Sharp and Fast Bounds for the Celis-Dennis-Tapia Problem

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SHARP AND FAST BOUNDS FOR THE CELIS-DENNIS-TAPIA PROBLEM*

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Abstract. In the Celis–Dennis–Tapia (CDT) problem a quadratic function is minimized over a region defined by two strictly convex quadratic constraints. In this paper we rederive a necessary and sufficient optimality condition for the exactness of the dual Lagrangian bound (equivalent to the Shor relaxation bound in this case). Starting from such a condition, we propose strengthening the dual Lagrangian bound by adding one or two linear cuts to the Lagrangian relaxation. Such cuts are obtained from supporting hyperplanes of one of the two constraints. Thus, they are redundant for the original problem, but they are not for the Lagrangian relaxation. The computational experiments show that the new bounds are effective and require limited computing times. In particular, one of the proposed bounds is able to solve all but one of the 212 hard instances of the CDT problem presented in [S. Burer and K. M. Anstreicher, {\it SIAM J. Optim.}, 23 (2013), pp. 432–451].

Key words. CDT problem, dual Lagrangian bound, linear cuts

MSC codes. 90C20, 90C22, 90C26

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1. Introduction.

The Celis–Dennis–Tapia problem (CDT problem in what follows) is defined as follows:

$$
p^\star = \min \quad \begin{array}{l} \mathbf{x}^\top \mathbf{Q}\mathbf{x} + \mathbf{q}^\top \mathbf{x} \\ \mathbf{x}^\top \mathbf{x} \le 1 \\ \mathbf{x}^\top \mathbf{A}\mathbf{x} + \mathbf{a}^\top \mathbf{x} \le a_0, \end{array} \tag{1.1}
$$

where $\mathbf{Q}, \mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{q}, \mathbf{a} \in \mathbb{R}^n$, $a_0 \in \mathbb{R}$, while $\mathbf{A}$ is assumed to be positive definite. We will denote by

$$
H = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^\top \mathbf{A}\mathbf{x} + \mathbf{a}^\top \mathbf{x} \le a_0\}
$$

the ellipsoid defined by the second constraint, by ∂H its border, and by $int(H)$ its interior. The CDT problem was originally proposed in [13] and has attracted a lot of attention in the last two decades. For some special cases a convex reformulation is available. For instance, in [26] it is shown that a semidefinite reformulation is possible when no linear terms are present, i.e., when $\mathbf{q} = \mathbf{a} = \mathbf{0}$. However, up to now no tractable convex reformulation of general CDT problems has been proposed in the literature. In spite of that, recently three different works [9, 14, 22] independently proved that the CDT problem is solvable in polynomial time. More precisely, in [14, 22] polynomial solvability is proved by identifying all KKT points through the solution of a bivariate polynomial system with polynomials of degree at most $2n$. The two unknowns are the Lagrange multipliers of the two quadratic constraints. Instead, in [9] an approach based on the solution of a sequence of feasibility problems for systems of quadratic inequalities is proposed. The systems are solved by a polynomial-time

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algorithm based on Barvinok's construction [6]. Though polynomial, all of these approaches are computationally demanding since the degree of the polynomial is quite large. Conditions guaranteeing that the classical Shor SDP relaxation or, equivalently in this case, the dual Lagrangian bound is exact, are discussed in [2, 7]. In particular, in [2] a necessary and sufficient condition is presented. It is shown that the lack of exactness is related to the existence of KKT points with the same Lagrange multipliers but two distinct primal solutions, both active at one of the two constraints but one violating and the other one fulfilling the other constraint. In [10] necessary and sufficient conditions for local and global optimality are discussed based on copositivity. In [11] an exactness condition is given for a copositive relaxation, also for the case with additional linear constraints. A trajectory following method to solve the CDT problem has been discussed in [26], while different branch-and-bound solvers are tested in [19].

Recently, different papers proposed valid bounds for the CDT problem. In [12] the Shor relaxation bound is strengthened by adding all RLT constraints obtained by **AQ1** supporting hyperplanes of the two ellipsoids. By fixing the supporting hyperplane for one ellipsoid, the RLT constraints obtained with all the supporting hyperplanes of the other can be condensed into a single `SOC-RLT` constraint. Varying the supporting hy- **AQ2** perplane of the first ellipsoid gives rise to an infinite number of `SOC-RLT` constraints which, however, can be separated in polynomial time. The addition of these constraints does not allow one to close the duality gap, but it is computationally shown that many instances which are not solved via the SDP bound, are solved with the addition of these `SOC-RLT` cuts. The authors generate 1000 random test instances for each $n = 5, 10, 20$, following a procedure described in [18] to generate trust-region problems with one local and nonglobal minimizer. The proposed bound based on `SOC-RLT` cuts allows for solving most instances except for 212 (38 for $n = 5$, 70 for $n = 70$, and 104 for $n = 20$). Such unsolved instances are considered as hard ones in subsequent works. In [25] lifted-RLT cuts are introduced and it is shown that the new constraints allow one to derive an exact bound for $n = 2$ but also to improve the bounds of [12] over the hard instances for $n > 2$. In [27] it is proved that the duality gap can be reduced to 0 by solving two subproblems with SOC constraints when the second constraint is the product of two linear functions and an exactness result is also provided for the case of problems with two variables. Due to its relations with the approach proposed in this work, we will further discuss the approach proposed in [27] at the end of section 3. In [3] cuts are introduced. These are Kronecker product con- **AQ3** straints which generalize both the classical RLT constraints obtained from two linear inequality constraints, and the `SOC-RLT` constraints obtained from one linear inequality constraint and a SOC constraint. Further hard instances from [12] are solved with the addition of these cuts. In the very recent paper [4] a branch and bound approach is proposed. The main feature of this approach is eigenvector branching, i.e., a branching rule based on the use of the eigenvector corresponding to the maximal eigenvalue of $\mathbf{X}^\star - \mathbf{x}^\star {\mathbf{x}^\star}^\top$, where $(\mathbf{X}^\star, \mathbf{x}^\star)$ is the optimal solution of an SDP relaxation.

In this paper we investigate ways to strengthen the dual Lagrangian bound through the addition of one or two linear cuts. In particular, the paper is structured as follows. In section 2 we derive some theoretical results for a class of problems with two constraints which includes the CDT problem as a special case. We develop a bisection technique to solve the dual Lagrangian relaxation for such class of problems. In the following sections we apply the results of section 2 to the CDT problem. In particular, in section 3 we introduce some results through which it will be possible to rederive the necessary and sufficient exactness condition discussed in [2] and we discuss how to improve the dual bound for the CDT problem by the addition of a linear cut. Next, in sections 4 and 5 we discuss techniques to further improve the bound. More precisely, in section 4 we still present a bound based on the addition of a linear cut, but we develop a technique to locally adjust a given linear cut, while in section 5 we consider a bound based on the addition of two linear cuts. Finally, in section 6 we present some computational experiments which show that the newly proposed bounds, in particular those based on two linear cuts, are both computationally cheap and effective. In particular, one of the bounds will be able to solve all but one of the hard instances from [12]. We also investigate which are the most challenging instances for the proposed bounds and, as we will see, the difficulties are related to the existence of multiple solutions of Lagrangian relaxations.

It is also worthwhile to highlight the contribution of this paper under another perspective. As previously discussed, while there is no known convex relaxation of the CDT problem, there are several problems, related to CDT, which do have exact SDP relaxations. These include the trust-region subproblem (TRS), the TRS with a single linear constraint (TRS1, see [12, 23]), and the TRS with two linear constraints, at least one of which is tight (TRS2eq, see [26]). This paper shows that such special cases, for which an exact convex relaxation exists, can be used to help solve the general CDT problem. Indeed, we first observe that the subproblems to be solved in section 3 to improve the dual Lagrangian bound, obtained by adding a linear cut corresponding to a supporting hyperplane for the second ellipsoidal constraint, turn out to be TRS1 problems. In section 4 we also discuss how to pick a "good" supporting hyperplane, i.e., one which leads to a good SDP relaxation and, in fact, we also provide a necessary and sufficient condition under which we can guarantee that the supporting hyperplane is the best one. Next, in section 5 we observe that the bound can be further improved by adding two linear cuts, one of which must be active, so that the subproblems to be solved in this case are TRS2eq problems.

2. Lower bounds obtained from the Lagrangian relaxation. The CDT problem (1.1) is a specific instance of the following, more general, one:

$$p^\star = \min_{\mathbf{x}\in\mathbb{R}^n}, \quad \begin{array}{l} f(\mathbf{x}), \\ g(\mathbf{x}) \le 0, \\ h(\mathbf{x}) \le 0. \end{array} \tag{2.1}$$

In this section, we discuss a class of lower bounds on the solution of problem (2.1) that can be obtained from its Lagrangian relaxation. In the next sections, we will apply these bounds to the specific case of the CDT problem (1.1). Throughout this and the following sections, we make the following assumptions.

Assumption 2.1. In problem (2.1)

(a) g, h are continuous;

(b) the set $\{\mathbf{x} \in \mathbb{R}^n : g(\mathbf{x}) \le 0\}$ is bounded;

(c) it holds that

$$h_0 = \min_{\mathbf{x}\ : g(\mathbf{x})\le 0} h(\mathbf{x}) < 0; \tag{2.2}$$

(d) the solution set of problem (2.1) without the last constraint, that is

$$\bar{P} = arg\ \min_{\mathbf{x}\in\mathbb{R}^n}, \quad \begin{array}{l} f(\mathbf{x}), \\ g(\mathbf{x}) \le 0, \end{array}$$

is such that $(\forall \mathbf{x} \in \bar{P})\ h(\mathbf{x}) > 0$.

Note that if the last condition in Assumption 2.1 is violated, we can find the solution of problem (2.1) by removing the last constraint and the relaxation discussed in this section is useless. Now, let $G = \{\mathbf{x} \in \mathbb{R}^n : g(\mathbf{x}) \leq 0\}$ and $H = \{\mathbf{x} \in \mathbb{R}^n : h(\mathbf{x}) \leq 0\}$. Let $X \supset H$ be a closed subset of $\mathbb{R}^n$ and for $\lambda \in \mathbb{R}$, with $\lambda \geq 0$, define the Lagrangian relaxation

$$p_X(\lambda) = \min_{\mathbf{x} \in X \cap G} f(\mathbf{x}) + \lambda h(\mathbf{x}), \tag{2.3}$$

and the corresponding solution set

$$P_X(\lambda) = arg \min_{\mathbf{x} \in X \cap G} f(\mathbf{x}) + \lambda h(\mathbf{x}).$$

Note that $P_X(\lambda)$ is compact, since $G \cap X$ is nonempty (in view of part (c)) of Assumption 2.1) and compact (in view of the compactness of G which follows from parts (a) and (b) of Assumption 2.1), and $f + \lambda h$ is continuous. Due to well-known properties of the Lagrangian relaxation, we have that function p_X is such that $(\forall \lambda \geq 0)\, p_X(\lambda) \leq p^*$, and is concave (it is the pointwise minimum of a set of functions linear in λ). The best bound that can be obtained as the solution of (2.3) is given by

$$\bar{p}_X = \max_{\lambda \geq 0} p_X(\lambda), \tag{2.4}$$

and corresponds to the solution of the dual Lagrangian problem. Note that function p_X depends on the choice of set X.

Now, we recall that the supergradient of a function $q : \mathbb{R} \to \mathbb{R}$ at $x \in \mathbb{R}$ is defined as

$$\partial q(x) = \{z \in \mathbb{R} : (\forall y \in \mathbb{R})\, q(y) - q(x) \leq z(y - x)\}.$$

Since p_X is concave, for any $\lambda \in \mathbb{R}$, the supergradient $\partial p_X(\lambda)$ is nonempty.

For $A \subset \mathbb{R}^n$ define the following subset of $\mathbb{R}$:

$$h(A) = \{h(\mathbf{x}) \ : \ \mathbf{x} \in A\}.$$

For $X \subset \mathbb{R}^n$, define a (set-valued) function $Q_X : \mathbb{R}_+ \to \mathcal{P}(\mathbb{R})$,

$$Q_X(\lambda) = h(P_X(\lambda)) \tag{2.5}$$

($\mathbb{R}_+$ denotes the set of nonnegative reals and $\mathcal{P}(\mathbb{R})$ is the power set of the set of real numbers). Also set $h_X^{\min}(\lambda) = \min Q_X(\lambda)$ and $h_X^{\max}(\lambda) = \max Q_X(\lambda)$. The following proposition shows that function Q_X is monotone nonincreasing (see Definition 3.5.1 of [5]) and upper semicontinuous (see Definition 1.4.1 of [5]). These two properties will play an important role in the computation of a lower bound for problem (2.1). Moreover, this proposition characterizes the supergradient of p_X at each $\lambda \geq 0$. In the proof of the proposition we will make use of Berge's maximum theorem (see [8]). In particular, we will consider the slightly different formulation presented as the corollary to Theorem 3 on page 30 of [15].

Corollary 2.1. *Let the correspondence (i.e., the set-valued function) β of S into T be compact-valued and continuous, and let $\phi : S \times T \to \mathbb{R}$ be a continuous function. Then, we have the following:*

(a) *The function $z \mapsto m(z) := max\{\phi(z,y) | y \in \beta(z)\}$ is continuous.*

(b) *The correspondence $z \mapsto \{y \in \beta(z) | \phi(z,y) = m(z)\}$ is nonempty and compact-valued and upper semicontinuous.*

PROPOSITION 2.2. *For any* $X \subset \mathbb{R}^n$,

(i) Q_X *is monotone non-increasing, that is if* $\lambda_1 \geq \lambda_2$, $y_1 \in Q_X(\lambda_1)$, $y_2 \in Q_X(\lambda_2)$, *then* $y_1 \leq y_2$.

(ii) Q_X *is upper semicontinuous, that is, if* $Q_X(\lambda) \subset U$, *where* U *is an open subset of* $\mathbb{R}$, *then there exists a neighborhood* V *of* λ *such that* $(\forall z \in V)\, Q_X(z) \subset U$.

(iii) $\partial p_X(\lambda) = [\min Q_X(\lambda), \max Q_X(\lambda)]$.

Proof. (i) Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ be such that $y_1 = h(\mathbf{x}_1)$ and $y_2 = h(\mathbf{x}_2)$, then $f(\mathbf{x}_1) + \lambda_1 h(\mathbf{x}_1) \leq f(\mathbf{x}_2) + \lambda_1 h(\mathbf{x}_2)$ and $f(\mathbf{x}_2) + \lambda_2 h(\mathbf{x}_2) \leq f(\mathbf{x}_1) + \lambda_2 h(\mathbf{x}_1)$. By adding up the previous inequalities, it follows that $(\lambda_1 - \lambda_2)(h(\mathbf{x}_1) - h(\mathbf{x}_2)) \leq 0$.

(ii) Apply Corollary 2.1 with $T = G \cap X$, $S = \mathbb{R}_+$, constant function $(\forall \lambda \in S)\, \beta(\lambda) = G \cap X$, $\phi(\lambda, \mathbf{x}) = -f(\mathbf{x}) - \lambda \cdot h(\mathbf{x})$. Since ϕ is continuous, set-valued function $P_X(\lambda) = \{\mathbf{x} \in G \cap X : \phi(\lambda, \mathbf{x}) = \max_{\mathbf{y} \in G \cap X} \phi(\lambda, \mathbf{y})\}$ is upper semicontinuous. Hence, also Q_X is upper semicontinuous, since it is obtained as the composition of P_X with h, which is continuous (see Theorem 1' on page 113 of [8]).

(iii) It is a consequence of Theorem 4.4.2 in [16], being G compact. ∎

The next proposition characterizes the optimal solution of the dual Lagrangian problem (2.4).

PROPOSITION 2.3. *Under Assumption* 2.1*, the optimal value of the dual Lagrangian problem* (2.4) *is:* (i) *either attained at* $\lambda_X = 0$ *in case* $\partial p_X(0) \cap \mathbb{R}_- \neq \emptyset$, *where* $\mathbb{R}_-$ *denotes the set of nonpositive real numbers;* (ii) *or is attained at* $\lambda_X > 0$ *such that* $0 \in \partial p_X(\lambda_X)$. *In the former case,* $\bar{p}_X = p^\star$ *holds.*

Proof. The proposition, apart from the last statement, is a direct consequence of the optimality conditions for the maximum of concave functions (see Theorem 1.1.1 in Chapter 7 of [16]). To prove the last statement, namely that if (i) holds $\bar{p}_X = p^\star$, note that, if $\partial p_X(0) \cap \mathbb{R}_- \neq \emptyset$, then, in view of part (iii) of Proposition 2.2, $h_X^{\min}(0) \leq 0$ and, thus, there exists an optimal solution $\mathbf{x}^\star$ of $\min_{\mathbf{x} \in X \cap G} f(\mathbf{x})$ such that $h(\mathbf{x}^\star) \leq 0$. This implies that $\mathbf{x}^\star$ is also an optimal solution of the original problem (2.1), so that $\bar{p}_X = p^\star$ holds. ∎

The following property shows that it is always possible to find a sufficiently high value of λ such that $P_X(\lambda) \subset H$, that is, the elements of $P_X(\lambda)$ are feasible solutions of problem (2.1).

LEMMA 2.4. *If*

$$\lambda \geq \hat{\lambda} = \frac{\max_{\mathbf{x} \in G \cap X} f(\mathbf{x}) - \min_{\mathbf{x} \in G \cap X} f(\mathbf{x})}{|h_0|}, \tag{2.6}$$

where h_0 *is defined in* (2.2)*, then* $P_X(\lambda) \subset H$.

Proof. By contradiction, assume that there exists $\mathbf{x} \in P_X(\lambda)$ such that $h(\mathbf{x}) > 0$, and let $\mathbf{x}_0 \in G \cap H$ be such that $h(\mathbf{x}_0) = h_0 < 0$; then $f(\mathbf{x}) + \lambda h(\mathbf{x}) \leq f(\mathbf{x}_0) + \lambda h(\mathbf{x}_0)$. Since $h(\mathbf{x}) > 0$, it follows that $\lambda \leq \frac{f(\mathbf{x}_0) - f(\mathbf{x})}{|h(\mathbf{x}_0)| + h(\mathbf{x})} < \frac{\max_{\mathbf{x} \in G \cap X} f(\mathbf{x}) - \min_{\mathbf{x} \in G \cap X} f(\mathbf{x})}{|h(\mathbf{x}_0)|}$, which contradicts the assumption on λ. ∎

The following proposition shows that if $0 \in Q_X(\lambda)$, then $p_X(\lambda)$ is equal to the optimal value of problem (2.1).

PROPOSITION 2.5. *Under Assumption* 2.1*, the following statements are equivalent for* $\lambda > 0$*:*

(i) $0 \in Q_X(\lambda)$,

(ii) $p^* = p_X(\lambda)$ *and there exists* $\bar{\mathbf{x}} \in$ *arg* $\min_{\mathbf{x} \in G \cap H} f(\mathbf{x})$ *such that* $h(\bar{\mathbf{x}}) = 0$.

Proof. (i) $\Rightarrow$ (ii). Let $\bar{\mathbf{x}}$ be such that $h(\bar{\mathbf{x}}) \in Q_X(\lambda)$ and $h(\bar{\mathbf{x}}) = 0$. Let $\mathbf{x}^*$ be a solution of (2.1). Then, $p_X(\lambda) = f(\bar{\mathbf{x}}) + \lambda h(\bar{\mathbf{x}}) = f(\bar{\mathbf{x}}) \leq f(\mathbf{x}^*) + \lambda h(\mathbf{x}^*) \leq f(\mathbf{x}^*)$, hence $p_X(\lambda) \leq p^*$. Moreover, $p_X(\lambda) = f(\bar{\mathbf{x}}) \geq \min_{\mathbf{x} \in G \cap H} f(\mathbf{x}) = p^*$.

(ii) $\Rightarrow$ (i). Assume that $p_X(\lambda) = p^*$, and let $\mathbf{x} \in P_X(\lambda)$. Then, by (ii), $f(\mathbf{x}) + \lambda h(\mathbf{x}) = f(\bar{\mathbf{x}}) = f(\bar{\mathbf{x}}) + \lambda h(\bar{\mathbf{x}})$. It follows that $\bar{\mathbf{x}} \in P_X(\lambda)$ and $Q_X(\lambda) \ni h(\bar{\mathbf{x}}) = 0$. ∎

Remark 2.6. If $0 \in Q_X(\lambda)$, by point (iii) of Proposition 2.2, $\partial p_X(\lambda) \ni 0$, so that λ corresponds to a maximizer of the dual Lagrangian. Note that equation $0 \in Q_X(\lambda)$ always admits a solution if Q_X is continuous. However, in the general case, Q_X is only upper semicontinuous. In this case, the value of λ for which $\partial p_X(\lambda) \ni 0$ may not satisfy $0 \in Q_X(\lambda)$. Thus, the optimal value of the dual Lagrangian (2.4) is not equal to the optimal value of (2.1) but it represents a lower bound of it.

In order to evaluate a numerical solution algorithm, we define the following weak solution of (2.1).

DEFINITION 2.7. $\mathbf{x}$ *is an* η*-solution of* (2.1) *if* $\mathbf{x} \in G \cap H$ *and* $f(\mathbf{x}) - p^* \leq \eta$.

The following proposition presents a bound on the error committed on the estimation of p^*.

PROPOSITION 2.8. *For any* $\lambda \geq 0$ *such that* $P_X(\lambda) \cap H \neq \emptyset$*, and for any* $\mathbf{x} \in P_X(\lambda) \cap H$*, it holds that* $f(\mathbf{x}) - p^* \leq \lambda |h(\mathbf{x})|$*, i.e.,* $\mathbf{x}$ *is an* η*-solution of problem of* (2.1) *with* $\eta = \lambda |h(\mathbf{x})|$.

Proof. Since $\mathbf{x} \in P_X(\lambda)$ and observing that $\mathbf{x}^* \in G \cap X$ for any $X \supset H$, $f(\mathbf{x}) + \lambda h(\mathbf{x}) \leq f(\mathbf{x}^*) + \lambda h(\mathbf{x}^*) \leq f(\mathbf{x}^*)$, from which $f(\mathbf{x}) - f(\mathbf{x}^*) \leq \lambda |h(\mathbf{x})|$. ∎

Now we introduce Algorithm 2.1 which is based on a binary search through different λ values and is able to return the solution of the dual Lagrangian problem, i.e., the maximum of function $p_X(\lambda)$ and, in some cases, even the solution of problem (2.1). The algorithm also returns a point $\mathbf{z}_1(\lambda^{\max}) \in H$ and (possibly) a point $\mathbf{z}_2(\lambda^{\min}) \notin H$. Note that according to Proposition 2.8, point $\mathbf{z}_1(\lambda^{\max})$ is an η-solution of problem (2.1) with $\eta = \lambda |h(\mathbf{z}_1(\lambda^{\max}))|$.

The algorithm starts with an initial interval of λ values $[\lambda^{\min}, \lambda^{\max}] = [0, \lambda^{\texttt{init}}]$, where $\lambda^{\texttt{init}}$ is a suitably large value and can be set equal to $\hat{\lambda}$ as defined in Lemma 2.4. At each iteration the algorithm halves such interval by evaluating the set Q_X^λ at $\lambda = (\lambda^{\max} + \lambda^{\min})/2$. Then, the algorithm sets- are $\lambda^{\min} = \lambda$, if $h_X^{\min}(\lambda) > 0$; $\lambda^{\max} = \lambda$ if $h_X^{\max}(\lambda) < 0$. Instead, if $0 \in \partial p_X(\lambda) = [h_X^{\min}(\lambda), h_X^{\max}(\lambda)]$, the algorithm sets $\lambda^{\max} = \lambda^{\min} = \lambda$ and exits the loop.

The following proposition characterizes Algorithm 2.1.

PROPOSITION 2.9. (i) *Algorithm* 2.1 *terminates in a finite number of iterations:*

(ii) *at each iteration* $\lambda^{\min} \leq \lambda_X \leq \lambda^{\max}$*,*

(iii) *at termination* $|\lambda^{\max} - \lambda_X| \leq \epsilon$*,*

(iv) *at each iteration, if* $\lambda^{\min} < \lambda_X < \lambda^{\max}$*, then* $[h_X^{\max}(\lambda^{\max}), h_X^{\min}(\lambda^{\min})] \supset \partial p_X(\lambda_X)$*,*

(v) *point* $\mathbf{z}_1(\lambda^{\max}) \in P_X(\lambda^{\max}) \cap H$ *is an* η*-solution of* (2.1) *with* $\eta = \lambda_{\max}| h(\mathbf{z}_1(\lambda^{\max}))|$.

Proof. (i) At each iteration the length of the interval $[\lambda^{\min}, \lambda^{\max}]$ is halved. Hence, in a sufficient large number of iterations, the termination condition of the main loop will be satisfied.

(ii) At the beginning of the algorithm we have that $\lambda^{\min} \leq \lambda_X \leq \lambda^{\max}$. Every time $\lambda^{\min}$ is updated, we set $\lambda^{\min} = \lambda$ if condition $h_X^{\min}(\lambda) > 0$ holds. Since $h_X^{\min}(\lambda_X) \leq 0$,

Algorithm 2.1 Binary search algorithm for the solution of the dual Lagrangian problem for (1.1).

```
DualLagrangian(X, λ_init)
  Set λ_min = 0, λ^max = λ^init
  while λ^max − λ^min > ε do
    Set λ = (λ^max + λ^min)/2
    Solve problem (2.3), and let P_X(λ) be its set of optimal solutions
    Compute the set Q_X(λ) and the values h_X^min(λ), h_X^max(λ)
    if h_X^min(λ) > 0 then
      Set λ_min = λ
    else if h_X^max(λ) < 0 then
      Set λ^max = λ
    else
      Set λ^max = λ^min = λ
    end if
  end while
  Set Lb = p_X(λ^max), and let z_1(λ^max) be some point in P_X(λ^max) ∩ H and
z_2(λ^min) be some point (if any) in P_X(λ^min) \ H
  return Lb, λ^max, z_1(λ^max), z_2(λ^min)
```

by the monotonicity of function $h_X^{\min}$, which is a consequence of the monotonicity of function Q_X, condition $\lambda^{\min} \leq \lambda_X$ is maintained. The same reasoning can be used to prove that $\lambda^{\max} \geq \lambda_X$.

(iii) It is a consequence of (ii) and the termination condition.

(iv) $\partial p_X(\lambda_X) = [h_X^{\min}(\lambda_X), h_X^{\max}(\lambda_X)] \subset [h_X^{\max}(\lambda^{\max}), h_X^{\min}(\lambda^{\min})]$, due to point (ii) and the monotonicity of functions $h_X^{\max}$ and $h_X^{\min}$, which is a consequence of the monotonicity of function Q_X.

(v) It is a consequence of Proposition 2.8. ◻

The following property is a direct consequence of the upper semicontinuity of Q_X.

PROPOSITION 2.10. *Let $X \supset H$ be such that $\sup Q_X(\lambda) < 0$; then there exists a neighborhood U of λ such that $(\forall \eta \in U)\ \max Q_X(\eta) < 0$.*

As a consequence of the previous proposition, it is possible to improve the lower bound on problem (2.1), obtained as the solution of (2.3), by replacing set X with a different set $Y \supset H$ fulfilling a given condition.

PROPOSITION 2.11. *Let $Y \supset H$ be such that $\max Q_Y(\lambda_X) \leq 0$ or, equivalently, $P_Y(\lambda_X) \setminus H = \emptyset$, and assume that $\bar{p}_X = p_X(\lambda_X) < p^*$. Then $\bar{p}_Y = p_Y(\lambda_Y) > \bar{p}_X$.*

Proof. Note that, by Proposition 2.3, $\bar{p}_X = p_X(\lambda_X) < p^*$ implies $\lambda_X > 0$. Now, in case $\max Q_Y(\lambda_X) = 0$, then $0 \in Q_Y(\lambda_X)$ and, by Proposition 2.5, $\bar{p}_Y = p^* > \bar{p}_X$. Thus, we only consider the case $\max Q_Y(\lambda_X) < 0$. In such case, by Proposition 2.10, $\lambda_Y < \lambda_X$. If $\lambda_Y = 0$, by Proposition 2.3 we have that $\bar{p}_Y = p^* > \bar{p}_X$ and we are done. Otherwise, if $\lambda_Y > 0$, again by Proposition 2.3 we have that $0 \in [h_Y^{\min}(\lambda_Y), h_Y^{\max}(\lambda_Y)]$, and, consequently, there exists $\mathbf{y} \in P_Y(\lambda_Y)$ such that $h(\mathbf{y}) \leq 0$. Note that $\bar{p}_Y = f(\mathbf{y}) + \lambda_Y h(\mathbf{y})$. If $h(\mathbf{y}) = 0$, then, by Proposition 2.5, $p_Y(\lambda_Y) = p^*$, so that the thesis is satisfied in view of $\bar{p}_X < p^*$. Otherwise, if $h(\mathbf{y}) < 0$, let $\mathbf{x} \in \mathbb{R}^n$ be such that $\bar{p}_X = f(\mathbf{x}) + \lambda_X h(\mathbf{x})$. Then $\bar{p}_X = f(\mathbf{x}) + \lambda_X h(\mathbf{x}) \leq f(\mathbf{y}) + \lambda_X h(\mathbf{y}) < f(\mathbf{y}) + \lambda_Y h(\mathbf{y}) = \bar{p}_Y$, where we used the facts that $h(\mathbf{y}) < 0$ and that $\lambda_Y < \lambda_X$. ◻

The following proposition deals with the special case of the previous result when $Y \subset X$.

PROPOSITION 2.12. *Let $X \supset Y \supset H$ be such that $Y \cap (P_X(\lambda_X) \setminus H) = \emptyset$, and assume that $\bar{p}_X = p_X(\lambda_X) < p^*$. Then $\bar{p}_Y = p_Y(\lambda_Y) > \bar{p}_X$.*

Proof. Since $h_X^{\min}(\lambda_X) < 0$ we have that $P_X(\lambda_X) \cap H \neq \emptyset$ and, consequently, since $Y \supset H$, also $Y \cap P_X(\lambda_X) \neq \emptyset$. Then, $Y \subset X$ implies $P_Y(\lambda_X) = Y \cap P_X(\lambda_X)$. Moreover, if $Y \cap (P_X(\lambda_X) \setminus H) = \emptyset$, then the condition $\max Q_Y(\lambda_X) \leq 0$ is satisfied and the result follows from Proposition 2.11. ∎

Stated in another way, the previous propositions show that, in case the lower bound $\bar{p}_X$ is not exact, we are able to improve (increase) it if we are able to replace set X with a new set Y which cuts away all members of $P_X(\lambda_X)$ outside H.

Remark 2.13. Up to now we have not discussed the difficulty of computing the values of function p_X or, equivalently, the difficulty of solving problem (2.3). Such difficulty is strictly related to the specific problem (i.e., to the specific functions f, g, h), and also to the specific set X. In the next sections we apply the general theory developed in this section to the CDT problem. We show that for suitably defined sets X (defined by one or two linear cuts), the computation of function p_X can be done efficiently, and, moreover, the corresponding lower bounds $\bar{p}_X$ improve the standard dual Lagrangian bound, corresponding to the case $X = \mathbb{R}^n$.

Remark 2.14. In principle one could also define a cutting algorithm where a sequence of sets $\{X_k\}$ is generated such that (i) $X_k \supset X_{k+1} \supset H$ for all k; (ii) $X_{k+1} \cap (P_{X_k}(\lambda_{X_k}) \setminus H) = \emptyset$; (iii) $\cap_{k=1}^{\infty} X_k = H$. The corresponding sequence of lower bounds $\{\bar{p}_{X_k}\}$ is strictly increasing in view of Proposition 2.12, and converges to p^*. However, the difficulty related to such an algorithm is that forcing (ii) may not be trivial and, moreover, as already commented in Remark 2.13, computing p_{X_k} may be computationally demanding.

The following algorithm, Algorithm 2.2, in principle, is able to always find an approximate solution of (2.1). The algorithm is based on an iterative reduction of set X, in order to eliminate its elements in which function h is positive. In practice, Algorithm 2.2 could be unimplementable. Indeed, it may require a large number of cuts on set X and each added cut may increase the complexity of the optimization problem that we need to solve to evaluate **DualLagrangian**. In section 4, we will see that, to refine the lower bound on the solution of the CDT problem, it is computationally more convenient to adjust existing cuts instead of adding new ones. We stress that we will not actually use Algorithm 2.2 for the solution of the CDT problem. We present this algorithm just as a theoretical contribution.

PROPOSITION 2.15. *Algorithm* 2.2 *terminates and $\bar{\mathbf{x}}$ is such that $h(\bar{\mathbf{x}}) \leq \frac{\eta}{\lambda^{\max}}$ and $|\bar{f} - f^*| \leq \eta$.*

Proof. By contradiction, assume that the algorithm does not terminate. Let l_i be the value of $\lambda^{\min}$ returned by the ith call to **DualLagrangian**. Sequence l_i is monotone nonincreasing; moreover, the domain of the sequence is a subset of finite cardinality of interval $[0, \lambda^{\texttt{init}}]$ (its maximum cardinality depends on $\lambda^{\texttt{init}}$ and ϵ). Indeed, the termination condition of function **DualLagrangian** allows only for a finite number of divisions of the interval $[0, \lambda^{\texttt{init}}]$. Hence, sequence l_i converges in a finite number of iterations to its limit $l_\infty = \lim_{i \to \infty} l_i$ and there exists $\bar{i} \in \mathbb{N}$ such that $(\forall i \geq \bar{i})\ l_i = l_\infty$. By (iv) of Proposition 2.9, $h^{\max}(l_\infty) \geq 0$ and, since the

Algorithm 2.2 Bound improvement through redefinition of set X.

1: Set $X = \mathbb{R}^n$
2: Set $\lambda^{\max} = \lambda^{\texttt{init}}$
3: **repeat**
4: Let $[Lb, \lambda^{\min}, \lambda^{\max}, \mathbf{z}_1(\lambda^{\max}), \mathbf{z}_2(\lambda^{\min}), h_X^{\min}, h_X^{\max}] = \textbf{DualLagrangian}(X, \lambda^{\texttt{init}})$
5: Set $Z = \{\mathbf{x} \in P_X(\lambda^{\min}) : h(\mathbf{x}) > 0\}$
6: Redefine $X = Y$, where Y is such that $X \supset Y \supset H$ and $Z \cap Y = \emptyset$.
7: **until** $\min\{h_X^{\max}(\lambda^{\min}), -h_X^{\min}(\lambda^{\max})\}\lambda^{\max} \le \eta$
8: **return** $\bar{\mathbf{x}} \in P_X(\lambda_X^{min}) \cup P_X(\lambda_X^{max})$ with $|h(\bar{\mathbf{x}})| \le \eta$, $\bar{f} = f(\bar{\mathbf{x}})$.

algorithm does not terminate, $h^{\max}(l_\infty) \ge \eta$. At the $\bar{i}+1$-iteration, the algorithm calls **DualLagrangian**(X, l_∞), which returns the value $\lambda^{\min} = l_\infty$. Anyway, at the previous iteration $\bar{i}$, the elements $P_X(\lambda^{\min})$ at which function h is positive had already been removed from X. This implies that **DualLagrangian**(X, l_∞) cannot return the strictly positive value $\lambda^{\min} = l_\infty$, leading to a contradiction. Hence, the algorithm terminates and the stated bounds hold because of the termination condition and by Proposition 2.8. ∎

3. Dual Lagrangian bound and a possible improvement. In this section, we apply the general properties presented in section 2 to the CDT problem (1.1). In fact, the CDT problem is a specific instance of (2.1) in which $f(\mathbf{x}) = \mathbf{x}^\top \mathbf{Q}\mathbf{x} + \mathbf{q}^\top \mathbf{x}$, $g(\mathbf{x}) = \mathbf{x}^\top \mathbf{x} - 1$, $h(\mathbf{x}) = \mathbf{x}^\top \mathbf{A}\mathbf{x} + \mathbf{a}^\top \mathbf{x} - a_0$.

Note that the first two requirements of Assumption 2.1 are satisfied; in order to satisfy the third one we assume that

$$h_0 = \min_{\mathbf{x}\,:\,\mathbf{x}^\top\mathbf{x} \le 1} \mathbf{x}^\top \mathbf{A}\mathbf{x} + \mathbf{a}^\top \mathbf{x} - a_0 < 0, \tag{3.1}$$

i.e., the feasible region of (1.1) has a nonempty interior. Note that the assumption can be checked in polynomial time by the solution of a trust region problem. As before, we denote by $X \subseteq \mathbb{R}^n$ a closed set such that $X \supset H$, i.e., it contains the ellipsoid defined by the second constraint. For each $\lambda \ge 0$, the Lagrangian relaxation (2.3) takes on the form

$$p_X(\lambda) = \min_{\mathbf{x}\in X} \quad \begin{array}{l} \mathbf{x}^\top(\mathbf{Q} + \lambda\mathbf{A})\mathbf{x} + (\mathbf{q} + \lambda\mathbf{a})^\top \mathbf{x} - \lambda a_0 \\ \mathbf{x}^\top \mathbf{x} \le 1. \end{array} \tag{3.2}$$

If $X = \mathbb{R}^n$, this is the standard Lagrangian relaxation of problem (1.1) and it can be solved efficiently since it is a trust region problem. Following the notation of section 2, let

$$P_X(\lambda) = arg \min_{\mathbf{x}\in X\,:\,\mathbf{x}^\top\mathbf{x} \le 1} \mathbf{x}^\top(\mathbf{Q} + \lambda\mathbf{A})\mathbf{x} + (\mathbf{q} + \lambda\mathbf{a})^\top \mathbf{x}$$

be the set of optimal solutions of (3.2). To apply Algorithm 2.1 to the CDT problem with $X = \mathbb{R}^n$, we need to characterize the set of optimal solutions $P_{\mathbb{R}^n}(\lambda)$ of problem (3.2) with $X = \mathbb{R}^n$, which is a trust region problem. The set of optimal solutions of a trust region problem has been derived, e.g., in [1, 20, 21]. Here we briefly recall the different cases. For simplicity, let $\mathbf{S}_\lambda = \mathbf{Q} + \lambda\mathbf{A}$ and $\mathbf{s}_\lambda = \mathbf{q} + \lambda\mathbf{a}$. We distinguish the following cases:

Case 1 If $\mathbf{S}_\lambda \succ \mathbf{O}$ and $\left\| -\frac{1}{2}\mathbf{S}_\lambda^{-1}\mathbf{s}_\lambda \right\| \leq 1$, then $-\frac{1}{2}\mathbf{S}_\lambda^{-1}\mathbf{s}_\lambda$ is the unique optimal solution of (3.2);

Case 2 Let $\mathbf{u}_j$ be the orthonormal eigenvectors of matrix $\mathbf{S}_\lambda$, and let γ_j be the corresponding eigenvalues. Let $\gamma_{\min} = \min_j \gamma_j$ and $J_\lambda = \mathit{arg}\ \min_j \gamma_j$. For each γ such that $(\forall j)\ \gamma \neq \gamma_j$, let

$$\mathbf{y}(\gamma) = \mathbf{y}_1(\gamma) + \mathbf{y}_2(\gamma),$$

where

$$\mathbf{y}_1(\gamma) = -\sum_{j \notin J_\lambda} \frac{\mathbf{s}_\lambda^\top \mathbf{u}_j}{\gamma_j - \gamma}\mathbf{u}_j, \quad \mathbf{y}_2(\gamma) = -\sum_{j \in J_\lambda} \frac{\mathbf{s}_\lambda^\top \mathbf{u}_j}{\gamma_j - \gamma}\mathbf{u}_j.$$

Then, we have the following subcases.

Case 2.1 It holds that $\mathbf{s}_\lambda^\top \mathbf{u}_j \neq 0$ for some $j \in J_\lambda$. Then, there exists a unique $\gamma^* \in (-\gamma_{\min}, +\infty)$ such that $\|\mathbf{y}(\gamma^*)\| = 1$ and $\mathbf{y}(\gamma^*)$ is the unique optimal solution of (3.2).

Case 2.2 It holds that $\mathbf{s}_\lambda^\top \mathbf{u}_j = 0$ for all $j \in J_\lambda$ but $\|\mathbf{y}_1(\gamma_{\min})\| \geq 1$. In this case there exists a unique $\gamma^* \in [-\gamma_{\min}, +\infty)$ such that $\|\mathbf{y}_1(\gamma^*)\| = 1$ and $\mathbf{y}_1(\gamma^*)$ is the unique optimal solution of (3.2).

Case 2.3 It holds that $\mathbf{s}_\lambda^\top \mathbf{u}_j = 0$ for all $j \in J_\lambda$ and $\|\mathbf{y}_1(\gamma_{\min})\| < 1$. In this case we have that $P_{\mathbb{R}^n}(\lambda)$ is not a singleton and is made up by the following points:

$$P_{\mathbb{R}^n}(\lambda) = \left\{ \mathbf{y}_1(\gamma_{\min}) + \sum_{j \in J_\lambda} \xi_j \mathbf{u}_j \ :\ \sum_{j \in J_\lambda} \xi_j^2 = 1 - \|\mathbf{y}_1(\gamma_{\min})\|^2 \right\}. \tag{3.3}$$

Thus, we recognize two further subcases.

Case 2.3.1 $|J_\lambda| = 1$, in which case $P_{\mathbb{R}^n}(\lambda)$ contains exactly two distinct points.

Case 2.3.2 $|J_\lambda| \geq 2$, in which case the set $P_{\mathbb{R}^n}(\lambda)$ contains an infinite number of points and is a connected set.

Note that in Cases 2.3.1 and 2.3.2 we can compute the two values $h_{\mathbb{R}^n}^{\min}(\lambda), h_{\mathbb{R}^n}^{\max}(\lambda)$ by solving a trust region problem over the border of a $|J_\lambda|$-dimensional ball. More precisely, we need to solve the following problems:

$$\begin{array}{ll} \min/\max_{\boldsymbol{\xi}} & \mathbf{w}(\boldsymbol{\xi})^\top \mathbf{A}\mathbf{w}(\boldsymbol{\xi}) + \mathbf{a}^\top \mathbf{w}(\boldsymbol{\xi}) - a_0 \\ & \|\mathbf{w}(\boldsymbol{\xi})\|^2 = 1, \end{array} \tag{3.4}$$

where $\mathbf{w}(\boldsymbol{\xi}) = \mathbf{y}_1(\gamma_{\min}) + \sum_{j \in J_\lambda} \xi_j \mathbf{u}_j$. In these cases, where $P_{\mathbb{R}^n}(\lambda)$ is not a singleton, we also set

$$\begin{array}{ll} \mathbf{z}_1(\lambda) = \mathbf{w}(\boldsymbol{\xi}_1), & \boldsymbol{\xi}_1 \in \ \mathit{arg}\ \min_{\boldsymbol{\xi}\ :\ \|\mathbf{w}(\boldsymbol{\xi})\|=1} \mathbf{w}(\boldsymbol{\xi})^\top \mathbf{A}\mathbf{w}(\boldsymbol{\xi}) + \mathbf{a}^\top \mathbf{w}(\boldsymbol{\xi}) - a_0, \\ \mathbf{z}_2(\lambda) = \mathbf{w}(\boldsymbol{\xi}_2), & \boldsymbol{\xi}_2 \in \ \mathit{arg}\ \max_{\boldsymbol{\xi}\ :\ \|\mathbf{w}(\boldsymbol{\xi})\|=1} \mathbf{w}(\boldsymbol{\xi})^\top \mathbf{A}\mathbf{w}(\boldsymbol{\xi}) + \mathbf{a}^\top \mathbf{w}(\boldsymbol{\xi}) - a_0, \end{array} \tag{3.5}$$

while in all other cases, when $P_{\mathbb{R}^n}(\lambda) = \{\mathbf{z}^\star(\lambda)\}$ is a singleton, we set

$$\mathbf{z}_1(\lambda) = \mathbf{z}_2(\lambda) = \mathbf{z}^\star(\lambda). \tag{3.6}$$

The following statement is a direct consequence of Proposition 2.5.

PROPOSITION 3.1. *In the CDT problem* (1.1), *if* $\lambda > 0$ *and*

- $h_{\mathbb{R}^n}^{\min}(\lambda) = 0$*;*
- *or if* $h_{\mathbb{R}^n}^{\max}(\lambda) = 0$*;*
- *or* $h_{\mathbb{R}^n}^{\min}(\lambda) < 0 < h_{\mathbb{R}^n}^{\max}(\lambda)$ *and* $|J_\lambda| \geq 2$ *(i.e., we are in Ccase* 2.3.2*);*

then $p_{\mathbb{R}^n}(\lambda) = p^\star$.

Proof. Since $\{h^{\min}_{\mathbb{R}^n}(\lambda), h^{\max}_{\mathbb{R}^n}(\lambda)\} \in Q_X(\lambda)$, in the first two cases $0 \in Q_X(\lambda)$ and the thesis is a consequence of Proposition 2.5. If $h^{\min}_{\mathbb{R}^n}(\lambda) < 0 < h^{\max}_{\mathbb{R}^n}(\lambda)$ and $|J_\lambda| \geq 2$, we observed that $P_{\mathbb{R}^n}(\lambda)$ is a connected set. Then, there exists $\mathbf{x}^\star \in P_{\mathbb{R}^n}(\lambda)$ such that $\mathbf{x}^\star \in \partial H$. More precisely, $\mathbf{x}^\star$ is a point along the curve in $P_{\mathbb{R}^n}(\lambda)$ connecting points $\mathbf{z}_1(\lambda)$ and $\mathbf{z}_2(\lambda)$, defined in (3.5). Thus, the lower bound $p_{\mathbb{R}^n}(\lambda)$ is equal to the optimal value of problem (1.1). ∎

Note that the first two conditions of Proposition 3.1 imply exactness of the bound also for generic regions $X \supset H$, while the last condition is specific to the case $X = \mathbb{R}^n$. The following result is related to the necessary and sufficient condition under which the dual Lagrangian bound is not exact discussed in [2].

PROPOSITION 3.2. *In the CDT problem* (1.1)*,* $p_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}) \neq p^*$ *if and only if* $\lambda_{\mathbb{R}^n} > 0$*,* $P_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n})$ *contains exactly two points (Case* 2.3.1*), and* $0 \in (h^{\min}_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}), h^{\max}_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}))$*.*

Proof. First note that, in view of Proposition 2.3, the dual Lagrangian bound is always exact when $\lambda_{\mathbb{R}^n} = 0$. When $\lambda_{\mathbb{R}^n} > 0$, the result is a consequence of Proposition 3.1 and the fact that for $|J_{\lambda_{\mathbb{R}^n}}| = 1$ it holds that $Q_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}) = \{h^{\min}_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}), h^{\max}_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n})\} \not\ni 0$. ∎

Now, we introduce an example where $p_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}) \neq p^*$, that is the dual Lagrangian bound is not exact, which will also be helpful in the following sections.

Example 3.3. Let us consider the following example taken from [12]:

$$\mathbf{Q} = \begin{pmatrix} -4 & 1 \\ 1 & -2 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{q} = (1\ 1) \quad \mathbf{a} = (0\ 0), \quad a_0 = 2.$$

Such an instance has optimal value -4 attained at points $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ and $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. The maximizer of $p_{\mathbb{R}^2}(\lambda)$ is $\lambda_{\mathbb{R}^2} = 1$ for which we have

$$h^{\min}_{\mathbb{R}^2} \approx -0.66 < 0 < 0.66 \approx h^{\max}_{\mathbb{R}^2},$$

and, moreover, $|J_{\lambda_{\mathbb{R}^2}}| = 1$, so that we have exactly two optimal solutions of (3.2), one violating the second constraint, namely $\mathbf{z}_2(\lambda_{\mathbb{R}^2}) = (-0.911, 0.4114)$, point x_1 in Figure 1, displayed as $\circ$, the other in $int(H)$, point z_1 in Figure 1, displayed as $\times$. The lower bound is $p_{\mathbb{R}^2}(1) = -4.25$, which is not exact.

Now, let us assume that the dual Lagrangian bound is not exact, i.e., as previously stated in Proposition 3.2

$$0 \in \left(h^{\min}_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}), h^{\max}_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n})\right), \quad |J_{\lambda_{\mathbb{R}^n}}| = 1.$$

Recall that, by Proposition 3.2, in this case, there exists a single point $\mathbf{z}_1(\lambda_{\mathbb{R}^n}) \in P_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}) \cap H$ (actually $\mathbf{z}_1(\lambda_{\mathbb{R}^n}) \in int(H)$), and a single point $\mathbf{z}_2(\lambda_{\mathbb{R}^n}) \in P_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}) \setminus H$. Now we show that the dual Lagrangian bound can be strictly improved through the addition of a linear cut. We first observe that the optimal value of problem (1.1) does not change if we add constraints which are implied by the second one.

In the following proposition, we define a projection $\Pi_{\mathbf{A},\mathbf{a}} : \mathbb{R}^n \setminus H \to \partial H$ that maps $\mathbf{x} \notin H$ to the element of ∂H located on the segment that joins $\mathbf{x}$ to the center of the ellipsoid H (given by $\boldsymbol{\alpha} = -\frac{1}{2}\mathbf{A}^{-1}\mathbf{a}$).

PROPOSITION 3.4. *For* $\mathbf{x} \notin H$*, set* $\Pi_{\mathbf{A},\mathbf{a}}(\mathbf{x}) = \sqrt{\frac{-h(\boldsymbol{\alpha})}{h(\mathbf{x}) - h(\boldsymbol{\alpha})}}(\mathbf{x} - \boldsymbol{\alpha}) + \boldsymbol{\alpha}$*, where* $\boldsymbol{\alpha} = -\frac{1}{2}\mathbf{A}^{-1}\mathbf{a}$ *is the center of the ellipsoid. Then* $h(\Pi_{\mathbf{A},\mathbf{a}}(\mathbf{x})) = 0$*.*

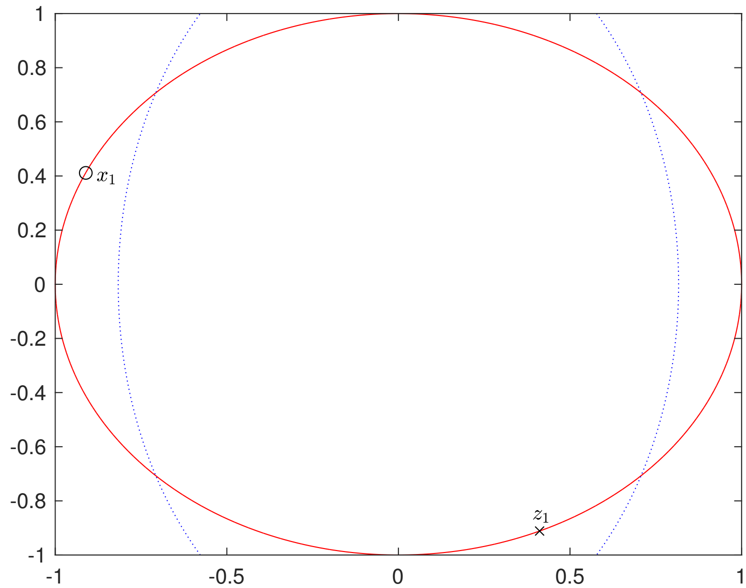


FIG. 1. *Optimal solutions of the dual Lagrangian bound outside H (x_1) and in int(H) (z_1), denoted by $\circ$ and $\times$, respectively. The continuous red curve is the border of the unit ball, while the dotted blue curve is the border of the ellipsoid H. (Figure in color online.)*

Proof. Note that $(\forall \beta \in \mathbb{R})\ h(\beta(\mathbf{x}-\boldsymbol{\alpha})+\boldsymbol{\alpha}) - h(\boldsymbol{\alpha}) = \beta^2(h(\mathbf{x})-h(\boldsymbol{\alpha}))$ (it is a consequence of the fact that function h is quadratic and it can be verified by direct substitution). Then $h(\Pi_{\mathbf{A},\mathbf{a}}(\mathbf{x})) = h\left(\sqrt{\frac{-h(\boldsymbol{\alpha})}{h(\mathbf{x})-h(\boldsymbol{\alpha})}}(\mathbf{x}-\boldsymbol{\alpha})+\boldsymbol{\alpha}\right) = \frac{-h(\boldsymbol{\alpha})}{h(\mathbf{x})-h(\boldsymbol{\alpha})}(h(\mathbf{x}) - h(\boldsymbol{\alpha})) + h(\boldsymbol{\alpha}) = 0.$ ∎

Given any $\bar{\mathbf{x}} \in \mathbb{R}^n$, it holds, by convexity, that

$$\mathbf{x}^\top \mathbf{A}\mathbf{x} + \mathbf{a}^\top \mathbf{x} \geq \bar{\mathbf{x}}^\top \mathbf{A}\bar{\mathbf{x}} + \mathbf{a}^\top \bar{\mathbf{x}} + (2\mathbf{A}\bar{\mathbf{x}}+\mathbf{a})^\top(\mathbf{x}-\bar{\mathbf{x}}).$$

Thus, the following linear constraint is implied by the second constraint in (1.1):

$$(2\mathbf{A}\bar{\mathbf{x}}+\mathbf{a})^\top \mathbf{x} - \bar{\mathbf{x}}^\top \mathbf{A}\bar{\mathbf{x}} \leq a_0, \tag{3.7}$$

and, consequently, it can be added to problem (1.1) without modifying its feasible region. In particular, if $\bar{\mathbf{x}} \in \partial H$, being $\bar{\mathbf{x}}^T \mathbf{A}\bar{\mathbf{x}} + \mathbf{a}^T \bar{\mathbf{x}} = a_0$, the linear constraint is

$$(2\mathbf{A}\bar{\mathbf{x}}+\mathbf{a})^\top(\mathbf{x}-\bar{\mathbf{x}}) \leq 0. \tag{3.8}$$

Due to the redundancy of the linear constraint for problem (1.1), we can define, for a given $\bar{\mathbf{x}} \in \partial H$, the new Lagrangian problem

$$\begin{array}{rl} p_X(\lambda) = \min_{\mathbf{x}} & \mathbf{x}^\top(\mathbf{Q}+\lambda\mathbf{A})\mathbf{x} + (\mathbf{q}+\lambda\mathbf{a})^\top \mathbf{x} - \lambda a_0 \\ & \mathbf{x}^\top \mathbf{x} \leq 1 \\ & (2\mathbf{A}\bar{\mathbf{x}}+\mathbf{a})^\top(\mathbf{x}-\bar{\mathbf{x}}) \leq 0, \end{array} \tag{3.9}$$

where

$$X = \Omega_{\bar{\mathbf{x}}} = \{\mathbf{x}\ :\ (2\mathbf{A}\bar{\mathbf{x}}+\mathbf{a})^\top(\mathbf{x}-\bar{\mathbf{x}}) \leq 0\} \supset H. \tag{3.10}$$

If we set $\bar{\mathbf{x}} = \Pi_{\mathbf{A},\mathbf{a}}(\mathbf{z}_2(\lambda_{\mathbb{R}^n}))$, i.e., $\bar{\mathbf{x}}$ is the projection over ∂H of the single point in $P_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}) \setminus H$, then $\mathbb{R}^n \supset X \supset H$ and, moreover, $X \cap (P_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}) \setminus H) = \emptyset$, so that, by Proposition 2.12, $\bar{p}_X > \bar{p}_{\mathbb{R}^n}$. Then, if we run again Algorithm 2.1 with input $X = \Omega_{\bar{\mathbf{x}}}$ defined in (3.10) and $\lambda_{\mathtt{init}} = \lambda_{\mathbb{R}^n}$ (or $\lambda_{\mathtt{init}} = \lambda_{\mathbb{R}^n}^{\max}$), we are able to improve strictly the dual Lagrangian bound. Note that problem (3.9), needed to compute function $p_{\Omega_{\bar{\mathbf{x}}}}$, can be solved in polynomial time according to the results proved in [12, 23]. But we also discuss an alternative way to solve problem (3.9), based on the solution of a trust region problem. For $\lambda = \lambda_{\mathbb{R}^n}$, after the addition of the linear cut, a unique optimal solution exists, lying in $int(H)$ and, consequently, in $int(\Omega_{\bar{\mathbf{x}}})$, since also the linear constraint in (3.9) is not active at it, being H a subset of the region **AQ4** defined by the linear cut. By continuity, for λ values smaller than but close to $\lambda_{\mathbb{R}^n}$, the unique optimal solution of (3.9) also lies in $int(H)$, i.e., $P_{\Omega_{\bar{\mathbf{x}}}}(\lambda) = \{\mathbf{z}_1(\lambda)\}$ with $\mathbf{z}_1(\lambda) \in int(H)$. Thus, such optimal solution must be a local and nonglobal optimal solution of the trust region problem (3.2) with $X = \mathbb{R}^n$. Indeed, the globally optimal solutions of this trust region problem always violate the second constraint in (1.1) for all $\lambda < \lambda_{\mathbb{R}^n}$. Now, for all $\lambda \in [0, \lambda_{\mathbb{R}^n})$, we first check whether a local and nonglobal optimal solution of problem (3.2) with $X = \mathbb{R}^n$ exists by exploiting the necessary and sufficient condition stated in [24]. Also recall that, if it exists, the local and nonglobal minimizer is unique. If it does not exist, then we set $f_1 = +\infty$. Otherwise, if it exists, we denote it by $\mathbf{z}_1(\lambda)$. If $\mathbf{z}_1(\lambda) \notin \Omega_{\bar{\mathbf{x}}}$, then we set again $f_1 = +\infty$, otherwise we denote by f_1 the value of the objective function of (3.9) evaluated at $\mathbf{z}_1(\lambda)$. If some globally optimal solution of the trust region problem (3.2) with $X = \mathbb{R}^n$ belongs to $\Omega_{\bar{\mathbf{x}}} \setminus H$, then it is also a solution of (3.9) and we set f_2 equal to the optimal value of this problem. Note that in this case $f_2 < f_1$, since f_1 is the function value at a local and nonglobal solution of the trust region problem. Then, Algorithm 2.1 sets $\lambda^{\min} = \lambda$. Instead, if all globally optimal solutions of the trust region problem do not belong to $\Omega_{\bar{\mathbf{x}}}$, we proceed as follows. We consider the best feasible solutions of problem (3.9) for which the linear constraint is imposed to be active. The resulting problem is converted into a trust region problem, after the change of variable $\mathbf{x} = \bar{\mathbf{x}} + \mathbf{V}\mathbf{z}$, where $\mathbf{V} \in \mathbb{R}^{n\times(n-1)}$ is a matrix whose columns form a basis for the null space of vector $2\mathbf{A}\bar{\mathbf{x}} + \mathbf{a}$. The resulting (trust region) problem is

$$
\begin{array}{ll}
\min_{\mathbf{w}\in\mathbb{R}^{n-1}} & \mathbf{w}^\top\mathbf{V}^\top(\mathbf{Q}+\lambda\mathbf{A})\mathbf{V}\mathbf{w} + \left[2\bar{\mathbf{x}}^\top(\mathbf{Q}+\lambda\mathbf{A})\mathbf{V} + (\mathbf{q}+\lambda\mathbf{a})^\top\right]\mathbf{w} + \ell(\bar{\mathbf{x}},\lambda) \\
 & \|\bar{\mathbf{x}} + \mathbf{V}\mathbf{w}\|^2 \le 1,
\end{array}
\tag{3.11}
$$

where $\ell(\bar{\mathbf{x}},\lambda) = \bar{\mathbf{x}}^\top(\mathbf{Q}+\lambda\mathbf{A})\bar{\mathbf{x}} + (\mathbf{q}+\lambda\mathbf{a})^\top\bar{\mathbf{x}} - \lambda a_0$ is constant with respect to the vector of variables $\mathbf{w}$. Let $W^\star(\lambda)$ be the set of optimal solutions of problem (3.11) and

$$P_1^\star(\lambda) = \{\bar{\mathbf{x}} + \mathbf{V}\mathbf{w}^\star \ : \ \mathbf{w}^\star \in W^\star(\lambda)\}.$$

Note that the set $W^\star(\lambda)$ can be computed through the procedure presented in section 3 with the different cases (namely, Cases 1, 2.1, 2.2, 2.3.1, 2.3.2) after rewriting it as a classical trust region problem. Moreover, let $f_2 < +\infty$ be the optimal value of problem (3.11). Now, after comparing f_1 and f_2, we are able to define the set $P_{\Omega_{\bar{\mathbf{x}}}}(\lambda)$ of optimal solutions for problem (3.9). More precisely, if $f_2 > f_1$, then $P_{\Omega_{\bar{\mathbf{x}}}}(\lambda) = \{\mathbf{z}_1(\lambda)\}$, i.e., $\mathbf{z}_1(\lambda)$ is the unique optimal solution of problem (3.9). In this case

$$h_{\Omega_{\bar{\mathbf{x}}}}^{\min}(\lambda) = h_{\Omega_{\bar{\mathbf{x}}}}^{\max}(\lambda) = \mathbf{z}_1(\lambda)^\top\mathbf{A}\mathbf{z}_1(\lambda) + \mathbf{a}^\top\mathbf{z}_1(\lambda) - a_0.$$

Instead, if $f_2 < f_1$, which always holds, e.g., if $f_1 = +\infty$, then $P_{\Omega_{\bar{\mathbf{x}}}}(\lambda) = P_1^\star(\lambda)$. Since all points in $P_1^\star(\lambda)$ lie over a supporting hyperplane of H, we must have that

$$h^{\min}_{\Omega_{\bar{\mathbf{x}}}}(\lambda) = \min_{\mathbf{x}\in P_1^\star(\lambda)} \mathbf{x}^\top \mathbf{A}\mathbf{x} + \mathbf{a}^\top \mathbf{x} - a_0 \geq 0,$$

and equality holds only if $\bar{\mathbf{x}} \in P_1^\star(\lambda)$. In the latter case, the bound is exact, otherwise Algorithm 2.1 sets $\lambda^{\min} = \lambda$. Finally, if $f_1 = f_2$, then $P_{\Omega_{\bar{\mathbf{x}}}}(\lambda) = P_1^\star(\lambda) \cup \{\mathbf{z}_1(\lambda)\}$ and in this case $0 \in [h^{\min}_{\Omega_{\bar{\mathbf{x}}}}(\lambda), h^{\max}_{\Omega_{\bar{\mathbf{x}}}}(\lambda)]$ and the algorithms exits the loop. The following result is a straightforward consequence of Proposition 2.12.

PROPOSITION 3.5. *Algorithm* 2.1 *with* $\varepsilon = 0$ *will stop after a finite number of iterations or will converge to some* $\lambda_{\Omega_{\bar{\mathbf{x}}}} < \lambda_{\mathbb{R}^n}$ *with a new lower bound* $\bar{p}_{\Omega_{\bar{\mathbf{x}}}} > \bar{p}_{\mathbb{R}^n}$.

Proof. Strict inequalities hold in view of Proposition 2.12 with $X = \mathbb{R}^n$ and $Y = \Omega_{\bar{\mathbf{x}}}$, since, as already observed, $\Omega_{\bar{\mathbf{x}}} \cap (P_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}) \setminus H) = \emptyset$. ∎

If the final bound is not exact, i.e., $\bar{p}_{\Omega_{\bar{\mathbf{x}}}} = p_{\Omega_{\bar{\mathbf{x}}}}(\lambda_{\Omega_{\bar{\mathbf{x}}}}) < p^\star$, at $\lambda_{\Omega_{\bar{\mathbf{x}}}}$ we have $f_1 = f_2$ and $P_{\Omega_{\bar{\mathbf{x}}}}(\lambda_{\Omega_{\bar{\mathbf{x}}}})$ contains multiple optimal solutions, in particular, one in *int*(H) and the other(s) outside H, more precisely on $\partial\Omega_{\bar{\mathbf{x}}} \setminus H$. We illustrate all this on Example 3.3.

Example 3.6. The optimal solution of (3.2) with $X = \mathbb{R}^n$ for $\lambda_{\mathbb{R}^n} = 1$ which violates the second constraint is $\mathbf{z}_2(\lambda_{\mathbb{R}^n}) = (-0.911, 0.4114)$. The lower bound is $p_{\mathbb{R}^n}(1) = -4.25$. After the addition of the linear inequality (3.7) obtained with $\bar{\mathbf{x}} = \Pi_{\mathbf{A},\mathbf{a}}(\mathbf{z}_2(\lambda_{\mathbb{R}^n}))$, equal to the projection of $\mathbf{z}_2(\lambda_{\mathbb{R}^n})$ over the boundary of the second constraint, we can run again Algorithm 2.1 with $X = \Omega_{\bar{\mathbf{x}}}$ and we get to $\lambda_{\Omega_{\bar{\mathbf{x}}}} \approx 0.726$ and $p_{\Omega_{\bar{\mathbf{x}}}}(\lambda_{\Omega_{\bar{\mathbf{x}}}}) \approx -4.097$, which improves the previous lower bound. In Figure 2 we show the linear cut and the two new optimal solutions outside H and in *int*(H) (x_2 and z_2, respectively) obtained at $\lambda_{\Omega_{\bar{\mathbf{x}}}}$. In the same figure we also display the previous pair of optimal solutions in order to show the progress of the algorithm.

It is worthwhile to discuss at this point the relations between the approach proposed in this work and the one proposed in [27], where the classical SDP relaxation

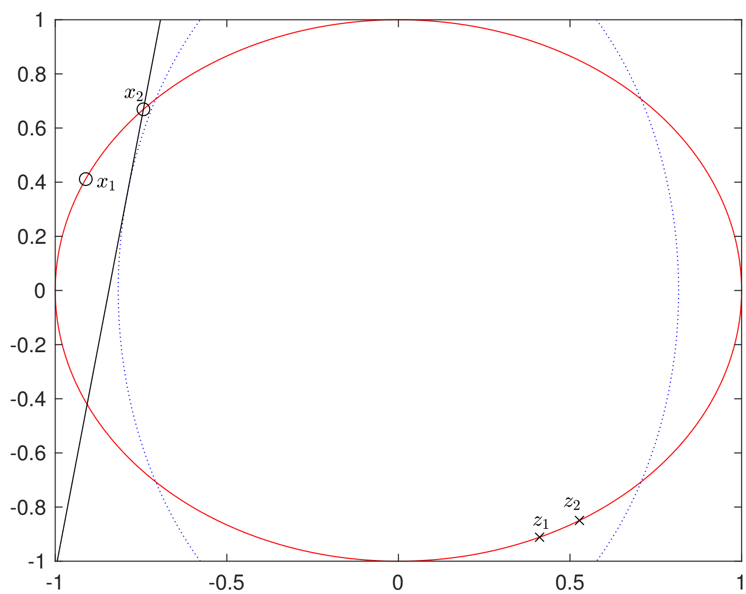


FIG. 2. *First linear cut and the two optimal solutions lying outside* H (x_2) *and in int*(H) (z_2), *denoted by* $\circ$ *and* $\times$, *respectively.*

of problem (1.1) is considered. Both approaches stem from the necessary and sufficient condition under which the dual Lagrangian bound is not exact discussed in [2], namely the existence of two distinct optimal solutions, one belonging to $int(H)$ and the other outside H. In both cases it is observed that, in order to improve the bound, it is necessary to separate such optimal solutions. But the way the separation is carried on in the two approaches is different. Following the terminology employed in Integer Programming, in [27] the separation is performed through a branching operation, while in this work it is performed through the addition of a cutting plane. Indeed, in [27] first, a hyperplane $\mathbf{w}^T\mathbf{x} = v$ separating the two optimal solutions is introduced; then, two distinct subproblems are solved, one by adding the inequality $\mathbf{w}^T\mathbf{x} \leq v$ (converted into an SOCP constraint) to the SDP relaxation, the other by adding the inequality $\mathbf{w}^T\mathbf{x} \geq v$ to the SDP relaxation; finally, the new bound is set equal to the minimum of the bounds over the two subregions into which the original feasible region has been split. Note that one of the two subregions may be empty, in which case its corresponding lower bound is set equal to $+\infty$ and the linear inequality is a separating hyperplane between H and the optimal solution outside H. In this paper the separation is performed through the addition of a linear cut and a single subproblem is solved. Moreover, in [27] it is observed that one could search for an ‘‘optimal‘‘ hyperplane separating the two optimal solutions, namely one which leads to the best possible bound. In that paper such a hyperplane is derived in the special case when the function h is the product of two affine functions and an exactness result is also provided for the case of problems with two variables, but the question about how to characterize an ''optimal‘‘ affine function is left open in the general case. In the next section we will be able to provide a necessary and sufficient condition for a linear cut to be the one delivering the best bound (Proposition 4.2). Based on this condition, we will also be able to propose a procedure to improve the bound by local adjustments of the linear cut. Finally, in this paper we will also show in section 5 that the bound can be further improved through the addition of a second linear cut, possibly followed by a local adjustment of the two linear cuts. The experiments in section 6 will show that the bound obtained by the addition of two linear cuts is quite a good one, allowing one to solve all except one of the 212 hardest instances introduced in [12]. As a final remark, we observe that the approach presented in [27] and the one discussed in this paper could actually be combined by performing a branching operation (as in [27]) followed by the addition of a linear cut (as in this work) in each branch. Borrowing again from the terminology of Integer Programming, this can be viewed as a branch-and-cut approach.

4. Improving the bound by local adjustments of the linear cut. In the previous section we proposed to set $\bar{\mathbf{x}}$ equal to the projection over ∂H of $\mathbf{z}_2(\lambda_{\mathbb{R}^n})$, the optimal solution of problem (3.2) with $X = \mathbb{R}^n$ lying outside H. However, this point can be improved by some local adjustment. We first give a necessary and sufficient condition under which the current point $\bar{\mathbf{x}}$ cannot be improved. The proof will also suggest how to improve the point (and the bound) when the condition is not fulfilled. Let

$$r(\mathbf{w},\mathbf{x}) = (2\mathbf{A}\mathbf{w}+\mathbf{a})^\top(\mathbf{x}-\mathbf{w})+\mathbf{w}^\top\mathbf{A}\mathbf{w}+\mathbf{a}^\top\mathbf{w}-a_0 = (2\mathbf{A}\mathbf{w}+\mathbf{a})^\top\mathbf{x}-\mathbf{w}^\top\mathbf{A}\mathbf{w}-a_0 \tag{4.1}$$

be the linearization of the ellipsoid constraint at $\mathbf{w}$. Note that constraint (3.7) can be written as $r(\bar{\mathbf{x}},\mathbf{x}) \leq 0$. Also note that for each $\mathbf{x}$, r is a concave function with respect to $\mathbf{w}$. Next, we set

$$p(\lambda, \mathbf{w}) = p_{\Omega_{\mathbf{w}}}(\lambda),$$

in order to highlight the dependency of the bound not only on λ but also on $\mathbf{w}$. Then, in order to maximize the lower bound, we need to solve the following problem:

$$\max_{\lambda \geq 0,\ \mathbf{w}} p(\lambda, \mathbf{w}).$$

As before, we denote by $P_{\Omega_{\mathbf{w}}}(\lambda)$ the optimal set of problem (3.9) with $\bar{\mathbf{x}} = \mathbf{w}$, while we denote by $P^1_{\Omega_{\mathbf{w}}}(\lambda) = P_{\Omega_{\mathbf{w}}}(\lambda) \setminus int(H)$ the set of optimal solutions of the same problem lying outside the interior of the ellipsoid H. We will need the following lemma.

Lemma 4.1. *Set-valued functions $\mathbf{w} \rightsquigarrow P_{\Omega_{\mathbf{w}}}(\lambda)$ and $\mathbf{w} \rightsquigarrow P^1_{\Omega_{\mathbf{w}}}(\lambda)$ are upper semicontinuous for any $\lambda \geq 0$.*

Proof. Upper semicontinuity of $\mathbf{w} \rightsquigarrow P_{\Omega_{\mathbf{w}}}(\lambda)$ follows from the maximum theorem (see, for instance, Theorem 1.4.16 of [5]), while upper semicontinuity of $\mathbf{w} \rightsquigarrow P^1_{\Omega_{\mathbf{w}}}(\lambda)$ follows from the fact that $P^1_{\Omega_{\mathbf{w}}}(\lambda)$ is obtained by intersecting the upper semicontinuous function $\mathbf{w} \rightsquigarrow P_{\Omega_{\mathbf{w}}}(\lambda)$ with the compact set $\{\mathbf{x} : \|\mathbf{x}\| \leq 1\} \setminus int(H)$ (see, for instance, Proposition 1.4.9 of [5]). ∎

Now, the following proposition characterizes the maxima of p.

Proposition 4.2. *Let $(\lambda^*, \mathbf{w}^*)$ be such that $\mathbf{w}^* \in \partial H$, $\lambda^* > 0$, $0 \in (h^{\min}_{\Omega_{\mathbf{w}^*}}(\lambda^*), h^{\max}_{\Omega_{\mathbf{w}^*}}(\lambda^*))$, and $0 \notin Q_{\Omega_{\mathbf{w}^*}}(\lambda^*)$. Assume also that $(\forall \mathbf{v} \in P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*))\ r(\mathbf{w}^*, \mathbf{v}) = 0$. Then, the following statements are equivalent:*

(i) $(\lambda^*, \mathbf{w}^*) = \mathrm{argmax}_{(\lambda \geq 0, \mathbf{w})} p(\lambda, \mathbf{w})$.

(ii) $(\forall \mathbf{w} \in \mathbb{R}^n)\ P^1_{\Omega_{\mathbf{w}}}(\lambda^*) \neq \emptyset$.

(iii)

$$(\forall \mathbf{d} \in \mathbb{R}^n)\,(\exists \mathbf{v} \in P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*))\ :\ -\mathbf{d}^\top \mathbf{A} \mathbf{d} + 2\mathbf{d}^\top \mathbf{A}(\mathbf{v} - \mathbf{w}^*) \leq 0. \tag{4.2}$$

Proof. Before proving the result we make some remarks. First, note that $0 \in (h^{\min}_{\Omega_{\mathbf{w}^*}}(\lambda^*), h^{\max}_{\Omega_{\mathbf{w}^*}}(\lambda^*))$ implies that $\lambda^* = \lambda_{\Omega_{\mathbf{w}^*}}$. Moreover, since $0 \notin Q_{\Omega_{\mathbf{w}^*}}(\lambda^*)$ and $\lambda^* > 0$, by Propositions 2.3 and 2.5, $p_{\Omega_{\mathbf{w}^*}}(\lambda^*) < p^*$ (i.e., the bound is not exact). If the bound were exact, the current pair $(\mathbf{w}^*, \lambda^*)$ would obviously be optimal. Also note that $h^{\min}_{\Omega_{\mathbf{w}^*}}(\lambda^*) < 0 < h^{\max}_{\Omega_{\mathbf{w}^*}}(\lambda^*)$ implies that $P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*) \neq \emptyset$. Finally, condition $(\forall \mathbf{v} \in P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*))\ r(\mathbf{w}^*, \mathbf{v}) = 0$ means that $P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*) \subset \partial \Omega_{\mathbf{w}^*}$, which, according to the discussion about the optimal solutions of problem (3.9), holds true provided that $\mathbf{z}_2(\lambda_{\mathbb{R}^n}) \notin \Omega_{\mathbf{w}^*}$.

(i) → (ii) By contradiction, let $\mathbf{w}$ be such that $P^1_{\Omega_{\mathbf{w}}}(\lambda^*) = \emptyset$. Then, $P_{\Omega_{\mathbf{w}}}(\lambda^*) \subset int(H)$. Therefore, function $\lambda \rightsquigarrow p_{\Omega_{\mathbf{w}}}(\lambda)$ is strictly decreasing at λ^*. As a consequence, there exists $0 \leq \bar{\lambda} < \lambda^*$ such that $p(\bar{\lambda}, \mathbf{w}) = p_{\Omega_{\mathbf{w}}}(\bar{\lambda}) > p_{\Omega_{\mathbf{w}}}(\lambda^*) = p(\lambda^*, \mathbf{w})$. Moreover, $p(\lambda^*, \mathbf{w}^*) = p(\lambda^*, \mathbf{w})$. Indeed, since $0 \in (h^{\min}_{\Omega_{\mathbf{w}^*}}(\lambda^*), h^{\max}_{\Omega_{\mathbf{w}^*}}(\lambda^*))$, we have that $h^{\min}_{\Omega_{\mathbf{w}^*}}(\lambda^*) < 0$, so that $P_{\Omega_{\mathbf{w}^*}}(\lambda^*) \cap int(H)$ is not empty and is equal to $P_{\Omega_{\mathbf{w}}}(\lambda^*)$. Hence, $p(\bar{\lambda}, \mathbf{w}) > p(\lambda^*, \mathbf{w}^*)$, which contradicts (i).

(ii) → (iii) By contradiction, let us assume that there exists $\mathbf{d} \in \mathbb{R}^n$ such that

$$(\forall \mathbf{v} \in P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*))\ -\mathbf{d}^\top \mathbf{A} \mathbf{d} + 2\mathbf{d}^\top \mathbf{A}(\mathbf{v} - \mathbf{w}^*) > 0. \tag{4.3}$$

By continuity of the left-hand side of the inequality in (4.3) with respect to $\mathbf{v}$, there exists a neighborhood B_1 of $P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*)$ such that

$$(\forall \mathbf{v} \in B_1)\ -\mathbf{d}^\top \mathbf{A} \mathbf{d} + 2\mathbf{d}^\top \mathbf{A}(\mathbf{v} - \mathbf{w}^*) > 0,$$

which implies that

$$(\forall \mathbf{v} \in B_1)\, 2\mathbf{d}^\top \mathbf{A}(\mathbf{v} - \mathbf{w}^*) > 0. \tag{4.4}$$

By upper semicontinuity of set-valued function $\mathbf{w} \rightsquigarrow P^1_{\Omega_{\mathbf{w}}}(\lambda^*)$ (see Lemma 4.1), there exists a neighborhood B_2 of $\mathbf{w}^*$ such that $(\forall \mathbf{w} \in B_2)\, P^1_{\Omega_{\mathbf{w}}}(\lambda^*) \subset B_1$. Let $\bar{\eta} > 0$ be such that $\mathbf{w}^* + \bar{\eta}\mathbf{d} \in B_2$ and consider function $\rho : [0, \bar{\eta}] \times \mathbb{R}^n \to \mathbb{R}$, $\rho(\eta, \mathbf{v}) = r(\mathbf{w}^* + \eta\mathbf{d}, \mathbf{v})$. Then, by definition of r in (4.1),

$$\rho(\eta, \mathbf{v}) = -\eta^2 \mathbf{d}^\top \mathbf{A}\mathbf{d} + 2\eta\mathbf{d}^\top \mathbf{A}(\mathbf{v} - \mathbf{w}^*) + \rho(0, \mathbf{v}).$$

By (4.4), $(\forall \mathbf{v} \in B_1)\, \partial_\eta \rho(0, \mathbf{v}) > 0$, where ∂_η denotes the partial derivative with respect to η. By continuity of ρ, there exists a continous function $\hat{\eta} : B_1 \to (0, \bar{\eta}]$ such that

$$(\forall \mathbf{v} \in B_1, \eta \in (0, \hat{\eta}(\mathbf{v})])\, \rho(\eta, \mathbf{v}) > \rho(\mathbf{0}, \mathbf{v}). \tag{4.5}$$

Hence, since B_1 is a compact set and $\hat{\eta}$ is continuous and strictly positive, setting $\tilde{\eta} = \min_{\mathbf{v} \in B_1} \hat{\eta}(\mathbf{v})$, it follows that $(\forall \eta \in [0, \tilde{\eta}])\, P^1_{\Omega_{\mathbf{w}^* + \eta\mathbf{d}}}(\lambda^*) \subseteq P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*)$. Moreover, since, by assumption, $(\forall \mathbf{v} \in P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*))\, r(\mathbf{w}^*, \mathbf{v}) = 0$,

$$(\forall \mathbf{v} \in P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*))\, \rho(0, \mathbf{v}) = 0.$$

Hence, (4.5) implies that

$$(\forall \mathbf{v} \in P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*))\, \rho(\tilde{\eta}, \mathbf{v}) > 0. \tag{4.6}$$

Being $P^1_{\Omega_{\mathbf{w}^* + \tilde{\eta}\mathbf{d}}}(\lambda^*) \subseteq P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*)$, (4.6) implies that $P^1_{\Omega_{\mathbf{w}^* + \tilde{\eta}\mathbf{d}}}(\lambda^*) = \emptyset$, which contradicts (ii).

(iii) $\to$ (i) By contradiction, there exists a couple $(\bar{\lambda}, \bar{\mathbf{w}})$ such that $p(\bar{\lambda}, \bar{\mathbf{w}}) > p(\lambda^*, \mathbf{w}^*)$. In particular, we can take $\bar{\lambda} = \lambda_{\Omega_{\bar{\mathbf{w}}}}$. In case $\bar{\lambda} = 0$, by assumption $\bar{\lambda} < \lambda^*$. Now we show that the same inequality holds true also when $\bar{\lambda} > 0$. If $\bar{\lambda} > 0$, then $\bar{\lambda} = \lambda_{\Omega_{\bar{\mathbf{w}}}}$ implies that $0 \in [h^{\min}_{\Omega_{\bar{\mathbf{w}}}}(\bar{\lambda}), h^{\max}_{\Omega_{\bar{\mathbf{w}}}}(\bar{\lambda})]$. Since, by assumption, $0 \in (h^{\min}_{\Omega_{\mathbf{w}^*}}(\lambda^*), h^{\max}_{\Omega_{\mathbf{w}^*}}(\lambda^*))$, we have that both $P_{\Omega_{\bar{\mathbf{w}}}}(\bar{\lambda}) \cap H \neq \emptyset$ and $P_{\Omega_{\mathbf{w}^*}}(\lambda^*) \cap H \neq \emptyset$ (i.e., the minimum values $p(\lambda^*, \mathbf{w}^*)$ and $p(\bar{\lambda}, \bar{\mathbf{w}})$ are both attained in H). Hence, $p(\bar{\lambda}, \bar{\mathbf{w}}) > p(\lambda^*, \mathbf{w}^*)$ implies that $\bar{\lambda} < \lambda^*$. Indeed, let us assume that $\bar{\lambda} \geq \lambda^*$ and let $\mathbf{z} \in P_{\Omega_{\mathbf{w}^*}}(\lambda^*) \cap H$. Then,

$$\begin{aligned} p(\lambda^*, \mathbf{w}^*) &= \mathbf{z}^\top \mathbf{Q}\mathbf{z} + \mathbf{q}^\top \mathbf{z} + \lambda^*(\mathbf{z}^\top \mathbf{Q}\mathbf{z} + \mathbf{q}^\top \mathbf{z} - a_0) \\ &\geq \mathbf{z}^\top \mathbf{Q}\mathbf{z} + \mathbf{q}^\top \mathbf{z} + \bar{\lambda}(\mathbf{z}^\top \mathbf{Q}\mathbf{z} + \mathbf{q}^\top \mathbf{z} - a_0) \geq p(\bar{\lambda}, \bar{\mathbf{w}}), \end{aligned}$$

which is a contradiction. Then, function $p(\lambda^*, \bar{\mathbf{w}})$ must be decreasing at λ^* or, equivalently, $P_{\Omega_{\bar{\mathbf{w}}}}(\lambda^*) \subset int(H)$ and $P_{\Omega_{\bar{\mathbf{w}}}}(\lambda^*) \neq \emptyset$. Since $h^{\min}_{\Omega_{\mathbf{w}^*}}(\lambda^*) < 0$, then $P_{\Omega_{\bar{\mathbf{w}}}}(\lambda^*)$ and $P_{\Omega_{\mathbf{w}^*}}(\lambda^*)$ have a common nonempty intersection within H and, consequently, $p(\lambda^*, \bar{\mathbf{w}}) = p(\lambda^*, \mathbf{w}^*)$ holds. This implies that $P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*) \cap \Omega_{\bar{\mathbf{w}}} = \emptyset$. Indeed, assume there exists $\mathbf{v} \in P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*) \cap \Omega_{\bar{\mathbf{w}}}$. Note that $\mathbf{v} \notin int(H)$ and, since $P^1_{\Omega_{\bar{\mathbf{w}}}}(\lambda^*) \subset P_{\Omega_{\bar{\mathbf{w}}}}(\lambda^*)$, $\mathbf{v}$ would also belong to $P_{\Omega_{\bar{\mathbf{w}}}}(\lambda^*)$ which, however, contradicts $P_{\Omega_{\bar{\mathbf{w}}}}(\lambda^*) \subset int(H)$.

Condition $P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*) \cap \Omega_{\bar{\mathbf{w}}} = \emptyset$ is equivalent to

$$(\forall \mathbf{v} \in P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*))\, r(\bar{\mathbf{w}}, \mathbf{v}) > 0.$$

Note that, by assumption, $\mathbf{v} \in P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*)$ implies $\mathbf{v} \in \partial\Omega_{\mathbf{w}^*}$. Moreover,

$$\begin{aligned} r(\bar{\mathbf{w}}, \mathbf{v}) = r((\bar{\mathbf{w}} - \mathbf{w}^*) + \mathbf{w}^*, \mathbf{v}) &= -(\bar{\mathbf{w}} - \mathbf{w}^*)^\top \mathbf{A}(\bar{\mathbf{w}} - \mathbf{w}^*) \\ &\quad + 2(\bar{\mathbf{w}} - \mathbf{w}^*)^\top \mathbf{A}(\mathbf{v} - \mathbf{w}^*) + r(\mathbf{w}^*, \mathbf{v}) > 0. \end{aligned}$$

Being $\mathbf{w}^* \in \partial H$ and $\mathbf{v} \in \partial\Omega_{\mathbf{w}^*}$, we have that $r(\mathbf{w}^*, \mathbf{v}) = 0$. Then, by taking $\mathbf{d} = \bar{\mathbf{w}} - \mathbf{w}^*$, (iii) is contradicted. ∎

Given the current point $\bar{\mathbf{x}}$ with $\lambda_{\Omega_{\bar{\mathbf{x}}}} > 0$, the question now is either to find a direction $\mathbf{d}$ fulfilling

$$(\forall \mathbf{v} \in P_1^\star(\lambda_{\Omega_{\bar{\mathbf{x}}}}))\ -\mathbf{d}^\top \mathbf{A}\mathbf{d} + 2\mathbf{d}^\top \mathbf{A}(\mathbf{v} - \bar{\mathbf{x}}) > 0 \tag{4.7}$$

or to establish that it does not exist. In case it does not exist,

$$p(\bar{\mathbf{x}}, \lambda_{\Omega_{\bar{\mathbf{x}}}}) = \max_{\lambda \geq 0,\ \mathbf{w}} p(\lambda, \mathbf{w}).$$

Otherwise, direction $(\mathbf{d}, -1)$ is an increasing direction for function p. We discuss different cases depending on the cardinality of $P_1^\star(\lambda_{\Omega_{\bar{\mathbf{x}}}})$ (see the cases discussed in section 3 for the trust region problem).

4.1. $|P_1^\star(\lambda_{\Omega_{\bar{x}}})| = 1$. In this case, let $\boldsymbol{v}$ be the unique point in $P_1^\star(\lambda_{\Omega_{\bar{\mathbf{x}}}})$. Then we need to solve the following convex optimization problem:

$$\max_{\mathbf{d} \in \mathbb{R}^n} -\mathbf{d}^\top \mathbf{A}\mathbf{d} + 2\mathbf{d}^\top \mathbf{A}(\mathbf{v} - \bar{\mathbf{x}}),$$

whose optimal solution is $\mathbf{d} = \mathbf{v} - \bar{\mathbf{x}}$ and its optimal value is $(\mathbf{v} - \bar{\mathbf{x}})^\top \mathbf{A}(\mathbf{v} - \bar{\mathbf{x}}) > 0$. Therefore, if $|P_1^\star(\lambda_{\Omega_{\bar{\mathbf{x}}}})| = 1$, we are always able to locally adjust the current point $\bar{\mathbf{x}}$ in such a way that the bound can be improved.

4.2. $|P_1^\star(\lambda_{\Omega_{\bar{x}}})| = 2$. In this case, let $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ be the two optimal points in $P_1^\star(\lambda_{\Omega_{\bar{x}}})$. Then, we need to solve the following optimization problem:

$$\max_{\boldsymbol{d} \in \mathbb{R}^n} \min\{-\boldsymbol{d}^\top \boldsymbol{A}\boldsymbol{d} + 2\boldsymbol{d}^\top \boldsymbol{A}(\boldsymbol{v}_1 - \bar{\boldsymbol{x}}), -\boldsymbol{d}^\top \boldsymbol{A}\boldsymbol{d} + 2\boldsymbol{d}^\top \boldsymbol{A}(\boldsymbol{v}_2 - \bar{\boldsymbol{x}})\}, \tag{4.8}$$

or, equivalently

$$\begin{array}{ll} \max & v \\ & v \leq -\boldsymbol{d}^\top \boldsymbol{A}\boldsymbol{d} + 2\boldsymbol{d}^\top \boldsymbol{A}(\boldsymbol{v}_1 - \bar{\boldsymbol{x}}) \\ & v \leq -\boldsymbol{d}^\top \boldsymbol{A}\boldsymbol{d} + 2\boldsymbol{d}^\top \boldsymbol{A}(\boldsymbol{v}_2 - \bar{\boldsymbol{x}}). \end{array}$$

This is a convex optimization problem, whose solution can be obtained in closed form. Indeed, by imposing the KKT conditions, it can be seen that the optimal solution has the following form:

$$\mathbf{d} = \beta(\mathbf{v}_1 - \bar{\mathbf{x}}) + (1 - \beta)(\mathbf{v}_2 - \bar{\mathbf{x}}), \quad \beta \in [0, 1]. \tag{4.9}$$

Now, let

$$\begin{aligned} a &= (\mathbf{v}_1 - \bar{\mathbf{x}})^\top \mathbf{A}(\mathbf{v}_1 - \bar{\mathbf{x}}) > 0, \\ b &= (\mathbf{v}_2 - \bar{\mathbf{x}})^\top \mathbf{A}(\mathbf{v}_2 - \bar{\mathbf{x}}) > 0, \\ c &= (\mathbf{v}_1 - \bar{\mathbf{x}})^\top \mathbf{A}(\mathbf{v}_2 - \bar{\mathbf{x}}). \end{aligned}$$

By replacing (4.9) in the objective function of (4.8), we have that (4.8) can be rewritten as

$$\max_{\beta \in [0,1]} \min \left\{ (-\beta^2 + 2\beta)a - (1 - \beta)^2 b + 2(1 - \beta)^2 c, -\beta^2 a + (1 - \beta^2) b + 2\beta^2 c \right\}.$$

The optimal solution of this problem is

$$\beta^\star = \begin{cases} 0 & \text{if } b \le c, \\ 1 & \text{if } a \le c, \\ \frac{b-c}{a+b-2c} & \text{otherwise.} \end{cases}$$

Then, the optimal value is

$$\begin{cases} b & \text{if } b \le c, \\ a & \text{if } a \le c, \\ \frac{ab-c^2}{a+b-2c} & \text{otherwise.} \end{cases}$$

We notice that $a, b > 0$,

$$a + b - 2c = (\mathbf{v}_1 - \mathbf{v}_2)^\top \mathbf{A}(\mathbf{v}_1 - \mathbf{v}_2) > 0,$$

and, by the Cauchy–Schwarz inequality,

$$ab - 2c^2 \ge 0,$$

and equality holds if and only if $(\mathbf{v}_1 - \bar{\mathbf{x}})$ and $(\mathbf{v}_2 - \bar{\mathbf{x}})$ are linearly dependent. Thus, the optimal value of (4.8) is always strictly positive unless the two vectors $(\mathbf{v}_1 - \bar{\mathbf{x}})$ and $(\mathbf{v}_2 - \bar{\mathbf{x}})$ lie along the same direction. More precisely, the optimal value is null only if the two vectors have the same direction but opposite sign. Indeed, let

$$\mathbf{v}_1 - \bar{\mathbf{x}} = \gamma(\mathbf{v}_2 - \bar{\mathbf{x}}).$$

Then, we have $b = \gamma^2 a$ and $c = \gamma a$. If γ is positive, then either $b \le c$ (if $\gamma \le 1$), or $a \le c$ (if $\gamma \ge 1$) occurs, so that the optimal value is equal to a or b and is, thus, positive. If $(\mathbf{v}_1 - \bar{\mathbf{x}})$ is not a negative multiple of $(\mathbf{v}_2 - \bar{\mathbf{x}})$, we are able to locally adjust $\bar{\mathbf{x}}$ along direction

$$\mathbf{d} = \beta^\star(\mathbf{v}_1 - \bar{\mathbf{x}}) + (1 - \beta^\star)(\mathbf{v}_2 - \bar{\mathbf{x}}).$$

4.3. $P_1^\star(\lambda_{\Omega_{\bar{\mathbf{x}}}})$ is an infinite connected set. In this case we need to solve the following optimization problem:

$$\max_{\mathbf{d} \in \mathbb{R}^n} \min_{\mathbf{v} \in P_1^\star(\lambda_{\Omega_{\bar{\mathbf{x}}}})} -\mathbf{d}^\top \mathbf{A}\mathbf{d} + 2\mathbf{d}^\top \mathbf{A}(\mathbf{v} - \bar{\mathbf{x}}). \tag{4.10}$$

An improving direction exists if and only if the optimal value of this problem is strictly positive (note that the optimal value is always nonnegative since the inner minimization problem has optimal value 0 for $\mathbf{d} = \mathbf{0}$). We first remark that the problem is convex. Indeed, for each fixed $\mathbf{v}$, we have a concave function with respect to $\mathbf{d}$, and the minimum of an infinite set of concave functions is itself a concave function (to be maximized, so that the problem is convex). The inner minimization problem can be solved in closed form. After removing the terms which do not depend on $\mathbf{v}$, the inner problem to be solved is

$$\min_{\mathbf{v} \in P_1^\star(\lambda_{\Omega_{\bar{\mathbf{x}}}})} 2\mathbf{d}^\top \mathbf{A}\mathbf{v}.$$

According to Case 2.3.2 in section 3, $P_1^\star(\lambda_{\Omega_{\bar{\mathbf{x}}}})$ can be written as in (3.3) and the minimization problem can be reduced to the computation of the minimum of a linear function over the unit sphere:

$$\min_{\boldsymbol{\xi}\in\mathbb{R}^q\,:\,\|\boldsymbol{\xi}\|^2=1} \bar{\mathbf{c}}(\mathbf{d})^\top \boldsymbol{\xi},$$

where $\bar{\mathbf{c}}(\mathbf{d})$ is some linear function of $\mathbf{d}$ and $q \geq 2$ is the multiplicity of the minimum eigenvalue of the matrix $\mathbf{V}^\top(\mathbf{Q}+\lambda\mathbf{A})\mathbf{V}$, corresponding to the Hessian of the objective function of problem (3.11). The optimal solution of this problem is

$$\boldsymbol{\xi}^\star = -\frac{\bar{\mathbf{c}}(\mathbf{d})}{\|\bar{\mathbf{c}}(\mathbf{d})\|},$$

while the optimal value is $-\|\bar{\mathbf{c}}(\mathbf{d})\|$.

4.4. An algorithm for the refinement of the bound. Let $\bar{\mathbf{x}}$ and $\lambda_{\mathbb{R}^n}$ be defined as in section 3. We propose Algorithm 4.1 for a bound based on successive local adjustments of the linear cut. In line 2, Algorithm 2.1 is run with input $X = \Omega_{\bar{\mathbf{x}}}$ and $\lambda_{\mathbb{R}^n}$. Note that with a slight abuse here we are assuming that the algorithm returns $\lambda_{\Omega_{\bar{\mathbf{x}}}}$ and the related points $\mathbf{z}_1$ and $\mathbf{z}_2$, while in practice close approximations of these quantities are returned, namely $\lambda^{\max}$, $\mathbf{z}_1(\lambda^{\max})$, and $\mathbf{z}_2(\lambda^{\min})$. In line 3, $\mathbf{z}$ is initialized with the input point $\bar{\mathbf{x}}$ itself and the direction $\mathbf{d}^\star$, following the discussion in section 4.1, is set equal to the difference between $\mathbf{z}_2(\lambda_{\Omega_{\bar{\mathbf{x}}}})$, the point outside H returned by Algorithm 2.1, and $\bar{\mathbf{x}}$. The outer while loop of the algorithm (lines 4–20) is repeated until the bound is improved by at least a tolerance value *tol*. Inside this loop, in line 5 the initial step size $\eta = 1$ is set and a new incumbent $\mathbf{y} \in \partial H$ is computed. The inner while loop (lines 7–15) computes the step size: until the optimal value of problem (3.9) with $\bar{\mathbf{x}} = \mathbf{y}$ and $\lambda = \lambda_{\Omega_{\mathbf{z}}}$, denoted by *opt*, is lower than the current lower bound *Lb*, we need to decrease the step size and recompute a new incumbent $\mathbf{y}$ (lines 10–11). If the step size falls below a given tolerance value, we exit the inner loop and also the outer one. Otherwise, we have identified a new valid incumbent and we set to 1 the exit flag *stop* for the inner loop (line 13), so that, later on, a new linear inequality (3.7) with $\bar{\mathbf{x}} = \mathbf{y}$ will be computed. Then, at line 17 we run Algorithm 2.1 with input $X = \Omega_{\mathbf{y}}$ and $\lambda_{\Omega_{\mathbf{z}}}$. Finally, in line 18, we update point $\mathbf{z}$ and the direction $\mathbf{d}^\star$. We remark that at each iteration $\mathbf{z}_2(\lambda_{\Omega_{\mathbf{z}}})$ is *one* optimal solution of the current subproblem (3.9) with $\lambda = \lambda_{\Omega_{\mathbf{z}}}$ lying outside H and at which the linear cut of the subproblem is active, i.e., $\mathbf{z}_2(\lambda_{\Omega_{\mathbf{z}}}) \in P_1^\star(\lambda_{\Omega_{\mathbf{z}}})$. As seen in section 4.1, if $|P_1^\star(\lambda_{\Omega_{\mathbf{z}}})| = 1$, i.e., $\mathbf{z}_2(\lambda_{\Omega_{\mathbf{z}}})$ is the unique optimal solution of the current subproblem (3.9) with $\lambda = \lambda_{\Omega_{\mathbf{z}}}$ lying outside H, then, in view of Proposition 2.11, the local adjustment employed in Algorithm 4.1 is guaranteed to improve the bound. However, as seen in sections 4.2 and 4.3, if $P_1^\star(\lambda_{\Omega_{\mathbf{z}}})$ contains more than one point, than the proposed local adjustment is not guaranteed to improve the bound. Sections 4.2 and 4.3 suggest how to define perturbing directions which still allow one to improve the bound, in case they exist. However, as we will see through the computational experiments, Algorithm 4.1 turns out to be time-consuming, and it is more convenient to improve the bound by adding a further linear cut, as we do in section 5, rather than further locally adjusting the current linear cut. In order to clarify this point, we can make a comparison with Integer Linear Programming (ILP). In ILP problems, once a linear relaxation is solved, a valid cut removes *one* optimal solution of the relaxation. If the optimal solution is unique, then after the addition of the valid cut, the bound improves. But if the linear relaxation has got multiple solutions, then the valid cut is not guaranteed to remove all of them and, thus, the bound may not improve. It is possible to try to strengthen the valid cut in such a way

Algorithm 4.1 Bound improvement through a local adjustment of the linear cut.

Input: $\bar{\mathbf{x}}, \lambda_{\mathbb{R}^n}$

Set $Lb_{old} = -\infty$
Let $[Lb, \lambda_{\Omega_{\bar{\mathbf{x}}}}, \mathbf{z}_1(\lambda_{\Omega_{\bar{\mathbf{x}}}}), \mathbf{z}_2(\lambda_{\Omega_{\bar{\mathbf{x}}}})] = \mathbf{DualLagrangian}(\Omega_{\bar{\mathbf{x}}}, \lambda_{\mathbb{R}^n})$
Set $\mathbf{z} = \bar{\mathbf{x}}$ and $\mathbf{d}^\star = \mathbf{z}_2(\lambda_{\Omega_{\bar{\mathbf{x}}}}) - \bar{\mathbf{x}}$
while $Lb - Lb_{old} > tol$ **do**
  Set $Lb_{old} = Lb$, $\eta = 1$ and $\mathbf{y} = \Pi_{\mathbf{A},\mathbf{a}}(\mathbf{z} + \mathbf{d}^\star) \in \partial H$
  Set $stop = 0$
  while $stop = 0$ and $\eta > \varepsilon$ **do**
    Solve problem (3.9) with $\bar{\mathbf{x}} = \mathbf{y}$ and $\lambda = \lambda_{\Omega_{\mathbf{z}}}$, and let opt be its optimal value
    if $opt < Lb$ **then**
      Set $\eta = \eta/2$
      Set $\mathbf{y} = \Pi_{\mathbf{A},\mathbf{a}}(\mathbf{z} + \eta\mathbf{d}^\star) \in \partial H$
    else
      Set $stop = 1$
    end if
  end while
  if $stop = 1$ **then**
    Let $[Lb, \lambda_{\Omega_{\mathbf{y}}}, \mathbf{z}_1(\lambda_{\Omega_{\mathbf{y}}}), \mathbf{z}_2(\lambda_{\Omega_{\mathbf{y}}})] = \mathbf{DualLagrangian}(\Omega_{\mathbf{y}}, \lambda_{\Omega_{\mathbf{z}}})$
    Set $\mathbf{z} = \mathbf{y}$, $\mathbf{d}^\star = \mathbf{z}_2(\lambda_{\Omega_{\mathbf{y}}}) - \mathbf{z}$
  end if
end while
return Lb

that all optimal solutions of the linear relaxations are removed. But, more commonly, new linear cuts are added.

Now we apply Algorithm 4.1 to our example.

Example 4.3. We have that $\mathbf{z}$ is initialized with $(-0.7901, 0.3565)$ and Lb with -4.0971. During the execution of Algorithm 4.1, $\mathbf{z}$ and Lb are updated as indicated in Table 1.

Interestingly, the best bound obtained in the example is exactly the one obtained for the same problem by the approach proposed in [12], based on the addition of SOC-RLT constraints. Figure 3 displays the situation at the last iteration of Algorithm 4.1. Problem (3.9) has three optimal solutions, one in $int(H)$ and two outside H. The

TABLE 1
Iterations of Algorithm 4.1 over the example.

Iteration	$\mathbf{z}$	Lb
1	$(-0.7204, 0.6658)$	-4.0850
2	$(-0.7742, 0.4493)$	-4.0638
3	$(-0.7481, 0.5665)$	-4.0477
4	$(-0.7607, 0.5136)$	-4.0416
5	$(-0.7556, 0.5361)$	-4.0378
6	$(-0.7571, 0.5296)$	-4.0364
7	$(-0.7568, 0.5309)$	-4.0362

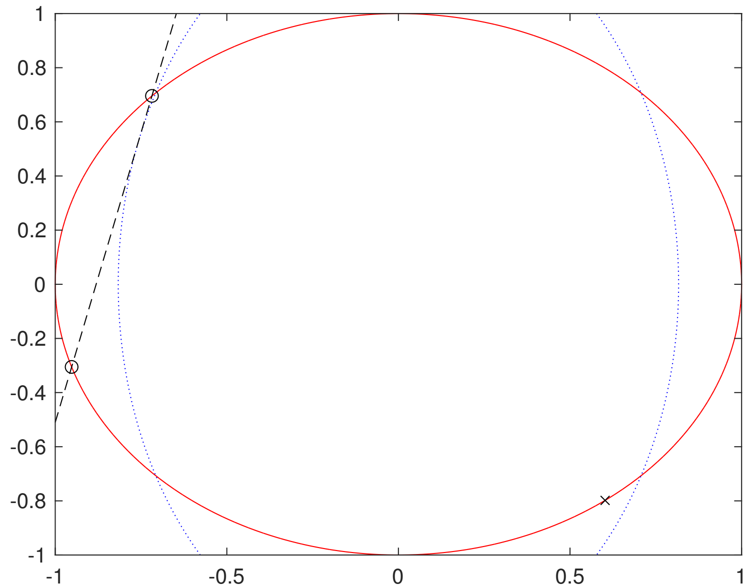


FIG. 3. *Final linear cut after running Algorithm* 4.1. *Problem* (3.9) *has three optimal solutions, one in int*(H) *and two outside* H. *The latter solutions are opposite to each other with respect to the final vector* $\mathbf{z}$.

two optimal solutions outside H are opposite to each other with respect to the final vector $\mathbf{z}$, so that, as discussed in section 4.2, no further local adjustment is possible to improve the bound in this case.

5. Bound improvement through the addition of a further linear cut. Another possible way to improve the bound is by adding a further linear cut to (3.9). Let $\bar{\mathbf{x}}$ and $\lambda_{\mathbb{R}^n}$ be defined as in section 3. In line 2 of Algorithm 4.1, we compute $[Lb, \lambda_{\Omega_{\bar{\mathbf{x}}}}, \mathbf{z}_1(\lambda_{\Omega_{\bar{\mathbf{x}}}}), \mathbf{z}_2(\lambda_{\Omega_{\bar{\mathbf{x}}}})] = \textbf{DualLagrangian}(\Omega_{\bar{\mathbf{x}}}, \lambda_{\mathbb{R}^n})$, and, later on, we try to locally adjust $\bar{\mathbf{x}}$. Rather than doing that, we can add a further linear cut, cutting $\mathbf{z}_2(\lambda_{\Omega_{\bar{\mathbf{x}}}}) \notin H$ away. In particular, we add the one obtained through the projection over ∂H of $\mathbf{z}_2(\lambda_{\Omega_{\bar{\mathbf{x}}}})$. Let $\tilde{\mathbf{x}} = \Pi_{\mathbf{A},\mathbf{a}}(\mathbf{z}_2(\lambda_{\Omega_{\bar{\mathbf{x}}}})) \in \partial H$ be such projection. Then, we define the following problem:

$$(5.1)\qquad \begin{array}{ll} \min_{\mathbf{x}} & \mathbf{x}^\top(\mathbf{Q}+\lambda\mathbf{A})\mathbf{x} + (\mathbf{q}+\lambda\mathbf{a})^\top\mathbf{x} - \lambda a_0 \\ & \mathbf{x}^\top\mathbf{x} \le 1 \\ & (2\mathbf{A}\bar{\mathbf{x}}+\mathbf{a})^\top(\mathbf{x}-\bar{\mathbf{x}}) \le 0 \\ & (2\mathbf{A}\tilde{\mathbf{x}}+\mathbf{a})^\top(\mathbf{x}-\tilde{\mathbf{x}}) \le 0, \end{array}$$

which is equivalent to problem (3.2) where

$$X = \Omega_{\bar{\mathbf{x}}} \cap \Omega_{\tilde{\mathbf{x}}} = \{\mathbf{x} \,:\, (2\mathbf{A}\bar{\mathbf{x}}+\mathbf{a})^\top(\mathbf{x}-\bar{\mathbf{x}}) \le 0, \;\; (2\mathbf{A}\tilde{\mathbf{x}}+\mathbf{a})^\top(\mathbf{x}-\tilde{\mathbf{x}}) \le 0\} \supset H.$$

A convex reformulation as the one proposed in [12, 23] for problem (3.9) is not available in this case (unless the two linear inequalities do not intersect in the interior of the unit ball). But in this case the alternative procedure discussed in section 3 turns out to be useful. As before, for each value λ in the while loop of Algorithm 2.1 we can first check whether a local and nonglobal optimal solution of problem (3.2) with $X = \mathbb{R}^n$ exists, by exploiting the necessary and sufficient condition stated in [24]. If it exists, and belongs to $\Omega_{\bar{\mathbf{x}}} \cap \Omega_{\tilde{\mathbf{x}}}$, we denote it by $\mathbf{z}_1(\lambda)$. Next, we need to compute the optimal value of (5.1) when at least one of the two linear constraints is active, i.e., we need to solve the following problem:

$$\begin{array}{ll} \min_{\mathbf{x}} & \mathbf{x}^\top(\mathbf{Q}+\lambda\mathbf{A})\mathbf{x} + (\mathbf{q}+\lambda\mathbf{a})^\top\mathbf{x} - \lambda a_0 \\ & \mathbf{x}^\top\mathbf{x} \le 1 \\ & (2\mathbf{A}\bar{\mathbf{x}}+\mathbf{a})^\top(\mathbf{x}-\bar{\mathbf{x}}) \le 0 \\ & (2\mathbf{A}\tilde{\mathbf{x}}+\mathbf{a})^\top(\mathbf{x}-\tilde{\mathbf{x}}) \le 0 \\ & \left[(2\mathbf{A}\bar{\mathbf{x}}+\mathbf{a})^\top(\mathbf{x}-\bar{\mathbf{x}})\right]\left[(2\mathbf{A}\tilde{\mathbf{x}}+\mathbf{a})^\top(\mathbf{x}-\tilde{\mathbf{x}})\right] = 0. \end{array} \tag{5.2}$$

A convex reformulation of this problem has been proposed in [26]. Alternatively, one can solve two distinct problems, each imposing that one of the two linear inequalities is active. Each of these problems can be converted into a trust region problem with an additional linear inequality, which can be solved in polynomial time through the already mentioned convex reformulation proposed in [12, 23]. Thus, we compute the set $P_1^\star(\lambda) \subseteq \partial\Omega_{\bar{\mathbf{x}}} \cap \Omega_{\tilde{\mathbf{x}}}$ of optimal solutions of (5.1) for which the first linear cut is active, and then the set $P_2^\star(\lambda) \subseteq \Omega_{\bar{\mathbf{x}}} \cap \partial\Omega_{\tilde{\mathbf{x}}}$ of optimal solutions of (5.1) for which the second linear cut is active. Finally, the optimal values of these problems are compared with the value of the local and nonglobal minimizer (if it exists) in order to identify the set $P_X(\lambda)$ of optimal solutions of (5.1). At this point we are able to compute $h_X^{\min}(\lambda), h_X^{\max}(\lambda)$ and update $\lambda^{\min}$ and $\lambda^{\max}$ accordingly. If for some λ we have that $\mathbf{z}_1(\lambda) \in P_X(\lambda)$ and $P_X(\lambda) \cap [P_1^\star(\lambda) \cup P_2^\star(\lambda)] \neq \emptyset$, i.e., problem (5.1) has an optimal solution in $int(H)$ and (at least) one optimal solution outside H, then $0 \in [h_X^{\min}(\lambda), h_X^{\max}(\lambda)]$ and Algorithm 2.1 stops. We illustrate all this on Example 3.3.

Example 5.1. We add a second linear cut obtained through the projection over ∂H of the optimal solution of problem (3.9) with $\lambda_{\Omega_{\bar{\mathbf{x}}}} = 0.726$ outside H. This leads to a further improvement with $\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}} \approx 0.39$ and $p_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}}(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}}) \approx -4.005$, which almost closes the gap. In Figure 4 we show the two linear cuts and the two new optimal

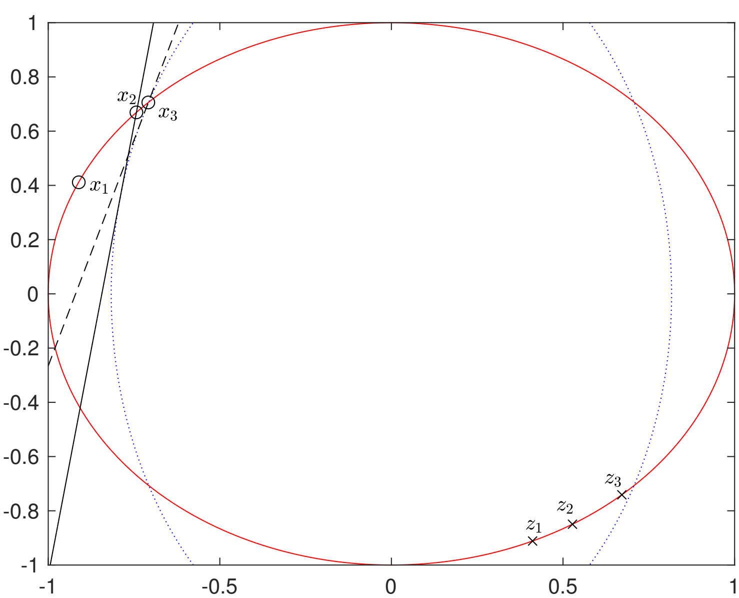


FIG. 4. *Two linear cuts and the two optimal solutions outside H (x_3) and in $int(H)$ (z_3), denoted by $\circ$ and $\times$, respectively.*

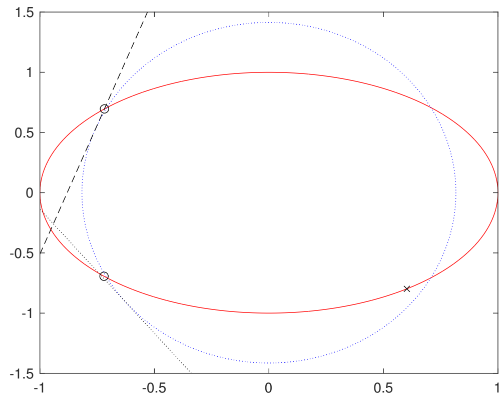
solutions, one outside H and one belonging to $int(H)$ (x_3 and z_3, respectively). Again, we also report the previous pairs of optimal solutions in order to show the progress.

Now, assume that the returned bound is not exact. Also in this case $\bar{\mathbf{x}}$ and $\tilde{\mathbf{x}}$ can be locally adjusted. One can combine the techniques presented in section 4 and in the current section, by using a technique similar to the one described in the former section to improve the pair of points $\bar{\mathbf{x}}$ and $\tilde{\mathbf{x}}$. In particular, at $\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}}$ we have one optimal solution of problem 5.1 belonging to $int(H)$, namely the local and nonglobal optimal solution of problem (3.2) with $X=\mathbb{R}^n$, and at least another one outside H. We denote the latter by $\mathbf{v}$ and we observe that at least one of the two linear cuts is active at this point, i.e., either $\mathbf{v}\in\partial\Omega_{\bar{\mathbf{x}}}$ or $\mathbf{v}\in\partial\Omega_{\tilde{\mathbf{x}}}$ (or both). Then, if only the first cut is active at $\mathbf{v}$, we update $\bar{\mathbf{x}}$ as follows: $\bar{\mathbf{x}}'=\Pi_{\mathbf{A},\mathbf{a}}(\bar{\mathbf{x}}+\eta(\mathbf{v}-\bar{\mathbf{x}}))$ for a sufficiently small η value, while $\tilde{\mathbf{x}}'=\tilde{\mathbf{x}}$. If only the second cut is active, we update $\tilde{\mathbf{x}}$ as follows: $\tilde{\mathbf{x}}'=\Pi_{\mathbf{A},\mathbf{a}}(\tilde{\mathbf{x}}+\eta(\mathbf{v}-\tilde{\mathbf{x}}))$ for a sufficiently small η value, while $\bar{\mathbf{x}}'=\bar{\mathbf{x}}$. Finally, if both are active we select one of the two cuts and perturb it. After the perturbation, we run again Algorithm 2.1 with input $X=\Omega_{\bar{\mathbf{x}}'}\cap\Omega_{\tilde{\mathbf{x}}'}$ and $\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}}$, and we repeat this procedure until there is a significant reduction of the bound. Note, however, that it might happen that no improvement is possible. In case $|P_1^\star(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}})|=1$ and $P_2^\star(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}})=\emptyset$ (similar for $|P_2^\star(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}})|=1$ and $P_1^\star(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}})=\emptyset$), then the proposed perturbation $\bar{\mathbf{x}}'=\Pi_{\mathbf{A},\mathbf{a}}(\bar{\mathbf{x}}+\eta(\mathbf{v}-\bar{\mathbf{x}}))$ for η sufficiently small allows improvement of the bound. Indeed, in such cases the local adjustment is able to cut the unique solution outside H away. In order to illustrate other different cases we employ Figures 5a–5c. As usual, in these figures the point in $int(H)$ is denoted by $\times$, while the others (outside H) are denoted by $\circ$. If $|P_1^\star(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}})|=|P_2^\star(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}})|=1$ and $P_1^\star(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}})\cap P_2^\star(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}})=\emptyset$ (see Figure 5a), or $|P_1^\star(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}})|=2, |P_2^\star(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}})|=1$, and $P_1^\star(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}})\cap P_2^\star(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}})\neq\emptyset$ (see Figure 5b), then it is not possible to remove all the solutions outside H by perturbing a single linear cut. Indeed, in both cases the perturbation of a single linear cut is able to remove just one of the two optimal solutions outside H. But it is possible to remove both by perturbing both linear cuts. Instead, Figure 5c illustrates a case where $|P_1^\star(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}})|=|P_2^\star(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}})|=2$ and $P_1^\star(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}})\cap P_2^\star(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}})\neq\emptyset$. In this case even the perturbation of both linear cuts is unable to remove all three solutions outside H. The only way to remove all three solutions outside H is through the addition of a further linear cut, but, of course, this leads to a more complex problem with one trust region constraint and three linear inequalities.

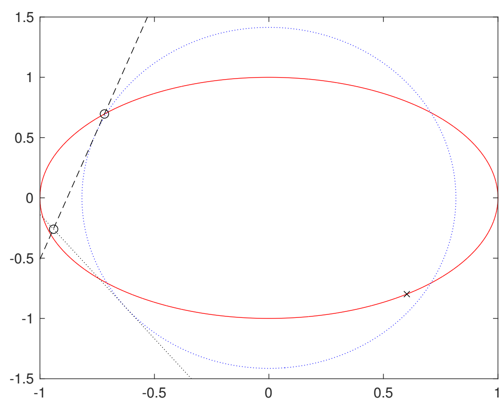
Example 5.2. In our example, this refinement is finally able to close the gap and return the exact optimal value -4. In Figure 6 we report the result of the first perturbation of the linear cuts. Since only the second linear cut is active at x_3, in this case the second linear cut is slightly perturbed and becomes equivalent to the tangent to H at the optimal solution $(-\sqrt{2}/2, \sqrt{2}/2)$ of the original problem (1.1). It is interesting to note that the new optimal solution outside H, indicated by x_4, lies in a different region with respect to the previous ones and is further from ∂H with respect to x_2 and x_3 (the reduction of λ reduces the penalization of points outside H). Such a solution is cut by the new linear inequality, obtained by a (not so small) perturbation of the first linear cut, displayed in Figure 7, together with the two new optimal solutions (x_5 and z_5), now corresponding to the two optimal solutions of problem (1.1).

6. Computational experiments.

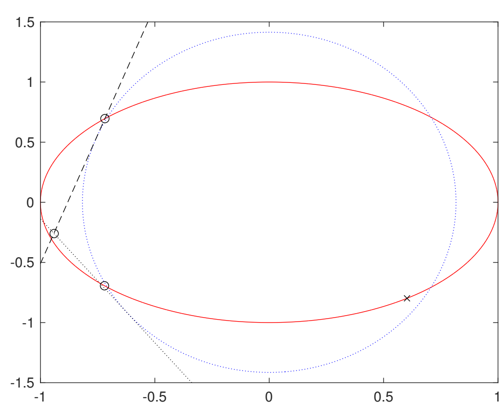
In this section we report the computational results for the proposed bounds over the set of hard instances selected from the random ones generated in [12] and inspired by [18]. More precisely, in [12] 1000 random



(a) Three optimal solutions, none with both linear cuts active.



(b) Three optimal solutions, one with both linear cuts active.



(c) Four optimal solutions.

Fig. 5.

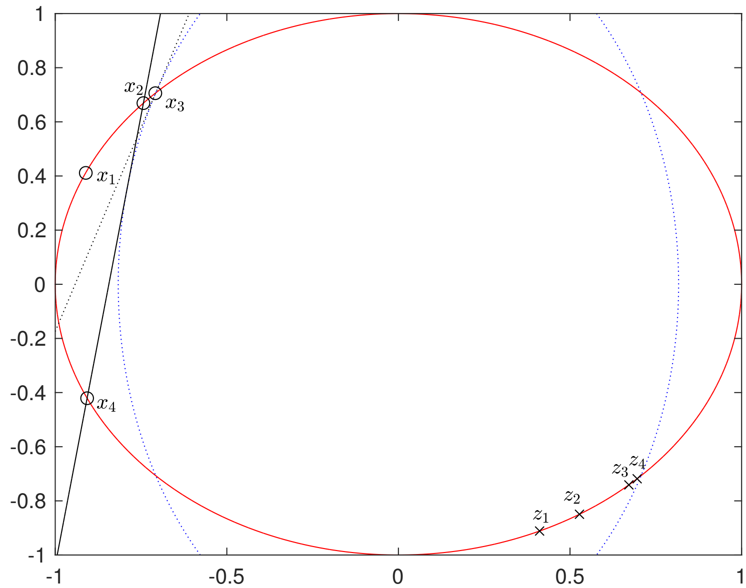


Fig. 6. *Perturbation of the second linear cut and the two new optimal solutions outside H (x_4) and in $int(H)$ (z_4), denoted by $\circ$ and $\times$, respectively.*

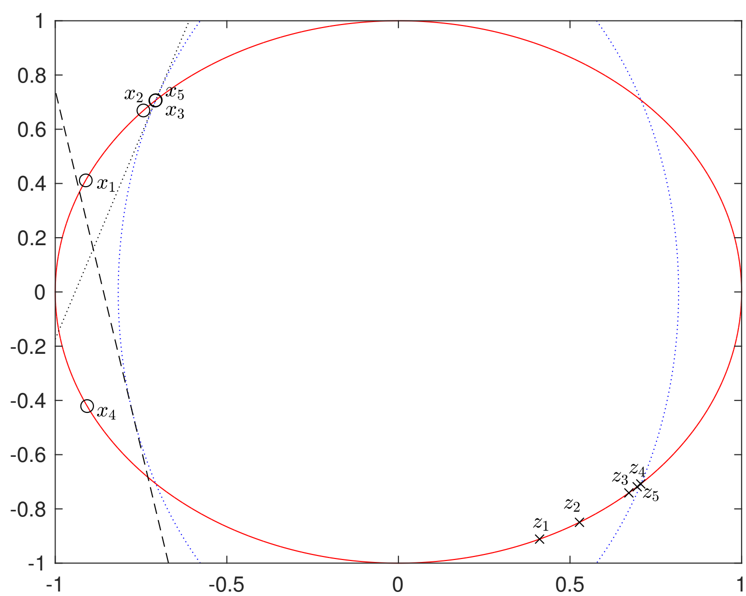


Fig. 7. *Perturbation of the first linear cut and the two optimal solutions outside H (x_5) and in $int(H)$ (z_5), denoted by $\circ$ and $\times$, respectively.*

instances were generated for each size $n = 5, 10, 20$. Some of these instances have been declared hard ones, namely those for which the bound obtained by adding `SOC-RLT` constraints was not exact. In particular, these are 38 instances with $n = 5$, 70 instances with $n = 10$, and 104 instances with $n = 20$. Such instances have been made available in `GAMS`, `AMPL`, and `COCOUNT` formats in [19]. We tested our bounds on such instances.

All tests have been performed on an Intel Core i7 running at 1.8 GHz with 16 GB of RAM. All bounds have been coded in MATLAB.

We computed the following bounds:

- LbDual, the dual Lagrangian bound computed through Algorithm 2.1 with input $X = \mathbb{R}^n$;
- LbOneCut, the bound obtained by adding a single linear cut and computed through Algorithm 2.1 with input $X = \Omega_{\bar{\mathbf{x}}}$;
- LbOneAdj, the bound obtained by local adjustments of the added linear cut as indicated in Algorithm 4.1;
- LbTwoCut, the bound obtained by adding two linear cuts;
- LbTwoAdj, the bound obtained by adjusting the two linear cuts.

According to what was done in [3, 12, 25], an instance is considered to be "solved" when the relative gap between the lower bound, say LB, and the upper bound, say UB, is not larger than 10^{-4}, i.e.,

$$\frac{UB - LB}{|UB|} \leq 10^{-4}.$$

We set UB equal to the lowest value obtained by running, after the addition of the first linear cut, two local searches for the original problem (1.1), one from the optimal solution $\mathbf{z}_1(\lambda_{\Omega_{\bar{\mathbf{x}}}}) \in int(H)$ of (3.9) returned at the end of Algorithm 2.1, and the other from an optimal solution of the same problem outside H. In Tables 2–4 we report the average and maximum relative gaps for each bound, and the average and maximum computing times for $n = 5, 10, 20$, respectively. In the last line of the tables we also report the same values for the SOC-RLT bound presented in [12], computed by Mosek.[1] Note that the average gap is taken only over the instances which were *not* solved (in the sense specified above) by a given bound. Moreover, the average computing time for bound LbTwoAdj is computed only over the instances (87 overall, as we will see) which are *not* solved by bound LbTwoCut.

We remark that the bound LbTwoCut is computed by adding the first cut as in bound LbOneCut, i.e., the supporting hyperplane at $\bar{\mathbf{x}} \in \partial H$, and then adding a further linear cut through the projection of an optimal solution outside H obtained when computing bound LbOneCut, i.e., point $\mathbf{z}_2(\lambda_{\min})$ returned by procedure DualLagrangian with input $X = \Omega_{\bar{\mathbf{x}}}$. We could as well choose the adjusted cut computed by bound LbOneAdj as the first cut for bound LbTwoCut, but we observed that with this choice no improvement over LbOneAdj is obtained. This is related to what already observed

TABLE 2
Average and maximum relative gaps and computing times (in seconds) for the instances with $n = 5$.

Bound	Average relative gap (%)	Max relative gap (%)	Average time	Max time
LbDual	0.90 %	2.97 %	0.013	0.015
LbOneCut	0.31 %	1.27 %	0.035	0.040
LbOneAdj	0.130 %	0.548 %	0.266	0.388
LbTwoCut	0.07 %	0.21 %	0.089	0.108
LbTwoAdj	0 %	0 %	0.146	0.281
SOC-RLT	0.131 %	0.548 %	1.435	2.080

[1]The authors are very grateful to Professor Samuel Burer for providing the MATLAB code for the computation of the SOC-RLT bound.

Table 3
Average and maximum relative gaps and computing times (in seconds) for the instances with $n = 10$.

Bound	Average relative gap (%)	Max relative gap (%)	Average time	Max time
`LbDual`	0.41 %	1.57 %	0.014	0.022
`LbOneCut`	0.14 %	0.81 %	0.039	0.057
`LbOneAdj`	0.073 %	0.478 %	0.339	0.574
`LbTwoCut`	0.05 %	0.24 %	0.101	0.173
`LbTwoAdj`	0 %	0 %	0.197	0.670
`SOC-RLT`	0.073 %	0.474%	1.228	2.898

Table 4
Average and maximum relative gaps and computing times (in seconds) for the instances with $n = 20$.

Bound	Average relative gap (%)	Max relative gap (%)	Average time	Max time
`LbDual`	0.20 %	0.59 %	0.019	0.027
`LbOneCut`	0.08 %	0.29 %	0.057	0.079
`LbOneAdj`	0.054 %	0.166 %	0.539	0.926
`LbTwoCut`	0.03 %	0.09 %	0.148	0.199
`LbTwoAdj`	0.05 %	0.05 %	0.350	1.574
`SOC-RLT`	0.053 %	0.166%	2.266	3.983

in Figure 3: bound `LbOneAdj` cannot be improved any more when there are (at least) two optimal solutions outside H (besides the one in $int(H)$). Thus, the second cut is able to remove one of such optimal solutions but not the other, so that the bound cannot be improved. Similarly, for bound `LbTwoAdj` the two initial cuts are the ones computed for bound `LbTwoCut`.

For what concerns the computing times, we observe that these are lower than those reported in [25] for the bound obtained by adding lifted-RLT cuts (around 92s for an instance with $n = 20$). They are also lower than those reported in [3] for the bound obtained by adding KSOC cuts (up to 2s for $n = 20$ instances). For the sake of correctness, we point out that the computing times reported in those papers have been obtained with different processors. However, such processors have comparable performance with respect to the one employed for the computational experiments in this paper. In general, the proposed bounds are very cheap. Only for two instances with $n = 20$, `LbTwoAdj` required times above 1s (around 1.5s in both cases). Usually the computing times are (largely) below 1s. Both the dual Lagrangian bound and the bound obtained by a single linear cut are pretty cheap but with poorer performance in terms of relative gap. The bound obtained by Algorithm 4.1 with a local adjustment of the linear cut is better than the two previous ones in terms of gap but is also more expensive (although still cheap). The bound `LbTwoCut` offers a good combination between quality and cheap computing time. But a more careful choice of the two linear cuts, through a local adjustment, improves the quality without compromising the computing times. This is confirmed by the results reported for `LbTwoAdj`. Although this bound is more expensive than the others, the additional search for adjusted linear cuts further increases the quality of the bound. In Table 5 we report the number of solved instances for `LbTwoCut` and `LbTwoAdj`. According to what was reported in [3], the total number of unsolved instances out of the 212 hard

TABLE 5
Number of solved instances for the bounds LbTwoCut and LbTwoAdj.

Bound	$n = 5$ (out of 38)	$n = 10$ (out of 70)	$n = 20$ (out of 104)
`LbTwoCut`	14	41	70
`LbTwoAdj`	38	70	103

TABLE 6
Minimum, average, and maximum $PercDiff$ values, where $PercDiff$ is defined in (6.1).

n	Minimum	Average	Maximum
5	−0.0020%	0.0009 %	0.0058 %
10	−0.0046 %	−0.0001 %	0.0024 %
20	−0.0022 %	−0.0003 %	0.0004 %

instances is equal to the following: 133 for the bound proposed in [25] (18 with $n = 5$, 49 with $n = 10$, and 66 with $n = 20$); 85 for the bound proposed in [3] (18 with $n = 5$, 22 with $n = 10$, and 45 with $n = 20$); 56 by considering the best bound between the one in [25] and the one in [3] (10 with $n = 5$, 15 with $n = 10$, and 31 with $n = 20$). For bound `LbTwoCut` the total number of unsolved instances reduces to 87 (24, 29, and 34 for $n = 5$, $n = 10$, and $n = 20$, respectively). Finally, for bound `LbTwoAdj` we have the remarkable outcome that there is just one unsolved instance (namely, instance˙20˙628). For the sake of correctness, we should warn that the value UB in [3, 25] is not computed by running two local searches as done in this paper. It is instead computed from the final solution of the relaxed problem, so that it could be slightly worse and justify the larger number of unsolved instances. All the same, the quality of the proposed bounds appears to be quite good.

We still need to compare our bounds with the `SOC-RLT` bound (last line in Tables 2–4). In terms of computing times we notice that both the average and the maximum computing times of the `SOC-RLT` bound are larger than those of all the proposed bounds. But we believe that the most interesting observation is that, in terms of average and maximum gap, the `SOC-RLT` bound is almost identical to the `LbOneAdj` bound. In order to better investigate the relation between the two bounds, in Table 6 we report the minimum, average, and maximum percentages difference between the two bounds, i.e., the quantity

$$PercDiff = 100 * \frac{\texttt{LbOneAdj} - \texttt{SOC} - \texttt{RLT}}{|\texttt{LbOneAdj}|}\ \%. \tag{6.1}$$

We notice that the difference is sometimes positive and sometimes negative, suggesting that none of the two bounds dominate the other. But the differences are also so small (below the tolerance value under which an instance is declared to be ‘’solved‘’ by a given bound) that they could also be numerical differences due to the tolerance values employed in the solvers. We believe that an interesting question for future research is to establish whether these two bounds are, in fact, equivalent, which would lead to a new interpretation of the `SOC-RLT` bound proposed in [12].

6.1. Investigating the hardest instance. As a final experiment, we investigate the behavior of bound `LbTwoAdj` over the hardest instance with $n = 20$, the one for which the relative error is above 10^{-4}. For this instance, at the last iteration we recorded the following objective function values, corresponding to values of local minimizers of problem (5.1), which certainly include the global minimizer(s) of such a problem:

- the value at the optimal solution of problem (5.1) belonging to $int(H)$;
- the value at a globally optimal solution of the trust region problem obtained by fixing in problem (5.1) the first linear cut to an equality, in case such solution fulfills the second linear cut, or, alternatively, the value at the local and nonglobal solution of the same problem, in case such solution exists and fulfills the second linear cut (if the global minimizer does not fulfill the second linear cut and the local and nonglobal minimizer does not exist or does not fulfill the second linear cut, then the value is left undefined);
- the same value as above but after fixing the second linear cut to an equality in problem (5.1);
- the value at a globally optimal solution of the trust region problem obtained by fixing both cuts to equalities in problem (5.1).

Note that two of the four values must be equal. In particular, one of the two equal values is always the first one, attained in $int(H)$. But for the hardest instance we observed that all four values are very close to each other and all of them are lower than the UB value. Thus, it appears that for this instance a situation like the one displayed in Figure 5c occurs. In this case even the perturbation of both linear cuts is unable to remove all of the three solutions outside H.

7. Conclusions. In this paper we discussed the CDT problem. First, we derived some theoretical results for a class of problems which includes the CDT problem as a special case. Then, from the theory developed for such class, we have rederived a necessary and sufficient condition for the exactness of the Shor relaxation and of the equivalent dual Lagrangian bound for the CDT problem. The condition is based on the existence of multiple solutions for a Lagrangian relaxation. Based on such condition, we proposed to strengthen the dual Lagrangian bound by adding one or two linear cuts. These cuts are based on supporting hyperplanes of one of the two quadratic constraints, and they are, thus, redundant for the original CDT problem (1.1). However, the cuts are not redundant for the Lagrangian relaxation and their addition allows one to improve the bound. We ran different computational experiments over the 212 hard test instances selected from the three thousand ones randomly generated in [12], reporting gaps and computing times. We have shown that the bounds are computationally cheap and are quite effective. In particular, one of them, based on the addition of two linear cuts, is able to solve all but one of the hard instances. We have also investigated more in detail such hardest instance for which the bound is not exact (though quite close to the optimal value). An interesting topic for future research could be that of establishing the relations between the bounds proposed in this work and those presented in the recent literature (in particular, as already mentioned, it would be interesting to establish whether bound `LbOneAdj` is equivalent to the `SOC-RLT` bound introduced in [12]). Moreover, it would also be interesting to develop procedures which are able to generate CDT instances for which the bound `LbTwoAdj` is unable to return the optimal value. Finally, it would be interesting to see if the results presented in this work could be extended to QP problems with more **AQ5** than two constraints. Some preliminary studies, which will appear elsewhere, show that for such problems it is sometimes possible to improve the dual Lagrangian bound with the addition of a linear cut, but it may be hard to identify it and it is not even guaranteed to exist.

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