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SHARP AND FAST BOUNDS FOR THE CELIS-DENNIS-TAPIA PROBLEM^{*}

LUCA CONSOLINI[†] AND MARCO LOCATELLI[†]

3 Abstract. In the Celis–Dennis–Tapia (CDT) problem a quadratic function is minimized over $\frac{4}{5}$ a region defined by two strictly convex quadratic constraints. In this paper we rederive a necessary and sufficient optimality condition for the exactness of the dual Lagrangian bound (equivalent to the 6 Shor relaxation bound in this case). Starting from such a condition, we propose strengthening the 7dual Lagrangian bound by adding one or two linear cuts to the Lagrangian relaxation. Such cuts are obtained from supporting hyperplanes of one of the two constraints. Thus, they are redundant for 8 9 the original problem, but they are not for the Lagrangian relaxation. The computational experiments 10show that the new bounds are effective and require limited computing times. In particular, one of the 11 proposed bounds is able to solve all but one of the 212 hard instances of the CDT problem presented 12 in [S. Burer and K. M. Anstreicher, {\it SIAM J. Optim.}, 23 (2013), pp. 432-451].

13 Key words. CDT problem, dual Lagrangian bound, linear cuts

14 MSC codes. 90C20, 90C22, 90C26

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15 **1. Introduction.** The Celis–Dennis–Tapia problem (CDT problem in what fol-16 lows) is defined as follows:

(1.1)
$$p^{\star} = \min \quad \mathbf{x}^{\top} \mathbf{Q} \mathbf{x} + \mathbf{q}^{\top} \mathbf{x} \\ \mathbf{x}^{\top} \mathbf{x} \le 1 \\ \mathbf{x}^{\top} \mathbf{A} \mathbf{x} + \mathbf{a}^{\top} \mathbf{x} \le a_0,$$

where $\mathbf{Q}, \mathbf{A} \in \mathbb{R}^{n \times n}, \mathbf{q}, \mathbf{a} \in \mathbb{R}^n, a_0 \in \mathbb{R}$, while \mathbf{A} is assumed to be positive definite. We will denote by

$$H = \{ \mathbf{x} \in \mathbb{R}^n : \mathbf{x}^\top \mathbf{A} \mathbf{x} + \mathbf{a}^\top \mathbf{x} \le a_0 \}$$

19the ellipsoid defined by the second constraint, by ∂H its border, and by int(H) its 20interior. The CDT problem was originally proposed in [13] and has attracted a lot of attention in the last two decades. For some special cases a convex reformulation 21is available. For instance, in [26] it is shown that a semidefinite reformulation is 22possible when no linear terms are present, i.e., when $\mathbf{q} = \mathbf{a} = \mathbf{0}$. However, up to now 23no tractable convex reformulation of general CDT problems has been proposed in the 24literature. In spite of that, recently three different works [9, 14, 22] independently 25proved that the CDT problem is solvable in polynomial time. More precisely, in [14, 2622] polynomial solvability is proved by identifying all KKT points through the solution 27of a bivariate polynomial system with polynomials of degree at most 2n. The two 28unknowns are the Lagrange multipliers of the two quadratic constraints. Instead, in 29[9] an approach based on the solution of a sequence of feasibility problems for systems 30 of quadratic inequalities is proposed. The systems are solved by a polynomial-time 31

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32algorithm based on Barvinok's construction [6]. Though polynomial, all of these 33 approaches are computationally demanding since the degree of the polynomial is quite large. Conditions guaranteeing that the classical Shor SDP relaxation or, equivalently 34in this case, the dual Lagrangian bound is exact, are discussed in [2, 7]. In particular, 35in [2] a necessary and sufficient condition is presented. It is shown that the lack of 36exactness is related to the existence of KKT points with the same Lagrange multipliers 37 but two distinct primal solutions, both active at one of the two constraints but one 3839violating and the other one fulfilling the other constraint. In [10] necessary and sufficient conditions for local and global optimality are discussed based on copositivity. 40In [11] an exactness condition is given for a copositive relaxation, also for the case 41 42with additional linear constraints. A trajectory following method to solve the CDT 43problem has been discussed in [26], while different branch-and-bound solvers are tested in [19]. 44

Recently, different papers proposed valid bounds for the CDT problem. In [12] 45the Shor relaxation bound is strengthened by adding all RLT constraints obtained by AQ1 46supporting hyperplanes of the two ellipsoids. By fixing the supporting hyperplane for 4748one ellipsoid, the RLT constraints obtained with all the supporting hyperplanes of the other can be condensed into a single SOC-RLT constraint. Varying the supporting hy-AQ2 49perplane of the first ellipsoid gives rise to an infinite number of SOC-RLT constraints 50which, however, can be separated in polynomial time. The addition of these con-51straints does not allow one to close the duality gap, but it is computationally shown 5253that many instances which are not solved via the SDP bound, are solved with the 54addition of these SOC-RLT cuts. The authors generate 1000 random test instances for each n = 5, 10, 20, following a procedure described in [18] to generate trust-region 55problems with one local and nonglobal minimizer. The proposed bound based on 56SOC-RLT cuts allows for solving most instances except for 212 (38 for n = 5, 70 for 5758n = 70, and 104 for n = 20). Such unsolved instances are considered as hard ones 59in subsequent works. In [25] lifted-RLT cuts are introduced and it is shown that the new constraints allow one to derive an exact bound for n = 2 but also to improve the 60 bounds of [12] over the hard instances for n > 2. In [27] it is proved that the duality 61gap can be reduced to 0 by solving two subproblems with SOC constraints when the 6263 second constraint is the product of two linear functions and an exactness result is also provided for the case of problems with two variables. Due to its relations with the 6465approach proposed in this work, we will further discuss the approach proposed in [27] at the end of section 3. In [3] cuts are introduced. These are Kronecker product con- AQ3 66 67 straints which generalize both the classical RLT constraints obtained from two linear inequality constraints, and the SOC-RLT constraints obtained from one linear inequal-68 69 ity constraint and a SOC constraint. Further hard instances from [12] are solved with the addition of these cuts. In the very recent paper [4] a branch and bound approach is 7071proposed. The main feature of this approach is eigenvector branching, i.e., a branching rule based on the use of the eigenvector corresponding to the maximal eigenvalue 72of $\mathbf{X}^{\star} - \mathbf{x}^{\star} \mathbf{x}^{\star +}$, where $(\mathbf{X}^{\star}, \mathbf{x}^{\star})$ is the optimal solution of an SDP relaxation. 73

In this paper we investigate ways to strengthen the dual Lagrangian bound 7475through the addition of one or two linear cuts. In particular, the paper is structured 76as follows. In section 2 we derive some theoretical results for a class of problems with two constraints which includes the CDT problem as a special case. We develop a bi-7778section technique to solve the dual Lagrangian relaxation for such class of problems. In the following sections we apply the results of section 2 to the CDT problem. In 7980 particular, in section 3 we introduce some results through which it will be possible to rederive the necessary and sufficient exactness condition discussed in [2] and we 81

82discuss how to improve the dual bound for the CDT problem by the addition of a linear cut. Next, in sections 4 and 5 we discuss techniques to further improve the 83 bound. More precisely, in section 4 we still present a bound based on the addition of 84 a linear cut, but we develop a technique to locally adjust a given linear cut, while in 85 86 section 5 we consider a bound based on the addition of two linear cuts. Finally, in 87 section 6 we present some computational experiments which show that the newly proposed bounds, in particular those based on two linear cuts, are both computationally 88 89 cheap and effective. In particular, one of the bounds will be able to solve all but one of the hard instances from [12]. We also investigate which are the most challenging 90 instances for the proposed bounds and, as we will see, the difficulties are related to 9192the existence of multiple solutions of Lagrangian relaxations.

93 It is also worthwhile to highlight the contribution of this paper under another perspective. As previously discussed, while there is no known convex relaxation of 94the CDT problem, there are several problems, related to CDT, which do have exact 95SDP relaxations. These include the trust-region subproblem (TRS), the TRS with a 96 single linear constraint (TRS1, see [12, 23]), and the TRS with two linear constraints, 9798at least one of which is tight (TRS2eq, see [26]). This paper shows that such special cases, for which an exact convex relaxation exists, can be used to help solve the general 99CDT problem. Indeed, we first observe that the subproblems to be solved in section 3 100to improve the dual Lagrangian bound, obtained by adding a linear cut corresponding 101 to a supporting hyperplane for the second ellipsoidal constraint, turn out to be TRS1 102103problems. In section 4 we also discuss how to pick a "good" supporting hyperplane, 104i.e., one which leads to a good SDP relaxation and, in fact, we also provide a necessary and sufficient condition under which we can guarantee that the supporting hyperplane 105is the best one. Next, in section 5 we observe that the bound can be further improved 106 by adding two linear cuts, one of which must be active, so that the subproblems to 107 108be solved in this case are TRS2eq problems.

109 **2. Lower bounds obtained from the Lagrangian relaxation.** The CDT 110 problem (1.1) is a specific instance of the following, more general, one:

(2.1)
$$p^{\star} = \min_{\mathbf{x} \in \mathbb{R}^{n}}, \quad f(\mathbf{x}), \\ g(\mathbf{x}) \le 0, \\ h(\mathbf{x}) \le 0.$$

111 In this section, we discuss a class of lower bounds on the solution of problem (2.1) 112 that can be obtained from its Lagrangian relaxation. In the next sections, we will 113 apply these bounds to the specific case of the CDT problem (1.1). Throughout this 114 and the following sections, we make the following assumptions.

445 Assumption 2.1. In problem (2.1)

- 117 (a) g, h are continuous;
- 118 (b) the set $\{\mathbf{x} \in \mathbb{R}^n : g(\mathbf{x}) \le 0\}$ is bounded;
- 119 (c) it holds that

(2.2)
$$h_0 = \min_{\mathbf{x} : g(\mathbf{x}) \le 0} h(\mathbf{x}) < 0;$$

(d) the solution set of problem (2.1) without the last constraint, that is

$$\bar{P} = \arg \min_{\mathbf{x} \in \mathbb{R}^n}, \quad f(\mathbf{x}), \\ g(\mathbf{x}) \le 0$$

121 is such that $(\forall \mathbf{x} \in \bar{P}) h(\mathbf{x}) > 0$.

122 Note that if the last condition in Assumption 2.1 is violated, we can find the 123 solution of problem (2.1) by removing the last constraint and the relaxation discussed 124 in this section is useless. Now, let $G = \{\mathbf{x} \in \mathbb{R}^n : g(\mathbf{x}) \leq 0\}$ and $H = \{\mathbf{x} \in \mathbb{R}^n :$ 125 $h(\mathbf{x}) \leq 0\}$. Let $X \supset H$ be a closed subset of \mathbb{R}^n and for $\lambda \in \mathbb{R}$, with $\lambda \geq 0$, define the 126 Lagrangian relaxation

(2.3)
$$p_X(\lambda) = \min_{\mathbf{x} \in X \cap G} f(\mathbf{x}) + \lambda h(\mathbf{x})$$

127 and the corresponding solution set

$$P_X(\lambda) = \arg \min_{\mathbf{x} \in X \cap C} f(\mathbf{x}) + \lambda h(\mathbf{x}).$$

128 Note that $P_X(\lambda)$ is compact, since $G \cap X$ is nonempty (in view of part (c)) 129 of Assumption 2.1) and compact (in view of the compactness of G which follows 130 from parts (a) and (b) of Assumption 2.1), and $f + \lambda h$ is continuous. Due to well-131 known properties of the Lagrangian relaxation, we have that function p_X is such that 132 $(\forall \lambda \ge 0) p_X(\lambda) \le p^*$, and is concave (it is the pointwise minimum of a set of functions 133 linear in λ). The best bound that can be obtained as the solution of (2.3) is given by

(2.4)
$$\bar{p}_X = \max_{\lambda > 0} p_X(\lambda),$$

134 and corresponds to the solution of the dual Lagrangian problem. Note that function 135 p_X depends on the choice of set X.

136 Now, we recall that the supergradient of a function $q : \mathbb{R} \to \mathbb{R}$ at $x \in \mathbb{R}$ is defined 137 as

$$\partial q(x) = \{ z \in \mathbb{R} : (\forall y \in \mathbb{R}) \ q(y) - q(x) \le z(y - x) \}.$$

138 Since p_X is concave, for any $\lambda \in \mathbb{R}$, the supergradient $\partial p_X(\lambda)$ is nonempty. 139 For $A \subset \mathbb{R}^n$ define the following subset of \mathbb{R} :

$$h(A) = \{h(\mathbf{x}) \ : \ \mathbf{x} \in A\}$$

140 For $X \subset \mathbb{R}^n$, define a (set-valued) function $Q_X : \mathbb{R}_+ \to \mathcal{P}(\mathbb{R})$,

 $(\mathbb{R}_+ \text{ denotes the set of nonnegative reals and } \mathcal{P}(\mathbb{R})$ is the power set of the set of real 141numbers). Also set $h_X^{\min}(\lambda) = \min Q_X(\lambda)$ and $h_X^{\max}(\lambda) = \max Q_X(\lambda)$. The following 142proposition shows that function Q_X is monotone nonincreasing (see Definition 3.5.1) 143of [5]) and upper semicontinuous (see Definition 1.4.1 of [5]). These two properties 144 will play an important role in the computation of a lower bound for problem (2.1). 145146Moreover, this proposition characterizes the supergradient of p_X at each $\lambda \geq 0$. In the proof of the proposition we will make use of Berge's maximum theorem (see [8]). In 147148particular, we will consider the slightly different formulation presented as the corollary to Theorem 3 on page 30 of [15]. 149

150 COROLLARY 2.1. Let the correspondence (i.e., the set-valued function) β of S 151 into T be compact-valued and continuous, and let $\phi : S \times T \to \mathbb{R}$ be a continuous 152 function. Then, we have the following:

(a) The function $z \mapsto m(z) := max\{\phi(z, y) | y \in \beta(z)\}$ is continuous.

154 (b) The correspondence $z \mapsto \{y \in \beta(z) | \phi(z, y) = m(z)\}$ is nonempty and compact-155 valued and upper semicontinuous.

- 156 PROPOSITION 2.2. For any $X \subset \mathbb{R}^n$,
- 157 (i) Q_X is monotone non-increasing, that is if $\lambda_1 \ge \lambda_2$, $y_1 \in Q_X(\lambda_1)$, $y_2 \in Q_X(\lambda_2)$, 158 then $y_1 \le y_2$.
- 159 (ii) Q_X is upper semicontinuous, that is, if $Q_X(\lambda) \subset U$, where U is an open subset 160 of \mathbb{R} , then there exists a neighborhood V of λ such that $(\forall z \in V) Q_X(z) \subset U$.
- 161 (iii) $\partial p_X(\lambda) = [\min Q_X(\lambda), \max Q_X(\lambda)].$

162 Proof. (i) Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ be such that $y_1 = h(\mathbf{x}_1)$ and $y_2 = h(\mathbf{x}_2)$, then $f(\mathbf{x}_1) + 163$ 163 $\lambda_1 h(\mathbf{x}_1) \leq f(\mathbf{x}_2) + \lambda_1 h(\mathbf{x}_2)$ and $f(\mathbf{x}_2) + \lambda_2 h(\mathbf{x}_2) \leq f(\mathbf{x}_1) + \lambda_2 h(\mathbf{x}_1)$. By adding up the 164 previous inequalities, it follows that $(\lambda_1 - \lambda_2)(h(\mathbf{x}_1) - h(\mathbf{x}_2)) \leq 0$.

- 165 (ii) Apply Corollary 2.1 with $T = G \cap X$, $S = \mathbb{R}_+$, constant function ($\forall \lambda \in S$) $\beta(\lambda) = G \cap X$, $\phi(\lambda, \mathbf{x}) = -f(\mathbf{x}) \lambda \cdot h(\mathbf{x})$. Since ϕ is continuous, set-valued function 167 $P_X(\lambda) = \{\mathbf{x} \in G \cap X : \phi(\lambda, \mathbf{x}) = \max_{\mathbf{y} \in G \cap X} \phi(\lambda, \mathbf{y})\}$ is upper semicontinuous. Hence, 168 also Q_X is upper semicontinuous, since it is obtained as the composition of P_X with 169 h, which is continuous (see Theorem 1' on page 113 of [8]).
- 170 (iii) It is a consequence of Theorem 4.4.2 in [16], being G compact. \Box
- 171 The next proposition characterizes the optimal solution of the dual Lagrangian prob-172 lem (2.4).
- 173 PROPOSITION 2.3. Under Assumption 2.1, the optimal value of the dual La-174 grangian problem (2.4) is: (i) either attained at $\lambda_X = 0$ in case $\partial p_X(0) \cap \mathbb{R}_- \neq \emptyset$, 175 where \mathbb{R}_- denotes the set of nonpositive real numbers; (ii) or is attained at $\lambda_X > 0$ 176 such that $0 \in \partial p_X(\lambda_X)$. In the former case, $\bar{p}_X = p^*$ holds.

177 Proof. The proposition, apart from the last statement, is a direct consequence of 178 the optimality conditions for the maximum of concave functions (see Theorem 1.1.1 179 in Chapter 7 of [16]). To prove the last statement, namely that if (i) holds $\bar{p}_X = p^*$, 180 note that, if $\partial p_X(0) \cap \mathbb{R}_- \neq \emptyset$, then, in view of part (iii) of Proposition 2.2, $h_X^{\min}(0) \leq 0$ 181 and, thus, there exists an optimal solution \mathbf{x}^* of $\min_{\mathbf{x} \in X \cap G} f(\mathbf{x})$ such that $h(\mathbf{x}^*) \leq 0$. 182 This implies that \mathbf{x}^* is also an optimal solution of the original problem (2.1), so that 183 $\bar{p}_X = p^*$ holds.

184 The following property shows that it is always possible to find a sufficiently high 185 value of λ such that $P_X(\lambda) \subset H$, that is, the elements of $P_X(\lambda)$ are feasible solutions 186 of problem (2.1).

187 LEMMA 2.4. If

(2.6)
$$\lambda \ge \hat{\lambda} = \frac{\max_{\mathbf{x} \in G \cap X} f(\mathbf{x}) - \min_{\mathbf{x} \in G \cap X} f(\mathbf{x})}{|h_0|},$$

188 where h_0 is defined in (2.2), then $P_X(\lambda) \subset H$.

189 Proof. By contradiction, assume that there exists $\mathbf{x} \in P_X(\lambda)$ such that $h(\mathbf{x}) > 0$, 190 and let $\mathbf{x}_0 \in G \cap H$ be such that $h(\mathbf{x}_0) = h_0 < 0$; then $f(\mathbf{x}) + \lambda h(\mathbf{x}) \le f(\mathbf{x}_0) + \lambda h(\mathbf{x}_0)$. 191 Since $h(\mathbf{x}) > 0$, it follows that $\lambda \le \frac{f(\mathbf{x}_0) - f(\mathbf{x})}{|h(\mathbf{x}_0)| + h(\mathbf{x})} < \frac{\max_{\mathbf{x} \in G \cap X} f(\mathbf{x}) - \min_{\mathbf{x} \in G \cap X} f(\mathbf{x})}{|h(\mathbf{x}_0)|}$, which 192 contradicts the assumption on λ .

- 193 The following proposition shows that if $0 \in Q_X(\lambda)$, then $p_X(\lambda)$ is equal to the 194 optimal value of problem (2.1).
- 195 PROPOSITION 2.5. Under Assumption 2.1, the following statements are equiva-196 lent for $\lambda > 0$:
- 197 (i) $0 \in Q_X(\lambda)$,
- 198 (ii) $p^* = p_X(\lambda)$ and there exists $\bar{\mathbf{x}} \in \arg \min_{\mathbf{x} \in G \cap H} f(\mathbf{x})$ such that $h(\bar{\mathbf{x}}) = 0$.

199*Proof.* (i) \Rightarrow (ii). Let $\bar{\mathbf{x}}$ be such that $h(\bar{\mathbf{x}}) \in Q_X(\lambda)$ and $h(\bar{\mathbf{x}}) = 0$. Let \mathbf{x}^* be a solution of (2.1). Then, $p_X(\lambda) = f(\bar{\mathbf{x}}) + \lambda h(\bar{\mathbf{x}}) = f(\bar{\mathbf{x}}) \leq f(\mathbf{x}^*) + \lambda h(\mathbf{x}^*) \leq f(\mathbf{x}^*)$, hence 200 $p_X(\lambda) \leq p^*$. Moreover, $p_X(\lambda) = f(\bar{\mathbf{x}}) \geq \min_{\mathbf{x} \in G \cap H} f(\mathbf{x}) = p^*$. 201

(ii) \Rightarrow (i). Assume that $p_X(\lambda) = p^*$, and let $\mathbf{x} \in P_X(\lambda)$. Then, by (ii), $f(\mathbf{x}) + f(\mathbf{x}) = p^*$. 202 $\lambda h(\mathbf{x}) = f(\bar{\mathbf{x}}) = f(\bar{\mathbf{x}}) + \lambda h(\bar{\mathbf{x}})$. It follows that $\bar{\mathbf{x}} \in P_X(\lambda)$ and $Q_X(\lambda) \ni h(\bar{\mathbf{x}}) = 0$. 203

204Remark 2.6. If $0 \in Q_X(\lambda)$, by point (iii) of Proposition 2.2, $\partial p_X(\lambda) \ni 0$, so that λ corresponds to a maximizer of the dual Lagrangian. Note that equation $0 \in Q_X(\lambda)$ 205always admits a solution if Q_X is continuous. However, in the general case, Q_X is 206only upper semicontinuous. In this case, the value of λ for which $\partial p_X(\lambda) \ge 0$ may not 207208satisfy $0 \in Q_X(\lambda)$. Thus, the optimal value of the dual Lagrangian (2.4) is not equal to the optimal value of (2.1) but it represents a lower bound of it. 209

- 210In order to evaluate a numerical solution algorithm, we define the following weak solution of (2.1). 211
- DEFINITION 2.7. **x** is an η -solution of (2.1) if $\mathbf{x} \in G \cap H$ and $f(\mathbf{x}) p^* \leq \eta$. 212
- 213The following proposition presents a bound on the error committed on the estimation of p^* . 214

PROPOSITION 2.8. For any $\lambda \geq 0$ such that $P_X(\lambda) \cap H \neq \emptyset$, and for any $\mathbf{x} \in \mathcal{A}$ 215 $P_X(\lambda) \cap H$, it holds that $f(\mathbf{x}) - p^* \leq \lambda |h(\mathbf{x})|$, i.e., \mathbf{x} is an η -solution of problem of 216217(2.1) with $\eta = \lambda |h(\mathbf{x})|$.

218*Proof.* Since $\mathbf{x} \in P_X(\lambda)$ and observing that $\mathbf{x}^* \in G \cap X$ for any $X \supset H$, $f(\mathbf{x}) + f(\mathbf{x}) \in G$ $\lambda h(\mathbf{x}) \leq f(\mathbf{x}^*) + \lambda h(\mathbf{x}^*) \leq f(\mathbf{x}^*)$, from which $f(\mathbf{x}) - f(\mathbf{x}^*) \leq \lambda |h(\mathbf{x})|$. 219П

Now we introduce Algorithm 2.1 which is based on a binary search through differ-220ent λ values and is able to return the solution of the dual Lagrangian problem, i.e., the 221222maximum of function $p_X(\lambda)$ and, in some cases, even the solution of problem (2.1). The algorithm also returns a point $\mathbf{z}_1(\lambda^{\max}) \in H$ and (possibly) a point $\mathbf{z}_2(\lambda^{\min}) \notin H$. 223Note that according to Proposition 2.8, point $\mathbf{z}_1(\lambda^{\max})$ is an η -solution of problem 224225(2.1) with $\eta = \lambda |h(\mathbf{z}_1(\lambda^{\max}))|$.

The algorithm starts with an initial interval of λ values $[\lambda^{\min}, \lambda^{\max}] = [0, \lambda^{\text{init}}],$ 226227where λ^{init} is a suitably large value and can be set equal to $\hat{\lambda}$ as defined in Lemma 2.4. At each iteration the algorithm halves such interval by evaluating the set Q_X^{λ} 228at $\lambda = (\lambda^{\max} + \lambda^{\min})/2$. Then, the algorithm sets- are $\lambda^{\min} = \lambda$, if $h_X^{\min}(\lambda) > 0$; 229 $\lambda^{\max} = \lambda$ if $h_X^{\max}(\lambda) < 0$. Instead, if $0 \in \partial p_X(\lambda) = [h_X^{\min}(\lambda), h_X^{\max}(\lambda)]$, the algorithm 230sets $\lambda^{\max} = \lambda^{\min} = \lambda$ and exits the loop. 231

The following proposition characterizes Algorithm 2.1.

PROPOSITION 2.9. (i) Algorithm 2.1 terminates in a finite number of iterations: 233(ii) at each iteration $\lambda^{\min} < \lambda_X < \lambda^{\max}$, 234

- (iii) at termination $|\lambda^{\max} \lambda_X| \leq \epsilon$,

(iv) at each iteration, if $\lambda^{\min} < \lambda_X < \lambda^{\max}$, then $[h_X^{\max}(\lambda^{\max}), h_X^{\min}(\lambda^{\min})] \supset$ 236 $\partial p_X(\lambda_X),$ 237

(v) point $\mathbf{z}_1(\lambda^{\max}) \in P_X(\lambda^{\max}) \cap H$ is an η -solution of (2.1) with $\eta = \lambda_{\max}$ 238 $h(\mathbf{z}_1(\lambda^{\max}))|.$ 239

Proof. (i) At each iteration the length of the interval $[\lambda^{\min}, \lambda^{\max}]$ is halved. Hence, 240241in a sufficient large number of iterations, the termination condition of the main loop 242will be satisfied.

(ii) At the beginning of the algorithm we have that $\lambda^{\min} \leq \lambda_X \leq \lambda^{\max}$. Every time 243 λ^{\min} is updated, we set $\lambda^{\min} = \lambda$ if condition $h_X^{\min}(\lambda) > 0$ holds. Since $h_X^{\min}(\lambda_X) \le 0$, 244

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Algorithm 2.1 Binary search algorithm for the solution of the dual Lagrangian problem for (1.1).

DualLagrangian (X, λ_{init}) Set $\lambda_{\min} = 0$, $\lambda^{\max} = \lambda^{\text{init}}$ while $\lambda^{\max} - \lambda^{\min} > \varepsilon$ do Set $\lambda = (\lambda^{\max} + \lambda^{\min})/2$ Solve problem (2.3), and let $P_X(\lambda)$ be its set of optimal solutions Compute the set $Q_X(\lambda)$ and the values $h_X^{\min}(\lambda), h_X^{\max}(\lambda)$ if $h_X^{\min}(\lambda) > 0$ then Set $\lambda_{\min} = \lambda$ else if $h_X^{\max}(\lambda) < 0$ then Set $\lambda^{\max} = \lambda$ else Set $\lambda^{\max} = \lambda^{\min} = \lambda$ end if end while Set $Lb = p_X(\lambda^{\max})$, and let $\mathbf{z}_1(\lambda^{\max})$ be some point in $P_X(\lambda^{\max}) \cap H$ and $\mathbf{z}_2(\lambda^{\min})$ be some point (if any) in $P_X(\lambda^{\min}) \setminus H$ return $Lb, \lambda^{\max}, \mathbf{z}_1(\lambda^{\max}), \mathbf{z}_2(\lambda^{\min})$

by the monotonicity of function h_X^{\min} , which is a consequence of the monotonicity of function Q_X , condition $\lambda^{\min} \leq \lambda_X$ is maintained. The same reasoning can be used to prove that $\lambda^{\max} \geq \lambda_X$.

(iii) It is a consequence of (ii) and the termination condition.

249 (iv) $\partial p_X(\lambda_X) = [h_X^{\min}(\lambda_X), h_X^{\max}(\lambda_X)] \subset [h_X^{\max}(\lambda^{\max}), h_X^{\min}(\lambda^{\min})]$, due to point 250 (ii) and the monotonicity of functions h_X^{\max} and h_X^{\min} , which is a consequence of the 251 monotonicity of function Q_X .

(v) It is a consequence of Proposition 2.8.

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The following property is a direct consequence of the upper semicontinuity of Q_X .

255 PROPOSITION 2.10. Let $X \supset H$ be such that $\sup Q_X(\lambda) < 0$; then there exists a 256 neighborhood U of λ such that $(\forall \eta \in U) \max Q_X(\eta) < 0$.

As a consequence of the previous proposition, it is possible to improve the lower bound on problem (2.1), obtained as the solution of (2.3), by replacing set X with a different set $Y \supset H$ fulfilling a given condition.

260 PROPOSITION 2.11. Let $Y \supset H$ be such that $\max Q_Y(\lambda_X) \leq 0$ or, equivalently, 261 $P_Y(\lambda_X) \setminus H = \emptyset$, and assume that $\bar{p}_X = p_X(\lambda_X) < p^*$. Then $\bar{p}_Y = p_Y(\lambda_Y) > \bar{p}_X$.

Proof. Note that, by Proposition 2.3, $\bar{p}_X = p_X(\lambda_X) < p^*$ implies $\lambda_X > 0$. Now, 262in case max $Q_Y(\lambda_X) = 0$, then $0 \in Q_Y(\lambda_X)$ and, by Proposition 2.5, $\bar{p}_Y = p^* > \bar{p}_X$. 263Thus, we only consider the case $\max Q_Y(\lambda_X) < 0$. In such case, by Proposition 2.10, 264265 $\lambda_Y < \lambda_X$. If $\lambda_Y = 0$, by Proposition 2.3 we have that $\bar{p}_Y = p^* > \bar{p}_X$ and we are done. Otherwise, if $\lambda_Y > 0$, again by Proposition 2.3 we have that $0 \in [h_V^{\min}(\lambda_Y), h_V^{\max}(\lambda_Y)]$, 266and, consequently, there exists $\mathbf{y} \in P_Y(\lambda_Y)$ such that $h(\mathbf{y}) \leq 0$. Note that $\bar{p}_Y =$ 267 $f(\mathbf{y}) + \lambda_Y h(\mathbf{y})$. If $h(\mathbf{y}) = 0$, then, by Proposition 2.5, $p_Y(\lambda_Y) = p^*$, so that the thesis 268is satisfied in view of $\bar{p}_X < p^*$. Otherwise, if $h(\mathbf{y}) < 0$, let $\mathbf{x} \in \mathbb{R}^n$ be such that $\bar{p}_X =$ 269 $f(\mathbf{x}) + \lambda_X h(\mathbf{x})$. Then $\bar{p}_X = f(\mathbf{x}) + \lambda_X h(\mathbf{x}) \le f(\mathbf{y}) + \lambda_X h(\mathbf{y}) < f(\mathbf{y}) + \lambda_Y h(\mathbf{y}) = \bar{p}_Y$, 270where we used the facts that $h(\mathbf{y}) < 0$ and that $\lambda_Y < \lambda_X$. 271Π 272 The following proposition deals with the special case of the previous result when 273 $Y \subset X$.

PROPOSITION 2.12. Let $X \supset Y \supset H$ be such that $Y \cap (P_X(\lambda_X) \setminus H) = \emptyset$, and assume that $\bar{p}_X = p_X(\lambda_X) < p^*$. Then $\bar{p}_Y = p_Y(\lambda_Y) > \bar{p}_X$.

276 Proof. Since $h_X^{\min}(\lambda_X) < 0$ we have that $P_X(\lambda_X) \cap H \neq \emptyset$ and, consequently, since 277 $Y \supset H$, also $Y \cap P_X(\lambda_X) \neq \emptyset$. Then, $Y \subset X$ implies $P_Y(\lambda_X) = Y \cap P_X(\lambda_X)$. Moreover, 278 if $Y \cap (P_X(\lambda_X) \setminus H) = \emptyset$, then the condition $\max Q_Y(\lambda_X) \le 0$ is satisfied and the result 279 follows from Proposition 2.11.

280 Stated in another way, the previous propositions show that, in case the lower 281 bound \bar{p}_X is not exact, we are able to improve (increase) it if we are able to replace 282 set X with a new set Y which cuts away all members of $P_X(\lambda_X)$ outside H.

Remark 2.13. Up to now we have not discussed the difficulty of computing the 283values of function p_X or, equivalently, the difficulty of solving problem (2.3). Such 284difficulty is strictly related to the specific problem (i.e., to the specific functions f, g, h), 285and also to the specific set X. In the next sections we apply the general theory 286developed in this section to the CDT problem. We show that for suitably defined sets 287X (defined by one or two linear cuts), the computation of function p_X can be done 288efficiently, and, moreover, the corresponding lower bounds \bar{p}_X improve the standard 289dual Lagrangian bound, corresponding to the case $X = \mathbb{R}^n$. 290

291 Remark 2.14. In principle one could also define a cutting algorithm where a 292 sequence of sets $\{X_k\}$ is generated such that (i) $X_k \supset X_{k+1} \supset H$ for all k; (ii) 293 $X_{k+1} \cap (P_{X_k}(\lambda_{X_k}) \setminus H) = \emptyset$; (iii) $\bigcap_{k=1}^{\infty} X_k = H$. The corresponding sequence of lower 294 bounds $\{\bar{p}_{X_k}\}$ is strictly increasing in view of Proposition 2.12, and converges to p^* . 295 However, the difficulty related to such an algorithm is that forcing (ii) may not be 296 trivial and, moreover, as already commented in Remark 2.13, computing p_{X_k} may be 297 computationally demanding.

298The following algorithm, Algorithm 2.2, in principle, is able to always find an 299approximate solution of (2.1). The algorithm is based on an iterative reduction of set X, in order to eliminate its elements in which function h is positive. In practice, 300 Algorithm 2.2 could be unimplementable. Indeed, it may require a large number of 301302cuts on set X and each added cut may increase the complexity of the optimization 303 problem that we need to solve to evaluate **DualLagrangian**. In section 4, we will see 304that, to refine the lower bound on the solution of the CDT problem, it is computationally more convenient to adjust existing cuts instead of adding new ones. We stress 305306 that we will not actually use Algorithm 2.2 for the solution of the CDT problem. We present this algorithm just as a theoretical contribution. 307

308 PROPOSITION 2.15. Algorithm 2.2 terminates and $\bar{\mathbf{x}}$ is such that $h(\bar{\mathbf{x}}) \leq \frac{\eta}{\lambda^{\max}}$ 309 and $|\bar{f} - f^*| \leq \eta$.

Proof. By contradiction, assume that the algorithm does not terminate. Let l_i 310be the value of λ^{\min} returned by the *i*th call to **DualLagrangian**. Sequence l_i is 311monotone nonincreasing; moreover, the domain of the sequence is a subset of finite 312cardinality of interval $[0, \lambda^{\text{init}}]$ (its maximum cardinality depends on λ^{init} and ϵ). 313Indeed, the termination condition of function **DualLagrangian** allows only for a 314finite number of divisions of the interval $[0, \lambda^{\text{init}}]$. Hence, sequence l_i converges in 315a finite number of iterations to its limit $l_{\infty} = \lim_{i \to \infty} l_i$ and there exists $\bar{i} \in \mathbb{N}$ such 316that $(\forall i \geq \overline{i}) \ l_i = l_{\infty}$. By (iv) of Proposition 2.9, $h^{\max}(l_{\infty}) \geq 0$ and, since the 317

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Algorithm 2.2 Bound improvement through redefinition of set X.

Set $X = \mathbb{R}^n$ 1:

- Set $\lambda^{\max} = \lambda^{\text{init}}$ 2:
- 3: repeat
- 4: Let

$$\begin{split} & [Lb,\lambda^{\min},\lambda^{\max},\mathbf{z}_1(\lambda^{\max}),\mathbf{z}_2(\lambda^{\min}),h_X^{\min},h_X^{\max}] = \mathbf{DualLagrangian}(X,\lambda^{\mathtt{init}}) \\ & \text{Set } Z = \{\mathbf{x} \in P_X(\lambda^{\min}) : h(\mathbf{x}) > 0\} \end{split}$$

5:

Redefine X = Y, where Y is such that $X \supset Y \supset H$ and $Z \cap Y = \emptyset$. 6:

- **until** $\min\{h_X^{\max}(\lambda^{\min}), -h_X^{\min}(\lambda^{\max})\}\lambda^{\max} \leq \eta$ 7:
- return $\mathbf{\bar{x}} \in P_X(\lambda_X^{min}) \cup P_X(\lambda_X^{max})$ with $|h(\mathbf{\bar{x}})| \le \eta$, $\bar{f} = f(\mathbf{\bar{x}})$. 8:

algorithm does not terminate, $h^{\max}(l_{\infty}) \geq \eta$. At the $\bar{i} + 1$ -iteration, the algorithm 318calls **DualLagrangian** (X, l_{∞}) , which returns the value $\lambda^{\min} = l_{\infty}$. Anyway, at the 319previous iteration \bar{i} , the elements $P_X(\lambda^{\min})$ at which function h is positive had already 320 been removed from X. This implies that **DualLagrangian** (X, l_{∞}) cannot return the 321strictly positive value $\lambda^{\min} = l_{\infty}$, leading to a contradiction. Hence, the algorithm 322 terminates and the stated bounds hold because of the termination condition and by 323 Proposition 2.8. 324Π

3. Dual Lagrangian bound and a possible improvement. In this section, 325we apply the general properties presented in section 2 to the CDT problem (1.1). In 326 fact, the CDT problem is a specific instance of (2.1) in which $f(\mathbf{x}) = \mathbf{x}^{\top} \mathbf{Q} \mathbf{x} + \mathbf{q}^{\top} \mathbf{x}$, 327 328 $g(\mathbf{x}) = \mathbf{x}^{\top}\mathbf{x} - 1, \ h(\mathbf{x}) = \mathbf{x}^{\top}\mathbf{A}\mathbf{x} + \mathbf{a}^{\top}\mathbf{x} - a_0.$

Note that the first two requirements of Assumption 2.1 are satisfied; in order to 329satisfy the third one we assume that 330

(3.1)
$$h_0 = \min_{\mathbf{x} : \mathbf{x}^\top \mathbf{x} \le 1} \mathbf{x}^\top \mathbf{A} \mathbf{x} + \mathbf{a}^\top \mathbf{x} - a_0 < 0,$$

i.e., the feasible region of (1.1) has a nonempty interior. Note that the assumption can 331 be checked in polynomial time by the solution of a trust region problem. As before, 332333 we denote by $X \subseteq \mathbb{R}^n$ a closed set such that $X \supset H$, i.e., it contains the ellipsoid defined by the second constraint. For each $\lambda \geq 0$, the Lagrangian relaxation (2.3) 334335takes on the form

(3.2)
$$p_X(\lambda) = \min_{\mathbf{x} \in X} \quad \mathbf{x}^\top (\mathbf{Q} + \lambda \mathbf{A}) \mathbf{x} + (\mathbf{q} + \lambda \mathbf{a})^\top \mathbf{x} - \lambda a_0 \\ \mathbf{x}^\top \mathbf{x} \le 1.$$

336 If $X = \mathbb{R}^n$, this is the standard Lagrangian relaxation of problem (1.1) and it can be solved efficiently since it is a trust region problem. Following the notation of section 337 338 2, let

$$P_X(\lambda) = \arg \min_{\mathbf{x} \in X : \mathbf{x}^\top \mathbf{x} \leq 1} \mathbf{x}^\top (\mathbf{Q} + \lambda \mathbf{A}) \mathbf{x} + (\mathbf{q} + \lambda \mathbf{a})^\top \mathbf{x}$$

339 be the set of optimal solutions of (3.2). To apply Algorithm 2.1 to the CDT problem with $X = \mathbb{R}^n$, we need to characterize the set of optimal solutions $P_{\mathbb{R}^n}(\lambda)$ of problem 340 (3.2) with $X = \mathbb{R}^n$, which is a trust region problem. The set of optimal solutions of a 341342 trust region problem has been derived, e.g., in [1, 20, 21]. Here we briefly recall the 343different cases. For simplicity, let $\mathbf{S}_{\lambda} = \mathbf{Q} + \lambda \mathbf{A}$ and $\mathbf{s}_{\lambda} = \mathbf{q} + \lambda \mathbf{a}$. We distinguish the following cases: $\frac{344}{5}$

Case 1 If $\mathbf{S}_{\lambda} \succ \mathbf{O}$ and $\left\| -\frac{1}{2} \mathbf{S}_{\lambda}^{-1} \mathbf{s}_{\lambda} \right\| \leq 1$, then $-\frac{1}{2} \mathbf{S}_{\lambda}^{-1} \mathbf{s}_{\lambda}$ is the unique optimal solution 346347 of (3.2);

Case 2 Let \mathbf{u}_j be the orthonormal eigenvectors of matrix \mathbf{S}_{λ} , and let γ_j be the cor-348responding eigenvalues. Let $\gamma_{\min} = \min_j \gamma_j$ and $J_{\lambda} = \arg \min_j \gamma_j$. For each γ 349such that $(\forall j) \ \gamma \neq \gamma_j$, let 350

$$\mathbf{y}(\gamma) = \mathbf{y}_1(\gamma) + \mathbf{y}_2(\gamma)$$

351where

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$$\mathbf{y}_1(\gamma) = -\sum_{j \notin J_\lambda} \frac{\mathbf{s}_\lambda^\top \mathbf{u}_j}{\gamma_j - \gamma} \mathbf{u}_j, \quad \mathbf{y}_2(\gamma) = -\sum_{j \in J_\lambda} \frac{\mathbf{s}_\lambda^\top \mathbf{u}_j}{\gamma_j - \gamma} \mathbf{u}_j.$$

352	Then, we have the following subcases.
354	Case 2.1 It holds that $\mathbf{s}_{\lambda}^{\top} \mathbf{u}_{j} \neq 0$ for some $j \in J_{\lambda}$. Then, there exists a unique
355	$\gamma^* \in (-\gamma_{\min}, +\infty)$ such that $\ \mathbf{y}(\gamma^*)\ = 1$ and $\mathbf{y}(\gamma^*)$ is the unique
356	optimal solution of (3.2) .
357	Case 2.2 It holds that $\mathbf{s}_{\lambda}^{\top}\mathbf{u}_{j} = 0$ for all $j \in J_{\lambda}$ but $\ \mathbf{y}_{1}(\gamma_{\min})\ \geq 1$. In this
358	case there exists a unique $\gamma^* \in [-\gamma_{\min}, +\infty)$ such that $\ \mathbf{y}_1(\gamma^*)\ = 1$

Case 2.2 It holds that
$$\mathbf{s}_{\lambda}^{\top} \mathbf{u}_{j} = 0$$
 for all $j \in J_{\lambda}$ but $\|\mathbf{y}_{1}(\gamma_{\min})\| \ge 1$. In this case there exists a unique $\gamma^{*} \in [-\gamma_{\min}, +\infty)$ such that $\|\mathbf{y}_{1}(\gamma^{*})\| = 1$ and $\mathbf{y}_{1}(\gamma^{*})$ is the unique optimal solution of (3.2).

Case 2.3 It holds that $\mathbf{s}_{\lambda}^{\dagger} \mathbf{u}_{j} = 0$ for all $j \in J_{\lambda}$ and $\|\mathbf{y}_{1}(\gamma_{\min})\| < 1$. In this case we have that $P_{\mathbb{R}^n}(\lambda)$ is not a singleton and is made up by the following points:

(3.3)
$$P_{\mathbb{R}^n}(\lambda) = \left\{ \mathbf{y}_1(\gamma_{\min}) + \sum_{j \in J_\lambda} \xi_j \mathbf{u}_j : \sum_{j \in J_\lambda} \xi_j^2 = 1 - \|\mathbf{y}_1(\gamma_{\min})\|^2 \right\}.$$

Thus, we recognize two further subcases. 363

- **Case 2.3.1** $|J_{\lambda}| = 1$, in which case $P_{\mathbb{R}^n}(\lambda)$ contains exactly two distinct points.
- **Case 2.3.2** $|J_{\lambda}| \geq 2$, in which case the set $P_{\mathbb{R}^n}(\lambda)$ contains an infinite number of points and is a connected set.
- Note that in Cases 2.3.1 and 2.3.2 we can compute the two values $h_{\mathbb{R}^n}^{\min}(\lambda), h_{\mathbb{R}^n}^{\max}(\lambda)$ 369 370by solving a trust region problem over the border of a $|J_{\lambda}|$ -dimensional ball. More 371precisely, we need to solve the following problems:

(3.4)
$$\min / \max_{\boldsymbol{\xi}} \quad \mathbf{w}(\boldsymbol{\xi})^{\top} \mathbf{A} \mathbf{w}(\boldsymbol{\xi}) + \mathbf{a}^{\top} \mathbf{w}(\boldsymbol{\xi}) - a_{0} \\ \|\mathbf{w}(\boldsymbol{\xi})\|^{2} = 1,$$

372 where $\mathbf{w}(\boldsymbol{\xi}) = \mathbf{y}_1(\gamma_{\min}) + \sum_{j \in J_{\lambda}} \xi_j \mathbf{u}_j$. In these cases, where $P_{\mathbb{R}^n}(\lambda)$ is not a singleton, 373we also set

(3.5)
$$\mathbf{z}_{1}(\lambda) = \mathbf{w}(\boldsymbol{\xi}_{1}), \quad \boldsymbol{\xi}_{1} \in arg \min_{\boldsymbol{\xi} : \|\mathbf{w}(\boldsymbol{\xi})\| = 1} \mathbf{w}(\boldsymbol{\xi})^{\top} \mathbf{A} \mathbf{w}(\boldsymbol{\xi}) + \mathbf{a}^{\top} \mathbf{w}(\boldsymbol{\xi}) - a_{0}, \\ \mathbf{z}_{2}(\lambda) = \mathbf{w}(\boldsymbol{\xi}_{2}), \quad \boldsymbol{\xi}_{2} \in arg \max_{\boldsymbol{\xi} : \|\mathbf{w}(\boldsymbol{\xi})\| = 1} \mathbf{w}(\boldsymbol{\xi})^{\top} \mathbf{A} \mathbf{w}(\boldsymbol{\xi}) + \mathbf{a}^{\top} \mathbf{w}(\boldsymbol{\xi}) - a_{0},$$

while in all other cases, when $P_{\mathbb{R}^n}(\lambda) = \{\mathbf{z}^*(\lambda)\}\$ is a singleton, we set 374

(3.6)
$$\mathbf{z}_1(\lambda) = \mathbf{z}_2(\lambda) = \mathbf{z}^*(\lambda)$$

375The following statement is a direct consequence of Proposition 2.5.

PROPOSITION 3.1. In the CDT problem (1.1), if $\lambda > 0$ and

• $h_{\mathbb{R}^n}^{\min}(\lambda) = 0;$ 378

- or if $h_{\mathbb{R}^n}^{\max}(\lambda) = 0;$
- or $h_{\mathbb{R}^n}^{\min}(\lambda) < 0 < h_{\mathbb{R}^n}^{\max}(\lambda)$ and $|J_{\lambda}| \ge 2$ (i.e., we are in Ccase 2.3.2); 380
- 381then $p_{\mathbb{R}^n}(\lambda) = p^*$.

382 Proof. Since $\{h_{\mathbb{R}^n}^{\min}(\lambda), h_{\mathbb{R}^n}^{\max}(\lambda)\} \in Q_X(\lambda)$, in the first two cases $0 \in Q_X(\lambda)$ and 383 the thesis is a consequence of Proposition 2.5. If $h_{\mathbb{R}^n}^{\min}(\lambda) < 0 < h_{\mathbb{R}^n}^{\max}(\lambda)$ and $|J_{\lambda}| \ge 2$, 384 we observed that $P_{\mathbb{R}^n}(\lambda)$ is a connected set. Then, there exists $\mathbf{x}^* \in P_{\mathbb{R}^n}(\lambda)$ such 385 that $\mathbf{x}^* \in \partial H$. More precisely, \mathbf{x}^* is a point along the curve in $P_{\mathbb{R}^n}(\lambda)$ connecting 386 points $\mathbf{z}_1(\lambda)$ and $\mathbf{z}_2(\lambda)$, defined in (3.5). Thus, the lower bound $p_{\mathbb{R}^n}(\lambda)$ is equal to the 387 optimal value of problem (1.1).

Note that the first two conditions of Proposition 3.1 imply exactness of the bound also for generic regions $X \supset H$, while the last condition is specific to the case $X = \mathbb{R}^n$. The following result is related to the necessary and sufficient condition under which the dual Lagrangian bound is not exact discussed in [2].

392 PROPOSITION 3.2. In the CDT problem (1.1), $p_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}) \neq p^*$ if and only if $\lambda_{\mathbb{R}^n} >$ 393 0, $P_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n})$ contains exactly two points (Case 2.3.1), and $0 \in (h_{\mathbb{R}^n}^{\min}(\lambda_{\mathbb{R}^n}), h_{\mathbb{R}^n}^{\max}(\lambda_{\mathbb{R}^n}))$.

394 Proof. First note that, in view of Proposition 2.3, the dual Lagrangian bound is 395 always exact when $\lambda_{\mathbb{R}^n} = 0$. When $\lambda_{\mathbb{R}^n} > 0$, the result is a consequence of Proposition 396 3.1 and the fact that for $|J_{\lambda_{\mathbb{R}^n}}| = 1$ it holds that $Q_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}) = \{h_{\mathbb{R}^n}^{\min}(\lambda_{\mathbb{R}^n}), h_{\mathbb{R}^n}^{\max}(\lambda_{\mathbb{R}^n})\} \not\ni$ 397 0.

Now, we introduce an example where $p_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}) \neq p^*$, that is the dual Lagrangian bound is not exact, which will also be helpful in the following sections.

400 *Example 3.3.* Let us consider the following example taken from [12]:

$$\mathbf{Q} = \begin{pmatrix} -4 & 1 \\ 1 & -2 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{q} = (1 \ 1) \quad \mathbf{a} = (0 \ 0), \quad a_0 = 2.$$

401 Such an instance has optimal value -4 attained at points $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ and $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. 402 The maximizer of $p_{\mathbb{R}^2}(\lambda)$ is $\lambda_{\mathbb{R}^2} = 1$ for which we have

$$h_{\mathbb{R}^2}^{\min} \approx -0.66 < 0 < 0.66 \approx h_{\mathbb{R}^2}^{\max}$$

- 403 and, moreover, $|J_{\lambda_{\mathbb{R}^2}}| = 1$, so that we have exactly two optimal solutions of (3.2), 404 one violating the second constraint, namely $\mathbf{z}_2(\lambda_{\mathbb{R}^2}) = (-0.911, 0.4114)$, point x_1 in 405 Figure 1, displayed as \circ , the other in int(H), point z_1 in Figure 1, displayed as \times . 406 The lower bound is $p_{\mathbb{R}^2}(1) = -4.25$, which is not exact.
- 407 Now, let us assume that the dual Lagrangian bound is not exact, i.e., as previously408 stated in Proposition 3.2

$$0 \in \left(h_{\mathbb{R}^n}^{\min}(\lambda_{\mathbb{R}^n}), h_{\mathbb{R}^n}^{\max}(\lambda_{\mathbb{R}^n})\right), \quad |J_{\lambda_{\mathbb{R}^n}}| = 1.$$

409 Recall that, by Proposition 3.2, in this case, there exists a single point $\mathbf{z}_1(\lambda_{\mathbb{R}^n}) \in$ 410 $P_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}) \cap H$ (actually $\mathbf{z}_1(\lambda_{\mathbb{R}^n}) \in int(H)$), and a single point $\mathbf{z}_2(\lambda_{\mathbb{R}^n}) \in P_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}) \setminus H$. 411 Now we show that the dual Lagrangian bound can be strictly improved through the 412 addition of a linear cut. We first observe that the optimal value of problem (1.1) does 413 not change if we add constraints which are implied by the second one.

414 In the following proposition, we define a projection $\Pi_{\mathbf{A},\mathbf{a}} : \mathbb{R}^n \setminus H \to \partial H$ that 415 maps $\mathbf{x} \notin H$ to the element of ∂H located on the segment that joins \mathbf{x} to the center 416 of the ellipsoid H (given by $\boldsymbol{\alpha} = -\frac{1}{2}\mathbf{A}^{-1}\mathbf{a}$).

417 PROPOSITION 3.4. For $\mathbf{x} \notin H$, set $\Pi_{\mathbf{A},\mathbf{a}}(\mathbf{x}) = \sqrt{\frac{-h(\alpha)}{h(\mathbf{x})-h(\alpha)}}(\mathbf{x}-\alpha) + \alpha$, where 418 $\boldsymbol{\alpha} = -\frac{1}{2}\mathbf{A}^{-1}\mathbf{a}$ is the center of the ellipsoid. Then $h(\Pi_{\mathbf{A},\mathbf{a}}(\mathbf{x})) = 0$.



FIG. 1. Optimal solutions of the dual Lagrangian bound outside $H(x_1)$ and in $int(H)(z_1)$, denoted by \circ and \times , respectively. The continuous red curve is the border of the unit ball, while the dotted blue curve is the border of the ellipsoid H. (Figure in color online.)

119 Proof. Note that
$$(\forall \beta \in \mathbb{R}) h(\beta(\mathbf{x} - \alpha) + \alpha) - h(\alpha) = \beta^2(h(\mathbf{x}) - h(\alpha))$$
 (it is a

420 consequence of the fact that function
$$h$$
 is quadratic and it can be verified by direct
421 substitution). Then $h(\Pi_{\mathbf{A},\mathbf{a}}(\mathbf{x})) = h\left(\sqrt{\frac{-h(\alpha)}{h(\mathbf{x})-h(\alpha)}}(\mathbf{x}-\alpha) + \alpha\right) = \frac{-h(\alpha)}{h(\mathbf{x})-h(\alpha)}(h(\mathbf{x}) - h(\alpha))$

422
$$h(\boldsymbol{\alpha})) + h(\boldsymbol{\alpha}) = 0.$$

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423 Given any $\bar{\mathbf{x}} \in \mathbb{R}^n$, it holds, by convexity, that

$$\mathbf{x}^{\top}\mathbf{A}\mathbf{x} + \mathbf{a}^{\top}\mathbf{x} \ge \bar{\mathbf{x}}^{\top}\mathbf{A}\bar{\mathbf{x}} + \mathbf{a}^{\top}\bar{\mathbf{x}} + (2\mathbf{A}\bar{\mathbf{x}} + \mathbf{a})^{\top}(\mathbf{x} - \bar{\mathbf{x}}).$$

_

424 Thus, the following linear constraint is implied by the second constraint in (1.1):

(3.7)
$$(2\mathbf{A}\bar{\mathbf{x}} + \mathbf{a})^{\top}\mathbf{x} - \bar{\mathbf{x}}^{\top}\mathbf{A}\bar{\mathbf{x}} \le a_0,$$

425 and, consequently, it can be added to problem (1.1) without modifying its feasible 426 region. In particular, if $\bar{\mathbf{x}} \in \partial H$, being $\bar{\mathbf{x}}^T \mathbf{A} \bar{\mathbf{x}} + \mathbf{a}^T \bar{\mathbf{x}} = a_0$, the linear constraint is

(3.8)
$$(2\mathbf{A}\bar{\mathbf{x}} + \mathbf{a})^{\top} (\mathbf{x} - \bar{\mathbf{x}}) \le 0.$$

427 Due to the redundancy of the linear constraint for problem (1.1), we can define, for a 428 given $\bar{\mathbf{x}} \in \partial H$, the new Lagrangian problem

(3.9)
$$p_X(\lambda) = \min_{\mathbf{x}} \quad \mathbf{x}^\top (\mathbf{Q} + \lambda \mathbf{A}) \mathbf{x} + (\mathbf{q} + \lambda \mathbf{a})^\top \mathbf{x} - \lambda a_0$$
$$\mathbf{x}^\top \mathbf{x} \le 1$$
$$(2\mathbf{A}\bar{\mathbf{x}} + \mathbf{a})^\top (\mathbf{x} - \bar{\mathbf{x}}) \le 0,$$

429 where

(3.10)
$$X = \Omega_{\bar{\mathbf{x}}} = \{ \mathbf{x} : (2\mathbf{A}\bar{\mathbf{x}} + \mathbf{a})^{\top} (\mathbf{x} - \bar{\mathbf{x}}) \le 0 \} \supset H.$$

430If we set $\bar{\mathbf{x}} = \prod_{\mathbf{A},\mathbf{a}}(\mathbf{z}_2(\lambda_{\mathbb{R}^n}))$, i.e., $\bar{\mathbf{x}}$ is the projection over ∂H of the single point in $P_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}) \setminus H$, then $\mathbb{R}^n \supset X \supset H$ and, moreover, $X \cap (P_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}) \setminus H) = \emptyset$, so that, 431by Proposition 2.12, $\bar{p}_X > \bar{p}_{\mathbb{R}^n}$. Then, if we run again Algorithm 2.1 with input 432 $X = \Omega_{\bar{\mathbf{x}}}$ defined in (3.10) and $\lambda_{\text{init}} = \lambda_{\mathbb{R}^n}$ (or $\lambda_{\text{init}} = \lambda_{\mathbb{R}^n}^{\max}$), we are able to improve 433 strictly the dual Lagrangian bound. Note that problem (3.9), needed to compute 434435function $p_{\Omega_{\mathbf{x}}}$, can be solved in polynomial time according to the results proved in [12, 23]. But we also discuss an alternative way to solve problem (3.9), based on the 436437solution of a trust region problem. For $\lambda = \lambda_{\mathbb{R}^n}$, after the addition of the linear cut, a unique optimal solution exists, lying in int(H) and, consequently, in $int(\Omega_{\bar{\mathbf{x}}})$, since 438also the linear constraint in (3.9) is not active at it, being H a subset of the region AQ4 439defined by the linear cut. By continuity, for λ values smaller than but close to $\lambda_{\mathbb{R}^n}$, 440the unique optimal solution of (3.9) also lies in int(H), i.e., $P_{\Omega_{\bar{\mathbf{x}}}}(\lambda) = \{\mathbf{z}_1(\lambda)\}$ with 441 $\mathbf{z}_1(\lambda) \in int(H)$. Thus, such optimal solution must be a local and nonglobal optimal 442solution of the trust region problem (3.2) with $X = \mathbb{R}^n$. Indeed, the globally optimal 443solutions of this trust region problem always violate the second constraint in (1.1) for 444 all $\lambda < \lambda_{\mathbb{R}^n}$. Now, for all $\lambda \in [0, \lambda_{\mathbb{R}^n})$, we first check whether a local and nonglobal 445446optimal solution of problem (3.2) with $X = \mathbb{R}^n$ exists by exploiting the necessary and sufficient condition stated in [24]. Also recall that, if it exists, the local and nonglobal 447 minimizer is unique. If it does not exist, then we set $f_1 = +\infty$. Otherwise, if it exists, 448 we denote it by $\mathbf{z}_1(\lambda)$. If $\mathbf{z}_1(\lambda) \notin \Omega_{\bar{\mathbf{x}}}$, then we set again $f_1 = +\infty$, otherwise we denote 449by f_1 the value of the objective function of (3.9) evaluated at $\mathbf{z}_1(\lambda)$. If some globally 450451optimal solution of the trust region problem (3.2) with $X = \mathbb{R}^n$ belongs to $\Omega_{\bar{\mathbf{x}}} \setminus H$, then 452it is also a solution of (3.9) and we set f_2 equal to the optimal value of this problem. Note that in this case $f_2 < f_1$, since f_1 is the function value at a local and nonglobal 453solution of the trust region problem. Then, Algorithm 2.1 sets $\lambda^{\min} = \lambda$. Instead, 454if all globally optimal solutions of the trust region problem do not belong to $\Omega_{\bar{\mathbf{x}}}$, we 455456proceed as follows. We consider the best feasible solutions of problem (3.9) for which the linear constraint is imposed to be active. The resulting problem is converted into 457a trust region problem, after the change of variable $\mathbf{x} = \bar{\mathbf{x}} + \mathbf{V}\mathbf{z}$, where $\mathbf{V} \in \mathbb{R}^{n \times (n-1)}$ 458is a matrix whose columns form a basis for the null space of vector $2\mathbf{A}\mathbf{\bar{x}} + \mathbf{a}$. The 459resulting (trust region) problem is 460

(3.11)

$$\min_{\mathbf{w} \in \mathbb{R}^{n-1}} \quad \mathbf{w}^\top \mathbf{V}^\top (\mathbf{Q} + \lambda \mathbf{A}) \mathbf{V} \mathbf{w} + \left[2 \bar{\mathbf{x}}^\top (\mathbf{Q} + \lambda \mathbf{A}) \mathbf{V} + (\mathbf{q} + \lambda \mathbf{a})^\top \right] \mathbf{w} + \ell(\bar{\mathbf{x}}, \lambda) \\ \| \bar{\mathbf{x}} + \mathbf{V} \mathbf{w} \|^2 \le 1,$$

461 where $\ell(\bar{\mathbf{x}}, \lambda) = \bar{\mathbf{x}}^{\top} (\mathbf{Q} + \lambda \mathbf{A}) \bar{\mathbf{x}} + (\mathbf{q} + \lambda \mathbf{a})^{\top} \bar{\mathbf{x}} - \lambda a_0$ is constant with respect to the 462 vector of variables \mathbf{w} . Let $W^{\star}(\lambda)$ be the set of optimal solutions of problem (3.11) 463 and

$$P_1^{\star}(\lambda) = \{ \bar{\mathbf{x}} + \mathbf{V}\mathbf{w}^{\star} : \mathbf{w}^{\star} \in W^{\star}(\lambda) \}.$$

464 Note that the set $W^{\star}(\lambda)$ can be computed through the procedure presented in section 465 3 with the different cases (namely, Cases 1, 2.1, 2.2, 2.3.1, 2.3.2) after rewriting it as a 466 classical trust region problem. Moreover, let $f_2 < +\infty$ be the optimal value of problem 467 (3.11). Now, after comparing f_1 and f_2 , we are able to define the set $P_{\Omega_{\bar{\mathbf{x}}}}(\lambda)$ of optimal 468 solutions for problem (3.9). More precisely, if $f_2 > f_1$, then $P_{\Omega_{\bar{\mathbf{x}}}}(\lambda) = \{\mathbf{z}_1(\lambda)\}$, i.e., 469 $\mathbf{z}_1(\lambda)$ is the unique optimal solution of problem (3.9). In this case

$$h_{\Omega_{\bar{\mathbf{z}}}}^{\min}(\lambda) = h_{\Omega_{\bar{\mathbf{z}}}}^{\max}(\lambda) = \mathbf{z}_1(\lambda)^{\top} \mathbf{A} \mathbf{z}_1(\lambda) + \mathbf{a}^{\top} \mathbf{z}_1(\lambda) - a_0$$

470 Instead, if $f_2 < f_1$, which always holds, e.g., if $f_1 = +\infty$, then $P_{\Omega_{\bar{\mathbf{x}}}}(\lambda) = P_1^{\star}(\lambda)$. Since 471 all points in $P_1^{\star}(\lambda)$ lie over a supporting hyperplane of H, we must have that

$$h_{\Omega_{\bar{\mathbf{x}}}}^{\min}(\lambda) = \min_{\mathbf{x} \in P_{1}^{\star}(\lambda)} \mathbf{x}^{\top} \mathbf{A} \mathbf{x} + \mathbf{a}^{\top} \mathbf{x} - a_{0} \ge 0$$

472 and equality holds only if $\bar{\mathbf{x}} \in P_1^{\star}(\lambda)$. In the latter case, the bound is exact, otherwise 473 Algorithm 2.1 sets $\lambda^{\min} = \lambda$. Finally, if $f_1 = f_2$, then $P_{\Omega_{\bar{\mathbf{x}}}}(\lambda) = P_1^{\star}(\lambda) \cup \{\mathbf{z}_1(\lambda)\}$ and 474 in this case $0 \in [h_{\Omega_{\bar{\mathbf{x}}}}^{\min}(\lambda), h_{\Omega_{\bar{\mathbf{x}}}}^{\max}(\lambda)]$ and the algorithms exits the loop. The following 475 result is a straightforward consequence of Proposition 2.12.

476 PROPOSITION 3.5. Algorithm 2.1 with $\varepsilon = 0$ will stop after a finite number of 477 iterations or will converge to some $\lambda_{\Omega_{\bar{x}}} < \lambda_{\mathbb{R}^n}$ with a new lower bound $\bar{p}_{\Omega_{\bar{x}}} > \bar{p}_{\mathbb{R}^n}$.

478 Proof. Strict inequalities hold in view of Proposition 2.12 with $X = \mathbb{R}^n$ and 479 $Y = \Omega_{\bar{\mathbf{x}}}$, since, as already observed, $\Omega_{\bar{\mathbf{x}}} \cap (P_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}) \setminus H) = \emptyset$.

480 If the final bound is not exact, i.e., $\bar{p}_{\Omega_{\bar{\mathbf{x}}}} = p_{\Omega_{\bar{\mathbf{x}}}}(\lambda_{\Omega_{\bar{\mathbf{x}}}}) < p^*$, at $\lambda_{\Omega_{\bar{\mathbf{x}}}}$ we have $f_1 = f_2$ and 481 $P_{\Omega_{\bar{\mathbf{x}}}}(\lambda_{\Omega_{\bar{\mathbf{x}}}})$ contains multiple optimal solutions, in particular, one in int(H) and the 482 other(s) outside H, more precisely on $\partial\Omega_{\bar{\mathbf{x}}} \setminus H$. We illustrate all this on Example 3.3.

484*Example* 3.6. The optimal solution of (3.2) with $X = \mathbb{R}^n$ for $\lambda_{\mathbb{R}^n} = 1$ which violates the second constraint is $\mathbf{z}_2(\lambda_{\mathbb{R}^n}) = (-0.911, 0.4114)$. The lower bound is 485 $p_{\mathbb{R}^n}(1) = -4.25$. After the addition of the linear inequality (3.7) obtained with $\bar{\mathbf{x}} =$ 486 $\Pi_{\mathbf{A},\mathbf{a}}(\mathbf{z}_2(\lambda_{\mathbb{R}^n}))$, equal to the projection of $\mathbf{z}_2(\lambda_{\mathbb{R}^n})$ over the boundary of the second 487 constraint, we can run again Algorithm 2.1 with $X = \Omega_{\bar{\mathbf{x}}}$ and we get to $\lambda_{\Omega_{\bar{\mathbf{x}}}} \approx 0.726$ 488489and $p_{\Omega_{\star}}(\lambda_{\Omega_{\star}}) \approx -4.097$, which improves the previous lower bound. In Figure 2 we show the linear cut and the two new optimal solutions outside H and in int(H) (x₂ 490and z_2 , respectively) obtained at $\lambda_{\Omega_{\bar{\mathbf{x}}}}$. In the same figure we also display the previous 491pair of optimal solutions in order to show the progress of the algorithm. 492

It is worthwhile to discuss at this point the relations between the approach proposed in this work and the one proposed in [27], where the classical SDP relaxation



FIG. 2. First linear cut and the two optimal solutions lying outside $H(x_2)$ and in $int(H)(z_2)$, denoted by \circ and \times , respectively.

495of problem (1.1) is considered. Both approaches stem from the necessary and sufficient condition under which the dual Lagrangian bound is not exact discussed in 496[2], namely the existence of two distinct optimal solutions, one belonging to int(H)497and the other outside H. In both cases it is observed that, in order to improve the 498bound, it is necessary to separate such optimal solutions. But the way the separation 499is carried on in the two approaches is different. Following the terminology employed 500 in Integer Programming, in [27] the separation is performed through a branching op-501eration, while in this work it is performed through the addition of a cutting plane. 502Indeed, in [27] first, a hyperplane $\mathbf{w}^T \mathbf{x} = v$ separating the two optimal solutions is 503introduced; then, two distinct subproblems are solved, one by adding the inequality 504 $\mathbf{w}^T \mathbf{x} \leq v$ (converted into an SOCP constraint) to the SDP relaxation, the other by 505adding the inequality $\mathbf{w}^T \mathbf{x} \geq v$ to the SDP relaxation; finally, the new bound is set 506equal to the minimum of the bounds over the two subregions into which the original 507feasible region has been split. Note that one of the two subregions may be empty, in 508509which case its corresponding lower bound is set equal to $+\infty$ and the linear inequality is a separating hyperplane between H and the optimal solution outside H. In this 510511paper the separation is performed through the addition of a linear cut and a single subproblem is solved. Moreover, in [27] it is observed that one could search for an 512"optimal" hyperplane separating the two optimal solutions, namely one which leads 513to the best possible bound. In that paper such a hyperplane is derived in the special 514case when the function h is the product of two affine functions and an exactness result 515516is also provided for the case of problems with two variables, but the question about 517how to characterize an 'optimal' affine function is left open in the general case. In the next section we will be able to provide a necessary and sufficient condition for a 518linear cut to be the one delivering the best bound (Proposition 4.2). Based on this 519condition, we will also be able to propose a procedure to improve the bound by local 520521adjustments of the linear cut. Finally, in this paper we will also show in section 5 522that the bound can be further improved through the addition of a second linear cut, 523possibly followed by a local adjustment of the two linear cuts. The experiments in 524section 6 will show that the bound obtained by the addition of two linear cuts is quite a good one, allowing one to solve all except one of the 212 hardest instances intro-525526duced in [12]. As a final remark, we observe that the approach presented in [27] and 527the one discussed in this paper could actually be combined by performing a branching 528operation (as in [27]) followed by the addition of a linear cut (as in this work) in each branch. Borrowing again from the terminology of Integer Programming, this can be 529viewed as a branch-and-cut approach. 530

4. Improving the bound by local adjustments of the linear cut. In the previous section we proposed to set $\bar{\mathbf{x}}$ equal to the projection over ∂H of $\mathbf{z}_2(\lambda_{\mathbb{R}^n})$, the optimal solution of problem (3.2) with $X = \mathbb{R}^n$ lying outside H. However, this point can be improved by some local adjustment. We first give a necessary and sufficient condition under which the current point $\bar{\mathbf{x}}$ cannot be improved. The proof will also suggest how to improve the point (and the bound) when the condition is not fulfilled. Let

(4.1)
$$r(\mathbf{w}, \mathbf{x}) = (2\mathbf{A}\mathbf{w} + \mathbf{a})^{\top}(\mathbf{x} - \mathbf{w}) + \mathbf{w}^{\top}\mathbf{A}\mathbf{w} + \mathbf{a}^{\top}\mathbf{w} - a_0 = (2\mathbf{A}\mathbf{w} + \mathbf{a})^{\top}\mathbf{x} - \mathbf{w}^{\top}\mathbf{A}\mathbf{w} - a_0$$

be the linearization of the ellipsoid constraint at \mathbf{w} . Note that constraint (3.7) can be written as $r(\bar{\mathbf{x}}, \mathbf{x}) \leq 0$. Also note that for each \mathbf{x} , r is a concave function with respect to \mathbf{w} . Next, we set

$$p(\lambda, \mathbf{w}) = p_{\Omega_{\mathbf{w}}}(\lambda)$$

541 in order to highlight the dependency of the bound not only on λ but also on **w**. Then, 542 in order to maximize the lower bound, we need to solve the following problem:

$$\max_{\lambda \ge 0} p(\lambda, \mathbf{w}).$$

543 As before, we denote by $P_{\Omega_{\mathbf{w}}}(\lambda)$ the optimal set of problem (3.9) with $\bar{\mathbf{x}} = \mathbf{w}$, while we 544 denote by $P_{\Omega_{\mathbf{w}}}^1(\lambda) = P_{\Omega_{\mathbf{w}}}(\lambda) \setminus int(H)$ the set of optimal solutions of the same problem 545 lying outside the interior of the ellipsoid H. We will need the following lemma.

546 LEMMA 4.1. Set-valued functions $\mathbf{w} \sim P_{\Omega_{\mathbf{w}}}(\lambda)$ and $\mathbf{w} \sim P_{\Omega_{\mathbf{w}}}^1(\lambda)$ are upper semi-547 continuous for any $\lambda \geq 0$.

548 Proof. Upper semicontinuity of $\mathbf{w} \sim P_{\Omega_{\mathbf{w}}}(\lambda)$ follows from the maximum theorem 549 (see, for instance, Theorem 1.4.16 of [5]), while upper semicontinuity of $\mathbf{w} \sim P_{\Omega_{\mathbf{w}}}^1(\lambda)$ 550 follows from the fact that $P_{\Omega_{\mathbf{w}}}^1(\lambda)$ is obtained by intersecting the upper semicontinuous 551 function $\mathbf{w} \sim P_{\Omega_{\mathbf{w}}}(\lambda)$ with the compact set $\{\mathbf{x} : \|\mathbf{x}\| \leq 1\} \setminus int(H)$ (see, for instance, 552 Proposition 1.4.9 of [5]).

553 Now, the following proposition characterizes the maxima of p.

554 PROPOSITION 4.2. Let $(\lambda^*, \mathbf{w}^*)$ be such that $\mathbf{w}^* \in \partial H$, $\lambda^* > 0$, $0 \in (h_{\Omega_{\mathbf{w}^*}}^{\min}(\lambda^*))$, 555 $h_{\Omega_{\mathbf{w}^*}}^{\max}(\lambda^*))$, and $0 \notin Q_{\Omega_{\mathbf{w}^*}}(\lambda^*)$. Assume also that $(\forall \mathbf{v} \in P_{\Omega_{\mathbf{w}^*}}^1(\lambda^*)) r(\mathbf{w}^*, \mathbf{v}) = 0$. Then, 556 the following statements are equivalent:

(i)
$$(\lambda^*, \mathbf{w}^*) = \operatorname{argmax}_{(\lambda \ge 0, \mathbf{w})} p(\lambda, \mathbf{w}).$$

(ii)
$$(\forall \mathbf{w} \in \mathbb{R}^n) P^1_{\Omega_{\mathbf{w}}}(\lambda^*) \neq \emptyset$$

(iii)

559

557 558

(4.2)
$$(\forall \mathbf{d} \in \mathbb{R}^n) (\exists \mathbf{v} \in P^1_{\Omega_{m*}}(\lambda^*)) : -\mathbf{d}^\top \mathbf{A} \mathbf{d} + 2\mathbf{d}^\top \mathbf{A} (\mathbf{v} - \mathbf{w}^*) \le 0.$$

560 Proof. Before proving the result we make some remarks. First, note that $0 \in (h_{\Omega_{\mathbf{w}^*}}^{\min}(\lambda^*), h_{\Omega_{\mathbf{w}^*}}^{\max}(\lambda^*))$ implies that $\lambda^* = \lambda_{\Omega_{\mathbf{w}^*}}$. Moreover, since $0 \notin Q_{\Omega_{\mathbf{w}^*}}(\lambda^*)$ and $\lambda^* > 0$, by Propositions 2.3 and 2.5, $p_{\Omega_{\mathbf{w}^*}}(\lambda^*) < p^*$ (i.e., the bound is not exact). If the bound were exact, the current pair $(\mathbf{w}^*, \lambda^*)$ would obviously be optimal. Also note that $h_{\Omega_{\mathbf{w}^*}}^{\min}(\lambda^*) < 0 < h_{\Omega_{\mathbf{w}^*}}^{\max}(\lambda^*)$ implies that $P_{\Omega_{\mathbf{w}^*}}^1(\lambda^*) \neq \emptyset$. Finally, condition $(\forall \mathbf{v} \in P_{\Omega_{\mathbf{w}^*}}^1(\lambda^*)) r(\mathbf{w}^*, \mathbf{v}) = 0$ means that $P_{\Omega_{\mathbf{w}^*}}^1(\lambda^*) \subset \partial\Omega_{\mathbf{w}^*}$, which, according to the discussion about the optimal solutions of problem (3.9), holds true provided that 567 $\mathbf{z}_2(\lambda_{\mathbb{R}^n}) \notin \Omega_{\mathbf{w}^*}$.

 $\begin{array}{ll} \text{568} \qquad (\mathrm{i}) \to (\mathrm{ii}) \text{ By contradiction, let } \mathbf{w} \text{ be such that } P_{\Omega_{\mathbf{w}}}^{1}(\lambda^{*}) = \emptyset. \text{ Then, } P_{\Omega_{\mathbf{w}}}(\lambda^{*}) \subset int(H). \\ \text{569} \qquad \text{Therefore, function } \lambda \rightsquigarrow p_{\Omega_{\mathbf{w}}}(\lambda) \text{ is strictly decreasing at } \lambda^{*}. \text{ As a consequence,} \\ \text{570} \qquad \text{there exists } 0 \leq \bar{\lambda} < \lambda^{*} \text{ such that } p(\bar{\lambda}, \mathbf{w}) = p_{\Omega_{\mathbf{w}}}(\bar{\lambda}) > p_{\Omega_{\mathbf{w}}}(\lambda^{*}) = p(\lambda^{*}, \mathbf{w}). \text{ More-} \\ \text{571} \qquad \text{over, } p(\lambda^{*}, \mathbf{w}^{*}) = p(\lambda^{*}, \mathbf{w}). \text{ Indeed, since } 0 \in (h_{\Omega_{\mathbf{w}}^{*}}^{\min}(\lambda^{*}), h_{\Omega_{\mathbf{w}}^{*}}^{\max}(\lambda^{*})), \text{ we have that} \\ \text{572} \qquad h_{\Omega_{\mathbf{w}}^{*}}^{\min}(\lambda^{*}) < 0, \text{ so that } P_{\Omega_{\mathbf{w}}^{*}}(\lambda^{*}) \cap int(H) \text{ is not empty and is equal to } P_{\Omega_{\mathbf{w}}}(\lambda^{*}). \\ \text{573} \qquad \text{Hence, } p(\bar{\lambda}, \mathbf{w}) > p(\lambda^{*}, \mathbf{w}^{*}), \text{ which contradicts (i).} \end{array}$

574 (ii) \rightarrow (iii) By contradiction, let us assume that there exists $\mathbf{d} \in \mathbb{R}^n$ such that

(4.3)
$$(\forall \mathbf{v} \in P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*)) - \mathbf{d}^\top \mathbf{A} \mathbf{d} + 2\mathbf{d}^\top \mathbf{A}(\mathbf{v} - \mathbf{w}^*) > 0.$$

575 By continuity of the left-hand side of the inequality in (4.3) with respect to **v**, there 576 exists a neighborhood B_1 of $P^1_{\Omega_{\dots*}}(\lambda^*)$ such that

$$(\forall \mathbf{v} \in B_1) - \mathbf{d}^\top \mathbf{A} \mathbf{d} + 2\mathbf{d}^\top \mathbf{A} (\mathbf{v} - \mathbf{w}^*) > 0,$$

577 which implies that

(4.4)
$$(\forall \mathbf{v} \in B_1) \, 2\mathbf{d}^\top \mathbf{A} (\mathbf{v} - \mathbf{w}^*) > 0$$

578 By upper semicontinuity of set-valued function $\mathbf{w} \sim P_{\Omega_{\mathbf{w}}}^{1}(\lambda^{*})$ (see Lemma 4.1), 579 there exists a neighborhood B_{2} of \mathbf{w}^{*} such that $(\forall \mathbf{w} \in B_{2}) P_{\Omega_{\mathbf{w}}}^{1}(\lambda^{*}) \subset B_{1}$. Let $\bar{\eta} > 0$ 580 be such that $\mathbf{w}^{*} + \bar{\eta} \mathbf{d} \in B_{2}$ and consider function $\rho : [0, \bar{\eta}] \times \mathbb{R}^{n} \to \mathbb{R}, \ \rho(\eta, \mathbf{v}) =$ 581 $r(\mathbf{w}^{*} + \eta \mathbf{d}, \mathbf{v})$. Then, by definition of r in (4.1),

$$\rho(\eta, \mathbf{v}) = -\eta^2 \mathbf{d}^\top \mathbf{A} \mathbf{d} + 2\eta \mathbf{d}^\top \mathbf{A} (\mathbf{v} - \mathbf{w}^*) + \rho(0, \mathbf{v})$$

582 By (4.4), $(\forall \mathbf{v} \in B_1) \partial_\eta \rho(0, \mathbf{v}) > 0$, where ∂_η denotes the partial derivative with respect 583 to η . By continuity of ρ , there exists a continuous function $\hat{\eta} : B_1 \to (0, \bar{\eta}]$ such that

(4.5)
$$(\forall \mathbf{v} \in B_1, \eta \in (0, \hat{\eta}(\mathbf{v})]) \ \rho(\eta, \mathbf{v}) > \rho(\mathbf{0}, \mathbf{v})$$

Hence, since B_1 is a compact set and $\hat{\eta}$ is continuous and strictly positive, setting

585 $\tilde{\eta} = \min_{\mathbf{v} \in B_1} \hat{\eta}(\mathbf{v})$, it follows that $(\forall \eta \in [0, \tilde{\eta}]) P^1_{\Omega_{\mathbf{w}^* + \eta \mathbf{d}}}(\lambda^*) \subseteq P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*)$. Moreover, 586 since, by assumption, $(\forall \mathbf{v} \in P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*)) r(\mathbf{w}^*, \mathbf{v}) = 0$,

$$(\forall \mathbf{v} \in P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*)) \rho(0, \mathbf{v}) = 0.$$

587 Hence, (4.5) implies that

(4.6)
$$(\forall \mathbf{v} \in P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*)) \,\rho(\tilde{\eta}, \mathbf{v}) > 0.$$

588 Being $P^1_{\Omega_{\mathbf{w}^*+\tilde{\eta}\mathbf{d}}}(\lambda^*) \subseteq P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*)$, (4.6) implies that $P^1_{\Omega_{\mathbf{w}^*+\tilde{\eta}\mathbf{d}}}(\lambda^*) = \emptyset$, which contradicts 589 (ii).

 $\begin{array}{ll} 590 \qquad (\mathrm{iii}) \to (\mathrm{i}) \text{ By contradiction, there exists a couple } (\bar{\lambda}, \bar{\mathbf{w}}) \text{ such that } p(\lambda, \bar{\mathbf{w}}) > p(\lambda^*, \mathbf{w}^*). \\ 591 \qquad \mathrm{In \ particular, \ we \ can \ take \ \bar{\lambda} = \lambda_{\Omega_{\bar{\mathbf{w}}}}. \ \mathrm{In \ case \ \bar{\lambda}} = 0, \ \mathrm{by \ assumption \ \bar{\lambda}} < \lambda^*. \ \mathrm{Now \ we} \\ 592 \qquad \mathrm{show \ that \ the \ same \ inequality \ holds \ true \ also \ when \ \bar{\lambda} > 0. \ \mathrm{If \ \bar{\lambda}} > 0, \ \mathrm{then \ \bar{\lambda}} = \lambda_{\Omega_{\bar{\mathbf{w}}}} \\ 593 \qquad \mathrm{implies \ that \ } 0 \in [h_{\Omega_{\bar{\mathbf{w}}}}^{\min}(\bar{\lambda}), h_{\Omega_{\bar{\mathbf{w}}}}^{\max}(\bar{\lambda})]. \ \mathrm{Since, \ by \ assumption, \ } 0 \in (h_{\Omega_{\mathbf{w}}}^{\min}(\lambda^*), h_{\Omega_{\mathbf{w}}}^{\max}(\lambda^*)), \\ 594 \qquad \mathrm{we \ have \ that \ both \ } P_{\Omega_{\bar{\mathbf{w}}}}(\bar{\lambda}) \cap H \neq \emptyset \ \mathrm{and \ } P_{\Omega_{\mathbf{w}}*}(\lambda^*) \cap H \neq \emptyset \ \mathrm{(i.e., \ the \ minimum \ values} \\ 595 \qquad p(\lambda^*, \mathbf{w}^*) \ \mathrm{and} \ p(\bar{\lambda}, \bar{\mathbf{w}}) \ \mathrm{are \ both \ attained \ in \ } H). \ \mathrm{Hence, \ } p(\bar{\lambda}, \bar{\mathbf{w}}) > p(\lambda^*, \mathbf{w}^*) \ \mathrm{implies} \\ 596 \qquad \mathrm{that \ } \bar{\lambda} < \lambda^*. \ \mathrm{Indeed, \ let \ us \ assume \ that \ } \bar{\lambda} \geq \lambda^* \ \mathrm{and \ let \ } \mathbf{z} \in P_{\Omega_{\mathbf{w}}*}(\lambda^*) \cap H. \ \mathrm{Then}, \end{array}$

$$p(\lambda^*, \mathbf{w}^*) = \mathbf{z}^\top \mathbf{Q} \mathbf{z} + \mathbf{q}^\top \mathbf{z} + \lambda^* (\mathbf{z}^\top \mathbf{Q} \mathbf{z} + \mathbf{q}^\top \mathbf{z} - a_0)$$

$$\geq \mathbf{z}^\top \mathbf{Q} \mathbf{z} + \mathbf{q}^\top \mathbf{z} + \bar{\lambda} (\mathbf{z}^\top \mathbf{Q} \mathbf{z} + \mathbf{q}^\top \mathbf{z} - a_0) \geq p(\bar{\lambda}, \bar{\mathbf{w}}),$$

597 which is a contradiction. Then, function $p(\lambda^*, \bar{\mathbf{w}})$ must be decreasing at λ^* or, equiv-598 alently, $P_{\Omega_{\bar{\mathbf{w}}}}(\lambda^*) \subset int(H)$ and $P_{\Omega_{\bar{\mathbf{w}}}}(\lambda^*) \neq \emptyset$. Since $h_{\Omega_{\bar{\mathbf{w}}^*}}^{\min}(\lambda^*) < 0$, then $P_{\Omega_{\bar{\mathbf{w}}}}(\lambda^*)$ 599 and $P_{\Omega_{\mathbf{w}^*}}(\lambda^*)$ have a common nonempty intersection within H and, consequently, 600 $p(\lambda^*, \bar{\mathbf{w}}) = p(\lambda^*, \mathbf{w}^*)$ holds. This implies that $P_{\Omega_{\mathbf{w}^*}}^1(\lambda^*) \cap \Omega_{\bar{\mathbf{w}}} = \emptyset$. Indeed, assume 601 there exists $\mathbf{v} \in P_{\Omega_{\mathbf{w}^*}}^1(\lambda^*) \cap \Omega_{\bar{\mathbf{w}}}$. Note that $\mathbf{v} \notin int(H)$ and, since $P_{\Omega_{\bar{\mathbf{w}}}}^1(\lambda^*) \subset P_{\Omega_{\bar{\mathbf{w}}}}(\lambda^*)$, 602 \mathbf{v} would also belong to $P_{\Omega_{\bar{\mathbf{w}}}}(\lambda^*)$ which, however, contradicts $P_{\Omega_{\bar{\mathbf{w}}}}(\lambda^*) \subset int(H)$.

603 Condition $P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*) \cap \Omega_{\bar{\mathbf{w}}} = \emptyset$ is equivalent to

$$(\forall \mathbf{v} \in P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*)) r(\bar{\mathbf{w}}, \mathbf{v}) > 0.$$

604 Note that, by assumption, $\mathbf{v} \in P^1_{\Omega_{\mathbf{w}^*}}(\lambda^*)$ implies $\mathbf{v} \in \partial \Omega_{\mathbf{w}^*}$. Moreover,

$$r(\bar{\mathbf{w}}, \mathbf{v}) = r((\bar{\mathbf{w}} - \mathbf{w}^*) + \mathbf{w}^*, \mathbf{v}) = -(\bar{\mathbf{w}} - \mathbf{w}^*)^\top \mathbf{A}(\bar{\mathbf{w}} - \mathbf{w}^*) + 2(\bar{\mathbf{w}} - \mathbf{w}^*)^\top \mathbf{A}(\mathbf{v} - \mathbf{w}^*) + r(\mathbf{w}^*, \mathbf{v}) > 0.$$

- 605 Being $\mathbf{w}^* \in \partial H$ and $\mathbf{v} \in \partial \Omega_{\mathbf{w}^*}$, we have that $r(\mathbf{w}^*, \mathbf{v}) = 0$. Then, by taking $\mathbf{d} = \bar{\mathbf{w}} \mathbf{w}^*$, 606 (iii) is contradicted.
- 607 Given the current point $\bar{\mathbf{x}}$ with $\lambda_{\Omega_{\bar{\mathbf{x}}}} > 0$, the question now is either to find a direction 608 **d** fulfilling

(4.7)
$$(\forall \mathbf{v} \in P_1^{\star}(\lambda_{\Omega_{\bar{\mathbf{x}}}})) - \mathbf{d}^{\top} \mathbf{A} \mathbf{d} + 2\mathbf{d}^{\top} \mathbf{A} (\mathbf{v} - \bar{\mathbf{x}}) > 0$$

609 or to establish that it does not exist. In case it does not exist,

$$p(\bar{\mathbf{x}}, \lambda_{\Omega_{\bar{\mathbf{x}}}}) = \max_{\lambda \ge 0, \mathbf{w}} p(\lambda, \mathbf{w}).$$

610 Otherwise, direction $(\mathbf{d}, -1)$ is an increasing direction for function p. We discuss 611 different cases depending on the cardinality of $P_1^{\star}(\lambda_{\Omega_{\bar{\mathbf{x}}}})$ (see the cases discussed in 612 section 3 for the trust region problem).

613 **4.1.** $|P_1^{\star}(\lambda_{\Omega_{\bar{x}}})| = 1$. In this case, let v be the unique point in $P_1^{\star}(\lambda_{\Omega_{\bar{x}}})$. Then 614 we need to solve the following convex optimization problem:

$$\max_{\mathbf{d}\in\mathbb{R}^n} -\mathbf{d}^\top \mathbf{A} \mathbf{d} + 2\mathbf{d}^\top \mathbf{A} (\mathbf{v} - \bar{\mathbf{x}}),$$

615 whose optimal solution is $\mathbf{d} = \mathbf{v} - \bar{\mathbf{x}}$ and its optimal value is $(\mathbf{v} - \bar{\mathbf{x}})^{\top} \mathbf{A} (\mathbf{v} - \bar{\mathbf{x}}) > 0$.

616 Therefore, if $|P_1^{\star}(\lambda_{\Omega_{\bar{\mathbf{x}}}})| = 1$, we are always able to locally adjust the current point $\bar{\mathbf{x}}$ 617 in such a way that the bound can be improved.

618 **4.2.** $|P_1^{\star}(\lambda_{\Omega_{\bar{x}}})| = 2$. In this case, let v_1 and v_2 be the two optimal points in 619 $P_1^{\star}(\lambda_{\Omega_{\bar{x}}})$. Then, we need to solve the following optimization problem:

(4.8)
$$\max_{\boldsymbol{d}\in\mathbb{R}^n} \min\{-\boldsymbol{d}^{\top}\boldsymbol{A}\boldsymbol{d} + 2\boldsymbol{d}^{\top}\boldsymbol{A}(\boldsymbol{v}_1-\bar{\boldsymbol{x}}), -\boldsymbol{d}^{\top}\boldsymbol{A}\boldsymbol{d} + 2\boldsymbol{d}^{\top}\boldsymbol{A}(\boldsymbol{v}_2-\bar{\boldsymbol{x}})\},\$$

620 or, equivalently

$$\begin{array}{ll} \max & v \\ & v \leq -\boldsymbol{d}^{\top}\boldsymbol{A}\boldsymbol{d} + 2\boldsymbol{d}^{\top}\boldsymbol{A}(\boldsymbol{v}_{1} - \bar{\boldsymbol{x}}) \\ & v \leq -\boldsymbol{d}^{\top}\boldsymbol{A}\boldsymbol{d} + 2\boldsymbol{d}^{\top}\boldsymbol{A}(\boldsymbol{v}_{2} - \bar{\boldsymbol{x}}) \end{array}$$

621 This is a convex optimization problem, whose solution can be obtained in closed form.

Indeed, by imposing the KKT conditions, it can be seen that the optimal solution hasthe following form:

(4.9)
$$\mathbf{d} = \beta(\mathbf{v}_1 - \bar{\mathbf{x}}) + (1 - \beta)(\mathbf{v}_2 - \bar{\mathbf{x}}), \quad \beta \in [0, 1].$$

624 Now, let

$$a = (\mathbf{v}_1 - \bar{\mathbf{x}})^\top \mathbf{A}(\mathbf{v}_1 - \bar{\mathbf{x}}) > 0,$$

$$b = (\mathbf{v}_2 - \bar{\mathbf{x}})^\top \mathbf{A}(\mathbf{v}_2 - \bar{\mathbf{x}}) > 0,$$

$$c = (\mathbf{v}_1 - \bar{\mathbf{x}})^\top \mathbf{A}(\mathbf{v}_2 - \bar{\mathbf{x}}).$$

By replacing (4.9) in the objective function of (4.8), we have that (4.8) can be rewritten as

$$\max_{\beta \in [0,1]} \min \left\{ (-\beta^2 + 2\beta)a - (1-\beta)^2 b + 2(1-\beta)^2 c, -\beta^2 a + (1-\beta^2)b + 2\beta^2 c \right\}.$$

627 The optimal solution of this problem is

$$\beta^{\star} = \begin{cases} 0 & \text{if } b \leq c, \\ 1 & \text{if } a \leq c, \\ \frac{b-c}{a+b-2c} & \text{otherwise.} \end{cases}$$

628 Then, the optimal value is

$$\begin{cases} b & \text{if } b \leq c, \\ a & \text{if } a \leq c, \\ \frac{ab-c^2}{a+b-2c} & \text{otherwise.} \end{cases}$$

We notice that a, b > 0, 629

$$a+b-2c = (\mathbf{v}_1 - \mathbf{v}_2)^\top \mathbf{A} (\mathbf{v}_1 - \mathbf{v}_2) > 0,$$

630

and, by the Cauchy-Schwarz inequality,

$$ab - 2c^2 \ge 0,$$

and equality holds if and only if $(\mathbf{v}_1 - \bar{\mathbf{x}})$ and $(\mathbf{v}_2 - \bar{\mathbf{x}})$ are linearly dependent. Thus, 631632 the optimal value of (4.8) is always strictly positive unless the two vectors $(\mathbf{v}_1 - \bar{\mathbf{x}})$ 633 and $(\mathbf{v}_2 - \bar{\mathbf{x}})$ lie along the same direction. More precisely, the optimal value is null only if the two vectors have the same direction but opposite sign. Indeed, let 634

$$\mathbf{v}_1 - \bar{\mathbf{x}} = \gamma (\mathbf{v}_2 - \bar{\mathbf{x}}).$$

Then, we have $b = \gamma^2 a$ and $c = \gamma a$. If γ is positive, then either $b \leq c$ (if $\gamma \leq 1$), or $a \leq c$ 635 (if $\gamma \geq 1$) occurs, so that the optimal value is equal to a or b and is, thus, positive. If 636 $(\mathbf{v}_1 - \bar{\mathbf{x}})$ is not a negative multiple of $(\mathbf{v}_2 - \bar{\mathbf{x}})$, we are able to locally adjust $\bar{\mathbf{x}}$ along 637 direction 638

$$\mathbf{d} = \beta^{\star} (\mathbf{v}_1 - \bar{\mathbf{x}}) + (1 - \beta^{\star}) (\mathbf{v}_2 - \bar{\mathbf{x}}).$$

4.3. $P_1^{\star}(\lambda_{\Omega_{\bar{x}}})$ is an infinite connected set. In this case we need to solve the 639 640 following optimization problem:

(4.10)
$$\max_{\mathbf{d}\in\mathbb{R}^n}\min_{\mathbf{v}\in P_1^{\star}(\lambda_{\Omega_{\bar{\mathbf{x}}}})} -\mathbf{d}^{\top}\mathbf{A}\mathbf{d} + 2\mathbf{d}^{\top}\mathbf{A}(\mathbf{v}-\bar{\mathbf{x}}).$$

641 An improving direction exists if and only if the optimal value of this problem is strictly positive (note that the optimal value is always nonnegative since the inner 642643 minimization problem has optimal value 0 for $\mathbf{d} = \mathbf{0}$). We first remark that the problem is convex. Indeed, for each fixed \mathbf{v} , we have a concave function with respect 644 to \mathbf{d} , and the minimum of an infinite set of concave functions is itself a concave 645 646 function (to be maximized, so that the problem is convex). The inner minimization problem can be solved in closed form. After removing the terms which do not depend 647 on \mathbf{v} , the inner problem to be solved is 648

$$\min_{\mathbf{v}\in P_1^\star(\lambda_{\Omega_{\bar{\mathbf{x}}}})} 2\mathbf{d}^\top \mathbf{A}\mathbf{v}$$

According to Case 2.3.2 in section 3, $P_1^{\star}(\lambda_{\Omega_{\bar{\mathbf{x}}}})$ can be written as in (3.3) and the 649 650minimization problem can be reduced to the computation of the minimum of a linear function over the unit sphere: 651

$$\min_{\boldsymbol{\xi} \in \mathbb{R}^q : \|\boldsymbol{\xi}\|^2 = 1} \bar{\mathbf{c}}(\mathbf{d})^\top \boldsymbol{\xi}$$

652	where $\bar{\mathbf{c}}(\mathbf{d})$ is some linear function of \mathbf{d} and $q \ge 2$ is the multiplicity of the minimum
653	eigenvalue of the matrix $\mathbf{V}^{\top}(\mathbf{Q}+\lambda\mathbf{A})\mathbf{V}$, corresponding to the Hessian of the objective
654	function of problem (3.11) . The optimal solution of this problem is

$$\boldsymbol{\xi}^{\star} = -\frac{\bar{\mathbf{c}}(\mathbf{d})}{\|\bar{\mathbf{c}}(\mathbf{d})\|},$$

655 while the optimal value is $-\|\bar{\mathbf{c}}(\mathbf{d})\|$.

4.4. An algorithm for the refinement of the bound. Let $\bar{\mathbf{x}}$ and $\lambda_{\mathbb{R}^n}$ be 656defined as in section 3. We propose Algorithm 4.1 for a bound based on successive 657local adjustments of the linear cut. In line 2, Algorithm 2.1 is run with input $X = \Omega_{\bar{\mathbf{x}}}$ 658 and $\lambda_{\mathbb{R}^n}$. Note that with a slight abuse here we are assuming that the algorithm 659returns $\lambda_{\Omega_{\mathbf{x}}}$ and the related points \mathbf{z}_1 and \mathbf{z}_2 , while in practice close approximations of these quantities are returned, namely λ^{\max} , $\mathbf{z}_1(\lambda^{\max})$, and $\mathbf{z}_2(\lambda^{\min})$. In line 3, \mathbf{z} is 660 661initialized with the input point $\bar{\mathbf{x}}$ itself and the direction \mathbf{d}^{\star} , following the discussion 662 in section 4.1, is set equal to the difference between $\mathbf{z}_2(\lambda_{\Omega_{\bar{\boldsymbol{x}}}})$, the point outside H 663 returned by Algorithm 2.1, and $\bar{\mathbf{x}}$. The outer while loop of the algorithm (lines 4– 664 66520) is repeated until the bound is improved by at least a tolerance value tol. Inside 666 this loop, in line 5 the initial step size $\eta = 1$ is set and a new incumbent $\mathbf{y} \in \partial H$ is computed. The inner while loop (lines 7–15) computes the step size: until the 667 optimal value of problem (3.9) with $\bar{\mathbf{x}} = \mathbf{y}$ and $\lambda = \lambda_{\Omega_{\mathbf{x}}}$, denoted by *opt*, is lower 668 than the current lower bound Lb, we need to decrease the step size and recompute a 669 670 new incumbent y (lines 10-11). If the step size falls below a given tolerance value, 671we exit the inner loop and also the outer one. Otherwise, we have identified a new valid incumbent and we set to 1 the exit flag stop for the inner loop (line 13), so 672that, later on, a new linear inequality (3.7) with $\bar{\mathbf{x}} = \mathbf{y}$ will be computed. Then, at 673 line 17 we run Algorithm 2.1 with input $X = \Omega_{\mathbf{y}}$ and $\lambda_{\Omega_{\mathbf{z}}}$. Finally, in line 18, we 674update point \mathbf{z} and the direction \mathbf{d}^* . We remark that at each iteration $\mathbf{z}_2(\lambda_{\Omega_{\mathbf{z}}})$ is 675 one optimal solution of the current subproblem (3.9) with $\lambda = \lambda_{\Omega_z}$ lying outside H 676 677 and at which the linear cut of the subproblem is active, i.e., $\mathbf{z}_2(\lambda_{\Omega_z}) \in P_1^*(\lambda_{\Omega_z})$. As seen in section 4.1, if $|P_1^{\star}(\lambda_{\Omega_z})| = 1$, i.e., $\mathbf{z}_2(\lambda_{\Omega_z})$ is the unique optimal solution of the 678current subproblem (3.9) with $\lambda = \lambda_{\Omega_z}$ lying outside H, then, in view of Proposition 679 2.11, the local adjustment employed in Algorithm 4.1 is guaranteed to improve the 680 681 bound. However, as seen in sections 4.2 and 4.3, if $P_1^{\star}(\lambda_{\Omega_z})$ contains more than one point, than the proposed local adjustment is not guaranteed to improve the bound. 682683 Sections 4.2 and 4.3 suggest how to define perturbing directions which still allow one to improve the bound, in case they exist. However, as we will see through the 684 computational experiments, Algorithm 4.1 turns out to be time-consuming, and it 685686is more convenient to improve the bound by adding a further linear cut, as we do in section 5, rather than further locally adjusting the current linear cut. In order to 687 clarify this point, we can make a comparison with Integer Linear Programming (ILP). 688 In ILP problems, once a linear relaxation is solved, a valid cut removes one optimal 689 solution of the relaxation. If the optimal solution is unique, then after the addition 690 of the valid cut, the bound improves. But if the linear relaxation has got multiple 691solutions, then the valid cut is not guaranteed to remove all of them and, thus, the 692 bound may not improve. It is possible to try to strengthen the valid cut in such a way

 $\label{eq:algorithm} \textbf{Algorithm 4.1} \ \text{Bound improvement through a local adjustment of the linear cut.}$

Input: $\bar{\mathbf{x}}, \lambda_{\mathbb{R}^n}$ 1: Set $Lb_{old} = -\infty$ 2: Let $[Lb, \lambda_{\Omega_{\bar{\mathbf{x}}}}, \mathbf{z}_1(\lambda_{\Omega_{\bar{\mathbf{x}}}}), \mathbf{z}_2(\lambda_{\Omega_{\bar{\mathbf{x}}}})] = \mathbf{DualLagrangian}(\Omega_{\bar{\mathbf{x}}}, \lambda_{\mathbb{R}^n})$ Set $\mathbf{z} = \bar{\mathbf{x}}$ and $\mathbf{d}^{\star} = \mathbf{z}_2(\lambda_{\Omega_{\bar{\mathbf{x}}}}) - \bar{\mathbf{x}}$ 3: 4: while $Lb - Lb_{old} > tol$ do Set $Lb_{old} = Lb$, $\eta = 1$ and $\mathbf{y} = \Pi_{\mathbf{A},\mathbf{a}}(\mathbf{z} + \mathbf{d}^{\star}) \in \partial H$ 5:6: Set stop = 07: while stop = 0 and $\eta > \varepsilon$ do 8: Solve problem (3.9) with $\bar{\mathbf{x}} = \mathbf{y}$ and $\lambda = \lambda_{\Omega_{\mathbf{z}}}$, and let *opt* be its optimal value 9: if opt < Lb then 10: Set $\eta = \eta/2$ Set $\mathbf{y} = \Pi_{\mathbf{A},\mathbf{a}}(\mathbf{z} + \eta \mathbf{d}^{\star}) \in \partial H$ 11: 12:else 13:Set stop = 1end if 14: 15:end while 16:if stop = 1 then 17:Let $[Lb, \lambda_{\Omega_{\mathbf{y}}}, \mathbf{z}_1(\lambda_{\Omega_{\mathbf{y}}}), \mathbf{z}_2(\lambda_{\Omega_{\mathbf{y}}})] = \mathbf{DualLagrangian}(\Omega_{\mathbf{y}}, \lambda_{\Omega_{\mathbf{z}}})$ 18:Set $\mathbf{z} = \mathbf{y}, \mathbf{d}^{\star} = \mathbf{z}_2(\lambda_{\Omega_{\mathbf{y}}}) - \mathbf{z}$ 19:end if 20:end while 21: return Lb

- that all optimal solutions of the linear relaxations are removed. But, more commonly,new linear cuts are added.
- 695 Now we apply Algorithm 4.1 to our example.

696 Example 4.3. We have that \mathbf{z} is initialized with (-0.7901, 0.3565) and Lb with 697 -4.0971. During the execution of Algorithm 4.1, \mathbf{z} and Lb are updated as indicated 698 in Table 1.

699 Interestingly, the best bound obtained in the example is exactly the one obtained 700 for the same problem by the approach proposed in [12], based on the addition of 701 SOC-RLT constraints. Figure 3 displays the situation at the last iteration of Algorithm 702 4.1. Problem (3.9) has three optimal solutions, one in int(H) and two outside H. The

Iteration	\mathbf{Z}	Lb
1	(-0.7204, 0.6658)	-4.0850
2	(-0.7742, 0.4493)	-4.0638
3	(-0.7481, 0.5665)	-4.0477

(-0.7607, 0.5136)

(-0.7556, 0.5361)

(-0.7571, 0.5296)

(-0.7568, 0.5309)

45

6

7

-4.0416

-4.0378

-4.0364

-4.0362

TABLE 1Iterations of Algorithm 4.1 over the example.



FIG. 3. Final linear cut after running Algorithm 4.1. Problem (3.9) has three optimal solutions, one in int(H) and two outside H. The latter solutions are opposite to each other with respect to the final vector \mathbf{z} .

703two optimal solutions outside H are opposite to each other with respect to the final704vector \mathbf{z} , so that, as discussed in section 4.2, no further local adjustment is possible705to improve the bound in this case.

706 5. Bound improvement through the addition of a further linear cut. 707 Another possible way to improve the bound is by adding a further linear cut to (3.9). Let $\bar{\mathbf{x}}$ and $\lambda_{\mathbb{R}^n}$ be defined as in section 3. In line 2 of Algorithm 4.1, we compute 708 $[Lb, \lambda_{\Omega_{\bar{\mathbf{x}}}}, \mathbf{z}_1(\lambda_{\Omega_{\bar{\mathbf{x}}}}), \mathbf{z}_2(\lambda_{\Omega_{\bar{\mathbf{x}}}})] = \mathbf{DualLagrangian}(\Omega_{\bar{\mathbf{x}}}, \lambda_{\mathbb{R}^n}), \text{ and, later on, we try to}$ 709 locally adjust $\bar{\mathbf{x}}$. Rather than doing that, we can add a further linear cut, cutting 710711 $\mathbf{z}_2(\lambda_{\Omega_{\bar{\mathbf{x}}}}) \notin H$ away. In particular, we add the one obtained through the projection over ∂H of $\mathbf{z}_2(\lambda_{\Omega_{\bar{\mathbf{x}}}})$. Let $\tilde{\mathbf{x}} = \prod_{\mathbf{A},\mathbf{a}}(\mathbf{z}_2(\lambda_{\Omega_{\bar{\mathbf{x}}}})) \in \partial H$ be such projection. Then, we define the 712following problem: 713

(5.1)
$$\min_{\mathbf{x}} \quad \mathbf{x}^{\top} (\mathbf{Q} + \lambda \mathbf{A}) \mathbf{x} + (\mathbf{q} + \lambda \mathbf{a})^{\top} \mathbf{x} - \lambda a_{0} \\ \mathbf{x}^{\top} \mathbf{x} \leq 1 \\ (2\mathbf{A}\bar{\mathbf{x}} + \mathbf{a})^{\top} (\mathbf{x} - \bar{\mathbf{x}}) \leq 0 \\ (2\mathbf{A}\tilde{\mathbf{x}} + \mathbf{a})^{\top} (\mathbf{x} - \tilde{\mathbf{x}}) \leq 0, \end{cases}$$

714 which is equivalent to problem (3.2) where

$$X = \Omega_{\bar{\mathbf{x}}} \cap \Omega_{\tilde{\mathbf{x}}} = \{ \mathbf{x} : (2\mathbf{A}\bar{\mathbf{x}} + \mathbf{a})^{\top} (\mathbf{x} - \bar{\mathbf{x}}) \le 0, \ (2\mathbf{A}\tilde{\mathbf{x}} + \mathbf{a})^{\top} (\mathbf{x} - \tilde{\mathbf{x}}) \le 0 \} \supset H.$$

715 A convex reformulation as the one proposed in [12, 23] for problem (3.9) is not available 716 in this case (unless the two linear inequalities do not intersect in the interior of the 717 unit ball). But in this case the alternative procedure discussed in section 3 turns out 718 to be useful. As before, for each value λ in the while loop of Algorithm 2.1 we can first 719 check whether a local and nonglobal optimal solution of problem (3.2) with $X = \mathbb{R}^n$ 720 exists, by exploiting the necessary and sufficient condition stated in [24]. If it exists, 721 and belongs to $\Omega_{\tilde{\mathbf{x}}} \cap \Omega_{\tilde{\mathbf{x}}}$, we denote it by $\mathbf{z}_1(\lambda)$. Next, we need to compute the optimal 722 value of (5.1) when at least one of the two linear constraints is active, i.e., we need to 723 solve the following problem:

(5.2)

$$\min_{\mathbf{x}} \quad \mathbf{x}^{\top} (\mathbf{Q} + \lambda \mathbf{A}) \mathbf{x} + (\mathbf{q} + \lambda \mathbf{a})^{\top} \mathbf{x} - \lambda a_{0} \\
\mathbf{x}^{\top} \mathbf{x} \leq 1 \\
(2\mathbf{A}\bar{\mathbf{x}} + \mathbf{a})^{\top} (\mathbf{x} - \bar{\mathbf{x}}) \leq 0 \\
(2\mathbf{A}\tilde{\mathbf{x}} + \mathbf{a})^{\top} (\mathbf{x} - \tilde{\mathbf{x}}) \leq 0 \\
[(2\mathbf{A}\bar{\mathbf{x}} + \mathbf{a})^{\top} (\mathbf{x} - \bar{\mathbf{x}})] [(2\mathbf{A}\tilde{\mathbf{x}} + \mathbf{a})^{\top} (\mathbf{x} - \tilde{\mathbf{x}})] = 0$$

724 A convex reformulation of this problem has been proposed in [26]. Alternatively, one 725can solve two distinct problems, each imposing that one of the two linear inequalities is active. Each of these problems can be converted into a trust region problem with 726an additional linear inequality, which can be solved in polynomial time through the 727 already mentioned convex reformulation proposed in [12, 23]. Thus, we compute the 728set $P_1^{\star}(\lambda) \subseteq \partial \Omega_{\bar{\mathbf{x}}} \cap \Omega_{\bar{\mathbf{x}}}$ of optimal solutions of (5.1) for which the first linear cut is 729 active, and then the set $P_2^{\star}(\lambda) \subseteq \Omega_{\bar{\mathbf{x}}} \cap \partial \Omega_{\bar{\mathbf{x}}}$ of optimal solutions of (5.1) for which 730 the second linear cut is active. Finally, the optimal values of these problems are 731compared with the value of the local and nonglobal minimizer (if it exists) in order 732 to identify the set $P_X(\lambda)$ of optimal solutions of (5.1). At this point we are able to 733 compute $h_X^{\min}(\lambda), h_X^{\max}(\lambda)$ and update λ^{\min} and λ^{\max} accordingly. If for some λ we 734have that $\mathbf{z}_1(\lambda) \in P_X(\lambda)$ and $P_X(\lambda) \cap [P_1^{\star}(\lambda) \cup P_2^{\star}(\lambda)] \neq \emptyset$, i.e., problem (5.1) has 735an optimal solution in int(H) and (at least) one optimal solution outside H, then 736 $0 \in [h_X^{\min}(\lambda), h_X^{\max}(\lambda)]$ and Algorithm 2.1 stops. We illustrate all this on Example 3.3. 738

739 Example 5.1. We add a second linear cut obtained through the projection over 740 ∂H of the optimal solution of problem (3.9) with $\lambda_{\Omega_{\tilde{\mathbf{x}}}} = 0.726$ outside H. This leads 741 to a further improvement with $\lambda_{\Omega_{\tilde{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}} \approx 0.39$ and $p_{\Omega_{\tilde{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}}(\lambda_{\Omega_{\tilde{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}}) \approx -4.005$, which 742 almost closes the gap. In Figure 4 we show the two linear cuts and the two new optimal



FIG. 4. Two linear cuts and the two optimal solutions outside $H(x_3)$ and in $int(H)(z_3)$, denoted by \circ and \times , respectively.

solutions, one outside H and one belonging to int(H) (x_3 and z_3 , respectively). Again, we also report the previous pairs of optimal solutions in order to show the progress.

Now, assume that the returned bound is not exact. Also in this case $\bar{\mathbf{x}}$ and $\tilde{\mathbf{x}}$ can be 745746 locally adjusted. One can combine the techniques presented in section 4 and in the current section, by using a technique similar to the one described in the former section 747748to improve the pair of points $\bar{\mathbf{x}}$ and $\tilde{\mathbf{x}}$. In particular, at $\lambda_{\Omega_{\bar{\mathbf{x}}} \cap \Omega_{\bar{\mathbf{x}}}}$ we have one optimal solution of problem 5.1 belonging to int(H), namely the local and nonglobal optimal 749solution of problem (3.2) with $X = \mathbb{R}^n$, and at least another one outside H. We denote 750the latter by \mathbf{v} and we observe that at least one of the two linear cuts is active at this 751752point, i.e., either $\mathbf{v} \in \partial \Omega_{\tilde{\mathbf{x}}}$ or $\mathbf{v} \in \partial \Omega_{\tilde{\mathbf{x}}}$ (or both). Then, if only the first cut is active at \mathbf{v} , we update $\bar{\mathbf{x}}$ as follows: $\bar{\mathbf{x}}' = \prod_{\mathbf{A},\mathbf{a}}(\bar{\mathbf{x}} + \eta(\mathbf{v} - \bar{\mathbf{x}}))$ for a sufficiently small η value, while 753754 $\tilde{\mathbf{x}}' = \tilde{\mathbf{x}}$. If only the second cut is active, we update $\tilde{\mathbf{x}}$ as follows: $\tilde{\mathbf{x}}' = \prod_{\mathbf{A},\mathbf{a}} (\tilde{\mathbf{x}} + \eta(\mathbf{v} - \tilde{\mathbf{x}}))$ for a sufficiently small η value, while $\bar{\mathbf{x}}' = \bar{\mathbf{x}}$. Finally, if both are active we select one 755of the two cuts and perturb it. After the perturbation, we run again Algorithm 2.1 756with input $X = \Omega_{\mathbf{x}'} \cap \Omega_{\mathbf{x}'}$ and $\lambda_{\Omega_{\mathbf{x}} \cap \Omega_{\mathbf{x}}}$, and we repeat this procedure until there is a significant reduction of the bound. Note, however, that it might happen that no 757 758improvement is possible. In case $|P_1^{\star}(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\bar{\mathbf{x}}}})| = 1$ and $P_2^{\star}(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\bar{\mathbf{x}}}}) = \emptyset$ (similar 759for $|P_2^{\star}(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\bar{\mathbf{x}}}})| = 1$ and $P_1^{\star}(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\bar{\mathbf{x}}}}) = \emptyset$, then the proposed perturbation $\bar{\mathbf{x}}' =$ 760 $\Pi_{\mathbf{A},\mathbf{a}}(\bar{\mathbf{x}}+\eta(\mathbf{v}-\bar{\mathbf{x}}))$ for η sufficiently small allows improvement of the bound. Indeed, 761in such cases the local adjustment is able to cut the unique solution outside H away. 762In order to illustrate other different cases we employ Figures 5a–5c. As usual, in these 763 figures the point in int(H) is denoted by \times , while the others (outside H) are denoted 764765by \circ . If $|P_1^{\star}(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\bar{\mathbf{x}}}})| = |P_2^{\star}(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\bar{\mathbf{x}}}})| = 1$ and $P_1^{\star}(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\bar{\mathbf{x}}}}) \cap P_2^{\star}(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\bar{\mathbf{x}}}}) = \emptyset$ (see Figure 5a), or $|P_1^{\star}(\lambda_{\Omega_{\tilde{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}})| = 2, |P_2^{\star}(\lambda_{\Omega_{\tilde{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}})| = 1, \text{ and } \hat{P}_1^{\star}(\lambda_{\Omega_{\tilde{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}}) \cap P_2^{\star}(\lambda_{\Omega_{\tilde{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}}) \neq \emptyset$ 766(see Figure 5b), then it is not possible to remove all the solutions outside H by 767perturbing a single linear cut. Indeed, in both cases the perturbation of a single 768linear cut is able to remove just one of the two optimal solutions outside H. But it is 769 possible to remove both by perturbing both linear cuts. Instead, Figure 5c illustrates 770 771 a case where $|P_1^{\star}(\lambda_{\Omega_{\mathbf{\tilde{x}}}\cap\Omega_{\mathbf{\tilde{x}}}})| = |P_2^{\star}(\lambda_{\Omega_{\mathbf{\tilde{x}}}\cap\Omega_{\mathbf{\tilde{x}}}})| = 2$ and $P_1^{\star}(\lambda_{\Omega_{\mathbf{\tilde{x}}}\cap\Omega_{\mathbf{\tilde{x}}}}) \cap P_2^{\star}(\lambda_{\Omega_{\mathbf{\tilde{x}}}\cap\Omega_{\mathbf{\tilde{x}}}}) \neq \emptyset$. In this case even the perturbation of both linear cuts is unable to remove all three 772773solutions outside H. The only way to remove all three solutions outside H is through the addition of a further linear cut, but, of course, this leads to a more complex 774problem with one trust region constraint and three linear inequalities. 775

776 *Example* 5.2. In our example, this refinement is finally able to close the gap 777 and return the exact optimal value -4. In Figure 6 we report the result of the first perturbation of the linear cuts. Since only the second linear cut is active at x_3 , in 778this case the second linear cut is slightly perturbed and becomes equivalent to the 779tangent to H at the optimal solution $(-\sqrt{2}/2,\sqrt{2}/2)$ of the original problem (1.1). It 780is interesting to note that the new optimal solution outside H, indicated by x_4 , lies 781in a different region with respect to the previous ones and is further from ∂H with 782respect to x_2 and x_3 (the reduction of λ reduces the penalization of points outside 783H). Such a solution is cut by the new linear inequality, obtained by a (not so small) 784perturbation of the first linear cut, displayed in Figure 7, together with the two new 785optimal solutions $(x_5 \text{ and } z_5)$, now corresponding to the two optimal solutions of 786problem (1.1). 787

6. Computational experiments. In this section we report the computational
results for the proposed bounds over the set of hard instances selected from the random
ones generated in [12] and inspired by [18]. More precisely, in [12] 1000 random



(a) Three optimal solutions, none with both linear cuts active.



(b) Three optimal solutions, one with both linear cuts active.



(c) Four optimal solutions.





FIG. 6. Perturbation of the second linear cut and the two new optimal solutions outside $H(x_4)$ and in $int(H)(z_4)$, denoted by \circ and \times , respectively.



FIG. 7. Perturbation of the first linear cut and the two optimal solutions outside $H(x_5)$ and in int(H) (z_5), denoted by \circ and \times , respectively.

791 instances were generated for each size n = 5, 10, 20. Some of these instances have been 792 declared hard ones, namely those for which the bound obtained by adding SOC-RLT 793 constraints was not exact. In particular, these are 38 instances with n = 5, 70 instances 794 with n = 10, and 104 instances with n = 20. Such instances have been made available 795 in GAMS, AMPL, and COCOUNT formats in [19]. We tested our bounds on such instances.

- All tests have been performed on an Intel Core i7 running at 1.8 GHz with 16 GB ofRAM. All bounds have been coded in MATLAB.
- 799 We computed the following bounds:

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- LbDual, the dual Lagrangian bound computed through Algorithm 2.1 with input X = Rⁿ;
- LbOneCut, the bound obtained by adding a single linear cut and computed through Algorithm 2.1 with input $X = \Omega_{\bar{\mathbf{x}}}$;
 - LbOneAdj, the bound obtained by local adjustments of the added linear cut as indicated in Algorithm 4.1;
 - LbTwoCut, the bound obtained by adding two linear cuts;
 - LbTwoAdj, the bound obtained by adjusting the two linear cuts.

According to what was done in [3, 12, 25], an instance is considered to be 'solved' when the relative gap between the lower bound, say LB, and the upper bound, say UB, is not larger than 10^{-4} , i.e.,

$$\frac{UB - LB}{|UB|} \le 10^{-4}.$$

We set UB equal to the lowest value obtained by running, after the addition of the 811 812 first linear cut, two local searches for the original problem (1.1), one from the optimal 813 solution $\mathbf{z}_1(\lambda_{\Omega_{\bar{\boldsymbol{z}}}}) \in int(H)$ of (3.9) returned at the end of Algorithm 2.1, and the other from an optimal solution of the same problem outside H. In Tables 2–4 we report the 814 815 average and maximum relative gaps for each bound, and the average and maximum 816 computing times for n = 5, 10, 20, respectively. In the last line of the tables we also report the same values for the SOC-RLT bound presented in [12], computed by Mosek.¹ 817 Note that the average gap is taken only over the instances which were *not* solved (in 818 the sense specified above) by a given bound. Moreover, the average computing time 819 820 for bound LbTwoAdj is computed only over the instances (87 overall, as we will see) which are *not* solved by bound LbTwoCut. 821

We remark that the bound LbTwoCut is computed by adding the first cut as in bound LbOneCut, i.e., the supporting hyperplane at $\bar{\mathbf{x}} \in \partial H$, and then adding a further linear cut through the projection of an optimal solution outside H obtained when computing bound LbOneCut, i.e., point $\mathbf{z}_2(\lambda_{\min})$ returned by procedure DualLagrangian with input $X = \Omega_{\bar{\mathbf{x}}}$. We could as well choose the adjusted cut computed by bound LbOneAdj as the first cut for bound LbTwoCut, but we observed that with this choice no improvement over LbOneAdj is obtained. This is related to what already observed

TABLE 2

Average and maximum relative gaps and computing times (in seconds) for the instances with n = 5.

Bound	Average relative gap $(\%)$	Max relative gap $(\%)$	Average time	Max time
LbDual	0.90~%	2.97~%	0.013	0.015
LbOneCut	0.31~%	1.27~%	0.035	0.040
LbOneAdj	0.130~%	0.548~%	0.266	0.388
LbTwoCut	0.07~%	0.21~%	0.089	0.108
LbTwoAdj	0 %	0 %	0.146	0.281
SOC-RLT	0.131~%	0.548~%	1.435	2.080

¹The authors are very grateful to Professor Samuel Burer for providing the MATLAB code for the computation of the SOC-RLT bound.

TABLE 3

Average and maximum relative gaps and computing times (in seconds) for the instances with n = 10.

	A 1.41 (07)	M 1 (07)	A	
Bound	Average relative gap (%)	Max relative gap $(\%)$	Average time	Max time
LbDual	0.41~%	1.57~%	0.014	0.022
LbOneCut	0.14 %	0.81~%	0.039	0.057
LbOneAdj	0.073~%	0.478~%	0.339	0.574
LbTwoCut	0.05~%	0.24~%	0.101	0.173
LbTwoAdj	0 %	0 %	0.197	0.670
SOC-RLT	0.073~%	0.474%	1.228	2.898

TABLE 4

Average and maximum relative gaps and computing times (in seconds) for the instances with n = 20.

Bound	Average relative gap $(\%)$	Max relative gap $(\%)$	Average time	Max time
LbDual	0.20~%	0.59~%	0.019	0.027
LbOneCut	0.08~%	0.29~%	0.057	0.079
LbOneAdj	0.054~%	0.166~%	0.539	0.926
LbTwoCut	0.03~%	0.09~%	0.148	0.199
LbTwoAdj	0.05~%	0.05~%	0.350	1.574
SOC-RLT	0.053~%	0.166%	2.266	3.983

in Figure 3: bound LbOneAdj cannot be improved any more when there are (at least) two optimal solutions outside H (besides the one in int(H)). Thus, the second cut is able to remove one of such optimal solutions but not the other, so that the bound cannot be improved. Similarly, for bound LbTwoAdj the two initial cuts are the ones computed for bound LbTwoCut.

834 For what concerns the computing times, we observe that these are lower than those reported in [25] for the bound obtained by adding lifted-RLT cuts (around 835 92s for an instance with n = 20). They are also lower than those reported in [3] 836 837 for the bound obtained by adding KSOC cuts (up to 2s for n = 20 instances). For 838 the sake of correctness, we point out that the computing times reported in those papers have been obtained with different processors. However, such processors have 839 comparable performance with respect to the one employed for the computational 840 experiments in this paper. In general, the proposed bounds are very cheap. Only 841 842 for two instances with n = 20, LbTwoAdj required times above 1s (around 1.5s in both cases). Usually the computing times are (largely) below 1s. Both the dual 843 844 Lagrangian bound and the bound obtained by a single linear cut are pretty cheap but with poorer performance in terms of relative gap. The bound obtained by Algorithm 845 4.1 with a local adjustment of the linear cut is better than the two previous ones in 846 847 terms of gap but is also more expensive (although still cheap). The bound LbTwoCut 848 offers a good combination between quality and cheap computing time. But a more careful choice of the two linear cuts, through a local adjustment, improves the quality 849 850 without compromising the computing times. This is confirmed by the results reported 851 for LbTwoAdj. Although this bound is more expensive than the others, the additional search for adjusted linear cuts further increases the quality of the bound. In Table 5 852853 we report the number of solved instances for LbTwoCut and LbTwoAdj. According to what was reported in [3], the total number of unsolved instances out of the 212 hard 854

TABLE 5 Number of solved instances for the bounds LbTwoCut and LbTwoAdj.

Bound	n = 5 (out of 38)	n = 10 (out of 70)	n = 20 (out of 104)
LbTwoCut	14	41	70
LbTwoAdj	38	70	103

TABLE 6 Minimum, average, and maximum PercDiff values, where PercDiff is defined in (6.1).

\overline{n}	Minimum	Average	Maximum
5	-0.0020%	0.0009~%	0.0058~%
10	-0.0046~%	-0.0001~%	0.0024~%
20	-0.0022~%	-0.0003~%	0.0004~%

855 instances is equal to the following: 133 for the bound proposed in [25] (18 with n=5, 49 with n = 10, and 66 with n = 20; 85 for the bound proposed in [3] (18 with n = 5, 856 22 with n = 10, and 45 with n = 20; 56 by considering the best bound between the 857 one in [25] and the one in [3] (10 with n = 5, 15 with n = 10, and 31 with n = 20). 858 For bound LbTwoCut the total number of unsolved instances reduces to 87 (24, 29, 859 860 and 34 for n = 5, n = 10, and n = 20, respectively). Finally, for bound LbTwoAdj we have the remarkable outcome that there is just one unsolved instance (namely, 861 instance 20.628). For the sake of correctness, we should warn that the value UB in [3, 862 25] is not computed by running two local searches as done in this paper. It is instead 863 computed from the final solution of the relaxed problem, so that it could be slightly 864 865 worse and justify the larger number of unsolved instances. All the same, the quality of the proposed bounds appears to be quite good. 866

We still need to compare our bounds with the SOC-RLT bound (last line in Ta-867 bles 2–4). In terms of computing times we notice that both the average and the 868 869 maximum computing times of the SOC-RLT bound are larger than those of all the pro-870 posed bounds. But we believe that the most interesting observation is that, in terms 871 of average and maximum gap, the SOC-RLT bound is almost identical to the LbOneAdj 872 bound. In order to better investigate the relation between the two bounds, in Table 6 we report the minimum, average, and maximum percentages difference between the 873 874 two bounds, i.e., the quantity

(6.1)
$$PercDiff = 100 * \frac{LbOneAdj - SOC - RLT}{|LbOneAdj|} \%.$$

We notice that the difference is sometimes positive and sometimes negative, suggesting that none of the two bounds dominate the other. But the differences are also so small (below the tolerance value under which an instance is declared to be 'solved' by a given bound) that they could also be numerical differences due to the tolerance values employed in the solvers. We believe that an interesting question for future research is to establish whether these two bounds are, in fact, equivalent, which would lead to a new interpretation of the SOC-RLT bound proposed in [12].

6.1. Investigating the hardest instance. As a final experiment, we investigate the behavior of bound LbTwoAdj over the hardest instance with n = 20, the one for which the relative error is above 10^{-4} . For this instance, at the last iteration we recorded the following objective function values, corresponding to values of local minimizers of problem (5.1), which certainly include the global minimizer(s) of such a problem:

- the value at the optimal solution of problem (5.1) belonging to int(H);
- the value at a globally optimal solution of the trust region problem obtained by fixing in problem (5.1) the first linear cut to an equality, in case such solution fulfills the second linear cut, or, alternatively, the value at the local and nonglobal solution of the same problem, in case such solution exists and fulfills the second linear cut (if the global minimizer does not fulfill the second linear cut and the local and nonglobal minimizer does not exist or does not fulfill the second linear cut, then the value is left undefined);
 - the same value as above but after fixing the second linear cut to an equality in problem (5.1);
 - the value at a globally optimal solution of the trust region problem obtained by fixing both cuts to equalities in problem (5.1).

901 Note that two of the four values must be equal. In particular, one of the two 902 equal values is always the first one, attained in int(H). But for the hardest instance 903 we observed that all four values are very close to each other and all of them are lower 904 than the UB value. Thus, it appears that for this instance a situation like the one 905 displayed in Figure 5c occurs. In this case even the perturbation of both linear cuts 906 is unable to remove all of the three solutions outside H.

7. Conclusions. In this paper we discussed the CDT problem. First, we derived 907 908 some theoretical results for a class of problems which includes the CDT problem as a special case. Then, from the theory developed for such class, we have rederived a 909 necessary and sufficient condition for the exactness of the Shor relaxation and of the 910 911equivalent dual Lagrangian bound for the CDT problem. The condition is based on 912 the existence of multiple solutions for a Lagrangian relaxation. Based on such con-913 dition, we proposed to strengthen the dual Lagrangian bound by adding one or two 914 linear cuts. These cuts are based on supporting hyperplanes of one of the two quadratic constraints, and they are, thus, redundant for the original CDT problem (1.1). 915 916 However, the cuts are not redundant for the Lagrangian relaxation and their addition 917 allows one to improve the bound. We ran different computational experiments over 918 the 212 hard test instances selected from the three thousand ones randomly generated in [12], reporting gaps and computing times. We have shown that the bounds 919 920 are computationally cheap and are quite effective. In particular, one of them, based on the addition of two linear cuts, is able to solve all but one of the hard instances. 921 922 We have also investigated more in detail such hardest instance for which the bound 923 is not exact (though quite close to the optimal value). An interesting topic for future research could be that of establishing the relations between the bounds proposed 924925 in this work and those presented in the recent literature (in particular, as already 926 mentioned, it would be interesting to establish whether bound LbOneAdj is equivalent 927 to the SOC-RLT bound introduced in [12]). Moreover, it would also be interesting to 928 develop procedures which are able to generate CDT instances for which the bound LbTwoAdj is unable to return the optimal value. Finally, it would be interesting to 929 see if the results presented in this work could be extended to QP problems with more AQ5 930 than two constraints. Some preliminary studies, which will appear elsewhere, show 931that for such problems it is sometimes possible to improve the dual Lagrangian bound 932 933 with the addition of a linear cut, but it may be hard to identify it and it is not even 934 guaranteed to exist.

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