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Energy methods for local or nonlocal problems in sub-Riemannian setting

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"Do not look upon this world with fear and loathing.
Bravely face whatever the gods offer."

— Morihei Ueshiba

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CHAPTER 1

Introduction

This thesis deals with energy minimization problems related to either local or nonlocal Euler-Lagrange equations invariant with respect to a non-commutative group of translations and homogeneous with respect to a family of anisotropic dilations, in turn equipping the underlying background manifolds with a Lie group structure.

In Part 2 we present the effects of the lack of compactness in the critical Folland-Stein-Sobolev embedding in the Heisenberg group $\mathbb{H}^n$,

$$\|u\|^{2^*}_{L^{2^*}(\mathbb{H}^n)} \leq S^* \|D_H u\|^{2^*}_{L^2(\mathbb{H}^n)}\,, \tag{1.1}$$

where $2^* = 2Q/(Q-2)$ is the related critical Sobolev exponent, with $Q = 2n+2$ being the usual homogeneous dimension of $\mathbb{H}^n$.

Since the pioneering work of Jerison and Lee ([94]), it is well-known that optimal functions to (1.1) – up to a multiplicative constant – are of the form

$$U(\xi) := \Big((1+|z|^2)^2 + |t|^2\Big)^{\frac{2-Q}{4}} \qquad \forall \xi := (z,t) \in \mathbb{H}^n. \tag{1.2}$$

Functions (1.2) seems in analogy to the one available in the Euclidean case of $\mathbb{R}^n$. However, note that the level surfaces of U are neither isoperimetric surfaces ([32]) nor the level sets of the fundamental solution of the Kohn Laplacian. This discrepancy is connected with the fact that the $2n+1$ real coordinates ξ, which parametrize $\mathbb{H}^n$, do not all appear with the same degree, as conversely do the $\mathbb{R}^n$ coordinates in the norm $|x|^2 = \sum |x_i|^2$; see the forthcoming expression of the Korányi norm in (2.3).

For this, several techniques involving symmetric decreasing rearrangements are not expected to work in the noncommutative setting of the Heisenberg group, in turn making it a very difficult and challenging habitat. Hence, various results, somewhat expected – available in the Euclidean setting – are still missing in the Heisenberg framework, despite the incredible amount of literature dealing with this subject.

In this regard, we prove that *a famous conjecture of Brezis and Peletier* ([22]) does still hold in the Heisenberg framework. In particular, in Chapter 4 we show that optimal functions u_ε for the following natural subcritical approximation of the Sobolev quotient

$$S_\varepsilon := \sup\Big\{F_\varepsilon(u) :\ u \in S^1_0(\Omega),\ \|D_H u\|^2_{L^2(\Omega)} \leq 1\Big\}\,,$$
$$\text{with } F_\varepsilon(u) := \int_\Omega |u|^{2^*-\varepsilon}\,\mathrm{d}\xi\,,$$

actually do concentrate energy at one point (see Theorem 4.2 in Chapter 4),

$$|D_H u_\varepsilon|^2\,\mathrm{d}\xi \overset{*}{\rightharpoonup} \delta_{\xi_\circ} \qquad \text{for } \xi_\circ \in \overline{\Omega}\,.$$

Also, in the particular case when the domain satisfies natural geometric and regularity features such as a measure theoretic density on the boundary and a uniformly starlikness along its characteristic set – expressed via the vector filed

$\mathcal{D}$ (7.2) generating of the group anisotropic dilations – the concentration point of the subcritical energies can be localized via the Robin function associated to the involved domain (see Theorem 7.3 in Chapter 7). In particular, we prove that u_ε concentrates at some point $\xi_{\rm o} \in \Omega$ such that

$$D_H \mathcal{R}_\Omega(\xi_{\rm o}) = 0\,,$$

with $\mathcal{R}_\Omega$ being the Robin function associated to Ω, i. e. $\mathcal{R}_\Omega(\xi) := H(\xi;\xi)$ with $H(\cdot;\cdot)$ being the regular part of the Green function

$$H(\xi;\eta) = G_\Omega(\xi;\eta) - \frac{1}{C_Q |\eta^{-1} \circ \xi|_{\mathcal{H}}^{Q-2}}$$

In order to achieve the aforementioned result, several proper estimates and tools are combined to attack the related CR Yamabe equation with novel feasible ingredients in PDEs and Calculus of Variations which also aim to constitute general independent results in the Heisenberg framework, as a De Giorgi's Γ-convergence approach to provide fine energy approximations in very general (possible non-smooth) domains (see Theorem 4.1) in Chapter 4; integral and boundary regularity estimates for solutions to subcritical equations; the asymptotic control of the optimal functions via the Jerison and Lee optimal functions (see Theorem 6.1 in Chapter 6); *the celebrated Global Compactness result* (see Theorem 5.1 in Chapter 5) extended in the Heisenberg group via a completely different approach with respect to the original one by Struwe ([126]).

Part 3 deals with regularity and related properties of weak solutions to

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega\,, \\ u = g & \text{in } \mathbb{H}^n \setminus \Omega\,, \end{cases} \tag{1.3}$$

where the main operator $\mathcal{L}$ belongs to a wide class of (possibly degenerate) integro-differential operators, whose explicitly expression is given by

$$\mathcal{L}u(\xi) := \lim_{\varepsilon \to 0^+} \int_{\mathbb{H}^n \setminus B_\varepsilon(\xi)} \frac{|u(\xi) - u(\eta)|^{p-2}(u(\xi) - u(\eta))}{d_{\rm o}(\xi,\eta)^{Q+sp}}\, \mathrm{d}\eta \quad \xi \in \mathbb{H}^n\,, \tag{1.4}$$

where $d_{\rm o}$ is a homogeneous norm in $\mathbb{H}^n$, according to Definition 2.1, $Q := 2n+2$ is the homogeneous dimension in the Heisenberg group, $p \in (1,\infty)$ and $s \in (0,1)$.

Operators such as (1.4) arise as a nonlinear version of the conformally invariant fractional subLaplacian $(-\Delta_H)^s$ on $\mathbb{H}^n$ which was introduced firstly in [20] via the spectral formula

$$(-\Delta_H)^s := 2^s |T|^s \frac{\boldsymbol{\Gamma}\Big(-\frac{1}{2}\Delta_H |T|^{-1} + \frac{1+s}{2}\Big)}{\boldsymbol{\Gamma}\Big(-\frac{1}{2}\Delta_H |T|^{-1} + \frac{1-s}{2}\Big)} \qquad \forall s \in (0,1),$$

with $\boldsymbol{\Gamma}(x) := \int_0^\infty t^{x-1} e^{-t}\, \mathrm{d}t$ being the Euler Gamma function.

Problem (1.3) inherits both the difficulties arising from the non-Euclidean geometrical structure and those naturally arising from the nonlocal and nonlinear character of the involved integro-differential operators $\mathcal{L}$. In particular, several techniques such as Dirichlet-to-Neumann maps ([65, 66]), in the same spirit as the celebrated Caffarelli-Silvestre s-harmonic extension ([29]), the approach via Fourier representation, the pseudo-differential commutator compactness ([111]), the commutator estimates ([120]), and many others, seem not to be easily adaptable to the framework we are considering. Moreover, we cannot simply attack this problem by standard classical techniques from the Calculus of Variations, since in most of the needed estimates we have to deal with precise error terms naturally arising from the fractional energies. In order to deal with such difficulties coming

from the nonlocality, we introduced in the Heisenberg framework the *the nonlocal tail of a function*

$$\mathrm{Tail}(u;\xi_0,R):=\left(R^{sp}\int_{\mathbb{H}^n\setminus B_R(\xi_0)}|u(\xi)|^{p-1}|\xi_0^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}\,\mathrm{d}\xi\right)^{\frac{1}{p-1}}.$$

This is a key-point to carefully quantify the fractional long-range interactions and, despite its youngness, it has already opened the way to a series of subsequent related investigations in the field of integro-differential equations on more general Carnot groups; see for example [64, 53, 54, 69, 75].

In Chapter 10 we extend and complete *the classical De Giorgi-Nash-Moser regularity theory* to the nonlinear nonlocal Heisenberg framework establishing an Hölder continuity regularity results for weak solutions to (1.3); see Theorem 10.6 there. The starting point in proving such regularity estimate for weak solutions is the interpolative local boundedness estimate proved in Theorem 10.5 in Chapter 10

$$\begin{aligned}\sup_{B_{r/2}}u\quad\leq\quad&\delta\,\mathrm{Tail}(u_+;\xi_0,r/2)+c\,\delta^{-\frac{(p-1)Q}{sp^2}}\left(\fint_{B_r}u_+^p\,\mathrm{d}\xi\right)^{\frac{1}{p}}\\&+r^{\frac{sp}{p-1}}\|f\|_{L^\infty(B_r)}^{\frac{1}{p-1}}\qquad\forall\delta\in(0,1]\,.\end{aligned}$$

The arbitrariness in the choice of the parameter δ in the above supremum estimate allowed us to combine standard covering arguments with a fractional expansion of positivity in order to prove the following nonlocal version of the Harnack estimate in Theorem 11.1

$$\sup_{B_r}u\;\leq\;c\inf_{B_r}u+c\left(\frac{r}{R}\right)^{\frac{sp}{p-1}}\mathrm{Tail}(u_-;\xi_0,R)+c\,r^{\frac{sp}{p-1}}\|f\|_{L^\infty(B_R)}^{\frac{1}{p-1}}.$$

This is the fractional counterpart of the results achieved in [31] for nonlinear subelliptic equations and the nonlinear extension of the fractional Harnack inequality obtained in [56] with a Dirichlet-to-Neumann extension map.

Starting form the regularity estimates obtained in Chapters 10 and 11, we deal with the nonlinear fractional subelliptic obstacle problem in Chapter 12. Several breakthrough results for the study of subelliptic obstacle problems have been achieved in [42, 41]. Here we prove existence and uniqueness of a solution to the subelliptic nonlinear fractional obstacle problem related to $\mathcal{L}$; see Theorem 12.2 in Chapter 12. Moreover, we show that, under a very general measure theoretic assumption on the boundary of Ω, the solution to the obstacle problem inherits (up to the boundary) the regularity of the obstacle; both in case of Hölder continuity and continuity.

All the aforementioned results were then employed in Chapter 13 to prove the existence of a generalized solution to (1.3) in the sense of Perron-Wiener-Brelot for very general boundary data.

In the last Part 3 we attack very general nonlocal kinetic equations of the type

$$\Big(\partial_t+v\cdot\nabla_x\Big)f=\mathcal{L}_v f\qquad(t,x,v)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}\,,\tag{1.5}$$

which includes, for instance, kinetic Kolmogorov-Fokker-Plank equations and even some Boltzmann-type equations. Above, the diffusion operator $\mathcal{L}_v$ acts like a fractionalLaplacian in the v-variables

$$\mathcal{L}f(t,x,v):=\lim_{\varepsilon\to0^+}\int_{\mathbb{R}^n\setminus B_\varepsilon(v)}\Big(f(t,x,v)-f(t,x,w)\Big)K(v,w)\,\mathrm{d}w\,,$$

where the measurable kernel K satisfies uniformly elliptic-type upper and lower estimates in terms of the Gagliardo kernel, i.e. $K(v,w) \approx |v-w|^{-n-2s}$. Notice that the natural setting in order to deal with equations in (1.5) is noncommutative, given their invariance with respect to a Lie group law and its homogeneity, so that the underlying geometry reassembles to some extent the aforementioned sub-Riemannian one.

In Chapter 16 we prove new boundedness estimates for weak solutions to (1.5). More than this, we establish a very useful nonlocal interpolating $L^\infty - L^2$-estimate, by properly quantifying the fractional contribution, and by combining together a new suitable gain of integrability which relies on the fundamental solutions of the fractional Kolmogorov equations, with appropriate general Caccioppoli-type inequalities with tail. For this, we introduce in the kinetic framework a new definition of *kinetic tail of a function*.

It is worth mentioning that the local boundedness result is completely new, since boundedness of the solutions has been always taken as extra assumption for similar class of kinetic equations; see, e. g., [**124**, **89**, **98**].

By making use of the approaches and the estimates in the nonlocal kinetic framework, we were subsequently able to prove that weak solutions to such a wide class of hypoelliptic equations do satisfy the classical Harnack inequality. This has been a long-standing open problem for any class of kinetic integro-differential equations; see Part 4.

The thesis is based on the works [**107**, **109**, **110**, **5**, **4**] and on the papers [**116**, **106**, **101**].

Other works and papers are not part of the present thesis but are the outcome of some fruitful collaborations I have done during my Ph.-D. are listed below.

Firstly, in [**43**, **117**, **44**] we investigate some regularity issues for variational integrals and nonuniformly elliptic systems. In particular, in [**43**] we obtain everywhere Lipschitz continuity under borderline Lorentz assumptions on the forcing term for solutions to nonhomogeneous, nonuniformly elliptic systems with zero boundary conditions imposed only on some part of the boundary of convex domains. This result positively answers to the optimality issue raised in [**18**]. In [**117**] we show that local minimizers of nonhomogeneous quasiconvex variational integrals with standard p-growth feature almost everywhere BMO-regular gradient provided that the forcing term belongs to the borderline Marcinkiewicz space $L(n,\infty)$. Moreover, in [**44**] we prove that minima of the log-multiphase variational integral

$$w \mapsto \int_\Omega \left[|Dw| \log(1+|Dw|) + a(x)|Dw|^q + b(x)|Dw|^s \right] \mathrm{d}x \quad \Omega \subset \mathbb{R}^n \,,$$

have locally Hölder continuous gradient under sharp quantitative bounds linking the growth powers (q,s) to the Hölder exponents of the modulating coefficients $a(\cdot)$ and $b(\cdot)$ respectively.

Lastly, in [**52**] we investigate the existence, uniqueness and asymptotic behaviour of minimizers for very general Allen-Cahn-type energies with possible degenerate potentials (and this is, in our knowledge, a veritable novelty in the fluid-fluid phase transitions theory as well as in the dislocations in crystals theory)

$$\frac{1}{4} \iint_{\mathbb{R}^{2n} \setminus (\mathscr{C}\Omega)^2} |u(x)-u(y)|^2 K(x-y)\, \mathrm{d}x\mathrm{d}y + \int_\Omega W(u)\, \mathrm{d}x, \quad \Omega \subset \mathbb{R}^n \,,$$

where – as said – W is a $C^{2,\beta}_{\mathrm{loc}}$ *degenerate* double-well potential and K belongs to a wide class of measurable kernels which includes as a particular case the

fractional Gagliardo kernel $K(x-y) \equiv |x-y|^{-n-2s}$ for any $s \in (0,1)$. Given the degeneracy of the involved potential, we cannot trivially extend the proofs available in literature, for this we have to employ new techniques suitably adapted to the framework we are dealing with. The methods used in order to establish our main results are based on "classical" properties of minimizers of nonlocal energies suitably modified in order to take in consideration the degeneracy as well as the generality of the involved potential and kernel. Moreover, in the particular case when one restricts himself to the case of non-degenerate potentials, he recovers the same results already available in the non-degenerate scenario, established by Cabré and Sire [24], Dipierro Et Al. [50, 51] and Palatucci Et Al. [114].

Part 1

Preliminaries

CHAPTER 2

The Heisenberg group

In this chapter we introduce the underlying geometrical framework of the problems we are going to deal with in the next Chapter 2 and 3. More specifically, we first introduce the Heisenberg group which is the most simple example of a Carnot group of step 2. Then, we recall some classical regularity estimates and properties (up to the boundary) of weak solutions to subelliptic equations. We then conclude with the definition and properties of the H-Kelvin transform and of the Taylor polynomials.

For a more precise treatment of the forthcoming results we refer the interested reader to the monographs [19, 91].

1. The geometry of the Heisenberg group

We denote points ξ in $\mathbb{C}^n \times \mathbb{R} \simeq \mathbb{R}^{2n+1}$ by

$$\xi := (z,t) = (x+iy,t) \simeq (x_1,\dots,x_n,y_1,\dots,y_n,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}.$$

For any $\xi, \xi' \in \mathbb{R}^{2n+1}$, the *group multiplication law* "$\circ$" is defined by

$$\xi \circ \xi' := (x+x',\, y+y',\, t+t'+2\langle y,x'\rangle - 2\langle x,y'\rangle).$$

Given $\xi' \in \mathbb{H}^n$, we can thus define the *left translation* $\tau_{\xi'}$ as follows

$$\tau_{\xi'}(\xi) := \xi' \circ \xi \qquad \forall \xi \in \mathbb{H}^n. \tag{2.1}$$

The group of non-isotropic *dilations* $\{\delta_\lambda\}_{\lambda>0}$ on $\mathbb{R}^{2n+1}$ is defined by

$$\xi \mapsto \delta_\lambda(\xi) := (\lambda x,\, \lambda y,\, \lambda^2 t), \tag{2.2}$$

and, as customary, $Q \equiv 2n+2$ denotes the *homogeneous dimension of* $\mathbb{R}^{2n+1}$ with respect to $\{\delta_\lambda\}_{\lambda>0}$, so that the Heisenberg-Weyl group $\mathbb{H}^n := (\mathbb{R}^{2n+1}, \circ, \delta_\lambda)$ is a homogeneous Lie group.

The Jacobian base of the Heisenberg Lie algebra $\mathscr{H}^n$ is given by

$$Z_j := \partial_{x_j} + 2y_j\partial_t, \quad Z_{n+j} := \partial_{y_j} - 2x_j\partial_t, \quad 1 \le j \le n, \quad T := \partial_t.$$

Since $[Z_j, Z_{n+j}] = -4\partial_t$ for every $1 \le j \le n$, it plainly follows that

$$\operatorname{rank}\Big(\operatorname{Lie}\{Z_1,\dots,Z_{2n},T\}(0,0)\Big) \;=\; 2n+1,$$

so that $\mathbb{H}^n$ is a Carnot group with the following stratification of the algebra

$$\mathscr{H}^n = \operatorname{span}\{Z_1,\dots,Z_{2n}\} \oplus \operatorname{span}\{T\}.$$

Letting $\Omega \subset \mathbb{H}^n$ being a domain, for any $u \in C^1(\Omega)$ we define the *horizontal gradient* $D_H u$ by

$$D_H u(\xi) := \big(Z_1 u(\xi), \dots, Z_{2n} u(\xi)\big).$$

The Kohn Laplacian (or sub-Laplacian) Δ_H on $\mathbb{H}^n$ is the second order operator invariant with respect to the left-translations $\tau_{\xi'}$ defined in (2.1) and homogeneous

of degree 2 with respect to the dilations δ_λ defined in (2.2),

$$\Delta_H := \sum_{j=1}^{2n} Z_j^2.$$

Definition 2.1. *A* homogeneous norm *on $\mathbb{H}^n$ is a continuous function (with respect to the Euclidean topology) $d_o : \mathbb{H}^n \to [0, +\infty)$ such that:*

(1) *$d_o(\delta_\lambda(\xi)) = \lambda d_o(\xi)$, for every $\lambda > 0$ and every $\xi \in \mathbb{H}^n$;*
(2) *$d_o(\xi) = 0$ if and only if $\xi = 0$.*

Moreover, we say that the homogeneous norm d_o is symmetric *if*

$$d_o(\xi^{-1}) = d_o(\xi) \qquad \forall \xi \in \mathbb{H}^n.$$

If d_o is a homogeneous norm on $\mathbb{H}^n$, then the function $(\xi, \eta) \mapsto |\eta^{-1} \circ \xi|_H$ is a pseudometric on $\mathbb{H}^n$. In particular, we will work with the standard homogeneous norm on $\mathbb{H}^n$, also known as *Korányi gauge*,

$$|\xi|_{\mathbb{H}^n} := \left(|z|^4 + t^2\right)^{\frac{1}{4}} \qquad \forall \xi = (z, t) \in \mathbb{H}^n. \tag{2.3}$$

As customary, we will denote by $B_\varrho \equiv B_\varrho(\eta_o)$ the ball with center $\eta_o \in \mathbb{H}^n$ and radius $\varrho > 0$ given by

$$B_\varrho(\eta_o) := \left\{\xi \in \mathbb{H}^n : |\eta_o^{-1} \circ \xi|_{\mathbb{H}^n} < \varrho\right\}.$$

We conclude this section with some properties of the homogeneous norm on $\mathbb{H}^n$ that will be useful in the rest of the paper.

Proposition 2.2 (Proposition 5.1.4 in [19])**.** *Let d_o be a homogeneous norm on $\mathbb{H}^n$. Then there exists a constant $c > 0$ such that*

$$\Lambda^{-1}|\xi|_{\mathbb{H}^n} \le d_o(\xi) \le \Lambda|\xi|_{\mathbb{H}^n} \qquad \forall \xi \in \mathbb{H}^n. \tag{2.4}$$

In view of the preceding proposition, in most of the forthcoming proofs one can simply take into account the Korányi norm defined with no modifications at all.

Proposition 2.3 (Proposition 5.1.7 in [19])**.** *Let d_o be a homogeneous norm on $\mathbb{H}^n$. Then there exists a constant $c > 0$ such that the following statements are satisfied:*

(1) *$d_o(\xi \circ \eta) \le c(d_o(\xi) + d_o(\eta))$;*
(2) *$d_o(\xi \circ \eta) \ge \frac{1}{c} d_o(\xi) - d_o(\eta^{-1})$;*
(3) *$d_o(\xi \circ \eta) \ge \frac{1}{c} d_o(\xi) - c d_o(\eta)$.*

We remark that in the case when the homogeneous norm d_o does reduce to the standard Korányi norm $|\cdot|_{\mathbb{H}^n}$, the constant c given by Proposition 2.3 can be chosen equal to 1. For the proof, we refer to [40]; see also Example 5.1 in [10].

2. Properties of weak solutions to subelliptic equation

We now prove some well expected regularity estimate for weak solutions to classical subelliptic equations. For related results on Carnot groups and Carnot-Carathéodory space we refer the interested reader to [31, 30, 67, 129, 99] and the references therein.

First thing first we consider the standard Folland-Stein-Sobolev space $S_0^1(\mathbb{H}^n)$ defined as the completion of $C_0^\infty(\mathbb{H}^n)$ with respect to the homogeneous subgradient norm $\|D_H \cdot\|_{L^2(\mathbb{H}^n)}$. We also recall that the following Sobolev-type embedding does hold in S_0^1 for some positive constant c,

$$\|u\|^{2^*}_{L^{2^*}(\mathbb{H}^n)} \le c\|D_H u\|^{2^*}_{L^2(\mathbb{H}^n)} \quad \forall u \in S_0^1(\mathbb{H}^n),$$

where the number $2^* := 2Q/(Q-2)$ is the critical Sobolev exponent; see for example [91].

We start recalling a Caccioppoli-type estimate. The proof is analogous as [73, Lemma 6]

Lemma 2.4. *Let us consider the following problem*

$$\begin{cases} -\Delta_H u = a(\xi) u^{p-1}, \\ u \in S^1_0(\Omega), \ u \geq 0. \end{cases}$$

where $a \in L^\infty(\Omega)$, $2 < p_0 < p \leq 2^*$. Then, for any $1 < q \leq 2^* - 1$, there exists υ_0 and ϱ_0 depending only on $\|a\|_{L^\infty(\Omega)}$, n, p and q such that, for any ξ with

$$\int_{\Omega \cap B_{2\varrho}(\xi)} u^p \, \mathrm{d}\xi \leq \upsilon_0 \quad \text{for any } \varrho \leq \varrho_0, \tag{2.5}$$

we have

$$\|u\|_{L^{\frac{(q+1)2^*}{2}}(\Omega \cap B_\varrho(\xi))} \leq \frac{c}{\varrho^{2/(q+1)}} \|u\|_{L^{q+1}(\Omega \cap B_\varrho(\xi))},$$

for a constant $c > 0$ depending on the dimension n only.

Moreover, the following L^∞-estimate holds true.

Lemma 2.5. *Let us consider the following problem*

$$\begin{cases} -\Delta_H u = f(\xi, u) u, \\ u \in S^1_0(\Omega), \ u \geq 0. \end{cases}$$

If $\xi \mapsto f(\xi, \cdot) \in L^{q/2}(\Omega)$, for a. e. $\xi \in \Omega$ and for some $q > Q$, then it holds

$$\sup_{B_\varrho} u \leq c \left(\frac{1}{\varrho^Q} \int_{\Omega \cap B_{2\varrho}} u^{2^*} \right)^{\frac{1}{2^*}} \qquad \forall B_{2\varrho} \subset \Omega, \tag{2.6}$$

where $c \equiv c(n, q, \|f\|_{L^{q/2}(\Omega)}) > 0$.

Proof. With no loss of generality, we assume that $\varrho = 1$. Consider for $p > 1$, $B_4(0) \equiv B_4$ and a nonnegative cut-off function $\varphi \in C^\infty_0(B_4)$, the test function $\phi := \varphi^2 u^p$. So that

$$\int_\Omega D_H u \cdot D_H(\varphi^2 u^p) \, \mathrm{d}\xi = \int_\Omega f(\xi, u) \varphi^2 u^{p+1} \, \mathrm{d}\xi. \tag{2.7}$$

Integrating by parts the first integral on the left-hand side in (2.7) and using Young's Inequality yield

$$\begin{aligned} &\int_\Omega D_H u \cdot D_H(\varphi^2 u^p) \, \mathrm{d}\xi \\ &\qquad \geq -2 \int_\Omega \varphi u^p |D_H u| |D_H \varphi| \, \mathrm{d}\xi + p \int_\Omega \varphi^2 u^{p-1} |D_H u|^2 \, \mathrm{d}\xi \\ &\qquad \geq \frac{p}{2} \int_\Omega \varphi^2 u^{p-1} |D_H u|^2 \, \mathrm{d}\xi - \frac{2}{p} \int_\Omega u^{p+1} |D_H \varphi|^2 \, \mathrm{d}\xi. \end{aligned}$$

Putting all together we have that

$$\int_\Omega \varphi^2 u^{p-1} |D_H u|^2 \, \mathrm{d}\xi \leq \frac{2}{p} \int_\Omega f(\xi, u) \varphi^2 u^{p+1} \, \mathrm{d}\xi + \frac{4}{p^2} \int_\Omega u^{p+1} |D_H \varphi|^2 \, \mathrm{d}\xi.$$

Applying now Sobolev's Inequality yields that

$$\|\varphi u^{\frac{p+1}{2}}\|^2_{L^{2^*}(\Omega)} \leq 2(S^*)^{\frac{2}{2^*}} \int_\Omega \varphi^2 u^{p-1} |D_H u|^2 \, \mathrm{d}\xi + 2(S^*)^{\frac{2}{2^*}} \int_\Omega u^{p+1} |D_H \varphi|^2 \, \mathrm{d}\xi$$

$$\leq \underbrace{2(S^*)^{\frac{2}{2^*}}\frac{(1+p)^2+3}{p^2}}_{=:\bar{c}}\int_\Omega \Big(f(\xi,u)\varphi^2+|D_H\varphi|^2\Big)u^{p+1}\,\mathrm{d}\xi, \tag{2.8}$$

where the constant $\bar{c}$ is bounded for any p greater than 1.

Using Hölder's Inequality with $q/2$ and $q/(q-2)$ and an interpolative inequality, we obtain

$$\begin{aligned}\int_\Omega f(\xi,u)\Big(\varphi u^{\frac{p+1}{2}}\Big)^2\,\mathrm{d}\xi &\leq \|f\|_{L^{q/2}(\Omega)}\|\varphi u^{\frac{p+1}{2}}\|^2_{L^{2q/(q-2)}(\Omega)}\\ &\leq \|f\|_{L^{q/2}(\Omega)}\Big(\epsilon\|\varphi u^{\frac{p+1}{2}}\|_{L^{2^*}(\Omega)}+\epsilon^{-\frac{Q}{q-Q}}\|\varphi u^{\frac{p+1}{2}}\|_{L^2(\Omega)}\Big)^2,\end{aligned}$$

Thus, by choosing ϵ sufficiently small and absorbing the terms in the left-hand side in (2.8), we have

$$\|\varphi u^{\frac{\vartheta}{2}}\|_{L^{2(2^*/2)}(\Omega)}\leq c\,\vartheta^{\frac{q}{q-Q}}\|(\varphi+|D_H\varphi|)u^{\frac{\vartheta}{2}}\|_{L^2(\Omega)}, \tag{2.9}$$

where we denoted by $\vartheta:=p+1$. We specify now the cut-off function φ. Let $1\leq\sigma<\varrho\leq 3$ and choose φ such that

$$0\leq\varphi\leq 1,\quad \varphi\equiv 1\,\text{on}\ B_\sigma,\quad \varphi\equiv 0\,\text{on}\ \mathbb{H}^n\setminus B_\varrho,\quad |D_H\varphi|\leq\frac{2}{\varrho-\sigma}.$$

With such a choice of φ, the estimate in (2.9) becomes

$$\|u^{\vartheta/2}\|_{L^{2(2^*/2)}(B_\sigma)}\leq\frac{c\,\vartheta^{\frac{q}{q-Q}}}{\varrho-\sigma}\|u^{\vartheta/2}\|_{L^2(B_\varrho)}. \tag{2.10}$$

Thus, once defined $A_{\mathfrak{q},s}:=\Big(\int_{B_s}u^{\mathfrak{q}}\,\mathrm{d}\xi\Big)^{\frac{1}{\mathfrak{q}}}$, we have that the inequality in (2.10) becomes

$$A_{\frac{2^*\vartheta}{2},\sigma}\leq\left(\frac{c\,\vartheta^{\frac{q}{q-Q}}}{\varrho-\sigma}\right)^{\frac{2}{\vartheta}}A_{\vartheta,\varrho}. \tag{2.11}$$

We are finally in the position to start a classical iteration method in order to get the desired supremum estimate. Taking $\vartheta_j:=(2^*/2)^j\vartheta$ and $\varrho_j:=1+2^{-j}$, for $j=0,1,\dots$, we prove that

$$A_{\vartheta_N,\varrho_N}\leq\prod_{j=0}^N\left(\frac{c\,\vartheta_j^{\frac{q}{q-Q}}}{\varrho_j-\varrho_{j+1}}\right)^{\frac{2}{\vartheta_j}}A_{\vartheta,2}\qquad\text{for any } N\geq 1. \tag{2.12}$$

Clearly the case $N\equiv 1$ follows from (2.11). We now assume that the estimate above holds for N and prove it for $N+1$. Indeed, recalling (2.11) we have

$$\begin{aligned}A_{\vartheta_{N+1}\vartheta,\varrho_{N+1}} &\leq \left(\frac{c\,\vartheta_N^{\frac{q}{q-Q}}}{\varrho_N-\varrho_{N+1}}\right)^{\frac{2}{\vartheta_N}}A_{\vartheta_N,\varrho_N}\\ &\leq \left(\frac{c\,\vartheta_N^{\frac{q}{q-Q}}}{\varrho_N-\varrho_{N+1}}\right)^{\frac{2}{\vartheta_N}}\prod_{j=0}^N\left(\frac{c\,\vartheta_j^{\frac{q}{q-Q}}}{\varrho_j-\varrho_{j+1}}\right)^{\frac{2}{\vartheta_j}}A_{\vartheta,2}\\ &= \prod_{j=0}^{N+1}\left(\frac{c\,\vartheta_j^{\frac{q}{q-Q}}}{\varrho_j-\varrho_{j+1}}\right)^{\frac{2}{\vartheta_j}}A_{\vartheta,2}\,,\end{aligned}$$

and the induction step does follow.

Moreover, note that

$$\prod_{j=0}^{\infty}\left(\frac{c\,\vartheta_j^{\frac{q}{q-Q}}}{\varrho_j-\varrho_{j+1}}\right)^{\frac{2}{\vartheta_j}}=\left(\frac{c\,\vartheta^{\frac{q}{q-Q}}}{\varrho_0-\varrho_1}\right)^{\frac{2}{\vartheta}}\mathrm{e}^{\sum_j\frac{2\log\left(c\,2^{j+1}\vartheta_j^{\frac{q}{q-Q}}\right)}{\vartheta_j}},$$

where we also used that $\displaystyle\sum_{j=1}^{\infty}\frac{2\log\left(c\,2^{j+1}\vartheta_j^{\frac{q}{q-Q}}\right)}{\vartheta_j}<\infty$.

Thus, letting N going to infinity in (2.12), we eventually arrive at

$$\sup_{B_1}u\le c\left(\int_{B_2}u^{\vartheta}\,\mathrm{d}\xi\right)^{\frac{1}{\vartheta}},$$

which gives the desired estimate (2.6) choosing $\vartheta=2^*$. □

Let us recall now the definition of intrinsic Hölder classes $\Gamma^{m,\alpha}$ introduced by Folland and Stein in [59, 60].

Definition 2.6. *Let $\alpha\in(0,1)$ and $\Omega\subset\mathbb{H}^n$. A function $u:\Omega\to\mathbb{R}$ belongs to $\Gamma^{0,\alpha}(\Omega)$ if there exists a positive constant c such that*

$$[u]_{\Gamma^{0,\alpha}(\Omega)}:=\sup_{\substack{\xi\neq\eta\\ \xi,\eta\in\Omega}}\frac{|u(\xi)-u(\eta)|}{|\eta^{-1}\circ\xi|_H^{\alpha}}\le c.$$

For any $m\in\mathbb{N}$, we say that $u\in\Gamma^{m,\alpha}(\Omega)$ if $Z_iu\in\Gamma^{m-1,\alpha}(\Omega)$, for any $i=1,\dots,2n$.

The intrinsic Hölder class $\Gamma^{0,\alpha}(\Omega)$ is as expected a Banach space once it is endowed with the norm

$$\|u\|_{\Gamma^{0,\alpha}(\Omega)}:=\|u\|_{L^\infty(\Omega)}+[u]_{\Gamma^{0,\alpha}(\Omega)}.$$

We conclude this section with some fundamental boundary regularity estimate. In contrast with the Euclidean framework when dealing with boundary regularity in the Heisenberg group (and in more general Carnot groups) one has to take in consideration the presence of the so-called *characteristic set* $\Sigma(\Omega)$ of a smooth domain Ω, that is

$$\Sigma(\Omega):=\Big\{\xi\in\partial\Omega\,|\,Z_j(\xi)\in T_\xi(\partial\Omega),\text{ for }j=1,\dots,2n\Big\}.$$

At such points the vector fields forming the principal part of the relevant operator Δ_H become tangent to the boundary. Thus, near a characteristic points, as pointed out by Jerison in his breakthrough papers [92, 93], even harmonic functions can encounter a sudden loss of regularity and the best boundary regularity achievable is $\Gamma^{0,\alpha}$.

Away from the characteristic set several boundary regularity estimates, e. g. Schauder-type estimates hold true. Indeed, the first result in this direction is due to Jerison.

Theorem 2.7 (Theorem 7.1 in [92])**.** *Let Ω be a smooth bounded domain of $\mathbb{H}^n$ and let $\phi\in C_0^\infty(\mathbb{H}^n)$ be supported in a small neighborhood of a non-characteristic point $\xi\in\partial\Omega$. Given $f\in\Gamma^{m,\alpha}(\overline{\Omega})$, $m\in\mathbb{N}\cup\{0\}$, $0<\alpha<1$, then for the unique solution u to*

$$-\Delta_{\mathbb{H}^n}u=f\ \text{ in }\ \Omega,\qquad u=0\ \text{ in }\ \partial\Omega,$$

one has $\phi u\in\Gamma^{m+2,\alpha}(\overline{\Omega})$.

Theorem 2.8 (Theorem 1.1 in [11]). *Let Ω be a bounded subset of $\mathbb{H}^n$ of class $C^{1,\beta}$ for some $\beta \in (0,1)$ and assume that the set $B_\varrho \cap \partial\Omega$ is non-characteristic. Let $u \in S^1_0(B_\varrho \cap \Omega) \cap C(\overline{B_\varrho \cap \Omega})$ be a weak solution to*

$$-\Delta_{\mathbb{H}^n} u = f \ \text{ in } B_\varrho \cap \Omega, \qquad u_{|B_\varrho \cap \partial\Omega} = \phi,$$

with $f \in L^\infty(\overline{B_\varrho \cap \Omega})$ and $\phi \in \Gamma^{1,\beta}(\overline{B_\varrho \cap \Omega})$. Then, $u \in \Gamma^{1,\beta}(\overline{B_{\varrho/2} \cap \Omega})$, and the following estimate does hold,

$$[\nabla_{\mathbb{H}^n} u]_{\Gamma^{0,\beta}(B_\varrho \cap \Omega)} \leq c\varrho^{-1-\beta}\Big(\|u\|_{L^\infty(B_\varrho \cap \Omega)} + \varrho^2 \|f\|_{L^\infty(B_\varrho \cap \Omega)} + \varrho^{1+\beta}[\phi]_{\Gamma^{1,\beta}(\overline{B_\varrho \cap \Omega})}\Big).$$

For others Schauder-type estimates on the boundary we refer the interested reader to [9, 37].

Moreover, the following negligibility of the characteristic set with respect to the surface measure will turn out useful in the proof of Theorem 7.3 in Chapter 7.

Theorem 2.9 (See Theorem 1.2 in [45]). *Let $\Omega \subset \mathbb{H}^n$ be a C^∞ domain and let $\Sigma(\Omega)$ be its characteristic set. Then, being H^{Q-2} the $(Q-2)$-dimensional Hausdorff measure, it holds that*

$$H^{Q-2}(\Sigma(\Omega)) = 0.$$

3. H-Kelvin transform

We briefly recall the definition of *H-Kelvin transform* which will play an important role in the forthcoming Chapter 2.

Definition 2.10. *For any $\xi = (z,t) \in \mathbb{H}^n \setminus \{0\}$ we call H-*inversion *the map*

$$\begin{aligned} &\boldsymbol{i}_H : \mathbb{H}^n \setminus \{0\} \longmapsto \mathbb{H}^n \setminus \{0\}, \\ &\boldsymbol{i}_H(\xi) := \left(\frac{z}{|z|^2 + it}, \frac{t}{|z|^4 + t^2}\right). \end{aligned}$$

*With $\boldsymbol{i}_H$ defined above, given a function $u : \mathbb{H}^n \to \mathbb{R}$ we denote by $u^\sharp$ its H-*Kelvin transform *defined by*

$$\begin{aligned} &u^\sharp : \mathbb{H}^n \setminus \{0\} \to \mathbb{R}, \\ &u^\sharp(\xi) := |\xi|_H^{-(Q-2)}\, u\Big(\boldsymbol{i}_H(\xi)\Big). \end{aligned}$$

It can be easily verified that

$$\boldsymbol{i}_H(\boldsymbol{i}_H(\xi)) = g, \quad \text{and} \quad |\boldsymbol{i}_H(\xi)|_H = |\xi|_H^{-1}. \tag{2.13}$$

We would now need to present a few properties of the H-Kelvin transform adapted to our framework.

Proposition 2.11 (Theorem 2.3.5 in [91]). *Let Ω be a domain and denote by $\Omega^\sharp$ the image of Ω under the H-inversion $\boldsymbol{i}_H$. Then, we have that the H-Kelvin transform is an isometry between $S^1_0(\Omega)$ and $S^1_0(\Omega^\sharp)$.*

Proposition 2.12 (Lemma 2.3.6 in [91]). *Let u be a solution to*

$$\begin{cases} -\Delta_H u = u^p, \\ u \in S^1_0(\Omega), \quad u \geq 0, \end{cases}$$

for some positive exponent $p > 0$. Then, its H-Kelvin transform $u^\sharp$ satisfies

$$\begin{cases} -\Delta_H u^\sharp(\xi) = |\xi|_H^{p(Q-2)-(Q+2)} u^\sharp(\xi)^p, \\ u^\sharp \in S^1_0(\Omega^\sharp), \quad u^\sharp \geq 0. \end{cases}$$

4. Taylor polynomials in the Heisenberg group

We now conclude the first chapter introducing an analogous of the Taylor polynomials in the sub-Riemannian setting of the Heisenberg group. The forthcoming results will be helpful in Section 4, when we prove that the fractional power of the subLaplacian $(-\Delta_H)^s$ does converge to Δ_H, for $s \to 1^-$.

Definition 2.13 (**MacLaurin polynomial in $\mathbb{H}^n$**). *Let $u \in C^\infty(\mathbb{H}^n)$. Then, for any $m \in \mathbb{N} \cup \{0\}$ there exists a unique polynomial P being δ_λ-homogeneous of degree at most m such that*

$$(Z_1, \dots, Z_{2n}, T)^\beta P(0) = (Z_1, \dots, Z_{2n}, T)^\beta u(0)$$

for any multi-index $\beta = (\beta_1, \dots, \beta_{2n+1})$ with $|\beta|_H := \beta_1 + \cdots + \beta_{2n} + 2\beta_{2n+1} \leq m$. We say that $P := P_m(u, 0)(\xi)$ is "the MacLaurin polynomial *of δ_λ-degree m associated to u".*

The MacLaurin polynomial of δ_λ-degree 2 can be explicitly written. Indeed, we have that

$$P_2(u, 0)(z, t) := u(0) + D_H u(0) \cdot z + \partial_t u(0) \cdot t + \frac{1}{2}\langle x, D_H^{2,*} u(0) \cdot x \rangle,$$

where $D_H u$ is the horizontal gradient whereas $D_H^{2,*}$ is the symmetrized horizontal Hessian matrix; that is,

$$D_H^{2,*} u(\xi) := \left(\frac{1}{2}\Big(Z_i Z_j u(\xi) + Z_j Z_i u(\xi) \Big) \right)_{i,j=1,\dots,2n}. \tag{2.14}$$

With this bit of notation we can now introduce the Taylor polynomials in $\mathbb{H}^n$ also stating an approximation result for sufficiently regular functions.

Definition 2.14 (**Taylor polynomial in $\mathbb{H}^n$**). *Let $u \in C^\infty(\mathbb{H}^n)$, $\xi_0 \in \mathbb{H}^n$, and $m \in \mathbb{N} \cup \{0\}$. Let us consider the MacLaurin polynomial $P_m(u(\xi_0 \circ \cdot), 0)$ of the function $\xi \longmapsto u(\xi_0 \circ \xi)$ The polynomial*

$$P_m(u, \xi_0)(\xi) := P_m(u(\xi_0 \circ \cdot), 0)(\xi_0^{-1} \circ \xi),$$

is the Taylor polynomial *of δ_λ-degree m centered at ξ_0 associated to u.*

As in the the Euclidean framework sufficiently regular functions can be approximated by their Taylor polynomial paying the price of considering an error term.

Proposition 2.15 (Corollary 20.3.5 in [19]). *For every $u \in C^{m+1}(\mathbb{H}^n)$, $\xi_0 \in \mathbb{H}^n$ and $m \in \mathbb{N} \cup \{0\}$, we have that*

$$u(\xi) = P_m(u, \xi_0)(\xi) + o(|\xi_0^{-1} \circ \xi|_H^{m+1}).$$

Part 2

Critical Sobolev embedding in the Heisenberg group

CHAPTER 3

Setting of the problem

In this part we are going to investigate some of the effects of the lack of compactness in the critical Folland-Stein-Sobolev embedding in general domains. We briefly recall the definition of the standard Folland-Stein-Sobolev space $S_0^1(\mathbb{H}^n)$ defined as the completion of $C_0^\infty(\mathbb{H}^n)$ with respect to the homogeneous subgradient norm $\|D_H \cdot\|_{L^2(\mathbb{H}^n)}$. As is well known, the following Sobolev-type inequality, first proved in [60], asserts that for a positive constant S^* it holds

$$\|u\|^{2^*}_{L^{2^*}(\mathbb{H}^n)} \leq S^* \|D_H u\|^{2^*}_{L^2(\mathbb{H}^n)} \quad \forall u \in S_0^1(\mathbb{H}^n)\,, \tag{3.1}$$

where we denote by $2^* = 2^*(Q) := 2Q/(Q-2)$ the Folland-Stein-Sobolev critical exponent; see also Chapter 2.

Over the past decades, the critical Sobolev inequality (3.1) has garnered significant attention as a captivating subject of study. Given the extent of the literature, providing an exhaustive treatment here is not feasible. We just mention the highly significant papers [94, 63, 95, 67, 39], the recent book [91], and the references therein for in-depth insights. The main reasons for studying the critical Sobolev inequality lie in its close association with the lack of compactness of the related critical Folland-Stein-Sobolev embedding and its connection to the corresponding Euler-Lagrange equation. This, in turn, addresses the significant CR Yamabe problem. Numerous results have been established, indicating compliance with the classical critical inequality in the Euclidean framework, despite the complexities posed by the sub-Riemannian geometry of the Heisenberg setting. However, many expected results remain open due to substantial differences from the Euclidean framework, stemming from the complex metric structure of the Heisenberg setting and the presence of characteristic points in the domains involved.

We want to investigate a natural approximation to (3.1) via subcritical Sobolev inequalities. Our approach is variational. For this, it is convenient to consider the following maximization problem,

$$S^* := \sup\left\{ \int_{\mathbb{H}^n} |u(\xi)|^{2^*} \,\mathrm{d}\xi \,:\, u \in S_0^1(\mathbb{H}^n), \int_{\mathbb{H}^n} |D_H u(\xi)|^2 \mathrm{d}\xi \leq 1 \right\}. \tag{3.2}$$

Clearly, the validity of (3.1) is equivalent to the finiteness of the constant S^*, defined in (3.2). The existence of maximizers in (3.2) poses a challenging problem due to the intrinsic dilations and translations invariance of this inequality, analogous to the difficulties encountered in the classical critical Sobolev inequality. However, the situation becomes even more delicate in this context, given the underlying non-Euclidean geometry of the Heisenberg group and the obstacles arising from non-commutativity. The groundbreaking paper by Jerison and Lee [94] has provided explicit forms of the maximizers, along with the computation of the optimal constant in (3.2). Additionally, related investigations on the Moser-Trudinger inequality in the sub-Riemannian setting can be found in [20].

Similarly, for any bounded domain $\Omega \subset \mathbb{H}^n$, we consider the following Sobolev embedding in the same variational form as the one in (3.2),

$$S^*_\Omega := \sup\left\{\int_\Omega |u(\xi)|^{2^*}\,\mathrm{d}\xi \,:\, u \in S^1_0(\Omega), \int_\Omega |D_H u(\xi)|^2 \mathrm{d}\xi \le 1\right\}, \tag{3.3}$$

where now the Folland-Stein-Sobolev space $S^1_0(\Omega)$ is given as the closure of $C^\infty_0(\Omega)$ with respect to the homogeneous L^2-subgradient norm in Ω. It can be verified that $S^*_\Omega \equiv S^*$ using a standard scaling argument on compactly supported smooth functions. However, in contrast to the critical case presented in (3.2), the variational problem (3.3) has no maximizers, as evidenced by the explicit form of the optimal functions in (3.2) (see, for instance, Theorem 4.3). The scenario changes drastically for the subcritical embeddings. In this case, due to the boundedness of Ω, the embedding $S^1_0(\Omega) \hookrightarrow L^{2^*-\varepsilon}(\Omega)$ becomes compact, given that $\varepsilon < 2^* - 2$, ensuring the existence of a maximizer $u_\varepsilon \in S^1_0(\Omega)$ for the associated variational problem

$$S^*_\varepsilon := \sup\left\{\int_\Omega |u(\xi)|^{2^*-\varepsilon}\,\mathrm{d}\xi \,:\, u \in S^1_0(\Omega), \int_\Omega |D_H u(\xi)|^2 \mathrm{d}\xi \le 1\right\}. \tag{3.4}$$

In the same fashion, in the Euler-Lagrange equation for the energy functionals in (3.4), that is

$$-\Delta_H u_\varepsilon = \lambda |u_\varepsilon|^{2^*-\varepsilon-2} u_\varepsilon \ \text{ in } (S^1_0(\Omega))', \tag{3.5}$$

where λ is a suitable Lagrange multiplier, we spot a different behaviour for $\varepsilon > 0$ and $\varepsilon = 0$. Whereas when $\varepsilon > 0$ the problem above has a solution u_ε, it becomes very delicate when $\varepsilon = 0$: even the existence of solutions is not granted. In particular, the existence and various properties of the solutions strongly depend on both the geometry and the topology of the domain Ω. We refer to the following papers: [95] for nonexistence of nonnegative solutions when Ω is a certain half-space; [63] for existence and nonexistence results for even more general nonlinearity; [39] where the authors show the existence of a solution in the case when the domain Ω has at least a nontrivial suitable homology group.

In view of such a qualitative change when $\varepsilon = 0$ in both (3.4) and (3.5), in the present part we analyze the asymptotic behavior as ε goes to 0 of both the subcritical Sobolev constant S^*_ε in the Heisenberg group given in (3.4) and of the corresponding optimal functions u_ε of the embedding $S^1_0(\Omega) \hookrightarrow L^{2^*-\varepsilon}(\Omega)$.

Regarding *the Euclidean counterpart* of this investigation, several results have been obtained, mostly via fine estimates and a standard regularity elliptic approach. These investigations primarily focus on a special class of solutions of equation (3.5), which serve as maximizers for the corresponding Sobolev embedding. Energy concentration results for these sequences, along with the subsequent localization of such concentration on specific points, have been established in [22, 118, 73], among other works.
However, the situation is quite different in *the Heisenberg panorama* due to the numerous inherent challenges in this framework. Indeed, the sub-Riemannian geometry prevents a straightforward generalization of several tools and symmetrization techniques, as well as regularity approximations. As a result, the Heisenberg setting remains relatively unexplored in comparison, leaving the field with significant research opportunities.

In this part, by pursuing a new approach in the Heisenberg group (relying on De Giorgi's Γ-convergence) we prove that a sequence of optimal functions for (3.4) do concentrate energy at one point $\xi_o \in \overline{\Omega}$ in very general (possibly not-smooth) domains; see Chapter 4. Moreover, in Chapter 7 we localize the

concentration point ξ_o in terms of the Green function associated to the involved domain and in clear accordance with the underlying sub-Riemannian geometry – and consequently a new suitable definition of domains *geometrical regular near their characteristic set* is given. Moreover, several independent results are proved first such as the celebrated Struwe's Global Compactness in Chapter 5 and a fine asymptotic control of the optimal functions via the Jerison & Lee extremals realizing the equality in the critical Sobolev inequality ([**94**]) in Chapter 6.

The results presented in this part of the thesis are contained in [**109**, **107**] and they have been announced in the preliminary research report [**110**].

CHAPTER 4

Γ-convergence and concentration

In this chapter, relying on De Giorgi's Γ-convergence, we establish the energy concentration for a sequence of maximizers of (3.4).

Our first result on the subcritical approximation of the Sobolev quotient S^* in the Heisenberg group reads as follows.

Theorem 4.1 (Γ-convergence)**.** *Let $\Omega \subseteq \mathbb{H}^n$ be a bounded domain, and denote by $\mathcal{M}(\overline{\Omega})$ the set of nonnegative Radon measures in Ω. Let $X = X(\Omega)$ be the space*

$$X := \left\{ (u,\mu) \in S_0^1(\Omega) \times \mathcal{M}(\overline{\Omega}) : \mu \geq |D_H u|^2 \mathrm{d}\xi,\ \mu(\overline{\Omega}) \leq 1 \right\},$$

endowed with the product topology $\boldsymbol{t}$ such that

$$(u_k,\mu_k) \xrightarrow{\boldsymbol{t}} (u,\mu) \overset{def}{\Leftrightarrow} \begin{cases} u_k \rightharpoonup u \ \text{ in } L^{2^*}(\Omega), \\ \mu_k \overset{*}{\rightharpoonup} \mu \ \text{ in } \mathcal{M}(\overline{\Omega}). \end{cases} \tag{4.1}$$

Let us consider the following family of functionals,

$$\mathcal{F}_\varepsilon(u,\mu) := \int_\Omega |u|^{2^*-\varepsilon} \mathrm{d}\xi \quad \forall (u,\mu) \in X\,. \tag{4.2}$$

Then, as $\varepsilon \to 0$, the Γ^+-limit of the family of functionals $\mathcal{F}_\varepsilon$ with respect to the topology $\boldsymbol{t}$ given by (4.1) is the functional $\mathcal{F}$ defined by

$$\mathcal{F}(u,\mu) = \int_\Omega |u|^{2^*} \mathrm{d}\xi + S^* \sum_{j=1}^{\infty} \mu_j^{\frac{2^*}{2}} \quad \forall (u,\mu) \in X. \tag{4.3}$$

Here S^* is the best Sobolev constant in $\mathbb{H}^n$, $2^* = 2Q/(Q-2)$ is the Folland-Stein-Sobolev critical exponent, and the numbers μ_j are the coefficients of the atomic part of the measure μ.

In order to prove such a result in the most general situation, i. e. without requiring no additional regularity assumptions nor special geometric features on the involved domains, we pursuit a new approach relying on De Giorgi's Γ-convergence techniques. This is in the same spirit of previous results regarding the classical Sobolev embedding in the Euclidean framework, as seen in [**3**, **104**, **105**]. However, our main proof deviates significantly, as we explicitly construct the optimal recovery sequences: a departure from the implicit existence results demonstrated in the aforementioned papers, which rely on compactness and locality properties of the Γ-limit energy functional. Interestingly, our strategy bears resemblance to the *fractional Sobolev spaces* scenario ([**113**]), although notable differences inevitably arise due to the natural discrepancy between the involved frameworks.

As a corollary of Theorem 4.1 above, we can deduce that the sequences of maximizers $\{u_\varepsilon\}$ for the subcritical Sobolev quotient S_ε^* do concentrate energy at

one point $\xi_o \in \overline{\Omega}$, in clear accordance with the analogous result in the Euclidean case.

Theorem 4.2 (Energy concentration)**.** *Let $\Omega \subset \mathbb{H}^n$ be a bounded domain and let $u_\varepsilon \in S_0^1(\Omega)$ be a maximizer for S_ε^*. Then, as $\varepsilon = \varepsilon_k \to 0$, up to subsequences, we have that there exists $\xi_o \in \overline{\Omega}$ such that*

$$u_k = u_{\varepsilon_k} \rightharpoonup 0 \text{ in } L^{2^*}(\Omega),$$

and

$$|D_H u_k|^2 \mathrm{d}\xi \overset{*}{\rightharpoonup} \delta_{\xi_o} \text{ in } \mathcal{M}(\overline{\Omega}),$$

with δ_{ξ_o} being the Dirac mass at ξ_o.

The concentration result above in very general domains was in fact one of our first goals, as well as the veritable guideline to the choice of the topology in the Γ^+-convergence Theorem 4.1 in order to provide a convergence of functionals which guarantees that maximizers of the approximating functionals converge to maximizers of a precise limit functional. Indeed, at this stage the proof of Theorem 4.2 does reduce to a fine analysis of the form of the maxima of the functional $\mathcal{F}$ defined by (4.2); see Section 3.

1. Preliminaries on the critical Sobolev embedding

In this section we recall some useful results about the lack of compactness in the critical Sobolev embedding in $\mathbb{H}^n$.

Firstly, we state (in the form adapted to our framework) the aforementioned pioneering result by Jerison and Lee [94] which gives the optimal constant in the critical Sobolev inequality together with the explicit expression of the functions giving the equality in (3.2).

Theorem 4.3 (Corollary C in [94])**.** *Let $2^* = 2Q/(Q-2)$. Then for any $\lambda > 0$ and any $\xi_o \in \mathbb{H}^n$, the function U_{λ,ξ_o} defined by*

$$U_{\lambda,\xi_o} := U\left(\delta_{\frac{1}{\lambda}}\left(\tau_{\xi_0^{-1}}(\xi)\right)\right), \tag{4.4}$$

where

$$U(\xi) = c_0\left(\left(1+|z|^2\right)^2 + t^2\right)^{-\frac{Q-2}{4}} \qquad \forall \xi \in \mathbb{H}^n, \tag{4.5}$$

is solution to the variational problem (3.2).

As in the classical Euclidean setting, the Concentration-compactness alternative in the Heisenberg framework, see Theorem 4.4 below, has been shown to be crucial for analyzing the asymptotic behaviour of bounded sequences in the Folland-Stein space.

Theorem 4.4 (Lemma 1.4.5 in [91])**.** *Let $\Omega \subseteq \mathbb{H}^n$ be an open subset and let $\{u_k\}$ be a sequence in $S_0^1(\Omega)$ weakly converging to u as $k \to \infty$ and such that*

$$|D_H u_k|^2 \mathrm{d}\xi \overset{*}{\rightharpoonup} \mu \quad \text{and} \quad |u_k|^{2^*} \mathrm{d}\xi \overset{*}{\rightharpoonup} \nu \quad \text{in } \mathcal{M}(\overline{\Omega}).$$

Then, either $u_k \to u$ in $L^{2^*}_{\mathrm{loc}}(\Omega)$ or there exists a (at most countable) set of distinct points $\{\xi_j\}_{j\in J}$ and positive numbers $\{\nu_j\}_{j\in J}$ such that we have

$$\nu = |u|^{2^*} \mathrm{d}\xi + \sum_j \nu_j \delta_{\xi_j}.$$

Moreover, there exist a positive measure $\tilde{\mu} \in \mathcal{M}(\overline{\Omega})$ *with* $\operatorname{spt} \tilde{\mu} \subset \overline{\Omega}$ *and positive numbers* $\{\mu_j\}_{j\in J}$ *such that*

$$\mu = |D_H u|^2 \mathrm{d}\xi + \tilde{\mu} + \sum_j \mu_j \delta_{\xi_j}, \quad \nu_j \leq S^* (\mu_j)^{\frac{2^*}{2}}.$$

We conclude this section by presenting an expected convergence result for the Sobolev quotients.

Proposition 4.5. *Let* S^*_ε *be defined by* (3.4) *and let* S^* *be defined by* (3.3). *Then*

$$\lim_{\varepsilon\to 0} S^*_\varepsilon = S^*.$$

Proof. We were not able to find a precise reference to the literature in the Heisenberg setting, thus we reproduce here a classical elementary proof. Firstly, since Ω is bounded, the Hölder inequality yields

$$\limsup_{\varepsilon\to 0} S^*_\varepsilon \leq S^*. \tag{4.6}$$

Indeed, take the maximizers $u_\varepsilon \in S^1_0(\Omega)$ for S^*_ε; one has

$$\begin{aligned} S^*_\varepsilon &= \int_\Omega |u_\varepsilon|^{2^*-\varepsilon} \mathrm{d}\xi = \left(\int_\Omega |u_\varepsilon|^{2^*} \mathrm{d}\xi\right)^{\frac{2^*-\varepsilon}{2^*}} |\Omega|^{\frac{\varepsilon}{2^*}} \\ &\leq (S^*)^{\frac{2^*-\varepsilon}{2^*}} |\Omega|^{\frac{\varepsilon}{2^*}}. \end{aligned}$$

It suffices to pass to the limit as ε goes to zero, and the inequality in (4.6) will follow.

The remaining inequality is a mere consequence of the pointwise convergence of the energy functional $\int |\cdot|^{2^*-\varepsilon}$ to $\int |\cdot|^{2^*}$. This is standard: for every $\upsilon > 0$ there exists $u_\upsilon \in S^1_0(\Omega)$ such that $\|D_H u_\upsilon\|^2_{L^2(\Omega)} \leq 1$ and

$$\int_\Omega |u_\upsilon|^{2^*} \mathrm{d}\xi > S^* - \upsilon. \tag{4.7}$$

Clearly, for such a function u_υ, one has

$$S^*_\varepsilon \geq \int_\Omega |u_\upsilon|^{2^*-\varepsilon} \mathrm{d}\xi.$$

Then, combining the preceeding inequality with (4.7) and passing to the limit as ε goes to zero, we get

$$\begin{aligned} \liminf_{\varepsilon\to 0} S^*_\varepsilon &\geq \lim_{\varepsilon\to 0} \int_\Omega |u_\upsilon|^{2^*-\varepsilon} \mathrm{d}\xi = \int_\Omega |u_\upsilon|^{2^*} \mathrm{d}\xi \\ &\geq S^* - \upsilon, \end{aligned}$$

which gives the desired inequality in view of the arbitrariness of υ. □

2. Proof of the Γ-convergence result

First, we present the functional setting in which we will perform the asymptotic analysis of the Sobolev embeddings via Γ-convergence.

2.1. The functional setting. Let us describe the asymptotic behavior as ε goes to 0 of the maximizers of the Sobolev constant S_ε^* associated to the embedding $S_0^1(\Omega) \hookrightarrow L^{2^*-\varepsilon}$. It is convenient to restate the variational form of the problem from the introduction; that is,

$$S_\varepsilon^* = \sup\left\{\mathcal{F}_\varepsilon(u) \,:\, u \in S_0^1(\Omega), \int_\Omega |D_H u|^2 \mathrm{d}\xi \leq 1\right\}, \tag{4.8}$$

where $\mathcal{F}_\varepsilon$ denotes the following family of energy functionals,

$$\mathcal{F}_\varepsilon(u) := \int_\Omega |u|^{2^*-\varepsilon}\mathrm{d}\xi,$$

on the set $\left\{u \in S_0^1(\Omega),\ \int_{\mathbb{H}^n} |D_H u|^2 \mathrm{d}\xi \leq 1\right\}$.

As mentioned, the main tool will be the Γ-convergence in the sense of De Giorgi. For this, it is crucial to introduce a convenient functional setting, that is, the functional space X with the choice of the topology $\boldsymbol{t}$ as presented in Theorem 4.1. In particular, the topology has to be sufficiently weak to assure the convergence of maximizing sequences and sufficiently strong to allow us to find the desired energy concentration.

The reason for the choice of such a space X is very natural. First of all, we are interested in the asymptotic behavior of the sequence $\{\mathcal{F}_\varepsilon(u_\varepsilon)\}$ for every sequence $\{u_\varepsilon\}$ such that $\|D_H u_\varepsilon\|^2_{L^2(\Omega)} \leq 1$. Such a constraint on the *horizontal Dirichlet energy* of u_ε implies that, up to subsequences, there exists $\mu \in \mathcal{M}(\overline{\Omega})$ and $u \in S_0^1(\Omega)$ such that $\mu(\overline{\Omega}) \leq 1$, $|D_H u_\varepsilon|^2 \mathrm{d}\xi \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\overline{\Omega})$ and $u_\varepsilon \rightharpoonup u$ in S_0^1. Clearly, by the Sobolev embedding, we also get $u_\varepsilon \rightharpoonup u$ in $L^{2^*}(\Omega)$. By Fatou's Lemma, we can then deduce that $\mu \geq |D_H u|^2 \mathrm{d}\xi$, so that we can always decompose μ as follows,

$$\mu = |D_H u|^2 \mathrm{d}\xi + \tilde{\mu} + \sum_{j=1}^{\infty} \mu_j \delta_{\xi_j},$$

where $\mu_j \in [0,1]$ and $\{\xi_j\} \subseteq \overline{\Omega}$ are distinct points; the positive measure $\tilde{\mu}$ can be viewed as the *non-atomic part* of the measure $(\mu - |D_H u|^2 \mathrm{d}\xi)$. In view of such a decomposition, the definition of the space X given in Theorem 4.1 is extremally natural. Also, the space X is sequentially compact in the topology $\boldsymbol{t}$. Indeed, if we take a sequence $\{(u_k, \mu_k)\} \subseteq X$, then $\{u_k\}$ is bounded in $S_0^1(\Omega)$. Up to subsequences, $\mu_k \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\overline{\Omega})$ and $u_k \rightharpoonup u$ in $S_0^1(\Omega)$ (and in $L^{2^*}(\Omega)$ by the Sobolev embedding) and the inequalities defining X do still hold for (u, μ) by weak lower semicontinuity. In view of the previous analysis, the space X appears as a sort of completion of the unit ball of $S_0^1(\Omega)$ in the weak topology of the product $L^{2^*}(\Omega) \times \mathcal{M}(\overline{\Omega})$.

We now recall the definition of Γ^+-convergence adapted to our specific framework.

Definition 4.6. *We say that the family $\{\mathcal{F}_\varepsilon\}$ Γ^+-converges to a functional $\mathcal{F} : X \to [0,\infty)$ as $\varepsilon \to 0$ if for every $(u,\mu) \in X$ the following conditions hold:*

(i) *for every sequence $\{(u_\varepsilon, \mu_\varepsilon)\} \subset X$ such that $u_\varepsilon \rightharpoonup u$ in $L^{2^*}(\Omega)$ and $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\overline{\Omega})$*

$$\mathcal{F}(u,\mu) \geq \limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u_\varepsilon, \mu_\varepsilon);$$

(ii) *there exists a sequence* $\{\bar u_\varepsilon, \bar\mu_\varepsilon)\} \subset X$ *such that* $\bar u_\varepsilon \rightharpoonup u$ *in* $L^{2^*}(\Omega)$, $\bar\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ *in* $\mathcal{M}(\overline\Omega)$ *and*

$$\mathcal{F}(u,\mu) \le \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(\bar u_\varepsilon, \bar\mu_\varepsilon).$$

2.2. Proof of the Γ^+-limsup inequality. The Γ^+-limsup inequality (i) follows from the Concentration-compactness alternative via plain application of the Hölder inequality; we have the following

Proposition 4.7. *Let* $\{\mathcal{F}_\varepsilon\}$ *be the family defined in* (4.2)*, and let* $\mathcal{F}$ *be the functional defined in* (4.3)*. Then, for every* $(u,\mu) \in X$ *and for every sequence* $\{(u_\varepsilon,\mu_\varepsilon)\} \subset X$ *such that* $(u_\varepsilon,\mu_\varepsilon) \overset{\boldsymbol{t}}{\to} (u,\mu)$,

$$\mathcal{F}(u,\mu) \ge \limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon,\mu_\varepsilon).$$

Proof. Let $\{(u_\varepsilon,\mu_\varepsilon)\}$ be a sequence in X such that $(u_\varepsilon,\mu_\varepsilon) \overset{\boldsymbol{t}}{\to} (u,\mu)$. This yields $\mu = |D_H u|^2 \mathrm{d}\xi + \tilde\mu + \sum_j \mu_j \delta_{\xi_j}$, for some $\tilde\mu \in \mathcal{M}_+(\overline\Omega)$, $\{\mu_j\} \subseteq (0,1]$ and $\{\xi_j\} \subseteq \mathbb{H}^n$. Then, up to subsequences, there exists a measure $\nu \in \mathcal{M}(\overline\Omega)$ such that $|u_\varepsilon|^{2^*}\mathrm{d}\xi \overset{*}{\rightharpoonup} \nu$, and by the Concentration-compactness alternative in Theorem 4.4 there exist nonnegative numbers $\{\nu_j\}$ such that (up to reordering the points $\{\xi_j\}$ and the $\{\mu_j\}$)

$$\nu = |u|^{2^*}\mathrm{d}\xi + \sum_j \nu_j \delta_{\xi_j} \quad \text{and} \quad \nu_j \le S^* \mu_j^{\frac{2^*}{2}}. \tag{4.9}$$

We now apply the Hölder inequality, which yields

$$\mathcal{F}_\varepsilon(u_\varepsilon,\mu_\varepsilon) \;\equiv\; \int_\Omega |u_\varepsilon|^{2^*-\varepsilon}\mathrm{d}\xi \;\le\; \left(\int_\Omega |u_\varepsilon|^{2^*}\mathrm{d}\xi\right)^{\frac{2^*-\varepsilon}{2^*}} |\Omega|^{\frac{\varepsilon}{2^*}}.$$

Thus, in view of the definition of ν and the decomposition (4.9) in Theorem 4.4, we have

$$\begin{aligned}\limsup_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon,\mu_\varepsilon) \;&\le\; \limsup_{\varepsilon\to 0}\left(\int_\Omega |u_\varepsilon|^{2^*}\mathrm{d}\xi\right)^{\frac{2^*-\varepsilon}{2^*}} |\Omega|^{\frac{\varepsilon}{2^*}} \;\le\; \nu(\overline\Omega)\\ &\le\; \int_\Omega |u|^{2^*}\mathrm{d}\xi + S^*\sum_{i=1}^\infty \mu_i^{\frac{2^*}{2}} \;\le\; \mathcal{F}(u,\mu),\end{aligned}$$

which gives the desired Γ^+-limsup inequality (i). □

2.3. Proof of the Γ^+-liminf inequality. The proof of the Γ^+-liminf inequality (ii) is extremely delicate, and it differs considerably from the Euclidean case (see, e. g., [3, 104]) where it has been proven via compactness and locality properties of the Γ^+-limit. Here, we are closer to the strategy presented in the fractional framework in [113], though in the various steps of our proof we need to deal with the Heisenberg framework, and we cannot make use of some fractional workarounds and well-established results for the fractional optimal maximizers.

For any fixed $N \in \mathbb{N}$, the proof will be performed in the auxiliary space of configurations $X_N \subseteq X$ defined as follows,

$$X_N := \left\{ (u,\mu) \in S^1_0(\Omega) \times \mathcal{M}(\overline\Omega) \,:\, \mu = |D_H u|^2 \mathrm{d}\xi + \tilde\mu + \sum_{j=1}^N \mu_j \delta_{\xi_j},\ \mu(\overline\Omega) < 1 \right\}. \tag{4.10}$$

Since X_N is $\boldsymbol{t}$-sequentially dense in X by a precise approximation, and $\mathcal{F}$ is continuous with respect to such an approximation – see the computations below – the Γ^+-liminf inequality will immediately follow by a standard diagonal argument.

Indeed, for any pair $(u,\mu)\in X$, it suffices to consider the sequence $\{(u_N,\mu_N)\}$ defined as follows,

$$u_N := a_N u \quad \text{and} \quad \mu_N := a_N^2|D_H u|^2\mathrm{d}\xi + a_N^2\tilde{\mu} + a_N^2\sum_{j=1}^N \mu_j\delta_{\xi_j}\,,$$

where $\{a_N\}\subset(0,1)$ is any increasing sequence such that $a_N\to 1$ as $N\to\infty$.

By definition, the sequence $\{(u_N,\mu_N)\}$ belongs to X_N, since, for any $N\in\mathbb{N}$, $u_N\in S^1_0(\Omega)$ and μ_N is a measure with a finite number of atoms such that $\mu_N(\overline{\Omega}) \le a_N^2\,\mu(\overline{\Omega}) \le a_N^2 < 1$. Also, $(u_N,\mu_N)\overset{\boldsymbol{t}}{\to}(u,\mu)$ as $N\to\infty$, because $u_N\to u$ in $S^1_0(\Omega)$ (hence weakly in $L^{2^*}(\Omega)$), and, for any $\phi\in C^0_0(\overline{\Omega})$,

$$\begin{aligned}\int_{\overline{\Omega}}\phi\mathrm{d}\mu_N &= \int_\Omega \phi a_N^2|D_H u|^2\mathrm{d}\xi + \int_\Omega \phi a_N^2\mathrm{d}\tilde{\mu} + a_N^2\sum_{j=1}^N\mu_j\phi(\xi_j)\\ &= a_N^2\left(\int_\Omega \phi|D_H u|^2\mathrm{d}\xi + \int_\Omega\phi\mathrm{d}\tilde{\mu} + \sum_{j=1}^N\mu_j\phi(\xi_j)\right) \overset{N\to\infty}{\longrightarrow} \int_\Omega\phi\mathrm{d}\mu\,.\end{aligned}$$

With such an approximation in mind, we also have

$$\begin{aligned}\mathcal{F}(u_N,\mu_N) &= \int_\Omega a_N^{2^*}|u|^{2^*}\mathrm{d}\xi + S^*\sum_{j=1}^N(a_N^2\mu_j)^{\frac{2^*}{2}}\\ &= a_N^{2^*}\left(\int_\Omega|u|^{2^*}\mathrm{d}\xi + S^*\sum_{j=1}^N\mu_j^{\frac{2^*}{2}}\right) \to \mathcal{F}(u,\mu), \quad \text{as } n\to\infty.\end{aligned}$$

We are in the position to perform the Γ^+-liminf inequality in the auxiliary space X_N; i. e.,

Proposition 4.8. *Let $\{\mathcal{F}_\varepsilon\}$ be the family defined in (4.2), and let $\mathcal{F}$ be the functional defined in (4.3). Then, for every (u,μ) in the auxiliary space X_N defined in (4.10) there exists a sequence $\{(\bar{u}_\varepsilon,\bar{\mu}_\varepsilon)\}\subset X$ such that $(\bar{u}_\varepsilon,\bar{\mu}_\varepsilon)\overset{\boldsymbol{t}}{\to}(u,\mu)$ and*

$$\mathcal{F}(u,\mu)\le\liminf_{\varepsilon\to 0}\mathcal{F}_\varepsilon(\bar{u}_\varepsilon,\bar{\mu}_\varepsilon). \tag{4.11}$$

The proof consists of the following steps:

- For any point $\xi_j\in\overline{\Omega}$, we construct a sequence $\{u_\varepsilon^{(j)}\}$ that concentrates energy at ξ_j; see Lemma 4.9.
- For any pair $(0,\mu)$ such that $\mu=\sum_j\mu_j$ is purely atomic, we show how to glue the concentrating sequences of the previous step into an approximating sequence

  $$\{(u_\varepsilon^{(\Sigma)},|D_H u_\varepsilon^{(\Sigma)}|^2\mathrm{d}\xi)\}_\varepsilon.$$

  See Lemma 4.10 below.
- We present a cut-off argument in order to finally construct the desired recovery sequence $(\bar{u}_\varepsilon,\bar{\mu}_\varepsilon)$ for any pair (u,μ) in the auxiliary space X_N.

Lemma 4.9. *For any $\xi_j \in \overline{\Omega}$ there exists a sequence $\{u_\varepsilon^{(j)}\} \subset S_0^1(\Omega)$ such that*

$$X \supseteq \left\{(u_\varepsilon^{(j)}, |D_H u_\varepsilon^{(j)}|^2 \mathrm{d}\xi)\right\} \ \boldsymbol{t}\text{-converges to } (0, \delta_{\xi_j}) \text{ as } \varepsilon \to 0. \tag{4.12}$$

Moreover, the sequence $\{u_\varepsilon^{(j)}\}$ satisfies

$$\lim_{\varepsilon\to 0} \int_\Omega |u_\varepsilon^{(j)}|^{2^*-\varepsilon} \, \mathrm{d}\xi = S^*, \tag{4.13}$$

with S^* being the best Sobolev constant in $\mathbb{H}^n$.

Proof. We begin by assuming that ξ_j belongs to the interior of Ω, and we modify the extremal functions U given by Theorem 4.3. Define

$$w_\varepsilon(\xi) := \varepsilon^{-\frac{Q-2}{2}} U_{\varepsilon,\xi_j}(\xi), \qquad \forall \xi \in \mathbb{H}^n.$$

It follows by the very definition that, for any $\varrho > 0$, the function w_ε satisfies

$$w_\varepsilon \to 0 \quad \text{in } L^{2^*}(\mathbb{H}^n \setminus \overline{B_\varrho(\xi_j)}) \tag{4.14}$$

and

$$|D_H w_\varepsilon|^2 \to 0 \quad \text{in } L^1(\mathbb{H}^n \setminus \overline{B_\varrho(\xi_j)}) \tag{4.15}$$

whenever $\varepsilon \to 0$. Indeed, a change of variables yields

$$\begin{aligned}
\int_{\mathbb{H}^n\setminus\overline{B_\varrho(\xi_j)}} |w_\varepsilon|^{2^*} \, \mathrm{d}\xi &= \int_{\mathbb{H}^n\setminus\overline{B_\varrho(\xi_j)}} \varepsilon^{-Q} \left|U(\delta_{\frac{1}{\varepsilon}}(\xi_j^{-1}\circ\xi))\right|^{2^*} \, \mathrm{d}\xi \\
&= \int_{\mathbb{H}^n\setminus\overline{B_\varrho(0)}} \varepsilon^{-Q} \left|U\left(\delta_{\frac{1}{\varepsilon}}(\eta)\right)\right|^{2^*} \, \mathrm{d}\eta \\
&= \int_{\mathbb{H}^n\setminus\overline{B_{\frac{\varrho}{\varepsilon}}(0)}} \left|U(\eta')\right|^{2^*} \, \mathrm{d}\eta' \xrightarrow{\varepsilon\to 0} 0.
\end{aligned}$$

In a similar way, the convergence $|D_H w_\varepsilon|^2 \to 0$ in $L^1(\mathbb{H}^n \setminus \overline{B_\varrho(\xi_j)})$ follows by a change of variables and recalling the one δ_λ-homogeneity of the horizontal gradient.

Fix $\varrho > 0$ and take $\varphi \in C_0^\infty(\mathbb{H}^n)$ such that $\varphi \equiv 0$ on $\mathbb{H}^n \setminus B_{2\varrho}(\xi_j)$, $\varphi \equiv 1$ on $B_\varrho(\xi_j)$ and $0 \le \varphi \le 1$. We use the cut-off function φ to localize w_ε in a smaller neighborhood of ξ_j. In particular, we set

$$\tilde{u}_\varepsilon^{(j)}(\xi) := \varphi(\xi) w_\varepsilon(\xi)\,,$$

and we claim that as $\varepsilon \to 0$ the follow convergence results do hold,

$$\tilde{u}_\varepsilon^{(j)} \rightharpoonup 0 \quad \text{in } L^{2^*}(\Omega)\,, \tag{4.16}$$

$$\|D_H \tilde{u}_\varepsilon^{(j)}\|_{L^2(\mathbb{H}^n)} \to 1\,, \tag{4.17}$$

and

$$\int_\Omega |\tilde{u}_\varepsilon^{(j)}|^{2^*-\varepsilon} \, \mathrm{d}\xi \to S^*. \tag{4.18}$$

The first convergence in (4.16) plainly follows from (4.14). Indeed, for $\phi \in L^{(2^*)'}(\Omega)$, we have

$$\int_\Omega \phi \tilde{u}_\varepsilon^{(j)} \, \mathrm{d}\xi = \int_{B_\varrho(\xi_j)} \phi \tilde{u}_\varepsilon^{(j)} \, \mathrm{d}\xi + \int_{\Omega\setminus B_\varrho(\xi_j)} \phi \tilde{u}_\varepsilon^{(j)} \, \mathrm{d}\xi, \tag{4.19}$$

where the first integral can be estimated as follows,

$$\int_{B_\varrho(\xi_j)} |\phi \tilde{u}_\varepsilon^{(j)}| \, \mathrm{d}\xi = \varepsilon^{-\frac{Q-2}{2}} \int_{B_\varrho(\xi_j)} |\phi(\xi) U_{\varepsilon,\xi_j}(\xi)| \, \mathrm{d}\xi$$

$$\leq \varepsilon^{\frac{Q+2}{2}} \|\phi\|_{L^{(2^*)'}(\Omega)} \|U\|_{L^{2^*}(\mathbb{H}^n)}$$

so that it converges to 0 as $\varepsilon \to 0$. The second integral in (4.19) can be treated in a similar fashion via the Hölder inequality and the convergence in (4.15).

In order to prove the claim in (4.17), it suffices to carefully estimate the horizontal Dirichlet energy; we have

$$\begin{aligned}\int_{\mathbb{H}^n} |D_H \tilde{u}_\varepsilon^{(j)}|^2 \,\mathrm{d}\xi &= \int_{B_{2\varrho}(\xi_j)\setminus B_\varrho(\xi_j)} |D_H \tilde{u}_\varepsilon^{(j)}|^2 \,\mathrm{d}\xi + \int_{B_\varrho(\xi_j)} |D_H w_\varepsilon|^2 \,\mathrm{d}\xi \\ &= \int_{B_{2\varrho}(\xi_j)\setminus B_\varrho(\xi_j)} |D_H \tilde{u}_\varepsilon^{(j)}|^2 \,\mathrm{d}\xi + \int_{B_{\frac{\varrho}{\varepsilon}}(0)} |D_H U(\xi)|^2 \,\mathrm{d}\xi\,,\end{aligned}$$

where the last identity plainly follows in view of the δ_λ-homogeneity of the horizontal gradient. Hence, (4.17) is satisfied once we proved that the first integral in the preceding inequality goes to 0 whenever $\varepsilon \to 0$. We have

$$\begin{aligned}&\int_{B_{2\varrho}(\xi_j)\setminus B_\varrho(\xi_j)} |D_H \tilde{u}_\varepsilon^{(j)}|^2 \,\mathrm{d}\xi \\ &\leq 2\|\varphi\|_{L^\infty}^2 \int_{B_{2\varrho}(\xi_j)\setminus B_\varrho(\xi_j)} |D_H w_\varepsilon|^2 \,\mathrm{d}\xi + 2\|D_H\varphi\|_{L^\infty}^2 \int_{B_{2\varrho}(\xi_j)\setminus B_\varrho(\xi_j)} |w_\varepsilon|^2 \,\mathrm{d}\xi \\ &\leq c\int_{\mathbb{H}^n\setminus B_\varrho(\xi_j)} |D_H w_\varepsilon|^2 \,\mathrm{d}\xi + c\varepsilon^2 \int_{\mathbb{H}^n\setminus B_{\frac{\varrho}{\varepsilon}}(0)} |U|^2 \,\mathrm{d}\xi \xrightarrow{\varepsilon\to 0} 0.\end{aligned}$$

As for the claim in (4.18), it is convenient to split the integral as follows,

$$\begin{aligned}\int_\Omega |\tilde{u}_\varepsilon^{(j)}|^{2^*-\varepsilon} \,\mathrm{d}\xi &= \int_{\Omega\cap\{w_\varepsilon<1\}} |\tilde{u}_\varepsilon^{(j)}|^{2^*-\varepsilon} \,\mathrm{d}\xi + \int_{\Omega\cap\{w_\varepsilon\geq 1\}} |\tilde{u}_\varepsilon^{(j)}|^{2^*-\varepsilon} \,\mathrm{d}\xi \\ &=: I_{1,\varepsilon} + I_{2,\varepsilon}. \qquad (4.20)\end{aligned}$$

Firstly, note that $|\varphi(\xi)w_\varepsilon(\xi)|^{2^*-\varepsilon} < 1$ on $\Omega\cap\{w_\varepsilon < 1\}$ uniformly with respect to ε, and $\varphi(\xi)w_\varepsilon(\xi) = \mathrm{o}(\varepsilon^{Q-2})$ as $\varepsilon \to 0$, since in $\Omega\cap\{w_\varepsilon < 1\}$ it is away from the concentration point. This yields

$$I_{1,\varepsilon} \to 0 \text{ as } \varepsilon \to 0\,. \qquad (4.21)$$

Now, recall that for ε sufficiently small, $\Omega\cap\{w_\varepsilon \geq 1\} \subset B_\varrho(\xi_j)$ and

$$\frac{\varphi^{2^*-\varepsilon}}{w_\varepsilon^\varepsilon} - 1 = \frac{1}{w_\varepsilon^\varepsilon} - 1\,. \qquad (4.22)$$

Also, on $\Omega\cap\{w_\varepsilon \geq 1\}$ we have

$$1 \leq w_\varepsilon^\varepsilon \leq (\max w_\varepsilon)^\varepsilon \leq c_0^\varepsilon \varepsilon^{-\varepsilon\frac{Q-2}{2}} \xrightarrow{\varepsilon\to 0} 1\,. \qquad (4.23)$$

We are finally ready to estimate the integral $I_{2,\varepsilon}$ in (4.20); we have

$$\begin{aligned}I_{2,\varepsilon} &\leq \left\| \frac{\varphi^{2^*-\varepsilon}}{w_\varepsilon^\varepsilon} - 1 \right\|_{L^\infty(\{w_\varepsilon\geq 1\})} \int_{\Omega\cap\{w_\varepsilon\geq 1\}} |w_\varepsilon|^{2^*} \,\mathrm{d}\xi \\ &\quad + \int_{\Omega\cap\{w_\varepsilon\geq 1\}} |w_\varepsilon|^{2^*} \,\mathrm{d}\xi \xrightarrow{\varepsilon\to 0} S^*, \qquad (4.24)\end{aligned}$$

where we used (4.22)-(4.23) and also the definition of w_ε in view of the optimality of the extremal functions U in Theorem 4.3.

Passing to the limit as $\varrho \to 0$, by a diagonal argument, for any $\varepsilon > 0$, we now set $u_\varepsilon^{(j)} := \tilde{u}_\varepsilon^{(j)} / \|D_H \tilde{u}_\varepsilon^{(j)}\|_{L^2(\mathbb{H}^n)}$. Clearly, it holds that the pair $(u_\varepsilon^{(j)}, |D_H u_\varepsilon^{(j)}|^2 \mathrm{d}\xi)$

belongs to the space X. Indeed,

$$(u_\varepsilon^{(j)}, |D_H u_\varepsilon^{(j)}|^2 \mathrm{d}\xi) \in S_0^1(\Omega) \times \mathcal{M}(\overline{\Omega}), \quad \text{and} \quad \int_{\overline{\Omega}} |D_H u_\varepsilon^{(j)}|^2 \mathrm{d}\xi \leq 1.$$

All in all, combining (4.16)-(4.18) and (4.20) with (4.21) and (4.24), the desired Γ^+-liminf result does follow in the case when ξ_j is an interior point.

The case when ξ_j belongs to $\partial\Omega$ can be recovered by considering an approximating sequence of interior points ξ_j^k converging to ξ_j and their corresponding optimal functions. □

We are now ready to construct a suitable recovery sequence for pairs $(0, \mu)$ having a purely atomic measure component.

Lemma 4.10. *For any finite set of distinct points $\{\xi_1, \xi_2, ..., \xi_N\} \subset \overline{\Omega}$ and for any set of positive numbers $\{\mu_1, \mu_2, ..., \mu_N\} \subseteq \mathbb{R}$ such that $\sum_j \mu_j < 1$, there exists a sequence $\{u_\varepsilon^{(\Sigma)}\} \subset S_0^1(\Omega)$ such that*

$$X \supseteq \left\{ \left(u_\varepsilon^{(\Sigma)}, |D_H u_\varepsilon^{(\Sigma)}|^2 \mathrm{d}\xi \right) \right\} \ \boldsymbol{t}\text{-converges to } \left(0, \sum_{j=1}^N \mu_j \delta_{\xi_j} \right) \ \text{as } \varepsilon \to 0. \tag{4.25}$$

Moreover, the sequence $\{u_\varepsilon^{(\Sigma)}\}$ satisfies

$$\lim_{\varepsilon \to 0} \int_\Omega |u_\varepsilon^{(\Sigma)}|^{2^* - \varepsilon} \, \mathrm{d}\xi = S^* \sum_{j=1}^N \mu_j^{\frac{2^*}{2}} . \tag{4.26}$$

Proof. Put $B_j := B_{r_j}(\xi_j) \cap \Omega$ for any $j = 1, 2, \ldots, N$, for suitable radii r_j and r_i such that $\mathrm{dist}(B_j, B_i) > 0$. By Lemma 4.9 there exists a sequence $\{u_\varepsilon^{(j)}\} \subset S_0^1(\Omega)$ such that $(u_\varepsilon^{(j)}, |D_H u_\varepsilon^{(j)}|^2 \mathrm{d}\xi) \overset{\boldsymbol{t}}{\to} (0, \delta_{\xi_j})$ as $\varepsilon \to 0$. Moreover, from the proof of Lemma 4.9 we immediately deduce that $\mathrm{spt}\, u_\varepsilon^{(j)} \subset B_j$, $\mathrm{dist}\Big(\mathrm{supp}\, u_\varepsilon^{(j)}, \{\xi_j\}\Big) \to 0$ as $\varepsilon \to 0$, and

$$\int_\Omega |u_\varepsilon^{(j)}|^{2^* - \varepsilon} \xrightarrow{\varepsilon \to 0} S^* \quad \text{for } j = 1, 2, \ldots, N.$$

Let us set $u_\varepsilon^{(\Sigma)} := \sum_{j=1}^N \sqrt{\mu_j} u_\varepsilon^{(j)}$. A simple computation gives

$$\begin{aligned} \int_{\mathbb{H}^n} |D_H u_\varepsilon^{(\Sigma)}|^2 \mathrm{d}\xi &= \sum_{j=1}^N \mu_j \int_{\mathbb{H}^n} |D_H u_\varepsilon^{(j)}|^2 \mathrm{d}\xi + 2 \sum_{\substack{i,j=1 \\ i<j}}^N \sqrt{\mu_i \mu_j} \int_{\mathbb{H}^n} D_H u_\varepsilon^{(i)} D_H u_\varepsilon^{(j)} \, \mathrm{d}\xi \\ &= \sum_{j=1}^N \mu_j \int_{\mathbb{H}^n} |D_H u_\varepsilon^{(j)}|^2 \mathrm{d}\xi, \end{aligned} \tag{4.27}$$

since $\int D_H u_\varepsilon^{(i)} D_H u_\varepsilon^{(j)} \, \mathrm{d}\xi = 0$ for $i \neq j$, being $u_\varepsilon^{(i)}$ and $u_\varepsilon^{(j)}$ with disjoint support whenever $i \neq j$. Combining (4.27) with the convergence in (4.12), we deduce that

$$|D_H u_\varepsilon^{(j)}|^2 \mathrm{d}\xi \overset{*}{\rightharpoonup} \sum_{j=1}^N \mu_j \delta_{\xi_j} \ \text{in } \mathcal{M}(\overline{\Omega}).$$

Finally, since $\sum_j \mu_j < 1$, we deduce by (4.27) that $\int_{\mathbb{H}^n} |D_H u_\varepsilon^{(\Sigma)}|^2 \mathrm{d}\xi \leq 1$ for ε small enough. Consequently, $\{(u_\varepsilon^{(\Sigma)}, |D_H u_\varepsilon^{(\Sigma)}|^2 \mathrm{d}\xi)\} \subset X$, as required. This completes

the proof of (4.25). The last claim given in (4.26) is a plain consequence of (4.13); we have

$$\int_\Omega |u_\varepsilon^{(\Sigma)}|^{2^*-\varepsilon}\,\mathrm{d}\xi \;=\; \sum_{j=1}^N \mu_j^{\frac{2^*-\varepsilon}{2}} \int_\Omega |u_\varepsilon^{(j)}|^{2^*-\varepsilon}\,\mathrm{d}\xi \;\xrightarrow{\varepsilon\to 0}\; \sum_{j=1}^N \mu_j^{\frac{2^*}{2}} S^*\,. \qquad \square$$

Proof of Proposition 4.8. Fix $(u,\mu)\in X_N$; i. e., $u\in L^{2^*}(\Omega)$ and

$$\mu = |D_H u|^2\mathrm{d}\xi + \tilde\mu + \sum_{j=1}^N \mu_j \delta_{\xi_j} \in \mathcal{M}(\overline\Omega),$$

such that $\mu(\overline\Omega) < 1$, and consider radius ϱ converging to 0 such that the balls $B_{2\varrho}(\xi_j)\cap B_{2\varrho}(\xi_i)=\emptyset$, for any $i,j\in\{1,\dots,N\}$ with $i\neq j$. Denote with

$$B_{\varrho,j} := B_\varrho(\xi_j)\cap\overline\Omega \quad\text{and}\quad \varphi_\varrho(\xi) = 1 - \sum_{j=1}^N \psi\Big(\delta_{\frac{1}{\varrho}}(\tau_{\xi_j}^{-1}(\xi))\Big),$$

where $\psi\in C_0^\infty(B_2)$, $\psi\equiv 1$ on $\overline{B_1}$, $0\le\psi\le 1$, with $B_2\equiv B_2(0)$. Note that $\varphi_\varrho\equiv 0$ in $\overline{B_{\varrho,j}}$, for $j=1,\dots,N$, and $\varphi_\varrho\equiv 1$ in $\overline\Omega\setminus\bigcup_{j=1}^N B_{2\varrho,j}$.

Now denote with $u_\varepsilon^{(\Sigma)}$ the sequence given by Lemma 4.10 and define

$$\bar u_\varepsilon := u\varphi_\varrho + u_\varepsilon^{(\Sigma)}, \qquad \bar\mu_\varepsilon := \tilde\mu + |D_H(u\varphi_\varrho + u_\varepsilon^{(\Sigma)})|^2\mathrm{d}\xi.$$

Taking first the limit as $\varepsilon\to 0$ and then for $\varrho\to 0$, by a diagonal argument, we prove that the above sequence is a recovery sequence for (u,μ).

First we prove that $(\bar u_\varepsilon,\bar\mu_\varepsilon)$ for ε and ϱ sufficiently small belongs to the space X. By construction we are only left to prove that

$$\bar\mu_\varepsilon(\overline\Omega)\le 1. \tag{4.28}$$

Indeed, for any $\varepsilon,\varrho>0$ we have that

$$\begin{aligned}\bar\mu_\varepsilon(\overline\Omega) &= \tilde\mu(\overline\Omega) + \int_\Omega |D_H(u\varphi_\varrho + u_\varepsilon^{(\Sigma)})|^2\mathrm{d}\xi\\ &= \tilde\mu(\overline\Omega) + \int_\Omega |D_H(u\varphi_\varrho)|^2\mathrm{d}\xi + \int_\Omega |D_H u_\varepsilon^{(\Sigma)}|^2\mathrm{d}\xi,\end{aligned}$$

where we have used the fact that $\int_\Omega D_H u\varphi_\varrho\cdot D_H u_\varepsilon^{(\Sigma)}\,\mathrm{d}\xi = 0$, for ε sufficiently small. Moreover, by Lemma 4.10 we have that

$$\lim_{\varepsilon\to 0}\int_\Omega |D_H u_\varepsilon^{(\Sigma)}|^2\,\mathrm{d}\xi = \sum_{j=1}^N \mu_j. \tag{4.29}$$

On the other hand, we have that

$$\begin{aligned}\int_\Omega |D_H(u\varphi_\varrho)|^2\,\mathrm{d}\xi &= \int_{\Omega\setminus\bigcup_j B_{2\varrho,j}} |D_H u|^2\,\mathrm{d}\xi + \sum_{j=1}^N \int_{B_{2\varrho,j}\setminus B_{\varrho,j}} |D_H(u\varphi_\varrho)|^2\,\mathrm{d}\xi\\ &\to \int_{\overline\Omega} |D_H u|^2\,\mathrm{d}\xi \text{ as } \varrho\to 0\,,\end{aligned} \tag{4.30}$$

since for any $j\in\{1,\dots,N\}$ the integral $\int_{B_{2\varrho,j}\setminus B_{\varrho,j}}|D_H(u\varphi_\varrho)|^2\,\mathrm{d}\xi$ tends to 0 with respect to ϱ.
Indeed, since $\psi\in C_0^\infty(B_2)$, we have

$$\begin{aligned}&\int_{B_{2\varrho,j}\setminus B_{\varrho,j}} |D_H(u\varphi_\varrho)|^2\,\mathrm{d}\xi\\ &\qquad\le 2\|D_H u\|^2_{L^2(B_{2\varrho,j}\setminus B_{\varrho,j})} + 2\varrho^{\frac{Q-2}{2}}\|u\|_{L^2(\Omega)}\|D_H\psi\|_{L^2(B_2\setminus B_1)}\,,\end{aligned}$$

which goes to 0 as $\varrho \to 0$. It is now sufficient to combine (4.29) with (4.30) to get that (4.28) holds true.

We show now that $\{(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon)\}$ $\boldsymbol{t}$-converges to (u, μ); i. e.,

$$\bar{u}_\varepsilon \rightharpoonup u \quad \text{in } L^{2^*}(\Omega) \quad \text{and} \quad \bar{\mu}_\varepsilon \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(\overline{\Omega}). \tag{4.31}$$

Note that $(4.31)_1$ follows from the fact that $u_\varepsilon^{(\Sigma)} \rightharpoonup 0$ in $L^{2^*}(\Omega)$, as shown in Lemma 4.10, and $u\varphi_\varrho$ converges (strongly) to u as $\varrho \to 0$. Indeed, on Ω we have that $|u\varphi_\varrho - u|^{2^*} = |1 - \varphi_\varrho|^{2^*}|u|^{2^*} \leq |u|^{2^*}$, thus by Lebesgue's Dominated Convergence Theorem we obtain that $u\varphi_\varrho \to u$ in $L^{2^*}(\Omega)$.

The second convergence in $(4.31)_2$ follows also from Lemma 4.10. Indeed, once fixed $\phi \in C_0^0(\overline{\Omega})$, we get that

$$\begin{aligned}
\lim_{\varrho\to 0}\lim_{\varepsilon\to 0}\int_{\overline{\Omega}} \phi \,\mathrm{d}\bar{\mu}_\varepsilon &= \int_{\overline{\Omega}} \phi \,\mathrm{d}\tilde{\mu} + \lim_{\varrho\to 0}\lim_{\varepsilon\to 0}\int_\Omega \phi\, |D_H(u\varphi_\varrho + u_\varepsilon^{(\Sigma)})|^2 \mathrm{d}\xi \\
&= \int_{\overline{\Omega}} \phi \,\mathrm{d}\tilde{\mu} + \lim_{\varrho\to 0}\int_\Omega \phi\, |D_H(u\varphi_\varrho)|^2 \mathrm{d}\xi \\
&\quad + \lim_{\varepsilon\to 0}\int_\Omega \phi\, |D_H u_\varepsilon^{(\Sigma)}|^2 \mathrm{d}\xi \\
&= \int_{\overline{\Omega}} \phi \,\mathrm{d}\tilde{\mu} + \int_\Omega \phi\, |D_H u|^2 \mathrm{d}\xi + \sum_{j=1}^N \int_{\overline{\Omega}} \phi \mu_j \,\mathrm{d}\delta_{\xi_j} \\
&= \int_{\overline{\Omega}} \phi \,\mathrm{d}\mu,
\end{aligned}$$

where we also used (4.29) and (4.30).

It remains to prove the liminf inequality in (4.11). Firstly, note that the integral in the definition of $\mathcal{F}_\varepsilon(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon)$ can be splitted as follows

$$\mathcal{F}_\varepsilon(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon) = \int_\Omega |u\varphi_\varrho|^{2^*-\varepsilon} \,\mathrm{d}\xi + \int_\Omega |u_\varepsilon^{(\Sigma)}|^{2^*-\varepsilon} \,\mathrm{d}\xi\,,$$

Moreover, as in the proof of (4.31), we can deduce that

$$\lim_{\varrho\to 0}\lim_{\varepsilon\to 0}\int_\Omega |u\varphi_\varrho|^{2^*-\varepsilon} \,\mathrm{d}\xi = \int_\Omega |u|^{2^*} \,\mathrm{d}\xi.$$

By combining all the previous results, up to a diagonal argument, we finally get

$$\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon) = \int_\Omega |u|^{2^*} \,\mathrm{d}\xi + S^* \sum_{j=1}^N \mu_j^{\frac{2^*}{2}} \equiv \mathcal{F}(u, \mu)$$

as desired. □

3. Energy concentration of subcritical extremals

In this section we will prove Theorem 4.2 showing that, due to the Γ^+-convergence result, the maximizers $\{u_\varepsilon\}$ for the variational problem (4.8) concentrate energy at one point $\xi_o \in \overline{\Omega}$ when ε goes to zero.

Firstly, notice that, since the embeddings $S_0^1(\Omega) \hookrightarrow L^{2^*-\varepsilon}(\Omega)$ are compact, the functionals $\mathcal{F}_\varepsilon$ as extended to X by (4.2) are continuous. As a consequence, we will have that the Γ^+-convergence of functionals in this space implies the convergence of maximizers $\{(u_\varepsilon, \mu_\varepsilon)\}$ of $\mathcal{F}_\varepsilon$ to the maxima of $\mathcal{F}$. This will be the final step in the alternative proof of the concentration result presented at the end this section. The first step is in Lemma 4.11 below where we show that

there are no further maximizers in the space X other then the pairs of the form $(u_\varepsilon, |D_H u_\varepsilon|^2 \mathrm{d}\xi)$.

Lemma 4.11. *For any $\varepsilon > 0$, let $(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon) \in X$ be such that*

$$\sup_{(u,\mu)\in X} \mathcal{F}_\varepsilon(u,\mu) = \mathcal{F}_\varepsilon(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon).$$

Then $\bar{\mu}_\varepsilon = |D_H \bar{u}_\varepsilon|^2 \mathrm{d}\xi$.

Proof. The present proof is a plain generalization of the one presented in the *fractional* (Euclidean) framework in [113], since it is based on the intrinsic scaling properties of the involved functional, and the underlying geometry does not interfere.

Firstly, notice that the supremum is attained at some $(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon)$ because X is sequentially compact and $\mathcal{F}_\varepsilon$ is sequentially continuous (due to the compact embedding $S^1_0(\Omega) \hookrightarrow L^{2^*-\varepsilon}(\Omega)$). Clearly, it is not restrictive to assume that $\bar{\mu}_\varepsilon(\overline{\Omega}) = 1$. Indeed, if one has $\bar{\mu}_\varepsilon(\overline{\Omega}) < 1$, then it suffices to consider the pair $(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon/\bar{\mu}_\varepsilon(\overline{\Omega}))$ which still belongs to the space X, it satisfies $(\bar{\mu}_\varepsilon/\bar{\mu}_\varepsilon(\overline{\Omega}))(\overline{\Omega}) = 1$, and

$$\mathcal{F}_\varepsilon(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon/\bar{\mu}_\varepsilon(\overline{\Omega})) = \mathcal{F}_\varepsilon(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon) = \max_{(u,\mu)\in X} \mathcal{F}_\varepsilon(u,\mu).$$

Since $\bar{u}_\varepsilon \neq 0$, by the very definition of the space X, it follows $0 < \|D_H \bar{u}_\varepsilon\|_{L^2} \leq 1$. Thus, by setting

$$b = b(\varepsilon) := \frac{1}{\|\bar{u}_\varepsilon\|^2_{S^1_0(\Omega)}} \geq 1, \tag{4.32}$$

one can consider a new pair $(\tilde{u}_\varepsilon, \tilde{\mu}_\varepsilon)$ given as follows,

$$\tilde{u}_\varepsilon := b^{\frac{1}{2}} \bar{u}_\varepsilon \quad \text{and} \quad \tilde{\mu}_\varepsilon := b|D_H \bar{u}_\varepsilon|^2 \mathrm{d}\xi.$$

Notice that $(\tilde{u}_\varepsilon, \tilde{\mu}_\varepsilon)$ belongs to the space X and it satisfies

$$\mathcal{F}_\varepsilon(\tilde{u}_\varepsilon, \tilde{\mu}_\varepsilon) = b^{\frac{2^*-\varepsilon}{2}} \mathcal{F}_\varepsilon(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon) = b^{\frac{2^*-\varepsilon}{2}} \max_{(u,\mu)\in X} \mathcal{F}_\varepsilon(u,\mu). \tag{4.33}$$

So that (4.32) and (4.33) yield $b = 1$, $(\bar{u}_\varepsilon, |D_H \bar{u}_\varepsilon|^2 \mathrm{d}\xi)$ is a maximizer, and $\|\bar{u}_\varepsilon\|_{S^1_0(\Omega)} = 1$. Since $1 = \int |D_H \bar{u}_\varepsilon|^2 \mathrm{d}\xi \leq \bar{\mu}_\varepsilon(\overline{\Omega}) = 1$, one has $\bar{\mu}_\varepsilon = |D_H \bar{u}_\varepsilon|^2 \mathrm{d}\xi$. □

The second step consists in Lemma 4.12 below where we prove an optimal upper bound for the limit functional $\mathcal{F}$ on the space X. This will be the keypoint of the proof of the concentration result.

Lemma 4.12. *Let $\mathcal{F}$ be the functional defined in (4.3). Then, for every $(u,\mu) \in X$, we have*

$$\mathcal{F}(u,\mu) \leq S^*, \tag{4.34}$$

and the equality holds if and only if $(u,\mu) = (0, \delta_{\xi_o})$ for some $\xi_o \in \overline{\Omega}$.

Proof. We adapt the argument in the proof of [3, Lemma 3.6] for the Euclidean case in $H^1_0(\Omega)$, which essentially relies on the well-known "convexity trick".

For every $(u,\mu) \in X$, the Sobolev inequality yields

$$\mathcal{F}(u,\mu) \equiv \int_\Omega |u|^{2^*} \mathrm{d}\xi + S^* \sum_{j=1}^\infty \mu_j^{\frac{2^*}{2}} \leq S^* \left(\int_\Omega |D_H u|^2 \mathrm{d}\xi \right)^{\frac{2^*}{2}} + S^* \sum_{j=1}^\infty \mu_j^{\frac{2^*}{2}}.$$

Now, by the convexity of the function $t \mapsto t^{\frac{2^*}{2}} \equiv t^{1+\frac{1}{n}}$, for every fixed $n \geq 1$, it follows

$$\begin{aligned} \mathcal{F}(u,\mu) &\leq S^* \left(\int_\Omega |D_H u|^2 \mathrm{d}\xi \right)^{\frac{2^*}{2}} + S^* \sum_{j=1}^{\infty} \mu_j^{\frac{2^*}{2}} \\ &\leq S^* \left(\int_\Omega |D_H u|^2 \mathrm{d}\xi + \sum_{j=1}^{\infty} \mu_j \right)^{\frac{2^*}{2}} \leq S^* \Big(\mu(\overline{\Omega}) \Big)^{\frac{2^*}{2}} \leq S^* , \end{aligned} \tag{4.35}$$

which proves (4.34), and the equality clearly holds if $(u,\mu) = (0, \delta_{\xi_o})$, for some $\xi_o \in \overline{\Omega}$. Indeed, let assume that the equality in (4.34) holds for some pair $(u,\mu) \in X$. Then, each inequality in (4.35) is in fact an equality. This plainly implies that $\tilde{\mu} = 0$. If $u \neq 0$ then by convexity we also deduce that $\mu_j = 0$ for every j. In turn, this fact yields $\mu = |D_H u|^2 \mathrm{d}\xi$ and $u \in S_0^1(\Omega)$ is optimal in the Sobolev inequality, which contradicts Theorem 4.3. Thus, $u = 0$, the equation in (4.35) and the strict convexity implies that $\mu = \delta_{\xi_o}$ for some $\xi_o \in \overline{\Omega}$ as desired. □

We are finally in the position to prove the expected concentration result.

Proof of Theorem 4.2. As mentioned at the beginning of the present section, by Theorem 4.1 and standard Γ^+-convergence properties, it follows that every sequence of maximizers of $\mathcal{F}_\varepsilon$, which is in fact in the form $\{(u_\varepsilon, |D_H u_\varepsilon|^2 \mathrm{d}\xi)\}$ in view of Lemma 4.11, must converge (up to subsequences) to a pair $(u,\mu) \in X$ which is a maximum for $\mathcal{F}$; that is,

$$(u_\varepsilon, |D_H u_\varepsilon|^2 \mathrm{d}\xi) \overset{\boldsymbol{t}}{\to} (u,\mu), \qquad \text{with } \mathcal{F}(u,\mu) = \max_{X(\overline{\Omega})} \mathcal{F}.$$

Thanks to Lemma 4.12, we know that $\mathcal{F}(u,\mu) \leq S^*$ for every $(u,\mu) \in X$ and that the equality is achieved if and only if $(u,\mu) = (0, \delta_{\xi_o})$ for some $\xi_o \in \overline{\Omega}$. Hence, it follows that $(u_\varepsilon, |D_H u_\varepsilon|^2 \mathrm{d}\xi) \overset{\boldsymbol{t}}{\to} (0, \delta_{\xi_o})$, which is the desired concentration property for the energy density. □

CHAPTER 5

Struwe's Global Compactness in $\mathbb{H}^n$

In this chapter, we investigate a way to circumvent the lack of compactness in the critical Folland-Stein-Sobolev embedding, by proving the validity of the so-called Global Compactness in the Heisenberg framework.

Since the seminal paper [126] by Struwe, the celebrated Global Compactness in the Sobolev space H^1 has become a fundamental tool in Analysis. It has played a crucial role in numerous existence results, such as e. g. for ground states solutions for nonlinear Schrödinger equations, for solutions of Yamabe-type equations in conformal geometry, for prescribing Q-curvature problems, harmonic maps from Riemann surfaces into Riemannian manifolds, Yang-Mills connections over four-manifolds, planar Toda systems, etc. The involved literature is really too wide to attempt any reasonable account here. In Theorem 5.1 below, we state the counterpart of Struwe's Global Compactness in the Heisenberg group setting.

In order to precisely state our result, consider for any fixed $\lambda \in \mathbb{R}$ the problem,

$$-\Delta_H u - \lambda u - |u|^{2^*-2}u = 0 \qquad \text{in } (S_0^1(\Omega))', \tag{P_λ}$$

together with its corresponding Euler–Lagrange functional $\mathcal{E}_\lambda : S_0^1(\Omega) \to \mathbb{R}$ given by

$$\mathcal{E}_\lambda(u) = \frac{1}{2}\int_\Omega |D_H u|^2 \,\mathrm{d}\xi - \frac{\lambda}{2}\int_\Omega |u|^2 \,\mathrm{d}\xi - \frac{1}{2^*}\int_\Omega |u|^{2^*} \,\mathrm{d}\xi.$$

Consider also the following limiting problem,

$$-\Delta_H u - |u|^{2^*-2}u = 0 \qquad \text{in } (S_0^1(\Omega_o))', \tag{P_0}$$

where Ω_o is either a half-space or the whole $\mathbb{H}^n$; i. e., the Euler-Lagrange equation corresponding to the energy functional $\mathcal{E}^* : S_0^1(\Omega_o) \to \mathbb{R}$,

$$\mathcal{E}^*(u) = \frac{1}{2}\int_{\Omega_o} |D_H u|^2 \,\mathrm{d}\xi - \frac{1}{2^*}\int_{\Omega_o} |u|^{2^*} \,\mathrm{d}\xi.$$

We have the following

Theorem 5.1 (Global Compactness in the Heisenberg group)**.** *Let $\{u_k\} \subset S_0^1(\Omega)$ be a Palais-Smale sequence for $\mathcal{E}_\lambda$; i. e., such that*

$$\mathcal{E}_\lambda(u_k) \le c \quad \text{for all } k, \tag{5.1}$$

$$\mathrm{d}\mathcal{E}_\lambda(u_k) \to 0 \quad \text{as } k \to \infty \quad \text{in } (S_0^1(\Omega))'. \tag{5.2}$$

Then, there exists a (possibly trivial) solution $u^{(0)} \in S_0^1(\Omega)$ to (P_λ) such that, up to a subsequence, we have

$$u_k \rightharpoonup u^{(0)} \quad \text{as } k \to \infty \quad \text{in } S_0^1(\Omega).$$

Moreover, either the convergence is strong or there is a finite set of indexes $I = \{1, \dots, J\}$ such that for all $j \in I$ there exist a nontrivial solution $u^{(j)} \in S_0^1(\Omega_o^{(j)})$

to (P_0) *with* $\Omega_o^{(j)}$ *being either a half-space or the whole* $\mathbb{H}^n$*, a sequence of nonnegative numbers* $\{\lambda_k^{(j)}\}$ *converging to zero and a sequences of points* $\{\xi_k^{(j)}\} \subset \Omega$ *such that, for a renumbered subsequence, we have for any* $j \in I$

$$u_k^{(j)}(\cdot) := {\lambda_k^{(j)}}^{\frac{Q-2}{2}} u_k\Big(\tau_{\xi_k^{(j)}}\Big(\delta_{\lambda_k^{(j)}}(\cdot)\Big)\Big) \rightharpoonup u^{(j)}(\cdot) \quad \text{in } S_0^1(\mathbb{H}^n) \qquad \text{as } \to \infty.$$

In addition, as $k \to \infty$ *we have*

$$u_k(\cdot) = u^{(0)}(\cdot) + \sum_{j=1}^{J} {\lambda_k^{(j)}}^{\frac{2-Q}{2}} u_k\Big(\delta_{1/\lambda_k^{(j)}}\Big(\tau^{-1}_{\xi_k^{(j)}}(\cdot)\Big)\Big) + o(1) \qquad \text{in } S_0^1(\mathbb{H}^n); \tag{5.3}$$

$$\left|\log \frac{\lambda_k^{(i)}}{\lambda_k^{(j)}}\right| + \left|\delta_{1/\lambda_k^{(j)}}\Big({\xi_k^{(j)}}^{-1} \circ \xi_k^{(i)}\Big)\right|_{\mathbb{H}^n} \to \infty \quad \text{for } i \neq j, \;\; i,j \in I; \tag{5.4}$$

$$\|u_k\|^2_{S_0^1} = \sum_{j=0}^{J} \|u^{(j)}\|^2_{S_0^1} + o(1); \tag{5.5}$$

$$\mathcal{E}_\lambda(u_k) = \mathcal{E}_\lambda(u^{(0)}) + \sum_{j=1}^{J} \mathcal{E}^*(u^{(j)}) + o(1). \tag{5.6}$$

Before going into the proof of Theorem 5.1 some comments must be made. First of all, it is worth stressing that the limiting domain Ω_o in the Global Compactness Theorem above can be either the whole $\mathbb{H}^n$ or a half-space. On the contrary, in the original proof in the Euclidean case by Struwe ([126]) one can exclude the existence of nontrivial solutions to the limiting problems in the half-space by Pohozaev identity and unique continuation. This is still the case when dealing with *nonnegative* solutions in the fractional framework H_0^s; see [112]. However, in all the remaining Euclidean cases (as, e. g., in $W^{s,p}$, in H_0^2, and so on), this possibility cannot be a priori excluded. The same happens in the sub-Riemannian setting, even in the very special case when a complete characterization of the limiting sets is possible under further regularity assumptions on Ω (see forthcoming Lemma 5.5). In such a Heisenberg framework, a very few nonexistence results are known, basically only in the case when the domain reduces to a half-plane parallel or perpendicular to the group center; see [39]. Further nonexistence results are known in the class of groups of Iwasawa-type (see [67]).

Moreover, the original proof by Struwe in [126] consists in a subtle analysis concerning how the Palais-Smale condition does fail for the functional $\mathcal{E}^*$, based on rescaling arguments, used in an iterated way to extract convergent subsequences with nontrivial limit, together with some slicing and extension procedures on the sequence of approximate solutions to (P_λ). Such a proof revealed to be very difficult to extend to different frameworks, and the aforementioned strategy seems even more cumbersome to be adapted to the Heisenberg framework considered here. However, several remarkable investigations regarding the behavior of Palais-Smale sequences for the critical energy $\mathcal{E}^*(\cdot)$ can be still found in [36] where the author proves an analogous representation result for nonnegative Palais-Smale sequences in order to prove existence results of positive solutions to a class of semilinear Dirichlet problem involving critical growth, in the same spirit of the well-known Brezis-Nirenberg result for the classical Laplacian. Moreover, it is worth mentioning the relevant paper [71], where the authors deduce the desired Global Compactness in the important case of critical energies on the $(2n+1)$-dimensional sphere (equipped with the CR structure) associated to

the sub-elliptic intertwining operator $\mathcal{L}_{2\kappa}$ of order 2κ, with $\kappa \in \mathbb{R}$ being such that $0 < 2\kappa < Q$; also covering fractional CR Yamabe energies.

In the present manuscript, we completely changes the approach to the problem, and we will be able to show how to deduce the Global Compactness result in Theorem 5.1 for very general domains $\Omega \subset \mathbb{H}^n$ in quite a simple way by means of the so-called *Profile Decomposition*, firstly proven by Gérard in [68] for bounded sequences in the fractional (Euclidean) space H^s. The aforementioned result by Gérard was successfully extended in the Heisenberg framework in [14][1]

Theorem 5.2 (Theorem 1.1 in [14]). *Let $\{u_k\}$ be a bounded sequence in $S^1_0(\mathbb{H}^n)$. Then, for any $j \in \mathbb{N}$ there exist a sequence of numbers $\{\lambda_k^{(j)}\} \subset (0,\infty)$, a sequence of points $\{\xi_k^{(j)}\} \subset \mathbb{H}^n$ and a function $\psi^{(j)} \in S^1_0(\mathbb{H}^n)$ such that as $k \to \infty$*

$$\begin{cases} \left|\log \frac{\lambda_k^{(i)}}{\lambda_k^{(j)}}\right| \to \infty, & \text{if } \lambda_k^{(i)} \neq \lambda_k^{(j)} \\ \left|\delta_{1/\lambda_k^{(j)}}({\xi_k^{(j)}}^{-1} \circ \xi_k^{(i)})\right|_{\mathbb{H}^n} \to \infty & \text{if } \lambda_k^{(i)} = \lambda_k^{(j)}, \end{cases} \tag{5.7}$$

and such that, for any integer $\ell \in \mathbb{N}$,

$$u_k(\cdot) = \sum_{j=1}^{\ell} {\lambda_k^{(j)}}^{\frac{-Q}{2^*}} \psi^{(j)}(\delta_{1/\lambda_k^{(j)}}(\tau^{-1}_{\xi_k^{(j)}}(\cdot))) + r_k^{(\ell)}(\cdot); \tag{5.8}$$

$$\|u_k\|^2_{S^1_0} = \sum_{j=1}^{\ell} \|\psi^{(j)}\|^2_{S^1_0} + \|r_k^{(j)}\|^2_{S^1_0(\mathbb{H}^n)} + o(1) \quad \text{as } k \to \infty; \tag{5.9}$$

$$\|r_k^{(\ell)}\|_{L^{2^*}} \to 0 \quad \text{as } k \to \infty. \tag{5.10}$$

Remark 5.3. *Before going straight into the proof a remark is needed. Given a bounded domain with smooth boundary $\Omega \subseteq \mathbb{H}^n$ we extend all functions of $C^\infty_0(\Omega)$ to the whole $\mathbb{H}^n$ putting them equal zero outside Ω. Then we can regard them as functions defined on the whole $\mathbb{H}^n$. Now, define the Folland-Stein Sobolev homogeneous space $S^1_0(\Omega)$ as the completion of $C^\infty_0(\Omega)$ with respect to the norm $\|D_H \cdot \|_{L^2(\mathbb{H}^n)}$.*

Proof of Theorem 5.1. We divide the proof into several steps.

Step 1. The sequence $\{u_k\}$ is bounded in $S^1_0(\Omega)$. Since $\{u_k\}$ is a Palais-Smale sequence for $\mathcal{E}_\lambda$, we have

$$\begin{aligned} \left(\frac{1}{2} - \frac{1}{2^*}\right) \int_\Omega |u_k|^{2^*} \, \mathrm{d}\xi &= \mathcal{E}_\lambda(u_k) - \frac{1}{2}\langle \mathrm{d}\mathcal{E}_\lambda(u_k), u_k\rangle \\ &\leq c + c|\langle \mathrm{d}\mathcal{E}_\lambda(u_k), u_k\rangle| \\ &\leq c + c\varepsilon \|u_k\|^2_{S^1_0} + c_\varepsilon o(1)\,. \end{aligned}$$

Now, since Ω is bounded, we have as $k \to \infty$

$$\begin{aligned} \int_\Omega |u_k|^2 \, \mathrm{d}\xi &\leq |\Omega|^{2/Q} \left(\int_\Omega |u_k|^{2^*} \, \mathrm{d}\xi\right)^{2/2^*} \quad \text{as } k \to \infty \\ &\leq c + c_\varepsilon o(1) + c\varepsilon \|u_k\|^2_{S^1_0}, \end{aligned}$$

where we also used Hölder's Inequality and Young's Inequality.

[1] See also the recent alternative proof in [111] based upon the results in the relevant book [127] where an abstract approach in general Hilbert spaces can be found.

Hence, by combining the estimates above, we have that

$$\begin{aligned}\|u_k\|^2_{S^1_0} &= 2\mathcal{E}_\lambda(u_k) + \lambda\int_\Omega |u_k|^2\,\mathrm{d}\xi + \frac{2}{2^*}\int_\Omega |u_k|^{2^*}\,\mathrm{d}\xi \\ &\leq c + c\varepsilon\|u_k\|^2_{S^1_0} + c_\varepsilon o(1) \quad \text{as } k\to\infty\,.\end{aligned}$$

So that, up to choosing $\varepsilon > 0$ sufficiently small and reabsorbing the S^1_0-norm of u_k in the left hand-side, we have

$$\|u_k\|^2_{S^1_0} \leq c + o(1) \quad \text{as } k\to\infty\,,$$

which proves the claim.

Step 2. The weak limit $u^{(0)}$ solves (P_λ). From Step 1., up to a subsequence not relabelled, there exists $u^{(0)} \in S^1_0(\Omega)$ such that $u_k \rightharpoonup u^{(0)}$ in $S^1_0(\Omega)$ as $k\to\infty$. Hence, by the compact embedding theorem (see Theorem 3.2 of [63]), as $k\to\infty$

$$\begin{aligned} &u_k \rightharpoonup u^{(0)} && \text{weakly in } L^{2^*}(\Omega),\\ &|u_k|^{2^*-2}u_k \to |u^{(0)}|^{2^*-2}u^{(0)} && \text{strongly in } L^1(\Omega),\\ &|u_k|^{2^*-2}u_k \rightharpoonup |u^{(0)}|^{2^*-2}u^{(0)} && \text{weakly in } L^{(2^*)'}(\Omega),\\ &u_k \to u^{(0)} && \text{strongly in } L^2_{\mathrm{loc}}(\mathbb{H}^n).\end{aligned} \tag{5.11}$$

Thus, the weak continuity of the functional $\mathrm{d}\mathcal{E}_\lambda$ yields that $u^{(0)}$ is a solution to (P_λ). If the convergence of u_k to $u^{(0)}$ is strong we are done.

Step 3. If the convergence is not strong, then $\{u_k\}$ contains further profiles. Assume that u_k does not converge strongly to $u^{(0)}$ in $S^1_0(\Omega)$. Then, we can apply Theorem 5.2 to the sequence $\{u_k\}$, obtaining a profile decomposition for a renumbered subsequence, that is for any integer $\ell \in \mathbb{N}$ we can write

$$u_k(\cdot) = \sum_{j=1}^{\ell} {\lambda_k^{(j)}}^{\frac{-Q}{2^*}} \psi^{(j)}(\delta_{1/\lambda_k^{(j)}}(\tau^{-1}_{\xi_k^{(j)}}(\cdot))) + r_k^{(\ell)}(\cdot).$$

Let $\mathrm{I} = \{j\in\mathbb{N}\,|\,\psi^{(j)} \neq u^{(0)}\}$ and set $u^{(j)} = \psi^{(j)}$ for $j\in\mathrm{I}$. Note that

- By the definition of profiles, for any $j\in\mathrm{I}$ there exist a sequence of numbers $\{\lambda_k^{(j)}\}\subset(0,\infty)$, a sequence of points $\{\xi_k^{(j)}\}\subset\mathbb{H}^n$ such that
$$u_k(\tau_{\xi_k^{(j)}}(\delta_{\lambda_k^{(j)}}(\cdot))) \rightharpoonup u^{(j)}(\cdot) \quad \text{in } S^1_0(\mathbb{H}^n). \tag{5.12}$$
- The limit function $u^{(0)}$ can be regarded as the profile associated to the trivial dilations and translations.

Hence, if we show that $\mathrm{I}\neq\emptyset$, we are done proving that $\{u_k\}$ contains further profiles. Therefore, assume by contradiction that $\mathrm{I}=\emptyset$ and so $\{u_k\}$ does not contain profiles different from $u^{(0)}$. Then, the sum in (5.8) reduces to the single term $u^{(0)}$ and (5.8) gives

$$u_k - u^{(0)} = u^{(0)} + r_k^{(0)} - u^{(0)} = r_k^{(0)},$$

which in turn implies by (5.10)

$$\|u_k - u^{(0)}\|_{2^*} = \|r_k^{(0)}\|_{2^*} \to 0 \quad \text{as } k\to\infty.$$

Hence, $u_k\to u^{(0)}$ strongly in $L^{2^*}(\Omega)$. On the other hand, as $k\to\infty$

$$\begin{aligned}\|u_k - u^{(0)}\|^2{}_{S^1_0(\Omega)} &= \langle u_k, u_k - u^{(0)}\rangle - \langle u^{(0)}, u_k - u^{(0)}\rangle \\ &= \langle \mathrm{d}\mathcal{E}_\lambda(u_k), u_k - u^{(0)}\rangle - \langle \mathrm{d}\mathcal{E}_\lambda(u^{(0)}), u_k - u^{(0)}\rangle + \lambda\|u_k - u^{(0)}\|_2^2 \\ &\quad + \int_\Omega \left(|u_k|^{2^*-2}u_k - |u^{(0)}|^{2^*-2}u^{(0)}\right)\left(u_k - u^{(0)}\right)\,\mathrm{d}\xi = o(1),\end{aligned}$$

by (5.2) and the convergence properties of u_k in (5.11). This gives the desired contradiction and so $\mathrm{I} \neq \emptyset$; i. e., $\{u_k\}$ contains further profiles.

Step 4. The scaling parameters $\{\lambda_k^{(j)}\}$ from Theorem 5.2 satisfy $\lambda_k^{(j)} \to 0$ and $\operatorname{dist}(\xi_k^{(j)}, \Omega) = \mathrm{O}(\lambda_k^{(j)})$ as $k \to \infty$ for any $j \in \mathbb{N}$. Arguing again by contradiction we first assume that, up to a subsequence, $\lambda_k^{(j)} \to \infty$ as $k \to \infty$. Note that the convergence in (5.12) is actually strong in $L^2_{\mathrm{loc}}(\mathbb{H}^n)$ by Rellich's Theorem. Then, since $u^{(j)} \neq 0$ we get by Fatou's lemma

$$\begin{aligned} 0 < \int_{\mathbb{H}^n} |u^{(j)}|^2 \,\mathrm{d}\xi &\leq \liminf_{k\to\infty} {\lambda_k^{(j)}}^{Q-2} \int_{\mathbb{H}^n} \left| u_k(\tau_{\xi_k^{(j)}}(\delta_{\lambda_k^{(j)}}(\xi))) \right|^2 \,\mathrm{d}\xi \\ &= \liminf_{k\to\infty} {\lambda_k^{(j)}}^{-2} \int_{\Omega} |u_k|^2 \,\mathrm{d}\xi = 0, \end{aligned}$$

which together with the fact that $\lambda_k^{(j)} \to \infty$ and (5.11) gives a contradiction.

Therefore, there exists $c^{(j)} > 0$ such that $\lambda_k^{(j)} \leq c^{(j)}$ for all k. Now assume that, up to a subsequence, $\lambda_k^{(j)} \geq c_j > 0$ for each k. Then, again up to subsequences, $|\xi_k^{(j)}|_H \to \infty$, which is impossible since the profiles $u^{(j)}$ and $u^{(0)}$, for $j \in \mathrm{I}$, must be attained along different sequence of parameters and points satisfying conditions (5.7).

Hence, we conclude that $|\xi_k^{(j)}|_H \to \infty$, but then $u_k(\tau_{\xi_k^{(j)}}(\delta_{\lambda_k^{(j)}}(\cdot))) \rightharpoonup 0$, since for any fixed k, each function is supported in $\delta_{1/\lambda_k^{(j)}}(\tau_{\xi_k^{(j)-1}}(\Omega))$, and so for k sufficiently large, they are zero on each compact set in $\mathbb{H}^n$, Ω being bounded and $\lambda_k^{(j)} \geq c_j > 0$. Since $u^{(j)} \neq 0$ we obtain a contradiction and therefore $\lambda_k^{(j)} \to 0$ as $k \to \infty$.

Let us now deal with the second part of the statement. Assume that

$$\frac{\operatorname{dist}(\xi_k^{(j)}, \Omega)}{\lambda_k^{(j)}} \to \infty \qquad \text{as } k \to \infty.$$

This yields, possibly along a subsequence, $\operatorname{dist}(\mathbf{0}, \delta_{1/\lambda_k^{(j)}}(\tau_{\xi_k^{(j)-1}}(\Omega))) \to \infty$ (where we denoted with $\mathbf{0}$ the identity element in $\mathbb{H}^n$) and still, by arguing as before, we obtain that $u_k(\tau_{\xi_k^{(j)}}(\delta_{\lambda_k^{(j)}}(\cdot))) \rightharpoonup 0$. This is impossible and so $\operatorname{dist}(\xi_k^{(j)}, \Omega) = \mathrm{O}(\lambda_k^{(j)})$ as $k \to \infty$.

Step 5. The profiles $u^{(j)}$ solve (P_0) either in a half-space or in the whole $\mathbb{H}^n$, and we may assume $\{\xi_k^{(j)}\} \subset \Omega$. According to Step 4, we shall distinguish two cases:
Case I: $\operatorname{dist}(\xi_k^{(j)}, \partial\Omega) = \mathrm{O}(\lambda_k^{(j)})$ as $k \to \infty$.
Case II: $\dfrac{\operatorname{dist}(\xi_k^{(j)}, \partial\Omega)}{\lambda_k^{(j)}} \to \infty$ as $k \to \infty$ (This only happens when $\xi_k^{(j)} \subset \Omega$ stays inside the domain or approaches the boundary slower than $\lambda_k^{(j)}$).

We know by (5.12) that $u_k^{(j)}(\cdot) := u_k(\tau_{\xi_k^{(j)}}(\delta_{\lambda_k^{(j)}}(\cdot))) \rightharpoonup u^{(j)}(\cdot)$ in $S_0^1(\mathbb{H}^n)$, and that $\operatorname{supp} u_k^{(j)} \subset \delta_{1/\lambda_k^{(j)}}(\tau_{\xi_k^{(j)-1}}(\Omega)))$. Note that, because of the condition on the scaling parameters,

$$\delta_{1/\lambda_k^{(j)}}(\tau_{\xi_k^{(j)-1}}(\Omega)) \to \Omega_{\mathrm{o}}^{(j)} \text{ as } k \to \infty,$$

where $\Omega_{\mathrm{o}}^{(j)}$ is either an open half-space or the entire space (in Case II).

Fix $\varphi \in C_0^\infty(\Omega_{\rm o}^{(j)})$. By the invariance of the S_0^1 and the L^{2^*} norms with respect to the scaling

$$u(\cdot) \mapsto \widetilde{u}(\cdot) = \eta^{\frac{2-Q}{2}} u(\delta_{1/\eta}(\tau_{(\xi)^{-1}}(\cdot))), \tag{5.13}$$

it is clear that the rescaled function $\widetilde{\varphi}(\cdot) := {\lambda_k^{(j)}}^{\frac{2-Q}{2}} \varphi(\delta_{1/\lambda_k^{(j)}}(\tau^{-1}_{\xi_k^{(j)}}(\cdot))) \in C_0^\infty(\Omega)$ and

$$\langle {\rm d}\mathcal{E}_\lambda(u_k), \widetilde{\varphi}\rangle = \langle {\rm d}\mathcal{E}^*(u_k^{(j)}), \varphi\rangle - \lambda (\lambda_k^{(j)})^{1+Q/2} \int_{\mathbb{H}^n} u_k^{(j)} \varphi \, {\rm d}\xi. \tag{5.14}$$

Now, by (5.2),

$$\langle {\rm d}\mathcal{E}_\lambda(u_k), \widetilde{\varphi}\rangle = {\lambda_k^{(j)}}^{\frac{2-Q}{2}} \langle {\rm d}\mathcal{E}_\lambda(u_k), \varphi\Big(\delta_{1/\lambda_k^{(j)}}(\tau^{-1}_{\xi_k^{(j)}}(\cdot))\Big)\rangle \to 0 \quad \text{as } k \to \infty.$$

On the other hand, since φ is compactly supported, $u_k^{(j)} \to u^{(j)}$ strongly in $L^2_{\rm loc}(\mathbb{H}^n)$ and $\lambda_k^{(j)} \to 0$ by Step 4 we get

$$(\lambda_k^{(j)})^{1+Q/2} \int_{\mathbb{H}^n} u_k^{(j)} \varphi \, {\rm d}\xi \to 0 \quad \text{as } k \to \infty. \tag{5.15}$$

Combining (5.14)–(5.15) with the weak continuity of $d\mathcal{E}^*$ we get $\langle d\mathcal{E}^*(u^{(j)}), \varphi\rangle = 0$ for all $\varphi \in C_0^\infty(\Omega_{\rm o}^{(j)})$. Hence, the claim follows by density.

In remains to show that $\{\xi_k^{(j)}\} \subset \Omega$. As already observed, this is obvious in Case II, and so we only need to prove it in Case I, that is when $\operatorname{dist}(\xi_k^{(j)}, \partial\Omega) = \mathrm{O}(\lambda_k^{(j)})$ as $k \to \infty$. In that case, fix $\bar{\xi}^{(j)} \in \Omega_{\rm o}^{(j)}$ and set

$$\begin{aligned} &\bar{\xi}_k^{(j)} := \tau_{\xi_k^{(j)}}(\delta_{\lambda_k^{(j)}}(\bar{\xi}^{(j)})), \\ &\bar{u}_k^{(j)}(\cdot) = u_k(\tau_{\bar{\xi}_k^{(j)}}(\delta_{\lambda_k^{(j)}}(\cdot)) = u_k(\tau_{\xi_k^{(j)}}(\delta_{\lambda_k^{(j)}}(\tau_{\bar{\xi}^{(j)}}(\cdot)))). \end{aligned}$$

Then, $\{\bar{\xi}_k^{(j)}\} \subset \Omega$ for k sufficiently large and $\bar{u}_k^{(j)}(\cdot) \rightharpoonup u^{(j)}(\tau_{\bar{\xi}^{(j)}}(\cdot))$ in $S_0^1(\mathbb{H}^n)$ as $k \to \infty$. Therefore, the claim follows by taking $\bar{\xi}_k^{(j)}$ as translation parameter and $\bar{u}^{(j)}(\cdot) = u^{(j)}(\tau_{\bar{\xi}^{(j)}}(\cdot))$.

Step 6. The profiles $u^{(j)}$ are in finite number. By (5.9) of Theorem 5.2, it suffices to show that the S_0^1 norms of the nontrivial profiles $u^{(j)}$, $j \in \mathbb{N}$, are uniformly bounded from below. To this aim, let us test (P_0) with $\varphi = u^{(j)}$. Thus, from the Folland-Stein inequality (3.1), we get

$$\|u^{(j)}\|^2_{S_0^1} \;=\; \|u^{(j)}\|^{2^*}_{L^{2^*}} \;\le\; S^* \Big(\|u^{(j)}\|^{2^*}_{S_0^1}\Big)^{\frac{2^*}{2}}.$$

Consequently, $\|u^{(j)}\|_{S_0^1} \ge (S^*)^{\frac{2}{2-2^*}}$ and so the claim is proved.

Step 7. The sequence of remainders $\{r_k^{(j)}\}$ strongly converges to 0 in $S_0^1(\mathbb{H}^n)$. First, by the scaling invariance we can easily see that

$$\begin{aligned}
\|r_k^{(j)}\|_{S_0^1}^2 &= \|u_k - u^{(0)}\|_{S_0^1}^2 + \sum_{j=1}^{J} \left\| \lambda_k^{(j)\,\frac{2-Q}{2}} u^{(j)}(\delta_{1/\lambda_k^{(j)}}(\tau_{\xi_k^{(j)}}^{-1}(\cdot))) \right\|_{S_0^1}^2 \\
&\quad -2\left(u_k - u^{(0)}, \sum_{j=1}^{J} \lambda_k^{(j)\,\frac{2-Q}{2}} u^{(j)}(\delta_{1/\lambda_k^{(j)}}(\tau_{\xi_k^{(j)}}^{-1}(\cdot))) \right) + o(1) \\
&= \|u_k - u^{(0)}\|_{S_0^1}^2 + \sum_{j=1}^{J} \|u^{(j)}\|_{S_0^1}^2 - 2\sum_{j=1}^{J} (u_k^{(j)}, u^{(j)}) \\
&\quad +2\sum_{j=1}^{I} \left(u^{(0)}, \lambda_k^{(j)\,\frac{2-Q}{2}} u^{(j)}(\delta_{1/\lambda_k^{(j)}}(\tau_{\xi_k^{(j)}}^{-1}(\cdot))) \right) + o(1) \\
&= \|u_k - u^{(0)}\|_{S_0^1}^2 - \sum_{j=1}^{J} \int_{\mathbb{H}^n} |\widetilde{u}^{(j)}|^{2^*}\, \mathrm{d}\xi + o(1), \qquad (5.16)
\end{aligned}$$

where we used the fact that $u^{(j)}$ solves (P_0) by Step 5 for any $j \in \mathrm{I}$. Now, arguing as in Step 3, as $k \to \infty$

$$\|u_k - u^{(0)}\|^2{}_{S_0^1(\Omega)} = \int_\Omega \left(|u_k|^{2^*-2}u_k - |u^{(0)}|^{2^*-2}u^{(0)} \right)\left(u_k - u^{(0)} \right) \mathrm{d}\xi = o(1),$$

and, using again (5.11), the previous equality becomes, as $k \to \infty$,

$$\|u_k - u^{(0)}\|_{S_0^1(\Omega)}^2 = \|u_k\|_{L^{2^*}(\Omega)}^{2^*} - \|u^{(0)}\|_{L^{2^*}(\Omega)}^{2^*} + o(1),$$

which yields in turn, thanks to (5.16)

$$\lim_{k\to\infty} \|r_k^{(j)}\|_{S_0^1}^2 = \lim_{k\to\infty} \|u_k\|_{L^{2^*}(\Omega)}^{2^*} - \|u^{(0)}\|_{L^{2^*}(\Omega)}^{2^*} - \sum_{j=1}^{J} \|\widetilde{u}^{(j)}\|_{L^{2^*}(\mathbb{H}^n)}^{2^*}. \qquad (5.17)$$

Hence, we are done if we show that

$$\lim_{k\to\infty} \|u_k\|_{L^{2^*}(\Omega)}^{2^*} = \|u^{(0)}\|_{L^{2^*}(\Omega)}^{2^*} + \sum_{j=1}^{J} \|\widetilde{u}^{(j)}\|_{L^{2^*}(\mathbb{H}^n)}^{2^*}.$$

By (5.8), we know that $u_k(\cdot) = \sum_{j=1}^{\ell} \lambda_k^{(j)\,\frac{-Q}{2^*}} \psi^{(j)}(\delta_{1/\lambda_k^{(j)}}(\tau_{\xi_k^{(j)}}^{-1}(\cdot))) + r_k^{(j)}(\cdot)$. Thus, by using iteratively the Brezis-Lieb lemma and the invariance of the L^{2^*} norm with respect to (5.13) together with (5.9), we obtain the desired conclusion.

Step 8. Completion of the proof. Clearly, (5.4) follows straightly from (5.7), while and (5.3) and (5.5) come from (5.9), thanks to Step 7. Hence, it remains to show (5.6). To this aim, let us recall that by (5.11), $u_k \to u^{(0)}$ in $L^2(\Omega)$, where and $u^{(0)}$ solves (P_λ) thanks to Step 2. Moreover, because of Step 5, we know that $\|u^{(j)}\|_{S_0^1}^2 = \|u^{(j)}\|_{L^{2^*}}^{2^*}$. Combining these two facts with (5.5), (5.17) and the definition of $\mathcal{E}_\lambda$, we obtain the validity of (5.6). This completes the proof. □

It is now worth remarking that in the special case when further regularity is assumed on the set Ω we can give a complete characterization of the limiting sets $\{\Omega_{\mathrm{o}}^{(j)}\}_{j\in\mathrm{I}}$. Let us recall the definition of H-flatness of Ω at a boundary point.

Definition 5.4 (H-flat domains)**.** *Let Ω be a smooth bounded domain of $\mathbb{H}^n$ and let $\xi \in \partial\Omega$. Assume $\Phi \in C^\infty(\mathbb{H}^n)$ be a defining function for the boundary of Ω in a neighborhood of ξ (that is, $\Phi : B_\varrho(\xi) \to \mathbb{R}$ with $\Phi \equiv 0$ on $B_\varrho(\xi) \cap \partial\Omega$ and $\Phi > 0$ on $B_\varrho(\xi) \cap \Omega$). If ξ is a characteristic point, then we say that Ω is H-*flat *at ξ if*

$$q_H\Phi(\xi) = 0,$$

where $q_H : \mathbb{R}^{2n} \to \mathbb{R}$ is the quadratic form associated with the Hessian matrix $D_H^2\Phi$, that is

$$\Big(q_H\Phi(\eta)\Big)(z) := \sum_{i,j=1}^{2n} \Big((D_H)_i(D_H)_j\Phi\Big)(\eta)z_iz_j \qquad \forall z \in \mathbb{R}^{2n}.$$

We say that Ω is H-flat if it is H-flat at any characteristic point of its boundary.

We have the following

Lemma 5.5 (Lemma 3.4 in [39])**.** *Let Ω be a smooth bounded domain in $\mathbb{H}^n$ and, for any sequence $\{\xi_k\}$ in Ω and for any divergent series $\{\lambda_k\} \in \mathbb{R}^+$, let*

$$\Omega_k := \delta_{\lambda_k}(\tau_{\xi_k^{-1}}(\Omega)).$$

Then, up to subsequences, $\Omega_K \to \Omega_o$ where

(i) *if $\lambda_k \mathrm{dist}(\xi_k, \partial\Omega)$ is unbounded then Ω_o is the whole space $\mathbb{H}^n$;*
(ii) *if $\lambda_k \mathrm{dist}(\xi_k, \partial\Omega)$ is bounded, then there exists $\mathfrak{a}$, $\mathfrak{b} \in \mathbb{R}^n$ and $\mathfrak{c} \in \mathbb{R}$ and a quadratic form q on $\mathbb{R}^{2n}$ such that, up to translations,*

$$\Omega_o = \Big\{(x, y, t) \in \mathbb{H}^n \,|\, \mathfrak{a} \cdot x + \mathfrak{b} \cdot y + \mathfrak{c} \cdot t + q(x, y) > 0\Big\};$$

(iii) *if $\lambda_k \mathrm{dist}(\xi_k, \partial\Omega)$ is bounded and Ω is H-flat, according to* Definition 5.4*, then Ω_o is a half-space of $\mathbb{H}^n$.*

The main consequence of Lemma 5.5 is that for bounded C^∞ domains in $\mathbb{H}^n$ we explicitly know the limiting sets $\Omega_o^{(j)}$. Moreover, if we consider in Theorem 5.1 only positive Palais-Smale sequences for $\mathcal{E}_\lambda$ on H-flat domains, then we have that the limiting spaces $\Omega_o^{(j)}$ are always the whole space $\mathbb{H}^n$. This follows from the nonexistence results proven in [95].

CHAPTER 6

Asymptotic control via the Jerison & Lee extremals

This chapter is devoted to the proof of an asymptotic control of the maximizing sequence u_ε for S^*_ε in (3.4) via the Jerison & Lee extremals. This result will play a major role in the proof of the localization result in Theorem 7.3 in Chapter 7. Moreover, Theorem 6.1 below reveals to be an independent result which could be also useful to investigate further properties related to the subcritical Folland-Stein embedding.

Theorem 6.1. *Let $\Omega \subset \mathbb{H}^n$ be a smooth bounded domain such that*

$$\liminf_{\varrho \to 0^+} \frac{|(\mathbb{H}^n \setminus \Omega) \cap B_\varrho(\xi)|}{|B_\varrho(\xi)|} > 0 \qquad \forall \xi \in \partial\Omega. \tag{6.1}$$

Then, for each $0 < \varepsilon < 2^* - 2$ letting $u_\varepsilon \in S^1_0(\Omega)$ being a maximizer for S^*_ε, there exist $\{\eta_\varepsilon\} \subset \Omega$, $\{\lambda_\varepsilon\} \subset \mathbb{R}^+$ such that, up to choosing ε sufficiently small, we have that

$$u_\varepsilon(\xi) \lesssim \lambda_\varepsilon^{-\frac{Q-2}{2}} U\Big(\delta_{\lambda_\varepsilon^{\frac{Q-2}{4}\varepsilon-1}}(\tau_{\eta_\varepsilon^{-1}}(\xi))\Big) \qquad \text{on } \Omega, \tag{6.2}$$

where $U(\cdot)$ are the Jerison & Lee extremal functions given in (4.5) and the sequences $\{\eta_\varepsilon\}$ and $\{\lambda_\varepsilon\}$ satisfy

$$\lim_{\varepsilon \to 0^+} \eta_\varepsilon = \xi_{\mathrm{o}} \qquad \text{and} \qquad |\lambda_\varepsilon^\varepsilon - 1| = \mathrm{O}(\lambda_\varepsilon^\varepsilon \varepsilon |\ln \varepsilon|), \tag{6.3}$$

with ξ_{o} being the concentration point.

The result above reminds somehow to the literature following the pioneering work in the Euclidean setting due to Aubin and Talenti, and this is one of the key-points in the subtle proof of the related conjecture by Han in [73]. Here, we have also to deal with the fact that, in strong contrast with the Euclidean setting, the Jerison & Lee extremals cannot be reduced to functions depending only on the standard Korányi gauge. For this, we need to pursuit a delicate strategy which makes use and refines the concentration result obtained in [109] via the Γ-convergence approach in order to detect the right scalings λ_ε and η_ε above. Also, we would make use of the Global Compactness-type result in $\mathbb{H}^n$ established in Chapter 5

Proof of Theorem 6.1. Consider a maximizing sequence $\{u_\varepsilon\}$ of (3.4). Then it holds that

$$\int_\Omega |u_\varepsilon|^{2^*-\varepsilon} \, \mathrm{d}\xi = S^* + o(1), \qquad \text{as } \varepsilon \to 0, \tag{6.4}$$

where we also used Proposition 2.5 in [109].

Step 1. The sequence of the supremum norms $\|u_\varepsilon\|_{L^\infty}$ diverges; i. e.,

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \to \infty, \qquad \textbf{as } \varepsilon \to 0. \tag{6.5}$$

By contradiction assume that there exists a sequence $\{\varepsilon_k\}_k$ for which u_{ε_k} remains bounded in Ω for $\varepsilon_k \to 0^+$ as $k \to +\infty$. This allows to apply the local regularity

theory firstly developed by Folland and Stein in [59, 60] to conclude that $u_\varepsilon \in C^\infty(\Omega)$. Moreover, by taking into account the regularity assumptions on the set Ω as in particular the measure density condition (6.1), the classical argument via Moser's Iteration leads to $u_\varepsilon \in \Gamma^{0,\alpha_{\varepsilon_k}}(\partial\Omega)$, for some exponent $1 > \alpha_{\varepsilon_k} > \alpha_{\bar\varepsilon} > 0$ for any $\varepsilon_k < \bar\varepsilon$. Hence, up to subsequences, we can assume that $u_{\varepsilon_k} \to v \not\equiv \infty$ uniformly on Ω.

If the limit function $v \equiv 0$, then by (6.4) we have reached a contradiction since $S^* \neq 0$. On the other hand, if $v \neq 0$ we would have obtained a maximizer of (3.3) which is a contradiction as well. Thus, (6.5) holds true.

Choose now a sequence of points $\{\eta_\varepsilon\} \subseteq \Omega$ and a sequence of numbers $\{\lambda_\varepsilon\} \subseteq \mathbb{R}^+$ such that

$$u_\varepsilon(\eta_\varepsilon) = \lambda_\varepsilon^{-\frac{Q-2}{2}} \equiv \|u_\varepsilon\|_{L^\infty(\Omega)}. \tag{6.6}$$

Consider the function

$$v_\varepsilon(\xi) := \lambda_\varepsilon^{\frac{Q-2}{2}} u_\varepsilon\left(\tau_{\eta_\varepsilon}\left(\delta_{\lambda_\varepsilon^{1-\frac{Q-2}{4}\varepsilon}}(\xi)\right)\right), \tag{6.7}$$

which is a weak solution to

$$\begin{cases} -\Delta_H v_\varepsilon = v_\varepsilon^{2^*-1-\varepsilon} & \text{in } \Omega_\varepsilon := \delta_{\lambda_\varepsilon^{\frac{Q-2}{4}\varepsilon-1}}\left(\tau_{\eta_\varepsilon}^{-1}(\Omega)\right) \\ v_\varepsilon(0) = 1, \\ 0 \le v_\varepsilon \le 1. \end{cases} \tag{6.8}$$

Indeed, by the homogeneity of the sub-Laplacian we get

$$\begin{aligned} -\Delta_H v_\varepsilon(\xi) &= \lambda_\varepsilon^{\frac{Q-2}{2}} \lambda_\varepsilon^{2-\frac{Q-2}{2}\varepsilon}(-\Delta_H u_\varepsilon)\left(\tau_{\eta_\varepsilon}\left(\delta_{\lambda_\varepsilon^{1-\frac{Q-2}{4}\varepsilon}}(\xi)\right)\right) \\ &= \lambda_\varepsilon^{\frac{Q+2}{2}-\frac{Q-2}{2}\varepsilon} u_\varepsilon^{2^*-1-\varepsilon}\left(\tau_{\eta_\varepsilon}\left(\delta_{\lambda_\varepsilon^{1-\frac{Q-2}{4}\varepsilon}}(\xi)\right)\right) \\ &= \lambda_\varepsilon^{\frac{Q-2}{2}\left(\frac{Q+2}{Q-2}-\varepsilon\right)} u_\varepsilon^{2^*-1-\varepsilon}\left(\tau_{\eta_\varepsilon}\left(\delta_{\lambda_\varepsilon^{1-\frac{Q-2}{4}\varepsilon}}(\xi)\right)\right) = v_\varepsilon^{2^*-1-\varepsilon}. \end{aligned}$$

Also, recalling the choice of $\{\eta_\varepsilon\}$ and $\{\lambda_\varepsilon\}$ in (6.6), we have that

$$v_\varepsilon(0) = \lambda_\varepsilon^{\frac{Q-2}{2}} u_\varepsilon(\eta_\varepsilon) = 1,$$

and that $0 \le v_\varepsilon \le 1$ in Ω_ε.

Now, since the sequence $\{v_\varepsilon\}$ is bounded it is equicontinuous on compact subset of $\mathbb{H}^n$, and by Ascoli-Arzelà's Theorem, up to subsequences, there exists a function $v_\infty \not\equiv 0$ such that $v_\varepsilon \to v_\infty$ uniformly on compact set.

Step 2. The sequence $\{\lambda_\varepsilon\}$ satisfies

$$0 < c \le \lambda_\varepsilon^\varepsilon \le 1, \qquad \textbf{with } c \equiv c(n). \tag{6.9}$$

First of all, since $\lambda_\varepsilon \to 0$ when $\varepsilon \to 0$, by (6.6), it trivially follows that $\lambda_\varepsilon^\varepsilon < 1$, for ε sufficiently small. Moreover, since the function v_ε defined in (6.7) tends to $v_\infty \not\equiv 0$ uniformly in $B_1(0)$, we have that there exists a constant c such that

$$\int_{B_1(0)} v_\varepsilon^{2^*-\varepsilon}\, \mathrm{d}\xi \ge c.$$

Then, we get

$$\begin{aligned}\int_{B_1(0)} v_\varepsilon^{2^*-\varepsilon}\,\mathrm{d}\xi &= \int_{B_1(0)} \lambda_\varepsilon^{\frac{Q-2}{2}\left(\frac{2Q}{Q-2}-\varepsilon\right)} u_\varepsilon^{\frac{2Q}{Q-2}-\varepsilon}\left(\tau_{\eta_\varepsilon}\left(\delta_{\lambda_\varepsilon^{1-\frac{Q-2}{4}\varepsilon}}(\xi)\right)\right)\mathrm{d}\xi \\ &= \lambda_\varepsilon^{\frac{(Q-2)^2}{4}\varepsilon}\int_{|\xi\circ\eta_\varepsilon^{-1}|_H\le\lambda_\varepsilon^{1-\frac{Q-2}{4}\varepsilon}} u_\varepsilon^{\frac{2Q}{Q-2}-\varepsilon}(\xi)\,\mathrm{d}\xi \\ &\le \lambda_\varepsilon^{\frac{(Q-2)^2}{4}\varepsilon}\int_\Omega u_\varepsilon^{2^*-\varepsilon}(\xi)\,\mathrm{d}\xi,\end{aligned}$$

which, together with (6.4), gives the desired estimate for a proper constant $c \equiv c(n) > 0$.

Step 3. The asymptotics in (6.3) for the sequence $\{\eta_\varepsilon\}$ chosen in Step 1 holds true. Recall that $|D_H u_\varepsilon|^2\,\mathrm{d}\xi \overset{*}{\rightharpoonup} \delta_{\xi_\mathrm{o}}$ in $\mathcal{M}(\overline{\Omega})$ for a given point $\xi_\mathrm{o} \in \overline{\Omega}$. Then, considering the function v_ε in (6.7), which converges to v_∞ uniformly on compact set, it yields

$$\begin{aligned}0 < \int_{B_\varrho(0)} |D_H v_\infty|^2\,\mathrm{d}\xi &= \lim_{\varepsilon\to 0^+}\int_{B_\varrho(0)} |D_H v_\varepsilon|^2\,\mathrm{d}\xi \\ &= \lim_{\varepsilon\to 0^+}\left(\lambda_\varepsilon^\varepsilon\right)^{\frac{(Q-2)^2}{4}}\int_{B_{\lambda_\varepsilon^{1-\frac{Q-2}{4}\varepsilon}R}(\eta_\varepsilon)} |D_H u_\varepsilon|^2\,\mathrm{d}\xi,\end{aligned}$$

which, in view of the fact that $0 < c \le \lambda_\varepsilon^\varepsilon \le 1$, gives a contradiction if η_ε does not converge to ξ_o.

Step 4. The concentration point ξ_o is away from the boundary of $\partial\Omega$. In order to prove this result, we will employ the maximum principle stated in Proposition 1.3 in [15].

First, let us show that there exists a direction ς, $|\varsigma|_H = 1$, along which u_ε decreases, i. e. for any $\xi \in \Omega$, it holds

$$u_\varepsilon(\delta_{\lambda_1}(\varsigma)\circ\xi) - u_\varepsilon(\delta_{\lambda_2}(\varsigma)\circ\xi) < 0 \quad \text{for any } 0 < \lambda_2 < \lambda_1 < \lambda_\mathrm{o}\,,$$

for some $\lambda_\mathrm{o} > 0$. By contradiction assume that there is no such direction. Then, by choosing ς with $|\varsigma|_H = 1$, and denoting $u_{\lambda,\varepsilon} := u_\varepsilon(\tau_{\delta_\lambda(\varsigma)}(\cdot))$, we have that $u_{\lambda,\varepsilon}$ is such that $u_{\lambda,\varepsilon} \ge u_\varepsilon$ and it solves the following problem,

$$\begin{cases} -\Delta_H u_{\lambda,\varepsilon} = u_{\lambda,\varepsilon}^{2^*-1-\varepsilon} & \text{in } \tau_{(\delta_\lambda(\varsigma))^{-1}}(\Omega) \\ u_{\lambda,\varepsilon} = 0 & \text{on } \partial\tau_{(\delta_\lambda(\varsigma))^{-1}}(\Omega). \end{cases}$$

Clearly, because of the boundedness of Ω one has that there exists $\lambda_\mathrm{o} \ge 0$ such that $\tau_{(\delta_{\lambda_\mathrm{o}}(\varsigma))^{-1}}(\Omega)\cap\Omega = \emptyset$ and $\Omega\cap\tau_{(\delta_\lambda(\varsigma))^{-1}}(\Omega) \ne \emptyset$ for any $\lambda \in [0,\lambda_\mathrm{o})$. We immediately notice that in the case when $\lambda = 0$, one has $\tau_{(\delta_\lambda(\varsigma))^{-1}}(\Omega) = \Omega$. For any $\lambda < \lambda_\mathrm{o}$ let us consider the function $w_{\lambda,\varepsilon}$ defined as follows,

$$w_{\lambda,\varepsilon} := u_{\lambda,\varepsilon} - u_\varepsilon.$$

Such a function is the solution of the following problem,

$$\begin{cases} \Delta_H w_{\lambda,\varepsilon} - f(\xi) w_{\lambda,\varepsilon} \le 0 & \text{in } \tau_{(\delta_\lambda(\varsigma))^{-1}}(\Omega)\cap\Omega, \\ w_{\lambda,\varepsilon} = 0 & \text{on } \partial(\tau_{(\delta_\lambda(\varsigma))^{-1}}(\Omega)\cap\Omega), \end{cases}$$

where $f \in L^\infty(\tau_{(\delta_\lambda(\varsigma))^{-1}}(\Omega)\cap\Omega)$. Letting $\delta > 0$ be the one given by Proposition 1.3 in [15] and choosing λ_δ such that

$$\left|\tau_{(\delta_\lambda(\varsigma))^{-1}}(\Omega)\cap\Omega\right| < \delta \quad \text{for any } \lambda > \lambda_\delta, \tag{6.10}$$

we can apply the maximum principle to get that

$$w_{\lambda,\varepsilon} \geq 0 \quad \text{in } \tau_{(\delta_\lambda(\varsigma))^{-1}}(\Omega) \cap \Omega.$$

Moreover, by the strong maximum principle, we get in particular that

$$w_{\lambda,\varepsilon} > 0 \quad \text{in } \tau_{(\delta_\lambda(\varsigma))^{-1}}(\Omega) \cap \Omega.$$

Now, define

$$\lambda_1 := \inf \left\{ \lambda > 0 : w_{s,\varepsilon} > 0 \text{ for any } s > \lambda \right\} \geq 0.$$

We show that $\lambda_1 = 0$. By contradiction assume that $\lambda_1 > 0$. Note that $w_{\lambda_1} \geq 0$ on $\tau_{(\delta_{\lambda_1}(\varsigma))^{-1}}(\Omega) \cap \Omega$ and, by the strong maximum principle, $w_{\lambda_1,\varepsilon} > 0$.

Choose a compact set $K \subset \tau_{(\delta_{\lambda_1}(\varsigma))^{-1}}(\Omega) \cap \Omega$ such that

$$|(\tau_{(\delta_{\lambda_1}(\varsigma))^{-1}}(\Omega) \cap \Omega) \setminus K| < \delta/3\,,$$

and

$$w_{s,\varepsilon} \geq 0 \text{ on } K \quad \text{for } s < \lambda_1. \tag{6.11}$$

Fix now $0 < \lambda < \lambda_1$ such that

$$K \subset \tau_{(\delta_\lambda(\varsigma))^{-1}}(\Omega) \cap \Omega, \quad \text{and} \quad |(\tau_{(\delta_\lambda(\varsigma))^{-1}}(\Omega) \cap \Omega) \setminus K| < \delta, \tag{6.12}$$

where δ is the one appearing in (6.10).

Choose $\lambda < \lambda_1$ sufficiently near to λ_1 such that (6.11) and (6.12) are satisfied for any $s \in (\lambda, \lambda_1)$. This plainly leads to a contradiction. Indeed, $w_{s,\varepsilon} \geq 0$ on $(\tau_{(\delta_s(\varsigma))^{-1}}(\Omega) \cap \Omega) \setminus K$ by Proposition 1.3 in [15] which together with (6.11) does imply that $w_{s,\varepsilon} \geq 0$ on $\tau_{(\delta_s(\varsigma))^{-1}}(\Omega) \cap \Omega$. Moreover, recalling that $w_{s,\varepsilon} > 0$ since $s > \lambda > 0$, from the strong maximum principle it follows that $w_{s,\varepsilon} > 0$ for some $s < \lambda_1$, which is a contradiction. Therefore, we have that $u_\varepsilon(\tau_{\delta_\lambda(\varsigma)}(\cdot))$ is decreasing for anyλ in $(0, \lambda_o)$.

Now, for any $\xi \in \{\xi' \in \overline{\Omega} : \mathrm{dist}(\xi', \partial\Omega) < \vartheta\}$, up to choosing a constant $\vartheta > 0$ small enough, we can find K_ξ such that

$$|K_\xi| > 0, \quad K_\xi \subset \left\{ \xi' \in \Omega : \mathrm{dist}(\xi', \partial\Omega) > \frac{\vartheta}{2} \right\}, \quad \text{and } u_\varepsilon(\xi') > u_\varepsilon(\xi),\ \forall \xi' \in K_\xi.$$

Thus, being $\{u_\varepsilon\}$ a maximizing sequence for S^*_ε, we finally arrive at

$$u_\varepsilon(\xi) < \fint_{K_\xi} u_\varepsilon \, \mathrm{d}\xi \leq |K_\xi|^{-\frac{1}{2^*-\varepsilon}} \|u_\varepsilon\|_{L^{2^*-\varepsilon}(\Omega)} = |K_\xi|^{-\frac{1}{2^*-\varepsilon}} (S^*)^{\frac{1}{2^*-\varepsilon}} + \mathrm{o}(1),$$

Then, since $u_\varepsilon(\eta_\varepsilon) \to \infty$, it must follow that η_ε is away from the boundary (up to choosing ε small enough).

Step 5. The function v_ε in (6.7) converges to U in (4.5) locally uniformly on compact sets. As proven above we have that the sequence $\{v_\varepsilon\}$ converges uniformly on compact set to v_∞.

Moreover, if we denote with Ω_o the limiting set for $\varepsilon \to 0$ of Ω_ε, we have that $\Omega_o \equiv \mathbb{H}^n$. Indeed, thanks to Step 2 and Step 3 we have that the sequence

$$\left\{ \lambda_\varepsilon^{\frac{Q-2}{4}\varepsilon - 1} \mathrm{dist}(\eta_\varepsilon, \partial\Omega) \right\}_{\varepsilon > 0}$$

is unbounded. Then, recalling [39, Lemma 3.4] we have that the limiting space Ω_o does coincide with $\mathbb{H}^n$.

This yields that v_∞ is a solution to

$$\begin{cases} -\Delta_H v_\infty = v_\infty^{2^*-1}, & \text{in } \mathbb{H}^n, \\ v_\infty(0) = 1, \\ 0 \le v_\infty \le 1, \end{cases}$$

which implies that v_∞ coincides with the function U, defined in (4.5).

Step 6. The sequence $\{\lambda_\varepsilon\}$ satisfies

$$|\lambda_\varepsilon^\varepsilon - 1| = \lambda_\varepsilon^{\bar{t}\varepsilon}\varepsilon|\ln \lambda_\varepsilon|\,, \tag{6.13}$$

for some $\bar{t} \in (0,1)$. By means of the Mean Value Theorem we have that there exists $\vartheta \in (\lambda_\varepsilon^\varepsilon, 1)$ such that

$$\frac{1}{\vartheta} = \frac{1}{1-\lambda_\varepsilon^\varepsilon}\int_{\lambda_\varepsilon^\varepsilon}^1 \frac{1}{s}\,\mathrm{d}s = \frac{\ln \lambda_\varepsilon^\varepsilon}{\lambda_\varepsilon^\varepsilon - 1}.$$

Hence, considering $\bar{t} := \ln\vartheta/\ln\lambda_\varepsilon^\varepsilon \in (0,1)$ we have that

$$|\lambda_\varepsilon^\varepsilon - 1| = \lambda_\varepsilon^{\bar{t}\varepsilon}\varepsilon|\ln\lambda_\varepsilon|,$$

by (6.9), which gives the desired result.

Step 7. The sequence $\{v_\varepsilon\}$ in (6.7) converges to U in $S_0^1(\Omega)$. We show now that the sequence $\{v_\varepsilon\}$ is a Palais-Smale sequence for $\mathcal{E}_0$; i. e., (5.1) and (5.2) hold true. We start by showing the bound in (5.1). For this, we recall the homogeneity of the horizontal gradient D_H and the result in (6.3); we have

$$\begin{aligned} \int_{\Omega_\varepsilon} |D_H v_\varepsilon|^2\,\mathrm{d}\xi &= \int_{\Omega_\varepsilon} \lambda_\varepsilon^{Q-\frac{Q-2}{2}\varepsilon}\left|(D_H u_\varepsilon)\left(\tau_{\eta_\varepsilon}\left(\delta_{\lambda_\varepsilon^{1-\frac{Q-2}{4}\varepsilon}}(\xi)\right)\right)\right|^2 \mathrm{d}\xi \\ &= \lambda_\varepsilon^{\frac{(Q-2)^2}{4}\varepsilon}\int_\Omega |D_H u_\varepsilon|^2\,\mathrm{d}\xi \\ &= \lambda_\varepsilon^{\frac{(Q-2)^2}{4}\varepsilon} < c. \end{aligned} \tag{6.14}$$

In a similar fashion, by means of (3.1) and (6.3), one has

$$\begin{aligned} \int_{\Omega_\varepsilon} |v_\varepsilon|^{2^*}\,\mathrm{d}\xi &= \lambda_\varepsilon^{\frac{Q(Q-2)}{4}\varepsilon}\int_\Omega |u_\varepsilon|^{2^*}\,\mathrm{d}\xi \\ &\le \lambda_\varepsilon^{\frac{Q(Q-2)}{4}\varepsilon} S^* \|D_H u_\varepsilon\|^{2^*}_{L^2(\Omega)} < c. \end{aligned} \tag{6.15}$$

In view of the estimate (6.14) and (6.15) above, condition (5.1) plainly follows. Condition (5.2) can be deduced by Ascoli-Arzelà's Theorem recalling the definition of U. Thus, thanks to the Global Compactness Theorem 5.1 we have that the $v_\varepsilon \to U$ in $S_0^1(\Omega)$.

Step 8. The asymptotic estimate (6.2) holds true. We start by observing that proving (6.2) is equivalent to show that the following estimate holds,

$$v_\varepsilon(\xi) \lesssim U(\xi) \text{ in } \Omega_\varepsilon \quad \text{as } \varepsilon \to 0^+. \tag{6.16}$$

Consider the H-Kelvin transform $v_\varepsilon^\sharp$ of v_ε, according to Definition 2.10; we now have that the proof of (6.16) is equivalent to show that

$$v_\varepsilon^\sharp(\xi) \le c \text{ in } \Omega_\varepsilon^\sharp \quad \text{as } \varepsilon \to 0^+, \tag{6.17}$$

for a suitable constant $c \equiv c(n)$ and where $\Omega_\varepsilon^\sharp$ is the H-Kelvin transformed of Ω_ε. Also, $v_\varepsilon \le 1$ yields that $v_\varepsilon^\sharp \le |\xi|_H^{-(Q-2)}$, and this will reduce the estimate for $v_\varepsilon^\sharp$ to be proven just near the origin.

Let us now focus on the local estimate in (6.17) near the origin. Start noticing that, it suffices to prove it in $B_{\lambda_\varepsilon^{(Q-2)\varepsilon/4-1}} \equiv B_{\lambda_\varepsilon^{(Q-2)\varepsilon/4-1}}(\mathbf{0})$ (which with no loss of generality we can assumed being contained in Ω_ε). Applying the H-inversion map, we have that $B_{\lambda_\varepsilon^{(Q-2)\varepsilon/4-1}}$ is mapped into $\mathbb{H}^n \setminus B_{\lambda_\varepsilon^{1-(Q-2)\varepsilon/4}}$, moreover, since v_ε satisfies (6.8), by Proposition 2.12, its H-Kelvin transform is a solution to

$$\begin{cases} -\Delta_H v_\varepsilon^\sharp(\xi) = |\xi|_H^{-(Q-2)\varepsilon}\Big(v_\varepsilon^\sharp(\xi)\Big)^{2^*-1-\varepsilon} & \text{in } \mathbb{H}^n \setminus B_{\lambda_\varepsilon^{1-(Q-2)\varepsilon/4}}, \\ v_\varepsilon^\sharp \geq 0. \end{cases}$$

Now, calling, $a(\xi) := |\xi|_{\mathbb{H}^n}^{-(Q-2)\varepsilon}$, we have that (by (6.9) $a(\cdot)$ is uniformly bounded in $\mathbb{H}^n \setminus B_{\lambda_\varepsilon^{1-(Q-2)\varepsilon/4}}$ independently of ε. Moreover, we get, for $\sigma \leq \sigma_0$ and $B_{2\sigma} \equiv B_{2\sigma}(\mathbf{0})$,

$$\begin{aligned} &\int_{(\mathbb{H}^n \setminus B_{\lambda_\varepsilon^{1-(Q-2)\varepsilon/4}}) \cap B_{2\sigma}} \Big(v_\varepsilon^\sharp(\xi)\Big)^{2^*-\varepsilon} \mathrm{d}\xi \\ &\leq \left(\int_{(\mathbb{H}^n \setminus B_{\lambda_\varepsilon^{1-(Q-2)\varepsilon/4}}) \cap B_{2\sigma}} \Big(v_\varepsilon^\sharp(\xi)\Big)^{2^*} \mathrm{d}\xi \right)^{\frac{2^*-\varepsilon}{2^*}} |\Omega_\varepsilon^\sharp \cap B_{2\sigma}|^{\frac{\varepsilon}{2^*}} \\ &\leq \left(\int_{\mathbb{H}^n \setminus B_{\lambda_\varepsilon^{1-(Q-2)\varepsilon/4}}} \Big(v_\varepsilon^\sharp(\xi)\Big)^{2^*} \mathrm{d}\xi \right)^{\frac{2^*-\varepsilon}{2^*}} |B_{2\sigma}|^{\frac{\varepsilon}{2^*}} \\ &= \left(\int_{B_{\lambda_\varepsilon^{(Q-2)\varepsilon/4-1}}} v_\varepsilon^{2^*} \mathrm{d}\xi \right)^{\frac{2^*-\varepsilon}{2^*}} |B_{2\sigma}|^{\frac{\varepsilon}{2^*}} \\ &\leq \left(\lambda_\varepsilon^{\frac{Q(Q-2)}{4}\varepsilon} S^* \|D_H u_\varepsilon\|_{L^2(\Omega)}^{2^*} \right)^{\frac{2^*-\varepsilon}{2^*}} |B_{2\sigma_0}|^{\frac{\varepsilon}{2^*}} \end{aligned}$$

where we have used Proposition 2.11 and the estimate (6.15) in the previous step.

By taking σ_0 sufficiently small we have that (2.5) is satisfied for some υ_0, who can be chosen independently of ε and u_ε (recalling that $\|D_H u_\varepsilon\|_{L^2(\Omega)}^2 \leq 1$).

An application of Lemma 2.4 (with $q = 2^* - 1$ there) yields

$$\int_{(\mathbb{H}^n \setminus B_{\lambda_\varepsilon^{1-(Q-2)\varepsilon/4}}) \cap B_\sigma} (v_\varepsilon^\sharp)^{\frac{(2^*)^2}{2}} \mathrm{d}\xi \leq c, \tag{6.18}$$

In particular, the above constant stays bounded as $\varepsilon \to 0^+$. Moreover, by (6.18) we get $\xi \mapsto f(\xi, v_\varepsilon^\sharp) \equiv |\xi|_H^{-(Q-2)\varepsilon} (v_\varepsilon^\sharp)^{2^*-2-\varepsilon} \in L^{q/2}\Big((\mathbb{H}^n \setminus B_{\lambda_\varepsilon^{1-(Q-2)\varepsilon/4}}) \cap B_\sigma\Big)$, a. e., once chosen

$$\frac{q}{2} := \frac{(2^*)^2}{2(2^*-2-\varepsilon)} > \frac{(2^*)^2}{2(2^*-2)} = \frac{Q}{Q-2} \cdot \frac{Q}{2} > \frac{Q}{2}.$$

Indeed, by (6.18), we have

$$\int_{(\mathbb{H}^n \setminus B_{\lambda_\varepsilon^{1-(Q-2)\varepsilon/4}}) \cap B_\sigma} |f(\xi, v_\varepsilon^\sharp)|^{\frac{q}{2}} \mathrm{d}\xi \leq c \int_{(\mathbb{H}^n \setminus B_{\lambda_\varepsilon^{1-(Q-2)\varepsilon/4}}) \cap B_\sigma} (v_\varepsilon^\sharp)^{\frac{(2^*)^2}{2}} \mathrm{d}\xi \leq c,$$

by (6.18), with c not depending on ε. Thus, Lemma 2.5 yields that

$$\sup_{(\mathbb{H}^n \setminus B_{\lambda_\varepsilon^{1-(Q-2)\varepsilon/4}}) \cap B_\sigma} v_\varepsilon^\sharp \leq c, \tag{6.19}$$

where, by Proposition 2.11 and estimate (6.15), the right-hand side in (6.19) is bounded as $\varepsilon \to 0^+$. Hence, by a diagonal argument we eventually arrive at

$$\sup_{B_\sigma \setminus \{\mathbf{0}\}} v_\varepsilon^\sharp \leq c \text{ as } \varepsilon \to 0^+,$$

which finally gives the desired estimate (6.17).

Step 9. Conclusion. Recalling the definition of v_ε in (6.7) and the estimate (6.16), we obtain that

$$u_\varepsilon(\xi) \lesssim \lambda_\varepsilon^{-\frac{Q-2}{2}} U\Big(\delta_{\lambda_\varepsilon^{\frac{Q-2}{4}\varepsilon-1}}(\tau_{\eta_\varepsilon^{-1}}(\xi))\Big),$$

which is (6.2).

□

CHAPTER 7

Localization of the concentration point

A natural question arises: can we localize the blowing up; i. e., is the concentration point ξ_o, whose existence was proven in Theorem 4.2, related to some extent to the geometry of the domain Ω?

In the Euclidean framework, Atkinson and Peletier, via ODEs methods, porved in [8] that the blow up (as $\varepsilon \to 0^+$) of solutions to the critical equation

$$-\Delta u = u^{\frac{n+2}{n-2}-\varepsilon} \quad \text{in } (H_0^1(\Omega))', \tag{7.1}$$

in the case when $\Omega \subset \mathbb{R}^3$ does coincide with a ball, satisfy

$$\lim_{\varepsilon\to 0^+} \varepsilon u_\varepsilon^2(0) = \frac{32}{\pi} \quad \text{and} \quad \lim_{\varepsilon\to 0^+} \frac{u_\varepsilon(x)}{\sqrt{\varepsilon}} = \frac{1}{4}\sqrt{\frac{\pi}{2}}\left(\frac{1}{|x|}-1\right).$$

Subsequently, relying on purely variational techniques, Brezis and Peletier in [22] extended such results to the case of spherical domains. In particular, in [22] it has been showed that subcritical solutions concentrate at one special point of the domain. Moreover, the authors also conjectured that an analogous result should hold for non spherical domains. This conjecture was later proved in the case of smooth domains by Han [73] and Rey [118] by showing that the solutions of (7.1), with maximal Sobolev energy, concentrate at one point which can be localized via the Green's function associated with the underlying domain. The involved proofs strongly rely on the regularity of Euclidean domains, which is in clear contrast with the complexity of the underlying sub-Riemannian geometry here. As is well known, even if the domain Ω is C^∞, the situation is dramatically different because of the possible presence of characteristic points on the boundary $\partial\Omega$. At such points the vector fields forming the principal part of the relevant operator Δ_H become tangent to the boundary. Hence, near those characteristic points – as firstly discovered by Jerison [92, 93] – even harmonic functions on the Heisenberg group can encounter a sudden loss of regularity. Also, one did not want to work in the restricted class of domains not having characteristic points; that is, by still including interesting sets as e. g. the torus obtained by revolting the sphere $\mathbb{S}^{2n}$ around the t-axis, but unfortunately excluding an extremely wide class of regular sets which play a pervasive role in several fundamental problems in the Heisenberg group, as e. g. the level sets of the Jerison & Lee extremal functions (4.5) and those of the Folland fundamental solutions; i. e., the Korányi balls. For this reason, and in order to deal with the aforementioned difficulties, it is then quite natural to work under the assumption that the domain Ω is *geometrical regular near its characteristic set* in the sense of the definitions firstly provided in [67] by Garofalo and Vassilev.

Before giving the definition of geometrical regular set near its characteristic set, some further notations is needed.

We denote by $\mathcal{D}$ the infinitesimal generator of the one-parameter group of non-isotropic dilations $\{\delta_\lambda\}_{\lambda>0}$ in (2.2)

$$\mathcal{D} := \sum_{j=1}^{n} \Big(x_j \partial_{x_j} + y_j \partial_{y_j} \Big) + 2t\partial_t. \tag{7.2}$$

Definition 7.1 (δ_λ**-starlike sets).** *Let Ω be a C^1 connected open set of $\mathbb{H}^n$ containing the group identity $\mathbf{0}$. We say that Ω is* δ_λ-starlike *(with respect to the identity $\mathbf{0}$) along a subset $K \subseteq \partial\Omega$ if*

$$\langle \mathcal{D}, \mathfrak{n} \rangle(\eta) \geq 0,$$

at every $\eta \in K$; in the display above $\mathfrak{n}$ indicates the exterior unit normal to $\partial\Omega$.

We say that Ω is uniformly δ_λ-starlike *(with respect to the identity $\mathbf{0}$) along K if there exists $\kappa_\Omega > 0$ such that, at every $\eta \in K$,*

$$\langle \mathcal{D}, \mathfrak{n} \rangle(\eta) \geq \kappa_\Omega.$$

A domain as above Ω is δ_λ-starlike (uniformly δ_λ-starlike, respectively) with respect to one of its point $\zeta \in \Omega$ along K if $\tau_{\zeta^{-1}}(\Omega)$ is δ_λ-starlike (uniformly δ_λ-starlike, respectively) with respect to the group identity $\mathbf{0}$ along $\tau_{\zeta^{-1}}(K)$.

We finally are in the position to introduce a natural class of regular sets that we take the liberty to christening for shorten; we refer to [67] for further details.

Definition 7.2 (Geometrical regular domains near their characteristic set)**.** *A smooth domain $\Omega \subset \mathbb{H}^n$ such that $\partial\Omega$ is an orientable hypersurface is* "geometrical regular near its characteristic set" *if the following conditions hold true,*

(Ω1) *There exist $\Phi \in C^\infty(\mathbb{H}^n)$, $c_\Omega > 0$ and $\varrho_\Omega \in \mathbb{R}$ such that*

$$\Omega := \Big\{ \Phi < \varrho_\Omega \Big\}, \quad \text{and} \quad |D\Phi| \geq c_\Omega.$$

(Ω2) *For any $\xi \in \partial\Omega$ it holds*

$$\liminf_{\varrho \to 0^+} \frac{|(\mathbb{H}^n \setminus \Omega) \cap B_\varrho(\xi)|}{|B_\varrho(\xi)|} > 0.$$

(Ω3) *There exist M_Ω such that*

$$\Delta_H \Phi \geq \frac{4|z|}{M_\Omega} \langle D_H \Phi, D_H |z| \rangle \quad \text{in } \omega,$$

where ω is an interior neighborhood of $\Sigma(\Omega)$.

(Ω4) *Ω is δ_λ-starlike with respect to one of its point $\zeta_o \in \Omega$ and uniformly δ_λ-starlike with respect to ζ_o along $\Sigma(\Omega)$.*

We are eventually able to deal with the aforementioned sub-Riemannian framework obstacles by proving the desired localization result for the concentration point ξ_o of the maximizing sequence u_ε in terms of the Green function associated with the domain Ω, in turn establishing the validity of the aforementioned Brezis & Peletier conjecture in the Heisenberg group. We have the following

Theorem 7.3. *Let $\Omega \subset \mathbb{H}^n$ be geometrical regular near its characteristic set and let $u_\varepsilon \in S_0^1(\Omega)$ be maximizer for S_ε^*. Then, up to subsequences,*

(i) *u_ε concentrates at some point $\xi_o \in \Omega$ such that*

$$D_H \mathcal{R}_\Omega(\xi_o) = 0\,, \tag{7.3}$$

with $\mathcal{R}_\Omega$ being the Robin function associated to Ω, i. e. $\mathcal{R}_\Omega(\xi) := H(\xi;\xi)$ with $H(\cdot;\cdot)$ being the regular part of the Green function

$$H(\xi;\eta) = G_\Omega(\xi;\eta) - \frac{1}{C_Q|\eta^{-1}\circ\xi|_{\mathcal{H}}^{Q-2}}$$

(ii)

$$\lim_{\varepsilon\to 0+} \varepsilon\|u_\varepsilon\|^2_{L^\infty(\Omega)} = \frac{2Q\bar{c}}{(Q-2)S^*}|\mathcal{R}_\Omega(\xi_0)|\,, \tag{7.4}$$

with S^* being the best Sobolev constant and $\bar{c}$ as in (7.24).

(iii) *for any $\xi \in \Omega\setminus\{\xi_0\}$*

$$\lim_{\varepsilon\to 0+}\frac{u(\xi)}{\sqrt{\varepsilon}} = \left[\frac{\bar{c}(Q-2)S^*}{2Q}\right]^{\frac{1}{2}}\frac{G_\Omega(\xi;\xi_o)}{\sqrt{|\mathcal{R}_\Omega(\xi_0)|}}\,,$$

The proof of our main result stated above will be postponed in the final section of the present manuscript, because it involves several new results – see in particular the theorem stated below – together with various general tools in the sub-Riemaniann framework, as e. g., maximum principles, Caccioppoli-type estimates, H-Kelvin transform, boundary Schauder-type regularity estimates, as well as with a fine boundary analysis for the subcritical CR Yamabe equation.

We remark that still as in [73], one could deduce Theorem 7.3 under the slightly weaker assumption that u_ε solves (3.5) (with a suitable choice of λ) and approximate the Sobolev quotient. Moreover, in the same flavour of subcritical approximations, still in clear accordance with the Euclidean framework studied in [22, 118, 73], one can consider to investigate the asymptotic behaviour of the sequences approaching the critical Sobolev inequality which solve the auxiliary family of equations $-\Delta_H u_\varepsilon = \lambda u_\varepsilon^{2^*-1} + \varepsilon u_\varepsilon$, so that the lack of compactness does similarly come into play when ε goes to 0.

1. Boundary behaviour for the subcritical CR Yamabe equation

In view of the assumptions (Ω1)–(Ω4), one can build fine subelliptic barriers as firstly seen in [67].

Lemma 7.4 (See Theorem 4.3 in [67]). *Let Ω be a smooth bounded domain of $\mathbb{H}^n$ satisfying (Ω1)–(Ω4). For any $\alpha\in(0,1]$ define*

$$\Psi_\alpha := (\varrho_\Omega - \Phi)^\alpha e^{-|z|^2/M_\Omega}.$$

Given a neighborhood K of the characteristic set $\Sigma(\Omega)$ such that

$$\overline{K} \subset \left\{|z|^2 \le \frac{nM_\Omega}{2}\right\}, \tag{7.5}$$

we have that

$$\Delta_H\Psi_\alpha \le -\frac{2n}{M_\Omega}\Psi_\alpha \quad \text{on } \omega := \Omega\cap K.$$

Furthermore, there exist $c_1, c_2 > 0$ such that for any $\eta_\lambda \in \partial\Omega\cap K$ and any $\lambda\in[\lambda_o,1]$ it holds

$$c_1(1-\lambda)^\alpha \le \Psi_\alpha(\eta_\lambda) \le c_2(1-\lambda)^\alpha.$$

We now prove the main result of this section. We remark that we denote by K an open neighborhood of the characteristic set $\Sigma(\Omega)$ not containing the concentration point ξ_o and with $\omega := \Omega\cap K$.

Theorem 7.5. *Let Ω be a smooth bounded domain of $\mathbb{H}^n$ satisfying (Ω1)–(Ω4), and let $u_\varepsilon \in S^1_0(\Omega)$ be a positive maximizer for S^*_ε. Then, there exists $\bar\varepsilon \equiv \bar\varepsilon(n) > 0$ such that, for any $\varepsilon < \bar\varepsilon$,*

$$\int_\gamma |D_H u_\varepsilon|^2 \langle \mathcal{D}, \mathfrak{n}\rangle \, dH^{Q-2} \le c \sup_{\eta\in\partial\Omega} \langle \mathcal{D}, \mathfrak{n}\rangle(\eta) H^{Q-2}(\gamma), \tag{7.6}$$

where γ is any hypersurface contained in ω and $c \equiv c(n) > 0$.

Let us remark that by the uniformly δ_λ-starlikeness of Ω along its characteristic set $\Sigma(\Omega)$ in (Ω4), up to taking a smaller neighborhood K, we get

$$\mathcal{D}\Phi(\eta) \ge c > 0, \qquad \forall \eta \in \partial\Omega \cap K, \tag{7.7}$$

where Φ is the defining function of Ω in (Ω1) and $\mathcal{D}$ is defined in (7.2).

Condition (7.7) implies that the trajectories of $\mathcal{D}$ starting from $\partial\Omega \cap K$ fill a full open set interior of Ω. Indeed, considering $\eta \in \partial\Omega \cap K$ and taking the Taylor expansion of $f(\lambda) := \Phi(\delta_\lambda(\eta))$ around $\lambda = 1$ we obtain that

$$\varrho_\Omega - \Phi(\delta_\lambda(\eta)) = \mathcal{D}\Phi(\eta)(1-\lambda) + o(1-\lambda) \ge c(1-\lambda).$$

Hence, with no loss of generality, up to further shrinking K, we assume there exists λ_o such that

$$\delta_\lambda(\eta) \in \Omega \cap K, \qquad \text{for } \lambda_o < \lambda < 1. \tag{7.8}$$

Proof of Theorem 7.5. Consider an open neighborhood K of the characteristic set $\Sigma(\Omega)$ not containing ξ_o and such that (7.8) holds true for any $\eta \in \partial\Omega \cap K$.

With the notation above, we prove that

$$u_\varepsilon(\delta_\lambda(\eta)) \le c(1-\lambda) \ \text{ for any } \eta \in \partial\Omega \cap K, \tag{7.9}$$

for any $\varepsilon < \bar\varepsilon(n)$; with λ as above.

Indeed, fix $\varepsilon \in (0, 2^* - 2)$. By Theorem 6.1 we get that there exists $\bar\varepsilon \equiv \bar\varepsilon(n) > 0$ such that

$$u_\varepsilon \lesssim U_{\lambda_\varepsilon, \eta_\varepsilon} \ \text{ on } \omega := \Omega \cap K, \ \text{ for any } \varepsilon < \bar\varepsilon.$$

Hence, $u_\varepsilon \in L^\infty(\omega)$, since $U_{\lambda_\varepsilon,\eta_\varepsilon}$ stays bounded in ω, giving that $\xi_o \notin \omega$. As explained in the proof of Theorem 6.1, form this we can deduce that $u_\varepsilon \in C^\infty(\omega)$ for any $\varepsilon < \bar\varepsilon$. Moreover, using condition (Ω2) one can adapt to the present setting the classical Moser iteration argument to get $u_\varepsilon \in \Gamma^{0,\alpha_\varepsilon}(\partial\Omega \cap K)$, for some exponent $1 > \alpha_\varepsilon > 0$ for any $\varepsilon < \bar\varepsilon$, up to scaling down $\bar\varepsilon$. Thus,

$$u_\varepsilon \in \Gamma^{0,\alpha_\varepsilon}(\overline{\omega}) \cap C^\infty(\omega).$$

Now, since $u_\varepsilon \equiv 0$ on $\partial\Omega$, we have that for any $\eta \in \partial\Omega$

$$u_\varepsilon(\eta_\lambda) \lesssim |\eta_\lambda \circ \eta|_H^{\alpha_\varepsilon} \ \text{ for any } \varepsilon < \bar\varepsilon.$$

Since homogeneous norms can be controlled up to a constant $c \equiv c(\Omega)$ in the following way as shown in Lemma 4.1 in [67],

$$|\eta_\lambda \circ \eta|_H \le c(1-\lambda)^{1/2},$$

where we denoted by $\eta_\lambda := \delta_\lambda(\eta)$, we obtain that

$$u_\varepsilon(\eta_\lambda) \le c(1-\lambda)^{\frac{\alpha_\varepsilon}{2}}, \qquad \forall \eta \in \partial\Omega \cap K, \forall \lambda \in [\lambda_o, 1]. \tag{7.10}$$

Noting that, for any $\varepsilon < \min\left\{\bar\varepsilon, 4/(Q-2)\right\}$, we have that $2^* - 1 - \varepsilon > 1$ we choose $m \in \mathbb{N}$ such that $(2^* - 1 - \varepsilon)^{-m} \le \alpha_\varepsilon/2$. Since the estimate in (7.10) does

hold for any $\eta \in \partial\Omega \cap K$ and any $\lambda \in [\lambda_o, 1]$, we can assume that the points η_λ cover $\overline{\omega}$. Thus,

$$u_\varepsilon \leq c(1-\lambda)^{(2^*-1-\varepsilon)^{-m}} \text{ on } \overline{\omega}, \tag{7.11}$$

up to relabelling $\bar{\varepsilon}$ as above. Hence, up to taking a smaller neighborhood K such that (7.5) is satisfied, we have that for η_λ, by (7.11) and Lemma 7.4, it holds

$$\begin{aligned} -\Delta_H u_\varepsilon(\eta_\lambda) &= u(\eta_\lambda)^{2^*-1-\varepsilon} \\ &= c^{2^*-1-\varepsilon}(1-\lambda)^{(2^*-1-\varepsilon)^{1-m}} \\ &\leq c^{2^*-1-\varepsilon} c_1^{-1} \Psi_{(2^*-1-\varepsilon)^{1-m}}(\eta_\lambda) \\ &\leq -c^{2^*-1-\varepsilon} c_1^{-1} \frac{M_\Omega}{2n} \Delta_H \Psi_{(2^*-1-\varepsilon)^{1-m}}(\eta_\lambda) \\ &\leq -\Delta_H (c_o \Psi_{(2^*-1-\varepsilon)^{1-m}})(\eta_\lambda). \end{aligned}$$

Thus,

$$\Delta_H (c_o \Psi_{(2^*-1-\varepsilon)^{1-m}} - u_\varepsilon) \leq 0 \text{ in } \omega. \tag{7.12}$$

Moreover, proceeding as in [129, Theorem 5.14], we get

$$c_o \Psi_{(2^*-1-\varepsilon)^{1-m}} \geq u_\varepsilon \text{ on } \partial\omega. \tag{7.13}$$

Combining together (7.12) with (7.13), by Bony's Maximum Principle we obtain

$$u_\varepsilon(\eta_\lambda) \leq c_o \Psi_{(2^*-1-\varepsilon)^{1-m}} \leq c_o c_2 (1-\lambda)^{(2^*-1-\varepsilon)^{1-m}} \text{ on } \overline{\omega},$$

which is a refinement of (7.11). Iterating this procedure m-times yields

$$u_\varepsilon(\delta_\lambda(\eta)) \leq c(1-\lambda) \text{ for any } \eta \in \partial\Omega \cap K.$$

Fix now an hypersurface $\gamma \subset \omega$ and cover it with a family $\{B_\varrho^{(i)}\}_{i=1}^{\vartheta}$ such that $B_\varrho(\xi_i) \equiv B_\varrho^{(i)} \subset \omega$, for any $i = 1, \dots, \vartheta$. By the interior estimate of [11, Theorem 3.1], we have that on every ball $B_\varrho^{(i)}$ it holds

$$\begin{aligned} \|D_H u_\varepsilon\|_{L^\infty(B_\varrho^{(i)})} &\leq \frac{2c}{\varrho}\left(\|u_\varepsilon\|_{L^\infty(\omega)} + \varrho^2 \|u_\varepsilon\|_{L^\infty(\omega)}^{2^*-1-\varepsilon}\right) \\ &\leq \frac{c}{\varrho}\left((1-\lambda) + \varrho^2 (1-\lambda)^{2^*-1-\varepsilon}\right), \end{aligned} \tag{7.14}$$

by (7.9), for any $\lambda \in [\lambda_o, 1]$. Note that the estimate above is bounded for any $\lambda \in [\lambda_o, 1]$ and for any value of ε. Moreover, by the δ_λ-starlikeness we have that

$$0 < \int_\gamma \langle \mathcal{D}, \mathfrak{n} \rangle \, dH^{Q-2} \leq \sup_{\zeta \in \partial\Omega} \langle \mathcal{D}, \mathfrak{n} \rangle(\zeta) H^{Q-2}(\gamma). \tag{7.15}$$

Thus, combining (7.14) and (7.15) with a standard covering argument yields

$$\int_\gamma |D_H u_\varepsilon|^2 \langle \mathcal{D}, \mathfrak{n} \rangle dH^{Q-2} \leq c \sup_{\zeta \in \partial\Omega} \langle \mathcal{D}, \mathfrak{n} \rangle(\zeta) H^{Q-2}(\gamma) \text{ for any } \varepsilon < \bar{\varepsilon},$$

where c does not depend on γ. □

We conclude this section by noticing that, in view of the hypotheses as in (Ω1)–(Ω4), it readily follows the boundedness of the $\mathcal{D}$-derivatives using the same argument developed in [67].

Theorem 7.6. *Let Ω be a smooth bounded domain of $\mathbb{H}^n$ satisfying (Ω1)–(Ω4), and let $u_\varepsilon \in S_0^1(\Omega)$ be a positive maximizer for S_ε^*. Then, there exists $\bar{\varepsilon} \equiv \bar{\varepsilon}(n)$ such that, for any $\varepsilon < \bar{\varepsilon}$ (uniformly),*

$$\mathcal{D} u_\varepsilon \in L^\infty(\overline{\omega}), \tag{7.16}$$

with $\omega := \Omega \cap K$, where K is an open neighborhood of the characteristic set $\Sigma(\Omega)$ not containing ξ_o.

2. Proof of the localization result via Green function

We are finally in the position to present the proof of Theorem 7.3, whose argument involves different techniques and results such as the asymptotic control via the Jerison & Lee optimal functions established in Theorem 6.1, the Pohozaev identity, the regularity theory for the subcritical CR Yamabe equation in Theorem 7.6 as well as the integral estimate in Theorem 7.5. Moreover, we will use the negligibility of the characteristic set, established in Theorem 2.9

Proof of Theorem 7.3. For the sake of readability, we divide the proof into several steps.

Step. 1 For any $\varepsilon \in (0, \bar{\varepsilon})$ it holds

$$\frac{\varepsilon(Q-2)}{2^*-\varepsilon}\int_\Omega u_\varepsilon^{2^*-\varepsilon}\,\mathrm{d}\xi = \int_{\partial\Omega} |D_H u_\varepsilon|^2 \langle \mathcal{D}, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2}. \tag{7.17}$$

We start recalling that since Ω is a smooth domain the characteristic set $\Sigma(\Omega)$ is compact. Moreover, thanks to Theorem 2.9 and Theorem 2.8, we consider an exhaustion of Ω of C^∞ connected open sets $\{\Omega_i\}$ such that $\Omega_i \uparrow \Omega$, $u_\varepsilon \in C^2(\overline{\Omega_i})$ and $\partial\Omega_i = \gamma_i^{(1)} \cup \gamma_i^{(2)}$ with $\gamma_i^{(1)} \subset \partial\Omega \setminus \Sigma(\Omega)$, $\gamma_i^{(1)} \uparrow \partial\Omega \setminus \Sigma(\Omega)$ and $\mathcal{H}^{Q-2}(\gamma_i^{(2)}) \to 0$.

We apply the Pohozaev identity on u_ε in Ω_i, getting

$$\begin{aligned}
&\int_{\Omega_i} \Big(\frac{2Q}{2^*-\varepsilon} u_\varepsilon^{2^*-\varepsilon} - (Q-2) u_\varepsilon^{2^*-\varepsilon}\Big)\,\mathrm{d}\xi \\
&\quad = 2\sum_{j=1}^{2n} \int_{\gamma_i^{(1)}\cup\gamma_i^{(2)}} \mathcal{D} u_\varepsilon \, Z_j u_\varepsilon \langle Z_j, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2} - \int_{\gamma_i^{(1)}\cup\gamma_i^{(2)}} |D_H u_\varepsilon|^2 \langle \mathcal{D}, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2} \\
&\quad = 2\sum_{j=1}^{2n} \int_{\gamma_i^{(1)}} \mathcal{D} u_\varepsilon \, Z_j u_\varepsilon \langle Z_j, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2} - \int_{\gamma_i^{(1)}} |D_H u_\varepsilon|^2 \langle \mathcal{D}, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2} \\
&\qquad + 2\sum_{j=1}^{2n} \int_{\gamma_i^{(2)}} \mathcal{D} u_\varepsilon \, Z_j u_\varepsilon \langle Z_j, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2} - \int_{\gamma_i^{(2)}} |D_H u_\varepsilon|^2 \langle \mathcal{D}, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2}.
\end{aligned} \tag{7.18}$$

Note that since $u_\varepsilon > 0$ in Ω_i and $u_\varepsilon = 0$ on $\gamma_i^{(1)}$, then there exists a function $w \le 0$ such that $Du_\varepsilon = w\boldsymbol{n}$ on $\gamma_i^{(1)}$. Hence,

$$\begin{aligned}
\sum_{j=1}^{2n} \mathcal{D} u_\varepsilon \, Z_j u_\varepsilon \langle Z_j, \boldsymbol{n}\rangle &= w\langle \mathcal{D}, \boldsymbol{n}\rangle \sum_{j=1}^{2n} Z_j u_\varepsilon \langle Z_j, \boldsymbol{n}\rangle \\
&= \langle \mathcal{D}, \boldsymbol{n}\rangle \sum_{j=1}^{2n} Z_j u_\varepsilon \langle Z_j, w\boldsymbol{n}\rangle \\
&= \langle \mathcal{D}, \boldsymbol{n}\rangle \sum_{j=1}^{2n} Z_j u_\varepsilon \underbrace{\langle Z_j, Du_\varepsilon\rangle}_{=: Z_j u_\varepsilon} = |D_H u_\varepsilon|^2 \langle \mathcal{D}, \boldsymbol{n}\rangle.
\end{aligned}$$

Then, putting the computation above inside (7.18) yields

$$\frac{\varepsilon(Q-2)}{2^*-\varepsilon}\int_{\Omega_i} u_\varepsilon^{2^*-\varepsilon}\,\mathrm{d}\xi = \int_{\gamma_i^{(1)}} |D_H u_\varepsilon|^2 \langle \mathcal{D}, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2}$$

$$+2\sum_{j=1}^{2n}\int_{\gamma_i^{(2)}}\mathcal{D}u_\varepsilon\, Z_j u_\varepsilon\langle Z_j,\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2}$$
$$-\int_{\gamma_i^{(2)}}|D_H u_\varepsilon|^2\langle\mathcal{D},\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2}.$$

Now recalling Theorem 7.5 we have that

$$\int_{\gamma_i^{(2)}}|D_H u_\varepsilon|^2\langle\mathcal{D},\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2}\approx\mathcal{H}^{Q-2}(\gamma_i^{(2)}),$$

and by Theorem 7.6 the $\mathcal{D}$-derivative of u_ε stays bounded, uniformly in ε near the characteristic set. Thus, passing to the limit as $i\to\infty$, recalling the δ_λ-starlikeness of Ω and estimate (7.14), by the Dominate Convergence and the Monotone Convergence theorem we eventually arrive at the desired estimate (7.17).

Step 2. The following limit holds

$$\lim_{\varepsilon\to 0^+}\lambda_\varepsilon^\varepsilon=1. \tag{7.19}$$

We start proving that there exists a constant $c\equiv c(n,\Omega)>0$ such that

$$\int_{\partial\Omega}|D_H u_\varepsilon|^2\langle\mathcal{D},\boldsymbol{n}\rangle\,\mathrm{d}H^{Q-2}\le c\lambda_\varepsilon^{Q-2-\frac{(Q-2)^2}{2}\varepsilon}. \tag{7.20}$$

Consider the Taylor polynomial $P_1(u_\varepsilon,\xi)$ of u_ε with center in $\xi\in\partial\Omega\setminus\Sigma(\Omega)$. As proven in [11, Formula (4.55)] for any point $\varsigma\in\Omega$ such that

$$|\varsigma^{-1}\circ\xi|_H=\mathrm{dist}(\varsigma,\partial\Omega)\sim\bar\sigma,$$

for $\bar\sigma>0$, it holds

$$\begin{aligned}|D_H u_\varepsilon(\xi)| &\le |D_H u_\varepsilon(\varsigma)-D_H P_1(u_\varepsilon,\xi)|+|D_H u_\varepsilon(\varsigma)|\\ &\le \frac{c}{\bar\sigma}\Big(\|u_\varepsilon-P_1(u_\varepsilon,\xi)\|_{L^\infty(B_{a\bar\sigma}(\varsigma))}+\bar\sigma^2\|u_\varepsilon\|^{2^*-1-\varepsilon}_{L^\infty(B_{a\bar\sigma}(\varsigma))}\Big)+|D_H u_\varepsilon(\varsigma)|,\end{aligned}$$

where, for $a>0$, $B_{a\bar\sigma}$ is a non-tangential ball from inside of Ω.

Such a ball can be constructed since we are considering non-characteristic points.

Indeed, up to left translations we assume that $\xi\equiv\mathbf{0}$. Moreover, since $\mathbf{0}$ is not characteristic there exists $j\in\{1,\dots,2n\}$ such that

$$\langle Z_j,\boldsymbol{n}\rangle(\mathbf{0})\neq 0.$$

Now, by an orthogonal transformation and the implicit function theorem we can assume the existence of $\varrho_0>0$ such that $\Omega\cap B_{\varrho_0}$ can be represented as

$$\Big\{x_n>\widetilde\Phi(x',y,t)\Big\}\ \text{ where } x':=(x_1,\dots,x_{n-1})\in\mathbb{R}^{n-1},$$

for a Lipschitz function $\widetilde\Phi$ such that $\widetilde\Phi(\mathbf{0})=0$, $\nabla_{x'}\widetilde\Phi(\mathbf{0})=0$. In view of the Lipschitz continuity of the function $\widetilde\Phi$ we can assert that, up to taking λ sufficiently small, the ball $B_{\lambda s}(\delta_\lambda(e_n,0))$ is strictly contained in Ω, for $s>0$ small enough. We also refer the reader to the proof of Proposition 3.3 in [11] where a non-tangential ball from outside was determined, and to the proof of Theorem 7.6 in [41].

Now, we take $\bar\sigma$ sufficiently small such that $\xi_o\notin B_{a\bar\sigma}$ and such that

$$\frac{c}{\bar\sigma}\|u_\varepsilon-P_1(u_\varepsilon,\xi)\|_{L^\infty(B_{a\bar\sigma}(\varsigma))}\le c\bar\sigma^\beta\le\frac14|D_H u_\varepsilon(\xi)|. \tag{7.21}$$

Note that the first estimate in the display above comes from the proof of Theorem 1.1 in [11]; see in particular Page 26 there.

Recalling now that, choosing $\phi \in C_0^\infty(B_\varrho)$ with $\xi_o \notin B_\varrho$ and $B_\varrho \subset \overline{\Omega}$, the concentration result of Theorem 4.2 yields

$$\lim_{\varepsilon\to 0^+} \int_{B_\varrho} \phi |D_H u_\varepsilon|^2 \, \mathrm{d}\xi = 0.$$

Thus, up to taking a smaller ball $B_\sigma \subset B_\varrho$ and ϕ such that $\phi \equiv 1$ on B_σ, we have that $\|D_H u_\varepsilon\|_{L^2(B_\sigma)} \to 0^+$ as $\varepsilon \to 0^+$. Hence, up to subsequences, we have that $|D_H u_\varepsilon(\varsigma)| \to 0^+$ as $\varepsilon \to 0^+$, for a. e. $\varsigma \neq \xi_o$. Then, take ε sufficiently small such that, still up to subsequences,

$$|D_H u_\varepsilon(\varsigma)| \leq \frac{1}{4} |D_H u_\varepsilon(\xi)| \ \text{ for a. e. } \varsigma \neq \xi_o \,. \tag{7.22}$$

All in all, by inserting (7.21) and (7.22) in (7.20) and recalling the interior asymptotic estimate (6.2) implies that $u_\varepsilon = o(\lambda_\varepsilon^{\frac{Q-2}{2} - \frac{(Q-2)^2}{4}\varepsilon})$ when $\xi \neq \xi_o$, we get that for ε sufficiently small

$$|D_H u_\varepsilon(\xi)| = o\Big(\lambda_\varepsilon^{\frac{Q-2}{2} - \frac{(Q-2)^2}{4}\varepsilon}\Big)$$

for a. e. $\xi \in \partial\Omega \setminus \Sigma(\Omega)$ and for some $c > 0$ bounded uniformly as $\varepsilon \to 0^+$, since $\xi_o \notin B_{a\bar{\sigma}}(\varsigma)$, which implies the desired estimate in (7.20), recalling that

$$\int_{\partial\Omega} \langle \mathcal{D}, \boldsymbol{n} \rangle \, \mathrm{d}H^{Q-2} = Q|\Omega|,$$

by the divergence theorem (since $\operatorname{div} \mathcal{D} = Q$).

Now, by (7.17) we obtain that

$$\varepsilon \leq \lambda_\varepsilon^{Q-2-\frac{(Q-2)^2}{2}\varepsilon},$$

which gives that

$$|\lambda_\varepsilon^\varepsilon - 1| = O(\lambda_\varepsilon^{Q-2} |\ln \lambda_\varepsilon|)\,,$$

one recalled (6.13) and (6.9). Hence, the desired (7.19) follows.

Step 3. The following limit holds true

$$\|u_\varepsilon\|_{L^\infty(\Omega)} u_\varepsilon \xrightarrow{\varepsilon\to 0} \bar{c}\, G_\Omega(\cdot, \xi_o) \ \textbf{in } \Gamma^{1,\beta}(\partial\Omega \setminus \Sigma(\Omega)), \tag{7.23}$$

for some $\beta \in (0,1)$ **and** $\bar{c}$ **is given by**

$$\bar{c} := \int_{\mathbb{H}^n} U^{\frac{Q+2}{Q-2}} \, \mathrm{d}\xi. \tag{7.24}$$

with U **defined in** (4.5)**.**

Define the function

$$w_\varepsilon := \|u_\varepsilon\|_{L^\infty(\Omega)} u_\varepsilon^{2^*-1-\varepsilon} \equiv \lambda_\varepsilon^{-\frac{Q-2}{2}} u_\varepsilon^{2^*-1-\varepsilon}.$$

We show that $w_\varepsilon \to \bar{c}\delta_{\xi_o}$, in the sense of distribution. For any $\phi \in C_0^\infty(\Omega)$, we have that

$$\begin{aligned}
\int_\Omega \phi w_\varepsilon \, \mathrm{d}\xi &= \lambda_\varepsilon^{-\frac{Q-2}{2}} \int_\Omega \phi u_\varepsilon^{2^*-1-\varepsilon} \, \mathrm{d}\xi \\
&= \lambda_\varepsilon^{-\left(\frac{Q-2}{2}\right)^2 \varepsilon} \int_{\Omega_\varepsilon} \phi(\tau_{\eta_\varepsilon}(\delta_{\lambda_\varepsilon^{1-\frac{Q-2}{4}\varepsilon}}(\eta)) v_\varepsilon^{2^*-1-\varepsilon}(\eta) \, \mathrm{d}\eta \\
&\to \phi(\xi_o) \int_{\mathbb{H}^n} U^{\frac{Q+2}{Q-2}} \, \mathrm{d}\xi \equiv \phi(\xi_o)\bar{c},
\end{aligned} \tag{7.25}$$

where we have used the definition in (6.7), the fact that $v_\varepsilon \to U$ uniformly on compact set (by Ascoli-Arzelà's Theorem), $v_\varepsilon \to 0$ when $|\xi|_H \to \infty$ uniformly in ε, $\Omega_\varepsilon \to \mathbb{H}^n$ and the asymptotic (7.19) for $\lambda_\varepsilon^\varepsilon$.

Moreover, note that when $\xi \neq \xi_o$, by (6.9) of Theorem 6.1, we get that

$$w_\varepsilon \lesssim \lambda_\varepsilon^{-\frac{Q-2}{2}-\frac{Q-2}{2}(2^*-1-\varepsilon)} U\Big(\delta_{\lambda_\varepsilon^{\frac{Q-2}{4}\varepsilon-1}}(\tau_{\eta_\varepsilon^{-1}}(\xi))\Big)^{2^*-1-\varepsilon} = \mathrm{o}(\lambda_\varepsilon)\,,$$

which yields that

$$w_\varepsilon \to 0 \qquad \text{as } \varepsilon \to 0^+. \tag{7.26}$$

Thus, combining (7.25) and (7.26) we obtain that $w_\varepsilon \to \bar{c}\delta_{\xi_o}$ in the sense of distribution.

Furthermore, note that the function $\|u_\varepsilon\|_{L^\infty(\Omega)}u_\varepsilon$ is a solution to

$$\begin{cases} -\Delta_H\Big(\|u_\varepsilon\|_{L^\infty(\Omega)}u_\varepsilon\Big) = \|u_\varepsilon\|_{L^\infty(\Omega)}u_\varepsilon^{2^*-1-\varepsilon} & \text{in } \Omega\cap\omega, \\ \|u_\varepsilon\|_{L^\infty(\Omega)}u_\varepsilon = 0 & \text{in } \partial\Omega\cap\omega, \end{cases}$$

where ω is an interior neighborhood of $\partial\Omega \setminus \Sigma(\Omega)$.

Consider a ball B_ϱ such that $B_\varrho \cap (\partial\Omega \setminus \Sigma(\Omega))$ is not characteristic and choose $r > 0$ sufficiently small such that $B_\varrho \cap \Omega$ does not contain ξ_o. By (7.26) since $\|u_\varepsilon\|_{L^\infty(\Omega)}u_\varepsilon = 0$ on $\partial\Omega$, we have that the hypotheses in Theorem 2.8 are satisfied. Thus, the sequence $\{\|u_\varepsilon\|_{L^\infty(\Omega)}u_\varepsilon\}$ is compact in $\Gamma^{1,\beta}(\overline{B_\varrho\cap\Omega})$, for some $\beta \in (0,1)$. Moreover, from (7.25) and (7.26) it has to converge to $\bar{c}G_\Omega(\cdot,\xi_o)$. Indeed, for any test function $\phi \in C_0^\infty(B_\varrho\cap\Omega)$ it holds

$$\begin{aligned} -\int_{B_\sigma\cap\Omega} \|u_\varepsilon\|_{L^\infty(\Omega)}u_\varepsilon\Big(\Delta_H\phi\Big)\,\mathrm{d}\xi &= -\int_{B_\sigma\cap\Omega}\Big(\Delta_H\|u_\varepsilon\|_{L^\infty(\Omega)}u_\varepsilon\Big)\phi\,\mathrm{d}\xi \\ &= \int_{B_\sigma\cap\Omega} w_\varepsilon\phi\,\mathrm{d}\xi \\ &\to \int_{B_\sigma\cap\Omega}\bar{c}\delta_{\xi_o}\phi\,\mathrm{d}\xi = -\int_{B_\sigma\cap\Omega}\bar{c}G_\Omega(\cdot,\xi_o)\Big(\Delta_H\phi\Big)\,\mathrm{d}\xi. \end{aligned}$$

Hence, (7.23) follows taking an open cover of $\partial\Omega\setminus\Sigma(\Omega)$.

Step 4. For any $k \in \{1,\dots,2n\}$ it holds

$$\int_{\partial\Omega} |D_H u_\varepsilon|^2\langle Z_k, \boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2} = 0. \tag{7.27}$$

Let us consider the exhaustion of Ω of C^∞ connected open sets $\{\Omega_i\}$ of the previous step, i. e. such that $\Omega_i \uparrow \Omega$, $u_\varepsilon \in C^2(\overline{\Omega_i})$ and $\partial\Omega_i = \gamma_i^{(1)} \cup \gamma_i^{(2)}$ with $\gamma_i^{(1)} \subset \partial\Omega\setminus\Sigma(\Omega)$, $\gamma_i^{(1)} \uparrow \partial\Omega\setminus\Sigma(\Omega)$ and $\mathcal{H}^{Q-2}(\gamma_i^{(2)}) \to 0$. Test the equation with $Z_k u_\varepsilon$. We have

$$\begin{aligned} &-\int_{\Omega_i} \Delta_H u_\varepsilon Z_k u_\varepsilon\,\mathrm{d}\xi \\ &= -\sum_{j=1}^{2n}\int_{\Omega_i} Z_j^2 u_\varepsilon Z_k u_\varepsilon\,\mathrm{d}\xi \\ &= \sum_{j=1}^{2n}\int_{\Omega_i} Z_j u_\varepsilon Z_j Z_k u_\varepsilon\,\mathrm{d}\xi - \sum_{j=1}^{2n}\int_{\partial\Omega_i} Z_j u_\varepsilon Z_k u_\varepsilon\langle Z_j,\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2} \\ &= \sum_{j=1}^{2n}\int_{\Omega_i} Z_j u_\varepsilon Z_k Z_j u_\varepsilon\,\mathrm{d}\xi - \sum_{j=1}^{2n}\int_{\partial\Omega_i} Z_j u_\varepsilon Z_k u_\varepsilon\langle Z_j,\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2} \end{aligned}$$

$$
\begin{aligned}
&-4\sum_{j=1}^{2n}\int_{\Omega_i} Z_j u_\varepsilon T u_\varepsilon \,\mathrm{d}\xi \\
&= \frac{1}{2}\sum_{j=1}^{2n}\int_{\Omega_i} Z_k\Big(Z_j u_\varepsilon\Big)^2 \,\mathrm{d}\xi - \sum_{j=1}^{2n}\int_{\partial\Omega_i} Z_j u_\varepsilon Z_k u_\varepsilon \langle Z_j, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2} \\
&\quad + 4\sum_{j=1}^{2n}\int_{\Omega_i} T Z_j u_\varepsilon u_\varepsilon \,\mathrm{d}\xi - 4\sum_{j=1}^{2n}\int_{\partial\Omega_i} Z_j u_\varepsilon u_\varepsilon \langle T, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2},
\end{aligned} \tag{7.28}
$$

where we have integrated by parts and used the fact that $[Z_k, Z_j] = -4T$ whenever $|i-j| = n$, and $[Z_k, Z_j] = 0$ otherwise. Again, since $u_\varepsilon > 0$ in Ω and $u_\varepsilon = 0$ on $\partial\Omega_i$, then there exists a function $w \le 0$ such that $Du_\varepsilon = w\boldsymbol{n}$ on $\gamma_i^{(1)}$. Hence,

$$
\begin{aligned}
\sum_{j=1}^{2n} Z_j u_\varepsilon \, Z_k u_\varepsilon \langle Z_j, \boldsymbol{n}\rangle &= w\langle Z_k, \boldsymbol{n}\rangle \sum_{j=1}^{2n} Z_j u_\varepsilon \langle Z_j, \boldsymbol{n}\rangle \\
&= \langle Z_k, \boldsymbol{n}\rangle \sum_{j=1}^{2n} Z_j u_\varepsilon \langle Z_j, w\boldsymbol{n}\rangle \\
&= \langle Z_k, \boldsymbol{n}\rangle \sum_{j=1}^{2n} Z_j u_\varepsilon \underbrace{\langle Z_j, Du_\varepsilon\rangle}_{=: Z_j u_\varepsilon} = |D_H u_\varepsilon|^2 \langle Z_k, \boldsymbol{n}\rangle.
\end{aligned}
$$

Moreover, since $[T, Z_j] = 0$ for any $j \in \{1, \dots, 2n\}$ and $u = 0$ on the boundary $\gamma_i^{(1)}$, form (7.28) we get

$$
\begin{aligned}
&-\int_{\Omega_i} \Delta_H u_\varepsilon Z_k u_\varepsilon \,\mathrm{d}\xi \\
&= \frac{1}{2}\sum_{j=1}^{2n}\int_{\Omega_i} Z_k\Big(Z_j u_\varepsilon\Big)^2 \,\mathrm{d}\xi - \int_{\gamma_i^{(1)}} |D_H u_\varepsilon| \langle Z_k, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2} \\
&\quad -4\sum_{j=1}^{2n}\int_{\gamma_i^{(2)}} Z_j u_\varepsilon u_\varepsilon \langle T, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2} - \sum_{j=1}^{2n}\int_{\gamma_i^{(2)}} Z_j u_\varepsilon Z_k u_\varepsilon \langle Z_j, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2} \\
&= \frac{1}{2}\int_{\gamma_i^{(1)}} |D_H u_\varepsilon| \langle Z_k, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2} - \int_{\gamma_i^{(1)}} |D_H u_\varepsilon| \langle Z_k, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2} \\
&\quad -4\sum_{j=1}^{2n}\int_{\gamma_i^{(2)}} Z_j u_\varepsilon u_\varepsilon \langle T, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2} - \sum_{j=1}^{2n}\int_{\gamma_i^{(2)}} Z_j u_\varepsilon Z_k u_\varepsilon \langle Z_j, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2} \\
&\quad +\frac{1}{2}\int_{\gamma_i^{(2)}} |D_H u_\varepsilon|^2 \langle Z_k, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2} \\
&= -\frac{1}{2}\int_{\gamma_i^{(1)}} |D_H u_\varepsilon| \langle Z_k, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2} - 4\sum_{j=1}^{2n}\int_{\gamma_i^{(2)}} Z_j u_\varepsilon u_\varepsilon \langle T, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2} \\
&\quad -\sum_{j=1}^{2n}\int_{\gamma_i^{(2)}} Z_j u_\varepsilon Z_k u_\varepsilon \langle Z_j, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2} + \frac{1}{2}\int_{\gamma_i^{(2)}} |D_H u_\varepsilon|^2 \langle Z_k, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2}
\end{aligned}
$$

after integrating by parts. Hence, passing to the limit for $i \to \infty$, by an analogous argument as in Step 2. we obtain

$$
-\int_\Omega \Delta_H u_\varepsilon Z_k u_\varepsilon \,\mathrm{d}\xi = -\frac{1}{2}\int_{\partial\Omega} |D_H u_\varepsilon| \langle Z_k, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2}. \tag{7.29}
$$

Moreover, denoting with $f(u) := u_\varepsilon^{2^*-1-\varepsilon}$ and with $F(u) := \int_0^u f(\tau)\,\mathrm{d}\tau$, we have

$$\begin{aligned}\int_\Omega f(u) Z_k u \,\mathrm{d}\xi &= \int_\Omega Z_k\Big(F(u)\Big)\,\mathrm{d}\xi \\ &= \int_{\partial\Omega} F(u)\langle Z_k, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2} = 0\,, \end{aligned} \tag{7.30}$$

recalling that $u_\varepsilon = 0$ on the boundary $\partial\Omega$.

Then, (7.27) follows combining together (7.29) with (7.30).

Step 5. Let us prove that, for any $k \in \{1,\dots,2n\}$,

$$\int_{\partial\Omega} |D_H G_\Omega(\cdot,\xi_\mathrm{o})|^2 \langle Z_k, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2} = -Z_k \mathcal{R}_\Omega(\xi_\mathrm{o})\,, \tag{7.31}$$

where $\mathcal{R}_\Omega(\cdot)$ is the Robin function.

With no loss of generality let $\xi_\mathrm{o} = \mathbf{0}$. For any $\varrho > 0$ consider $\delta^{(\varrho)} := \chi_{B_\varrho}/|B_\varrho|$. Clearly, this converges weakly* to $\delta_{\mathbf{0}}$. Indeed, for any $\phi \in C_0^0(\Omega)$ we have that

$$\lim_{\varrho\to 0+} \int_\Omega \delta^{(\varrho)}\phi \,\mathrm{d}\xi = \lim_{\varrho\to 0+} \fint_{B_\varrho(\mathbf{0})} \phi \,\mathrm{d}\xi = \phi(\mathbf{0})\,,$$

for the Lebsegue-Besicovithc Theorem. Now let $v_\varrho(r)$ be solution to

$$\begin{cases} -\Delta_H v = \delta^{(\varrho)} & \text{in } \Omega \\ u = \Gamma(\cdot;\mathbf{0}) & \text{in } \partial\Omega \end{cases}$$

where $\Gamma \equiv \Gamma(\xi;\mathbf{0})$ is Folland's fundamental solution with pole in $\mathbf{0}$. Note that the sequence $\{v_\varrho\}_\varrho$ converge weakly in the sense of distributions to $\Gamma(\cdot;\mathbf{0})$ in Ω. Indeed, for any test function $\phi \in C_0^\infty(\Omega)$ it holds

$$\begin{aligned} -\int_\Omega v_\varrho\Big(\Delta_H\phi\Big)\,\mathrm{d}\xi &= -\int_\Omega \Big(\Delta_H v_\varrho\Big)\phi\,\mathrm{d}\xi = \int_\Omega \delta^{(\varrho)}\phi\,\mathrm{d}\xi \\ &\to -\phi(\mathbf{0}) = -\int_{B_\sigma\cap\Omega} \Gamma(\cdot,\mathbf{0})\Big(\Delta_H\phi\Big)\,\mathrm{d}\xi. \end{aligned}$$

Now, consider the solution u_ϱ to

$$\begin{cases} -\Delta_H u = \delta^{(\varrho)} & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases} \tag{7.32}$$

By the same computation as Step 3 before. In a similar fashion we can deduce that an analogous estimate such as (7.29) holds for u_ϱ as well

$$-\int_\Omega \Delta_H u_\varrho Z_k u_\varrho \,\mathrm{d}\xi = -\frac{1}{2}\int_{\partial\Omega} |D_H u_\varrho| \langle Z_k, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2}. \tag{7.33}$$

Indeed, considering the same exhaustion of Ω of C^∞ connected open sets $\{\Omega_i\}$ of the previous step, we have that

$$\begin{aligned} &-\int_{\Omega_i} \Delta_H u_\varrho Z_k u_\varepsilon \,\mathrm{d}\xi \\ &= -\frac{1}{2}\int_{\gamma_i^{(1)}} |D_H u_\varrho|\langle Z_k, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2} - 4\sum_{j=1}^{2n}\int_{\gamma_i^{(2)}} Z_j u_\varrho u_\varrho \langle T, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2} \\ &\quad -\sum_{j=1}^{2n}\int_{\gamma_i^{(2)}} Z_j u_\varrho Z_k u_\varrho \langle Z_j, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2} + \frac{1}{2}\int_{\gamma_i^{(2)}} |D_H u_\varrho|^2 \langle Z_k, \boldsymbol{n}\rangle \,\mathrm{d}\mathcal{H}^{Q-2}. \end{aligned}$$

Now, recalling that by [129, Theorem 5.7] (up to taking ϱ sufficiently small such that in an interior neighborhood ω of the characteristic set $\delta^{(\varrho)} = 0$) u_ϱ

has bounded horizontal gradient, we can pass to the limit in the above inequality getting (7.33), by Theorem 2.9. Moreover, note that the sequence $\{u_\varrho\}_\varrho$ converge weakly in the sense of distributions to $G_\Omega(\cdot;\mathbf{0})$. Also, converge weakly in the sense of distribution to $G_\Omega(\cdot;\mathbf{0})$, away from the characteristic set. Indeed, taking a ball B_σ, such that $B_\sigma\cap\partial\Omega$ is non-characteristic, for any test function $\phi\in C_0^\infty(B_\sigma\cap\Omega)$ it holds

$$\begin{aligned}-\int_{B_\sigma\cap\Omega} u_\varrho\Big(\Delta_H\phi\Big)\,\mathrm{d}\xi &= -\int_{B_\sigma\cap\Omega}\Big(\Delta_H u_\varrho\Big)\phi\,\mathrm{d}\xi\\ &= \int_{B_\sigma\cap\Omega}\delta^{(\varrho)}\phi\,\mathrm{d}\xi\\ &\to \int_{B_\sigma\cap\Omega}\delta_{\mathbf{0}}\phi\,\mathrm{d}\xi = -\int_{B_\sigma\cap\Omega} G_\Omega(\cdot,\mathbf{0})\Big(\Delta_H\phi\Big)\,\mathrm{d}\xi.\end{aligned}$$

Hence, it is enough to take an open cover of $\partial\Omega\setminus\Sigma(\Omega)$, to get that $u_\varrho\to G_\Omega(\cdot;\mathbf{0})$ as $\varrho\to 0^+$. Moreover, such convergence is actually in $\Gamma^{1,\beta}(\partial\Omega\setminus\Sigma(\Omega))$, for some $\beta\in(0,1)$, since the sequence $\{u_\varrho\}_\varrho$ is compact in $\Gamma^{1,\beta}(\partial\Omega\setminus\Sigma(\Omega))$, by Theorem 2.8; since (up to taking ϱ sufficiently small $\delta^{(\varrho)}\equiv 0$ near $\partial\Omega$, since the $\xi_{\mathrm{o}}\equiv\mathbf{0}$ is away from the boundary).

As for the left-hand side we have

$$\int_\Omega \delta^{(\varrho)} Z_k u_\varrho\,\mathrm{d}\xi = \int_\Omega \delta^{(\varrho)} Z_k(u_\varrho - v_\varrho)\,\mathrm{d}\xi + \int_\Omega \delta^{(\varrho)} Z_k v_\varrho\,\mathrm{d}\xi\,, \tag{7.34}$$

where the last integral is 0 by symmetry, since is an integral over B_ϱ.

By (7.32) we have that

$$\begin{cases}-\Delta_H(u_\varrho - v_\varrho) = 0 & \text{in } \Omega\,,\\ u_\varrho - v_\varrho = \Gamma(\cdot;\mathbf{0}) & \text{in } \partial\Omega.\end{cases}$$

So that, the sequence $\{u_\varrho - v_\varrho\}_\varrho$ is compact in $C^k(\Omega)$, for any $k\in\mathbb{N}$. Also, the sequence $\{u_\varrho - v_\varrho\}_\varrho$ is compact in $\Gamma^{k,\beta}(\partial\Omega\setminus\Sigma(\Omega))$, for any $k\in\mathbb{N}$, by [12, Theorem 1.1] (this can be shown taking an open covering of the boundary made of $B_\sigma\cap\overline{\Omega}$ so that $B_\sigma\cap\partial\Omega$ is non-characteristic and then applying Theorem 1.1 in [12]) and converges to $H(\xi,\mathbf{0})$, with $H(\xi,\mathbf{0})$ being the regular part of the Green function. Hence, passing to the limit in (7.34) as $\varrho\to 0$, by Lebesgue-Besicovitch differentiation lemma, we have that

$$\lim_{\varrho\to 0^+}\int_\Omega \delta^{(\varrho)} Z_k(u_\varrho - v_\varrho)\,\mathrm{d}\xi = Z_k H(\mathbf{0},\mathbf{0}) = \frac{1}{2} Z_k\mathcal{R}_\Omega(\mathbf{0})\,,$$

Since, by symmetry of the function $H(\xi,\eta)$ we have

$$Z_k^{(\xi)} H(\mathbf{0},\mathbf{0}) = Z_k^{(\xi)} H(\mathbf{0},\mathbf{0}) + Z_k^{(\eta)} H(\mathbf{0},\mathbf{0}) = 2Z_k^{(\xi)} H(\mathbf{0},\mathbf{0})\,,$$

with $Z_k^{(\xi)}$, being the Z_k-derivative with respect the ξ variables. Thus, passing also to the limit on (7.33) yields the desired (7.31).

Step 6. Let us prove that

$$\int_{\partial\Omega} |D_H G_\Omega(\cdot,\xi_{\mathrm{o}})|^2\langle\mathcal{D},\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2} = -(Q-2)\mathcal{R}_\Omega(\xi_{\mathrm{o}}). \tag{7.35}$$

As in the previous Steps, consider the approximation of Ω by exhaustion made of $\{\Omega_i\}$. We apply the Pohozaev identity in [67, Crollary 3.3] on $\Omega_i\setminus B_\varrho(\mathbf{0})$, to

the Green function $G_\Omega(;\mathbf{0})$. We get

$$
\begin{aligned}
&2\sum_{j=1}^{2n}\int_{\partial(\Omega_i\setminus B_\varrho(\mathbf{0}))} \mathcal{D}G_\Omega(;\mathbf{0})Z_jG_\Omega(;\mathbf{0})\langle Z_j,\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2}\\
&\qquad +(Q-2)\int_{\Omega_i\setminus B_\varrho(\mathbf{0})}|D_HG_\Omega(\cdot;\mathbf{0})|^2\,\mathrm{d}\xi\\
&\qquad =\int_{\partial(\Omega_i\setminus B_\varrho(\mathbf{0}))}|D_HG_\Omega(;\mathbf{0})|^2\langle\mathcal{D},\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2}\,. \qquad (7.36)
\end{aligned}
$$

Now, note that

$$
\begin{aligned}
&\int_{\Omega_i\setminus B_\varrho(\mathbf{0})}|D_HG_\Omega(\cdot;\mathbf{0})|^2\,\mathrm{d}\xi\\
&\quad=\sum_{j=1}^{2n}\int_{\Omega_i\setminus B_\varrho(\mathbf{0})}\Big(Z_jG_\Omega(\cdot;\mathbf{0})\Big)^2\,\mathrm{d}\xi\\
&\quad=-\sum_{j=1}^{2n}\int_{\Omega_i\setminus B_\varrho(\mathbf{0})}Z_j^2G_\Omega(\cdot;\mathbf{0})\cdot G_\Omega(\cdot;\mathbf{0})\,\mathrm{d}\xi\\
&\qquad+\sum_{j=1}^{2n}\int_{\partial(\Omega_i\setminus B_\varrho(\mathbf{0}))}Z_jG_\Omega(\cdot;\mathbf{0})\cdot G_\Omega(\cdot;\mathbf{0})\langle Z_j,\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2}\\
&\quad=\sum_{j=1}^{2n}\int_{\partial(\Omega_i\setminus B_\varrho(\mathbf{0}))}Z_jG_\Omega(\cdot;\mathbf{0})\cdot G_\Omega(\cdot;\mathbf{0})\langle Z_j,\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2}\,, \qquad (7.37)
\end{aligned}
$$

since $\Delta_HG_\Omega(\cdot;\mathbf{0})=0$ in $\Omega_i\setminus B_\varrho(\mathbf{0})$. Hence, putting (7.37) in (7.36) yields

$$
\begin{aligned}
&2\sum_{j=1}^{2n}\int_{\partial(\Omega_i\setminus B_\varrho(\mathbf{0}))} \mathcal{D}G_\Omega(;\mathbf{0})Z_jG_\Omega(;\mathbf{0})\langle Z_j,\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2}\\
&\qquad +(Q-2)\sum_{j=1}^{2n}\int_{\partial(\Omega_i\setminus B_\varrho(\mathbf{0}))}Z_jG_\Omega(\cdot;\mathbf{0})\cdot G_\Omega(\cdot;\mathbf{0})\langle Z_j,\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2}\\
&\qquad =\int_{\partial(\Omega_i\setminus B_\varrho(\mathbf{0}))}|D_HG_\Omega(;\mathbf{0})|^2\langle\mathcal{D},\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2}\,. \qquad (7.38)
\end{aligned}
$$

Passing now to the limit as $i\to\infty$, recalling that the Green function has bounded horizontal gradient near the characteristic set by [**129**, Theorem 5.7] and that, by Theorem 2.9, the characteristic set has negligible surface measure, we get

$$
\begin{aligned}
&2\sum_{j=1}^{2n}\int_{\partial(\Omega\setminus B_\varrho(\mathbf{0}))} \mathcal{D}G_\Omega(;\mathbf{0})Z_jG_\Omega(;\mathbf{0})\langle Z_j,\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2}\\
&\qquad -(Q-2)\sum_{j=1}^{2n}\int_{\partial B_\varrho(\mathbf{0})}Z_jG_\Omega(\cdot;\mathbf{0})\cdot G_\Omega(\cdot;\mathbf{0})\langle Z_j,\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2}\\
&\qquad =\int_{\partial(\Omega\setminus B_\varrho(\mathbf{0}))}|D_HG_\Omega(;\mathbf{0})|^2\langle\mathcal{D},\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2}\,.
\end{aligned}
$$

since $G_\Omega(\cdot;\mathbf{0}) = 0$ on $\partial\Omega$. Thus, we can rewrite the above identity as follows

$$
\begin{aligned}
&2\sum_{j=1}^{2n}\int_{\partial\Omega} \mathcal{D}G_\Omega(;\mathbf{0})Z_jG_\Omega(;\mathbf{0})\langle Z_j,\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2}\\
&\qquad -\int_{\partial\Omega}|D_HG_\Omega(;\mathbf{0})|^2\langle\mathcal{D},\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2}\\
&\qquad =-\int_{\partial B_\varrho(\mathbf{0})}|D_HG_\Omega(;\mathbf{0})|^2\langle\mathcal{D},\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2}\\
&\qquad +2\sum_{j=1}^{2n}\int_{\partial B_\varrho(\mathbf{0})} \mathcal{D}G_\Omega(;\mathbf{0})Z_jG_\Omega(;\mathbf{0})\langle Z_j,\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2}\\
&\qquad +(Q-2)\sum_{j=1}^{2n}\int_{\partial B_\varrho(\mathbf{0})} Z_jG_\Omega(\cdot;\mathbf{0})\cdot G_\Omega(\cdot;\mathbf{0})\langle Z_j,\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2}\,.
\end{aligned}
\tag{7.39}
$$

Starting with the left-hand part of (7.39), we note that $DG_\Omega(\xi;\mathbf{0}) = g\boldsymbol{n}$, for some function g, being $G_\Omega(\cdot;\mathbf{0})$ on $\partial\Omega$, so that

$$
\mathcal{D}G_\Omega(\xi;\mathbf{0})\langle Z_j,\boldsymbol{n}\rangle = g\langle\mathcal{D},\boldsymbol{n}\rangle\langle Z_j,\boldsymbol{n}\rangle = Z_jG_\Omega(\xi;\mathbf{0})\langle\mathcal{D},\boldsymbol{n}\rangle\,,
$$

which in turn yields

$$
\begin{aligned}
&2\sum_{j=1}^{2n}\int_{\partial\Omega} \mathcal{D}G_\Omega(;\mathbf{0})Z_jG_\Omega(;\mathbf{0})\langle Z_j,\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2}\\
&\qquad -\int_{\partial\Omega}|D_HG_\Omega(;\mathbf{0})|^2\langle\mathcal{D},\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2}\\
&\qquad =\int_{\partial\Omega}|D_HG_\Omega(;\mathbf{0})|^2\langle\mathcal{D},\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2}.
\end{aligned}
\tag{7.40}
$$

As for the right-hand side term in (7.39), we first note that on $\partial B_\varrho(\mathbf{0})$ the fundamental solutions $\Gamma(\cdot;\mathbf{0}) = (c_Q)^{-1}\varrho^{2-Q}$, so $\mathcal{D}\Gamma(\cdot;\mathbf{0}) = 0$. Moreover, except for the term $H(\cdot;\mathbf{0})Z_j\Gamma(\cdot;\mathbf{0})$, we can consider any other quantity as $o(\varrho)$, since both the $\mathcal{D}$-derivative and the Z_j-derivative of $G_\Omega(\cdot;\mathbf{0})$ do coincide with the analogous one of H, given that $\Gamma(\cdot;\mathbf{0}) = (c_Q)^{-1}\varrho^{2-Q}$. Thus, the right-hand side of (7.39) behave as $\varrho\to 0^+$ as

$$
(Q-2)\sum_{j=1}^{2n}\int_{\partial B_\varrho(\mathbf{0})} Z_j\Gamma(\cdot;\mathbf{0})H(;\mathbf{0})\langle Z_j,\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2}+\mathrm{o}(\varrho).
$$

We rewrite the above integral in a more convenient way now. Let us star noting that

$$
\begin{aligned}
&Z_j\Gamma(\cdot;\mathbf{0})\langle Z_j,\boldsymbol{n}\rangle\\
&\quad =-\frac{Q-2}{c_Q\varrho^{Q-1}}\langle Z_j,D|\xi|_{\mathbb{H}^n}\rangle\left\langle Z_j,\frac{D|\xi|_{\mathbb{H}^n}}{|D|\xi|_{\mathbb{H}^n}|}\right\rangle\\
&\quad =-\frac{Q-2}{c_Q\varrho^{Q-1}}\frac{\left(Z_j(|\xi|_{\mathbb{H}^n})\right)^2}{|D|\xi|_{\mathbb{H}^n}|}.
\end{aligned}
$$

Thus,

$$(Q-2)\sum_{j=1}^{2n}\int_{\partial B_\varrho(\mathbf{0})} Z_j\Gamma(\cdot;\mathbf{0})H(;\mathbf{0})\langle Z_j,\boldsymbol{n}\rangle\,\mathrm{d}\mathcal{H}^{Q-2}$$
$$=-\frac{(Q-2)^2}{c_Q\varrho^{Q-1}}\int_{\partial B_\varrho(\mathbf{0})} H(\cdot;\mathbf{0})\frac{|D_H(|\xi|_{\mathbb{H}^n})|^2}{|D|\xi|_{\mathbb{H}^n}|}\,\mathrm{d}\mathcal{H}^{Q-2}=-(Q-2)H(\mathbf{0},\mathbf{0})\,,$$

by the representation of harmonic functions with their surface measure; see for example [19, Theorem 5.5.4].

Then, (7.35) follows passing to the limit in $\varrho\to 0^+$ in all the above estimate and recalling (7.40).

Step 7. Conclusion The thesis follows by the Pohozaev identity obtained in (7.17) and (7.27), the identity (7.31) and (7.35) and the limiting behaviour in (7.23).

□

Part 3

Nonlinear fractional equations in the Heisenberg group

CHAPTER 8

Setting of the problem

In this part we focus on establishing "classical" regularity estimates as well as other related existence and qualitative properties of weak solutions to the following class of problems

$$\begin{cases} \mathcal{L}u = f(u) & \text{in } \Omega, \\ u = g & \text{in } \mathbb{H}^n \setminus \Omega, \end{cases} \tag{8.1}$$

where the nonlocal boundary datum g belongs to the fractional space $W^{s,p}(\mathbb{H}^n)$ (see the forthcoming Section 9), the datum $f \equiv f(\cdot, u) \in L^\infty_{\mathrm{loc}}(\mathbb{H}^n)$ locally uniformly in Ω, and the leading operator $\mathcal{L}$ is an integro-differential operator of differentiability exponent $s \in (0,1)$ and summability exponent $p > 1$ given by

$$\mathcal{L}u(\xi) = P.V. \int_{\mathbb{H}^n} \frac{|u(\xi) - u(\eta)|^{p-2}(u(\xi) - u(\eta))}{d_{\mathrm{o}}(\eta^{-1} \circ \xi)^{Q+sp}} \, \mathrm{d}\eta, \qquad \xi \in \mathbb{H}^n, \tag{8.2}$$

with d_{o} being a generic homogeneous norm on $\mathbb{H}^n$ and where the symbol $P.V.$ stands for "in the principal value sense".

In the linear case when $p = 2$ operators as in (8.2) do arise as a generalization of the classical *conformally invariant* fractional subLaplacian $(-\Delta_H)^s$ on $\mathbb{H}^n$ which was first introduced in [20] via the spectral formula

$$(-\Delta_H)^s := 2^s |T|^s \frac{\boldsymbol{\Gamma}\Big(-\frac{1}{2}\Delta_H |T|^{-1} + \frac{1+s}{2} \Big)}{\boldsymbol{\Gamma}\Big(-\frac{1}{2}\Delta_H |T|^{-1} + \frac{1-s}{2} \Big)}, \qquad \forall s \in (0,1),$$

with $\boldsymbol{\Gamma}(x) := \int_0^\infty t^{x-1} e^{-t} \, \mathrm{d}t$ being the Euler Gamma function.

As proven in [119, Proposition 4.1] the following representation formula holds true

$$(-\Delta_H)^s u(\xi) := C(n,s) \, P.V. \int_{\mathbb{H}^n} \frac{u(\xi) - u(\eta)}{|\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{Q+2s}} \, \mathrm{d}\eta, \qquad \xi \in \mathbb{H}^n, \tag{8.3}$$

where $C(n,s)$ is a positive constant which depends only on n and s.

Notice that the operator defined in Formula (8.3) above does not coincide with the classical power of the subLaplacian defined via the heat semigroup built by Folland ([58]).

In the last decades, many relevant results have been achieved in the study of fractional operators in the Heisenberg group both for the conformally and not conformally invariant fractional subLaplacian. As an example, we just recall the aforementioned paper [119], [34] for related Hardy and uncertainty inequalities on general stratified Lie groups involving fractional powers of the subLaplacian, and [2], where Sobolev and Morrey-type embeddings are derived for fractional order Sobolev spaces. Moreover, very important results have been obtained based on the construction of fractional operators via a Caffarelli-Silvestre-type extension as seen in the Euclidean framework in [29]. In particular, we mention the recent series of papers [65, 66] where the authors used ad-hoc Dirichlet-to-Neumann

map to built the explicit fundamental solution to (8.3), and to prove some intertwining formulas for $(-\Delta_H)^s$. Also, a Liouville-type theorem was proven in [35]; Harnack and Hölder results in Carnot groups in [56]; whereas the connection with the fractional perimeters of sets in Carnot group can be found in [57].

In this part we deal with more general operators where a p-growth exponent is involved. Nonlinear equations on stratified Lie group both in their nonlocal and local formulation are connected with several concrete models arising from many different contexts where the analysis in sub-Riemannian geometry revealed to be decisive; see for example [13, 38, 102, 131] and the references therein. Here, we present some very recent results regarding the De Giorgi-Nash-Moser theory for the weak solutions to general equations led by integro-differential operators $\mathcal{L}$ defined in (8.2), and we conclude by recalling a Perron-type resolutivity theorem for the related Dirichlet problem. All the forthcoming theorems are the natural extension in the Heisenberg setting of the Euclidean counterpart proven for a class of very general p-fractional Laplacian-type operators in [47, 48, 85, 86, 97]. For further results on equations where the leading operator is such as the one in (8.2), we also refer to [75, 81, 82, 130].

The main difficulty in dealing with the operator $\mathcal{L}$ in (8.2) lies in its own definition. Indeed, we have to deal both with its nonlocal and nonlinear structure and with the underlying non-Euclidean geometry of the Heisenberg group. Moreover, since the integrability exponent p can differ from 2, most of the classical tools successfully extended in the Heisenberg framework in the linear case such as the Dirichlet-to-Neumann map in [62, 65, 66], the approach via (non-commutative) Fourier representation, and many others, are not trivially applicable. As natural when dealing with nonlocal operators we have to consider a tail-type contribution which gives a precise control on the long-range interactions that naturally occur.

Definition 8.1 (Nonlocal tail)**.** *For any $p \in (1,\infty)$ and $s \in (0,1)$ we call* nonlocal tail *of a function u in a ball $B_R(\xi_0) \subset \mathbb{H}^n$ of radius $R > 0$ and centred in $\xi_0 \in \mathbb{H}^n$ the quantity defined as follows*

$$\operatorname{Tail}(u;\xi_0,R) := \left(R^{sp} \int_{\mathbb{H}^n \setminus B_R(\xi_0)} |u(\xi)|^{p-1} |\xi_0^{-1} \circ \xi|_{\mathbb{H}^n}^{-Q-sp} \, \mathrm{d}\xi \right)^{\frac{1}{p-1}}. \tag{8.4}$$

In the Euclidean framework the nonlocal tail has been firstly introduced in the study of nonlocal operators in [47, 48] and then subsequently used to derive fine properties of solutions to nonlocal equations; see e. g. [21, 85, 86, 87] and the references therein.

In the considered Heisenberg framework, the nonlocal tail has been firstly introduced in the papers [106, 101], and – despite its young age – it has been already proven to be a fundamental tool in order to achieve several results, as for instance the ones regarding the regularity for the obstacle problem and the Perron method ([116]), and those involving the fractional critical (Folland-Stein)-Sobolev embedding ([64]).

In clear accordance with Definition 8.1, we can consider as done in [85, 86] the associated *tail space* L^{p-1}_{sp} given by

$$L^{p-1}_{sp}(\mathbb{H}^n) := \left\{ v \in L^{p-1}_{\mathrm{loc}}(\mathbb{H}^n) : \int_{\mathbb{H}^n} \frac{|v(\xi)|^{p-1}}{(1+|\xi|_{\mathbb{H}^n})^{Q+sp}} \, \mathrm{d}\xi < \infty \right\}.$$

The previous definitions are very natural when dealing with operators of the type (8.2).

In Chapter 10 we present the De Giorgi-Nash-Moser Theory for the nonlinear fractional problems led by the operator $\mathcal{L}$. Chapter 11 is devoted to the proof of the nonlocal Harnack inequalities, whereas Chapter 12 treats the fractional obstacle problem for $\mathcal{L}$ and the regularity up to the boundary for weak solutions to (8.1). Lastly, in Chapter 13 we solve the Dirichlet problem related to $\mathcal{L}$ in the sense of Perron in general measurable open set and with very general boundary data g.

All the forthcoming results are contained in [**116**, **106**, **101**]. For a summary of the results contained in the aforementioned papers and related open problems for nonlocal operators in the Heisenberg group we also refer the reader to the survey article [**108**].

CHAPTER 9

Fractional Sobolev spaces

In this chapter we recall some basic definition and properties of fractional Sobolev spaces in $\mathbb{H}^n$ as well as precisely giving the setting of problem (8.1).

Let $p \geq 1$ and $s \in (0,1)$, and let $u : \mathbb{H}^n \to \mathbb{R}$ be a measurable function; we define the Gagliardo (semi)norm of u as follows,

$$[u]_{W^{s,p}(\mathbb{H}^n)} := \left(\int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \frac{|u(\xi) - u(\eta)|^p}{|\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{Q+sp}} \, \mathrm{d}\xi \, \mathrm{d}\eta \right)^{\frac{1}{p}}.$$

The fractional Sobolev spaces $W^{s,p}(\mathbb{H}^n)$ on the Heisenberg group is defined as

$$W^{s,p}(\mathbb{H}^n) := \left\{ u \in L^p(\mathbb{H}^n) : [u]_{W^{s,p}(\mathbb{H}^n)} < +\infty \right\},$$

endowed with the natural fractional norm

$$\|u\|_{W^{s,p}(\mathbb{H}^n)} := \left(\|u\|^p_{L^p(\mathbb{H}^n)} + [u]^p_{W^{s,p}(\mathbb{H}^n)} \right)^{\frac{1}{p}}, \qquad u \in W^{s,p}(\mathbb{H}^n).$$

Similarly, given a domain $\Omega \subset \mathbb{H}^n$, one can define the fractional Sobolev space $W^{s,p}(\Omega)$ in the natural way, as follows

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \left(\int_\Omega \int_\Omega \frac{|u(\xi) - u(\eta)|^p}{|\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{Q+sp}} \, \mathrm{d}\xi \, \mathrm{d}\eta \right)^{\frac{1}{p}} < +\infty \right\}$$

endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + \int_\Omega \int_\Omega \frac{|u(\xi) - u(\eta)|^p}{|\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{Q+sp}} \, \mathrm{d}\xi \, \mathrm{d}\eta \right)^{\frac{1}{p}}.$$

By $W^{s,p}_0(\Omega)$ we denote the closure of $C^\infty_0(\Omega)$ in $W^{s,p}(\mathbb{H}^n)$. Conversely, if $v \in W^{s,p}(\Omega')$ with $\Omega \Subset \Omega'$ and $v = 0$ outside of Ω almost everywhere, then v has a representative in $W^{s,p}_0(\Omega)$ as well.

As expected, one can prove a fractional Sobolev embedding on the Heisenberg group. We have the following

Theorem 9.1 (Theorem 2.5 in [81]). *Let $p > 1$ and $s \in (0,1)$ such that $sp < Q$. For any measurable compactly supported function $u : \mathbb{H}^n \to \mathbb{R}$ there exists a positive constant $c \equiv c(n,p,s)$ such that*

$$\|u\|^p_{L^{p^*_s}(\mathbb{H}^n)} \leq c \, [u]^p_{W^{s,p}(\mathbb{H}^n)},$$

where $p^*_s = Qp/(Q - sp)$ is the critical Sobolev exponent.

In the proof of the theorem above the authors extended the strategy in the standard Euclidean settings as seen in [114, 49].

Moreover, applying the same technique used in [7] we can prove a fractional Poincaré-type inequality

Proposition 9.2. *Let $p \geq 1$ and $s \in (0,1)$ and $u \in W^{s,p}_{\mathrm{loc}}(\Omega)$. Then, for any $B_r \equiv B_r(\xi_0) \Subset \Omega$ we have that*

$$\int_{B_r} |u-(u)_r|^p \,\mathrm{d}\xi \leq c\, r^{sp} \int_{B_r}\int_{B_r} \frac{|u(\xi)-u(\eta)|^p}{|\eta^{-1}\circ\xi|^{Q+sp}_{\mathbb{H}^n}}\,\mathrm{d}\xi\mathrm{d}\eta,$$

where $c \equiv c(n,p) > 0$ and $(u)_r = \fint_{B_r} u\,\mathrm{d}\xi$.

As in the classical case with s being an integer, the space $W^{s,p}$ is continuously embedded in $W^{s_1,p}$ when $s_1 \leq s$, as the result below points out.

Proposition 9.3. *Let $p > 1$ and $0 < s_1 \leq s < 1$. Let Ω be an open subset of $\mathbb{H}^n$, and let $u \in W^{s,p}(\Omega)$. Then*

$$\|u\|_{W^{s_1,p}(\Omega)} \leq c\|u\|_{W^{s,p}(\Omega)},$$

for some suitable positive constant c depending only on n, p and s_1.

Proof. We extend the strategy in the proof in the fractional Euclidean framework; see [49, Proposition 2.1].

Firstly, we can control the size of the nonlocal tail of u by its L^p-norm. We have

$$\begin{aligned}\int_\Omega \int_{\Omega\cap\{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}\geq 1\}} \frac{|u(\xi)|^p}{|\eta^{-1}\circ\xi|^{Q+s_1p}_{\mathbb{H}^n}}\,\mathrm{d}\xi\,\mathrm{d}\eta &\leq \int_\Omega \left(\int_{\mathbb{H}^n\setminus B_1(0)} \frac{1}{|\tilde{\xi}|^{Q+s_1p}_{\mathbb{H}^n}} d\tilde{\xi}\right)|u(\xi)|^p\,\mathrm{d}\xi \\ &\leq c(n,p,s_1)\|u\|^p_{L^p(\Omega)},\end{aligned}$$

and thus

$$\begin{aligned}&\int_\Omega \int_{\Omega\cap\{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}\geq 1\}} \frac{|u(\xi)-u(\eta)|^p}{|\eta^{-1}\circ\xi|^{Q+s_1p}_{\mathbb{H}^n}}\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\leq 2^{p-1}\int_\Omega \int_{\Omega\cap\{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}\geq 1\}} \frac{|u(\xi)|^p+|u(\eta)|^p}{|\eta^{-1}\circ\xi|^{Q+s_1p}_{\mathbb{H}^n}}\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\leq c\|u\|^p_{L^p(\Omega)}, \qquad (9.1)\end{aligned}$$

up to relabelling the constant c.

On the other hand,

$$\begin{aligned}&\int_\Omega \int_{\Omega\cap\{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}<1\}} \frac{|u(\xi)-u(\eta)|^p}{|\eta^{-1}\circ\xi|^{Q+s_1p}_{\mathbb{H}^n}}\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\leq \int_\Omega \int_{\Omega\cap\{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}<1\}} \frac{|u(\xi)-u(\eta)|^p}{|\eta^{-1}\circ\xi|^{Q+sp}_{\mathbb{H}^n}}\,\mathrm{d}\xi\,\mathrm{d}\eta. \qquad (9.2)\end{aligned}$$

Combining (9.1) with (9.2), we finally get

$$\int_\Omega\int_\Omega \frac{|u(\xi)-u(\eta)|^p}{|\eta^{-1}\circ\xi|^{Q+s_1p}_{\mathbb{H}^n}}\,\mathrm{d}\xi\,\mathrm{d}\eta \leq c\|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\Omega)},$$

which yields

$$\|u\|^p_{W^{s_1,p}(\Omega)} \leq (c+1)\|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\Omega)} \leq c\|u\|^p_{W^{s,p}(\Omega)},$$

again up to relabelling the constant c. □

We conclude this chapter by providing the definition of weak solution to the class of fractional problem we deal with.

Let Ω be a bounded open set in $\mathbb{H}^n$ and $g \in W^{s,p}(\mathbb{H}^n)$, we are interested in the weak solutions to the following integro-differential problems,

$$\begin{cases} \mathcal{L}u = f(u) & \text{in } \Omega, \\ u = g & \text{in } \mathbb{H}^n \setminus \Omega, \end{cases} \qquad (9.3)$$

where the datum $f \equiv f(\cdot, u) \in L^{\infty}_{\mathrm{loc}}(\mathbb{H}^n)$ locally uniformly in Ω, and the leading operator $\mathcal{L}$ is an integro-differential operator of differentiability exponent $s \in (0,1)$ and summability exponent $p > 1$ given by

$$\mathcal{L}u(\xi) = P.V. \int_{\mathbb{H}^n} \frac{|u(\xi) - u(\eta)|^{p-2}(u(\xi) - u(\eta))}{d_{\mathrm{o}}(\eta^{-1} \circ \xi)^{Q+sp}} \, \mathrm{d}\eta, \qquad \xi \in \mathbb{H}^n,$$

with d_{o} being a homogeneous norm on $\mathbb{H}^n$ in accordance with Definition 2.1.

We now need to introduce some further notation. For any $g \in W^{s,p}(\mathbb{H}^n)$ the classes $\mathcal{K}^{\pm}_g(\Omega)$ of suitable fractional functions are defined by

$$\mathcal{K}^{\pm}_g(\Omega) := \Big\{ v \in W^{s,p}(\mathbb{H}^n) : (g - v)_{\pm} \in W^{s,p}_0(\Omega) \Big\},$$

and

$$\mathcal{K}_g(\Omega) := \mathcal{K}^{+}_g(\Omega) \cap \mathcal{K}^{-}_g(\Omega) = \Big\{ v \in W^{s,p}(\mathbb{H}^n) : v - g \in W^{s,p}_0(\Omega) \Big\}.$$

We have the following

Definition 9.4. *A function $u \in \mathcal{K}^{-}_g(\Omega)$ ($\mathcal{K}^{+}_g(\Omega)$, respectively) is a* weak subsolution (supersolution, *resp.*) *to (9.3) if*

$$\int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \frac{|u(\xi) - u(\eta)|^{p-2}(u(\xi) - u(\eta))(\psi(\xi) - \psi(\eta))}{d_o(\eta^{-1} \circ \xi)^{Q+sp}} \, \mathrm{d}\xi \, \mathrm{d}\eta$$
$$\leq \Big(\geq, \textit{resp.} \Big) \int_{\mathbb{H}^n} f(\xi, u(\xi)) si(\xi) \, \mathrm{d}\xi,$$

for any nonnegative $\psi \in W^{s,p}_0(\Omega)$.
A function u is a weak solution *to (9.3) if it is both a weak sub- and supersolution. In particular, u belongs to $\mathcal{K}_g(\Omega)$ and it satisfies*

$$\int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \frac{|u(\xi) - u(\eta)|^{p-2}(u(\xi) - u(\eta))(\psi(\xi) - \psi(\eta))}{d_o(\eta^{-1} \circ \xi)^{Q+sp}} \, \mathrm{d}\xi \, \mathrm{d}\eta = \int_{\mathbb{H}^n} f(\xi, u(\xi)) \psi(\xi) \, \mathrm{d}\xi,$$

for any $\psi \in W^{s,p}_0(\Omega)$.

Recalling Definition 8.1 of nonlocal tail of a functions, few observations are in order.

Firstly, we notice that, by Hölder's Inequality, since $u \in L^p(\mathbb{H}^n)$ and $R > 0$, we have that $\mathrm{Tail}(u; \xi_0, R) < +\infty$.

Second, we have the following

Remark 9.5. *The requirement on the boundary datum g to be in the whole $W^{s,p}(\mathbb{H}^n)$ can be weakened by assuming only a local fractional differentiability, namely $g \in W^{s,p}_{\mathrm{loc}}(\Omega)$, in addition to the boundedness of its nonlocal tail.*

Remark 9.6. *The presence of the datum f is a novelty with respect to the Euclidean counterpart studied in* [48] *where the authors assume the right-hand side in (9.3) to be zero. However, as we basically will prove, the techniques there can be applied also to more general framework.*

In addition, in accordance with the classical elliptic theory, with no important modifications in the forthcoming proofs, one could consider the case when the local boundedness assumption on the datum f is replaced by a uniformly growth control from above, as, e. g.,

$$|f(\xi, u)| \leq a + b|u|^q \quad \textit{for almost everywhere } \xi \in \Omega \textit{ and any } u \in \mathbb{R},$$

for some suitable choice of the exponent $q = q(n, p, s)$.

Before going into the proofs, it is worth pointing that we will only consider the case when the structural parameters n, s and p are such that $sp \leq Q$. This is

not a restriction, since, in the remaining case when $sp > Q$, the desired boundedness and Hölder continuity results are assured by the fractional Morrey embedding in the Heisenberg group; for the proof we refer for instance to Theorem 1.5 in [2].

CHAPTER 10

De Giorgi-Nash-Moser theory for nonlinear fractional equations

1. Fundamental estimates

The aim of this section is to give a full proof of some fundamental estimates such as a Caccioppoli-type inequality with tail and a fractional Logarithmic Lemma. These will turn out to be the starting point in establishing the sought De Giorgi-Nash-Moser regularity result for weak solution to (8.1).

We start stating and proving the Caccioppoli estimate with tail for weak subsolution to problem (8.1).

Theorem 10.1 (Caccioppoli estimates with tail)**.** *Let $s \in (0,1)$, $p \in (1,\infty)$, and let $u \in W^{s,p}(\mathbb{H}^n)$ be a weak subsolution to (8.1). Then, for any $B_r \equiv B_r(\xi_0) \subset \Omega$ and any nonnegative $\varphi \in C_0^\infty(B_r)$, the following estimate holds true*

$$
\begin{aligned}
&\int_{B_r}\int_{B_r} \frac{|w_+(\xi)\varphi(\xi) - w_+(\eta)\varphi(\eta)|^p}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\leq c\int_{B_r}\int_{B_r} \frac{w_+^p(\xi)|\varphi(\xi)-\varphi(\eta)|^p}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\quad + c\int_{B_r} w_+(\xi)\varphi^p(\xi)\,\mathrm{d}\xi\left(\sup_{\eta\in\operatorname{supp}\varphi}\int_{\mathbb{H}^n\setminus B_r}\frac{w_+^{p-1}(\xi)}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}\,\mathrm{d}\xi + \|f\|_{L^\infty(B_r)}\right)
\end{aligned}
\tag{10.1}
$$

where $w_+ := (u-k)_+$ with $k \in \mathbb{R}$, and $c \equiv c(n,p,s)$.

We would stress that, as in the classical Euclidean case, in both the entire framework and the fractional one, the *Caccioppoli estimates with tail* (10.1) encode all the needed information to derive the desired Hölder continuity from the minimum properties of the solutions, and it is an independent result which could be very useful in order to detect further regularity properties of the solutions to general fractional problems.

In the proof above, we made use of the following small inequality, which will be useful in the next section, as well; the proof is straightforward and follows via convexity and a standard iteration process ([48]).

Lemma 10.2. *Let $p \geq 1$ and $\varepsilon \in (0,1]$. Then*

$$
|a|^p \leq |b|^p + c_p\varepsilon|b|^p + (1+c_p\varepsilon)\varepsilon^{1-p}|a-b|^p, \qquad c_p := (p-1)\Gamma\big(\max\{1,\,p-2\}\big),
$$

holds for every $a,b \in \mathbb{R}^m$, $m \geq 1$. Here Γ stands for the standard Gamma function.

Proof of Theorem 10.1. Let u be a weak subsolution. We firstly choose as a test function in 9.4 the function

$$
\psi := w_+\varphi^p \equiv (u-k)_+\varphi^p, \qquad \text{for } k \in \mathbb{R},
$$

where φ is any nonnegative function in $C_0^\infty(B_r)$. We get

$$\begin{aligned} 0 \;\geq\; & \int_{B_r}\int_{B_r} \frac{|u(\xi)-u(\eta)|^{p-2}(u(\xi)-u(\eta))(w_+(\xi)\varphi^p(\xi)-w_+(\eta)\varphi^p(\eta))}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}\,\mathrm{d}\xi\mathrm{d}\eta \\ & +2\int_{\mathbb{H}^n\setminus B_r}\int_{B_r} \frac{|u(\xi)+u(\eta)|^{p-2}(u(\xi)-u(\eta))w_+(\xi)\varphi^p(\xi)}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}\,\mathrm{d}\xi\mathrm{d}\eta \\ & -\int_{B_r} f(\xi,u(\xi))w_+(\xi)\varphi^p(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{10.2}$$

Note that ψ is an admissible test function since truncations of functions in $W^{s,p}(\mathbb{H}^n)$ still belongs to $W^{s,p}(\mathbb{H}^n)$.

Let us begin by estimating the first integral on the right-hand side in (10.2). Without loss of generality, we assume that $u(\xi)\geq u(\eta)$; otherwise it just suffices to interchange the roles of ξ and η below. We have

$$\begin{aligned} &|u(\xi)-u(\eta)|^{p-2}(u(\xi)-u(\eta))(w_+(\xi)\varphi^p(\xi)-w_+(\eta)\varphi^p(\eta)) \\ &\quad=(u(\xi)-u(\eta))^{p-1}((u(\xi)-k)_+\varphi^p(\xi)-(u(\eta)-k)_+\varphi^p(\eta)) \\ &\quad=\begin{cases} (w_+(\xi)-w_+(\eta))^{p-1}(w_+(\xi)\varphi^p(\xi)-w_+(\eta)\varphi^p(\eta)) & \text{for } u(\xi),\,u(\eta)>k, \\ (u(\xi)-u(\eta))^{p-1}w_+(\xi)\varphi^p(\xi) & \text{for } u(\xi)>k,\;u(\eta)\leq k, \\ 0 & \text{otherwise} \end{cases} \\ &\quad\geq\;(w_+(\xi)-w_+(\eta))^{p-1}(w_+(\xi)\varphi^p(\xi)-w_+(\eta)\varphi^p(\eta)). \end{aligned}$$

For the second term on the right-hand side in (10.2) we have

$$\begin{aligned} |u(\xi)-u(\eta)|^{p-2}(u(\xi)-u(\eta))w_+(\xi) &\geq -(u(\eta)-u(\xi))_+^{p-1}(u(\xi)-k)_+ \\ &\geq -(u(\eta)-k)_+^{p-1}(u(\xi)-k)_+ \\ &= -w_+^{p-1}(\eta)w_+(\xi)\,, \end{aligned}$$

which yields

$$\begin{aligned} &\int_{\mathbb{H}^n\setminus B_r}\int_{B_r} \frac{|u(\xi)-u(\eta)|^{p-2}(u(\xi)-u(\eta))w_+(\xi)\varphi^p(\xi)}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\quad\geq -\int_{\mathbb{H}^n\setminus B_r}\int_{B_r} \frac{w_+^{p-1}(\eta)w_+(\xi)\varphi^p(\xi)}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\quad\geq -\int_{B_r} w_+(\xi)\varphi^p(\xi)\,\mathrm{d}\xi\left(\sup_{\xi\in\operatorname{supp}\varphi}\int_{\mathbb{H}^n\setminus B_r}\frac{w_+^{p-1}(\eta)}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}\,\mathrm{d}\eta\right). \end{aligned} \tag{10.3}$$

From (10.2)-(10.3), we deduce

$$\begin{aligned} 0 \;\geq\; & \int_{B_r}\int_{B_r} \frac{|w_+(\xi)-w_+(\eta)|^{p-1}(w_+(\xi)\varphi^p(\xi)-w_+(\eta)\varphi^p(\eta))}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}\,\mathrm{d}\xi\mathrm{d}\eta \\ & -2\int_{B_r} w_+(\xi)\varphi^p(\xi)\,\mathrm{d}\xi\left(\sup_{\xi\in\operatorname{supp}\varphi}\int_{\mathbb{H}^n\setminus B_r}\frac{w_+^{p-1}(\eta)}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}\,\mathrm{d}\eta\right) \\ & -\int_{B_r} f(\xi,u)w_+(\xi)\varphi^p(\xi)\;\mathrm{d}\xi. \end{aligned} \tag{10.4}$$

Let us consider the first term in (10.4). In the case when $w_+(\xi)\geq w_+(\eta)$ and $\varphi(\xi)\leq\varphi(\eta)$, we can use forthcoming Lemma 10.2 to obtain

$$\varphi^p(\xi)\;\geq\;(1-c_p\varepsilon)\varphi^p(\eta)-(1+c_p\varepsilon)\varepsilon^{1-p}|\varphi(\xi)-\varphi(\eta)|^p,\qquad \varepsilon\in(0,1].$$

Choosing

$$\varepsilon := \frac{1}{\max\{1, 2c_p\}} \frac{w_+(\xi) - w_+(\eta)}{w_+(\xi)} \in (0, 1]$$

we have that

$$\begin{aligned}(w_+(\xi) - w_+(\eta))^{p-1} w_+(\xi)\varphi^p(\xi) \quad \geq \quad & (w_+(\xi) - w_+(\eta))^{p-1} w_+(\xi)(\max\{\varphi(\xi), \varphi(\eta)\})^p \\ & -\frac{1}{2}(w_+(\xi) - w_+(\eta))^p (\max\{\varphi(\xi), \varphi(\eta)\})^p \\ & -c(\max\{w_+(\xi), w_+(\eta)\})^p |\varphi(\xi) - \varphi(\eta)|^p,\end{aligned}$$

where c depends only on p. We now recall that it has been assumed that $\varphi(\xi) \leq \varphi(\eta)$. On the other hand, if $w_+(\xi) = w_+(\eta) = 0$, or if $w_+(\xi) \geq w_+(\eta)$ and $\varphi(\xi) \geq \varphi(\eta)$, then the estimate above trivially follows. Hence, we have

$$\begin{aligned}&(w_+(\xi) - w_+(\eta))^{p-1}(w_+(\xi)\varphi^p(\xi) - w_+(\eta)\varphi^p(\eta)) \\ &\quad \geq (w_+(\xi) - w_+(\eta))^{p-1}(w_+(\xi)(\max\{\varphi(\xi), \varphi(\eta)\})^p - w_+(\eta)\varphi^p(\eta)) \\ &\qquad -\frac{1}{2}(w_+(\xi) - w_+(\eta))^p (\max\{\varphi(\xi), \varphi(\eta)\})^p \\ &\qquad -c(\max\{w_+(\xi), w_+(\eta)\})^p |\varphi(\xi) - \varphi(\eta)|^p \\ &\quad \geq \frac{1}{2}(w_+(\xi) - w_+(\eta))^p (\max\{\varphi(\xi), \varphi(\eta)\})^p \\ &\qquad -c(\max\{w_+(\xi), w_+(\eta)\})^p |\varphi(\xi) - \varphi(\eta)|^p,\end{aligned}$$

whenever $w_+(\xi) \geq w_+(\eta)$. In the case when the opposite inequality holds, again it just suffices to interchange the roles of ξ and η. Therefore, we have

$$\begin{aligned}&\int_{B_r}\int_{B_r} \frac{|w_+(\xi) - w_+(\eta)|^{p-1}(w_+(\xi)\varphi^p(\xi) - w_+(\eta)\varphi^p(\eta))}{|\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{Q+sp}} \,\mathrm{d}\xi \mathrm{d}\eta \\ &\quad \geq \frac{1}{2}\int_{B_r}\int_{B_r} \frac{(w_+(\xi) - w_+(\eta))^p (\max\{\varphi(\xi), \varphi(\eta)\})^p}{|\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{Q+sp}} \,\mathrm{d}\xi \,\mathrm{d}\eta \\ &\qquad - c\int_{B_r}\int_{B_r} \frac{(\max\{w_+(\xi), w_+(\eta)\})^p |\varphi(\xi) - \varphi(\eta)|^p}{|\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{Q+sp}} \,\mathrm{d}\xi \,\mathrm{d}\eta.\end{aligned} \tag{10.5}$$

Now, we note that

$$\begin{aligned}|w_+(\xi)\varphi(\xi) - w_+(\eta)\varphi(\eta)|^p \quad \leq \quad & 2^{p-1}|w_+(\xi) - w_+(\eta)|^p (\max\{\varphi(\xi), \varphi(\eta)\})^p \\ & + 2^{p-1}(\max\{w_+(\xi), w_+(\eta)\})^p |\varphi(\xi) - \varphi(\eta)|^p.\end{aligned}$$

Hence, combining the preceding inequality with (10.4) and (10.5), it follows

$$\begin{aligned}0 \quad \geq \quad & \int_{B_r}\int_{B_r} \frac{|w_+(\xi)\varphi(\xi) - w_+(\eta)\varphi(\eta)|^p}{|\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{Q+sp}} \,\mathrm{d}\xi \,\mathrm{d}\eta \\ & -c\int_{B_r}\int_{B_r} \frac{(\max\{w_+(\xi), w_+(\eta)\})^p |\varphi(\xi) - \varphi(\eta)|^p}{|\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{Q+sp}} \,\mathrm{d}\xi \,\mathrm{d}\eta \\ & -2\int_{B_r} w_+(\xi)\varphi^p(\xi) \,\mathrm{d}\xi \left(\sup_{\xi \in \operatorname{supp}\varphi} \int_{\mathbb{H}^n \setminus B_r} \frac{w_+^{p-1}(\eta)}{|\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{Q+sp}} \,\mathrm{d}\eta \right) \\ & -\int_{B_r} f(\xi, u) w_+(\xi)\varphi^p(\xi) \,\mathrm{d}\xi \,.\end{aligned} \tag{10.6}$$

Moreover, in the second integral in the right-hand side in the display above, we can suppose that $w_+(\xi) \geq w_+(\eta)$ up to interchanging ξ with η; recall that $|\eta^{-1} \circ$

$\xi|_{\mathbb{H}^n} = |\xi^{-1}\circ\eta|_{\mathbb{H}^n}$ is symmetric. The last integral in (10.6) can be finally estimated thanks to the assumption on f. We have

$$\int_{B_r} f(\xi,u(\xi))w_+(\xi)\varphi^p(\xi)\,\mathrm{d}\xi \le \|f\|_{L^\infty(B_r)}\int_{B_r} w_+(\xi)\varphi^p(\xi)\,\mathrm{d}\xi. \tag{10.7}$$

The desired estimate in (10.1) is thus a plain consequence of the estimates in (10.6) and (10.7). □

Subsequently the proof of Theorem 10.1 we focus on establishing another fundamental tool in regularity theory, namely a Fractional Logarithmic Lemma.

Lemma 10.3 (Logarithmic Lemma)**.** *Let $s\in(0,1)$, $p\in(1,\infty)$, and let $u\in W^{s,p}(\mathbb{H}^n)$ be a weak solution to (8.1) such that $u\ge 0$ in $B_R\equiv B_R(\xi_0)\subset\Omega$. Then, the following estimate holds for any $B_r\equiv B_r(\xi_0)\subset B_{\frac{R}{2}}(\xi_0)$ and any $d>0$,*

$$\begin{aligned}&\int_{B_r}\int_{B_r}\left|\log\left(\frac{u(\xi)+d}{u(\eta)+d}\right)\right|^p\frac{\mathrm{d}\xi\,\mathrm{d}\eta}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}+\int_{B_r}f_+(\xi,u)(u(\xi)+d)^{1-p}\,\mathrm{d}\xi\\ &\le c\,r^{Q-sp}+cd^{1-p}\frac{r^Q}{R^{sp}}\Big(\big[\mathrm{Tail}(u_-;\xi_0,R)\big]^{p-1}+1\Big)\\ &\quad+c\,\|f\|_{L^\infty(B_r)}\int_{B_{2r}}(u(\xi)+d)^{1-p}\,\mathrm{d}\xi.\end{aligned}\tag{10.8}$$

where $\mathrm{Tail}(u_-;\xi_0,R)$ *is defined in (8.4),* $u_-:=\max\{-u,0\}$ *and c depends only on n,p and s.*

Proof. Choose $r>0$ such that $B_r\equiv B_r(\xi_0)\subset B_{\frac{R}{2}}\equiv B_{\frac{R}{2}}(\xi_0)$ and consider a smooth cut-off function $\phi\in C_0^\infty(B_{3r/2})$ such that

$$0\le\phi\le 1,\qquad \phi\equiv 1\ \text{in}\ B_r\qquad\text{and}\qquad |D_H\phi|\le c\,r^{-1}\ \text{in}\ B_{3r/2}.$$

Now, take the following test function ψ in Definition 9.4,

$$\psi=(u+d)^{1-p}\phi^p.$$

We have

$$\begin{aligned}&\int_{B_{2r}} f(\xi,u(\xi))(u(\xi)+d)^{1-p}\varphi^p(\xi)\,\mathrm{d}\xi\\ &=\int_{B_{2r}}\int_{B_{2r}}|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}|u(\xi)-u(\eta)|^{p-2}(u(\xi)-u(\eta))\\ &\qquad\qquad\times\left[\frac{\phi^p(\xi)}{(u(\xi)+d)^{p-1}}-\frac{\phi^p(\eta)}{(u(\eta)+d)^{p-1}}\right]\mathrm{d}\xi\,\mathrm{d}\eta\\ &\quad+2\int_{\mathbb{H}^n\setminus B_{2r}}\int_{B_{2r}}|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}|u(\xi)-u(\eta)|^{p-2}\frac{u(\xi)-u(\eta)}{(u(\xi)+d)^{p-1}}\phi^p(\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta\\ &=:I_1+I_2.\end{aligned}\tag{10.9}$$

Using the very definition of the function ϕ and the assumption on the datum f, we can estimate the left-hand side of (10.9) as follows,

$$\begin{aligned}&\int_{B_{2r}} f(\xi,u(\xi))(u(\xi)+d)^{1-p}\varphi^p(\xi)\,\mathrm{d}\xi\\ &\qquad=\int_{B_{2r}}(f(\xi,u))_+(u(\xi)+d)^{1-p}\varphi^p(\xi)\,\mathrm{d}\xi-\int_{B_{2r}}(f(\xi,u))_-(u(\xi)+d)^{1-p}\varphi^p(\xi)\,\mathrm{d}\xi\\ &\qquad\ge\int_{B_r}(f(\xi,u))_+(u(\xi)+d)^{1-p}\,\mathrm{d}\xi-\|f\|_{L^\infty(B_{2r})}\int_{B_{2r}}(u(\xi)+d)^{1-p}\,\mathrm{d}\xi.\end{aligned}$$

We now focus on the remaining integrals on the right-hand side of (10.9).

We start with I_1. If $u(\xi) > u(\eta)$ then we can apply Lemma 10.2, by choosing there

$$a = \phi(\xi), \quad b = \phi(\eta),$$

and

$$\varepsilon = \delta \frac{u(\xi) - u(\eta)}{u(\xi) + d} \in (0,1), \qquad \delta \in (0,1);$$

since $u \geq 0$ in $B_{2r} \subset B_R$, we can therefore estimate the integrand in I_1 as follows,

$$\begin{aligned}
&|\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{-Q-sp} |u(\xi) - u(\eta)|^{p-2} (u(\xi) - u(\eta)) \left[\frac{\phi^p(\xi)}{(u(\xi)+d)^{p-1}} - \frac{\phi^p(\eta)}{(u(\eta)+d)^{p-1}} \right] \\
&\leq |\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{-Q-sp} \left(\frac{u(\xi) - u(\eta)}{u(\xi) + d} \right)^{p-1} \phi^p(\eta) \left[1 + c_p \delta \frac{u(\xi) - u(\eta)}{u(\xi) + d} - \left(\frac{u(\xi) + d}{u(\eta) + d} \right)^{p-1} \right] \\
&\quad + c_p \delta^{1-p} |\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{-Q-sp} |\phi(\xi) - \phi(\eta)|^p \\
&=: J_1 + J_2.
\end{aligned}$$

Now, in order to estimate the contribution J_1 we will follow the strategy in the proof of Lemma 1.3 in [48]. We firstly notice that

$$J_1 \equiv |\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{-Q-sp} \left(\frac{u(\xi) - u(\eta)}{u(\xi) + d} \right)^p \phi^p(\eta) \left[\frac{1 - \left(\frac{u(\eta)+d}{u(\xi)+d} \right)^{1-p}}{1 - \frac{u(\eta)+d}{u(\xi)+d}} + c_p \delta \right] \tag{10.10}$$

Secondly, we can consider the real function g given by

$$g(t) := \frac{1 - t^{1-p}}{1 - t} = -\frac{p-1}{1-t} \int_t^1 \tau^{-p} \mathrm{d}\tau, \qquad \forall t \in (0,1).$$

Since g is an increasing function, we have

$$g(t) \leq -(p-1) \quad \forall t \in (0,1).$$

Moreover, for any $t \leq 1/2$,

$$g(t) \leq -\frac{p-1}{2p} \frac{t^{1-p}}{1-t}.$$

Therefore, in the case when

$$t = \frac{u(\eta) + d}{u(\xi) + d} \in \left(0, \frac{1}{2} \right];$$

i. e.,

$$u(\eta) + d \leq \frac{u(\xi) + d}{2},$$

then, since $\Big(u(\xi) - u(\eta)\Big)\Big(u(\eta) + d\Big)^{p-1} / \Big(u(\xi) + d\Big)^p \leq 1$, we get

$$J_1 \leq |\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{-Q-sp} \left(c_p \delta - \frac{p-1}{2^p} \right) \left[\frac{u(\xi) - u(\eta)}{u(\eta) + d} \right]^{p-1} \phi^p(\eta). \tag{10.11}$$

Hence, it suffices to choose the following suitable δ in the preceding inequality,

$$\delta = \frac{p-1}{2^{p+1} c_p},$$

to get

$$J_1 \leq -|\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{-Q-sp} \frac{p-1}{2^{p+1}} \left[\frac{u(\xi) - u(\eta)}{u(\eta) + d} \right]^{p-1} \phi^p(\eta).$$

We consider case when

$$u(\eta) + d > \frac{u(\xi) + d}{2};$$

i. e.,

$$t = \frac{u(\eta)+d}{u(\xi)+d} \in \left(\frac{1}{2}, 1\right),$$

we can choose the parameter δ as in (1), and we have

$$J_1 \le -|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}\frac{(2^{p+1}-1)(p-1)}{2^{p+1}}\left[\frac{u(\xi)-u(\eta)}{u(\xi)+d}\right]^p \phi^p(\eta). \tag{10.12}$$

Furthermore, if $2(u(\eta)+d) < u(\xi)+d$, then

$$\left[\log\left(\frac{u(\xi)+d}{u(\eta)+d}\right)\right]^p \le c_p\left[\frac{u(\xi)-u(\eta)}{u(\eta)+d}\right]^{p-1} \tag{10.13}$$

holds, where we used the fact that $(\log x)^p \le c(x-1)^{p-1}$ when $x > 2$.

On the other hand, if $2(u(\eta)+d) \ge u(\xi)+d$, then – recalling that we have assumed $u(\xi) > u(\eta)$ – we have

$$\left[\log\left(\frac{u(\xi)+d}{u(\eta)+d}\right)\right]^p = \left[\log\left(1+\frac{u(\xi)-u(\eta)}{u(\eta)+d}\right)\right]^p \le 2^p\left[\frac{u(\xi)-u(\eta)}{u(\xi)+d}\right]^p, \tag{10.14}$$

where we used

$$\log(1+x) \le x, \quad \forall\, x \ge 0, \qquad \text{with } x = \frac{u(\xi)-u(\eta)}{u(\eta)+d} \le \frac{2[u(\xi)-u(\eta)]}{u(\xi)+d}.$$

Hence, combining (10.11) with (10.12), (10.13), and (10.14), we can conclude with

$$\begin{aligned}
&|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}|u(\xi)-u(\eta)|^{p-2}\Big(u(\xi)-u(\eta)\Big)\left[\frac{\phi^p(\xi)}{(u(\xi)+d)^{p-1}} - \frac{\phi^p(\eta)}{(u(\eta)+d)^{p-1}}\right]\\
&\le -\frac{1}{c_p}|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}\left[\log\left(\frac{u(\xi)+d}{u(\eta)+d}\right)\right]^p\phi^p(\eta)\\
&\quad + c_p\,\delta^{1-p}|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-ps}|\phi(\xi)-\phi(\eta)|^p.
\end{aligned}$$

Notice that if $u(\xi) = u(\eta)$, then the same estimate above does trivially hold. If $u(\eta) > u(\xi)$ we can interchange the roles of ξ and η in the computation above. Finally, we get the estimate for the integral I_1 in (10.10),

$$\begin{aligned}
I_1 \le\ & -\frac{1}{c_{p,\delta}}\int_{B_{2r}}\int_{B_{2r}}|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}\left|\log\left(\frac{u(\xi)+d}{u(\eta)+d}\right)\right|^p\phi^p(\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&+ c_{p,\delta}\int_{B_{2r}}\int_{B_{2r}}|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}|\phi(\xi)-\phi(\eta)|^p\,\mathrm{d}\xi\,\mathrm{d}\eta.
\end{aligned} \tag{10.15}$$

For the second contribution in (10.10), namely I_2, we can proceed as follows. Firstly, we notice that it $\eta \in B_R$, then $u(\eta) \ge 0$, and so

$$\frac{(u(\xi)-u(\eta))_+^{p-1}}{(u(\xi)+d)^{p-1}} \le 1 \qquad \forall \xi \in B_{2r}, \eta \in B_R.$$

Moreover, when $\eta \in \mathbb{H}^n \setminus B_R$,

$$(u(\xi)-u(\eta))_+^{p-1} \le 2^{p-1}[u^{p-1}(\xi) + (u(\eta))_-^{p-1}], \qquad \forall \xi \in B_{2r}.$$

Then, since $u(\xi) \geq 0$ and $\phi(\xi) \leq 1$ on B_{2r}, the integral I_2 can be estimated as follows,

$$\begin{aligned} I_2 &\leq 2\int_{B_R\setminus B_{2r}}\int_{B_{2r}} |\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}(u(\xi)-u(\eta))_+^{p-1}(u(\xi)+d)^{1-p}\phi^p(\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\quad +2\int_{\mathbb{H}^n\setminus B_R}\int_{B_{2r}} |\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}(u(\xi)-u(\eta))_+^{p-1}(u(\xi)+d)^{1-p}\phi^p(\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\leq c_p\int_{B_R\setminus B_{2r}}\int_{B_{2r}} |\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}\phi^p(\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\quad +c_p\int_{\mathbb{H}^n\setminus B_R}\int_{B_{2r}} |\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}(u^{p-1}(\xi)+(u(\eta))_-^{p-1})(u(\xi)+d)^{1-p}\phi^p(\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta\,, \end{aligned}$$

and thus

$$\begin{aligned} I_2 &\leq c_p\int_{B_R\setminus B_{2r}}\int_{B_{2r}} |\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}\phi^p(\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\quad + c_p d^{1-p}\int_{\mathbb{H}^n\setminus B_R}\int_{B_{2r}} |\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}\phi^p(\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\quad + c_p d^{1-p}\int_{\mathbb{H}^n\setminus B_R}\int_{B_{2r}} |\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}(u(\eta))_-^{p-1}\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &=: I_{2,1}+I_{2,2}+I_{2,3}. \end{aligned} \tag{10.16}$$

Let us estimate the contribution in the right-side of (10.16). The integral $I_{2,1}$ can be easily estimated by recalling the definition of the cut-off function ϕ; we have

$$I_{2,1} \leq cr^Q\int_{B_R\setminus B_{2r}} \sup_{\xi\in B_{3r/2}} |\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}\,\mathrm{d}\eta \leq cr^{Q-sp}, \tag{10.17}$$

where $c\equiv c(n,p)$.

For any $\xi\in B_{2r}$, and any $\eta\in\mathbb{H}^n\setminus B_R$, in view of the symmetry of $|\cdot|_{\mathbb{H}^n}$, we have

$$\begin{aligned} \frac{|\eta^{-1}\circ\xi_0|_{\mathbb{H}^n}}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}} &\leq \frac{|\xi^{-1}\circ\xi_0|_{\mathbb{H}^n}+|\eta^{-1}\circ\xi|_{\mathbb{H}^n}}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}} \\ &= 1+\frac{|\xi^{-1}\circ\xi_0|_{\mathbb{H}^n}}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}} \\ &\leq 1+\frac{2r}{|\eta^{-1}\circ\xi_0|_{\mathbb{H}^n}-|\xi^{-1}\circ\xi_0|_{\mathbb{H}^n}} \\ &\leq 1+\frac{2r}{R-2r} := C, \end{aligned}$$

where $C>0$ since $R>2r$. This yields

$$I_{2,2} \leq cd^{1-p}\,r^Q\int_{\mathbb{H}^n\setminus B_R} |\eta^{-1}\circ\xi_0|_{\mathbb{H}^n}^{-Q-sp}\,\mathrm{d}\eta \leq cd^{1-p}\,\frac{r^Q}{R^{sp}}, \tag{10.18}$$

where the constant c depending only on n,p and s.

For what concerns the integral $I_{2,3}$ in (10.16), we have

$$\begin{aligned} I_{2,3} &= c_p d^{1-p} \int_{\mathbb{H}^n \setminus B_R} \int_{B_{2r}} |\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{-Q-sp} (u(\eta))_-^{p-1} \, \mathrm{d}\xi \, \mathrm{d}\eta \\ &\leq c d^{1-p} |B_{2r}| \int_{\mathbb{H}^n \setminus B_R} |\eta^{-1} \circ \xi_0|_{\mathbb{H}^n}^{-Q-sp} (u(\eta))_-^{p-1} \, \mathrm{d}\eta \\ &\leq c d^{1-p} \frac{r^Q}{R^{sp}} [\mathrm{Tail}(u_-; \xi_0, R)]^{p-1}, \end{aligned} \tag{10.19}$$

as long as we enlarge the constant $c \equiv c(n,p,s)$. Combining (10.16),(10.17),(10.18) with (10.19) we finally obtain

$$I_2 \leq c r^{Q-sp} + c d^{1-p} r^Q R^{-sp} + c r^Q d^{1-p} R^{-sp} [\mathrm{Tail}(u_-; \xi_0, R)]^{p-1}. \tag{10.20}$$

Combining, now, (10.9)) with (10.15) and (10.20) we have

$$\begin{aligned} &\int_{B_r} \int_{B_r} |\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{-Q-sp} \left| \log \left(\frac{u(\xi)+d}{u(\eta)+d} \right) \right|^p \phi^p(\eta) \, \mathrm{d}\xi \, \mathrm{d}\eta \\ &\leq c \int_{B_{2r}} \int_{B_{2r}} |\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{-Q-sp} |\phi(\xi) - \phi(\eta)|^p \, \mathrm{d}\xi \, \mathrm{d}\eta \\ &\quad + c r^{Q-sp} + c d^{1-p} r^Q R^{-sp} + c r^Q d^{1-p} R^{-sp} [\mathrm{Tail}(u_-; \xi_0, R)]^{p-1}, \end{aligned} \tag{10.21}$$

where c depends only on n, p and s. Let us consider the first integral on the right-hand side in (10.21). Fixed $\eta \in B_{2r}$ there exists a vector $\tilde{\xi}$ such that $\xi = \eta \circ \tilde{\xi}$. Then we can rewrite

$$|\phi(\xi) - \phi(\eta)| = |\phi(\eta \circ \tilde{\xi}) - \phi(\eta)|.$$

Thanks to Theorem 1.41 in [61] we have that the previous quantity can be estimated as follows,

$$|\phi(\eta \circ \tilde{\xi}) - \phi(\eta)| \leq c |\tilde{\xi}|_{\mathbb{H}^n} \sup_{B_{3r/2}} |D_H \phi|$$

Hence, changing again variables and recalling the estimate from above on $|D_H \phi|$, we get

$$\begin{aligned} &\int_{B_{2r}} \int_{B_{2r}} |\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{-Q-sp} |\phi(\xi) - \phi(\eta)|^p \, \mathrm{d}\xi \, \mathrm{d}\eta \\ &\leq c r^{-p} \int_{B_{2r}} \int_{B_{2r}} |\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{-Q-sp+p} \, \mathrm{d}\xi \, \mathrm{d}\eta \,. \end{aligned} \tag{10.22}$$

For any $\eta \in B_{2r}$ note that a simple application of the triangular inequality yields

$$|\eta^{-1} \circ \xi|_{\mathbb{H}^n} \leq |\eta^{-1} \circ \xi_0|_{\mathbb{H}^n} + |\xi^{-1} \circ \xi_0|_{\mathbb{H}^n} \leq 4r, \qquad \forall \xi \in B_{2r},$$

so that $B_{2r} \subset B_{4r}(\eta)$. Hence, as seen in the local framework in [72, 128], one can apply Proposition 5.4.4 in [19] to get

$$\begin{aligned} \int_{B_{2r}} \int_{B_{2r}} |\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{-Q-sp+p} \, \mathrm{d}\xi \, \mathrm{d}\eta &\leq \int_{B_{2r}} \int_{B_{4r}(\eta)} |\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{-Q-sp+p} \, \mathrm{d}\xi \, \mathrm{d}\eta \\ &\leq Q |B_1(0)| \int_{B_{2r}} \int_0^{4r} \varrho^{p-sp-1} \, \mathrm{d}\varrho \, \mathrm{d}\eta \\ &\leq \frac{c r^{Q+p-sp}}{p-sp}. \end{aligned}$$

The estimate in (10.22) thus becomes

$$\int_{B_{2r}} \int_{B_{2r}} |\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{-Q-sp} |\phi(\xi) - \phi(\eta)|^p \, \mathrm{d}\xi \, \mathrm{d}\eta \leq c r^{Q-sp},$$

and the proof is complete. □

We conclude this section by presenting an important consequence of the Logarithmic Lemma, which will prove extremely useful in Section 3.

Corollary 10.4. *Let $s \in (0,1)$, $p \in (1,\infty)$, and let $u \in W^{s,p}(\mathbb{H}^n)$ be the solution to problem (8.1) such that $u \geq 0$ in $B_R \equiv B_R(\xi_0) \subset \Omega$. Let $a, d > 0, b > 1$ and define*

$$v := \min\Big\{(\log(a+d) - \log(u+d))_+, \log(b)\Big\}.$$

Then, the following estimates holds true, for any $B_r \equiv B_r(\xi_0) \subset B_{R/2}(\xi_0) \subset B_{R/2}(\xi_0)$,

$$\begin{aligned} &\fint_{B_r} |v - (v)_{B_r}|^p \,\mathrm{d}\xi \\ &\qquad \leq c + cd^{1-p}\left(\frac{r}{R}\right)^{sp}\left(1 + \Big[\mathrm{Tail}(u_-;\xi_0,R)\Big]^{p-1}\right) \\ &\qquad\quad + c\, r^{sp}\|f\|_{L^\infty(B_{2r})} \fint_{B_{2r}} (u(\xi)+d)^{1-p}\,\mathrm{d}\xi. \end{aligned} \tag{10.23}$$

where c depends only on n, p, s.

Proof. The estimate in (10.23) is a plain consequence of the Logarithmic Lemma 10.3. Firstly, we need to apply the fractional Poincaré inequality and obtain

$$\fint_{B_r} |v - (v)_{B_r}|^p \,\mathrm{d}\xi \;\leq\; cr^{sp-Q}\int_{B_r}\int_{B_r} |\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}|v(\xi) - v(\eta)|^p \,\mathrm{d}\xi\,\mathrm{d}\eta,$$

where $c \equiv c(n,p,s)$.

Now, it is sufficient to observe that v is precisely a truncation of the sum of a constant and $\log(u+d)$. For this, we have

$$\begin{aligned} &\int_{B_r}\int_{B_r} |\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}|v(\xi) - v(\eta)|^p \,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\leq \int_{B_r}\int_{B_r} |\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}\left|\log\left(\frac{u(\eta)+d}{u(\xi)+d}\right)\right|^p \,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\quad + \int_{B_r} (f(\xi,u))_+(u(\xi)+d)^{1-p}\,\mathrm{d}\xi, \end{aligned}$$

so that the desired estimate plainly follow by applying the estimate in (10.8). □

2. Local boundedness

The aim of this section is to prove the following local boundedness result:

Theorem 10.5 (Local boundedness)**.** *Let $s \in (0,1)$ and $p \in (1,\infty)$, let $u \in W^{s,p}(\mathbb{H}^n)$ be a weak subsolution to (8.1), and let $B_r \equiv B_r(\xi_0) \subset \Omega$. Then the following estimate holds true, for any $\delta \in (0,1]$,*

$$\begin{aligned} \sup_{B_{r/2}} u \;\leq\; &\delta\,\mathrm{Tail}(u_+;\xi_0,r/2) + c\,\delta^{-\frac{(p-1)Q}{sp^2}}\left(\fint_{B_r} u_+^p \,\mathrm{d}\xi\right)^{\frac{1}{p}} \\ &+ r^{\frac{sp}{p-1}}\|f\|_{L^\infty(B_r)}^{\frac{1}{p-1}}, \end{aligned} \tag{10.24}$$

where $\mathrm{Tail}(u_+;\xi_0,r/2)$ is defined in (8.4), $u_+ := \max\{u, 0\}$ is the positive part of the function u, and the constant c depends only on n, p, s.

In our knowledge, such a result is new even in the case of the pure *conformally invariant* fractional subLaplacian on the Heisenberg group. Here we are able to prove that, via careful estimates based on the nonlocal tail of the solutions together with the Caccioppoli inequality of Theorem 10.1 one can extend the approach firstly seen in [48] for the counterpart in the Euclidean case.

We would also like to stress that the presence of the parameter δ in (10.24) allows an interpolation between the local and nonlocal terms.

Proof of Theorem 10.5. Before starting, we would need to define a few quantities. For any $j \in \mathbb{N}$ and $r > 0$ such that $B_r(\xi_0) \equiv B_r \subset \Omega$,

$$r_j = \frac{1}{2}(1 + 2^{-j})r, \qquad \tilde{r}_j = \frac{r_j + r_{j+1}}{2},$$

$$B_j = B_{r_j}(\xi_0), \qquad \tilde{B}_j = B_{\tilde{r}_j}(\xi_0).$$

Also,

$$\varphi_j \in C_0^\infty(\tilde{B}_j), \quad 0 \le \varphi_j \le 1, \quad \varphi_j \equiv 1 \text{ on } B_{j+1} \text{ and } |D_H \varphi_j| < 2^{j+3}/r,$$

$$k_j = k + (1 - 2^{-j})\tilde{k}, \quad \tilde{k}_j = \frac{k_{j+1} + k_j}{2}, \quad \tilde{k} \in \mathbb{R}^+ \text{ and } k \in \mathbb{R},$$

$$\tilde{w}_j = (u - \tilde{k}_j)_+ \qquad w_j = (u - k_j)_+.$$

We divide the proof into two steps. Firstly, we consider the subcritical case when $sp < Q$. Recalling the fractional Sobolev exponent $p_s^* = \frac{Qp}{Q-sp}$, we have

$$\frac{p}{p_s^*} = \frac{p}{\frac{Qp}{Q-sp}} = \frac{Qp - sp^2}{Qp} = 1 - \frac{sp}{Q}.$$

Consequently,

$$\left(\frac{1}{|B_j|}\right)^{\frac{p}{p_s^*}} = \frac{1}{|B_j|}|B_j|^{\frac{sp}{Q}} = \frac{cr_j^{sp}}{|B_j|}.$$

We now apply the Sobolev inequality in Theorem 9.1 to the function $\tilde{w}_j\varphi_j$. It yields

$$\left(\fint_{B_j} |\tilde{w}_j(\xi)\varphi_j(\xi)|^{p_s^*}\,\mathrm{d}\xi\right)^{\frac{p}{p_s^*}} \le cr_j^{sp} \fint_{B_j}\int_{B_j} \frac{|\tilde{w}_j(\xi)\varphi_j(\xi) - \tilde{w}_j(\eta)\varphi_j(\eta)|^p}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

which, combined with the nonlocal Caccioppoli inequality (10.1), gives

$$\begin{aligned}
&\left(\fint_{B_j} |\tilde{w}_j(\xi)\varphi_j(\xi)|^{p_s^*}\,\mathrm{d}\xi\right)^{\frac{p}{p_s^*}} \\
&\le cr_j^{sp} \fint_{B_j}\int_{B_j} \frac{\tilde{w}_j^p(\xi)|\varphi_j(\xi) - \varphi_j(\eta)|^p}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\quad + cr_j^{sp} \fint_{B_j} \tilde{w}_j(\eta)\varphi_j^p(\eta)\,\mathrm{d}\eta \left(\sup_{\eta\in\mathrm{spt}\ \varphi_j} \int_{\mathbb{H}^n\setminus B_j} \frac{\tilde{w}_j^{p-1}(\xi)}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}\,\mathrm{d}\xi\right) \\
&\quad + cr_j^{sp}\|f\|_{L^\infty(B_j)} \fint_{B_j} \tilde{w}_j(\eta)\varphi_j^p(\eta)\,\mathrm{d}\eta \\
&=: I_1 + I_2 + I_3\,.
\end{aligned} \tag{10.25}$$

We begin by estimating the first contribution in the right-hand side of (10.25). By applying the same strategy as in the proof of the Logarithmic Lemma in Section 1, we have

$$\begin{aligned} I_1 &\le c2^{pj} r_j^{sp-p} \fint_{B_j}\int_{B_j} |\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp+p} \tilde{w}_j^p(\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\le c2^{pj} r_j^{sp-p} \fint_{B_j} \tilde{w}_j^p(\xi)\int_{B_j} |\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp+p}\,\mathrm{d}\eta\,\mathrm{d}\xi \\ &\le c2^{pj} \fint_{B_j} \tilde{w}_j^p(\xi)\,\mathrm{d}\xi. \end{aligned}$$

Now, we note that $\tilde{w}_j \le w_j^p/(\tilde{k}_j - k_j)^{p-1}$. Moreover, since $\eta \in \operatorname{spt}\varphi_j \subseteq \tilde{B}_j$ and $\xi \in \mathbb{H}^n \setminus B_j$, we have

$$\frac{|\xi^{-1}\circ\xi_0|_{\mathbb{H}^n}}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}} \le \frac{|\eta^{-1}\circ\xi_0|_{\mathbb{H}^n} + |\eta^{-1}\circ\xi|_{\mathbb{H}^n}}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}} \le 1 + \frac{\tilde{r}_j}{r_j - \tilde{r}_j} \le 2^{j+4}.$$

For this,

$$\begin{aligned} I_2 &\le c2^{j(Q+sp)} r_j^{sp} \fint_{B_j} \frac{w_j^p(\eta)}{(\tilde{k}_j - k_j)^{p-1}}\,\mathrm{d}\eta \int_{\mathbb{H}^n\setminus B_j} \frac{w_j^{p-1}(\xi)}{|\xi^{-1}\circ\xi_0|_{\mathbb{H}^n}^{Q+sp}}\,\mathrm{d}\xi \\ &\le \frac{c2^{j(Q+sp+p-1)}}{\tilde{k}^{p-1}} [\mathrm{Tail}(w_0;\xi_0,r/2)]^{p-1} \fint_{B_j} w_j^p(\eta)\,\mathrm{d}\eta. \end{aligned}$$

Since $\tilde{w}_j \le 2^{j(p-1)}\tilde{k}^{1-p} w_j$, $r_j \le r$ and $\varphi \le 1$, the third integral in (10.25) can be easily estimated as follows

$$I_3 \le 2^{j(p-1)} \frac{r^{sp}\|f\|_{L^\infty(B_r)}}{\tilde{k}^{p-1}} \fint_{B_j} w_j^p(\eta)\,\mathrm{d}\eta.$$

For what concerns the left-hand side in (10.25), we have

$$\begin{aligned} \left(\fint_{B_j} |\tilde{w}_j(\xi)\varphi_j(\xi)|^{p_s^*}\,\mathrm{d}\xi\right)^{\frac{p}{p_s^*}} &\ge (k_{j+1} - \tilde{k}_j)^{\frac{(p^*-p)p}{p_s^*}} \left(\fint_{B_{j+1}} w_{j+1}^p(\xi)\,\mathrm{d}\xi\right)^{\frac{p}{p_s^*}} \\ &= \left(\frac{\tilde{k}}{2^{j+2}}\right)^{\frac{(p_s^*-p)p}{p_s^*}} \left(\fint_{B_{j+1}} w_{j+1}^p(\xi)\,\mathrm{d}\xi\right)^{\frac{p}{p_s^*}}. \end{aligned}$$

Then, calling $A_j := \left(\fint_{B_j} w_j^p(\xi)\,\mathrm{d}\xi\right)^{\frac{1}{p}}$ we obtain

$$\begin{aligned} &\left(\frac{\tilde{k}^{1-\frac{p}{p_s^*}}}{2^{(j+2)\frac{p_s^*-p}{p_s^*}}}\right)^p A_{j+1}^{\frac{p^2}{p_s^*}} \\ &\quad\le c2^{j(Q+sp+p-1)} \left(1 + \frac{(\mathrm{Tail}(w_0;\xi_0,r/2))^{p-1} + r^{sp}\|f\|_{L^\infty(B_r)}}{\tilde{k}^{p-1}}\right) A_j^p. \end{aligned}$$

Now, by taking

$$\tilde{k} \ge \delta\mathrm{Tail}(w_0;\xi_0,r/2) + r^{\frac{sp}{p-1}}\|f\|_{L^\infty(B_r)}^{\frac{1}{p-1}}, \qquad \delta \in (0,1], \tag{10.26}$$

we obtain

$$\left(\frac{A_{j+1}}{\tilde{k}}\right)^{\frac{p}{p_s^*}} \le \delta^{\frac{1-p}{p}} c^{\frac{p}{p_s^*}} 2^{j\left(\frac{sp}{Q} + \frac{Q+sp+p-1}{p}\right)} \frac{A_j}{\tilde{k}}, \tag{10.27}$$

where $\bar c \equiv c^{\frac{p_s^*}{p^2}}(\delta^{p-1}+1)^{\frac{p_s^*}{p^2}} 2^{\frac{2(p_s^*-p)}{p}}$. We set

$$C := 2^{\frac{sp}{Q-sp}+\frac{Q(Q+sp+p-1)}{p(Q-sp)}} > 1, \qquad \beta = \frac{p_s^*}{p} - 1,$$

so that the estimate in (10.27) can be rewritten as

$$\frac{A_{j+1}}{\tilde k} \le \delta^{\frac{p_s^*(1-p)}{p^2}} \bar c C^j \left(\frac{A_j}{\tilde k}\right)^{1+\beta}.$$

Thus it suffices to prove that the following estimate does hold,

$$\frac{A_0}{\tilde k} \le \delta^{\frac{p_s^*(p-1)}{p^2\beta}} \bar c^{-\frac{1}{\beta}} C^{-\frac{1}{\beta^2}}, \tag{10.28}$$

and by a standard iteration argument, it will follow that $A_j \to 0$ as $j \to \infty$. Since

$$\frac{p_s^*(p-1)}{p^2\beta} = \frac{(p-1)Q}{sp^2},$$

we then choose

$$\tilde k := \delta \mathrm{Tail}(w_0;\xi_0,r/2) + \delta^{-\frac{(p-1)Q}{sp^2}} H A_0 + r^{\frac{sp}{p-1}} \|f\|_{L^\infty(B_r)}^{\frac{1}{p-1}}, \qquad H = \bar c^{\frac{1}{\beta}} C^{\frac{1}{\beta^2}},$$

which is in accordance with (10.26). It follows

$$\begin{aligned}
\sup_{B_{r/2}} u &\le k + \tilde k \\
&= k + \delta \mathrm{Tail}((u-k)_+;\xi_0,r/2) + \delta^{-\frac{(p-1)Q}{sp^2}} H \left(\fint_{B_r} (u-k)_+^p \right)^{\frac{1}{p}} \\
&\quad + r^{\frac{sp}{p-1}} \|f\|_{L^\infty(B_r)}^{\frac{1}{p-1}},
\end{aligned}$$

which gives the desired result by choosing $k = 0$. The proof is complete in the case when $sp < Q$.

We now turn to the borderline case, when $sp = Q$, which in the Euclidean case in [48] is mentioned but not even sketched. We fill this gap here, by providing all the details to investigate such a case. Choose $0 < s_1 < s < 1$; in particular $\tilde w_j\varphi_j - (\tilde w_j\varphi_j)_{B_j} \in W^{s,p}(B_j) \subseteq W^{s_1,p}(B_j)$. Clearly, $s_1 p < sp = Q$, and we have the right to apply the fractional Sobolev inequality which gives, for any $p < q < p_{s_1}^* := \frac{Qp}{Q-s_1p}$,

$$\|\tilde w_j\varphi_j - (\tilde w_j\varphi_j)_{B_j}\|_{L^q(B_j)} \le |B_j|^{\frac{p_{s_1}^*-q}{qp_{s_1}^*}} \|\tilde w_j\varphi_j - (\tilde w_j\varphi_j)_{B_j}\|_{L^{p_{s_1}^*}(B_j)}.$$

Thus, we can write

$$\begin{aligned}
&\left| \left(\fint_{B_j} |\tilde w_j(\xi)\varphi_j(\xi)|^q \,\mathrm{d}\xi \right)^{\frac{1}{q}} - \left| \fint_{B_j} \tilde w_j(\xi)\varphi_j(\xi)\,\mathrm{d}\xi \right| \right|^p \\
&\qquad \le c \left(\fint_{B_j} |\tilde w_j(\xi)\varphi_j(\xi) - (\tilde w_j\varphi_j)_{B_j}|^q \,\mathrm{d}\xi \right)^{\frac{p}{q}} \\
&\qquad \le c|B_j|^{\frac{(p_{s_1}^*-q)p}{qp_{s_1}^*}-\frac{p}{q}} \left(\int_{B_j} |\tilde w_j(\xi)\varphi_j(\xi) - (\tilde w_j\varphi_j)_{B_j}|^{p_{s_1}^*} \,\mathrm{d}\xi \right)^{\frac{p}{p_{s_1}^*}} \\
&\qquad \le c\frac{r_j^{s_1p}}{r_j^Q} \int_{B_j}\int_{B_j} \frac{|\tilde w_j(\xi)\varphi_j(\xi) - \tilde w_j(\eta)\varphi_j(\eta)|^p}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+s_1p}} \,\mathrm{d}\xi\,\mathrm{d}\eta,
\end{aligned}$$

where in the last inequality we also used that

$$|B_j|^{\frac{(p^*_{s_1}-q)p}{qp^*_{s_1}}-\frac{p}{q}} = |B_j|^{-\frac{p}{p^*_{s_1}}} = |B_j|^{\frac{s_1 p}{Q}-1} = c\frac{r_j^{s_1 p}}{r_j^Q}.$$

In addition, we notice that

$$\left|\left(\fint_{B_j}|\tilde{w}_j(\xi)\varphi_j(\xi)|^q\,\mathrm{d}\xi\right)^{\frac{1}{q}} - \left|\fint_{B_j}\tilde{w}_j(\xi)\varphi_j(\xi)\,\mathrm{d}\xi\right|\right|^p \geq \left(\fint_{B_j}|\tilde{w}_j(\xi)\varphi_j(\xi)|^q\,\mathrm{d}\xi\right)^{\frac{p}{q}} - \fint_{B_j}|\tilde{w}_j(\xi)\varphi_j(\xi)|^p d\xi.$$

We are now in a position to apply the nonlocal Caccioppoli-type inequality in (10.1), as done before in the subcritical case. Similarly, we get

$$\left(\fint_{B_j}|\tilde{w}_j(\xi)\varphi_j(\xi)|^q\,\mathrm{d}\xi\right)^{\frac{p}{q}} \leq c\,2^{j(Q+sp+p-1)}\left(1+\frac{[\mathrm{Tail}(w_0;\xi_0,r/2)]^{p-1}+r^{sp}\|f\|_{L^\infty(B_r)}}{\tilde{k}^{p-1}}\right)\fint_{B_j}w_j^p(\eta)\,\mathrm{d}\eta.$$

Moreover, the term on the left-hand side in the inequality above can be estimated as follows

$$\left(\fint_{B_j}|\tilde{w}_j(\xi)\varphi_j(\xi)|^q\,\mathrm{d}\xi\right)^{\frac{p}{q}} \geq \left(\frac{\tilde{k}^{1-\frac{p}{q}}}{2^{(j+2)\frac{(q-p)}{q}}}\right)^p\left(\fint_{B_{j+1}}w_{j+1}^p\,\mathrm{d}\xi\right)^{\frac{p}{q}}.$$

We set $A_j := \left(\fint_{B_j}w_j^p(\xi)\,\mathrm{d}\xi\right)^{\frac{1}{p}}$ and we choose $\tilde{k}$ as in (10.26). It yields

$$\frac{A_{j+1}}{\tilde{k}} \leq \delta^{\frac{q(1-p)}{p^2}}\bar{c}C^j\left(\frac{A_j}{\tilde{k}}\right)^{1+\beta},$$

where

$$\bar{c} := c^{\frac{q}{p^2}}\left(\delta^{p-1}+1\right)^{\frac{q}{p^2}}2^{\frac{2(q-p)}{p}},$$

$$C := 2^{\frac{q(Q+sp+p-1)}{p^2}+\frac{q-p}{p}} > 1, \quad \text{and} \quad \beta := \frac{q}{p}-1.$$

We can now estimate the term A_0 as in (10.28), by replacing p_s^* with q, and considering as $\bar{c}, C$ and β the quantities defined in the display above. Then,

$$A_j \to 0 \text{ as } j \to +\infty.$$

Taking

$$\tilde{k} := \delta\mathrm{Tail}(w_0;\xi_0,r/2) + \delta^{-\frac{(p-1)q}{p(q-p)}}HA_0 + r^{\frac{sp}{p-1}}\|f\|^{\frac{1}{p-1}}_{L^\infty(B_r)}, \qquad H = \bar{c}^{\frac{1}{\beta}}C^{\frac{1}{\beta^2}},$$

which is in clear accordance with (10.26), we finally deduce that

$$\sup_{B_{r/2}} u \leq k+\delta\mathrm{Tail}((u-k)_+;\xi_0,r/2)+\delta^{-\frac{(p-1)q}{p(q-p)}}H\left(\fint_{B_r}(u-k)_+^p\right)^{\frac{1}{p}}+r^{\frac{sp}{p-1}}\|f\|^{\frac{1}{p-1}}_{L^\infty(B_r)},$$

which gives the desired result by choosing $k=0$. □

3. Hölder continuity

In this section we proof the main regularity results for weak solutions to (8.1). The theorem below provides an extension of the classical results by De Giorgi-Nash-Moser to the nonlocal framework on the Heisenberg group. In the linear case, when $p = 2$, for what concerns classical Hölder regularity results for linear integro-differential operators in a very wide class of metric measure spaces, we refer to the important paper by Chen et Kumagai [33]. Still in the linear case, it is also worth mentioning some related regularity results in [56], where the authors deal with linear operators related to (8.2) by making use of the Neumann-to-Dirichlet extension, which is not applicable in our nonlinear setting.

Our main result reads as follow.

Theorem 10.6 (Hölder continuity)**.** *Let $s \in (0,1)$, $p \in (1,\infty)$, and let $u \in W^{s,p}(\mathbb{H}^n)$ be a solution to (8.1). Then u is locally Hölder continuous in Ω. In particular, there are constants $\alpha < sp/(p-1)$ and $c > 0$, both depending only on n, p and s, such that if $B_{2r}(\xi_0) \subset \Omega$ then*

$$\operatorname*{osc}_{B_\varrho(\xi_0)} u \leq c \left(\frac{\varrho}{r}\right)^\alpha \left(\operatorname{Tail}(u;\xi_0,r) + \left(\fint_{B_{2r}(\xi_0)} |u|^p \,\mathrm{d}\xi \right)^{\frac{1}{p}} + r^{\frac{sp}{p-1}} \|f\|_{L^\infty(B_{2r})}^{\frac{1}{p-1}} \right),$$

for every $\varrho \in (0,r)$.

An iteration lemma is the keypoint of the proof, and, as before, we would have to handle the nonlocality of the involved operator, together with the geometry of our settings. A certain care is required and in the proof below all the estimates proven in the previous sections will appear.

We need to fix some notation. For any $j \in \mathbb{N}$, let $0 < r < R/2$ for some R such that $B_R(\xi_0) \subset \Omega$,

$$r_j := \sigma^j \frac{r}{2}, \quad \sigma \in \left(0, \frac{1}{4}\right], \quad B_j := B_{r_j}(\xi_0).$$

Moreover, let us define

$$\frac{1}{2}\omega(r_0) := \frac{1}{2}\omega(r) = \operatorname{Tail}(u;\xi_0,r/2) + c \left(\fint_{B_r} u_+^p \,\mathrm{d}\xi \right)^{\frac{1}{p}} + r^{\frac{sp}{p-1}} \|f\|_{L^\infty(B_r)}^{\frac{1}{p-1}},$$

with c as in Theorem 10.5 and

$$\omega(r_j) := \left(\frac{r_j}{r_0}\right)^\alpha \omega(r_0) \qquad \text{for some } \alpha < \frac{sp}{p-1}.$$

In order to prove the Theorem 10.6 it suffices to prove the following

Lemma 10.7. *Under the notation introduced above, let $u \in W^{s,p}(\mathbb{H}^n)$ be a solution to problem (8.1). Then*

$$\operatorname*{osc}_{B_j} u \leq \omega(r_j), \qquad \forall j = 0,1,2,\dots \tag{10.29}$$

Proof. We will proceed by induction. For this, note that by the definition of $\omega(r_0)$ and Theorem 10.5 (with $\delta = 1$ there), the estimates (10.29) does trivially hold for $j = 0$, since both the functions $(u)_+$ and $(-u)_+$ are weak subsolution.

Now, we make a strong induction assumption and assume that (10.29) is valid for all $i \in \{1,\dots,j\}$ for some $j \geq 0$, and then prove that it holds also for

$j+1$. We have that either

$$\frac{|2B_{j+1}\cap\{u\geq \inf_{B_j} u+\omega(r_j)/2\}|}{|2B_{j+1}|}\ \geq\ \frac{1}{2}, \tag{10.30}$$

or

$$\frac{|2B_{j+1}\cap\{u\leq \inf_{B_j} u+\omega(r_j)/2\}|}{|2B_{j+1}|}\ \geq\ \frac{1}{2}, \tag{10.31}$$

must hold. If (10.30) holds, then we set $u_j := u-\inf_{B_j} u$; if 10.31 holds, then we set $u_j := \omega(r_j)-(u-\inf_{B_j} u)$. Consequently, in all the cases we have that $u_j\geq 0$ in B_j and the following estimate holds,

$$\frac{|2B_{j+1}\cap\{u_j\geq \omega(r_j)/2\}|}{|2B_{j+1}|}\geq\frac{1}{2}. \tag{10.32}$$

Moreover, u_j is a weak solution which satisfies

$$\sup_{B_i}|u_j|\leq 2\omega(r_i),\qquad \forall i\in\{1,\dots,j\}. \tag{10.33}$$

We now claim that under the induction assumption we have

$$[\mathrm{Tail}(u_j;\xi_0,r_j)]^{p-1}\ \leq\ c\sigma^{-\alpha(p-1)}[\omega(r_j)]^{p-1}, \tag{10.34}$$

where c depends only on n,p,s and α; in particular it is independent of σ. Indeed,

$$\begin{aligned}
[\mathrm{Tail}(u_j;\xi_0,r_j)]^{p-1} &= r_j^{sp}\sum_{i=1}^{j}\int_{B_{i-1}\setminus B_i}|u_j(\xi)|^{p-1}|\xi_0^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}\,\mathrm{d}\xi\\
&\quad + r_j^{sp}\int_{\mathbb{H}^n\setminus B_0}|u_j(\xi)|^{p-1}|\xi_0^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}\,\mathrm{d}\xi\\
&\leq r_j^{sp}\sum_{i=1}^{j}[\sup_{B_{i-1}}|u_j|]^{p-1}\int_{\mathbb{H}^n\setminus B_i}|\xi_0^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}\,\mathrm{d}\xi\\
&\quad + r_j^{sp}\int_{\mathbb{H}^n\setminus B_0}|u_j(\xi)|^{p-1}|\xi_0^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}\,\mathrm{d}\xi\\
&\leq c\sum_{i=1}^{j}\left(\frac{r_j}{r_i}\right)^{sp}[\omega(r_{i-1})]^{p-1}, \qquad (10.35)
\end{aligned}$$

where in the last line we also used (10.33) and the fact that

$$\begin{aligned}
&\int_{\mathbb{H}^n\setminus B_0}|u_j(\xi)|^{p-1}|\xi_0^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}\,\mathrm{d}\xi\\
&\quad\leq cr_0^{-sp}\sup_{B_0}|u|^{p-1}+cr_0^{-sp}[\omega(r_0)]^{p-1}+c\int_{\mathbb{H}^n\setminus B_0}|u(\xi)|^{p-1}|\xi_0^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}\,\mathrm{d}\xi\\
&\quad\leq cr_1^{-sp}[\omega(r_0)]^{p-1}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\sum_{i=1}^{j}\left(\frac{r_j}{r_i}\right)^{sp}[\omega(r_{i-1})]^{p-1}\\
&\qquad=[\omega(r_0)]^{p-1}\left(\frac{r_j}{r_0}\right)^{\alpha(p-1)}\sum_{i=1}^{j}\left(\frac{r_{i-1}}{r_i}\right)^{\alpha(p-1)}\left(\frac{r_j}{r_i}\right)^{sp-\alpha(p-1)}\\
&\qquad=[\omega(r_j)]^{p-1}\sigma^{-\alpha(p-1)}\sum_{i=0}^{j-1}\sigma^{i(sp-\alpha(p-1))}\\
&\qquad\le[\omega(r_j)]^{p-1}\frac{\sigma^{-\alpha(p-1)}}{1-\sigma^{sp-\alpha(p-1)}}\\
&\qquad\le\frac{4^{sp-\alpha(p-1)}}{\log(4)(sp-\alpha(p-1))}\sigma^{-\alpha(p-1)}[\omega(r_j)]^{p-1} \qquad (10.36)
\end{aligned}$$

where we used the fact that $\sigma\le\frac14\le\frac14$ and $\alpha<sp/(p-1)$. Combining (10.35) with (10.36) yields the desired estimate in (10.34).

Next, let us consider the function v defined by

$$v:=\min\left\{\left(\log\left(\frac{\omega(r_j)/2+d}{u_j+d}\right)\right)_+,k\right\},\qquad k>0. \qquad (10.37)$$

Since we have chosen $\sigma\le\frac14$ we have that $2B_{j+1}\subset B_{\frac{r_j}{2}}\subset B_j$. Indeed,

$$2r_{j+1}=2\sigma^{j+1}\frac{r}{2}\le\frac12\sigma^j\frac{r}{2}=\frac{r_j}{2}.$$

Hence, we can apply Corollary 10.4, with $a\equiv\omega(r_j)/2$ and $b\equiv\exp(k)$ there, obtaining that

$$\begin{aligned}
\fint_{2B_{j+1}}|v-(v)_{2B_{j+1}}|^p\,\mathrm{d}\xi &\le c+cd^{1-p}\left(\frac{r_{j+1}}{r_j}\right)^{sp}\left(1+[\mathrm{Tail}(u_j;\xi_0,r_j)]^{p-1}\right)\\
&\quad+c(2r_{j+1})^{sp}\|f\|_{L^\infty(B_r)}\fint_{4B_{j+1}}(u_j(\xi)+d)^{1-p}\,\mathrm{d}\xi\\
&=: I_1+I_2.
\end{aligned}$$

In view of the estimate in (10.34), we can estimate the first term in the right-hand side of the display above as follows

$$\begin{aligned}
I_1 &\le c+c\sigma^{sp}d^{1-p}\{1+\sigma^{-\alpha(p-1)}[\omega(r_j)]^{p-1}\}\\
&= c+c\sigma^{sp}d^{1-p}\{1+\sigma^{\alpha(p-1)(j-1)}[\omega(r_0)]^{p-1}\},
\end{aligned}$$

where c depends only on n,p,s and α. Moreover, choosing

$$d=\sigma^{\frac{sp}{p-1}}.$$

we arrive at

$$I_1\le c+c\sigma^{\alpha(p-1)(j-1)}[\omega(r_0)]^{p-1}=:C_1<\infty, \qquad (10.38)$$

where we used the fact that $\alpha<sp/(p-1)$, $\sigma\le\frac14\le\frac14$, $j>1$, and $\omega(r_0)<\infty$.

Let us estimate the second integral I_2. Notice that since $\sigma<1/4$, we have

$$4r_{j+1}=4\sigma^{j+1}\frac{r}{2}\le\sigma^j\frac{r}{2}=r_j,$$

so that $4B_{j+1} \subseteq B_j$, and consequently $u_j \geq 0$ on $4B_{j+1}$. Then, in view of the choice of d, we also have $(u_j + d)^{1-p} \leq \sigma^{-sp}$. We arrive at

$$I_2 \leq c\sigma^{sp(j-1)}(r/4)^{sp}\|f\|_{L^\infty(B_r)} \leq c\sigma^{sp(j-1)}\omega(r_0)^{p-1} < \infty. \tag{10.39}$$

Hence from (10.38) and (10.39) we obtain

$$\fint_{2B_{j+1}} |v - (v)_{2B_{j+1}}|^p \,\mathrm{d}\xi \leq C. \tag{10.40}$$

for a suitable positive constant C.

Denote by shortness $\tilde{B} = 2B_{j+1}$. In accordance with (10.32) and (10.37) we can write

$$\begin{aligned} k &= \frac{1}{|\tilde{B} \cap \{u_j \geq \omega(r_j)/2\}|} \int_{\tilde{B}\cap\{u_j \geq \omega(r_j)/2\}} k \,\mathrm{d}\xi \\ &= \frac{1}{|\tilde{B} \cap \{u_j \geq \omega(r_j)/2\}|} \int_{\tilde{B}\cap\{v=0\}} k \,\mathrm{d}\xi \\ &\leq \frac{2}{|\tilde{B}|} \int_{\tilde{B}} (k - v) \,\mathrm{d}\xi = 2[k - (v)_{\tilde{B}}]\,. \end{aligned}$$

We then integrate the inequality above over the set $\tilde{B} \cap \{v = k\}$ to get

$$\begin{aligned} \frac{|\tilde{B} \cap \{v = k\}|}{|\tilde{B}|} k &\leq \frac{2}{|\tilde{B}|} \int_{\tilde{B}\cap\{v=k\}} [k - (v)_{\tilde{B}}] \,\mathrm{d}\xi \\ &\leq \frac{2}{|\tilde{B}|} \int_{\tilde{B}} |v - (v)_{\tilde{B}}| \,\mathrm{d}\xi \leq c, \end{aligned}$$

where we also used (10.40).

We now take

$$\begin{aligned} k &= \log\left(\frac{\omega(r_j)/2 + \sigma^{\frac{sp}{p-1}}}{2\sigma^{\frac{sp}{p-1}}\omega(r_j) + \sigma^{\frac{sp}{p-1}}}\right) \\ &= \log\left(\frac{\omega(r_j)}{4\sigma^{\frac{sp}{p-1}}\omega(r_j) + 2\sigma^{\frac{sp}{p-1}}} + \frac{\sigma^{\frac{sp}{p-1}}}{2\sigma^{\frac{sp}{p-1}}\omega(r_j) + \sigma^{\frac{sp}{p-1}}}\right) \approx \log\left(\frac{1}{\sigma^{\frac{sp}{p-1}}}\right), \end{aligned}$$

so that

$$\frac{|\tilde{B} \cap \{v = k\}|}{|\tilde{B}|} k \leq c$$

gives

$$\frac{|\tilde{B} \cap \{u_j \leq 2d\omega(r_j)\}|}{|\tilde{B}|} \leq \frac{c}{k} \leq \frac{c_{\log}}{\log\left(\frac{1}{\sigma}\right)}; \tag{10.41}$$

the constant $c_{\log}$ depending only on n, p and s.

We are now in a position to start a suitable iteration to deduce the desired oscillation reduction. First, for any $i = 0, 1, 2, \ldots$, we define

$$\varrho_i = r_{j+1} + 2^{-i} r_{j+1}, \quad \tilde{\varrho}_i := \frac{\varrho_{i+1} + \varrho_i}{2}, \quad B^i = B_{\varrho_i}, \quad \tilde{B}^i = B_{\tilde{\varrho}_i},$$

and corresponding cut-off functions

$$\phi_i \in C_0^\infty(\tilde{B}^i), \quad 0 \leq \phi_i \leq 1, \quad \phi_i \equiv 1 \text{ on } B^{i+1} \quad \text{and } |D_H \phi_i| \leq c\varrho_i^{-1}.$$

Furthermore, set

$$k_i = (1 + 2^{-i})d\omega(r_j), \qquad w_i := (k_i - u_j)_+,$$

and

$$A_i = \frac{|B^i \cap \{u_j \le k_i\}|}{|B^i|} = \frac{|B^i \cap \{w_i > 0\}|}{|B^i|}.$$

Hence, the Caccioppoli inequality (10.1) yields

$$\begin{aligned}
&\int_{B^i}\int_{B^i} \frac{|w_i(\xi)\phi_i(\xi) - w_i(\eta)\phi_i(\eta)|^p}{|\eta^{-1}\circ\xi|_H^{Q+sp}}\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\le c\int_{B^i}\int_{B^i} \frac{w_i^p(\xi)|\phi_i(\xi) - \phi(\eta)|^p}{|\eta^{-1}\circ\xi|_H^{Q+sp}}\,\mathrm{d}\xi\,\mathrm{d}\eta \qquad (10.42)\\
&\quad + c\int_{B^i} w_i(\xi)\phi_i^p(\xi)\,\mathrm{d}\xi \left(\sup_{\eta\in\tilde{B}_i}\int_{\mathbb{H}^n\setminus B^i} \frac{w_i^{p-1}(\xi)}{|\eta^{-1}\circ\xi|_H^{Q+sp}}\,\mathrm{d}\xi \;+\; \|f\|_{L^\infty(B_r)}\right).
\end{aligned}$$

We now focus on the subcritical case when $sp < Q$. We make use of the fractional Sobolev inequality, in order to estimate the first term on the right-hand side as follows

$$\begin{aligned}
A_{i+1}^{\frac{p}{p_s^*}}(k_i - k_{i+1})^p &= \frac{1}{|B^{i+1}|^{\frac{p}{p_s^*}}}\left(\int_{B^{i+1}\cap\{u_j\le k_{i+1}\}} (k_i - k_{i+1})^{p_s^*}\phi_i^{p_s^*}(\xi)\,\mathrm{d}\xi\right)^{\frac{p}{p_s^*}} \\
&\le \frac{1}{|B^{i+1}|^{\frac{p}{p_s^*}}}\left(\int_{B^i} w_i^{p_s^*}(\xi)\phi_i^{p_s^*}(\xi)\,\mathrm{d}\xi\right)^{\frac{p}{p_s^*}} \\
&\le c r_{j+1}^{sp-Q}\int_{B^i}\int_{B^i} \frac{|w_i(\xi)\phi_i(\xi) - w_i(\eta)\phi_i(\eta)|^p}{|\eta^{-1}\circ\xi|_H^{Q+sp}}\,\mathrm{d}\xi\,\mathrm{d}\eta. \qquad (10.43)
\end{aligned}$$

Recalling that $|D_H\phi_i| \le c2^i r_{j+1}^{-1}$, we can treat the first term in the right-hand side in (10.42) as follows

$$\begin{aligned}
&r_{j+1}^{sp}\int_{B^i}\int_{B^i} \frac{w_i^p(\xi)|\phi_i(\xi) - \phi(\eta)|^p}{|\eta^{-1}\circ\xi|_H^{Q+sp}}\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\le c2^{ip} r_{j+1}^{-p} r_{j+1}^p \int_{B^i\cap\{u_j\le k_i\}} w_i^p(\xi)\,\mathrm{d}\xi \\
&\le c2^{ip}[d\omega(r_j)]^p |B^i\cap\{u_j\le k_i\}|. \qquad (10.44)
\end{aligned}$$

Moreover,

$$\int_{B^i} w_i(\xi)\phi_i^p(\xi)\,\mathrm{d}\xi \;\le\; c[d\omega(r_j)]|B^i\cap\{u_j\le k_i\}|, \qquad (10.45)$$

holds.

Now, we notice that for any $\eta\in\tilde{B}^i$, we have

$$\frac{|\xi^{-1}\circ\xi_0|_{\mathbb{H}^n}}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}} \le \frac{|\eta^{-1}\circ\xi_0|_{\mathbb{H}^n} + |\eta^{-1}\circ\xi|_{\mathbb{H}^n}}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}} \le 1 + \frac{\tilde{\varrho}_i}{\varrho_i - \tilde{\varrho}_i} \le c2^i$$

for all $\xi\in\mathbb{H}^n\setminus B^i$ and that

$$B_{r_{j+1}} \equiv B_{j+1} \subset B^i \;\Rightarrow\; \mathbb{H}^n\setminus B^i \subset \mathbb{H}^n\setminus B_{j+1}.$$

Then we have

$$\begin{aligned}
&r_{j+1}^{sp}\left(\sup_{\eta\in\tilde{B}_i}\int_{\mathbb{H}^n\setminus B^i} \frac{w_i^{p-1}(\xi)}{|\eta^{-1}\circ\xi|_H^{Q+sp}}\,\mathrm{d}\xi\right) \\
&\le c2^{i(Q+sp)}[\mathrm{Tail}(w_i;\xi_0,r_{j+1})]^{p-1} \qquad (10.46)
\end{aligned}$$

Recalling (10.34) and the facts that $w_i \le 2d\omega(r_j)$ in B_j and $w_i \le |u_j| + 2d\omega(r_j)$ in $\mathbb{H}^n$, we further get

$$\begin{aligned}
&\big[\mathrm{Tail}(w_i;\xi_0,r_{j+1})\big]^{p-1}\\
&\quad\le cr_{j+1}^{sp}\int_{B_j\setminus B_{j+1}} w_i^{p-1}(\xi)|\xi_0^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}\,\mathrm{d}\xi + c\left(\frac{r_{j+1}}{r_j}\right)^{sp}[\mathrm{Tail}(w_i;\xi_0,r_j)]^{p-1}\\
&\quad\le cd^{p-1}\omega(r_j)^{p-1} + c\sigma^{sp}[\mathrm{Tail}(u_j;\xi_0,r_j)]^{p-1}\\
&\quad\le c\left(1+\frac{\sigma^{sp-\alpha(p-1)}}{d^{p-1}}\right)[d\omega(r_j)]^{p-1}\\
&\quad\le c[d\omega(r_j)]^{p-1},
\end{aligned}$$

where c depends only on α, p and s. Combining the estimate above with (10.46) we get

$$r_{j+1}^{sp}\left(\sup_{\eta\in\tilde{B}^i}\int_{\mathbb{H}^n\setminus B^i} w_i^{p-1}(\xi)|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}\,\mathrm{d}\xi\right) \le c2^{i(Q+sp)}[d\omega(r_j)]^{p-1}. \tag{10.47}$$

Putting together (10.42), (10.43), (10.44), (10.45) and (10.47), we obtain

$$A_{i+1}^{\frac{p}{p_s^*}}(k_i-k_{i+1})^p \;\le\; c2^{i(Q+sp+p)}[d\omega(r_j)]^p A_i,$$

which yields, recalling that $r_{j+1}<r$

$$A_{i+1} \le c2^{i(Q+(2+s)p)\frac{p_s^*}{p}}A_i^{1+\beta}$$

with $\beta := sp/(Q-sp)$ by the definition of k_i's.

Now, if we will prove the following estimate on A_0,

$$\begin{aligned}
A_0 &= \frac{|\tilde{B}\cap\{u_j\le 2d\omega(r_j)\}|}{|\tilde{B}|}\\
&\le c^{-1/\beta}2^{-\frac{(Q+(2+s)p)p_s^*}{p\beta^2}} =: \nu^*, \qquad (10.48)
\end{aligned}$$

then we can deduce that

$$A_i\to 0 \quad\text{as}\quad i\to\infty.$$

Indeed, the condition (10.48) it is guaranteed by (10.41) choosing

$$\sigma = \min\left\{\frac{1}{4},\, \mathrm{e}^{-\frac{c_{\log}}{\nu^*}}\right\},$$

which depends only on n, p, s and α. We have hence shown that

$$\operatorname*{osc}_{B_{j+1}} u \;\le\; (1-d)\omega(r_j) \;=\; (1-d)\left(\frac{r_j}{r_{j+1}}\right)^{\alpha}\omega(r_{j+1}) \;=\; (1-d)\sigma^{-\alpha}\omega(r_{j+1}).$$

Taking $\alpha\in\left(0,\frac{sp}{p-1}\right)$ small enough that

$$\sigma^\alpha \ge 1-d = 1-\sigma^{\frac{sp}{p-1}}$$

leads to

$$\operatorname*{osc}_{B_{j+1}} u \le \omega(r_{j+1}),$$

and the proof is complete in the case when $sp<Q$.

For the remaining case, namely when $sp = Q$, we can proceed as in the proof of the local boundedness. Consider $0 < s_1 < s < 1$; the fractional Sobolev inequality gives

$$\|w_i\varphi_i\|^p_{L^{p^*_{s_1}}(B^i)} \le c[w_i\varphi_i]^p_{s_1,p}$$

with $p_{s_1} := \frac{Qp}{Q-s_1p}$. Note also that

$$\|w_i\varphi_i\|_{L^q(B^i)} \le |B^i|^{\frac{p^*_{s_1}-q}{qp^*_{s_1}}} \|w_i\varphi_i\|_{L^{p^*_{s_1}}(B^i)},$$

and thus,

$$\begin{aligned} A_{i+1}^{\frac{p}{q}}(k_i-k_{i+1})^p &= \frac{1}{|B^{i+1}|^{\frac{p}{q}}}\left(\int_{B^{i+1}\cap\{u_j\le k_{i+1}\}}(k_i-k_{i+1})^q\phi_i^q(\xi)\,\mathrm{d}\xi\right)^{\frac{p}{q}} \\ &\le \frac{1}{|B^{i+1}|^{\frac{p}{q}}}\left(\int_{B^i} w_i^q(\xi)\phi_i^q(\xi)\,\mathrm{d}\xi\right)^{\frac{p}{q}} \\ &\le \frac{|B^i|^{\frac{(p^*_{s_1}-q)p}{qp^*_{s_1}}}}{|B^{i+1}|^{\frac{p}{q}}}\left(\int_{B^i} w_i^{p^*_{s_1}}(\xi)\phi_i^{p^*_{s_1}}(\xi)\,\mathrm{d}\xi\right)^{\frac{p}{p^*_{s_1}}} \\ &\le c\frac{r_{j+1}^{s_1p}}{r_{j+1}^{Q}}\int_{B^i}\int_{B^i}\frac{|w_i(\xi)\phi_i(\xi)-w_i(\eta)\phi_i(\eta)|^p}{|\eta^{-1}\circ\xi|_H^{Q+sp}}\,\mathrm{d}\xi\,\mathrm{d}\eta. \end{aligned} \tag{10.49}$$

The term in the right-hand side of (10.49) can be estimated using the nonlocal Caccioppoli inequality as in (10.42) and recalling the estimates (10.45) and (10.47).

All in all, we have

$$A_{i+1}^{\frac{p}{q}}(k_i-k_{i+1})^p \le c2^{i(Q+sp+p)}[d\omega(r_j)]^pA_i,$$

which yields

$$A_{i+1} \le c2^{i\frac{(Q+(2+s)p)q}{p}}A_i^{1+\beta},$$

where we denoted by $\beta := q/p-1 > 0$. We now choose the following σ in (10.41),

$$\sigma := \min\left\{\frac{1}{4},\, \mathrm{e}^{-\frac{c_{\log}}{\bar\nu}}\right\}$$

with $\bar\nu := c^{-\frac{1}{\beta}}(1+r^{sp}\|f\|_{L^\infty(B_r)})^{-\frac{q}{p\beta}}2^{-\frac{(Q+(2+s)p)q}{p\beta^2}}$, in order to deduce that

$$A_0 = \frac{|\tilde B\cap\{u_j\le 2d\omega(r_j)\}|}{|\tilde B|} \le c^{-\frac{1}{\beta}}2^{-\frac{(Q+(2+s)p)q}{p\beta^2}}.$$

Hence,

$$A_i \to 0, \qquad \text{as} \qquad i\to\infty.$$

and the proof is complete, by proceeding exactly as in the case when $sp < Q$. □

CHAPTER 11

Nonlocal Harnack inequalities

In this section we establish general nonlocal Harnack inequalities for weak solutions to (8.1). Our results read as follow.

Theorem 11.1 (Nonlocal Harnack inequality)**.** *For any $s \in (0,1)$ and any $p \in (1,\infty)$, let $u \in W^{s,p}(\mathbb{H}^n)$ be a weak solution to (8.1) such that $u \geq 0$ in $B_R \equiv B_R(\xi_0) \subset \Omega$. Then, for any B_r such that $B_{6r} \subset B_R$, it holds*

$$\sup_{B_r} u \;\leq\; c \inf_{B_r} u + c\left(\frac{r}{R}\right)^{\frac{sp}{p-1}} \mathrm{Tail}(u_-;\xi_0,R) + c\, r^{\frac{sp}{p-1}} \|f\|^{\frac{1}{p-1}}_{L^\infty(B_R)}\,, \tag{11.1}$$

where $\mathrm{Tail}(\cdot)$ *is defined in (8.4),* $u_- := \max\{-u,0\}$ *is the negative part of the function u, and c depends only on n, s, p, and the structural constant Λ defined in (2.4).*

Notice that in the case when u is nonnegative in the whole $\mathbb{H}^n$, the formulation in (11.1) does reduce to that of the classical Harnack inequality.

In the particular situation when u is merely a weak supersolution to Problem (8.1), still in analogy with the classical case when $s = 1$, a weak Harnack inequality can be proven, as stated in the following

Theorem 11.2 (Nonlocal weak Harnack inequality)**.** *For any $s \in (0,1)$ and any $p \in (1,\infty)$, let $u \in W^{s,p}(\mathbb{H}^n)$ be a weak supersolution to (8.1) such that $u \geq 0$ in $B_R \subseteq \Omega$. Then, for any B_r such that $B_{6r} \subset B_R$, it holds*

$$\left(\fint_{B_r} u^t \,\mathrm{d}\xi\right)^{\frac{1}{t}} \;\leq\; c \inf_{B_{\frac{3}{2}r}} u \,+ c\left(\frac{r}{R}\right)^{\frac{sp}{p-1}} \mathrm{Tail}(u_-;\xi_0,R) \,+ c\chi,$$

where

$$\chi = \begin{cases} r^{\frac{Qsp}{t(Q-sp)}} \|f\|^{\frac{Q}{t(Q-sp)}}_{L^\infty(B_R)} & \text{for } t < \frac{Q(p-1)}{Q-sp} \qquad \text{if } sp < Q, \\ r^{\frac{Q(s-\varepsilon)}{t\varepsilon}} \|f\|^{\frac{s}{t\varepsilon}}_{L^\infty(B_R)} & \text{for any } s - Q/p < \varepsilon < s \text{ and } t < \frac{(p-1)s}{\varepsilon} \qquad \text{if } sp \geq Q, \end{cases}$$

$\mathrm{Tail}(\cdot)$ *is defined in (8.4), and* $u_- := \max\{-u,0\}$ *is the negative part of the function u. The constant c depends only on n, s, p, and the structural constant Λ defined in (2.4).*

In the linear case when $p = 2$ a first version of the Harnack inequality was proved by Ferrari and Franchi ([56]) for entire nonnegative solutions to fractional equations via a Dirichlet-to-Neumann map. However, our problem is extremely different than the one treated in [56]. First of all, given our nonlinear setting the apporach via Caffarelli-Silvestre-type extensions can not be pursued. Moreover, we do not want to assume extra positivity assumptions on the involved solutions. In order to partially consider mild conditions on the sign of the solution u to (8.1), as shown by Kassmann in his breakthrough papers [76, 77], a natural tail-type contribution can be extepectedd even for the fractional Laplacian. Thus, we are able to extend part of the strategy developed in [47] where nonlocal Harnack

inequalities have been proven for the homogeneous version of the analogue of problem (8.1) in the Euclidean framework. Further efforts are also needed due to the presence of the non homogeneous datum f, as well as in order to deal with the limit case when $sp = Q$, both of them are novelty even with respect to the results proven in the Euclidean framework in [47].

1. Expansion of positivity

In this section we prove a careful estimate of the weak supersolutions to (8.1), by generalizing the original strategy applied in the local framework as well as that in the fractional Euclidean one.

Lemma 11.3. *Let with $s \in (0,1)$ and $p \in (1,\infty)$, and let $u \in W^{s,p}(\mathbb{H}^n)$ be a weak supersolution to problem (8.1) such that $u \geq 0$ in $B_R \equiv B_R(\xi_0) \subset \Omega$. Let $k \geq 0$, and suppose that there exists $\sigma \in (0,1]$ such that*

$$|B_{6r} \cap \{u \geq k\}| \geq \sigma |B_{6r}|, \tag{11.2}$$

for some $r > 0$ such that $B_{8r} \subset B_R$. Then there exists a constant $\bar{c} \equiv \bar{c}(n,p,s,\|f\|_{L^\infty(B_R)},\Lambda)$ such that

$$\left| B_{6r} \cap \left\{ u \leq 2\delta k - \frac{1}{2}\left(\frac{r}{R}\right)^{\frac{sp}{p-1}} \mathrm{Tail}(u_-;\xi_0,R) \right\} \right| \leq \frac{\bar{c}}{\sigma \log\left(\frac{1}{2\delta}\right)} |B_{6r}| \tag{11.3}$$

holds for all $\delta \in (0,1/4)$, where $\mathrm{Tail}(\cdot)$ is defined in (8.4).

Proof. We firstly notice that with no loss of generality one can suppose $\mathrm{Tail}(u_-;\xi_0,R) > 0$, otherwise (11.3) plainly follows from (11.2) by choosing the constant $\bar{c}$ large enough.

Take a smooth function $\phi \in C_0^\infty(B_{7r})$ such that

$$0 \leq \varphi \leq 1, \quad \varphi \equiv 1 \quad \text{in } B_{6r}, \quad \text{and} \quad |D_H \varphi| \leq c/r\,,$$

an set

$$d := \frac{1}{2}\left(\frac{r}{R}\right)^{\frac{sp}{p-1}} \mathrm{Tail}(u_-;\xi_0,R), \quad \text{and} \quad \tilde{u} := u + d\,. \tag{11.4}$$

Now, choose $\psi := \tilde{u}^{1-p}\varphi^p$ as a test function in Definition 9.4 by making use of the fact that u is a supersolution. It follows

$$\begin{aligned}
&\int_{\mathbb{H}^n} f(\xi,u)\tilde{u}^{1-p}(\xi)\varphi^p(\xi)\mathrm{d}\xi \\
&\quad\leq \int_{B_{8r}}\int_{B_{8r}} |\tilde{u}(\xi)-\tilde{u}(\eta)|^{p-2}(\tilde{u}(\xi)-\tilde{u}(\eta))(\tilde{u}^{1-p}(\xi)\varphi^p(\xi)-\tilde{u}^{1-p}(\eta)\varphi^p(\eta))\,\mathrm{d}\nu \\
&\qquad+\int_{\mathbb{H}^n\setminus B_{8r}}\int_{B_{8r}} |\tilde{u}(\xi)-\tilde{u}(\eta)|^{p-2}(\tilde{u}(\xi)-\tilde{u}(\eta))\tilde{u}^{1-p}(\xi)\varphi^p(\xi)\,\mathrm{d}\nu \\
&\qquad-\int_{B_{8r}}\int_{\mathbb{H}^n\setminus B_{8r}} |\tilde{u}(\xi)-\tilde{u}(\eta)|^{p-2}(\tilde{u}(\xi)-\tilde{u}(\eta))\tilde{u}^{1-p}(\eta)\varphi^p(\eta)\,\mathrm{d}\nu \\
&(11.5) =: \ I_1 + I_2 + I_3\,,
\end{aligned}$$

where we have also used the definition of $\tilde{u}$ and the translation invariant property of the fractional seminorms.

We start by estimating the integral contribution in the left-hand side. Thanks to the hypothesis on f and in view of the definition of ψ, we have get

$$\int_{\mathbb{H}^n} f(\xi,u)\tilde{u}^{1-p}(\xi)\varphi^p(\xi)\,\mathrm{d}\xi \geq \int_{B_{6r}} f_+(\xi,u)\tilde{u}^{1-p}(\xi)\,\mathrm{d}\xi - cr^Q d^{1-p}\|f\|_{L^\infty(B_R)}.$$

Now, we focus on the terms on the right-hand side. We can treat the first integral I_1 as in the proof of the Logarithmic Lemma 10.3 so that

$$I_1 \leq -\frac{1}{c}\int_{B_{6r}}\int_{B_{6r}}\left|\log\left(\frac{\tilde{u}(\xi)}{\tilde{u}(\eta)}\right)\right|^p \,\mathrm{d}\nu + cr^{Q-sp}\,.$$

We now proceed to estimating the second integral I_2 in (11.5), in turn obtaining an estimate for I_3 as well. We split I_2 as follows,

$$\begin{aligned} I_2 \;=\; & \int_{\mathbb{H}^n\setminus B_{8r}\cap\{\tilde{u}(\eta)<0\}}\int_{B_{8r}} |\tilde{u}(\xi)-\tilde{u}(\eta)|^{p-2}(\tilde{u}(\xi)-\tilde{u}(\eta))\tilde{u}^{1-p}(\xi)\varphi^p(\xi)\,\mathrm{d}\nu \\ & + \int_{\mathbb{H}^n\setminus B_{8r}\cap\{\tilde{u}(\eta)\geq 0\}}\int_{B_{8r}} |\tilde{u}(\xi)-\tilde{u}(\eta)|^{p-2}(\tilde{u}(\xi)-\tilde{u}(\eta))\tilde{u}^{1-p}(\xi)\varphi^p(\xi)\,\mathrm{d}\nu \\ (11.6)\eqqcolon:\quad & I_{2,1}+I_{2,2}. \end{aligned}$$

The contribution in $I_{2,1}$ can be estimated as follows

$$\begin{aligned} I_{2,1} &= \int_{\mathbb{H}^n\setminus B_{8r}}\int_{B_{8r}} |\tilde{u}(\xi)+(\tilde{u}(\eta))_-|^{p-1}\tilde{u}^{1-p}(\xi)\varphi^p(\xi)\,\mathrm{d}\nu \\ &\leq cr^Q\int_{\mathbb{H}^n\setminus B_{8r}}\left(1+\frac{(u(\eta))_-}{d}\right)^{p-1}|\eta^{-1}\circ\xi_0|_{\mathbb{H}^n}^{-Q-sp}\,\mathrm{d}\eta \\ &\leq cr^{Q-sp}+cr^Q d^{1-p}R^{-sp}\mathrm{Tail}(u_-;\xi_0,R)^{p-1} \\ &= c(n,s,p,\Lambda)r^{Q-sp}\,, \end{aligned}$$

where we have used that, for any $\xi\in B_{7r}$ and any $\eta\in\mathbb{H}^n\setminus B_{8r}$,

$$\begin{aligned} \frac{|\eta^{-1}\circ\xi_0|_{\mathbb{H}^n}}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}} &\leq \frac{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}+|\xi\circ\xi_0|_{\mathbb{H}^n}}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}} \\ &\leq 1+\frac{7r}{|\eta^{-1}\circ\xi_0|_{\mathbb{H}^n}-|\xi^{-1}\circ\xi_0|_{\mathbb{H}^n}} \;\leq\; 8. \end{aligned} \tag{11.7}$$

For $I_{2,2}$, notice that $\tilde{u}\geq 0$ in B_{7r}. Then, $\tilde{u}(\xi)-\tilde{u}(\eta)\leq\tilde{u}(\xi)$, for every $\eta\in\mathbb{H}^n\setminus B_{8r}\cap\{\tilde{u}(\eta)\geq 0\}$ and every $\xi\in B_{7r}$; we get

$$I_{2,2}\leq cr^{Q-sp}. \tag{11.8}$$

Thus, by combining (11.6) with (11.7) and (11.8), we arrive at

$$I_2+I_3\leq cr^{Q-sp}\,,$$

for a constant c depending only on n, p, s and Λ. All in all,

$$\int_{B_{6r}}\int_{B_{6r}}\left|\log\left(\frac{\tilde{u}(\xi)}{\tilde{u}(\eta)}\right)\right|^p \mathrm{d}\nu+\int_{B_{6r}} f_+(\xi,u)\tilde{u}^{1-p}(\xi)\mathrm{d}\xi \;\leq\; cr^{Q-sp}+cr^Q d^{1-p}\|f\|_{L^\infty(B_R)}\,.$$

Now, recalling the particular choice of d in (11.4), one can deduce the existence of a constant $c:=c(n,p,s,\|f\|_{L^\infty(B_R)},\Lambda)$ such that

$$\int_{B_{6r}}\int_{B_{6r}}\left|\log\left(\frac{\tilde{u}(\xi)}{\tilde{u}(\eta)}\right)\right|^p \mathrm{d}\nu+\int_{B_{6r}}(f(\xi,u))_+\tilde{u}^{1-p}(\xi)\,\mathrm{d}\xi \;\leq\; cr^{Q-sp}. \tag{11.9}$$

For any $\delta\in(0,1/4)$, define

$$v:=\left(\min\left\{\log\frac{1}{2\delta},\log\frac{k+d}{\tilde{u}}\right\}\right)_+.$$

Since v is a truncation of $\log(k+d)-\log(\tilde u)$, the estimate in (11.9) yields

$$\int_{B_{6r}}\int_{B_{6r}}|v(\xi)-v(\eta)|^p\,\mathrm{d}\nu\le\int_{B_{6r}}\int_{B_{6r}}\left|\log\left(\frac{\tilde u(\xi)}{\tilde u(\eta)}\right)\right|^p\,\mathrm{d}\nu\ \le\ cr^{Q-sp}$$

By the fractional Poincaré inequality of Proposition 9.2, we have

$$\begin{aligned}\int_{B_{6r}}|v(\xi)-(v)_{B_{6r}}|^p\,\mathrm{d}\xi\ &\le\ cr^{sp}\int_{B_{6r}}\int_{B_{6r}}|v(\xi)-v(\eta)|^p\,\mathrm{d}\nu\\ &\le\ cr^{Q}\,.\end{aligned}\tag{11.10}$$

Notice that, in view of the definitions of v and $\tilde u$, we have

$$\{v=0\}=\{\tilde u\ge k+d\}=\{u\ge k\}\,.$$

Therefore, the inequality in (11.2) yields

$$|B_{6r}\cap\{v=0\}|\ge\sigma|B_{6r}|.$$

Also, notice that

$$\begin{aligned}\log\frac{1}{2\delta}\ &=\ \frac{1}{|B_{6r}\cap\{v=0\}|}\int_{B_{6r}\cap\{v=0\}}\left(\log\frac{1}{2\delta}-v(\xi)\right)\mathrm{d}\xi\\ &\le\ \frac{1}{\sigma}\left[\log\frac{1}{2\delta}-(v)_{B_{6r}}\right]\,.\end{aligned}$$

which integrated on $B_{6r}\cap\{v=\log(1/2\delta)\}$ gives

$$\begin{aligned}\left|B_{6r}\cap\left\{v=\log\left(\frac{1}{2\delta}\right)\right\}\right|\log\left(\frac{1}{2\delta}\right)\ &\le\ \frac{1}{\sigma}\int_{B_{6r}}|v(\xi)-(v)_{B_{6r}}|\,\mathrm{d}\xi\\ &\le\ \frac{1}{\sigma}|B_{6r}|^{\frac{1}{p'}}\left(\int_{B_{6r}}|v(\xi)-(v)_{B_{6r}}|^p\,\mathrm{d}\xi\right)^{\frac{1}{p}}\\ &\le\ \frac{c}{\sigma}|B_{6r}|\,,\end{aligned}$$

where we also used (11.10) and the Hölder inequality.

From all the previous estimates we finally arrive at

$$\left|B_{6r}\cap\left\{\tilde u\le 2\delta(k+d)\right\}\right|\ \le\ \frac{c}{\sigma}\frac{1}{\log\frac{1}{2\delta}}|B_{6r}|\,,$$

so that the desired inequality plainly follows by inserting the definition of d given in (11.4). □

We are now in the position to refine the previous estimates in order to prove the main result of this section; i. e.,

Lemma 11.4. *Let $s\in(0,1)$ and $p\in(1,\infty)$ and let $u\in W^{s,p}(\mathbb{H}^n)$ be a weak supersolution to problem (8.1) such that $u\ge 0$ in $B_R(\xi_0)\subset\Omega$.*

Let $k\ge 0$ and suppose that there exists $\sigma\in(0,1]$ such that

$$|B_{6r}\cap\{u\ge k\}|\ge\sigma|B_{6r}|,$$

for some r satisfying $0<6r<R$. Then, there exists a constant $\delta\in(0,1/4)$ depending on n,s,p, σ and Λ, such that

$$\inf_{B_{4r}}u\ \ge\ \delta k-\left(\frac{r}{R}\right)^{\frac{sp}{p-1}}\mathrm{Tail}(u_-;\xi_0,R)\,.\tag{11.11}$$

Proof. We immediately notice that in the case when $k = 0$, the inequality (11.11) does trivially hold, since $u \geq 0$ in B_R. Also, with no loss of generality, we can assume that for $\delta > 0$

$$\left(\frac{r}{R}\right)^{\frac{sp}{p-1}} \operatorname{Tail}(u_-;\xi_0,R) \leq \delta k. \tag{11.12}$$

Now, for any $r \leq \varrho \leq 6r$, take a smooth function $\varphi \in C_0^\infty(B_\varrho)$ such that $0 \leq \varphi \leq 1$, and consider the test function $\psi := w_-\varphi^p$, where $w_- := (\ell - u)_+$, for any $\ell \in (\delta k, 2\delta k)$. Testing Definition 9.4 with such a smooth function ψ yields

$$\begin{aligned}
&\int_{B_\varrho} f(\xi,u)w_-(\xi)\varphi^p(\xi)\,\mathrm{d}\xi \\
&\quad\leq \int_{B_\varrho}\int_{B_\varrho} |u(\xi)-u(\eta)|^{p-2}(u(\xi)-u(\eta))(w_-(\xi)\varphi^p(\xi)-w_-(\eta)\varphi^p(\eta))\,\mathrm{d}\nu \\
&\qquad+\int_{\mathbb{H}^n\setminus B_\varrho}\int_{B_\varrho} |u(\xi)-u(\eta)|^{p-2}(u(\xi)-u(\eta))w_-(\xi)\varphi^p(\xi)\,\mathrm{d}\nu \\
&\qquad-\int_{B_\varrho}\int_{\mathbb{H}^n\setminus B_\varrho} |u(\xi)-u(\eta)|^{p-2}\Big(u(\xi)-u(\eta)\Big)w_-(\eta)\varphi^p(\eta)\,\mathrm{d}\nu \\
&\quad=: J_1+J_2+J_3.
\end{aligned} \tag{11.13}$$

We begin to estimate the term on the left-hand side. As done in our proof of Lemma 11.3, we obtain that

$$\begin{aligned}
\int_{B_\varrho} f(\xi,u)w_-(\xi)\varphi^p(\xi)\,\mathrm{d}\xi &= \int_{B_\varrho} f_+(\xi,u)w_-(\xi)\varphi^p(\xi)\mathrm{d}\xi - \int_{B_\varrho} f_-(\xi,u)w_-(\xi)\varphi^p(\xi)\mathrm{d}\xi \\
&\geq -\ell\|f\|_{L^\infty(B_R)}|B_\varrho\cap\{u<\ell\}|.
\end{aligned}$$

Now we focus on the right-hand side of (11.13). It is convenient to split J_2 as follows

$$\begin{aligned}
J_2 &= \int_{\mathbb{H}^n\setminus B_\varrho\cap\{u(\eta)<0\}}\int_{B_\varrho} |u(\xi)-u(\eta)|^{p-2}(u(\xi)-u(\eta))w_-(\xi)\varphi^p(\xi)\,\mathrm{d}\nu \\
&\quad+\int_{\mathbb{H}^n\setminus B_\varrho\cap\{u(\eta)\geq 0\}}\int_{B_\varrho} |u(\xi)-u(\eta)|^{p-2}(u(\xi)-u(\eta))w_-(\xi)\varphi^p(\xi)\,\mathrm{d}\nu \\
&=: J_{2,1}+J_{2,2}.
\end{aligned}$$

Now, notice that

$$\begin{aligned}
&|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}|u(\xi)-u(\eta)|^{p-2}\Big(u(\xi)-u(\eta)\Big)w_-(\xi)\varphi^p(\xi) \\
&\qquad\leq (\ell+(u(\eta))_-)^{p-1}\,\ell\big(\sup_{\xi\in\operatorname{supp}\varphi}|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}\big)\chi_{B_\varrho\cap\{u<\ell\}}(\xi)\,,
\end{aligned}$$

which yields

$$J_{2,1} \leq \ell\left(\sup_{\xi\in\operatorname{supp}\varphi}\int_{\mathbb{H}^n\setminus B_\varrho}(\ell+(u(\eta))_-)^{p-1}|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}\,d\eta\right)\Big|B_\varrho\cap\Big\{u<\ell\Big\}\Big|.$$

For $J_{2,2}$, since $u \geq 0$ in B_ϱ, we can write

$$\begin{aligned}
&|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}|u(\xi)-u(\eta)|^{p-2}\Big(u(\xi)-u(\eta)\Big)w_-(\xi)\varphi^p(\xi) \\
&\qquad\leq \ell^p\big(\sup_{\xi\in\operatorname{supp}\varphi}|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}\big)\chi_{B_\varrho\cap\{u<\ell\}}(\xi)\,.
\end{aligned}$$

By reasoning as above for the integral J_3, we finally arrive at

$$J_2+J_3 \;\le\; c\ell\left(\sup_{\xi\in\operatorname{supp}\varphi}\int_{\mathbb{H}^n\setminus B_\varrho}(\ell+(u(\eta))_-)^{p-1}|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}\,\mathrm{d}\eta\right)|B_\varrho\cap\{u<\ell\}|\,.$$

It remains to estimate the contribution J_1, and for this one can proceed as seen in the Euclidean setting (see Theorem 1.4 in [48]); it follows

$$\begin{aligned}J_1 \;\le\; & -c\int_{B_\varrho}\int_{B_\varrho}|w_-(\xi)\varphi(\xi)-w_-(\eta)\varphi(\eta))|^p\,\mathrm{d}\nu\\ & +\int_{B_\varrho}\int_{B_\varrho}\Big(\max\Big\{w_-(\xi),\,w_-(\eta)\Big\}\Big)^p|\varphi(\xi)-\varphi(\eta)|^p\,\mathrm{d}\nu.\end{aligned}$$

Combining all the above estimates, we get

$$\begin{aligned}&\int_{B_\varrho}\int_{B_\varrho}|w_-(\xi)\varphi(\xi)-w_-(\eta)\varphi(\eta))|^p\,\mathrm{d}\nu\\ (11.14)\;&\le c\int_{B_\varrho}\int_{B_\varrho}(\max\{w_-(\xi),w_-(\eta)\})^p|\varphi(\xi)-\varphi(\eta)|^p\,\mathrm{d}\nu\\ &\quad+c\ell|B_\varrho\cap\{u<\ell\}|\bigg(\sup_{\xi\in\operatorname{supp}\varphi}\int_{\mathbb{H}^n\setminus B_\varrho}\frac{(\ell+(u(\eta))_-)^{p-1}}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}\,\mathrm{d}\eta+\|f\|_{L^\infty(B_R)}\bigg).\end{aligned}$$

At this point an iteration argument is needed. To this aim, define

$$\ell\equiv\ell_j:=\delta k+2^{-j-1}\delta k,\quad \varrho\equiv\varrho_j:=4r+2^{1-j}r,\quad\text{and}\quad\tilde\varrho_j:=\frac{\varrho_{j+1}+\varrho_j}{2},\quad\forall j=0,1,...$$

Note that both $4r<\varrho_j,\tilde\varrho_j<6r$ and

$$\ell_j-\ell_{j+1}\;=\;2^{-j-2}\delta k\;\ge\;2^{-j-3}\ell_j.$$

Moreover, in view of (11.12), we have

$$\ell_0\;=\;\frac{3}{2}\delta k\;\le\;2\delta k-\frac{1}{2}\left(\frac{r}{R}\right)^{\frac{sp}{p-1}}\operatorname{Tail}(u_-;\xi_0,R)\,,$$

which yields

$$\Big\{u<\ell_0\Big\}\subset\left\{u<2\delta k-\frac{1}{2}\left(\frac{r}{R}\right)^{\frac{sp}{p-1}}\operatorname{Tail}(u_-;\xi_0,R)\right\}.$$

We are now in the position to apply the estimate in Lemma 11.3; we arrive at

$$\frac{|B_{6r}\cap\{u<\ell_0\}|}{|B_{6r}|}\;\le\;\frac{\bar c}{\sigma\log\frac{1}{2\delta}}.$$

Now,

$$w_-\equiv w_j=(\ell_j-u)_+\;\ge\;(\ell_j-\ell_{j+1})\chi_{B_\varrho\cap\{u<\ell_{j+1}\}}\;\ge\;2^{-j-3}\ell_j\chi_{B_\varrho\cap\{u<\ell_{j+1}\}},\quad\forall j=0,1,...$$

Denote by $B_j:=B_{\varrho_j}(\xi_0)$ and let $\varphi_j\in C_0^\infty(B_{\tilde\varrho_j})$ be such that

$$0\le\varphi_j\le1\qquad\varphi_j\equiv1\text{ on }B_{j+1},\qquad|D_H\varphi_j|\le2^{j+3}/r.$$

We have

$$
\begin{aligned}
(\ell_j-\ell_{j+1})^p\left(\frac{|B_{j+1}\cap\{u<\ell_{j+1}\}|}{|B_{j+1}|}\right)^{\frac{p}{p_s^*}} &\le \left(\fint_{B_{j+1}} w_j^{p_s^*}\varphi_j^{p_s^*}\,\mathrm{d}\xi\right)^{\frac{p}{p_s^*}}\\
&\le c\left(\fint_{B_j} w_j^{p_s^*}\varphi_j^{p_s^*}\,\mathrm{d}\xi\right)^{\frac{p}{p_s^*}}\\
&\le cr^{sp}\fint_{B_j}\int_{B_j}|w_j(\xi)\varphi_j(\xi)-w_j(\eta)\varphi_j(\eta)|^p\,\mathrm{d}\nu, \qquad (11.15)
\end{aligned}
$$

where in the last inequality we have used the Sobolev embedding with $p_s^*=Qp/(Q-sp)$.

Let us estimate (11.15) with the aid of (11.14). Firstly, by the particular choice of φ, we have

$$
\begin{aligned}
&c\int_{B_j}\int_{B_j}(\max\left\{w_j(\xi),\,w_j(\eta)\right\})^p|\varphi_j(\xi)-\varphi_j(\eta)|^p\,\mathrm{d}\nu\\
&\le c2^{jp}\ell_j^p r^{-p}\int_{B_j}\int_{B_j\cap\{u<\ell_j\}}|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp+p}\,\mathrm{d}\xi\mathrm{d}\eta\\
&\le c2^{jp}\ell_j^p r^{-sp}|B_j\cap\{u<\ell_j\}|\,,
\end{aligned}
$$

where, proceeding as in the proof of the Logarithmic Lemma 10.3, we have that

$$
\int_{B_j}|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{p-Q-sp}\,\mathrm{d}\eta\le cr^{p-sp}.
$$

Now, notice that, for any $\eta\in\mathbb{H}^n\setminus B_j$ and any $\xi\in\operatorname{supp}\varphi_j\subset B_{\tilde{\varrho}_j}$, it holds

$$
\frac{|\eta^{-1}\circ\xi_0|_{\mathbb{H}^n}}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}}\le\frac{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}+|\xi^{-1}\circ\xi_0|_{\mathbb{H}^n}}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}}\le c2^j.
$$

Thus,

$$
\sup_{\xi\in\operatorname{supp}\varphi_j}|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}\le c2^{j(Q+sp)}|\eta^{-1}\circ\xi_0|_{\mathbb{H}^n}^{-Q-sp},\quad\forall\eta\in\mathbb{H}^n\setminus B_j\,,
$$

which yields

$$
\begin{aligned}
&\sup_{\xi\in\operatorname{supp}\varphi_j}\int_{\mathbb{H}^n\setminus B_j}(\ell_j+(u(\eta))_-)^{p-1}|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}\,\mathrm{d}\eta\\
&\le c2^{j(Q+sp)}\ell_j^{p-1}r^{-sp}+c2^{j(Q+sp)}r^{-sp}\left(\frac{r}{R}\right)^{sp}[\mathrm{Tail}(u_-;\xi_0,R)]^{p-1}\\
&\le c2^{j(Q+sp)}r^{-sp}\ell_j^{p-1}, \qquad (11.16)
\end{aligned}
$$

where we have also used the fact that $u\ge0$ in B_R, the estimate in (11.12), and the fact that $\delta k<\ell_j$.

From (11.14), (11.15), and (11.16), we arrive at

$$
\left(\frac{|B_{j+1}\cap\{u<\ell_{j+1}\}|}{|B_{j+1}|}\right)^{\frac{p}{p_s^*}}\le c2^{j(Q+p+sp)}\frac{\max\{\ell_j^p,\ell_j\}}{(\ell_j-\ell_{j+1})^p}(1+r^{sp}\|f\|_{L^\infty(B_R)})\frac{|B_j\cap\{u<\ell_j\}|}{|B_j|}\,.
$$

We are finally in the position to apply a classic iteration Lemma. We set

$$
A_j:=\frac{|B_j\cap\{u<\ell_j\}|}{|B_j|}
$$

the previous estimate can be rewritten as follows

$$A_{j+1}^{\frac{p}{p_s^*}} \le c\frac{2^{j(Q+p+sp)}\max\left\{\ell_j^p,\ell_j\right\}}{\left(\ell_j-\ell_{j+1}\right)^p}\left(1+r^{sp}\|f\|_{L^\infty(B_R)}\right)A_j\,.$$

Also, since

$$\frac{\max\{\ell_j^p,\ell_j\}}{(\ell_j-\ell_{j+1})^p}\le c_p 2^{jp}\max\{1,\frac{3}{2}(\delta k)^{1-p}\}\le c2^{jp},$$

it follows

$$A_{j+1}\le c_1 2^{j(\frac{Qp_s^*}{p}+2p_s^*+sp_s^*)}A_j^{1+\beta},\qquad \text{with } \beta=\frac{sp}{Q-sp},$$

and $c_1^{\frac{p}{p_s^*}}=c\Big(1+r^{sp}\|f\|_{L^\infty(B_R)}\Big)$. Choosing $\delta>0$ as follows,

$$\delta:=\frac{1}{4}\exp\left\{-\frac{\bar{c}c_1^{\frac{Q-sp}{sp}}2^{(\frac{Q}{p}+s+2)\frac{Q(Q-sp)}{ps^2}}}{\sigma}\right\}<\frac{1}{4}\,,$$

and using the previous estimate, we arrive at

$$A_0=\frac{|B_{6r}\cap\{u<\ell_0\}|}{|B_{6r}|}\ \le\ c_1^{-\frac{Q-sp}{sp}}2^{-(\frac{Q}{p}+s+2)\frac{Q(Q-sp)}{ps^2}},$$

which gives

$$\lim_{j\to\infty}A_j=0,$$

so that $\inf_{B_{4r}}u\ge\delta k$, and hence (11.11) plainly follows.

We consider now the case when $sp=Q$. For this, we choose $0<\varepsilon<s$ and, calling $s_\varepsilon:=s-\varepsilon$, we have that $s_\varepsilon p<Q$. Then, for $p<q<p_\varepsilon^*:=\frac{Qp}{Q-s_\varepsilon p}$, we apply the Sobolev inequality in Theorem 9.1; it follows

$$\left(\fint_{B_j}w_j^q\,\varphi_j^q\mathrm{d}\xi\right)^{\frac{p}{q}}\le|B_j|^{\frac{(p_\varepsilon^*-q)p}{qp_\varepsilon^*}-\frac{p}{q}}\left(\int_{B_j}w_j^{p_\varepsilon^*}\,\varphi_j^{p_\varepsilon^*}\mathrm{d}\xi\right)^{\frac{p}{p_\varepsilon^*}}\le c\frac{r^{s_\varepsilon p}}{r_j^Q}\left(\int_{B_j}w_j^{p_\varepsilon^*}\,\varphi_j^{p_\varepsilon^*}\mathrm{d}\xi\right)^{\frac{p}{p_\varepsilon^*}}.$$

Thus, by the inequality above we get

$$\begin{aligned}
&(\ell_j-\ell_{j+1})^p\left(\frac{|B_{j+1}\cap\{u<\ell_{j+1}\}|}{|B_{j+1}|}\right)^{\frac{p}{q}}\\
&\le c\left(\fint_{B_j}w_j^q\varphi_j^q\,\mathrm{d}\xi\right)^{\frac{p}{q}}\\
&\le c\frac{r^{s_\varepsilon p}}{r_j^Q}\left(\int_{B_j}w_j^{p_\varepsilon^*}\,\varphi_j^{p_\varepsilon^*}\,\mathrm{d}\xi\right)^{\frac{p}{p_\varepsilon^*}}\\
&\le cr^{s_\varepsilon p}\fint_{B_j}\int_{B_j}\frac{|w_j(\xi)\varphi_j(\xi)-w_j(\eta)\varphi_j(\eta)|^p}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+s_\varepsilon p}}\,\mathrm{d}\xi\mathrm{d}\eta\\
&\le cr^{s_\varepsilon p}\left(\fint_{B_j}w_j^p\,\varphi_j^p\,\mathrm{d}\xi\fint_{B_j}\int_{B_j}\frac{|w_j(\xi)\varphi_j(\xi)-w_j(\eta)\varphi_j(\eta)|^p}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}\,\mathrm{d}\xi\mathrm{d}\eta\right),
\end{aligned}\tag{11.17}$$

where in the last inequality we have split the seminorm $[w_j\varphi_j]^p_{W^{s_\varepsilon,p}}$ in the following way

$$\begin{aligned}\int_{B_j}\int_{B_j}&\frac{|w_j(\xi)\varphi_j(\xi)-w_j(\eta)\varphi_j(\eta)|^p}{|\eta^{-1}\circ\xi|^{Q+s_\varepsilon p}_{\mathbb{H}^n}}\,\mathrm{d}\xi\mathrm{d}\eta\\ &=\int_{B_j}\int_{B_j\cap\{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}\geq 1\}}\frac{|w_j(\xi)\varphi_j(\xi)-w_j(\eta)\varphi_j(\eta)|^p}{|\eta^{-1}\circ\xi|^{Q+s_\varepsilon p}_{\mathbb{H}^n}}\,\mathrm{d}\xi\mathrm{d}\eta\\ &\quad+\int_{B_j}\int_{B_j\cap\{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}<1\}}\frac{|w_j(\xi)\varphi_j(\xi)-w_j(\eta)\varphi_j(\eta)|^p}{|\eta^{-1}\circ\xi|^{Q+s_\varepsilon p}_{\mathbb{H}^n}}\,\mathrm{d}\xi\mathrm{d}\eta,\end{aligned}$$

and we have estimated the two integrals above as done in [101, Proposition 2.8].

The first term on the right-hand side of (11.17) can be treated as follows

$$c\,r^{s_\varepsilon p}\fint_{B_j} w_j^p\,\mathrm{d}\xi\;\leq\; c\,r^{s_\varepsilon p}\,\ell_j^p\,\frac{|B_j\cap\{u<\ell_j\}|}{|B_j|}.$$

On the other hand, using the same techniques applied in the subcritical case when $sp<Q$, we have that the second term on the right-hand side in (11.17) becomes

$$\begin{aligned}cr^{s_\varepsilon p}&\fint_{B_j}\int_{B_j}|w_j(\xi)\varphi_j(\xi)-w_j(\eta)\varphi_j(\eta)|^p|\eta^{-1}\circ\xi|^{-Q-sp}_{\mathbb{H}^n}\,\mathrm{d}\xi\mathrm{d}\eta\\ &\leq c\,r^{-\varepsilon p}\,2^{j(Q+p+sp)}\max\{\ell_j^p,\ell_j\}(1+r^Q\,\|f\|_{L^\infty(B_R)})\frac{|B_j\cap\{u<\ell_j\}|}{|B_j|}\,,\end{aligned}$$

Setting, as before,

$$A_j:=\frac{|B_j\cap\{u<\ell_j\}|}{|B_j|},$$

we get

$$A_{j+1}\leq c_1 2^{j\left(\frac{Qq}{p}+2q+sq\right)}A_j^{1+\beta},$$

with $c_1^{\frac{p}{q}}:=c\,r^{-\varepsilon p}\,(1+r^Q+r^Q\|f\|_{L^\infty(B_R)})$ and $\beta:=\frac{q-p}{p}$. Choosing now

$$0<\delta:=\frac{1}{4}\exp\left\{-\frac{\bar{c}c_1^{\frac{p}{q-p}}2^{\left(\frac{Q}{p}+s+2\right)\frac{qp^2}{(q-p)^2}}}{\sigma}\right\}<\frac{1}{4},$$

hence we get that

$$A_0=\frac{|B_{6r}\cap\{u<\ell_0\}|}{|B_{6r}|}\leq c_1^{-\frac{p}{q-p}}2^{-(\frac{Qq}{p}+2q+sq)\frac{p^2}{(q-p)^2}}.$$

Then, a classical iteration lemma, with

$$c_0:=c_1\qquad\text{and}\qquad b:=2^{\frac{Qq}{p}+2q+sq},$$

yields

$$\lim_{j\to\infty}A_j=0;$$

which implies $\inf_{B_{4r}}u\geq\delta k$ and hence (11.11) when $sp=Q$.

The case when $sp>Q$ can be deduced as in the latter, without relevant modifications, choosing the parameter $\varepsilon>(s-Q/p)$. With such a choice in hand, we can use the Sobolev embedding for $W^{s_\varepsilon,p}$ in Theorem 9.1 and the desired result plainly follows as in the previous case when $sp=Q$. □

2. Proof of the nonlocal Harnack inequality

In order to derive the Harnack inequalities with tail, we firstly need the estimate (11.18) below, which is a straightforward consequence of the refined positivity expansion proven in the previous section, together with the classical Krylov-Safonov covering lemma (whose proof can be found for instance in [84, Lemma 7.2]), which can be adjusted to our framework thanks to the role of the nonlocal tail, as shown in the Euclidean framework in the proof of [47, Lemma 4.1].

Lemma 11.5. *Let $s \in (0,1)$, $p \in (1,\infty)$ and let $u \in W^{s,p}(\mathbb{H}^n)$ be a weak supersolution to (8.1) such that $u \geq 0$ in $B_R \equiv B_R(\xi_0) \subset \Omega$. Then, for any $B_{6r} \equiv B_{6r}(\xi_0) \subset B_R$, there exist constants $\varepsilon \in (0,1)$ and $c \equiv c(n,s,p,\Lambda) \geq 1$ such that*

$$\left(\fint_{B_r} u^{\varepsilon} \, \mathrm{d}\xi \right)^{\frac{1}{\varepsilon}} \leq c \inf_{B_r} u + c \left(\frac{r}{R} \right)^{\frac{sp}{p-1}} \mathrm{Tail}(u_-; \xi_0, R), \tag{11.18}$$

where Tail$(\cdot)$ *is defined in (8.4).*

In the next lemma, we prove that the tail of the positive part of the weak solutions to (8.1) can be controlled in a precise way.

Lemma 11.6. *Let $s \in (0,1)$, $p \in (1,\infty)$ and $u \in W^{s,p}(\mathbb{H}^n)$ be a weak solution to (8.1) such that $u \geq 0$ in $B_R(\xi_0) \subset \Omega$. Then, for any $0 < r < R$,*

$$\mathrm{Tail}(u_+; \xi_0, r) \leq c \sup_{B_r} u + c \left(\frac{r}{R} \right)^{\frac{sp}{p-1}} \mathrm{Tail}(u_-; \xi_0, R) + c r^{\frac{sp}{p-1}} \|f\|_{L^\infty(B_R)}^{\frac{1}{p-1}}, \tag{11.19}$$

where Tail$(\cdot)$ *is defined in (8.4) and $c \equiv c(n,s,p,\Lambda)$.*

Proof. Set $k := \sup_{B_r} u$ and choose a cut-off function $\varphi \in C_0^\infty(B_r)$ such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $B_{r/2}$ and $|D_H \varphi| \leq 8/r$. Take now the test function $\psi := (u - 2k)\varphi^p$. We have

$$\begin{aligned}
&\int_{B_r} f(\xi,u)(u(\xi) - 2k)\varphi^p(\xi) \mathrm{d}\xi \\
&\quad = \int_{B_r} \int_{B_r} |u(\xi) - u(\eta)|^{p-2}(u(\xi) - u(\eta)) \Big((u(\xi) - 2k)\varphi^p(\xi) - (u(\eta) - 2k)\varphi^p(\eta) \Big) \, \mathrm{d}\nu \\
&\qquad + \int_{\mathbb{H}^n \setminus B_r} \int_{B_r} |u(\xi) - u(\eta)|^{p-2}(u(\xi) - u(\eta))(u(\xi) - 2k)\varphi^p(\xi) \, \mathrm{d}\nu \\
&\qquad - \int_{B_r} \int_{\mathbb{H}^n \setminus B_r} |u(\xi) - u(\eta)|^{p-2}(u(\xi) - u(\eta))(u(\eta) - 2k)\varphi^p(\eta) \, \mathrm{d}\nu \\
&\quad =: H_1 + H_2 + H_3.
\end{aligned} \tag{11.20}$$

The last two integral in the identity above can be estimated as in the proof of Lemma 4.2 in [47]; we have

$$\begin{aligned}
H_2 + H_3 \geq\ & ck|B_r| r^{-sp} [\mathrm{Tail}(u_+; \xi_0, r)]^{p-1} - ck^p r^{-sp} |B_r| \\
& - ck|B_r| R^{-sp} [\mathrm{Tail}(u_-; \xi_0, R)]^{p-1}.
\end{aligned} \tag{11.21}$$

For what concerns the contribution H_1 in (11.20), we have

$$\begin{aligned}
H_1 &\geq -ck^p r^{-p} \int_{B_r} \int_{B_r} |\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{p-Q-sp} \, \mathrm{d}\xi \mathrm{d}\eta \\
&\geq -ck^p r^{-sp} |B_r|,
\end{aligned} \tag{11.22}$$

where we argued as in the proof of Lemma 10.3.

The contribution given by the datum f can be easily estimated as follows,

$$\int_{B_r} f(\xi,u)(u(\xi)-2k)\varphi^p(\xi)\,\mathrm{d}\xi \;\le\; k|B_r|\|f\|_{L^\infty(B_R)}. \tag{11.23}$$

Finally, combining (11.20) with (11.21), (11.22), and (11.23), we obtain

$$\mathrm{Tail}(u_+;\xi_0,r) \;\le\; ck + c\left(\frac{r}{R}\right)^{\frac{sp}{p-1}}\mathrm{Tail}(u_-;\xi_0;R) + cr^{\frac{sp}{p-1}}\|f\|^{\frac{1}{p-1}}_{L^\infty(B_R)},$$

which gives the desired inequality by recalling the definition of k. □

Armed with the tail estimate in Lemma 11.6, and the interpolative inequality given by Theorem 10.5, we are ready to complete the proof of the Harnack inequality with tail in (11.1). The strategy does generalize that successfully applied in [47] in the analysis of the homogeneous case in the Euclidean framework.

Proof of Theorem 11.1. Combining the supremum estimate (10.24) in Theorem 10.5 with the tail estimate (11.19), we get

$$\begin{aligned}\sup_{B_{\varrho/2}} u \;\le\; & c\delta^{-\gamma}\left(\fint_{B_\varrho} u_+^p\,\mathrm{d}\xi\right)^{\frac{1}{p}} + c\delta\sup_{B_\varrho} u\\ & + c\delta\left(\frac{\varrho}{R}\right)^{\frac{sp}{p-1}}\mathrm{Tail}(u_-;\xi_0,R) \;+\; c\varrho^{\frac{sp}{p-1}}\|f\|^{\frac{1}{p-1}}_{L^\infty(B_R)}.\end{aligned}$$

We now set $\varrho := (\sigma-\sigma')r$, with $1/2 \le \sigma' < \sigma \le 1$, so that

$$\begin{aligned}\sup_{B_{\sigma' r}} u \;\le\; & c\frac{\delta^{-\gamma}}{(\sigma-\sigma')^{\frac{Q}{p}}}\left(\sup_{B_{\sigma r}} u\right)^{\frac{p-\varepsilon}{p}}\left(\fint_{B_{\sigma r}} u^\varepsilon\,\mathrm{d}\xi\right)^{\frac{1}{p}} + c\delta\sup_{B_{\sigma r}} u\\ & + c\delta\left(\frac{r}{R}\right)^{\frac{sp}{p-1}}\mathrm{Tail}(u_-;\xi_0,R) + c\delta r^{\frac{sp}{p-1}}\|f\|^{\frac{1}{p-1}}_{L^\infty(B_R)},\end{aligned}$$

where $\varepsilon \in (0,1)$ is the one given by Lemma 11.5. We choose the interpolation parameter $\delta = (4c)^{-1}$, and we obtain

$$\begin{aligned}\sup_{B_{\sigma' r}} u \;\le\; & \frac{1}{2}\sup_{B_{\sigma r}} u + \frac{c}{(\sigma-\sigma')^{\frac{Q}{\varepsilon}}}\left(\fint_{B_r} u^\varepsilon\,\mathrm{d}\xi\right)^{\frac{1}{\varepsilon}}\\ & + c\left(\frac{r}{R}\right)^{\frac{sp}{p-1}}\mathrm{Tail}(u_-;\xi_0,R) + cr^{\frac{sp}{p-1}}\|f\|^{\frac{1}{p-1}}_{L^\infty(B_R)},\end{aligned} \tag{11.24}$$

where we also used a suitable Young inequality. Note that by Jensen's inequality, recalling that the exponent ε given by Lemma 11.5 is in $(0,1)$, we have that

$$\left(\fint_{B_r} u^\varepsilon\,\mathrm{d}\xi\right)^{\frac{1}{\varepsilon}} = \left(\fint_{B_r} u^{\frac{\varepsilon}{p}p}\,\mathrm{d}\xi\right)^{\frac{1}{\varepsilon}} \le \left(\fint_{B_r} u^p\,\mathrm{d}\xi\right)^{\frac{1}{p}} < \infty,$$

since $u \in L^p(B_r)$. Thus the right-hand term in (11.24) is finite.

Now let us recall the following well known iteration lemma

Lemma 11.7. *Let $g = g(t)$ be a nonnegative bounded function defined for $0 \le T_0 \le t \le T_1$. Suppose that for $T_0 \le t < \tau \le T_1$ we have*

$$g(t) \le c_1(\tau-t)^{-\theta} + c_2 + \zeta g(\tau),$$

where c_1, c_2, θ and $\zeta < 1$ are nonnegative constant. Then, there exists a constant c depending only on θ and ζ, such that for every $\varrho, R, T_0 \le \varrho < R \le T_1$, we have

$$g(\varrho) \le c\Big(c_1(R-\varrho)^{-\theta} + c_2\Big).$$

Applying Lemma 11.7 with $g(t) := \sup_{B_t} u$, $\tau = \sigma r$, $t := \sigma' r$, $\theta := \frac{Q}{\varepsilon}$, and $\zeta := \frac{1}{2}$, yields

$$\sup_{B_r} u \;\le\; c\left(\fint_{B_r} u^{\varepsilon}\,\mathrm{d}\xi\right)^{\frac{1}{\varepsilon}} + c\left(\frac{r}{R}\right)^{\frac{sp}{p-1}} \mathrm{Tail}(u_-;\xi_0,R) + cr^{\frac{sp}{p-1}}\|f\|_{L^\infty(B_R)}^{\frac{1}{p-1}},$$

which gives the desired inequality (11.1) thanks to the result in Lemma 11.5. □

3. Proof of the nonlocal weak Harnack inequality

Let $1/2 < \sigma' < \sigma \le 3/4$, and let $\varphi \in C_0^\infty(B_{\sigma r})$ be such that $\varphi \equiv 1$ on $B_{\sigma' r}$, and $|D_H\varphi| \le 4/(\sigma-\sigma')r$.

We firstly deal with the case when $sp < Q$. In such a case one can apply Theorem 9.1 to the function $w\varphi$, with $w := \tilde{u}^{\frac{p-q}{p}} = (u+d)^{\frac{p-q}{p}}$, to get

$$\left(\fint_{B_r} |w(\xi)\varphi(\xi)|^{p_s^*}\mathrm{d}\xi\right)^{\frac{p}{p_s^*}} \;\le\; c\frac{r^{sp}}{r^Q}\int_{B_r}\int_{B_r} |w(\xi)\varphi(\xi) - w(\eta)\varphi(\eta)|^p \mathrm{d}\nu. \tag{11.25}$$

Now, notice that, by the very definition of φ, it follows

$$|\varphi(\xi)-\varphi(\eta)|^p \;\le\; , c|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^p \sup_{B_{\sigma r}} |D_H\varphi|^p \;\le\; , \frac{c}{\Big((\sigma-\sigma')r\Big)^p}|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^p,$$

which yields

$$\begin{aligned}&\int_{B_r}\int_{B_r}\Big(\max\{w(\xi),w(\eta)\}\Big)^p|\varphi(\xi)-\varphi(\eta)|^p\mathrm{d}\nu\\ &\qquad\le\; \frac{cr^{-sp}}{(\sigma-\sigma')^p}\int_{B_{\sigma r}} w^p(\eta)\,\mathrm{d}\eta\,,\end{aligned} \tag{11.26}$$

where we have also used the estimate below

$$\int_{B_r} d_{\mathrm{o}}(\eta^{-1}\circ\xi)^{-Q-sp+p}\mathrm{d}\xi \;\le\; , cr^{p-sp}.$$

Collecting the estimates (11.25) and (11.26) with the Caccioppoli inequality of Theorem 10.1, we obtain

$$\begin{aligned}&\left(\fint_{B_r} |w(\xi)\varphi(\xi)|^{p_s^*}\mathrm{d}\xi\right)^{\frac{p}{p_s^*}}\\ &\le c\left(\frac{c}{(\sigma-\sigma')^p} + d^{1-p}\left(\frac{r}{R}\right)^{sp}[\mathrm{Tail}(u_-;\xi_0,R)]^{p-1}\right)\fint_{B_r} w^p(\xi)\varphi^p(\xi)\,\mathrm{d}\xi\\ &\quad + cd^{1-q}r^{sp}\|f\|_{L^\infty(B_R)}\,,\end{aligned}$$

where we have used the fact that

$$\sup_{\xi\in\mathrm{supp}\,\varphi}\int_{\mathbb{H}^n\setminus B_r} d_{\mathrm{o}}(\eta^{-1}\circ\xi)^{-Q-sp}\mathrm{d}\eta \le cr^{-sp}.$$

Now, we choose d as in the proof of Lemma 11.3; see (11.4) there. It follows

$$\left(\fint_{B_{\sigma' r}} \tilde{u}^{(p-q)\frac{Q}{Q-sp}}\,\mathrm{d}\xi\right)^{\frac{Q-sp}{Q}} \;\le\; \frac{c}{(\sigma-\sigma')^p}\fint_{B_{\sigma r}} \tilde{u}^{p-q}\,\mathrm{d}\xi + cr^{sp}\|f\|_{L^\infty(B_R)}\,, \tag{11.27}$$

where c depends only on n, p, s and the structural constant Λ defined in (2.4).

Let $t = (p-q)Q/(Q-sp)$ for any $q \in (1,p)$. Thanks to a standard finite Moser iteration, the inequality in (11.27) becomes

$$\left(\fint_{B_{\frac{r}{2}}} u^t \mathrm{d}\xi \right)^{\frac{1}{t}} \leq c \left(\fint_{B_{\frac{3r}{4}}} \tilde{u}^{t'} \mathrm{d}\xi \right)^{\frac{1}{t'}} + c r^{\frac{Qsp}{t(Q-sp)}} \|f\|_{L^\infty(B_R)}^{\frac{Q}{t(Q-sp)}},$$

for any $0 < t' < t < (p-1)Q/(Q-sp)$.

Since $6r < R$, we can apply Lemma 11.5 with $\varepsilon = t'$ there; it follows

$$\left(\fint_{B_{\frac{r}{2}}} u^t \mathrm{d}\xi \right)^{\frac{1}{t}} \leq c \inf_{B_{\frac{3r}{4}}} u + c \left(\frac{r}{R} \right)^{\frac{sp}{p-1}} \mathrm{Tail}(u_-; \xi_0, R) + c r^{\frac{Qsp}{t(Q-sp)}} \|f\|_{L^\infty(B_R)}^{\frac{Q}{t(Q-sp)}},$$

which provides the desired inequality, up to relabelling r.

We investigate now the case when $sp = Q$. Fix $0 < \varepsilon < s$, and set $s_\varepsilon := s - \varepsilon$. Since $s_\varepsilon p < sp = Q$, we can make use of the Sobolev inequality for the function $w\varphi$, to get

$$\begin{aligned}
\left(\fint_{B_r} |w(\xi)\varphi(\xi)|^{p^*_\varepsilon} \mathrm{d}\xi \right)^{\frac{p}{p^*_\varepsilon}} &\leq c \frac{r^{s_\varepsilon p}}{r^Q} \int_{B_r} \int_{B_r} |w(\xi)\varphi(\xi) - w(\eta)\varphi(\eta)|^p |\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{-Q-s_\varepsilon p} \mathrm{d}\xi \mathrm{d}\eta \\
&\leq c\, r^{s_\varepsilon p} \Bigg(\fint_{B_r} |w(\xi)\varphi(\xi)|^p \mathrm{d}\xi \\
&\qquad + \fint_{B_r} \int_{B_r} |w(\xi)\varphi(\xi) - w(\eta)\varphi(\eta)|^p |\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{-Q-sp} \mathrm{d}\xi \mathrm{d}\eta \Bigg), \qquad (11.28)
\end{aligned}$$

where we have obtained the last inequality by following by the same argument used at the end of the proof of Lemma 11.4. Now, we choose d as in (11.4), so that the inequality in (11.28) becomes

$$\begin{aligned}
&\left(\fint_{B_r} |w(\xi)\varphi(\xi)|^{p^*_\varepsilon} \mathrm{d}\xi \right)^{\frac{p}{p^*_\varepsilon}} \\
&\leq c\, r^{-\varepsilon p} \left(\frac{c}{(\sigma - \sigma')^p} + d^{1-p} \left(\frac{r}{R} \right)^{sp} [\mathrm{Tail}(u_-; \xi_0, R)]^{p-1} + r^Q \right) \fint_{B_r} |w(\xi)\varphi(\xi)|^p \mathrm{d}\xi \\
&\quad + c d^{1-q} r^{(s-\varepsilon)p} \|f\|_{L^\infty(B_R)} \\
&\leq \frac{c}{(\sigma - \sigma')^p} \fint_{B_r} |w(\xi)\varphi(\xi)|^p \mathrm{d}\xi + c r^{(s-\varepsilon)p} \|f\|_{L^\infty(B_R)}.
\end{aligned}$$

Thus, recalling the definition of w and that of the cut-off function φ, we have that

$$\left(\fint_{B_{\sigma' r}} \tilde{u}^{(p-q)\frac{s}{\varepsilon}} \mathrm{d}\xi \right)^{\frac{\varepsilon}{s}} \leq \frac{c}{(\sigma - \sigma')^p} \fint_{B_{\sigma r}} \tilde{u}^{p-q} \mathrm{d}\xi + c\, r^{(s-\varepsilon)p} \|f\|_{L^\infty(B_R)}.$$

Set $t = (p-q)s/\varepsilon$, for any $q \in (1,p)$; a standard application of the finite Moser iteration yields

$$\left(\fint_{B_{\frac{r}{2}}} u^t \mathrm{d}\xi \right)^{\frac{1}{t}} \leq c \left(\fint_{B_{\frac{3r}{4}}} \tilde{u}^{t'} \mathrm{d}\xi \right)^{\frac{1}{t'}} + c r^{\frac{Q(s-\varepsilon)}{t\varepsilon}} \|f\|_{L^\infty(B_R)}^{\frac{s}{t\varepsilon}},$$

for any $0 < t' < t < (p-1)s/\varepsilon$.

Finally, we can conclude as in the proof in the case when $sp < Q$; that is, it suffices to apply Lemma 11.5, with $\varepsilon = t'$ there, in order to get

$$\left(\fint_{B_{\frac{r}{2}}} u^t \mathrm{d}\xi \right)^{\frac{1}{t}} \leq c \inf_{B_{\frac{3r}{4}}} u + c \left(\frac{r}{R} \right)^{\frac{Q}{p-1}} \mathrm{Tail}(u_-; \xi_0, R) + c\, r^{\frac{Q(s-\varepsilon)}{t\varepsilon}} \|f\|_{L^\infty(B_R)}^{\frac{s}{t\varepsilon}},$$

which provides the desired inequality, up to relabelling r.

The case when $sp > Q$ can be deduced as in the latter, without relevant modifications, choosing the parameter $\varepsilon > (s - Q/p)$. With such a choice in hand, it plainly follows that $s_\varepsilon p < Q$, and thus one can use the Sobolev embedding in Theorem 9.1 and proceed as done in the limit case when $sp = Q$. □

4. Robustness of the Harnack inequalities

Let now focus on the linear case when $p = 2$ when the datum f in the right-hand side of the equation in (8.1) is zero. We investigate the special linear case in which $\mathcal{L}$ does reduce to the pure fractional subLaplacian. Firstly, as expected, we prove that the conformally invariant fractional subLaplacian $(-\Delta_H)^s$ effectively converges to the standard subLaplacian $-\Delta_H$ as s goes to 1, as stated in Proposition 11.8 below. For this, we shall carefully estimate the weighted second order integral form of the fractional subLaplacian with the aim of a suitable Mac-Laurin-type expansion in the Heisenberg group.

Proposition 11.8. *For any $u \in C_0^\infty(\mathbb{H}^n)$ the following statement holds true*

$$\lim_{s \to 1^-} (-\Delta_H)^s u = -\Delta_H u,$$

Notice that a proof of the result above for fractional subLaplacian on Carnot groups can be found in the relevant paper [57], via heat kernel characterization.

Before going in the proof of Proposition 11.8 we recall the precise definition of the fractional subLaplacian operator; that is,

$$(-\Delta_H)^s u(\xi) = C(n,s)\ P.V. \int_{\mathbb{H}^n} \frac{u(\xi) - u(\eta)}{|\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{Q+2s}} \mathrm{d}\eta, \qquad \forall \xi \in \mathbb{H}^n,$$

where $|\cdot|_{\mathbb{H}^n}$ is the norm defined in (2.3), and $C(n,s)$ is given by

$$C(n,s) = \frac{c_1(n,s)\omega_{2n}}{n c_2(n,s)}, \tag{11.29}$$

with

(11.30)

$$c_1(n,s) = \left(\int_{\mathbb{R}^{2n+1}} \frac{1 - \cos(x_1)}{\|\eta\|^{1+2(n+s)}} \mathrm{d}\eta \right)^{-1} \quad \text{and} \quad c_2(n,s) = \int_{\partial B_1} \frac{x_1^2}{|\eta|_{\mathbb{H}^n}^{Q+2s}} \mathrm{d}\mathcal{H}^{Q-2}(\eta),$$

for $\eta := (x_1, \cdots, x_{2n}, t)$. In the display above, we denote by $\|\cdot\|$ the standard Euclidean norm on $\mathbb{R}^{2n+1}$, and by $\mathcal{H}^{Q-2}$ the surface measure on ∂B_1; see, e. g., Proposition 1.15 in [61].

Proof of Proposition 11.8. For the sake of readability, we denote the points ξ in $\mathbb{H}^n$ as follows,

$$\xi := (x_1, \ldots, x_{2n}, t).$$

Also, it is convenient to use the weighted second order integral definition of the fractional sublaplacian,

$$(-\Delta_H)^s u(\xi) = -\frac{1}{2} C(n,s) \int_{\mathbb{H}^n} \frac{u(\xi \circ \eta) + u(\xi \circ \eta^{-1}) - 2u(\xi)}{|\eta|_{\mathbb{H}^n}^{Q+2s}} \mathrm{d}\eta, \quad \forall \xi \in \mathbb{H}^n;$$

see, e. g., [57, Proposition 1.4] and [49, Proposition 3.2]. We also recall that, given $D_H^{2,*}u(\xi)$ in (2.14), one has

$$\Delta_H u \equiv \mathrm{Tr}(D_H^{2,*}u) = \sum_{i=1}^{2n} X_i^2 u\,.$$

As the computation below shows, we have no contribution outside the unit ball in the limit as s goes to 1^-,

$$\begin{aligned}\left|\int_{\mathbb{H}^n\setminus B_1} \frac{u(\xi\circ\eta)+u(\xi\circ\eta^{-1})-2u(\xi)}{|\eta|_{\mathbb{H}^n}^{Q+2s}}\,\mathrm{d}\eta\right| &\le 4\|u\|_{L^\infty(\mathbb{H}^n)}\int_{\mathbb{H}^n\setminus B_1}\frac{1}{|\eta|_{\mathbb{H}^n}^{Q+2s}}\,\mathrm{d}\eta\\ &\le 4c\|u\|_{L^\infty(\mathbb{H}^n)}.\end{aligned}$$

Hence, recalling (11.29) and (11.32), it follows

$$\lim_{s\to 1^-} -\frac{C(n,s)}{2}\int_{\mathbb{H}^n\setminus B_1(0)} \frac{u(\xi\circ\eta)+u(\xi\circ\eta^{-1})-2u(\xi)}{|\eta|_{\mathbb{H}^n}^{Q+2s}}\,\mathrm{d}\eta \;=\; 0.$$

It remain to estimate the integral contribution in the unit ball. In view of Proposition 2.15, for any $\eta=(x,t)$, one get

$$u(\xi\circ\eta^{-1}) = P_2(u,\xi)(\xi\circ\eta^{-1}) + \mathrm{o}(|\eta|_{\mathbb{H}^n}^3) \quad \text{as } |\eta|_{\mathbb{H}^n}\to 0\,, \tag{11.31}$$

where $P_2(u,\xi)$ is the Taylor polynomial of $\mathbb{H}^n$-degree 2 associated to u and centered at ξ presented in Chapter 2.

Also, by the very definition of Taylor polynomial, it follows

$$P_2(u,\xi)(\xi\circ\eta^{-1}) \;=\; P_2\Big(u(\xi\circ\cdot),0\Big)(\eta^{-1}) \;=\; u(\xi)-\Big(\nabla_{\mathbb{H}^n}u(\xi),\,\partial_t u(\xi)\Big)\cdot\eta+\frac{1}{2}\langle x, D_H^{2,*}u(\xi)\cdot x\rangle.$$

Thus, inequality (11.31) yields

$$u(\xi\circ\eta^{-1}) = u(\xi)-\Big(\nabla_{\mathbb{H}^n}u(\xi),\,\partial_t u(\xi)\Big)\cdot\eta+\frac{1}{2}\langle x, D_H^{2,*}u(\xi)\cdot x\rangle+\mathrm{o}(|\eta|_{\mathbb{H}^n}^3) \quad \text{as } |\eta|_{\mathbb{H}^n}\to 0\,.$$

Using again the result in Proposition 2.15, we arrive at

$$\begin{aligned}&\left|\int_{B_1}\frac{u(\xi\circ\eta)+u(\xi\circ\eta^{-1})-2u(\xi)-\langle x, D_H^{2,*}u(\xi)\cdot x\rangle}{|\eta|_{\mathbb{H}^n}^{Q+2s}}\,\mathrm{d}\eta\right|\\ &\qquad\le \int_{B_1}\frac{|u(\xi\circ\eta)-P_2(u(\xi\circ\cdot),0)(\eta)|+\mathrm{o}(|\eta|_{\mathbb{H}^n}^3)}{|\eta|_{\mathbb{H}^n}^{Q+2s}}\,\mathrm{d}\eta\\ &\qquad\le \int_{B_1}\frac{\mathrm{o}(|\eta|_{\mathbb{H}^n}^3)}{|\eta|_{\mathbb{H}^n}^{Q+2s}}\mathrm{d}\eta\\ &\qquad\le \int_{B_1}\frac{1}{|\eta|_{\mathbb{H}^n}^{Q-1}} \;=:\; c(n).\end{aligned}$$

The preceeding estimate yields

$$\begin{aligned}&\lim_{s\to 1^-} -\frac{C(n,s)}{2}\int_{B_1}\frac{u(\xi\circ\eta)+u(\xi\circ\eta^{-1})-2u(\xi)}{|\eta|_{\mathbb{H}^n}^{Q+2s}}\,\mathrm{d}\eta\\ &\qquad= \lim_{s\to 1^-} -\frac{C(n,s)}{2}\int_{B_1}\frac{\langle x, D_H^{2,*}u(\xi)\cdot x\rangle}{|\eta|_{\mathbb{H}^n}^{Q+2s}}\,\mathrm{d}\eta.\end{aligned}$$

Now, notice that for any $i\neq j$ it holds

$$\int_{B_1}\left(\frac{1}{2}(X_iX_ju(\xi)+X_jX_iu(\xi))\right)x_i\cdot x_j\,\mathrm{d}\eta = -\int_{B_1}\left(\frac{1}{2}(X_iX_ju(\xi)+X_jX_iu(\xi))\right)\tilde{x}_i\cdot\tilde{x}_j\,\mathrm{d}\tilde{\eta},$$

where $\tilde{x}_i = x_i$, for $i \neq j$, and $\tilde{x}_j = -x_j$. Therefore,

$$\int_{B_1} \left(\frac{1}{2}(X_i X_j u(\xi) + X_j X_i u(\xi)) \right) x_i \cdot x_j \mathrm{d}\eta = 0, \qquad \text{for } i \neq j.$$

Moreover, for any fixed index i, making using of the polar coordinates, namely Proposition 1.15 in [61], we get that, up to permutations,

$$\begin{aligned}
\int_{B_1} \frac{X_i^2 u(\xi) x_i^2}{|\eta|_{\mathbb{H}^n}^{Q+2s}} \mathrm{d}\eta &= X_i^2 u(\xi) \int_0^1 \int_{\partial B_1} \frac{x_1^2 r^{Q+1}}{|\Phi_r(\eta)|_{\mathbb{H}^n}^{Q+2s}} \mathrm{d}\mathcal{H}^{Q-2}(\eta)\, \mathrm{d}r \\
&= X_i^2 u(\xi) \int_{\partial B_1} \frac{x_1^2}{|\eta|_{\mathbb{H}^n}^{Q+2s}} \mathrm{d}\mathcal{H}^{Q-2}(\eta) \int_0^1 \frac{1}{r^{2s-1}} \mathrm{d}r \\
&= \frac{c_2(n,s)}{2(1-s)} X_i^2 u(\xi).
\end{aligned}$$

where $c_2(n,s)$ is defined in (11.30).

Hence, recalling the result in [49, Corollary 4.2] which shows that

$$\lim_{s \to 1^-} \frac{c_1(n,s)}{s(1-s)} = \frac{4n}{\omega_{2n}}, \tag{11.32}$$

where ω_{2n} denotes the $(2n)$-dimensional Lebesgue measure of the unit sphere $\mathbb{S}^{2n}$, we finally obtain that

$$\begin{aligned}
\lim_{s \to 1^-} (-\Delta_H)^s u(\xi) &= \lim_{s \to 1^-} -\frac{C(n,s)}{2} \int_{B_1} \frac{\langle x, D_H^{2,*} u(\xi) \cdot x \rangle}{|\eta|_{\mathbb{H}^n}^{Q+2s}} \mathrm{d}\eta \\
&= \lim_{s \to 1^-} -\frac{C(n,s)}{2} \sum_{i=1}^{2n} \int_{B_1} \frac{X_i^2 u(\xi) x_i^2}{|\eta|_{\mathbb{H}^n}^{Q+2s}} \mathrm{d}\eta \\
&= \lim_{s \to 1^-} -\frac{c_1(n,s)\omega_{2n}}{4n(1-s)} \sum_{i=1}^{2n} X_i^2 u(\xi) \equiv -\Delta_H u(\xi),
\end{aligned}$$

as desired. □

In the linear case when $p = 2$ and in the particular situation in which the datum $f \equiv 0$ we consider a variant of problem (8.1) when the operator $\mathcal{L}$ in (8.2) does coincide with the conformally invariant fractional subLaplacian $(-\Delta_H)^s$ on $\mathbb{H}^n$, so that problem (8.1) does reduce to

$$\begin{cases} (-\Delta_H)^s u = 0 & \text{in } \Omega \subset \mathbb{H}^n, \\ u = g & \text{in } \mathbb{H}^n \setminus \Omega, \end{cases} \tag{11.33}$$

where $g \in H^s(\mathbb{H}^n) \equiv W^{s,2}(\mathbb{H}^n)$.

Now, we revisit the proofs of Theorems 11.1-11.2 by taking care of the dependance of the differentiability exponent s in all the estimates, so that we are eventually able to obtain the results below in clear accordance with the analogous ones in the Euclidean framework [77], by proving that the nonlocal tail term will vanish when s goes to 1, in turn recovering the classical Harnack formulation.

Theorem 11.9. *For any $s \in (0,1)$ let $u \in H^s(\mathbb{H}^n)$ be a weak solution to (11.33) such that $u \geq 0$ in $B_R(\xi_0) \subset \Omega$. Then, the following estimate holds true for any B_r such that $B_{6r} \subset B_R$,*

$$\sup_{B_r} u \leq c \inf_{B_r} u + c\,(1-s) \left(\frac{r}{R} \right)^{2s} \mathrm{Tail}(u_-; \xi_0, R),$$

where $\mathrm{Tail}(\cdot)$ *is defined in* (8.4) *by taking* $p = 2$ *there,* $u_- := \max\{-u, 0\}$ *is the negative part of the function* u*, and* $c \equiv c(n, s)$*.*

Theorem 11.10. *For any* $s \in (0, 1)$ *let* $u \in H^s(\mathbb{H}^n)$ *be a weak supersolution to* (11.33)*, such that* $u \geq 0$ *in* $B_R(\xi_0) \subseteq \Omega$*. Then, the following estimate holds true for any* B_r *such that* $B_{6r} \subset B_R$*, and any* $t < Q/(Q - 2s)$*,*

$$\left(\fint_{B_r} u^t \, \mathrm{d}\xi \right)^{\frac{1}{t}} \leq c \inf_{B_{\frac{3r}{2}}} u + c\,(1 - s) \left(\frac{r}{R} \right)^{2s} \mathrm{Tail}(u_-; \xi_0, R),$$

where $\mathrm{Tail}(\cdot)$ *is defined in* (8.4) *by taking* $p = 2$ *there,* $u_- := \max\{-u, 0\}$ *is the negative part of the function* u*, and* $c \equiv c(n, s)$*.*

The proofs of Theorem 11.9 and Theorem 11.10 can be plainly deduced from the ones in Chapter 11 for the subcritical case, by taking there $p = 2$, $f \equiv 0$, and the Korányi-Folland norm in place of the generic homogeneous norm d_{o}.

CHAPTER 12

The subelliptic obstacle problem

In the present paper we investigate a class of obstacle problems where the leading operator $\mathcal{L}$ defined in (8.2).

In order to state our main results about the obstacle problem related to $\mathcal{L}$ some further notation is needed. Let $\Omega \Subset \Omega'$ be bounded open sets, $h : \mathbb{H}^n \to [-\infty, \infty)$ be an extended real-valued function, called the *obstacle*, and $g \in W^{s,p}(\mathbb{H}^n)$. Define the class

$$\mathcal{K}_{g,h}(\Omega, \Omega') := \Big\{ u \in W^{s,p}(\Omega') : u \geq h \text{ a. e. in } \Omega,\ u = g \text{ a. e. in } \mathbb{H}^n \setminus \Omega \Big\}.$$

and consider the functional $\boldsymbol{A} : \mathcal{K}_{g,h}(\Omega, \Omega') \to [W^{s,p}(\Omega')]'$ given by

$$\begin{aligned} \langle \boldsymbol{A}(u), v \rangle \quad := \quad & \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \frac{|u(\xi) - u(\eta)|^{p-2}(u(\xi) - u(\eta))(v(\xi) - v(\eta))}{d_{\mathrm{o}}(\eta^{-1} \circ \xi)^{Q+sp}} \, \mathrm{d}\xi \mathrm{d}\eta \\ & - \int_{\Omega} f(\xi, u) v(\xi) \, \mathrm{d}\xi. \end{aligned} \tag{12.1}$$

where the function $f : \mathbb{H}^n \times \mathbb{R} \to \mathbb{R}$ satisfies

(12.2)
$f(\cdot, x)$ is measurable for all $x \in \mathbb{R}$ and $f(\xi, \cdot)$ is continuous for a. e. $\xi \in \mathbb{H}^n$;

(12.3)
$f \equiv f(\cdot, x) \in L^{\infty}_{\mathrm{loc}}(\mathbb{H}^n), \qquad$ for any $x \in \mathbb{R}$, uniformly in Ω;

(12.4)
$\big(f(\xi, x_1) - f(\xi, x_2)\big)(x_1 - x_2) \leq 0, \qquad \forall x_1, x_2 \in \mathbb{R}$ and for a. e. $\xi \in \mathbb{H}^n$.

With this bit of notations we can give the definition of solution to the obstacle problem related to $\mathcal{L}$.

Definition 12.1. *We say that $u \in \mathcal{K}_{g,h}(\Omega, \Omega')$ is a* solution to the obstacle problem in $\mathcal{K}_{g,h}(\Omega, \Omega')$ *if*

$$\langle \boldsymbol{A}(u), v - u \rangle \geq 0 \qquad \forall v \in \mathcal{K}_{g,h}(\Omega, \Omega').$$

Our first result regards the existence and uniqueness of a solution u to the obstacle problem.

Theorem 12.2 (Existence and uniqueness)**.** *Under conditions* (12.2), (12.3) *and* (12.4)*, if the class $\mathcal{K}_{g,h}(\Omega, \Omega')$ is not empty, then there exists a unique solution to the obstacle problem in $\mathcal{K}_{g,h}(\Omega, \Omega')$.*

Moreover, we can also relate the solution to the obstacle problem in $\mathcal{K}_{g,h}(\Omega, \Omega')$ with the energy solution of the Dirichlet problem (8.1). In particular, the following corollary holds

Corollary 12.3. *Let u be the solution to the obstacle problem in $\mathcal{K}_{g,h}(\Omega, \Omega')$. Then u is weak supersolution to* (8.1) *in Ω, according to Definition* 9.4*.*
Moreover, if $B_r \equiv B_r(\xi_0) \subset \Omega$ is such that

$$\inf_{B_r}(u - h) > 0,$$

then u is a weak solution to (8.1) in B_r. In particular, if u is lower semicontinuous and h is upper semicontinuous in Ω, then u is a weak solution to (8.1) in $\{u > h\} \cap \Omega$.

We now investigate the regularity of the solution to the obstacle problem in $\mathcal{K}_{g,h}(\Omega, \Omega')$. In the same spirit of regularity results up to the boundary for nonlocal equation ([**85**, **86**]) a natural measure theoretic condition on the boundary of Ω has to be assumed; that is, there exists $\delta_\Omega \in (0,1)$ and $r_0 > 0$ such that for every $\xi_0 \in \partial\Omega$

$$\inf_{r\in(0,r_0)} \frac{|(\mathbb{H}^n \setminus \Omega) \cap B_r(\xi_0)|}{|B_r(\xi_0)|} \geq \delta_\Omega. \tag{12.5}$$

Thus, we have that the following results holds true.

Theorem 12.4 (Regularity up to the boundary)**.** *Let $s \in (0,1)$, $p \in (1,\infty)$ and, under assumptions (12.2), (12.3) and (12.4), let u be a solution to the obstacle problem in $\mathcal{K}_{g,h}(\Omega, \Omega')$ with $g \in \mathcal{K}_{g,h}(\Omega, \Omega')$ and Ω satisfying condition (12.5). If g is locally Hölder continuous (resp. continuous) in Ω' and h is locally Hölder continuous (rips. continuous) in Ω or $h \equiv -\infty$, then u is locally Hölder continuous (resp. continuous) in Ω'.*

Let us note that if we assume the obstacle $h \equiv -\infty$, from Corollary 12.3, we can see the theorem above as a boundary regularity result for weak solution to (8.1), which completes the properties obtained in Section 10.

Theorems 12.4 extends to the Heisenberg setting the results obtained by Korvenpää, Kuusi and Palatucci in [**85**] for the obstacle problem related to a class of integro-differential operators modeled on the fractional p-Laplacian, as well as it generalizes to the fractional setting the regularity properties obtained for the obstacle problem in Carnot groups; see for instance [**42**, **41**] and the references therein. Nevertheless, the involved functional and geometrical settings are different than the ones considered in the aforementioned papers. Indeed, given the nonlinearity $f \equiv f(\cdot, u)$, one has take into account the monotonicity assumption in (12.4) in order to extend the theory presented in [**85**] to a more general nonhomogeneous framework. Moreover, conditions (12.2) and (12.3) are crucial in order to apply the regularity properties of weak solutions to (8.1) such as boundedness, Hölder continuity and Harnack inequality established in the previous sections.

Lastly, we remark that very general boundary regularity is in most cases a very challenging task in $\mathbb{H}^n$. We would just mention the relevant counterexample proven by Jerison in the seminal papers [**92**, **93**]. For what concerns Theorem 12.4, we stress that we proved the transferring of regularity from the obstacle (whether is not identically equals to $-\infty$) and boundary datum to the solution u of the obstacle problem. In the same vein of further developments, it is quite natural to wonder whether or not the Hölder-regularity obtained in Theorem 12.4 can be improved (under further assumptions on the data) at some extent: at least in the linear case, when $p = 2$, to the best of our knowledge no further regularity is known in the fractional case even though several breakthrough results have been achieved in the Euclidean case; see [**28**, **122**] for $C^{1,s}$-regularity estimates and related results.

1. Existence of the solution

This section is devoted to the proof of Theorem 12.2 and Corollary 12.3.

In order to prove Theorem 12.2 it will be helpful to split the operator $\boldsymbol{A}$ in (12.1) as follows

$$\langle \boldsymbol{A}_1(u), v\rangle := \int_{\Omega'}\int_{\Omega'} \frac{L(u(\xi),u(\eta))(v(\xi)-v(\eta))}{d_{\mathrm{o}}(\eta^{-1}\circ\xi)^{Q+sp}}\,\mathrm{d}\xi\mathrm{d}\eta - \int_{\Omega} f(\xi,u)v(\xi)\,\mathrm{d}\xi,$$

and

$$\langle \boldsymbol{A}_2(u), v\rangle := 2\int_{\mathbb{H}^n\setminus\Omega'}\int_{\Omega} \frac{L(u(\xi),g(\eta))v(\xi)}{d_{\mathrm{o}}(\eta^{-1}\circ\xi)^{Q+sp}}\,\mathrm{d}\xi\mathrm{d}\eta\,,$$

where we have adopted the notation

$$L(a,b) := |a-b|^{p-2}(a-b), \qquad a,b\in\mathbb{R}.$$

The following observation is useful in estimating the second term $\boldsymbol{A}_2$.

Remark 12.5. *Since $\Omega \Subset \Omega'$ there exist $\xi_0\in\Omega$ and $r\in(0,r_0)$, with $r_0 := \operatorname{dist}(\Omega,\partial\Omega')$, such that $\Omega\Subset B_r(\xi_0)\Subset\Omega'$. Hence, for any $\eta\in\mathbb{H}^n\setminus B_r(\xi_0)$ and any $\xi\in\Omega$, it holds*

$$\frac{|\eta^{-1}\circ\xi_0|_{\mathbb{H}^n}}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}} \le 1 + \frac{|\xi^{-1}\circ\xi_0|_{\mathbb{H}^n}}{|\eta^{-1}\circ\xi_0|_{\mathbb{H}^n} - |\xi^{-1}\circ\xi_0|_{\mathbb{H}^n}} \le c.$$

Thus, by the inequality above, considering a function $g\in L^{p-1}_{sp}(\mathbb{H}^n)$ and a function $v\in L^p(\Omega)$, we have that

$$\begin{aligned}
\int_{\mathbb{H}^n\setminus\Omega'}\int_{\Omega} |g(\eta)|^{p-1}|v(\xi)||\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{-Q-sp}\,\mathrm{d}\xi\mathrm{d}\eta&\\
&\le c\int_{\mathbb{H}^n\setminus B_r(\xi_0)}\int_{\Omega} |g(\eta)|^{p-1}|v(\xi)||\eta^{-1}\circ\xi_0|_{\mathbb{H}^n}^{-Q-sp}\,\mathrm{d}\xi\mathrm{d}\eta\\
&\le \int_{\mathbb{H}^n\setminus B_r(\xi_0)} |g(\eta)|^{p-1}|\eta^{-1}\circ\xi_0|^{-Q-sp}\,\mathrm{d}\eta\int_{\Omega}|v(\xi)|\,\mathrm{d}\xi\\
&\le c\,r^{-sp}\mathrm{Tail}(g;\xi_0,r)^{p-1}\|v\|_{L^p(\Omega)},
\end{aligned}$$

by Jensen's Inequality, with $c\equiv c(p,\Omega)>0$.

Proof of Theorem 12.2. We begin proving that the operator $\boldsymbol{A}$, defined in (12.1), is monotone, weakly continuous and coercive on the set $\mathcal{K}_{g,h}(\Omega,\Omega')$. We first prove that the operator $\boldsymbol{A}$ in (12.1) is monotone, weakly continuous and coercive.

First we prove that $\boldsymbol{A}$ is monotone, i. e.

$$\langle \boldsymbol{A}(u)-\boldsymbol{A}(v), u-v\rangle \ge 0 \qquad \text{for every } u,v\in\mathcal{K}_{g,h}(\Omega,\Omega'). \tag{12.6}$$

Let us fix two functions $u,v\in\mathcal{K}_{g,h}(\Omega,\Omega')$ and consider separately $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$. For $\boldsymbol{A}_1$ we have

$$\begin{aligned}
\langle \boldsymbol{A}_1(u)-\boldsymbol{A}_1(v), u-v\rangle = \int_{\Omega'}\int_{\Omega'} &(L(u(\xi),u(\eta))-L(v(\xi),v(\eta)))\\
&\times (u(\xi)-u(\eta)-v(\xi)+v(\eta))d_{\mathrm{o}}(\eta^{-1}\circ\xi)^{-Q-sp}\,\mathrm{d}\xi\mathrm{d}\eta\\
&-\int_{\Omega}(f(\xi,u)-f(\xi,v))(u(\xi)-v(\xi))\,\mathrm{d}\xi.
\end{aligned}$$

By the inequality

$$\frac{1}{c} \le \frac{(|a|^{p-2}a-|b|^{p-2}b)(a-b)}{(|a|+|b|)^{p-2}(a-b)^2} \le c,$$

when $a,b\in\mathbb{R}$, $a\neq b$, we have that we only need to prove that

$$\int_{\Omega}(f(\xi,u)-f(\xi,v))(u(\xi)-v(\xi))\,\mathrm{d}\xi \le 0. \tag{12.7}$$

But by condition (12.4) there exists a negligible set E such that for any $\xi \in \Omega \setminus E$ it holds

$$\Big(f(\xi, u(\xi)) - f(\xi, v(\xi))\Big)\Big(u(\xi) - v(\xi)\Big) \leq 0.$$

Hence (12.7) follows. Similarly, proceeding as in [85], it follows that $\langle \boldsymbol{A}_2(u) - \boldsymbol{A}_2(v), u - v\rangle \geq 0$. Hence, (12.6) is proven.

Now, let us show that $\boldsymbol{A}$ is weakly continuous. Let $\{u_j\}$ be a sequence in $\mathcal{K}_{g,h}(\Omega, \Omega')$ converging to $u \in \mathcal{K}_{g,h}(\Omega, \Omega')$ in $W^{s,p}(\Omega')$. We want to prove that

$$\langle \boldsymbol{A}(u_j) - \boldsymbol{A}(u), v\rangle \to 0 \qquad \text{for all } v \in W^{s,p}(\Omega'). \tag{12.8}$$

Let $v \in W^{s,p}(\Omega')$. We estimate separately $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$.

$$\begin{aligned}
&|\langle \boldsymbol{A}_1(u_j) - \boldsymbol{A}_1(u), v\rangle| \\
&\quad \leq \int_{\Omega'}\int_{\Omega'} |L(u_j(\xi), u_j(\eta)) - L(u(\xi), u(\eta))||v(\xi) - v(\eta)| d_{\mathrm{o}}(\eta^{-1} \circ \xi)^{-Q-sp}\, \mathrm{d}\xi \mathrm{d}\eta \\
&\qquad + \int_{\Omega} |f(\xi, u_j) - f(\xi, u)||v(\xi)|\, \mathrm{d}\xi \\
&\quad =: I_1 + I_2.
\end{aligned}$$

Consider I_2. Since u_j converges to u in $W^{s,p}(\Omega')$, it also converges to u in $L^p(\Omega)$.

Thus, there exists a subsequence $\{u_{j_k}\}_k$ which converges to u almost everywhere in Ω. Moreover, since by condition (12.2) $x \mapsto f(\xi, x)$ is continuous for a. e. $\xi \in \Omega$, we have that

$$f(\xi, u_{j_k}) \to f(\xi, u) \qquad \text{for a. e. } \xi \in \Omega.$$

Hence, since the sequence $\{f(\cdot, u_{j_k})\}$ has $L^{p/(p-1)}(\Omega)$-norms uniformly bounded thanks to (12.3), we have that $f(\xi, u_{j_k}) \to f(\xi, u)$ weakly in $L^{p/(p-1)}(\Omega)$. Since the weak limit is independent of the choice of the subsequence it follows that $f(\xi, u_j) \to f(\xi, u)$ weakly in $L^{p/(p-1)}(\Omega)$. Then,

$$I_2 \xrightarrow{j\to\infty} 0.$$

Let us focus on I_1. Applying the following Lemma

Lemma 12.6. *Let $1 < p \leq 2$ and $a, b, a', b' \in \mathbb{R}$. Then,*

$$|L(a, b) - L(a', b')| \leq 4|a - a' - b + b'|^{p-1}.$$

Let $p \geq 2$ and $a, b, a', b' \in \mathbb{R}$. Then,

$$|L(a, b) - L(a', b')| \leq c\,|a - a'|^{p-1} + c\,|a - a'||a - b|^{p-2},$$

and

$$|L(a, b) - L(a', b')| \leq c\,|b - b'|^{p-1} + c\,|b - b'||a - b|^{p-2},$$

where c depends only on p.

together with Hölder's Inequality we obtain, for $1 < p \leq 2$,

$$\begin{aligned}
I_1 &\leq c \int_{\Omega'}\int_{\Omega'} |u_j(\xi) - u(\xi) - (u_j(\eta) - u(\eta))|^{p-1}|v(\xi) - v(\eta)| \frac{\mathrm{d}\xi \mathrm{d}\eta}{|\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{Q+sp}} \\
&\leq c\, \|u_j - u\|_{W^{s,p}(\Omega')}^{p-1} \|v\|_{W^{s,p}(\Omega')} \xrightarrow{j\to\infty} 0.
\end{aligned}$$

When $p \geq 2$ by (12.6), proceeding as done in [85], we eventually arrive at

$$I_1 \leq c\,(\|u_j\|_{W^{s,p}(\Omega')} + \|u\|_{W^{s,p}(\Omega)})^{p-2} \|u_j - u\|_{W^{s,p}(\Omega')} \|v\|_{W^{s,p}(\Omega')},$$

which also vanishes when $j \to \infty$. Also for $\boldsymbol{A}_2$ we split the case $1 < p \leq 2$ and $p > 2$. For $1 < p \leq 2$ using again Lemma 12.6, we obtain

$$|\langle \boldsymbol{A}_2(u_j) - \boldsymbol{A}_2(u), v\rangle|$$

$$
\begin{aligned}
&\le 2\int_{\mathbb{H}^n\setminus\Omega'}\int_{\Omega}|L(u_j(\xi),g(\eta))-L(u(\xi),g(\eta))||v(\xi)|d_{\mathrm{o}}(\eta^{-1}\circ\xi)^{-Q-sp}\,\mathrm{d}\xi\mathrm{d}\eta\\
&\le c\int_{\mathbb{H}^n\setminus\Omega'}\int_{\Omega}|u_j(\xi)-u(\xi)|^{p-1}|v(\xi)|\frac{\mathrm{d}\xi\mathrm{d}\eta}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}\\
&\le c\,\|u_j-u\|^{p-1}_{W^{s,p}(\Omega')}\|v\|_{W^{s,p}(\Omega')}\xrightarrow{j\to\infty}0.
\end{aligned}
$$

When $p\ge 2$, proceeding as done in [85], by Remark 12.5, we have that

$$
\begin{aligned}
|\langle \boldsymbol{A}_2(u_j)-\boldsymbol{A}_2(u),v\rangle| &\le c\,\|u_j-u\|^{p-1}_{W^{s,p}(\Omega')}\|v\|_{W^{s,p}(\Omega')}\\
&\quad+c\,\|u_j-u\|_{W^{s,p}(\Omega')}\|u\|^{p-2}_{W^{s,p}(\Omega')}\|v\|_{W^{s,p}(\Omega')}\\
&\quad+c\,r^{-sp}\mathrm{Tail}(g;\xi_0,r)^{p-2}\|u_j-u\|_{W^{s,p}(\Omega')}\|v\|_{W^{s,p}(\Omega')}\xrightarrow{j\to\infty}0.
\end{aligned}
$$

Thus, (12.8) holds.

In the end, let us prove the coercivity, i. e. we prove that there exists a function $v\in\mathcal{K}_{g,h}(\Omega,\Omega')$ such that

$$
\frac{\langle \boldsymbol{A}(u)-\boldsymbol{A}(v),u-v\rangle}{\|u-v\|_{W^{s,p}(\Omega')}}\to\infty\quad\text{as}\quad\|u\|_{W^{s,p}(\Omega')}\to\infty. \tag{12.9}
$$

Since we have assumed that the class $\mathcal{K}_{g,h}(\Omega,\Omega')$ is not empty, there exists at least one $v\in\mathcal{K}_{g,h}(\Omega,\Omega')$. Let us fix such v. Following the same chain of estimates used in [85] it can be proved that there exists a constant c independent of u such that

$$
|\langle \boldsymbol{A}(u)-\boldsymbol{A}(v),v\rangle|\le c\,\|u\|^{p-1}_{W^{s,p}(\Omega')}+c.
$$

We prove that the contribution from $\langle \boldsymbol{A}(u)-\boldsymbol{A}(v),u\rangle$ dominates when $\|u\|_{W^{s,p}(\Omega')}$ is large. As customary we split the operator $\boldsymbol{A}$ into $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$. Then,

$$
\begin{aligned}
&\langle \boldsymbol{A}_1(u)-\boldsymbol{A}_1(v),u\rangle\\
&\quad=\int_{\Omega'}\int_{\Omega'}(L(u(\xi),u(\eta))-L(v(\xi),v(\eta)))(u(\xi)-u(\eta))d_{\mathrm{o}}(\eta^{-1}\circ\xi)^{-Q-sp}\,\mathrm{d}\xi\mathrm{d}\eta\\
&\qquad-\int_{\Omega}(f(\xi,u)-f(\xi,v))u(\xi)\,\mathrm{d}\xi\\
&\quad\ge c\int_{\Omega'}\int_{\Omega'}\frac{|u(\xi)-u(\eta)|^p}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}\,\mathrm{d}\xi\mathrm{d}\eta\\
&\qquad-c\int_{\Omega'}\int_{\Omega'}|v(\xi)-v(\eta)|^{p-1}|u(\xi)-u(\eta)|\frac{\mathrm{d}\xi\mathrm{d}\eta}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}\\
&\qquad-\|f\|_{L^\infty(\Omega)}\int_{\Omega'}|u(\xi)|\,\mathrm{d}\xi\\
&\quad\ge c\,[u-g]^p_{W^{s,p}(\Omega')}-c\,[g]^p_{W^{s,p}(\Omega')}-c\,[v]^{p-1}_{W^{s,p}(\Omega')}[u]_{W^{s,p}(\Omega')}-c\,\|u\|^{p-1}_{L^p(\Omega')}\\
&\quad\ge c\,\|u\|^p_{W^{s,p}(\Omega')}-c\,\|g\|^p_{W^{s,p}(\Omega')}-c\,\|v\|^{p-1}_{W^{s,p}(\Omega')}\|u\|_{W^{s,p}(\Omega')}-c\,\|u\|^{p-1}_{W^{s,p}(\Omega')}
\end{aligned}
$$

where we have used Hölder's Inequality and the Sobolev embedding Theorem 9.1 and where the constant c depends on $n,p,s,\|f\|_{L^\infty(\Omega)}$ and Ω' and the structural constant Λ. On the other hand, proceeding as in [85] and recalling Remark 12.5, for $\boldsymbol{A}_2$, we have

$$
\begin{aligned}
&\langle \boldsymbol{A}_2(u)-\boldsymbol{A}_2(v),u\rangle\\
&\qquad\ge -c\,\|u\|^{p-1}_{L^p(\Omega')}\|v\|_{L^p(\Omega')}-c\,r^{-sp}\mathrm{Tail}(g,\xi_0,r)^{p-1}\|v\|_{L^p(\Omega')}-\|v\|^p_{L^p(\Omega')}.
\end{aligned}
$$

Combining all previous estimates we obtain that

$$\langle \boldsymbol{A}(u) - \boldsymbol{A}(v), u - v\rangle \geq c\ \|u\|^{p}_{W^{s,p}(\Omega')} - c\ \|u\|^{p-1}_{W^{s,p}(\Omega')} - c\ \|u\|_{W^{s,p}(\Omega)} - c,$$

where the constant c is independent of u. Hence, the proof of (12.9) is complete.

Then, since $\boldsymbol{A}$ in (12.1) is monotone, weakly continuous and coercive in $\mathcal{K}_{g,h}(\Omega,\Omega')$, the existence and uniqueness of the solution u to the obstacle problem can be proved extending to our setting the same argument used in Theorem 1 in [85]. □

We now prove Corollary 12.3.

Proof of Corollary 12.3. Let us prove that the solution u to the obstacle problem in $\mathcal{K}_{g,h}(\Omega,\Omega')$ is a weak supersolution to (8.1) according to Definition 9.4. First of all note that clearly $u \in W^{s,p}_{\mathrm{loc}}(\Omega) \cap L^{p-1}_{sp}(\mathbb{H}^n)$. Moreover, for any nonnegative $\psi \in C^\infty_0(\Omega)$ the function $v := u + \psi \in \mathcal{K}_{g,h}(\Omega,\Omega')$. Hence,

$$\begin{aligned} 0 &\leq \langle \boldsymbol{A}(u), v - u\rangle \\ &= \int_{\mathbb{H}^n}\int_{\mathbb{H}^n} L(u(\xi),u(\eta))(\psi(\xi)-\psi(\eta)) d_{\mathrm{o}}(\eta^{-1}\circ\xi)^{-Q-sp}\,\mathrm{d}\xi\mathrm{d}\eta - \int_{\mathbb{H}^n} f(\xi,u)\psi(\xi)\,\mathrm{d}\xi. \end{aligned}$$

Then, u is a weak supersolution according to Definition 9.4. The last statement of Corollary 12.3 follows repeating the argument of Corollary 1 in [85]. □

2. Regularity of the solution of the obstacle problem

In this section we prove the boundary regularity result of Theorem 12.4. We divide the proof in two cases: the case of the interior regularity and the case of the boundary regularity.

2.1. Interior regularity. In order to prove that solutions to the obstacle problem are Hölder continuous or simply continuous inside Ω we begin proving a local boundedness result.

Theorem 12.7. *Let $s \in (0,1)$, $p \in (1,\infty)$ and, under hypothesis (12.2), (12.3) and (12.4), let u be the solution to the obstacle problem in $\mathcal{K}_{g,h}(\Omega,\Omega')$. Assume that $B_r \equiv B_r(\xi_0) \subset \Omega'$ and set*

$$M := \max\Big(\sup_{B_r\cap\Omega} h,\ \sup_{B_r\smallsetminus\Omega} g\Big).$$

Here the interpretation is that $\sup_A \psi = -\infty$ if $A = \emptyset$. If M is finite, then u is bounded from above in $B_{r/2} \equiv B_{r/2}(\xi_0)$ and

$$\begin{aligned} \sup_{B_{r/2}} (u-m)_+ \quad &\leq \quad \delta\ \mathrm{Tail}((u-m)_+;\xi_0,r/2) + c\ \delta^{-\gamma}\left(\fint_{B_r}(u-m)_+^t\,\mathrm{d}\xi\right)^{\frac{1}{t}} \\ &\qquad + r^{\frac{sp}{p-1}}\|f\|^{\frac{1}{p-1}}_{L^\infty(B_r)}, \end{aligned} \tag{12.10}$$

holds for all $m \geq M$, $t \in (0,p)$ and $\delta \in (0,1]$ with constant $\gamma \equiv \gamma(n,p,s,t) > 0$ and $c \equiv c(n,p,s,\Lambda) > 0$.

Proof. Let us suppose that $M < \infty$ and let us choose $k \geq 0$, $m \geq M$ and a test function $\varphi \in C^\infty_0(B_r)$, such that $0 \leq \varphi \leq 1$. Denote with $v := u - (u-m-k)_+\varphi^p \in \mathcal{K}_{h,g}(\Omega,\Omega')$. Since u solves the obstacle problem, denoting with $u_m := u - m$, we have that

$$\begin{aligned} 0 \leq &\int_{\mathbb{H}^n}\int_{\mathbb{H}^n} \frac{L(u(\xi),u(\eta))(v(\xi)-u(\xi)-v(\eta)+u(\eta))}{d_{\mathrm{o}}(\eta^{-1}\circ\xi)^{Q+sp}}\,\mathrm{d}\xi\mathrm{d}\eta \\ &- \int_{\mathbb{H}^n} f(\xi,u)(v(\xi)-u(\xi))\,\mathrm{d}\xi \end{aligned}$$

$$= -\int_{\mathbb{H}^n}\int_{\mathbb{H}^n} \frac{L(u_m(\xi),u_m(\eta))((u_m(\xi)-k)_+\varphi^p(\xi)-(u_m(\eta)-k)_+\varphi^p(\eta))}{d_{\mathrm{o}}(\eta^{-1}\circ\xi)^{Q+sp}}\,\mathrm{d}\xi\mathrm{d}\eta$$
$$+\int_{\mathbb{H}^n} f(\xi,u)(u_m(\xi)-k)_+\varphi^p(\xi)\,\mathrm{d}\xi,$$

which yields

$$0 \geq \int_{\mathbb{H}^n}\int_{\mathbb{H}^n} \frac{L(u_m(\xi),u_m(\eta))((u_m(\xi)-k)_+\varphi^p(\xi)-(u_m(\eta)-k)_+\varphi^p(\eta))}{d_{\mathrm{o}}(\eta^{-1}\circ\xi)^{Q+sp}}\,\mathrm{d}\xi\mathrm{d}\eta$$
$$-\|f\|_{L^\infty(B_r)}\int_{\mathbb{H}^n}(u_m(\xi)-k)_+\varphi^p(\xi)\,\mathrm{d}\xi.$$

Referring to the proof of Theorem 10.1 the above inequality is enough to establish the Caccioppoli estimate in (10.1), which subsequently can be used to prove a local boundedness result like Theorem 10.5, which yields

$$\sup_{B_{\varrho/2}(\eta)}(u_m)_+ \leq \tilde{\delta}\,\mathrm{Tail}((u_m)_m;\eta,\varrho/2)+c\tilde{\delta}^{-\tilde{\gamma}}\left(\fint_{B_\varrho(\eta)}(u_m)_+^p\,\mathrm{d}\xi\right)^{\frac{1}{p}} + \varrho^{\frac{sp}{p-1}}\|f\|^{\frac{1}{p-1}}_{L^\infty(B_\varrho(\eta))}, \tag{12.11}$$

where $B_\varrho(\eta)\subset B_r$, $\tilde{\delta}\in(0,1]$, the constant $\tilde{\gamma}=\tilde{\gamma}(n,p,s)>0$ and c depending on n,p,s, Λ and the function f.

Now we apply a covering argument. Let us set

$$\varrho=(\sigma-\sigma')r, \qquad \frac{1}{2}\leq\sigma'<\sigma\leq 1, \qquad \text{and} \qquad \eta\in B_{\sigma' r}.$$

We then proceed to estimate the nonlocal contribution in (12.11). Let us first note that for any $\xi\in\mathbb{H}^n\setminus B_{\sigma r}$ we have that, by the choice of η, it holds

$$\frac{|\xi^{-1}\circ\xi_0|_{\mathbb{H}^n}}{|\xi^{-1}\circ\eta|_{\mathbb{H}^n}}\leq 1+\frac{|\eta^{-1}\circ\xi_0|_{\mathbb{H}^n}}{|\xi^{-1}\circ\xi_0|_{\mathbb{H}^n}-|\eta^{-1}\circ\xi_0|_{\mathbb{H}^n}}\leq 1+\frac{\sigma' r}{\sigma r-\sigma' r}=\frac{\sigma r}{(\sigma-\sigma')r} \tag{12.12}$$

Then, by inequality (12.12) we have that the nonlocal tail on the righthand side of (12.11) can be treated as follows

$$\begin{aligned}
&\mathrm{Tail}((u_m)_+;\eta,\varrho/2)^{p-1}\\
&\quad\leq\left(\frac{\varrho}{2}\right)^{sp}\sup_{B_{\sigma r}}(u_m)_+^{p-1}\int_{B_{\sigma r}\setminus B_\varrho(\eta)}|\xi^{-1}\circ\eta|_{\mathbb{H}^n}^{-Q-sp}\,\mathrm{d}\xi\\
&\qquad+c\,(\sigma-\sigma')^{-Q}\,\mathrm{Tail}((u_m)_+;\xi_0,\sigma r)^{p-1}\\
&\quad\leq c\sup_{B_{\sigma r}}(u_m)_+^{p-1}+c\,(\sigma-\sigma')^{-Q}\,\mathrm{Tail}((u_m)_+;\xi_0,r/2)^{p-1}.
\end{aligned} \tag{12.13}$$

Let us study the local contribution in (12.11). Applying Young's Inequality, with exponent $p/(p-t)$ and p/t, we obtain that

$$\begin{aligned}
\tilde{\delta}^{-\tilde{\gamma}}\left(\fint_{B_\varrho(\eta)}(u_m)_+^p\,\mathrm{d}\xi\right)^{\frac{1}{p}} &\leq \tilde{\delta}^{-\tilde{\gamma}}\sup_{B_\varrho(\eta)}(u_m)_+^{\frac{p-t}{p}}\left(\fint_{B_\varrho(\eta)}(u_m)_+^t\,\mathrm{d}\xi\right)^{\frac{1}{p}}\\
&\leq\frac{1}{4}\sup_{B_{\sigma r}}(u_m)_+ + c\;\tilde{\delta}^{-\frac{\tilde{\gamma}p}{t}}\left(\fint_{B_\varrho(\eta)}(u_m)_+^t\,\mathrm{d}\xi\right)^{\frac{1}{t}}\\
&\leq\frac{1}{4}\sup_{B_{\sigma r}}(u_m)_+ + c\;\tilde{\delta}^{-\frac{\tilde{\gamma}p}{t}}(\sigma-\sigma')^{-\frac{Q}{t}}\left(\fint_{B_r}(u_m)_+^t\,\mathrm{d}\xi\right)^{\frac{1}{t}}.
\end{aligned} \tag{12.14}$$

Thus, by combining (12.11) with (12.13) and (12.14) with $\tilde{\delta} \leq 1/4c$, we eventually arrive at

$$\sup_{B_{\sigma' r}} (u_m)_+ \leq \frac{1}{2} \sup_{B_{\sigma r}} (u_m)_+ + c\,\tilde{\delta}^{-\frac{\tilde{\gamma} p}{t}} (\sigma - \sigma')^{-\frac{Q}{t}} \left(\fint_{B_r} (u_m)_+^t \, \mathrm{d}\xi \right)^{\frac{1}{t}} + c\,\tilde{\delta} (\sigma - \sigma')^{-\frac{Q}{p-1}} \operatorname{Tail}((u_m)_+; \xi_0, r/2) + r^{\frac{sp}{p-1}} \|f\|_{L^\infty(B_r)}^{\frac{1}{p-1}}.$$

Extending the same argument of the iteration Lemma 11.7 and taking in consideration the datum f and the non-Euclidean structure of our problem we finally obtain the desired inequality (12.10). □

Theorem 12.8. *Let $s \in (0,1)$, $p \in (1,\infty)$ and let us suppose that h is locally Hölder continuous in Ω or $h \equiv -\infty$. Then, under assumptions (12.2), (12.3) and (12.4), the solution u to the obstacle problem in $\mathcal{K}_{g,h}(\Omega, \Omega')$ is locally Hölder continuous in Ω as well. Moreover, for every $\xi_0 \in \Omega$ there is $r_0 > 0$ such that, for any $B_r \equiv B_r(\xi_0) \Subset B_{r_0} \equiv B_{r_0}(\xi_0)$, it holds, for any $\varrho \in (0, r/3]$,*

$$\operatorname*{osc}_{B_\varrho} u \leq c \left(\frac{\varrho}{r}\right)^\alpha \left[\operatorname{Tail}(u - h(\xi_0); \xi_0, r) + \left(\fint_{B_r} |u - h(\xi_0)|^p \, \mathrm{d}\xi \right)^{\frac{1}{p}} \right] + c \int_\varrho^r \left(\frac{\varrho}{\tau}\right)^\alpha \left[\omega_h\left(\frac{\tau}{\sigma}\right) + \Psi\left(\frac{\tau}{\sigma}\right) \right] \frac{\mathrm{d}\tau}{\tau}, \tag{12.15}$$

where $\omega_h(\varrho) := \operatorname{osc}_{B_\varrho} h$, the function Ψ is given by

$$\chi = \begin{cases} \tau^{\frac{sp}{p-1}} \|f\|_{L^\infty(B_r)}^{\frac{Q}{t(Q-sp)}}, \ \forall t < \min\left(p, \frac{Q(p-1)}{Q-sp}\right) & \text{if } sp < Q, \\ \tau^{\frac{p(s-\epsilon)}{p-1}} \|f\|_{L^\infty(B_r)}^{\frac{\epsilon}{t\epsilon}}, \ \forall 0 < \epsilon < s \ \ t < \min\left(p, \frac{(p-1)s}{\epsilon}\right) & \text{if } sp = Q, \end{cases} \tag{12.16}$$

and α, σ and c are positive constants depending only on n, p, s and the structural constant Λ in (2.4).

Proof. With no loss of generality we assume that $sp \leq Q$, since if the reverse inequality holds true we have that u is Hölder continuous, as proven in [2].

We divide the proof into two cases: when ξ_0 belongs to the contact set, i. e. when it is such that for every $r \in (0, R)$, with $R := \operatorname{dist}(\xi_0, \partial\Omega)$, we have for $B_r \equiv B_r(\xi_0)$,

$$\inf_{B_r} (u - h) = 0,$$

and when ξ_0 does not belongs to the contact set.

Note that, in this second scenario, we can find $r_0 \in (0, R)$ such that

$$\inf_{B_{r_0}} (u - h) > 0.$$

By Corollary 12.3 it holds that u is a weak solution to (8.1) in B_{r_0}. Hence, an application of Theorem 10.6 yields

$$\operatorname*{osc}_{B_\varrho} u \leq c \left(\frac{\varrho}{r}\right)^\alpha \left(\operatorname{Tail}(u; \xi_0, r/2) + \left(\fint_{B_r} |u|^p \, \mathrm{d}\xi \right)^{\frac{1}{p}} + r^{\frac{sp}{p-1}} \|f\|_{L^\infty(B_{2r})}^{\frac{1}{p-1}} \right),$$

for any $r \in (0, r_0)$, $\varrho \in (0, r/2)$. Thus, the desired inequality (12.15) follows.

Let us fix ξ_0 on the contact set. We want to show that, for any $r \in (0, R)$, there exist two positive constants $\sigma \in (0,1)$ and c such that, for $B_r \equiv B_r(\xi_0)$,

$$\operatorname*{osc}_{B_{\sigma r}} u + \operatorname{Tail}(u - h(\xi_0); \xi_0, \sigma r)$$

$$\leq \frac{1}{2}\left(\operatorname*{osc}_{B_r} u + \operatorname{Tail}(u-h(\xi_0);\xi_0,r)\right) + c\,\omega_h(r) + c\,\chi. \tag{12.17}$$

with χ defined in (12.16). Let us then begin observing that $u \geq d := h(\xi_0) - \omega_h(r)$ almost everywhere in B_r. So let us set $u_d := u - d$. By Theorem 12.2 we have that u_d is a nonnegative weak supersolution in B_r with datum $f_d(\xi,\cdot) := f(\xi,\cdot + d)$. We can proceed as in Theorem 12.7 to derive the boundedness estimate (12.10) for u_d, which, with $m = d + 2\omega_h(r) \geq \sup_{B_{2\varrho}} h$, gives

$$\sup_{B_\varrho} u_d \leq 2\omega_h(r) + \delta\ \operatorname{Tail}((u_d)_+;\xi_0,\varrho) + c\ \delta^{-\gamma}\left(\fint_{B_{2\varrho}} u_d^t\,\mathrm{d}\xi\right)^{\frac{1}{t}} + \varrho^{\frac{sp}{p-1}}\|f\|_{L^\infty(B_{2\varrho})}^{\frac{1}{p-1}}, \tag{12.18}$$

for $\varrho \in (0,r]$, $t \in (0,p)$ and $\delta \in (0,1]$. Moreover, since u_d is a nonegative weak supersolution of (8.1) with datum f_d and $\|f_d\|_{L^\infty(B_r)} = \|f\|_{L^\infty(B_r)}$, Theorem 11.2 yields, exchanging the role of f_d and f,

$$\left(\fint_{B_{2\varrho}} u_d^t\,\mathrm{d}\xi\right)^{\frac{1}{t}} \leq c\inf_{B_{3\varrho}} u_d + c\left(\frac{\varrho}{r}\right)^{\frac{sp}{p-1}} \operatorname{Tail}((u_d)_-;\xi_0,r) + c\,\chi(\varrho),$$

where χ is defined in (12.16) and given by Theorem 11.2, $\varrho \in (0,r/3)$ and c depending on n, p, s and the structural constant Λ. Recalling that $\inf_{B_\varrho} u_d \leq \omega_h(r)$, due to $\inf_{B_r}(u-h) = 0$, and that $u_d \geq 0$ on B_r, combining the previous estimates we arrive at

$$\begin{aligned}\operatorname*{osc}_{B_\varrho} u &\leq c\,\delta^{-\gamma}\omega_h(r) + c\,\delta\operatorname{Tail}(u_d;\xi_0,\varrho)\\ &\quad + c\,\delta^{-\gamma}\left(\frac{\varrho}{r}\right)^{\frac{sp}{p-1}}\operatorname{Tail}(u_d;\xi_0,r) + c\,\delta^{-\gamma}\chi(\varrho),\end{aligned}$$

for $\varrho \in (0,r/3)$, $\delta \in (0,1)$, $c \equiv c(n,p,s,\Lambda) > 0$ and χ given by (12.16).

Observe now that

$$\operatorname{Tail}(u_d;\xi_0,\varrho) \leq c\sup_{B_r}|u_d| + c\left(\frac{\varrho}{r}\right)^{\frac{sp}{p-1}}\operatorname{Tail}(u_d;\xi_0,r), \tag{12.19}$$

and we can also estimate

$$\sup_{B_r}|u_d| = \sup_{B_r}|u - h(\xi_0) + \omega_h(r)| \leq \operatorname*{osc}_{B_r} u + 2\omega_h(r).$$

Hence, putting together all the estimates above we eventually obtain

$$\begin{aligned}\operatorname*{osc}_{B_\varrho} u &\leq c\,\delta\operatorname*{osc}_{B_r} u + c\,(\delta^{-\gamma}+2\delta)\omega_h(r)\\ &\quad + c\,\delta^{-\gamma}\left(\frac{\varrho}{r}\right)^{\frac{sp}{p-1}}\operatorname{Tail}(u_d;\xi_0,r)\\ &\quad + c\,\delta^{-\gamma}\chi(\varrho).\end{aligned}$$

For any $\varepsilon \in (0,1)$ we choose δ and $\tilde\sigma \in (0,1)$ such that

$$c\,\delta \leq \frac{\varepsilon}{2}, \qquad \text{and} \qquad c\,\delta^{-\gamma}\tilde\sigma^{\frac{sp}{p-1}} \leq \frac{\varepsilon}{2}.$$

Thus, for $\varrho := \tilde\sigma r$ it holds

$$\operatorname*{osc}_{B_{\tilde\sigma r}} u \leq \varepsilon\left(\operatorname*{osc}_{B_r} u + \operatorname{Tail}(u-h(\xi_0);\xi_0,r)\right) + c\,\omega_h(r) + c\,\chi(r) \tag{12.20}$$

Applying estimate (12.20) with (12.19), we can estimate for any $\sigma \in (0,\tilde\sigma)$

$$\operatorname{Tail}(u-h(\xi_0);\xi_0,\sigma r) \leq c\operatorname*{osc}_{B_{\tilde\sigma r}} u + c\left(\frac{\sigma}{\tilde\sigma}\right)^{\frac{sp}{p-1}}\operatorname{Tail}(u-h(\xi_0);\xi_0,\tilde\sigma r) + c\,\omega_h(r)$$

$$\leq c\,\varepsilon\left(\operatorname*{osc}_{B_r} u + \operatorname{Tail}(u-h(\xi_0);\xi_0,r)\right) + c\,\omega_h(r) + c\,\chi(r)$$
$$+ c\left(\frac{\sigma}{\tilde\sigma}\right)^{\frac{sp}{p-1}}\left(\operatorname*{osc}_{B_r} u + \operatorname{Tail}(u-h(\xi_0);\xi_0,r)\right).$$

Adding now (12.20) and choosing σ and ε such that

$$c\left(\frac{\sigma}{\tilde\sigma}\right)^{\frac{sp}{p-1}} \leq \varepsilon \quad \text{and} \quad (c+2)\varepsilon \leq \frac{1}{2},$$

we obtain (12.17).

Iterating (12.17) we have that, for any $k \in \mathbb{N}$, it holds

$$\operatorname*{osc}_{B_{\sigma^k r}} u + \operatorname{Tail}(u-h(\xi_0);\xi_0,\sigma^k r)$$
$$\leq 2^{1-k}\left(\operatorname*{osc}_{B_{\sigma r}} u + \operatorname{Tail}(u-h(\xi_0);\xi_0,\sigma r)\right)$$
$$+ c\sum_{j=0}^{k-2} 2^{-j}\omega_h(\sigma^{k-j-1}r) + c\sum_{j=0}^{k-2} 2^{-j}\chi(\sigma^{k-j-1}r).$$

After noticing that $\operatorname{osc}_{B_r} u = \operatorname{osc}_{B_r} u_d \leq \sup_{B_r} u_d$ and using the supremum estimate (12.18), the inequality above yields

$$\operatorname*{osc}_{B_{\sigma^k r}} u + \operatorname{Tail}(u-h(\xi_0);\xi_0,\sigma^k r)$$
$$\leq c\,2^{1-k}\left(\operatorname{Tail}(u-h(\xi_0);\xi_0,r) + \left(\fint_{B_r} |u-h(\xi_0)|^t\,\mathrm{d}\xi\right)^{\frac{1}{t}}\right)$$
$$+ c\sum_{j=0}^{k-1} 2^{-j}\omega_h(\sigma^{k-j-1}r) + c\sum_{j=0}^{k-1} 2^{-j}\chi(\sigma^{k-j-1}r).$$

Thus, estimate (12.15) follows in the case of the contact set for some exponent $\alpha \leq -\log 2/\log\sigma$ and some easy manipulation. □

The main consequence of the theorem above is the following corollary whose proof follows directly from the one in [85, Theorem 4] using (12.15).

Corollary 12.9. *Let $s \in (0,1)$ and $p \in (1,\infty)$ and let us suppose that h is continuous in Ω or $h \equiv -\infty$. Then, under assumptions (12.2), (12.3) and (12.4), the solution to the obstacle problem in $\mathcal{K}_{g,h}(\Omega,\Omega')$ is continuous in Ω as well.*

2.2. Boundary regularity. We now prove that (up to the boundary) the solution to the obstacle problem inherits the regularity of the obstacle.

We begin stating some helpful lemmas. The first one is a Caccioppoli-type inequality whose proof is a verbatim repetition of the one presented in Theorem 10.1, whose proof strategy goes back to [47, 48].

Lemma 12.10. *Let $s \in (0,1)$, $p \in (1,\infty)$, and, under assumptions (12.2), (12.3) and (12.4), let $u \in \mathcal{K}_{g,h}(\Omega,\Omega')$ be the solution to the obstacle problem in $\mathcal{K}_{g,h}(\Omega,\Omega')$. Let $\xi_0 \in \partial\Omega$ and let $r \in (0, \operatorname{dist}(\xi_0,\partial\Omega'))$, and suppose that, for $B_r \equiv B_r(\xi_0)$,*

$$k_+ \geq \max\left(\sup_{B_r} g,\ \sup_{B_r\cap\Omega} h\right) \quad \text{and} \quad k_- \leq \inf_{B_r} g.$$

Then, for $w_\pm := (u-k_\pm)_\pm$, *we have*

$$\int_{B_r}\int_{B_r}\frac{|w_\pm(\xi)\varphi(\xi)-w_\pm(\eta)\varphi(\eta)|^p}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}\,\mathrm{d}\xi\mathrm{d}\eta$$
$$\le c\int_{B_r}\int_{B_r}\frac{w_\pm^p(\xi)|\varphi(\xi)-\varphi(\eta)|^p}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}\,\mathrm{d}\xi\mathrm{d}\eta$$
$$+c\int_{B_r}w_\pm(\xi)\varphi^p(\xi)\,\mathrm{d}\xi\left(\sup_{\eta\in\operatorname{supp}\varphi}\int_{\mathbb{H}^n\setminus B_r}\frac{w_\pm^{p-1}(\xi)}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}\,\mathrm{d}\xi+\|f\|_{L^\infty(B_r)}\right),$$

for any $\varphi\in C_0^\infty(B_r)$ *with* $0\le\varphi\le 1$ *and* c *depending on* n, p, s *and the structural constant* Λ.

We remark that if $\max(\sup_{B_r} g, \sup_{B_r\cap\Omega} h)$ is infinite or $\inf_{B_r} g = -\infty$ then the interpretation is that no test function of the type $w_\pm$ exists.

Let us note that since we have a Caccioppoli-type estimate on the boundary of the set Ω we can prove a local boundedness estimate following the same iteration argument already used for the proof of Theorem 10.5.

Theorem 12.11. *Suppose that, under assumptions* (12.2), (12.3) *and* (12.4), $u\in\mathcal{K}_{g,h}(\Omega,\Omega')$ *solves the obstacle problem in* $\mathcal{K}_{g,h}(\Omega,\Omega')$ *and let* $s\in(0,1)$ *and* $p\in(1,\infty)$. *Let* $\xi_0\in\partial\Omega$ *and* $r\in(0,\operatorname{dist}(\xi_0,\partial\Omega)$. *Assume that, for* $B_r\equiv B_r(\xi_0)$,

$$\sup_{B_r} g+\sup_{B_r\cap\Omega} h<\infty\quad\text{and}\quad\inf_{B_r} g>-\infty.$$

Then, u *is essentially bounded close to* ξ_0.

In order to prove the Hölder continuity of the solution to the obstacle problem on the boundary a logarithmic-type estimate is needed. Then, following the same argument of Lemma 10.3, we have

Lemma 12.12. *Suppose that, under assumptions* (12.2), (12.3) *and* (12.4), $u\in\mathcal{K}_{g,h}(\Omega,\Omega')$ *solves the obstacle problem in* $\mathcal{K}_{g,h}(\Omega,\Omega')$ *with* $s\in(0,1)$ *and* $p\in(1,\infty)$. *Let* $B_R\equiv B_R(\xi_0)\Subset\Omega'$, $B_r\subset B_{R/2}$ *and assume that*

$$+\infty>k_+\ge\max\left(\sup_{B_R} g,\sup_{B_R\cap\Omega} h\right)\quad\text{and}\quad-\infty<k_-\le\inf_{B_R} g.$$

Then the functions

$$w_\pm := \sup_{B_R}(u-k_\pm)_\pm-(u-k_\pm)_\pm+\varepsilon$$

satisfy the following estimate

$$\int_{B_r}\int_{B_r}\left|\log\frac{w_\pm(\xi)}{w_\pm(\eta)}\right|^p d_o(\eta^{-1}\circ\xi)^{-Q-sp}\,\mathrm{d}\xi\mathrm{d}\eta$$
$$\le c\,r^{Q-sp}\left(1+r^{sp}\varepsilon^{1-p}\|f\|_{L^\infty(B_R)}+\varepsilon^{1-p}\left(\frac{r}{R}\right)^{sp}\operatorname{Tail}((w_\pm)_-;\xi_0,R)^{p-1}\right),$$

for every $\varepsilon>0$ *and* c *depending on* n, p, s *and the structural constant* Λ.

We can now prove the following result.

Lemma 12.13. *Assume that* $\xi_0=0\in\partial\Omega$ *and* $g(0)=0$, *where* Ω *satisfies* (12.5) *for all* $r\le R$. *Suppose that, under assumptions* (12.2), (12.3) *and* (12.4), $u\in\mathcal{K}_{g,h}(\Omega,\Omega')$ *solves the obstacle problem in* $\mathcal{K}_{g,h}(\Omega,\Omega')$ *with* $s\in(0,1)$ *and* $p\in(1,\infty)$. *Let* $\omega>0$. *Then, there exist* $\tau_0\in(0,1)$, $\sigma\in(0,1)$ *and* $\vartheta\in(0,1)$, *all depending on* n, p, s *and* δ_Ω, *such that if*

$$\operatorname*{osc}_{B_R} u+\sigma\operatorname{Tail}(u;0,R)\le\omega,\quad\operatorname*{osc}_{B_R} g\le\frac{\omega}{8},\quad\text{and}\quad\|f\|_{L^\infty(B_R)}\le(\vartheta\omega)^{p-1},\tag{12.21}$$

hold for $B_R \equiv B_R(0)$, then the following decay estimate

$$\operatorname*{osc}_{B_{\tau R}} u + \sigma \operatorname{Tail}(u; 0, \tau R) \leq (1-\vartheta)\omega,$$

holds as well for every $\tau \in (0, \tau_0]$.

We will use the following lemma, whose proof can be done extending the one presented in [85, Lemma 7] and using the Poincaré inequality in Proposition 9.2.

Lemma 12.14. *Let us consider a set Ω which satisfies condition (12.5) for $r_0 > 0$ and $\delta_\Omega > 0$ and let $B_r \equiv B_r(\xi_0)$ with $\xi_0 \in \partial\Omega$ and $r \in (0, r_0)$. Let us suppose that $\psi \in W^{s,p}(B_r)$ and $\psi \equiv 0$ in $B_r \setminus \Omega$. Then*

$$\fint_{B_r} |\psi|^p \, \mathrm{d}\xi \leq c \left(1 - (1-\delta_\Omega)^{1-1/p}\right)^{-p} r^{sp} \int_{B_r} \fint_{B_r} \frac{|\psi(\xi) - \psi(\eta)|^p}{|\eta^{-1} \circ \xi|_{\mathbb{H}^n}^{Q+sp}} \, \mathrm{d}\xi \mathrm{d}\eta.$$

Proof of Lemma 12.13. Let us begin denoting with $H = \vartheta/\sigma$ and $B \equiv B_R(0)$. We can estimate the tail term as follows

$$\begin{aligned}
\sigma^{p-1} \operatorname{Tail}(u; 0, \tau R)^{p-1} &= \sigma^{p-1} (\tau R)^{sp} \int_{B \setminus \tau B} \frac{|u(\xi)|^{p-1}}{|\xi|_{\mathbb{H}^n}^{Q+sp}} \, \mathrm{d}\xi \\
&\quad + \sigma^{p-1} \tau^{sp} \operatorname{Tail}(u; 0, R)^{p-1} \\
&\leq c \, \sigma^{p-1} \omega^{p-1} + \tau^{sp} \omega^{p-1}.
\end{aligned}$$

Hence,

$$\sigma \operatorname{Tail}(u; 0, \tau R) \leq c \left(\frac{\vartheta}{H} + \tau^{\frac{sp}{p-1}} \right) \omega \leq \frac{2c\vartheta}{H} \omega \leq \vartheta \omega$$

where we have chosen $\tau_0 \leq \sigma^{(p-1)/(sp)}$ and $H = 2c$. Thus, we only have to show

$$\operatorname*{osc}_{\tau B} u \leq (1 - 2\vartheta)\omega, \qquad \forall \tau \leq \tau_0. \tag{12.22}$$

Let us adopt the following notation

$$k_+ := \sup_B u - \frac{\omega}{4}, \quad , k_- := \inf_B u + \frac{\omega}{4}, \quad \varepsilon := \vartheta\omega,$$

and

$$w_\pm := \sup_B (u - k_\pm)_\pm - (u - k_\pm)_\pm + \varepsilon, \quad \tilde{w}_\pm := \frac{w_\pm}{\sup_B w_\pm}.$$

Clearly we can assume that $\sup_B u \geq \frac{3}{8}\omega$ and $\inf_B u \leq -\frac{3}{8}\omega$ since if the reverse inequalities hold true we have $\operatorname{osc}_B u \leq \frac{3}{4}\omega$ and (12.22) follows choosing $\vartheta \leq 1/8$.

We consider the case $\sup_B u \geq \frac{3}{8}\omega$ being the other one symmetric. Let us note that due to the condition $u = g \leq \omega/8$ in $B \setminus \Omega$ then $\tilde{w}_+ = 1$ in $B \setminus \Omega$. Hence, we apply Lemma 12.14 and Lemma 12.12 together with (12.21), with $r = 2\tau R$ and $\tau_0 \leq \min(1/4, \sigma^{2(p-1)/sp})$, obtaining

$$\begin{aligned}
\fint_{2\tau B} |\log \tilde{w}_+|^p \, \mathrm{d}\xi &\leq c \, (\tau R)^{sp} \int_{2\tau B} \fint_{2\tau B} \left| \log \frac{\tilde{w}_+(\xi)}{\tilde{w}_+(\eta)} \right|^p d_{\mathrm{o}}(\eta^{-1} \circ \xi)^{-Q-sp} \, \mathrm{d}\xi \mathrm{d}\eta \\
&\leq c \left(1 + (2\tau R)^{sp} \varepsilon^{1-p} \|f\|_{L^\infty(2\tau B)} + (\vartheta\omega)^{1-p} (2\tau)^{sp} \operatorname{Tail}((\tilde{w}_+)_-; 0, R)^{p-1} \right) \\
&\leq c.
\end{aligned}$$

Combining Chebyshev's Inequality with the previous estimate we get

$$\begin{aligned}
\frac{|2\tau B \cap \{|\log \tilde{w}_+| \geq |\log(20\vartheta)|\}|}{|2\tau B|} &\leq |\log(20\vartheta)|^{-p} \fint_{2\tau B} |\log \tilde{w}_+|^p \, \mathrm{d}\xi \\
&\leq c \, |\log(20\vartheta)|^{-p}. \qquad (12.23)
\end{aligned}$$

We want to estimate now the left-hand side of (12.23). Recall that, by definition $0 < \tilde{w}_+ \le 1$ and $\sup_B (u-k_+)_+ = \omega/4$, we obtain that, when $\vartheta < 1/20$,

$$\begin{aligned}\{|\log \tilde{w}_+| \ge |\log(20\vartheta)|\} &= \{\tilde{w}_+ \le 20\vartheta\} \\ &= \left\{ \frac{\omega}{4} - (u-k_+)_+ + \varepsilon \le 20\vartheta \left(\frac{\omega}{4} + \varepsilon\right) \right\} \supset \left\{ u \ge \sup_B u - 4\vartheta\omega \right\}.\end{aligned}$$

Hence, denoting with $\tilde{k} := \sup_B u - 4\vartheta\omega$ we eventually arrive at

$$\left(\fint_{2\tau B} (u-\tilde{k})_+^p \, \mathrm{d}\xi \right)^{\frac{1}{p}} \le 4\vartheta\omega \left(\frac{|2\tau B \cap \{u \ge \sup_B u - 4\vartheta\omega\}|}{|2\tau B|} \right)^{\frac{1}{p}} \le \frac{c\vartheta\omega}{|\log(20\vartheta)|}.$$

Since $\tilde{k} \ge \sup_B g$ by Theorem 12.11 we obtain that

$$\sup_{\tau B} (u-\tilde{k})_+ \le \delta \operatorname{Tail}((u-\tilde{k})_+; 0, \tau R) + c\,\delta^{-\gamma} \left(\fint_{2\tau B} (u-\tilde{k})_+^p \, \mathrm{d}\xi \right)^{\frac{1}{p}},$$

for any $\delta \in (0,1]$. Thus,

$$\sup_{\tau B} u \le \sup_B u - 4\vartheta\omega + \delta \operatorname{Tail}((u-\tilde{k})_+; 0, \tau R) + \frac{c\delta^{-\gamma}\vartheta\omega}{|\log(20\vartheta)|}. \tag{12.24}$$

We are left to estimating the tail term

$$\begin{aligned}\operatorname{Tail}((u-\tilde{k})_+; 0, \tau R)^{p-1} &\le (\tau R)^{sp} \int_{B \setminus \tau B} \frac{(u-\tilde{k})_+^{p-1}}{|\xi|_{\mathbb{H}^n}^{Q+sp}} \, \mathrm{d}\xi \\ &\quad + \tau^{sp} \operatorname{Tail}((u-\tilde{k})_+; 0, R)^{p-1} \\ &\le c\,(\vartheta\omega)^{p-1} \left(1 + \frac{\tau^{sp}}{\vartheta^{p-1}\sigma^{p-1}} \right) \le c(\vartheta\omega)^{p-1},\end{aligned}$$

where we have taken $\tau^{sp} \le \tau_0^{sp} \le (\sigma\vartheta)^{p-1}$ and we have used condition (12.21). Then, up to taking δ and ϑ sufficiently small, from (12.24), we have that

$$\sup_{\tau B} \le \sup_B u - 2\vartheta\omega,$$

and condition (12.22) follows as desired. □

We are now able to prove our main result.

Theorem 12.15. *Under assumptions (12.2), (12.3) and (12.4) let $u \in \mathcal{K}_{g,h}(\Omega, \Omega')$ be the solution to the obstacle problem in $\mathcal{K}_{g,h}(\Omega, \Omega')$ with $s \in (0,1)$ and $p \in (1, \infty)$ and assume $\xi_0 \in \partial\Omega$ and $B_{2R}(\xi_0) \subset \Omega'$. If $g \in \mathcal{K}_{g,h}(\Omega, \Omega')$ is Hölder continuous in $B_R(\xi_0)$ and Ω satisfies (12.5) for all $r \le R$, then u is Hölder continuous in Ω as well.*

Proof. With no loos of generality we may assume that $\xi_0 = 0$ and $g(0) = 0$. Moreover, we can suppose that there exists R_0 such that $\operatorname{osc}_{B_0} g \le \operatorname{osc}_{B_0} u$ for $B_0 \equiv B_{R_0}(0)$ since otherwise the thesis follows by the Hölder continuity of g.

Let us define now, for $\vartheta \in (0,1)$,

$$\omega_0 := 8 \left(\operatorname*{osc}_{B_0} u + \operatorname{Tail}(u; 0, R_0) + \frac{\|f\|_{L^\infty(B_0)}^{\frac{1}{p-1}}}{\vartheta} \right). \tag{12.25}$$

By Lemma 12.13 we have that there exist τ_0, σ and ϑ depending only on n, p, s and δ_Ω such that if

$$\operatorname*{osc}_{B_r(0)} u + \sigma \operatorname{Tail}(u; 0, r) \le \omega \quad \operatorname*{osc}_{B_r(0)} g \le \frac{\omega}{8} \quad \text{and} \quad \|f\|_{L^\infty(B_r(0))} \le (\vartheta\omega)^{p-1}, \tag{12.26}$$

hold for a ball $B_r(0)$ and for $\omega > 0$, then

$$\operatorname*{osc}_{B_{\tau r}(0)} u + \sigma \operatorname{Tail}(u; 0, \tau r) \leq (1-\vartheta)\omega, \tag{12.27}$$

holds for every $\tau \in (0, \tau_0]$. Let us take $\tau \leq \tau_0$ such that

$$\operatorname*{osc}_{\tau^j B_0} g \leq (1-\vartheta)^j \frac{\omega_0}{8}, \quad \text{and} \qquad \|f\|_{L^\infty(\tau^j B_0)} \leq \Big(\vartheta(1-\vartheta)^j \omega_0\Big)^{p-1}, \quad \text{for every } j = 0, 1, \dots \tag{12.28}$$

We now iterate (12.27) with (12.26) and (12.28), noticing that the initial condition is satisfied if we choose ω_0 as in (12.25). Thus, we obtain that

$$\operatorname*{osc}_{\tau^j B_0} u \leq (1-\vartheta)^j \omega_0 \quad \text{for every } j = 0, 1, \dots$$

from which follows that $u \in C^{0,\alpha}_{\mathrm{loc}}(B_0)$ with exponent $\alpha = \log(1-\vartheta)/\log \tau \in (0,1)$. □

From the previous theorem we can easily show that the following result holds true as well.

Theorem 12.16. *Let $s \in (0,1)$, $p \in (1, \infty)$ and, under assumptions (12.2), (12.3) and (12.4), let u solves the obstacle problem in $\mathcal{K}_{g,h}(\Omega, \Omega')$. Assume $\xi_0 \in \partial\Omega$ and $B_{2R}(\xi_0) \subset \Omega'$. If $g \in \mathcal{K}_{g,h}(\Omega, \Omega')$ is continuous in $B_R(\xi_0)$ and Ω satisfies condition (12.5) for all $r \leq R$, then u is continuous in $B_R(\xi_0)$ as well.*

Combining together Theorem 12.8, Corollary 12.9, Theorem 12.15 and Theorem 12.16 we obtain Theorem 12.4.

CHAPTER 13

Perron method for the Dirichlet problem

Armed with the results established in the previous chapters now we focus on extending the classical Perron Method for the Dirichlet problem (8.1).

As is well known, this method is a classical procedure in Potential Theory which provides the existence of a solution for the Dirichlet problem related to the Laplace equation in an arbitrary open set $\Omega \subset \mathbb{R}^n$ and for any boundary datum g, without "regularity assumption". It simply works by finding the least superharmonic function greater or equal to the datum g on the boundary $\partial\Omega$ and it can be applied once knowing some helpful properties of superharmonic functions, such as comparison and maximum principles, and some barriers techniques. Some generalization of the Perron method are available in the literature when one considers instead of the classical Laplace operator its nonlinear counterpart. In this sense, we refer the reader to the paper [70] by Granlund, Lindqvist and Martio and to the monograph [74] by Heinonen, Kilpeläinen and Martio and the references therein, for the study of more general nonlinear operators, whose prototype is the p-Laplacian.

We aim to extend the existence results for the generalized Perron solution proved in [97, 86] for nonlocal and nonlinear operators in the Heisenberg group.

We start giving the important definition of Perron generalized solutions.

Definition 13.1 (Perron solutions)**.** *Let Ω be an open and bounded subset of $\mathbb{H}^n$ and assume that $g \in W^{s,p}(\mathbb{H}^n)$. We define the* upper class $\mathscr{U}_g$ *as the family of all functions u such that*

(i) $u : \mathbb{H}^n \to (-\infty, +\infty]$ *is lower semicontinuous in Ω and there exists $h \in L^{p-1}_{sp}(\mathbb{H}^n)$ such that $u \le h$;*

(ii) *Given $D \Subset \Omega$ and $v \in C(\overline{D})$ a weak solution in D to (13.5) such that $v \le u$ almost everywhere in $\mathbb{H}^n \setminus D$ and*

$$v(\xi) \le \liminf_{\eta \to \xi} u(\eta), \qquad \forall \xi \in \partial D.$$

Then, $v \le u$ a. e. in D as well;

(iii) $\liminf_{\substack{\eta \to \xi \\ \eta \in \Omega}} u(\eta) \ge \operatorname{ess\,lim\,sup}_{\substack{\eta \to \xi \\ \eta \in \mathbb{H}^n \setminus \Omega}} g(\eta)$ *for all $\xi \in \partial\Omega$;*

(iv) $u \ge g$ *a. e. in $\mathbb{H}^n \setminus \Omega$.*

The lower class *is* $\mathscr{L}_g := \Big\{u : -u \in \mathscr{U}_{-g}\}$. *The function* $\overline{H}_g := \inf\{u : u \in \mathscr{U}_g\}$ *is the* upper Perron solution *with boundary datum g in Ω and* $\underline{H}_g := \sup\{u : u \in \mathscr{L}_g\}$ *is the* lower Perron solution *with boundary datum g in Ω.*

Some comments about the definition above must be made. First, let us note that by (i) it follows that $|u| \neq \infty$ a. e. in $\mathbb{H}^n$, otherwise it can not be $h \in L^{p-1}_{sp}(\mathbb{H}^n)$ such that $u \le h$ in $\mathbb{H}^n$. Moreover, the upper class $\mathscr{U}_g$ contains all lower semicontinuous weak supersolutions satisfying the boundary data. Analogously the same holds true for weak subsolutions and for the lower class $\mathscr{L}_g$.

In Definition 13.1 we have assumed the boundary datum g in $W^{s,p}(\mathbb{H}^n)$. In the lemma below we show that we can require less regularity on g in order to have a not empty upper class $\mathscr{U}_g$.

Lemma 13.2. *If $g \in L^{p-1}_{sp}(\mathbb{H}^n) \cap L^\infty(\mathbb{H}^n)$, then the upper class $\mathscr{U}_g$ is not empty.*

Proof. Let us choose a constant $M > 0$ such that

$$\sup_{\mathbb{H}^n} g + c\,\|f\|^{\frac{1}{p-1}}_{L^\infty(\Omega)} \le M < \infty, \tag{13.1}$$

with the constant c defined as follows

$$c^{-\frac{1}{p-1}} := 2 \min_{\xi\in\Omega} \int_{\mathbb{H}^n \setminus B_r} d_{\mathrm{o}}(\eta^{-1}\circ\xi)^{-Q-sp}\,\mathrm{d}\eta, \qquad \text{for } B_r : \Omega \Subset B_r.$$

Note that in (13.1) in the choice of the constant M we have also taken in consideration the non-homogeneity of the problem given by the presence of the supremum norm of the function f.

Take

$$u := M\,\chi_\Omega + g\,\chi_{\mathbb{H}^n\setminus\Omega} \in W^{s,p}_{\mathrm{loc}}(\Omega) \cap L^{p-1}_{sp}(\mathbb{H}^n).$$

Clearly Definition 13.1(i) and (iii-iv) are satisfied. In order to prove condition (ii) we show that u is a weak supersolution according to Definition 9.4.

Testing against a nonnegative test function $\psi \in C^\infty_0(\Omega)$ gives

$$\begin{aligned}
&\int_{\mathbb{H}^n}\int_{\mathbb{H}^n} \frac{|u(\xi)-u(\eta)|^{p-2}(u(\xi)-u(\eta))(\psi(\xi)-\psi(\eta))}{d_{\mathrm{o}}(\eta^{-1}\circ\xi)^{Q+sp}}\,\mathrm{d}\xi\mathrm{d}\eta \\
&\qquad = 2\int_{\mathbb{H}^n\setminus\Omega}\int_\Omega \frac{|M-u(\eta)|^{p-2}(M-u(\eta))\psi(\xi)}{d_{\mathrm{o}}(\eta^{-1}\circ\xi)^{Q+sp}}\,\mathrm{d}\xi\mathrm{d}\eta \\
&\qquad \ge 2\int_{\mathbb{H}^n\setminus B_r}\int_\Omega c^{p-1}\,\|f\|_{L^\infty(\Omega)}\psi(\xi) d_{\mathrm{o}}(\eta^{-1}\circ\xi)^{-Q-sp}\,\mathrm{d}\xi\mathrm{d}\eta \\
&\qquad \ge \int_\Omega f(\xi,u)\psi(\xi)\,\mathrm{d}\xi.
\end{aligned}$$

Hence, u is a weak supersolution in Ω to (8.1). Thus, it also satisfies the comparison principle in Ω. Then, $u \in \mathscr{U}_g$. □

The main result of this chapter will be the following one.

Theorem 13.3 (Resolutivity)**.** *Let $s \in (0,1)$ and $p \in (1,\infty)$ and assume that*

(13.2)
$f(\cdot,x)$ is measurable for all $x \in \mathbb{R}$ and $f(\xi,\cdot)$ is continuous for a. e. $\xi \in \mathbb{H}^n$;

(13.3)
$f \equiv f(\cdot,x) \in L^\infty_{\mathrm{loc}}(\mathbb{H}^n)$, for any $x \in \mathbb{R}$, uniformly in Ω;

(13.4)
$\big(f(\xi,x_1) - f(\xi,x_2)\big)(x_1-x_2) \le 0$, $\forall x_1,x_2 \in \mathbb{R}$ and for a. e. $\xi \in \mathbb{H}^n$.

Then, it holds that $\overline{H}_g = \underline{H}_g = H_g$, with $\overline{H}_g, \underline{H}_g$.
Moreover, H_g is a continuous weak solution in Ω to

$$\begin{cases} \mathcal{L}u = f(\cdot,u) & \text{in } \Omega, \\ u = g & \text{in } \mathbb{H}^n \setminus \Omega, \end{cases} \tag{13.5}$$

according to Definition 9.4.

A few further comments are in order. We remark that for the proof of Theorem 13.3 above an approach similar to the one used in [86], where the class of (s,p)-superharmonic functions is defined, is not possible. Indeed, due to the presence of the function $f \equiv f(\cdot, u)$ in the Dirichlet problem (13.5), most of the properties of (s,p)-superharmonic functions, holding in the classical Euclidean and homogeneous case, are not easily extendable to our context. This has led us to generalized the strategy used in [97] where the author studied the non-homogeneous problem when $f = f(x)$. The dependence of the function f of the solution u is a novelty with respect to the case considered in [97] and this feature changes drastically the background of the problem we are dealing with. Indeed, we are forced to assume (13.4) in order to prove some classical basic technique in Potential Theory, such as comparison principle (see Proposition 13.7) and to apply the regularity results for weak solutions up to the boundary in Theorem 12.4. We remind that if $f = f(\xi)$ condition (13.4) is trivially satisfied and we plainly obtain the same results as the ones presented in [97]. Hence, our method is consistent.

1. Properties of fractional weak supersolutions

In this section we prove that weak supersolutions to (13.5) enjoy some properties such as boundedness estimate and comparison principle. Moreover, we give a characterization of their pointwise behaviour and recall some helpful facts about convergence of sequences of weak supersolutions that will be used in the proof of Theorem 13.3.

All the forthcoming results are of independent interest in the analysis of nonlinear fractional equations in the Heisenberg group. They are the natural extension of the ones proven in [86] for the homogeneous end Euclidean case.

We always assume that f satisfies conditions (12.2), (12.3), and (12.4). However, in most of the forthcoming properties condition (13.4) is not strictly necessary, thus most of the following propositions simply work requiring only the regularity hypothesis in (12.2) and (13.3). When not explicitly specified in the statements of the results we only assume that only this regularity conditions hold true.
Moreover, for the sake of readability in most of the forthcoming proofs we will adopt the following notation where we have adopted the notation

$$L(a,b) := |a-b|^{p-2}(a-b), \qquad a,b \in \mathbb{R}.$$

1.1. A priori bounds for weak supersolutions. We show that weak supersolutions satisfy a Caccioppoli-type inequality with tail (Lemma 13.5 below). This estimate will then imply an upper bound on the fractional norm for weak supersolutions, namely Lemma 13.6.

Let us recall the following result whose proof is omitted being a plain extension of its Euclidean counterpart [86, Lemma 3] and of Theorem 10.5.

Lemma 13.4. *Let $s \in (0,1)$, $p \in (1,\infty)$. Let u be a weak supersolution in Ω, $h \in L^{p-1}_{sp}(\mathbb{H}^n)$ and assume that $h \le u \le 0$ almost everywhere in $\mathbb{H}^n$. Then, for all $D \Subset \Omega$ there is a constant $c \equiv c(n,p,s,\Lambda,\Omega,D,h) > 0$ such that*

$$\inf_D u \ge -c.$$

From Theorem 10.1 we can deduce another Caccioppoli-type estimate for weak supersolutions.

Lemma 13.5. *Let $s \in (0,1)$, $p \in (1,\infty)$ and $M > 0$. Suppose that u is a weak supersolution in $B_{2r} \equiv B_{2r}(\xi_0)$ such that $u \le M$ in $B_{3r/2}$. Then, for a positive constant $c \equiv c(n,p,s,\Lambda)$, it holds*

$$\int_{B_r} \fint_{B_r} \frac{|u(\xi)-u(\eta)|^p}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}\,\mathrm{d}\xi\mathrm{d}\eta \le c\,\max(r^{-sp},1)\,H^p + c\,\|f\|_{L^\infty(B_{3r/2})}\,H,$$

where

$$H := M + \left(\fint_{B_{3r/2}} u_-^p(\xi)\,\mathrm{d}\xi\right)^{\frac{1}{p}} + \mathrm{Tail}(u_-;\xi_0,3r/2) + \left(\fint_{B_{3r/2}} f_-^{\frac{p}{p-1}}(\xi,u)\,\mathrm{d}\xi\right)^{\frac{1}{p}}.$$

Proof. Let us choose a cut-off function $\varphi \in C_0^\infty(B_{4r/3})$ such that $0 \le \varphi \le 1$ and $\varphi \equiv 1$ in B_r and $|D_H\varphi| \le c/r$. Setting $w := 2H - u$ and choosing as test function $\psi := w\varphi^p$ in Definition 9.4, we get

$$\begin{aligned}
&\frac{1}{|B_r|}\int_{\mathbb{H}^n} f(\xi,u)w(\xi)\varphi^p(\xi)\,\mathrm{d}\xi\\
&\le \frac{1}{|B_r|}\int_{\mathbb{H}^n}\int_{\mathbb{H}^n} \frac{L(u(\xi),u(\eta))\,\big(w(\xi)\varphi^p(\xi) - w(\eta)\varphi^p(\eta)\big)}{d_{\mathrm{o}}(\eta^{-1}\circ\xi)^{Q+sp}}\,\mathrm{d}\xi\mathrm{d}\eta\\
&= -\frac{1}{|B_r|}\int_{B_{3r/2}}\int_{B_{3r/2}} \frac{L(w(\xi),w(\eta))(w(\xi)\varphi^p(\xi) - w(\eta)\varphi^p(\eta))}{d_{\mathrm{o}}(\eta^{-1}\circ\xi)^{Q+sp}}\,\mathrm{d}\xi\mathrm{d}\eta\\
&\quad + \frac{2}{|B_r|}\int_{\mathbb{H}^n\setminus B_{3r/2}}\int_{B_{3r/2}} \frac{L(u(\xi),u(\eta))w(\xi)\varphi^p(\xi)}{d_{\mathrm{o}}(\eta^{-1}\circ\xi)^{Q+sp}}\,\mathrm{d}\xi\mathrm{d}\eta\\
&=: -I_1 + 2I_2. \qquad (13.6)
\end{aligned}$$

We estimate separately the integrals I_1 and I_2 in the righthand side of (13.6). We start with I_1. As in the proof of Theorem 10.1, assuming without losing generality, $w(\xi) \ge w(\eta)$ we can deduce that

$$\begin{aligned}
I_1 &\ge \frac{1}{c}\int_{B_{3r/2}} \fint_{B_{3r/2}} \frac{|u(\xi)-u(\eta)|^p}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}\,\big(\max(\varphi(\xi),\varphi(\eta))\big)^p\,\mathrm{d}\xi\mathrm{d}\eta\\
&\quad - c\int_{B_{3r/2}} \fint_{B_{3r/2}} (2H-u(\xi))^p \frac{|\varphi(\xi)-\varphi(\eta)|^p}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}\,\mathrm{d}\xi\mathrm{d}\eta\\
&\ge \frac{1}{c}\int_{B_{3r/2}} \fint_{B_{3r/2}} \frac{|u(\xi)-u(\eta)|^p}{|\eta^{-1}\circ\xi|_{\mathbb{H}^n}^{Q+sp}}\,\big(\max(\varphi(\xi),\varphi(\eta))\big)^p\,\mathrm{d}\xi\mathrm{d}\eta - c\,r^{-sp}H^p, \qquad (13.7)
\end{aligned}$$

where in the last inequality we have estimated the singular integral given by the homogeneous norm $|\cdot|_{\mathbb{H}^n}$ in the same spirit of Lemma 10.3.

For the second integral I_2 we have

$$\begin{aligned}
I_2 &\le c\int_{\mathbb{H}^n\setminus B_{3r/2}}\int_{B_{4r/3}} (H^{p-1} + u_-^{p-1}(\eta))(2H + u_-(\xi))|\eta^{-1}\circ\xi_0|_{\mathbb{H}^n}^{-Q-sp}\,\mathrm{d}\xi\mathrm{d}\eta\\
&\le c\,r^{-sp}H^p + c\,H\int_{\mathbb{H}^n\setminus B_{3r/2}} u_-^{p-1}(\eta)|\eta^{-1}\circ\xi_0|_{\mathbb{H}^n}^{-Q-sp}\,\mathrm{d}\eta\\
&\le c\,r^{-sp}H^p. \qquad (13.8)
\end{aligned}$$

We are left with estimating the term on the lefthand side of (13.6)

$$\begin{aligned}
\frac{1}{|B_r|}\int_{\mathbb{H}^n} f(\xi,u)w(\xi)\varphi^p(\xi)\,\mathrm{d}\xi &\ge -\frac{1}{|B_r|}\int_{B_{3r/2}} f_-(\xi,u)(2H+u_-(\xi))\,\mathrm{d}\xi\\
&\ge -c\left(H\fint_{B_{3r/2}} f_-(\xi,u)\,\mathrm{d}\xi + \fint_{B_{3r/2}} f_-(\xi,u)u_-(\xi)\,\mathrm{d}\xi\right)\\
&\ge -c\,H^p - c\,\|f\|_{L^\infty(B_{3r/2})}H. \qquad (13.9)
\end{aligned}$$

Combining (13.7), (13.8) and (13.9) we obtain the desired inequality. □

Putting together all the results above we can extend with no difficulty the proof used in [86, Lemma 5] to show a uniform bound in $W^{s,p}$ for weak supersolutions.

Lemma 13.6. *Let $s \in (0,1)$, $p \in (1,\infty)$, $M > 0$ and $h \in L^{p-1}_{sp}(\mathbb{H}^n)$ with $h \le M$. Let u be a weak supersolution in Ω such that $u \ge h$ almost everywhere in $\mathbb{H}^n$ and $u \le M$ almost everywhere in Ω. Then, for all $D \Subset \Omega$ there is a constant $c = c(n,p,s,\Lambda,\Omega,D,M,h) > 0$ such that*

$$\int_D \int_D \frac{|u(\xi)-u(\eta)|^p}{|\eta^{-1}\circ\xi|^{Q+sp}_{\mathbb{H}^n}}\,\mathrm{d}\xi\mathrm{d}\eta \le c.$$

1.2. Comparison principle for weak supersolutions. We show now that weak supersolutions enjoy a comparison principle. This result constitutes a fundamental tool in the whole PDE theory. The proof of the forthcoming Proposition differs to the class of problem considered in [86] due to the presence of the datum f. In particular, as already carefully explained in the Introduction, the monotonicity hypothesis (13.4) is necessary in order to derive the desired comparison principle.

Proposition 13.7 (Comparison principle)**.** *Let Ω be a bounded open subsets of $\mathbb{H}^n$. Let $u \in W^{s,p}(\Omega)$ be a weak supersolution to (13.5) in Ω and let $v \in W^{s,p}(\Omega)$ be a weak subsolution to (13.5) in Ω such that $v \le u$ almost everywhere in $\mathbb{H}^n \setminus \Omega$ and*

$$\limsup_{\eta\to\xi} v(\eta) \le \liminf_{\eta\to\xi} u(\eta), \qquad \text{for any } \xi \in \partial\Omega.$$

Then, under assumptions (13.2), (13.3) and (13.4), we have that $v \le u$ a. e. in Ω as well.

Proof. We begin noticing that there exists a compact set K such that $\{v > u\} \Subset K \Subset \Omega$. Hence, the function $\psi := (v-u)_+ \in W^{s,p}_0(K)$ is a proper test function. Considering the weak formulation both for v and u in Definition 9.4 with ψ defined above, we obtain

$$\int_{\mathbb{H}^n} f(\xi,v)\psi(\xi)\,\mathrm{d}\xi \ge \int_{\mathbb{H}^n}\int_{\mathbb{H}^n} \frac{L(v(\xi),v(\eta))(\psi(\xi)-\psi(\eta))}{d_{\mathrm{o}}(\eta^{-1}\circ\xi)^{Q+sp}}\,\mathrm{d}\xi\mathrm{d}\eta, \tag{13.10}$$

and

$$\int_{\mathbb{H}^n} f(\xi,u)\psi(\xi)\,\mathrm{d}\xi \le \int_{\mathbb{H}^n}\int_{\mathbb{H}^n} \frac{L(u(\xi),u(\eta))(\psi(\xi)-\psi(\eta))}{d_{\mathrm{o}}(\eta^{-1}\circ\xi)^{Q+sp}}\,\mathrm{d}\xi\mathrm{d}\eta. \tag{13.11}$$

Then, summing together (13.10) with (13.11) we eventually arrive at

$$\begin{aligned} 0 \le{}& \int_{\{v>u\}} (f(\xi,v)-f(\xi,u))(v(\xi)-u(\xi))\,\mathrm{d}\xi \\ &+ \iint_{\{v>u\}} \frac{\Big(L(u(\xi),u(\eta))-L(v(\xi),v(\eta))\Big)\Big(v(\xi)-u(\xi)-v(\eta)+u(\eta)\Big)}{d_{\mathrm{o}}(\eta^{-1}\circ\xi)^{Q+sp}}\,\mathrm{d}\xi\mathrm{d}\eta \\ &+ 2\int_{\{v\le u\}}\int_{\{v>u\}} \frac{\Big(L(u(\xi),u(\eta))-L(v(\xi),v(\eta)\Big)\Big(v(\xi)-u(\xi)\Big)}{d_{\mathrm{o}}(\eta^{-1}\circ\xi)^{Q+sp}}\,\mathrm{d}\xi\mathrm{d}\eta \\ &\le 0 \end{aligned} \tag{13.12}$$

By the monotonicity of the function $L(a,b)$ and by condition (12.4). Thus, we obtain that all terms in (13.12) must be 0. This implies that $\psi \equiv 0$ almost everywhere on $\{v > u\}$. Hence, we obtain that $|\{v > u\}| = 0$. □

1.3. Pointwise behaviour of weak supersolutions. Let us focus now on the pointwise behaviour of weak supersolutions. Before stating our main result, i. e. Theorem 13.8, some considerations are needed. First of all, due to the presence of the datum f, we could not trivially prove that the function $v := u(\xi_0) - u$ for any $\xi_0 \in \Omega$ is a weak subsolution to (13.5), as done for the homogeneous case in [86]. Nevertheless such a function satisfies a Caccioppoli-type inequality with tail which is the starting point in proving a boundedness estimate as Theorem 10.5. Inequality (10.24) will have a central role in proving the forthcoming theorem. Indeed, thanks to the its ductility, we can choose a proper interpolation parameter $\delta \in (0, 1]$ in order to carefully estimate the nonlocal contributions and the local ones. Extending the procedure used in [86] we have

Theorem 13.8. *Let $s \in (0, 1)$ and $p \in (1, \infty)$ and let u be a weak supersolution in Ω to problem (13.5). Then we have*

$$u(\xi) := \operatorname{ess}\liminf_{\eta \to \xi} u(\eta), \qquad \text{for a. e. } \xi \in \Omega.$$

In particular, u has a lower semicontinuous representative.

1.4. Convergence result for weak supersolutions. We conclude this section recalling some convergence results for sequences of weak supersolutions from the Euclidean setting. This properties will turn out to be of crucial importance in the following section.

Theorem 13.9. *Let $s \in (0, 1)$ and $p \in (1, \infty)$. Consider $g, h \in L^{p-1}_{sp}(\mathbb{H}^n)$ such that $h \le g$ in $\mathbb{H}^n$. Let $\{u_j\}$ be a sequence of weak supersolutions in Ω such that $h \le u_j \le g$ almost everywhere in $\mathbb{H}^n$ and u_j is uniformly essentially bounded from above in Ω. Let*

$$u(\xi) := \lim_{j \to \infty} u_j(\xi), \qquad \text{for a. e. } \xi \in \mathbb{H}^n.$$

Then, under assumptions (13.2), (13.3) and (13.4), u is a weak supersolution in Ω.

The proof of the previous theorem is omitted since it is a verbatim repetition of the one presented in [86, Theorem 10]. Indeed, thanks to the regularity assumption made on the datum f in (13.2) and (13.3), we can easily pass to the limit on the righthand side of the integral inequality in Definition 9.4. The lefthhand side can then be treated as in the Euclidean case presented in [86]. As main consequence of the previous theorem we have

Corollary 13.10. *Let $s \in (0, 1)$, $p \in (1, \infty)$, h, $g \in L^{p-1}_{sp}(\mathbb{H}^n)$ and let $\{u_j\}$ be a sequence of continuous weak solutions in Ω such that $h \le u_j \le g$ and $u := \lim_{j \to \infty} u_j$ exists almost everywhere in $\mathbb{H}^n$. Then, under assumptions (13.2), (13.3) and (13.4), u exists at every point of Ω and is a continuous weak solution in Ω.*

2. Perron's Method

We have everything needed to prove Theorem 13.3. Let us remark that we assume as main conditions $s \in (0, 1)$, $p \in (1, \infty)$ and f satisfying conditions (13.2), (13.3) and (13.4), without recalling them every time in the statements of the results presented below.

The first result is proving that our method is consistent. Indeed, let us show that if problem (13.5) admits a weak solution h_g in Ω, then

$$\overline{H}_g = h_g = \underline{H}_g.$$

Lemma 13.11. *Let us assume that $h_g \in C(\overline{\Omega})$ is a weak solution in Ω to (13.5) such that*

$$\lim_{\substack{\eta\to\xi\\ \eta\in\Omega}} h_g(\eta) = g(\xi) \text{ for every } \xi\in\partial\Omega \qquad \text{and} \qquad h_g = g \text{ a. e. in } \mathbb{H}^n\setminus\Omega,$$

for some $g\in C(\Omega')\cap L^{p-1}_{sp}(\mathbb{H}^n)$ with $\Omega \Subset \Omega'$. Then, $\overline{H}_g = h_g = \underline{H}_g$.

Proof. We prove the lemma for the upper solution $\overline{H}_g$ being the other case symmetric. Clearly h_g belongs to $\mathscr{U}_g$. Hence, $h_g \geq \overline{H}_g$. To show that the reverse inequality holds true let $u\in\mathscr{U}_g$. Then, for any $\varepsilon>0$ there exists an open set $D\Subset\Omega$ such that $u+\varepsilon > h_g$ almost everywhere in $\mathbb{H}^n\setminus D$ and $h_g \leq \liminf_{\eta\to\xi} u(\eta)+\varepsilon$ for any $\xi\in\partial D$. Hence, by comparison principle we have that $u+\varepsilon\geq h_g$ almost everywhere in D as well. Taking the infimum over $\varepsilon>0$ and then over $u\in\mathscr{U}_g$ we have that $\overline{H}_g\geq h_g$ for a. e. $\xi\in\Omega$. By the continuity of h_g and $\overline{H}_g$ (Proposition 13.14 below) we have $\overline{H}_g\geq h_g$ for any $\xi\in\Omega$. □

In order to prove Theorem 13.3, we have to introduce the *Poisson modification* of a function u in a set D.

Definition 13.12 (Poisson modification of a continuous function; Definition 5.4 in [116]). *Let us consider a set $D\Subset\Omega$ such that $\mathbb{H}^n\setminus D$ satisfies the measure density condition (12.5) and let $u\in C(\overline{D})\cap L^{p-1}_{sp}(\mathbb{H}^n)$. We call* Poisson modification *of u in the domain D the solution to the obstacle problem in D with boundary datum u, i. e. $\mathcal{P}_{u,D}\in\mathcal{K}_{u,-\infty}(D,\Omega)\cap C(\overline{D})$.*

Note that by Corollary 12.3, we have that $\mathcal{P}_{u,D}$ is the weak solution of the following problem

$$\begin{cases} \mathcal{L}w = f, & \text{in } D,\\ w = u, & \text{in } \mathbb{H}^n\setminus D. \end{cases}$$

By a density argument we extend the Definition 13.12 to all functions belonging to $\mathscr{U}_g$.

Proposition 13.13. *Let $u\in\mathscr{U}_g$. Let $D\Subset\Omega$ be an open subset of Ω such that $\mathbb{H}^n\setminus D$ satisfies the measure density condition (12.5). Then, the Poisson modification of u in D satisfies*

(i) $\mathcal{P}_{u,D}\leq u$;
(ii) $\mathcal{P}_{u,D}$ *is a continuous weak solution to (13.5) in D;*
(iii) $\mathcal{P}_{u,D}\in\mathscr{U}_g$.

Proof. Applying the same procedure in [86, Lemma 9] we can construct an increasing sequence $\{u_j\}$, $u_j\in C(\overline{D})$, of weak supersolutions to (13.5) in D, converging to u pointwise in $\mathbb{H}^n$. Clearly by Definition 13.1(i) we have $u_j\leq u\leq h$, with $h\in L^{p-1}_{sp}(\mathbb{H}^n)$.

Define the Poisson modification of the function u_j in the set D, $\mathcal{P}_{u_j,D}\in\mathcal{K}_{u_j,-\infty}(D,\Omega)\cap C(\overline{D})$ and denote with

$$\mathcal{P}_{u,D} := \lim_{j\to\infty}\mathcal{P}_{u_j,D}.$$

As consequence of the comparison principle in Proposition 13.7 and the fact that the sequence $\{u_j\}$ is increasing we have that $\{\mathcal{P}_{u_j,D}\}$ is increasing as well and $\mathcal{P}_{u_j,D}\leq u$.

Hence, the pointwise limit

$$\mathcal{P}_{u,D}(\xi) = \lim_{j\to\infty}\mathcal{P}_{u_j,D}(\xi),$$

exists and satisfies $\mathcal{P}_{u,D}\leq u$. Moreover, the pointwise limit is lower semicontinuous.

Since the sequence $\{\mathcal{P}_{u_j,D}\}$ is increasing and bounded above by u, we have that for any $j \in \mathbb{N}$, $\mathcal{P}_{u_1,D} \le \mathcal{P}_{u_j,D} \le h$, with $\mathcal{P}_{u_1,D}, h \in L^{p-1}_{sp}(\mathbb{H}^n)$. Thus, since is the limit of an increasing sequence of weak solutions in D, by Corollary 13.10, we obtain that $\mathcal{P}_{u,D}$ is a continuous weak solution in D. Hence, Proposition 13.13(i-ii) follow.

We have to prove that $\mathcal{P}_{u,D} \in \mathscr{U}_g$. Note that we only have to prove Definition 13.1(ii). Take an open set $U \Subset \Omega$ and consider a weak solution $v \in C(\overline{U})$ to problem (13.5) such that $v \le \mathcal{P}_{u,D}$ almost everywhere in $\mathbb{H}^n \setminus U$ and

$$v(\xi) \le \liminf_{\eta\to\xi} \mathcal{P}_{u,D}(\eta), \qquad \forall \xi \in \partial U.$$

We claim that $v \le \mathcal{P}_{u,D}$ almost everywhere in U. Since $\mathcal{P}_{u,D} \le u$ we have that $v \le u$ a. e. in $\mathbb{H}^n \setminus U$ and

$$v(\xi) \le \liminf_{\eta\to\xi} \mathcal{P}_{u,D}(\eta) \le \liminf_{\eta\to\xi} u(\eta), \qquad \forall \xi \in \partial U.$$

Hence, $v \le u$ almost everywhere in U as well. Now, if $U \cap D = \emptyset$ we are done since $\mathcal{P}_{u,D} \equiv u$ outside D. Assume $U \cap D \neq \emptyset$. We have that $\mathcal{P}_{u,D}$ is a weak solution in $U \cap D$. In the set $\mathbb{H}^n \setminus U$, it holds $v \le \mathcal{P}_{u,D}$ almost everywhere and in $U \cap (\mathbb{H}^n \setminus D)$ we have that $v \le u \equiv \mathcal{P}_{u,D}$ as shown above.

Hence, in $\mathbb{H}^n \setminus (U \cap D)$ we have that $v \le \mathcal{P}_{u,D}$ almost everywhere and on the boundary it holds

$$\begin{cases} v(\xi) \le \liminf\limits_{\substack{\eta\to\xi \\ \eta\in U\cap D}} \mathcal{P}_{u,D}(\eta), & \text{when } \xi \in \partial U \cap D, \\ v(\xi) \le u(\xi) \le \liminf\limits_{\substack{\eta\to\xi \\ \eta\in U\cap D}} \mathcal{P}_{u,D}(\eta), & \text{when } \xi \in U \cap \partial D. \end{cases}$$

Hence by the comparison principle of Proposition 13.7 it follows that $v \le \mathcal{P}_{u,D}$ a. e. in $U \cap D$ as well. □

Using Poisson modification and the Hölder continuity result of Theorem 10.6 by repeating the procedure used in [97], we can prove the following proposition.

Proposition 13.14. *The upper and lower Perron solutions $\overline{H}_g$ and $\underline{H}_g$ are continuous weak solution to (13.5) in Ω, according to Definition 9.4.*

Finally we can prove our last main result.

Proof of Theorem 13.3. We divide the proof into two parts. Firstly we show that

$$\overline{H}_g \ge \underline{H}_g. \tag{13.13}$$

Let us choose $\varepsilon > 0$ and consider two function $u \in \mathscr{U}_g$ and $v \in \mathcal{L}_g$ such that

$$\overline{H}_g + \varepsilon \ge u, \qquad \text{and} \qquad \underline{H}_g - \varepsilon \le v.$$

By Definition 13.1(iii) we have that

$$\liminf_{\substack{\eta\to\xi \\ \eta\in\Omega}} \overline{H}_g(\eta) \ge \limsup_{\substack{\eta\to\xi \\ \eta\in\Omega}} \underline{H}_g(\eta), \qquad \text{for all } \xi \in \partial\Omega.$$

Hence, since by Proposition 13.14, both Perron solutions are weak solutions in Ω to 13.5 and $\overline{H}_g \ge \underline{H}_g$ a. e. in $\mathbb{H}^n \setminus \Omega$, we have that the desired inequality (13.13) follows by the comparison principle of Proposition 13.7.

We are left to prove the reverse inequality of (13.13) in Ω. With no loss of generality we may assume that the function $g < \infty$ almost everywhere in Ω. Then, let us consider the solution to the obstacle problem $u \in \mathcal{K}_{g,g}(\Omega, \Omega')$, with $\Omega \Subset \Omega'$. Recalling that we assume the boundary datum g belonging to $W^{s,p}(\mathbb{H}^n)$ we have that, by Theorem 12.2, u is a weak supersolution in Ω and $u \in W^{s,p}(\mathbb{H}^n)$. Hence, $u \in \mathscr{U}_g$.

Let us consider an exhaustion of Ω with open sets $\{D_j\}$:

$$D_1 \subset D_2 \subset \ldots \subset \Omega = \bigcup_{j=1}^{\infty} D_j,$$

such that $\mathbb{H}^n \setminus D_j$ satisfies the measure density condition (12.5), for any $j \in \mathbb{N}$, and take the sequence $\{\mathcal{P}_{u,D_j}\}$ of the Poisson modifications of u in $\{D_j\}$.

Clearly, by Proposition 13.13, we have that $\mathcal{P}_{u,D_j}$ is a weak solution in D_j, $\mathcal{P}_{u,D_j} \equiv g$ in $\mathbb{H}^n \setminus \Omega$ and $\mathcal{P}_{u,D_j} \in \mathscr{U}_g$.

Moreover, $\{\mathcal{P}_{u,D_j}\}$ is a non-increasing sequence by the comparison principle of Proposition 13.7 and

$$u \geq \mathcal{P}_{u,D_1} \geq \mathcal{P}_{u,D_2} \geq \ldots$$
$$\|u\|_{L^p(\mathbb{H}^n)} \geq \|P_{u,D_1}\|_{L^p(\mathbb{H}^n)} \geq \|\mathcal{P}_{u,D_2}\|_{L^p(\mathbb{H}^n)} \geq \ldots$$

Hence, the pointwise limit $\mathcal{P}_{u,\Omega} = \lim_j \mathcal{P}_{u,D_j}$ does exist almost everywhere and belongs to $L^p(\mathbb{H}^n)$.

Then, by in Definition 13.1(i) we have that $\mathcal{P}_{u,\Omega} \leq \mathcal{P}_{u,D_j} \leq h$ with $\mathcal{P}_{u,\Omega}, h \in L^{p-1}_{sp}(\mathbb{H}^n)$. By Corollary 13.10 it follows that $\mathcal{P}_{u,\Omega}$ is a weak solution in Ω to (13.5).

Let us note that the function $\mathcal{P}_{u,\Omega}$ may not belong to the upper class $\mathscr{U}_g$. Nevertheless, since $\mathcal{P}_{u,D_j} \in \mathscr{U}_g$, we have that for any $\xi \in \mathbb{H}^n$

$$\overline{H}_g(\xi) \leq \mathcal{P}_{u,D_j}(\xi), \qquad \overline{H}_g(\xi) \leq \lim_{j\to\infty} \mathcal{P}_{u,D_j}(\xi) = \mathcal{P}_{u,\Omega}(\xi).$$

Thus, $\overline{H}_g \leq \mathcal{P}_{u,\Omega}$.

Repeating the same argument for the lower Perron solution we have that $\underline{H}_g \geq \mathcal{P}_{u,\Omega}$. Hence we have show that $\overline{H}_g \leq \underline{H}_g$.

We have that $\overline{H}_g = \underline{H}_g = H_g$ and the proof ends applying Proposition 13.14. □

Part 4

Harnack inequalities for kinetic integral equations

CHAPTER 14

Setting of the problem

In this part we deal with a wide class of kinetic Kolmogorov-Fokker-Planck equations, whose diffusion part is an integro-differential operator of differentiability order $s \in (0,1)$.

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$$\partial_t f + v \cdot \nabla_x f = \mathcal{L}_v f \quad \text{in } \Omega \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \,, \tag{14.1}$$

where the diffusion term $\mathcal{L}_v$ is given by

$$\mathcal{L}_v f(t,x,v) = \text{ p. v.} \int_{\mathbb{R}^n} \Big(f(t,x,w) - f(t,x,v) \Big) K(w,v) \, \mathrm{d}w \,, \tag{14.2}$$

with $K = K(w,v) \approx |w-v|^{-n-2s}$ being a symmetric kernel of order $s \in (0,1)$ with merely measurable coefficients, whose prototype is the classical fractional Laplacian operator $(-\Delta_v)^s$, with respect to the v-variables, given by

$$(-\Delta_v)^s f(t,x,v) := c_{n,s} \text{ p. v.} \int_{\mathbb{R}^n} \frac{f(t,x,v) - f(t,x,w)}{|v-w|^{n+2s}} \, \mathrm{d}w \,. \tag{14.3}$$

In the display above, $c_{n,s}$ is a positive constant only depending on the dimension n and the differentiability exponent s; see [49, Section 2] for further details. We also notice that the integrals in (14.2)-(14.3) may be singular at the origin and they must be interpreted in the appropriate sense. Since we assume that the coefficients in (14.2) are merely measurable, the related equation has to have a suitable weak formulation.

During the last century the validity of the classical Harnack inequality has been an open problem in the nonlocal setting, and more in general for integro-differential operators. The first answer, for the purely fractional Dirichlet equation, has been eventually given by Kaßmann in his breakthrough papers [76, 77], where the strong Harnack inequality is proven to be still valid by adding an extra term, basically a natural tail-type contribution on the right-hand side which cannot be dropped nor relaxed even in the most simple case when $\mathcal{L}_v$ does coincide with the fractional Laplacian operator $(-\Delta_v)^s$ in (14.3); see Theorem 1.2 in [76]. Such an extra term does completely disappear in the case of nonnegative weak solutions; see Theorem 3.1 in [77], so that one falls in the classical strong Harnack formulation.

After the breakthrough results by Kaßmann, Harnack-type inequalities and a quite comprehensive nonlocal De Giorgi-Nash-Moser theory have been presented in more general *integro-differential elliptic frameworks*, even for nonlinear fractional equations. The literature is really too wide to attempt any comprehensive list here; we refer to [47, 87, 48, 21, 23, 83, 55] and the references therein; it is worth presenting also the important Harnack inequalities in [79], which deals with very irregular integro-differential kernels so that a link to Boltzmann-type collision kernels seems veritably close.

The situation becomes more convoluted in the *integro-differential parabolic framework*. Indeed, in order to prove Harnack-type results, the intrinsic scaling

of the involving cylinders will depend not only on the time variable t, as in the classical pioneering work by DiBenedetto [46], but also on the differentiability order s. This is not for free, as one can imagine, even for the purely p-fractional heat equation. A few fractional parabolic Harnack inequalities are however still available, in the case when the kernel of the leading operator is a sort of $(s,2)$-Gagliardo-type one, as, e. g., in [125] in part extending the results in the elliptic counterpart in [47]; see also [78], and the very recent paper [80] together with the aforementioned paper [79] dealing with very intricate irregular kernels. Nevertheless, notable differences in such a parabolic framework inevitably arise, and the validity of a (*strong*) *Harnack inequality could fail* depending on the specific assumptions on the involved kernels, even when starting from bounded solutions; see, e. g., [16, 17].

As noticed above, already in the nonlocal parabolic framework – that to some extents should be seen as the space homogeneous version of (14.1) – one needs new strategies and ideas (see, as a concrete example, the fine analysis in [123]), and strong Harnack inequalities are still not assured (as in the case of the aforementioned counter-examples). More specifically, even for purely kinetic equations with fractional diffusion as in (14.1) *the validity or not of a strong Harnack inequality has been an open problem*. This is not a surprise because of the very form of the equations in (14.1) which also involves a transport term, and the nonlocality in velocity has to be dealt keeping into account the involuted intrinsic scalings naturally arising. Indeed, to our knowledge, there is still no strong Harnack inequality in the whole *integro-differential kinetic panorama*, even when the nonnegative solutions are assumed to be bounded a priori, and/or under other assumptions in clear accordance (or not) with some related physical models.

In order to clarify the current situation, it is enlightening to focus on a fundamental class of nonlocal kinetic equations; i. e., those modeling the Boltzmann problem without cut-off, for which very important estimates and regularity results have been recently proven, via fine variational techniques and radically new approaches. An inspiring step in such an advance in the regularity theory relies in the approach proposed in the breakthrough paper [88], where the authors, amongst other results, are able to derive a weak Harnack inequality for solutions to a very large class of kinetic integro-differential equations as in (14.1) with very mild assumptions on the integral diffusion in velocity having degenerate kernels K in (14.2) which are not symmetric (not in the usual way) nor pointwise bounded by Gagliardo-type kernels; see Theorem 1.6 there. Under a coercivity condition on $\mathcal{L}_v$ and other natural assumptions (see Section 1.1 in [88]), the same result for the Boltzmann equation mentioned above follows as a corollary. Further related regularity estimates under conditional assumptions on the solutions f have been subsequently proven in [90]. Despite the fine estimates and the new approach in [88, 90], a strong Harnack inequality is still missing. A very recent step in this direction, which is worth to be presented, is the following inequality obtained in the interesting paper [98] via a quantitative De Giorgi-type approach based on (local) trajectories, by assuming the solutions f to be globally bounded a priori,

$$\sup_{\widetilde{Q}^-} f \;\le\; c\Big(\inf_{\widetilde{Q}^+} f\Big)^{\beta}. \tag{14.4}$$

This is a nontrivial result ([98, Theorem 1.3]), but the exponent $0<\beta<1$ in the estimate above is in fact a root, and thus a strong Harnack inequality cannot be deduced. Also, the cylinders $\widetilde{Q}^{\pm}$ in (14.4) – which are naturally *slanted* in

order to deal with the underlying kinetic geometry – are not the expected ones because there is a substantial gap in time which seems to be related to the not optimal expansion of positiveness in the proofs there (see in particular Figure 2 in [98]); compare in fact with the slanted cylinders in [124, 88, 90] as well as with the sharp ones in our forthcoming Theorems 14.2 and 14.3 here, also pictured in forthcoming Figure 14. Again, for integro-differential equations, the situation is different than for classical second order equations. For this, we take the liberty to quote the clarifying explanation by the authors in [88, Pag. 548], «`It is not true that the maximum of a nonnegative subsolution can be bounded above by a multiple of its` L^2 `norm. One needs to impose an extra global restriction (in this case we assume` $0 \leq f \leq 1$ `globally). This is because of nonlocal effects, since the positive values of the function outside the domain of the equation may pull the maximum upwards.`»

In order to overcome the nonlocality issues mentioned above which will also prevent an almost direct strong Harnack inequality from Hölder estimates, in the present Part 4 we found a way to present a totally new *δ-interpolation L^∞-inequality with tail* for weak subsolutions to (14.1) which are not required to being nonnegative. The parameter $0 < \delta \leq 1$ in such a boundedness estimate can suitably chosen in order to balance in a quantitative way the local contributions and the nonlocal ones; see in particular in the right-side of the inequality (14.5) in the theorem below the nonlocal kinetic tail Tail_p quantity, for which we immediately refer to forthcoming Definition 14.5 in Section 2 where one can find also related observations on the fractional framework and the underlying hypoelliptic geometry. We are ready to state the aforementioned local boundedness result. We have the following

Theorem 14.1 (δ-interpolative L^∞-L^2 estimate)**.** *For any $s \in (0,1)$, let $f \in \mathcal{W}$ be a weak subsolution to (14.1) in Ω and let $Q_1 \equiv Q_1(\mathbf{0}) \subset \Omega$. Then, for any $Q_{\frac{r}{2}} \equiv Q_{\frac{r}{2}}(\mathbf{0})$, any $\delta \in (0,1]$, and any $p > p^\star$, it holds*

$$\sup_{Q_{\frac{r}{2}}} f \leq c\,(\delta r^{4s})^{-\frac{1}{2\varepsilon_*}} \Big(\int_{Q_r} f_+^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \Big)^{\frac{1}{2}} + \delta \mathrm{Tail}_p(f_+; \mathbf{0}, r, r/2)\,, \tag{14.5}$$

where $Q_r := (-r^{2s}, 0] \times B_{r^{1+2s}} \times B_r$, and the quantity $\varepsilon_* > 0$ and $p^\star > 2$ depend only on the dimension n and the exponent s, whereas the positive constant c depends also on the kernel structural constant Λ in (14.12).

The proof will rely on a precise energy estimate which will essentially involve a Caccioppoli inequality with tail combined with a summability gain result for subsolutions – in turn based on the fundamental solution for the fractional Kolmogorov equation; see forthcoming Lemma 15.1 – as well as with a fine iterative argument taking into account both the Tail_p term and the desired interpolative effect.

As expected, the result in Theorem 14.1 above will permit us to bypass the aforementioned extra global restriction usually assumed in the previous literature, in turn being fundamental in order to prove the other main results in the present Part 4. We start with a weak Harnack inequality in the case when the function f is merely a weak supersolution to (14.1).

Theorem 14.2 (The weak Harnack inequality)**.** *For any $s \in (0,1)$, let $f \in \mathcal{W}$ be a nonnegative weak supersolution to (14.1) in Ω and let $Q_1(\mathbf{0}) \equiv Q_1 \subset \Omega$. Then, there exist r_0, c and ζ depending on s and the dimension n such that*

$$\Big(\fint_{Q_{r_0}^-} f(t,x,v)^\zeta \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \Big)^{\frac{1}{\zeta}} \leq c \inf_{Q_{r_0}^+} f\,, \tag{14.6}$$

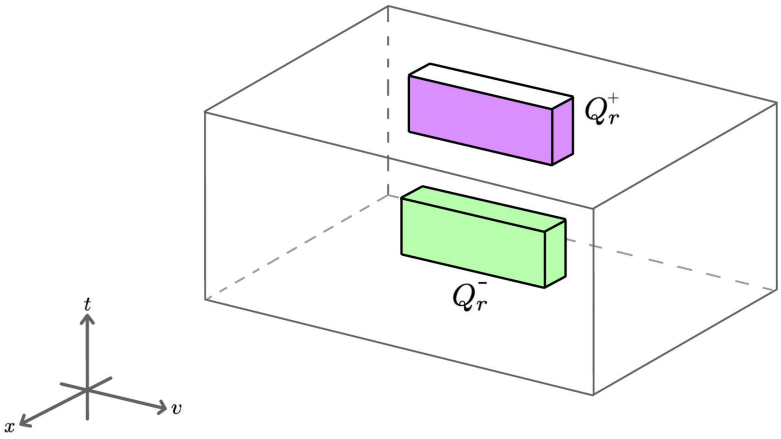

FIGURE 1. The geometry of the Harnack inequalities for kinetic equations with integro-differential diffusion.

where

$$Q_{r_0}^+ := (-{r_0}^{2s}, 0] \times B_{{r_0}^{1+2s}} \times B_{r_0}$$
$$\text{and} \quad Q_{r_0}^- := (-1, -1 + {r_0}^{2s}] \times B_{{r_0}^{1+2s}} \times B_{r_0}. \tag{14.7}$$

The proof of Theorem 14.2 will be finalized by extending the De Giorgi approach; that is, by proving both a new suitable Intermediate Value lemma and a Measure-to-Pointwise lemma. This is in the same spirit of the pioneering work [25], as well as of recent parabolic results for fractional heat equations (as seen, e. g., in [125, 96]. See, also, [26] for related regularity estimates for the fractional parabolic obstacle problem). However, because of the difficulties naturally arising in the hypoelliptic framework here, we have to deal with the intrinsic peculiarities following by the natural scalings trichotomy: time, space and velocity. In addition, since we are seeking for clean Harnack inequalities on *optimal* (*local*) *slanted cylinders* with no gap in time, we will handle all the related nonlocal estimates in a possibly sharp way by taking into account the tail contributions in this sort of expansion of positivity of suitable subsolutions. The δ-interpolative L^∞-L^2 inequality in Theorem 14.1 will be in fact applied to a suitable (not positive) sequence $g = g_k$ approximating an auxiliary subsolutions to (14.1) which is built starting from the supersolutions f. Also, whereas we have to operate several modifications in usual fractional estimates, we still cannot apply the standard Krylov-Safonov covering lemma in the framework we are dealing with. For these reasons, we eventually complete the proof of Theorem 14.2 thanks to an applications of the new Ink-spots Theorem by Imbert and Silvestre in [88, Section 9] which will allow us to deal with the naturally *slanted* kinetic cylinders; see Section 1 below.

Finally, still without requiring no additional boundedness assumptions for solutions f to (14.1), we are able to prove the very first strong Harnack inequality for kinetic equations with nonlocal diffusion in velocity. Our main result reads as follows,

Theorem 14.3 (The Harnack inequality)**.** *For any $s \in (0,1)$, let $f \in \mathcal{W}$ be a nonnegative weak solution to (14.1) in Ω and let $Q_1(\mathbf{0}) \equiv Q_1 \subset \Omega$. Then, there exist r_0, and c depending on s and the dimension n such that, for any $0 < r < r_0$,*

it holds

$$\sup_{Q_r^-} f \;\leq\; c \inf_{Q_r^+} f\,, \tag{14.8}$$

where the slanted cylinders $Q^{\pm}$ *are those defined in* (14.7).

The proof will follow by combining all the previous results, together with a precise control of the Tail of the solution in accord with the summability exponent $p^\star$ in Theorem 14.1; see the beginning of Section 2. In this respect, it is worth mentioning that our approach steps outside from the nonlocal parabolic counterpart where one can usually take care of the nonlocality by a sort of time slicing via a *supremum tail*. Here, since a (t,x)-freezing is basically not admissible because of the transport term in the very form of the kinetic equation in (14.1), we introduced and made use of the *integral* Tail_p quantity.

We then will combine the weak Harnack inequality (14.6) in Theorem 14.2 with an application of the local boundedness inequality in (14.5) by suitably choosing the interpolation parameter δ there. A new iterative argument taking into account the involved transport and diffusion radii is finally applied in order to complete the estimate in (14.8). This will rely on a new Besicovitch's covering lemma for slanted cylinders; see forthcoming Lemma 18.1, which reminds to the classical covering argument appearing in the last step of most regularity results for both local and/or nonlocal elliptic or parabolic problems via the usual variational approach. We believe that the latter is a very general argument that will have to be taken into account in other results and extensions in the kinetic frameworks. In this section we fix notation, and we briefly recall the necessary underlying framework in which one needs to work in order to deal with the class of nonlocal kinetic equations as in (14.1). For a more comprehensive analysis of Lie groups in the kinetic setting we refer the reader to the surveys [**6**, **5**] and the references therein; the interested reader could also refer to the recent paper [**100**] which deals with the class of operators in (14.2) by presenting an intrinsic Taylor formula in our framework.

The results presented in this part of the thesis are contained in [4].

1. The underlying geometry

We start by endowing $\mathbb{R}^{1+2n} = \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$ of the group law $\circ$ given by

$$(t,x,v)\circ(t_0,x_0,v_0) := \Big(t+t_0,\, x+x_0+tv_0,\, v+v_0\Big), \tag{14.9}$$

so that $(\mathbb{R}^{1+2n},\circ)$ is a Lie group with identity element $\mathbf{0}\equiv(0,0,0)$ and inverse element for $(t,x,v)\in\mathbb{R}^{1+2n}$ given by $(-t,\,-x+tv,\,-v)$.

For any $r>0$, consider the usual anisotropic dilation $\delta_r : \mathbb{R}^{1+2n}\mapsto\mathbb{R}^{1+2n}$ defined by

$$\delta_r(t,x,v) := \Big(r^{2s}t,\, r^{1+2s}x,\, rv\Big), \tag{14.10}$$

so that if u is a solution to (14.1), then the function $u_r = u_r(t,x,v)$ such that

$$u_r(t,x,v) := u\Big(r^{2s}t,\, r^{1+2s}x,\, rv\Big)$$

does satisfy the same equation in a suitably rescaled domain.

As customary in the hypoelliptic setting, we need to introduce a family of fractional kinetic cylinders respecting the invariant transformations defined above.

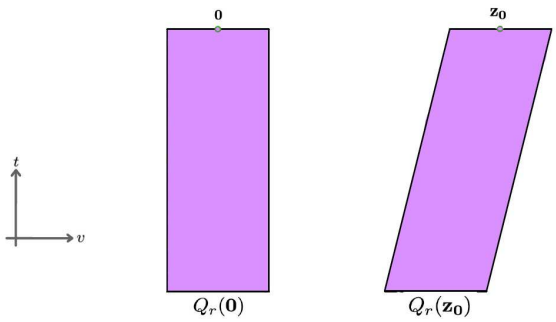

FIGURE 2. On the left the cylinder $Q_r(\mathbf{0})$ centred at the origin; on the right a *slanted* cylinder $Q_r(t_0, x_0, v_0)$ according to the invariant transformation given in (14.11).

For every $(t_0, x_0, v_0) \in \mathbb{R}^{1+2n}$ and for every $r > 0$, the *slanted* cylinder $Q_r = Q_r(t_0, x_0, v_0)$ is defined as follows,

$$
\begin{aligned}
Q_r(t_0, x_0, v_0) &:= (t_0, x_0, v_0) \circ \delta_r\Big((-1, 0] \times B_1 \times B_1\Big) \\
&\equiv \Big\{(t, x, v) \in \mathbb{R}^{1+2n} : t_0 - r^{2s} < t \le t_0, \\
&\qquad\qquad \big|(x - x_0) - (t - t_0)v_0\big| < r^{1+2s},\ |v - v_0| < r\Big\}. \qquad (14.11)
\end{aligned}
$$

In order to simplify the notation, we denote by Q_r a cylinder centred in $(0, 0, 0)$ of radius r; that is,

$$
Q_r \equiv Q_r(\mathbf{0}) := U_r(0, 0) \times B_r(0) = (-r^{2s}, 0] \times B_{r^{1+2s}}(0) \times B_r(0)\,.
$$

Now, we recall a suitable covering argument in the same flavour of the Krylov-Safonov Ink-spots theorem. Indeed, in our framework one cannot plainly apply the usual Calderón-Zygmund decomposition, because there is no space to tile slanted cylinders with varying slopes. This is a major difficulty in the nonlocal kinetic framework which has been firstly addressed in an original way by Imbert and Silvestre in [88], who were able to state and prove a custom version of the ink-spots theorem. Such a result, that we will present right below in Theorem 14.4, will allow us to conclude the proof of the weak Harnack inequality. Instead, for what concerns the strong Harnack inequality in (14.8), as mentioned in the Introduction, we will state and prove a new Besicovitch-type covering, which is also suitable for very general kinetic-type frameworks when slanted cylinders do naturally lead the involved geometry; see forthcoming Section 1. In order to state Imbert-Silvestre's Ink-Spots Theorem, we need to introduce the *stacked (and slanted) cylinders* $\bar{Q}_r^m$ for some given $m \in \mathbb{N}$. We have

$$
\begin{aligned}
\bar{Q}_r^m(t_0, x_0, v_0) := \Big\{&(t, x, v) \in \mathbb{R}^{1+2n} : 0 < t - t_0 \le m r^{2s}, \\
&\big|(x - x_0) - (t - t_0)v_0\big| < (m + 2)r^{1+2s},\ |v - v_0| < r\Big\}.
\end{aligned}
$$

Notice that the cylinder $\bar{Q}_r^m$ starts at the end (in time) of Q_r and its duration (still in time) is exactly m-times the one of Q_r, whereas its spatial radius is $m + 2$-times the one of Q_r; see Figure 1 below.

Then we have the following

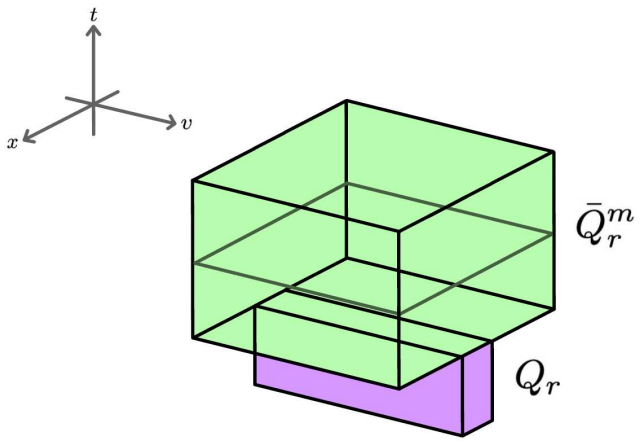

FIGURE 3. A stacked (and slanted) cylinder $\bar{Q}_r^m$. We refer to Section 10 in [88] for a very detailed analysis and further related results.

Theorem 14.4 (the Ink-spots Theorem with leakage; see [88, Corollary 10.2])**.** *Let $E \subset F$ be bounded measurable sets. Assume that for some constant $\mu \in (0,1)$,*

(i) *$E \subset Q_1$,*
(ii) *if a cylinder $Q \subset Q_1$ satisfies $|Q \cap E| \geq (1-\mu)|Q|$, then $\bar{Q}^m \subset F$ and also $Q = Q_\sigma(t_0, x_0, v_0)$ for some $\sigma \in (0, r_0)$.*

Then,

$$|E| \leq \frac{m+1}{m}(1 - c\mu)\left(|F \cap Q_1| + Cmr_0^{2s}\right)$$

for some constants c and C depending only on n and s.

It is worth noticing that related Krylov-Safonov-type results both in the local and nonlocal kinetic framework can be also found in [115, 121]. We would also mention that it was used in [1] for dealing with solutions to a class of Kolmogorov-Fokker-Planck equations in non-divergence form.

2. The nonlocal energy setting

We now introduce our fractional functional setting. Let Ω_v be an open subset of $\mathbb{R}^n$; for $s \in (0,1)$ recall the definition of the classical fractional Sobolev spaces $H^s(\Omega_v)$; i. e.,

$$H^s(\Omega_v) \equiv W^{s,2}(\Omega_v) := \left\{ f \in L^2(\Omega_v) \ : \ [f]_{H^s(\Omega_v)} < +\infty \right\},$$

where the fractional seminorm $[f]_{H^s(\Omega_v)}$ is the usual one via Gagliardo kernels,

$$[f]_{H^s(\Omega_v)} := \left(\int_{\Omega_v} \int_{\Omega_v} \frac{|f(v) - f(w)|^2}{|v-w|^{n+2s}} \, \mathrm{d}v \, \mathrm{d}w \right)^{1/2}.$$

A norm of $H^s(\Omega_v)$ is given by

$$\|f\|_{H^s(\Omega_v)} := \|f\|_{L^2(\Omega_v)} + [f]_{H^s(\Omega_v)}.$$

A function f belongs to $H^s_{\mathrm{loc}}(\Omega_v)$ if $f \in H^s(\Omega_v')$ whenever $\Omega_v' \Subset \Omega_v$.

As mentioned in the Introduction, the kernel $K : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ is a measurable kernel having s-differentiability for any $s \in (0,1)$; that is, there exists a positive constant Λ such that

$$\Lambda^{-1}|v-w|^{-n-2s} \leq K(v,w) \leq \Lambda|v-w|^{-n-2s}, \quad \text{for a. e. } v,w \in \mathbb{R}^n, \tag{14.12}$$

though most of the estimates in the rest of the Part 4 will still work by weakening such a pointwise control from above, by assuming appropriate coercivity, local integral boundedness and cancelation properties. The pointwise control from below by a Gagliardo-type kernel, on the contrary, is strongly used in some of the needed estimates. However, for the sake of simplicity, we prefer to present our results for the class of measurable kernel as in (14.12), so that the reader has just to keep in mind the case when the diffusion in velocity is a purely fractional Laplacian with coefficients. Consequently, no precise dependance on the constant Λ will be explicitly written when not needed.

As expected when dealing with nonlocal operators, the long-range contributions must be taken into account.

Definition 14.5 (The kinetic nonlocal tail)**.** *Let $(t_0,x_0,v_0) \in \Omega := (t_1,t_2) \times \Omega_x \times \Omega_v \subset \mathbb{R}^{1+2n}$ and $R > 0$ be such that $U_R(t_0,x_0) \subset (t_1,t_2) \times \Omega_x$, and let f be a measurable function on $(t_1,t_2) \times \Omega_x \times \mathbb{R}^n$. For any $p \in [1,\infty]$ the* "kinetic nonlocal tail of f centred in (t_0,x_0,v_0) of transport radius R and diffusion radius r" *is the quantity* $\mathrm{Tail}_p \equiv \mathrm{Tail}_p(f;(t_0,x_0,v_0),R,r)$ *given by*

$$\begin{aligned} &\mathrm{Tail}_p(f;(t_0,x_0,v_0),R,r) \\ &\qquad := \left[\int_{U_R(t_0,x_0)} \left(\int_{\mathbb{R}^n \setminus B_r(v_0)} \frac{|f(t,x,v)|}{|v_0-v|^{n+2s}} \,\mathrm{d}v \right)^p \mathrm{d}x \,\mathrm{d}t \right]^{\frac{1}{p}}. \end{aligned} \tag{14.13}$$

Despite being not needed in the present manuscript, it is worth mentioning that in the case when $p = \infty$ the kinetic tail definition is meant to be given by taking the L^∞-norm in (t,x) instead of the L^p one in (14.13).

The kinetic nonlocal tail reminds of the nonlocal tail quantity firstly defined in the purely p-fractional elliptic setting in [47, 48] and subsequently proven to be decisive in the analysis of many other nonlocal problems when a fine quantitative control of the naturally arising long-range interactions is needed; see, e. g. [87, 21, 125, 96, 23] and the references therein. Several tail related properties of nonlocal harmonic functions are naturally expected – as for instance the fact that their tail is finite, and that their tail is controlled by that of their negative part, and so on – and they are proven in [86]. However, their kinetic counterparts are not for free, and we need to operate step by step in the forthcoming proofs here.

It is also worth noticing that it is usually the nonnegativeness of solutions to interfere with the validity of Harnack inequalities in fractional settings, and $\mathrm{Tail}((f)_-)$ is the decisive player in such a game, as it has been firstly showed by Kaßmann in [76, 77] and then confirmed in the many subsequently related results. On the contrary, our strategy to make use of a nonlocal L^∞-L^2-type estimate does involve an auxiliary (possibly not positive) subsolution g whose error term will be controlled by the kinetic nonlocal tail of its positive part $(g)_+$; see the formulation in (14.5) and the details in the related proofs in the rest of the present Part 4.

Given $\Omega := (t_1,t_2) \times \Omega_x \times \Omega_v \subset \mathbb{R}^{1+2n}$ we denote by $\mathcal{W}$ the natural functions space where weak solutions to (14.1) are taken. We have

$$\mathcal{W} := \Big\{ f \in L^2\Big((t_1,t_2)\times\Omega_x;\, H^s(\mathbb{R}^n)\Big) \\ : [\partial_t + v\cdot\nabla_x]\, f \in L^2\Big((t_1,t_2)\times\Omega_x;\, H^{-s}(\mathbb{R}^n)\Big)\Big\}. \tag{14.14}$$

Furthermore, we denote by $\mathcal{E}(\cdot)$ the nonlocal energy associated with our diffusion term $\mathcal{L}_v$ in (14.3)

$$\mathcal{E}(f(t,x,\cdot),\phi(t,x,\cdot)) \\ := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Big(f(t,x,v)-f(t,x,w)\Big)\Big(\phi(t,x,v)-\phi(t,x,w)\Big)K(v,w)\,\mathrm{d}v\,\mathrm{d}w\,,$$

for any test function ϕ smooth enough. We are now in the position to recall the definition of weak sub- and supersolution.

Definition 14.6. *A function $f \in \mathcal{W}$ is a* weak subsolution (*resp.,* supersolution) *to* (14.1) *in* Ω *if*

$$\int_{t_1}^{t_2}\int_{\Omega_x} \mathcal{E}\left(f(t,x,\cdot),\phi(t,x,\cdot)\right)\,\mathrm{d}x\,\mathrm{d}t \\ + \int_{t_1}^{t_2}\int_{\Omega_x} \langle(\partial_t f + v\cdot\nabla_x f)\,|\,\phi\rangle\,\mathrm{d}x\,\mathrm{d}t \;\le\; 0 \quad \Big(\ge 0,\ \textit{resp.}\Big),$$

for any nonnegative $\phi \in \mathcal{W}$ such that $\operatorname{supp}\phi \Subset \Omega$*; in the display above we denote by $\langle\cdot\,|\,\cdot\rangle$ the usual duality paring between $H^s(\mathbb{R}^n)$ and $H^{-s}(\mathbb{R}^n)$.*

A function $f \in \mathcal{W}$ is a weak solution *to* (14.1) *if it is both a weak sub- and supersolution.*

CHAPTER 15

Interpolative L^∞-L^2-type estimate

This section is devoted to the proof of the local boundedness estimate with tail for subsolutions to (14.1) with no a priori sign assumptions, as stated in Theorem 14.1.

1. Kinetic Energy estimates with tail

Firstly, we need a precise energy estimate which will require to prove a Caccioppoli-type estimate with (kinetic) tail, and a Gehring-type one for subsolutions to (14.1). We have the following

Lemma 15.1 (Gain of integrability for subsolutions)**.** *Let f be a weak subsolution in Ω according to* Definition 14.6 *and let $Q_1 \equiv Q_1(\mathbf{0}) \subset \Omega$. For any $Q_r \equiv Q_r(\mathbf{0}) \subset Q_1$, any $q \in [2, q^\star)$, where $q^\star = q^\star(n,s) > 2$, and any $\varrho < r$, the following estimate does hold,*

$$\begin{aligned}\|\omega\|^2_{L^q(Q_\varrho)} \;\le\; & c\int_{U_r}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\omega(v)\omega(w)|\varphi(v)-\varphi(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t \\ & + c\int_{Q_r}(|v\cdot\nabla_x(\varphi^2)| + |v\cdot\nabla_x\varphi|^2)\omega^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + c\,r^{2s}\int_{Q_r}(\omega\varphi)^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\,. \end{aligned} \tag{15.1}$$

where $\omega := (f-k)_+$, for any $k \in \mathbb{R}$, and where $\varphi = \varphi(x,v)$ is a cut-off function such that $\varphi(x,v) \equiv 1$ in $B_{\varrho^{1+2s}} \times B_\varrho$ and $\varphi \equiv 0$ outside $B_{r^{1+2s}} \times B_r$.

Proof. For the sake of the reader, it is convenient to divide the present proof in two separate steps.

Step 1: Kinetic Caccioppoli inequality with tail Up to regularizing by mollification, for any fixed $t \in (-r^{2s}, 0]$ we can assume that $\omega\varphi^2$ is sufficiently regular in order to be an admissible test function compactly supported in the cylinder $(Q_r)^t := \{(v,x) \in \mathbb{R}^{2n} : (t,x,v) \in Q_r\}$. Consider now the weak formulation in Definition 14.6 by choosing as a test function $\phi \equiv \omega\varphi^2$ there; for a. e. $t \in (-r^{2s}, 0]$ it yields

$$\begin{aligned} 0 \;\ge\; & \int_{(Q_r)^t}(\partial_t f + v\cdot\nabla_x f)\omega\varphi^2\,\mathrm{d}x\,\mathrm{d}v \\ & + \int_{B_{r^{1+2s}}} \mathcal{E}(f,\omega\varphi^2)\,\mathrm{d}x \;=:\; I_1 + I_2. \end{aligned} \tag{15.2}$$

We start by considering I_1. Using the fact that $\partial_t\varphi = 0$, we have that

$$I_1 \;\ge\; \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{(Q_r)^t}(\omega\varphi)^2\,\mathrm{d}x\,\mathrm{d}v - \frac{1}{2}\int_{(Q_r)^t}|v\cdot\nabla_x(\varphi^2)|\omega^2\,\mathrm{d}x\,\mathrm{d}v\,. \tag{15.3}$$

For what concern I_2 we note that

$$\begin{aligned}\Big(f(v)-f(w)\Big)\Big(\omega\varphi^2(v)-\omega\varphi^2(w)\Big)&\\ &= \Big((f(v)-k)-(f(w)-k)\Big)\Big(\omega\varphi^2(v)-\omega\varphi^2(w)\Big)\\ &\geq \Big(\omega\varphi(v)-\omega\varphi(w)\Big)^2-\omega(v)\omega(w)|\varphi(v)-\varphi(w)|^2\,,\end{aligned}$$

which yields

$$\begin{aligned}I_2 \;\geq\; &\int_{B_{r^{1+2s}}}[\omega\varphi]^2_{H^s(\mathbb{R}^n)}\,\mathrm{d}x\\ &-\int_{B_{r^{1+2s}}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\omega(v)\omega(w)|\varphi(v)-\varphi(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,,\end{aligned}\tag{15.4}$$

where we also used the definition of the kernel K in (14.12) by neglecting a constant depending on Λ there, for the sake of simplicity. Combining (15.3) and (15.4) with (15.2), it yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{(Q_r)^t}(\omega\varphi)^2\,\mathrm{d}x\,\mathrm{d}v+\int_{B_{r^{1+2s}}}[\omega\varphi]^2_{H^s(\mathbb{R}^n)}\,\mathrm{d}x\\ &\qquad\leq c\int_{B_{r^{1+2s}}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\omega(v)\omega(w)|\varphi(v)-\varphi(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\\ &\qquad\quad+c\int_{(Q_r)^t}|v\cdot\nabla_x(\varphi^2)|\omega^2\,\mathrm{d}v\,\mathrm{d}x\,,\end{aligned}\tag{15.5}$$

Then, by integrating (15.5) in $[\tau_1,\tau_2]$, for $-r^{2s}\leq\tau_1<\tau_2\leq 0$, we get

$$\begin{aligned}&\int_{B_{r^{1+2s}}\times B_r}(\omega\varphi)^2(\tau_2,x,v)\,\mathrm{d}x\,\mathrm{d}v+\int_{\tau_1}^{\tau_2}\int_{B_{r^{1+2s}}}[\omega\varphi]^2_{H^s(\mathbb{R}^n)}\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\leq c\int_{U_r}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\omega(v)\omega(w)|\varphi(v)-\varphi(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\quad+c\int_{Q_r}|v\cdot\nabla_x(\varphi^2)|\omega^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t+c\int_{B_{r^{1+2s}}\times B_r}(\omega\varphi)^2(\tau_1,x,v)\,\mathrm{d}x\,\mathrm{d}v\,.\end{aligned}\tag{15.6}$$

Taking the supremum over τ_2 on the left-hand side and the average integral over $\tau_1\in[-r^{2s},0]$ on both sides of the inequality, we get

$$\begin{aligned}&\sup_{t\in[-r^{2s},0]}\int_{B_{r^{1+2s}}\times B_r}(\omega\varphi)^2\,\mathrm{d}v\,\mathrm{d}x\\ &\qquad\leq c\int_{U_r}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\omega(v)\omega(w)|\varphi(v)-\varphi(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\quad+c\int_{Q_r}|v\cdot\nabla_x(\varphi^2)|\,\omega^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t+c\,r^{2s}\int_{Q_r}(\omega\varphi)^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}\tag{15.7}$$

Reconsidering (15.6) and evaluating with $\tau_1 = -r^{2s}$ and $\tau_2 = 0$ there, we then have

$$\begin{aligned}
&\int_{U_r} [\omega\varphi]^2_{H^s(\mathbb{R}^n)}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leq c\int_{U_r}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\omega(v)\omega(w)|\varphi(v)-\varphi(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + c\int_{Q_r} |v\cdot\nabla_x(\varphi^2)|\omega^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + c\,r^{2s}\int_{Q_r}(\omega\varphi)^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

Thus, combining the display above with (15.7), we obtain the following Caccioppoli-type estimate,

$$\begin{aligned}
&\sup_{t\in[-r^{2s},0]}\int_{B_{r^{1+2s}}\times B_r}(\omega\varphi)^2\,\mathrm{d}v\,\mathrm{d}x + \int_{U_r}[\omega\varphi]^2_{H^s(\mathbb{R}^n)}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leq c\int_{U_r}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\omega(v)\omega(w)|\varphi(v)-\varphi(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + c\int_{Q_r} |v\cdot\nabla_x(\varphi)^2|\omega^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + cr^{2s}\int_{Q_r}(\omega\varphi)^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned} \tag{15.8}$$

Step 2: Local L^q-estimate for subsolutions Now, an L^q-estimate whose proof is in the same spirit of the original result for solutions to the Boltzmann equation without cut-off by Imbert and Silvestre – see in particular Lemma 6.1 and Proposition 2.2 in [88] – which in turn reminds to the strategy in [115] and to classical Gehring-type results; see in particular Sections 2 and 3 there. Nevertheless, for the sake of the reader, we sketch the proof below.

Furthermore, it is worth noticing that the exponent $p^\star$ in the statement of Theorem 14.1 will basically show up here, being linked to the maximal gain in summability in forthcoming formula (15.9).

Consider the smooth function $\varphi = \varphi(x,v)$ such that $\varphi(x,v)\equiv 1$ in $B_{\varrho^{1+2s}}\times B_\varrho$ and $\varphi\equiv 0$ outside $B_{r^{1+2s}}\times B_r$, with $\varrho < r < 1$.

Then, the function $g := \omega\varphi$ satisfies the following

$$[\partial_t + v\cdot\nabla_x]\ g - \mathcal{L}_v g \ \leq\ \big([\partial_t + v\cdot\nabla_x]\omega\big)\,\varphi + \omega\,\big([\partial_t + v\cdot\nabla_x]\varphi\big) - \Lambda^{-1}(-\Delta_v)^s g\,;$$

where Λ is the kernel structural constant in (14.12). Now, we can apply the result in Lemma 6.1 in[88] – by taking $H_2 := \Lambda^{-1}(-\Delta_v)^{\frac{s}{2}} g$ and $H_1 := \big([\partial_t + v\cdot\nabla_x]\omega\big)\,\varphi + \omega\,\big([\partial_t + v\cdot\nabla_x]\varphi\big)$ there – and we get that, for any $q\geq 2$ such that $q < q^\star$ with

$$q^\star := 2 + \frac{2s}{n(1+s)}\,, \tag{15.9}$$

the following estimate holds,

$$\begin{aligned}
&\|g\|_{L^q([-r^{2s},0]\times\mathbb{R}^{2n})} \\
&\quad\leq c\Big(\|g(-r^{2s},\cdot,\cdot)\|_{L^2(\mathbb{R}^{2n})} + \|([\partial_t + v\cdot\nabla_x]\omega)\varphi\|_{L^2([-r^{2s},0]\times\mathbb{R}^{2n})} \\
&\qquad + \|\omega([\partial_t + v\cdot\nabla_x]\varphi)\|_{L^2([-r^{2s},0]\times\mathbb{R}^{2n})} + \int_{U_r}[g]^2_{H^s(\mathbb{R}^n)}\,\mathrm{d}x\,\mathrm{d}t\Big) \\
&\quad\leq c\left(\|g(-r^{2s},\cdot,\cdot)\|_{L^2(\mathbb{R}^{2n})} + \|\omega(v\cdot\nabla_x\varphi)\|_{L^2([-r^{2s},0]\times\mathbb{R}^{2n})} + \int_{U_r}[g]^2_{H^s(\mathbb{R}^n)}\,\mathrm{d}x\,\mathrm{d}t\right),
\end{aligned}$$

where we have integrated by parts in the second integral and the positive constant c also depends on n and s; recall Proposition 3.6 in [49].

Now, by recalling the definition of φ, we restate the latter estimate as follows,

$$\begin{aligned}\|\omega\|_{L^q(Q_\varrho)} \;\leq\; & c\,\|\omega\varphi(-r^{2s},\cdot,\cdot)\|_{L^2(B_{r^{1+2s}}\times B_r)} \\ & + c\,\|\omega(v\cdot\nabla_x\varphi)\|_{L^2(Q_r)} + c\int_{U_r}[\omega\varphi]^2_{H^s(\mathbb{R}^n)}\,\mathrm{d}x\,\mathrm{d}t\,. \end{aligned} \tag{15.10}$$

Conclusion It finally suffice to combine (15.10) with (15.8). □

We are now in the position to prove the δ-interpolative L^∞-L^2 inequality.

Proof of Theorem 14.1. Let $r>0$ and, for any $j\in\mathbb{N}$, define a decreasing family of positive radii $r_j := \frac{1}{2}(1+2^{-j})r$ and a family of slanted cylinders $Q_j \equiv Q_{r_j}(\mathbf{0})$ such that $Q_{j+1}\Subset Q_j$ for every $j\in\mathbb{N}$. We will denote with $U_j := (-r_j^{2s},0]\times B_{r_j^{1+2s}}$, so that $Q_j := U_j\times B_{r_j}$.

Consider a family $\{\varphi_j\}_{j\in\mathbb{N}}$ of test functions $\varphi_j\equiv\varphi_j(x,v)\in C_0^\infty(B_{r_j^{1+2s}}\times B_{r_j})$, such that $0\leq\varphi_j\leq 1$, $\varphi_j\equiv 1$ on $B_{r_{j+1}^{1+2s}}\times B_{r_{j+1}}$, $\varphi_j(x,\cdot)=0$ outside $B_{(r_j+r_{j+1})/2}$, $|\nabla_v\varphi_j|\leq c2^{j+2}/r$, and $|v\cdot\nabla_x\varphi_j|\leq c2^{(j+1)(1+2s)}/r^{2s}$. For any $j\in\mathbb{N}$, let $k_j := (1-2^{-j})k_0$, with $k_0>0$ which will be fixed later on, and define $\omega_j := (f-k_j)_+$.

An application of Lemma 15.1 yields, for some $q\equiv q(n,s)>2$ that

$$\begin{aligned}&\int_{Q_{j+1}}\omega_{j+1}^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &\leq\left(\int_{Q_{j+1}}\omega_{j+1}^q\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{2}{q}}|Q_{j+1}\cap\{f\geq k_{j+1}\}|^{1-\frac{2}{q}} \\ &\leq c\left(\int_{U_j}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\omega_{j+1}(v)\omega_{j+1}(w)|\varphi_j(v)-\varphi_j(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t\right. \\ &\quad\left.+\int_{Q_j}(|v\cdot\nabla_x(\varphi_j^2)|+|v\cdot\nabla_x\varphi_j|^2)\,\omega_{j+1}^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t+r_j^{2s}\int_{Q_j}(\omega_{j+1}\varphi_j)^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right) \\ &\quad\times|Q_{j+1}\cap\{f\geq k_{j+1}\}|^{1-\frac{2}{q}} \\ &=: (I_1+I_2+I_3)|Q_{j+1}\cap\{f\geq k_{j+1}\}|^{1-\frac{2}{q}} \end{aligned} \tag{15.11}$$

We estimate separately the last three integrals. Starting from I_1, we get that

$$\begin{aligned} I_1 \;=\; & c\int_{Q_j}\int_{B_j}\frac{\omega_{j+1}(v)\omega_{j+1}(w)|\varphi_j(v)-\varphi_j(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t \\ & + c\int_{Q_j}\int_{\mathbb{R}^n\setminus B_j}\frac{\omega_{j+1}(v)\omega_{j+1}(w)\varphi_j^2(v)}{|v-w|^{n+2s}}\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ =: \; & I_{1,1}+I_{1,2}\,. \end{aligned} \tag{15.12}$$

The first integral $I_{1,1}$ can be treated assuming $\omega_{j+1}(v)\geq\omega_{j+1}(w)$, noticing that the reverse inequality holds true when one exchanges the roles of v and w,

as follows,

$$\begin{aligned} I_{1,2} &\leq c2^{2(j+2)}r^{-2}\int_{Q_j}\omega_{j+1}^2\left(\int_{2B_j(v)}\frac{\mathrm{d}w}{|v-w|^{n-2(1-s)}}\right)\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &\leq c2^{2(j+2)}r^{-2}\int_{Q_j}\omega_{j+1}^2\left(\int_0^{2r}\varrho^{2(1-s)-1}\,\mathrm{d}\varrho\right)\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &\leq \frac{c2^{2(j+2)}}{2(1-s)r^{2s}}\int_{Q_j}\omega_j^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\,. \end{aligned}$$

As for the second integral $I_{1,2}$, we have by Hölder's Inequality for some $p>2$ that

$$\begin{aligned} I_{1,2} &\leq c\int_{Q_j\cap\operatorname{supp}\ \varphi_j}\int_{\mathbb{R}^n\setminus B_j}\frac{\omega_{j+1}(v)\omega_{j+1}(w)}{|v-w|^{n+2s}}\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &\leq \left(\int_{Q_j}\omega_{j+1}^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac12} \\ &\quad\times\left[\int_{Q_j\cap\operatorname{supp}\ \varphi_j}\left(\int_{\mathbb{R}^n\setminus B_j}\frac{\omega_{j+1}(w)}{|v-w|^{n+2s}}\,\mathrm{d}w\right)^2\chi_{\{f(v)>k_{j+1}\}}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac12} \\ &\leq c2^{j(n+2s)}\left(\int_{Q_j}\omega_j^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac12}\operatorname{Tail}_p(f_+;\mathbf{0},r,r/2)|Q_j\cap\{f>k_{j+1}\}|^{1-\frac2p} \\ &\leq \frac{c2^{j(n+2s+\frac{2p-4}{p})}k_0^2}{\delta}\left(\int_{Q_j}\frac{\omega_j^2}{k_0^2}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{3p-4}{2p}}\,, \end{aligned}$$

up to choosing

$$k_0\geq\delta\operatorname{Tail}_p(f_+;\mathbf{0},r,r/2)\qquad\text{for }\delta\in(0,1]\,, \tag{15.13}$$

and where we have used that, for $w\in\mathbb{R}^n\setminus B_j$ and $v\in\operatorname{supp}\varphi_j$ (recalling that the support in the v-variable of φ_j is contained in $B_{(r_j+r_{j+1})/2}$)

$$\frac{|w|}{|v-w|}\leq 1+\frac{|v|}{|w|-|v|}\leq 1+\frac{r_j+r_{j+1}}{r_j-r_{j+1}}\leq 2^{j+4}\,.$$

as well as

$$\begin{aligned} |Q_j\cap\{f>k_{j+1}\}| &= |Q_j\cap\{f-k_j>k_{j+1}-k_j\}| \\ &\leq |Q_j\cap\{f-k_j>2^{-j-1}k_0\}| \\ &\leq 2^{2j+2}\int_{Q_j}\frac{\omega_j^2}{k_0^2}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\,. \end{aligned}$$

Combining all the previous estimate for $I_{1,1}$ and $I_{1,2}$ together we get that

$$I_1\leq\frac{c2^{j(n+2s+\frac{2p-4}{p}+2)}k_0^2}{\delta r^{2s}}\left[\int_{Q_j}\frac{\omega_j^2}{k_0^2}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t+\left(\int_{Q_j}\frac{\omega_j^2}{k_0^2}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{3p-4}{2p}}\right]. \tag{15.14}$$

In a similar way, recalling the particular choice of the test function φ_j, we have that

$$\begin{aligned} I_2 + I_3 &\leq \frac{c2^{2j(1+2s)}}{r^{4s}} \int_{Q_j} \omega_j^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &\leq \frac{c2^{2j(1+2s)}}{r^{4s}} \int_{Q_j} \omega_j^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t = \frac{c2^{2j(1+2s)}k_0^2}{r^{4s}} \int_{Q_j} \frac{\omega_j^2}{k_0^2} \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\,. \end{aligned} \tag{15.15}$$

where we have used the fact that

$$\begin{aligned} |v\cdot\nabla_x(\varphi_j^2)| + |v\cdot\nabla_x\varphi_j|^2 &= 2|\varphi_j||v\cdot\nabla_x\varphi_j| + |v\cdot\nabla_x\varphi_j|^2 \\ &\leq \frac{c2^{j(1+2s)}}{r^{2s}} + \frac{c2^{2j(1+2s)}}{r^{4s}} \leq \frac{c2^{2j(1+2s)}}{r^{4s}}\,, \end{aligned}$$

recalling that $r<1$.

Moreover, the measure of the superlevel set in (15.11) can be estimated as follows,

$$\begin{aligned} |Q_{j+1}\cap\{f>k_{j+1}\}|^{1-\frac{2}{q}} &= |Q_{j+1}\cap\{f-k_j>k_{j+1}-k_j\}|^{1-\frac{2}{q}} \\ &\leq |Q_j\cap\{f-k_j>2^{-j-1}k_0\}|^{1-\frac{2}{q}} \\ &\leq 2^{j\frac{2q-4}{q}}\left(\int_{Q_j}\frac{\omega_j^2}{k_0^2}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right)^{1-\frac{2}{q}}. \end{aligned} \tag{15.16}$$

We now define A_j

$$A_j := \int_{Q_j}\frac{\omega_j^2}{k_0^2}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\,,$$

so that, by putting (15.14), (15.15) and (15.16) in (15.11), we get

$$A_{j+1} \leq \frac{c2^{j(n+2s+\frac{2p-4}{2p}+2+2(1+2s)+\frac{2q-4}{q})}}{\delta r^{4s}}\left(A_j^{1+\frac{q-2}{q}} + A_j^{1+\frac{3p-4}{2p}-\frac{2}{q}}\right). \tag{15.17}$$

Now, note that $1-2/q>0$, given that $q>2$, and that $\frac{3p-4}{2p}-\frac{2}{q}>0$, which is in fact possible for p large enough, say $p>p^\star=p^\star(n,s)$, where clearly the latter depends on the growth power $q^\star$ defined in (15.9).

Thus, we can rewrite the inequality in (15.17) as follows,

$$A_{j+1} \leq \frac{cb^j}{\delta r^{4s}}A_j^{1+\varepsilon_*}\,,$$

for some $\varepsilon_*\equiv\varepsilon_*(n,s)$ and $b>1$. Then, up to choosing

$$k_0 := (\delta r^{4s})^{-\frac{1}{2\varepsilon_*}} c^{\frac{1}{2\varepsilon_*}} b^{\frac{1}{2\varepsilon_*^2}}\left(\int_{Q_r} f_+^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} + \delta\,\mathrm{Tail}_p(f_+;\mathbf{0},r,r/2)\,,$$

which is in clear accordance with (15.13). A standard iteration argument yields that $A_j\to 0$ as $j\to\infty$, which finally gives the desired result. □

CHAPTER 16

Towards a Harnack inequality

This section is devoted to the proof of the main ingredients required to obtain Harnack inequalities in (14.6) and (14.8); i. e., the De Giorgi Intermediate Values lemma and the Measure-to-pointwise one, which in turn does also rely on a suitable application of the δ-interpolative L^∞ inequality in Theorem 14.1 by carefully estimating the tail contributions; see forthcoming Section 2.

1. De Giorgi's Intermediate Values Lemma

Our strategy will extend that in the pioneering paper [25], which will help in some of the estimates on the nonlocal energy terms arising from the diffusion in velocity. However, some decisive modifications need to be carried out because of our kinetic framework; that is, the novel presence of the transport term in (14.1). Also, it is worth noticing that our methods are feasible of further generalizations when more spatial commutators are involved.

Consider $\mu < 1$, $r_3 > r_2 > 0$ and a cut-off function $\varphi \equiv \varphi(x) \in C_0^\infty(\mathbb{R}^n)$ such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ in $B_{r_2^{1+2s}}$ and $\varphi \equiv 0$ outside $B_{r_3^{1+2s}}$. Then define the following three auxiliary functions $F_i = F_i(v)$,

$$
\begin{aligned}
F_0(v) &:= \frac{1}{r_3} \max\left\{ -r_3, \, \frac{\min\left\{0, \, |v|^2 - 2r_3^2\right\}}{r_3} \right\}, \\
F_1(v) &:= \frac{1}{r_3} \max\left\{ -r_3, \, \frac{\min\left\{0, \, |v|^2 - r_3^2\right\}}{r_3} \right\}, \\
F_2(v) &:= \frac{1}{r_2} \max\left\{ -r_2, \, \frac{\min\left\{0, \, |v|^2 - r_2^2\right\}}{r_2} \right\}.
\end{aligned}
\tag{16.1}
$$

The underling kinetic geometry is coming up; compare indeed with the single Lipschitz function F which does the job in the purely fractional parabolic setting in [25, Section 4]. Accordingly, we would need the three following consecutive functions φ_i depending on F_i,

$$
\varphi_i = \varphi_i(x,v) := 2 - \varphi(x) + \mu^i F_i(v), \qquad \text{for } i = 0,1,2. \tag{16.2}
$$

We can state the following

Theorem 16.1 (De Giorgi's Intermediate Values Lemma)**.** *Let f be a weak subsolution to (14.1) in Ω according to* Definition 14.6 *such that $f \le 1$ and let $Q_1 \equiv Q_1(\mathbf{0}) \subset \Omega$. Consider $0 < r_1, r_2 < r_3 < 1$ and $0 > t_2 > t_1 > -1$. Define now*

$$
Q^{(1)} := (-1, t_1] \times B_{r_1^{1+2s}} \times B_{r_1}, \qquad Q^{(2)} := (t_2, 0] \times B_{r_2^{1+2s}} \times B_{r_2}
$$

$$
Q^{(3)} := (-1, 0] \times B_{r_3^{1+2s}} \times B_{r_3}.
$$

Given $\delta_1, \delta_2 \in (0,1)$ there exist $\nu, \mu \equiv \nu, \mu(\delta_1, \delta_2, r_1, r_2, r_3, s, n)$ such that if it holds

$$|\{f \le \varphi_0\} \cap Q^{(1)}| \ge \delta_1 |Q^{(1)}| \quad \text{and} \quad |\{f \ge \varphi_2\} \cap Q^{(2)}| \ge \delta_2 |Q^{(2)}|\,, \tag{16.3}$$

then f satisfies

$$|\{\varphi_0 < f < \varphi_2\} \cap Q^{(3)}| \ge \nu |Q^{(3)}|\,, \tag{16.4}$$

where φ_i are defined in (16.2) for $i = 1, 2, 3$.

Proof. The proof requires a sort of both nonlocal and kinetic approach based on suitable choices also in order to estimate all the energy contributions by tracking explicit dependencies on the involved quantities so that the forthcoming Harnack inequalities will not depend on the local slanted cylinders. We divide it into three steps.

Step 1: The energy estimate Up to regularizing by mollification, for any fixed $t \in (-1, 0]$ we can assume that $(f - \varphi_1)_+$ is sufficiently regular in order to be an admissible test function compactly supported in the cylinder $(Q^{(3)})^t := \{(v, x) \in \mathbb{R}^{2n} : (t, x, v) \in Q^{(3)}\}$. Consider now the weak formulation in Definition 14.6 by choosing as a test function $\phi \equiv (f - \varphi_1)_+ \eta^2$ there, with $\eta \in C_0^\infty(B_{(2r_3)^{1+2s}})$ be a cut-off function so that $0 \le \eta \le 1$ and $\eta \equiv 1$ on $B_{r_2^{1+2s}}$; for a. e. $t \in (-1, 0]$ it yields

$$\begin{aligned} 0 \;\ge\; & \int_{(Q^{(3)})^t} (\partial_t f + v \cdot \nabla_x f)(f - \varphi_1)_+ \eta^2 \,\mathrm{d}x\,\mathrm{d}v \\ & + \int_{B_{r_3^{1+2s}}} \mathcal{E}\Big(f, (f - \varphi_1)_+ \eta^2\Big)\,\mathrm{d}x \;=:\; I_1 + I_2. \end{aligned} \tag{16.5}$$

We start by considering I_1. Using the fact that $\partial_t \varphi_1 = 0$ and that $\nabla_x \varphi_1 \neq 0$ only on $B_{r_3^{1+2s}} \setminus B_{r_2^{1+2s}}$, we have that

$$\begin{aligned} I_1 \;=\; & \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{(Q^{(3)})^t} \Big((f - \varphi_1)_+ \eta\Big)^2 \,\mathrm{d}x\,\mathrm{d}v \\ & - \frac{1}{2} \int_{(Q^{(3)})^t} v \cdot \nabla_x (\eta^2)(f - \varphi_1)_+^2 \,\mathrm{d}x\,\mathrm{d}v \\ & + \int_{(Q^{(3)})^t} v \cdot \nabla_x \varphi_1 (f - \varphi_1)_+ \,\mathrm{d}x\,\mathrm{d}v \\ \ge\; & \frac{1}{2} \min_{B_{r_3^{1+2s}}} (\eta^2) \frac{\mathrm{d}}{\mathrm{d}t} \int_{(Q^{(3)})^t} (f - \varphi_1)_+^2 \,\mathrm{d}x\,\mathrm{d}v \\ & - \frac{|Q_1|(2\|v \cdot \nabla_x(\varphi_1)\|_{L^\infty} - \|v \cdot \nabla_x(\eta^2)\|_{L^\infty})}{2} \mu^2\,, \end{aligned} \tag{16.6}$$

where we have used the fact that $(f - \varphi_1)_+ \le \mu$. Moreover, choosing a proper cut-off function η in order to control the second term in the right-hand side of (16.6), we get

$$I_1 \gtrsim \frac{\mathrm{d}}{\mathrm{d}t} \int_{(Q^{(3)})^t} \Big((f - \varphi_1)_+ \eta\Big)^2 \,\mathrm{d}x\,\mathrm{d}v - \mu^2\,. \tag{16.7}$$

We now consider the integral I_2. We start noticing that by the linearity of the involved energy $\mathcal{E}(\cdot)$, we have that

$$\begin{aligned} & \mathcal{E}(f, (f - \varphi_1)_+) \\ & \quad = [(f - \varphi_1)_+]_{H^s}^2 - \mathcal{E}((f - \varphi_1)_-, (f - \varphi_1)_+) + \mathcal{E}(\varphi_1, (f - \varphi_1)_+). \end{aligned} \tag{16.8}$$

Moreover, we note that

$$\begin{aligned}\mathcal{E}(\varphi_1,(f-\varphi_1)_+) &\le \frac{1}{2}[(f-\varphi_1)_+]^2_{H^s}\\ &\quad+2\mu^2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|F_1(v)-F_1(w)|^2}{|v-w|^{n+2s}}(\chi_{B_{r_3}}(v)+\chi_{B_{r_3}}(w))\,\mathrm{d}v\,\mathrm{d}w\\ &\le \frac{1}{2}[(f-\varphi_1)_+]^2_{H^s}+c\,\mu^2\iint_{B_{r_3}\times B_{r_3}}\frac{\mathrm{d}v\,\mathrm{d}w}{|v-w|^{n-2(1-s)}}\\ &\le \frac{1}{2}[(f-\varphi_1)_+]^2_{H^s}+c\,\mu^2\iint_{B_{r_3}\times B_{2r_3}(v)}\frac{\mathrm{d}v\,\mathrm{d}w}{|v-w|^{n-2(1-s)}}\\ &\le \frac{1}{2}[(f-\varphi_1)_+]^2_{H^s}+c\mu^2\,, \qquad (16.9)\end{aligned}$$

where we have used the Lipschitz continuity of F_1 – recall its definition in (16.1) – alongside with the definition of $\mathcal{E}$.

Recalling (16.7) and the fact that $-I_1\ge I_2$ by (16.5), it yields

$$c\mu^2\ge I_2+\frac{\mathrm{d}}{\mathrm{d}t}\int_{(Q^{(3)})^t}(f-\varphi_1)^2_+\,\mathrm{d}x\,\mathrm{d}v. \qquad (16.10)$$

Moreover, by combining (16.8) with (16.9), we obtain that

$$\begin{aligned}I_2 &\le \frac{3}{2}\int_{B_{r_3^{1+2s}}}[(f-\varphi_1)_+]^2_{H^s}\eta^2\,\mathrm{d}x\\ &\quad-\int_{B_{r_3^{1+2s}}}\mathcal{E}((f-\varphi_1)_-,(f-\varphi_1)_+)\eta^2\,\mathrm{d}x+c\,\mu^2.\end{aligned} \qquad (16.11)$$

Then, by summing (16.11) with (16.10), and recalling that $\mu<1$, it follows

$$\begin{aligned}c\mu^2-2I_2 &\ge \frac{\mathrm{d}}{\mathrm{d}t}\int_{(Q^{(3)})^t}(f-\varphi_1)^2_+\,\mathrm{d}x\,\mathrm{d}v\\ &\quad-\frac{3}{2}\int_{B_{r_3^{1+2s}}}[(f-\varphi_1)_+]^2_{H^s}\eta^2\,\mathrm{d}x\\ &\quad+\int_{B_{r_3^{1+2s}}}\mathcal{E}((f-\varphi_1)_-,(f-\varphi_1)_+)\eta^2\,\mathrm{d}x\,;\end{aligned}$$

so that, also in view of (16.8), we finally arrive at

$$\begin{aligned}c\mu^2 &\ge \frac{\mathrm{d}}{\mathrm{d}t}\int_{(Q^{(3)})^t}(f-\varphi_1)^2_+\,\mathrm{d}x\,\mathrm{d}v\\ &\quad+\frac{1}{2}\int_{B_{r_3^{1+2s}}}[(f-\varphi_1)_+]^2_{H^s}\eta^2\,\mathrm{d}x-\int_{B_{r_3^{1+2s}}}\mathcal{E}((f-\varphi_1)_-,(f-\varphi_1)_+)\eta^2\,\mathrm{d}x\\ &\quad+2\int_{B_{r_3^{1+2s}}}\mathcal{E}(\varphi_1,(f-\varphi_1)_+)\eta^2\,\mathrm{d}x.\end{aligned}$$

We now estimate the energy contribution $\mathcal{E}(\varphi_1,(f-\varphi_1)_+)$. We firstly split the contribution given by the nonlocal term as follows,

$$\begin{aligned}
&\mathcal{E}(\varphi_1,(f-\varphi_1)_+)\\
&= \iint_{|v-w|\geq 1} \frac{\Big(\varphi_1(v)-\varphi_1(w)\Big)\Big((f-\varphi_1)_+(v)-(f-\varphi_1)_+(w)\Big)}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w \\
&\quad + \iint_{|v-w|<1} \frac{\Big(\varphi_1(v)-\varphi_1(w)\Big)\Big((f-\varphi_1)_+(v)-(f-\varphi_1)_+(w)\Big)}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w.
\end{aligned} \tag{16.12}$$

Using the very definition of φ_1 and the boundedness of the auxiliary function F_1 in (16.1), we obtain that

$$\begin{aligned}
&\left|\iint_{|v-w|\geq 1} \frac{(\varphi_1(v)-\varphi_1(w))(f-\varphi_1)_+(v)}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\right| \\
&\qquad \leq c\mu \int_{|v-w|\geq 1} \frac{\mathrm{d}w}{|v-w|^{n+2s}} \int_{\mathbb{R}^n} (f-\varphi_1)_+(v)\,\mathrm{d}v \\
&\qquad \leq c\,\mu^2 \int_1^\infty \sigma^{-1-2s}\,\mathrm{d}\sigma \;=\; c\,\mu^2,
\end{aligned}$$

where we have also used that $(f-\varphi_1)_+\leq \mu$ and it is compactly supported. Then, the first integral in the right-hand side of (16.12) can be estimated as follows

$$\left|\iint_{|v-w|\geq 1} \frac{(\varphi_1(v)-\varphi_1(w))((f-\varphi_1)_+(v)-(f-\varphi_1)_+(w))}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\right| \leq c\,\mu^2.$$

Let us consider now the second integral in the right-hand side of (16.12). By suitable applying the Hölder inequality and the Young inequality, we can deduce that

$$\begin{aligned}
&\left|\iint_{|v-w|<1} \frac{(\varphi_1(v)-\varphi_1(w))((f-\varphi_1)_+(v)-(f-\varphi_1)_+(w))}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\right| \\
&\qquad \leq \varepsilon \iint_{|v-w|<1} \frac{|(f-\varphi_1)_+(v)-(f-\varphi_1)_+(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w \\
&\qquad\quad + \frac{1}{\varepsilon}\iint_{|v-w|<1} \frac{|\varphi_1(v)-\varphi_1(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\,,
\end{aligned}$$

for some $\varepsilon>0$ which will be fixed later on. Now, the second integral in the right-hand side of the display above can be estimated via the definition of φ_1 as well as done in previous estimate (16.11), so that

$$\begin{aligned}
\iint_{|v-w|<1} \frac{|\varphi_1(v)-\varphi_1(w)|^2\,\mathrm{d}v\,\mathrm{d}w}{|v-w|^{n+2s}} &\leq \mu^2 \iint_{|v-w|<1} \frac{|F_1(v)-F_1(w)|^2}{|v-w|^{n+2s}}\chi_{B_{r_3}}(v)\,\mathrm{d}v\,\mathrm{d}w \\
&\leq \mu^2 \int_{B_{r_3}} \int_{|v-w|<1} \frac{\mathrm{d}w}{|v-w|^{n-2(1-s)}}\,\mathrm{d}v \\
&\leq c\,\mu^2 \int_0^1 \sigma^{1+2s}\,\mathrm{d}\sigma \int_{B_{r_3}} \mathrm{d}v \\
&\leq \frac{c\,\mu^2 |B_{r_3}|}{2(1+s)}\,.
\end{aligned}$$

All in all, from (16.12) the following estimate has been obtained,

$$\big|\mathcal{E}(\varphi_1,(f-\varphi_1)_+)\big| \,\le\, c(\varepsilon)\,\mu^2 + \varepsilon\,\big[(f-\varphi_1)_+\big]^2_{H^s}\,,$$

for a suitable positive constant $c=c(\varepsilon)$. Choosing ε sufficiently small in order to absorb the seminorm $\big[(f-\varphi_1)_+\big]_{H^s}$ in the right-hand side of the display above, it yields

$$\begin{aligned} c\mu^2 \,\ge\, & \frac{\mathrm{d}}{\mathrm{d}t}\int_{(Q^{(3)})^t}(f-\varphi_1)_+^2\,\mathrm{d}x\,\mathrm{d}v \\ & + \int_{B_{r_3^{1+2s}}}\big[(f-\varphi_1)_+\big]^2_{H^s}\,\eta^2\,\mathrm{d}x \\ & - \int_{B_{r_3^{1+2s}}}\mathcal{E}\Big((f-\varphi_1)_-,(f-\varphi_1)_+\Big)\eta^2\,\mathrm{d}x. \end{aligned} \tag{16.13}$$

Moreover, recalling that $(f-\varphi_1)_+(f-\varphi_1)_-=0$, we obtain that

$$\mathcal{E}((f-\varphi_1)_-,(f-\varphi_1)_+) = -2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(f-\varphi_1)_+(v)(f-\varphi_1)_-(w)}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w.$$

Hence, the inequality in (16.13), also by recalling the fact that $\eta\in C_0^\infty(B_{(2r_3)^{1+2s}})$, can be written as follows,

$$\begin{aligned} c\mu^2 \,\ge\, & \frac{\mathrm{d}}{\mathrm{d}t}\int_{(Q^{(3)})^t}(f-\varphi_1)_+^2\,\mathrm{d}x\,\mathrm{d}v \\ & + \int_{B_{r_3^{1+2s}}}\big[(f-\varphi_1)_+\big]^2_{H^s}\,\eta^2\,\mathrm{d}x \\ & + \int_{B_{r_3^{1+2s}}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(f-\varphi_1)_+(v)(f-\varphi_1)_-(w)}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x. \end{aligned} \tag{16.14}$$

Notice that both the second and the third term in the inequality above in (16.14) are nonnegative, once we define

$$\mathcal{H}(t) := \int_{(Q^{(3)})^t}(f-\varphi_1)_+^2(t,x,v)\,\mathrm{d}x\,\mathrm{d}v\,, \tag{16.15}$$

it yields that for $-1<t\le 0$ we have

$$\mathcal{H}'(t)\le c\mu^2\,.$$

Moreover, let us note that $(f-\varphi_1)_+ \le \mu\mathbb{1}_{Q^{(3)}}$, hence $\mathcal{H}(t)\le c\,\mu^2$. Then, integrating inequality (16.14) in time for $-1<\tau_1<\tau_2\le 0$, we finally get

$$\begin{aligned} \int_{\tau_1}^{\tau_2}\int_{B_{r_3^{1+2s}}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(f-\varphi_1)_+(v)(f-\varphi_1)_-(w)}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t & \\ \le c\,\mu^2(\tau_2-\tau_1) + \big|\mathcal{H}(\tau_2)-\mathcal{H}(\tau_1)\big| & \\ \le c\,\mu^2(\tau_2-\tau_1) + c\,\mu^2 \;\le\; c\,\mu^2\,. & \end{aligned} \tag{16.16}$$

Step 2: Estimating time and space slices Starting from the assumption in (16.3), let us call Σ the set of times in $(-1,t_1]$ defined as follows,

$$\Sigma := \left\{ t\in(-1,t_1] \,:\, \Big|\big\{f(t,\cdot,\cdot)\le\varphi_0\big\}\cap(Q^{(3)})^t\Big| \ge \frac{\delta_1|Q^{(1)}|}{4}\right\}.$$

Such a set Σ satisfies the following estimate,

$$|\Sigma| \geq \frac{\delta_1}{2}\Big(\frac{r_1}{r_3}\Big)^{2(n+s)}(t_1+1)\,.$$

Indeed, a plain computation leads to

$$\begin{aligned}\Big|\{f\leq\varphi_0\}\cap\Big((Q^{(3)})^t\times(-1,t_1]\Big)\Big| \\ = \Big|\big\{f(t,\cdot,\cdot)\leq\varphi_0\big\}\cap(Q^{(3)})^t\Big|\,|\Sigma| + \Big|\big\{f(t,\cdot,\cdot)\leq\varphi_0\big\}\cap(Q^{(3)})^t\Big|\,\Big|\mathscr{C}_{(-1,t_1]}(\Sigma)\Big|\,,\end{aligned}$$

where, as customary, by $\mathscr{C}_{(-1,t_1]}(\Sigma)$ we denoted the complementary set of Σ in $(-1,t_1]$. Thus, we get

$$\begin{aligned}|\{f(t,\cdot,\cdot)\leq\varphi_0\}\cap(Q^{(3)})^t||\Sigma| \\ &\geq |\{f\leq\varphi_0\}\cap Q^{(1)}| - |\{f(t,\cdot,\cdot)\leq\varphi_0\}\cap(Q^{(3)})^t||\mathscr{C}_{(-1,t_1]}(\Sigma)| \\ &\geq \delta_1|Q^{(1)}| - \frac{\delta_1|Q^{(1)}|}{4}|\mathscr{C}_{(-1,t_1]}(\Sigma)| \\ &\geq \delta_1|Q^{(1)}| - \frac{\delta_1|Q^{(1)}|}{4} \\ &\geq \frac{3}{4}\delta_1|Q^{(1)}| > \frac{1}{2}\delta_1|Q^{(1)}|\,.\end{aligned}$$

Then, dividing the previous inequality on both sides by $|\{f(t,\cdot,\cdot)\leq\varphi_0\}\cap(Q^{(3)})^t|$ yields

$$\begin{aligned}|\Sigma| &\geq \frac{\delta_1}{2}\frac{|Q^{(1)}|}{|\{f(t,\cdot,\cdot)\leq\varphi_0\}\cap(Q^{(3)})^t|} \\ &\geq \frac{\delta_1}{2}\frac{|Q^{(1)}|}{|Q^{(3)}|} \geq \frac{\delta_1}{2}\Big(\frac{r_1}{r_3}\Big)^{2(n+s)}(t_1+1)\,,\end{aligned}$$

as claimed.

Starting again from estimate (16.16), we have

$$\begin{aligned}c\,\mu^2 &\geq \int_{-1}^0\int_{B_{r_3^{1+2s}}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(f-\varphi_1)_+(v)(f-\varphi_1)_-(w)}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t \\ &\geq c\int_\Sigma\int_{B_{r_3^{1+2s}}}\iint_{\mathbb{R}^n\times B_{r_3}}(f-\varphi_1)_+(v)(f-\varphi_1)_-(w)\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t \\ &\geq c\int_\Sigma\int_{B_{r_3^{1+2s}}}\int_{(\{f(\cdot,x,t)\leq\varphi_0\}\cap B_{r_3})\times B_{r_3}}(f-\varphi_1)_+(v)(\varphi_1-\varphi_0)_-(w)\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t \\ &\geq c\int_\Sigma\int_{B_{r_3^{1+2s}}}\int_{(\{f(\cdot,x,t)\leq\varphi_0\}\cap B_{r_3})\times B_{r_3}}(f-\varphi_1)_+(v)(\mu F_1-F_0)_+(w)\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t \\ &\geq c(1-\mu)\int_\Sigma\int_{B_{r_3^{1+2s}}}\left(\int_{B_{r_3}}(f-\varphi_1)_+(v)\,\mathrm{d}v\right)\Big|\big\{f(\cdot,x,t)\leq\varphi_0\big\}\cap B_{r_3}\Big|\,\mathrm{d}x\,\mathrm{d}t \\ &\geq \frac{c(1-\mu)|Q^{(1)}|\delta_1}{4\mu}\int_\Sigma\int_{(Q^{(3)})^t}(f-\varphi_1)_+^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\,, \qquad (16.17)\end{aligned}$$

where we have also used the following estimates,

$$\begin{aligned}
&\Big|\big\{f(\cdot,x,t)\le\varphi_0\big\}\cap B_{r_3}\Big| \;\ge\; \frac{\delta_1|Q^{(1)}|}{4} \quad \text{a. e. on } B_{r_3^{1+2s}} \text{ and for } t\in\Sigma\,,\\
&(f-\varphi_1)_+ \;\le\; \mu\,,\\
&(\varphi_1-\varphi_0)_- \;=\; (\mu F_1-F_0)_+ \;\ge\; 1-\mu \quad \text{on } B_{r_3}\,,
\end{aligned}$$

and

$$\inf_{B_{r_3}\times B_{r_3}} |v-w|^{-n-2s} \;\ge\; c>0.$$

Hence, we eventually get

$$\int_\Sigma\int_{(Q^{(3)})^t}(f-\varphi_1)_+^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \;\le\; \frac{c\mu^3}{(1-\mu)\delta_1} \;\le\; \mu^{3-1/8}\,,$$

if μ is sufficiently small. In particular, we have that

$$\int_{(Q^{(3)})^t}(f-\varphi_1)_+^2\,\mathrm{d}v\,\mathrm{d}x \;\le\; \mu^{3-1/4} \tag{16.18}$$

does hold for any $t\in(-1,t_1]$ except on a set Υ for which we have that, by Chebychev's Inequality,

$$|\Upsilon| \;:=\; \left|\left\{t\in(-1,t_1]\,:\,\|(f-\varphi_1)_+(t,\cdot)\|^2_{L^2((Q^{(3)})^t)}>\mu^{3-1/4}\right\}\right| \le \mu^{1/8}.$$

Taking a smaller μ such that

$$\mu\le\left(\frac{\delta_1}{4}\right)^8, \tag{16.19}$$

we can finally have that (16.18) holds on a set $t\in(-1,t_1]$ of measure greater than $\frac{3}{4}\delta_1$.

Step 3: The intermediate set for f Assume now that there exists a time $\tau_0\in(t_2,0)$ such that

$$|\{(v,x)|(f-\varphi_2)_+(\tau_0,\cdot,\cdot)>0\}\cap Q^{(2)}|>\frac{\delta_2}{2}|Q^{(2)}|.$$

Thus, at time τ_0 we have

$$\begin{aligned}
\mathcal{H}(\tau_0) \;&=\; \int_{(Q^{(3)})^{\tau_0}}(f-\varphi_1)_+^2(v,x,\tau_0)\,\mathrm{d}x\,\mathrm{d}v\\
&\ge\; \int_{(Q^{(3)})^{\tau_0}}(\varphi_2-\varphi_1)^2(v,x,\tau_0)\chi_{\{(f-\varphi_2)_+(\cdot,\tau_0)>0\}}\,\mathrm{d}x\,\mathrm{d}v\\
&\ge\; \int_{(Q^{(3)})^{\tau_0}}(\mu^2F_2-\mu F_1)^2(v,x,\tau_0)\chi_{\{(f-\varphi_2)_+(\cdot,\tau_0)>0\}}\,\mathrm{d}x\,\mathrm{d}v\\
&\ge\; \int_{(Q^{(2)})^{\tau_0}}\mu^2(\mu F_2-F_1)^2(v)\,\chi_{\{(f-\varphi_2)_+(\tau_0,\cdot,\cdot)>0\}}\,\mathrm{d}x\,\mathrm{d}v\\
&\overset{(16.19)}{\ge}\; \frac{\mu^2}{2}\min_{\substack{v\in B_{r_2}\\ \mu\le(\frac{\delta_1}{4})^8}}(\mu F_2-F_1)^2\left|\big\{(x,v)\,:\,(f-\varphi_2)_+(\tau_0,\cdot,\cdot)>0\big\}\cap Q^{(2)}\right|\\
&\ge\; \mathfrak{c}\frac{\mu^2}{4}\delta_2|Q^{(2)}|\,, \qquad (16.20)
\end{aligned}$$

where the positive constant $\mathfrak{c}$ depends only on F_1, F_2 and δ_1.

Moreover, consider a time $\overline{\tau} \leq \tau_0$ such that $\overline{\tau} \in (-1, t_1]$ such that

$$\mathcal{H}(\overline{\tau}) = \int_{(Q^{(3)})^{\overline{\tau}}} (f - \varphi_1)_+^2(v, x, \overline{\tau}) \, \mathrm{d}v \, \mathrm{d}x \, \leq \, \mu^{3-1/4}.$$

In this way, we choose μ sufficiently small (up to shrinking a smaller δ_2 if needed) such that

$$\mu^{1-1/4} \geq \mathfrak{c} \frac{|Q^{(2)}|\delta_2}{16}. \tag{16.21}$$

and thus the energy $\mathcal{H}(\cdot)$ of $(f - \varphi_1)_+^2(t, \cdot, \cdot)$ passes through the range of times

$$D := \left\{ \tau \in (\overline{\tau}, \tau_0) \, : \, \mathfrak{c} \frac{|Q^{(2)}|\mu^2}{16} \delta_2 < \mathcal{H}(\tau) < \mathfrak{c} \frac{|Q^{(2)}|\mu^2}{4} \delta_2 \right\}.$$

In such a range of times we have that

$$\left| \left\{ (f - \varphi_2)_+(\tau, \cdot, \cdot) > 0 \right\} \cap (Q^{(3)})^\tau \right| \leq \frac{\delta_2}{2} |Q^{(2)}|. \tag{16.22}$$

Indeed, by contradiction assume that the reverse inequality holds true for some $\tau \in D$. Hence, at such a time slice τ, going through the same computation as in (16.20), we will arrive at $\mathcal{H}(\tau) \geq \mathfrak{c} \frac{|Q^{(2)}|\mu^2}{4} \delta_2$ which is in contradiction with the fact that $\tau \in D$.

Thus, up to choosing δ_2 sufficiently small, we have that the measure of the set appearing in (16.22) is negligible.

Now, we estimate the size of the set U of times slice of D for which

$$\left| \left\{ (f - \varphi_0)_+(\tau, \cdot, \cdot) \leq 0 \right\} \cap (Q^{(3)})^t \right| \, \geq \, \delta_1 |Q^{(2)}|.$$

By (16.16) we have

$$\begin{aligned} c\,\mu^2 \quad &\geq \quad \int_{-1}^0 \int_{B_{r_3^{1+2s}}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f - \varphi_1)_+(v)(f - \varphi_1)_-(w)}{|v - w|^{n+2s}} \, \mathrm{d}v \, \mathrm{d}w \, \mathrm{d}x \, \mathrm{d}t \\ &\geq \quad \frac{c\delta_1 |Q^{(2)}|}{\mu} \int_U \int_{B_{r_3^{1+2s}}} \int_{\mathbb{R}^n} (f - \varphi_1)^2 \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t \\ &\geq \quad \frac{c\delta_1\delta_2 |U| \mu |Q^{(2)}|}{16}, \end{aligned}$$

where we have followed a similar reasoning as in (16.17)–(16.20), and in the last line we have also used the fact that $\tau \in U \subset D$. Therefore, we obtain that

$$|U| \, \leq \, \frac{c\mu}{\delta_1\delta_2 |Q^{(2)}|}.$$

Then, by choosing

$$\mu \, \leq \, \delta_1\delta_2 |D| |Q^{(2)}|/(2c) \tag{16.23}$$

we plainly deduce that

$$|U| \, \leq \, \frac{|D|}{2}.$$

Consider now those times $\tau \in D$ which are not in U. Hence, we have

$$\left| \left\{ \varphi_0 < f(\tau, \cdot, \cdot) < \varphi_2 \right\} \cap (Q^{(3)})^\tau \right| \geq \frac{|Q^{(3)}|}{2}, \tag{16.24}$$

Indeed,

$$\begin{aligned}\Big|\big\{\varphi_0 < f(\tau,\cdot,\cdot) < \varphi_2\big\} \cap (Q^{(3)})^\tau\Big| &\geq |(Q^{(3)})^\tau| - \Big|\big\{\varphi_0 > f(\tau,\cdot,\cdot)\big\} \cap (Q^{(3)})^\tau\Big| - \Big|\big\{f(\tau,\cdot,\cdot) > \varphi_2\big\} \cap (Q^{(3)})^\tau\Big| \\ &\geq |(Q^{(3)})^\tau| - \delta_1|Q^{(2)}| - \frac{\delta_2}{2}|Q^{(2)}| \\ &\geq \frac{|Q^{(3)}|}{2},\end{aligned}$$

up to choosing δ_1 and δ_2 small enough.

Hence, by (16.24), we finally deduce

$$\begin{aligned}|\{\varphi_0 < f < \varphi_2\} \cap Q^{(3)}| &= \int_{-1}^{0} |\{\varphi_0 < f(\tau,\cdot,\cdot) < \varphi_2\} \cap (Q^{(3)})^\tau| \,\mathrm{d}\tau \\ &\geq \int_{D\setminus U} |\{\varphi_0 < f(\tau,\cdot,\cdot) < \varphi_2\} \cap (Q^{(3)})^\tau| \,\mathrm{d}\tau \\ &\geq \frac{|D||Q^{(3)}|}{4} \geq \nu|Q^{(3)}|,\end{aligned}$$

up to choosing a constant $\nu \equiv \nu(\delta_1, \delta_2)$ sufficiently small, as desired. □

2. The Measure-to-pointwise Lemma

In order to prove a Measure-to-point-wise-type lemma, we will employ Theorem 16.1 established in the previous section, together with the nonlocal L^∞-L^2-type estimate with tail (14.5) stated in the Introduction.

Theorem 16.2 (Measure-to-pointwise Lemma)**.** *Let $\mathring{\delta} \in (0,1)$ and for $0 < r_1, r_2 < 1$ and $0 > t_2 > t_1 > -1$ consider*

$$Q^{(1)} := (-1, t_1] \times B_{r_1^{1+2s}} \times B_{r_1}, \qquad Q^{(2)} := (t_2, 0] \times B_{r_2^{1+2s}} \times B_{r_2},.$$

Let g be a weak subsolution to (14.1) in Ω such that $g \leq 1$ in Ω and

$$|\{g \leq 0\} \cap Q^{(1)}| \geq \mathring{\delta}|Q^{(1)}|. \tag{16.25}$$

Then there exists a real number $\vartheta \equiv \vartheta(\mathring{\delta}, \mu, \nu) \in (0,1)$ such that

$$g \leq 1 - \vartheta \qquad \text{in} \quad Q_\varrho(t_0, x_0, v_0), \tag{16.26}$$

for any $Q_\varrho(t_0, x_0, v_0) \subseteq Q^{(2)}$, with $\varrho \leq \min\{|t_2|, r_2\}/2$, where ν and μ are the constants introduced in Theorem 16.1.

Proof. Let us consider $0 < \mu < 1$ as in (16.2) and let us define a sequence of functions

$$g_k := \frac{1}{\mu^{2k}}\Big(g - (1 - \mu^{2k})\Big) = 1 - \frac{1-g}{\mu^{2k}}, \qquad k \geq 0.$$

Given g is a subsolution, then also g_k is a subsolution to (14.1) for every $k \geq 0$. Additionally, since $g \leq 1$, then $g_k \leq 1$ for every $k \geq 0$, but notice that it may also be negative.

Moreover, it is true that

$$\begin{aligned}&\{\varphi_0 < g_i < \varphi_2\} \cap \{\varphi_0 < g_j < \varphi_2\} = \emptyset \qquad \forall i, j \geq 0, \text{ with } i \neq j, \\ &\text{and} \qquad |\{g_k \leq 0\} \cap Q^{(1)}| \geq \mathring{\delta}|Q^{(1)}|.\end{aligned} \tag{16.27}$$

The first relation in the display above descends from the very definition of g_k and (16.2). Indeed, for any $k \geq 0$ it holds

$$\{\varphi_0 < g_k < \varphi_2\}$$
$$:= \left\{1 + \mu^{2k}\left(1 - \varphi(x) + F_0(v)\right) < g < 1 + \mu^{2k}\left(1 - \varphi(x) + \mu^2 F_2(v)\right)\right\}.$$

Hence, it is clear that if we consider $i, j \geq 0$, with $i \neq j$ the claim is proved. As far as we are concerned with the second relation in (16.27), for every $k \geq 0$ the set $\{g_k \leq 0\}$ is equivalent to the set $\{g \leq 1 - \mu^{2k}\}$. Then, considering that $1 - \mu^{2k} \geq 0$ together with (16.25), the claim is proved for every $k \geq 0$.

Now, we apply the boundedness estimate (14.5) to every g_k, with $k \geq 0$ in $Q^{(2)}$, and we obtain

$$\sup_{Q_\varrho(t_0,x_0,v_0)} g_k \leq c\,\delta^{-\frac{1}{2\varepsilon_*}}\left(\int_{Q^{(2)}} (g_k)_+^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} + \delta \mathrm{Tail}_p((g_k)_+; \mathbf{0}, r, r/2)\,,$$

for any $Q_\varrho(t_0, x_0, v_0) \subset Q^{(2)}$, with $\varrho \leq \min\{|t_2|, r_2\}/2$.

We now observe that if for some $\bar{k}$ the following inequality does hold true,

$$c\,\delta^{-\frac{1}{2\varepsilon_*}}\left(\int_{Q^{(2)}} (g_{\bar{k}})_+^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} + \delta \mathrm{Tail}_p((g_{\bar{k}})_+; \mathbf{0}, r, r/2) < \frac{1}{2}\,,$$

then $g_{\bar{k}} \leq 1/2$, and hence $g \leq 1 - \mu^{2\bar{k}}/2$ implying the thesis with $\vartheta := \mu^{2\bar{k}}/2$.

Hence, we are left with the proof of our statement when there exists $k_0 \geq 0$ such that

$$c\,\delta^{-\frac{1}{2\varepsilon_*}}\left(\int_{Q^{(2)}} (g_k)_+^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} + \delta \mathrm{Tail}_p((g_k)_+; \mathbf{0}, r, r/2)$$
$$> \frac{1}{2} \quad \forall k \text{ s. t. } 0 \leq k \leq k_0 - 1\,, \tag{16.28}$$

Then, for every $k \in \mathbb{R}$ such that $0 \leq k \leq k_0 - 1$ it holds

$$|\{g_k \leq \varphi_0\} \cap Q^{(1)}| = |\{g \leq 1 - \mu^{2k}\varphi(x)\} \cap Q^{(1)}| \geq \mathring{\delta}|Q^{(1)}|\,,$$

because $0 < \mu < 1$, $F_0(v) = -1$ in B_r, $\varphi(x) \in [0,1]$, and thus $1 - \mu^{2k}\varphi(x) \geq 0$, allowing us to employ (16.27). Now, if we recall that $\varphi_2 = 1 - \mu^2$ in $Q^{(2)}$ by definition, we also have that, choosing δ sufficiently small,

$$\begin{aligned}
\frac{\left|\{g_k > \varphi_2\} \cap Q^{(2)}\right|}{|Q^{(2)}|} &\geq \frac{|\{g_k > 1 - \mu^2\} \cap Q_r^+|}{|Q_1(\mathbf{0})|} \\
&= c|\{g_{k+1} > 0\} \cap Q^{(2)}| \\
&\geq c\int_{Q^{(2)}} (g_{k+1})_+^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\
&> c\,\delta^{\frac{1}{\varepsilon_*}}\left(\frac{1}{2} - \delta \mathrm{Tail}_p((g_k)_+; \mathbf{0}, r, r/2)\right)^2 \\
&> c\,\delta^{\frac{1}{\varepsilon_*}}\left(\frac{1}{2} - \frac{\delta\omega_n|U_1(0,0)|^{\frac{1}{p}}}{2s}\right)^2 =: \delta_2\,.
\end{aligned}$$

The estimate above comes from the fact that $g_{k+1} \leq 1$, implying $0 \leq (g_{k+1})_+ \leq 1$, combined with the definition of the indicator function $\mathbb{1}_{\{g_{k+1} \geq 0\}}$, the estimate in (16.28).

Finally, thanks to Theorem 16.1 applied to every g_k, with $0 \le k \le k_0 - 1$, with the choice $\delta_1 = \mathring{\delta}$ there, we can deduce the existence of a constant $\nu \equiv \nu(\mathring{\delta}, n, s) > 0$ (recalling the dependencies of δ_2) such that

$$\left|\{\varphi_0 < g_k < \varphi_2\} \cap Q^{(3)}\right| \ge \nu |Q^{(3)}|.$$

Bearing in mind the sets $|\{\varphi_0 < g_k < \varphi_2\}|$ are disjoint (compare with (16.27)), we then obtain

$$|Q^{(3)}| \ge \sum_{k=0}^{k_0-1} \left|\{\varphi_0 < g_k < \varphi_2\} \cap Q^{(3)}\right| \ge k_0\, \nu |Q^{(3)}|.$$

Hence, it holds $k_0 \le 1/\nu$ and, recalling that $g_{k_0+1} \le 1/2$ in $Q_\varrho(t_0, x_0, v_0)$ by definition of k_0 (see (16.28)), it yields

$$g \le 1 - \frac{\mu^{2k_0+2}}{2} \le 1 - \frac{\mu^{\frac{2}{\nu}+2}}{2} \text{ in } Q_\varrho(t_0, x_0, v_0).$$

Eventually, the claim follows by taking $\vartheta := \mu^{\frac{2}{\nu}+2}/2$. □

CHAPTER 17

Proof of the weak Harnack inequality

In view of the results in the proceeding Section, it suffices to apply the final strategy in [88] with no fundamental modifications, except than the fact that we can rely on our Measure-to-point Lemma, in turn relying in Theorem 14.1, and thus we will not need the whole architecture running the propagation of minima argument there (see, in particular, [88, Sections 6 and 9]. For the sake of the reader, we will present the whole proof in a few steps right below.

Step 1: Propagation in measure As a consequence of the result in Theorem 16.2, we can prove that there exist two constants $M > 1$ and $\delta > 0$ such that if

$$|\{f \geq M\} \cap Q_1(\mathbf{0})| \geq (1-\delta)|Q_1(\mathbf{0})|\,, \tag{17.1}$$

then $f \geq 1$ on $\mathcal{Q} := [0, 2^{2s}] \times B_{2^{1+2s}} \times B_2$. Notice that this is equivalent to prove that if $|\{f \leq 1\} \cap Q_1(\mathbf{0})| < \delta$ then $f \geq 1/M$.

For this, we apply the measure-to-pointwise lemma to the function $f(t + 2^{2s}, x, v)$ in appropriated cylinders shifted in time, so that we can deduce

$$f \geq 1/M \quad \text{in } \mathcal{Q}, \tag{17.2}$$

by choosing $M = 1/\vartheta$, where ϑ is the one in Formula (16.26).

Step 2: Stacked propagation By the same argument as in [88, Corollary 9.2], for $k \geq 1$, $T_k = \sum_{i=1}^{k} 2^{2si}$, if f satisfies

$$|\{f \geq M^k\} \cap Q_1(\mathbf{0})| \geq (1-\delta)|Q_1(\mathbf{0})|\,,$$

for M and δ given by Step 1 above, then $f \geq 1$ in $Q[k] := [T_{k-1}, T_k] \times B_{2^{(1+2s)k}} \times B_{2^k}$; see Figure 17.

Step 3: Proof of the weak Harnack inequality (14.6) We prove that, for any $k \geq 1$ and for some fixed $r_0 \in (0,1)$ which will be chosen later on, it holds

$$|\{f > \bar{M}^k\} \cap Q^-_{r_0}| \leq \bar{c}(1-\bar{\delta})^k\,, \tag{17.3}$$

for some constants $\bar{M}$, $\bar{c}$ and $\bar{\delta} \in (0,1)$.

We start by induction. For $k = 1$, we simply choose $\bar{c}$ and $\bar{\delta}$ so that

$$|Q^-_{r_0}| \leq \bar{c}(1-\delta) \quad \text{and} \quad \bar{\delta} \leq \delta.$$

Assume now that (17.3) holds true up to rank k and prove it for $k+1$. We want to apply the Ink-spot Theorem 14.4 with $\mu = \delta$, with some integer m (which will be fixed later on) and with $\bar{M} := M^m$, with M and δ being the constant give by Step 1.

Let us consider the following sets,

$$E := \{f \geq \bar{M}^{k+1}\} \cap Q^-_{r_0} \quad \text{and} \quad F := \{f > \bar{M}^k\} \cap Q_1(\mathbf{0}).$$

Clearly, by recalling the definition (14.7), we infer the sets $E \subset F \subset Q_1(\mathbf{0})$ are bounded and measurable. Let us assume that for any cylinder $Q_\sigma(t_0, x_0, v_0) \subset$

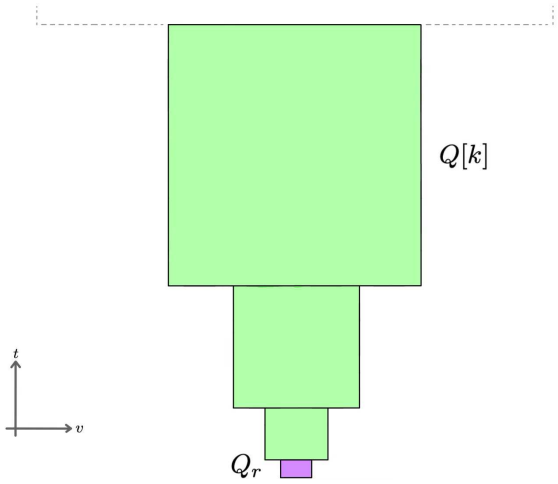

FIGURE 1. The stacked propagation geometry in the proof of the weak Harnack inequality in Theorem 14.2, as introduced in [88, Section 9].

$Q^-_{r_0}$, for some $\sigma \in (0, r_0)$, it holds

$$|Q_\sigma(t_0, x_0, v_0) \cap E| > (1-\delta)|Q_\sigma(t_0, x_0, v_0)|.$$

Hence,

$$|\{f \geq \bar{M}^{k+1}\} \cap Q_\sigma(t_0, x_0, v_0)| > (1-\mu)|Q_\sigma(t_0, x_0, v_0)|. \tag{17.4}$$

We apply now the Measure-to-pointwise Lemma, Theorem 16.2, to the subsolution g defined as follows,

$$g := 1 - f\left((t_0 + 1 - \sigma^{2s}, x_0, v_0) \circ \cdot\right) / \bar{M}^{k+1},$$

and we get that

$$f\left((t_0 + 1 - \sigma^{2s}, x_0, v_0) \circ \cdot\right) \geq \vartheta \bar{M}^{k+1} \qquad \text{on } Q^+_{\sigma/2}.$$

Thus, choosing $\mathfrak{q} := -\log\vartheta / \log(2^{2s}) > 0$ we get (recalling that we can assume f having infimum less or equal than 1)

$$1 \geq \vartheta \bar{M}^{k+1} = \left(\frac{1}{2}\right)^{2s\mathfrak{q}} \bar{M}^{k+1} \geq \left(\frac{\sigma}{2}\right)^{2s\mathfrak{q}} \bar{M}^{k+1}.$$

Then, we get that

$$r_0 := 2\bar{M}^{-\frac{k}{2s\mathfrak{q}}}.$$

Now to prove that $\bar{Q}^m_\sigma(t_0, x_0, v_0) \subset F$, that is $\bar{Q}^m_\sigma(t_0, x_0, v_0) \subset \{f \geq \bar{M}^k\}$, we apply the result of Step 2 with $k = m$ to $\bar{M}^{-k} f((t_0, x_0, v_0) \circ \cdot)$.

Thus, by the Ink-spot Theorem 14.4 (with $\mu \equiv \delta$ and $r_0 \equiv 2\bar{M}^{-\frac{k}{2sq}}$ there) we get

$$
\begin{aligned}
&|\{f \geq \bar{M}^{k+1}\} \cap Q^-_{r_0}| \\
&\quad \leq \frac{1+m}{m}(1-c\delta)\Big(|\{f > \bar{M}^k\} \cap Q^-_{r_0}| + Cmr_0^{2s}\Big) \\
&\quad \leq \frac{1+m}{m}(1-c\delta)\Big(\bar{c}(1-\bar{\delta})^k + cm\bar{M}^{-\frac{k}{q}}\Big) \quad \Big(\text{by the induction step (17.3)}\Big) \\
&\quad \leq \bar{c}\frac{1+m}{m}(1-c\delta)\left(1+\frac{cm}{\bar{c}}\right)(1-\bar{\delta})^k \quad \Big(\text{choosing } 1-\bar{\delta} > \bar{M}^{-1/q}\Big) \\
&\quad \leq \bar{c}(1-\bar{\delta})^{k+1}\,,
\end{aligned}
$$

up to choose m and consequently $\bar{c}$ large enough. This proves the desired induction step. The proof of estimate (14.6) will then follow by a standard argument via the layer-cake formula.

CHAPTER 18

Proof of the strong Harnack inequality

1. A new Besicovitch-type covering for slanted cylinders

In particular, as mentioned in the Introduction, we will rely on a new covering argument for the involved slanted cylinders. Such a general Besicovitch-type result will be presented right below.

The following properties for the slanted cylinders in (14.11) do hold true:

(1) **(Monotonicity)** Given a slanted cylinder $Q_\sigma(t,x,v)$, and $\varrho>0$, there exist a point (t',x',v') and two constant $\varkappa,\varepsilon>0$ such that

$$\|(t,x,v)^{-1}\circ(t',x',v')\|_{\rm kin}\,\le\,\varepsilon\Big(\frac{\varkappa\sigma}{\varrho}\Big)\,,$$
$$\text{and}\quad Q_{\frac{\varkappa\sigma}{\varrho}}(t',x',v')\subset Q_\sigma(t,x,v)\subset Q_{\frac{\sigma}{\varkappa\varrho}}(t',x',v')\,.$$

(2) **(Exclusion)** There exists $\gamma>0$ such that for any $Q_\varrho(t_0,x_0,v_0)$ and $(t,x,v)\notin Q_\varrho(t_0,x_0,v_0)$ it holds

$$Q_{\epsilon^\gamma}(t,x,v)\cap Q_{(1-\epsilon)\varrho}(t_0,x_0,v_0)=\emptyset\quad\text{for any }0<\epsilon<1.$$

(3) **(Inclusion)** There exists $\wp>1$ such that for $0<\sigma<\varrho<1$ and $(t,x,v)\in Q_\varrho(t_0,x_0,v_0)$ it holds

$$Q_{(\varrho-\sigma)^\wp}(t,x,v)\subset Q_\varrho(t_0,x_0,v_0).$$

(4) **(Engulfment)** There exists a constant $\kappa\equiv\kappa(s)$ such that for any $Q_\varrho(t_0,x_0,v_0)$ and $Q_\sigma(t,x,v)$ with

$$Q_\varrho(t_0,x_0,v_0)\cap Q_\sigma(t,x,v)\neq\emptyset\quad\text{and}\quad 2\varrho\ge\sigma\,,$$

it holds that

$$Q_\sigma(t,x,v)\subset\kappa Q_\varrho(t_0,x_0,v_0)\,,$$

with

$$\kappa Q_\varrho(t_0,x_0,v_0):=\Big\{(t,x,v)\ :\ -\frac{\kappa^{2s}+1}{2}\varrho^{2s}<t-t_0\le\frac{\kappa^{2s}-1}{2}\varrho^{2s}$$
$$|v-v_0|<\kappa\varrho,\ \big|x-x_0-(t-t_0)v_0\big|<(\kappa\varrho)^{1+2s}\Big\}.$$

The quantity $\|\cdot\|_{\rm kin}$ is that obtained via the customary *kinetic distance*, firstly seen in [89] for proving Schauder estimates for Boltzmann equations; that is,

$$\|(t,x,v)\|_{\rm kin}:=\max\Big\{|t|^{\frac{1}{2s}},\,|x|^{\frac{1}{1+2s}},\,|v|\Big\}.$$

Compare, also, our *Engulfment* with Lemma 10.4 there.

We can now state and prove the following

Lemma 18.1 (Besicovitch's covering Lemma for slanted cylinders)**.** *Let $\Omega \subset \mathbb{R}^{2n+1}$ be a bounded set. Assume that for any $(t,x,v) \in \Omega$ there exists a family of slanted cylinders $Q_r(t,x,v)$ with $r \leq R$, for some $R > 0$. Then, there exists a countable family $\Im := \{Q_{r_k}(t_k,x_k,v_k)\}_{k\in\mathbb{N}}$ with the following properties*

(i) $\Omega \subset \bigcup_{k=1}^{\infty} Q_{r_k}(t_k,x_k,v_k)$.

(ii) $(t_k,x_k,v_k) \notin \bigcup_{j<k} Q_{r_k}(t_j,x_j,v_j)$, *for any* $k \geq 2$.

(iii) *For $\epsilon > 0$ depending only on the* Monotonicity *constants $\varkappa$ and ε, the family $\Im_\epsilon := \left\{Q_{(1-\epsilon)r_k}(t_k,x_k,v_k)\right\}_{k\in\mathbb{N}}$ has bounded overlaps. Moreover,*

$$\sum_{k=1}^{\infty} \chi_{Q_{(1-\epsilon)r_k}(t_k,x_k,v_k)}(t,x,v) \;\leq\; c\log\left(\frac{1}{\epsilon}\right),$$

where the constant c depends only on $\varkappa$, ε, and the Exclusion *constant γ.*

Proof. Let us assume with no loss of generality that $R := \sup\{r : Q_r(t,x,v) \in \Im\}$. We set

$$\Im_0 := \left\{Q_r(t,x,v) : \frac{R}{2} < r \leq R,\ Q_r(t,x,v) \in \Im\right\},$$

and

$$O_0 := \left\{(t,x,v) : Q_r(t,x,v) \in \Im_0\right\}.$$

Let us choose $Q_{r_1}(t_1,x_1,v_1) \in \Im_0$. If $O_0 \subset Q_{r_1}(t_1,x_1,v_1)$, then we stop. Otherwise, let us choose $Q_{r_2}(t_2,x_2,v_2)$ so that

- $Q_{r_2}(t_2,x_2,v_2) \in \Im_0$;
- $(t_2,x_2,v_2) \in O_0 \setminus Q_{r_1}(t_1,x_1,v_1)$.

Now, if $O_0 \subset Q_{r_1}(t_1,x_1,v_1) \cup Q_{r_2}(t_2,x_2,v_2)$, then we stop, otherwise we continue to iterate such a process. In such a way, we build a subfamily $\Im_0' := \left\{Q_{r_j^0}(t_j^0,x_j^0,v_j^0)\right\}_{j\in\mathbb{N}}$ such that $(t_k^0,x_k^0,v_k^0) \in O_0 \setminus \bigcup_{j<k} Q_{r_j^0}(t_j^0,x_j^0,v_j^0)$.

Now, we consider the following families,

$$\Im_1 := \left\{Q_r(t,x,v) : \frac{R}{4} < r \leq \frac{R}{2},\ Q_r(t,x,v) \in \Im\right\},$$

and

$$O_1 := \left\{(t,x,v) \,:\, Q_r(t,x,v) \in \Im_1 \text{ and } (t,x,v) \notin \bigcup_{j=1}^{\infty} Q_{r_j^0}(t_j^0,x_j^0,v_j^0)\right\}.$$

In a similar fashion as above we build a family $\Im_1' := \left\{Q_{r_j^1}(t_j^1,x_j^1,v_j^1)\right\}_{j\in\mathbb{N}}$ such that $(t_k^1,x_k^1,v_k^1) \in O_1 \setminus \bigcup_{j<k} Q_{r_j^1}(t_j^1,x_j^1,v_j^1)$.

By iterating this process up to the k^{th}-stage, we obtain the following two families,

$$\Im_k := \left\{Q_r(t,x,v) : \frac{R}{2^{k+1}} < r \leq \frac{R}{2^k},\ Q_r(t,x,v) \in \Im\right\},$$

and

$$O_k := \left\{(t,x,v) : Q_r(t,x,v) \in \Im_k \text{ and } (t,x,v) \notin \bigcup_{i=0}^{k-1}\bigcup_{j=1}^{\infty} Q_{r_j^i}(t_j^i,x_j^i,v_j^i)\right\}.$$

From this, we get a family of cylinders $\Im'_k := \left\{Q_{r^k_j}(t^k_j, x^k_j, v^k_j)\right\}_{j\in\mathbb{N}}$ so that $(t^k_\ell, x^k_\ell, v^k_\ell) \in O_k \setminus \bigcup_{j<\ell} Q_{r^k_j}(t^k_j, x^k_j, v^k_j)$.

We now are in the position to prove that the collection of all slanted cylinders in all $\Im'_k$ do satisfy the conditions of Lemma 18.1.

We start by proving that each family $\Im'_i$ has bounded overlapping. For this, suppose that

$$(t,x,v) \in Q_{r^i_{j_1}}(t^i_{j_1}, x^i_{j_1}, v^i_{j_1}) \cap \ldots \cap Q_{r^i_{j_m}}(t^i_{j_m}, x^i_{j_m}, v^i_{j_m}),$$

with $Q_{r^i_{j_\ell}}(t^i_{j_\ell}, x^i_{j_\ell}, v^i_{j_\ell}) \in \Im'_i$. Now, let $Q_{r^i_0}(t^i_0, x^i_0, v^i_0)$ be the cylinder with $r^i_0 := \max\left\{r^i_{j_\ell} : 1 \le \ell \le m\right\}$. Note that, by construction, we can also assume that $(t^i_{j_N}, x^i_{j_N}, v^i_{j_N}) \notin Q_{r^i_{j_\ell}}(t^i_{j_\ell}, x^i_{j_\ell}, v^i_{j_\ell})$, for $\ell < N$.

In view of the *Monotonicity* property of the slanted cylinder, we have that there exist $\varkappa$, $\varepsilon > 0$ such that

$$\begin{aligned} &\|(t^i_{j_\ell}, x^i_{j_\ell}, v^i_{j_\ell})^{-1} \circ (t'_\ell, x'_\ell, v'_\ell)\|_{\text{kin}} \le \varepsilon\Big(\frac{\varkappa r^i_{j_\ell}}{r^i_0}\Big), \\ \text{and}\quad &Q_{\frac{\varkappa r^i_{j_\ell}}{r^i_0}}(t'_\ell, x'_\ell, v'_\ell) \subset Q_{r^i_{j_\ell}}(t^i_{j_\ell}, x^i_{j_\ell}, v^i_{j_\ell}) \subset Q_{\frac{r^i_{j_\ell}}{\varkappa r^i_0}}(t'_\ell, x'_\ell, v'_\ell), \end{aligned} \tag{18.1}$$

for any $1 \le \ell \le m$. Recalling that $(t^i_{j_N}, x^i_{j_N}, v^i_{j_N}) \notin Q_{r^i_{j_\ell}}(t^i_{j_\ell}, x^i_{j_\ell}, v^i_{j_\ell})$, we get by (18.1) that, for $N > \ell$,

$$(t^i_{j_N}, x^i_{j_N}, v^i_{j_N}) \notin Q_{\frac{\varkappa r^i_{j_\ell}}{r^i_0}}(t'_\ell, x'_\ell, v'_\ell). \tag{18.2}$$

Then, by combining (18.1) with (18.2) we have that

$$\begin{aligned} &\|(t^i_{j_N}, x^i_{j_N}, v^i_{j_N})^{-1} \circ (t^i_{j_\ell}, x^i_{j_\ell}, v^i_{j_\ell})\|_{\text{kin}} \\ &\quad \ge \|(t^i_{j_N}, x^i_{j_N}, v^i_{j_N})^{-1} \circ (t'_\ell, x'_\ell, v'_\ell)\|_{\text{kin}} - \|(t^i_{j_\ell}, x^i_{j_\ell}, v^i_{j_\ell})^{-1} \circ (t'_\ell, x'_\ell, v'_\ell)\|_{\text{kin}} \\ &\quad > \frac{\varkappa r^i_{j_\ell}}{r^i_0}\Big(1-\varepsilon\Big) > c(\varkappa,\varepsilon) > 0, \end{aligned} \tag{18.3}$$

since $R2^{-(i+1)} < r^i_{j_\ell} \le r^i_0 \le R2^{-i}$. Moreover, by taking into account (18.2), we have that $Q_{r^i_{j_\ell}}(t^i_{j_\ell}, x^i_{j_\ell}, v^i_{j_\ell})$ is contained in a slanted cylinder $Q_{\mathfrak{R}}(\mathbf{0})$, with the radius $\mathfrak{R}$ depending only on $\varkappa$ and ε. By (18.3), proceeding as in [27, Lemma 1], we obtain that the overlapping in each family $\Im'_i$ is at most α, with α depending only on κ, ε and the dimension n only.

Now, we prove that the family $\Im'_i$ is finite. Since Ω is bounded and $R2^{-(i+1)} < r^i_j \le R2^{-i}$, there exists a constant $C > 0$ such that $O_i \subset Q_{Cr^i_1}(t^i_1, x^i_1, v^i_1)$ and $Cr^i_1 \ge R2^{-i}$. Then, for any $Q_{r^i_j}(t^i_j, x^i_j, v^i_j) \in \Im'_i$ we get

$$\begin{aligned} Q_{\frac{\varkappa}{2C}}(t'_j, x'_j, v'_j) &\subset Q_{r^i_j}(t^i_j, x^i_j, v^i_j) \\ &\subset Q_{\frac{r^i_j}{C\varkappa r^i_1}}(t'_j, x'_j, v'_j) \\ &\subset Q_{\frac{1}{C\varkappa}}(t'_j, x'_j, v'_j) \subset Q_{\mathfrak{R}}(\mathbf{0}), \end{aligned} \tag{18.4}$$

with $\mathfrak{R}$ depending only on C and $\varkappa$.

Since $\Im'_i$ has overlapping bounded by α, we get

$$\sum_{j=1}^{\infty} \chi_{Q_{r_j^i}(t_j^i,x_j^i,v_j^i)}(t,x,v) \le \alpha\,,$$

which, in view of (18.4), implies

$$\sum_{j=1}^{\infty} \chi_{Q_{\frac{\varkappa}{2C}}(t'_j,x'_j,v'_j)}(t,x,v) \;\le\; \alpha\chi_{Q_{\Re}(\mathbf{0})}\,.$$

Hence, iterating the sums above we deduce that $\Im'_i$ has a finite number of cylinders.

We now estimate the boundedness of overlapping between different generators of the families $\Im'_i$. We start by shrinking the selected cylinders

$$(t_0,x_0,v_0) \in \bigcap_{i=1}^{\infty} Q_{(1-\epsilon)r_{j_i}^{e_i}}(t_{j_i}^{e_i},x_{j_i}^{e_i},v_{j_i}^{e_i})\,, \tag{18.5}$$

with $e_1 < e_2 < \ldots$, $R2^{-(e_i+1)} < r_{j_i}^{e_i} \le R2^{-e_i}$. Fix now, i and $\ell > i$, let us measure the gap between e_i and e_ℓ. Since $r_{j_\ell}^{e_\ell} < r_{j_i}^{e_i}$ we have that

$$Q_{\frac{\varkappa(1-\epsilon)r_{j_\ell}^{e_\ell}}{r_{j_i}^{e_i}}}(t',x',v') \subset Q_{r_{j_\ell}^{e_\ell}}(t_{j_\ell}^{e_\ell},x_{j_\ell}^{e_\ell},v_{j_\ell}^{e_\ell}) \subset Q_{\frac{(1-\epsilon)r_{j_\ell}^{e_\ell}}{\varkappa r_{j_i}^{e_i}}}(t',x',v').$$

Moreover, by the *Exclusion* property we have that

$$Q_{\epsilon^\gamma}(t_{j_\ell}^{e_\ell},x_{j_\ell}^{e_\ell},v_{j_\ell}^{e_\ell}) \cap Q_{(1-\epsilon)r_{j_i}^{e_i}}(t_{j_i}^{e_i},x_{j_i}^{e_i},v_{j_i}^{e_i}) = \emptyset.$$

Thus,

$$\begin{aligned}
0 \;&<\; \epsilon^\gamma \\
&<\; \|(t_{j_\ell}^{e_\ell},x_{j_\ell}^{e_\ell},v_{j_\ell}^{e_\ell})^{-1}\circ(t_0,x_0,v_0)\|_{\mathrm{kin}} \\
&\le\; \|(t',x',v')^{-1}\circ(t_0,x_0,v_0)\|_{\mathrm{kin}} + \|(t_{j_\ell}^{e_\ell},x_{j_\ell}^{e_\ell},v_{j_\ell}^{e_\ell})^{-1}\circ(t',x',v')\|_{\mathrm{kin}} \\
&\le\; \frac{(1-\epsilon)r_{j_\ell}^{e_\ell}}{\varkappa r_{j_i}^{e_i}} + \varepsilon\frac{\varkappa(1-\epsilon)r_{j_\ell}^{e_\ell}}{r_{j_i}^{e_i}} \;\lesssim\; 2^{e_i-e_\ell}\,,
\end{aligned}$$

which yields

$$e_\ell - e_i \lesssim \log_2\left(\frac{1}{\epsilon}\right),$$

up to a constant, depending only on ε, $\varkappa$, and the *Exclusion* constant γ.

All in all, the number of cylinders in (18.5) is bounded by a multiple of $\log_2\left(\frac{1}{\epsilon}\right)$, up to a multiplicative constant which – we recall – will depend only on ε, $\varkappa$ and γ.

Now consider the family $\Im' := \{\Im'_i\}_{i=1}^{\infty}$. Since any family $\Im'_i$ covers O_i, the family $\Im'$ cover Ω, so (i) follows. Moreover, up to relabelling the cylinders, one can deduce (ii). Finally, by the argument above, also (iii) is satisfied up to enlarging the constant. □

2. Proof of the strong Harnack inequality

This section is devoted to the completion of the proof of the strong Harnack inequality in Theorem 14.3. Armed with the weak Harnack estimate in (14.6) obtained in the preceding section, as well as with the feasibility of the L^∞-estimates in Section 15 via their δ-interpolative parameter, in order to concretize our final strategy we will also need a nonlocal tail control before going into the Besicovitch-type covering argument presented in Section 1. As already said, the whole strategy below seems feasible to attack several different kinetic integral problems when the related ingredients will be cooked as in the preceding sections.

Proof of Theorem 14.3. This proof is divided in a few steps, in which all the results in the rest of the paper will intervene.

Step 1: An estimate of the Tail$_p$ quantity There exists a dilation parameter $\lambda > 1$ such that the following estimate holds true for any $p > 1$

$$\mathrm{Tail}_p(f;(t_0,x_0,v_0),R,r) \leq c \sup_{U_R(t_0,x_0)\times B_{\lambda r}(v_0)} f\,, \tag{18.6}$$

where the positive constant c depends only on n, s.

Firstly, assume that $\mathrm{Tail}_p(f;(t_0,x_0,v_0),R,r) > 0$, otherwise the desired estimate in (18.6) trivially follows. Now, let us start noticing that, for a. e. $(t,x) \in U_R(t_0,x_0)$, we have (since $\lambda > 1$)

$$\begin{aligned}
&r^{2s}\int_{\mathbb{R}^n\setminus B_r(v_0)} \frac{f(t,x,v)}{|v-v_0|^{n+2s}}\,\mathrm{d}v\\
&r^{2s}\int_{\mathbb{R}^n\setminus B_r(v_0)} \frac{f(t,x,v)}{|v-v_0|^{n+2s}}\,\mathrm{d}v\\
&\quad= r^{2s}\int_{B_{\lambda r}(v_0)\setminus B_r(v_0)} \frac{f(t,x,v)}{|v-v_0|^{n+2s}}\,\mathrm{d}v + r^{2s}\int_{\mathbb{R}^n\setminus B_{\lambda r}(v_0)} \frac{f(t,x,v)}{|v-v_0|^{n+2s}}\,\mathrm{d}v\\
&\quad\leq r^{2s}\Big(\sup_{U_R(t_0,x_0)\times B_{\lambda r}(v_0)} f\Big)\int_{\mathbb{R}^n\setminus B_r(v_0)} \frac{\mathrm{d}v}{|v-v_0|^{n+2s}}\\
&\qquad+(\lambda r)^{2s}\int_{\mathbb{R}^n\setminus B_{\lambda r}(v_0)} \frac{f(t,x,v)}{|v-v_0|^{n+2s}}\,\mathrm{d}v\\
&\quad\leq c\sup_{U_R(t_0,x_0)\times B_{\lambda r}(v_0)} f + (\lambda r)^{2s}\int_{\mathbb{R}^n\setminus B_{\lambda r}(v_0)} \frac{f(t,x,v)}{|v-v_0|^{n+2s}}\,\mathrm{d}v\,,
\end{aligned} \tag{18.7}$$

where the constant c depends only on n and s.

Finally, in view of the strictly positiveness of $\mathrm{Tail}_p(f;(t_0,x_0,v_0),R,r)$, we notice that one can enlarge the dilation parameter λ, so that the second term in the right hand-side of (18.7) can be controlled as follows, a. e., in $U_R(t_0,x_0)$,

$$(\lambda r)^{2s}\int_{\mathbb{R}^n\setminus B_{\lambda r}(v_0)} \frac{f(t,x,v)}{|v-v_0|^{n+2s}}\,\mathrm{d}v \leq \frac{9}{10}\left(r^{2s}\int_{\mathbb{R}^n\setminus B_r(v_0)} \frac{f(t,x,v)}{|v-v_0|^{n+2s}}\,\mathrm{d}v\right).$$

By contradiction, if the opposite holds true for any $\lambda > 1$, then by letting $\lambda \to +\infty$ one would deduce that $\mathrm{Tail}_p(f;(t_0,x_0,v_0),R,r) = 0$, which is a contradiction. Indeed, note that using that $f(t,x,\cdot) \in L^2(\mathbb{R}^n)$ and that $|v-v_0|^{-n-2s} \leq (\lambda r)^{-n-2s}$

on $\mathbb{R}^n \setminus B_{\lambda r}(v_0)$, and thus Hölder's Inequality yields

$$\begin{aligned}
&\frac{9}{10}\left(r^{2s}\int_{\mathbb{R}^n\setminus B_r(v_0)}\frac{f(t,x,v)}{|v-v_0|^{n+2s}}\,\mathrm{d}v\right)\\
&< (\lambda r)^{2s}\int_{\mathbb{R}^n\setminus B_{\lambda r}(v_0)}\frac{f(t,x,v)}{|v-v_0|^{n+2s}}\,\mathrm{d}v\\
&\le (\lambda r)^{2s}\Big(\int_{\mathbb{R}^n\setminus B_{\lambda r}(v_0)}\frac{f(t,x,v)^2}{|v-v_0|^{n+2s}}\,\mathrm{d}v\Big)^{\frac12}\Big(\int_{\mathbb{R}^n\setminus B_{\lambda r}(v_0)}\frac{\mathrm{d}v}{|v-v_0|^{n+2s}}\Big)^{\frac12}\\
&\le c(\lambda r)^{s-\frac{n}{2}}\|f(t,x,\cdot)\|_{L^2(\mathbb{R}^n\setminus B_{\lambda r}(v_0))}\Big(\int_{\lambda r}^{\infty}\varrho^{-2s-1}\,\mathrm{d}\varrho\Big)^{\frac12}\\
&\le c(\lambda r)^{-\frac{n}{2}}\|f(t,x,\cdot)\|_{L^2(\mathbb{R}^n\setminus B_{\lambda r}(v_0))}\to 0\,,\quad \text{as } \lambda\to+\infty\,,
\end{aligned}$$

which is a contradiction in view of the positivity of $\mathrm{Tail}_p(f;(t_0,x_0,v_0),R,r)$.

Thereby, we have

$$\begin{aligned}
&\mathrm{Tail}_p(f;(t_0,x_0,v_0),R,r)\\
&\qquad\le c\sup_{U_R(t_0,x_0)\times B_{\lambda r}(v_0)} f+\frac{9}{10}\mathrm{Tail}_p(f;(t_0,x_0,v_0),R,r)\,,
\end{aligned}$$

which gives the desired estimate in (18.6), up to relabeling c.

Step 2: An application of the interpolative boundedness estimate and the kinetic covering Now, let us set $1/2\le\sigma'<\sigma\le 1$, $\varrho:=(1-\epsilon)[(\sigma-\sigma')r_0]^{\wp}$, with $\wp$ being the *Inclusion* exponent in Section 1, r_0 being the radius given by the weak Harnack inequality in Theorem 14.2 and ϵ given by Lemma 18.1 (iii), depending only on the *Monotonicity constants*. Consider the cylinders $Q_{(1-\epsilon)[(\sigma-\sigma')r_0]^{\wp}}(t_0,x_0,v_0)$, for any $(t_0,x_0,v_0)\in Q^-_{\sigma' r_0}$. By taking into account the *Inclusion* property, any cylinder of this family is contained in $Q^-_{\sigma r}$.

Now, note that, recalling the definition of $\varrho:=(1-\epsilon)[(\sigma-\sigma')r_0]^{\wp}$

$$\begin{aligned}
&\Big(\frac{\varrho}{2}\Big)^{2s}\int_{\mathbb{R}^n\setminus B_{\frac{\varrho}{2}}(v_0)}\frac{f(t,x,v)\,\mathrm{d}v}{|v-v_0|^{n+2s}}\\
&\le\Big(\frac{\varrho}{2}\Big)^{2s}\int_{B_{(\sigma r_0)^{\wp}}(v_0)\setminus B_{\frac{\varrho}{2}}(v_0)}\frac{f(t,x,v)\,\mathrm{d}v}{|v-v_0|^{n+2s}}\,\mathrm{d}v+(\sigma r_0)^{2s\wp}\int_{\mathbb{R}^n\setminus B_{(\sigma r_0)^{\wp}}(v_0)}\frac{f(t,x,v)\,\mathrm{d}v}{|v-v_0|^{n+2s}}\\
&\le c\sup_{U_{\sigma r_0}(-1+(\sigma r_0)^{2s},0)\times B_{(\sigma r_0)^{\wp}}(v_0)} f+(\sigma r_0)^{2s\wp}\int_{\mathbb{R}^n\setminus B_{(\sigma r_0)^{\wp}}(v_0)}\frac{f(t,x,v)\,\mathrm{d}v}{|v-v_0|^{n+2s}}\,,
\end{aligned}$$

where we have also used that $Q^-_{\sigma r_0}:=U_{\sigma r_0}(-1+(\sigma r_0)^{2s},0)\times B_{\sigma r_0}$.

Hence, recalling the definition of Tail_p and (18.6), we get

$$\begin{aligned}
&\mathrm{Tail}_p(f;(t_0,x_0,v_0),\varrho,\varrho/2)\\
&\qquad\le c\sup_{U_{\sigma r_0}(-1+(\sigma r_0)^{2s},0)\times B_{(\sigma r_0)^{\wp}}(v_0)} f+\mathrm{Tail}_p(f;(t_0,x_0,v_0),[(\sigma-\sigma')r_0]^{\wp},(\sigma r_0)^{\wp})\\
&\qquad\le c\sup_{U_{\sigma r_0}(-1+(\sigma r_0)^{2s},0)\times B_{\lambda(\sigma r_0)^{\wp}}(v_0)} f\\
&\qquad\le c\sup_{Q^-_{\sigma r_0}} f\,,
\end{aligned}\tag{18.8}$$

up to enlarging $\wp$, so that $B_{\lambda(\sigma r_0)^{\wp}}\subset B_{\sigma r_0}$.

We now apply Theorem 14.1 and, thanks to our Besicovitch-type covering presented in Section 1, up to renumbering the family, we can cover $Q^-_{\sigma' r_0}$ by a

countable family of slanted cylinders $\{Q^{(k)} \equiv Q_{\varrho_k}(t_k, x_k, v_k)\}_{k\in\mathbb{N}}$ with radii ϱ_k such that $\varrho_k \sim (1-\epsilon)[(\sigma-\sigma')r_0]^{\wp}/2^k$. Moreover, since the covering has bounded overlaps we get, by Lemma 18.1 (iii), that, for a. e. $(t_0, x_0, v_0) \in Q^-_{\sigma' r_0}$ it holds that

$$\#\left\{k \in \mathbb{N} : (t_0, x_0, v_0) \in Q^{(k)}\right\} \leq c \log\left(\frac{1}{\epsilon}\right),$$

with a slight abuse of notation, where c depends only on *Monotonicity* constants $\varkappa$, ε and the *Exclusion* constant γ. Thereby, for a. e. $(t_0, x_0, v_0) \in Q^-_{\sigma' r_0}$, we have

$$\begin{aligned}
f(t_0,x_0,v_0) &\leq c \sum_{j=1}^{\#\{k\in\mathbb{N}:(t_0,x_0,v_0)\in Q^{(k)}\}} (\delta \varrho_j^{4s})^{-\frac{1}{2\varepsilon_*}} \left(\int_{Q^{(j)}} f_+^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{2}} \\
&\quad + \sum_{j=1}^{\#\{k\in\mathbb{N}:(t_0,x_0,v_0)\in Q^{(k)}\}} \delta \mathrm{Tail}_p(f;(t_j,x_j,v_j),\varrho_j,\varrho_j/2) \\
&\leq \frac{c\,\delta^{-\frac{1}{2\varepsilon_*}}}{[(\sigma-\sigma')r_0]^{\frac{2s\wp}{\varepsilon_*}}} \left(\int_{Q^-_{\sigma r_0}} f_+^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{2}} + c\delta \sup_{Q^-_{\sigma r_0}} f \\
&\leq \frac{c\,\delta^{-\frac{1}{2\varepsilon_*}}}{[(\sigma-\sigma')r_0]^{\frac{2s\wp}{\varepsilon_*}}} \left(\sup_{Q^-_{\sigma r_0}} f \right)^{\frac{2-\zeta}{2}} \left(\int_{Q^-_{\sigma r_0}} f_+^{\zeta} \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{2}} + c\delta \sup_{Q^-_{\sigma r_0}} f \\
&\leq \frac{2-\zeta}{2} \sup_{Q^-_{\sigma r_0}} f + \frac{c\,\delta^{-\frac{1}{2\varepsilon_*}}}{[(\sigma-\sigma')r_0]^{\frac{4s\wp}{\zeta\varepsilon_*}}} \left(\int_{Q^-_{\sigma r_0}} f_+^{\zeta} \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{\zeta}} + c\delta \sup_{Q^-_{\sigma r_0}} f, \qquad (18.9)
\end{aligned}$$

by also making use of an application of Young's Inequality (with exponents $2/\zeta$ and $2/(2-\zeta)$). Now, we choose δ such that

$$c\delta < \frac{\zeta}{2},$$

which together with (18.9) yields

$$\sup_{Q^-_{\sigma' r_0}} f \leq \xi \sup_{Q^-_{\sigma r_0}} f + \frac{c}{[(\sigma-\sigma')r_0]^{\frac{4s\wp}{\zeta\varepsilon_*}}} \left(\fint_{Q^-_{r_0}} f_+^{\zeta} \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{\zeta}},$$

for some $\xi \in (0,1)$.

Step 3: An application of the weak Harnack inequality Hence, a final application of a classical iterative argument together with the weak Harnack inequality in Theorem 14.2 yields the desired estimate (14.8). To conclude, we just notice that in the case when $\zeta \geq 2$ there is no need to apply Young's Inequality as above, being actually enough to choose $c\delta < 1/2$ in order to apply the aforementioned iteration lemma. □

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