{1,3}$, $P_2 + P_4$, or P_6, and the mim-width of the class of (K_r, H)-free graphs is unbounded; or*
- *$H = sP_1 + tP_2 + uP_3$, where $u \geq 1$ and $t + u \geq 2$.*

Proof. By [6, Theorem 31-(i)], if $H \notin \mathcal{S}$, then the mim-width of the class of (K_r, H)-free graphs is unbounded. So we may assume that H is a forest of paths and subdivided claws. By [6, Theorem 31-(iii)], if H contains a $K_{1,3}$, then the mim-width is again unbounded. So we may assume that H is a linear forest. If $H \subseteq_i sP_1 + P_5$ or $H \subseteq_i tP_2$, then mim-width is bounded and quickly computable by parts (xii) and (xiv) of [6, Theorem 30]. So we may assume that H is a linear forest containing $P_2 + P_3$. By [6, Theorem 31-(viii)], we may also assume H contains neither $P_2 + P_4$ nor P_6, otherwise the mim-width is again unbounded. It now follows that $H \subseteq_i tP_2 + uP_3$ for some u, t such that $u \geq 1$ and $t + u \geq 2$. Therefore, $H = sP_1 + tP_2 + uP_3$, for $u \geq 1$ and $t + u \geq 2$. □

5.1. Unboundedness results

Similarly to Section 4.2, the unboundedness results for $(K_r, sP_1 + tP_2 + uP_3)$-free graphs in this section (Theorem 35 for $r = 5$ and Theorem 38 for $r = 4$) are obtained by applying Lemma 25. However, in the case of Theorem 38, only certain types of edges are added inside each colour class; this is to avoid creating copies of K_4. We will also make use of the following two results.

Lemma 33 *(Vatshelle [41]). Let G be a graph and $v \in V(G)$. Then $\text{mimw}(G) \geq \text{mimw}(G - v)$.*

Lemma 34 *(Brettell et al. [6]). Let G be a graph and let G' be the graph obtained by 1-subdividing an edge of G. Then $\text{mimw}(G') \geq \text{mimw}(G)$.*

Theorem 35. *The class of $(K_5, P_3 + P_2 + P_1)$-free graphs has unbounded mim-width.*

Proof. Consider first a $2n \times 2n$-grid G_{2n} with vertex set $\{(i, j) : 1 \leq i, j \leq 2n\}$. Consider the set of vertices $S = \{(i, j) : i + j \equiv 1 \pmod 2\}$ and the set of edges $T = \{(i, j)(i, j - 1) : (i, j) \in S\}$. We define the graph W_n as $W_n = (V(G_n), E(G_n) \setminus T)$. Since

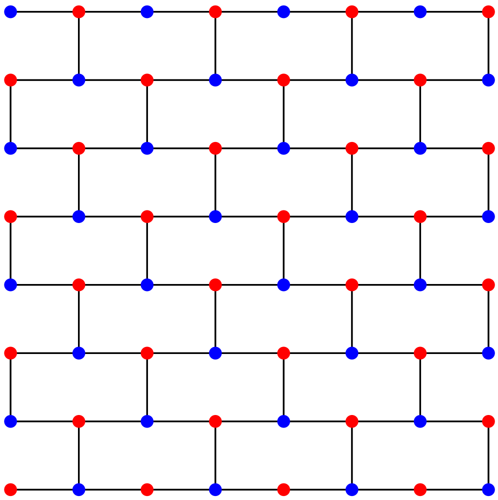


Fig. 5. The graph W_4 with the red-blue colouring as in the proof of Theorem 35.

W_n contains the elementary $n \times n$ wall as an induced subgraph, Theorem 24 and Lemma 33 imply that the class of graphs $\{W_n : n \geq 1\}$ has unbounded mim-width. Given W_n, we now consider the following partition of its vertices:

$$A = \{(i, j) : i + j \equiv 0 \pmod 2,\ i \equiv 1 \pmod 3\}$$
$$B = \{(i, j) : i + j \equiv 0 \pmod 2,\ i \equiv 2 \pmod 3\}$$
$$C = \{(i, j) : i + j \equiv 0 \pmod 2,\ i \equiv 0 \pmod 3\}$$
$$D = \{(i, j) : i + j \equiv 1 \pmod 2,\ i \equiv 1 \pmod 3\}$$
$$E = \{(i, j) : i + j \equiv 1 \pmod 2,\ i \equiv 2 \pmod 3\}$$
$$F = \{(i, j) : i + j \equiv 1 \pmod 2,\ i \equiv 0 \pmod 3\}.$$

We then colour in red the vertices in $A \cup B \cup C$, and in blue the vertices in $D \cup E \cup F$ (see Fig. 5). This gives a proper 2-colouring of W_n and, in particular, each partition class defined above forms an independent set in W_n. Observe that each vertex is adjacent to at most one vertex from each partition class of the opposite colour. That is, each vertex in $A \cup B \cup C$ is adjacent to at most one vertex from each of D, E and F, and each vertex in $D \cup E \cup F$ is adjacent to at most one vertex from each of A, B and C.

We now build the graph W'_n by adding all edges between different partition classes of the same colour. That is, we make A, B, C pairwise complete and D, E, F pairwise complete. No other edges are added. In particular, $W'_n[A \cup B \cup C]$ and $W'_n[D \cup E \cup F]$ are complete tripartite graphs.

Applying Lemma 25 to the bipartition $(A \cup B \cup C, D \cup E \cup F)$ of $V(W_n)$, we obtain that $\text{mimw}(W'_n) \geq \text{mimw}(W_n)/2$. Hence, the class of graphs $\{W'_n : n \geq 1\}$ has unbounded mim-width. It is then enough to show that W'_n is K_5-free and $(P_3 + P_2 + P_1)$-free.

Claim 36. *W'_n is K_5-free.*

Proof of Claim 36. Suppose, to the contrary, that $\{v_1, v_2, v_3, v_4, v_5\}$ induces a copy of K_5 in W'_n. Since each of A, B, C, D, E, F is an independent set, the v_i's belong to different partition classes. In particular, without loss of generality, v_1, v_2, v_3 are red and v_4, v_5 blue, or vice versa. Since no edges between red and blue vertices are added when constructing W'_n, we have that $\{v_1, v_2, v_3\}$ is complete to $\{v_4, v_5\}$ in W_n. But this contradicts the fact that in W_n no two vertices have two common neighbours. ⋄

Claim 37. *W'_n is $(P_3 + P_2 + P_1)$-free.*

Proof of Claim 37. Suppose, to the contrary, that $\{v_1, \ldots, v_6\}$ induces a copy of $P_3 + P_2 + P_1$ in W'_n, where $W'_n[v_1, v_2, v_3] \cong P_3$ with v_2 adjacent to both v_1 and v_3, $\{v_4, v_5\}$ is anticomplete to $\{v_1, v_2, v_3\}$ and induces a copy of P_2, and $\{v_6\}$ is anticomplete to $\{v_1, \ldots, v_5\}$. Suppose, without loss of generality, that v_2 is red.

Case 1: Both v_1 and v_3 are blue. Since v_4 is adjacent to at most one vertex from each blue partition class, we have that v_1 and v_3 belong to different blue partition classes. By construction, these partition classes are complete, contradicting the fact that v_1 is non-adjacent to v_3 in W'_n.

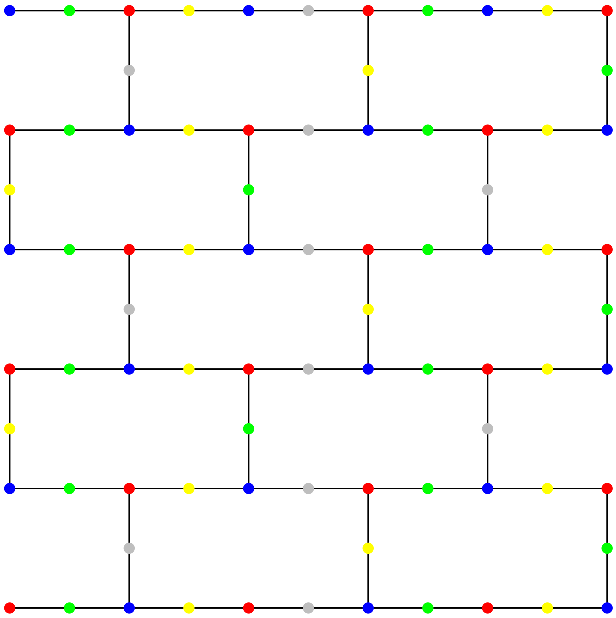


Fig. 6. The graph W'_3 in the proof of Theorem 38, together with a proper 5-colouring: blue vertices correspond to X, red vertices to Y, grey vertices to A, yellow vertices to B and green vertices to C.

Case 2: At least one of v_1 and v_3 is red. Without loss of generality, v_1 is red. Since each partition class forms an independent set in W'_n, we have that v_1 does not belong to the class of v_2. But then $\{v_1, v_2\}$ dominates the red vertices and so v_4, v_5, v_6 are all blue. By a similar reasoning, v_4, v_5, v_6 all belong to the same blue partition class, or else there exists a vertex in $\{v_4, v_5, v_6\}$ dominating the remaining two. But each partition class is an independent set, contradicting the fact that v_4 is adjacent to v_5. ◇

This concludes the proof of Theorem 35. □

Theorem 38. *The class of $(K_4, P_3 + 2P_2 + P_1, 2P_3 + P_2)$-free graphs has unbounded mim-width.*

Proof. Let W_n be the graph defined in the proof of Theorem 35. Given W_n, we subdivide every edge $(i_1, j_1)(i_2, j_2)$ by adding a new vertex $(\frac{i_1+i_2}{2}, \frac{j_1+j_2}{2})$. We then multiply the coordinates of all vertices by 2 (so, e.g., $(4, 5.5)$ becomes $(8, 11)$) and preserve the adjacencies between vertices in order to obtain a new graph W'_n. By Lemma 34, $\text{mimw}(W'_n) \geq \text{mimw}(W_n)$. We now define a partition of the vertices of W'_n as follows (see Fig. 6):

$$X = \{(i, j) : i + j \equiv 2 \pmod 4\}$$

$$Y = \{(i, j) : i + j \equiv 0 \pmod 4\}$$

$$A = \{(i, j) : i + j \text{ is odd}, \ i \equiv 1 \pmod 3\}$$

$$B = \{(i, j) : i + j \text{ is odd}, \ i \equiv 2 \pmod 3\}$$

$$C = \{(i, j) : i + j \text{ is odd}, \ i \equiv 0 \pmod 3\}$$

Note that X and Y consist of the vertices of W_n, and A, B and C consist of the new vertices introduced after edge subdivisions. In particular, each partition class is an independent set. Moreover, X is anticomplete to Y, and A, B, C are pairwise anticomplete. Since W'_n has no cycle of length 4, each $x \in X$ and $y \in Y$ have at most one common neighbour in $A \cup B \cup C$.

Observation 39. *Let $u_1 = (i_1, j_1)$ and $u_2 = (i_2, j_2)$ be two vertices belonging to the same partition class in $\{A, B, C\}$. The following hold:*

- *3 divides $|i_1 - i_2|$;*
- *If $i_1 = i_2$, then 2 divides $|j_1 - j_2|$.*

We now proceed to the construction of the graph W''_n, obtained as follows. Start from W'_n and

- Add all edges between X and Y;
- For each pair of distinct sets R and S in $\{A, B, C\}$ and $r = (i_r, j_r) \in R$ and $s = (i_s, j_s) \in S$, add the edge rs, unless $j_r = j_s$ and $|i_r - i_s| = 2$, that is, unless r and s are the "right neighbour" and the "left neighbour" of a vertex in $X \cup Y$.

The edges left out in the second step above avoid the creation of copies of K_4, as will be shown shortly.

Since $(X \cup Y, A \cup B \cup C)$ is a bipartition of $V(W''_n)$, Lemma 25 implies that $\text{mimw}(W''_n) \geq \text{mimw}(W'_n)/2$. Hence the class of graphs $\{W''_n : n \geq 1\}$ has unbounded mim-width. It is then enough to show that W''_n does not contain any graph in $\{K_4, P_3 + 2P_2 + P_1, 2P_3 + P_2\}$ as an induced subgraph. This will be done in a series of claims.

Claim 40. *W''_n is K_4-free.*

Proof of Claim 40. Suppose, to the contrary, that $\{v_1, v_2, v_3, v_4\}$ induces a copy of K_4 in W''_n. Since each of X, Y, A, B and C is an independent set, the four vertices belong to four different partition classes.

Suppose first that exactly one of $\{v_1, v_2, v_3, v_4\}$ belongs to $X \cup Y$. Without loss of generality, $v_1 \in X$ and $v_2, v_3, v_4 \in A \cup B \cup C$. Since no edges between X and $A \cup B \cup C$ are added to $E(W'_n)$ in order to build W''_n, the vertices v_2, v_3, v_4 are adjacent to v_1 in W'_n. Suppose that $v_1 = (i, j)$. Then, up to relabelling, we must have that $v_2 = (i-1, j)$, $v_3 = (i+1, j)$ and $v_4 = (i, j \pm 1)$. In other words, v_2 and v_3 are the left neighbour and right neighbour of v_1, respectively. But then, by construction, $v_2v_3 \notin E(W''_n)$, a contradiction.

Suppose finally that exactly two vertices of $\{v_1, v_2, v_3, v_4\}$ belong to $X \cup Y$. Without loss of generality, $v_1, v_2 \in X \cup Y$, and $v_3, v_4 \in A \cup B \cup C$. Since no edges between $X \cup Y$ and $A \cup B \cup C$ are added to $E(W'_n)$ in order to build W''_n, both v_1 and v_2 are adjacent to v_3 and v_4 in W'_n, contradicting the fact that W'_n does not contain any cycle of length 4. $\diamond$

Claim 41. *Let u_1, u_2 be two distinct vertices from the same partition class in $\{A, B, C\}$. Let u_3 be a vertex from a partition class in $\{A, B, C\}$ different from that of u_1 and u_2. Then u_3 is adjacent to at least one of u_1 and u_2.*

Proof of Claim 41. Let $u_1 = (i_1, j_1)$, $u_2 = (i_2, j_2)$ and $u_3 = (i_3, j_3)$. Suppose, to the contrary, that u_3 is non-adjacent to both u_1 and u_2. By construction of W''_n, this implies that $j_1 = j_3 = j_2$ and $|i_1 - i_3| = 2 = |i_2 - i_3|$. Since u_1 and u_2 are distinct, $i_1 \neq i_2$, which implies that $|i_1 - i_2| = 4$, contradicting the first part of Observation 39. $\diamond$

We now prove that W''_n is $(P_3 + 2P_2 + P_1)$-free and $(2P_3 + 2P_2)$-free. The following result will be used as the backbone of both proofs.

Claim 42. *Suppose that $\{v_1, \ldots, v_7\}$ induces a copy of $P_3 + 2P_2$, where v_2 is adjacent to v_1 and v_3, v_4 is adjacent to v_5, v_6 is adjacent to v_7 and no other edges are present in $W''_n[\{v_1, \ldots, v_7\}]$. Then the following hold:*

1. *At least one of v_1 and v_3 belongs to $X \cup Y$;*
2. *$v_2 \in X \cup Y$.*

Proof of Claim 42. We first show that at least one of v_1 and v_3 belongs to $X \cup Y$. Suppose, to the contrary, that both v_1 and v_3 belong to $A \cup B \cup C$. Since A, B and C are pairwise disjoint, $v_1, v_3 \in S \cup T$ for some distinct $S, T \in \{A, B, C\}$.

Observe that at least two of v_4, v_5, v_6, v_7, say v_i and v_j, belong to $A \cup B \cup C$, or else at least three vertices among v_4, v_5, v_6, v_7 belong to $X \cup Y$ and so $W''_n[X \cup Y]$ contains a copy of $P_2 + P_1$, contradicting the fact that $W''_n[X \cup Y]$ is a complete bipartite graph.

Observe now that, by Claim 41 and the previous paragraph, v_1 and v_3 belong the same partition class. Without loss of generality, $v_1, v_3 \in S$. Since S is an independent set, $v_2 \notin S$, and since each vertex in $X \cup Y$ has at most one neighbour in each of A, B and C, we have that $v_2 \notin X \cup Y$. Moreover, by Claim 41, v_i and v_j both belong to S. But this contradicts Claim 41, as $v_2 \in (A \cup B \cup C) \setminus S$.

We finally show that $v_2 \in X \cup Y$. Suppose, to the contrary, that $v_2 \in R$, for some $R \in \{A, B, C\}$. Since R is an independent set, $v_1, v_3 \notin R$. In view of part 1, we distinguish two cases, according to which partition classes v_1 and v_3 belong. Let S and T be the two distinct partition classes in $\{A, B, C\} \setminus R$.

Case 1: v_1 and v_3 both belong to $X \cup Y$.

Since each vertex in R is adjacent to at most one vertex in X and at most one vertex in Y, one of v_1 and v_3 belongs to X and the other to Y, contradicting the fact that X is complete to Y.

Case 2: One of v_1 and v_3 belongs to $X \cup Y$ and the other to $S \cup T$.

Without loss of generality, $v_1 \in S$ and $v_3 \in X$. Since X is complete to Y, $v_4, v_5, v_6, v_7 \notin Y$. Since X is an independent set, at most one of v_4 and v_5 belongs to X and at most one of v_6 and v_7 belongs to X. Without loss of generality, $v_4, v_6 \in A \cup B \cup C$. If both v_4 and v_6 belong to R, then $v_1 \in S$ is non-adjacent to both $v_4, v_6 \in R$, contradicting Claim 41. If exactly one of v_4 and v_6 belongs to R, say without loss of generality $v_4 \in R$ and $v_6 \notin R$, then $v_6 \in (A \cup B \cup C) \setminus R$ is non-adjacent to $v_2 \in R$ and $v_4 \in R$, contradicting Claim 41. Therefore, none of v_4 and v_6 belongs to R. If v_4 and v_6 belong to the same partition class in $(A \cup B \cup C) \setminus R$, then $v_2 \in R$ being non-adjacent to both of them contradicts Claim 41. Finally, if v_4 and v_6 belong to different partition classes in $(A \cup B \cup C) \setminus R$, then one of them belongs to the partition class S of v_1, say without loss of generality $v_4 \in S$. But then v_6 being non-adjacent to both v_1 and v_4 contradicts Claim 41. ◇

Claim 43. *W''_n is $(P_3 + 2P_2 + P_1)$-free.*

Proof of Claim 43. Suppose, to the contrary, that $\{v_1, \dots, v_8\}$ induces a copy of $P_3 + 2P_2 + P_1$, where v_2 is adjacent to v_1 and v_3, v_4 is adjacent to v_5, v_6 is adjacent to v_7 and no other edges are present in $W''_n[\{v_1, \dots, v_8\}]$ (hence v_8 is the isolated vertex). By Claim 42, $v_2 \in X \cup Y$ and at least one of v_1 and v_3 belongs to $X \cup Y$. Without loss of generality, $v_2 \in X$ and $v_1 \in X \cup Y$. Since X is an independent set, $v_1 \in Y$. Since X is complete to Y, we have that $\{v_1, v_2\}$ dominates $X \cup Y$ and so $\{v_4, \dots, v_8\} \subseteq A \cup B \cup C$. By the pigeonhole principle, there exists two vertices among v_4, v_5, v_6, v_7 that belong to the same partition class in $\{A, B, C\}$. Since these classes form independent sets, the two vertices are non-adjacent. Without loss of generality, $v_4, v_6 \in R$ for some $R \in \{A, B, C\}$. If $v_8 \in (A \cup B \cup C) \setminus R$, then v_8 is non-adjacent to both $v_4, v_6 \in R$, contradicting Claim 41. Therefore, $v_8 \in R$. Since R is an independent set, $v_4 \in R$ implies that $v_5 \in (A \cup B \cup C) \setminus R$ and v_5 is non-adjacent to both $v_6, v_8 \in R$, contradicting Claim 41. ◇

Claim 44. *W''_n is $(2P_3 + P_2)$-free.*

Proof of Claim 44. Suppose, to the contrary, that $\{v_1, \dots, v_8\}$ induces a copy of $2P_3 + P_2$, where v_2 is adjacent to v_1 and v_3, v_4 is adjacent to v_5, v_7 is adjacent to v_6 and v_8, and no other edges are present in $W''_n[\{v_1, \dots, v_8\}]$. By Claim 42, $v_2 \in X \cup Y$ and at least one of v_1 and v_3 belongs to $X \cup Y$. Without loss of generality, $v_2 \in X$ and $v_1 \in X \cup Y$. As in the proof of Claim 43, $\{v_1, v_2\}$ dominates $X \cup Y$ and so $\{v_4, \dots, v_8\} \in A \cup B \cup C$.

Suppose first that at least two vertices among v_6, v_7 and v_8 belong to the same partition class in $\{A, B, C\}$. These two vertices are non-adjacent, as A, B, C are independent sets, and so they must be v_6 and v_8. Without loss of generality, $v_6, v_8 \in R$ for some $R \in \{A, B, C\}$. Similarly, at least one of v_4 and v_5 does not belong to R, say $v_4 \in (A \cup B \cup C) \setminus R$. Then v_4 is non-adjacent to both $v_6, v_8 \in R$, contradicting Claim 41.

Therefore, v_6, v_7 and v_8 belong to distinct partition classes in $\{A, B, C\}$. By Claim 41, none of v_4 and v_5 belongs to the partition class of either v_6 or v_8. But then v_4 and v_5 both belong to the partition class of v_7, contradicting the fact that every class is an independent set. ◇

This concludes the proof of Theorem 38. □

5.2. Summary results

With the aid of Theorems 35 and 38, we can finally show Theorems 9 and 10, which we restate for convenience.

Theorem 9. *Let $r \geq 5$ be an integer and let $H = sP_1 + tP_2 + uP_3$, for $s, t, u \geq 0$. Then exactly one of the following holds:*

- *$H \subseteq_i sP_1 + tP_2$, or $H \subseteq_i sP_1 + P_3$, and the mim-width of the class of (K_r, H)-free graphs is bounded and quickly computable;*
- *$H \supseteq_i P_3 + P_2 + P_1$, and the mim-width of the class of (K_r, H)-free graphs is unbounded;*
- *$H = 2P_3$, or $H = P_3 + P_2$.*

Proof. By Theorem 35, if H contains $P_3 + P_2 + P_1$, then the mim-width of the class of (K_r, H)-free graphs is unbounded. So we may assume that $u \leq 2$. If $u = 0$, then the mim-width is bounded by [6, Theorem 30-(xiv)]. If $u = 1$, then the mim-width is unbounded for $t \geq 2$ and $s \geq 0$ or $t = 1$ and $s \geq 1$ (Theorem 35), and bounded for $t = 0$ ([6, Theorem 30-(xii)]). This leaves open the case $H = P_3 + P_2$. Finally, if $u = 2$, then the mim-width is unbounded if one of t and s is at least 1. This leaves open the case $H = 2P_3$. □

Theorem 10. *Let $r = 4$ and let $H = sP_1 + tP_2 + uP_3$, for $s, t, u \geq 0$. Then exactly one of the following holds:*

- *$H \subseteq_i sP_1 + tP_2$, or $H \subseteq_i sP_1 + P_3$, and the mim-width of the class of (K_r, H)-free graphs is bounded and quickly computable;*
- *$H \supseteq_i P_3 + 2P_2 + P_1$, or $2P_3 + P_2$, and the mim-width of the class of (K_r, H)-free graphs is unbounded;*
- *$H = P_3 + 2P_2$, or $H = P_3 + P_2 + sP_1$, or $H = 2P_3 + sP_1$.*

Proof. By Theorem 38, if H contains $P_3 + 2P_2 + P_1$ or $2P_3 + P_2$, then the mim-width of the class of (K_r, H)-free graphs is unbounded. So we may assume that $u \leq 2$. If $u = 0$, then the mim-width is bounded by [6, Theorem 30-(xiv)]. If $u = 1$, then the mim-width is bounded for $t = 0$ ([6, Theorem 30-(xii)]), and unbounded for $t \geq 2$ and $s \geq 1$ or $t \geq 3$ and $s \geq 0$ (Theorem 38). This leaves open the cases $H = P_3 + 2P_2$ and $P_3 + P_2 + sP_1$. Finally, if $u = 2$, then the mim-width is unbounded for $t \geq 1$. This leaves open the case $H = 2P_3 + sP_1$. □

6. Concluding remarks and open problems

In view of Corollary 7, we believe that the main open problem related to algorithmic applications of sim-width is whether Independent Set is polynomial-time solvable for graph classes whose sim-width is bounded and quickly computable (this was first formulated in [32]). We highlight a possible connection. In [19], it is asked whether there exists a (tw, ω)-bounded graph class for which Independent Set is NP-hard. In view of these two open problems, it would be interesting to determine whether every (tw, ω)-bounded graph class has bounded sim-width (the converse does not hold, as mentioned in Section 1.1).

Cameron and Hell [10] showed that Independent $\mathcal{H}$-packing is polynomial-time solvable for weakly chordal graphs, a superclass of chordal graphs, and for AT-free graphs, a superclass of co-comparability graphs. Both chordal graphs and co-comparability graphs have sim-width at most 1 and in [32] it is asked whether weakly chordal graphs and AT-free graphs have bounded sim-width. We believe that Corollary 7 also provides strong motivation for studying the sim-width of weakly chordal and AT-free graphs.

Finally, we conclude by asking to classify the mim-width for the remaining open cases in Theorems 9 and 10. A particularly interesting open case is the mim-width of $(K_r, 2P_3)$-free graphs, for $r \geq 5$. In view of Theorem 4, this is related to the open problem in [24] of whether there exists $k \in \mathbb{N}$ for which List k-Colouring restricted to uP_3-free graphs is NP-hard for some $u \in \mathbb{N}$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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